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Lie Groups and Lie Algebras III

Structure of Lie Groups
and Lie Algebras

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Encyclopaedia of Mathematical Sciences

Volume 41

Editor-in-Chief: R.V. Gamkrelidze

Structure of Lie Groups and Lie Algebras

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Translated from the Russian
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Introduction

This article builds on Vinberg and Onishchik [1988] and is devoted to an exposition of the main results on the structure of Lie groups and finite-dimensional Lie algebras. The greater part of the article is concerned with theorems on the structure and classification of semisimple Lie groups (algebras) and their subgroups (subalgebras). The tables given at the end of the article can be used as reference material in any work on Lie groups.

We consider only the results of the classical theory of Lie groups. Some classes of infinite-dimensional Lie groups and Lie algebras, as well as Lie supergroups and superalgebras, will be dealt with in special articles of one of the following volumes of this series. The same applies to the theory of Lie algebras over fields of finite characteristic. However, the results on Lie algebras given in the present article can be extended to more general fields of characteristic 0 (e.g., the field $\mathbb{C}$ of complex numbers can be replaced by any algebraically closed field of characteristic 0).

For the theory of linear representations of Lie groups and algebras, the reader is referred to the volumes especially devoted to this theory, although we had to include in this article some classical theorems on finite-dimensional representations, which form an inseparable part of the structural theory. We also use some results from the theory of algebraic groups. Almost all of them can be found in Springer [1989], and some in Chap. 1, Sect. 6. On the other hand, the results on complex and real algebraic groups contained in Springer [1989] can be treated as results on Lie groups. Some of them (e.g. the Bruhat decomposition) are not dealt with in this volume.

The authors have tried, whenever possible, to give the reader the ideas of the proofs.

The terminology and notation of the article follow that of Vinberg and Onishchik [1988]. In particular, Lie groups are denoted by upper-case Roman letters, and their tangent algebras by lower-case Gothic.

Chapter 1
General Theorems

All vector spaces and Lie algebras considered in this chapter are assumed to be finite-dimensional. The ground field is denoted by K, which is either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers.

§ 1. Lie's and Engel's Theorems

1.1. Lie's Theorem. Denote by $T_n(K)$ the subgroup of $\mathrm{GL}_n(K)$ consisting of all nondegenerate upper triangular matrices, and by $\mathfrak{t}_n(K)$ the subalgebra of the Lie algebra $\mathfrak{gl}_n(K)$ consisting of all triangular matrices. The group $T_n(K)$ (respectively, Lie algebra $\mathfrak{t}_n(K)$) can be interpreted as a subgroup of the full linear group $\mathrm{GL}(V)$ (respectively, subalgebra of the full linear algebra $\mathfrak{gl}(V)$), where V is an n-dimensional vector space over K consisting of operators preserving some full flag, i.e. a set of subspaces $V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V$, where $\dim V_i = i$. The group $T_n(K)$ and the Lie algebra $\mathfrak{t}_n(K)$ are solvable (see Vinberg and Onishchik [1988], Chap. 2, Sect. 5.5). The following theorem, first proved by Sophus Lie, shows that the subgroup $T_n(\mathbb{C})$ (subalgebra $\mathfrak{t}_n(\mathbb{C})$) is, up to conjugation, the only maximal connected solvable Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ (respectively, maximal solvable subalgebra of $\mathfrak{gl}_n(\mathbb{C})$).

Theorem 1.1 (see Bourbaki [1975], Jacobson [1962]). (1) *Let $R: G \to \mathrm{GL}(V)$ be a complex linear representation of a connected solvable Lie group G. Then there is a full flag in V invariant under $R(G)$.*

(2) *Let $\mathfrak{g}$ be a solvable Lie algebra, and $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ a complex linear representation of it. Then there is a full flag in V invariant under $\rho(\mathfrak{g})$.*

Because of the correspondence between solvable Lie groups and Lie algebras (see Vinberg and Onishchik [1988], Chap. 2, Sect. 5.5), statements (1) and (2) of the theorem are equivalent. We now give an outline of the proof of statement (1).

We start with some definitions and simple auxiliary statements.

Let $R: G \to \mathrm{GL}(V)$ be a linear representation of a group G over an arbitrary field K. For any character χ of the group G, i.e. a homomorphism $\chi: G \to K^\times$, where $K^\times$ is the multiplicative group of the field K, we set

$$V_\chi = V_\chi(G) = \{v \in V | R(g)v = \chi(g)v \quad \text{for all} \quad g \in G\}.$$

If $V_\chi \neq 0$, then the character χ is said to be a *weight* of the representation R, the subspace V_χ is called the *weight subspace*, and its nonzero vectors the *weight vectors* corresponding to the weight χ. Similarly, for any linear representation ρ of the Lie algebra $\mathfrak{g}$ over the field K and any linear form

$\lambda \in \mathfrak{g}^*$ let

$$V_\lambda(\mathfrak{g}) = \{v \in V | \rho(x)v = \lambda(x)v \quad \text{for all} \quad x \in \mathfrak{g}\}.$$

If $V_\lambda(\mathfrak{g}) \neq 0$, then the form λ is said to be a *weight* of the representation ρ, the subspace $V_\lambda(\mathfrak{g})$ is called the *weight subspace*, and its nonzero vectors the *weight vectors* corresponding to the weight λ.

Weight subspaces corresponding to different weights are linearly independent. Thus a finite-dimensional linear representation may have only finitely many weights.

The proof of Lie's theorem is based on the following property of weight subspaces.

Lemma 1.1. *Let H be a normal subgroup of the group G, χ the character of H, and $R: G \to \mathrm{GL}(V)$ a linear representation. Then for any $g \in G$ we have*

$$R(g)V_\chi(H) = V_{\chi^g}(H),$$

where $\chi^g(h) = \chi(g^{-1}hg)$ $(h \in H)$.

Outline of the proof of Theorem 1.1. First, one shows by induction on $\dim G$ that R has at least one weight in V. For $\dim G = 1$ the statement is evident. In the general case, the definition of a solvable Lie group implies that there is a virtual normal Lie subgroup H of G of codimension 1. Clearly, $G = CH$, where C is a connected virtual one-dimensional Lie subgroup. By the inductive hypothesis, $V_\chi(H) \neq 0$ for some character χ of the group H. In view of Lemma 1.1, the operators $R(g)$, $g \in G$, permute the weight subspaces of the group H. Since G is connected, $V_\chi(H)$ is invariant under $R(G)$.

The one-dimensional subgroup C has a one-dimensional invariant subspace in $V_\chi(H)$, which is evidently invariant under the action of the entire group G.

Thus, there is a one-dimensional subspace in V invariant under G. The existence of a full flag in V invariant under G is then proved by induction on $\dim V$. □

Corollary 1. *Any irreducible complex linear representation of a connected solvable Lie group or a solvable Lie algebra is one-dimensional.*

Corollary 2. *Let $G \subset \mathrm{GL}(V)$ be a connected irreducible complex linear Lie group. Then either G is semisimple, or* $\mathrm{Rad}\, G = \{cE | c \in \mathbb{C}^\times\}$.

Proof. Suppose that G is not semisimple. Consider the vector subspace $W = V_\chi(\mathrm{rad}\, G) \neq 0$. Lemma 1.1 implies that it is invariant under G. Hence $W = V$, i.e. $\mathrm{Rad}\, G$ contains scalar operators only. □

Corollary 3. *A Lie algebra $\mathfrak{g}$ over $K = \mathbb{C}$ or $\mathbb{R}$ is solvable if and only if the Lie algebra $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}_n(K)$ is nilpotent.*

Proof. If $\mathfrak{g} = \mathfrak{t}_n(K)$, then $[\mathfrak{g}, \mathfrak{g}]$ is the nilpotent Lie algebra of all upper diagonal matrices with zeros on the diagonal. In the general case one can assume, using the complexification procedure if necessary, that $K = \mathbb{C}$. We now see, by Lie's theorem, that if $\mathfrak{g}$ is solvable, then the Lie algebra $\operatorname{ad}[\mathfrak{g},\mathfrak{g}] = [\operatorname{ad}\mathfrak{g}, \operatorname{ad}\mathfrak{g}]$ is nilpotent and therefore $\mathfrak{g}$ is also nilpotent. □

1.2. Generalizations of Lie's Theorem. First we consider the possibilities of generalizing Lie's theorem to Lie algebras over an arbitrary field K. If a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{g}$ over K has an invariant full flag, then the characteristic numbers of all operators $\rho(x)$, $x \in \mathfrak{g}$, must belong to the field K, which is far from being always true if K is not algebraically closed. If $\operatorname{char} K = 0$, then the above mentioned property of the operators $\rho(x)$, $x \in \mathfrak{g}$, turns out to be also sufficient for the existence of an invariant flag.

Theorem 1.2. *Let $\mathfrak{g}$ be a solvable Lie algebra over a field K of characteristic 0 and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a linear representation of it over K. If all characteristic numbers of all operators $\rho(x)$, $x \in \mathfrak{g}$, belong to K, then there is a full flag in V invariant under $\rho(\mathfrak{g})$.*

The proof is similar to that of Theorem 1.1, and makes use of the following analogue of Lemma 1.1.

Lemma 1.2. *Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of a Lie algebra $\mathfrak{g}$ over a field K of characteristic 0, $\mathfrak{h}$ an ideal in $\mathfrak{g}$, and $V_\lambda(\mathfrak{h})$ a weight subspace of the representation $\rho|\mathfrak{h}$. Then the following two equivalent statements hold: (1) $V_\lambda(\mathfrak{h})$ is invariant under $\rho(\mathfrak{g})$; (2) $\lambda(z) = 0$ for any $z \in [\mathfrak{g},\mathfrak{h}]$.*

Corollary 3 to Theorem 1.1 is extended to the case of an arbitrary field of characteristic 0. If a field of characteristic 0 is algebraically closed, then the analogues of Corollaries 1 and 2 hold.

The condition imposed by Theorem 1.2 on the characteristic is essential, as the following example shows.

Example. If $\operatorname{char} K = 2$, then the Lie algebra $\mathfrak{gl}_2(K)$ is solvable, but its identity representation in K^2 has no weight vectors.

Without going into details, we note that Lie's theorem can be extended to connected solvable linear algebraic groups over an algebraically closed field of arbitrary characteristic. This follows from Borel's fixed point theorem (see Springer [1989], Chap. 1, Sect. 3.5). We also state the following simple theorem on representations of abstract solvable groups.

Theorem 1.3 (see Merzlyakov [1987]). *Let G be a solvable group, and $R\colon G \to \mathrm{GL}(V)$ a complex linear representation of it. Then there is a full flag in V invariant under a subgroup of finite index $G_1 \subset G$.*

Proof. Consider the algebraic closure $H = {}^aR(G)$ of the subgroup $R(G)$ of $\mathrm{GL}(V)$. The solvable linear algebraic group H has a finite number of connected components. According to Theorem 1.1, there is a full flag in V invariant under H^0. But then it is also invariant under the subgroup $G_1 = R^{-1}(H^0)$, which is of finite index in G. □

In addition to the main statement of Theorem 1.3 one can also show that the subgroup G_1 can be chosen in such a way that its index does not exceed a number depending on $\dim V$ only (see Merzlyakov [1987]).

1.3. Engel's Theorem and Corollaries to It. The cornerstone in the theory of nilpotent Lie algebras and Lie groups is the following theorem first proved by F. Engel.

Theorem 1.4 (see Bourbaki [1975], Jacobson [1955]). *Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of a Lie algebra $\mathfrak{g}$ over an arbitrary field K. Suppose that for each $x \in \mathfrak{g}$ the linear operator $\rho(x)$ is nilpotent. Then there is a basis in V with respect to which the operators $\rho(x)$, $x \in \mathfrak{g}$, are represented by upper triangular matrices with zeros on the diagonal. In particular, the Lie algebra $\rho(\mathfrak{g})$ is nilpotent.*

Proof. As for Lie's theorem, induction on $\dim V$ reduces the theorem to the proof of the existence of a weight vector (with the weight 0). The latter is achieved by induction on $\dim \mathfrak{g}$. For $\dim \mathfrak{g} = 1$ the statement is evident. Suppose that the statement holds for all Lie algebras of dimension less than m, and let $\dim \mathfrak{g} = m$. It follows from the statement of the theorem and the inductive hypothesis that there is an ideal $\mathfrak{h}$ of codimension 1 in $\mathfrak{g}$ (one can take for $\mathfrak{h}$ any maximal subalgebra of $\mathfrak{g}$). Then $\mathfrak{g} = \mathfrak{h} + \langle y \rangle$, where $y \in \mathfrak{g}$. Consider the weight subspace $V_0(\mathfrak{h}) \neq 0$. Since $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, Lemma 1.2 implies that $V_0(\mathfrak{h})$ is invariant under $\mathfrak{g}$. The operator $\rho(y)$ is nilpotent, whence there is a vector $v_0 \in V_0(\mathfrak{h})$, $v_0 \neq 0$, such that $\rho(y)v_0 = 0$. Evidently, v_0 is the desired weight vector with respect to $\mathfrak{g}$. $\square$

Corollary 1. *If under the conditions of Theorem* 1.4 *the representation ρ is irreducible, then it is trivial and one-dimensional.*

An application of Engel's theorem to the adjoint representation easily yields the following corollary.

Corollary 2. *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if either of the following two conditions is satisfied:*

(1) *For any $x \in \mathfrak{g}$ the operator* $\operatorname{ad} x$ *is nilpotent.*
(2) *There is a basis $\{e_i\}$ in $\mathfrak{g}$ such that $[e_i, e_j]$ is a linear combination of the elements $e_k, e_{k+1}, \dots, e_m$, where $k = \max(i,j) + 1$.*

A Lie algebra $\mathfrak{g}$ is said to be *engelian* if all the operators $\operatorname{ad} x$, $x \in \mathfrak{g}$, are nilpotent. Corollary 2 implies that a finite-dimensional Lie algebra is engelian if and only if it is nilpotent. For an infinite-dimensional Lie algebra this statement does not hold, in general. If, however, $\mathfrak{g}$ is finitely generated and $(\operatorname{ad} x)^k = 0$ for some $k \in \mathbb{N}$ and all $x \in \mathfrak{g}$, then $\mathfrak{g}$ is nilpotent.

We also note that a stronger version of Engel's theorem is also valid, namely its conclusion holds for linear representations ρ of a Lie algebra $\mathfrak{g}$ such that $\rho(\mathfrak{g})$ is generated (as a Lie algebra) by a set of nilpotent operators closed under the commutator.

The next theorem lists other important properties of nilpotent Lie algebras proved with the use of Engel's theorem.

Theorem 1.5 (see Bourbaki [1975], Jacobson [1955], Serre [1987]). *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then*

(i) $\operatorname{codim}[\mathfrak{g}, \mathfrak{g}] \geq 2$.
(ii) *If $\mathfrak{a}$ is a subspace in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} + [\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{a}$ generates $\mathfrak{g}$ as a Lie algebra.*
(iii) *If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then $\mathfrak{h} \cap \mathfrak{z}(\mathfrak{g}) \neq 0$.*
(iv) *If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then its normalizer $\mathfrak{n}(\mathfrak{h})$ strictly contains $\mathfrak{h}$.*

Finally, we note the following application of Engel's theorem to the theory of nilpotent Lie groups.

Theorem 1.6. *A connected Lie group G is nilpotent if and only if all operators $\operatorname{Ad} g$ ($g \in G$) are unipotent. Any compact subgroup of a connected nilpotent Lie group G is contained in $Z(G)$.*

Proof. The first statement follows from Corollary 2 to Theorem 1.4 and the correspondence between nilpotent Lie groups and Lie algebras (see Vinberg and Onishchik [1988], Chap. 2, Theorem 5.13). To prove the second statement, consider the restriction R of the representation Ad to a compact subgroup $L \subset G$. Since R is completely reducible (see below Chap. 4, Corollary to Proposition 2.1), Corollary 1 to Theorem 1.4 implies that R is trivial. Hence $L \subset \operatorname{Ker} \operatorname{Ad} = Z(G)$.

1.4. An Analogue of Engel's Theorem in Group Theory. The following theorem can be considered as a group-theoretical analogue of Engel's theorem. It is not a formal consequence of Engel's theorem because it applies to groups that are not necessarily Lie groups.

Theorem 1.7 (Kolchin, see Merzlyakov [1987], Serre [1987]). *Let G be a group, and $R: G \to \mathrm{GL}(V)$ a linear representation of it over a field K. Suppose that $V \neq 0$ and all operators $R(g)$, $g \in G$, are unipotent. Then $\chi \equiv 1$ is a weight of the representation R.*

Proof. Consider the system of linear equations $(R(g) - E)v = 0$, where g runs over the entire group G. Since we are looking for nontrivial solutions of the system, the field K can be assumed to be algebraically closed. Replacing V by its minimal nonzero invariant subspace, one can also assume that R is irreducible. The Burnside theorem (see Kirillov [1987]) implies that the operators $R(g)$, $g \in G$, generate $\mathfrak{gl}(V)$ as a vector space.

On the other hand, let $Z = R(g) - E$. Then $\operatorname{tr} R(g) = \operatorname{tr} E + \operatorname{tr} Z = \dim \mathfrak{g}$ does not depend on $g \in G$. If $g,\ g' \in G$, then

$$\operatorname{tr}(ZR(g')) = \operatorname{tr}((R(g) - E)R(g')) = \operatorname{tr} R(gg') - \operatorname{tr} R(g') = 0.$$

Hence $\operatorname{tr}(ZX) = 0$ for any $X \in \mathfrak{gl}(V)$, whence $Z = 0$, i.e. $\rho(g) = E$. □

§ 2. The Cartan Criterion

2.1. Invariant Bilinear Forms. Let G be a Lie group over a field K. A bilinear form b on the tangent algebra $\mathfrak{g}$ of the group G is said to be *invariant* if

$$b((\operatorname{Ad} g)x,\ (\operatorname{Ad} g)y) = b(x,y) \tag{1}$$

for all $g \in G$, $x, y \in \mathfrak{g}$. It follows from formula (18) in Vinberg and Onishchik [1988], Chap. 2 that the invariant form b satisfies the relation

$$b([x,y],z) + b(y,[x,z]) = 0 \tag{2}$$

for all $x, y, z \in \mathfrak{g}$. Conversely, relation (2) implies (1) if G is connected. A bilinear form b on an arbitrary Lie algebra $\mathfrak{g}$ is said to be *invariant* if it satisfies property (2).

Example 1. Let E be a three-dimensional Euclidean space with the scalar product $(\ ,\ )$. Fix an orientation in E and consider the vector product in E. Then E becomes a Lie algebra over $\mathbb{R}$ such that the form $(\ ,\ )$ is invariant.

Example 2. In the Lie algebra $\mathfrak{gl}(V)$ of linear transformations of a vector space V over K there is an invariant bilinear form

$$b(X,Y) = \operatorname{tr}(XY). \tag{3}$$

Example 3. Let $\mathfrak{g}$ be a Lie algebra over K, and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a linear representation of it. Then the symmetric bilinear form

$$b_\rho(x,y) = \operatorname{tr}(\rho(x)\rho(y))$$

is invariant on $\mathfrak{g}$. In particular, there is an invariant bilinear form

$$k_{\mathfrak{g}}(x,y) = b_{\mathrm{ad}}(x,y) = \operatorname{tr}((\operatorname{ad} x)(\operatorname{ad} y)),$$

called the *Killing form* of the algebra $\mathfrak{g}$.

In what follows we always assume that an invariant bilinear form b on a Lie algebra $\mathfrak{g}$ is symmetric. The following assertions are proved without difficulty.

Proposition 2.1. *Let b be an invariant bilinear form on a Lie algebra $\mathfrak{g}$ and $\mathfrak{a}$ an ideal in $\mathfrak{g}$. Then $\mathfrak{a}^\perp = \{x \in \mathfrak{g} \mid b(x,y) = 0 \quad \forall y \in \mathfrak{a}\}$ is also an ideal in $\mathfrak{g}$. If $\mathfrak{a} = [\mathfrak{g},\mathfrak{g}]$, then $\mathfrak{a}^\perp \supset \mathfrak{z}(\mathfrak{g})$, and if b is nondegenerate, then $\mathfrak{a}^\perp = \mathfrak{z}(\mathfrak{g})$.*

Proposition 2.2 *The Killing form $k = k_{\mathfrak{g}}$ of any Lie algebra $\mathfrak{g}$ satisfies the relation*

$$k(a(x), a(y)) = k(x,y)$$

for all $x, y \in \mathfrak{g}$ and any $a \in \operatorname{Aut} \mathfrak{g}$. If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, then the restriction of the form $k_{\mathfrak{g}}$ to $\mathfrak{a}$ coincides with $k_{\mathfrak{a}}$.

2.2. Criteria of Solvability and Semisimplicity. In this section we denote by b the invariant bilinear form in $\mathfrak{gl}(V)$ defined by formula (3).

Theorem 2.1. *A subalgebra* $\mathfrak{g} \subset \mathfrak{gl}(V)$ *is solvable if and only if* $b([X,Y],Z) = 0$ *for all* $X, Y, Z \in \mathfrak{g}$.

Proof. For the proof of Theorem 2.1 one can assume that $K = \mathbb{C}$ (the real case is reduced to the complex one by considering the complexification, i.e. the Lie algebra $\mathfrak{g}(\mathbb{C}) = \mathfrak{g} + i\mathfrak{g} \subset \mathfrak{gl}(V(\mathbb{C}))$ (see Sect. 7)). For any $X \in \mathfrak{gl}(V)$ denote by X_s and X_n the semisimple and nilpotent components respectively in the additive Jordan decomposition $X = X_s + X_n$, (see Springer [1989], Sect. 3.1.1). Denote by $\overline{X}_s$ the semisimple operator having the same eigenvectors as X_s but with the complex conjugate eigenvalues. Let $b([X,Y],Z) = 0$ for all $X, Y, Z \in \mathfrak{g}$. By virtue of Engel's theorem, it is sufficient to prove that $X_s = 0$ for any $X \in [\mathfrak{g}, \mathfrak{g}]$. Write $X = \sum_{i=1}^{p} [X_i, Y_i]$, where $X_i, Y_i \in \mathfrak{g}$. Then

$$b(X, \overline{X}_s) = \sum_{i=1}^{p} b([X_i, Y_i], \overline{X}_s) = \sum_{i=1}^{p} b(Y_i, [\overline{X}_s, X_i]).$$

The relation $(\operatorname{ad} X)(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$ and the equality $\operatorname{ad} \overline{X}_s = \overline{(\operatorname{ad} X)}_s$ imply that $(\operatorname{ad} \overline{X}_s)(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}]$. Hence $b(X, \overline{X}_s) = \operatorname{tr}(X\overline{X}_s) = 0$, whence $X_s = 0$. The converse statement easily follows from Lie's theorem. □

Corollary. *A Lie algebra* $\mathfrak{g}$ *is solvable if and only if* $k_{\mathfrak{g}}([x,y],z) = 0$ *for all* $x, y, z \in \mathfrak{g}$ *or if* $k_{\mathfrak{g}}(x,y) = 0$ *for all* $x, y \in [\mathfrak{g}, \mathfrak{g}]$.

Theorem 2.2. *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if the Killing form* $k_{\mathfrak{g}}$ *is nondegenerate.*

Proof. Let $\mathfrak{u} = \{x \in \mathfrak{g} | k_{\mathfrak{g}}(x,y) = 0 \quad \forall y \in \mathfrak{g}\}$. By virtue of Proposition 2.1, $\mathfrak{u}$ is an ideal in $\mathfrak{g}$, while Theorem 2.1 implies that the Lie algebra $\operatorname{ad} \mathfrak{u}$ is solvable. Since $\operatorname{ad} \mathfrak{u} \simeq \mathfrak{u}$, we have $\mathfrak{u} = 0$ if $\mathfrak{g}$ is semisimple. Conversely, if $\mathfrak{g}$ is not semisimple, and $\mathfrak{a}$ is its nonzero abelian ideal, then $\mathfrak{a} \subset \mathfrak{u}$ because $((\operatorname{ad} x)(\operatorname{ad} y))^2 = 0$ for all $x \in \mathfrak{a}$, $y \in \mathfrak{g}$. □

Remark. A similar proof yields the following assertion: if ρ is a faithful linear representation of a semisimple Lie algebra $\mathfrak{g}$, then the form b_ρ (see Example 3) is nondegenerate on $\mathfrak{g}$.

Corollary. *If* $\mathfrak{g}$ *is semisimple, then* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

2.3. Factorization into Simple Factors

Proposition 2.3. *If* $\mathfrak{g}$ *is semisimple, and* $\mathfrak{a}$ *is an ideal in* $\mathfrak{g}$, *then* $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{a}^\perp$. *Any ideal* $\mathfrak{a} \subset \mathfrak{g}$ *and the quotient algebra* $\mathfrak{g}/\mathfrak{a}$ *are semisimple.*

Proof. As in the proof of Theorem 2.2, one can verify that $\mathfrak{a} \cap \mathfrak{a}^\perp$ is a solvable ideal in $\mathfrak{g}$. □

The following theorem is now derived without difficulty.

Theorem 2.3. *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if* $\mathfrak{g}$ *can be decomposed into the direct sum*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s, \tag{4}$$

where $\mathfrak{g}_i$ *are simple noncommutative ideals. Any ideal of the algebra* $\mathfrak{g}$ *is a sum of some of the ideals* $\mathfrak{g}_i$. *In particular, the decomposition* (4) *into simple ideals is unique.*

We now restate Theorem 2.3 in the language of Lie groups. A Lie group G is said to be decomposable into the *locally direct product* of normal subgroups $G_1, \ldots, G_s$ if $G = G_1 G_2 \ldots G_s$ and all intersections $G_i \cap (G_1 \ldots G_{i-1} G_{i+1} \ldots G_s)$ $(i = l, \ldots, s)$ are discrete.

Theorem 2.4. *A connected Lie group* G *is semisimple if and only if it is decomposable into the locally direct product of connected simple normal Lie subgroups:* $G = G_1 \ldots G_s$. *Given such a decomposition, any connected normal Lie subgroup of* G *is the product of some of the subgroups* G_i.

§ 3. Complete Reducibility of Representations and Triviality of the Cohomology of Semisimple Lie Algebras

3.1. Cohomological Criterion of Complete Reducibility. Let $\mathfrak{g}$ be a Lie algebra over K. As usual, a $\mathfrak{g}$-*module* is a vector space V over K together with a given linear representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$. As before, the space V is assumed to be finite-dimensional.

Proposition 3.1. *A linear representation of a Lie algebra* $\mathfrak{g}$ *is completely reducible if and only if* $H^1(\mathfrak{g}, V) = 0$ *for any* $\mathfrak{g}$-*module* V.

Proof. The proof of the proposition is based on a well-known bijection between the space $H^1(\mathfrak{g}, \mathrm{Hom}\,(V, W))$, where V, W are $\mathfrak{g}$-modules, and the classes of extensions of the $\mathfrak{g}$-module V by W. Here the zero cohomology class corresponds to the class of split (nonessential) extensions (see Feigin and Fuks [1988], Chap. 2, Sect. 2.1 D). □

3.2. The Casimir Operator. Let b be a nondegenerate invariant bilinear form on a Lie algebra $\mathfrak{g}$. Using the isomorphism $\mathfrak{g}^* \to \mathfrak{g}$ generated by the form b, one can identify $b \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ with an element of the space $\mathfrak{g} \otimes \mathfrak{g}$. The corresponding element c of the universal enveloping algebra $U(\mathfrak{g})$ (see Vinberg and Onishchik [1988], Chap. 3, Sect. 1) is said to be the *Casimir element* corresponding to the form b. Since b is invariant, the element c belongs to the centre of the algebra $U(\mathfrak{g})$. The element c can be found as follows. Let $e_1, \ldots, e_n$ be an arbitrary basis of the algebra $\mathfrak{g}$, and $f_1, \ldots, f_n$ the basis dual to it with respect to b (i.e. $b(e_i, f_j) = \delta_{ij}$). Then $c = \sum_{i=1}^{n} e_i f_i$. For any linear representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ one defines the operator

$$C = \rho(c) = \sum_{i=1}^{n} \rho(e_i)\rho(f_i)$$

in the space V, which is called the *Casimir operator* of the representation ρ and commutes with all $\rho(x)$, $x \in \mathfrak{g}$.

In particular, let $\mathfrak{g}$ be semisimple, and ρ its faithful representation. According to the remark following Theorem 2.2 the invariant form b_ρ is nondegenerate on $\mathfrak{g}$. Denote the corresponding Casimir operator in the space V by C_ρ.

Proposition 3.3. *We have* $\operatorname{tr} C_\rho = \dim \mathfrak{g}$. *If ρ is irreducible, then the operator C_ρ is nondegenerate.*

3.3. Theorems on the Triviality of Cohomology

Theorem 3.1. *Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a nonzero irreducible representation of a semisimple Lie algebra $\mathfrak{g}$. Then $H^p(\mathfrak{g}, V) = 0$ for all $p \geq 0$.*

Proof. Let $\mathfrak{g}_2 = \operatorname{Ker} \rho$, $\mathfrak{g}_1$ be the ideal in $\mathfrak{g}$ complementary to $\mathfrak{g}_2$, $\rho_1 = \rho|\mathfrak{g}_1$, $e_1, \ldots, e_n$ the basis in $\mathfrak{g}_1$, $f_1, \ldots, f_n$ the basis dual to it with respect to b_{ρ_1}, and $C \in \mathrm{GL}(V)$ the Casimir operator of the representation ρ_1. Define $h\colon C^p(\mathfrak{g}, V) \to C^{p-1}(\mathfrak{g}, V) (p \geq 1)$ by the formula

$$h(c)(x_1, \ldots, x_{p-1}) = \sum_{i=1}^{h} \rho(e_i)c(f_i, x_1, \ldots, x_{p-1}).$$

Then $d \circ h + h \circ d = C$ on $C^p(\mathfrak{g}, V)$, $p \geq 1$, where d is the coboundary operator in the complex $C^*(\mathfrak{g}, V)$, and the Casimir operator C acts on the cochain values. In other words, there is a chain homotopy between C and 0. It follows from this and the equality $C \circ d = d \circ C$ that $H^p(\mathfrak{g}, V) = 0$ for $p > 0$. The case $p = 0$ is evident. □

Theorem 3.2. *If $\mathfrak{g}$ is a semisimple Lie algebra, then $H^1(\mathfrak{g}, V) = H^2(\mathfrak{g}, V) = 0$ for any $\mathfrak{g}$-module V.*

Proof. Using the exact cohomology sequence (see Feigin and Fuks [1988], Chap. 1, Sect. 2.1) and induction on $\dim V$, one can reduce the statement to the case where V is irreducible. If V is nontrivial, the assertion follows from Theorem 2.1. The equality $H^1(\mathfrak{g}, K) = 0$ follows from the corollary to Theorem 1.2, and the equality $H^2(\mathfrak{g}, K) = 0$ from Theorem 1 of Feigin and Fuks [1988], Chap. 3, Sect. 13. □

Corollary. *If $\mathfrak{g}$ is semisimple, then* $\operatorname{Der} \mathfrak{g} = \operatorname{ad} \mathfrak{g}$ *and* $\operatorname{Int} \mathfrak{g} = (\operatorname{Aut} \mathfrak{g})^0$.

The proof of the corollary makes use of the isomorphism $\operatorname{Der} \mathfrak{g}/\operatorname{ad} \mathfrak{g} \simeq H^1(\mathfrak{g}, \mathfrak{g})$ (see Feigin and Fuks [1988], Chap. 2, Sect. 1.2 B). □

3.4. Complete Reducibility of Representations. The following statement provides a cohomological criterion for a Lie algebra to be semisimple.

Theorem 3.3. *The following properties of a Lie algebra* $\mathfrak{g}$ *are equivalent:*
(a) $\mathfrak{g}$ *is semisimple;*
(b) $H^1(\mathfrak{g}, V) = 0$ *for any* $\mathfrak{g}$*-module* V*;*
(c) *any linear representation of* $\mathfrak{g}$ *is completely reducible.*

Proof. (a)⇒(b) by virtue of Theorem 3.2 and (b)⇒(c) by Proposition 3.1. Suppose that any representation of $\mathfrak{g}$ is completely reducible, and that $\mathfrak{a}$ is a commutative ideal in $\mathfrak{g}$. The complete reducibility of the adjoint representation implies that there is a homomorphism $\pi: \mathfrak{g} \to \mathfrak{a}$ that is the identity on $\mathfrak{a}$. If $\mathfrak{a} \neq 0$, then $\mathfrak{a}$ admits a representation ρ that is not completely reducible. Thus, (c)⇒(a). □

Corollary. *Any linear representation of a semisimple Lie algebra or a connected semisimple Lie group is completely reducible.*

Another proof of complete reducibility uses the unitary trick (see Kirillov [1987], Chap. 5, Sect. 63) and is based on the existence of a compact real form in any complex semisimple Lie Algebra (see Chap. 4, Theorem 1.1, and the corollary to Proposition 2.1).

We also note that the assertion of the corollary is extended to semisimple Lie groups with finitely many connected components.

3.5. Reductive Lie Algebras. Consider the following important class of Lie algebras, which includes both semisimple and commutative algebras. A Lie algebra $\mathfrak{g}$ is said to be *reductive* if its radical $\operatorname{rad}\mathfrak{g}$ coincides with the centre $\mathfrak{z}(\mathfrak{g})$. For example, the Lie algebra $\mathfrak{gl}(V) \simeq \mathfrak{gl}_n(K)$, where $n = \dim V$, is reductive.

Theorem 3.4. *The following properties of a Lie algebra* $\mathfrak{g}$ *are equivalent:*
(a) $\mathfrak{g}$ *is reductive;*
(b) $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_0$, *where* $\mathfrak{g}_0$ *is a semisimple ideal;*
(c) *the adjoint representation* $\mathfrak{g}$ *is completely reducible;*
(d) $\mathfrak{g}$ *can be decomposed into a direct sum of simple ideals;*
(e) *the Lie algebra* $[\mathfrak{g}, \mathfrak{g}]$ *is semisimple.*

Proof. The implications (a)⇒(b) and (b)⇒(c) follow from the corollary to Theorem 3.3. It follows from Theorem 2.2 that if condition (e) is satisfied, then the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ with respect to $k_{\mathfrak{g}}$ is the ideal complementary to $[\mathfrak{g}, \mathfrak{g}]$. This ideal evidently coincides with $\mathfrak{z}(\mathfrak{g})$, whence (e)⇒(a). □

The corollary to Theorem 3.3 easily implies that a linear representation ρ of a reductive complex Lie algebra $\mathfrak{g}$ is completely reducible if and only if the operator $\rho(x)$ is semisimple for all $x \in \mathfrak{z}(\mathfrak{g})$. One also has the following generalization of Corollary 2 to Theorem 1.1.

Proposition 3.4. *A Lie algebra* $\mathfrak{g}$ *is reductive if and only if it admits a faithful completely reducible linear representation.*

Proof. One can assume that $K = \mathbb{C}$. Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a completely reducible linear representation. It follows from Lie's theorem that there exists a weight subspace V_λ in V (with respect to the representation $\rho|\operatorname{rad}\mathfrak{g}$). It is invariant under ρ, whence $V = V_\lambda \oplus V'$, where V' is invariant under ρ. An iterated application of the same procedure yields the decomposition $V = \bigoplus_{i=1}^{p} V_{\lambda_i}$, where V_{λ_i} are the weight subspaces of $\rho|\operatorname{rad}\mathfrak{g}$ invariant under ρ. Thus $\rho(\operatorname{rad}\mathfrak{g}) \subset \mathfrak{z}(\rho(\mathfrak{g}))$. □

Note that Theorem 3.1 is easily generalized to the case of reductive $\mathfrak{g}$. This implies the following theorem.

Theorem 3.5. *Let $\mathfrak{g}$ be a reductive Lie algebra and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a completely reducible representation of it. Then for any $p > 0$ we have*

$$H^p(\mathfrak{g}, V) = H^p(\mathfrak{g}, V^\mathfrak{g}) \simeq H^p(\mathfrak{g}, K) \otimes V^\mathfrak{g},$$

where $V^\mathfrak{g} = H^0(\mathfrak{g}, V) = \{v \in V | \rho(x)v = 0 \quad \forall x \in \mathfrak{g}\}$.

§ 4. Levi Decomposition

4.1. Levi's Theorem. Let $\mathfrak{g}$ be a reductive Lie algebra over a field K. A subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is said to be a *Levi subalgebra* if $\mathfrak{g}$ can be decomposed into the semidirect sum

$$\mathfrak{g} = \operatorname{rad}\mathfrak{g} \ni \mathfrak{l}. \tag{5}$$

The decomposition (5) is called the *Levi decomposition* of $\mathfrak{g}$.

A Levi subalgebra $\mathfrak{l}$ in $\mathfrak{g}$ is always semisimple because the natural homomorphism $\mathfrak{g} \to \mathfrak{g}/\operatorname{rad}\mathfrak{g}$ maps $\mathfrak{l}$ isomorphically onto the semisimple Lie algebra $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$. Evidently, $\mathfrak{l}$ is a maximal semisimple subalgebra. As we will see in Sect. 4.2, the converse statement is also true.

Theorem 4.1 (Levi). *In any Lie algebra $\mathfrak{g}$ over K there exists a Levi subalgebra.*

Proof. Induction on $\dim\operatorname{rad}\mathfrak{g}$ reduces the proof of the theorem to the case where $\operatorname{rad}\mathfrak{g}$ is commutative. Then the adjoint representation of $\mathfrak{g}$ in $\operatorname{rad}\mathfrak{g}$ defines a representation of the semisimple Lie algebra $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ in this space. Now, by Theorem 3.2, $H^2(\mathfrak{g}/\operatorname{rad}\mathfrak{g}, \operatorname{rad}\mathfrak{g}) = 0$, and the well-known connection between extensions and cohomology (see Feigin and Fuks [1988], p. 159, Theorem 2′) implies that the extension $0 \to \operatorname{rad}\mathfrak{g} \to \mathfrak{g} \to \mathfrak{g}/\operatorname{rad}\mathfrak{g} \to 0$ is split, i.e. there exists a subalgebra of $\mathfrak{g}$ complementary to $\operatorname{rad}\mathfrak{g}$. □

For a proof of Levi's theorem which formally makes no use of cohomology and utilizes the complete reducibility of representations of a semisimple Lie algebra, see Serre [1987] and also Onishchik and Vinberg [1990].

Corollary. *Let G be a connected Lie group. There exists a virtual connected Lie subgroup $L \subset G$ such that*

$$G = (\operatorname{Rad} G)L, \quad \dim (\operatorname{Rad} G) \cap L = 0. \tag{6}$$

The subgroup L is semisimple.

A connected virtual Lie subgroup $L \subset G$ having the property (6) is said to be a *Levi subgroup* of the group G, and the decomposition (6) is said to be a *Levi decomposition* of this group.

If G is simply-connected, then $\operatorname{Rad} G$ and the Levi subgroup L are also simply-connected and the Levi decomposition (6) is a semidirect one. This follows, for example, from the proof of Theorem 4.2 (see below). In particular, in this case L is a Lie subgroup of G. The following example shows that in general this does not hold.

Example. Let $\mathcal{A}$ be a simply-connected covering group for $\mathrm{SL}_2(\mathbb{R})$. As is known, $Z(\mathcal{A}) \simeq \mathbb{Z}$ (see Chap. 4, Sect. 3.2, Example 4). Let z_0 be a generator of the group $Z(\mathcal{A})$, and let $t_0 \in \mathbb{T} = \mathrm{SO}_2$ be a rotation through an angle incommensurable with π. Then $\Gamma = \langle (t_0, z_0) \rangle$ is a discrete normal subgroup of $\mathbb{T} \times \mathcal{A}$. Let $G = (\mathbb{T} \times \mathcal{A})/\Gamma$. Then the image L of the subgroup $\mathcal{A}$ under the natural homomorphism $\mathbb{T} \times \mathcal{A} \to G$ is a Levi subgroup of G. The fact that the subgroup $\langle t_0 \rangle$ is dense in $\mathbb{T}$ implies that L is dense in G, i.e. it is not a Lie subgroup.

4.2. Existence of a Lie Group with a Given Tangent Algebra. Let $\mathfrak{g}$ be a Lie algebra over a field K. Our aim is to prove the existence of a Lie group G over K whose tangent algebra is isomorphic to $\mathfrak{g}$. In the case where $\mathfrak{g}$ is solvable, this fact is established by induction on $\dim \mathfrak{g}$ (see Chap. 2, Theorem 5.10). On the other hand, if $\mathfrak{g}$ is a Lie algebra with trivial centre, then $\mathfrak{g} \simeq \operatorname{ad} \mathfrak{g}$, and one can take for G the group $\operatorname{Int} \mathfrak{g}$ of inner automorphisms of the algebra $\mathfrak{g}$.

Theorem 4.2 (Cartan). *For any Lie algebra $\mathfrak{g}$ over the field K there exists a simply-connected Lie group G over K whose tangent algebra is isomorphic to $\mathfrak{g}$.*

Proof. Consider the Levi decomposition (5). It follows from what was said above that, there exist simply-connected Lie groups R and L with the tangent algebras $\operatorname{rad} \mathfrak{g}$ and $\mathfrak{l}$, respectively. Let $\beta\colon \mathfrak{l} \to \operatorname{Der}(\operatorname{rad} \mathfrak{g})$ be the adjoint representation of the subalgebra $\mathfrak{l}$ in $\operatorname{rad} \mathfrak{g}$. Then there exists a homomorphism $B\colon L \to \operatorname{Aut}(\operatorname{rad} \mathfrak{g})$ such that $dB = \beta$. By virtue of Proposition 4.2 of Vinberg and Onishchik [1988], Chap. 2, the semidirect product $G = R \rtimes_B L$ defined by the homomorphism B is the desired simply-connected Lie group. $\square$

The proof just outlined essentially coincides with the original proof of É. Cartan [1930]. Direct proofs of Theorem 4.2 making no use of Levi's theorem can be found in Gorbatsevich [1974b], van Est [1988]. Theorem 4.2 is also derived without difficulty from Ado's theorem (see Sect. 5.3).

4.3. Malcev's Theorem. It turns out that a Levi subalgebra of an arbitrary Lie algebra is unique up to conjugacy. This is implied by the following assertion.

Theorem 4.3 (Malcev). *Let $\mathfrak{l}$ be a Levi subalgebra of a Lie algebra $\mathfrak{g}$. For any semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$ there exists $\varphi \in \operatorname{Int} \mathfrak{g}$ such that $\varphi(\mathfrak{s}) \subset \mathfrak{l}$.*

Various proofs of this theorem can be found in Onishchik and Vinberg [1990], Bourbaki [1975], and Serre [1987]. Note that the automorphism φ can be chosen in the form $\varphi = \exp \operatorname{ad} z$, where z belongs to the nilpotent radical of the algebra $\mathfrak{g}$.

Corollary 1. *Any two Levi subalgebras of a Lie algebra can be taken into one another by an inner automorphism of this algebra.*

Corollary 2. *Any maximal semisimple subalgebra of a Lie algebra is a Levi subalgebra.*

Corollary 3. *Let G be a connected Lie group, and L a Levi subgroup of it. Any connected semisimple virtual Lie subgroup of G is conjugate to a subgroup of L. Any two Levi subgroups of G are conjugate. A connected virtual Lie subgroup S of G is a Levi subgroup if and only if S is a maximal semisimple subgroup of G.*

4.4. Classification of Lie Algebras with a Given Radical. Given a solvable Lie algebra $\mathfrak{r}$ over the field $K = \mathbb{C}$ or $\mathbb{R}$, consider the problem of classifying (up to an isomorphism) all Lie algebras $\mathfrak{g}$ for which $\operatorname{rad} \mathfrak{g} = \mathfrak{r}$.

According to Theorem 4.1, a Lie algebra $\mathfrak{g}$ with the radical $\mathfrak{r}$ is of the form $\mathfrak{g} = \mathfrak{l} \ni \mathfrak{r}$, where $\mathfrak{l}$ is a semisimple subalgebra of $\mathfrak{g}$. The algebra $\mathfrak{g}$ will be called *faithful* if the adjoint representation of the subalgebra $\mathfrak{l}$ in $\mathfrak{r}$ is faithful. This condition is equivalent to the fact that $\mathfrak{g}$ contains no nonzero semisimple ideals. Indeed, by virtue of Theorem 4.3, a semisimple ideal $\mathfrak{s}$ of $\mathfrak{g}$ is contained in any Levi subalgebra $\mathfrak{l}$, and evidently $[\mathfrak{s}, \mathfrak{r}] = 0$. Any Lie algebra with the radical $\mathfrak{r}$ can be decomposed into the direct sum of a semisimple Lie algebra and a faithful Lie algebra with the same radical. Therefore we can assume that $\mathfrak{g}$ is a faithful Lie algebra.

Let $\mathfrak{l}_0$ be a fixed Levi subalgebra of the Lie algebra $\operatorname{Der} \mathfrak{r}$. We call the faithful Lie algebra $\mathfrak{g}_0 = \mathfrak{l}_0 \in_{\mathrm{id}} \mathfrak{r}$, where id is the identity representation of the Lie algebra $\mathfrak{l}_0$ in $\mathfrak{r}$, the *universal Lie algebra* with radical $\mathfrak{r}$.

Theorem 4.4 (Malcev [1944]). *Any faithful Lie algebra with radical $\mathfrak{r}$ is isomorphic to a Lie algebra of the form $\mathfrak{l} \in_{\mathrm{id}} \mathfrak{r}$, where $\mathfrak{l}_1$, $\mathfrak{l}_2$ are semisimple subalgebras of $\mathfrak{l}_0$. The algebras $\mathfrak{g}_1 = \mathfrak{l}_1 \in_{\mathrm{id}} \mathfrak{r}$ and $\mathfrak{g}_2 = \mathfrak{l}_2 \in_{\mathrm{id}} \mathfrak{r}$, where $\mathfrak{l}_1$, $\mathfrak{l}_2$ are semisimple subalgebras of $\mathfrak{l}_0$, are isomorphic if and only if $\mathfrak{l}_2 = A\mathfrak{l}_1 A^{-1}$, where A is an automorphism of $\mathfrak{r}$ such that $A\mathfrak{l}A^{-1} = \mathfrak{l}$.*

Thus the classification problem under consideration reduces to the analysis of derivations and automorphisms of the Lie algebra $\mathfrak{l}$. Consider some examples of universal Lie algebras studied in Onishchik and Hakimjanov [1975].

Example. (a) If $\mathfrak{r}$ is abelian, then $\operatorname{Aut}\mathfrak{r} = \mathrm{GL}(\mathfrak{r})$, and $\mathfrak{g}_0 = \mathfrak{sl}(\mathfrak{r}) \in_{\mathrm{id}} \mathfrak{r}$. By virtue of Theorem 4.4, the classification of Lie algebras with the radical $\mathfrak{r}$ is reduced to the classification of semisimple subalgebras of $\mathfrak{sl}(\mathfrak{r})$ up to conjugacy.

(b) Let $\mathfrak{r} = \mathfrak{t}_n(K)$ be the algebra of upper triangular matrices of order n. Then $\operatorname{Der}\mathfrak{r} = \operatorname{ad}\mathfrak{r}$, so $\mathfrak{g}_0 = \mathfrak{r}$. Any Lie algebra with the radical $\mathfrak{r}$ admits a direct Levi decomposition. The same holds in the case where $\mathfrak{r}$ is the Borel (i.e. maximal solvable) subalgebra of a semisimple Lie algebra (see Tolpygo [1972]).

(c) Let $\mathfrak{r} = \mathfrak{h}_k(K)$ be the Heisenberg algebra of dimension $2k+1$ (see Chap. 2, Sect. 4.1, Example 2). Then $\mathfrak{g}_0 = \mathfrak{sp}_{2k}(K) \in \mathfrak{h}_k$ can be embedded in $\mathfrak{sp}_{2k+2}(K)$ as the subalgebra of all operators annihilating, for example, the vector e_1 of the standard basis (Onishchik and Hakimjanov [1975]). Thus any faithful Lie algebra with radical $\mathfrak{h}_k$ is isomorphic to a subalgebra of $\mathfrak{sp}_{2k+2}(K)$ containing $\mathfrak{h}_k$. Onishchik and Hakimjanov [1975] also studied the case where $\mathfrak{r}$ is the nilradical of an arbitrary parabolic subalgebra of a semisimple Lie algebra (see Chap. 6, Sect. 1.3).

§ 5. Linear Lie Groups

5.1. Basic Notions. A Lie group G (either real or complex) is said to be *linear* if it is a virtual Lie subgroup of the group $\mathrm{GL}(V)$ of linear transformations of a finite-dimensional vector space (over the field $K = \mathbb{R}$ or $\mathbb{C}$, respectively).

A Lie group (over a field K) is said to be *linearizable* if it is isomorphic to a linear Lie group, or, which is the same, if it admits a faithful finite-dimensional linear representation (over the field K).

Linear Lie groups can be regarded as linearizable Lie groups with a fixed finite-dimensional linear representation. Allowing a certain degree of inaccuracy, we will sometimes identify the notions of linearity and linearizability, calling a group linear if it admits a faithful finite-dimensional linear representation.

The study of linear Lie groups is connected, on the one hand, with a more general problem of studying arbitrary linear groups (that are not necessarily Lie groups and may be defined over arbitrary fields). For more on such groups see, for example, Merzlyakov [1987], Suprunenko [1979]. On the other hand, linear Lie groups are closely connected with the algebraic groups over the fields $\mathbb{R}$ and $\mathbb{C}$ (on which see §6 below). Namely, each algebraic, i.e. closed in the Zariski topology in $\mathrm{GL}(V)$, linear group is a linear algebraic group. Algebraic linear groups are the best studied class of linear groups.

Note that the group $\mathrm{GL}_n(\mathbb{C})$ can be regarded as a real algebraic subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$, so any complex linear Lie group or an algebraic linear group can also be regarded as a real one.

5.2. Some Examples. *Example* 1. The classical groups described in Vinberg and Onishchik [1988], Chap. 1, Sect. 1.2 (such as $\mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)$, $\mathrm{Sp}_n(K)$, $\mathrm{O}_n(K)$, etc., where $K = \mathbb{R}$ or $\mathbb{C}$), are linear algebraic groups.

Other linear algebraic groups are $T_n(K)$, $N_n(K)$, as well as the group $D_n(K)$ of all diagonal nonsingular matrices over the field K.

Example 2. The group $\mathrm{Aff}\, K^n$ of all nonsingular affine transformations of the space K^n is linearizable. Its embedding in $\mathrm{GL}_{n+1}(K)$ is of the form

$$\mathrm{Aff}\, K^n = \mathrm{GL}_n(K) \cdot K^n \ni (A, v) \mapsto \left[\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right] \in \mathrm{GL}_{n+1}(K),$$

where $A \in \mathrm{GL}_n(K)$, $v = (v_1, \dots, v_n)^\top \in K^n$. Clearly, the image of the group $\mathrm{Aff}\, K^n$ under this embedding is an algebraic subgroup of $\mathrm{GL}_{n+1}(K)$.

Example 3. Consider the Lie group $G = K \ltimes_\varphi K^n$, the semidirect product corresponding to some homomorphism $\varphi\colon K \to \mathrm{GL}_n(K)$. The Lie group G is simply-connected, solvable, and has the following faithful linear representation:

$$G \ni (t, v) \mapsto \left[\begin{array}{c|c} \varphi(t) & v \\ \hline 0 & 1 \end{array}\right] \in \mathrm{GL}_{n+1}(K),$$

where $t \in K$, $\varphi(t) \in \mathrm{GL}_n(K)$, $v = (v_1, \dots, v_n)^\top \in K^n$. Therefore the group G is linear. It is a linear algebraic group if and only if the one-parameter subgroup $\varphi(t)$ is an algebraic subgroup of $\mathrm{GL}_n(K)$. The last assertion holds if and only if one of the following conditions is satisfied:

(i) the subgroup $\varphi(t)$ is unipotent (i.e. the matrices $\varphi(t)$ are unipotent for all $t \in K$);

(ii) the subgroup $\varphi(t)$ consists of semisimple elements and is conjugate to the subgroup of the form $\exp X \cdot t$, where $X = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$ and all λ_i are integers.

Example 4. The subgroup $G_\alpha = \{\mathrm{diag}\,(e^t, e^{\alpha t}) | t \in K\}$ is linear for any $\alpha \in K$, but it is an algebraic linear group if and only if $\alpha \in \mathbb{Q}$.

For irrational values of the parameter $\alpha \in K$ the group G_α is isomorphic to an algebraic linear subgroup of $\mathrm{GL}_2(K)$, namely, the subgroup $N_2(K) \subset \mathrm{GL}_2(K)$. The next example shows that there exist linear Lie groups nonisomorphic to any linear algebraic group.

Example 5. Consider the Lie group $G = \mathbb{R} \ltimes_\varphi \mathbb{R}^2$, where the homomorphism φ is of the form

$$\varphi(t) = \begin{bmatrix} e^t & t \\ 0 & e^t \end{bmatrix}, \qquad t \in \mathbb{R}.$$

This Lie group G is linearizable but it is not isomorphic to any real linear algebraic group. The reason is that for $t \neq 0$ the matrix $\varphi(t)$ is neither unipotent nor semisimple, so the one-parameter subgroup $\varphi(t)$ is not an algebraic subgroup of $\mathrm{GL}_2(\mathbb{R})$. If G is isomorphic to a linear algebraic group, then G admits a Chevalley decomposition $G = H \cdot U$, where H is a reductive subgroup (algebraic torus), and U a unipotent radical (see Sect. 5). Since $U \supset (G, G)$ (see Chap. 2, Sect. 5.3), and $\dim (G, G) = 2$, it is clear that $\dim U = 2$ and $U \simeq \mathbb{R}^2$. The image of the torus H under conjugations acting on U must coincide (up to a similarity) with the subgroup $\varphi(t)$ of $\mathrm{GL}_2(\mathbb{R})$. But then the one-parameter subgroup $\varphi(t)$ must be algebraic, which, as we have seen, is false.

Consider now some examples of nonlinearizable Lie groups. Many such examples owe their origin to the following statement.

Proposition 5.1. *Let G be a connected semisimple linear Lie group (either real or complex). Then its centre $Z(G)$ is finite.*

Proof. This is a consequence of the complete reducibility of linear semisimple Lie groups (see Sect. 3.4), the Schur lemma, and the discreteness of the group $Z(G)$. □

Example 6. Consider the Lie group $\mathcal{A} = \widetilde{\mathrm{SL}}_2(\mathbb{R})$, the simply-connected universal covering of the group $\mathrm{SL}_2(\mathbb{R})$. The group $\mathcal{A}$ is nonlinearizable because $Z(\mathcal{A}) \simeq \mathbb{Z}$ (see Chap. 4, Sect. 3.2, Example 4).

A similar argument shows that the nonlinearizable Lie groups $\widetilde{\mathrm{SO}}(p, 2)$ $(p \geq 1)$ and $\widetilde{\mathrm{SU}}(p, q)$ $(p \geq q \geq 1)$ are simply-connected coverings for the groups $\mathrm{SO}(p, 2)$ and $\mathrm{SU}(p, q)$, respectively. However, there are nonlinearizable semisimple Lie groups whose centres are finite. For example, none of the Lie groups $\mathcal{A}_n$ covering the group $\mathrm{SL}_2(\mathbb{R})$ (with the multiplicity $n > 1$) is linearizable, while $Z(\mathcal{A}_n) \simeq \mathbb{Z}_{2n}$ (see Chap. 4, Sect. 3.6).

Example 7. Consider the group $N = N_3(\mathbb{R})$, and its subgroup $Z = Z(N)(\mathbb{Z})$, which is the group of integer points of the centre $Z(N_3(\mathbb{R})) \simeq \mathbb{R}$:

$$Z(N)(\mathbb{Z}) = \left\{ \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \; k \in \mathbb{Z} \right\}$$

$$\subset N_3(\mathbb{R}) = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \middle| \; x, y, z \in \mathbb{R} \right\}.$$

Let $G = N/Z$. We will show that the resulting nilpotent Lie group G is nonlinearizable. If T is a maximal compact subgroup in G, then T is a one-dimensional torus, and $T = (G, G)$. If the group G were isomorphic to a linear one, we would have $T \cap (G, G) = \{e\}$ (see Sect. 5.4). The resulting contradiction proves that G is nonlinearizable.

5.3. Ado's Theorem. As we have just seen, not every connected Lie group is linearizable. However, it turns out that a Lie group is always locally isomorphic to a linear Lie group. This follows from Ado's theorem, one of the main results of the Lie algebra theory.

Theorem 5.1 (Ado, see Bourbaki [1975], Jacobson [1962], Serre [1987]). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K of characteristic* 0. *Then the Lie algebra $\mathfrak{g}$ has a faithful finite-dimensional linear representation over K.*

Proof. If the centre $\mathfrak{z}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is trivial, then the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is faithful. If $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$, one has to find a representation ρ_1 of the Lie algebra $\mathfrak{g}$ that is faithful on $\mathfrak{z}(\mathfrak{g})$. Then $\rho = \rho_1 \oplus \mathrm{ad}$ is a faithful representation of the Lie algebra $\mathfrak{g}$.

A representation ρ_1 that is faithful on $\mathfrak{z}(\mathfrak{g})$ is constructed as follows. Choose a faithful representation of the centre $\mathfrak{z}(\mathfrak{g})$. Then, using the Levi decomposition for $\mathfrak{g}$ and induction on $\dim \mathfrak{g}$, construct an extension of this representation to a representation ρ_1 of the entire Lie algebra $\mathfrak{g}$, which is the desired one. □

Ado's theorem admits various refinements. For example, the faithful representation ρ of the Lie algebra $\mathfrak{g}$ can be chosen in such a way that the image of the nilradical $\mathfrak{n}$ of the Lie algebra $\mathfrak{g}$ (see Chap. 2, Sect. 5.1) consists of nilpotent matrices. In particular, this means that any finite-dimensional nilpotent Lie algebra is isomorphic to a subalgebra of the Lie algebra $\mathfrak{n}_n(K)$ of all nilpotent triangular matrices of some order $n \in \mathbb{N}$. Other refinements of Ado's theorem concern the upper bounds of the minimal dimension of the representation ρ (see, for example, Sect. 5.4 below and also Reed [1964]).

Corollary. *Let G be a simply-connected Lie group over $K = \mathbb{R}$ or $\mathbb{C}$. Then there exists a discrete central subgroup $Z \subset Z(G)$ such that the group G/Z is linearizable over K.*

Thus any Lie group is locally isomorphic to a linear Lie group.

5.4. Criteria of Linearizability for Lie Groups. Linearizer. In this section we consider only real Lie groups.

Theorem 5.2 (Malcev [1943]). *Let G be a connected Lie group, and L its Levi subgroup. The Lie group is linearizable if and only if so are the groups L and* $\mathrm{Rad}\, G$.

This result reduces the general problem of linearizability of Lie groups to the cases where the group is either semisimple or solvable. The criterion of linearizability for solvable Lie groups will be given in Chap. 2 (see Theorem

7.1). From this theorem and Theorem 5.2 one can derive another form of the linearizability criterion.

Theorem 5.3 (see Goto [1950]). *Let G be a connected Lie group, and L its Levi subgroup. The group G is linearizable if and only if L is linearizable and the group* $\operatorname{Rad}(G, G)$ *is simply-connected.*

Another approach to the problem of linearizability makes use of the notion of linearizer. Let G be a connected Lie group. Denote by $\Lambda(G)$ the intersection of the kernels of all (finite-dimensional) linear representations of G. The subgroup $\Lambda(G)$ (called the *linearizer*) is the central Lie subgroup of G. In general it is nondiscrete. For example, if $G = N/Z$ is the Lie group of Sect. 5.2, Example 6, then $\Lambda(G) = Z(N)/Z \simeq \mathrm{SO}_2$.

Theorem 5.4 (see Malcev [1943], Hochschild [1960]). *Let G be a connected Lie group; and then $G_{\mathrm{lin}} = G/\Lambda(G)$ is a linearizable Lie group.*

If $R: G \to \mathrm{GL}(V)$ is a finite-dimensional representation of a connected Lie group G, then it can be factorized via the natural epimorphism $\mathrm{lin}: G \to G/\Lambda(G)$, and induces the representation $R': G_{\mathrm{lin}} \to \mathrm{GL}(V)$ of the linear Lie group G_{lin}.

$$\begin{tikzcd} G \arrow[rr, "R"] \arrow[dr, "\mathrm{lin}"'] & & GL(V) \\ & G_{\mathrm{lin}} \arrow[ur, "R'"'] & \end{tikzcd}$$

If the Lie group G is simply-connected, then it is not difficult to deduce from Theorem 5.2 that $\Lambda(G) = \Lambda(L)$, where L is the Levi subgroup of G. In particular, in this case the subgroup $\Lambda(G)$ is discrete. For the description of $\Lambda(G)$ in the general case see Hochschild [1960].

The linearizers of semisimple Lie groups are studied in Chap. 4, Sect. 3.6.

The knowledge of the linearizer of semisimple Lie groups together with Theorem 7 of Chap. 2 makes it possible to check without difficulty whether an arbitrary connected Lie group G is linearizable (without computing $\Lambda(G)$ in the general case).

5.5. Sufficient Linearizability Conditions. In many particular cases, it is sometimes convenient to carry out the verification of linearizability of Lie groups by means of special methods without resorting to the general criteria of Sect. 5.4. For example, the following assertions (following from Theorem 7.1 of Chap. 2 and Theorem 5.4, respectively) are often used. Let G, G' be connected Lie groups and $\alpha: G \to G'$ an epimorphism. Then

(1) if the Lie group G is semisimple and linearizable, then G' is also linearizable (and, of course, semisimple);

(2) if $\operatorname{Ker} \alpha$ is discrete, and G' is solvable and linearizable, then G is linearizable (and solvable).

For arbitrary connected Lie groups G, G' the statement analogous to (1) (as well as (2)) does not hold in the general case. However, if the kernel of the epimorphism is connected, the following analogue of statement (1) holds.

Proposition 5.2 (Gorbatsevich [1985]). *Let G, G' be connected Lie groups, and $\alpha: G \to G'$ an epimorphism. If G is linearizable, and the kernel* $\operatorname{Ker}\alpha$ *is connected, then G' is also linearizable.*

Consider now the question of the linearizability of the holomorph $\operatorname{Hol} G$ of a connected Lie group G. Recall that the *holomorph* of a group G is the semidirect product $\operatorname{Aut} G \ltimes G$. The holomorph of a connected Lie group (even of a linear one) is not necessarily linearizable.

Example. Let $G = \mathbb{R} \times \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is a one-dimensional torus. Consider the following one-parameter subgroup $A = \{\varphi(t)\}$ in $\operatorname{Aut} G$:

$$\mathbb{R} \times \mathbb{T} \ni (\lambda, z) \underset{\varphi(t)}{\mapsto} (\lambda, e^{i\lambda t} \cdot z) \in \mathbb{R} \times \mathbb{T},$$

where λ, $t \in \mathbb{R}$, $z \in \mathbb{C}$, $|z| = 1$. The semidirect product $F = A \ltimes G$ is naturally embedded in $\operatorname{Hol} G$. If $\operatorname{Hol} G$ is linearizable, so is F. The group F is evidently solvable, whence, by virtue of Theorem 7.1 of Chap. 2, we must have $T \cap (F, F) = \{e\}$, where $T = \mathbb{T}$ is the maximal torus in F. However, $(F, F) = T$, so $\operatorname{Hol} G$ is not linearizable.

The Lie group G in this example is abelian but not simply-connected. The role of the latter fact is clarified by the following result.

Theorem 5.5 (see Greenleaf and Moskowitz [1980], Hochschild [1978]). *Let G be a connected linearizable Lie group, and N its nilradical. If N is simply-connected, then the Lie group* $\operatorname{Hol} G$ *is linearizable.*

Now let G be an arbitrary (not necessarily connected) Lie group. We can verify the linearizability of its connected component G^0 of the identity with the use of the above results. If G has finitely many connected components, then the groups G and G^0 can be linearizable only simultaneously. This is a consequence of the following general result.

Proposition 5.3. *Let Π be an arbitrary group, and Π_0 a normal subgroup of finite index in Π. The group Π_0 is linearizable if and only if so is Π.*

Proof. If $\rho: \Pi \to \mathrm{GL}(V)$ is a faithful finite-dimensional linear representation, then $\rho|\Pi_0$ is faithful on Π_0. Conversely, let $\rho_0: \Pi_0 \to \mathrm{GL}(W)$ be a faithful finite-dimensional linear representation of the subgroup Π_0. If n is the index of the subgroup Π_0 in Π, then $\Gamma = \Pi/\Pi_0$ is a subgroup of order n and is therefore naturally embedded in the group S_n of permutations of degree n. But then the group Π can be embedded in the wreath product $\Pi_0 \wr \Gamma$ of the groups Π_0 and Γ. The group $\Pi_0 \wr S_n$ has a faithful representation in the space $V = \underbrace{W \oplus \ldots \oplus W}_{n}$ (for more details on that see Kargapolov and Merzlyakov [1982]). But $\Pi_0 \wr \Gamma$ is a subgroup of $\Pi_0 \wr S_n$ and therefore has a faithful representation in V. $\square$

5.6. Structure of Linear Lie Groups

Theorem 5.6. *Let G be a connected linear Lie group. Then there exists a decomposition $G = S \cdot T \cdot F$, where S is a Levi subgroup of G, T is a torus, F a simply-connected solvable normal subgroup of G, such that $(S \cdot T) \cap F = \{e\}$, and the subgroup S centralizes the torus T.*

Proof. Let $G = S \cdot R$ be a Levi decomposition of the Lie group G. Since the radical R is linearizable, then, by virtue of Chap. 2 of Theorem 7.1, it has a decomposition $R = T \cdot F$, where T is a torus (coinciding with the maximal compact subgroup in R), and F is a simply-connected Lie subgroup of R such that $T \cap F = \{e\}$. Clearly, $F \supset (R, R)$. Consider the abelian group $A = R/(R, R)$ wherein T can be embedded. The action of S on A induced by the action of S on R by conjugations preserves A. Hence it follows that the torus $T \subset G$ can be chosen in such a way that S centralizes it. Since $Z(S)$ is finite (see Proposition 5.1), we can easily prove that $(S \cdot T) \cap F = \{e\}$. □

The subgroup $T \cdot F$ in the decomposition of the Lie group G from Theorem 5.6 coincides with the radical R, so this decomposition can be regarded as a refined version of the Levi decomposition in the case of linear Lie groups. Another refinement of the Levi decomposition can be carried out in the case of algebraic linear groups (see Sect. 6.5 below).

An arbitrary connected virtual Lie subgroup $G \subset \mathrm{GL}(V)$ is not necessarily closed in $\mathrm{GL}(V)$ (even in the Euclidean topology, to say nothing of the Zariski topology). A description of the closure $\overline{G}$ (in the Euclidean topology) of the subgroup $G \subset \mathrm{GL}(V)$ is given in the following theorem.

Theorem 5.7 (Goto [1973]). *Let G be a connected virtual subgroup of $\mathrm{GL}(V)$, where V is a finite-dimensional vector space over $\mathbb{R}$. Then there exists a closed (in $\mathrm{GL}(V)$) normal subgroup $F \triangleleft G$ and a subgroup $A \subset G$ isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$, whose closure $\overline{A}$ is a torus, such that $G = A \ltimes F$, and $\overline{G} = \overline{A} \cdot F$ is an almost semidirect product (i.e. F is normal in $\overline{G}$ and $\overline{A} \cap F$ is finite).*

Let G be a linearizable Lie group. If $R\colon G \to \mathrm{GL}(V)$ is a faithful finite-dimensional linear representation of it, then in general $R(G)$ is not closed in $\mathrm{GL}(V)$.

Theorem 5.8 (Djokovic [1976]). *Let G be a connected linearizable Lie group. Then there exists a faithful finite-dimensional linear representation $R\colon G \to \mathrm{GL}(V)$ such that the subgroup $R(G)$ is a Lie subgroup of $\mathrm{GL}(V)$.*

§ 6. Lie Groups and Algebraic Groups

6.1. Complex and Real Algebraic Groups. In this article we mean by an *algebraic group* over the field $K = \mathbb{C}$ or $\mathbb{R}$ a linear algebraic group over K, i.e. an affine algebraic variety G over K endowed with the group structure for which the mapping $(a, b) \mapsto ab^{-1}$ of the variety $G \times G$ into G is polynomial. In terms of Springer [1989] (which the reader may consult for the general theory of algebraic groups), an algebraic group G over $\mathbb{R}$ (or a real linear algebraic group) is the group of $\mathbb{R}$-points of some linear algebraic $\mathbb{R}$-group over $\mathbb{C}$, which we will denote by $G(\mathbb{C})$. Any algebraic group is isomorphic to an algebraic linear group in the sense of Sect. 5.1 (see Springer [1989], Sect. 2.3.4). We also note that any complex algebraic group can be considered as a real one.

Theorem 6.1 (see Onishchik and Vinberg [1990]). *Any algebraic group over a field K is a Lie group over K. Polynomial homomorphisms of algebraic groups are homomorphisms of Lie groups, and algebraic (i.e. closed in the Zariski topology) subgroups are Lie subgroups. A complex algebraic group is irreducible (or connected) in the Zariski topology if and only if it is connected as a complex Lie group.*

Example. The last assertion of Theorem 6.1 does not hold for real algebraic groups. For example, the real algebraic group $\mathrm{GL}_n(\mathbb{R})$ $(n \geq 1))$ is irreducible, but is not connected as a real Lie group, its identity component being the subgroup $\mathrm{GL}_n^+(\mathbb{R})$ of matrices with positive determinant.

One can easily see that the Lie algebra of an algebraic group G defined in Springer [1989], Sect. 2.4 is naturally identified with the tangent algebra of the Lie group G in the sense of Vinberg and Onishchik [1988], Chap. 2.

6.2. Algebraic Subgroups and Subalgebras. Let G be an algebraic group over the field $K = \mathbb{C}$ or $\mathbb{R}$. As the example of Sect. 6.1 and Example 4 of Sect. 5.2 show, a Lie subgroup of G is not necessarily algebraic. A subalgebra $\mathfrak{h}$ of the tangent algebra $\mathfrak{g}$ of G is said to be *algebraic* if $\mathfrak{h}$ is a tangent algebra of some algebraic subgroup $H \subset G$. In the case $K = \mathbb{C}$, this is equivalent to the fact that the connected virtual Lie subgroup of G corresponding to the subalgebra $\mathfrak{h}$ is algebraic (see Theorem 6.1). If $\mathfrak{h}$ is an arbitrary subalgebra of $\mathfrak{g}$, then there exists a smallest algebraic subalgebra ${}^a\mathfrak{h} \subset \mathfrak{g}$ containing $\mathfrak{h}$. The algebraic subgroup corresponding to ${}^a\mathfrak{h}$ is the closure (in the Zariski topology) of the connected virtual Lie subgroup corresponding to $\mathfrak{h}$. The subalgebra ${}^a\mathfrak{h}$ is said to be the *algebraic closure* of $\mathfrak{h}$. The properties of the algebraic closure are analogous to those of the Malcev closure (see Vinberg and Onishchik [1988], Chap. 2, Sect. 5.2). In particular, the following theorem holds.

Theorem 6.2. *If $\mathfrak{h}$ is a subalgebra of the tangent algebra $\mathfrak{g}$ of the algebraic group G, then $[{}^a\mathfrak{h}, {}^a\mathfrak{h}] = [\mathfrak{h}, \mathfrak{h}]$.*

Corollary 1. *If $\mathfrak{h}$ is commutative (or solvable), then ${}^a\mathfrak{h}$ is also commutative (respectively, solvable).*

One can easily see that if $\mathfrak{h}$ is an ideal, so is ${}^a\mathfrak{h}$. We therefore have the following corollary.

Corollary 2. *The radical* $\operatorname{rad}\mathfrak{g}$ *is an algebraic subalgebra of* $\mathfrak{g}$. *The radical of a complex algebraic group is an algebraic subgroup.*

As is known, the commutator subgroup of a connected complex algebraic group is an algebraic subgroup of it (see Onishchik and Vinberg [1990], Sect. 3.1.4). This implies the next corollary.

Corollary 3. *For any subalgebra* $\mathfrak{h} \subset \mathfrak{g}$, *the subalgebra* $[\mathfrak{h}, \mathfrak{h}]$ *is algebraic. In particular, any subalgebra* $\mathfrak{h}$ *coinciding with its derived algebra (for example, any semisimple subalgebra) is algebraic.*

A linear Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(V)$ is said to be *algebraic* if $\mathfrak{h}$ is an algebraic subalgebra of $\mathfrak{gl}(V)$, i.e. if $\mathfrak{h}$ is the tangent algebra of some algebraic linear group $H \subset \mathrm{GL}(V)$.

Corollary 4. *If a linear Lie algebra* $\mathfrak{h}$ *coincides with* $[\mathfrak{h}, \mathfrak{h}]$, *then* $\mathfrak{h}$ *is algebraic. A connected complex linear Lie group coinciding with its derived algebra is an algebraic linear group.*

Let G be a real linear Lie group, and $\mathfrak{h}$ a semisimple subalgebra of its tangent algebra $\mathfrak{g}$. It follows from the above considerations and Corollary 1 to Theorem 1.2 that $\mathfrak{h}$ is the tangent algebra of some Lie subgroup $H \subset G$. Replacing H by H^0, one can assume that H is connected. The example of Sect. 4.1 shows that a connected semisimple virtual Lie subgroup of an arbitrary Lie group G is not necessarily closed. However, the following proposition holds.

Proposition 6.1. *Let* H *be a connected virtual semisimple Lie subgroup of the Lie group* G. *If* G *is simply-connected or* H *has a finite centre, then* H *is a Lie subgroup of* G.

6.3. Semisimple and Reductive Algebraic Groups. Let G be a Lie group over $K = \mathbb{C}$ or $\mathbb{R}$. In what follows, we will consider the structures of an algebraic group over K on G inducing the given Lie group structure on G. A necessary condition (which is not a sufficient one, see Example 1 below) for such a structure to exist is that G admits a faithful linear representation over the field K. The next theorem singles out a class of connected complex Lie groups admitting a unique structure of a complex algebraic group.

Theorem 6.3. *Let* G *be a connected complex Lie group, coinciding with its commutator subgroup and admitting a faithful linear representation. Then there exists a complex algebraic group structure on* G. *Any holomorphic linear representation of* G *is polynomial. In particular, the structure of a complex algebraic group on* G *is unique.*

Proof. The existence of an algebraic structure follows from Corollary 4 to Theorem 6.2. Let $R: G \to \mathrm{GL}(V)$ be a holomorphic linear representation, and let $\Gamma = \{(g, R(g)) \in G \times \mathrm{GL}(V) | g \in G\}$ be its graph. Corollary 3 to Theorem 6.2 implies that Γ is an algebraic subgroup of $G \times \mathrm{GL}(V)$. Since the projection $\Gamma \to G$ is polynomial, the inverse homomorphism $g \mapsto (g, R(g))$ is also polynomial (see Onishchik and Vinberg [1990], Sect. 3.1.4). □

Taking into account the corollary to Theorem 2.2, one obtains the following statement.

Corollary 1. *Any connected semisimple complex Lie group admitting a faithful linear representation has a unique complex algebraic group structure.*

Actually, in the situation of Corollary 1, the condition requiring the existence of a faithful linear representation is always satisfied (see Chap. 3, Theorem 2.7).

An element x of a Lie algebra $\mathfrak{g}$ is said to be *semisimple* (*nilpotent*) if the operator $\operatorname{ad} x$ in the space $\mathfrak{g}$ is semisimple (respectively, nilpotent).

Corollary 2. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra, and ρ a linear representation of it. If $x \in \mathfrak{g}$ is semisimple (nilpotent), then the operator $\rho(x)$ is semisimple (nilpotent).*

Note that for connected real Lie groups (even those admitting a faithful linear representation), the statements on the existence and uniqueness of an algebraic structure do not hold.

Example 1. Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then the complex Lie algebra $\mathfrak{g}(\mathbb{C})$ is also semisimple (see Sect. 7.1). By virtue of the corollary to Theorem 3.2, the connected complex linear Lie group $\operatorname{Int}\mathfrak{g}(\mathbb{C})$ is semisimple, and therefore algebraic. Thus, $Q = \operatorname{Int}\mathfrak{g}(\mathbb{C}) \cap \operatorname{Aut}\mathfrak{g}$ is a real algebraic linear group. The connected component of unity in the group Q is $\operatorname{Int}\mathfrak{g}$, and its tangent algebra is $\operatorname{ad}\mathfrak{g}$. One can easily see that $\operatorname{Int}\mathfrak{g}$ is algebraic if and only if $\operatorname{Int}\mathfrak{g} = Q$. For example, this implies that the linear group $G = \operatorname{Int}\mathfrak{sl}_2(\mathbb{R})$ is not algebraic. One can also prove that G admits no structure of a real algebraic group whatsoever.

Example 2. It follows from the argument of the preceding example that the linear group $G = \operatorname{Int}\mathfrak{sl}_3(\mathbb{R})$ is algebraic. The adjoint representation $\operatorname{Ad}: \mathrm{SL}_3(\mathbb{R}) \to G$ is polynomial, and establishes an isomorphism of abstract groups because $\mathrm{SL}_3(\mathbb{R})$ has a trivial centre. At the same time, the homomorphism $\operatorname{Ad}^{-1}: G \to \mathrm{SL}_3(\mathbb{R})$ is not polynomial, since it does not extend to a homomorphism of the corresponding complex algebraic groups. Since all automorphisms of the group G are polynomial, there is no polynomial isomorphism between the groups $\mathrm{SL}_3(\mathbb{R})$ and G. Thus, there are at least two nonisomorphic real algebraic structures on $\mathrm{SL}_3(\mathbb{R})$.

Recall (see Springer [1989], Chap. 1, Sect. 3.6.61) that a complex algebraic group G is said to be *reductive* if $\operatorname{Rad} G$ consists of semisimple elements. In fact the radical of a reductive algebraic group is an algebraic torus (i.e. it is isomorphic to $(\mathbb{C}^\times)^m$), and if G is connected, then $\operatorname{Rad} G \subset Z(G)$. In particular, the tangent algebra of a reductive algebraic group is reductive.

In order to facilitate the verification of reductivity one can use the following proposition.

Proposition 6.2 (see Onishchik and Vinberg [1990], Sect. 4.1.1). *A complex algebraic linear Lie algebra* $\mathfrak{g} \subset \mathfrak{gl}(V)$ *is the tangent algebra of a reductive algebraic linear group* $G \subset \mathrm{GL}(V)$ *if and only if the bilinear form* (3) *is nondegenerate on* $\mathfrak{g}$.

The assertion of Corollary 1 to Theorem 6.3 about the uniqueness of an algebraic structure extends to reductive groups.

Theorem 6.4. *Let G be a reductive complex algebraic group. Any holomorphic linear representation of G is polynomial. If one considers G as a complex Lie group, the algebraic structure on G is unique.*

Example 3. Consider the complex algebraic group $G = \mathbb{C} \times \mathbb{R}^\times = G_a \times G_m$. For each $c \in \mathbb{C}$ we define the mapping $\varphi_c : G \to G$ by the formula $\varphi_c(z, e^w) = (z, e^{w+cz})$ $(z, w \in \mathbb{C})$. Evidently, φ_c, $c \neq 0$, are nonpolynomial automorphisms of the complex Lie group G. These automorphisms generate the continuum of different complex algebraic group structures on G.

A complex Lie group will be called *reductive* if it admits the structure of a reductive complex algebraic group. For the intrinsic characterization of such groups see Chap. 4, Sect. 2.5.

A real algebraic group G is said to be *reductive* if so is the complex algebraic group $G(\mathbb{C})$.

The following assertion is close to the corollary to Theorem 3.3, and to Proposition 3.4 (see Onishchik and Vinberg [1990], Sect. 5.2.6).

Theorem 6.5. *Any linear representation of a reductive complex algebraic group is completely reducible. An algebraic (either complex or real) linear group is reductive if and only if it is completely reducible.*

6.4. Polar Decomposition. As is known in linear algebra, any invertible linear operator $A \in \mathrm{GL}(E)$, where E is a Euclidean or Hermitian space, can be uniquely represented in the form $A = XY$, where X is an orthogonal (or unitary) operator, and Y is a positive definite self-adjoint operator. In this section we consider the class of linear algebraic groups for which an analogous theorem holds.

Denote by $\mathrm{O}(E)$ (respectively, $\mathrm{U}(E)$) the group of all orthogonal (unitary) operators in the Euclidean (Hermitian) space E. For any $g \in \mathrm{GL}(E)$ we denote by g^* the adjoint operator in E. We denote by $S(E)$ and $P(E)$ the subspace in $L(E)$ and the submanifold in $\mathrm{GL}(V)$ consisting, respectively, of all self-adjoint and positive definite operators. A subgroup $G \subset \mathrm{GL}(E)$ is said to be *self-adjoint* if $g^* \in G$ for any $g \in G$.

Theorem 6.6. *Let $G \subset \mathrm{GL}(E)$ be a self-adjoint algebraic linear group (either real or complex), $K = G \cap \mathrm{O}(E)$ (respectively $G \cap \mathrm{U}(E)$), $P = G \cap P(E)$.*

Then

$$G = KP,$$

and the decomposition $g = kp$, where $k \in K$, $p \in P$, is unique for each $g \in G$. To be more precise, if $\mathfrak{p} = \mathfrak{g} \cap S(E)$, then the mapping $\varphi: K \times \mathfrak{p} \to G$ defined by the formula

$$\varphi(k, y) = k \exp y,$$

is a diffeomorphism of real manifolds. If G is a complex algebraic group, then $\mathfrak{p} = i\mathfrak{k}$.

Proof. For any $g \in G$ we have $q = g^*g \in P$. Then $q = \exp x$, where $x \in S(F)$, and $q^m \in G$ for all $m \in \mathbb{Z}$. Using the fact that the group G is algebraic, we deduce that $\exp tx \in G$ for all $t \in \mathbb{R}$. Thus, $x \in \mathfrak{p}$. Let $p = \exp \frac{1}{2}x \in P$. One can easily see that $k = gp^{-1} \in K$, so $g = kp \in KP$. □

Corollary 1. *Under the assumptions of Theorem* 6.6 *the group G is diffeomorphic to $K \times \mathbb{R}^s$, where $s = \dim \mathfrak{p}$. In particular, G is connected if and only if K is connected, and in this case $\pi_1(G) \simeq \pi_1(K)$. If G is a complex algebraic group, then $s = \dim K = \dim_{\mathbb{C}} G$.*

Corollary 2. *Under the assumptions of Theorem* 6.6,

$$Z(G) = (Z(G) \cap K) \times (Z(G) \cap P),$$

and $Z(G) \cap P \simeq \mathbb{R}^u$ for some $u \geq 0$. If G is semisimple, then $Z(G) \subset Z(K)$, and for a complex semisimple group G we have $Z(G) = Z(K)$.

Using Proposition 6.2, one obtains the following corollary.

Corollary 3. *Any self-adjoint algebraic linear group (either complex or real) is reductive.*

The converse statement is also true (see Chap. 4, Sect. 2.2).

6.5. Chevalley Decomposition. Let G be an algebraic group over the field $K = \mathbb{C}$ or $\mathbb{R}$. The *unipotent radical* of G is its largest normal subgroup $\mathrm{Rad}_u G$ consisting of unipotent elements. The subgroup $\mathrm{Rad}_u G$ is connected and coincides with the set of all unipotent elements of the radical $\mathrm{Rad}\, G$ of the group G. The group G is reductive if and only if $\mathrm{Rad}_u G = \{e\}$. It turns out that an arbitrary algebraic group admits a decomposition into a semidirect product of the unipotent radical and a reductive algebraic subgroup (it is called the *Chevalley decomposition*, or the *algebraic Levi decomposition*). There is also an analogous decomposition of Lie algebras (see Chevalley [1955], Chap. 5, Sect. 4).

A *reductive Levi subgroup* of an algebraic group G is an algebraic subgroup $H \subset G$ such that $G = \mathrm{Rad}_u G \rtimes H$. Evidently, any reductive Levi subgroup is a maximal reductive subgroup of G.

Theorem 6.7 (see Onishchik and Vinberg [1990], Chap. 6). *In any algebraic group there exists a reductive Levi subgroup.*

Theorem 6.8 (see Onishchik and Vinberg [1990], Chap. 6). *Let H be a reductive Levi subgroup of an algebraic group G, and let S be a reductive algebraic subgroup of G. Then there exists $g \in \mathrm{Rad}_u G$ such that $gSg^{-1} \subset H$.*

Corollary 1. *Any two reductive Levi subgroups of an algebraic group are conjugate.*

Corollary 2. *An algebraic subgroup is a reductive Levi subgroup if and only if it is a maximal reductive subgroup.*

§ 7. Complexification and Real Forms

7.1. Complexification and Real Forms of Lie Algebras. Let $\mathfrak{g}$ be a real Lie algebra. The vector space $\mathfrak{g} \oplus \mathbb{C}$ is naturally equipped with the structure of a Lie algebra. The resulting complex Lie algebra is denoted by $\mathfrak{g}(\mathbb{C})$ and is called the *complexification* of $\mathfrak{g}$. For example, if $\mathfrak{g} = \mathfrak{n}_n(\mathbb{R})$ is the Lie algebra of all real nilpotent upper triangular matrices of order n, then $\mathfrak{g}(\mathbb{C}) = \mathfrak{n}_n(\mathbb{C})$ is the Lie algebra of all complex nilpotent upper triangular matrices of order n. An arbitrary element $z \in \mathfrak{g}(\mathbb{C})$ is uniquely represented in the form $z = x + iy$, where $x, y \in \mathfrak{g}$. Let $\overline{z} = x - iy$; the correspondence $z \mapsto \overline{z}$ is an antilinear automorphism of the Lie algebra $\mathfrak{g}(\mathbb{C})$, i.e.

$$\overline{z' + z''} = \overline{z'} + \overline{z''}, \qquad \text{where} \quad z', z'' \in \mathfrak{g}(\mathbb{C}),$$

$$\overline{\lambda z} = \overline{\lambda} \cdot \overline{z}, \quad \lambda \in \mathbb{C}, \ z \in \mathfrak{g}(\mathbb{C}).$$

The operation inverse (in a certain sense) to complexification is the *realification* of a complex Lie algebra. If $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$, denote by $\mathfrak{g}_{\mathbb{R}}$ the same Lie algebra, but considered over the field $\mathbb{R}$; $\mathfrak{g}_{\mathbb{R}}$ is then called the realification of the complex Lie algebra $\mathfrak{g}$.

In the Lie algebra $\mathfrak{g}$ the operator of the complex structure is defined, i.e. the operator of multiplication by i:

$$Ix = i \cdot x, \qquad \text{where} \quad x \in \mathfrak{g}_{\mathbb{R}}.$$

The operator I is linear over the field $\mathbb{R}$ and satisfies the relations

$$I^2 = -E,$$

$$I([x, y]) = [x, Iy], \qquad \text{where} \quad x, y \in \mathfrak{g}_{\mathbb{R}}.$$

Generally, given a real Lie algebra $\mathfrak{g}$, a *complex structure* on it is a linear operator I satisfying these two relations. Setting $(\alpha + i \cdot \beta)x = \alpha x + \beta \cdot I(x)$, we obtain the structure of a complex Lie algebra on $\mathfrak{g}$ (whose realification evidently coincides with the initial Lie algebra $\mathfrak{g}$).

If I is a complex structure on a real Lie algebra, then $-I$ is evidently also a complex structure. The complex structures I and $-I$ are said to be conjugate. For example, for an arbitrary complex Lie algebra $\mathfrak{g}$, by reversing the sign of the operator of the complex structure we obtain another structure of a complex Lie algebra. We denote this algebra by $\overline{\mathfrak{g}}$, and it is said to be conjugate to $\mathfrak{g}$. If $\lambda = \alpha + i\beta \in \mathbb{C}$, and $x \in \mathfrak{g}$, then the operation of multiplication by the number λ in the Lie algebra $\overline{\mathfrak{g}}$ is of the form $\lambda \cdot x = (\alpha - i\beta) \cdot x$, where the right-hand side features the scalar multiplication in $\mathfrak{g}$. Clearly, $\mathfrak{g}_{\mathbb{R}} \simeq \overline{\mathfrak{g}}_{\mathbb{R}}$. The Lie algebras $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ are isomorphic (over $\mathbb{C}$) if and only if $\mathfrak{g}$ admits an antilinear automorphism.

Let $\mathfrak{g}$ be an arbitrary complex Lie algebra. Then, as one can easily verify, $(\mathfrak{g}_{\mathbb{R}})(\mathbb{C}) \simeq \mathfrak{g} \oplus \overline{\mathfrak{g}}$. In particular, if $\overline{\mathfrak{g}} \simeq \mathfrak{g}$, then $(\mathfrak{g}_{\mathbb{R}})(\mathbb{C}) \simeq \mathfrak{g} \oplus \mathfrak{g}$.

A real subalgebra $\mathfrak{h}$ of a complex Lie algebra $\mathfrak{g}$ is said to be a *real form* for $\mathfrak{g}$ if the natural embedding $\mathfrak{h} \subset \mathfrak{g}$ (it would be more appropriate to write $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{R}}$) can be extended to an isomorphism $\mathfrak{h}(\mathbb{C}) \simeq \mathfrak{g}$ of complex Lie algebras. For example, $\mathfrak{n}_n(\mathbb{R})$ is a real form of $\mathfrak{n}_n(\mathbb{C})$. Another example: $\mathfrak{h} = \mathfrak{su}_2$ is a real form for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, and $\mathfrak{h}' = \mathfrak{sl}_2(\mathbb{R})$ is another real form for $\mathfrak{g}$. It is clear that two real forms $\mathfrak{h}'$, $\mathfrak{h}''$ for $\mathfrak{g}$ are isomorphic (over $\mathbb{R}$) if and only if there exists an automorphism $\varphi \in \operatorname{Aut}_{\mathbb{C}}\mathfrak{g}$ such that $\varphi(\mathfrak{h}') = \mathfrak{h}''$.

Let $\mathfrak{h}$ be a real form of a complex Lie algebra $\mathfrak{g}$. Then the natural operation of complex conjugation on $\mathfrak{h}(\mathbb{C})$ generates an involutory antilinear automorphism σ of $\mathfrak{g}$ such that $\mathfrak{h}$ coincides with the set $\mathfrak{g}^\sigma$ of its fixed points. Conversely, if σ is an arbitrary involutory antilinear automorphism of a complex Lie algebra $\mathfrak{g}$, then $\mathfrak{g}^\sigma$ is a real form of $\mathfrak{g}$. Isomorphic real forms correspond to involutory antilinear homomorphisms conjugate by elements from $\operatorname{Aut}_{\mathbb{C}}\mathfrak{g}$. By a real structure in $\mathfrak{g}$ we will mean an arbitrary involutory antilinear automorphism of this algebra.

If a complex Lie algebra $\mathfrak{g}$ has a real form, then, as follows from above, $\overline{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}$. In particular, in this case we have $(\mathfrak{g}_{\mathbb{R}})(\mathbb{C}) \simeq \mathfrak{g} \oplus \mathfrak{g}$.

Any complex semisimple Lie algebra has a real form (see Chap. 4, Sect. 1.1, Example 6), and may have even more than one. However, there are complex Lie algebras having no real forms.

Example 1. Let $\mathfrak{g} = \mathbb{C} \in_{\varphi} \mathbb{C}^2$ be the semidirect sum of Lie algebras corresponding to a homomorphism $\varphi\colon \mathbb{C} \to \mathfrak{gl}_2(\mathbb{C})$ such that $\varphi(1) = \operatorname{diag}(1, 2i)$. Suppose that $\mathfrak{g}$ has a real form $\mathfrak{h}$; then $\mathfrak{h} \simeq \mathbb{R} \in_{\psi} \mathbb{R}^2$, where $\psi\colon \mathbb{C} \to \mathfrak{gl}_2(\mathbb{R})$ is a homomorphism. Let λ_1, λ_2 be the eigenvalues of the operator $\psi(1)$, $\lambda_1 \cdot \lambda_2 \neq 0$. The isomorphism between $\mathfrak{h} \otimes \mathbb{C}$ and $\mathfrak{g}$ implies that $\lambda_1 = \lambda \cdot 1$, $\lambda_2 = \lambda \cdot 2i$ for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Since λ_1, λ_2 cannot be both real (otherwise $\lambda \in \mathbb{R}$, and therefore $2i \in \mathbb{R}$), they must be complex conjugates, in particular $|\lambda_1| = |\lambda_2|$. But then $|\lambda| = |2i\lambda|$, which is impossible. Thus, the Lie algebra $\mathfrak{g}$ has no real form.

The operation of complexification preserves basic algebraic constructions in Lie algebras. For example, $[\mathfrak{g}(\mathbb{C}), \mathfrak{g}(\mathbb{C})] = [\mathfrak{g}, \mathfrak{g}](\mathbb{C})$, from which we can deduce that the Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is solvable (nilpotent) if and only if the complex Lie algebra $\mathfrak{g}(\mathbb{C})$ is solvable (nilpotent). Next, $(\operatorname{rad}\mathfrak{g})(\mathbb{C}) = \operatorname{rad}(\mathfrak{g}(\mathbb{C}))$, which implies that the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}(\mathbb{C})$ can be semisimple only simultaneously. Finally, if $\mathfrak{g} = \mathfrak{s} + \mathfrak{r}$ is a Levi decomposition of a real Lie algebra

$\mathfrak{g}$ ($\mathfrak{r}$ is the radical, $\mathfrak{s}$ a Levi subalgebra), then $\mathfrak{g}(\mathbb{C}) = \mathfrak{s}(\mathbb{C}) + \mathfrak{r}(\mathbb{C})$ is a Levi decomposition of the Lie algebra $\mathfrak{g}(\mathbb{C})$ over $\mathbb{C}$.

7.2. Complexification and Real Forms of Lie Groups. Let G be a real Lie group. By analogy with the case of Lie algebras, we can consider the question of constructing the complexification $G(\mathbb{C})$ of G. If G is algebraic (to be more precise, G is isomorphic to the group of $\mathbb{R}$-points of some algebraic group defined over $\mathbb{R}$), then it is natural to mean by its complexification the group of its G-points $G(\mathbb{C})$. However, if G is an arbitrary Lie group, then there is no candidate naturally entitled to the position of the complexification of G. This is not a mere accident, because if we understand the complexification as an embedding in a complex Lie group with some additional properties, such a complexification does not necessarily exist. For example, it is not difficult to show that the Lie group $\mathcal{A} = \widetilde{\mathrm{SL}}_2(\mathbb{R})$ is not isomorphic to any virtual Lie subgroup of a complex Lie group (this is related to the fact that the group $\mathcal{A}$ is simple and its centre is infinite; see Example 6 of Sect. 5.2 above).

For linear Lie groups a very natural complexification can be constructed. Let G be a virtual Lie subgroup of $\mathrm{GL}(V)$, where V is a finite-dimensional vector space over $\mathbb{R}$. Embedding $\mathrm{GL}(V)$ into $\mathrm{GL}(V(\mathbb{C}))$, we denote by $G(\mathbb{C})$ the intersection of all complex virtual Lie subgroups of $\mathrm{GL}(V(\mathbb{C}))$ containing G. In other words, $G(\mathbb{C})$ is the smallest complex virtual Lie subgroup of $\mathrm{GL}(V(\mathbb{C})$ containing G. The subgroup $G(\mathbb{C})$ can be described constructively as follows. Consider a tangent subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, together with its complexification $\mathfrak{g}(\mathbb{C}) \subset \mathfrak{gl}(V(\mathbb{C}))$. To the subalgebra $\mathfrak{g}(\mathbb{C})$ there corresponds a connected virtual Lie subgroup $\hat{G} \subset \mathrm{GL}(V(\mathbb{C}))$. We can easily verify that $G(\mathbb{C})^0 = \hat{G}$, and $G(\mathbb{C}) = G \cdot \hat{G}$. The operation of complex conjugation induces an involution in $G(\mathbb{C})$ whose set of fixed points coincides with G. If G is a real algebraic group, then $G(\mathbb{C})$ coincides with the set of all $\mathbb{C}$-points of G.

For a complex Lie group G we naturally define its realification $G_{\mathbb{R}}$.

Now consider the notion of a real form of a complex Lie group. Let G be a complex Lie group, and H a real Lie subgroup of it (i.e. H is a Lie subgroup of $G_{\mathbb{R}}$). The Lie subgroup H is said to be a *real form* of G if its tangent algebra $\mathfrak{h}$ is a real form of the Lie algebra $\mathfrak{g}$, and $G = H \cdot G^0$ (the last condition means that H intersects any connected component of G). If G_1 is a real form of a linear Lie group, then it is a real form of $G_1(\mathbb{C})$ if G_1 is closed in $G_1(\mathbb{C})$ (which is not necessarily the case).

A *real structure* in a complex Lie group G is an involutory smooth (over $\mathbb{R}$) automorphism $s\colon G \to G$ such that its differential ds is a real structure in the tangent Lie algebra $\mathfrak{g}$ of the Lie group G. For example, if G is an algebraic group defined over $\mathbb{R}$, then the complex conjugation induces a real structure on the set $G(\mathbb{C})$ of all its $\mathbb{C}$-points.

If s is a real structure on the Lie group G, then the set G^s of fixed points of the involution s is, as can be easily verified, a real form of G. If G is algebraic, then G^s is also algebraic.

Example. The operation of complex conjugation $A \mapsto \overline{A}$ in the group $\mathrm{GL}_n(\mathbb{C})$ is a real structure on $\mathrm{GL}_n(\mathbb{C})$, the corresponding real form being $\mathrm{GL}_n(\mathbb{R})$. A number of similar examples for complex semisimple Lie groups can be found in Chap. 4, Sect. 1.1.

7.3. Universal Complexification of a Lie Group. As noted above, for an arbitrary Lie group a complexification considered as an embedding in a complex Lie group does not necessarily exist. Following Hochschild [1966], we now make the notion of the complexification more precise.

Let G be a real Lie group. A *universal complexification* of G is a homomorphism $\gamma_G: G \to G^+$ into a connected complex Lie group G^+ such that for any homomorphism $\varphi: G \to G_1$ into a connected complex Lie group G_1 there exists a unique homomorphism $\varphi^+: G^+ \to G_1$ for which $\varphi^+ \circ \gamma_G = \varphi$, i.e. the diagram

$$\begin{tikzcd} & G^+ \arrow[d, "\varphi^+"] \\ G \arrow[ur, "\gamma_G"] \arrow[r, "\varphi"'] & G_1 \end{tikzcd}$$

commutes.

The homomorphism γ_G is the universal object in the category of homomorphisms of G into connected complex Lie groups. Two universal complexifications of a Lie group are, evidently, equivalent in a natural (say, categorical) sense of the word.

We now show that an arbitrary connected Lie group has a universal complexification. Let $\pi: \widetilde{G} \to G$ be a simply-connected covering of the Lie group G, and $C = \operatorname{Ker} \pi$. Consider a tangent Lie algebra $\mathfrak{g}$ of the Lie group G and its embedding in the complexification $f: \mathfrak{g} \hookrightarrow \mathfrak{g}(\mathbb{C})$. Denote by $\widetilde{G}^{\mathbb{C}}$ the simply-connected group corresponding to the Lie algebra $\mathfrak{g}(\mathbb{C})$. The embedding f induces the homomorphism $\varphi: G \to \widetilde{G}^{\mathbb{C}}$ of simply-connected Lie groups. We can easily verify that $\varphi(C) \subset Z(\widetilde{G}^{\mathbb{C}})$. Let C^* be the smallest complex Lie subgroup of $\widetilde{G}^{\mathbb{C}}$ containing $\varphi(C)$ (C^* coincides with the intersection of all complex Lie subgroups of $\widetilde{G}^{\mathbb{C}}$ containing $\varphi(C)$). Let $G^+ = \widetilde{G}^{\mathbb{C}}/C^*$. The homomorphism φ induces the homomorphism $G \to G^+$, which, as we can easily see, provides the universal complexification of G.

Example 2. If $\gamma: \widetilde{SL}_2(\mathbb{R}) \to G^+$ is a universal complexification of the Lie group $\mathcal{A} = \widetilde{\mathrm{SL}}_2(\mathbb{R})$, then $G^+ \simeq \mathrm{SL}_2(\mathbb{C})$, and $\operatorname{Ker}\gamma = \{z^2 | z \in Z(\mathcal{A})\} = \Lambda(\mathcal{A}) \simeq \mathbb{Z}$.

We can see from this example that in general the universal complexification γ_G is not an injection. In the general case it is not even locally injective (see below). Clearly, $\operatorname{Ker}\gamma_G \subset Z(G)$.

If a Lie group G is linear, then its complexification $G(\mathbb{C})$ (see Sect. 7.2) is not necessarily isomorphic to the group G^+. For example, if G is a one-parameter subgroup $\varphi(t) = e^t$ of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, then $G(\mathbb{C}) = \mathbb{C}^*$, and $G^+ \simeq \mathbb{C}$.

Proposition 7.1 (Hochschild [1966]). *Let G be a connected Lie group G. If G admits a faithful (locally faithful) finite-dimensional linear representation, then any universal complexification $\gamma_G: G \to G^+$ is an injective (respectively, locally injective) mapping.*

Consider two special cases. If a connected Lie group G is semisimple, then $\operatorname{Ker}\gamma_G$ coincides with the linearizer $L(G)$. Hence it follows that γ_G is injective if and only if the semisimple Lie group G is linear. Now let G be a connected solvable Lie group. We will show that in this case γ_G is injective. With that in mind, consider the subgroup $\varphi(C)$ appearing in the construction of a universal complexification. Since the simply-connected Lie group $\widetilde{G^{\mathbb{C}}}$ in the case under consideration is solvable, its subgroup $\varphi(G)$ is closed in $\widetilde{G^{\mathbb{C}}}$ (see Chap. 2, Sect. 3.4). But then $\varphi(C)$ is a discrete subgroup of $\widetilde{G^{\mathbb{C}}}$, and hence $C^* = C$. This means that $\operatorname{Ker}\gamma_G = \{e\}$, i.e. γ_G is injective.

Theorem 7.1 (Hochschild [1966]). *Let $\gamma_G: G \to G^+$ be a universal complexification of a connected Lie group $\widetilde{G}$. Then $\gamma_G(G)$ is a real form of G^+.*

Theorem 7.2 (Hochschild [1966]). *Let G be a connected Lie group, $G = S \cdot R$ its Levi decomposition (where R is the radical, S a Levi subgroup), and $\gamma_G: G \to G^+$ a universal complexification. Then the following statements are equivalent:*

(i) *γ_G is locally injective;*

(ii) *the restriction of γ_G to R is locally injective;*

(iii) $\operatorname{Ker}\gamma_G = \operatorname{Ker}\gamma_S$ *(where $\gamma_S: S \to S^+$ is a universal complexification for S);*

(iv) *the subgroup S is closed in G;*

(v) *the subgroup $S \cap \operatorname{Ker}\gamma_G$ is closed in G.*

Here if γ_G is locally injective, then the natural mapping $S^+ \to G^+$ (induced by the embedding $S \hookrightarrow G$) is an embedding of S^+ in G^+ as a closed Levi subgroup.

We have given above (see Sect. 4.1) an example of a Lie group G for which a Levi subgroup S is not closed in G. By virtue of Theorem 7.2, for such a Lie group G the mapping γ_G is not locally injective, i.e. $\dim\operatorname{Ker}\gamma_G > 0$.

Theorem 7.3 (Hochschild [1966]). *Let $\gamma_G: G \to G^+$ be a universal complexification of a connected Lie group G. If G has a locally faithful finite-dimensional linear representation, then G^+ is linearizable.*

§ 8. Splittings of Lie Groups and Lie Algebras

Ado's theorem (see Sect. 5.3) makes it possible to regard any Lie algebra as a linear one. But this fact does not introduce any substantial simplification into the analysis of Lie algebras (and groups). However, one class of linear Lie groups and algebras can be described in detail. It is the class of algebraic linear Lie groups and algebras (see Sect. 6, and Springer [1989]). The methods of the theory of algebraic groups turn out to be applicable to the analysis of arbitrary Lie groups and algebras. Such applications are related to the notions of splittability and Malcev splittability.

8.1. Malcev Splittable Lie Groups and Lie Algebras. Here we will mainly consider real Lie groups and algebras (the analysis of the complex case is similar but even simpler). Let G be a real algebraic linear group. Consider its Chevalley decomposition $G = F \cdot U$, where $U = \mathrm{Rad}_u G$ is a unipotent radical, and F a reductive Levi subgroup ($F = T \cdot S$, where S is a Levi subgroup G, and T is an abelian Lie subgroup consisting of semisimple elements). The radical of the Lie group G coincides with $T \cdot U$, whence the Levi decomposition for G is of the form $G = S \cdot (T \cdot U)$. A similar decomposition $\mathfrak{g} = \mathfrak{s} + \mathfrak{t} + \mathfrak{u}$ holds for the tangent Lie algebra $\mathfrak{g}$ of the Lie group G. In order to transfer some useful properties of algebraic groups and their tangent Lie algebras to a wider class of Lie groups and algebras, we introduce the notions of a Malcev splittable Lie group and Lie algebra.

A Lie algebra $\mathfrak{g}$ is said to be *Malcev splittable* if it admits a decomposition of the form $\mathfrak{g} = \mathfrak{t} + \mathfrak{s} + \mathfrak{n}$, where $\mathfrak{n}$ is a nilpotent ideal, $\mathfrak{s}$ a semisimple Lie subalgebra, $\mathfrak{t}$ an abelian Lie subalgebra such that $[\mathfrak{t}, \mathfrak{s}] = \{0\}$, and the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$ is faithful and completely reducible. Evidently (see Chap. 2, Sect. 5.1) $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$, $\mathfrak{s}$ is a Levi subalgebra, and $\mathfrak{t}+\mathfrak{n}$ is the radical of $\mathfrak{g}$.

A connected Lie group G is said to be *Malcev splittable* if it admits a decomposition of the form $G = T \cdot S \cdot N$, where N is a connected normal nilpotent Lie subgroup, S a semisimple Lie subgroup, T a connected abelian Lie subgroup such that $(T, S) = \{e\}$, and the adjoint action of T on the tangent Lie algebra $\mathfrak{g}$ of the Lie group G is locally faithful and semisimple. The subgroup N is the nilradical of G, S is a Levi subgroup, and $T \cdot N$ is the radical of G.

It follows from the definition of a splittable Lie algebra $\mathfrak{g}$ that $\mathfrak{t} \cap \mathfrak{n} = \{0\}$ and that $\mathfrak{t}$ is isomorphic to a subalgebra of $\mathrm{Der}\, \mathfrak{n}$, the Lie algebra of derivations of $\mathfrak{n}$. Similarly, for a splittable Lie group G, the subgroup $T \cap N$ is discrete, and T is locally isomorphic to a subgroup of $\mathrm{Aut}\, N$. As follows from above, any connected algebraic linear Lie group is splittable, and its tangent Lie algebra is a splittable Lie algebra. There exist also nonalgebraic splittable Lie groups (and Lie algebras). For example, the Lie group of the form $\mathbb{R} \ltimes_{\varphi} \mathbb{R}^n$ is splittable if the image of the homomorphism $\varphi \colon \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$ consists of semisimple matrices (here $S = \{e\}$, $T \simeq \mathbb{R}$, $N \simeq \mathbb{R}^n$), but if the subgroup $\operatorname{Im}\varphi$ is not algebraic in $\mathrm{GL}_n(\mathbb{R})$, then the Lie group G is not isomorphic to any real algebraic group.

On the other hand, there also exist nonsplittable Lie groups. For example, such is the simply-connected solvable Lie group $\mathbb{R} \ltimes_\varphi \mathbb{R}^2$ of Sect. 5.2, Example 5. Furthermore, its Lie algebra is also nonsplittable.

8.2. Definition of Splitttings of Lie Groups and Lie Algebras. Starting with the pioneering work of Malcev [1945a], a number of papers consider the construction of embeddings of Lie groups (and Lie algebras) in splittable ones. These embeddings are chosen in such a way that they satisfy some additional restrictions (for example, that the image of the embedding is minimal, or universal, or dense in some sense, etc.). Such embeddings are called splittings.

Definition. A *splitting* of a Lie group (Lie algebra) is its embedding in a splittable Lie group (Lie algebra).

One can prove that any connected Lie group or Lie algebra (over $K = \mathbb{R}$ or $\mathbb{C}$) has at least one splitting. In fact, a single Lie algebra or a connected Lie group has many different splittings. We consider only some of them, and in the case of a Lie group restrict our attention to simply-connected groups only.

The first of the splittings considered here for solvable Lie algebras over the field $\mathbb{C}$ was constructed by Malcev [1945a]. Later other constructions of the same splitting were given (see, for example, Auslander and Brezin [1968], Florac [1971]). For our treatment of this splitting we choose an axiomatic approach.

Let $\mathfrak{g}$ be a real Lie algebra. A Malcev splitting for $\mathfrak{g}$ is an embedding $\alpha\colon \mathfrak{g} \hookrightarrow M(\mathfrak{g})$ in a splittable Lie algebra $M(\mathfrak{g}) = \mathfrak{t} + \mathfrak{s} + \mathfrak{n}$ such that $\alpha(\mathfrak{g})$ is an ideal in $M(\mathfrak{g})$, and $M(\mathfrak{g}) = \alpha(\mathfrak{g}) \ni \mathfrak{t} = \alpha(\mathfrak{g}) + \mathfrak{n}$.

For a connected Lie group G, a *Malcev splitting* is an embedding $\alpha\colon G \hookrightarrow M(G)$ in a splittable Lie group $M(G) = T \cdot S \cdot N$ such that $\alpha(G)$ is a normal Lie subgroup of $M(G)$, and $M(G) = T \ltimes \alpha(G) = \alpha(G) \cdot N$. If $M(G)$ is simply-connected, the splitting is said to be simply-connected.

Theorems on the existence and uniqueness of the Malcev decomposition are given below. In the meantime, we consider another splitting of Lie groups and algebras.

An *algebraic splitting* of a Lie algebra $\mathfrak{g}$ is an embedding $\alpha_a\colon \mathfrak{g} \hookrightarrow A(\mathfrak{g})$ in a linear algebraic Lie algebra $A(\mathfrak{g}) = \mathfrak{t}_a + \mathfrak{s}_a + \mathfrak{n}_a$ such that $\alpha_a(\mathfrak{g})$ is an ideal of $A(\mathfrak{g})$, $A(\mathfrak{g}) = \mathfrak{t}_a + \alpha_a(\mathfrak{g})$, and ${}^a(\alpha_a(\mathfrak{g})) = A(\mathfrak{g})$, where ${}^a(\alpha_a(\mathfrak{g}))$ is the algebraic closure of the subalgebra $\alpha_a(\mathfrak{g})$ in $A(\mathfrak{g})$.

An algebraic splitting for Lie groups is defined similarly, but in a more complicated way (see Gorbatsevich [1979]). In what follows we show that the algebraic splitting is closely connected with the Malcev splitting.

8.3. Theorem on the Existence and Uniqueness of Splittings

Theorem 8.1 (see Gorbatsevich [1979]). *A Malcev splitting exists for any Lie algebra and any simply-connected Lie group over* $\mathbb{R}$.

Proof. The cases of Lie algebras and simply-connected Lie groups are equivalent, so we consider just splittings of simply-connected Lie groups. There are two essentially different constructions of the Malcev splitting. One of them originates in the initial construction of Malcev [1945a] (improved by Auslander, see Auslander [1973]). Another is based on the use of generating functions and goes back to Hochschild and Mostow (see Hochschild and Mostow [1978]). We present the splitting construction scheme following the ideas of Malcev.

Suppose that $G = S \cdot R$ is a Levi decomposition of a simply-connected Lie group G. Let $G^* = \mathrm{Ad}_G(G)$, where Ad_G is the adjoint representation, $G^* \subset \mathrm{GL}(\mathfrak{g})$. Denote by ${}^aG^*$ the algebraic closure of G^* in $\mathrm{GL}(\mathfrak{g})$, and consider the Chevalley decomposition ${}^aG^* = F^* \cdot N^*$, where $F^* = T^* \cdot S^*$ is reductive, and N^* is a unipotent radical. The subgroup $T^* \cap (S^* \cdot N^*)$ is finite. Consider the natural epimorphism $t^*: {}^*G^a \to {}^aG^*/S^* \cdot N^* = \hat{T}$ with the kernel $S^* \cdot N^*$. The group G is connected, whence $t^*(\mathrm{Ad}_G(G)) \subset (\hat{T})^0$. The homomorphism $t^* \circ \mathrm{Ad}_G: G \to (T)^0$ can be lifted to the homomorphism $\hat{t}: G \to \widetilde{T}$, where $\widetilde{T}$ is the simply-connected covering over $(\hat{T})^0$. Let $T = \widetilde{t}(G)$.

Since $\mathrm{Aut}\,\mathfrak{g}$ is algebraic and $G^* \subset \mathrm{Aut}\,\mathfrak{g}$, T^* is a subgroup of $\mathrm{Aut}\,\mathfrak{g}$. This makes it possible to consider the action of T on G by automorphisms (since $\mathrm{Aut}\,\mathfrak{g} \simeq \mathrm{Aut}\,G$). The corresponding semidirect product $T \ltimes G$ yields the required Lie group $M(G)$. The natural embedding of G in $T \ltimes G = M(G)$ is the Malcev splitting for G. □

If $G = S \cdot R$ is a Levi decomposition of a simply-connected Lie group, one can show that $M(G) = S \cdot M(R)$. Therefore the study of the Malcev splitting for solvable Lie groups (and algebras) is of the utmost importance.

Example 1. Consider the Lie group $G = \mathbb{R} \ltimes_\varphi \mathbb{R}^n$, i.e. the semidirect product corresponding to the homomorphism $\varphi: \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$ of the form $\varphi(t) = \exp(t \cdot X)$, where X is some matrix from $\mathfrak{gl}_n(\mathbb{R})$, $X \neq 0$. Let $X = X_s + X_n$. This is the Jordan decomposition (X_s is a semisimple matrix, X_n is nilpotent, $[X_s, X_n] = 0$). Then $M(G) = \mathbb{R}^2 \ltimes_{\varphi_s \times \varphi_n} \mathbb{R}^n$, where $\varphi_s \times \varphi_n: \mathbb{R}^2 \to \mathrm{GL}_n(\mathbb{R})$ is the homomorphism defined as follows:

$$(\varphi_s \times \varphi_n)(s, t) = \exp(X_s \cdot s) \cdot \exp(X_n \cdot t) = \exp(s \cdot X_s + t \cdot X_n).$$

In particular, if the matrix X is semisimple, then $M(G) \simeq G \times \mathbb{R}$.

Two splittings $\alpha: G \hookrightarrow M$, and $\alpha': G \hookrightarrow M'$ (where M, M' are splittable Lie groups) are said to be *isomorphic* if there exists an isomorphism $\varphi: M \to M'$ such that $\alpha = \varphi \circ \alpha$. The isomorphy of splittings for Lie algebras is defined in a similar manner.

Theorem 8.2 (see Gorbatsevich [1979]). (i) *A Malcev splitting for a real Lie algebra is unique up to isomorphism.*

(ii) *A simply-connected Malcev splitting for a simply-connected Lie group is unique up to isomorphism.*

This theorem implies, in particular, that different constructions of the Malcev splitting cited above yield the same result.

We now proceed to the questions of existence and uniqueness of an algebraic splitting. Let $\mathfrak{g}$ be a real Lie algebra and $\alpha: \mathfrak{g} \hookrightarrow M(\mathfrak{g}) = \mathfrak{t} + \mathfrak{s} + \mathfrak{n}$ its Malcev splitting. The adjoint representation of the Lie algebra $M(\mathfrak{g})$ induces the homomorphism $f: \mathfrak{t} \to \operatorname{Der}(\mathfrak{s} + \mathfrak{n})$. Let ${}^{a}(\operatorname{Im} f)$ be the algebraic closure of the Lie subalgebra $\operatorname{Im} f$ in the algebraic Lie algebra $\operatorname{Der}(\mathfrak{s} + \mathfrak{n})$. Let $A(\mathfrak{g}) = {}^{a}(\operatorname{Im} f) \in (\mathfrak{s}+\mathfrak{n})$. We can easily show that the composition of the embedding α and the natural embedding $M(\mathfrak{g}) \subset A(\mathfrak{g})$ yields the algebraic splitting $\alpha_a: \mathfrak{g} \hookrightarrow A(\mathfrak{g})$. This proves the following theorem.

Theorem 8.3. *Any real Lie algebra has an algebraic splitting.*

A construction of the algebraic splitting for Lie groups is given in Gorbatsevich [1979]. The above considerations reveal a close connection between the Malcev splitting and the algebraic splitting. The next theorem is proved in complete analogy with Theorem 8.2.

Theorem 8.4. *An algebraic splitting of a Lie algebra is unique up to isomorphism.*

Example 2. Let $\mathfrak{g} = \mathbb{R} \in_{\varphi} \mathbb{R}^n$ be the semidirect sum corresponding to the homomorphism $\varphi: \mathbb{R} \to \mathfrak{gl}_n(\mathbb{R})$. Consider the one-dimensional subalgebra $\operatorname{Im}\varphi$ of $\mathfrak{gl}_n(\mathbb{R})$, and let $\mathfrak{t} = {}^{a}(\operatorname{Im}\varphi)$ be its algebraic closure. Then $\mathfrak{t}$ is an abelian algebraic subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ containing, in particular, the components $\varphi(1)_s$ and $\varphi(1)_n$ of the Jordan decomposition for the matrix $\varphi(1)$. The Lie algebra $\mathfrak{t} \in \mathbb{R}^n$ is algebraic, and the natural embedding of the Lie algebra $\mathbb{R} \in_{\varphi} \mathbb{R}^n$ into $\mathfrak{t} \in \mathbb{R}^n$ is an algebraic splitting of the Lie algebra $\mathfrak{g}$.

One application of notion of the splitting of a Lie algebra is the reduction of all solvalbe Lie algebras to the classification of nilpotent Lie algebras (this, however, gives rise to the additional problem of describing orbits of some unipotent groups), see Malcev [1945a]. Another application of the notion of splitting is to the study of homogeneous spaces of Lie groups (see Gorbatsevich and Onishchik [1988]) and flows on them.

§ 9. Cartan Subalgebras and Subgroups. Weights and Roots

9.1. Representations of Nilpotent Lie Algebras. This section is devoted to the description of linear representations of nilpotent Lie algebras over the field $\mathbb{C}$ (or over any algebraically closed field of characteristic 0). As is known from linear algebra, to any linear operator $X \in \mathfrak{gl}(V)$ there corresponds the decomposition of the space V into root subspaces

$$V^{\lambda}(X) = \{v \in V | (X - \lambda E)^m v = 0 \quad \text{for some} \quad m \in \mathbb{N}\}.$$

The operator X defines a representation of the one-dimensional Lie algebra $\mathbb{C}$ in the space V. It turns out that a similar decomposition into "root subspaces" exists for any linear representation of a nilpotent complex Lie algebra.

Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of the Lie algebra $\mathfrak{g}$, and $\lambda \in \mathfrak{g}^*$. We set

$$V^{\lambda}(\mathfrak{g}) = \{v \in V | \exists m \in \mathbb{N} \quad \text{such that} \quad (\rho(x) - \lambda(x)E)^m v = 0 \quad \text{for all} \quad x \in \mathfrak{g}\}.$$

If $V^{\lambda}(\mathfrak{g}) \neq 0$, then $V^{\lambda}(\mathfrak{g})$ is said to be a *root subspace* of the representation ρ. In this case λ is a weight, and the weight subspace $V_{\lambda}(\mathfrak{g})$ is contained in $V^{\lambda}(\mathfrak{g})$. Root subspaces corresponding to different weights of the representation are linearly independent.

Theorem 9.1 (see Goto and Grosshans [1978], Jacobson [1962]). *Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of a nilpotent complex Lie algebra. Then*

$$V = \bigoplus_{i=1}^{s} V^{\lambda_i}(\mathfrak{g}), \tag{7}$$

where $\lambda_i \in \mathfrak{g}^*$ are different weights of the representation ρ. In an appropriate basis of the space V the matrices of the operators $\rho(x)$ are of the form

$$\rho(x) = \begin{pmatrix} \begin{array}{ccc} \lambda_1(x) & & * \\ & \ddots & \\ 0 & & \lambda_1(x) \end{array} & & 0 \\ & \ddots & \\ 0 & & \begin{array}{ccc} \lambda_s(x) & & * \\ & \ddots & \\ 0 & & \lambda_s(x) \end{array} \end{pmatrix} \tag{8}$$

Proof. The condition that the Lie algebra $\mathfrak{g}$ is nilpotent implies that for any $x \in \mathfrak{g}$ the root spaces $V^{\lambda}(\rho(x))$ are invariant under $\rho(\mathfrak{g})$. The theorem is then proved by induction on $\dim V$. If for any $x \in \mathfrak{g}$ the operator $\rho(x)$ has a unique eigenvalue $\lambda(x)$, then, by virtue of Lie's theorem, there exists a basis in V in which all $\rho(x)$ have matrices of the form (8) with a single box ($s = 1$). Therefore $\lambda \in \mathfrak{g}^*$ is the only weight of the representation, and $V = V^{\lambda}(\mathfrak{g})$. If there exists $x_0 \in \mathfrak{g}$ such that $V = \bigoplus_{i=1}^{p} V^{\mu_i}(\rho(x_0))$, where μ_i are eigenvalues of the operator $\rho(x_0)$, and $p > 1$, then $V^{\mu_i}(\rho(x_0))$ are invariant under $\rho(\mathfrak{g})$, which enables one to apply the inductive hypothesis. □

9.2. Weights and Roots with Respect to a Nilpotent Subalgebra. Let $\mathfrak{g}$ be a complex Lie algebra, $\mathfrak{h}$ a nonzero nilpotent subalgebra of it, and $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ a complex linear representation. Applying Theorem 9.1 to the representation $\rho|\mathfrak{h}$, we obtain the decomposition

$$V = \bigoplus_{i=1}^{s} V^{\lambda_i}, \tag{9}$$

where $\lambda_i: \mathfrak{h} \to \mathbb{C}$ $(i = 1, \dots, s)$ are different linear forms, and V^{λ_i} are the corresponding root subspaces. The forms $\lambda_1, \dots, \lambda_s$ are different weights of the representation $\rho|\mathfrak{h}$; they are also called the *weights of the representation* ρ *with respect to* $\mathfrak{h}$. We shall write $\Phi_\rho(\mathfrak{h}) = \{\lambda_1, \dots, \lambda_s\} \subset \mathfrak{h}^*$.

The above considerations can be applied to the adjoint representation $\rho = \operatorname{ad}$ of the Lie algebra $\mathfrak{g}$. The fact that $\mathfrak{h}$ is nilpotent implies that $\mathfrak{h} \subset \mathfrak{g}^0$. Thus, $0 \in \Phi_{\mathrm{ad}}(\mathfrak{h})$. Nonzero weights of the representation ad with respect to $\mathfrak{h}$ are said to be the *roots* of the Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{h}$. Denote the system of all roots by $\Delta_{\mathfrak{g}}(\mathfrak{h}) = \Phi_{\mathrm{ad}}(\mathfrak{h}) \setminus \{0\}$. The decomposition (9) in this case is of the form

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi_{\mathrm{ad}}(\mathfrak{h})} \mathfrak{g}^\alpha = \mathfrak{g}^0 \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathfrak{g}}(\mathfrak{h})} \mathfrak{g}^\alpha \right) \tag{10}$$

and is called the *root decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Proposition 9.1. *Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation. If $\alpha \in \Phi_{\mathrm{ad}}(\mathfrak{h})$, $\lambda \in \Phi_\rho(\mathfrak{h})$, then for all $x \in \mathfrak{g}^\alpha$ we have*

$$\rho(x)(V^\lambda) \begin{cases} \subset V^{\lambda+\alpha} & \text{if } \lambda + \alpha \in \Phi_\rho(\mathfrak{h}), \\ = 0 & \text{otherwise}. \end{cases}$$

In particular, for all $\alpha, \beta \in \Phi_{\mathrm{ad}}(\mathfrak{h})$

$$[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \begin{cases} \subset \mathfrak{g}^{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi_{\mathrm{ad}}(\mathfrak{h}), \\ = 0 & \text{otherwise}. \end{cases}$$

Thus, the subspace $\mathfrak{g}^0$ is a subalgebra, and each V^λ is invariant with respect to $\mathfrak{g}^0$.

Proposition 9.2. *If $\alpha, \beta \in \Phi_{\mathrm{ad}}(\mathfrak{h})$ and $\alpha + \beta \neq 0$, then $k_{\mathfrak{g}}(x, y) = 0$ for all $x \in \mathfrak{g}^\alpha$, $y \in \mathfrak{g}^\beta$.*

Proof. Proposition 9.1 and the finiteness of the system of weights imply that the operator $(\operatorname{ad} x)(\operatorname{ad} y)$ is nilpotent. □

9.3. Cartan Subalgebras. Let $\mathfrak{g}$ be a Lie algebra over the field $K = \mathbb{R}$ or $\mathbb{C}$. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a *Cartan subalgebra* if $\mathfrak{h}$ is nilpotent and coincides with its normalizer. Any Cartan subalgebra is a maximal nilpotent subalgebra but the converse statement does not hold. If, as in Sect. 9.2, we set

$$\mathfrak{g}^0 = \{y \in \mathfrak{g} | (\operatorname{ad} x)^m y = 0 \quad \text{for some} \quad m > 0 \quad \text{and all} \quad x \in \mathfrak{h}\},$$

then the nilpotent subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra if and only if $\mathfrak{g}^0 = \mathfrak{h}$. Thus, in the case $K = \mathbb{C}$ the root decomposition (10) with respect to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathfrak{g}}(\mathfrak{h})} \mathfrak{g}^\alpha \right). \tag{11}$$

Let $x \in \mathfrak{g}$. We set

$$\mathfrak{g}_x^0 = \{y \in \mathfrak{g} | (\operatorname{ad} x)^m y \quad \text{for some} \quad m > 0\}.$$

The number $l_x = \dim \mathfrak{g}_x^0 > 0$ coincides with the multiplicity of the zero root of the characteristic polynomial of the operator $\operatorname{ad} x$ in $\mathfrak{g}$. The number $\operatorname{rk} \mathfrak{g} = \min\limits_{x \in \mathfrak{g}} l_x$ is called the *rank* of the Lie algebra $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is said to be *regular* if $l_x = \operatorname{rk} \mathfrak{g}$ and *singular* if $l_x > \operatorname{rk} \mathfrak{g}$. Regular elements form a subset in $\mathfrak{g}$ open in the Zariski topology.

Theorem 9.2. *If x is a regular element of the Lie algebra $\mathfrak{g}$, then $\mathfrak{h} = \mathfrak{g}_x^0$ is a Cartan subalgebra of $\mathfrak{g}$, and* $\dim \mathfrak{h} = \operatorname{rk} \mathfrak{g}$.

Corollary. *In any Lie algebra there exists a Cartan subalgebra.*

Theorem 9.3. *All Cartan subalgebras of a complex Lie algebra $\mathfrak{g}$ are conjugate.*

We give an outline of the proof from Serre [1987].

First one proves that any Cartan subalgebra $\mathfrak{h}$ is of the form $\mathfrak{h} = \mathfrak{g}_x^0$, where x is a regular element of the algebra $\mathfrak{g}$. Here one takes for x a regular element $x \in \mathfrak{h}$ inducing an invertible linear operator in $\mathfrak{g}/\mathfrak{h}$. Next it is proved that if the subalgebras $\mathfrak{g}_x^0$ and $\mathfrak{g}_y^0$, where x and y are regular elements of $\mathfrak{g}$, are conjugate, so are the subalgebras $\mathfrak{g}_u^0$ and $\mathfrak{g}_v^0$, where u, v are regular elements sufficiently close to x and y. Since the set of all regular elements is connected, this yields the statement of the theorem. $\square$

Remark. In fact, one can prove (see Bourbaki [1975], Chap. 7, Sect. 3) that any two Cartan subalgebras of a complex Lie algebra $\mathfrak{g}$ are conjugate with respect to the subgroup of $\operatorname{Int} \mathfrak{g}$ generated by the elements $e^{\operatorname{ad} x}$, where $x \in \mathfrak{g}$ and the operator $\operatorname{ad} x$ is nilpotent.

It is easy to see that a subalgebra $\mathfrak{h}$ of a real Lie algebra $\mathfrak{g}$ is a Cartan subalgebra if and only if $\mathfrak{h}(\mathbb{C})$ is a Cartan subalgebra of $\mathfrak{g}(\mathbb{C})$. The following example shows that the conjugacy theorem does not hold for real Lie algebras.

Example. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. One can easily see that the subalgebra $\mathfrak{h}$ of all diagonal matrices is a Cartan subalgebra of $\mathfrak{g}$. We have $\operatorname{rk} \mathfrak{g} = n$. Regular elements of $\mathfrak{g}$ are semisimple (diagonalizable) matrices all characteristic roots of which are distinct.

Consider the real form $\mathfrak{g}_0 = \mathfrak{gl}_n(\mathbb{R})$ of $\mathfrak{g}$. The set $\mathfrak{h}_0$ of all real diagonizable matrices is a Cartan subalgebra of it. However, for $n > 1$ there exist Cartan subalgebras nonconjugate to $\mathfrak{h}_0$. For example, for $n = 2$ such is the subalgebra of all matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where $a, b \in \mathbb{R}$.

Nevertheless, in solvable real Lie algebras Cartan subalgebras are also conjugate. To be more precise, the following theorem holds.

Theorem 9.4 (Bourbaki [1975]). *Let $\mathfrak{h}$ be a solvable Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$, and $\mathfrak{g}_\infty$ the intersection of all terms of its descending central series* (see Vinberg and Onishchik [1988], Chap. 2, Sect. 5.7). *Then any two Cartan subalgebras of $\mathfrak{g}$ are taken into each other by an automorphism of the form $e^{\operatorname{ad} x}$, where $x \in \mathfrak{g}_\infty$.*

Consider the relation between Cartan subalgebras and the Levi decomposition of the Lie algebra $\mathfrak{g}$. We say that a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is compatible with the Levi decomposition (5) if $\mathfrak{h} = \mathfrak{a} + \mathfrak{b}$, where $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{l}$, and $\mathfrak{b} \subset \operatorname{rad} \mathfrak{g}$.

Theorem 9.5 (Dixmier [1956]). *If $\mathfrak{h} = \mathfrak{a} + \mathfrak{b}$ is a Cartan subalgebra of a Lie algebra $\mathfrak{g}$ over K compatible with its Levi decomposition* (5), *then $\mathfrak{b}$ is a Cartan subalgebra of the centralizer of the subalgebra $\mathfrak{a}$ of* $\operatorname{rad} \mathfrak{g}$. *Conversely, if $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{l}$, and $\mathfrak{b}$ is a Cartan subalgebra of its centralizer in* $\operatorname{rad} \mathfrak{g}$, *then $\mathfrak{h} = \mathfrak{a} + \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$. Any Cartan subalgebra of $\mathfrak{g}$ is compatible with a Levi decomposition of it.*

The following proposition also holds.

Proposition 9.3. *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a decomposition of a Lie algebra $\mathfrak{g}$ into a direct sum of ideals, then Cartan subalgebras of $\mathfrak{g}$ are precisely subalgebras of the form $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}_i$ $(i = 1, 2)$.*

9.4. Cartan Subalgebras and Root Decompositions of Semisimple Lie Algebras. In this section we consider the case where a Lie algebra $\mathfrak{g}$ is semisimple (or reductive).

Proposition 9.4. *Let $\mathfrak{h}$ be a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$. Then, for any $\alpha \in \Delta_\mathfrak{g}(\mathfrak{h})$ we have $-\alpha \in \Delta_\mathfrak{g}(\mathfrak{h})$. The Killing form $k_\mathfrak{g}$ is nondegenerate on $\mathfrak{h}$ and on $\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$ for any $\alpha \in \Delta_\mathfrak{g}(\mathfrak{h})$.*

Proposition 9.2 implies that $\mathfrak{g}$ can be decomposed into the orthogonal (with respect to $k_\mathfrak{g}$) direct sum of the spaces $\mathfrak{h}$ and $\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$ $(\alpha \in \Delta_\mathfrak{g}(\mathfrak{h}))$. Using Theorem 2.2, one can see that $k_\mathfrak{g}$ is nondegenerate both on $\mathfrak{h}$ and on $\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$. Since $k_\mathfrak{g}|\mathfrak{g}^\alpha = 0$, we have $\mathfrak{g}^{-\alpha} \neq 0$. □

Theorem 9.6. *Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ over $K = \mathbb{R}$ or $\mathbb{C}$. Then $\mathfrak{h}$ is commutative, consists of semisimple elements, and coincides with its normalizer. The form $k_\mathfrak{g}|\mathfrak{h}$ is nondegenerate.*

Proof. It is sufficient to consider the case $K = \mathbb{C}$. Nondegeneracy of $k_{\mathfrak{g}}$ on $\mathfrak{h}$ is proved in Proposition 9.4. Theorem 2.1 shows that $[\mathfrak{h}, \mathfrak{h}] = 0$. Semisimplicity of the elements is also derived from the fact that the form $k_{\mathfrak{g}}|\mathfrak{h}$ is nondegenerate. $\square$

Corollary 1. *A Cartan subalgebra of a semisimple Lie algebra is a maximal commutative subalgebra of it, consisting of semisimple elements. Conversely, any such subalgebra is a Cartan subalgebra.*

Taking into account Corollary 2 to Theorem 6.3, one obtains the following assertion.

Corollary 2. *If $\mathfrak{h}$ is a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$, and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex linear representation of it, then the operator $\rho(x)$ is semisimple for any $x \in \mathfrak{h}$. If $\lambda \in \Phi_\rho(\mathfrak{h})$, then V^λ coincides with the weight subspace V_λ. In particular, we have*

$$\mathfrak{g}^\alpha = \mathfrak{g}_\alpha = \{y \in \mathfrak{g} | [x, y] = \alpha(x)y \quad \text{for all} \quad x \in \mathfrak{h}\}$$

for any $\alpha \in \Phi_{\mathrm{ad}}(\mathfrak{h})$.

Corollary 3. *Let $\mathfrak{h}$ be a Cartan subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$. Then any semisimple element of $\mathfrak{g}$ is conjugate to an element $\mathfrak{h}$. The elements $x, y \in \mathfrak{h}$ are conjugate in $\mathfrak{g}$ if and only if $y = \varphi(x)$, where φ is an inner automorphism of $\mathfrak{g}$ taking $\mathfrak{h}$ into itself.*

Example. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Consider the basis $\{\mathbf{h}, \mathbf{e}, \mathbf{f}\}$ in $\mathfrak{g}$, where $\mathbf{h} = E_{11} - E_{22}$, $\mathbf{e} = E_{12}$, $\mathbf{f} = E_{21}$. Then $\mathfrak{h} = \langle \mathbf{h} \rangle$ is a Cartan subalgebra of $\mathfrak{g}$. The corresponding system of roots is $\Delta = \{\alpha, -\alpha\}$, where $\alpha(\mathbf{h}) = 2$. We have $\mathfrak{g}_\alpha = \langle \mathbf{e} \rangle$, $\mathfrak{g}_{-\alpha} = \langle \mathbf{f} \rangle$.

Now let $\mathfrak{g}$ be a reductive Lie algebra. Then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_0$, where $\mathfrak{g}_0 = [\mathfrak{g}, \mathfrak{g}]$ is semisimple, and Cartan subalgebras of $\mathfrak{g}$ are of the form $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}_0$, where $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$. Therefore all the statements of Theorem 9.6, except the last one, as well as those of Corollary 1, hold in the case where $\mathfrak{g}$ is reductive. The last statement of the theorem can be replaced by the following more general one: any nondegenerate symmetric invariant bilinear form on a reductive Lie algebra $\mathfrak{g}$ is nondegenerate on $\mathfrak{h}$. Proposition 9.4 also holds if one considers, instead of the Killing form, an arbitrary invariant nondegenerate symmetric bilinear form. The assertion of Corollary 2 is valid for completely reducible linear representations of reductive Lie algebras.

9.5. Cartan Subgroups. A *Cartan subgroup* of an abstract group G is defined as a maximal nilpotent subgroup $H \subset G$ such that any normal subgroup H' in H of finite index has a finite index in $N_G(H')$.

Theorem 9.7 (Chevalley [1955a], Chap. 6). *Let G be a connected Lie group. Then any Cartan subgroup $H \subset G$ is a Lie subgroup (a complex one if G is a complex Lie group), and its tangent algebra is a Cartan subalgebra of the*

Lie algebra $\mathfrak{g}$. For any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ there exists a unique Cartan subgroup $H \subset G$ for which $\mathfrak{h}$ is the tangent algebra.

Corollary. *Cartan subgroups of a connected complex Lie group are conjugate.*

The *rank* of a Lie group G is the dimension $\operatorname{rk} G$ of an arbitrary Cartan subgroup of it (in the case of a complex Lie group it is the complex dimension). Clearly, $\operatorname{rk} G = \operatorname{rk} \mathfrak{g}$, where $\mathfrak{g}$ is the tangent algebra of the group G.

We now describe Cartan subgroups of connected complex algebraic groups. An *algebraic torus* is an algebraic group isomorphic to a direct product of several groups $\mathbf{G}_m = \mathbb{C}^\times$. A *maximal torus* of an algebraic group is a maximal algebraic subgroup, that is a torus.

Theorem 9.8 (Borel [1969], Chap. 4, Sect. 12). *Let G be a connected algebraic group, and H a subgroup of it. The following conditions are equivalent:*
(1) *H is a Cartan subgroup of G;*
(2) *H is a nilpotent algebraic subgroup coinciding with $N_G(H)^0$;*
(3) *H is the centralizer of a maximal torus of G.*

With the use of Theorem 9.6, this implies the following corollary.

Corollary. *Cartan subgroups of a connected reductive complex algebraic group coincide with its maximal tori.*

Example. One can take for a Cartan subgroup (or a maximal torus) of the subgroup $\mathrm{GL}_n(\mathbb{C})$ the subgroup $\mathrm{D}_n(\mathbb{C})$ of all diagonal matrices of this group. Correspondingly, the subalgebra $\mathfrak{d}(\mathbb{C})$ of all diagonal matrices in $\mathfrak{gl}_n(\mathbb{C})$ is a Cartan subalgebra of $\mathfrak{gl}_n(\mathbb{C})$. The subgroup $\mathrm{D}_n(\mathbb{C}) \cap \mathrm{SL}_n(\mathbb{C})$ is a Cartan subgroup of $\mathrm{SL}_n(\mathbb{C})$. Similarly, Cartan subgroups of the classical groups $G = \mathrm{SO}_{2l}(\mathbb{C})$, $\mathrm{SO}_{2l+1}(\mathbb{C})$, $\mathrm{Sp}_{2l}(\mathbb{C})$ are given by the subgroups $G \cap \mathrm{D}_n(\mathbb{C})$, where G consists of linear representations of the space $\mathbb{C}^n$ preserving the bilinear form with the matrix

$$\begin{pmatrix} 0 & E_l \\ E_l & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & E_l & 0 \\ E_l & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & E_l \\ -E_l & 0 \end{pmatrix},$$

respectively. The Cartan subalgebras of $\mathfrak{so}_{2l}(\mathbb{C})$, $\mathfrak{so}_{2l+1}(\mathbb{C})$, $\mathfrak{sp}_{2l}(\mathbb{C})$ corresponding to these groups consist of all matrices of the form

$$\operatorname{diag}(x_1, \ldots, x_l, -x_1, \ldots, -x_l),$$

$$\operatorname{diag}(x_1, \ldots, x_l, -x_1, \ldots, -x_l, 0),$$

$$\operatorname{diag}(x_1, \ldots, x_l, -x_1, \ldots, x_l),$$

respectively.

We now go back, but on the global level, to the examination of weights of a linear representation. Let T be an n-dimensional complex algebraic torus. Denote by $\mathfrak{X}(T)$ the group of its *characters*, i.e. polynomial homomorphisms

$T \to \mathbb{C}^\times$. As is known, $\mathfrak{X}(T)$ is a free abelian group of rank n. Associating with each $\lambda \in \mathfrak{X}(T)$ its differential $d\lambda \in \mathfrak{t}^*$, we obtain an injective homomorphism of the group $\mathfrak{X}(T)$ into the space $\mathfrak{t}(\mathbb{R})^*$, where $\mathfrak{t}(\mathbb{R}) = \{x \in \mathfrak{t} | \lambda(x) \in \mathbb{R} \quad \text{for all} \quad \lambda \in \mathfrak{X}(T)\}$, under which $\mathfrak{X}(T)$ is identified with a lattice in the space $\mathfrak{t}(\mathbb{R})^*$ generated by some basis in this space. In what follows we will usually identify the character λ with the linear form $d\lambda$. Also let $\mathfrak{t}(\mathbb{Z}) = \{x \in \mathfrak{t} | \lambda(x) \in \mathbb{Z} \quad \text{for all} \quad \lambda \in \mathfrak{X}(T)\}$.

Let G be a reductive complex algebraic group and H a maximal torus of it. For any complex linear representation $R: G \to \mathrm{GL}(V)$, we have the decomposition

$$V = \bigoplus_{i=1}^{s} V_{\lambda_i},$$

where $\lambda_i \in \mathfrak{X}(H)$ and $V_{\lambda_i} = V_{\lambda_i}(H)$ are the corresponding weight subspaces of the representation $R|H$. The characters λ_i are called the *weights* of the representation R with respect to H, and V_{λ_i} the corresponding *weight subspaces*. We write $\Phi_R(H) = \{\lambda_1, \dots, \lambda_s\} \subset \mathfrak{h}(\mathbb{R})^*$. Evidently, the weights of the representation R are identified with the weights of the corresponding representation $dR: \mathfrak{g} \to \mathfrak{gl}(V)$, and the weight subspaces of the representations R and dR coincide.

Chapter 2
Solvable Lie Groups and Lie Algebras

A Lie group (Lie algebra) is said to be *solvable* if its iterated commutator groups (respectively, iterated derived algebras) become trivial after finitely many steps (see Vinberg and Onishchik [1988], Chap. 2, Sect. 5.5).

§ 1. Examples

Example 1. The Lie groups $T_n(K)$ of nonsingular upper triangular matrices over the fields $K = \mathbb{C}$, $\mathbb{R}$ are solvable. Note that $T_n(K)$ can be regarded as a subgroup of $\mathrm{GL}_n(K)$ preserving the standard full flag $\{0\} \subset K^1 \subset K^2 \subset \dots \subset K^n$ in K^n.

Similarly, Lie algebras $\mathfrak{t}_n(K)$ (of triangular matrices over the fields $K = \mathbb{C}$, $\mathbb{R}$) are solvable.

Example 2. Consider the Lie group $G = A \ltimes_\varphi B$, i.e. the semidirect product of abelian connected Lie groups A and B corresponding to the homomorphism

$\varphi: A \to \operatorname{Aut} B$. The Lie group G is connected and solvable. It is simply-connected if and only if the Lie groups A and B are simply-connected.

In particular, the Lie groups of the form $G = \mathbb{R} \ltimes_\varphi \mathbb{R}^n$, where $\varphi: \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$ is a homomorphism, are solvable. The subgroup $\operatorname{Im} \varphi$ is a one-parameter subgroup of $\mathrm{GL}_n(\mathbb{R})$ and therefore $\varphi(t) = \exp(X \cdot t)$, where X is a matrix from $\mathfrak{gl}_n(\mathbb{R})$.

Similarly, the semidirect sum $\mathfrak{a} \in_\varphi \mathfrak{b}$ of two abelian Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$ is a solvable Lie algebra. In particular, the Lie algebras of the form $\mathfrak{g} = \mathbb{R} \in_\varphi \mathbb{R}^n$, where the homomorphism φ is uniquely defined by the matrix $X = \varphi(1) \in \mathfrak{gl}_n(\mathbb{R})$, are solvable. Here the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ coincides with the image of the linear operator X.

Example 3. Let G be the group $\operatorname{Aff} \mathbb{R}^1$ of affine transformations of the real line $\mathbb{R}^1$. Then the connected component G^0 of unity is the semidirect product $G^0 = H \cdot T$ of the normal subgroup $T = \mathbb{R}$, consisting of translations of the real line $\mathbb{R}^1$, and the subgroup $H \simeq \mathbb{R}^{>0}$ of homothetic transformations $x \mapsto \lambda x$, where $x \in \mathbb{R}^1$, $\lambda \in \mathbb{R}^{>0}$. The Lie group G^0 is solvable (and so is the disconnected Lie group G). The tangent algebra of the Lie group $\operatorname{Aff} \mathbb{R}^1$, which we denote by $\mathfrak{r}_2 = \mathfrak{r}_2(\mathbb{R})$, is a unique (up to isomorphism) two-dimensional nonabelian Lie algebra. In an appropriate basis X, Y (X is a nonzero element in $\mathfrak{h}$, and Y is a nonzero element in $\mathfrak{t}$, where $\mathfrak{h}$, $\mathfrak{t}$ are the tangent algebras for the Lie groups H, T, respectively) the algebra $\mathfrak{t}_2$ is defined by the commutation relation $[X, Y] = Y$.

The group $\operatorname{Aff} \mathbb{R}^1$ has a unique matrix representation

$$\operatorname{Aff} \mathbb{R}^1 \simeq \left\{ \begin{bmatrix} \lambda & a \\ 0 & 1 \end{bmatrix} \middle| \quad a, \lambda \in \mathbb{R}, \lambda \neq 0 \right\};$$

Similarly, for its Lie algebra we have

$$\mathfrak{t}_1 \simeq \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \middle| \quad x, y \in \mathbb{R} \right\}.$$

Example 4. Let G be the group $E(2)$ of motions of the Euclidean plane E^2; then $G^0 = E^0(2)$ is the subgroup of proper motions. The Lie group $E^0(2)$ is decomposed into the semidirect product $E^0(2) = C \ltimes T$, where $T \simeq \mathbb{R}^2$ is the normal subgroup consisting of translations and $C \simeq \mathrm{SO}_2$ is the subgroup of rotations about the origin. Since both C and T are abelian, $E^0(2)$ is a connected solvable Lie group.

§ 2. Triangular Lie Groups and Lie Algebras

One of the basic results about solvable Lie algebras over an algebraically closed field is Lie's theorem (see Chap. 1, Sect. 1.1). No such statement holds for arbitrary solvable Lie algebras over a field that is not algebraically closed (for example, over the field $\mathbb{R}$). There is, however, a class of solvable Lie algebras over an arbitrary field of characteristic 0, for which the analogue of the Lie theorem is valid.

A Lie algebra $\mathfrak{g}$ over a field k is said to be *triangular* (over k) if for any $X \in \mathfrak{g}$ all eigenvalues of the linear operator $\operatorname{ad} X$ belong to k. Lie algebras triangular over $k = \mathbb{R}$ are usually called simply triangular. Triangular Lie algebras are sometimes also called *completely solvable* (all of them are solvable, see below), or *Lie algebras of type* (R).

A Lie group is said to be *triangular* (or *completely solvable*, or of *type* (R)) if for any $g \in G$ all eigenvalues of the linear operator $\operatorname{Ad} g$ are real. Clearly, a connected Lie group G is triangular if and only if so is its Lie algebra. For a connected triangular Lie group G the eigenvalues of the operators $\operatorname{Ad} g$, $g \in G$, are not only real but even positive.

Example 2.1. The Lie algebra $\mathfrak{t}_n(\mathbb{R})$ and the Lie group $T_n(\mathbb{R})$ are triangular.

Example 2.2. A Lie algebra of the form $\mathfrak{g} = \mathbb{R} \in_{\varphi} \mathbb{R}^n$, i.e. the semidirect sum corresponding to the homomorphism $\varphi\colon \mathbb{R} \to \mathfrak{gl}_n(\mathbb{R})$, is triangular if and only if all eigenvalues of the matrix $\varphi(1)$ are real.

Theorem 2.1. *Let* $\mathfrak{g}$ *be a triangular Lie algebra. Then*

(i) $\mathfrak{g}$ *is solvable*;

(ii) *any subalgebra and quotient algebra of* $\mathfrak{g}$ *are triangular.*

Similar statements also hold for connected triangular Lie groups.

If $\mathfrak{g}$ is a finite-dimensional triangular Lie algebra, then one can apply Theorem 1.2 of Chap. 1 to its adjoint representation. In general, many properties of triangular Lie algebras are similar to those of solvable Lie algebras over an algebraically closed field of characteristic 0. In particular, any finite-dimensional triangular Lie algebra is isomorphic to a subalgebra of $\mathfrak{t}_n(\mathbb{R})$ for some $n \in \mathbb{N}$.

§ 3. Topology of Solvable Lie Groups and Their Subgroups

3.1. Canonical Coordinates. There are two different ways of introducing a canonical coordinate system in a neighborhood of unity of a Lie group, yielding canonical coordinate systems of the first and second kind (see Vinberg and Onishchik [1988], Chap. 2, Sect. 3.2). For solvable Lie groups we consider some special coordinate systems of the second kind, which we will call simply *canonical*.

Theorem 3.1. *Let G be a connected real Lie group.* (i) *If there is a connected normal Lie subgroup G_1 of G of codimension* 1, *then there exists a one-dimensional Lie subgroup C of G such that $G = C \ltimes G_1$, the semidirect* *product of C and G_1.* (ii) *If the Lie group G is solvable, then it has a connected normal Lie subgroup of codimension* 1.

Proof. (i) The Lie group G/G_1 is connected and one-dimensional, and therefore abelian. Hence $G_1 \supset (G, G)$. Using induction on $\dim G$, we can reduce the general case to that of the group G, which is abelian and either simply-connected or a torus. In these particular cases the construction of a Lie subgroup C complementary to G_1 is carried out without difficulty.

(ii) Consider the Lie subgroup $G_1' = \overline{(G, G)}$, the closure of the commutator group (G, G). The solvability of the Lie group G implies that $G_1' \neq G$. The group G/G_1' is abelian, and evidently has a connected Lie subgroup A' of codimension 1. If $\pi: G \to G/G_1'$ is the natural epimorphism, then $G_1 = \pi^{-1}(A')$ is the desired normal Lie subgroup of G. □

Corollary 1 (see Malcev [1945b], Bourbaki [1975]). *Let G be a connected solvable real Lie group. Then there exist one-parameter Lie subgroups $C_i = \{c_i(t)\}$, $1 \leq i \leq n = \dim G$, of G such that*

(1) *$G = C_1 \cdot C_2 \cdot \ldots \cdot C_n$, where each element $g \in G$ is uniquely represented in the form $g = g_1 \ldots g_n$, where $g_i \in C_i$;*
(2) *if the Lie group G is simply-connected, then $C_i \simeq \mathbb{R}$, $1 \leq i \leq n$;*
(3) *let $G_0 = \{e\}$, $G_i = C_{n-i+1} \cdot \ldots \cdot C_n$, $1 \leq i < n$; then the G_i are Lie subgroups of G, $\dim G_i = i$, and G_i is normal in G_{i+1} for $1 \leq i < n$.*

As a result, if a Lie group G is simply-connected, we obtain the decomposition $G = C_1 \cdot \ldots \cdot C_n$ into the product of one-parameter subgroups $C_i \simeq \mathbb{R}$. For each $g \in G$ there exist uniquely defined $t_i = t_i(g) \in R$ such that $g = c_1(t_1) \ldots c_n(t_n)$. The functions $t_i = t_i(g)$ are smooth on G, and the mapping

$$G \mapsto \mathbb{R}^n,$$

$$g \mapsto (t_1(g), \ldots, t_n(g))$$

defined by them is a diffeomorphism. We have thus obtained a global coordinate system on G.

3.2. Topology of Solvable Lie Groups. The results of Sect. 3.1 imply the following assertion.

Corollary 2. *A simply-connected solvable Lie group is diffeomorphic to $\mathbb{R}^n$, $n = \dim G$.*

In particular, a simply-connected solvable Lie group is contractible as a topological space. There exist nonsolvable contractible Lie groups. For example, the Lie group $A = \widetilde{\mathrm{SL}}_2(\mathbb{R})$, the simply-connected covering for $\mathrm{SL}_2(\mathbb{R})$, is diffeomorphic to $\mathbb{R}^3$. For more on *contractible* Lie groups see Sect. 3.3 below.

Theorem 3.1 also implies the following statement.

Corollary 3. *A connected solvable Lie group G is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^l$, $k + l = \dim G$.*

By virtue of Corollary 3, any compact connected solvable Lie group G is diffeomorphic to a torus $\mathbb{T}^k$. Actually, G is isomorphic to $\mathbb{T}^k$ (see Chap. 4, Proposition 1.2).

3.3. Aspherical Lie Groups

Theorem 3.2. *Let G be a connected Lie group. Then the following conditions are equivalent:*

(i) *the Lie group G is diffeomorphic to $\mathbb{R}^n$, $n = \dim G$;*
(ii) *the Lie group G is contractible;*
(iii) *the maximal compact subgroup of G is trivial;*
(iv) *the Levi decomposition for the group G is of the form $G = S \ltimes R$, where the radical R is simply-connected, and the Levi subgroup S is isomorphic to $\underbrace{\mathcal{A} \times \ldots \times \mathcal{A}}_{m}$, $\mathcal{A} = \mathrm{SL}_2(\mathbb{R})$ for some $m \geq 0$.*

Proof. The proof is based on the fact that the only contractible simple Lie group is $\mathcal{A}$ (this follows, for example, from the classification of all real simple Lie groups, see Chap. 4, Sect. 1.5). □

Let G be an arbitrary connected solvable Lie group. Then its simply-connected covering $\widetilde{G}$ is diffeomorphic to $\mathbb{R}^n$, and therefore all homotopy groups $\pi_i(G)$ for $i \geq 2$ are trivial (since for $i \geq 2$ we have $\pi_i(G) \simeq \pi_(\widetilde{G})$)). Thus, as a topological space, the Lie group G is an *aspherical* one (another name also in use is a *space of type* $K(\pi, 1)$). It follows from Corollary 3 that $\pi_1(G) \simeq \mathbb{Z}^k$ for some $k \geq 0$. Now let G be an arbitrary aspherical Lie group, then $\widetilde{G}$ is contractible, and one can apply Theorem 3.2 to it. In particular, this makes it possible to prove the following assertion.

Theorem 3.3. *Let G be an aspherical connected Lie group. Then any maximal compact subgroup of G is a torus and G is diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^l$, where $k + l = \dim G$.*

For example, the group $\mathrm{SL}_2(\mathbb{R})$ is aspherical and diffeomorphic to $\mathbb{T} \times \mathbb{R}^2$.

3.4. Topology of Subgroups of Solvable Lie Groups

Theorem 3.4 (see Malcev [1945b]). *Let G be a simply-connected solvable Lie group and H a connected virtual Lie subgroup of it. Then*

(1) *the subgroup H is closed in G (i.e., H is a Lie subgroup) and simply-connected;*
(2) *the quotient space G/H is diffeomorphic to $\mathbb{R}^m$ (where $m = \dim G - \dim H$).*

Proof. (1) If the Lie group G is abelian, then $G \simeq \mathbb{R}^n$, and then $H \simeq \mathbb{R}^k$ is evidently a linear subspace of $\mathbb{R}^n$, $k \leq n$. For arbitrary solvable simply-connected Lie groups G the closeness of H in G is proved by induction in the length of the series of iterated commutator groups. The fact that H is simply-connected can also be proved by induction, but another approach is possible. If L is a maximal compact subgroup of H, then, since $L \subset G$ and G is simply-connected and solvable, we have $L = \{e\}$, which, by virtue of Corollary 3 to Theorem 3.1, implies that H is simply-connected.

(2) If G is abelian, then G/H is evidently diffeomorphic to the space $\mathbb{R}^m$; in the general case the proof is achieved by induction on the length of the series of iterated commutator groups.

In particular, Theorem 3.4 implies that any solvmanifold, i.e. the homogeneous space of a solvable Lie group, is aspherical.

All the assertions of Theorem 3.4, as well as the fact that the homogeneous spaces of a Lie group G are aspherical, can also be proved in the case where G is an arbitrary contractible Lie group.

§ 4. Nilpotent Lie Groups and Lie Algebras

4.1. Definitions and Examples. Recall that a Lie group (Lie algebra) is said to be *nilpotent* if the lower (or descending) central series for it terminates after finitely many steps (see Vinberg and Onishchik [1988], Chap. 2, Sect. 5.7). Formerly nilpotent Lie groups and Lie algebras used to be called (this terminology is now obsolete) *special* Lie groups (algebras), or *groups (algebras) of rank* 0.

It is sometimes convenient to define the nilpotency in terms of *upper* central series instead of lower ones. For example, the *upper* (or *ascending*) *central series* $C^0\mathfrak{g} \subset C^1\mathfrak{g}C\ldots$ for a Lie algebra $\mathfrak{g}$ is defined as follows:

$$C^0\mathfrak{g} = \{0\}, C^1\mathfrak{g} = Z(\mathfrak{g}), C^{k+1}\mathfrak{g}/C^k\mathfrak{g} = Z(\mathfrak{g}/C^k\mathfrak{g}), k = 1, 2, \ldots$$

A Lie algebra $\mathfrak{g}$ is nilpotent if and only if $C^n\mathfrak{g} = \mathfrak{g}$ for some $m \geq 0$. The smallest number m such that $C^m\mathfrak{g} = \mathfrak{g}$ is said to be the *class* or the *degree of nilpotency* of the Lie algebra (this number coincides with that of the first trivial term of the lower central series of $\mathfrak{g}$). In the case of Lie groups the notion of the upper central series $\{C^kG\}$ is introduced in a similar manner. The tangent algebras to the groups C^kG are $C^k\mathfrak{g}$, the terms of the upper central series for $\mathfrak{g}$, the tangent algebra of the Lie group G.

Example 1. The Lie groups $N^n(K)$ consisting of all upper triangular unipotent matrices over the field $K = \mathbb{R}$ or $\mathbb{C}$ are nilpotent Lie groups (over K), their degree of nilpotency being equal to $n - 1$.

Similarly, the Lie algebras $\mathfrak{n}_n(K)$ consisting of all upper triangular nilpotent matrices over the field K are nilpotent.

The Lie algebra $\mathfrak{n}_3(k)$ is three-dimensional. It is the only (up to isomorphism) three-dimensional nonabelian nilpotent Lie algebra over the field K. In an appropriate basis X, Y, Z it is defined by the commutation relations $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$.

Example 2. The *Heisenberg group* $H_n(K)$ is the Lie group over the field $K = \mathbb{R}$ or $\mathbb{C}$ consisting of all matrices of the form

$$\begin{bmatrix} 1 & x_1 & x_2 & \dots & x_n & z \\ & 1 & & & & y_n \\ & & 1 & & 0 & \\ & & & \ddots & & \vdots \\ & & & & 1 & y_1 \\ & 0 & & & & 1 \end{bmatrix} \in GL_{n+1}(K), \qquad \text{where} \quad x_i,\, y_i,\, z \in K.$$

We have $C^1 H_n(K) = Z(H_n(K)) \simeq K$, $C^k H_n(K) = \{e\}$ for $k \geq 2$. The Heisenberg group is therefore nilpotent of class 2. Note that $\dim H_n(K) = 2n+1$, and that $H_1(K)$ is isomorphic to $N_3(K)$.

Similarly, the *Heisenberg algebra* $\mathfrak{h}_n(K)$ over a field K consists of all matrices of the form

$$\begin{bmatrix} 0 & x_1 & x_2 & \dots & x_n & z \\ & 0 & & & & y_n \\ & & & \ddots & 0 & \\ & 0 & & & 0 & y_1 \\ & & & & & 0 \end{bmatrix}, \qquad \text{where} \quad x_i, y_i, z \in K,$$

It is also nilpotent of class 2. In an appropriate basis $X_{1_j}, \dots, X_n, Y_1, \dots, Y_n, Z$ it is defined by the relations

$$\begin{gathered} [X_i, Y_i] = Z, \\ [X_i, Z] = [Y_i, Z] = 0 \qquad \text{for} \quad 1 \leq i \leq n, \\ [X_i, X_j] = 0 \qquad 1 \leq i < j \leq n. \end{gathered}$$

The tangent algebra to the Heisenberg group $H_n(K)$ is the Heisenberg algebra $\mathfrak{h}_n(K)$.

The notion of the Heisenberg group (and algebra) can be generalized as follows (see Reiter [1974]). Let A, B, C be three abelian topological groups (for example, Lie groups), and $\varphi\colon A \times B \to C$ a $\mathbb{Z}$-bilinear continuous mapping. Consider $G = A \times B \times C$, the product of A, B, C as topological spaces, and define an operation on G by the formula

$$(a, b, c) \cdot (a', b', c') = (a + a', b + b', c + c' + \varphi(a, b')),$$

where $a, a' \in A$, $b, b' \in B$, $c, c' \in C$. It is clear that G is a topological nilpotent group of class 2, and it is called a generalized Heisenberg group or an *HR*-group. The definition of a generalized Heisenberg algebra is analogous.

Evidently, virtual Lie subgroups (Lie subalgebras) and quotient groups (quotient algebras) of nilpotent Lie groups (Lie algebras) are also nilpotent. If $\{0\} \to \mathfrak{g}' \to \mathfrak{g} \to \mathfrak{g}'' \to \{0\}$ is a central extension (i.e. $\mathfrak{g}' \subset Z(\mathfrak{g})$) of a nilpotent Lie algebra $\mathfrak{g}''$, then $\mathfrak{g}$ is also nilpotent. A similar statement holds for nilpotent Lie groups.

A nilpotent Lie group (Lie algebra) is solvable.

4.2. Malcev Coordinates. Let G be an arbitrary connected nilpotent Lie group. Consider its decomposition into a product of one-parameter Lie subgroups $G = C_1 \cdot \ldots \cdot C_n$ introduced in Sect. 3.1. Since G is nilpotent, the Lie subgroups C_i can be chosen in such a way that the Lie subgroups $G_i = C_{n-i+1} \cdot \ldots \cdot C_n$ are normal in G (and not only in G_{i+1} as is the case of arbitrary solvable Lie groups). This can be proved by slightly modifying the proof of assertion (ii) of Theorem 3.1, on which Corollary 1 is based, namely, by replacing $\overline{(G_i, G_i)}$ in the construction of G_{i+1} by $\overline{(G, G_i)}$.

If the Lie group G is nilpotent and simply-connected, then we obtain a coordinate system $g \mapsto (t_1(g), \ldots, t_n(g))$ on it such that the Lie subgroups $G_i = \{x_{n-i+1}(t_{n-i+1}) \cdot \ldots \cdot x_n(t_n) | t_j \in \mathbb{R}\}$ are normal in G. We will call these coordinates $t_i = t_i(g)$ the *Malcev coordinates* (he used them very effectively in Malcev [1949] in order to study nilpotent Lie groups and their homogeneous spaces).

Let e_i be the vectors tangent to the one-parameter subgroups C_i, $1 \leq i \leq n$. By construction of Malcev coordinates, for any $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the tangent algebra of the Lie group G, the vector $[X, e_i]$ is a linear combination of $e_{i+1}, \ldots, e_n$. Consider two arbitrary elements $g, g' \in G$. Then $g = \exp X$, $g' = \exp X'$ for some $X, X' \in \mathfrak{g}$ (because for a simply-connected nilpotent Lie group exp is a diffeomorphism, see Sect. 6.4). Let $\{t_i\}$ be the Malcev coordinates for g, and $\{t'_i\}$ for g'. By the Campbell-Hausdorff formula (see Vinberg and Onishchik [1988], Chap. 3, Sect. 3.2), we have

$$g \cdot g' = e^Y, \qquad \text{where} \quad Y = X + X' + \frac{1}{2}[X, X'] + \ldots,$$

where, in view of the fact that the Lie algebra $\mathfrak{g}$ is nilpotent, only finitely many terms are nonzero. Therefore the coordinates $\{y_i\}$ of the vector Y in the basis $\{e_i\}$ are polynomials in $\{x_i\}$ and $\{x'_i\}$, the coordinates of the vectors X and X'.

We have $g = x_1(t_1) \cdot \ldots \cdot x_n(t_n) = e^{x_1 e_1 + \ldots + x_n e_n}$, where $t_i = t_i(g)$ are the Malcev coordinates of the element $g \in G$. Since $x_i(t) = e^{te_i}$, the Campbell-Hausdorff formula implies (because the Lie group G is nilpotent) that the coordinate x_i can be expressed in terms of $\{t_j\}$ as follows:

$$x_i = t_i + f_i(t_1, \ldots, t_{i-1}),$$

where $f_i(\ldots)$ are polynomials. Conversely, $t_i = x_i + \varphi_i(x_1, \ldots, x_{i-1})$, where $\varphi_i(\ldots)$ are polynomials. The situation for the element $g' \in G$ is similar. As a result, we deduce that if $\{s_i\}$ are the Malcev coordinates of the element $g \cdot g'$, then $s_i = t_i + t'_i + F_i(t_1, \ldots, t_{-1}, t'_1, \ldots, t'_{i-1})$, where F_i are some polynomials. This proves the main part of the following theorem.

Theorem 4.1 (see Malcev [1949], Merzlyakov [1987]). *Let G be a simply-connected nilpotent Lie group. Then the multiplication in the group G is polynomial with respect to the Malcev coordinates.*

Conversely, if the multiplication in a connected Lie group G is polynomial with respect to some local coordinate system, then G is nilpotent.

Theorem 4.1 implies, in particular, that a real simply-connected group G admits a natural structure of a linear algebraic group, which is unipotent (for complex Lie groups the situation is similar). In particular, the Zariski topology (besides the Euclidean one) is introduced on G.

Using Theorem 4.2 (see below) one can define an algebraic group structure on an arbitrary simply-connected nilpotent Lie group G. By Ado's theorem (see Chap. 2, Sect. 4.3), there exists a faithful unipotent representation $\varphi: G \to GL(V)$ of the group G (over the field $\mathbb{R}$). But then $\varphi(G) \subset N_n(\mathbb{R})$, and, by Theorem 4.2, the structure of an algebraic group is induced on $\varphi(G)$, which the mapping φ^{-1} takes to G. By virtue of the corollary, the resulting structure of an algebraic group on G coincides with that constructed above with the use of the Malcev coordinates.

The structure of an affine algebraic group on an arbitrary simply-connected nilpotent Lie group is often very useful (for example, in representation theory, in the study of homogeneous spaces, discrete subgroups, etc.).

Theorem 4.2. *Let $\varphi: G \to G'$ be a homomorphism of connected Lie groups, where G' is simply-connected and nilpotent. Then*

(i) *the subgroup $\varphi(G)$ is closed in the Zariski topology on G' (defined by the Malcev coordinates)*;

(ii) *if the Lie group G is also simply-connected and nilpotent, then the homomorphism φ is polynomial with respect to the Malcev coordinates on G and G'.*

Proof. Let $H = \varphi(G)$. Then H is a connected virtual Lie subgroup of the simply-connected nilpotent Lie group G'. By Theorem 3.4, H is a Lie subgroup of G'. We will show that H is closed in G in the Zariski topology. Note that the tangent subalgebra $\mathfrak{h}$ of the subgroup H is defined by linear equations in $\mathfrak{g}'$ (the tangent algebra of G'). Then the connection between the coordinates $\{t_i\}$ and $\{x_i\}$ described above implies that, with respect to the Malcev coordinates on G', the subgroup H is described by polynomial equations.

(ii) Consider the graph $\Gamma_\varphi \subset G \times G'$ of the homomorphism φ. Clearly, Γ_φ is a virtual Lie subgroup of the simply-connected nilpotent group $G \times G'$. In view of the already proved assertion (i), Γ_φ is an algebraic subgroup of $G \times G'$. This means that, with respect to the Malcev coordinates on G and G', the homomorphism φ is polynomial. □

Corollary. *The structure of an algebraic group on a nilpotent simply-connected Lie group is unique up to isomorphism.*

By virtue of this corollary, the structure on an algebraic group on a simply-connected nilpotent Lie group does not depend on the choice of Malcev coordinates.

4.3. Cohomology and Outer Automorphisms. Let $\varphi: \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ over a field k; then V can be considered as a $\mathfrak{g}$-module. We will say that a $\mathfrak{g}$-module W is contained in V if W is a quotient module of some submodule of V. Denote by $H^p(\mathfrak{g}, V)$ the space of p-dimensional cohomology of the Lie algebra $\mathfrak{g}$ with coefficients in the module V.

Theorem 4.3 (Dixmier, 1955). *Let $\varphi: \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional linear representation of a nilpotent Lie algebra $\mathfrak{g}$ (over an infinite field k). Then*

(i) *if the $\mathfrak{g}$-module V contains no trivial modules, then $H^p(\mathfrak{g}, V) = \{0\}$ for all $p \geq 0$;*

(ii) *if the $\mathfrak{g}$-module V contains a trivial module, then*

$$\dim H^p(\mathfrak{g}, V) \geq 2 \quad \text{for} \quad 0 < p < \dim \mathfrak{g},$$

$$\dim H^p(\mathfrak{g}, V) \geq 1 \quad \text{for} \quad p = 0, \dim \mathfrak{g}.$$

Now we mention an application of Theorem 4.3. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Consider the Lie algebra $\operatorname{Der} \mathfrak{g}$ of all derivations of the Lie algebra $\mathfrak{g}$ and the subalgebra $I(\mathfrak{g})$ of it consisting of all *inner derivations*, i.e. operators of the form $\operatorname{ad} X$, $X \in \mathfrak{g}$ (clearly, $I(\mathfrak{g}) \simeq \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$). One can easily verify that $I(\mathfrak{g})$ is an ideal in $\operatorname{Der} \mathfrak{g}$. The quotient algebra $\operatorname{Der} \mathfrak{g}/I(\mathfrak{g})$ is denoted by $\operatorname{Out}(\mathfrak{g})$, and is called the *Lie algebra of outer derivations of* $\mathfrak{g}$. As is known, $\operatorname{Out}(\mathfrak{g})$ can be identified with the one-dimensional cohomology space $H^1(\mathfrak{g}, \mathfrak{g})$ of the Lie algebra $\mathfrak{g}$, acting on itself by the adjoint representation (see Feigin and Fuks [1988], Chap. 2, Sect. 1.1). By virtue of assertion (ii) of Theorem 4.3, we have $\dim H^1(\mathfrak{g}, \mathfrak{g}) \geq 1$ (and even ≥ 2 if $\dim \mathfrak{g} > 1$). This means, in particular, that $\operatorname{Out} \mathfrak{g} \neq \{0\}$, i.e. a nilpotent finite-dimensional Lie algebra always has an outer derivation. This is not always the case for arbitrary Lie algebras, for example $\operatorname{Der} \mathfrak{r}_2 \simeq \mathfrak{r}_2 \simeq I(\mathfrak{r}_2)$.

We now give a direct construction of a nonzero element in $\operatorname{Out} \mathfrak{g}$ for a nilpotent Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_1$ be an ideal of codimension 1 in $\mathfrak{g}$. Then there exists $X \in \mathfrak{g}$, $X \neq 0$, such that $\mathfrak{g} = \langle x \rangle + \mathfrak{g}_1$. Consider the subalgebra $C = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}_1)$, the centralizer of the subalgebra $\mathfrak{g}_1$ in $\mathfrak{g}$. Since $\mathfrak{g}$ is nilpotent, we have $C \neq \{0\}$. There exists $n \in \mathbb{N}$ such that $C \subset \mathfrak{g}_n$ and $C \not\subset \mathfrak{g}_{n+1}$ (where $\mathfrak{g}_i$ are the terms of the lower central series). Let Y be an element in C such that $Y \notin \mathfrak{g}_{n+1}$. Denote by φ the endomorphism of the space $\mathfrak{g}$ taking X into Y, and $\mathfrak{g}_1$ into $\{0\}$. One can easily check that $\varphi \in \operatorname{Der} \mathfrak{g}$. If $\varphi \in I(\mathfrak{g})$, then $\varphi = \operatorname{ad} U$, where $U \in \mathfrak{g}$. But then $U \in C$ and $[U, X] = Y$, implying that $Y \in \mathfrak{g}_{n+1}$, which is an impossibility. Therefore φ gives rise to a nonzero element in $\operatorname{Out} \mathfrak{g}$. Thus the following theorem is proved (or follows from Theorem 4.9).

Theorem 4.4 (see Jacobson [1962]). *For each nilpotent Lie algebra there exists an outer derivation.*

§ 5. Nilpotent Radicals in Lie Algebras and Lie Groups

5.1. Nilradical

Lemma. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an arbitrary field k, and $\mathfrak{h}$, $\mathfrak{h}'$ nilpotent ideals in $\mathfrak{g}$. Then $\mathfrak{h} + \mathfrak{h}'$ is also a nilpotent ideal in $\mathfrak{g}$.*

Proof. By the hypothesis, $\mathfrak{h}_m = \{0\}$, where $\mathfrak{h}_m$ is the m-th term of the lower central series of $\mathfrak{h}$, and similarly $\mathfrak{h}'_m = \{0\}$, where $m, m' < \dim \mathfrak{g}$. Induction on k together with the Jacobi identity proves that $(\mathfrak{h}+\mathfrak{h}')_k \subset \mathfrak{h}_{[k/2]+1} + \mathfrak{h}'_{[k/2]+1}$, where $[k/2]$ is the largest integer in $k/2$. Taking $k = 2\dim \mathfrak{g}$, we arrive at the conclusion that the subalgebra $\mathfrak{h} + \mathfrak{h}'$, which is evidently an ideal in $\mathfrak{g}$, is nilpotent. $\square$

Let $\mathfrak{g}$ be an arbitrary finite-dimensional Lie algebra. By Lemma 5.1, the sum of all nilpotent ideals of $\mathfrak{g}$ is a nilpotent ideal in $\mathfrak{g}$, which is evidently the largest one. This nilpotent ideal is called the *nilradical* of the Lie algebra $\mathfrak{g}$. It is a characteristic subalgebra of $\mathfrak{g}$.

Example 1. The nilradical of the Lie algebra $\mathfrak{t}_n(k)$ coincides with $\mathfrak{n}_n(k)$.

The nilradical of a Lie group G is defined in the same way – it is the largest connected nilpotent normal subgroup of G. Its tangent Lie algebra coincides with the nilradical of the tangent Lie algebra of the Lie group $\mathfrak{g}$.

Example 2. The nilradical of the Lie group $E(2)$ of motions of the Euclidean plane coincides with the subgroup of translations $T \simeq \mathbb{R}^2$.

The nilradical $\mathfrak{n}$ of a Lie algebra $\mathfrak{g}$ is contained in the radical $\mathfrak{r}$ of this Lie algebra; $\mathfrak{n}$ coincides with the set of all nilpotent elements of the Lie algebra $\mathfrak{r}$, and is the nilradical of $\mathfrak{r}$.

Theorem 5.1 (Chevalley [1955], Jacobson [1962]). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\mathfrak{r}$ its radical, and $\mathfrak{n}$ its nilradical. Then $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$.*

In other words, the action of $\mathfrak{g}$ on $\mathfrak{r}/\mathfrak{n}$ induced by the adjoint representation is trivial.

Theorem 5.1 implies, in particular, that if $\mathfrak{g}$ is a solvable Lie algebra, then $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$. Thus $\dim \mathfrak{n} \geq \dim [\mathfrak{g}, \mathfrak{g}]$. Another bound for $\dim \mathfrak{n}$ is given by the following theorem.

Theorem 5.2 (Mubarakzianov [1966]) *Let $\mathfrak{n}$ be the nilradical of a solvable Lie algebra $\mathfrak{g}$. Then $\dim \mathfrak{n} \geq \frac{1}{2}(\dim \mathfrak{g} + \dim Z(\mathfrak{g}))$.*

5.2. Nilpotent Radical. Consider a Lie algebra $\mathfrak{g}$ over an arbitrary field k. The intersection of kernels the of all irreducible finite-dimensional linear representations of $\mathfrak{g}$ is called the *nilpotent radical* of $\mathfrak{g}$; we denote it by $\operatorname{rad}_n(\mathfrak{g})$. The nilpotent radical for Lie groups is defined similarly.

Theorem 5.3 (see Bourbaki [1975], Jacobson [1955]). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k of characteristic 0. Then the nilpotent radical $\mathrm{rad}_n\mathfrak{g}$ of $\mathfrak{g}$ coincides with $[\mathfrak{g},\mathfrak{g}] \cap \mathfrak{r}$, where $\mathfrak{r}$ is the radical of $\mathfrak{g}$.*

In particular, if $\mathfrak{g}$ is solvable, then $\mathrm{rad}_n\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$.

In view of Theorem 5.1, theorem 5.3 implies that $\mathrm{rad}_n\mathfrak{g} \subset \mathfrak{n}$. In particular, $\mathrm{rad}_n\mathfrak{g}$ is nilpotent (which explains its name).

5.3. Unipotent Radical. Let G be an arbitrary connected Lie group (either real or complex). As for Lie algebras, we introduce the notions of the nilradical and nilpotent radical of G. However, if G is an algebraic linear group, then one can consider one more nilpotent subgroup of G called the unipotent radical and denoted by $\mathrm{Rad}_u G$ (see Chap. 1, Sect. 6.5). In the general case, these three nilpotent subgroups of G are distinct, but $\mathrm{Rad}_u G \subset \mathrm{Rad}_n G \subset N$, where N is the nilradical and $\mathrm{Rad}_n G$ the nilpotent radical. If G is solvable, then $\mathrm{Rad}_u G \supset (G,G)$.

Example 3. Consider the algebraic linear group

$$G = \left\{ \begin{bmatrix} 1 & x & z & 0 \\ & 1 & y & 0 \\ & & 1 & 0 \\ 0 & & & \lambda \end{bmatrix} \middle| \; x, y, z, \lambda \in \mathbb{R},\ \lambda > 0 \right\}.$$

Then G is nilpotent and therefore coincides with its nilradical. On the other hand, by Theorem 5.3, $\mathrm{Rad}_n G = (G,G) \simeq \mathbb{R}$, and the unipotent radical is isomorphic to $N_3(\mathbb{R})$.

§ 6. Some Classes of Solvable Lie Groups and Lie Algebras

In this section we consider several special classes of nilpotent solvable Lie groups and Lie algebras (another class – that of triangular Lie groups and Lie algebras – was considered in Sect. 2).

6.1. Characteristically Nilpotent Lie Algebras. A Lie algebra $\mathfrak{g}$ is said to be *characteristically nilpotent* if the algebra of its derivations $\mathrm{Der}\,\mathfrak{g}$ consists of nilpotent operators (clearly, the Lie algebra $\mathfrak{g}$ is then nilpotent).

Occasionally this name is applied to a Lie algebra $\mathfrak{g}$ for which $\mathrm{Der}\,\mathfrak{g}$ is nilpotent (but does not necessarily consist of nilpotent transformations), and if $\mathrm{Der}\,\mathfrak{g}$ does consist of nilpotent transformations, then $\mathfrak{g}$ is called strictly characteristically nilpotent. There are also other versions of these definitions.

If $\mathfrak{g}$ is a finite-dimensional Lie algebra over the field $K = \mathbb{R}$ or $\mathbb{C}$, then it is characteristically nilpotent if and only if the Lie group $(\mathrm{Aut}\,\mathfrak{g})^0$ is unipotent. Subalgebras and quotient algebras of a characteristically nilpotent Lie algebra are not necessarily characteristically nilpotent, while the direct sum of such Lie algebras is characteristically nilpotent.

Example 1. Denote by $\mathfrak{g}_{2n}$ the algebra of dimension $2n$, $n \geq 4$, defined in the basis $X_1, \ldots, X_{2n}$ by the relations

$$[X_1, X_2] = X_{k+1} \qquad \text{for} \quad 2 \leq k \leq 2n-1,$$

$$[X_2, X_k] = X_{k+1} \qquad \text{for} \quad 3 \leq k \leq 2n-3,$$

$$[X_k, X_{2n-k}] = (-1)^{k+1} \cdot X_{2n} \qquad \text{for} \quad 2 \leq k \leq n-1,$$

(for $i < j$, all the remaining relations are of the form $[x_i, x_j] = 0$). It turns out that all the Lie algebras $\mathfrak{g}_{2n}$ are characteristically nilpotent. Moreover, not only $(\operatorname{Aut} \mathfrak{g}_{2n})^0$, but also the entire group $\operatorname{Aut} \mathfrak{g}_{2n}$ is unipotent (Yamaguchi [1981]). We note that Yamaguchi [1981] also considered the algebras $\mathfrak{g}_{2n+1}$ introduced by analogous relations. However, they are not Lie algebras (this fact was not noticed by the author), because for $n = 3$ the Jacobi identity is not satisfied, for example, for X_1, X_2, X_4.

Example 2 (Favre [1971]). Consider the Lie algebra $\mathfrak{g}_7$ defined in the basis $X_1, \ldots, X_7$ by the relations

$$[X_1, X_1] = X_{i+1}, \qquad 2 \leq i \leq 6,$$

$$[X_3, X_4] = X_7,$$

$$[X_2, X_3] = -X_6,$$

$$[X_2, X_4] = -X_7,$$

$$[X_2, X_5] = -X_7.$$

Then $\mathfrak{g}_7$ is characteristically nilpotent.

Suppose that the ground field is of characteristic 0. It follows from the classification of nilpotent Lie algebras of dimension ≤ 6 (see Chap. 7, Sect. 1.1 and Morozov [1958]) that there are no characteristically nilpotent Lie algebras of dimension ≤ 6. Thus Example 6.2 provides a characteristically nilpotent Lie algebra of the least possible dimension. In general, characteristically nilpotent Lie algebras are to be found in any dimension ≥ 7 (see, for example, Vergne [1970]). Recently, multiparameter families of characteristically nilpotent Lie algebras have been constructed (Hakimjanov [1990a, 1990b, 1991]).

In contrast to the case of a field of characteristic 0, for fields of finite characteristic there exist characteristically nilpotent Lie algebras in dimension 6 (there are no characteristically nilpotent Lie algebras of dimension ≤ 5 over any field).

Example 3 (Bratzlavsky [1973]). The Lie algebra $\mathfrak{g}_6$ of dimension 6 over the field $\mathbb{F}_2$ defined by the relations

$$[X_1, X_i] = X_{i+1}, \qquad 2 \leq i \leq 5$$

$$[X_2, X_3] = X_5 + X_6,$$

$$[X_2, X_4] = X_6,$$

(the remaining relations are $[X_i, X_j] = 0$ for $i < j$) is characteristically nilpotent.

Theorem 6.1. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k of characteristic* 0, *and $\mathfrak{n}$ the nilradical of $\mathfrak{g}$. Then if the Lie algebra $\mathfrak{n}$ is characteristically nilpotent, the Levi decomposition for $\mathfrak{g}$ is a direct one, and the radical is isomorphic to $\mathfrak{n}$ (i.e. $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$, where $\mathfrak{s}$ is a Levi subalgebra).*

Proof. The adjoint representation of the Lie algebra $\mathfrak{g}$ induces the homomorphism $\varphi\colon \mathfrak{s} \to \operatorname{Der}\mathfrak{n}$. By hypothesis, $\operatorname{Der}\mathfrak{n}$ is nilpotent but $\mathfrak{s}$ is semisimple, whence the action of $\mathfrak{s}$ on $\mathfrak{n}$ is trivial, i.e. $[\mathfrak{s}, \mathfrak{n}] = 0$. It follows from $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$ (see Sect. 5.1) that the action of $\mathfrak{s}$ on the radical $\mathfrak{r}$ is also trivial. The action of $\mathfrak{r}$ on $\mathfrak{n}$ by the adjoint representation must be nilpotent, whence $\mathfrak{r} = \mathfrak{n}$. Therefore $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$. □

6.2. Filiform Lie Algebras. A Lie Algebra $\mathfrak{g}$ is said to be filiform if $\operatorname{codim}_{\mathfrak{g}}\mathfrak{g}_k = k + 1$ for $2 \le i \le n - 1$ ($n = \dim \mathfrak{g}$), where $\mathfrak{g}_k$ are the terms of the lower central series for $\mathfrak{g}$. Filiform Lie algebras are nilpotent of class $n-1$ (the highest possible for a given n, which provides another characterization of them). The value of $\dim \mathfrak{g}_k$ for them is the largest possible for nilpotent Lie algebras of a given dimension. Filiform Lie algebras can therefore be regarded as the "most nonabelian" among nilpotent Lie algebras.

If $\mathfrak{g}^i$ are the terms of the upper central series, then for filiform Lie algebras we have $\dim \mathfrak{g}^i = i$ for $1 \le i \le n - 1$ (this is yet another characterization of filiform Lie algebras).

Example 4. Let $\mathfrak{g}_n^*$ be an n-dimensional Lie algebra defined in the basis $X_1, \dots, X_n$ by the relations $[X_1, X_j] = X_{i+1}$ for $2 \le i \le n-1$ (the remaining relations are $[X_i, X_j] = 0$ for $i < j$).

We have $\mathfrak{g}_r^{*i} = \langle X_{i+2}, \dots, X_n \rangle$ for $i \ge 1$, and therefore $\mathfrak{g}_n^*$ is filiform. Note that $\mathfrak{g}_3^* \simeq \mathfrak{n}_3(k)$.

Theorem 6.2 (Bratzlavsky [1974b]). *Let $\mathfrak{g}$ be an n-dimensional filiform Lie algebra. Then there exists a basis $\{X_i\}$ in $\mathfrak{g}$ such that $[X_1, X_i] = X_{i+1}$ for $2 \le i \le n$ (here we set $X_{n+1} = 0$).*

This statement forms the basis for the study of filiform Lie algebras. In the general case it remains to describe, in addition, the commutators $[X_i, X_j]$ for $j > i > 1$. In some particular cases this can be done explicitly, for example, if the subalgebra $[\mathfrak{g}, \mathfrak{g}]$ is abelian (Bratzlavsky [1974b]).

The classification of complex filiform Lie algebra is known for dimension ≤ 8. Those of dimension ≤ 6 among them are

$$\begin{array}{rl} \dim \mathfrak{g} = 1 & \mathfrak{g} \simeq \mathbb{C}^1, \\ 2 & \quad \mathbb{C}^2, \\ 3 & \quad \mathfrak{n}_3(\mathbb{C}), \\ 4 & \quad \mathfrak{n}_4(\mathbb{C}), \\ 5 & \quad \mathfrak{g}_{5,5} \text{ or } \mathfrak{g}_{5,6}, \\ 6 & \quad \mathfrak{g}_{6,19}, \mathfrak{g}_{6,20} \text{ or } \mathfrak{g}_{6,21}. \end{array}$$

Here we use the notation introduced in Chap. 7, Sect. 1.1. For $\dim \mathfrak{g} = 7, 8, \dots$ there are continuously many filiform Lie algebras $\mathfrak{g}$. There is precisely one (up to isomorphism) one-parameter family of complex filiform Lie algebras in dimension 7:

$$[X_1, X_i] = X_{i+1}, \quad 1 \le i \le 6,$$

$$[X_2, X_5] = \alpha X_7,$$

$$[X_3, X_4] = X_7,$$

$$[X_2, X_3] = (1+\alpha) X_5,$$

$$[X_2, X_4] = (1+\alpha) X_6, \quad \alpha \in \mathbb{C}.$$

Examples for $\dim \mathfrak{g} \ge 11$ see Hakimjanov [1990a].

Let $N_n(\mathbb{C})$ be the space of all n-dimensional complex nilpotent Lie algebras (see Chap. 7, Sect. 2.3). It turns out that for $n = 2, 3, 4, 5, 6, 8$ there exist rigid filiform Lie algebras in $N_n(\mathbb{C})$, while for $n = 7$ there is no such algebra (Goze and Ancochea-Bermudez [1985]). (For the definition of rigidity see Chap. 7, Sect. 2.4.)

6.3. Nilpotent Lie Algebras of Class 2. A Lie algebra $\mathfrak{g}$ is nilpotent of class 2 (sometimes it is also called metabelian) if $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$, or, equivalently, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g})$. Examples of such algebras are the Heisenberg algebras $\mathfrak{h}_n(k)$. We will show that the problem of classifying nilpotent Lie algebras of class 2 reduces to a problem of linear algebra which, however, at the time of writing (1992) is very far from having been solved. Namely, one has to describe the orbits of the natural linear action of the group $\mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ on some Zariski open subset of $\Lambda^2 k^m \otimes k^n$.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of class 2 over some field k. We set $V = [\mathfrak{g}, \mathfrak{g}]$, and let U be the subspace of $\mathfrak{g}$ complementary to V. We have $g = U + V$, where $[U, U] \subset V$, $[U, V] = [V, V] = \{0\}$. It is clear that the Lie algebra structure on $\mathfrak{g}$ is defined by the bilinear mapping $B: U \wedge U \to V$, $x \wedge y \mapsto [x, y]$, where $x, y \in U$, and $\operatorname{Im} B = V$. Conversely, let $\mathfrak{g} = U + V$ be a decomposition of the Lie algebra $\mathfrak{g}$ into the direct sum of two subspaces U and V such that $[U, U] = V$, $[U, V] = [V, V] = \{0\}$. Then $\mathfrak{g}$ is evidently a nilpotent Lie algebra of class 2, and $[\mathfrak{g}, \mathfrak{g}] = V$.

Thus the study of nilpotent Lie algebras of class 2 is reduced to the study of bilinear skew-symmetric mappings $B: U \wedge U \to V$ (where U and V are some vector spaces) such that $\operatorname{Im} B = V$. These mappings can be regarded as elements of $\Lambda^2 U \otimes V^*$ (where V^* is the space dual to V). In the general case this problem of linear algebra is very difficult. We consider only two simple special cases.

Consider nilpotent Lie algebras $\mathfrak{g}$ over an arbitrary field k for which $\dim[\mathfrak{g},\mathfrak{g}] = 1$. Clearly, such Lie algebras are always nilpotent of class 2. Hence $V = k$, and the structure of a Lie algebra on $\mathfrak{g}$ is defined by a skew-symmetric bilinear form $b: U \wedge U \to k$. It follows from the classification of such forms that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_k(k) \otimes k^l$ for some $k, l \in \mathbb{N}$ such that $2k+l-1 = \dim \mathfrak{g}$. The case $\dim[\mathfrak{g},\mathfrak{g}] = 2$ is reduced to the classification of a pair of skew-symmetric bilinear forms (defined by the mapping $B: \Lambda^2 U \to k^2$).

Let us go back to the general case. We denote by $N_{m,n}$ the subset of elements in $\Lambda^2 k^m \oplus k^n$ such that $\operatorname{Im} B = k^n$. Clearly, $N_{m,n}(2)$ is nonempty if and only if $m(m-1) \geq 2n$, and if this condition is satisfied, then $N_{m,n}(2)$ is a Zariski open subset in $\Lambda^2 k^m \otimes k^n \simeq k^r$, where $r = \frac{nm(m-1)}{2}$. It is clear that $N_{m,n}(2)$ parametrizes the set of those structures of nilpotent Lie algebras $\mathfrak{g}$ of class 2 for which $\dim[\mathfrak{g},\mathfrak{g}] = n$, $\dim \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = m$.

Consider the natural action of the group $G_{m,n} = \mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ on the space $\Lambda^2 k^m \otimes k^n$. The subset $N_{m,n}(2)$ is clearly invariant under the action of the group $G_{m,n}$, and orbits of the induced action of $G_{m,n}$ on $N_{m,n}(2)$ define isomorphic Lie algebras. Thus, the classification of nilpotent Lie algebras of class 2 reduces to the description of orbits of the action of group $G_{m,n}$ on $N_{m,n}(2)$ for all $m, n \in \mathbb{N}$ such that $m(m-1) \geq 2n$.

An analysis of the dimension of the orbits of the $G_{m,n}$-action on $N_{m,n}(2)$ makes it possible to prove, for example, that there exist continuously many nilpotent Lie algebras of class 2 and arbitrary dimension ≥ 10 (Chao [1962]) over the field $\mathbb{R}$.

6.4. Exponential Lie Groups and Lie Algebras. A Lie group G is said to be *exponential* if the exponential mapping $\exp: \mathfrak{g} \to G$ is a diffeomorphism. Such a Lie group is characterized by the following property: for each element $g \neq e$ there is precisely one-parameter subgroup passing through it.

A Lie algebra (over the field $\mathbb{R}$ or $\mathbb{C}$) is said to be *exponential* if so is the simply-connected Lie group corresponding to it.

Sometimes exponential Lie groups (algebras) are called Lie *groups* (*algebras of type* (E)). It is clear that any exponential Lie group G is isomorphic to $\mathbb{R}^n$, where $n = \dim G$ (in particular, G is simply-connected).

Example 5. Any simply-connected nilpotent Lie group is exponential. This follows, for example, from the fact that in the Malcev coordinates on G any exponential mapping is a polynomial one, as well as its inverse (see also Theorem 6.4).

Example 6. Any simply-connected triangular Lie group is exponential (this follows, for example, from Theorem 6.4, assertions (i) and (iv)).

Example 7. Let G be a three-dimensional simply-connected solvable Lie group (for the enumeration of such groups see Chap. 7, Sect. 1.2). If G is not isomorphic to the Lie group $\widetilde{E}^0(2)$, i.e. the simply-connected covering for the group $E^0(2)$, one can prove that G is exponential.

Consider the exponential mapping for the group $\widetilde{E}^0(2)$ in more detail. The group $\widetilde{E}^0(2)$ is the semidirect product $\mathbb{R} \ltimes_\varphi \mathbb{R}^2$, where the homomorphism $\varphi: \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ is of the form

$$\varphi(t) = \begin{bmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{bmatrix}, \quad t \in \mathbb{R}.$$

One can easily verify that the complement in $\widetilde{E}^0(2)$ of the image of the exponential mapping $\exp: \mathfrak{e}(2) \to \widetilde{E}^0(2)$ consists of elements of the form $(n, v) \in \mathbb{R} \ltimes_\varphi \mathbb{R}^2$ such that $n \in \mathbb{Z}$, $n \neq 0$, and $v \neq 0$. In particular, the group $\widetilde{E}^0(2)$ is not exponential.

Through the points (x, v) for $x \in \mathbb{Z}$, $v = 0$ there pass all the one-parameter subgroups of $\widetilde{E}^0(2)$. Note that if $g = (1, v)$, where $v \neq 0$, then g does not lie in $\exp e(2)$, neither does any power g^m, $m \in \mathbb{N}$, of the element g.

Example 8. The simple Lie group $\mathcal{A} = \widetilde{\mathrm{SL}_2}(\mathbb{R})$ is not exponential. For example, for any element of the form z^2, where $z \in Z(\mathcal{A})$, $z \neq e$, there are infinitely many one-parameter subgroups passing through it.

Theorem 6.3. *Any exponential Lie group is solvable.*

Proof. If G is an exponential Lie group, then it is diffeomorphic to $\mathbb{R}^n$, and therefore, by virtue of Theorem 3.2, its Levi decomposition is of the form $G = S \cdot R$, where the Levi subgroup S is either trivial or isomorphic to $\mathcal{A} \times \ldots \times \mathcal{A}$. But, as mentioned in Example 8, the restriction of the mapping $\exp: \mathfrak{g} \to G$ to S is not injective. Hence $S = \{e\}$, and $G = R$ is solvable. □

Exponential Lie groups and algebras can be characterized in many ways.

Theorem 6.4 (see Dixmier [1957], Nono [1960], Gorbatsevich [1974b]). *Let G be a simply-connected Lie group. Then the following conditions are equivalent:*

(i) *the Lie group is exponential;*
(ii) *the mapping* $\exp: \mathfrak{g} \to G$ *is injective;*
(iii) *the mapping* $\exp: \mathfrak{g} \to G$ *is locally injective;*
(iv) *for any element X of the tangent algebra $\mathfrak{g}$ of the Lie group G, the operator* $\operatorname{Ad} X$ *has no nonzero imaginary eigenvalues;*
(v) *for any $g \in G$ the operator* $\operatorname{Ad} g$ *has no eigenvalues equal to* 1 *in absolute value but different from* 1;
(vi) *there is no ideal $\mathfrak{h}$ in the algebra $\mathfrak{g}$ such that the quotient algebra $\mathfrak{g}/\mathfrak{h}$ has a subalgebra isomorphic to $\mathfrak{e}(2)$.*

Note that if $\exp: \mathfrak{g} \to G$ is a surjection, then the Lie algebra $\mathfrak{g}$ is not necessarily exponential. For example, an exponential mapping is surjective for any compact connected Lie group, while no such group is exponential.

Theorem 6.4 implies the following corollaries.

Corollary 1. *If a complex Lie group is exponential, then it is nilpotent.*

Corollary 2. *The class of exponential Lie algebras is closed with respect to taking subalgebras, quotient algebras, and central extensions.*

6.5. Lie Algebras and Lie Groups of Type (I). A Lie algebra $\mathfrak{g}$ (over the field $K = \mathbb{R}$ or $\mathbb{C}$) is called an *algebra of type* (I)[1] if for any $X \in \mathfrak{g}$ all eigenvalues of the operator $\operatorname{ad} g$ are pure imaginary (some of them may be equal to zero).

A Lie group is called a *group of type* (I) if its tangent algebra is of type (I). A connected Lie group of type G is characterized by the fact that for any $g \in G$ the absolute values of all eigenvalues of the operator $\operatorname{ad} g$ are equal to 1.

Example 9. Any nilpotent Lie group (algebra) is of type (I).

The class of Lie groups (algebras) of type (I) can be regarded in some sense as the opposite class for all triangular Lie groups (algebras). The intersection of the two classes is exactly the class of nilpotent Lie groups (algebras).

Example 10. Let N be a nilpotent connected Lie group, and K a compact subgroup of the group $\operatorname{Aut} N$. The semisimple product $K \ltimes_i N$ corresponding to the tautological embedding $i\colon K \hookrightarrow \operatorname{Aut} N$ is a Lie group of type (I).

In particular, the group $E^0(n) = \mathrm{SO}_n \cdot E^n$ of proper motions of the Euclidean space E^n is a group of type (I).

If one takes for K the torus in $\operatorname{Aut} N$, then $K \ltimes N$ is a solvable Lie group of type (I).

One can easily see that the class of Lie groups of type (I) is closed with respect to transitions to Lie subgroups and quotient groups. The situation for Lie algebras of type (I) is similar.

Theorem 6.5 (see Auslander, Green and Hahn [1963]). *Let G be a connected Lie group, and $G = S \cdot R$ its Levi decomposition (where R is a radical, S a Levi subgroup). The Lie group G is a group of type (I) if and only if S is compact and R is of type (I).*

A similar statement holds for Lie algebras. Theorem 6.5, the description of Lie groups (and algebras) reduces to the analysis of the solvable case. Moreover, it is sufficient to consider just simply-connected solvable Lie groups.

Theorem 6.6 (Auslander, Green and Hahn [1963]). *Let G be a simply-connected solvable Lie group of type (I). Then G is isomorphic to a virtual Lie subgroup of a Lie group of the form $T \ltimes N$, where N is a simply-connected nilpotent Lie group, and T is a torus in* $\operatorname{Aut} N$.

[1] In Auslander, Green and Hahn [1963] such Lie algebras are called Lie algebras of type (R), an abbreviation of (rigid). Our notation (I) seems to be more appropriate, recalling that the eigenvalues are imaginary.

Conversely, any virtual Lie subgroup of a Lie group of the form $T \ltimes N$ *is a group of type* (I).

Corollary. *Let* G *be a simply-connected Lie group of type* (I). *Then* G *is isomorphic to a virtual Lie subgroup of a Lie group of the form* $F = K \ltimes N$, *where* N *is a simply-connected nilpotent normal Lie group in* F *and* K *is a compact subgroup of* F.

§ 7. Linearizability Criterion for Solvable Lie Groups

Theorem 7.1 (see Malcev [1943]). *Let* G *be a connected solvable Lie group. Then the following conditions are equivalent:*

(1) *G is linearizable, i.e. admits a faithful finite-dimensional linear representation (see Chap.* 1, *Sect.* 5.1);
(2) *G can be decomposed into the semidirect product* $T \ltimes F$ *of a torus* T *and a simply-connected normal Lie subgroup* F;
(3) *the commutator group* (G, G) *of the Lie group* G *is simply-connected.*

Proof. (1)$\Rightarrow$(3). Suppose that the Lie group G is linearizable, i.e. isomorphic to a virtual Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ for some $n \in \mathbb{N}$. Then G is also isomorphic to a virtual Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$, and therefore, by virtue of Lie's theorem (see Chap. 1, Sect. 1.1), also in $T_n(\mathbb{C})$. This implies that the commutator group (G, G) is isomorphic to a virtual Lie subgroup of $(T_n(\mathbb{C}), T_n(\mathbb{C})) = N_n(\mathbb{C})$. Any connected virtual Lie subgroup of $N_n(\mathbb{C})$ is simply-connected (see Sect. 3.4). Therefore (G, G) is simply connected.

(3)$\Rightarrow$(2). Suppose that T is a maximal compact subgroup of G. Then T is a torus (see Proposition 1.1 of Chap. 4). The fact that (G, G) is simply-connected implies that $T \cap (G, G) = \{e\}$. From this one can easily deduce that there exists a closed normal subgroup F of G (containing (G, G)) such that $G = T \cdot F$ and $T \cap F = \{e\}$, which yields the desired decomposition.

The proof of the implication (2)$\Rightarrow$(1) is more complicated and is not given here. $\square$

Theorem 7.1 implies, in particular, that any simply-connected solvable Lie group G is linearizable. Moreover, one can show that such a Lie group G is isomorphic to a Lie subgroup of $T_m(\mathbb{C})$, where $m = n^n + n + 1$, $n = \dim G$ (for a similar assertion for Lie algebras see Reed [1969]). Note that an arbitrary simply-connected solvable Lie group is not necessarily isomorphic to a Lie subgroup (even a virtual one) of $T_m(\mathbb{R})$ for some $m \in \mathbb{N}$.

Theorem 7.2. *A connected Lie group* G *is isomorphic to a virtual subgroup of* $T_m(\mathbb{R})$ *for some* $m \in \mathbb{N}$ *if and only if* G *is simply-connected and triangular (for more details on triangular groups see Sect.* 2).

Proof. If G is isomorphic to a virtual subgroup of $T_m(\mathbb{R})$, then G is simply-connected because so is $T_m(\mathbb{R})$ (see Sect. 3.4). Since $T_m(\mathbb{R})$ is triangular, so is G (see Sect. 2). Conversely, let G be a simply-connected triangular Lie group. A modification of the proofs of the theorems of Ado and Lie shows that the tangent algebra $\mathfrak{g}$ of the Lie group G is isomorphic to a Lie subalgebra of $\mathfrak{t}_m(\mathbb{R})$ for some $m \in \mathbb{N}$. Since G is simply-connected, the induced homomorphism of Lie groups $G \to T_m(\mathbb{R})$ is an embedding. $\square$

If G is a simply-connected nilpotent Lie group, then it is isomorphic to a Lie subgroup of $N_m(\mathbb{R})$ for some $m \in \mathbb{N}$ (see Chap. 2, Sect. 5.3). Here one can take $m = 1 + n^k$, where $n = \dim G$ and k is the nilpotency class of G (for the case of Lie algebras see Reed [1964]).

In the case of nilpotent linear Lie groups, Theorem 7.2 yields the following statement.

Theorem 7.3 (see Merzlyakov [1987]). *Let G be a connected nilpotent Lie group. The group G is linearizable if and only if it can be decomposed into the direct product $G = T \times F$ of a torus T and a simply-connected nilpotent Lie group F.*

Proof. By virtue of Theorem 7.2, if a nilpotent group is linear, then it admits a decomposition $G = T \cdot F$, where T is a torus and F a simply-connected normal Lie subgroup of G. The action of the torus T on the tangent Lie algebra $\mathfrak{f}$ of the Lie group F (induced by the adjoint action of G) is both semisimple (since T is compact) and unipotent (since G is nilpotent). But then this action is trivial, and the semidirect product $T \cdot F$ is a direct one. The converse statement follows immediately from Theorem 7.2. $\square$

Chapter 3
Complex Semisimple Lie Groups and Lie Algebras

§ 1. Root Systems

As we have seen in Chap. 1, Sect. 9, one can associate with each complex Lie algebra $\mathfrak{g}$ the system of its roots $\Delta_{\mathfrak{g}}(\mathfrak{h})$ with respect to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. A root system is a finite system of vectors in the space $\mathfrak{h}^*$. If $\mathfrak{g}$ is semisimple, then the root system $\Delta_{\mathfrak{g}}(\mathfrak{h})$ can be described as a system of vectors in a Euclidean vector space possessing some remarkable symmetries and completely defining the Lie algebra $\mathfrak{g}$. The corresponding axiomatics leads to the notion of an abstract root system, which we will now define.

1.1. Abstract Root Systems. Let E be a finite-dimensional Euclidean space equipped with the scalar product $(,)$. For any $\alpha \in E$, $\alpha \neq 0$, denote by L_α the hyperplane in E orthogonal to α and by r_α the reflection in L_α. The reflection r_α can be written in the form

$$r_\alpha(\beta) = \beta - \langle\beta|\alpha\rangle\alpha \quad (\beta \in E),$$

where

$$\langle\beta|\alpha\rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Note that L_α, r_α and the function $\langle\beta|\alpha\rangle$ do not change if the scalar product in E is replaced by $c(,)$, where $c > 0$.

A subset $\Delta \subset E$ is called a *root system* in E if

(1) Δ is finite and consists of nonzero vectors;
(2) $r_\alpha(\Delta) = \Delta$ for any $\alpha \in \Delta$;
(3) $\langle\alpha|\beta\rangle \in \mathbb{Z}$ for any $\alpha, \beta \in \Delta$.

Denote by $\operatorname{rk}\Delta$ the rank of the system Δ, i.e. $\dim\langle\Delta\rangle$.

Let $\alpha \in \Delta$. It follows from (2) that $-\alpha \in \Delta$, and one can easily deduce from (3) that if $c \in \mathbb{R}$ and $c\alpha \in \Delta$, then $c \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. A root system is said to be *reduced* if the following condition holds:

(4) if $\alpha \in \Delta$ and $c\alpha \in \Delta$ for some $c \in \mathbb{R}$, then $c = \pm 1$.

Sometimes it is more convenient to use the following modification of this definition, which requires E to be a real vector space (with no Euclidean structure on it). Condition (1) remains unchanged, while conditions (2) and (3) are modified as follows:

(2′) for any $\alpha \in \Delta$ there is a linear transformation $r_\alpha \in \mathrm{GL}(E)$ such that $r_\alpha(\Delta) = \Delta$, $r_\alpha(\alpha) = -\alpha$, and the set $\{\beta \in E | r_\alpha(\beta) = \beta\}$ is a hyperplane in E complementary to $\langle\alpha\rangle$;
(3′) for any $\alpha, \beta \in \Delta$ we have $r_\alpha(\beta) - \beta = m\alpha$, where $m \in \mathbb{Z}$.

Note that if $\langle\Delta\rangle = E$, then condition (2′) defines the transformation r_α $(\alpha \in \Delta)$ uniquely. Transformations r_α generate a finite subgroup of $\mathrm{GL}(E)$, and therefore there is a scalar product in E invariant under all r_α. We thus arrive at the first definition of a root system.

Suppose that $\Delta_i \subset E_i$ $(i = 1, \ldots, s)$ are some root systems, and let $E = \bigoplus_{i=1}^{s} E_i$ be the orthogonal direct sum of the Euclidean spaces E_i. Then $\Delta = \bigcup_{i=1}^{s} \Delta_i$ is a root system in E called the *direct sum* of the systems Δ_i. A root system is said to be *indecomposable* if it cannot be represented as the union of two proper subsets orthogonal to each other. An arbitrary root system Δ can be uniquely decomposed into the direct sum of indecomposable root systems called *indecomposable components* of the system Δ.

Axiom (3) imposes rigorous constraints on the possible angles between roots and the ratios of their lengths.

Proposition 1.1. *Let α, β be nonzero vectors of a Euclidean space E and θ the angle between them. If $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$ are nonpositive integers and $|\beta| \geq |\alpha|$, then for θ, $\langle\alpha|\beta\rangle$, $\langle\beta|\alpha\rangle$, and $|\beta|^2/|\alpha|^2$ only the following values are possible*

θ	$\langle\alpha\|\beta\rangle$	$\langle\beta\|\alpha\rangle$	$\|\beta\|^2/\|\alpha\|^2$
$\pi/2$	0	0	
$2\pi/3$	-1	-1	1
$3\pi/4$	-1	-2	2
$5\pi/6$	-1	-3	3
π	-2	-2	1
π	-1	-4	4

Proof. The assertion follows from the equality $\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = 4\cos^2\theta$. $\square$

Let α, β be two nonproportional elements of a root system Δ. The set $\{\gamma \in \Delta | \gamma = \beta + k\alpha, k \in \mathbb{Z}\}$ is called the *α-string of roots containing β.*

Proposition 1.2. *The α-string of roots containing β is of the form $\{\beta + k\alpha | -p \leq k \leq q\}$, where $p, q \geq 0$ and $p - q = \langle\beta|\alpha\rangle$. If $(\beta, \alpha) < 0$, then $\beta + \alpha \in \Delta$, and under the condition that $\beta - \alpha \notin \Delta$ the converse statement also holds.*

Example. The following figure shows the root systems of rank 1 and 2. All of them are indecomposable (except $A_1 + A_1$), and reduced (except BC_1 and BC_2).

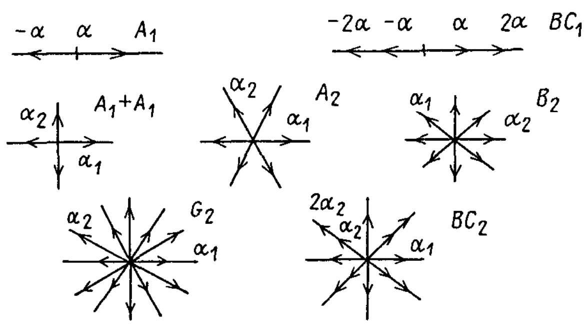


Fig. 1

Let $F = E^*$. Let us identify F^* with E with the help of the natural isomorphism $E \to (E^*)^* = F^*$. Denote by $\lambda \mapsto u_\lambda$ the isomorphism of vector spaces $E \to F$ defined by the scalar product in E, i.e. given by the formula

$$\lambda(u_\mu) = (\lambda, \mu) \quad (\lambda, \mu \in E).$$

This isomorphism takes the scalar product into F if we set

$$(u_\lambda, u_\mu) = (\lambda, \mu) = \lambda(u_\mu) = \mu(u_\lambda) \quad (\lambda, \mu \in E).$$

Let Δ be a root system in E. For any $\alpha \in \Delta$ we set

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)} u_\alpha.$$

Then $\mu(\alpha^\vee) = \langle \mu | \alpha \rangle \quad (\mu \in E)$ and

$$\langle \alpha^\vee | \beta^\vee \rangle = \langle \beta | \alpha \rangle \quad (\alpha, \beta \in \Delta).$$

Proposition 1.3. *If Δ is a root system in E, then $\Delta^\vee = \{\alpha^\vee | \alpha \in \Delta\}$ is a root system in F, reduced if and only if so is Δ. We have* $\operatorname{rk} \Delta = \operatorname{rk} \Delta^\vee$.

The root system $\Delta^\vee$ is called the *dual* of Δ. Objects related to this system will be labeled by $^\vee$. Note that the hyperplane $L_{\alpha^\vee} (\alpha \in \Delta)$ in F is defined by the equation $\alpha(x) = 0$.

1.2. Root Systems of Reductive Groups. Let G be a connected reductive complex algebraic group, H a Cartan subgroup of it (i.e. a maximal algebraic torus, see Chap. 1, Sect. 9.5), and $\mathfrak{g} \subset \mathfrak{h}$ the corresponding tangent algebras. The root system $\Delta(\mathfrak{h})$ is identified with the system of nonzero weights $\Delta_G(H)$ of the adjoint representation of the group G in the space $\mathfrak{g}$. According to Chap. 1, Sect. 9.5, $\Delta_G(H) = \Delta$ lies in the real vector space $E = \mathfrak{h}(\mathbb{R})^*$. Proposition 6.2 of Chap. 1 implies that there exists a nondegenerate invariant symmetric bilinear form $(\, , \,)$ in $\mathfrak{g}$ whose restriction to $F = \mathfrak{h}(\mathbb{R})$ is positive definite (for semisimple G one can take for such a form the Killing form k). As in Sect. 1.1, one can now transfer the Euclidean space structure from F to E.

An important role in the analysis of reductive groups G is played by the following construction of some subalgebras of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Consider the root decomposition of the algebra $\mathfrak{g}$ with respect to $\mathfrak{h}$ (see Chap. 1, Sect. 9.2). Let $\alpha \in \Delta$. Proposition 9.4 of Chap. 1 implies that for each nonzero $e_\alpha \in \mathfrak{g}_\alpha$ there exists $e_{-\alpha} \in \mathfrak{g}_\alpha$ such that $(e_\alpha, e_{-\alpha}) = \frac{2}{(\alpha,\alpha)}$. Then the elements e_α, $e_{-\alpha}$, and $h_\alpha = \frac{2}{(\alpha,\alpha)} u_\alpha \in \mathfrak{h}(\mathbb{R})$ satisfy the relations

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \quad [h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, e_{-\alpha}] = -2e_{-\alpha}. \tag{1}$$

Therefore the mapping $\varphi_\alpha : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ given by the formulae

$$\varphi_\alpha(\mathbf{e}) = e_\alpha, \quad \varphi_\alpha(\mathbf{f}) = e_{-\alpha}, \quad \varphi_\alpha(\mathbf{h}) = h_\alpha$$

(see the example of Chap. 1, Sect. 9.4) is an isomorphism of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ onto the subalgebra $\mathfrak{g}^{(\alpha)} = \langle e_\alpha, e_{-\alpha}, h_\alpha \rangle \subset \mathfrak{g}$. This isomorphism corresponds to the homomorphism $F_\alpha : \mathrm{SL}_2(\mathbb{C}) \to G$ such that $dF_\alpha = \varphi_\alpha$, and the subgroup $G^{(\alpha)} = \operatorname{Im} F_\alpha$. Let $n_\alpha = F_\alpha(E_{12} - E_{21}) \in G^{(\alpha)}$.

Proposition 1.4. *We have $n_\alpha \in N_G(H)$, and* $\operatorname{Ad} n_\alpha$ *induces a transformation on F conjugate to r_α, i.e. the reflection in the hyperplane given by the equation $\alpha(x) = 0$. Reflections r_α $(\alpha \in \Delta)$ take the system of weights of any linear representation of the group G into itself.*

Theorem 1.1. *The root system $\Delta = \Delta_G(H)$ of a reductive algebraic group G is a root system in the space $E = \mathfrak{h}(\mathbb{R})^*$ in the sense of Sect. 1.1. The dual root system $\Delta^\vee \subset F = \mathfrak{h}(\mathbb{R})$ consists of elements $\alpha^\vee = h_\alpha = \frac{2}{(\alpha,\alpha)} u_\alpha (\alpha \in \Delta)$.*

Proof. Proposition 1.4 implies that the system Δ satisfies condition (2) of the definition of the root system.

Since F_α is polynomial (see Theorem 6.3, Chap. 1), φ_α takes the subgroup $\mathfrak{t}(\mathbb{Z})$ of the tangent algebra of the maximal torus $T = D_2(\mathbb{C}) \cap \mathrm{SL}_2(\mathbb{C})$ of the group $\mathrm{SL}_2(\mathbb{C})$ into $\mathfrak{h}(\mathbb{Z})$. In particular, $h_\alpha = \varphi_\alpha(\mathbf{h}) \in \mathfrak{h}(\mathbb{Z})$, which implies that Δ satisfies condition (3) of Sect. 1.1. □

Proposition 1.4 also implies the following theorem.

Theorem 1.2. *The root system $\Delta = \Delta_G(H)$ of a complex reductive algebraic group G with respect to the maximal torus H is reduced. We have* $\dim \mathfrak{g}_\alpha = 1$ *for all $\alpha \in \Delta$. If $\alpha, \beta, \alpha + \beta \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

Corollary. *One can take for a basis of the Lie algebra $\mathfrak{g}$ the set of arbitrary nonzero elements $e_\alpha \in \mathfrak{g}_\beta$ $(\alpha \in \Delta)$ together with some basis $h_1, \dots, h_l$ of the space $\mathfrak{h}(\mathbb{R})$.*

Note that the rank of the root system $\Delta_G(H)$ coincides with the rank of the commutator group (G, G). In particular, G is semisimple if and only if $\langle \Delta_G(H) \rangle = E$, and G is an algebraic torus if and only if $\Delta_G(H) = \emptyset$.

Proposition 1.5. *The root system Δ of a semisimple complex Lie algebra $\mathfrak{g}$ is indecomposable if and only if $\mathfrak{g}$ is simple. In the general case, indecomposable components of the root system Δ naturally correspond to simple ideals of $\mathfrak{g}$.*

The above construction of three-dimensional subalgebras admits the following generalization due to Morozov [1960].

Theorem 1.3. *Let x be a nonzero nilpotent element of a semisimple Lie algebra $\mathfrak{g}$. Then there exist elements $y, z \in \mathfrak{g}$ and a homomorphism $\varphi \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$ such that $\varphi(\mathbf{e}) = x$, $\varphi(\mathbf{f}) = y$, $\varphi(\mathbf{h}) = z$. Therefore $\langle x, y, z \rangle$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.*

Proof. The nondegeneracy of the Killing form k of the algebra $\mathfrak{g}$ (see Theorem 2.2 of Chap. 1) implies that $\operatorname{Im} \operatorname{ad} x$ coincides with the orthogonal complement $\mathfrak{z}(x)^\perp$ of $\mathfrak{z}(x) = \operatorname{Ker} \operatorname{ad} x$ with respect to k. Since x is nilpotent, $x \in \mathfrak{z}(x)^\perp = \operatorname{Im} \operatorname{ad} x$. We deduce that there exists a nonzero algebraic commutative subalgebra $\mathfrak{f} \subset \mathfrak{g}$ consisting of semisimple elements with respect to which x is a root vector corresponding to some root $\alpha \neq 0$. Next, there exists a root vector y corresponding to the root $-\alpha$ with respect to $\mathfrak{f}$ such that $k(x, y) \neq 0$. Then the vectors $x, y, z = c[x, y]$, where $c \in \mathbb{C}$ is an appropriate nonzero coefficient, constitute the desired triple. □

1.3. Root Decompositions and Root Systems for Classical Complex Lie Algebras. We now present the explicit form of the root decompositions, root systems, and dual root systems for the classical Lie algebras $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C}), \mathfrak{sp}_n(\mathbb{C})$ with respect to the Cartan subalgebras selected in the example of Chap. 1, Sect. 9.5.

It is convenient to express roots of the classical Lie algebras $\mathfrak{g}$ in terms of the weights of the standard (identity) representation Id of the classical groups G. In particular, for $G = \mathrm{GL}_n(\mathbb{C})$, one takes for a maximal torus (or a Cartan subalgebra) in G the subgroup $\mathrm{D}_n(\mathbb{C})$ of all diagonal matrices. Then $\mathfrak{h}(\mathbb{R}) = \mathfrak{d}_n(\mathbb{R})$ is the algebra of all real diagonal matrices. The scalar product of two matrices $X = \operatorname{diag}(x_1, \ldots, x_n)$ and $Y = \operatorname{diag}(y_1, \ldots, y_n)$ in $\mathfrak{d}_n(\mathbb{R})$ is equal to

$$(X, Y) = \operatorname{tr}(X \cdot Y) = \sum_{i=1}^{n} x_i y_i.$$

The vectors e_i $(i = 1, \ldots, n)$ of a standard basis of the space $\mathbb{C}^n$ are weight vectors of the representation Id with respect to $\mathrm{D}_n(\mathbb{C})$. The corresponding weights $\varepsilon_i \in \mathfrak{h}(\mathbb{R})^*$ are of the form

$$\varepsilon_i(\operatorname{diag}(x_1, \ldots, x_n)) = x_i \quad (i = 1, \ldots, n).$$

In what follows we also denote by ε_i the restrictions of the linear forms ε_i to the Cartan subalgebra $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{d}_n(\mathbb{C})$ of a classical subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$. Denote by $H_1, \ldots, H_n$ the basis of the space $\mathfrak{d}_n(\mathbb{R})$ dual to $\varepsilon_1, \ldots, \varepsilon_n$:

$$H_i = \operatorname{diag}(0, \ldots, \underbrace{1}_{i}, \ldots, 0) = E_{ii}.$$

Example 1. For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{h} = \mathfrak{d}_n(\mathbb{C})$ we have

$$\Delta = \Delta_{\mathfrak{g}}(\mathfrak{h}) = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j; i, j = 1, \ldots, n\},$$
$$\mathfrak{g}_{\alpha_{ij}} = \langle E_{ij} \rangle,$$
$$h_{\alpha_{ij}} = H_i - H_j.$$

Example 2. In the case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ $(n \geq 2)$ the root system Δ is given by the same formula as in the preceding example (but the dimension of the space it lies in is $n-1$); $\mathfrak{g}_{\alpha_{ij}}$ and $h_{\alpha_{ij}}$ are of the same form. The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ (and its root system) is called the *Lie algebra* (*root system*) *of type* A_{n-1}.

In the simplest case $n = 2$ we have $\Delta_{\mathfrak{g}} = \{\alpha, -\alpha\}$, where $\alpha = \alpha_{12}$ (see the root system A_1 in the example of Sect. 1.1). Here $\mathfrak{g}_\alpha = \langle \mathbf{e} \rangle$, $\mathfrak{g}_{-\alpha} = \langle \mathbf{f} \rangle$, $h_\alpha = \langle \mathbf{h} \rangle$.

Example 3. In the case $\mathfrak{g} = \mathfrak{so}_{2l}(\mathbb{C})$ $(l \geq 2)$ we have

$$\Delta = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j \ (i \neq j),\ \beta_{ij} = \varepsilon_i + \varepsilon_j \ (i < j),\ -\beta_{ij} \,|\, i, j = 1, \ldots, l\},$$
$$\mathfrak{g}_{\alpha_{ij}} = \langle E_{ij} - E_{l+j,l+i} \rangle,$$

$$\mathfrak{g}_{\beta_{ij}} = \langle E_{i,l+j} - E_{j,l+i} \rangle, \quad \mathfrak{g}_{-\beta_{ij}} = \langle E_{l+i,j} - E_{l+j,i} \rangle,$$
$$h_{\alpha_{ij}} = H_i - H_j - H_{l+i} + H_{l+j}, \quad h_{\beta_{ij}} = H_i + H_j - H_{l+i} - H_{l+j}.$$

The Lie algebra $\mathfrak{so}_{2l}(\mathbb{C})$ (and its root system) is called the *Lie algebra (root system) of type D_l.*

One can easily see that in the case $l = 2$ the system Δ is the direct sum of the two systems of type A_1 (see the example of Sect. 1.1). By virtue of Proposition 1.5, the Lie algebra $\mathfrak{so}_4(\mathbb{C})$ is not simple (although it is semisimple).

Example 4. In the case $\mathfrak{g} = \mathfrak{so}_{2l+1}(\mathbb{C})$ $(l \geq 1)$ we have

$$\Delta = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j \ (i \neq j),\ \beta_{ij} = \varepsilon_i + \varepsilon_j \ (i < j),\ -\beta_{ij}, \varepsilon_i, -\varepsilon_i | i, j = 1, \ldots, l\};$$

$\mathfrak{g}_{\alpha_{ij}}$, $\mathfrak{g}_{\beta_{ij}}$, $\mathfrak{g}_{-\beta_{ij}}$ are given by the same formulae as in Example 3,

$$\mathfrak{g}_{\varepsilon_i} = \langle E_{i,2l+1} - E_{2l+1,l+i} \rangle, \quad \mathfrak{g}_{-\varepsilon_i} = \langle E_{l+i,2l+1} - E_{2l+1,i} \rangle,$$

$h_{\alpha_{ij}}$, $h_{\beta_{ij}}$ are given by the same formulae as in Example 3,

$$h_{\varepsilon_{ij}} = 2(H_i - H_{l+i}).$$

The Lie algebra $\mathfrak{so}_{2l+1}(\mathbb{C})$ (and its root system) is called the *Lie algebra (root system) of type B_l.*

Example 5. In the case $\mathfrak{g} = \mathfrak{sp}_{2l}$ $(l \geq 1)$, we have

$$\Delta = \{\alpha_{ij} = \varepsilon_i - \varepsilon_j \ (i \neq j), \quad \beta_{ij} = \varepsilon_i + \varepsilon_j \ (i < j), \quad -\beta_{ij} | i, j = 1, \ldots, l\},$$

$\mathfrak{g}_{\alpha_{ij}}$, $\mathfrak{g}_{\beta_{ij}}$, $\mathfrak{g}_{-\beta_{ij}}$ are the same as in Example 3,

$$\mathfrak{g}_{\beta_{ij}} = \langle E_{i,l+j} + E_{j,l+i} \rangle,$$
$$\mathfrak{g}_{-\beta_{ij}} = \langle E_{l+i,j} + E_{l+j,i} \rangle,$$
$$h_{\beta_{ij}} = H_i - H_{l+i}.$$

The Lie algebra $\mathfrak{sp}_{2l}(\mathbb{C})$ (and its root system) is called the *Lie algebra (root system) of type C_l.* Note that $\mathfrak{sp}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$.

1.4. Weyl Chambers and Simple Roots. Let $\Delta \subset E$ be a root system. The hyperplanes $L_{\alpha^\vee}$ $(\alpha \in \Delta)$ subdivide F into finitely many polyhedral convex cones. The elements of the set $F_{\text{reg}} = F \setminus \cup_{\alpha \in \Delta} L_{\alpha^\vee}$ are said to be *regular*. The connected components of this set are called (*open*) *Weyl chambers*, and their closures *closed Weyl chambers*.

If C is a Weyl chamber, then $-C = \{x \in F | -x \in C\}$ is also a Weyl chamber. It is called the Weyl chamber *opposite* to C. A hyperplane $P \subset F$ is called a *wall* of the Weyl chamber C if $P \cap C = \emptyset$ and $P \cap \overline{C}$ contains a nonempty subset open in P.

Under the natural isomorphism $\lambda \mapsto u_\lambda$ of the Euclidean spaces $E \to F$ the hyperplane L_α goes into $L_{\alpha^\vee}$. Thus the Weyl chambers of the system $\Delta^\vee$ are mapped onto the Weyl chambers of the system Δ.

A subsystem Π of a root system Δ is called a *system of simple roots* (or a *base*) of the system Δ if Π is linearly independent and each $\beta \in \Delta$ can be represented in the form

$$\beta = \sum_{\alpha \in \Pi} k_\alpha \alpha,$$

where k_α are integers, which are simultaneously either nonpositive or nonnegative. In the first case β is said to be *positive* ($\beta > 0$), in the second *negative* ($\beta < 0$) with respect to Π. Denote by Δ^+ and Δ^- the sets of positive and negative roots, respectively. Evidently, $\Delta^- = -\Delta^+$. The choice of a system of simple roots Π defines the following partial order in the space E: $\xi \geq \eta$ if $\xi - \eta = \sum_{\alpha \in \Pi} k_\alpha \alpha$, where $k_\alpha \geq 0$ for all α.

If $\Delta = \Delta_G = \Delta_{\mathfrak{g}}$ is a root system of a reductive complex algebraic group G or its tangent algebra $\mathfrak{g}$, then the system of simple roots $\Pi \subset \Delta$ will be denoted by Π_G or $\Pi_{\mathfrak{g}}$ and called a *system of simple roots of the group G* (or *Lie algebra* $\mathfrak{g}$).

Now we establish the correspondence between Weyl chambers and systems of simple roots. Let C be a Weyl chamber, and $\alpha \in \Delta$. Since C is connected, we have either $\alpha(x) > 0$ for all $x \in C$, or $\alpha(x) < 0$ for all $x \in C$. In the first case α is C-positive, in the second C-negative. Denote by $\Pi(C)$ the set of all C-positive roots α that are not representable in the form $\alpha = \beta + \gamma$, where β and γ are C-positive roots.

Theorem 1.4. *For any Weyl chamber C the system $\Pi(C)$ is a system of simple roots. The roots that are positive (negative) with respect to $Pi(C)$ coincide with C-positive (respectively, C-negative) roots. The correspondence $C \mapsto \Pi(C)$ between the Weyl chambers and systems of simple roots is bijective. For any Weyl chamber C, we have*

$$C = \{x \in F | \alpha(x) > 0 \, (\alpha \in \Pi(C))\},$$

$$\overline{C} = \{x \in F | \alpha(x) \geq 0 \, (\alpha \in \Pi(C))\}.$$

The walls of the Weyl chamber C are the hyperplanes $L_{\alpha^\vee}$, where $\alpha \in \Pi(C)$.

Corollary. *If $\langle \Delta \rangle = E$, then any closed Weyl chamber is a simplicial cone.*

We now present another way of constructing simple root systems. It is the method used by E.B. Dynkin in his paper Dynkin [1947], where the method of simple roots was presented in detail for the first time.

A real vector space E is said to be *ordered* if E is equipped with the order $>$ possessing the following properties:

(1) $\lambda > 0$, $\mu > 0 \Rightarrow \lambda + \mu > 0$;
(2) $\lambda > 0$, $c \in \mathbb{R}$, $c > 0 \Rightarrow c\lambda > 0$.

An example of such an order is a lexicographic order with respect to some basis in the space E. Let Δ be a system of roots in an ordered Euclidean space E. A root $\alpha > 0$ is said to be simple if $\alpha \neq \beta + \gamma$, where $\beta, \gamma \in \Delta$, $\beta > 0$, $\gamma > 0$. It turns out (see Dynkin [1947]) that the set of simple roots is the set of simple roots in the sense of the above definition, while the corresponding system Δ^+ coincides with the set $\{\alpha \in \Delta | \alpha > 0\}$.

Example. Using the lexicographic order with respect to the basis composed of the weights ε_i, one can easily construct systems of simple roots $\Pi_{\mathfrak{g}}$ in the root systems $\Delta_{\mathfrak{g}}$ of the classical Lie algebras $\mathfrak{g}$ (see Examples 1–5 of Sect. 1.3):

$$\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) \quad \text{or} \quad \mathfrak{sl}_n(\mathbb{C}) \quad (n \geq 2);$$

$$\Pi_{\mathfrak{g}} = \{\alpha_1, \ldots, \alpha_{n-1}\}, \quad \text{where} \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1},$$

$$\Delta_{\mathfrak{g}} = \{\varepsilon_i - \varepsilon_j | i < j;\ i, j = 1, \ldots, n\}.$$

The corresponding Weyl chamber consists of the set of diagonal matrices $\operatorname{diag}(x_1, \ldots, x_n)$ such that $x_1 > x_2 > \ldots > x_n$.

$$\mathfrak{g} = \mathfrak{so}_{2l}(\mathbb{C}), \quad l \geq 2:$$

$$\Pi_{\mathfrak{g}} = \{\alpha_1, \ldots, \alpha_l\}, \quad \text{where} \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}\,(i = 1, \ldots, l-1), \quad \alpha_l = \varepsilon_{l-1} + \varepsilon_l,$$

$$\Delta_{\mathfrak{g}}^+ = \{\varepsilon_i \pm \varepsilon_j | i < j;\ i, j = 1, \ldots, l\}.$$

$$\mathfrak{g} = \mathfrak{so}_{2l+1}(\mathbb{C}), \quad l \geq 1:$$

$$\Pi_{\mathfrak{g}} = \{\alpha_1, \ldots, \alpha_l\}, \quad \text{where} \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}\,(i = 1, \ldots, l-1), \quad \alpha_l = \varepsilon_l,$$

$$\Delta_{\mathfrak{g}}^+ = \{\varepsilon_i \pm \varepsilon_j, \varepsilon_i | i, j = 1, \ldots, l\}.$$

$$\mathfrak{g} = \mathfrak{sp}_{2l}(\mathbb{C}), \quad l \geq 1:$$

$$\Pi_{\mathfrak{g}} = \{\alpha_1, \ldots, \alpha_l\}, \quad \text{where} \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}\,(i = 1, \ldots, l-1), \quad \alpha_l = 2\varepsilon_l,$$

$$\Delta_{\mathfrak{g}}^+ = \{\varepsilon_i \pm \varepsilon_j, 2\varepsilon_i | i, j = 1, \ldots, l\}.$$

Proposition 1.6. *Let Π be a system of simple roots of a reduced root system Δ. Then $\Pi^\vee = \{\alpha^\vee | \alpha \in \Pi\} \Delta^\vee$ is a system of simple roots in.*

We now consider some properties of simple roots, which are often used in the theory of Lie algebras. With the use of Proposition 1.2 one can prove the following assertion.

Proposition 1.7. *Let Π be the system of simple roots in the root system Δ. Then*

(a) *if $\alpha, \beta \in \Pi$, $\alpha \neq \beta$, then $\alpha - \beta \notin \Delta$ and $(\alpha, \beta) \leq 0$;*
(b) *any $\alpha \in \Delta^+$ can be represented in the form $\alpha = \alpha_1 + \ldots + \alpha_s$, where $\alpha_i \in \Pi$ and $\alpha_1 + \ldots + \alpha_k \in \Delta$ for any $k = 1, \ldots, s$.*

Proposition 1.8. *A root system Δ is indecomposable if and only if so is its system of simple roots Π. If $\Delta = \Delta_1 \cup \ldots \cup \Delta_r$ is the decomposition of the system Δ into indecomposable components, then $\Pi = \Pi_1 \cup \ldots \cup \Pi_r$, where $\Pi_i = \Pi \cap \Delta_i$ is a system of simple roots of Δ_i.*

Propositions 1.5 and 1.8 imply the following theorem.

Theorem 1.5. *A semisimple complex Lie algebra $\mathfrak{g}$ is simple if and only if its system of simple roots $\Pi_{\mathfrak{g}}$ is indecomposable. If $\Pi_{\mathfrak{g}} = \Pi_1 \cup \ldots \cup \Pi_r$ is a* *decomposition of* $\Pi_{\mathfrak{g}}$ *into indecomposable components, then* $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$, *where* $\mathfrak{g}_i$ *is a simple ideal with the system of simple roots* Π_i.

One can easily verify that all systems of simple roots $\Pi_{\mathfrak{g}}$ of the example are indecomposable with the single exception of $\mathfrak{g} = \mathfrak{so}_4(\mathbb{C})$. Thus we arrive at the following corollary.

Corollary. *The Lie algebras* $\mathfrak{sl}_n(\mathbb{C})$ $(n \geq 2)$, $\mathfrak{so}_n(\mathbb{C})$ $(n \geq 3, n \neq 4)$, $\mathfrak{sp}_n(\mathbb{C})$ $(n = 2l \geq 2)$ *are simple.*

1.5. Borel Subgroups and Subalgebras. Let G be a real or complex Lie group. A *Borel subgroup* of G is any maximal connected solvable Lie subgroup of it (respectively, real or complex). Similarly, a *Borel subalgebra* of a Lie algebra is any maximal solvable subalgebra of it. One can easily see that there is a natural one-to-one correspondence between the Borel subgroups of a Lie group and the Borel subalgebras of its tangent algebra. A Borel subgroup of a complex algebraic group is always algebraic. The following theorem of Morozov and Borel holds.

Theorem 1.6. *All Borel subgroups (subalgebras) of a connected complex Lie group (respectively, complex Lie algebra) are conjugate to each other.*

Proof. The fact that Borel subgroups of a complex algebraic group are conjugate is proved in Springer [1989], Chap. 1, Sect. 3.5. For a semisimple complex Lie group G the conjugacy of the Borel subgroups follows from their conjugacy in the linear algebraic group $\operatorname{Ad} G$. The case of an arbitrary complex Lie group G is reduced (by means of the Levi decomposition) to the semisimple case because $\operatorname{Rad} G$ is contained in any Borel subgroup. $\square$

Now we establish a one-to-one correspondence between Borel subgroups of a reductive complex algebraic group containing maximal torus of it and the Weyl chambers of the corresponding root system.

Let G be a reductive complex algebraic group, H a maximal torus of it, $F = \mathfrak{h}(\mathbb{R})$, $C \subset F$ some Weyl chamber, and $\Delta = \Delta^+ \cup \Delta^-$ a decomposition of the system $\Delta = \Delta_G$ into C-positive and C-negative roots. Then the subspaces

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$$

are subalgebras of $\mathfrak{g}$. The subalgebras

$$\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha, \quad \mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$$

are constructed similarly and correspond to the opposite Weyl chamber $-C$.

Proposition 1.9 *The subalgebra* $\mathfrak{b}^+$ *is a Borel subalgebra of* $\mathfrak{g}$ *and coincides with its normalizer. The corresponding Borel subgroup* $B^+ \subset G$ *can be represented in the form* $B^+ H \ltimes N^+$, *where* N^+ *is the unipotent radical of the group* B^+, *and the tangent algebra of the subgroup* N^+ *coincides with* $\mathfrak{n}^+$.

Theorem 1.7. *The mapping $C \mapsto B^+$ constructed above is a bijection of the set of all Weyl chambers in F onto the set of all Borel subgroups of G containing H.*

Proof. First we prove that the mapping $C \mapsto B^+$ is surjective. Let B a Borel subgroup of G containing H. By Theorem 1.6, there exists $g \in G$ such that $gBg^{-1} = B^+$. The conjugacy of maximal tori in the group B^+ enables one to assume that $g \in N_G(H)$. Then one can easily see that the subgroup B corresponds to the Weyl chamber $w(C)$, where $w = \operatorname{Ad} g^{-1}|F$. □

The above construction of Borel subgroups is used in the proof of the following important theorem.

Theorem 1.8. *Let B be a Borel subgroup of a connected complex algebraic group G. Then $N_G(B) = B$. The algebraic variety G/B is projective and simply-connected.*

Proof. It is sufficient to consider the case of G semisimple. Consider the Grassmann variety $\mathrm{Gr}_k(\mathfrak{g})$ of k-dimensional subspaces in $\mathfrak{g}$, where $k = \dim B$. The adjoint representation of G in $\mathfrak{g}$ defines the polynomial action of G on the algebraic variety $\mathrm{Gr}_k(\mathfrak{g})$. The stabilizer of the point $\mathfrak{b}^+ \in \mathrm{Gr}_k(\mathfrak{g})$ in G is evidently $N_G(B^+)$. Let Q be the orbit $G(\mathfrak{b}^+) \subset \mathrm{Gr}_k(\mathfrak{g})$. One can show that G/B^+ is compact (see Proposition 1.7, Chap. 6). Hence it follows that Q is an algebraic subvariety in $\mathrm{Gr}_k(\mathfrak{g})$ and consequently a projective algebraic variety. Let N^- be a connected algebraic subgroup of G corresponding to $\mathfrak{n}^-$. Then $\mathfrak{g} = \mathfrak{b}^+ \oplus \mathfrak{n}^-$ (the direct sum of vector spaces). Since $N_G(B^+)^0 = B$ (see Proposition 1.9), it follows that the orbit $N^-(\mathfrak{b}^+)$ is Zariski open in Q. Now the fact that the subgroup N^- is unipotent implies that $N^- \cap N_G(B^+) = \{e\}$, so N^- acts simply transitively on $N^-(\mathfrak{b}^+)$. Therefore the manifold $N^-(\mathfrak{b}^+)$ is isomorphic to $\mathbb{C}^q$ and, in particular, is simply-connected. This implies that Q is also simply-connected (see Helgason [1978], Chap. 7, Sect. 12). Therefore the subgroup $N_G(B)$ is connected (see Vinberg and Onishchik [1988], Chap. 1, Sect. 4.5), i.e. $B = N_G(B)$. □

Example. Let $G = \mathrm{GL}_n(\mathbb{C})$, and let C be the Weyl chamber in $\mathfrak{d}_n(\mathbb{R})$ described in the example of Sect. 1.4. Then the corresponding Borel subgroups B^+ and B^- coincide, respectively, with the subgroups $\mathrm{T}_n^+(\mathbb{C})$ and $\mathrm{T}_n^-(\mathbb{C})$ of all upper and lower triangular matrices.

Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be one of the classical complex Lie groups, $H = G \cap \mathrm{D}_n(\mathbb{C})$ its maximal torus from the example of Chap. 1, Sect. 9.5, and C the Weyl chamber corresponding to the system of system roots chosen in Sect. 1.4. Then B^+ (B^-) coincides with $G \cap \mathrm{T}_n^+(\mathbb{C})$ (respectively, with $G \cap \mathrm{T}_n^-(\mathbb{C})$).

1.6. The Weyl Group. Let Δ be a root system in a space E. Denote by W and $W^\vee$ the groups of orthogonal transformations of the spaces $F = E^*$ and E generated by the reflections $r_{\alpha^\vee}$ and r_α ($\alpha \in \Delta$) respectively. The group W is called the *Weyl group* of the root system Δ. Clearly, $W^\vee$ is the Weyl group of the dual root system $\Delta^\vee$. Since $r_{\alpha^\vee} = r_\alpha^\top = (r_\alpha^\top)^{-1}$, the mapping $w \mapsto (w^\top)^{-1}$ defines an isomorphism of W onto $W^\vee$.

The definition of a root system implies that $W^\vee(\Delta) = \Delta$. Therefore W permutes the hyperplanes $L_{\alpha^\vee}$ ($\alpha \in \Delta$) and the Weyl chambers in F. In the basis consisting of simple roots each transformation from the group $W^\vee$ in the space $\langle\Delta\rangle$ is written as an integer matrix. Thus the Weyl group is finite.

Theorem 1.9. *The Weyl group W acts simply transitively on the set of all Weyl chambers in F, and so does the group $W^\vee$ on the set of all systems of simple roots in Δ. For a fixed system of simple roots $\Pi \subset \Delta$ the groups W and $W^\vee$ are generated by reflections $r_{\alpha^\vee}$ and r_α ($\alpha \in \Pi$), respectively, and for any $\alpha \in \Delta$ there exists $w \in W^\vee$ such that $w(\alpha) \in \Pi$ (or $\frac{1}{2}w(\alpha) \in \Pi$).*

Proof. If C and C' are two Weyl chambers, then there exists a sequence of Weyl chambers $C_0, C_1, \ldots, C_r$ such that $C = C_0$, $C' = C_r$, where C_i and C_{i+1} ($i = 0, \ldots, r-1$) lie on different sides of the hyperplane that is their common wall. This easily implies that the group W is transitive on the set of all Weyl chambers. Using Theorem 1.4, we deduce that W is generated by the reflections $r_{\alpha^\vee}$ ($\alpha \in \Pi$). $\square$

Theorem 1.10. *Any closed Weyl chamber is a fundamental set for W, i.e. it intersects the orbit $W(y)$ of any point $y \in F$ at a single point.*

Now consider the case where $\Delta = \Delta_G$ is the root system of a connected reductive algebraic group G with respect to a maximal torus H. The Weyl group W of the root system Δ_G is called the Weyl group of the reductive algebraic group G or its tangent algebra $\mathfrak{g}$. We now give a description of the Weyl group in group terms. Each element $n \in N_G(H)$ defines an automorphism $\operatorname{Ad} n$ of the Lie algebra $\mathfrak{g}$ taking $F = \mathfrak{h}(\mathbb{R})$ into itself and preserving the scalar product in F. This defines a homomorphism $\nu\colon N_G(H) \to \mathrm{O}(F)$.

Theorem 1.11. *We have $\operatorname{Ker}\nu = H$, $\operatorname{Im}\nu = W$. The homomorphism ν defines an isomorphism of the group $N_G(H)/H$ onto the Weyl group W.*

Proof. The equality $\operatorname{Ker}\nu = H$ follows from the equality $Z_G(H) = H$. One can see from Proposition 1.4 that $W \subset \operatorname{Im}\nu$. On the other hand, with the help of Theorem 1.8, one can show that the group $\operatorname{Im}\nu$ acts freely on the set of Weyl chambers. Since the group W is transitive on this set, we have $W = \operatorname{Im}\nu$. $\square$

Example. Let $G = \mathrm{GL}_n(\mathbb{C})$, $H = \mathrm{D}_n(\mathbb{C})$. Then $r_{\alpha_{ij}^\vee}$ transposes H_i with H_j and preserves the vectors H_k, $k \neq i, j$. Therefore the Weyl group $W \simeq S_n$ is the group of linear transformations of the space $\mathrm{D}_n(\mathbb{R})$ defined by all permutations of the basis elements $H_1, \ldots, H_n$. The group $N_G(H)$ is the group of all monomial matrices, i.e. matrices with exactly one nonzero element in each row and column.

For $G = \mathrm{SL}_n(\mathbb{C})$ $(n \geq 2)$, the Weyl group $W \simeq S_n$ is obtained from the linear group described above by the restriction to the invariant subspace $\mathrm{D}_n(\mathbb{R}) \cap \mathrm{SL}_n(\mathbb{C})$.

Let $G = \mathrm{SO}_{2l}(\mathbb{C})$ $(l \geq 2)$, $\mathrm{SO}_{2l+1}(\mathbb{C})$ $(l \geq 1)$, or $\mathrm{Sp}_{2l}(\mathbb{C})$ $(l \geq 1)$. Then $H_i - H_{i+1}$ $(i = 1, \dots, l)$ constitute the basis of the space $\mathfrak{h}(\mathbb{R})$. For $G = \mathrm{SO}_{2l+1}(\mathbb{C})$ and $\mathrm{Sp}_{2l}(\mathbb{C})$ the Weyl group W is generated by permutations and sign reversals of the vectors of the basis. In the case $G = \mathrm{SO}_{2l}(\mathbb{C})$ the Weyl group W is generated by permutations of the vectors of the basis and simultaneous sign reversals in pairs of these vectors.

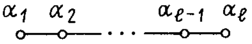
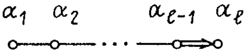
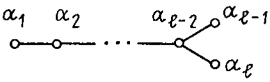
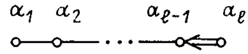
1.7. The Dynkin Diagram and the Cartan Matrix. Let $\Gamma = \{\gamma_1, \dots, \gamma_s\}$ be a system of nonzero vectors in a Euclidean space E. A graph may be assigned to Γ which clarifies how this system decomposes into indecomposable components. Namely, to each vector γ_i assign a vertex of the graph and join the vertices corresponding to the vectors γ_i and γ_j $(i \neq j)$ if and only if $(\gamma_i, \gamma_j) \neq 0$. The indecomposable components of the system Γ are evidently in one-to-one correspondence with the connected components of the resulting graph. The edges of the graph may be equipped with additional labels providing information about the angles between the vectors γ_i and the ratios of their lengths. We will carry out this construction for one special class of vector systems.

A system $\Gamma = \{\gamma_1, \dots, \gamma_s\}$ is said to be *admissible* if $a_{ij} = \langle \gamma_i | \gamma_j \rangle$ is a nonpositive integer for all $i \neq j$. The integer square matrix $A(\Gamma) = (a_{ij})$ is called the *matrix of the system* Γ. Let $m_{ij} = a_{ij}a_{ji}$ and let θ_{ij} be the angle between the vectors γ_i and γ_j $(i \neq j)$. Proposition 1.1 implies that for an admissible system Γ the numbers m_{ij} and the angles θ_{ij} can assume only the following values: $m_{ij} = 0, 1, 2, 3, 4$; $\theta_{ij} = \pi\left(1 - \frac{1}{n_{ij}}\right)$, where $n_{ij} = 2, 3, 4, 6, \infty$, respectively.

The *Dynkin diagram* of an admissible system of vectors is the graph described above in which the edge joining the vertices numbered by i and j $(i \neq j, m_{ij} > 0)$ is of multiplicity m_{ij}. If $|a_{ij}| < |a_{ji}|$, then the corresponding edge is oriented by an arrow pointing from the j-th vertex towards the i-th one.

By Proposition 1.7 the system of simple roots Π of any root system Δ is admissible and therefore has a Dynkin diagram. By Theorem 1.9 this diagram does not depend on the choice of the system Π. If $\Delta = \Delta_G$, where G is a reductive complex algebraic group, then the Dynkin diagram of the system Π_G is also called the *Dynkin diagram of* G or its tangent algebra $\mathfrak{g}$. Proposition 1.2 shows that the Dynkin diagram does not depend on the choice of an invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. By virtue of Theorem 1.5 a semisimple complex Lie algebra $\mathfrak{g}$ is simple if and only if its Dynkin diagram is connected, while in the general case connected components of the Dynkin diagram are Dynkin diagrams of simple ideals of the algebra $\mathfrak{g}$.

Example 1. The Dynkin diagrams of the classical simple Lie algebras $\mathfrak{g}$ are of the following form (the right column contains the standard notation for the Dynkin diagram, which is the same as that used for the corresponding root system, the number of vertices in each diagram being equal to $l = \operatorname{rk}\mathfrak{g}$):

$\mathfrak{sl}_{\ell+1}(\mathbb{C}),\ \ell \geq 1$		A_ℓ
$\mathfrak{so}_{2\ell+1}(\mathbb{C}),\ \ell \geq 2$		B_ℓ
$\mathfrak{so}_3(\mathbb{C})$		$B_1 = (= A_1 = C_1)$
$\mathfrak{so}_{2\ell}(\mathbb{C}),\ \ell \geq 3$		D_ℓ
$\mathfrak{sp}_{2\ell}(\mathbb{C}),\ \ell \geq 2$		C_ℓ

Example 2. If $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ is an admissible system of vectors in a Euclidean space E, then so is the system $\Gamma^\vee = \{\gamma_1^\vee, \ldots, \gamma_s^\vee\}$, where $\gamma_i^\vee = \frac{2}{(\gamma_i, \gamma_i)} u_{\gamma_i}$ $(i = 1, \ldots, s)$. Its Dynkin diagram is obtained from the Dynkin diagram of the system Γ by reversing all the arrows.

The geometric meaning of the Dynkin diagram can be clarified with the help of the following notion. Let Ω and Ω' be systems of vectors in Euclidean spaces E and E', respectively. An *isomorphism* of Ω onto Ω' is any isomorphism $\varphi\colon \langle\Omega\rangle \to \langle\Omega'\rangle$ of their linear spans (considered as vector spaces over $\mathbb{R}$) such that $\varphi(\Omega) = \Omega'$ and $\langle\alpha|\beta\rangle = \langle\varphi(\alpha)|\varphi(\beta)\rangle$ $(\alpha, \beta \in \Omega)$. In particular, this defines the isomorphism of root systems, and isomorphic root systems. Denote by $\operatorname{Aut}\Omega$ the group of automorphisms of a vector system Ω (i.e. of isomorphisms of the system onto itself).

Proposition 1.10. *Let $\Omega \subset E$ and $\Omega' \subset E'$ be two admissible systems of vectors. Then the following conditions are equivalent:*

(1) *the systems Ω and Ω' are isomorphic;*
(2) *the matrices of Ω and Ω' are obtained from each other by the same permutation of rows and columns;*
(3) *the Dynkin diagrams of the systems Ω and Ω' are isomorphic (in the natural sense).*

Systems of simple roots are linearly independent admissible systems. Now we will construct examples of linearly dependent admissible systems.

Let Π be a system of simple roots of a root system Δ. Consider the partial order in Δ defined by the system Π (see Sect. 1.4). Evidently, there are elements in Δ maximal with respect to this order.

Theorem 1.12. *Any maximal root $\delta \in \Delta$ satisfies the conditions $(\delta, \alpha) \geq 0$ for all $\alpha \in \Pi$, and $(\delta, \beta) > 0$ for some $\beta \in \Pi$. If Δ is indecomposable, then the maximal root is unique and $\delta = \sum_{\alpha \in \Pi} n_\alpha \alpha$, where n_α are positive integers.*

Corollary. *If Δ is an indecomposable root system, $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ its system of simple roots, δ the maximal root, and $\alpha_0 = -\delta$, then $\widetilde{\Pi} = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ is an indecomposable linearly dependent admissible system of vectors.*

The maximal root δ of an indecomposable root system is also called its *highest root*, and $-\delta$ its *lowest root*. The system $\widetilde{\Pi}$ is said to be the *extended system of simple roots* of the system Δ.

In the case when $\Delta = \Delta_G$ is the root system of a simple noncommutative complex Lie group G one speaks about the highest (lowest) root and extended system of simple roots of the group G (or its tangent algebra $\mathfrak{g}$). The Dynkin diagram of an extended system of simple roots is called the *extended Dynkin diagram* of G (or the algebra $\mathfrak{g}$).

Example 1. The extended Dynkin diagrams of simple classical Lie algebras are of the following form (each diagram contains $l + 1$ vertices, the right column lists the standard notation for each of the diagrams):

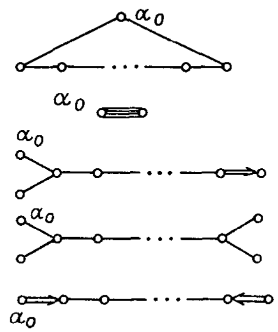
$\mathfrak{sl}_{l+1}(\mathbf{C})$, $l \geq 2$: $A_l^{(1)}$

$\mathfrak{sl}_2(\mathbf{C})$: $A_1^{(1)}$

$\mathfrak{so}_{2l+1}(\mathbf{C})$, $l \geq 3$: $B_l^{(1)}$

$\mathfrak{so}_{2l}(\mathbf{C})$, $l \geq 4$: $D_l^{(1)}$

$\mathfrak{sp}_{2l}(\mathbf{C})$, $l \geq 2$: $C_l^{(1)}$



Let us now find out which matrices may serve as matrices of admissible systems of vectors. The matrix $A(\Gamma) = (a_{ij})$ of an admissible system of vectors $\Gamma = \{\gamma_1, \ldots, \gamma_s\}$ has the following properties:

(1) $a_{ii} = 2\,(i = 1, \ldots, s)$;
(2) if $i \neq j$, then $a_{ij} \leq 0$, and if $a_{ij} = 0$, then $a_{ji} = 0$;
(3) $a_{ij} \in \mathbb{Z}$, and $m_{ij} = a_{ij}a_{ji} = 0, 1, 2, 3, \text{or } 4$;
(4) the matrix $G(A) = (g_{ij})$, where $g_{ii} = 1$ $(i, j = 1, \ldots, s)$, $g_{ij} = -\frac{1}{2}\sqrt{m_{ij}}$ $(i \neq j)$, is nonnegative definite.

Indeed, property (4) follows from the fact that $G(\Gamma)$ is the Gram matrix of the system of vectors $\left\{\frac{1}{|\gamma_1|}\gamma_1, \ldots, \frac{1}{|\gamma_s|}\gamma_s\right\}$.

A square matrix $A = (a_{ij})$ is said to be *admissible* if it satisfies properties (1)–(4). An admissible matrix A is called a *Cartan matrix* if $G(A)$ is positive definite (equivalent conditions: $\det G(A) > 0$, $\det A > 0$).

Theorem 1.13. *A matrix A is admissible if and only if $A = A(\Gamma)$, where Γ is an admissible system of vectors of a Euclidean vector space. Here Cartan matrices correspond to linearly independent admissible systems and vice versa.*

1.8. Classification of Admissible Systems of Vectors and Root Systems. We now present the classification results (up to an isomorphism) for indecomposable admissible systems of vectors. These results are given in the language of Dynkin diagrams.

Theorem 1.14. *Indecomposable admissible linearly independent systems of vectors are, up to an isomorphism, those corresponding to the Dynkin diagrams A_l $(l \geq 1)$, B_l $(l \geq 2)$, C_l $(l \geq 3)$, D_l $(l \geq 4)$, E_l $(l = 6,7,8)$, F_4, G_2* (see Table at the end of the book), *where different Dynkin diagrams correspond to nonisomorphic systems of vectors.*

Theorem 1.15. *Indecomposable admissible linearly dependent systems of vectors are, up to an isomorphism, those corresponding to the Dynkin diagrams $A_l^{(1)}$ $(l \geq 1)$, $B_l^{(1)}$ $(l \geq 2)$, $C_l^{(1)}$ $(l \geq 3)$, $D_l^{(1)}$ $(l \geq 4)$, $E_l^{(1)}$, $(l = 6,7,8)$, $F_4^{(1)}$, $G_2^{(1)}$, $A_{2l-1}^{(2)}$ $(l \geq 3)$, $A_{2l}^{(2)}$ $(l \geq 1)$, $D_{l+1}^{(2)}$ $(l \geq 2)$, $E_6^{(2)}$, $D_4^{(3)}$* (see Table 3). *Here different Dynkin diagrams correspond to nonisomorphic systems of vectors.*

Note that any indecomposable admissible linearly dependent system of s vectors is of rank $s-1$, and each subsystem of such a system is linearly independent.

We now proceed with the classification of root systems. It is based on the following assertion derived from Theorem 1.9.

Proposition 1.11. *Let $\Delta \subset E$, $\Delta' \subset E'$ be root systems, $\Pi \subset \Delta$ a system of simple roots, and $\varphi\colon \langle\Delta\rangle \to \langle\Delta'\rangle$ an isomorphism of the system Π onto the subsystem $\Pi' = \varphi(\Pi) \subset \Delta'$. If Δ is reduced, then φ is an isomorphism of the system Δ onto some root system $\varphi(\Delta) \subset \Delta'$. If Δ' is also reduced and Π' is a system of simple roots in Δ', then $\varphi(\Delta) = \Delta'$.*

Thus, the system of simple roots of a reduced root system defines the latter uniquely up to an isomorphism.

In Sect. 1.7, Example 1 we have constructed indecomposable reduced root systems of the types A_l $(l \geq 1)$, B_l $(l \geq 1)$, C_l $(l \geq 1)$, D_l $(l \geq 3)$. It turns out that there also exist reduced root systems whose systems of simple roots have the Dynkin diagrams E_l $(l = 6,7,8)$, F_4, and G_2. Such root systems can be constructed, for example, as root systems of exceptional simple complex Lie algebras (see Chap. 5). These root systems will be called *systems of types E_l,*

F_4, G_2. Their explicit form is given in Table 1. Theorem 1.14 and Proposition 1.11 imply the following statement.

Theorem 1.16. *Indecomposable reduced root systems are exhausted, up to an isomorphism, by the systems of types* A_l $(l \geq 1)$, B_l $(l \geq 2)$, C_l $(l \geq 3)$, D_l $(l \geq 4)$, E_l $(l = 6, 7, 8)$, F_4, G_2 *in Table* 1.

We conclude the classification by enumerating nonreduced indecomposable root systems.

Theorem 1.17. *Any nonreduced indecomposable system of roots of rank* $l \geq 1$ *is isomorphic to a root system of type* BC_l, *the union of the root systems* B_l *and* C_l, *i.e. the system* $\{\pm\varepsilon_i \pm \varepsilon_j \ (i \neq j), \pm\varepsilon_i, \pm 2\varepsilon_i\}$ (see Sect. 1.3, Examples 4 and 5). *The Weyl chambers, the system of simple roots, and the Weyl group of the system of type* BC_l *are the same as those of the system of type* B_l.

Theorems 1.16 and 1.9 imply the following proposition.

Proposition 1.12. *Any indecomposable reduced root system contains elements of at most two different lengths. The Weyl group acts transitively on the set of all roots of a given length.*

Note that the highest root always has the maximal length possible. Roots of different lengths exist in the root systems of types B_l $(l \geq 2)$, C_l $(l \geq 2)$, F_4, G_2. In each of them there is a largest (with respect to the partial order corresponding to the given system of simple roots) root ζ of minimal length. It is of the following form (in the notation of Table 1):

Type of Δ	ζ
B_l, $l \geq 2$	$\varepsilon_1 = \alpha_1 + \alpha_2 + \ldots + \alpha_l$
C_l, $l \geq 2$	$\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{l-1} + \alpha_l$
F_4	$\varepsilon_1 = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$
G_2	$\varepsilon_1 = 2\alpha_1 + \alpha_2$

Proposition 1.11 can be used for computing the group $\operatorname{Aut}\Delta$ of all automorphisms of a root system Δ. One can easily see that the Weyl group $W^\vee$ is a normal subgroup of $\operatorname{Aut}\Delta$. On the other hand, Proposition 1.11 implies that each automorphism of a system of simple roots $\Pi \subset \Delta$ is an automorphism of Δ. Thus, $\operatorname{Aut}\Pi$ is a subgroup of $\operatorname{Aut}\Delta$. Making use of Theorem 1.9, we obtain the following statement.

Theorem 1.18. $\operatorname{Aut}\Delta = W^\vee \rtimes \operatorname{Aut}\Pi$.

Note that the group $\operatorname{Aut}\Pi$ is naturally isomorphic to the group of automorphisms of the corresponding Dynkin diagram. For indecomposable root systems the groups $\operatorname{Aut}\Pi$ are given in Table 2.

1.9. Root and Weight Lattices. A *lattice* in a real vector space V is a free abelian subgroup of V whose basis is a linearly independent (over $\mathbb{R}$) system of vectors. As is known, any discrete subgroup of a vector group V is a lattice (see Bourbaki [1947]). Let Γ be a lattice in V. Then

$$\Gamma^* = \{\lambda \in V^* | \lambda(x) \in \mathbb{Z} \quad \text{for all } x \in \Gamma\}$$

is a subgroup of V^*. If $\langle \Gamma \rangle = V$, then Γ^* is also a lattice and $\langle \Gamma^* \rangle = V^*$. If $(e_1, \dots, e_n)$ is a basis of Γ, then the basis of Γ^* is the basis of the space V^* dual to $e_1, \dots, e_n$. The lattice Γ^* is naturally identified with the group $\operatorname{Hom}(\Gamma, \mathbb{Z})$; it is called the lattice *dual* to Γ. The lattice $(\Gamma^*)^*$ is identified with Γ.

Let Δ be a root system in a Euclidean space E, and Π its system of simple roots. Then the additive subgroup Q of the space E generated by Δ is a lattice with the basis Π. It is called the *root lattice*.

Suppose that $\langle \Delta \rangle = E$, and let

$$P = \{\gamma \in E | \langle \gamma | \alpha \rangle \in \mathbb{Z} \quad \text{for all } \alpha \in \Delta\}.$$

Given $\Pi = \{\alpha_1, \dots, \alpha_l\}$, define the elements $\pi_i \in P$ $(i = 1, \dots, l)$ by the formulae $\langle \pi_i | \alpha_j \rangle = \delta_{ij}$. Then P is a lattice with the basis $\{\pi_1, \dots, \pi_l\}$. This lattice is called the *weight lattice*, its elements are called *weights*, and the weights π_i are called *fundamental weights*. Simple roots are expressed in terms of π_i by the formulae

$$\alpha_i = \sum_{j=1}^{l} a_{ij} \pi_j, \tag{2}$$

where $A = (a_{ij})$ is the matrix of the system Π. The lattices Q and P are invariant under the Weyl group $W^\vee$.

The definition of a root system implies that $Q \subset P$. The group

$$\pi(\Delta) = P/Q$$

is called the *fundamental group* of the root system Δ. Formula (2) yields the following proposition.

Proposition 1.13. *The fundamental group $\pi(\Delta)$ is isomorphic to $\bigoplus_{i=1}^{s} \mathbb{Z}_{m_i}$, where $m_1, \dots, m_s$ are invariant factors of the matrix A of the system Π different from* 1. *In particular, $|\pi(\Delta)| = \det A$.*

The root system $\Delta^\vee \subset F = E^*$ dual to Δ also corresponds to the root and weight lattices $Q^\vee \subset P^\vee$ in the space F that are invariant under the Weyl group W.

Proposition 1.14. $Q^\vee = P^*$, $P^\vee = Q^*$, $\pi(\Delta^\vee) \simeq \pi(\Delta)$.

For the case when Δ is an indecomposable reduced root system the groups $\pi(\Delta^\vee)$ and their generators are given in Table 2. Note that the group $\pi(\Delta) \simeq \pi(\Delta^\vee)$ is trivial if Δ is a root system of the types E_8, F_4, G_2; it is a finite cyclic group if Δ is of type A_l, B_l, C_l, or D_l, l odd; it is isomorphic to

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if Δ is of type D_l, l even. The general case is easily reduced to the indecomposable one.

Now consider the case when $\Delta = \Delta_G$ is the root system of a semisimple complex algebraic group G with respect to a maximal torus H. As we have seen in Chap. 1, Sect. 9.5, the group $\mathfrak{X}(H)$ is identified with a lattice in the space $E = \mathfrak{h}(\mathbb{R})^*$. The lattice $\mathfrak{X}(H)^* \subset F = \mathfrak{h}(\mathbb{R})$ dual to it coincides with $\mathfrak{h}(\mathbb{Z})$.

Proposition 1.15. *We have: the following dual triples of lattices*

$$Q \subset \mathfrak{X}(H) \subset P,$$
$$Q^\vee \subset \mathfrak{h}(\mathbb{Z}) \subset P^\vee.$$

Note that the lattices Q, P, $Q^\vee$, $P^\vee$ are completely defined by the tangent algebra $\mathfrak{g}$ of the group G, while the lattices $\mathfrak{X}(H)$ and $\mathfrak{h}(\mathbb{Z})$ depend, in general, on the global structure of G. As we shall see in Sect. 2, a semisimple Lie algebra $\mathfrak{g}$ is defined (up to an isomorphism) by the root system Δ, and a connected semisimple algebraic group G by the root system $\Delta_G = \Delta$ and either of the lattices $\mathfrak{X}(H)$, $\mathfrak{h}(\mathbb{Z})$.

Note also that the system of weights Φ_ρ of any linear representation ρ of $\mathfrak{g}$ is contained in the weight lattice P.

Consider the reversal of formulae (2):

$$\pi_i = \sum_{j=1}^{l} b_{ij}\alpha_j,$$

where $B = (b_{ij}) = A^{-1}$ is a rational matrix.

Proposition 1.16 (see Humphreys [1972]). *We have $b_{ij} \geq 0$. If the root system Δ is indecomposable, then $b_{ij} > 0$ for all i, j.*

Let Δ be a reduced root system. In some questions of representation theory the element

$$\gamma = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$$

plays an important role.

Proposition 1.17. $\gamma = \pi_1 + \ldots + \pi_l$.

Proof. Since $(\alpha_i, \alpha_j) \leq 0$ $(i \neq j)$, any reflection $r_{\alpha_i}^\vee$ takes the system $\Delta^+ \setminus \{\alpha_i\}$ onto itself. Therefore $r_{\alpha_i}^\vee(\gamma) = \gamma - \alpha_i$, whence $\langle\gamma|\alpha_i\rangle = 1$ for all i. □

1.10. Chevalley Basis. Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra of it. Choose a nonzero vector e_α in each subspace $\mathfrak{g}_\alpha$ $(\alpha \in \Delta_\mathfrak{g})$. By Theorem 1.2,

$$[e_\alpha, e_\beta] = c_{\alpha\beta}e_{\alpha+\beta} \quad \text{if } \alpha + \beta \in \Delta_\mathfrak{g} \quad \text{(otherwise 0)},$$

where $c_{\alpha\beta} \in \mathbb{C}$. As we have seen in Sect. 1.2, e_α can be chosen in such a way that $[e_\alpha, e_{-\alpha}] = h_\alpha$ for $\alpha \in \Delta_{\mathfrak{g}}$. It turns out that one can also ensure that the relations $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$ hold. The union of the set of vectors e_α with an appropriate choice of basis in the space $\mathfrak{h}$ forms a basis in $\mathfrak{g}$ with respect to which the structure constants of $\mathfrak{g}$ are integers. More precisely, the following theorem holds.

Theorem 1.19 (see Humphreys [1972], Steinberg [1968]). *Suppose that $\Pi = \{\alpha_1, \dots, \alpha_l\}$ is a system of simple roots in $\Delta_{\mathfrak{g}}$, and let $h_i = h_{\alpha_i}$ $(i = 1, \dots, l)$. If the root vectors $e_\alpha \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta_{\mathfrak{g}})$ are chosen as above, then the basis $\{e_\alpha(\alpha \in \Delta_{\mathfrak{g}}),\ h_i(i = 1, \dots, l)\}$ of the algebra $\mathfrak{g}$ has the following properties:*

$[h_i, h_j] = 0 \qquad (i, j = 1, \dots, l)$;

$[h_i, e_\alpha] = \langle \alpha | \alpha_i \rangle e_\alpha \qquad (i = 1, \dots, l,\ \alpha \in \Delta_{\mathfrak{g}})$;

$[e_\alpha, e_{-\alpha}] = h_\alpha$ *is an integral linear combination of the vectors* $h_1, \dots, h_l$;

if $\alpha, \beta \in \Delta_{\mathfrak{g}}$, $\alpha + \beta \neq 0$, and $\beta - p\alpha, \dots, \beta + q\alpha$ is the α-string of roots containing β, then

$$[e_\alpha, e_\beta] = \begin{cases} 0 & \text{if } q = 0 \\ \pm(p+1)e_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta_{\mathfrak{g}} \ . \end{cases} \tag{3}$$

The basis of the algebra $\mathfrak{g}$ described in Theorem 1.19 is called the *Chevalley basis*, and it was first constructed in Chevalley [1955b]. An algorithm for choosing the signs in formula (3) is given in Samelson [1969] and Tits [1966b]. With the help of this basis, Chevalley constructed, for any semisimple Lie algebra $\mathfrak{g}$, analogues of the adjoint group $\operatorname{Int}\mathfrak{g}$ associated with arbitrary fields (the so-called *Chevalley groups*).

§ 2. Classification of Complex Semisimple Lie Groups and Their Linear Representations

This section contains the main classification theorems for semisimple complex Lie algebras and connected semisimple complex Lie groups. In particular, we shall see that for each Cartan matrix there exists a unique (up to an isomorphism) semisimple complex Lie algebra for which this matrix is the matrix of the system of simple roots. Thus, there is a one-to-one correspondence between semisimple complex Lie algebras, reduced root systems, linearly independent admissible systems of vectors, their Dynkin diagrams, and their Cartan matrices. We shall also describe finite-dimensional linear representations of connected semisimple complex Lie groups and semisimple complex Lie algebras. All Lie groups and Lie algebras, as well as their representations, considered in this section are assumed to be complex.

2.1. Uniqueness Theorems for Lie Algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of it, $\Delta \subset E = \mathfrak{h}(\mathbb{R})^*$ the system of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ a system of simple roots in Δ, and $F = \mathfrak{h}(\mathbb{R})$. Using the notation of Sect. 1.2, we set

$$h_i = h_{\alpha_i}, \quad e_i = e_{\alpha_i}, \quad f_i = e_{-\alpha_i} \quad (i = 1, \ldots, l).$$

Let $A = (a_{ij})$ be the matrix of the system Π; we will call it the *Cartan matrix* of the Lie algebra $\mathfrak{g}$.

Formulae (1), Theorem 1.2, and Proposition 1.6(b) imply the following proposition.

Proposition 2.1. *The elements h_i, e_i, f_i $(i = 1, \ldots, l)$ constitute the system of generators of the Lie algebra $\mathfrak{g}$ and satisfy the following relations:*

$$\begin{gathered}[h_i, h_j] = 0, \\ [h_i, e_j] - a_{ji}e_j = 0, \quad [h_i, f_j] + a_{ji}f_j = 0, \\ [e_i, f_i] - h_i = 0, \quad [e_i, f_j] = 0 \quad \textit{for} \quad i \neq j.\end{gathered} \tag{4}$$

The system of elements $\{h_i, e_i, f_i | i = 1, \ldots, l\}$ is called the *canonical system of generators* of the algebra $\mathfrak{g}$ associated with $\mathfrak{h}$ and Π.

Denote by $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}(A)$ the Lie algebra with generators $\hat{h}_i$, $\hat{e}_i$, $\hat{f}_i$ $(i = 1, \ldots l)$ and defining relations obtained from (4) by replacing h_i, e_i, f_i by $\hat{h}_i$, $\hat{e}_i$, $\hat{f}_i$, respectively. There is evidently a homomorphism $\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$ such that

$$\pi(\hat{h}_i) = h_i, \quad \pi(\hat{e}_i) = e_i, \quad \pi(\hat{f}_i) = f_i.$$

The subspace $\hat{\mathfrak{h}} = \langle \hat{h}_1, \ldots, \hat{h}_l \rangle$ is a commutative subalgebra of $\hat{\mathfrak{g}}$. Denote $\hat{\mathfrak{n}}^+$ (respectively, $\hat{\mathfrak{n}}^-$) the subalgebra of $\hat{\mathfrak{g}}$ generated by the elements $\hat{e}_1, \ldots, \hat{e}_l$ (respectively $\hat{f}_1, \ldots, \hat{f}_l$)). Then $\hat{\mathfrak{g}} = \hat{\mathfrak{h}} + \hat{\mathfrak{n}}^+ + \hat{\mathfrak{n}}^-$.

Theorem 2.1. *A semisimple Lie algebra is defined by its Cartan matrix (or the Dynkin diagram) uniquely up to isomorphism. More precisely, if $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are semisimple Lie algebras with canonical systems of generators $\{h_i, e_i, f_i | i = 1, \ldots, l\}$ and $\{\tilde{h}_i, \tilde{e}_i, \tilde{f}_i | i = 1, \ldots, l\}$ respectively, then there exists a (unique) isomorphism $\varphi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that $\varphi(h_i) = \tilde{h}_i$, $\varphi(e_i) = \tilde{e}_i$, $\varphi(f_i) = \tilde{f}_i$ $(i = 1, \ldots, l)$.*

Proof. One can prove that the set of ideals of the Lie algebra $\hat{\mathfrak{g}}$ constructed above containing none of $\hat{h}_i$ has the largest element $\mathfrak{m}$. Here $\mathfrak{m} = \operatorname{Ker} \pi$, so $\mathfrak{g} \simeq \hat{\mathfrak{g}}/\mathfrak{m}$. The statement of the theorem follows from the fact that $\hat{\mathfrak{g}}$ and $\mathfrak{m}$ are uniquely defined by the Cartan matrix. □

Example 1. Among the Dynkin diagrams of classical simple Lie algebras given in Sect. 1.7, Example 1 there are the following isomorphisms: $A_1 \simeq B_1 \simeq C_1$, $B_2 \simeq C_2$, $A_3 \simeq D_3$. Theorem 2.1 implies that $\mathfrak{sl}_2(\mathbb{C}) \simeq \mathfrak{so}_3(\mathbb{C})$, $\mathfrak{sp}_4(\mathbb{C}) \simeq \mathfrak{so}_5(\mathbb{C})$, $\mathfrak{sl}_4(\mathbb{C}) \simeq \mathfrak{so}_6(\mathbb{C})$. We now give an explicit form of the isomorphisms of these Lie algebras.

(a) Consider the adjoint representation ad of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Since the Killing form in $\mathfrak{sl}_2(\mathbb{C})$ is nondegenerate, one can assume that ad is an injective homomorphism $\mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{so}_3(\mathbb{C})$. Comparison of dimensions shows that it is an isomorphism. The adjoint representation Ad of the group $SL_2(\mathbb{C})$ is a surjective homomorphism $SL_2(\mathbb{C}) \to SO_3(\mathbb{C})$ but has a nontrivial kernel $Z(SL_2(\mathbb{C})) = \{E, -E\}$.

(b) Consider the exterior square $\Lambda^2\mathrm{Id}$ of the identity representation of the group $SL_4(\mathbb{C})$ acting in the space of bivectors $\Lambda^2\mathbb{C}^4$ of the space $\mathbb{C}_4$. The formula $u \wedge v = b(u, v)e_1 \wedge e_2 \wedge e_3 \wedge e_4$ $(u, v \in \Lambda^2\mathbb{C}^4)$ defines a nondegenerate symmetric bilinear form b in the space of bivectors invariant under $SL_4(\mathbb{C})$. Thus $\Lambda^2\mathrm{Id}$ is a homomorphism of the group $SL_4(\mathbb{C})$ into $SO_6(\mathbb{C})$; its kernel is $\{E, -E\}$. Comparison of dimensions shows that this homomorphism is surjective. The corresponding homomorphism of Lie algebras $\mathfrak{sl}_4(\mathbb{C}) \to \mathfrak{so}_6(\mathbb{C})$ is an isomorphism.

(c) The subgroup $Sp_4(\mathbb{C}) \subset SL_4(\mathbb{C})$ preserves a nondegenerate skew-symmetric bilinear form in $\mathbb{C}^4$. One can easily see that this form, considered as an element of the space $(\Lambda^2\mathbb{C}^4)^*$, defines a nonisotropic (with respect to b) vector in $\Lambda^2\mathbb{C}^4$ invariant under $Sp_4(\mathbb{C})$. Thus $Sp_4(\mathbb{C})$ is mapped onto $SO_5(\mathbb{C})$ with the kernel $\{E, -E\}$. The corresponding homomorphism $\mathfrak{sp}_4(\mathbb{C}) \to \mathfrak{so}_5(\mathbb{C})$ is an isomorphism.

Example 2. The example of Sect. 1.4 shows that the Dynkin diagram of the Lie algebra $\mathfrak{so}_4(\mathbb{C})$ is isomorphic to the Dynkin diagram of the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. An explicit isomorphism between these two Lie algebras can be constructed as follows. The group $SL_2(\mathbb{C})$ coincides with $Sp_2(\mathbb{C})$ because it preserves the nondegenerate skew-symmetric bilinear form $d(u, v) = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = u_1v_2 - u_2v_1$. Consider the tensor product $R = \mathrm{Id} \otimes \mathrm{Id}$ of the identity representations of two groups $SL_2(\mathbb{C})$. It is a representation of the group $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ acting in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \simeq \mathbb{C}^4$ by the formula

$$R(g, h)(u \otimes v) = (gu) \otimes (hv) \quad (g, h \in SL_2(\mathbb{C}),\ u, v \in \mathbb{C}^2).$$

The form $d \otimes d$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$ defined by the formula

$$(d \otimes d)(u_1 \otimes v_1, u_2 \otimes v_2) = d(u_1, u_2)d(v_1, v_2)$$

is symmetric, nondegenerate, and invariant under R. Hence it follows that R defines a surjective homomorphism $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to SO_4(\mathbb{C})$ with the kernel $\{(E, E), (-E, -E)\}$. This homomorphism corresponds to the isomorphism $dR\colon \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{so}_4(\mathbb{C})$.

2.2. Uniqueness Theorem for Linear Representations. We shall use the notation of Sect. 2.1. Let $P \subset E$ be a weight lattice of the root system Δ. Each weight λ is completely determined by the integers $\lambda(h_i) = \langle\lambda|\alpha_i\rangle = \Lambda_i$ $(i = 1, \dots, l)$, which are its coordinates in the basis of fundamental weights $\pi_1, \dots, \pi_l$ (see Sect. 1.9). The numbers Λ_i are called the *numerical labels* (or just the *labels*) of the weight λ. A weight $\lambda \in P$ is said to be *dominant* if its numerical labels are nonnegative. This means that $\lambda \in \overline{C}^\vee$, where $C^\vee \subset E$ is the Weyl chamber corresponding to the system of simple roots $\Pi^\vee$.

Now let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional linear representation, and $V = \bigoplus_{\lambda \in \Phi_\rho} V_\lambda$ the corresponding weight decomposition. A weight vector $v \in V_\lambda$ is called a *highest vector* if

$$\rho(e_i)v = 0 \quad (i = 1, \dots, l).$$

The corresponding weight $\Lambda \in \Phi_\rho$ is called a *highest weight* of the representation ρ.

Example 1. Let $\mathfrak{g}$ be a classical semisimple Lie algebra, and id its identity representation. A highest weight of the representation id is ε_1, and the corresponding highest vector is the first vector of the basis e_1 (see Sect. 1.3).

If $\rho = \mathrm{ad}$ is the adjoint representation of a simple Lie algebra $\mathfrak{g}$, then the root vector e_δ corresponding to the highest root δ (see Sect. 1.7) is a highest vector, and the highest root δ is a highest weight of the representation.

Theorem 2.2. *A highest weight of any linear representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is dominant. If ρ is irreducible, it has a unique highest weight Λ, where* $\dim V_\Lambda = 1$ *and any other weight $\lambda \in \Phi_\rho$ is of the form $\lambda = \Lambda - \alpha_{i_1} - \dots - \alpha_{i_k}$, where $\alpha_{i_j} \in \Pi$.*

Proof. Consider the $\mathfrak{g}$-invariant subspace $W \subset V$ generated by a highest vector v_Λ. By Proposition 9.1 of Chap. 1, any weight of the representation ρ in W different from Λ is of the form $\Lambda - \alpha_{i_1} - \dots - \alpha_{i_k}$, where $\alpha_{i_j} \in \Pi$. On the other hand, $r_{\alpha_i}(\Lambda) = \Lambda - \Lambda_i \alpha_i$ is a weight of the representation ρ in W (see Proposition 1.4). Hence $\Lambda_i \geq 0$. If ρ is irreducible, then $W = V$, which implies the remaining assertions. □

We will now prove that an irreducible representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is uniquely defined by its highest weight. For this we construct a linear representation of the Lie algebra $\hat{\mathfrak{g}}$ in the complex space $\hat{V}$ with the basis $\{\hat{v}_\emptyset, \hat{v}_{i_1 \dots i_k} | 1 \leq i_1, \dots, i_k \leq l,\ k \geq 1\}$, defining it on generators by the formulae

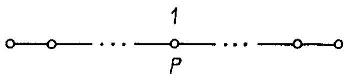
$$\begin{gathered}
\hat{\rho}(\hat{h}_i)\hat{v}_\emptyset = \Lambda_i \hat{v}_\emptyset, \\
\hat{\rho}(\hat{h}_i)\hat{v}_{i_1 \dots i_k} = (\Lambda_i - a_{i_1 i} - \dots - a_{i_k i})\hat{v}_{i_1 \dots i_k}, \\
\hat{\rho}(\hat{f}_i)\hat{v}_{i_1 \dots i_k} = \hat{v}_{i i_1 \dots i_k}, \\
\hat{\rho}(\hat{e}_i)\hat{v}_\emptyset = 0, \\
\hat{\rho}(\hat{e}_i)\hat{v}_{i_1 \dots i_k} = (\delta_{i i_1}\hat{\rho}(\hat{h}_i) + \hat{\rho}(\hat{f}_{i_1})\hat{\rho}(e_i))\hat{v}_{i_2 \dots i_k}
\end{gathered} \tag{5}$$

(the last formula should be considered as a recurrent definition). Among the invariant subspaces of $\hat{V}$ not coinciding with $\hat{V}$ there exists the largest subspace $M^{(\Lambda)}$. Now there exists a unique linear mapping $p\colon \hat{V} \to V$ such that $p(\hat{v}_\emptyset) = v_\Lambda$ and $p \circ \hat{\rho}(x) = \rho(\pi(x)) \circ p$ $(x \in \mathfrak{g})$. Here p is surjective and $\operatorname{Ker} p = M^{(\Lambda)}$, so $V \simeq \hat{V}/M^{(\Lambda)}$. It turns out that $\hat{\rho}$ induces an irreducible linear representation in $\hat{V}/M^{(\Lambda)}$ isomorphic to ρ. Since this induced representation is completely defined by the weight Λ, this implies the following theorem.

Theorem 2.3. *An irreducible finite-dimensional linear representation of a semisimple Lie algebra is defined by its highest weight uniquely up to an isomorphism.*

The irreducible representation with the highest weight Λ will be denoted by $\rho(\Lambda)$. The following method of describing the representation $\rho(\Lambda)$ is often used: the vertices of the Dynkin diagram of the algebra $\mathfrak{g}$ are marked by the corresponding numerical labels of the weight Λ (zero labels are omitted).

Example 2. Let $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$. Consider the representation $\rho(\pi_p)$ $(p = 1, \ldots, l)$ whose highest weight is the p-th fundamental weight. The corresponding numerical labels are of the form



For $\rho(\pi_p)$ one can take the representation $\Lambda^p\mathrm{id}$ induced by the identity representation id in the space of p-vectors $\Lambda^p\mathbb{C}^{l+1}$. Indeed, the weights of the representation $\Lambda^p\mathrm{id}$ are of the form $\varepsilon_{i_1} + \ldots + \varepsilon_{i_p}$, where $i_1 < \ldots < i_p$. The only dominant weight is the highest weight $\Lambda = \varepsilon_1 + \ldots + \varepsilon_p = \pi_p$. Therefore $\Lambda^p\mathrm{id}$ is irreducible and equivalent to $\rho(\pi_p)$.

It follows from Theorem 2.3 that the system of weights Φ_ρ of an irreducible representation ρ of a semisimple Lie algebra $\mathfrak{g}$ is completely determined by the highest weight $\Lambda \in \Phi_\rho$. We now present a method of explicitly determining Φ_ρ from the highest weight. Since the Weyl group takes Φ_ρ into itself, by virtue of Theorem 1.9 all weights of the representation ρ can be obtained from dominant weights of this representation with the help of the Weyl group. The dominant weights of the representation ρ can be found with the use of the following proposition derived from Proposition 1.2.

Proposition 2.2. *Suppose that the weight $\lambda = \Lambda - \alpha_{i_1} - \ldots - \alpha_{i_k}$, where Λ is a highest weight of the irreducible representation ρ and α_{i_j} are simple roots, is dominant. Then $\lambda \in \Phi_\rho$.*

2.3. Existence Theorems. Let $A = (a_{ij})$ be an arbitrary complex square matrix of order l. As in Sect. 2.1 we can construct the Lie algebra $\hat{\mathfrak{g}}(A)$ with generators $\hat{h}_i$, $\hat{e}_i$, $\hat{f}_i$ $(i = 1, \ldots, l)$ and defining relations obtained from (4) by writing ^ over the letters h, e, f. If the matrix A is nondegenerate, then, as in Sect. 2.1, there is a largest ideal $\mathfrak{m}$ in $\hat{\mathfrak{g}}(A)$ containing none of the elements $\hat{h}_i$. Let $\mathfrak{g}(A) = \hat{\mathfrak{g}}(A)/\mathfrak{m}$ and let $\pi\colon \hat{\mathfrak{g}}(A) \to \mathfrak{g}(A)$ be a natural homomorphism, $h_i = \pi(\hat{h}_i)$, $e_i = \pi(\hat{e}_i)$, $f_i = \pi(\hat{f}_i)$.

Theorem 2.4 (see Onishchik and Vinberg [1990], Serre [1987]). *If A is a Cartan matrix, then $\mathfrak{g}(A)$ is a finite-dimensional semisimple Lie algebra and $\{h_i, e_i, f_i | 1, \ldots, l\}$ is its canonical system of generators, and the corresponding Cartan matrix coincides with A. The algebra $\mathfrak{g}(A)$ can be described as the Lie algebra the generators h_i, e_i, f_i $(i = 1, \ldots, l)$ and defining relations consisting* *of those given by formulae* (4) *and the following relations:*

$$(\operatorname{ad} f_j)^{-a_{ij}+1} f_i = (\operatorname{ad} e_j)^{-a_{ij}+1} e_i = 0 \quad (i \neq j).$$

For any set of complex numbers $(\Lambda_1, \ldots, \Lambda_l)$ one can consider the linear representation $\hat{\rho}$ of the Lie algebra $\hat{\mathfrak{g}}(A)$ in the space $\hat{V}$ of Sect. 2.1 defined on the generators by formulae (5). In the space $\hat{V}$ there is the largest (other than $\hat{V}$) invariant subspace M. Let $V = \hat{V}/M$, and let $p: \hat{V} \to V$ be the natural mapping. Then $\hat{\rho}$ defines a linear representation ρ of the algebra $\mathfrak{g}(A)$ in the space V.

Theorem 2.5. *If A is a Cartan matrix and all the Λ_i are nonnegative integers, then ρ is an irreducible finite-dimensional linear representation whose highest weight has numerical labels $\Lambda_1, \ldots, \Lambda_l$ and the highest vector coincides with $p(\hat{v}_\emptyset)$. For any semisimple Lie algebra $\mathfrak{g}$ and any dominant weight $\Lambda \in P$ there exists an irreducible finite-dimensional linear representation of the algebra $\mathfrak{g}$ with the highest weight Λ.*

2.4. Global Structure of Connected Semisimple Lie Groups. Let G be a connected semisimple algebraic group, H its maximal torus, $Q \subset P$ the root and weight lattices of the root system Δ_G with respect to H.

Theorem 2.6. *The group G is simply-connected if and only if $\mathfrak{X}(H) = P$ or $\mathfrak{h}(\mathbb{Z}) = Q^\vee$.*

Proof. Let $\{\alpha_1, \ldots, \alpha_l\}$ be a system of simple roots in Δ_G, $G^{(k)} = G^{(\alpha_k)}$ the three-dimensional subgroup of G corresponding to the root α_k (see Sect. 1.2), and $H^{(k)}$ its maximal torus lying in H. If $\mathfrak{X}(H) = P$, then all the F_{α_k} are injective, so the groups $G^{(k)} \simeq \mathrm{SL}_2(\mathbb{C})$ are simply-connected, and $H = H^{(1)} \times \ldots \times H^{(l)}$. This implies that the homomorphism $i_*: \pi_1(H) \to \pi_1(G)$, where $i: H \to G$ is an embedding, sends everything to zero. Using Proposition 1.9 and Theorem 1.8, we deduce from the exact sequence of the fibration of G into cosets with respect to the Borel subgroup B^+ that $\pi_1(G) = \{e\}$. □

For the proof of the converse statement consider a linear representation ρ of the tangent algebra $\mathfrak{g}$ of the group G whose weights generate the lattice P (for example, the representation $\rho = \rho(\pi_1) + \ldots + \rho(\pi_l)$). If G is simply-connected, then there exists a representation R of the group G such that $dR = \rho$. Since $\Phi_R = \Phi_\rho$ generates P, we have $\mathfrak{X}(H) = P$. □

Theorem 2.7. *Any connected semisimple Lie group G admits a faithful linear representation and consequently the structure of a linear algebraic group.*

Proof. Let G be simply-connected. As the proof of the preceding theorem shows, there exists a linear representation R of the group G such that Φ_{dR} generates the weight lattice P of the tangent algebra $\mathfrak{g}$. Then the characters of the maximal torus of the algebraic group $R(G)$ generate P, and, by Theorem 2.6, $R(G)$ is simply-connected. Therefore $R: G \to R(G)$ is an isomorphism of complex Lie groups. For an arbitrary connected semisimple Lie group our statement follows from the fact that the quotient group of a linear algebraic group with respect to a normal algebraic subgroup is also algebraic (see Springer [1989], Chap. 1, Sect. 2.5). □

By Corollary 1 to Theorem 6.3 of Chap. 1, the algebraic structure on a semisimple complex Lie group is unique. In what follows we assume that a connected semisimple complex Lie group is equipped with this unique algebraic group structure.

We will now describe the centre $Z(G)$ and the fundamental group $\pi_1(G)$ of a connected semisimple Lie group G. Let H be a maximal torus in G. Then $Z(G) \subset H$, where $Z(G)$ is a zero-dimensional, i.e. finite, subgroup. The description of finite subgroups of H can be reduced to the description of certain lattices in the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For this consider the covering $\mathcal{E}: \mathfrak{h} \to H$ defined by the formula

$$\mathcal{E}(x) = \exp(2\pi i x). \tag{6}$$

One can easily see that $\operatorname{Ker} \mathcal{E} = \mathfrak{h}(\mathbb{Z})$. By associating with a finite subgroup $S \subset H$ the lattice $\mathcal{E}^{-1}(S) \subset \mathfrak{h}$ we obtain a one-to-one correspondence between finite subgroups of H and lattices in $\mathfrak{h}$ containing $\mathfrak{h}(\mathbb{Z})$, where $S \simeq \mathcal{E}^{-1}(S)/\mathfrak{h}(\mathbb{Z})$.

Consider, as in Sect. 1.9, the root and weight lattices $Q \subset P \subset \mathfrak{h}(\mathbb{R})^*$ and $Q^\vee \subset P^\vee \subset \mathfrak{h}(\mathbb{R})$.

Theorem 2.8. *Let G be a connected semisimple Lie group, and H a maximal torus of it. Then $\mathcal{E}^{-1}(Z(G)) = P^\vee$ and $Z(G) \simeq P^\vee/\mathfrak{h}(\mathbb{Z}) \simeq \mathfrak{X}(H)/Q$.*

The proof follows from the fact that $Z(G) = \{\exp x | x \in \mathfrak{h}, e^{\alpha(x)} = 1$ for all $\alpha \in \Delta_G\}$. □

Corollary 1. *If G is simply-connected, then $Z(G) \simeq \pi(\Delta_G)$.*

Corollary 2. *The group G is isomorphic to the adjoint linear group* $\operatorname{Ad} G = \operatorname{Int} \mathfrak{g}$ *if and only if $\mathfrak{X}(H) = Q$ or $\mathfrak{h}(\mathbb{Z}) = P^\vee$.*

Let $p: \widetilde{G} \to G$ be a simply-connected covering of a connected semisimple Lie group G. Then $\widetilde{H} = p^{-1}(H)$ is a maximal torus in $\widetilde{G}$. Denote by $\widetilde{\mathcal{E}}: \widetilde{\mathfrak{h}} \to \widetilde{H}$ the covering given by formula (5).

Theorem 2.9. *We have $\widetilde{\mathcal{E}}^{-1}(\operatorname{Ker} p) = \widetilde{\mathfrak{h}}(\mathbb{Z})$ and $\pi_1(G) \simeq \mathfrak{h}(\mathbb{Z})/Q^\vee \simeq P/\mathfrak{X}(H)$.*

The proof follows from the fact that $\pi_1(G) \simeq \operatorname{Ker} p \subset Z(\widetilde{G})$ and Theorem 2.6. □

2.5. Classification of Connected Semisimple Lie Groups. In this section we will show that any semisimple Lie group G is defined up to an isomorphism by its Dynkin diagram and the character lattice $\mathfrak{X}(H)$ of a maximal torus $H \subset G$, where $\mathfrak{X}(H)$ can be any lattice lying between the root and weight lattices.

Theorem 2.10. *Let G_1, G_2 be two connected semisimple Lie groups, $H_i \subset G_i$ their maximal tori, and $\Pi_i \subset \mathfrak{h}_i(\mathbb{R})^*$ systems of simple roots. For any isomorphism $\psi: \Pi_1 \to \Pi_2$ mapping $\mathfrak{X}(H_1)$ onto $\mathfrak{X}(H_2)$ there exists an isomorphism $\Phi: G_1 \to G_2$ taking H_1 into H_2 and inducing the isomorphism ψ.*

Proof. Theorem 2.1 implies the existence of an isomorphism $\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$ taking $\mathfrak{h}_1$ into $\mathfrak{h}_2$ and inducing the isomorphism ψ of the systems of simple roots. Let $\widetilde{G}_i$ be the simply-connected covering of the group G_i. Then there exists an isomorphism $\widetilde{\Phi}: \widetilde{G}_1 \to \widetilde{G}_2$ such that $d\widetilde{\Phi} = \varphi$. Using Theorem 2.9, we verify that $\widetilde{\Phi}$ induces the desired isomorphism $\Phi: G_1 \to G_2$. □

Theorem 2.11. *Let $\Delta \subset E$ be a reduced root system, and $Q \subset P$ its root and weight lattices. For any lattice $L \subset E$ satisfying the condition $Q \subset L \subset P$ there exist a connected semisimple Lie group G, a maximal torus $H \subset G$, and an isomorphism of the root systems $\Delta_G(H) \to \Delta$ taking $\mathfrak{X}(T)$ onto L.*

Proof. Using Theorem 2.4 and Proposition 1.11 one can assume that $\Delta = \Delta_\mathfrak{g}$ is the root system of a semisimple complex Lie algebra with respect to a Cartan subalgebra $\mathfrak{h}$. Consider the maximal torus $\widetilde{H}$ in the corresponding simply-connected Lie group $\widetilde{G}$, with the tangent algebra $\mathfrak{h}$. Let $\widetilde{\mathcal{E}}: \mathfrak{h} \to \widetilde{H}$ be the covering defined by formula (6). By setting $G = \widetilde{G}/N$, where $N = \widetilde{\mathcal{E}}(L^*)$, we obtain the desired Lie group. □

Corollary. *Let $\Delta \subset E$ be a reduced root system, $\Pi \subset \Delta$ the system of simple roots, and $Q \subset P$ the root and weight lattices. The classes of isomorphic connected semisimple Lie groups G for which $\Delta_G \simeq \Delta$ (or, equivalently, whose tangent algebras are isomorphic to a semisimple Lie algebra $\mathfrak{g}$ with the root system Δ) are in one-to-one correspondence with the lattices $L \subset E$ such that $Q \subset L \subset P$, considered up to a transformation from the group* Aut Π.

Note that a lattice L satisfying the condition $Q \subset L \subset P$ is uniquely defined by the subgroup L/Q of the finite group $P/Q = \pi(\Delta)$ (where, by Theorem 2.8, L/Q is isomorphic to the centre of the group G corresponding to L). Thus the classification of connected groups with a given root system Δ is equivalent to the classification of subgroups $Z \subset \pi(\Delta)$ up to automorphisms of the group $\pi(\Delta)$ induced by the group Aut Π. The following examples provide the classification of connected Lie groups with a given simple tangent algebra.

Example 1. Let $\mathfrak{g}$ be a simple noncommutative Lie algebra not of type D_{2s}, $s \geq 2$. Then $\pi(\Delta_\mathfrak{g})$ is a cyclic group, so any subgroup of it is invariant under all automorphisms. Therefore the connected Lie group G with the tangent algebra $\mathfrak{g}$ is defined, up to an isomorphism, by its centre $Z(G)$, which can be isomorphic to any subgroup of $\pi(\Delta_\mathfrak{g})$. The number of nonisomorphic groups G equals the number of natural divisors of the number $|\pi(\Delta_G)| = \det A$, where A is the Cartan matrix.

Example 2. Let $\mathfrak{g} = \mathfrak{so}_{4s}(\mathbb{C})$, $s \geq 2$, be a simple Lie algebra of type D_{2s}. Then $\pi(\Delta_G) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In the case $s \geq 3$ the group $\operatorname{Aut} \Pi$ is of order 2, and its generator permutes the terms in $\pi(\Delta_G)$. Therefore there are exactly two nonisomorphic connected Lie groups G with tangent algebra $\mathfrak{so}_{4s}(\mathbb{C})$, $s \geq 3$, and centre $Z(G) \simeq \mathbb{Z}_2$. The total number of groups G is 4. In the case $s = 2$ the group $\operatorname{Aut} \Pi \simeq S_3$ acts on $\pi(\Delta_G)$ as the full group of automorphisms. Therefore for $\mathfrak{g} = \mathfrak{so}_8(\mathbb{C})$ there is a unique connected Lie group G with a given centre. The total number of such groups is 3.

2.6. Linear Representations of Connected Reductive Algebraic Groups. Let G be a connected reductive algebraic group, H its maximal torus, Π a system of simple roots in the root system $\Delta_G(H)$, B^+ the corresponding Borel subgroup, and N^+ the unipotent radical of B^+ (see Sect. 1.5). Let $R: G \to \mathrm{GL}(V)$ be a linear representation of the group G. A weight vector $v \in V_\Lambda$ with respect to H is said to be a *highest* one if $R(g)v = v$ for all $g \in N^+$ or, equivalently, if v is a weight vector with respect to the entire subgroup B^+. The corresponding weight $\Lambda \in \Phi_R$ is said to be a *highest weight* of the representation R.

Recall that in Chap. 1, Sect. 9.5 we have identified the weights of a representation R of the group G with the weights of the representation $\rho = dR$ of its tangent algebra $\mathfrak{g}$. The highest vectors and the highest weights of the representation R are also called the highest vectors and the highest weights of the representation ρ of the reductive Lie algebra $\mathfrak{g}$. In the case where both G and $\mathfrak{g}$ are semisimple, this definition coincides with that given in Sect. 2.3. Theorem 2.2 implies that a highest weight is dominant and that an irreducible representation of the group G possesses a unique highest weight.

Theorem 2.12. *An irreducible finite-dimensional linear representation of a connected reductive algebraic group G is defined by its highest weight uniquely up to an isomorphism. For any dominant character $\Lambda \in \mathfrak{X}(H)$ there exists an irreducible finite-dimensional linear representation $R(\Lambda)$ of the group G with the highest weight Λ.*

Proof. If R is an irreducible representation of the group G, then, the Schur's lemma, its restriction to the algebraic torus $\operatorname{Rad} G = Z(G)^0 \subset H$ is one-dimensional. Thus, $R|\operatorname{Rad} G$ is a character of the torus coinciding with the restriction of the highest weight Λ. The proof therefore reduces to the case when G is semisimple, and one makes use of Theorems 2.3 and 2.9. $\square$

The irreducible representation with highest weight Λ is denoted by $R(\Lambda)$.

There is the following important *Weyl formula* expressing the dimension of an irreducible representation in terms of its highest weight (see Humphreys [1972], Jacobson [1962]):

$$\dim R(\Lambda) = \prod_{\alpha \in \Delta_G^+} \frac{(\Lambda + \gamma, \alpha)}{(\gamma, \alpha)}, \tag{7}$$

where γ is the sum half of the positive roots.

Example. Since the group $\mathrm{SL}_2(\mathbb{C})$ is simply-connected, any dominant weight of its representation can be a highest weight. Therefore irreducible representations of the group $\mathrm{SL}_2(\mathbb{C})$ are of the form $R(k\pi_1)$, where $k \geq 0$. An example of a realization of $R(k\pi_1)$ is the representation $S^k\mathrm{Id}$ induced by the identity representation in the space $S^k\mathbb{C}^2 = \mathbb{C}[u,v]_k$ of homogeneous polynomials of degree k in the elements of the standard basis $u = e_1$, $v = e_2$ in $\mathbb{C}^2$. The highest vector is u^k, $\dim \mathbb{C}[u,v]_k = k+1$, and the weight system is of the form $\{j\pi_1 | j = k, k-2, \ldots, -k\}$.

One can easily see that $-E \in \operatorname{Ker} R(k\pi_1)$ if and only if k is even. Thus irreducible representations of the group $\mathrm{SO}_3(\mathbb{C}) \simeq \mathrm{SL}_2(\mathbb{C})/\{E, -E\}$ are of the form $R(k\pi_1)$, where $k \geq 0$ is even.

Note that if a reductive group G is not semisimple, then the weight $\lambda \in P$ is not uniquely defined by its numerical labels. Therefore one cannot define an irreducible representation of the group G by the numerical labels of its highest weight. For example, the identity representation Id of the group $\mathrm{GL}_2(\mathbb{C})$ and the adjoint representation $\mathrm{Id}^*\colon g \mapsto (g^\top)^{-1}$ have highest weights ε_1 and $-\varepsilon_2$, respectively, and therefore are not equivalent, while the restrictions of these weights to $\mathfrak{d}_2(\mathbb{C}) \cap \mathfrak{sl}_2(\mathbb{C})$ coincide.

If the Borel subgroup B^+ appearing in the definition of a highest vector and a highest weight is replaced by the Borel subgroup B^- corresponding to the opposite Weyl chamber, one obtains the notion of a *lowest vector* and a *lowest weight*. For example, if δ is the highest root of a simple Lie algebra $\mathfrak{g}$, then the lowest root $\alpha_0 = -\delta$ is the lowest weight of the adjoint representation.

The highest and lowest vectors (weights) of a representation are related as follows. Theorem 1.9 implies that there exists a unique element w_0 of the Weyl group W taking the Weyl chamber C corresponding to the system Π into the opposite chamber $-C$. Clearly, $w_0^2 = \mathrm{id}$.

Proposition 2.3. *The transformation $w_0^\top$ takes the highest weights of the representation R into the lowest ones and vice versa. If $n_0 \in N_G(H)$ is an element such that $(\mathrm{Ad}\, n_0)|\mathfrak{h}(\mathbb{R}) = w_0$, then $R(n_0)$ takes the highest vectors into the lowest ones and vice versa.*

It remains to determine the transformation $w_0^\top$. Since $w_0^\top(\Pi) = -\Pi$, the transformation $\nu = -w_0^\top$ of the space $E = \mathfrak{h}(\mathbb{R})^*$ is an automorphism of the system Π. This automorphism (and the corresponding automorphism of the Dynkin diagrams) is called the *canonical involution*. Clearly, ν takes each irreducible component of the system Π into itself and induces a canonical involution there. It is therefore sufficient to consider the case where the group G is simple.

Theorem 2.13. *For all simple noncommutative Lie algebras $\mathfrak{g}$ other than $\mathfrak{sl}_n(\mathbb{C})$ $(n \geq 3)$, $\mathfrak{so}_{4n+2}(\mathbb{C})$ $(n \geq 1)$, and E_6, the canonical involution ν of the system of simple roots is the identity transformation. For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ $(n \geq 3)$,*

$\mathfrak{so}_{4n+2}(\mathbb{C})$ $(n \geq 1)$, E_6, *the involution* ν *coincides with the only nontrivial automorphism of the system* Π.

Proof. For the classical Lie algebras $\mathfrak{g}$ the statement of the theorem is an easy consequence of the explicit description of the Weyl group (see the example of Sect. 1.6). If $\mathfrak{g} = E_6$ and $\nu = \mathrm{id}$, then $-\mathrm{id} = w_0^\top \in W^\vee$. Therefore $w_0^\top$ induces a nontrivial automorphism of the group $\pi(\Delta_G) \simeq \mathbb{Z}_3$. On the other hand, the definition of the root system implies that any reflection $r_\alpha^\vee$ acts trivially on $\pi(\Delta_G) = P/Q$. □

It follows from Proposition 2.3 that the properties of lowest weights are completely analogous to those of highest weights. In particular, the lowest weight of an irreducible representation R is the least element of the system Φ_R with respect to the partial order defined by the system of simple roots Π and determines R uniquely up to equivalence.

2.7. Dual Representations and Bilinear Invariants. Let G be a reductive algebraic group and $R: G \to \mathrm{GL}(V)$ a linear representation of it. Denote by R^* the linear representation of the group G in the space V^* dual to R. One can easily see that $\Phi_{R^*} = -\Phi_R$. A representation R is said to be *self-dual* if $R \sim R^*$.

Theorem 2.14. *If* Λ *and* M *are the highest and the lowest weights of an irreducible representation* R, *then* R^* *has highest weight* $M = \nu(\Lambda)$ *and lowest weight* $-\Lambda = \nu(M)$, *where* ν *is the canonical involution. The representation* R *is self-dual if and only if* $M = -\Lambda$ *or, equivalently, if* $\nu(\Lambda) = \Lambda$.

Proof. $-M$ is the largest element of the system $\Phi_{R^*} = -\Phi_R$ and therefore the highest weight of the representation R^*. The assertion now follows from Proposition 2.3. □

Corollary. *Let* R *be an irreducible representation of a simple noncommutative Lie group* G. *If the tangent subalgebra of the group* G *is different from* $\mathfrak{sl}_n(\mathbb{C})$ $(n \geq 3)$, $\mathfrak{so}_{4n+2}$ $(n \geq 1)$, *and* E_6, *then* R *is self-dual. In the other cases* R *is self-dual if and only if the arrangement of numerical labels of its highest weight is symmetric.*

Since any linear mapping $V \to V^*$ can be considered as a bilinear form on the space V, the self-duality of a linear representation $R: G \to \mathrm{GL}(V)$ is equivalent to the existence of a nondegenerate bilinear form on V invariant under R. Schur's lemma implies that the space of invariant bilinear forms for an irreducible self-dual representation is one-dimensional and that all these forms are either symmetric or skew-symmetric. A linear representation is said to be *orthogonal* (*symplectic*) if it admits a symmetric (respectively, skew-symmetric) nondegenerate invariant bilinear form. We now give the criteria for a self-dual irreducible representation of a semisimple Lie group to be orthogonal or symplectic. First consider the simplest special case.

Example. As follows from Theorem 2.14, any linear representation of the group $\mathrm{SL}_2(\mathbb{C})$ is self-dual. Consider the irreducible representation $R(k\pi_1)$ of this group, which can be viewed as the representation $S^k\mathrm{Id}$ in the space of bilinear forms of degree k (see the example in Sect. 2.2). Consider the bilinear form β in the space $\mathbb{C}[u,v]_k$, which is uniquely determined by the equality

$$\beta((au+bu)^k,\ (cu+du)^k) = (ad-bc)^k.$$

One can verify that β is invariant under $R(k\pi_1)$. Therefore $R(k\pi_1)$ is orthogonal if k is even and symplectic if k is odd.

The case of an arbitrary connected semisimple Lie group G is reduced to the case $G = \mathrm{SL}_2(\mathbb{C})$ by means of the following construction. Let $\mathfrak{g}$ be a semisimple Lie algebra, and Π its system of simple roots. Consider the element $h = \sum\limits_{\alpha\in\Delta^+} h_\alpha$. An application of Proposition 1.17 to the root system $\Delta_{\mathfrak{g}}^\vee$ shows that $h = 2\sum\limits_{i=1}^{l} \pi_i^\vee$, where $\pi_i^\vee$ are fundamental weights of the root system $\Delta_{\mathfrak{g}}^\vee$. It is also clear that

$$h = \sum_{i=1}^{l} r_i h_i, \tag{8}$$

where r_i are positive integers. Let

$$l_+ = \sum_{i=1}^{l} \sqrt{r_i} e_i, \quad l_- = \sum_{i=1}^{l} \sqrt{r_i} f_i.$$

Then the subspace $\langle h, l_+, l_-\rangle$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ (it is called *a principal three-dimensional subalgebra*, see Chap. 6, Sect. 2.3).

Theorem 2.15. *Let $R\colon G \to \mathrm{GL}(V)$ be an irreducible self-dual linear representation of a connected semisimple Lie group G, Λ its highest weight, $\Lambda_i = \Lambda(h_i)$ $(i = 1,\ldots,l)$ the numerical labels of the weight Λ, and $m = \sum\limits_{i=1}^{l} r_i\Lambda_i$, where the numbers r_i are given by formula* (8). *The representation is orthogonal if m is even and symplectic if m is odd.*

Proof. Let $\mathfrak{g}_0$ be a principal three-dimensional subalgebra of the tangent algebra $\mathfrak{g}$ of the group G. The highest weight of the representation R generates the subspace W of V, which is invariant and irreducible under $dR(\mathfrak{g}_0)$ and contains both V_Λ and V_M. The representation $dR|\mathfrak{g}_0$ is defined by the numerical label $\Lambda(h) = m$ and is therefore orthogonal (symplectic) if m is even (respectively, odd) (see example). It follows from Theorem 2.14 that V_Λ and V_M are not orthogonal to each other with respect to the nondegenerate G-invariant bilinear form β in the space V, and consequently the restriction $\beta|W$ does not vanish. This implies our statement. $\square$

Note that the number $m = \Lambda(h)$ also coincides with $2\sum\limits_{i=1}^{l} a_i$, where $\Lambda = \sum\limits_{i=1}^{l} a_i\alpha_i$. An explicit calculation of m in terms of the numerical labels of the weight Λ yields the following statement.

Corollary. *If G is simple, then the self-dual irreducible representation $R(\Lambda)$ of the group G is orthogonal (symplectic) if and only if the number given in the following table in terms of the numerical labels Λ_i is even (respectively, odd). (The enumeration of simple roots is the same as in Table 1. For the groups of type other than those given in the table, a self-dual representation is always orthogonal.)*

A_{4q+1}	B_l $l = 4q+1,\ 4q+2$	C_l	D_{4q+2}	E_7
Λ_{2q+1}	Λ_l	$\Lambda_1 + \Lambda_3 +$ $+\Lambda_5 + \ldots$	$\Lambda_{4q+1} + \Lambda_{4q+2}$	$\Lambda_1 + \Lambda_6 + \Lambda_3$

In the general case the self-dual representation $R(\Lambda)$ is orthogonal or symplectic depending on the parity of the sum of the numbers corresponding to distinct connected components of the Dynkin diagram of G.

Remark. The orthogonality (symplecticity) criteria of Theorem 2.15 can also be recast in the following form. By Theorem 2.8, the element $z_0 = \mathcal{E}(\gamma^\vee)$, where $\gamma^\vee = \frac{1}{2}\sum\limits_{\alpha\in\Delta^+}\alpha^\vee$, belongs to the centre $Z(G)$ and $z_0^2 = e$. If R is an irreducible linear representation of the group G, then Schur's lemma implies that $R(z_0) = E$ or $-E$. The self-dual irreducible linear representation is orthogonal in the first case and symplectic in the second. For a direct proof of this (making use of Proposition 2.3) see Steinberg [1968b], Sect. 12, which considers representations of semisimple algebraic groups over an algebraically closed field of arbitrary characteristic.

Now let R be an arbitrary representation of a reductive group G and $R = R_1 + \ldots + R_s$ its decomposition into irreducible components. Then $R^* = R_1^* + \ldots + R_s^*$ is the decomposition of the representation R^* into irreducible components. The uniqueness of such a decomposition implies the following theorem.

Theorem 2.16. *A representation R of a reductive algebraic group G is self-dual if and only if its decomposition into irreducible components is of the form*

$$R = R_1 + \ldots + R_p + R_{p+1} + \ldots + R_{p+q} + R_{p+1}^* + \ldots + R_{p+q}^*,$$

where R_i $(i = 1, \ldots, p)$ are pairwise non-equivalent self-dual irreducible representations. The representation R is orthogonal (symplectic) if so are R_i $(i = 1, \ldots, p)$.

2.8. The Kernel and the Image of a Locally Faithful Linear Representation. Suppose that a representation R of the group G is locally faithful, i.e. that $\rho = dR$ is a faithful representation of the tangent algebra $\mathfrak{g}$. Then $\operatorname{Ker} R$ is a subgroup of $Z(G)$. We will now show a way of expressing this subgroup in terms of the weights of R. Denote by L_R the sublattice in $\mathfrak{X}(H)$ generated by the system Φ_R. The complete reducibility of R and Theorem 2.2 imply that L_R is generated by the lattice Q and the highest weights of R. We have

$$L_R^* = \{x \in P^\vee \,|\, \Lambda(x) \in \mathbb{Z} \quad \text{for all highest weights} \quad \Lambda \text{ of the representation } R\}.$$

If the space $\mathfrak{h}(\mathbb{R})^*$ is identified with its image under $(\rho^\top)^{-1}$, then the lattice L_R is identified with $\mathfrak{X}(R(H))$.

Theorem 2.17 (see Dynkin and Onishchik [1955]). *We have*

$$\operatorname{Ker} R \simeq L_R^*/\mathfrak{h}(\mathbb{Z}) \simeq \mathfrak{X}(H)/L_R,$$

$$Z(R(G)) \simeq P^\vee/L_R^* \simeq L_R/Q.$$

Corollary 1. *A representation R is faithful if and only if Φ_R generates $\mathfrak{X}(H)$ or if $L_R^* = \mathfrak{h}(\mathbb{Z})$.*

Corollary 2. *$Z(R(G)) = \{e\}$ if and only if $L_R = Q$. In the case where R is irreducible this is equivalent to either of the two following conditions:* (1) *the highest weight $\Lambda \in Q$;* (2) *Φ_R contains the zero weight.*

Proof. If $0 \in \Phi_R$, then $\Lambda \in Q$ by virtue of Theorem 2.2. The reverse statement follows from Propositions 1.15 and 2.2. □

Let R be a linear representation of a connected simple complex Lie group G. Theorem 2.5 makes it possible to compute the centre and the global type of the group $R(G)$ explicitly (see Sect. 2.5) in terms of the highest weights of the representation R. In the case of an irreducible representation one obtains the following result.

Theorem 2.18 (see Dynkin and Onishchik [1955]). *Let $R\colon G \to \mathrm{GL}(V)$ be an irreducible linear representation of a connected simple complex Lie group G, $\Lambda_1, \ldots, \Lambda_l$ the numerical labels of the highest weight Λ of the representation and $G' = R(G)$.*

Let $\mathfrak{g} = \mathfrak{sl}_{l+1}(\mathbb{C})$, $l \geq 1$. Then $|Z(G')| = \frac{l+1}{d}$, where $d = GCD\left(l+1, \sum_{i=1}^{l} i\Lambda_i\right)$.

Let $\mathfrak{g} = \mathfrak{so}_{2l+1}$, $l \geq 2$. Then $|Z(G')| = 2$ if Λ_l is odd and $|Z(G')| = 1$ if Λ_l is even.

Let $\mathfrak{g} = \mathfrak{sp}_{2l}(\mathbb{C})$, $l \geq 2$. Then $|Z(G')| = 2$ if $\Lambda_1 + \Lambda_3 + \Lambda_5 + \ldots$ is odd and $|Z(G')| = 1$ otherwise.

Let $\mathfrak{g} = \mathfrak{so}_{2l}(\mathbb{C})$, $l \geq 3$, l odd. Then $|Z(G')| = 4$ if $\Lambda_{l-1} + \Lambda_l$ is odd; $|Z(G')| = 2$ if $\Lambda_{l-1} + \Lambda_l$ is even but $\Lambda_1 + \Lambda_3 + \ldots + \Lambda_{l-2} + \frac{1}{2}(\Lambda_{l-1} + \Lambda_l)$ is *odd;* $|Z(G')| = 1$ *if both* $\Lambda_{l-1}+\Lambda_l$ *and* $\Lambda_1+\Lambda_3+\ldots+\Lambda_{l-2}+\frac{1}{2}(\Lambda_{l-1}+\Lambda_l)$ *are even.*

(e) *Let* $\mathfrak{g} = \mathfrak{so}_{2l}(\mathbb{C})$, $l \geq 6$, l *even. If* $\Lambda_{l-1} + \Lambda_l$ *is odd, then* $Z(G')$ *is the group of order 2 with the generator given by the element* $\frac{1}{2}(h_{l-1} + h_l)$. *If* $\Lambda_{l-1}+\Lambda_l$ *is even and* $\Lambda_1+\Lambda_3+\ldots+\Lambda_{l-1}$ *odd, then* $Z(G')$ *is the group of order 2 with the generator given by the element* $\frac{1}{2}(h_1 + h_3 + \ldots + h_{l-1}) \in P^\vee$. *If both* $\Lambda_{l-1}+\Lambda_l$ *and* $\Lambda_1+\Lambda_3+\ldots+\Lambda_{l-1}$ *are even, then* $|Z(G')| = 1$.

(f) *Let* $\mathfrak{g} = \mathfrak{so}_8(\mathbb{C})$. *If some of the numbers* Λ_1, Λ_3, Λ_4 *have different parities, then* $|Z(G')| = 2$. *If all three of them are of the same parity, then* $|Z(G')| = 1$.

(g) *Let* $\mathfrak{g} = E_6$. *If* $\Lambda_1 - \Lambda_2 + \Lambda_4 - \Lambda_5 \not\equiv 0 \pmod 3$, *then* $|Z(G')| = 3$; *if* $\Lambda_1 - \Lambda_2 + \Lambda_4 - \Lambda_5 \equiv 0 \pmod 3$, *then* $|Z(G')| = 1$.

(h) *Let* $\mathfrak{g} = E_7$. *If* $\Lambda_1 + \Lambda_3 + \Lambda_7$ *is odd, then* $|Z(G')| = 2$; *if* $\Lambda_1 + \Lambda_3 + \Lambda_7$ *is even, then* $|Z(G')| = 1$.

(i) *Let* $\mathfrak{g} = E_8, F_4, G_2$. *Then* $|Z(G')| = 1$.

2.9. The Casimir Operator and Dynkin Index. Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of a complex semisimple Lie algebra and C the Casimir operator in the space V (see Chap. 1, Sect. 3.2) associated with a nondegenerate invariant bilinear form $b(x,y) = (x,y)$ on $\mathfrak{g}$, which as usual will be assumed to be positive definite on $\mathfrak{h}(\mathbb{R})$. If ρ is irreducible, then Schur's lemma implies that C is a scalar operator.

Proposition 2.4. *If* ρ *is an irreducible representation with highest weight* Λ, *then* $C = (\Lambda, \Lambda + 2\gamma)E$.

Proof. Choose the basis $\{h_i\ (i = 1, \ldots, l),\ e_\alpha\ (\alpha \in \Delta_\mathfrak{g})\}$ in $\mathfrak{g}$ described in the corollary to Theorem 1.2. In this basis we have

$$C = \sum_{i=1}^{l} \rho(h_i)^2 + \sum_{\alpha\in\Delta_\mathfrak{g}} \frac{1}{(e_\alpha, e_{-\alpha})}\rho(e_\alpha)\rho(e_{-\alpha}).$$

Applying C to the highest vector v_Λ we verify that $Cv_\Lambda = (\Lambda, \Lambda+2\gamma)v_\Lambda$. □

Let $\mathfrak{g}$ be a simple noncommutative Lie algebra. Normalize the nondegenerate invariant bilinear form b on $\mathfrak{g}$ in such a way that $(\alpha,\alpha) = 2$ for the root $\alpha \in \Delta_\mathfrak{g}$ of largest length. Let $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ be a homomorphism of $\mathfrak{g}$ into another simple Lie algebra $\mathfrak{g}'$ equipped with the form b' chosen in a similar way. The uniqueness of an invariant bilinear form on $\mathfrak{g}$ (see Sect. 2.7) implies that there exists a number j_φ such that

$$(\varphi(x), \varphi(y)) = b'(\varphi(x), \varphi(y)) = j_\varphi b(x,y) = j_\varphi(x,y) \quad (x, y \in \mathfrak{g}).$$

The number j_φ is called the *Dynkin index* of the homomorphism φ. If $\mathfrak{g}$ is a noncommutative simple subalgebra of a simple Lie algebra $\mathfrak{g}'$, then the index of the embedding $\mathfrak{g} \to \mathfrak{g}'$ is called the *index* of the subalgebra $\mathfrak{g}$ in $\mathfrak{g}'$ and denoted by $j(\mathfrak{g}, \mathfrak{g}')$.

The simplest properties of the Dynkin index are as follows. Let $\alpha \in \operatorname{Aut} \mathfrak{g}'$, $\beta \in \operatorname{Aut} \mathfrak{g}$. Then

$$j_{\alpha\circ\varphi\circ\beta} = j_\varphi; \quad j(\alpha(\mathfrak{g}), \mathfrak{g}') = j(\mathfrak{g}, \mathfrak{g}').$$

Given two homomorphisms of simple Lie algebras $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ and $\psi\colon \mathfrak{g}' \to \mathfrak{g}''$, we have $j_{\psi\varphi} = j_\psi j_\varphi$. If $\mathfrak{g} \subset \mathfrak{g}' \subset \mathfrak{g}''$, then $j(\mathfrak{g}, \mathfrak{g}'') = j(\mathfrak{g}, \mathfrak{g}') \cdot j(\mathfrak{g}', \mathfrak{g}'')$.

The definition of the index easily implies that it is nonnegative. In fact the following statement, first noted in Dynkin [1952b], holds (see also Braden [1991]).

Proposition 2.5. *The Dynkin index of any injective homomorphism of simple Lie algebras (in particular, the index of any simple subalgebra) is a positive integer.*

Proof. Let $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ be an injective homomorphism of simple Lie algebras. Choose Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$, $\mathfrak{h}' \subset \mathfrak{g}'$ such that $\varphi(\mathfrak{h}) \subset \mathfrak{h}'$. Then $\varphi(\mathfrak{h}(\mathbb{R})) \subset \mathfrak{h}'(\mathbb{R})$. One can easily see that φ^{T} maps the weight lattice of the root system $\Delta_{\mathfrak{g}'}$ into the weight lattice of the system $\Delta_{\mathfrak{g}}$. By Proposition 1.15, $\varphi(Q^\vee) \subset Q^{\vee\prime}$, where $Q^\vee$, $Q^{\vee\prime}$ are the root lattices of the systems $\Delta_{\mathfrak{g}}^\vee$, $\Delta_{\mathfrak{g}'}^\vee$ respectively. Let $\varepsilon \in \Delta_{\mathfrak{g}}$ be the root of largest length. Then $(h_\alpha, h_\alpha) = 2$ and $j_\varphi = \frac{1}{\varphi}(\varphi(h_\alpha), \varphi(h_\alpha))$. It remains to note that the scalar product of any two elements of the lattice $Q^{\vee\prime}$ is even. Indeed, for any $\beta \in \Delta_{\mathfrak{g}}$ the number $(h_\beta, h_\beta) = \frac{4}{(\beta,\beta)}$ is even because the ratio of squares of root lengths is an integer, and one similarly verifies that $(h_\beta, h_\varepsilon) \in \mathbb{Z}$ for any $\beta, \varepsilon \in \Delta_{\mathfrak{g}'}$. $\square$

Now let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a linear representation of a noncommutative simple Lie algebra $\mathfrak{g}$. Then ρ can be considered as a homomorphism $\mathfrak{g} \to \mathfrak{sl}(V)$. Therefore one can speak of the *index* j_ρ of the representation ρ. It is defined by the formula

$$\operatorname{tr}(\rho(x)\rho(y)) = j_\rho(x, y) \quad (x, y \in \mathfrak{g}) \tag{9}$$

and, by Proposition 2.5, is a nonnegative integer. The index of a representation has the following properties:

$$j_{\rho_1+\rho_2} = j_{\rho_1} + j_{\rho_2},$$
$$j_{\rho_1\rho_2} = (\dim \rho_1) j_{\rho_2} + (\dim \rho_2) j_{\rho_1}.$$

The following theorem enables one to compute the index of an irreducible representation in terms of its highest weight.

Theorem 2.19. *The index of an irreducible representation $\rho(\Lambda)$ of a noncommutative simple Lie algebra $\mathfrak{g}$ is equal to*

$$j_{\rho(\Lambda)} = \frac{\dim \rho(\Lambda)}{\dim \mathfrak{g}} (\Lambda, \Lambda + 2\gamma).$$

Proof. The Casimir operator C of the representation ρ associated with the form b is given by $C = \sum_{i=1}^{n} \rho(v_i)\rho(v^i)$, where $v_1, \ldots, v_n$ is a basis of the algebra $\mathfrak{g}$ and $v^1, \dots, v^n$ the basis dual to it with respect to b. Hence $\operatorname{tr} C = \sum_{i=1}^{n} \operatorname{tr}(\rho(v_i)\rho(v^i)) = j_\rho \cdot n = j_\rho \dim \mathfrak{g}$ by formula (9). It remains to apply Proposition 2.4. □

Example. If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, then

$$j_{\rho(k\pi_1)} = \frac{1}{6} k(k+1)(k+2).$$

2.10. Spinor Group and Spinor Representation. As is known (see Vinberg and Onishchik [1988], Chap. 1, Sect. 4.5), $\pi_1(\mathrm{SO}_n(\mathbb{C})) \simeq \mathbb{Z}_2$ for $n \geq 3$. By virtue of Theorem 2.12, a simply-connected Lie group covering the group $\mathrm{SO}_n(\mathbb{C})$ admits a faithful linear representation. This section reproduces the classical construction of a simply-connected Lie group covering $\mathrm{SO}_n(\mathbb{C})$ (this covering is called the spinor group) and its faithful representation (called the spinor representation). For real orthogonal groups this construction is given in Shafarevich [1986], Sect. 15.

Consider the space $\mathbb{C}^n$ equipped with the standard symmetric bilinear form $\beta(x, y) = \sum_{i=1}^{h} x_i y_i$. Denote by C_n the corresponding Clifford algebra. In terms of the standard basis $e_1, \dots, e_n$ of the space $\mathbb{C}^n$ the algebra C_n can be defined as an associative algebra with unity, the generators $e_1, \dots, e_n$, and defining relations

$$e_i e_j + e_j e_i = 0 \ (i \neq j), \quad e_i^2 = 1 \ (i, j = 1, \dots, n).$$

For the basis of C_n one can take the set of elements of the form $e_{i_1} e_{i_2} \dots e_{i_s}$, where $i_1 < i_2 < \dots < i_s$, including unity, whence $\dim C_n = 2^n$. The algebra C_n admits a $\mathbb{Z}_2$-grading

$$C_n = C_n^{\bar{0}} \oplus C_n^{\bar{1}},$$

where $C_n^{\bar{0}}$ (respectively, $C_n^{\bar{1}}$) is the subspace spanned by the monomials $e_{i_1} e_{i_2} \dots e_{i_s}$ with s even (respectively, odd). Define an antiautomorphism $u \mapsto \overline{u}$ in the algebra C_n preserving the $\mathbb{Z}_2$-grading, by the formula

$$\overline{u_1 u_2 \dots u_s} = u_s \dots u_2 u_1 \quad (u_i \in \mathbb{C}^n).$$

Let

$$N(u) = \overline{u} u.$$

The group of invertible elements $C_n^\times$ of the algebra C_n naturally acquires the structure of a complex Lie group. Its tangent algebra is the algebra C_n in which multiplication is replaced by the commutator. The adjoint representation of the group $C_n^\times$ is of the form

$$(\operatorname{Ad} u)x = uxu^{-1} \quad (u \in C_n^\times, \ x \in C_n).$$

The group $(C_n^{\bar{0}})^\times$ is a Lie subgroup of $C_n^\times$. Consider the Lie subgroup $GC_n^+ = \{u \in (C_n^{\bar{0}})^\times | u\mathbb{C}^n u^{-1} = \mathbb{C}^n\}$. It turns out that the mapping N is a homomorphism of the group GC_n^+ into $\mathbb{C}^\times$. The Lie subgroup

$$\mathrm{Spin}_n(\mathbb{C}) = \mathrm{Ker}\, N = \{u \in GC_n^+ | N(u) = 1\}$$

is called the *complex spinor group.*

The mapping $A: u \mapsto (\mathrm{Ad}\, u)|\mathbb{C}^n$ defines a linear representation of the group $\mathrm{Spin}_n(\mathbb{C})$ in the space $\mathbb{C}^n$. We have $\mathrm{Im}\, A = \mathrm{SO}_n(\mathbb{C})$, $\mathrm{Ker}\, A = \{1, -1\}$. Furthermore, the group $\mathrm{Spin}_n(\mathbb{C})$ is connected. Thus $A: \mathrm{Spin}_n(\mathbb{C}) \to \mathrm{SO}_n(\mathbb{C})$ is a two-sheeted covering and therefore the group $\mathrm{Spin}_n(\mathbb{C})$ is simply-connected.

If $y \in Q^{n-1} = \{y \in \mathbb{C}^n | \beta(y, y) = 1\}$, then $y \in C_n^\times$. It turns out that $A(y)$ takes $\mathbb{C}^n$ into itself and induces in $\mathbb{C}^n$ the transformation $-r_y$, where r_y is the reflection in the hyperplane orthogonal to y. One deduces that the group $\mathrm{Spin}_n(\mathbb{C})$ consists of all possible products an of even number of elements of the quadric Q^{n-1}.

Proposition 2.6. *The tangent algebra* $\mathfrak{spin}_n(\mathbb{C})$ *of the group* $\mathrm{Spin}_n(\mathbb{C})$ *coincides with the subspace in* $C_n^{\bar{0}}$ *spanned by the elements* $e_i e_j$ $(i < j)$. *We have* $dA(e_i e_j) = 2(-E_{ij} + E_{ji})$.

If $n = 2l+1$ is odd, the algebra $C_n^{\bar{0}}$ is simple and therefore admits a unique (up to equivalence) irreducible linear representation S,

which acts in its simple left ideal I, $\dim I = 2^l$, and induces an irreducible linear representation S of the group $\mathrm{Spin}_{2l+1}(\mathbb{C})$ in the space I. This representation is called a *spinor* representation. If $n = 2l$ is even, then the algebra C_n is simple and has a unique irreducible representation S acting in its simple left ideal J, $\dim J = 2^l$. It induces a representation S of the group $\mathrm{Spin}_{2l}(\mathbb{C})$ in the space J, also called a *spinor* representation. Since $J = J^{\bar{0}} \oplus J^{\bar{1}}$, where $J^k = J \cap C_n^k$, and $\mathrm{Spin}_n(\mathbb{C}) \subset C_n^{\bar{0}}$, the representation S decomposes into the sum $S = S^{\bar{0}} + S^{\bar{1}}$, where S^k is a subrepresentation arising in J^k, $k = \bar{0}, \bar{1}$, $\dim J^k = 2^{l-1}$. The representations are irreducible and bear the name of *semispinor* representations. In both cases the spinor representation is faithful.

Let us describe the representation S in the case $n = 2l$. It is convenient to choose the basis $(u_1, \ldots, u_l, , v_1, \ldots, v_l)$ in the space $\mathbb{C}^n$ defined by the formulae

$$u_j = \frac{1}{2}(ie_j + e_{l+j}), \quad v_j = \frac{1}{2}(ie_j - e_{l+j}) \quad (j = 1, \ldots, l).$$

In this basis the matrix of the form β is $\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$, and the defining relations of the algebra C_n are of the form

$$u_j u_k + u_k u_j = v_j v_k + v_k v_j = 0 \quad (j, k = 1, \ldots, l),$$
$$u_j v_k + u_k v_j = 0 \quad (j \neq k),$$
$$u_j v_j + v_j u_j = 1 \quad (j, = 1, \ldots, l).$$

Hence $\Lambda(v_1, \ldots, v_l) \subset C_n$. The left ideal $J = C_n u_1 \ldots u_l$ coincides with $\Lambda(v_1 \ldots v_l) u_1 \ldots u_l$ and is simple. The mapping $w \mapsto w u_1 \ldots u_l$ defines an isomorphism of the vector space $\Lambda(v_1, \ldots, v_l)$ onto J. One can assume therefore that the representation S acts in the space $\Lambda(v_1, \dots, v_l)$, while the semispinor representations act in the even and odd components of the space $\Lambda(v_1, \dots, v_l)$ with respect to the natural $\mathbb{Z}_2$-grading. This action can be easily expressed in terms of multiplication in the Clifford algebra. By identifying the Lie algebra $\mathfrak{spin}_{2l}(\mathbb{C})$ with $\mathfrak{so}_{2l}(\mathbb{C})$ by means of dA (see Proposition 2.6), we obtain

$$\Delta_S = \left\{ \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \ldots \pm \varepsilon_l) \right\}, \tag{10}$$

where $\Delta_{S\bar{0}}$ and $\Delta_{S\bar{1}}$ consist of the weights with even and odd number of minuses, respectively. The highest weights of the semispinor representations are $\frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_l) = \pi_l$ and $\frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_{l-1} - \varepsilon_l) = \pi_{l-1}$.

One can see from Theorem 2.17 that the semispinor representations are faithful if l is odd and have kernels of order 2 if l is even; in the latter case the group $\operatorname{Spin}_{2l}(\mathbb{C})$ has no faithful irreducible representations at all.

The case $n = 2l+1$ is treated on the same lines. It turns out that here Δ_S is also of the form (10), and that the highest weight of the representation S is $\frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_l) = \pi_l$.

Example. As shown in Examples 1 and 2 of Sect. 2.1, the groups $SO_3(\mathbb{C})$, $SO_4(\mathbb{C})$, $SO_5(\mathbb{C})$, and $SO_6(\mathbb{C})$ are covered by the groups $SL_2(\mathbb{C})$, $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$, $Sp_4(\mathbb{C})$, and $SL_4(\mathbb{C})$, respectively. Since the four last groups are simply-connected, we have the isomorphisms $\operatorname{Spin}_3(\mathbb{C}) \simeq SL_2(\mathbb{C})$, $\operatorname{Spin}_4(\mathbb{C}) \simeq SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$, $\operatorname{Spin}_5(\mathbb{C}) \simeq Sp_4(\mathbb{C})$, $\operatorname{Spin}_6(\mathbb{C}) \simeq SL_4(\mathbb{C})$. It follows from the above description of the highest weights of spinor representations that for $n = 3$ and 5 the spinor representation is identified with the identity representation Id of the groups $SL_2(\mathbb{C})$ and $Sp_4(\mathbb{C})$. For $n = 4$ the semispinor representations are identified with the representations $\mathrm{Id} \otimes N$ and $N \otimes \mathrm{Id}$ of the group $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$, where N is the trivial one-dimensional representation. For $n = 6$ the semispinor representations are identified with the representations Id and Id^* of the group $SL_4(\mathbb{C})$.

§ 3. Automorphisms and Gradings

Throughout this section $\mathfrak{g}$ denotes a complex semisimple Lie algebra, G a connected complex Lie group (which in some cases will be specified) for which $\mathfrak{g}$ is the tangent algebra, and $(\ ,\ )$ a scalar product in $\mathfrak{g}$ invariant under all its automorphisms (for example, the Killing form). The group G will also be considered as an algebraic group (see Theorem 2.7).

3.1. Description of the Group of Automorphisms. The group $\operatorname{Aut}\mathfrak{g}$ of automorphisms of the algebra $\mathfrak{g}$ is an algebraic linear group (and thus a linear Lie group). Its tangent algebra is the algebra $\operatorname{Der}\mathfrak{g}$ of derivations of the algebra $\mathfrak{g}$, coinciding with $\operatorname{ad}(\mathfrak{g})$. The identity component of the group

$\operatorname{Aut}\mathfrak{g}$ coincides with the group of inner automorphisms $\operatorname{Int}\mathfrak{g} = \operatorname{Ad}(G)$ (see Chap. 1, Corollary to Theorem 3.2).

Let us now proceed to the description of outer automorphisms. Denote by e_i, f_i, h_i $(i = 1, \ldots, l)$ the canonical generators of the algebra $\mathfrak{g}$ associated with a Cartan subgroup H and the system Π of simple roots with respect to H (see Sect. 2.1). Consider the group $\operatorname{Aut}\Pi$ of automorphisms of the system Π in the sense of Sect. 1.7. For each $\sigma \in \operatorname{Aut}\Pi$ there exists a (unique) automorphism of the algebra $\mathfrak{g}$ taking e_i, f_i, h_i into $e_{\sigma i}, f_{\sigma i}, h_{\sigma i}$, respectively. Denote this automorphism by the same letter σ. Thus the group $\operatorname{Aut}\Pi$ becomes a subgroup of the group $\operatorname{Aut}\mathfrak{g}$.

Theorem 3.1. $\operatorname{Aut}\mathfrak{g} = \operatorname{Int}\mathfrak{g} \rtimes \operatorname{Aut}\Pi$.

For the proof see, e.g., Onishchik and Vinberg [1990], Springer [1989].

Thus the "group of outer automorphisms" $\operatorname{Aut}\mathfrak{g}/\operatorname{Int}\mathfrak{g}$ of the algebra $\mathfrak{g}$ is naturally isomorphic to the group $\operatorname{Aut}\Pi$.

3.2. Quasitori of Automorphisms and Gradings. A commutative complex algebraic group whose identity component is an (algebraic) torus is called an (algebraic) *quasitorus*. Any quasitorus is a direct product of a torus and a commutative finite group. An algebraic linear group is a quasitorus if and only if in some basis its elements can be expressed simultaneously by diagonal matrices.

The closure (in the Zariski topology) of the cyclic subgroup generated by any semisimple element of an algebraic group is a quasitorus with a cyclic group of components, and, conversely, in any quasitorus with the cyclic group of components there is an element whose powers constitute a dense subgroup.

Quasitori are objects dual to finitely generated commutative groups. Namely, the group $\mathfrak{X}(S)$ of characters of a quasitorus S is a finitely generated commutative group, and, conversely, the group of characters of any finitely generated commutative group is a quasitorus.

Let S be a quasitorus. For any homomorphism $\varphi: S \to \operatorname{Aut}\mathfrak{g}$ we have the decomposition of the algebra $\mathfrak{g}$ into the direct sum of the weight subspaces

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{X}(S)} \mathfrak{g}_\alpha, \tag{11}$$

$$\mathfrak{g}_\alpha = \{\xi \in \mathfrak{g} : \varphi(s)(\xi) = \alpha(s)\xi \quad \forall s \in S\}. \tag{12}$$

A standard calculation demonstrates that

$$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \quad \text{for} \quad \alpha + \beta \neq 0, \tag{13}$$

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \tag{14}$$

(we use the additive notation for the operation in $\mathfrak{X}(S)$). It follows from (13) that the subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are dual to each other with respect to the scalar product, and, in particular, the scalar product is nondegenerate on $\mathfrak{g}_0$.

Property (14) means that the decomposition (11) is an $\mathfrak{X}(S)$-grading of the algebra $\mathfrak{g}$. Conversely, let A be an arbitrary finitely generated commutative group, and let an *A-grading* of the algebra $\mathfrak{g}$ be given, i.e. a decomposition $\mathfrak{g} = \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$ satisfying condition (14). Then the formula

$$\varphi(s)(\xi) = s(\alpha)\xi \quad \text{for} \quad \xi \in \mathfrak{g}_\alpha \tag{15}$$

defines a homomorphism of the quasitorus $S = \mathfrak{X}(A)$ into the group $\operatorname{Aut}\mathfrak{g}$ such that the $\mathfrak{X}(S)$-grading of the algebra $\mathfrak{g}$ defined by this homomorphism coincides with the original A-grading under the natural identification of $\mathfrak{X}(S)$ with A.

Example. If $S = H$ is a Cartan subgroup (a maximal torus) of the group $\operatorname{Aut}\mathfrak{g}$, and φ the identity embedding, then the decomposition (11) is a root decomposition of the algebra $\mathfrak{g}$. It is a $\mathbb{Z}^l$-grading, where $l = \operatorname{rk}\mathfrak{g}$.

3.3. Homogeneous Semisimple and Nilpotent Elements. Consider a grading $\mathfrak{g} = \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$ of an algebra $\mathfrak{g}$ by means of a finitely generated commutative group A. An element $\xi \in \mathfrak{g}$ is said to be *homogeneous* if it belongs to one of the subspaces $\mathfrak{g}_\alpha$. There are the following graded modifications of the Jordan decomposition (see Springer [1989], Chap. 1, Sect. 3.1) and Morozov's theorem (see Theorem 1.3).

Theorem 3.3. *Any homogeneous element of the algebra $\mathfrak{g}$ is uniquely represented as a sum of commuting semisimple and nilpotent homogeneous elements.*

Proof. Let $\xi \in \mathfrak{g}_\alpha$ and let $\xi = \xi_s + \xi_n$ be the Jordan decomposition of ξ in the algebra $\mathfrak{g}$. The theorem will be proved if we show that $\xi_s, \xi_n \in \mathfrak{g}_\alpha$. Let $S = \mathfrak{X}(A)$ and suppose that $\varphi\colon S \to \operatorname{Aut}\mathfrak{g}$ is the homomorphism defined by the grading according to formula (15). Then for any $s \in S$ the decomposition $\varphi(s)(\xi) = \varphi(s)(\xi_s) + \varphi(s)(\xi_n)$ is the Jordan decomposition of the element $\varphi(s)(\xi) = s(\alpha)\xi$. Hence it follows that $\varphi(s)(\xi_s) = s(\alpha)\xi_s$ and $\varphi(s)(\xi_n) = s(\alpha)\xi_n$, i.e. $\xi_s, \xi_n \in \mathfrak{g}_\alpha$. □

Theorem 3.4. *For any nonzero nilpotent element $e \in \mathfrak{g}_\alpha$ there exist a semisimple element $h \in \mathfrak{g}_0$ and a nilpotent element $f \in \mathfrak{g}_{-\alpha}$ such that*

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{16}$$

The proof of the theorem and some useful refinements can be found in Vinberg [1979].

Corollary. *If $\mathfrak{g}_0 = 0$, then all homogeneous elements of the algebra $\mathfrak{g}$ are semisimple.*

On the other hand, the following proposition holds.

Proposition 3.5. *If $\alpha \in A$ is an element of infinite order, then all elements of the subspace $\mathfrak{g}_\alpha$ are nilpotent.*

Proof. For $\xi \in \mathfrak{g}_\alpha$ we have $\operatorname{ad}(\xi)\mathfrak{g}_\beta \subset \mathfrak{g}_{\beta+\alpha}$. Since only finitely many subspaces constituting the grading can be different from zero, for any $\beta \in A$ there is a positive integer k such that $\mathfrak{g}_{\beta+k\alpha} = 0$. This shows that the operator $\operatorname{ad}(\xi)$ is nilpotent. □

For more details on homogeneous semisimple and nilpotent elements of graded semisimple Lie algebras see Vinberg [1976] and [1979].

3.4. Fixed Points of Automorphisms. For any family S of automorphisms of an algebra $\mathfrak{g}$, denote by $\mathfrak{g}^S$ the subalgebra of its fixed points in $\mathfrak{g}$. If all automorphisms from S can be lifted to automorphisms of the group G (which we shall denote by the same letters), then we denote by G^S the subgroup of fixed points of the family S in the group G. Clearly, G^S is an algebraic subgroup and $\mathfrak{g}^S$ is its tangent algebra and therefore an algebraic subalgebra of $\mathfrak{g}$.

Proposition 3.6. *If $S \subset \operatorname{Aut}\mathfrak{g}$ is a reductive algebraic subgroup (for example, a quasitorus), then G^S is a reductive algebraic subgroup of G.*

Proof. Denote by $\mathfrak{g}_S$ the only S-invariant subspace complementary to $\mathfrak{g}^S$. It is generated by the elements of the form $s(\xi) - \xi$ $(s \in S, \xi \in \mathfrak{g})$. For $\eta \in \mathfrak{g}^S$ we have

$$(s(\xi) - \xi, \eta) = (\xi, s^{-1}(\eta) - \eta) = 0,$$

so $(\mathfrak{g}_S, \mathfrak{g}^S) = 0$. Hence it follows that the scalar product is nondegenerate on $\mathfrak{g}^S$, and this, by Proposition 6.2 of Chap. 1, means that G^S is a reductive subgroup. □

Theorem 3.7. *If $\mathfrak{g} \neq 0$ and $s \in \operatorname{Aut}\mathfrak{g}$ is a semisimple automorphism, then $\mathfrak{g}^s \neq 0$.*

Proof. Let $S \subset \operatorname{Aut}\mathfrak{g}$ be a quasitorus that is the closure of the cyclic subgroup $\langle s \rangle$. We set $A = \mathfrak{X}(S)$ and consider the A-grading of the algebra $\mathfrak{g}$ defined by the quasitorus S. In this grading we have $\mathfrak{g}_0 = \mathfrak{g}^s$. Suppose that $\mathfrak{g}^s = 0$. Making use of the corollary to Theorem 3.4 and Proposition 3.5, we deduce, first, that all homogeneous elements of the algebra $\mathfrak{g}$ are semisimple and, second, that A is a finite cyclic group. Let α be its generator, and p the least positive integer such that $\mathfrak{g}_{p\alpha} \neq 0$. For $\xi \in \mathfrak{g}_{p\alpha}$ we have $\operatorname{ad}(\xi)\mathfrak{g}_\beta \subset \mathfrak{g}_{\beta+p\alpha}$. For any $\beta \in A$ there is a positive integer k such that $\beta + kp\alpha = q\alpha$, where $0 \leq q < p$ and consequently $\mathfrak{g}_{q\alpha} = 0$. Therefore the operator $\operatorname{ad}(\xi)$ is nilpotent, which is impossible if $\xi \neq 0$. □

Note that the statement of the theorem also holds for any nonsemisimple automorphism θ. Indeed, such an automorphism is of the form $\theta = s \exp \operatorname{ad}(u)$, where $u \in \mathfrak{g}^s$ is a nonzero nilpotent element (the Jordan decomposition in the group $\operatorname{Aut}\mathfrak{g}$), and it is clear that $u \in \mathfrak{g}^\theta$.

Theorem 3.8. *If the group G is simply-connected and $s \in \operatorname{Aut}\mathfrak{g}$ $(= \operatorname{Aut} G)$ is a semisimple automorphism, then the subgroup G^s is connected.*

For the proof see, e.g., Steinberg [1968a] or Onishchik and Vinberg [1990]. Some of the arguments used in these proofs will be considered in Sect. 3.6.

In this connection we mention the following result by Rashevsky [1972]. Let G be any simply-connected (either complex or real) Lie group and $s \in \operatorname{Aut} \mathfrak{g}$ $(= \operatorname{Aut} G)$ a semisimple automorphism all of whose (complex) eigenvalues are equal to unity in absolute value. Then the subgroup G^s is connected.

3.5. One-dimensional Tori of Automorphisms and $\mathbb{Z}$-gradings. According to Sect. 3.2, defining a $\mathbb{Z}$-grading of an algebra $\mathfrak{g}$ is equivalent to defining a homomorphism $\varphi \colon \mathbb{C}^* \to \operatorname{Aut} \mathfrak{g}$. Such a homomorphism is defined by an element $h \in \mathfrak{g}$ for which $d\varphi(1) = \operatorname{ad}(h)$. This role can be played by any semisimple element satisfying the condition

$$\exp 2\pi i \operatorname{ad}(h) = \mathrm{id}. \tag{17}$$

Let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$, and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ a system of simple roots with respect to $\mathfrak{h}$. Acting by inner automorphisms, we can arrange that $h \in \mathfrak{h}$ and $\alpha_i(h) \geq 0$ for $i = 1, \dots, l$. Then condition (17) means that $\alpha_i(h) \in \mathbb{Z}$ for $i = 1, \dots, l$. We set $\alpha_i(h) = p_i$. One can easily see that the subspace $\mathfrak{g}_p$ $(p \in \mathbb{Z})$ of the grading is the sum of the root subspaces $\mathfrak{g}_\alpha$ corresponding to the roots $\alpha = \sum k_i \alpha_i$ with $\sum k_i p_i = p$ ($\mathfrak{g}_0$ also includes the Cartan subalgebra $\mathfrak{h}$).

Thus, a $\mathbb{Z}$-grading of the algebra $\mathfrak{g}$ is defined (up to an inner automorphism) by nonnegative integer labels assigned to the vertices of the Dynkin diagram. Two $\mathbb{Z}$-gradings can be taken into one another by an outer automorphism if and only if the corresponding sets of labels can be taken into one another by an automorphism of the Dynkin diagram.

In the above notation, let

$$\Pi_0 = \{\alpha_i \in \Pi \colon p_i = 0\}. \tag{18}$$

Then $\mathfrak{g}_0$ is a reductive subalgebra containing $\mathfrak{h}$ (for more on such subalgebras see Chap. 6, Sect. 1.2) with the system of simple roots Π_0. Its centre is

$$\mathfrak{z}_0 = \{z \in \mathfrak{h} \colon \alpha_i(z) = 0 \quad \text{for all} \quad \alpha_i \in \Pi_0\}. \tag{19}$$

Let us now describe linear representations of the algebra $\mathfrak{g}_0$ in the subspaces $\mathfrak{g}_p$ induced by the adjoint representation of the algebra $\mathfrak{g}$.

For this the following lemma will be useful.

Lemma 3.9. *Suppose that all weights of a linear representation ρ of a semisimple Lie algebra $\mathfrak{s}$ are of multiplicity 1 and are congruent modulo the root lattice. Then ρ is irreducible.*

Proof. It is sufficient to prove that the highest weight is unique. Suppose that there are two highest weights Λ, M. If $\{\gamma_1, \dots, \gamma_n\}$ is a system of simple roots of $\mathfrak{s}$, then

$$\Lambda - \sum_{i \in I} k_i \gamma_i = M - \sum_{j \in J} l_j \gamma_j = N,$$

where k_i, l_j are positive integers, and I, J are disjoint sets of indices. We will prove that N is a dominant linear form. For any s we have either $s \notin I$ or $s \notin J$. Suppose, to be definite, that $s \notin I$. Then

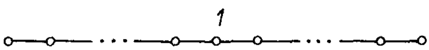
$$N(\gamma_s^\vee) = \Lambda(\gamma_s^\vee) - \sum_{i \in I} k_i \gamma_i(\gamma_s^\vee) \geq 0.$$

Therefore N is a weight of each of the irreducible representations with the highest weights Λ, M (see Proposition 2.2) and consequently is a weight of the representation ρ of multiplicity greater than 1. □

Decompose the space $\mathfrak{g}_p$ ($p \neq 0$) into the direct sum of the subspaces $\mathfrak{g}_p^{(\nu)}$, where each $\mathfrak{g}_p^{(\nu)}$ is the sum of root subspaces $\mathfrak{g}_\alpha$ corresponding to the roots $\alpha = \sum k_i \alpha_i$ with fixed coefficients at the simple roots $\alpha_i \notin \Pi_0$ (and with $\sum k_i p_i = p$). Clearly, each of the subspaces $\mathfrak{g}_p^{(\nu)}$ is invariant under $\mathfrak{g}_0$, and the elements of the centre $\mathfrak{z}_0$ act on each of them as scalars. Since weights of the representation $\mathfrak{g}_0$ in $\mathfrak{g}_p$ are roots of the algebra $\mathfrak{g}$, it follows from Lemma 3.9 applied to the commutator subalgebra of $\mathfrak{g}_0$ that the representation of the algebra $\mathfrak{g}_0$ in each $\mathfrak{g}_p^{(\nu)}$ is irreducible .

Note that the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_{-p}$ is dual to that in $\mathfrak{g}_p$.

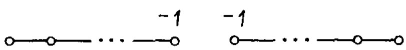
Example 1. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and suppose that the labels p_i are of the form



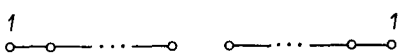
where zero labels are omitted, while a nonzero label is at the r-th vertex. In the matrix model the corresponding grading can be depicted as follows

$$\begin{array}{r l} r \; \{ & \begin{array}{|c|c|} \hline \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 \\ \hline \end{array} \\ n-r \; \{ & \end{array}$$

(For $|p| > 1$ we have $\mathfrak{g}_p = 0$.) The algebra $\mathfrak{g}_0$ has a one-dimensional centre and its semisimple part $\mathfrak{g}_0'$ is equal to $\mathfrak{sl}_r(\mathbb{C}) \oplus \mathfrak{sl}_{n-r}(\mathbb{C})$. The simple root α_r is clearly the lowest weight of the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$. Its labels on the diagram of simple roots of the algebra $\mathfrak{g}_0$ are of the form

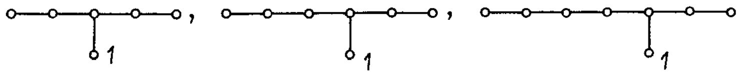


and, according to the general rule (see Sect. 2.6), the labels of the highest weight are obtained by multiplying them by -1 and applying the canonical involution of the Dynkin diagram, i.e. they are of the form



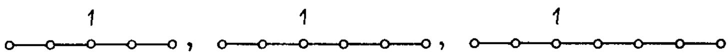
This means that the representation of $\mathfrak{g}_0'$ in $\mathfrak{g}_1$ is the tensor product of the identity representation of the algebra $\mathfrak{sl}_r(\mathbb{C})$ and the representation of the algebra $\mathfrak{sl}_{n-r}(\mathbb{C})$ dual to the identity, which is, however, evident from the matrix model.

Example 2. Let $\mathfrak{g}$ be a simple Lie algebra of type E_l ($l = 6, 7, 8$) and suppose that the labels p_i are of the form

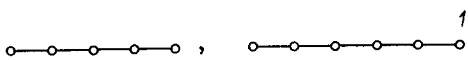


Then $\mathfrak{g}_0$ is an algebra with one-dimensional centre and semisimple part of type A_{l-1}, i.e., $\mathfrak{g}_0 \simeq \mathfrak{gl}(V)$, where $\dim V = l$. The space $\mathfrak{g}_p$ for $p > 0$ is the sum of the root subspaces $\mathfrak{g}_\alpha$ corresponding to the positive roots $\alpha = \sum k_i \alpha_i$ with $k_i = p$, where α_l is the distinguished simple root. The greatest value of p for which $\mathfrak{g}_p \neq 0$ is equal to the coefficient of α_l in the decomposition of the highest root, i.e., 2 for $l = 6, 7$ and 3 for $l = 8$ (see Table 3). Since α_l is the only simple root with the nonzero label, the representation of the algebra $\mathfrak{g}_0$ in each of the subspaces $\mathfrak{g}_p$ ($p \neq 0$) is irreducible.

Clearly, α_l is the lowest weight of the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$. As in Example 3.2, we obtain the following labels of the highest weight of the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$:



Hence it follows that the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ is isomorphic to the natural representation $\Lambda^3\mathrm{id}$ of the algebra $\mathfrak{gl}(V)$ in $\Lambda^3 V$ (see Example 2 of Sect. 2.2). For $l = 6, 7$ the highest root of the algebra $\mathfrak{g}$ is the highest weight of the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_2$. Its labels are of the form



This means that the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_2$ is isomorphic to the natural representation of $\mathfrak{gl}(V)$ in $\Lambda^6 V$. One can show that the same holds for $l = 8$, and the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_3$ in this case is isomorphic to the natural representation of $\mathfrak{gl}(V)$ in $V \otimes \Lambda^8 V$ (its highest weight is the highest root of the algebra $\mathfrak{g}$).

3.6. Canonical Form of an Inner Semisimple Automorphism. Two automorphisms of an algebra $\mathfrak{g}$ are said to be conjugate (respectively, conjugate in $\mathrm{Aut}\,\mathfrak{g}$) if they are conjugate by means of an element from $\mathrm{Int}\,\mathfrak{g}$ (respectively, from $\mathrm{Aut}\,\mathfrak{g}$). In this and the ensuing sections we will give the classification of semisimple automorphisms up to conjugacy.

We begin with a general remark on the reduction to the case of a simple algebra. If θ is an arbitrary automorphism of a semisimple Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ uniquely decomposes into the direct sum of θ-invariant subalgebras such that the automorphism θ cyclically permutes simple components of each of them. It is therefore sufficient to consider the case when $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$, where $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ are simple algebras and $\theta(\mathfrak{g}_i) = \mathfrak{g}_{i+1}$ (assuming that $\mathfrak{g}_{k+1} = \mathfrak{g}_1$). Then $\theta^k(\mathfrak{g}_i) = \mathfrak{g}_i$ and, if we set $\theta^k|_{\mathfrak{g}_1} = \theta_1$, then the pair $(\mathfrak{g}, \theta)$ is uniquely defined by the triple $(\mathfrak{g}_1, \theta_1, k)$, where one can take for θ_1 any automorphism of $\mathfrak{g}_1$. It is also clear that θ is semisimple if and only if so is θ_1.

Identifying the tangent algebra $\operatorname{ad}(\mathfrak{g})$ of the group $\operatorname{Int}\mathfrak{g}$ with $\mathfrak{g}$, we shall consider this group as the group G for which $\mathfrak{g}$ is its tangent algebra.

Let H be a Cartan subgroup (a maximal torus) of the group $G = \operatorname{Int}\mathfrak{g}$. Then any inner semisimple automorphism of $\mathfrak{g}$ is conjugate to an element of H (see Springer [1989], Chap. 1, Sect. 3.5.3). Two elements of H are conjugate if and only if they are equivalent under the action of the Weyl group $W = N/H$, where $N = N(H)$ is the normalizer of H in G.

In order to make this description more effective, consider the mapping

$$\mathcal{E}\colon \mathfrak{h} \to H, \quad x \mapsto \exp 2\pi i x. \tag{20}$$

Its kernel is the lattice $P^\vee$ dual to the root lattice Q of $\mathfrak{g}$ (see Corollary 2 to Theorem 2.8). It contains the lattice $Q^\vee$ dual to the weight lattice P of $\mathfrak{g}$ and generated by the elements $h_1, \ldots, h_l$, where the quotient group $P^\vee/Q^\vee$ is canonically isomorphic to $\pi_1(G)$ (Theorem 2.9).

Two elements $\mathcal{E}(x), \mathcal{E}(y) \in H$ are equivalent under the action of the group W if and only if the elements $x, y \in \mathfrak{h}$ are equivalent under the action of the group $\widetilde{W} = P^\vee \rtimes W$, where elements of the lattice $P^\vee$ act as parallel translations of the space $\mathfrak{h}$. We will now give a description of the latter group.

Proposition 3.10. (1) *The group $\widetilde{W}_r = Q^\vee \rtimes W$ considered as a group of motions of the Euclidean space $\mathfrak{h}(\mathbb{R})$ is a discrete group generated by reflections.*

(2) *If the algebra $\mathfrak{g}$ is simple and δ is its highest root, then the simplex*

$$D = \{x \in \mathfrak{h}(\mathbb{R})\colon \alpha_i(x) \geq 0 \ (i = 1, \ldots, l),\ \delta(x) \leq 1\} \tag{21}$$

is a fundamental polyhedron of the group $\widetilde{W}_r$ in $\mathfrak{h}(\mathbb{R})$.

(3) *The group $\widetilde{W} = P^\vee \rtimes W$ can be represented as the semidirect product $\widetilde{W}_r \rtimes \Gamma$, where $\Gamma \simeq \pi_1(G)$ is a finite group of symmetries of a fundamental polyhedron of the group $\widetilde{W}_r$.*

For the proof see, e.g., Onishchik and Vinberg [1990]. On discrete groups generated by reflections see Vinberg and Shvartsman [1988].

The group $\widetilde{W}_r$ is called the *extended Weyl group*, and its fundamental polyhedron lying in the Weyl chamber and containing zero is called the *restricted Weyl chamber* of the algebra $\mathfrak{g}$ (if $\mathfrak{g}$ is simple, this polyhedron is the simplex D described in Proposition 3.10).

Suppose that $\mathfrak{g}$ is simple and $\alpha_0 = -\delta$ is its lowest root. Recall (see Sect. 1.7) that the extended system of simple roots $\widetilde{\Pi} = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ is an admissible system of vectors and that its metric properties are described by the so-called extended Dynkin diagram of $\mathfrak{g}$. The elements of the system $\widetilde{\Pi}$ satisfy a linear relation with positive integer coefficients $n_0, n_1, \ldots, n_l$, where $n_0 = 1$. The extended Dynkin diagrams of the simple Lie algebras together with the coefficients $n_0, n_1, \ldots, n_l$ are listed in Table 3.

Any automorphism of the system Π can be extended to an automorphism of $\widetilde{\Pi}$ in such a way that $\operatorname{Aut}\Pi$ turns out to be a subgroup of $\operatorname{Aut}\widetilde{\Pi}$. The group Γ mentioned in Proposition 3.10 is naturally identified with a normal subgroup of $\operatorname{Aut}\widetilde{\Pi}$ acting simply transitively on the set of those roots $\alpha_i \in \widetilde{\Pi}$ for which $n_i = 1$. Then $\operatorname{Aut}\widetilde{\Pi} = \Gamma \rtimes \operatorname{Aut}\Pi$.

The element $x \in \mathfrak{h}$ can be defined by the numbers

$$x_0 = 1 - \delta(x), \quad x_1 = \alpha_1(x), \ldots, x_l = \alpha_l(x), \tag{22}$$

satisfying the condition

$$\sum_{i=0}^{l} n_i x_i = 1. \tag{23}$$

Admitting a certain abuse of language, we will call them the *barycentric coordinates* of x. Elements of the chamber D are characterized by the property that their barycentric coordinates are real and nonnegative, and the elements of the lattice $P^\vee$ by the property that their barycentric coordinates are integers.

It follows from the theory of groups generated by reflections that each element of the Cartan subalgebra $\mathfrak{h}$ is equivalent, under the action of the group $\widetilde{W}_r$, to the unique element x whose barycentric coordinates $x_0, x_1, \ldots, x_l$ satisfy the conditions:

(P1) $\operatorname{Re} x_i \geq 0$ for all i;

(P2) if $\operatorname{Re} x_i = 0$ for some i, then $\operatorname{Im} x_i \geq 0$.

If one is interested in equivalence with respect to the action of the group $\widetilde{W} = \widetilde{W}_r \rtimes \Gamma$, permutations of the barycentric coordinates belonging to the group Γ should also be allowed.

This yields the following classification of inner semisimple automorphisms.

Theorem 3.11. *Any inner semisimple automorphism of a simple Lie algebra* $\mathfrak{g}$ *is conjugate to an automorphism of the form* $\exp 2\pi i x$, *where* x *is the element of the Cartan subalgebra* $\mathfrak{h}$ *whose barycentric coordinates* $x_0, x_1, \ldots, x_l$ *satisfy conditions* (P1) *and* (P2). *Two automorphisms of such a form are conjugate if and only if the barycentric coordinates of the first can be obtained from those of the second by a permutation belonging to the group* Γ.

Note that, by ignoring the action of the group Γ, we obtain a classification of semisimple elements of the simply-connected Lie group having $\mathfrak{g}$ for its Lie algebra. On the other hand, considering, in place of Γ, the entire group $\operatorname{Aut}\widetilde{\Pi}$, we get a classification of inner semisimple automorphisms up to conjugacy in the group $\operatorname{Aut}\mathfrak{g}$.

3.7. Inner Automorphisms of Finite Order and $\mathbb{Z}_m$-gradings of Inner Type. Defining a $\mathbb{Z}_m$-grading of an algebra $\mathfrak{g}$ is equivalent to defining a homomorphism of a cyclic group of order m into $\operatorname{Aut}\mathfrak{g}$ or, which is the same, of an automorphism $\theta \in \operatorname{Aut}\mathfrak{g}$ satisfying the condition $\theta^m = \mathrm{id}$. In this section we consider the case when the automorphism θ is an inner one. We will call the corresponding gradings $\mathbb{Z}_m$-gradings of *inner type*.

According to Theorem 3.11, one can assume that $\theta = \exp 2\pi i x$, where x is an element of the Cartan subalgebra $\mathfrak{h}$ whose barycentric coordinates $x_0, x_1, \ldots, x_l$ satisfy conditions (P1) and (P2). The additional condition $\theta^m = \mathrm{id}$ implies that $x_i = \frac{p_i}{m}$, $p_i \in \mathbb{Z}$, $p_i \geq 0$ $(i = 0, 1, \ldots, l)$ and that condition (23) can be rewritten in the form

$$\sum_{i=0}^{l} n_i p_i = m. \tag{24}$$

Thus, a $\mathbb{Z}_m$-grading of inner type can be defined by the extended diagram of simple roots of the algebra $\mathfrak{g}$ with integer nonnegative labels p_i satisfying condition (24). Such a diagram is called a *Kac diagram*.

Two $\mathbb{Z}_m$-gradings can be taken into each other by an automorphism (respectively, inner automorphism) of the algebra $\mathfrak{g}$ if and only if their Kac diagrams can be transformed into each other by an automorphism of an extended Dynkin diagram (respectively, by a permutation belonging to the group Γ, see Proposition 3.10).

As we have already done in Sect. 3.5 for $\mathbb{Z}$-gradings, one can give simple rules for finding the subalgebra $\mathfrak{g}_0$ and its representation in the spaces $\mathfrak{g}_p$, $1 \leq p \leq m-1$ (here we consider the integers $0, 1, \ldots, m-1$ as elements of the group $\mathbb{Z}_m$) from the Kac diagram of a $\mathbb{Z}_m$-grading.

One can easily see that any root α of $\mathfrak{g}$ can be uniquely represented in the form

$$\alpha = \sum_{i=0}^{l} k_i \alpha_i, \quad 0 \leq k_i \leq n_i. \tag{25}$$

(If $\alpha > 0$, then $k_0 = 0$, and if $\alpha < 0$, then $k_0 = 1$.) The subspace $\mathfrak{g}_p$, $1 \leq p \leq m-1$, is the sum of the root subspaces $\mathfrak{g}_\alpha$ corresponding to the roots α with $\sum_{i=0}^{l} k_i p_i = p$. The subalgebra $\mathfrak{g}_0$ is the sum of the Cartan subalgebra $\mathfrak{h}$ and the root subspaces $\mathfrak{g}_\alpha$ corresponding to the roots α with $\sum_{i=0}^{l} k_i p_i = 0$ or m. We set

$$\widetilde{\Pi}_0 = \{\alpha_i \in \widetilde{\Pi} : p_i = 0\}; \tag{26}$$

then $\mathfrak{g}_0$ is the reductive subalgebra with Cartan subalgebra $\mathfrak{h}$ and the system of simple roots $\widetilde{\Pi}_0$. Its centre is of the form

$$\mathfrak{z}_0 = \{z \in \mathfrak{h} : \alpha_i(z) = 0 \quad \text{for all} \quad \alpha_i \in \widetilde{\Pi}_0\}, \tag{27}$$

so the dimension of $\mathfrak{z}_0$ is one less than the total number of nonzero labels.

Decompose the space $\mathfrak{g}_p$ $(1 \leq p \leq m-1)$ into the direct sum of subspaces $\mathfrak{g}_p^{(\nu)}$, where each $\mathfrak{g}_p^{(\nu)}$ is the sum of root subspaces $\mathfrak{g}_\alpha$ corresponding to the roots α with fixed coefficients of the roots $\alpha_i \notin \widetilde{\Pi}_0$ in the decomposition (25) (and with $\sum_{i=0}^{l} k_i p_i = p$). Each of the subspaces $\mathfrak{g}_p^{(\nu)}$ is invariant under $\mathfrak{g}_0$ with elements of the centre $\mathfrak{z}_0$ acting on it as scalars. It follows from Lemma 3.9

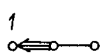
that the representation of $\mathfrak{g}_0$ in each $\mathfrak{g}_p^{(\nu)}$ is irreducible. The representation of $\mathfrak{g}_0$ in $\mathfrak{g}_{m-p}$ $(= \mathfrak{g}_{-p})$ is dual to the representation in $\mathfrak{g}_p$.

Note that for a description of the subspaces $\mathfrak{g}_p$ one may consider root representations of the form $\alpha = \sum\limits_{i=0}^{l} k_i\alpha_i$, where $k_i \in \mathbb{Z}$ but not necessarily $0 \leq k_i \leq n_i$. Then, in view of (24), $\mathfrak{g}_\alpha \subset \mathfrak{g}_p$ if $\sum\limits_{i=0}^{l} k_i p_i \equiv p \pmod{m}$.

The order of the automorphism θ is $\frac{m}{d}$, where d is the greatest common divisor of the labels $p_0, p_1, \ldots, p_l$. If $d > 1$, then the $\mathbb{Z}_m$-grading is "fictitious" and actually reduces to a $\mathbb{Z}_{m/d}$-grading.

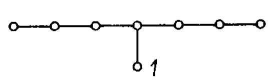
Example 1. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $m = 2$. Then, except for the fictitious $\mathbb{Z}_2$-grading, two of the labels p_i must equal 1 and the rest of them vanish. In this case the group Γ consists of all rotations of the extended Dynkin diagram, so we can assume that $p_0 = 1$. Suppose, in addition, that $p_r = 1$. Then the structure of the subalgebra $\mathfrak{g}_0$ is the same as that in Example 1 of Sect. 3.5, and the subspace $\mathfrak{g}_1$ is the sum of subspaces $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ of the same example. The representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ can be decomposed into the sum of two irreducible representations with lowest weights α_0 and α_r. Since the pair of roots $\{\alpha_0, \alpha_r\}$ can be taken into the pair $\{\alpha_{n-r}, \alpha_0\}$ by a rotation of the Dynkin diagram, the corresponding $\mathbb{Z}_2$-gradings can be transformed into each other by an inner isomorphism.

Example 2. Let $\mathfrak{g}$ be a simple Lie algebra of type G_2. The Kac diagram



defines a $\mathbb{Z}_3$-grading of the algebra $\mathfrak{g}$ for which $\mathfrak{g}_0$ is a simple Lie algebra of type A_2, i.e. $\mathfrak{g}_0 \simeq \mathfrak{sl}(V)$, where $\dim V = 3$. The labelled simple root serves as the lowest weight of the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$, so the space $\mathfrak{g}_1$ may be identified with V, and the space $\mathfrak{g}_{-1}$ with V^*, in such a way that the representations of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ coincide with the natural representations of the algebra $\mathfrak{sl}(V)$ in V and V^*, respectively.

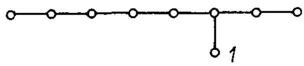
Example 3. Let $\mathfrak{g}$ be a simple Lie algebra of type E_7, and $m = 2$. Suppose that the labels p_i are of the form



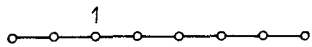
Then $\mathfrak{g}_0$ is a simple Lie algebra of type A_7. Since $p_7 = 1$ is the only nonzero label, the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ is irreducible. Therefore the labels of the highest weight of this representation are of the form

This means that if one identifies $\mathfrak{g}_0$ with the algebra $\mathfrak{sl}(V)$, where $\dim V = 8$, the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ becomes isomorphic to the natural representation of $\mathfrak{sl}(V)$ in $\Lambda^4 V$.

Example 4. Let $\mathfrak{g}$ be a simple Lie algebra of type E_8, $m = 3$. Suppose that the labels p_i are of the form



Then $\mathfrak{g}_0$ is a simple Lie algebra of type A_8, i.e. $\mathfrak{g}_0 \simeq \mathfrak{sl}(V)$, where $\dim V = 9$. The labels of the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ are of the form



This representation is isomorphic to the natural representation of $\mathfrak{sl}(V)$ in $\Lambda^3 V$, and the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_2 = \mathfrak{g}_{-1}$ is isomorphic to the natural representation of $\mathfrak{sl}(V)$ in the dual space $\Lambda^3 V^*$.

3.8. Quasitorus Associated with a Component of the Group of Automorphisms. The classification of outer semisimple automorphisms can be obtained on the same lines as that of inner ones, but requires a modification of the notions of a Cartan subgroup and the root decomposition. This method goes back to F.R. Gantmakher [1939], to whom the majority of main results of this section are due.

Fix an element $\sigma \in \operatorname{Aut} \Pi$ and consider the connected component σG of the group $\operatorname{Aut} \mathfrak{g}$. Let

$$l(\sigma) = \min_{s \in \sigma G} \dim \mathfrak{g}^s; \tag{28}$$

an automorphism $s \in \sigma G$ will be said to be regular if $\dim \mathfrak{g}^s = l(\sigma)$.

Let $s \in \sigma G$ be a semisimple automorphism. Consider a Cartan subgroup $H^{(\sigma)}$ of the group G^s and the quasitorus $S^{(\sigma)}$ generated by $H^{(\sigma)}$ and s. Set $M^{(\sigma)} = sH^{(\sigma)}$ and denote by $N^{(\sigma)}$ the normalizer of $M^{(\sigma)}$ in G (*a priori* it is a subgroup of finite index of the normalizer of $S^{(\sigma)}$ in G). The group $W^{(\sigma)} = N^{(\sigma)}/H^{(\sigma)}$ acts naturally on $M^{(\sigma)}$.

Theorem 3.12. (1) *The subset $M^{(\sigma)} \subset \sigma G$, and therefore the subgroups $H^{(\sigma)}$ and $S^{(\sigma)}$, depend, up to conjugacy, only on the element $\sigma \in \operatorname{Aut} \Pi$.*

(2) $\dim H^{(\sigma)} (= \dim S^{(\sigma)}) = l(\sigma)$.

(3) *Regular semisimple automorphisms form a nonempty open (in the Zariski topology) subset in σG.*

(4) *Any semisimple automorphism from σG is conjugate to an automorphism from $M^{(\sigma)}$.*

(5) *Two automorphisms from $M^{(\sigma)}$ are conjugate if and only if they are equivalent under the action of the group $W^{(\sigma)}$.*

Proof. Consider the weight decomposition of the algebra $\mathfrak{g}$ with respect to the quasitorus $S^{(\sigma)}$. In this decomposition $\mathfrak{g}_0 = \mathfrak{h}^{(\sigma)}$ (which is the tangent algebra of $H^{(\sigma)}$). Denote by $M_0^{(\sigma)}$ a (nonempty) open subset of $M^{(\sigma)}$ consisting of elements on which none of the nonzero weights reduces to one. For each $t \in M_0^{(\sigma)}$ we have $\mathfrak{g}^t = \mathfrak{g}_0 = \mathfrak{h}^{(\sigma)}$. One can easily deduce that the differential of the mapping

$$G \times M_0^{(\sigma)} \to \sigma G, \quad (g, t) \mapsto gtg^{-1},$$

is surjective at each point. Therefore the set of automorphisms of the form gtg^{-1} ($g \in G, t \in M_0^{(\sigma)}$) is open in σG. The same argument applied to another semisimple automorphism $\tilde{s} \in \sigma G$ shows that there exists an element $g \in G$ such that $g\widetilde{M}_0^{(\sigma)} g^{-1} \cap M_0^{(\sigma)} \neq \emptyset$, but then $g\widetilde{M}^{(\sigma)} g^{-1} = M^{(\sigma)}$. This implies statements (1)–(4) of the theorem. Statement (5) follows from the conjugacy of maximal tori in the group G^s. □

The following theorem provides a precise description of the torus $H^{(\sigma)}$.

Theorem 3.13. *The centralizer of the torus $H^{(\sigma)}$ in G is a Cartan subgroup H invariant under s and satisfying the condition $H^{(\sigma)} = H^s$. There exists a system of simple roots of the algebra $\mathfrak{g}$ with respect to H such that the action of s on $\mathfrak{h}$ is defined by the formulae*

$$sh_i = h_{\sigma i} \quad (i = 1, \dots, l). \tag{29}$$

Proof. By virtue of Theorem 3.12, the validity of the statements of the theorem does not depend on the choice of a semisimple automorphism $s \in \sigma G$. Let H and $\{\alpha_1, \dots, \alpha_l\}$ be a Cartan subgroup and a system of simple roots with which the embedding $\operatorname{Aut} \Pi \subset \operatorname{Aut} \mathfrak{g}$ is associated (see Sect. 3.1). Consider the element $u \in \mathfrak{h}$ defined by the conditions $\alpha_i(u) = 1$ $(i = 1, \dots, l)$. Clearly, $\sigma(u) = u$. Let $s = \sigma \exp tu$ ($t \in \mathbb{R}$, $t > 0$), where t is so large that all eigenvalues of s in the invariant subspace $\bigoplus\limits_{\alpha>0} \mathfrak{g}_\alpha$ (respectively, $\bigoplus\limits_{\alpha<0} \mathfrak{g}_\alpha$) are greater (respectively, less) than one in absolute value. Then $\mathfrak{g}^s = \mathfrak{h}^s = \mathfrak{h}^\sigma$, so $H^{(\sigma)}$ coincides with $(G^s)^0$ and has $\mathfrak{h}^\sigma$ for its tangent algebra. Since $\mathfrak{h}^\sigma$ contains a regular element u of the Cartan subalgebra $\mathfrak{h}$, we have $Z(H^{(\sigma)}) = H$. Formulae (29) hold by the definition of σ.

It remains to verify that the subgroup $H^s = H^\sigma$ is connected. Consider the mapping $\mathcal{E}\colon \mathfrak{h} \to H$ defined by formula (20). Its kernel is the lattice $P^\vee$ with basis $\{u_1, \dots, u_l\}$, where $\alpha_i(u_j) = \delta_{ij}$. Clearly, $\sigma(u_j) = u_{\sigma j}$. Let $\mathcal{E}\left(\sum\limits_j c_j u_j\right) \in H^\sigma$. Then

$$\sum_j c_j u_{\sigma j} - \sum_j c_j u_j = \sum_j (c_{\sigma^{-1} j} - c_j) u_j \in P^\vee,$$

i.e. $c_{\sigma j} \equiv c_j \pmod{\mathbb{Z}}$ for all j. Therefore there exist $c'_j \equiv c_j \pmod{\mathbb{Z}}$ such that $\sum\limits_j c'_j u_j \in \mathfrak{h}^\sigma$. Thus, $H^\sigma = \exp \mathfrak{h}^\sigma$. □

Corollary 1. *For any semisimple automorphism s of the algebra $\mathfrak{g}$ there exist a Cartan subgroup and a Borel subgroup containing it, that are both invariant under s.*

Corollary 2. *The dimension of $H^{(\sigma)}$ equals the number of orbits of the action of the cyclic group $\langle\sigma\rangle$ on the system of simple roots.*

Corollary 3. *$H^{(\sigma)} = S^{(\sigma)} \cap G$ and $M^{(\sigma)} = S^{(\sigma)} \cap \sigma G$.*

Corollary 4. *$N^{(\sigma)}$ coincides with the normalizer of $S^{(\sigma)}$ in G.*

Corollary 5. *The group $W^{(\sigma)}$ acts effectively on $M^{(\sigma)}$.*

Since $N(S^{(\sigma)}) \subset N(H^\sigma) \subset N(H)$, there is a natural homomorphism $W^{(\sigma)} \to W$, where W is the Weyl group of $\mathfrak{g}$ with respect to H. One can easily see that its image coincides with the subgroup W^σ of fixed points of W under the natural action of σ. Its kernel is the subgroup $W_0^{(\sigma)}$ of fixed points of the automorphism σ in H/H^σ, which is isomorphic to $\mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_\nu}$, where $k_1, \ldots, k_\nu$ are the orders of nontrivial orbits of the group $\langle\sigma\rangle$ on the system of simple roots. For more information on the group $W^{(\sigma)}$ see Sect. 3.9.

3.9. Generalized Root Decomposition. The weight decomposition of the algebra $\mathfrak{g}$ with respect to the quasitorus $S^{(\sigma)}$ constructed in the preceding section possesses many important properties of the classical root decomposition. We will call it the *σ-root decomposition*.

The quasitorus $S^{(\sigma)}$ is the direct product of the torus $H^{(\sigma)} = H^\sigma$ of dimension $l(\sigma)$ and the cyclic group $\langle\sigma\rangle$, the order of which will be denoted by q. The group $\mathfrak{X}(S^{(\sigma)})$ is consequently the direct product of the group $\mathfrak{X}(H^\sigma) \simeq \mathbb{Z}^{l(\sigma)}$ and the group $\mathbb{Z}_q$ (whose elements will be written as integers $0, 1, \ldots, q-1$). There is a homomorphism $\mathfrak{X}(S^{(\sigma)}) \to \mathfrak{X}(H^\sigma)$ associating with each character $\lambda \in \mathfrak{X}(S^{(\sigma)})$ its restriction to H^σ, which will be denoted by $\overline{\lambda}$. Thus any character $\lambda \in \mathfrak{X}(S^{(\sigma)})$ is a pair $(\overline{\lambda}, k)$, where $\overline{\lambda} \in \mathfrak{X}(H^\sigma)$, and $k \in \mathbb{Z}_q$.

Nonzero characters $\alpha \in \mathfrak{X}(S^{(\sigma)})$ for which $\mathfrak{g}_\alpha \neq 0$ will be called *σ-roots* of the algebra $\mathfrak{g}$ (with respect to $S^{(\sigma)}$). A σ-root α will be called *real* if $\overline{\alpha} \neq 0$, and *imaginary* otherwise (so that there are at most $q-1$ imaginary σ-roots). In the case $q = 1$ we obtain the usual root decomposition whose all roots are real.

The system of all σ-roots will be denoted by $\Delta^{(\sigma)}$, and the system of all real σ-roots by $\Delta_{\mathrm{re}}^{(\sigma)}$.

As in the classical situation (see Sect. 1.2), each real σ-root $\alpha = (\overline{\alpha}, k)$ is associated with a simple three-dimensional subalgebra $\mathfrak{g}^{(\alpha)} = \langle e_\alpha, f_\alpha, h_\alpha\rangle$, where $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in \mathfrak{h}_\alpha^{(\sigma)}$. The corresponding subgroup $G^{(\alpha)}$ of G contains the element n_α acting on $\mathfrak{h}^{(\sigma)}$ as a reflection of the following form:

$$R_{\overline{\alpha}}: x \mapsto x - \overline{\alpha}(x) h_\alpha. \tag{30}$$

(The element h_α depends only on $\overline{\alpha}$.) Considering the action of σ on $\mathfrak{g}^{(\alpha)}$, it is easy to deduce that

$$n_\alpha \sigma n_\alpha^{-1} = \sigma \exp\left(-2\pi i \frac{kh_\alpha}{q}\right). \tag{31}$$

Hence it follows that $n_\alpha \in N^{(\sigma)}$. Formulae (30) and (31) define the action of the element $r_\alpha = n_\alpha H^\sigma \in W^{(\sigma)}$ on $M^{(\sigma)}$.

(In contrast to the classical situation, the elements r_α, $\alpha \in \Delta_{\mathrm{re}}^{(\sigma)}$, do not necessarily generate the group $W^{(\sigma)}$, but in any case they do generate a group that is very close to $W^{(\sigma)}$.)

An examination of the adjoint action of the subalgebras $\mathfrak{g}^{(\alpha)}$ on $\mathfrak{g}$ yields the following properties of a σ-root decomposition.

Theorem 3.14. (1) $\dim \mathfrak{g}_\alpha = 1$ *for* $\alpha \in \Delta_{\mathrm{re}}^{(\sigma)}$.

(2) *If* $\alpha \in \Delta_{\mathrm{re}}^{(\sigma)}$, *then* $k\alpha \notin \Delta_{\mathrm{re}}^{(\sigma)}$ *for* $k \neq \pm 1$.

(3) *The set* $\overline{\Delta}^{(\sigma)}$ *of restrictions to* H^σ *of real* σ*-roots (ignoring their multiplicities) is a root system in the sense of the definition of Sect.* 1.1 (*this system is not necessarily reduced*).

The system of simple roots of the system $\overline{\Delta}^{(\sigma)}$ will be called the *system of simple σ-roots* of the algebra $\mathfrak{g}$, denoted by $\Pi^{(\sigma)}$.

Now suppose that $\mathfrak{g}$ is a simple Lie algebra. Then the element $\sigma \in \operatorname{Aut} \Pi$, $\sigma \neq \mathrm{id}$, is defined up to conjugacy in the group $\operatorname{Aut}\Pi$ by its order $q = 2$ or 3. Let $\mathfrak{g} = \bigoplus_k \mathfrak{g}_k$ be the $\mathbb{Z}_q$-grading of $\mathfrak{g}$ defined by the automorphism σ. The analysis of all possible cases shows that the algebra $\mathfrak{g}^\sigma = \mathfrak{g}_0$ is always simple, its representations in the spaces $\mathfrak{g}_k$ are irreducible, and in the case $q = 3$ (which is possible only if $\mathfrak{g}$ is an algebra of type D_4) the representations of $\mathfrak{g}^\sigma$ in $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are isomorphic. In addition, the highest weight η of the representation of $\mathfrak{g}^\sigma$ in $\mathfrak{g}_1$ is either a root or a double root of $\mathfrak{g}^\sigma$.

Hence it follows that $\Pi^{(\sigma)}$ is a system of simple roots of $\mathfrak{g}^\sigma$ and that if $\Pi^{(\sigma)} = \{\beta_1, \ldots, \beta_{l(\sigma)}\}$, then

$$\eta = \sum_{i=1}^{l(\sigma)} n_i' \beta_i, \quad n_i' \in \mathbb{Z}, \quad n_i' > 0.$$

Let us identify the roots β_i $(i = 1, \ldots, l(\sigma))$ of the algebra $\mathfrak{g}^\sigma$ with the σ-roots $(\beta_i, 0)$ of the algebra $\mathfrak{g}$ and adjoin to them the σ-root

$$\beta_0 = (-\eta, 1).$$

The resulting system $\widetilde{\Pi}^{(\sigma)} = \{\beta_0, \beta_1, \ldots, \beta_{l(\sigma)}\}$ of σ-roots will be called the *extended system of simple σ-roots* of the algebra $\mathfrak{g}$. It is an admissible system of vectors (see Sect. 1.7). There is a unique (up to a scalar multiple) linear relation

$$\sum_{i=0}^{l(\sigma)} n_i \beta_i = 0, \tag{32}$$

where $n_0 = q$, $n_i = qn_i'$ $(i = 1, \ldots, l(\sigma))$. One can show that any σ-root α

can be uniquely represented in the form

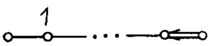
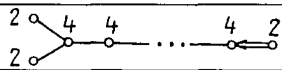
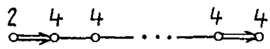
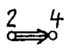
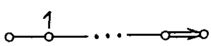
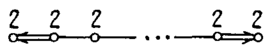
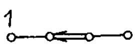
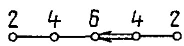
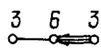
$$\alpha = \sum_{i=0}^{l(\sigma)} k_i \beta_i, \quad 0 \le k_i \le n_i. \tag{33}$$

Imaginary σ-roots are multiples of

$$\zeta = \sum_{i=0}^{l(\sigma)} n_i' \beta_i \quad (\text{where } n_0' = 1).$$

The following table lists, for each pair $(\mathfrak{g}, \sigma)$, the Dynkin diagram of the system $\Pi^{(\sigma)}$ of simple σ-roots and that of the extended system $\widetilde{\Pi}^{(\sigma)}$ of simple σ-roots. The first of them carries the labels of the highest weight η of the representation of $\mathfrak{g}^\sigma$ in $\mathfrak{g}_1$, and the second the coefficients n_i of the linear relation (32). (The scalar products of characters of $S^{(\sigma)}$ are defined as scalar products of their restrictions on H^σ.) The notation used for the weight η is standard (see Table 1 at the end of the book).

Let us now give a precise description of the *generalized Weyl group* $W^{(\sigma)}$ (see Sect. 3.8 for the definition). The table shows that the root system $\overline{\Delta}^{(\sigma)}$ has no outer automorphisms. Therefore any element of the group $W^{(\sigma)}$ acts on H^σ as an element of the Weyl group $W(\mathfrak{g}^\sigma)$ of the algebra $\mathfrak{g}^\sigma$. On the other hand, the latter group is naturally embedded in $W^{(\sigma)}$. Thus the homomorphism $W^{(\sigma)} \to W$ described at the end of Sect. 3.8 maps the group $W(\mathfrak{g}^\sigma)$ onto W^σ isomorphically, and the group $W^{(\sigma)}$ is the direct product of this group and the group $W_0^{(\sigma)} = (H/H^\sigma)^\sigma$.

Type of $\mathfrak{g}$	q	Type of $\overline{\Delta}^{(\sigma)}$	$\Pi^{(\sigma)}$	η	$\widetilde{\Pi}^{(\sigma)}$
A_{2n-1} $(n \ge 3)$	2	C_n		$\varepsilon_1 + \varepsilon_2$	
A_{2n} $(n \ge 2)$	2	BC_n		$2\varepsilon_1$	
A_2	2	BC_1		$4\varepsilon_1$	
D_{n+1} $(n \ge 2)$	2	B_n		ε_1	
E_6	2	F_4		ε_1	
D_4	3	G_2		ε_1	

3.10. Canonical Form of an Outer Semisimple Automorphism. Theorem 3.12 and the analysis of the generalized root decomposition carried out in Sect. 3.9 make is possible to obtain a classification of outer semisimple automorphisms, which is similar to the classification of inner ones given by Theorem 3.11.

Consider the covering

$$\mathcal{E}_\sigma : \mathfrak{h}^\sigma \to M^{(\sigma)}, \quad x \mapsto \sigma \exp 2\pi i x, \tag{34}$$

and denote by $\widetilde{W}^{(\sigma)}$ the group of all (affine) transformations of the space $\mathfrak{h}^\sigma$ covering transformations from the group $W^{(\sigma)}$ acting on $M^{(\sigma)}$. By virtue of Theorem 3.12, two automorphisms $\mathcal{E}_\sigma(x), \mathcal{E}_\sigma(y) \in M^{(\sigma)}$ are conjugate if and only if the elements $x, y \in \mathfrak{h}^\sigma$ are equivalent with respect to the group $\widetilde{W}^{(\sigma)}$.

Formulae (30) and (31) imply that the transformation $r_\alpha \in W^{(\sigma)}$, $\alpha = (\overline{\alpha}, k) \in \Delta_{\mathrm{re}}^{(\sigma)}$, is covered by the affine transformation

$$R_\alpha : x \mapsto x - \left(\overline{\alpha}(x) + \frac{k}{q}\right) h_\alpha, \tag{35}$$

which is the reflection in the hyperplane $\overline{\alpha}(x) + \frac{k}{q} = 0$. Denote by $\widetilde{W}_r^{(\sigma)}$ the subgroup of $\widetilde{W}^{(\sigma)}$ generated by all such reflections and parallel translations along the vectors h_α, $\alpha \in \Delta_{\mathrm{re}}^{(\sigma)}$.

Proposition 3.15. (1) *The group $\widetilde{W}_r^{(\sigma)}$ considered as the group of motions of the Euclidean space $\mathfrak{h}^\sigma(\mathbb{R})$ is a discrete group generated by reflections.*

(2) *The group $\widetilde{W}^{(\sigma)}$ is a semisimple product of $\widetilde{W}_r^{(\sigma)}$ and some (finite) group $\Gamma^{(\sigma)}$ of symmetries of the fundamental polyhedron of $\widetilde{W}_r^{(\sigma)}$ in $\mathfrak{h}^\sigma(\mathbb{R})$.*

(3) *If the algebra $\mathfrak{g}$ is simple and $\sigma \neq \mathrm{id}$, then the simplex*

$$D^{(\sigma)} = \left\{ x \in \mathfrak{h}^\sigma(\mathbb{R}) : \beta_i(x) \geq 0 \ (i = 1, \ldots, l(\sigma)), \ \eta(x) \leq \frac{1}{q} \right\} \tag{36}$$

is a fundamental polyhedron of the group $\widetilde{W}_r^{(\sigma)}$ in $\mathfrak{h}^\sigma(\mathbb{R})$ (in the notation of Sect. 3.9), and $\Gamma^{(\sigma)}$ is the group of all its symmetries, which is naturally isomorphic to the group of automorphisms of the extended system of simple σ-roots (and consists of at most two elements).

Let the algebra $\mathfrak{g}$ be simple. An element $x \in \mathfrak{h}^\sigma$ can be described by the numbers

$$x_0 = \frac{1}{q} - \eta(x), x_1 = \beta_1(x), \ldots, x_{l(\sigma)} = \beta_{l(\sigma)}(x) \tag{37}$$

satisfying the condition

$$\sum_{i=0}^{l(\sigma)} n_i x_i = 1. \tag{38}$$

These numbers will be called the *barycentric coordinates* of x.

Now we can formulate an analogue of Theorem 3.11.

Theorem 3.16. *Any outer semisimple automorphism of a simple Lie algebra $\mathfrak{g}$ belonging to $\sigma \mathrm{Int}\, \mathfrak{g}$ ($\sigma \in \mathrm{Aut}\, \Pi$) is conjugate to an automorphism of the form $\sigma \exp 2\pi i x$, where $x \in \mathfrak{h}^\sigma$ is an element whose barycentric coordinates $x_0, x_1, \ldots, x_{l(\sigma)}$ satisfy conditions* (P1) *and* (P2) *of Sect.* 3.6. *Two automorphisms of such a form are conjugate if and only if their barycentric* *coordinates can be obtained from each other by a permutation from the group* $\Gamma^{(\sigma)}$.

3.11. Outer Automorphisms of Finite Order and $\mathbb{Z}_m$-gradings of Outer Type. Any $\mathbb{Z}_m$-grading of an algebra $\mathfrak{g}$ defined by an outer automorphism θ (see Sect. 3.7) will be called a $\mathbb{Z}_m$-grading of *outer type*.

Let $\theta \in \sigma\mathrm{Int}\,\mathfrak{g}$ ($\sigma \in \mathrm{Aut}\,\Pi$). We can assume that $\theta = \sigma \exp 2\pi i x$, where $x \in \mathfrak{h}^{\sigma}$ is an element satisfying the conditions of Theorem 3.16. The additional condition $\theta^m = \mathrm{id}$ means that $q \mid m$ and $mx \in P^{\vee}$. Since the restriction of any root of $\mathfrak{g}$ to $\mathfrak{h}^{\sigma}$ is an integral linear combination of the simple σ-roots $\beta_1, \ldots, \beta_{l(\sigma)}$, the condition $mx \in P^{\vee}$ is equivalent to the fact that $m\beta_i(x) \in \mathbb{Z}$ ($i = 1, \ldots, l(\sigma)$). Hence it follows that the barycentric coordinates $x_0, x_1, \ldots, x_{(l(\sigma)}$ of an element x are of the form

$$x_i = \frac{p_i}{m},\ p_i \in \mathbb{Z},\ p_i \geq 0\ (i = 0, 1, \ldots, l(\sigma)),$$

where the condition

$$\sum_{i=0}^{l(\sigma)} n_i p_i = m \tag{39}$$

must be satisfied (cf. Sect 3.7). The order of the automorphism θ is equal to $\frac{m}{d}$, where d is the greatest common divisor of the numbers $p_0, p_1, \ldots, p_{l(\sigma)}$.

Thus the $\mathbb{Z}_m$-grading of outer type associated with an automorphism $\theta \in \sigma\mathrm{Int}\,\mathfrak{g}$ can be defined by the extended diagram of simple σ-roots of the algebra $\mathfrak{g}$ with integral nonnegative labels p_i satisfying condition (39) (which automatically implies that $q \mid m$). As in the case of gradings of inner type, such a diagram is called a *Kac diagram*. Two $\mathbb{Z}_m$-gradings of outer type can be taken into each other by an inner automorphism if and only if their Kac diagrams can be taken into each other by an automorphism of the extended diagram of simple σ-roots.

The description of the grading subspaces $\mathfrak{g}_p$ is similar to that of Sect. 3.7 for $\mathbb{Z}_m$-gradings of inner type, with the subalgebra $\mathfrak{h}$ replaced by $\mathfrak{h}^{\sigma}$ and the extended system of simple roots $\{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ by the extended system of simple σ-roots $\{\beta_0, \beta_1, \ldots, \beta_{l(\sigma)}\}$. In particular, $\mathfrak{g}_0$ is the reductive subalgebra with Cartan subalgebra $\mathfrak{h}^{\sigma}$ and the system of simple roots

$$\widetilde{\Pi}_0^{(\sigma)} = \{\beta_i \in \widetilde{\Pi}^{(\sigma)} : p_i = 0\}. \tag{40}$$

The dimension of its centre $\mathfrak{z}_0$ is one less than the number of nonzero labels p_i. (To be more precise, the roots of $\mathfrak{g}_0$ are not the σ-roots $\beta_i \in \widetilde{\Pi}_0^{(\sigma)}$ themselves but their restrictions $\overline{\beta}_i$ (in the notation of Sect. 3.9); however, since the latter are linearly independent, the mapping $\lambda \mapsto \overline{\lambda}$ is bijective on the subgroup generated by $\widetilde{\Pi}_0^{(\sigma)}$. This enables us, admitting a certain degree of imprecision, to identify the roots of $\mathfrak{g}_0$ with the corresponding integral linear combinations of σ-roots $\beta_i \in \widetilde{\Pi}_0^{(\sigma)}$.)

The description of representations of $\mathfrak{g}_0$ in the subspaces $\mathfrak{g}_p$ ((1 $\leq p \leq m-1$) is slightly different. Namely, the elements of the centre $\mathfrak{z}_0$ of $\mathfrak{g}_0$ still act on each of the subspaces $\mathfrak{g}_p^{(\nu)}$ (defined in a similar way) as scalars, but the representation of its semisimple part $\mathfrak{g}_0'$ in $\mathfrak{g}_p^{(\nu)}$ can now be reducible. This reducibility is, however, of a special character: the algebra $\mathfrak{g}_0'$ can be decomposed into the direct sum of ideals $\mathfrak{g}_0^{(0)}, \mathfrak{g}_0^{(1)}, \ldots, \mathfrak{g}_0^{(r)}$, and the space $\mathfrak{g}_p^{(\nu)}$ into the direct sum of invariant subspaces $\mathfrak{g}_p^{(\nu,1)}, \ldots, \mathfrak{g}_p^{(\nu,r)}$ in such a way that $[\mathfrak{g}_0^{(i)}, \mathfrak{g}_p^{(\nu,j)}] = 0$ for $i \neq j$ and the representation of $\mathfrak{g}_0^{(i)}$ in $\mathfrak{g}_0^{(\nu,i)}$ is faithful and irreducible.

Proof. Since the weights of the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_p^{(\nu)}$ are σ-roots of $\mathfrak{g}$ having fixed coefficients of $\beta_i \notin \widetilde{\Pi}_0^{(\sigma)}$ in the decomposition (33) (under the same provision as in the description of simple roots of $\mathfrak{g}_0$), it is clear that they are congruent modulo the root lattice of $\mathfrak{g}_0$. The corresponding weight subspaces are the σ-root subspaces of $\mathfrak{g}$. Those of them corresponding to the real σ-roots are one-dimensional (Theorem 3.14), while at most one weight subspace corresponds to an imaginary σ-root and the corresponding weight is zero. One therefore arrives at the conditions of Lemma 3.9 with the only difference that the multiplicity of the zero weight can be greater than one. The assertion now follows from the following generalization of Lemma 3.9. □

Lemma 3.17. *Suppose that all the weights of a linear representation ρ of a semisimple Lie algebra $\mathfrak{s}$ are congruent modulo the root lattice, and that all of them except the zero one are of multiplicity one. Then $\mathfrak{s}$ can be decomposed into the direct sum of ideals $\mathfrak{s}^{(0)}, \mathfrak{s}^{(1)}, \ldots, \mathfrak{s}^{(r)}$, and the representation space of ρ can be decomposed into the direct sum of invariant subspaces $V^{(1)}, \ldots, V^{(r)}$ in such a way that $\rho(\mathfrak{s}^{(i)})V^{(j)} = 0$ for $i \neq j$ and the representation of $\mathfrak{s}^{(i)}$ in $V^{(i)}$ is faithful and irreducible.*

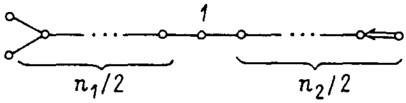
Proof. In the notation adopted in the proof of Lemma 3.9, we have to consider the case $N = 0$, that is $\Lambda = \sum_{i \in I} k_i \gamma_i$. For $s \notin I$ in this case one has $\Lambda(\gamma_s^\vee) = 0$ and $\gamma_i(\gamma_s^\vee) = 0$ for all $i \in I$. This easily implies the statement of the lemma. □

Example 1. Let $n = n_1 + n_2$, where n_2 is even. Suppose that an n-dimensional vector space V is decomposed into the direct sum $V = V_1 \oplus V_2$, where $\dim V_1 = n_1$, $\dim V_2 = n_2$. Let f be a nondegenerate bilinear form on V satisfying the conditions

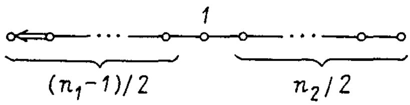
(1) $f(V_1, V_2) = f(V_2, V_1) = 0$;
(2) the form $f|_{V_1}$ is symmetric and $f|_{V_2}$ is skew-symmetric.

For each linear operator X in the space V we denote by X^* the operator conjugate to X with respect to f. Then the mapping $\theta: X \mapsto -X^*$ is an outer automorphism of order 4 of the algebra $\mathfrak{sl}(V)$. Its square is of the form $\theta^2: X \mapsto RXR^{-1}$, where R is the reflection in V_1 parallel to V_2;

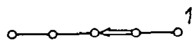
the corresponding $\mathbb{Z}_2$-grading coincides with that described in Sect. 3.7, Example 1. In the $\mathbb{Z}_4$-grading corresponding to the automorphism θ we have $\mathfrak{g}_0 = \mathfrak{so}(V_1) \oplus \mathfrak{sp}(V_2)$ (where the metric structures in V_1 and V_2 are defined by the restrictions of the form f); the representations of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are isomorphic to the tensor product of the identity representations of the algebras $\mathfrak{so}(V_1)$ and $\mathfrak{sp}(V_2)$, and the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_2$ is isomorphic to the direct sum of the natural irreducible representations of the algebras $\mathfrak{so}(V_1)$ and $\mathfrak{sp}(V_2)$ in the spaces of symmetric operators with zero trace in V_1 and V_2, respectively. Depending on the parity of n, the Kac diagram of this grading is of the form



or



Example 2. The Kac diagram



defines a $\mathbb{Z}_2$-grading of the simple Lie algebra $\mathfrak{g}$ of type E_6. Here $\mathfrak{g}_0$ is a simple Lie algebra of type C_4, i.e. $\mathfrak{g}_0 \simeq \mathfrak{sp}(V)$, where V is an 8-dimensional symplectic space. The representation of $\mathfrak{g}_0$ in $\mathfrak{g}_1$ has the labelled simple σ-root for its lowest weight and is isomorphic to the natural representation of $\mathfrak{sp}(V)$ in the space $\Lambda_0^4 V$ of 4-vectors whose contraction with the metric tensor of the space V vanishes.

The description of automorphisms of finite order and the corresponding gradings given in Sect. 3.7 and this one are due, except for some details, to V.G. Kac [1969]. His original idea was to associate with each simple $\mathbb{Z}_m$-graded finite-dimensional Lie algebra a simple $\mathbb{Z}$-graded infinite-dimensional Lie algebra of finite growth "covering" it and then use the theory of such infinite-dimensional Lie algebras, which he had developed in an earlier paper. For an exposition that is also based on the examination of the covering algebra but is closed in itself see Helgason [1978]. See also Vinberg [1976], which also contains numerous examples.

3.12. Jordan Gradings of Classical Lie Algebras. There are gradings of simple Lie algebras that owe their origin to finite commutative subgroups of the group of inner automorphisms with finite centralizer (which therefore contains no nontrivial torus).

Some remarkable subgroups of this kind can be found in the group $\mathrm{Int}\,\mathfrak{sl}_n(\mathbb{C}) = \mathrm{PGL}_n(\mathbb{C})$ (see Examples 2 and 3 below). They were studied by Suprunenko [1979] in connection with the problem of describing maximal solvable subgroups of the group $\mathrm{GL}_n(\mathbb{C})$. He took as a model the solution of the similar problem for the symmetric group obtained by Jordan in his famous treatise on substitutions. In the Jordan's treatise some special commutative subgroups naturally arose in the symmetric group and in classical linear groups over finite fields. For that reason A. Alekseevskij [1974] suggested the name of Jordan subgroups for the commutative subgroups studied by Suprunenko and similar subgroups of the groups of inner automorphisms of other simple complex Lie algebras. We now give a precise definition[2].

Definition 3.18. Let $\mathfrak{g}$ be a simple complex Lie algebra and $G = \operatorname{Int}\mathfrak{g}$. A finite commutative subgroup $A \subset G$ and the corresponding grading of the algebra $\mathfrak{g}$ are said to be *Jordan* if

(J1) the centralizer $Z(A)$ of A is finite;
(J2) A is a minimal normal subgroup of its normalizer $N(A)$;
(J3) $N(A)$ is maximal among normalizers of finite commutative subgroups satisfying conditions (J1) and (J2).

Condition (J2) implies that A is a p-elementary group for some prime number p. If one considers A as a vector space over the field $\mathbb{Z}_p$, then the group $W(A) = N(A)/Z(A)$ acting naturally on A can be considered as a linear group over the field $\mathbb{Z}_p$. Condition (J2) is then equivalent to the fact that $W(A)$ is an irreducible linear group. This linear group will be called the *Weyl group* of A (or, more appropriately, of the algebra $\mathfrak{g}$ with respect to A).

For classical algebras $\mathfrak{g}$ all Jordan subgroups and the corresponding gradings were known before the general notion of a Jordan subgroup was introduced. Let us describe them in a series of examples.

Example 1. Let A be the image of the subgroup of diagonal matrices with diagonal entries ± 1 and determinant 1 under the adjoint representation of the group $\mathrm{SO}_n(\mathbb{C})$ $(n \geq 3)$. The commutative subgroup $A \subset \operatorname{Int}\mathfrak{so}_n(\mathbb{C})$ (isomorphic to $\mathbb{Z}_2^{n-1}$ for odd n and to $\mathbb{Z}_2^{n-2}$ for even n) defines a grading of $\mathfrak{so}_n(\mathbb{C})$ such that the grading subspace corresponding to the zero character is zero, while the nonzero grading subspaces are one-dimensional subspaces of the form $\langle E_{ij} - E_{ji}\rangle$, where $i < j$. The centralizer of A in $\operatorname{Int}\mathfrak{so}_n(\mathbb{C})$ coincides with A, and its normalizer is the image of the subgroup of monomial matrices with nonzero elements ± 1 and determinant 1 under the adjoint representation. The Weyl group $W(A)$ is isomorphic to the symmetric group S_n and permutes transitively the nonzero subspaces of the grading.

Example 2. Let p be a prime number. Consider the group

$$\Gamma = \langle x, y, z \mid x^p = y^p = z^p = e,\ (x,z) = (y,z) = e,\ (x,y) = z\rangle.$$

It is a group of order p^3; its centre is the cyclic group $Z = \langle z\rangle$ of order p, and the quotient group Γ/Z is isomorphic to $\mathbb{Z}_p^2$. If one considers the group Z as the field $\mathbb{Z}_p$ and the group Γ/Z as a two-dimensional vector space over the field $\mathbb{Z}_p$, then the commutator operation defines a nondegenerate alternating

[2] There is a misprint in the definition of Jordan subgroups given in Alekseevskij [1974]: in the condition (3) $N_G(A)$ should be replaced by $N_{\hat{G}}(A)$. After this correction, his definition is slightly more general than ours, namely, there arises exactly one additional subgroup (of the group of inner automorphisms of the simple Lie algebra of type D_4).

bilinear form on Γ/Z. This form is preserved by any automorphism of Γ, that is the identity on Z; any automorphism that is the identity both on Z and on Γ/Z is an inner one. The quotient group of the group of automorphisms of Γ that are the identity on Z with respect to the subgroup of inner automorphisms is naturally isomorphic to the group $\mathrm{SL}_2(\mathbb{Z}_p)$ $(= \mathrm{Sp}_2(\mathbb{Z}_p))$. The group Γ has a unique p-dimensional irreducible linear representation R for which $R(z) = \omega E$, where $\omega = e^{\frac{2\pi i}{p}}$. In an appropriate basis

$$R(x) = \begin{pmatrix} 1 & & & & & & 0 \\ & \omega & & & & & \\ & & \omega^2 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & 0 & & & & \omega^{p-2} & \\ & & & & & & \omega^{p-1} \end{pmatrix},$$

$$R(y) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let A be the image of the group $R(\Gamma)$ under the adjoint representation of the group $\mathrm{GL}_p(\mathbb{C})$ in $\mathfrak{sl}_p(\mathbb{C})$. This is a subgroup of $\mathrm{Int}\,\mathfrak{sl}_p(\mathbb{C})$, isomorphic to $\mathbb{Z}_p^2$. It follows from the irreducibility of the representation R and the above properties of automorphisms of the group Γ that A coincides with its centralizer in $\mathrm{Int}\,\mathfrak{sl}_p(\mathbb{C})$, and that the Weyl group $W(A)$ is naturally isomorphic to $\mathrm{SL}_2(\mathbb{Z}_p)$ (Suprunenko [1979]).

Example 3. The preceding example can be generalized in the following way. Consider the p^n-dimensional irreducible linear representation $R \otimes \ldots \otimes R$ of the group $\Gamma \times \ldots \times \Gamma$. It gives rise to a representation $R^{(n)}$ of the group

$$\Gamma^{(n)} = \langle x_i, y_i\, (i = 1, \ldots, n), z | x_i^p = y_i^p = z^p = e,$$

$$(x_i, z) = (y_i, z) = e, (x_i, y_j) = e\, (i \neq j),\ (x_i, y_i) = z\rangle,$$

which is the unique p^n-dimensional irreducible linear representation of this group under which the central element z goes into ωE. The quotient group of $\Gamma^{(n)}$ with respect to its centre $Z = \langle z \rangle$ can naturally be considered as a $2n$-dimensional symplectic space over the field $\mathbb{Z}_p$. The image $A^{(n)}$ of the group $R^{(n)}(\Gamma^{(n)})$ under the adjoint representation of the group $\mathrm{GL}_{p^n}(\mathbb{C})$ in $\mathfrak{sl}_{p^n}(\mathbb{C})$ is isomorphic to $\mathbb{Z}_p^{2n}$ and coincides with its centralizer in $\mathrm{Int}\,\mathfrak{sl}_{p^n}(\mathbb{C})$. The Weyl group $W(A^{(n)})$ is naturally isomorphic to the group $\mathrm{Sp}_{2n}(\mathbb{Z}_p)$ (Suprunenko [1979]).

The grading of $\mathfrak{sl}_{p^n}(\mathbb{C})$ defined by the group $A^{(n)}$ has the same properties as the grading in Example 1. It can be described in terms of the generalized (usual when $p=2$) Clifford algebra (Morinaga and Nono [1952]).

Example 4. For $p=2$ the representation R, and consequently the representation $R^{(n)}$, is orthogonal. Moreover, $R^{(n)}(\Gamma^{(n)}) \subset \mathrm{SO}_{2^n}(\mathbb{C})$ for $n \geq 2$. Therefore for $n \geq 2$ the group $A^{(n)}$ can be considered as a subgroup of $\operatorname{Int}\mathfrak{sl}_{2^n}(\mathbb{C})$. Its Weyl group is naturally isomorphic to the group $\mathrm{O}_{2n}(f, \mathbb{Z}_2)$, where f is a nondegenerate quadratic form of index n over the field $\mathbb{Z}_2$. The corresponding grading of the algebra $\mathfrak{so}_{2^n}(\mathbb{Z})$ is described in terms of the Clifford algebra (Popovici [1970]).

Example 5. Denote by V the space of the representation R. For $p=2$ it is two-dimensional and therefore carries the canonical symplectic metric (defined up to a scalar multiple). The space of the representation $R^{(n)}$ is $\underbrace{V \otimes \ldots \otimes V}_{n}$. Define a symplectic metric in it as a tensor product of the symplectic metric in one of the factors and the orthogonal metrics invariant under R in the others. Then any operator of the representation $R^{(n)}$ is proportional to a symplectic one, i.e. $R^{(n)}(\Gamma^{(n)}) \subset \mathbb{C}^\times \mathrm{Sp}_{2^n}(\mathbb{C})$. Therefore the group A_n can be considered as a subgroup of $\operatorname{Int}\mathfrak{sl}_{2^n}(\mathbb{C})$. Its Weyl group is naturally isomorphic to the group $\mathrm{O}_{2n}(g, \mathbb{Z}_2)$, where g is a nondegenerate quadratic form of index $n-1$ over the field $\mathbb{Z}_2$. The corresponding grading of the algebra $\mathfrak{sp}_{2^n}(\mathbb{C})$ is also described in terms of the Clifford algebra (Popovici [1970]).

Theorem 3.19 (A. Alekseevskij [1974]). *Jordan subgroups of inner automorphisms of classical simple complex Lie algebras are precisely the subgroups described in Examples* 1–5.

Note that in all the above cases the following properties hold:

(1) $Z(A) = A$;
(2) the group $W(A)$ preserves a nondegenerate alternating bilinear form on A;
(3) the group $W(A)$ is generated by transvections, i.e. linear transformations of the form

$$r_{\alpha,e}\colon x \mapsto x - \alpha(x)e,$$

where e is a nonzero vector, α a nonzero linear form, and $\alpha(e)=0$;
(4) a linear form $\alpha \in A^*$ is a weight of the group A in $\mathfrak{g}$ if and only if $r_{\alpha,e} \in W(A)$ for some $e \in A$;
(5) the group $W(A)$ acts transitively on the set of weights;
(6) all weight subspaces are one-dimensional and consist of semisimple elements.

All these properties can be derived from the first two ones without any use of the classification (Alekseevskij [1974]).

3.13. Jordan Gradings of Exceptional Lie Algebras. A. Alekseevskij [1974] also classified all Jordan subgroups of groups of inner automorphisms of exceptional simple complex Lie algebras. They are listed in the following table. For all of them the Weyl group consists of all unimodular linear transformations, and the weights are all nonzero linear forms in the space A. The weight subspaces $\mathfrak{g}_\alpha$ ($\alpha \in A^*$, $\alpha \neq 0$) have the dimensions given in the table and are commutative subalgebras consisting of semisimple elements.

Type of $\mathfrak{g}$	A	$Z(A)/A$	$\dim \mathfrak{g}_\alpha$
G_2	$\mathbb{Z}_2^3$	e	2
F_4	$\mathbb{Z}_3^3$	e	2
E_8	$\mathbb{Z}_5^3$	e	2
E_6	$\mathbb{Z}_3^3$	$\mathbb{Z}_3^3$	3
E_8	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^{10}$	8

The Jordan subgroup given in the last row of the table defines the grading of the Lie algebra of type E_8 for which all nonzero subspaces of the grading are Cartan subalgebras. This grading is specifically considered in Thompson [1976]. All such gradings (not necessarily Jordan ones) for simple Lie algebras are found in Hesselink [1982]. Other types of gradings, close to Jordan ones, have been considered in A. Kostrikin, I. Kostrikin, and Ufnarovskij [1982] and Adams [1987]. Orthogonal decompositions into the sum of Cartan subalgebras that are not gradings have been studied in A. Kostrikin, I. Kostrikin, and Ufnarovskij [1981], [1984] and A. Kostrikin [1989].

Chapter 4
Real Semisimple Lie Groups and Lie Algebras

§ 1. Classification of Real Semisimple Lie Algebras

In this section we present the basic facts on the classification of real semisimple Lie algebras. As we have seen in Chap. 1, Sect. 7, a real Lie algebra is semisimple if and only if its complexification is simple. However, the correspondence between real and complex semisimple Lie algebras established with the help of complexification is not one-to-one. As it turns out, to describe the real forms of a given complex semisimple Lie algebra $\mathfrak{g}$ is the same as to classify the involutory automorphisms of $\mathfrak{g}$ up to conjugacy in $\operatorname{Aut} \mathfrak{g}$. The latter classification is easily obtained from the results of Chap. 3, Sect. 3.

Here an important role is played by the existence in $\mathfrak{g}$ of a unique (up to isomorphism) compact real form.

1.1. Real Forms of Classical Lie Groups and Lie Algebras. The notions of a real form and a real structure in a complex Lie group or Lie algebra were defined in Chap. 1, Sect. 7.1. In this section we specify several real forms of the classical complex groups $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$ and their tangent algebras. Actually, as we will see below, the real forms listed here exhaust (up to isomorphism) all real forms of the classical complex Lie algebras. It is easy to see that all real structures and real forms of classical groups listed below are algebraic.

Clearly, $\mathrm{GL}_n(\mathbb{R})$ is a real form of $\mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{R})$ is a real form of $\mathfrak{gl}_n(\mathbb{C})$. The corresponding real structure on $\mathrm{GL}_n(\mathbb{C})$ is the complex conjugation $S(A) = \overline{A}$.

Example 1. The complex conjugation $A \mapsto \overline{A}$ transforms each of the groups $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$ into itself and defines real structures in them. Thus the following real forms of classical groups are defined: $\mathrm{SL}_n(\mathbb{R}) \subset \mathrm{SL}_n(\mathbb{C})$, $\mathrm{O}_n \subset \mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n \subset \mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{R}) \subset \mathrm{Sp}_n(\mathbb{C})$. The corresponding real forms of the Lie algebras are

$$\mathfrak{sl}_n(\mathbb{R}) \subset \mathfrak{sl}_n(\mathbb{C}),\ \ \mathfrak{so}_n(\mathbb{R}) \subset \mathfrak{so}_n(\mathbb{C}),\ \ \mathfrak{sp}_n(\mathbb{R}) \subset \mathfrak{sp}_n(\mathbb{C}).$$

The next series of examples is related to quadratic forms. Consider the *pseudoorthogonal group* of signature (k,l) $\mathrm{O}_{k,l} \subset \mathrm{GL}_n(\mathbb{R})$, $n = k+l$, preserving the quadratic form

$$x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_n^2, \tag{1}$$

and the *special* pseudoorthogonal group $\mathrm{SO}_{k,l} = \mathrm{O}_{k,l} \cap \mathrm{SL}_n(\mathbb{R})$.

Let $I_{k,l} = \begin{pmatrix} E_k & 0 \\ 0 & -E_l \end{pmatrix}$ be the matrix of the form (1) and let $L_{k,l} = \begin{pmatrix} E_k & 0 \\ 0 & iE_l \end{pmatrix}$. Then $L_{k,l}^2 = I_{k,l}$.

Example 2. The transformation $S(A) = I_{k,l}\overline{A}I_{k,l}$ is a real structure on the complex Lie groups $G = \mathrm{O}_n(\mathbb{C}), \mathrm{SO}_n(\mathbb{C})$; the corresponding real forms G^S coincide with $L_{k,l}\mathrm{O}_{k,l}L_{k,l}^{-1}$ and $L_{k,l}\mathrm{SO}_{k,l}L_{k,l}^{-1}$, respectively. The corresponding real form $L_{k,l}\mathfrak{so}_{k,l}L_{k,l}^{-1}$ of the Lie algebra $\mathfrak{so}_n(\mathbb{C})$ consists of matrices of the form

$$\begin{array}{cc} & \begin{array}{cc} k & \ \ l \end{array} \\ \begin{array}{c} k \\ l \end{array} & \begin{pmatrix} X & iY \\ -iY^\top & Z \end{pmatrix} \end{array},$$

where X, Y, Z are real matrices, $X^\top = -X$, $Z^\top = -Z$.

Similarly, the *pseudounitary group* of signature (k,l) is defined as the group $\mathrm{U}_{k,l}$ of all linear transformations of $\mathbb{C}^n$, $n = k+l$, preserving the Hermitian quadratic form

$$|z_1|^2 + \ldots + |z_k|^2 - |z_{k+1}|^2 - \ldots - |z_n|^2.$$

In particular, the group $\mathrm{U}_n = \mathrm{U}_{n,0}$ is the group of *unitary* matrices (or the *unitary group*). The groups $\mathrm{SU}_{k,l} = \mathrm{U}_{k,l} \cap \mathrm{SL}_n(\mathbb{C})$ and $\mathrm{SU}_n = \mathrm{SU}_{n,0}$ are said to be *special* pseudounitary and unitary groups. The corresponding tangent algebras are denoted by $\mathfrak{u}_{k,l}$, $\mathfrak{u}_n$, $\mathfrak{su}_{k,l}$, $\mathfrak{su}_n$.

Example 3. The transformation $S(A) = I_{k,l}(A^\top)^{-1}I_{k,l}$ is a real structure on the complex Lie groups $G = \mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $n = k+l$; the corresponding real forms G^S coincide with $\mathrm{U}_{k,l}$ and $\mathrm{SU}_{k,l}$, respectively. The corresponding real forms of the Lie algebras are $\mathfrak{u}_{k,l} \subset \mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{su}_{k,l} \subset \mathfrak{sl}_n(\mathbb{C})$, consisting of matrices of the form

$$\begin{pmatrix} X & Y \\ \overline{Y}^\top & Z \end{pmatrix},$$

where $\overline{X}^\top = -X$, $\overline{Z}^\top = -Z$, and for $\mathfrak{su}_{k,l}$ the extra condition $\operatorname{tr} X + \operatorname{tr} Z = 0$ is satisfied.

The last group of examples is related to the existence of a quaternionic structure in $\mathbb{C}^{2m}$. Consider the right quaternion vector space $\mathbb{H}^m$ over the field of quaternions $\mathbb{H}$. Its linear transformations are identified with $m \times m$ matrices over $\mathbb{H}$. Let $\mathrm{GL}_m(\mathbb{H})$ be the group of invertible quaternion matrices. Its tangent algebra is the Lie algebra $\mathfrak{gl}_m(\mathbb{H})$ of all quaternion matrices.

Consider $\mathbb{C}$ as the subfield of $\mathbb{H}$ generated by the elements 1, i. Each vector $q \in \mathbb{H}^m$ can be uniquely represented in the form $q = z + jw$, where $z, w \in \mathbb{C}^m$. The correspondence $q \mapsto (z, w)$ is an isomorphism $\mathbb{H}^m \to \mathbb{C}^{2m}$ of vector spaces over $\mathbb{C}$ that maps qj into $(-\overline{w}, \overline{z})$. Therefore this isomorphism identifies $\mathfrak{gl}_m(\mathbb{H})$ with the subalgebra of $\mathfrak{gl}_{2m}(\mathbb{C})$ consisting of all transformations commuting with the antilinear transformation $J\colon \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ whose action is defined by the formula

$$J(z, w) = (-\overline{w}, \overline{z}).$$

Note that $J = S_m\tau$, where τ is the standard complex conjugation in $\mathbb{C}^{2m}$ and

$$S_m = \begin{pmatrix} 0 & -E_m \\ E_m & 0 \end{pmatrix}.$$

Example 4. The transformation $S(A) = JAJ^{-1} = -S_m\overline{A}S^m$ is a real structure on the complex Lie groups $G = \mathrm{GL}_{2m}(\mathbb{C}), \mathrm{SL}_{2m}(\mathbb{C}), \mathrm{SO}_{2m}(\mathbb{C})$. The corresponding real form of the Lie group $\mathrm{GL}_{2m}(\mathbb{C})$ is identified with $\mathrm{GL}_m(\mathbb{H})$. Denote the real forms G^S of the groups $G = \mathrm{SL}_{2m}(\mathbb{C}), \mathrm{SO}_{2m}(\mathbb{C})$ by $\mathrm{SL}_m(\mathbb{H})$, $\mathrm{U}^*_m(\mathbb{H})$, respectively. The latter notation is chosen since $\mathrm{U}^*_m(\mathbb{H})$ is identified with the subgroup of $\mathrm{GL}_m(\mathbb{H})$ consisting of all linear transformations C of $\mathbb{H}^m$ preserving the skew-Hermitian form

$$\sum_{r=1}^m \overline{q}_r j q_r,$$

i.e. satisfying the condition $\overline{C}(jE)C = jE$. Denote the tangent algebras of the groups $\mathrm{SL}_m(\mathbb{H})$, $\mathrm{U}^*_m(\mathbb{H})$ by $\mathfrak{sl}_m(\mathbb{H})$, $\mathfrak{u}^*_m(\mathbb{H})$. They are the real forms of the Lie algebras $\mathfrak{sl}_m(\mathbb{C})$, $\mathfrak{so}_{2m}(\mathbb{C})$. The Lie algebras $\mathfrak{gl}_m(\mathbb{H})$, $\mathfrak{sl}_m(\mathbb{H})$, $\mathfrak{u}^*_m(\mathbb{H})$ are subalgebras of $\mathfrak{gl}_{2m}(\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} X & Y \\ -\overline{Y} & \overline{X} \end{pmatrix},$$

where $X, Y \in \mathfrak{gl}_m(\mathbb{C})$ and satisfy the additional condition $\operatorname{tr} X + \operatorname{tr} \overline{X} = 0$ for $\mathfrak{sl}_m(\mathbb{H})$ and $X^\top = -X$, $Y^\top = \overline{Y}$ for $\mathfrak{u}^*_m(\mathbb{H})$.

Consider the subgroup $\mathrm{Sp}_{k,l}$, $k + l = m$, of $\mathrm{GL}_m(\mathbb{H})$ consisting of transformations preserving the Hermitian quadratic form

$$|q_1|^2 + \ldots + |q_k|^2 - |q_{k+l}|^2 - \ldots - |q_m|^2. \tag{2}$$

Under the isomorphism $\mathbb{H}^m \to \mathbb{C}^{2m}$ described above the form (2) goes into the Hermitian quadratic form

$$\sum_{i=1}^{k} |z_i|^2 - \sum_{j=k+1}^{m} |z_j|^2 + \sum_{i=1}^{k} |w_i|^2 - \sum_{j=k+1}^{m} |w_j|^2. \tag{3}$$

Therefore $\mathrm{Sp}_{k,l}$ is identified with the subgroup of $\mathrm{GL}_{2m}(\mathbb{C})$ consisting of matrices A satisfying the conditions

$$A = -S_m \overline{A} S_m, \quad \overline{A}^\top K_{k,l} A = K_{k,l},$$

where $K_{k,l} = \begin{pmatrix} I_{k,l} & 0 \\ 0 & I_{k,l} \end{pmatrix}$ is a matrix of the form (3). These conditions imply that $A(K_{k,l}S_m)A^\top = K^{k,l}S_m$, i.e. $\mathrm{Sp}_{k,l}$ is contained in the complex symplectic group corresponding to the form with matrix $K_{k,l}S_m$. Setting $M_{k,l} = \begin{pmatrix} L_{k,l} & 0 \\ 0 & L_{k,l} \end{pmatrix}$ (see Example 2), we see that the group $M_{k,l}\mathrm{Sp}_{k,l}M_{k,l}^{-1}$ is contained in the standard symplectic group $\mathrm{Sp}_{2m}(\mathbb{C})$ and coincides with the subgroup of all elements of the symplectic group preserving the form (3).

Example 5. The transformation $S(A) = K_{k,l}(\overline{A}^\top)^{-1}K_{k,l}$ is a real structure on the group $G = \mathrm{Sp}_{2m}(\mathbb{C})$ for which $G^S = M_{k,l}\mathrm{Sp}_{k,l}M_{k,l}^{-1}$. In what follows we will identify the subgroup $\mathrm{Sp}_{k,l}$ with G^S. The corresponding real form $\mathfrak{sp}_{k,l} \subset \mathfrak{sp}_{2m}(\mathbb{C})$ consists of matrices of the form

$$\begin{array}{c} \\ k \\ l \\ k \\ l \end{array}
\begin{array}{c}
\begin{array}{cccc} k \quad & \quad l \quad & \quad k \quad & \quad l \end{array} \\
\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ \overline{X}_{12}^\top & X_{22} & X_{14}^\top & X_{24} \\ -\overline{X}_{13} & \overline{X}_{14} & \overline{X}_{11} & -\overline{X}_{12} \\ \overline{X}_{14}^\top & -\overline{X}_{24} & -X_{12}^\top & \overline{X}_{22} \end{pmatrix}
\end{array},$$

where $\overline{X}_{11}^\top = -X_{11}$, $\overline{X}_{22}^\top = -X_{22}$, $X_{13}^\top = X_{13}$, $X_{24}^\top = X_{24}$.

In particular, the group $\mathrm{Sp}_{m,0}$ coincides with the group $\mathrm{Sp}_m = \mathrm{GL}_m(\mathbb{H}) \cap \mathrm{U}_{2m}$ of unitary quaternion matrices, and its tangent algebra $\mathfrak{sp}_{m,0}$ with the Lie algebra $\mathfrak{sp}_m = \mathfrak{gl}_m(\mathbb{H}) \cap \mathfrak{u}_{2m}$ (in this case $M_{k,l} = E$).

We now present a general method of constructing a real form of a complex semisimple Lie algebra.

Example 6. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\{h_i, e_i, f_i | i = 1, \dots, l\}$ its canonical system of generators (see Chap. 3, Sect. 2.1). Then the real subalgebra $\mathfrak{h} \subset \mathfrak{g}$ generated by the elements h_i, e_i, f_i is a real form of $\mathfrak{g}$. It is said to be the *normal* real form. The corresponding real structure, which is also called *normal*, takes each of the elements h_i, e_i, f_i into itself. It turns out that the following real forms of classical Lie algebras are normal (with respect to some canonical system of generators): $\mathfrak{sl}_n(\mathbb{R}) \subset \mathfrak{sl}_n(\mathbb{C})$ $(n \geq 2)$, $\mathfrak{so}_{l,l+1} \subset \mathfrak{so}_{2l+1}(\mathbb{C})$ $(l \geq 1)$, $\mathfrak{sp}_n(\mathbb{R}) \subset \mathfrak{sp}_n(\mathbb{C})$ $(n \geq 2)$, $\mathfrak{so}_{l,l} \subset \mathfrak{so}_{2l}(\mathbb{C})$ $(l \geq 2)$.

1.2. Compact Real Form. In this section we will show that each connected semisimple complex Lie group has a compact real form. This will enable us to establish a one-to-one correspondence between reductive complex algebraic groups and compact real Lie groups.

A finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is said to be *compact* if there exists a positive definite invariant scalar product in $\mathfrak{g}$. Clearly, any subalgebra of a compact Lie algebra is compact.

Proposition 1.1. *The tangent algebra $\mathfrak{g}$ of any compact Lie group G is compact.*

Proof. As is known, any linear representation of a compact Lie group over the field $\mathbb{R}$ preserves a positive definite scalar product (see corollary to Proposition 2.1 below). This statement must be applied to the adjoint representation of the group G. □

The definition of a compact Lie algebra easily implies the following proposition.

Proposition 1.2. *Any compact Lie algebra is reductive.*

The fact that the operators $\operatorname{ad} x$ $(x \in \mathfrak{g})$ on a compact Lie algebra $\mathfrak{g}$ are skew-symmetric implies the following proposition.

Proposition 1.3. *The Killing form of a compact Lie algebra is nonpositive definite. A real Lie algebra is semisimple and compact if and only if its Killing form is negative definite.*

Proposition 1.4. *For any compact Lie algebra $\mathfrak{g}$ there exists a connected compact Lie group G with tangent algebra $\mathfrak{g}$.*

Proof. If $\mathfrak{g}$ is semisimple, then the group $G = \operatorname{Int} \mathfrak{g}$ has the required properties. If $\mathfrak{g}$ is commutative, so is the compact torus $\mathbb{T}^m$ of dimension $m = \dim \mathfrak{g}$. In the general case the assertion follows from Proposition 1.2. □

Now let $\mathfrak{g}$ be an arbitrary complex Lie algebra and σ a real structure on $\mathfrak{g}$. Define the Hermitian form on $\mathfrak{g}$ by the formula

$$h_\sigma(x,y) = -k(x,\sigma(y)), \tag{4}$$

where k is the Killing form on $\mathfrak{g}$.

Proposition 1.5. *The form h_σ has the following properties:*

$$h_\sigma([z,x],y) + h_\sigma(x,[z,y]) = 0 \quad (x,y \in \mathfrak{g}, z \in \mathfrak{g}^\sigma);$$

$$h_\sigma(\gamma x, \gamma y) = h_\sigma(x,y) \quad (x,y \in \mathfrak{g})$$

if $\gamma \in \operatorname{Aut}\mathfrak{g}$ is an automorphism commuting with σ.

Propositions 1.1, 1.2, and 1.5 imply the following proposition.

Proposition 1.6. *Suppose that $\mathfrak{g}$ is semisimple and $\sigma = dS$, where S is a real structure on a connected Lie group G with tangent algebra $\mathfrak{g}$. Then the following conditions are equivalent:*
(1) *the group G^S is compact;*
(2) *the Lie algebra $\mathfrak{g}^\sigma$ is compact;*
(3) *the form h^σ is positive definite.*

This proposition is used in the proof of the following theorem.

Theorem 1.1. *Any connected semisimple complex Lie group G has a compact real form. The tangent algebra of this form is a compact real form of the tangent algebra of G.*

Proof. Fix a maximal torus $H \subset G$ and a system of simple roots $\{\alpha_1, \ldots, \alpha_l\}$ in the root system Δ_G with respect to H. Consider the canonical system of generators $\{h_i, e_i, f_i | i = 1, \ldots, l\}$ of the algebra $\mathfrak{g}$ (see Chap. 3, Sect. 2.1). Then $\{-h_i, -f_i, -e_i | i = 1, \ldots, l\}$ is the canonical system of generators related to the system of simple roots $\{-\alpha_1, \ldots, -\alpha_l\}$. By Theorem 2.1, Chap. 3 of there exists a unique automorphism μ of the algebra $\mathfrak{g}$ such that

$$\mu(h_i) = -h_i,\ \mu(e_i) = -f_i,\ \mu(f_i) = -e_i \quad (i = 1, \ldots, l).$$

We have $\mu^2 = e$. By setting $\sigma = \sigma_0\mu$, where σ_0 is a normal real structure (see Sect. 1.1, Example 6), we obtain a real structure σ on $\mathfrak{g}$ for which $\sigma(h_i) = -h_i$, $\sigma(e_i) = -f_i$, $\sigma(f_i) = -e_i$ $(i = 1, \ldots, l)$. There exists a real structure S on the group G such that $ds = \sigma$. Indeed, this is clear in the case when G is simply-connected. One can see from Chap. 3, Theorem 2.8 that s acts trivially on any subgroup $N \subset Z(G)$, and therefore defines a real structure on any quotient group G/N.

It follows from the definition of the form h_σ that the subspaces $\mathfrak{h}$ and $\mathfrak{g}_\alpha$ $(\alpha \in \Delta_G)$ are pairwise orthogonal with respect to h_σ and that h_σ is positive definite on any $\mathfrak{g}_{\alpha_i}$, $i = 1, \ldots, l$. Using the Weyl group, which, as it turns out, is induced on $\mathfrak{h}$ by the group $N_G(H) \cap G^S$, one can prove that h_σ is positive definite on all $\mathfrak{g}_\alpha$ $(\alpha \in \Delta_G)$ and consequently on the entire space $\mathfrak{g}$. □

1.3. Real Forms and Involutory Automorphisms. Let $\mathfrak{g}$ be a complex Lie algebra. Consider the problem of enumerating the real forms of $\mathfrak{g}$ up to isomorphism. It follows from Chap. 1, Sect. 7.1 that the classes of isomorphic real forms are in one-to-one correspondence with involutory antilinear automorphisms considered up to conjugacy by means of elements from $\operatorname{Aut}\mathfrak{g}$. In this section we will show that for a semisimple Lie algebra $\mathfrak{g}$ the antilinear automorphisms appearing in this classification can be replaced by linear ones.

Let σ and τ be two real structures on a Lie algebra $\mathfrak{g}$. The real forms $\mathfrak{g}^\sigma$ and $\mathfrak{g}^\tau$ are said to be *compatible* if $\sigma\tau = \tau\sigma$ or, equivalently, if the isomorphism $\theta = \sigma\tau$ is involutory. It follows from Proposition 1.6 that compatible compact real forms of a semisimple Lie algebra coincide.

Lemma 1.1. *Let $\mathfrak{a}$ and $\mathfrak{u}$ be real forms of a complex semisimple Lie algebra $\mathfrak{g}$, where $\mathfrak{u}$ is compact. Then there exists $\varphi \in \operatorname{Int}\mathfrak{g}$ such that $\mathfrak{a}$ and $\varphi(\mathfrak{u})$ are compatible.*

Proof. Let σ and τ be the real structures in $\mathfrak{g}$ corresponding to $\mathfrak{a}$ and $\mathfrak{u}$ and let $\theta = \sigma\tau$. Consider $\mathfrak{g}$ as a Hermitian space with respect to the positive definite Hermitian form h_τ defined by formula (4) (where σ is replaced by τ). Then $p = \theta^2$ is a positive definite self-adjoint operator in $\mathfrak{g}$. Write $p = \exp q$, where q is a self-adjoint operator and set $p^t = \exp(tq)$ $(t \in \mathbb{R})$. Since the group $\operatorname{Aut}\mathfrak{g}$ is algebraic, we have $p^t \in \operatorname{Int}\mathfrak{g}$ for all $t \in \mathbb{R}$. Therefore $\sigma p^t \sigma = p^{-t}$, which implies that the automorphism $\varphi = p^{\frac{1}{4}}$ satisfies the condition of the lemma. □

Theorem 1.2. *Any two compact real forms of a semisimple complex Lie algebra are conjugate. Any real form of $\mathfrak{g}$ is compatible with a compact real form. If a real form $\mathfrak{a}$ is compatible with two compact forms $\mathfrak{u}_1$ and $\mathfrak{u}_2$, then there exists an automorphism $\varphi \in \operatorname{Int}\mathfrak{g}$ such that $\varphi(\mathfrak{u}_1) = \mathfrak{u}_2$ and $\varphi(\mathfrak{a}) = \mathfrak{a}$.*

Proof. The first two statements follow directly from Lemma 1.1 and Theorem 1.1. Suppose that the form $\mathfrak{a}$ is compatible with the real forms $\mathfrak{u}_1$ and $\mathfrak{u}_2$. The proof of Lemma 1.1 shows that the automorphism $\varphi \in \operatorname{Int}\mathfrak{g}$ taking $\mathfrak{u}_1$ into $\mathfrak{u}_2$ can be chosen in such a way that $\varphi(\mathfrak{a}) = \mathfrak{a}$. □

Theorem 1.2 enables one to establish a correspondence between real forms of a semisimple complex Lie algebra and its involutory automorphisms. Namely, let σ be a real structure on $\mathfrak{g}$. According to Theorem 1.2, there exists a compact real structure τ commuting with σ. Then $\theta = \sigma\tau$ is an involutory automorphism of $\mathfrak{g}$. If τ_1 is another compact real structure commuting with σ, then, as follows easily from Theorem 1.2, the automorphisms θ and $\theta_1 = \sigma\tau_1$ are conjugate in the group $\operatorname{Aut}\mathfrak{g}$. Thus there is a mapping associating with each real structure (or a real form) in $\mathfrak{g}$ a class of conjugate involutory automorphism of $\mathfrak{g}$. An argument similar to that given above leads to the following theorem.

Theorem 1.3. *The constructed mapping defines a bijection of the set of classes of isomorphic real forms of* $\mathfrak{g}$ *onto the set of classes of conjugate involutory automorphisms of* $\mathfrak{g}$.

It is useful to give an explicit construction of the real form $\mathfrak{a}$ of the algebra $\mathfrak{g}$ corresponding to an involutory automorphism $\theta \in \operatorname{Aut}\mathfrak{g}$. For this it is convenient to fix a compact real form $\mathfrak{u}$ of $\mathfrak{g}$. Replacing θ by the conjugate automorphism, we may assume that $\theta(\mathfrak{u}) = \mathfrak{u}$. Let

$$\mathfrak{u} = \mathfrak{u}(1) \oplus \mathfrak{u}(-1)$$

be the decomposition of $\mathfrak{u}$ into the eigenspaces of the automorphism θ corresponding to the eigenvalues 1 and -1. By Theorem 1.3 the real form $\mathfrak{a}$ of the Lie algebra $\mathfrak{g}$ corresponding to the class of the automorphism θ is of the form

$$\mathfrak{a} = \mathfrak{u}(1) \oplus i\mathfrak{u}(-1). \tag{5}$$

In particular, the class corresponding to the identity automorphism $\theta = \mathrm{id}$ is that of compact real forms of $\mathfrak{g}$.

1.4. Involutory Automorphisms of Complex Simple Algebras. Here we describe the classes of conjugate involutory automorphisms of complex simple Lie algebras using the method described in Chap. 3, Sect. 3. Let $\mathfrak{g}$ be a noncommutative complex simple Lie algebra of type L_l. It suffices to consider non-identity involutory automorphisms $\theta \in \operatorname{Aut}\mathfrak{g}$, i.e. automorphisms θ of order 2. According to Chap. 3, Sect. 3.7 and 3.11 the classes of automorphisms of order 2 that are conjugate in $\operatorname{Aut}\mathfrak{g}$ are in one-to-one correspondence with the Kac diagrams (considered up to isomorphism) of types $L_l^{(k)}$, $k = 1, 2$, whose numerical labels p_j ($j = 0, 1, \dots, l$) are nonnegative integers, that are relatively prime and satisfy the equation

$$\sum_{j=0}^{l(\sigma)} n_j p_j = 2. \tag{6}$$

Here $n_0, n_1, \dots, n_{l(\sigma)}$ are the positive integers listed in Table 3.

One can easily see that Kac diagrams satisfying condition (2) belong to one of the following two types:

(I) $u_i = 0$ for all i except some $i = s$; $p_s = 1$, $n_s = 2$;

(II) $k = 1$, $u_i = 0$ for all i except some $i = s, t\ (s \neq t)$; $p_s = p_t = 1$; $n_s = n_t = 1$.

In the case (II) we may assume that $t = 0$ if the Kac diagrams are considered up to isomorphism.

Making use of Table 3, it is not difficult to list all (up to isomorphism) Kac diagrams satisfying condition (6). As an application, let us describe explicitly the classes of conjugate involutory automorphisms of simple complex Lie algebras. We use the notation of Sect. 1.1. The corresponding subalgebras

$\mathfrak{g}^\theta$ are also given. Note that $\mathfrak{g}^\theta$ is semisimple in the case (I) and has one-dimensional centre in the case (II).

Theorem 1.4. *The following automorphisms θ of simple classical complex Lie algebras $\mathfrak{g}$ form the complete system of representatives of classes of conjugate involutory automorphisms (for $\theta \neq \mathrm{id}$ the class of the corresponding Kac diagram is indicated):*

(1) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $n \geq 2$:
  (a) $\theta(X) = -X^\top$ (*type* (I) *for* $n > 2$; *type* (II) *for* $n = 2$), $\mathfrak{g}^\theta = \mathfrak{so}_n(\mathbb{C})$;
  (b) $\theta(X) = -\mathrm{Ad}\, S_m(X^\top)$, $n = 2m > 2$ (*type* (I)), $\mathfrak{g}^\theta = \mathfrak{sp}_n(\mathbb{C})$;
  (c) $\theta = \mathrm{Ad}\, I_{p,n-p}$ $(p = 0, 1, \ldots, [\frac{n}{2}])$ (*type* (II) *for* $p > 0$), $\mathfrak{g}^\theta = \mathfrak{g} \cap (\mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_{n-p}(\mathbb{C}))$.

(2) $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$, $n \geq 5$:
  (a) $\theta = \mathrm{Ad}\, I_{p,n-p}$ $(p = 0, 1, \ldots, [\frac{n}{2}])$ (*type* (I) *for* $p \neq 0, 2$; *type* (II) *for* $p = 2$), $\mathfrak{g}^\theta = \mathfrak{so}_p(\mathbb{C}) \oplus \mathfrak{so}_{n-p}(\mathbb{C})$;
  (b) $\theta = \mathrm{Ad}\, S_m$, $n = 2m$ (*type* (II)), $\mathfrak{g}^\theta = \mathfrak{gl}_m(\mathbb{C})$.

(3) $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C})$, $n = 2m \geq 2$:
  (a) $\theta = \mathrm{Ad}\, S_m$ (*type* (II)), $\mathfrak{g}^\theta = \mathfrak{gl}_m(\mathbb{C})$;
  (b) $\theta = \mathrm{Ad}\, K_{p,m-p}$ $(p = 0, 1, \ldots, [\frac{m}{2}])$ (*type* (I) *for* $p > 0$), $\mathfrak{g}^\theta = \mathfrak{sp}_{2p}(\mathbb{C}) \oplus \mathfrak{sp}_{2(m-p)}(\mathbb{C})$.

1.5. Classification of Real Simple Lie Algebras. The results of Sect. 1.3 and 1.4 enable one to list all (up to isomorphism) real forms of noncommutative complex simple Lie algebras. For classical Lie algebras this list is given by the following theorem.

Theorem 1.5. *Any real form of a classical simple complex Lie algebra $\mathfrak{g}$ is isomorphic to exactly one of the following real forms $\mathfrak{a} \subset \mathfrak{g}$:*

(1) $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $n \geq 2$:
  (a) $\mathfrak{a} = \mathfrak{sl}_n(\mathbb{R})$,
  (b) $\mathfrak{a} = \mathfrak{sl}_m(\mathbb{H})$, $n = 2m$,
  (c) $\mathfrak{a} = \mathfrak{su}_{p,n-p}$ $\quad(p = 0, 1, \ldots, [\frac{n}{2}])$;

(2) $\mathfrak{g} = \mathfrak{so}_n(\mathbb{C})$, $n = 3$ *or* $n \geq 5$:
  (a) $\mathfrak{a} = \mathfrak{so}_{p,n-p}$ $\quad(p = 0, 1, \ldots, [\frac{n}{2}])$,
  (b) $\mathfrak{a} = \mathfrak{u}^*_m(\mathbb{H})$, $n = 2m$,

(3) $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C})$, $n = 2m \geq 2$:
  (a) $\mathfrak{a} = \mathfrak{sp}_n(\mathbb{R})$,
  (b) $\mathfrak{a} = \mathfrak{sp}_{p,m-p}$ $\quad(p = 0, 1, \ldots, [\frac{m}{2}])$.

Noncompact real forms of exceptional simple complex Lie algebras are listed in Table 4.

We now state the final result of classification of real simple Lie algebras.

Theorem 1.6. *Noncommutative real simple Lie algebras are exhausted up to isomorphism by the real forms $\mathfrak{a}$ listed in Theorem 1.5, by the real forms* *of the exceptional simple complex Lie algebras, and by the Lie algebras* $\mathfrak{g}_{\mathbb{R}}$, *where* $\mathfrak{g}$ *are different noncommutative complex simple Lie algebras.*

Proof. One can easily see that if $\mathfrak{g}$ is a noncommutative complex simple Lie algebra, then any real form of $\mathfrak{g}$ and the algebra $\mathfrak{g}_{\mathbb{R}}$ is simple. If a real Lie algebra $\mathfrak{g}$ is simple but the algebra $\mathfrak{g}(\mathbb{C})$ is not simple, then $\mathfrak{g}(\mathbb{C}) = \mathfrak{a} \oplus \overline{\mathfrak{a}}$, where $\overline{\mathfrak{a}}$ is an ideal in $\mathfrak{g}(\mathbb{C})$. Setting $I(y + \overline{y}) = iy - i\overline{y}$ $(y \in \mathfrak{a})$, we obtain a complex structure on the Lie algebra $\mathfrak{a}$, which in this case becomes a simple complex Lie algebra. $\square$

Note that if $\mathfrak{g}$ is a complex Lie algebra and the algebra $\mathfrak{g}_{\mathbb{R}}$ is compact, then $\mathfrak{g}$ is commutative. Thus the operation of complexification defines a bijection between simple compact Lie algebras and simple complex Lie algebras (see Theorem 1.1 and 1.2).

Finally we remark that Theorem 1.6 completely solves the problem of classifying semisimple real Lie algebras, because by Chap. 1, Theorem 2.3 any semisimple Lie algebra uniquely decomposes into the direct sum of noncommutative simple ideals.

Example. There are the following isomorphisms between the classical Lie algebras of different series (see Sect. 1.1):

$$\begin{gathered}
\mathfrak{so}_3 \simeq \mathfrak{su}_2 = \mathfrak{sp}_1, \quad \mathfrak{so}_6 \simeq \mathfrak{su}_4, \\
\mathfrak{so}_{1,2} \simeq \mathfrak{su}_{1,1} \simeq \mathfrak{sl}_2(\mathbb{R}) = \mathfrak{sp}_2(\mathbb{R}), \quad \mathfrak{so}_{1,5} \simeq \mathfrak{sl}_2(\mathbb{H}), \\
\mathfrak{so}_4 \simeq \mathfrak{su}_2 \oplus \mathfrak{su}_2, \quad \mathfrak{so}_{2,4} \simeq \mathfrak{su}_{2,2}, \\
\mathfrak{so}_{1,3} \simeq \mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}}, \quad \mathfrak{so}_{3,3} \simeq \mathfrak{sl}_4(\mathbb{R}), \\
\mathfrak{so}_{2,2} \simeq \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R}), \quad \mathfrak{u}_2^*(\mathbb{H}) \simeq \mathfrak{su}_2 \oplus \mathfrak{sl}_2(\mathbb{R}), \\
\mathfrak{so}_5 \simeq \mathfrak{sp}_2, \quad \mathfrak{u}_3^*(\mathbb{H}) \simeq \mathfrak{su}_{1,3}, \\
\mathfrak{so}_{1,4} \simeq \mathfrak{sp}_{1,1}, \quad \mathfrak{u}_4^*(\mathbb{H}) \simeq \mathfrak{so}_{2,6}, \\
\mathfrak{so}_{2,3} \simeq \mathfrak{sp}_4(\mathbb{R}).
\end{gathered}$$

The classification of real simple algebras was obtained by É. Cartan at the beginning of this century. The approach to this classification outlined above is presented in detail in Helgason [1978], Onishchik and Vinberg [1990] and follows the ideas of F.R. Gantmakher [1939b]. Note that Yen Chih-ta [1959a] suggested using for the description of involutory automorphisms and real forms the diagrams essentially coinciding with the Kac diagrams corresponding to this case. In Yen Chih-ta [1959b] he used these diagrams to describe the group $\operatorname{Aut}\mathfrak{g}/\operatorname{Int}\mathfrak{g}$, where $\mathfrak{g}$ is a real semisimple Lie algebra.

§ 2. Compact Lie Groups and Complex Reductive Groups

The main goal of this section is to establish a one-to-one correspondence between compact Lie groups and reductive complex Lie groups, and also between homomorphisms of compact and reductive groups. In the language of category theory this means that there is an equivalence between the categories of compact Lie groups and reductive complex Lie groups. The results are applied to the structural theory of reductive groups and compact Lie groups.

2.1. Some Properties of Linear Representations of Compact Lie Groups. Let $R: K \to \mathrm{GL}(V)$ be a linear representation of a compact Lie group K over the field $\mathbb{R}$ or $\mathbb{C}$. As is known, the averaging operator $\mu: V \to V$ is associated with the representation R; this is a projection of the space V on the space V^G of invariant elements. The operator μ is defined by the formula

$$\mu v = \int_G R(g) v\, dg,$$

where the integral is taken with respect to the left-invariant density on G such that the volume of the entire group G is equal to 1 (see, for example, Gorbatsevich and Onishchik [1988], Chap. 2, Sect. 3.5). In this section we will derive some facts following from the existence of the averaging operator.

Proposition 2.1. *Let $\Omega \subset V$ be a nonempty convex set invariant under the representation $R: K \to \mathrm{GL}(V)$ of a compact Lie group K. Then K has a fixed point in Ω.*

Proof. If $v \in \Omega$, then $\mu v \in \Omega \cap V^k$. □

As an application, consider the representation of the group K in the space of symmetric bilinear (or, in the complex case, Hermitian) forms on V induced by the representation $R: K \to \mathrm{GL}(V)$. Applying Proposition 2.1 to the set of all positive definite forms, we obtain the following corollary.

Corollary (H. Weyl). *For any linear representation $R: K \to \mathrm{GL}(V)$ of a compact Lie group K there exists a scalar product in the space V with respect to which all the operators $R(g)$ $(g \in K)$ are orthogonal (or unitary). Any representation of a compact Lie group is completely reducible.*

Proposition 2.2. *Let S be a real affine space, K a compact Lie group, and $\varphi: K \to \mathrm{GA}(S)$ its affine representation. Then there exists a fixed point in S with respect to K.*

Proof. Let V be the vector space associated with S. Then S can be identified with the affine hyperplane $(V, 1)$ in the vector space $W = V \oplus \mathbb{R}$. Under this identification $\mathrm{GA}(S)$ is identified with the subgroup of $\mathrm{GL}(W)$ consisting of transformations taking V into itself and inducing the identity transformation on W/V. Therefore φ can be viewed as a linear representation in the space W taking V into itself. By the corollary to Proposition 2.1 there exists a K-invariant one-dimensional subspace in W complementary to V. Its common point with S is the desired fixed point. □

Now we will deduce from Proposition 2.1 that the orbits and the image of a linear representation of a compact group K are algebraic. Let $R: K \to \mathrm{GL}(V)$ be a real linear representation. Consider the algebra of polynomial functions $\mathbb{R}[V]$ on V. The representation R induces the representation R_* of the group K in $\mathbb{R}[V]$ given by the formula

$$(R_*(g)f)(x) = f(R(g^{-1})(x) \quad (g \in K,\ f \in \mathbb{R}[V],\ x \in V).$$

Any element from $\mathbb{R}[V]$ belongs to a finite-dimensional K-invariant subspace. Making use of Proposition 2.1 and the Weierstrass approximation theorem one can show without difficulty that for any two orbits of the group K in V there exists a K-invariant function in $\mathbb{R}[V]$ assuming different values on these orbits. This implies the following theorem.

Theorem 2.1. *The orbits of a finite-dimensional linear representation $R: K \to \mathrm{GL}(V)$ of a compact Lie group K are algebraic varieties in V. The image of $R(K)$ is an algebraic subgroup of* $\mathrm{GL}(V)$.

2.2. Self-adjointness of Reductive Algebraic Groups. In this section we give sufficient conditions for an algebraic linear group to be self-adjoint (see Chap. 1, Sect. 6.4). As we shall see, these conditions are actually equivalent to the fact that the group is reductive.

Proposition 2.3. *Let $G \subset \mathrm{GL}(V)$ be a complex algebraic linear group with a compact real form K. Then G is self-adjoint with respect to the Hermitian scalar product in V invariant under K, and $K = G \cap U(V)$ is an algebraic real form. The subgroup G^S, where S is any real structure in G taking K into itself, is also self-adjoint. Here $G^S \cap \mathrm{U}(V) = K^S$.*

Proof. Consider the automorphism $T: g \mapsto (g^*)^{-1}$ of $\mathrm{GL}(V)$ viewed as a real Lie group. Then $dT(\mathfrak{g}) = \mathfrak{g}$, whence $T(G^0) = G^0$. Since $G = G^0K$ and $K \subset \mathrm{U}(V)$, we have $T(G) = G$. The condition imposed on S implies that $TS = ST$. Hence $T(G^S) = G^S$, i.e. G^S is self-adjoint. □

Note, by way of application, the following result.

Theorem 2.2. *Let S be a real structure on a simply-connected semisimple complex Lie group G. Then the real form G^S is connected.*

Proof. According to Theorem 1.2, there exists a compact real form $K = G^T$ in G, where $TS = ST$. Then $\Theta = TS$ is an involutory automorphism of G. One can assume that G is linear. By Proposition 2.3, the group G^S is self-adjoint. Similarly, the group G^Θ is also self-adjoint and, by Chap. 3, Theorem 3.8, it is connected. In view of Corollary 1 to Theorem 6.6 of Chap. 1 the group $G^S \cap K = G^\Theta \cap K$ is connected, whence G^S is also connected. □

2.3. Algebraicity of a Compact Lie Group. It turns out that any compact Lie group admits a faithful linear representation and is therefore a real algebraic group. We start with the case when the group is simply-connected.

Proposition 2.4. *Any simply-connected Lie group K with a compact semisimple tangent algebra is isomorphic to a compact real form of some simply-connected complex semisimple Lie group. Any connected semisimple Lie group with a compact tangent algebra is compact and therefore has a finite centre.*

Proof. Consider a simply-connected complex Lie group G with tangent algebra $\mathfrak{k}(\mathbb{C})$. By virtue of Proposition 1.6, to a subalgebra $\mathfrak{k} \subset \mathfrak{k}(\mathbb{C})$ there corresponds a compact real form $K_1 \subset G$. According to Chap. 3, Theorem 2.7, we can assume that G is an algebraic subgroup of $\mathrm{GL}(V)$, where V is a complex vector space. By virtue of Proposition 2.3, the group G is self-adjoint and $K_1 = U(V) \cap G$. Corollary 1 to Theorem 6.6 of Chap. 1 shows that K_1 is simply-connected, whence $K = K_1$. □

Theorem 2.3. *Any compact Lie group K admits a faithful linear representation.*

Proof. By virtue of Chap. 1, Proposition 5.3, it is sufficient to prove the statement in the case where K is connected. Making use of Propositions 2.4 and 1.1, we see that K admits a finite covering by a group of the form $Z \times L$, where Z is a compact torus and L is a simply-connected semisimple compact Lie group. By virtue of Proposition 2.4, the latter group is a real form of some reductive complex algebraic group G. Hence K is a real form of some quotient group of G and is consequently also algebraic. □

Corollary. *On any compact Lie group there exists the structure of a real algebraic group.*

Theorem 2.3 and Proposition 1.1 imply the following theorem describing the structure of connected compact Lie groups.

Theorem 2.4. *Let K be a connected compact Lie group. Then the commutator $K_0 = (K, K)$ is a connected semisimple compact Lie subgroup of K, and K admits a locally direct decomposition $K = ZK_0$, where $Z = \mathrm{Rad}\, K$ is a compact torus coinciding with the identity component $Z(K)^0$ of the centre of K.*

2.4. Some Properties of Extensions of Compact Lie Groups. The main goal of this section is to prove the following theorem, which can be interpreted as the theorem on the triviality of the one-dimensional cohomology of a compact Lie group with values in an arbitrary linear representation.

Theorem 2.5. *Let G be a Lie group containing a vector normal subgroup A such that the group G/A is compact. Then $G = A \rtimes K$, where K is a compact subgroup of G.*

Proof. Since the automorphisms of A coincide with linear transformations of the corresponding vector space, the formula $R(g) = a(g)|A$ $(g \in G)$ defines a linear representation of the group in the space A. Let $\pi: G \to L = G/A$ be the natural homomorphism. Since $A \subset \operatorname{Ker} R$, a linear representation $R_0: L \to \mathrm{GL}(A)$ is defined such that $R = R_0\pi$. As is known (see Feigin and Fuks [1988], Chap. 2, Sect. 2.1), it is sufficient to prove that any smooth 1-cocycle f of the group L with values in R_0 is cohomologous to 0. In other words, for any smooth function $f: L \times L \to A$ such that

$$f(x, yz) + R_0(x)f(y, z) = f(xy, z) + f(x, y) \quad (x, y, z \in L) \tag{7}$$

there must exist a smooth function $h: L \to A$ satisfying the condition

$$f(x, y) = h(xy) - h(x) - R_0(x)h(y) \quad (x, y \in L). \tag{8}$$

The function h can be defined by the formula

$$h(x) = -\int_L f(x, y)\, dy,$$

where the integration is carried out with respect to the left-invariant metric on L for which the volume of the group is equal to 1. Indeed, integrating formula (7) with respect to y, one obtains (8). □

Note that the subgroup K in Theorem 2.5 is unique up to conjugacy in the group G. To be more precise, the following theorem holds.

Theorem 2.6. *Let $G = A \rtimes K$, where A is a vector group and K a compact Lie group. Then K is a maximal compact subgroup of G. Any maximal compact subgroup of G is conjugate to K.*

Proof. The first assertion is evident. To prove the second one, consider A as an affine space. Then an affine action $\widetilde{R}: G \to \mathrm{GA}(A)$ is defined such that $\widetilde{R}(k) = a(k)|A$ $(k \in K)$ and $\widetilde{R}(a)(b) = b + a$ $(a, b \in A)$. This action is transitive and the stabilizer of the point 0 coincides with K. If L is a compact subgroup of G, then, by virtue of Proposition 2.2, L has a fixed point in A, which implies that L is conjugate in G to a subgroup of K. □

We now use these results in order to obtain a description of a class of groups closest to compact ones, namely Lie groups having compact tangent algebra and finitely many connected components. The simplest among them are connected commutative Lie groups. Recall (see Vinberg and Onishchik [1988], Chap. 2, Theorem 2.12) that any connected commutative group G can be represented in the form $G = A \times B$, where A is a vector group and B a compact torus. Here B is the largest compact subgroup of G, which means that it contains all compact subgroups of this group.

Corollary. *Let G be a Lie group with finitely many connected components and compact tangent algebra. Suppose that $Z = \operatorname{Rad} G = Z(G^0)^0$. The non-compact part A of the group Z can be chosen in such a way that it is a normal* *subgroup of* G. *For any choice of such a subgroup* A *we have* $G = A \rtimes K$, *where* K *is a compact subgroup. Here* $G^0 = A \rtimes K^0$. *The subgroup* K *is a maximal compact one in* G, *and* K^0 *is maximal compact in* G^0. *Any maximal compact subgroup of* G *is conjugate to* K.

Proof. Making use of the complete reducibility of the adjoint representation of the group G in $\mathfrak{z}$, write $\mathfrak{z}$ in the form $\mathfrak{z} = \mathfrak{a} \oplus \mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are invariant under $\operatorname{Ad} G$ and correspond to the noncompact and compact A and B parts of the group Z, respectively. Then apply Theorems 2.5 and 2.6. □

2.5. Correspondence Between Real Compact and Complex Reductive Lie Groups. In this section we will prove that the operation of complexification of real algebraic groups establishes a one-to-one correspondence between compact Lie groups and reductive complex Lie groups (considered up to isomorphism).

Let G be a compact Lie group. The corollary to Theorem 2.3 implies that there exists a real algebraic group structure on K. *A priori* this structure depends on the choice of faithful representation, although, as we shall see below, it is in fact unique. Consider the complexification $K(\mathbb{C})$ of the compact algebraic group K, which can be assumed to be embedded in $\mathrm{GL}(V(\mathbb{C}))$ if $K \subset \mathrm{GL}(V)$. By the corollary to Proposition 2.1 all elements of the tangent algebra $\mathfrak{k}$ are skew-symmetric operators with respect to the scalar product in V. Therefore the bilinear form $b(X, Y) = \operatorname{tr} XY$ is negative definite on $\mathfrak{k}$ and nondegenerate on $\mathfrak{k}(\mathbb{C})$. It follows from Chap. 1, Proposition 6.2 that the algebraic group $K(\mathbb{C})$ is reductive.

Now we will show that the algebraic structure on K (and, of course, the group $K(\mathbb{C})$) are defined uniquely. This is a consequence of the following statement.

Proposition 2.5. *Let* K_1, K_2 *be compact real algebraic groups. Then each smooth homomorphism* $\varphi: K_1 \to K_2$ *can be uniquely extended to a polynomial homomorphism* $\varphi(\mathbb{C}): K_1(\mathbb{C}) \to K_2(\mathbb{C})$. *If* φ *is an isomorphism, so is* $\varphi(\mathbb{C})$.

Proof. Let $G_i = K_i(\mathbb{C})$ $(i = 1, 2)$. Then $G_1 \times G_2 = (K_1 \times K_2)(\mathbb{C})$. Let $\pi_i: G_1 \times G_2 \to G_i$ be a projection $(i = 1, 2)$. Consider the graph $\Gamma = \{(k, \varphi(k)) | k \in K_1\}$ of the homomorphism φ. By Theorem 2.1, Γ is an algebraic subgroup of $K_1 \times K_2$. The corresponding algebraic subgroup of $G_1 \times G_2$ is $\Gamma(\mathbb{C})$. The homomorphism $\pi_1: \Gamma(\mathbb{C}) \to G_1$ is injective on Γ. Making use of the polar decomposition, for example, one can deduce that π_1 is injective and is consequently a polynomial isomorphism of the group $\Gamma(\mathbb{C})$ on G_1. Then $\varphi(\mathbb{C}) = \pi_2 \pi_1^{-1}$ is the desired extension of the homomorphism φ. □

Remark. Let K be a compact Lie group. Then the algebra of polynomial functions $\mathbb{R}[K]$ corresponding to the unique real algebraic group structure on K coincides with the algebra of real representing functions on K (see Gorbatsevich and Onishchik [1988], Chap. 1, Sect. 1.5).

Now we formulate the final result.

Theorem 2.7. *Any reductive complex Lie group G has a compact real form, which is unique up to conjugacy. Conversely, any compact Lie group is a real form of some reductive complex Lie group. Two compact Lie groups are isomorphic if and only if so are the corresponding reductive complex Lie groups.*

Proof. It is sufficient to prove the first statement. For G semisimple and connected this was proved in Sect. 1.2. and 1.3, and if G is an algebraic torus, then it is evident. This implies that our statement holds for connected reductive Lie groups. In the general case it is proved by using the corollary to Theorems 2.5 and 2.6 (see Onishchik and Vinberg [1990], Sect. 5.2.3 for details). □

The following theorem provides the inner characterization of reductive complex Lie groups (i.e. complex Lie groups admitting the structure of a reductive algebraic group).

Theorem 2.8. *For a complex Lie group G the following conditions are equivalent:*

(a) *G is reductive;*
(b) *G has a compact real form;*
(c) *G has finitely many connected components and* $\mathrm{Rad}\, G \simeq (\mathbb{C}^\times)^m$ *for some* $m \geq 0$.

Proof. (a)⇒(b) by virtue of Theorem 2.7. (b)⇒(c) reduces to the easy case of a commutative group. (c)⇒(a) for a connected group G follows from Theorem 2.7 of Chap. 3, and in the general case from Proposition 5.3 of Chap. 1. □

2.6. Maximal Tori in Compact Lie Groups. In this section we consider Cartan subgroups of compact Lie groups, which, as it turns out, coincide with maximal tori. The term "torus" means a compact torus, i.e. a Lie group isomorphic to $\mathbb{T}^n$. Recall that any connected compact commutative Lie group is a torus. Let us first study Cartan subalgebras for compact Lie algebras.

Theorem 2.9. *The Cartan subalgebras of a compact Lie algebra $\mathfrak{k}$ coincide with maximal commutative subalgebras. Any two Cartan subalgebras of $\mathfrak{k}$ are conjugate.*

Proof. The first statement follows from the fact that the Lie algebra is reductive and all its elements are semisimple (see Chap. 1, Sect. 9.4).

To prove the second statement we first show that for any $x_1, x_2 \in \mathfrak{k}$ there exists $a \in \mathrm{Int}\, \mathfrak{k}$ such that $[a(x_1), x_2] = 0$. By Proposition 1.4, we can assume that $\mathfrak{k}$ is the tangent algebra of a connected compact Lie group K. Consider the function $f(k) = (x_1, (\mathrm{Ad}\, k)x_2)$ on K, where $(\ ,\ )$ is an invariant scalar product on $\mathfrak{k}$. Since K is compact, f has a minimum for some point k_0. Then for any $z \in \mathfrak{k}$ the function $\tilde{f}(t) = f(k_0 \exp tz)$ has a minimum for $t = 0$. Hence

$$0 = \tilde{f}'(0) = (x_1, \operatorname{Ad} k_0[z, x_2]) = ((\operatorname{Ad} k_0^{-1})x_1,\ [z, x_2])$$
$$= -([(\operatorname{Ad} K_0^{-1})x_1, x_2], z),$$

whence $[(\operatorname{Ad} k_0^{-1})x_1, x_2] = [x_1, (\operatorname{Ad} k_0)x_2] = 0$. Now let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be two Cartan subalgebras of K. Then $\mathfrak{a}_i = \mathfrak{z}_{x_i}$, where x_i is a regular element in $\mathfrak{a}_i$. Applying the above statement to x_1 and x_2 we see that $a(\mathfrak{a}_1) = \mathfrak{a}_2$, where $a \in \operatorname{Int} \mathfrak{k}$. □

Let K be a connected Lie group with compact tangent algebra $\mathfrak{k}$. Then maximal commutative subalgebras of $\mathfrak{k}$ correspond to maximal connected commutative Lie subgroups of K. If K is compact, then its maximal connected commutative subgroup is a torus; it is called the *maximal torus* of K. One can easily show that a compact subgroup A of a compact Lie group K is a (maximal) torus if and only if $A(\mathbb{C})$ is a (maximal) algebraic torus of the reductive complex algebraic group $K(\mathbb{C})$.

Theorem 2.10. *Let K be a connected Lie group with compact tangent algebra. Any maximal connected commutative Lie subgroup A of K coincides with its centre. In particular, A contains $Z(K)$ and is maximal among all commutative subgroups of K. Any element of K belongs to one of its maximal connected commutative subgroups. Maximal connected commutative Lie subgroups of K coincide with its Cartan subgroups. Such subgroups of K are conjugate.*

Proof. If K is compact, then the properties of its maximal tori follow from similar properties of algebraic maximal tori of the group $K(\mathbb{C})$ (see Springer [1989], Chap. 1, Sect. 3.5.5). In the general case $K = C \times L$, where C is a vector group and L is a connected compact Lie group (see the corollary in Sect. 2.4). Maximal connected commutative Lie subgroups of K are of the form $C \times A$, where A is a maximal torus in L, which implies our statements. □

§ 3. Cartan Decomposition

In this section we will study the so-called Cartan decomposition of real semisimple Lie groups. It is an analogue of the polar decomposition considered in Chap. 1, Sect. 6.4, and for semisimple algebraic groups these decompositions coincide. The Cartan decomposition leads to important theorems on the topological structure of connected Lie groups and conjugacy of their maximal compact subgroups. It also enables one to give a global classification of connected semisimple Lie groups.

3.1. Cartan Decomposition of a Semisimple Lie Algebra. Let $\mathfrak{g}$ be a real semisimple Lie algebra. A decomposition of $\mathfrak{g}$ into the direct sum of subspaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{9}$$

is called a *Cartan decomposition* if

(1) the transformation $\theta: x + y \mapsto x - y \, (x \in \mathfrak{k}, y \in \mathfrak{p})$ is an automorphism of $\mathfrak{g}$;

(2) the bilinear form

$$b_\theta(x, y) = -k(x, \theta y) \tag{10}$$

is positive definite on $\mathfrak{g}$.

Note that $\theta^2 = e$ and therefore the form b_θ is symmetric. One can easily check that conditions (1) and (2) are equivalent to the following ones:

(1′) $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$;

(2′) the form k is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

The latter conditions show that the subspace $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ in any Cartan decomposition (9). The subspace $\mathfrak{p}$ coincides with the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form and is called a the *Cartan subspace* in $\mathfrak{g}$.

Example 1. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ $(n \geq 2)$, $\mathfrak{k} = \mathfrak{so}_n$, and let $\mathfrak{p}$ be the space of all symmetric matrices of order n with zero trace. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition and $\theta(X) = -X^\top \, (X \in \mathfrak{g})$.

Example 2. Let $\mathfrak{u}$ be a compact real form of a semisimple complex Lie algebra $\mathfrak{g}$. Then the decomposition

$$\mathfrak{g}_\mathbb{R} = \mathfrak{u} \oplus I\mathfrak{u},$$

where I is the complex structure in $\mathfrak{g}$, is a Cartan decomposition. Here $\theta = \tau$ is the real structure corresponding to the real form $\mathfrak{u}$, and b_θ coincides with h_τ (see Sect. 1.2).

We will now describe all Cartan decompositions of a real semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{u}$ be a compact real form in $\mathfrak{g}(\mathbb{C})$ compatible with the real form $\mathfrak{g}$ (see Sect. 1.3). Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \text{where} \quad \mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}, \quad \mathfrak{p} = \mathfrak{g} \cap (i\mathfrak{u}). \tag{11}$$

Proposition 3.1. *The decomposition* (11) *is a Cartan one and* $\theta = \sigma\tau$, *where* σ *and* τ *are the real structures corresponding to* $\mathfrak{g}$ *and* $\mathfrak{u}$, *respectively. Any Cartan decomposition is of the form* (11), *where* $\mathfrak{u} = \mathfrak{k} \oplus (i\mathfrak{p})$ *is a compact real form compatible with* $\mathfrak{g}$.

Thus, there is a one-to-one correspondence between Cartan decompositions of $\mathfrak{g}$ and compact real forms of $\mathfrak{g}(\mathbb{C})$ compatible with $\mathfrak{g}$. Taking into account Theorem 1.2, one obtains the following statement.

Theorem 3.1. *Each real semisimple Lie algebra has a Cartan decomposition. Any two Cartan decompositions of* $\mathfrak{g}$ *can be taken into each other by an inner automorphism.*

Now we will state simple properties of Cartan decompositions of a semisimple Lie algebra $\mathfrak{g}$. Consider $\mathfrak{g}$ as a Euclidean space with the scalar product b_θ given by formula (10).

Proposition 3.2. *For any $x \in \mathfrak{g}$ we have $\operatorname{ad}\theta(x) = -(\operatorname{ad} x)^*$. In particular, the operator $\operatorname{ad} x$ is symmetric (skew-symmetric) if and only if $x \in \mathfrak{p}$ and if and only if $x \in \mathfrak{k}$.*

Proposition 3.3. *Let $\mathfrak{g} = \bigoplus_{i=1}^{s} \mathfrak{g}_i$, where $\mathfrak{g}_i$ are simple ideals, and let $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ $(i = 1, \dots, s)$ be their Cartan decompositions. Then $\mathfrak{k} = \bigoplus_{i=1}^{s} \mathfrak{k}_i$ and $\mathfrak{p} = \bigoplus_{i=1}^{s} \mathfrak{p}_i$ define a Cartan decomposition of $\mathfrak{g}$, and any Cartan decomposition of this algebra can be obtained in this way.*

Proposition 3.4. *If $\mathfrak{g}$ is simple, then the adjoint representation of the algebra $\mathfrak{k}$ in $\mathfrak{p}$ is irreducible and $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{p}$.*

Proof. Let $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $[\mathfrak{k}, \mathfrak{p}_i] \subset \mathfrak{p}_i$ $(i = 1, 2)$. Then $[\mathfrak{p}_1, \mathfrak{p}_2]$ is orthogonal to $\mathfrak{k}$ with respect to the Killing form, whence $[\mathfrak{p}_1, \mathfrak{p}_2] = 0$. If $\mathfrak{p}_1 \neq 0$, $\mathfrak{p}_2 \neq 0$, then $[\mathfrak{p}_1, \mathfrak{p}_1] \oplus \mathfrak{p}_1$ is an ideal in $\mathfrak{g}$ different from 0 and $\mathfrak{g}$. □

Proposition 3.5. *If $\mathfrak{g}$ contains no nonzero compact ideals, then $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ and the adjoint representation of the algebra $\mathfrak{k}$ in $\mathfrak{p}$ is faithful.*

Proof. The second statement follows from the first one and, by Proposition 3.3, it is sufficient to prove the first statement in the case where $\mathfrak{g}$ is simple and noncompact. One can easily see that $\mathfrak{g}_0 = [\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$ is an ideal in $\mathfrak{g}$. If $\mathfrak{g}$ is simple and $\mathfrak{p} \neq 0$, then $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$. □

3.2. Cartan Decomposition of a Semisimple Lie Group. Let G be a real semisimple Lie group (not necessarily connected) and let a Cartan decomposition (9) of its tangent algebra be given. In this section we will prove the existence of the corresponding global decomposition

$$G = KP,$$

where K is a Lie subgroup of G with tangent algebra $\mathfrak{k}$ and $P = \exp \mathfrak{p}$. This decomposition, described in Theorem 3.2, will be called the *Cartan decomposition* of G.

Denote by θ the involutory automorphism of $\mathfrak{g}$ corresponding to the decomposition (9), and consider $\mathfrak{g}$ as a Euclidean space with the scalar product b_θ defined by formula (10).

Theorem 3.2. *Let G be a real semisimple Lie group and let a Cartan decomposition (9) of its tangent algebra be given. We set $K = \{g \in G | \operatorname{Ad} g \in \mathrm{O}(\mathfrak{g})\}$, $P = \exp \mathfrak{p}$. Then $G = KP$ and every element $g \in G$ can be uniquely represented in the form $g = kp$, where $k \in K$, $p \in P$. The mapping $\varphi: K \times \mathfrak{p} \to G$ given by the formula*

$$\varphi(k, y) = k \exp y \qquad (k \in K, y \in \mathfrak{p})$$

is a diffeomorphism. The mapping $\Theta: kp \mapsto kp^{-1}$ *is an automorphism of* G *such that* $d\Theta = \theta$.

Proof. One can easily verify that $\theta a \theta^{-1} = (a^*)^{-1}$ for any $a \in \operatorname{Aut} \mathfrak{g}$. In particular, $\operatorname{Aut} \mathfrak{g}$ is a self-adjoint linear group. Consider its polar decomposition $\operatorname{Aut} \mathfrak{g} = \hat{K}\hat{P}$ (see Chap. 1, Theorem 6.6). Then $\operatorname{Ad} K \subset \hat{K}$, $\operatorname{Ad} P = \hat{P}$, and the statement of the theorem follows from the fact that the diagram

$$\begin{array}{ccccc} & K \times \mathfrak{p} & \xrightarrow{\varphi} & G & \\ \operatorname{Ad} \times \operatorname{ad} & \downarrow & & \downarrow & \operatorname{Ad} \\ & \hat{K} \times \hat{\mathfrak{p}} & \xrightarrow{\hat{\varphi}} & \operatorname{Aut} \mathfrak{g} & \end{array},$$

where $\hat{\varphi}$ defines the polar decomposition of the group $\operatorname{Aut} \mathfrak{g}$, commutes. □

Corollary 1. *The group* G *is diffeomorphic to* $K \times \mathbb{R}^m$, *where* $m = \dim \mathfrak{p}$.

Corollary 2. *The subgroup* K *coincides with the subgroup* $G^\Theta = \{g \in G | \Theta(g) = g\}$; *its tangent algebra is* $\mathfrak{k}$.

Corollary 3. *The subgroup* K *coincides with the normalizer* $N_G(K^0)$.

Corollary 4. $Z(G) \subset Z(K)$, $Z(G^0) \subset Z(K^0)$.

Corollary 5. *The subgroup* K *is compact if and only if* G *has finitely many connected components and the centre* $Z(G^0)$ *is finite.*

Example 1. Let $G \subset \operatorname{GL}(V)$ be a complex semisimple algebraic linear group, and K its compact real form. Then the Cartan decomposition of G corresponding to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ of its tangent algebra (see Sect. 3.1, Example 2) coincides with the polar decomposition described in Theorem 3.2.

Example 2. Let $G \subset \operatorname{GL}(V)$, where V is a vector space over $\mathbb{R}$, be a semisimple linear Lie group. Then the centre $Z(G^0)$ is finite, since it is contained in the centre of the connected semisimple complex algebraic group ${}^a(G^0) \subset \operatorname{GL}(V(\mathbb{C}))$. Therefore if G has finitely many connected components, then the subgroup K from Theorem 3.2 is compact (Corollary 5).

Example 3. If the subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is semisimple and G has finitely many connected components, then the subgroup K is compact. If $\mathfrak{k}$ is not semisimple, then by Corollary 1 of Theorem 3.2 applied to a simply-connected group G, the subgroup K is also simply-connected and consequently it is not compact. The simplest example of such a group is $G = \mathcal{A} = \widetilde{SL}_2(\mathbb{R})$. Here $\mathfrak{k} = \mathfrak{so}_2$, $K \simeq \mathbb{R}$, therefore by Corollary 1 the group $\mathcal{A}$ is diffeomorphic to $\mathbb{R}^3$.

Example 4. Let $G = \operatorname{PSL}_2(\mathbb{R}) = \operatorname{SL}_2(\mathbb{R})/\{E, -E\}$, and let $\pi: \operatorname{SL}_2(\mathbb{R}) \to \operatorname{PSL}_2(\mathbb{R})$ be the natural homomorphism. If $\operatorname{SL}_2(\mathbb{R}) = \operatorname{SO}_2 P$ is the Cartan decomposition, then $\operatorname{PSL}_2(\mathbb{R}) = \pi(\operatorname{SO}_2)\pi(P)$. This is the Cartan decomposition of the group $\operatorname{PSL}_2(\mathbb{R})$. Since $\pi(\operatorname{SO}_2) \simeq \operatorname{SO}_2$, we have $\pi_1(\operatorname{PSL}_2(\mathbb{R})) \simeq \mathbb{Z}$, which implies that $Z(\mathcal{A}) \simeq \mathbb{Z}$.

Suppose that $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{R}$ admitting no complex structure, i.e. a real form of a complex simple Lie algebra $\mathfrak{g}(\mathbb{C})$. Then the automorphism θ extended by linearity to $\mathfrak{g}(\mathbb{C})$ is the involutory automorphism of the algebra $\mathfrak{g}(\mathbb{C})$ corresponding, by Theorem 1.3, to the real form $\mathfrak{g}$, and the subalgebra $\mathfrak{k}(\mathbb{C})$ coincides with $\mathfrak{g}(\mathbb{C})^{\theta}$. According to the classification of Sect. 1.4, the case of a semisimple subalgebra $\mathfrak{k}$ corresponds to involutory automorphisms of type I, and that of a non-semisimple subalgebra to automorphisms of type II; in the latter case $\mathfrak{k}$ has a one-dimensional centre.

3.3. Conjugacy of Maximal Compact Subgroups of Semisimple Lie Groups. In this section we will describe maximal compact subgroups of semisimple Lie groups with finitely many connected components. In particular, we will prove that all these subgroups are conjugate. First, we consider the general case and formulate a conjugacy theorem for subgroups more general than compact ones.

A subgroup M of a semisimple Lie group G is called *pseudocompact* if the linear group $\operatorname{Ad} M \subset \mathrm{GL}(\mathfrak{g})$ is compact. Any compact group is pseudocompact.

Theorem 3.3. *Let $G = KP$ be a Cartan decomposition of a real semisimple Lie group G. Then K is a maximal pseudocompact subgroup of G. For any pseudocompact subgroup $M \subset G$ there exists $g \in P$ such that $gMg^{-1} \subset K$.*

Before we prove this theorem let us deduce several corollaries of it. If G has finitely many connected components, then, by Theorem 3.2, K has the same property. Since $\mathfrak{k}$ is compact, the corollary to Theorems 2.5 and 2.6 implies that $K = A \rtimes L$, where $A \simeq \mathbb{R}^s$ and L is a maximal compact subgroup of K, which is unique up to conjugacy.

Corollary 1. *If G has finitely many connected components, then any maximal compact subgroup L of K is a maximal compact subgroup of G. Any maximal compact subgroup of G is conjugate to L by an automorphism of the form $a(g)$, where $g \in G^0$.*

Corollary 2. *A semisimple Lie group G with finitely many connected components is diffeomorphic to $L \times \mathbb{R}^N$, where L is any maximal compact subgroup of G.*

Corollary 3. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and M a compact subgroup of* $\operatorname{Aut} \mathfrak{g}$. *Then $\mathfrak{g}$ admits a Cartan decomposition invariant under M.*

Let $\mathbb{E}$ be a finite-dimensional Euclidean space. Then the group $\mathrm{GL}(\mathbb{E})$ acts on the manifold $\mathrm{P}(\mathbb{E})$ of positive definite symmetric operators in $\mathbb{E}$ by the formula

$$\mathrm{Sq}(A)(X) = AXA^* (X \in \mathrm{P}(\mathbb{E}), A \in \mathrm{GL}(\mathbb{E})).$$

It is known from linear algebra that this action is transitive, and the stabilizer of the identity operator $E \in \mathrm{P}(\mathbb{E})$ is the orthogonal group $\mathrm{O}(\mathbb{E})$. Consider the differentiable function r of two variables on $\mathrm{P}(\mathbb{E})$ defined by the formula

$$r(X,Y) = \operatorname{tr}(XY^{-1}).$$

Clearly, r is invariant under the Sq action.

Let Ω be a compact set in $\mathrm{P}(\mathbb{E})$. Define a continuous function ρ on $\mathrm{P}(\mathbb{E})$ by the formula

$$\rho(X) = \max_{Y\in\Omega} r(X,Y). \tag{12}$$

Let $\mathrm{SP}(\mathbb{E}) = \mathrm{P}(\mathbb{E}) \cap \mathrm{SL}(\mathbb{E})$. Clearly, $\mathrm{SP}(\mathbb{E})$ is closed in $\mathrm{SL}(\mathbb{E})$ and is therefore closed in the space $\mathfrak{gl}(\mathbb{E})$ of all linear operators of the space $\mathbb{E}$.

Lemma 3.1. *For any compact set $\Omega \subset \mathrm{P}(\mathbb{E})$ the function ρ defined by formula* (12) *attains its minimum on any closed subset $F \subset \mathrm{SP}(\mathbb{E})$.*

Lemma 3.2. *For any fixed $X, Y \in \mathrm{P}(\mathbb{E})$, $X \neq E$, the functions*

$$f_{X,Y}(t) = r(X^t, Y),$$

$$\varphi_X(t) = \rho(X^t)$$

are strictly convex on the entire real axis.

We now go back to the situation of Theorem 3.3. Consider the tangent algebra $\mathfrak{g}$ of the group G as a Euclidean space with the scalar product (10) corresponding to our Cartan decomposition. Let $\hat{P} = \exp \operatorname{ad} \mathfrak{p}$. It is a closed submanifold in $\mathrm{SP}(\mathfrak{g})$ coinciding with the orbit of the point E under the action $\mathrm{Sq} \circ \mathrm{Ad}$ of the group G on $\mathrm{P}(\mathfrak{g})$. The stabilizer of E with respect to this action is K. One deduces from Lemmas 3.1 and 3.2 that for any compact set $\Omega \subset \mathrm{P}(\mathfrak{g})$ the function ρ on $\mathrm{P}(\mathfrak{g})$ defined by the formula (12) has a unique minimum point in $\hat{P}$.

Let M be a pseudocompact subgroup of G. Then the set $\Omega = \mathrm{Sq}(\mathrm{Ad}\, M)(E)$ is compact in $\mathrm{P}(\mathfrak{g})$. The function ρ corresponding to it is invariant under M. Therefore its unique minimum point in $\hat{P}$ is invariant under M. Since G^0 acts on transitively $\hat{P}$, this implies that $gMg^{-1} \subset K$ for some $g \in G^0$.

3.4. Topological Structure of Lie Groups. This section is devoted to a generalization of Corollaries 1 and 2 to Theorem 3.3 to the case of arbitrary Lie groups with finitely many connected components. Let G be a Lie group, and $G_1, \ldots, G_k$ its Lie subgroups. We say that G can be topologically decomposed into the direct product of the subgroups $G_1, \ldots, G_k$ if the mapping $\mu\colon (g_1, \ldots, g_k) \mapsto g_1 \ldots g_k$ of the manifold $G_1 \times \ldots \times G_k$ onto G is a diffeomorphism.

Theorem 3.4. *Any Lie group G with finitely many connected components can be topologically decomposed into the direct product of Lie subgroups K, $H_1, \ldots H_s$, where K is a maximal compact subgroup, and $H_1, \ldots, H_s$ are noncompact one-dimensional subgroups. In particular, the manifold G is diffeomorphic to $K \times \mathbb{R}^s$.*

Proof. Suppose that the theorem holds for all groups whose dimension is less than $\dim G$. If G is semisimple, then we apply Corollary 1 to Theorem 3.3. If the group G is not semisimple, then it contains a connected commutative normal Lie subgroup $N \neq \{e\}$, which is either compact or isomorphic to $\mathbb{R}^p$. The group G/N can be topologically decomposed into the direct product $G/N = K'H_1' \dots H_t'$, where K' is a maximal compact subgroup, and H_i' are one-dimensional Lie subgroups. Let $\pi: G \to G/N$ be the natural homomorphism, $L = \pi^{-1}(K')$, and H_i one-dimensional Lie subgroups of G such that $\pi(H_i) = H_i'$. Then G can be topologically decomposed into the direct product $G = LH_1 \dots H_t$. If N is compact, then L is a maximal compact subgroup of G. If N is a vector group, then we apply Theorem 2.5, which implies that $L = K \ltimes N$, where K is a maximal compact subgroup of L, and therefore also of G. □

Theorem 3.5. *Let G be a Lie group with finitely many connected components. Then any two maximal compact subgroups of G are conjugate.*

Proof. As in Theorem 3.4, the proof is achieved by induction on $\dim G$. If G is semisimple, we apply Corollary 2 to Theorem 3.3. If G is not semisimple, consider the natural homomorphism $\pi: G \to G/N$, where N is the normal Lie subgroup introduced in the proof of Theorem 3.3. If N is compact, then any maximal subgroup of G is of the form $\pi^{-1}(L)$, where L is a maximal compact subgroup of G/N. The validity of our theorem for G/N implies that it also holds for G. If N is a vector group, then one can prove, with the help of Theorem 2.5, that π defines a bijection between maximal compact subgroups of G and G/N. If K_1 and K_2 are maximal compact subgroups of G, one can assume that $\pi(K_1) = \pi(K_2) = K'$. An application of Theorem 2.6 to the group $L = \pi^{-1}(K')$ shows that K_1 and K_2 are conjugate. □

3.5. Classification of Connected Semisimple Lie Groups. This section is devoted to the global classification of connected real semisimple Lie groups. It turns out that as in the complex case this classification can be given in terms of tangent algebras and lattices in some commutative subalgebras of these algebras. By a "torus" we always mean a compact torus.

Let G be a connected semisimple Lie group. A connected subgroup $A \subset G$ will be called a *pseudotorus* if $\operatorname{Ad} A$ is a torus. It follows from Theorems 2.9 and 3.3 that all maximal pseudotori of G are conjugate. A commutative subalgebra $\mathfrak{a}$ is said to be *pseudotoric* if the subgroup $\exp \operatorname{ad} \mathfrak{a} \subset \operatorname{Int} \mathfrak{g}$ is compact, i.e. if it is a torus. A subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is (maximal) pseudotoric if and only if it is the tangent algebra of a maximal pseudotorus in G. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{g}$ is a Cartan decomposition, then any maximal commutative subalgebra of the algebra $\mathfrak{k}$ is a pseudotoric one.

Let A be a maximal pseudotorus of a connected semisimple Lie group G, and let $\mathfrak{a}$ be the corresponding maximal pseudotoric subalgebra of $\mathfrak{g}$. The kernel of the surjective homomorphism $\exp = \exp_G: \mathfrak{a} \to A$ is a lattice in $\mathfrak{a}$, which, as we shall see below, defines (together with the Lie algebra $\mathfrak{g}$)

the group G uniquely up to isomorphism. It is, however, more convenient to consider the lattice $L(G) = \operatorname{Ker} \mathcal{E} \subset \mathfrak{a}(\mathbb{C})$, where $\mathcal{E} = \mathcal{E}_G : i\mathfrak{a} \to G$ is the homomorphism defined by the formula

$$\mathcal{E}(x) = \exp 2\pi i x.$$

The lattice $L(G)$ is called the *characteristic lattice* of the group G.

It follows from Corollary 4 to Theorem 3.2 and Theorem 2.9 that the centre of a connected semisimple Lie group is contained in any maximal pseudotorus of it. This implies the following proposition.

Proposition 3.6. *Let G_1, G_2 be two connected semisimple Lie subgroups with the same tangent algebra $\mathfrak{g}$, and suppose that $\mathfrak{a} \subset \mathfrak{g}$ is a maximal pseudotoric algebra. The characteristic lattices of G_1 and G_2 satisfy the condition $L(G_1) \subset L(G_2)$ if and only if there exists a homomorphism $\pi: G_1 \to G_2$ such that $d\pi = e$. Here $\mathcal{E}_{G_1}^{-1}(\operatorname{Ker} \pi) = L(G_2)$, whence*

$$\operatorname{Ker} \pi \simeq L(G_2)/L(G_1).$$

The following statement follows from Theorem 2.10.

Theorem 3.6. *Let G_j $(j = 1, 2)$ be two connected semisimple Lie groups, $\mathfrak{a}_j \subset \mathfrak{g}_j$ maximal pseudotoric subalgebras of their tangent algebras, and $L(G_j) \subset i\mathfrak{a}_j$ their characteristic lattices. The groups G_1 and G_2 are isomorphic if and only if there exists an isomorphism $\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\varphi(\mathfrak{a}_1) = \mathfrak{a}_2$ and $\varphi(\mathbb{C})(L(G_1)) = L(G_2)$.*

To complete the classification one has to find out which lattices may serve as characteristic ones.

Again let G be a connected semisimple Lie group and $\mathfrak{a}$ a maximal pseudotoric subalgebra of $\mathfrak{g}$. The lattice $L_0 = L(\widetilde{G}) \subset i\alpha$ corresponds to the simply-connected covering $\widetilde{G}$ of the group G. On the other hand, the lattice $L_1 = L(\operatorname{Int} \mathfrak{g}) \subset i \operatorname{ad} \mathfrak{a} \subset i \operatorname{ad} \mathfrak{g}$ corresponds to the group $\operatorname{Int} \mathfrak{g}$. Identifying $\mathfrak{g}$ with $\operatorname{ad} \mathfrak{g}$ with the help of the isomorphism ad, we have $L_1 \subset i\mathfrak{a}$. Proposition 3.6 implies that

$$L_0 \subset L(G) \subset L_1.$$

Proposition 3.7. *The following relations hold: $\mathcal{E}^{-1}(Z(G)) = L_1$, $Z(G) = \mathcal{E}(L_1) \simeq L_1/L(G)$, $\pi_1(G) \simeq L(G)/L_0$. Any lattice L satisfying the condition $L_0 \subset L \subset L_1$ is the characteristic lattice of a connected Lie group with tangent algebra $\mathfrak{g}$.*

We will now describe the lattices L_0 and L_1. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{g}$. Denote by θ both the involutory automorphism of $\mathfrak{g}$ associated with this decomposition and the extension of this automorphism onto the complex semisimple Lie algebra $\mathfrak{g}(\mathbb{C})$. Then $\mathfrak{k}(\mathbb{C}) = \mathfrak{g}(\mathbb{C})^\theta$. We may assume that $\mathfrak{a}$ is a maximal commutative subalgebra of the Lie algebra $\mathfrak{k}$. We have $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{z}(\mathfrak{k})$, where $\mathfrak{k}_0 = [\mathfrak{k}, \mathfrak{k}]$, $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{z}(\mathfrak{k})$, and $\mathfrak{a}_0$ is a maximal commutative subalgebra of the Lie algebra $\mathfrak{k}_0$. The subalgebras $\mathfrak{t} = \mathfrak{a}(\mathbb{C})$ and $\mathfrak{t}_0 = \mathfrak{a}_0(\mathbb{C})$

are Cartan subalgebras of the reductive algebraic subalgebra $\mathfrak{k}(\mathbb{C}) \subset \mathfrak{g}(\mathbb{C})$ and the semisimple Lie algebra $\mathfrak{k}_0(\mathbb{C}) = [\mathfrak{k}(\mathbb{C}), \mathfrak{k}(\mathbb{C})]$, respectively.

Proposition 3.8. *The lattice L_0 coincides with $Q^\vee(\mathfrak{k}_0(\mathbb{C})) \subset i\mathfrak{a}_0$, where $Q^\vee(\mathfrak{k}_0(\mathbb{C}))$ is the dual root lattice of $\mathfrak{k}_0(\mathbb{C})$ with respect to $\mathfrak{t}_0$.*

By Chap. 3, Theorem 3.13, the centralizer $\mathfrak{h}$ of the subalgebra $\mathfrak{t}$ in $\mathfrak{g}(\mathbb{C})$ is the (unique) Cartan subalgebra of $\mathfrak{g}(\mathbb{C})$ containing $\mathfrak{t}$, and $\theta(\mathfrak{h}) = \mathfrak{h}$.

Proposition 3.9. *We have $L_1 = P^\vee \cap \mathfrak{t}$, where $P^\vee$ is the weight lattice of the dual root system $\Delta^\vee_{\mathfrak{g}(\mathbb{C})}$ of $\mathfrak{g}(\mathbb{C})$ with respect to $\mathfrak{h}$.*

The lattice L_1 may be expressed in terms of the automorphism θ. According to Theorem 3.13 of Chap. 3 there is a system of simple roots Π in $\Delta_{\mathfrak{g}(\mathbb{C})}$ invariant under $\theta^\top$. Let $\tau = (\theta^\top)^{-1} \in \operatorname{Aut}\Pi$. As in Chap. 3, Sect. 3.1, we denote by the same letter τ the automorphism of $\mathfrak{g}(\mathbb{C})$ defined by the corresponding permutation of canonical generators. Then $\mathfrak{t}$ is a Cartan subalgebra of the semisimple Lie algebra $\mathfrak{g}(\mathbb{C})^\tau$.

Proposition 3.10. *The lattice L_1 coincides with the weight lattice $P^\vee(\mathfrak{g}(\mathbb{C})^\tau)$ of the dual root system $\Delta^\vee_{\mathfrak{g}(\mathbb{C})^\tau}$ of $\mathfrak{g}(\mathbb{C})^\tau$ with respect to $\mathfrak{t}$.*

Propositions 3.7, 3.8, 3.9, 3.10 imply the following statements.

Theorem 3.7. *Let $\mathfrak{a}$ be a maximal commutative subalgebra of the algebra $\mathfrak{k}$. The lattice $L \subset i\mathfrak{a}$ is a characteristic lattice for a connected Lie group with tangent algebra $\mathfrak{g}$ if and only if*

$$Q^\vee(\mathfrak{k}_0(\mathbb{C})) \subset L \subset P^\vee(\mathfrak{g}(\mathbb{C})^\tau),$$

where τ is the automorphism of the system of simple roots Π induced by θ.

Theorem 3.8. *For any connected Lie group G with tangent algebra $\mathfrak{g}$ we have $\mathcal{E}^{-1}(Z(G)) = P^\vee(\mathfrak{g}(\mathbb{C})^\tau)$, whence*

$$Z(G) \simeq P^\vee(\mathfrak{g}(\mathbb{C})^\tau)/L(G).$$

We also have

$$\pi_1(G) \simeq L(G)/Q^\vee(\mathfrak{k}_0(\mathbb{C})).$$

In particular, for a simply-connected group $G = \widetilde{G}$ we have

$$Z(\widetilde{G}) \simeq P^\vee(\mathfrak{g}(\mathbb{C})^\tau)/Q^\vee(\mathfrak{k}_0(\mathbb{C})).$$

3.6. Linearizer of a Semisimple Lie Group. In this section we present (in terms of characteristic lattices) the explicit form of the linearizer of a connected semisimple Lie group (see Chap. 1, Sect. 5.4).

Let G be a connected semisimple Lie group, $\pi: \widetilde{G} \to G$ its simply-connected covering, $\Gamma = \operatorname{Ker}\pi$, F a simply-connected complex Lie group with tangent algebra $\mathfrak{g}(\mathbb{C})$, and $j: \widetilde{G} \to F$ a homomorphism such that dj is the identity embedding $\mathfrak{g} \to \mathfrak{g}(\mathbb{C})$. Then $j(\widetilde{G})$ is a real form of the group F with tangent algebra $\mathfrak{g}$ such that $j(\Gamma) \subset Z(F)$. Clearly, there exists a homomorphism $\Phi: G \to F/j(\Gamma)$ such that the diagram

$$\begin{array}{ccccc} & \widetilde{G} & \xrightarrow{j} & F & \\ \pi & \downarrow & & \downarrow & \tilde{\pi}\,, \\ & G & \xrightarrow{\Phi} & F/j(\Gamma) & \end{array} \tag{13}$$

where $\tilde{\pi}$ is the natural homomorphism, commutes. By Chap. 3, Theorem 2.7 $F/j(\Gamma)$ admits a faithful linear representation. This implies the following proposition.

Proposition 3.11. *The linearizer $\Lambda(G)$ is discrete and coincides with the kernel of the homomorphism Φ in the diagram* (13). *We have $G_{\text{lin}} \simeq \Phi(G)$.*

Corollary. *Let G be a real form of a simply-connected semisimple complex Lie group F, and $\pi: \widetilde{G} \to G$ its simply-connected covering. Then $\Lambda(\widetilde{G}) =$ $\operatorname{Ker}\pi$ and $\widetilde{G}_{\text{lin}} \simeq G$. If G is not simply-connected, then the group $\widetilde{G}$ is not linearizable.*

Proof. In our case $j = i\pi$, where $i: G \to F$ is an embedding, so $j(\Gamma) = \{e\}$ and Φ is identified with j. □

Example. Let $G = \mathrm{SL}_n(\mathbb{R})$. By the corollary to Proposition 3.11, we have $G_{\text{lin}} = \mathrm{SL}_n(\mathbb{R})$. Thus among all the groups locally isomorphic to $\mathrm{SL}_n(\mathbb{R})$ only $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{PSL}_n(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R})/Z(\mathrm{SL}_n(\mathbb{R}))$ are linearizable. If $n = 2$, then $Z(G) \simeq \mathbb{Z}_2$ and $Z(\widetilde{G}) \simeq \mathbb{Z}$ (see Sect. 3.2, Example 4), so $\Lambda(\widetilde{G}) = Z(\widetilde{G})^2 \simeq \mathbb{Z}$. If $n \geq 3$, then $\Lambda(\widetilde{G}) \simeq \mathbb{Z}_2$.

We will now express the linearizer $\Lambda(G)$ in terms of the characteristic lattice of the group G. Suppose, as in Sect. 3.5, that we are given a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and let $\mathfrak{a}$ be a maximal commutative subalgebra of the algebra $\mathfrak{k}$, $\mathfrak{t} = \mathfrak{a}(\mathbb{C}) \subset \mathfrak{k}(\mathbb{C})$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}(\mathbb{C})$ containing $\mathfrak{t}$.

Theorem 3.9. *For any connected Lie group G with tangent algebra $\mathfrak{g}$ we have*

$$\mathcal{E}^{-1}(\Lambda(G)) = L(G) + Q^\vee \cap \mathfrak{t},$$

where $Q^\vee$ is the dual root lattice of the Lie algebra $\mathfrak{g}(\mathbb{C})$ with respect to $\mathfrak{h}$. Therefore

$$\Lambda(G) \simeq (Q^\vee \cap \mathfrak{t})/(Q^\vee \cap L(G)).$$

In particular, for a simply-connected group $G = \widetilde{G}$ we have

$$\mathcal{E}^{-1}(\Lambda(\widetilde{G})) = Q^\vee \cap \mathfrak{t},$$

$$\Lambda(\widetilde{G}) \simeq (Q^\vee \cap \mathfrak{t})/Q^\vee(\mathfrak{k}_0(\mathbb{C})).$$

An explicit calculation of the linearizer for various simply-connected simple groups was carried out in Sirota and Solodovnikov [1963] (see also Table 10 in the book by Onishchik and Vinberg [1990]).

§ 4. Real Root Decomposition

In this section we consider the root decomposition of a real semisimple Lie algebra with respect to a maximal subalgebra expressed in the adjoint representation by diagonal matrices. The analysis of the corresponding root system enables one to assign to a real semisimple Lie algebra the so-called Satake diagram, which can be considered as a generalization of the Dynkin diagram. Satake diagrams can be used in the classification of real semisimple Lie algebras carried out in Sect. 1 by another method (see Araki [1962]). Another application of the real root decomposition is Iwasawa's theorem generalizing the classical Gram-Schmidt orthogonalization method.

4.1. Maximal $\mathbb{R}$-Diagonalizable Subalgebras. Let $\mathfrak{g}$ be a real Lie algebra. A subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is said to be $\mathbb{R}$-*diagonalizable* if there is a basis in $\mathfrak{g}$ with respect to which all operators $\operatorname{ad} x$ $(x \in \mathfrak{a})$ are expressed by diagonal matrices. In this case we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \right), \tag{13}$$

where Σ is a finite set of nonzero elements of the space $\mathfrak{a}^*$ and $\mathfrak{g}_\lambda = \mathfrak{g}_\lambda(\mathfrak{a})$ is the weight subspace with respect to the representation $\operatorname{ad}|\mathfrak{a}$. The set Σ is called the *root system* of the Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{a}$ and the decomposition (13) is called the *root decomposition.* As in the complex case, for any $\lambda, \mu \in \Sigma \cup \{0\}$ we have

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \begin{cases} \subset \mathfrak{g}_{\lambda+\mu} & \text{if } \lambda + \mu \in \Sigma \cup \{0\} \\ = 0 & \text{if } \lambda + \mu \notin \Sigma \cup \{0\} \end{cases}.$$

In particular, $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ (the centralizer of the subalgebra $\mathfrak{a}$).

Now suppose that $\mathfrak{g}$ is semisimple. Clearly, any $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is commutative. If $x \in \mathfrak{a}$ and $\alpha(x) = 0$ for all $\alpha \in \Sigma$, then $x \in \mathfrak{z}(\mathfrak{g})$ and therefore $x = 0$. This makes it evident that Σ generates the space $\mathfrak{a}^*$.

Proposition 4.1. *Let $\mathfrak{a}$ be a maximal diagonalizable subalgebra of a Lie algebra $\mathfrak{g}$. Then there is a Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{14}$$

such that $\mathfrak{a} \subset \mathfrak{p}$ and is maximal among the subalgebras of $\mathfrak{g}$ contained in $\mathfrak{p}$. Conversely, any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ contained in $\mathfrak{p}$ and maximal among such subalgebras is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$. The centralizer $\mathfrak{g}^0$ of such a subalgebra is of the form

$$\mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}, \tag{15}$$

where $\mathfrak{m} = \mathfrak{g}^0 \cap \mathfrak{k}$.

Let $\Sigma \subset \mathfrak{a}^*$ be the root system associated with a maximal diagonalizable subalgebra $\mathfrak{a}$. Note that $\Sigma \neq \emptyset$ if and only if $\mathfrak{a} \neq 0$. Any $\alpha \in \Sigma$ defines the hyperplane $P_\alpha = \operatorname{Ker} \alpha$ in $\mathfrak{a}$.

The elements of the nonempty open set

$$\mathfrak{a}_{\text{reg}} = \mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} P_\alpha$$

are said to be *regular*. The centralizer of any regular element of $\mathfrak{a}$ coincides with $\mathfrak{a}$.

Theorem 4.1. *Let K be the maximal compact subgroup of G corresponding to the subalgebra $\mathfrak{k}$ of the decomposition* (14). *Any two maximal subalgebras of $\mathfrak{p}$ are transformed into each other by an element of K. Any two maximal $\mathbb{R}$-diagonalizable subalgebras of $\mathfrak{g}$ are conjugate.*

Proof. The proof of the first statement is similar to that of Theorem 2.9, and the second reduces to the first one with the help of Proposition 4.1. □

The dimension of a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ of a real semisimple Lie algebra $\mathfrak{g}$ (independent of the choice of $\mathfrak{a}$, by Theorem 4.1) is called the *real rank* of $\mathfrak{g}$ and is denoted by $\operatorname{rk}_{\mathbb{R}}\mathfrak{g}$. The real rank is equal to 0 if and only if $\mathfrak{g}$ is compact.

If a real semisimple Lie algebra $\mathfrak{g}$ splits into the direct sum of ideals, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, then the maximal $\mathbb{R}$-diagonalizable subalgebras $\mathfrak{a}$ of $\mathfrak{g}$ are of the form $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$, where $\mathfrak{a}_i$ $(i = 1, 2)$ is an arbitrary maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}_i$. In particular,

$$\operatorname{rk}_{\mathbb{R}}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = \operatorname{rk}_{\mathbb{R}}\mathfrak{g}_1 + \operatorname{rk}_{\mathbb{R}}\mathfrak{g}_2.$$

Under the natural identification of the space $\mathfrak{a}^*$ with $\mathfrak{a}_1^* \oplus \mathfrak{a}_2^*$ the root system Σ of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is identified with $\Sigma_1 \cup \Sigma_2$, where $\Sigma_i \subset \mathfrak{a}_i^*$ is the root system of $\mathfrak{g}_i$ with respect to $\mathfrak{a}_i$.

4.2. Real Root Systems. Let $\mathfrak{g}$ be a real semisimple Lie algebra with a fixed decomposition (14), $\mathfrak{a} \subset \mathfrak{p}$ a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$, and Σ the corresponding root system. Note that $\Sigma \neq \emptyset$ if and only if $\mathfrak{g}$ is noncompact. The subspace $\mathfrak{a}$ is a Euclidean space with respect to the Killing form of $\mathfrak{g}$. Let us transfer the scalar product from $\mathfrak{a}$ to $\mathfrak{a}^*$ in the natural way.

Theorem 4.2. *The root system $\Sigma \subset \mathfrak{a}^*$ of a semisimple Lie algebra $\mathfrak{g}$ with respect to a maximal $\mathbb{R}$-diagonalizable subalgebra $\mathfrak{a}$ is a root system in the sense of Chap.* 3, *Sect.* 1.1 (*not necessarily reduced*).

The proof is similar to that of Chap. 3, Theorem 1.1 (see, for example, Onishchik and Vinberg [1990], Chap. 5, Sect. 4). □

One associates with the root system Σ the Weyl group W, i.e. the group of orthogonal transformations of the space $\mathfrak{a}$ generated by reflections in the hyperplanes P_α (see Sect. 4.1). As in the case of a complex semisimple Lie algebra (see Chap. 3, Sect. 1.6), the elements of the Weyl group are induced by inner automorphisms of $\mathfrak{g}$.

Proposition 4.2 (see Helgason [1978], Mostow [1971]). *Let G be a connected Lie group with tangent algebra $\mathfrak{g}$, and $G = KP$ its Cartan decomposition corresponding to the decomposition* (14). *Then W coincides with the group of transformations induced by the automorphisms* $\operatorname{Ad} k$ $(k \in N_K(\mathfrak{a}))$ *and also with the group of transformations induced by the automorphisms* $\operatorname{Ad} g$ $(g \in N_G(\mathfrak{a}))$. *Thus*

$$W \simeq N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \simeq N_G(\mathfrak{a})/Z_G(\mathfrak{a})$$

(here $N_K(\mathfrak{a})$, $Z_K(\mathfrak{a})$, $N_G(\mathfrak{a})$, $Z_G(\mathfrak{a})$ are normalizers and centralizers of the subalgebras $\mathfrak{a}$ in K and G).

We will now give another interpretation of the root system Σ. As follows from Proposition 4.1, the Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ that contain $\mathfrak{a}$ are of the form

$$\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{a}, \tag{16}$$

where $\mathfrak{h}^+$ is any Cartan subalgebra of $\mathfrak{m}$. Now take the complexification $\mathfrak{g}(\mathbb{C})$ of $\mathfrak{g}$ and consider its Cartan subalgebra

$$\mathfrak{c} = \mathfrak{h}(\mathbb{C}) = \mathfrak{h}^+(\mathbb{C}) \oplus \mathfrak{a}(\mathbb{C}). \tag{17}$$

Extend the automorphism θ to $\mathfrak{g}(\mathbb{C})$ by linearity. Denote by σ the complex conjugation in $\mathfrak{g}(\mathbb{C})$ with respect to $\mathfrak{g}$. Then the Cartan subalgebra $\mathfrak{c}$ is invariant under both σ and θ. The subalgebras $\mathfrak{c}^- = \mathfrak{a}(\mathbb{C})$ and $\mathfrak{c}^+ = \mathfrak{h}^+(\mathbb{C})$ are algebraic in $\mathfrak{g}(\mathbb{C})$ and consist of semisimple elements. Here $\mathfrak{c}^+$ is the Cartan subalgebra of the reductive algebraic subalgebra $\mathfrak{m}(\mathbb{C})$. We have

$$\mathfrak{c}(\mathbb{R}) = (i\mathfrak{h}^+) \oplus \mathfrak{a}. \tag{18}$$

Now consider the connection between the root system Σ and the root system $\Delta = \Delta(\mathfrak{c})$ of $\mathfrak{g}(\mathbb{C})$ with respect to $\mathfrak{c}$. Clearly, the restriction map $r\colon \mathfrak{c}(\mathbb{R})^* \to \mathfrak{c}^-(\mathbb{R})^* = \mathfrak{a}^*$ takes $\Delta \cup \{0\}$ onto $\Sigma \cup \{0\}$. Let

$$\Delta_0 = \{\alpha \in \Delta | r(\alpha) = 0\}, \quad \Delta_1 = \Delta \setminus \Delta_0.$$

Then

$$\mathfrak{m}(\mathbb{C}) = \mathfrak{c} \oplus \left(\bigoplus_{\alpha \in \Delta_0} \mathfrak{g}(\mathbb{C})_\alpha \right),$$

$$\mathfrak{g}_\lambda(\mathbb{C}) = \bigoplus_{\rho(\alpha)=\lambda} \mathfrak{g}(\mathbb{C})_\alpha \quad (\lambda \in \Sigma).$$

In particular, Δ_0 is the root system of the semisimple Lie algebra $\mathfrak{m}_0(\mathbb{C})$, where $\mathfrak{m}_0 = [\mathfrak{m}, \mathfrak{m}]$, with respect to $\mathfrak{c} \cap \mathfrak{m}_0(\mathbb{C})$. Now $\operatorname{Ker} r = \{\gamma \in \mathfrak{c}^* | \theta^\top(\gamma) = \gamma\}$, whence

$$\Delta_0 = \{\alpha \in \Delta | \theta^\top(\alpha) = \alpha\}.$$

Let

$$\sigma^\top(\gamma)(x) = \overline{\gamma(\sigma(x))} \quad (\gamma \in \mathfrak{c}^*, x \in \mathfrak{c}).$$

Then $\sigma^\top(\gamma) \in \mathfrak{c}^*$. This defines an antilinear transformation $\sigma^\top(\gamma)\colon \mathfrak{c}^* \to \mathfrak{c}^*$. It turns out that σ and $\sigma^\top$ take $\mathfrak{c}(\mathbb{R})$ and $\mathfrak{c}(\mathbb{R})^*$ into themselves and coincide on these subspaces with $-\theta$ and $-\theta^\top$, respectively. We have $\sigma(\mathfrak{g}(\mathbb{C})_\alpha) = \mathfrak{g}(\mathbb{C})_{\sigma^\top(\alpha)} = \mathfrak{g}(\mathbb{C})_{-\theta^\top(\alpha)}$ for all $\alpha \in \Delta$.

4.3. Satake Diagrams. We retain the notation of Sect. 4.2. Choose a basis $v_1, \dots, v_l$ in $\mathfrak{c}(\mathbb{R})$ such that $v_1, \dots, v_m$, $m \leq l$, is a basis in $\mathfrak{a}$, and consider the lexicographic orderings with respect to these bases in $\mathfrak{c}(\mathbb{R})^*$ and $\mathfrak{a}^*$ (see Chap. 3, Sect. 1.4). Then $\rho(\lambda) > 0$ implies that $\lambda > 0$ for $\lambda \in \mathfrak{c}(\mathbb{R})^*$. Denote by Δ^+, Σ^+ (respectively, Δ^-, Σ^-) the sets of positive (negative) roots with respect to these orderings. Let

$$\Delta_i^\pm = \Delta_i \cap \Delta^\pm, \qquad i = 0, 1.$$

Then $r(\Delta_1^\pm) = \Sigma^\pm$, $\theta^\top(\Delta_1^\pm) = \Delta_1^\mp$, $\sigma^\top(\Delta_1^\pm) = \Delta_1^\pm$. Let $\Pi \subset \Delta^+$ and $\Theta \subset \Sigma^+$ be systems of simple roots. We set $\Pi_i = \Delta_i \cap \Pi$, $i = 0, 1$. One can easily see that Π_0 is a system of simple roots in Δ_0 and that $r(\Pi_1) \supset \Theta$. Actually, as we shall see below, $r(\pi_1) = \Theta$. With this in mind, let us prove the following statement.

Proposition 4.2. *There exists an involutory transformation $\omega\colon \Pi_1 \to \Pi_1$ such that for any $\alpha \in \Pi_1$ we have*

$$\theta^\top(\alpha) = -\omega(\alpha) - \sum_{\gamma \in \Pi_0} c_{\alpha\gamma}\gamma,$$

where $c_{\alpha\gamma}$ are nonnegative integers.

Proof. For any $\alpha \in \Pi_1$ we have

$$\theta^\top(\alpha) = -\sum_{\beta \in \Pi_1} c_{\alpha\beta}\beta - \sum_{\gamma \in \Pi_0} c_{\alpha\gamma}\gamma,$$

where $c_{\alpha\beta}$, $c_{\alpha\gamma}$ are nonnegative integers. The fact the automorphism θ is involutory implies that the matrix $C = (c_{\alpha\beta})_{\alpha,\beta\in\Pi_1}$ satisfies the condition $C^2 = E$. This implies that C is a matrix corresponding to an involutory permutation ω of the elements of the system Π. □

It follows from Proposition 4.2 that $r(\alpha) = r(\beta)$ for $\alpha, \beta \in \Pi_1$ if and only if either $\alpha = \beta$ or $\alpha = \omega(\beta)$. The system $r(\Pi_1)$ is linearly independent and therefore coincides with Θ.

Proposition 4.2 enables one to assign to any real semisimple Lie algebra $\mathfrak{g}$ the *Satake diagram* obtained from the Dynkin diagram of the complex Lie algebra $\mathfrak{g}(\mathbb{C})$ as follows: the circles corresponding to the roots from Π_0 are blackened, and the pairs of different roots from Π_1 transformed into each other by an involution ω are joined by arrows.

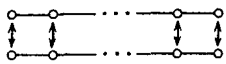
If $\mathfrak{g}_1, \mathfrak{g}_2$ are real semisimple Lie algebras, then the Satake diagram of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the disjoint union of the Satake diagrams of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. A real semisimple Lie algebra is simple if and only if its Satake diagram is connected. We note also that

$$\operatorname{rk}\mathfrak{g}(\mathbb{C}) = \operatorname{rk}_{\mathbb{R}}\mathfrak{g} + |\Pi_0| + s,$$

where s is the number of arrows on the Satake diagram.

Example 1. The Satake diagram of a semisimple compact Lie algebra $\mathfrak{g}$ is obtained from the Dynkin diagram of $\mathfrak{g}(\mathbb{C})$ by blackening all the circles representing the vertices. Any semisimple Lie algebra over $\mathbb{R}$ for which all vertices of the Satake diagram are black is compact.

Example 2. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Then the Satake diagram of the algebra $\mathfrak{g}_{\mathbb{R}}$ is obtained from the Dynkin diagram of $\mathfrak{g}$ by duplicating the latter and joining the corresponding vertices of the two diagrams by arrows. For instance, the Satake diagram of $\mathfrak{sl}_{l+1}(\mathbb{C})_{\mathbb{R}}$ contains $2l$ vertices and is of the form



Indeed, consider a compact real form $\mathfrak{u} \subset \mathfrak{g}$. If $\mathfrak{h}^+$ is a Cartan subalgebra of $\mathfrak{u}$, then $\mathfrak{h} = \mathfrak{h}^+(\mathbb{C})$ is a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{a} = I\mathfrak{h}^+$ is a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Furthermore, $\mathfrak{g}_{\mathbb{R}}(\mathbb{C})$ is identified with $\mathfrak{g} \oplus \mathfrak{g}$, and the Cartan subalgebra $\mathfrak{c} = \mathfrak{h}(\mathbb{C})$ of this algebra with $\mathfrak{h} \oplus \mathfrak{h}$. Moreover, $\sigma(x,y) = (\overline{y},\overline{x})\,(x,y \in \mathfrak{g})$, where $z \mapsto \overline{z}$ $(z \in \mathfrak{g})$ is the complex conjugation with respect to $\overline{\mathfrak{u}}$. The root system Δ of $\mathfrak{g}_{\mathbb{R}}(\mathbb{C})$ with respect to $\mathfrak{c}$ is of the form $\Delta = \Delta_{\mathfrak{g}} \cup \sigma^{\top}(\Delta_{\mathfrak{g}})$, where $\Delta_{\mathfrak{g}}$ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Similarly, $\Pi = \Pi_{\mathfrak{g}} \cup \sigma^{\top}(\Pi_{\mathfrak{g}})$, where $\Pi_{\mathfrak{g}} \subset \Delta_{\mathfrak{g}}$, $\Pi \subset \Delta$ are systems of simple roots, and $\omega = \sigma^{\top}$.

It clear from what was said above that to list Satake diagrams of semisimple Lie algebras $\mathfrak{g}$ we may confine ourselves to the case where $\mathfrak{g}$ is a noncompact real form of a simple Lie algebra $\mathfrak{g}(\mathbb{C})$. The Satake diagrams of all such Lie algebras $\mathfrak{g}$ are listed in Table 4, which also contains the Dynkin diagrams of the corresponding root systems Σ, the types of these systems, and the dimensions of the root subspaces $m_\lambda = \dim\mathfrak{g}_\lambda$, $\lambda \in \Sigma$. This table implies the following statement.

Theorem 4.3. *Two semisimple Lie algebras are isomorphic if and only if so are their Satake diagrams (in the natural sense).*

4.4. Split Real Semisimple Lie Algebras. A real semisimple Lie algebra is said to be *split* if any of its maximal $\mathbb{R}$-diagonalizable subalgebras is a Cartan subalgebra. This is equivalent to either of the following conditions: $\operatorname{rk}\mathfrak{g} = \operatorname{rk}\mathfrak{g}(\mathbb{C})$; the Satake diagram of $\mathfrak{g}$ contains neither black vertices nor arrows.

If $\mathfrak{g}$ is split, then in the notation of Sect. 4.2 we have $\mathfrak{m} = 0$, $\Delta = \Sigma$, $\mathfrak{g}(\mathbb{C})_\alpha = \mathfrak{g}_\alpha(\mathbb{C})$ for all $\alpha \in \Delta$. Thus $\dim\mathfrak{g}_\alpha = 1$ for all $\alpha \in \Sigma$. Any ideal of a split semisimple Lie algebra is split. The direct sum of two split Lie algebras is split.

Theorem 4.4. *Any semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a unique (up to isomorphism) split real form $\mathfrak{s}$, which is simple if and only if so is $\mathfrak{g}$.*

Proof. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. The normal real form of $\mathfrak{g}$ associated with an arbitrary canonical system of generators (see Sect. 1.1, Example 6) is split. One can show that any split real form of $\mathfrak{g}$ is normal with respect to some canonical system of generators, which implies that the split form is unique. □

Example. Simple split real Lie algebras are $\mathfrak{sl}_n(\mathbb{R})$ $(n \geq 2)$, $\mathfrak{so}_{k,k+1}$ $(k \geq 1)$, $\mathfrak{so}_{k,k}$ $(k \geq 3)$, $\mathfrak{sp}_n(\mathbb{R})$ $(n \geq 2)$, $E\,\mathrm{I}$, $E\,\mathrm{V}$, $E\,\mathrm{VIII}$, $F\,\mathrm{I}$, G. This is clear from the values of the real rank listed in Table 4.

Note that the correspondence established in Sect. 1.3 associates with the split real form of a complex semisimple Lie algebra $\mathfrak{g}$ a (unique up to conjugacy) involutory automorphism $\theta \in \operatorname{Aut}\mathfrak{g}$ such that $\theta(x) = -x$ for all elements x of a fixed Cartan subalgebra of $\mathfrak{g}$. Clearly, the automorphism of the system Π corresponding to θ is the automorphism ν defined in Chap. 3, Sect. 2.6.

4.5. Iwasawa Decomposition. Again let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of a real semisimple Lie algebra, $\mathfrak{a} \subset \mathfrak{p}$ a maximal $\mathbb{R}$-diagonalizable subalgebra, and Σ the root system with respect to $\mathfrak{a}$. Choose a system of simple roots Θ in Σ and denote by Σ^+ the corresponding subsystem of positive roots. The following construction is similar to that of a Borel subalgebra of a complex semisimple Lie algebra (see Chap. 3, Sect. 1.5). Namely, we set

$$\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda .$$

One can easily see that $\mathfrak{n}$ is a unipotent algebraic subalgebra of $\mathfrak{g}$. We have $[\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n}$, so $\mathfrak{d} = \mathfrak{a} \oplus \mathfrak{n}$ is a solvable algebraic subalgebra of $\mathfrak{g}$. The next theorem follows from (13), (15) and the inclusion $\mathfrak{g}_\lambda \subset \mathfrak{k} + \mathfrak{g}_{-\lambda}$ $(\lambda \in \Sigma)$.

Theorem 4.5. *The following decompositions into the direct sums of vector spaces hold:*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{k} \oplus \mathfrak{d}.$$

Now we want to construct topological direct decompositions of a connected semisimple Lie group into the products of its Lie subgroups corresponding to the decompositions of Theorem 4.5. Let G be a connected semisimple Lie group with tangent algebra $\mathfrak{g}$. As is shown in Sect. 3, there exists a connected Lie subgroup K of G with tangent algebra $\mathfrak{k}$. Furthermore, there are simply-connected Lie subgroups A, N, D of G with tangent algebras $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{d}$, respectively. Moreover,

$$D = A \ltimes N.$$

Let V be a real vector space. A linear group $G \subset \mathrm{GL}(V)$ (or a linear Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$) is said to be *triangular* if there is a basis in V with respect to which all operators from G (respectively $\mathfrak{g}$) are expressed by triangular matrices. These notions should not be confused with those of a triangular Lie group and Lie algebra introduced in Chap. 2, Sect. 2. Clearly, a connected linear Lie group is triangular if and only if so is its tangent algebra.

Proposition 4.3. *Let G be a connected real semisimple Lie group. Then for any real linear representation $R: G \to \mathrm{GL}(V)$ (or $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$) the linear group $R(D)$ (respectively, the linear algebra $\rho(\mathfrak{d})$) is triangular.*

Proof. Consider the complexification $\rho(\mathbb{C}): \mathfrak{g}(\mathbb{C}) \to \mathfrak{gl}(V(\mathbb{C}))$ of the representation ρ. It follows from (18) that all the weights of the representation $\rho(\mathbb{C})$ with respect to the Cartan subalgebra $\mathfrak{c}$ given by formula (17) are real on $\mathfrak{a}$. Chap. 1, Theorem 1.2 now implies that $\rho(\mathfrak{d})$ is triangular. □

We will now consider some properties of triangular linear groups.

Proposition 4.4. *Let $G \subset \mathrm{GL}(V)$ be a connected triangular linear Lie group. For any compact subgroup $K \subset \mathrm{GL}(V)$ we have $K \cap G = \{e\}$. The group G is diffeomorphic to $\mathbb{R}^s$ and solvable.*

Proof. If $g \in G$, then the diagonal elements $\lambda_1, \ldots, \lambda_n$ of the triangular matrix of g are positive. If, in addition, $g \in K$, then the set $\{g^m | m \in \mathbb{Z}\}$ is relatively compact in $\mathrm{GL}(V)$, whence all $\lambda_i = 1$. Since g is semisimple, we have $g = E$. In particular, G contains no nontrivial compact subgroups. The results of Sect. 3.2 and 3.6 now imply that G then contains no simple noncommutative subgroups. Hence G is solvable. □

Proposition 4.5. *Let G be a connected linear Lie group containing a compact Lie subgroup K and a connected triangular Lie subgroup D. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{d}$ (the direct sum of vector spaces) if and only if $G = KD$ (topological decomposition into the direct product).*

Proof. Consider the mapping $\mu: (k, d) \mapsto kd$ of the manifold $K \times D$ into D. Since $K \cap D = \{e\}$ (Proposition 4.4), μ is injective. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{d}$, then μ is a diffeomorphism of $K \times D$ onto the open subset $KD \subset G$. Since the set KD is closed, we have $KD = G$. The converse statement is evident. □

We will now return to global decompositions corresponding to decompositions of Theorem 4.5.

Theorem 4.6. *Let G be a connected semisimple real Lie group, and K, A, N, D its connected Lie subgroups defined above. Then $G = KD = KAN$ (topological decompositions into direct products of subgroups).*

Proof. It follows from Theorem 4.5 and Propositions 4.3 and 4.5 that $\mathrm{Int}\,\mathfrak{g} = (\mathrm{Ad}\,K)(\mathrm{Ad}\,D)$ (topological decomposition into the direct product). In order to pass from $\mathrm{Int}\,\mathfrak{g} = \mathrm{Ad}\,G$ to G one uses Corollary 4 to Theorem 3.2. □

Example. Let $G = \mathrm{SL}_n(\mathbb{R})$, $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$. Under an appropriate choice of a system of positive roots in $\Sigma = \Delta_{\mathfrak{sl}_n(\mathbb{C})}$ the subalgebra $\mathfrak{d}$ is the subalgebra of all upper triangular matrices with trace 0, and the subgroup D is the subgroup of all upper triangular matrices with determinant 1 and positive diagonal entries. The group K coincides with SO_n. Theorem 8.6 in this case follows easily from the classical theorem on the reduction of a positive definite quadratic form to the normal form with the use of a triangular change of basis.

In general, if G is a connected linear real semisimple Lie group, then K is a maximal compact subgroup of it (see Sect. 3.3) and D is a triangular linear group (see Proposition 4.3). This case of Theorem 4.6 admits the following generalization.

Theorem 4.7. *Let G be a real algebraic linear group. Then G^0 can be topologically decomposed into the direct product of the groups K and D, where K is compact and D is a triangular linear Lie group. Moreover, K is a maximal compact and D a maximal triangular subgroup of G.*

Proof. The direct decomposition $G^0 = KD$ for semisimple groups G has already been proved, while for arbitrary algebraic groups it follows from Sect. 1, Theorem 6.7. The last statement follows from Proposition 4.4. □

The decompositions of semisimple Lie algebras and Lie groups described in Theorems 4.5 and 4.6 are called *Iwasawa decompositions*.

The Iwasawa decomposition, like the Cartan decomposition, implies that a connected semisimple Lie group G is diffeomorphic to the direct product of a maximal compact subgroup of it and a Euclidean space. Moreover, if G has finite centre, then the subgroup K of Theorem 4.6 is compact, so one can take for the "Euclidean" term the subgroup D. In fact, the following statement holds.

Theorem 4.8 (Gorbatsevich [1974b]). *An arbitrary connected semisimple Lie group G can be topologically decomposed into the direct product $G = LS$, where L is a maximal compact subgroup, and S is a Lie subgroup* (*in general, nonsolvable*) *diffeomorphic to $\mathbb{R}^s$.*

4.6. Maximal Connected Triangular Subgroups. In this section we consider two classes of subgroups of connected real Lie groups similar to the class of Borel subgroups in the complex case (see Chap. 3, Sect. 1.5). The first of them is the class of maximal connected triangular subgroups of a linear real Lie group. The second class of subgroups is defined for an abstract Lie group.

Let G be a real Lie group. A subgroup $H \subset G$ is said to be *triangular in G* if the linear group $\mathrm{Ad}\, H \subset \mathrm{GL}(\mathfrak{g})$ is triangular. Subalgebras of a real Lie algebra $\mathfrak{g}$ that are triangular in $\mathfrak{g}$ are defined in a similar way. Clearly, connected virtual Lie subgroups that are triangular in G correspond to the subalgebras of its tangent algebra $\mathfrak{g}$ that are triangular in $\mathfrak{g}$. A maximal connected subgroup triangular in G (which is a Lie subgroup of G) corresponds to a subalgebra that is maximal triangular in $\mathfrak{g}$.

If G is a linear group, then any triangular subgroup of it is triangular in G. One can show (see Vinberg [1961]) that for semisimple linear groups the converse statement holds.

Our goal is to prove a real analogue of the Morozov-Borel theorem about the conjugacy of maximal connected triangular (in either sense) subgroups. We shall need the following auxiliary results.

Lemma 4.1. *Let V be a real vector space, and X a linear transformation of it whose characteristic roots are all real. For any point p of the projective space $\mathcal{P}(V)$ there exists the limit*

$$p_0 = \lim_{t\to\infty} (\exp tX)(p) \in \mathcal{P}(V).$$

The point p_0 is fixed under the action of the group $\{\exp tX | t \in \mathbb{R}\}$.

Proof. Express X by a triangular matrix in a basis of the space V. The diagonal entries of this matrix are the eigenvalues $\lambda_1, \ldots, \lambda_r$ of X (multiplicities counted). The entries of the matrix $\exp tX$ are functions in t of the form

$$\sum_{i=1}^{r} Q_i(t) e^{\lambda_i t},$$

where Q_i are polynomials. The coordinates of the vector $(\exp tX)v$, where $v \in V$ is a nonzero vector such that $\langle v \rangle = p$, are of the same form. Let Λ be the largest number λ_i appearing among the coordinates of this vector, and M the highest degree of the corresponding polynomials Q_i. Then $(\exp tX)v = t^M e^{\Lambda t}(v_0 + \varepsilon(t))$, where $v_0 \neq 0$, and $\varepsilon(t) \to 0$ as $t \to \infty$. Clearly, $\langle v_0 \rangle = \lim_{t\to\infty} (\exp tX)(p)$ and $p_0 = \langle v_0 \rangle$ is fixed under $\exp tX$ $(t \in \mathbb{R})$. □

Proposition 4.6. *Any connected triangular linear group $G \subset \mathrm{GL}(V)$ has a fixed point in any G-invariant closed subset of the space $\mathcal{P}(V)$, as well as in any G-invariant closed subset of the flag variety $\mathcal{F}(V)$ of the space V.*

Proof. Let $\Omega \subset \mathcal{P}(V)$ be a G-invariant closed set. The proof is achieved by induction on $\dim G$. It follows from Lemma 4.1 that the statement holds for $\dim G = 1$. Since G is solvable, we have $G = G_1 G_0$, where G_1, G_0 are connected, G_0 is normal in G, $\dim G_1 = 1$, $\dim G_0 = \dim G - 1$. By the induction hypothesis, the closed set $\Omega_0 = \{p \in \Omega | gp = p \quad \text{for all} \quad g \in G_0\}$ is nonempty. Since G_1 takes Ω_0 into itself, there is a point $p_0 \in \Omega_0$ such that $G_1(p_0) = p_0$. Then $G(p_0) = p_0$. In order to pass from $\mathcal{P}(V)$ to the flag variety, consider the standard embedding of $\mathcal{F}(V)$ into $\mathcal{P}(V) \times \mathcal{P}(\Lambda^2 V) \times \ldots \times \mathcal{P}(\Lambda^n V) \subset \mathcal{P}(V \otimes \Lambda^1 V \otimes \ldots \otimes \Lambda^n V)$, where $n = \dim V$. □

Theorem 4.9. *Let G be a connected real Lie group. If G is linear, then any two maximal connected triangular subgroups of G are conjugate. In the general case any two maximal connected subgroups triangular in G have the same property.*

Proof. Let $G \subset \mathrm{GL}(V)$ be a connected semisimple linear Lie group, and $G = KD$ its Iwasawa decomposition. By Proposition 4.3, there exists a flag $f_0 \in \mathcal{F}(\mathcal{V})$ such that $Df_0 = f_0$. We have $D = (G_{f_0})^0$ because D is a maximal connected triangular subgroup of G. The orbit $\Omega = G(f_0) = K(f_0)$ is compact. If C is a connected triangular subgroup of G, then, by Proposition 4.6, C has a fixed point in Ω. Therefore C is conjugate in G to a subgroup of D.

Let G be an arbitrary connected linear Lie group, and let $\mathfrak{g} = \mathfrak{l} \in \operatorname{rad}\mathfrak{g}$ be a Levi decomposition of its tangent algebra. Denote by $\mathfrak{t}_\mathfrak{r}$ the set of all operators in $\operatorname{rad}\mathfrak{g}$ having only real characteristic numbers. Since $[\mathfrak{g}, \operatorname{rad}\mathfrak{g}]$ consists of nilpotent operators, we have $[\mathfrak{g}, \operatorname{rad}\mathfrak{g}] \subset \mathfrak{t}_\mathfrak{r}$, whence $\mathfrak{t}_\mathfrak{r}$ is an ideal in $\mathfrak{g}$. Therefore any maximal triangular subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ contains $\mathfrak{t}_\mathfrak{r}$. One can prove that $\mathfrak{t} = \mathfrak{t}_\mathfrak{l} \in \mathfrak{t}_\mathfrak{r}$, where $\mathfrak{t}_\mathfrak{l}$ is a maximal triangular subalgebra of $\mathfrak{l}$. But, as shown above, all maximal triangular subalgebras of $\mathfrak{l}$ are conjugate.

The version of the theorem concerning subgroups triangular in G is proved similarly. □

4.7. Cartan Subalgebras of a Real Semisimple Lie Algebra. Recall (see Chap. 1, Sect. 9.3 and Sect. 2.2 of this chapter) that all Cartan subalgebras of a complex or compact Lie algebra are conjugate. In this section we describe the classes of conjugate Cartan subalgebras of a real semisimple Lie algebra. For the proof of the following statements see Sugiura [1959], where one can also find an explicit description of classes of conjugate Cartan subalgebras of simple Lie algebras.

Again let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}$, and $\mathfrak{a} \subset \mathfrak{p}$ a fixed maximal $\mathbb{R}$-diagonalizable subalgebra. A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called *standard* if $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$, where $\mathfrak{h}^+ \subset \mathfrak{k}$, $\mathfrak{h}^- \subset \mathfrak{a}$. The subspace $\mathfrak{h}^-$ is said to be the *vector part* of $\mathfrak{h}$.

Theorem 4.10. *Any Cartan subalgebra of $\mathfrak{g}$ is conjugate to a standard Cartan subalgebra. Two standard Cartan subalgebras are conjugate if and only if their vector parts are taken into each other by an element of the Weyl group W of the root system Σ. In particular, there are finitely many classes of conjugate Cartan subalgebras of $\mathfrak{g}$.*

Fix a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}$ containing $\mathfrak{a}$, i.e. having the form (16), and consider, as in Sect. 4.2, the root system Δ of $\mathfrak{g}(\mathbb{C})$ with respect to $\mathfrak{h}_0(\mathbb{C})$. A subsystem $\Gamma \subset \Delta$ will be called *standard* if $\theta^\top(\gamma) = -\gamma$ and $\gamma \pm \delta \notin \Delta \cup \{0\}$ for any $\gamma, \delta \in \Gamma$. Clearly, any standard system is contained in Δ_1.

Theorem 4.11. *A subspace $\mathfrak{a}_0 \subset \mathfrak{a}$ is a vector part of a standard Cartan subalgebra if and only if $\mathfrak{a}_0 = \{x \in \mathfrak{a} | \gamma(x) = 0 \quad \text{for all} \quad \gamma \in \Gamma\}$, where $\Gamma \subset \Delta$ is a standard subsystem. If two standard Cartan subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ have vector parts defined by standard subsystems Γ_1 and Γ_2, then $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are conjugate if and only if Γ_1 and Γ_2 are taken into each other by an element of the Weyl group of $\mathfrak{g}(\mathbb{C})$ that commutes with θ.*

§ 5. Exponential Mapping for Semisimple Lie Groups

5.1. Image of the Exponential Mapping. Let G be a connected Lie group (either real or complex) and $\exp\colon \mathfrak{g} \to G$ the exponential mapping. Denote its image by E_G and the complement $G \setminus E_G$ by E'_G. Clearly, if $g \in E_G$, then all elements conjugate to g in G also belong to E_G. Therefore E_G consists of classes of conjugate elements, and the same holds for E'_G. The classes of conjugate elements can be described in detail in the case when the Lie group G is semisimple, thus providing a key to the study of E_G and E'_G for such G. In the general case, the subset E_G is different from G; moreover, it is not necessarily dense in G.

Example 1. Let $G = \mathrm{SL}_2(\mathbb{R})$; then $E_G = \{g \in \mathrm{SL}_2(\mathbb{R}) | \operatorname{tr} g > -2\} \cup \left\{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right\}$. To verify this, consider the element $g = \exp X \in E_{G'}$, where $X \in \mathfrak{sl}_2(\mathbb{R})$. Let λ_1, λ_2 be the eigenvalues of X. Then $\lambda_1 + \lambda_2 = 0$, where λ_1, λ_2 are either both real or both pure imaginary. The examination of the values of $\operatorname{tr} g = e^{\lambda_1} + e^{\lambda_2}$ yields the above expression for E_G.

Clearly, the closure $\overline{E}_G$ of the subset E_G in G is of the form $\{g \in \mathrm{SL}_2(\mathbb{R}) | \operatorname{tr} g \geq -2\}$ and does not coincide with $\mathrm{SL}_2(\mathbb{R})$.

However, in some particular cases the subset E_G is dense in G (the corresponding groups are sometimes called *weakly exponential*). For example, this is the case if the Lie group G is connected and solvable (Hofmann and Mukherjea [1978]). Furthermore, an arbitrary complex semisimple Lie group G is also weakly exponential. This is implied by the fact that the regular elements of G forming the dense subset in G all belong to E_G. Moreover, the following theorem holds.

Theorem 5.1 (Hofmann and Mukherjea [1978]). *Let G be a connected complex Lie group. Then E_G is dense of G.*

For a semisimple complex Lie group G we can describe the subset E_G in some detail. We will use the Jordan decomposition $g = g_s g_u$ for elements g of the group G (which, by virtue of Chap. 3, Theorem 2.7, may be assumed to be algebraic).

Theorem 5.2 (Djokovic [1980]). *Let G be a connected complex semisimple Lie group and $g = g_s g_u$ the Jordan decomposition of its element g. Then the following statements are equivalent:*

(i) $g \in E_G$;
(ii) $g_s \in Z_G(g)^0$;
(iii) $g_s \in Z_G(g_u)^0$.

In particular, $g \in E'_G$ if and only if $g \notin Z_G(g_u)^0$ (here $Z_G(u)$ denotes the centralizer of the element $x \in G$).

This theorem implies that in order to describe subsets of E_G it is sufficient to enumerate all classes of conjugate unipotent elements $u \in G$ such that Z_G is not connected, and then to classify all $Z_G(u)$-conjugate semisimple classes in $Z_G(u) \setminus Z_G(u)^0$ (see Djokovic [1980]).

Example 2. Let $G = \mathrm{GL}_2(\mathbb{C})$. Let $u = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$; then $Z_G(u) = \left\{ \begin{bmatrix} \pm 1 & a \\ 0 & \pm 1 \end{bmatrix} \middle| a \in \mathbb{C} \right\}$. Taking $g_s = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and setting $g_u = u$, we obtain the element $g = g_s \cdot g_u = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \in E'_G$. We can easily show that E'_G consists only of elements conjugate to g.

5.2. Index of an Element of a Lie Group. Let G be a Lie group, $g \in G$. In general, $g \notin E_G$ but it may turn out that there exists a number $q > 0$ such that $g^q \in E_G$. For example, if $G = \mathrm{SL}_2(\mathbb{C})$ and $g = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ (see Example 2 above), then $g \notin E_G$, $g^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \in E_G$.

Definition. Let g be an element of the Lie group G. If there exist numbers $q > 0$ such that $g_q \in E_G$, then the *index* $\mathrm{ind}_G g$ of the element g in G is the smallest of these numbers q. If there are no such numbers, we set $\mathrm{ind}_G g = \infty$.

For example, $\mathrm{ind}_G g = 1$ if and only if $g \in E_G$. For the element g in Example 2 we have $\mathrm{ind}_{\mathrm{SL}_2(\mathbb{C})}(g) = 2$.

If there exist numbers $q > 0$ such that $g^q \in E_G$ for any element $g \in G$, then the smallest of these q is said to be the *index* $\mathrm{ind}\, G$ *of the Lie group* G. If there are no such numbers we set $\mathrm{ind}\, G = \infty$. Clearly, $\mathrm{ind}\, G = \mathrm{L.C.M.}(\mathrm{ind}_G g | g \in G)$.

Theorem 5.3 (Goto [1977], see also Lai [1978b]). *Let G be a linear algebraic Lie group (over the field $K = \mathbb{C}$ or $\mathbb{R}$). Then* $\mathrm{ind}\, G < \infty$.

Theorem 5.3 admits various generalizations. Thus it can be proved that if Γ is an arbitrary finitely generated nilpotent subgroup of an algebraic linear group G (over $\mathbb{C}$ or $\mathbb{R}$), then there exists a subgroup of finite index contained in a connected nilpotent subgroup $N \subset G$ (in particular, $\Gamma_1 \subset E_G$), see Gorbatsevich [1974a]. In Goto [1977] Theorem 5.3 is proved for linear Lie groups $G \subset \mathrm{GL}_n(\mathbb{R})$ such that G contains a maximal connected compact subgroup of an algebraic subgroup in $\mathrm{GL}_n(\mathbb{R})$ containing G.

Corollary. *Let G be a semisimple linear Lie group (either real or complex) with finitely many connected components. Then* $\mathrm{ind}\, G < \infty$.

Here the condition that the Lie group G is linear is essential.

Example 3. Let $G = \widetilde{\mathrm{SL}_2}(\mathbb{R})$ be the simply-connected covering for the Lie group $\mathrm{SL}_2(\mathbb{R})$. We will show that $\mathrm{ind}\, G = \infty$.

Let $n = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in \mathfrak{sl}_2(\mathbb{R}) = \mathfrak{g}$. Now we have $Z(G) \simeq \mathbb{Z}$ (see Chap. 4, Sect. 3.2). Let $a \in Z(G)$ be a generator. Set $g = a \cdot \exp n$. We can show without difficulty that $g_q \notin E_G$ for any $q > 0$, whence $\operatorname{ind}_G(g) = \infty$. Hence it follows that $\operatorname{ind}_G(g) = \infty$. This assertion agrees with the above corollary because the Lie group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is nonlinear (see Chap. 1, Sect. 5.1).

5.3. Indices of Simple Lie Groups. This section includes some computational results concerning the index of simple Lie groups. We begin with the case of simple complex Lie groups G. Let $\alpha = \sum_{i=1}^{i} n_i\alpha_i$ be the decomposition of the highest root α with respect to the simple roots α_i. The coefficients n_i of such a decomposition are listed in Table 3. Let $n_0 = 1$.

Theorem 5.4 (Lai [1978a]). *Let G be a simple complex Lie group and $Z(G) = \{e\}$. Then $\{\operatorname{ind}_G g | g \in G\} = \{n_0, n_1, \ldots, n_l\}$.*

Making use of the classification of simple complex Lie algebras, we obtain the following corollary to Theorem 5.4.

Corollary. *Let G be a simple complex Lie group and $Z(G) = \{e\}$. Then, depending on the type of g, we have the following values for $\{\operatorname{ind}_G g\}$ and $\operatorname{ind} G$:*

Type	A	B	C	D	G_2	F_4	E_6	E_7	E_8
$\{\operatorname{ind}_G g\}$	$\{1\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2$ $3,4\}$	$\{1,2,3\}$	$\{1,2$ $3,4\}$	$\{1,2,3$ $4,5\}$
$\operatorname{ind} G$	1	2	2	2	6	12	6	12	60

Theorem 5.5 (Lai [1978b]). *Let G be a simple complex Lie group. There exist numbers $r_0, r_1, \ldots, r_l \in \mathbb{N}$, $r_i \geq 1$ (depending on $Z(G)$) such that $\{\operatorname{ind}_G g | g \in G\} = \{$divisors of the numbers $r_j \cdot n_j,\ 0 \leq j \leq l\}$.*

For simply-connected simple complex Lie groups the numbers r_j are described as follows:

Type of G	r_j
A_l	$r_1 = r_2 = \ldots = r_l = l+1$
B_l	$r_j = 1$ if j is even $r_j = 2$ if j is odd
C_l	$r_l = 2$, $r_j = 1$ if $j < l$
D_l, l even	$r_j = 2$ if j is odd and $j < l-2$ or if $j = l-1$ $r_j = 1$ otherwise
D_l, l odd	$r_j = 2$ if j is odd and $j \leq l-2$ $r_{l-1} = r_l = 4$ $r_j = 1$ otherwise
E_6	$r_1 = r_2 = r_5 = r_6 = 3$, otherwise $r_j = 1$
E_7	$r_1 = r_3 = r_5 = 1$, otherwise $r_j = 1$

Note that for the Lie groups G of types G_2, F_4, E_8 all $r_j = 1$ because in this case one always has $Z(G) = \{e\}$.

Corollary. *Let G be a simply-connected simple complex Lie group. Then $\{\mathrm{ind}_G g\}$ and $\mathrm{ind}\, G$ are of the form*

Type of G	A_l	B_l	C_l	D_l	E_6	E_7
$\{\mathrm{ind}_G g\}$	{divisors of $l+1$}	$\{1,2,4\}$ for $l \geq 3$ $\{1,2\}$ for $l=2$	$\{1,2\}$	$\{1,2\}$ if l is even $\{1,2,4\}$ if l is odd	$\{1,2,3,6\}$	$\{1,2,3,4$ $6,12\}$
$\mathrm{ind}\, G$	$l+1$	4 for $l \geq 3$ 2 for $l=2$	2	2 if l is even 4 if l is odd	6	12

For the types G_2, F_4, E_8 the corollary to Theorem 5.4 applies.

We now consider real simple Lie groups. If G is compact, then $\mathrm{ind}\, G = 1$ (because in this case exp is surjective, see Chap. 4, Theorem 2.10.). Therefore we will consider only noncompact simple Lie groups.

Theorem 5.5 (Djokovic [1988]). *Let G be a classical noncompact simple Lie group. Then*

(i) *if $G = \mathrm{SO}_n(\mathbb{C}), \mathrm{SO}_{k,n-k}, \mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{R})$, then $\{\mathrm{ind}_G g\} = \{1,2\}$ and* $\mathrm{ind}\, G = 2$;

(ii) *if $G = \mathrm{SL}_n(H), \mathrm{Sp}_{k,n-k}, \mathrm{O}^*_{2n}$, then $E_G = G$ and* $\mathrm{ind}\, G = 1$;

(iii) *if $G = \mathrm{SL}_n(\mathbb{C})$, then* $\{\mathrm{ind}_G g\} = \{$divisors of $n\}$ *and* $\mathrm{ind}\, G = 1$;

(iv) *if $G = \mathrm{SU}_{k,n-k}$, then $\{\mathrm{ind}_G g\}$ is described as follows: consider partitions s of the number n into a sum $n = m_1 + \ldots + m_s$ such that $m_i \in \mathbb{N}$, $m_i > 0$ and $\sum\limits_{i=1}^{s} \left[\frac{m_i}{2}\right] \leq k$; then*

$$\{\mathrm{ind}_G g | g \in G\} = \mathrm{GCD}((m_1, \ldots, m_s) \text{ for all partitions } s).$$

Example 4. Let us find $\mathrm{SU}_{6,4}$. Here $n = 6$, $k = 2$. Therefore the possible partitions s are $(4,1,1)$, $(3,3)$, $(3,2,1)$, $(3,1,1,1)$, $(2,2,1,1)$, $(2,1,1,1,1,1)$, $(1,1,1,1,1,1)$. Clearly, $\mathrm{GCD}(m_1, \ldots, m_s) = 1$ or 3. Hence $\{\mathrm{ind}_{\mathrm{SU}_{6,4}} g\} = \{1,3\}$ and $\mathrm{ind}\, \mathrm{SU}_{6,4} = 3$.

Chapter 5
Models of Exceptional Lie Algebras

§ 1. Models Associated with the Cayley Algebra

In this chapter we shall often consider algebras defined "over an arbitrary field K", although in the context of Lie group theory we could confine ourselves to the cases $K = \mathbb{C}$ and $\mathbb{R}$. We will, however, assume without special mention that K is of characteristic zero. The reader interested in the case of positive characteristic can find the corresponding statements in the algebraic literature cited below.

1.1. Cayley Algebra. The classical (either real and complex) simple Lie algebras are associated with the fields $\mathbb{R}$ and $\mathbb{C}$ of real and complex numbers and the algebra of quaternions $\mathbb{H}$. Similarly, all the exceptional simple Lie algebras are in one way or another associated with the *Cayley algebra* (algebra of octonions).

The description of a standard (nonsplit real) Cayley algebra $\mathbb{O}$ can be found, e.g., in Shafarevich [1986] or Kantor and Solodovnikov [1973]. For a clearer understanding of the structure of this algebra, a brief survey of the general theory of composition algebras is in order.

Let K be an arbitrary field. An algebra over K is said to be *alternative* if the associator

$$[xyz] \doteqdot x(yz) - (xy)z \tag{1}$$

is skew-symmetric in x, y, z, or, equivalently, if the algebra generated by any pair of its elements is associative. An algebra with unit element over K is called a *composition* algebra if it admits a nondegenerate quadratic form N (called the norm) satisfying the property of multiplicativity:

$$N(xy) = N(x)N(y). \tag{2}$$

It is known (see Albert [1942]) that an algebra with unit element is a composition algebra if and only if it is alternative and admits a (unique) involutory antiautomorphism $x \mapsto \overline{x}$ (called conjugacy) whose space of fixed points coincides with K $(= K \cdot 1)$; in this case

$$N(x) = x\overline{x} = \overline{x}x. \tag{3}$$

In a composition algebra $\mathfrak{A}$ the polarization of the norm defines the scalar product

$$(x, y) = \frac{1}{2}(x\overline{y} + y\overline{x}) = \frac{1}{2}(\overline{x}y + \overline{y}x), \tag{4}$$

invariant under all automorphisms and antiautomorphisms of $\mathfrak{A}$. The elements $x \in \mathfrak{A}$ for which $\overline{x} = -x$ are said to be pure imaginary; they form a subspace $\mathfrak{A}^0$ coinciding with the orthogonal complement of K. The following properties of pure imaginary elements are formally derived from (3), (4) and the alternativity identity:

(I1) if $x \in \mathfrak{A}^0$, then $x^2 = -N(x) \in K$;

(I2) if $x, y \in \mathfrak{A}^0$ and $x \perp y$, then $xy = -yx \in \mathfrak{A}^0$ and $xy \perp x$, y;

(I3) if $x, y, z \in \mathfrak{A}^0$, then $(xy, z) = (x, yz)$;

(I4) if $x, y, z \in \mathfrak{A}^0$ are pairwise orthogonal and $z \perp xy$, then $x \perp yz$, $y \perp zx$, and $(xy)z = (yz)x = (zx)y \in \mathfrak{A}^0$.

Over an algebraically closed field K (in particular, over $\mathbb{C}$) there exist exactly four finite-dimensional composition algebras whose dimensions are 1, 2, 4, and 8 (Albert [1942]). Denote them by $\mathfrak{A}_0(K)$, $\mathfrak{A}_1(K)$, $\mathfrak{A}_2(K)$, and $\mathfrak{A}_3(K)$ (so $\dim \mathfrak{A}_p(K) = 2^p$). The algebras $\mathfrak{A}_0(K)$, $\mathfrak{A}_1(K)$, $\mathfrak{A}_2(K)$ are associative and have the following structure:

(0) $\mathfrak{A}_0(K) = K$, $\overline{x} = x$, $N(x) = x^2$;

(1) $\mathfrak{A}_1(K) = K \oplus K$; if $x = (a, b)$, then $\overline{x} = (b, a)$, $N(x) = ab$;

(2) $\mathfrak{A}_2(K) = L_2(K)$; if $x = \begin{pmatrix} a & c \\ d & b \end{pmatrix}$, then $\overline{x} = \begin{pmatrix} b & -c \\ -d & a \end{pmatrix}$, $N(x) = ab - cd (= \det x)$.

In order to describe the algebra $\mathfrak{A}_3(K)$ consider a 3-dimensional space V over K with a fixed volume form det. Define the vector product $V \times V \to V^*$ in the usual way by the formula

$$\langle v_1 \times v_2, v_3 \rangle = \det(v_1, v_2, v_3),$$

where $\langle\, , \rangle$ denotes the pairing of elements of the spaces V and V^*. Choose the volume form $\det^{-1}$ in V^* so that

$$\det(v_1, v_2, v_3) \det{}^{-1}(v_1^*, v_2^*, v_3^*) = \det(\langle v_i, v_j^* \rangle),$$

and define in a similar way the vector product $V^* \times V^* \to V$. Now we describe the algebra $\mathfrak{A}_3(K)$.

(3) $\mathfrak{A}_3(K)$ is the algebra of matrices of the form $\begin{pmatrix} a & v \\ v^* & b \end{pmatrix}$, where $a, b \in K$, $v \in V$, $v^* \in V^*$, with the operation

$$\begin{pmatrix} a_1 & v_1 \\ v_1^* & b_1 \end{pmatrix} \begin{pmatrix} a_2 & v_2 \\ v_2^* & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + \langle v_1, v_2^* \rangle & b_2 v_1 + a_1 v_2 - v_1^* \times v_2^* \\ a_2 v_1^* + b_1 v_2^* + v_1 \times v_2 & b_1 b_2 + \langle v_1^*, v_2 \rangle \end{pmatrix};$$

if $x = \begin{pmatrix} a & v \\ v^* & b \end{pmatrix}$, then $\overline{x} = \begin{pmatrix} b & -v \\ -v^* & a \end{pmatrix}$, $N(x) = ab - \langle v, v^* \rangle$.

The following embeddings are evident:

$$\mathfrak{A}_0(K) \subset \mathfrak{A}_1(K) \subset \mathfrak{A}_2(K) \subset \mathfrak{A}_3(K). \tag{5}$$

The composition property is preserved if the ground field is extended. Therefore any finite-dimensional composition algebra over an arbitrary field K is a K-form of one of the four finite-dimensional composition algebras over the algebraic closure of K and, in particular, has the same dimension. The above constructions make sense for an arbitrary field K and yield four finite-dimensional composition algebras called *split*. However, over an algebraically non-closed field there may also exist other finite-dimensional composition algebras. For example, nonsplit two-dimensional composition algebras are quadratic extensions of the ground field.

Going back to the case which is of interest in the first place, i.e. that of real and complex algebras, note that the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ (as algebras over $\mathbb{R}$) are composition ones, and the norm in each of them is a positive definite quadratic form. They are real forms of the algebras $\mathfrak{A}_0(\mathbb{C})$, $\mathfrak{A}_1(\mathbb{C})$, $\mathfrak{A}_2(\mathbb{C})$, and $\mathfrak{A}_3(\mathbb{C})$, respectively. For the sake of uniformity, we will sometimes denote them by $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$.

One can easily see that for a finite-dimensional real composition algebra $\mathfrak{A}$ the following conditions are equivalent:

(1) $\mathfrak{A}$ has no divisors of zero (or, which is the same, is a division algebra);
(2) the norm is a positive definite quadratic form.

Real composition algebras satisfying these two equivalent conditions are said to be *compact*. Thus $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{A}_3$ are compact composition algebras. Split real composition algebras, with the exception of $\mathfrak{A}_0(\mathbb{R}) = \mathbb{R}$, are not compact since they have divisors of zero.

According to the Hurwitz theorem (see, e.g., Kantor and Solodovnikov [1973]) any compact composition algebra is isomorphic to one of the algebras $\mathfrak{A}_p$ $(p = 0, 1, 2, 3)$. Any finite-dimensional real composition algebra is either split or compact (Albert [1942]). In other words each of the complex composition algebras $\mathfrak{A}_p(\mathbb{C})$ $(p = 1, 2, 3)$ has up to isomorphism exactly two real forms: $\mathfrak{A}_p(\mathbb{R})$ and $\mathfrak{A}_p$.

1.2. The Algebra G_2. The complex simple Lie algebra of type G_2 and both its real forms are realized as the algebras of derivations of the Cayley algebras $\mathfrak{A}_3(\mathbb{C})$, $\mathfrak{A}_3(\mathbb{R})$, and $\mathbb{O}$, and the Lie groups corresponding to them as the groups of automorphisms of the Cayley algebras.

Let K be an arbitrary field. The algebra $\mathfrak{A}_3(K)$ admits the following $\mathbb{Z}_3$-grading:

$$\mathfrak{A}_3(K) = V^* \oplus \mathfrak{A}_1(K) \oplus V, \tag{6}$$

where the subspaces on the right-hand side are of degrees $-1, 0, 1 \in \mathbb{Z}_3$ respectively, the algebra $\mathfrak{A}_1(K) = K \oplus K$ is assumed to be embedded in $\mathfrak{A}_3(K)$ as the subalgebra of diagonal matrices (in the model described in the preceding section), each vector $v \in V$ is identified with the matrix $\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}$, and each vector $v^* \in V^*$ with $\begin{pmatrix} 0 & 0 \\ v^* & 0 \end{pmatrix}$. Accordingly, the Lie algebra $\operatorname{Der}\mathfrak{A}_3(K)$ also acquires a $\mathbb{Z}_3$-grading if a homogeneous derivation of degree $l \in \mathbb{Z}_3$ is defined as a derivation taking each homogeneous element of degree k of $\mathfrak{A}_3(K)$ into a homogeneous element of degree $k + l$. One can easily see that derivations of degree 0 are of the form

$$D_A\colon \begin{pmatrix} a & v \\ v^* & b \end{pmatrix} \mapsto \begin{pmatrix} 0 & Av \\ -A^\top v^* & 0 \end{pmatrix} \quad (A \in \mathfrak{sl}(V)), \tag{7}$$

where $A^\top$ denotes the operator in the space V^* that is adjoint to A. Derivations of degree 1 are of the form

$$D_u\colon \begin{pmatrix} a & v \\ v^* & b \end{pmatrix} \mapsto \begin{pmatrix} -\langle u, v^*\rangle & (a-b)u \\ u\times v & \langle u, v^*\rangle \end{pmatrix} \quad (u \in V), \tag{8}$$

and derivations of degree -1 are of the form

$$D_{u^*}\colon \begin{pmatrix} a & v \\ v^* & b \end{pmatrix} \mapsto \begin{pmatrix} \langle u^*, v\rangle & -u^*\times v^* \\ (b-a)u^* & -\langle u^*, v\rangle \end{pmatrix} \quad (u^* \in V^*). \tag{9}$$

By identifying the derivation D_A with the operator $A \in \mathfrak{sl}(V)$, D_u with the vector $u \in V$, and D_{u^*} with the vector $u^* \in V^*$, one can write the grading of the algebra $\operatorname{Der}\mathfrak{A}_3(K)$ in the form

$$\operatorname{Der}\mathfrak{A}_3(K) = V^* \oplus \mathfrak{sl}(V) \oplus V. \tag{10}$$

Then the commutation rules are of the form

$$[A, u] = Au, \quad [A, u^*] = -A^\top u^*,$$

$$[u_1, u_2] = -2(u_1 \times u_2), \quad [u_1^*, u_2^*] = 2(u_1^* \times u_2^*),$$

$$[u, u^*] = 3(u \otimes u^*) - \langle u, u^*\rangle E, \quad [A, B] = AB - BA. \tag{11}$$

Let $\mathfrak{h} \subset \mathfrak{sl}(V)$ be the subalgebra consisting of the operators diagonal in a fixed basis of the space V. It follows from (11) that $\mathfrak{h}$ is a Cartan subalgebra of $\operatorname{Der}\mathfrak{A}_3(K)$ whose adjoint representation in $\operatorname{Der}\mathfrak{A}_3(K)$ is diagonalizable. Analyzing the root decomposition of $\operatorname{Der}\mathfrak{A}_3(K)$ with respect to $\mathfrak{h}$, one can easily see that $\operatorname{Der}\mathfrak{A}_3(K)$ is a split simple Lie algebra of type G_2.

We now state the result in the cases $K = \mathbb{C}$ and $\mathbb{R}$.

Theorem 1.1. *The algebra of derivations of* $\mathfrak{A}_3(\mathbb{C})$ (*respectively,* $\mathfrak{A}_3(\mathbb{R})$, $\mathbb{O}$) *is a complex* (*respectively, noncompact real, compact real*) *simple Lie algebra of type* G_2.

Proof. Clearly, derivations of any Cayley algebra annihilate the norm. Since the norm in the algebra $\mathbb{O}$ is a positive definite quadratic form, the Lie algebra $\operatorname{Der}\mathbb{O}$ is compact. Its complexification is the algebra $\operatorname{Der}\mathfrak{A}_3(\mathbb{C})$ and consequently $\operatorname{Der}\mathbb{O}$ is a simple Lie algebra of type G_2. □

Derivations of any eight-dimensional composition algebra $\mathfrak{A}$ (generalized Cayley algebra) can be described in intrinsic terms. Namely, for any $a, b \in \mathfrak{A}$ the mapping

$$D_{a,b}\colon x \mapsto [[ab]x] + 3[abx] \tag{12}$$

(where $[uv] \doteqdot uv - vu$) is a derivation of the algebra $\mathfrak{A}$ (Schafer [1966]). This defines a linear mapping $\Lambda^2\mathfrak{A} \to \operatorname{Der}\mathfrak{A}$, which evidently commutes with the action of $\operatorname{Der}\mathfrak{A}$. Since $\operatorname{Der}\mathfrak{A}$ is simple, this mapping is surjective (but not injective).

It is useful to note that such a description of derivations remains formally valid for associative finite-dimensional composition algebras. For four-dimensional algebras (generalized quaternion algebras) it turns into the well-known statement that all derivations of these algebras are inner ones (and form a three-dimensional simple Lie algebra), and for two- and one-dimensional algebras it simply means that there are no nontrivial derivations.

Now let us examine the automorphisms of generalized Cayley algebras. Let $\mathfrak{A}$ be such an algebra, and $e_1, e_2, e_3 \in \mathfrak{A}$ nonzero pure imaginary elements satisfying the conditions

$$e_1 \perp e_2, \quad e_3 \perp e_1, e_2, e_1 e_2. \tag{13}$$

The properties (I1)–(I4) of Sect. 5.1 easily imply that the elements

$$1,\ e_1,\ e_2,\ e_1e_2,\ e_3,\ e_1e_3,\ e_2e_3,\ (e_1e_2)e_3 \tag{14}$$

form an orthogonal basis of $\mathfrak{A}$, and the multiplication table in this basis depends just on the elements $N(e_1), N(e_2), N(e_3) \in K$. Therefore if $e_1', e_2', e_3' \in \mathfrak{A}$ are other nonzero pure imaginary elements satisfying (13) and the condition $N(e_i') = N(e_i)$ $(i = 1, 2, 3)$, then there exists a unique automorphism of $\mathfrak{A}$ taking e_i into e_i' $(i = 1, 2, 3)$.

In particular, if $\mathfrak{A} = \mathfrak{A}_3(K)$ is the split Cayley algebra, one can choose pure imaginary elements $e_1, e_2, e_3 \in \mathfrak{A}$ satisfying (13) such that $N(e_1) = N(e_2) = 1$, $N(e_3) = -1$. Such a triple $\{e_1, e_2, e_3\}$ will be called a canonical system of generators of $\mathfrak{A}_3(K)$.

Similarly, in the compact (real) Cayley algebra $\mathbb{O}$ one can find pure imaginary elements e_1, e_2, e_3 satisfying conditions (13) such that $N(e_1) = N(e_2) = N(e_3) = 1$. Such a triple $\{e_1, e_2, e_3\}$ will be called a canonical system of generators of $\mathbb{O}$.

It follows from above that the group of automorphisms of each of the algebras $\mathfrak{A}_3(\mathbb{C})$, $\mathfrak{A}_3(\mathbb{R})$, and $\mathbb{O}$ acts simply transitively on the set of canonical systems of generators. This enables one, in particular, to determine the topological structure of these groups.

Consider, for example, the algebra $\mathbb{O}$. The element e_1 in a canonical system of generators in $\mathbb{O}$ is chosen arbitrarily on the 6-dimensional sphere in the space of pure imaginary Cayley numbers, the element e_2 is chosen on the 5-dimensional sphere in the orthogonal complement of $\langle 1, e_1 \rangle$, and the element e_3 on the 3-dimensional sphere in the orthogonal complement of $\langle 1, e_1, e_2, e_1e_2 \rangle$. Therefore $\operatorname{Aut} \mathbb{O}$ is a connected compact Lie group of dimension $6 + 5 + 3 = 14$.

A similar argument proves the following proposition.

Proposition 1.2. *The group of automorphisms of any of the algebras* $\mathfrak{A}_3(\mathbb{C})$, $\mathfrak{A}_3(\mathbb{R})$, $\mathbb{O}$ *is connected.*

For more information see Freudenthal [1985], Schafer [1966], Jacobson [1971], Postnikov [1982].

1.3. Exceptional Jordan Algebra. The exceptional simple Lie algebras of types F_4, E_6, and E_7 can be described in terms of the exceptional Jordan algebra discovered by Albert in 1947.

Let K be an arbitrary field. A *Jordan algebra* over K is a commutative algebra for which the identity $[x^2yx] = 0$ is satisfied. A subalgebra generated by any element of a Jordan algebra is associative. Any associative algebra is a Jordan one with respect to the operation $x \circ y = \frac{1}{2}(xy + yx)$. The resulting algebras and their subalgebras are called *special Jordan algebras*.

Let $\mathfrak{A}$ be an algebra with the involution (involutory antiautomorphism) $x \mapsto \overline{x}$. For any matrix X of order n with entries in $\mathfrak{A}$ we set $X^* = \overline{X}^{\top}$. The formula $(XY)^* = Y^*X^*$ holds, so the mapping $X \mapsto X^*$ is an involution of the algebra of matrices of order n with elements in $\mathfrak{A}$. A matrix X is said to be Hermitian (respectively, skew-Hermitian) if $X^* = X$ (respectively, $X^* = -X$). Hermitian matrices of order n form an algebra with respect to the operation $X \circ Y = \frac{1}{2}(XY + YX)$. Denote this algebra by $\mathfrak{H}_n(\mathfrak{A})$.

If $\mathfrak{A}$ is an associative algebra, then $\mathfrak{H}_n(\mathfrak{A})$ is a (special) Jordan algebra. In the general case $\mathfrak{H}_n(\mathfrak{A})$ is not necessarily Jordan, of course. However, if $\mathfrak{A}$ is a generalized Cayley algebra, then $\mathfrak{H}_n(\mathfrak{A})$ is a Jordan algebra for $n \leq 3$. The algebra $\mathfrak{H}_3(\mathfrak{A})$ is of special interest. Its dimension is 27.

If the field K is algebraically closed, then $\mathfrak{H}_3(\mathfrak{A}_3(K))$ is the only simple finite-dimensional Jordan algebra over K, which is not a special one. This is what is called the *exceptional Jordan algebra* over K.

An exceptional Jordan algebra over any field K is defined as any form of the exceptional Jordan algebra over the algebraic closure of the field K. In particular, such is the algebra $\mathfrak{H}_3(\mathfrak{A})$, where $\mathfrak{A}$ is any generalized Cayley algebra over K. An exceptional Jordan algebra of the form $\mathfrak{H}_3(\mathfrak{A}_3(K))$ is said to be *split*.

The minimal polynomial of a generic element x of the exceptional Jordan algebra $\mathfrak{H}$ is of the form

$$f_x(t) = t^3 - T(x)t^2 + Q(x)t - N(x), \tag{15}$$

where T (the trace) is a linear form, Q is a quadratic form, and N (the norm) is a cubic form in the space $\mathfrak{H}$.

The forms T, Q, and N are clearly invariant under all automorphisms of the algebra $\mathfrak{H}$ and are therefore annihilated by all its derivations.

For any $x \in \mathfrak{H}$ the polynomial f_x is always annihilating. It is called the characteristic polynomial of the element x.

Let us introduce the (nondegenerate) scalar product in $\mathfrak{H}$:

$$(x, y) \doteqdot T(xy). \tag{16}$$

It has the following "associativity" property:

$$(xy, z) = (x, yz). \tag{17}$$

In the case when $\mathfrak{H} = \mathfrak{H}_3(\mathfrak{A})$, where $\mathfrak{A}$ is a generalized Cayley algebra, the forms T, Q, and N are described as follows. Let $X = (x_{ij}) \in \mathfrak{H}_3(\mathfrak{A})$ $(x_{ij} = \overline{x}_{ji} \in \mathfrak{A})$. Then

$$T(X) = x_{11} + x_{22} + x_{33}, \tag{18}$$

$$Q(X) = x_{22}x_{33} + x_{33}x_{11} + x_{11}x_{22} - \tag{19}$$

$$-x_{23}x_{32} - x_{31}x_{13} - x_{12}x_{21},$$

$$N(X) = x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{22}x_{31}x_{13} - \tag{20}$$

$$-x_{33}x_{12}x_{21} + (x_{12}x_{23})x_{31} + x_{13}(x_{32}x_{21});$$

i.e. T is the usual trace, Q is the usual sum of principal second-order minors, and N is the usual determinant of the matrix X, the only proviso being that the order of the factors and the position of the brackets in the last two terms is important. The scalar square of the matrix X is also given by the usual formula

$$(X, X) = \sum_{i,j} x_{ij}x_{ji} = \sum_{i,j} N(x_{ij}). \tag{21}$$

In conclusion we give a classification of exceptional Jordan algebras over $\mathbb{R}$, i.e. real forms of the complex algebra $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{C}))$. We already know two such algebras. They are the algebra $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{R}))$, which is a split one, and the algebra $\mathfrak{H}_3(\mathbb{O})$. A remarkable property of the latter is that the scalar product (16) in it is positive definite (see (21)). This algebra is said to be *compact.*

Another possibility is to consider pseudo-Hermitian matrices instead of Hermitian ones. Namely, a matrix X of order 3 with entries in $\mathfrak{A}$ and an involution is said to be pseudo-Hermitian if $X^* = FXF$, where

$$F = \operatorname{diag}(1, 1, -1). \tag{22}$$

Pseudo-Hermitian matrices also form an algebra with respect to the operation $X \circ Y = \frac{1}{2}(XY + YX)$. Denote this algebra by $\mathfrak{H}_{2,1}(\mathfrak{A})$. If $\mathfrak{A}$ is a generalized Cayley algebra, then $\mathfrak{H}_{2,1}(\mathfrak{A})$ is an exceptional Jordan algebra.

Up to isomorphism the exceptional Jordan algebras over $\mathbb{R}$ are exhausted by the algebras $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{R}))$, $\mathfrak{A}_3(\mathbb{O})$, and $\mathfrak{H}_{2,1}(\mathbb{O})$.

For the proofs of the above statements and additional information see Freudenthal [1985], Albert and Jacobson [1957], Braun and Koecher [1966], Jacobson [1971].

1.4. The Algebra F_4. The complex simple Lie algebra of type F_4 and all its three real forms can be realized as the algebras of derivations of the exceptional Jordan algebras $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{C}))$, $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{R}))$, $\mathfrak{H}_3(\mathbb{O})$, and $\mathfrak{H}_{2,1}(\mathbb{O})$, and the corresponding Lie groups are realized as the groups of automorphisms of the exceptional Jordan algebras.

Let K be an arbitrary field and $\mathfrak{A}$ a generalized Cayley algebra over K. We will now describe derivations of the exceptional Jordan algebra $\mathfrak{H}_3(\mathfrak{A})$.

Firstly, mappings of the form

$$\operatorname{ad}(X) : Y \mapsto XY - YX,$$

where X is any skew-Hermitian matrix with zero trace (and the entries in $\mathfrak{A}$), are derivations. Such derivations form a subspace (but not a subalgebra!) of the Lie algebra $\operatorname{Der}\mathfrak{H}_3(\mathfrak{A})$, which we will denote by $\operatorname{Der}_0\mathfrak{H}_3(\mathfrak{A})$.

Secondly, any derivation D of the algebra $\mathfrak{A}$ generates a derivation of the algebra $\mathfrak{H}_3(\mathfrak{A})$, the result of applying D to each entry of the matrix $X \in \mathfrak{H}_3(\mathfrak{A})$. Such derivations form a subalgebra, which can be identified with $\operatorname{Der}\mathfrak{A}$.

As a vector space, $\operatorname{Der}\mathfrak{H}_3(\mathfrak{A})$ is decomposed into the direct sum

$$\operatorname{Der}\mathfrak{H}_3(\mathfrak{A}) = \operatorname{Der}_0\mathfrak{H}_3(\mathfrak{A}) \oplus \operatorname{Der}\mathfrak{A}. \tag{23}$$

Here $[D, \operatorname{ad}(X)] = \operatorname{ad}(DX)$ and

$$[\operatorname{ad}(X), \operatorname{ad}(Y)] = \operatorname{ad}((XY - YX)_0) + \frac{1}{3}\sum_{i,j} D_{x_{ij},y_{ij}}, \tag{24}$$

where $C_0 = C - \frac{1}{3}(\operatorname{tr} C)E$ is the projection of the matrix C onto the space of zero trace matrices, x_{ij} and y_{ij} denotes entries of the matrices X and Y, respectively, and $D_{a,b} \in \operatorname{Der}\mathfrak{A}$ is defined by formula (12).

Let $\mathfrak{A} = \mathfrak{A}_3(K)$ be the split Cayley algebra and $e \in \mathfrak{A}$ a pure imaginary element with $N(e) = -1$. Denote by $\mathfrak{h}_0$ the (two-dimensional) space of derivations of the form $\operatorname{ad}(X)$, where X is a diagonal zero trace matrix whose elements are scalar multiples of e, and by $\mathfrak{h}$ the sum of $\mathfrak{h}_0$ and a diagonalizable Cartan subalgebra of $\operatorname{Der}\mathfrak{A}$. One can easily see that $\mathfrak{h}$ is a diagonalizable Cartan subalgebra of $\operatorname{Der}\mathfrak{H}_3(\mathfrak{A})$. An examination of the root decomposition of this algebra with respect to $\mathfrak{h}$ shows that $\operatorname{Der}\mathfrak{H}_3(\mathfrak{A})$ is a split simple Lie algebra of type F_4.

We now state the main result for the cases $K = \mathbb{C}$ and $\mathbb{R}$.

Theorem 1.3. *The algebra of derivations of the algebra $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{C}))$ (respectively, $\mathfrak{H}_3(\mathbb{O})$) is a complex (respectively, real compact) simple Lie algebra of type F_4. The algebra of derivations of the algebra $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{R}))$ (respectively, $\mathfrak{H}_{2,1}(\mathbb{O})$) is a noncompact real simple Lie algebra of type $F\,\mathrm{I}$ (respectively, $F\,\mathrm{II}$).*

(For the notation of noncompact real simple Lie algebras see Table 4.)

Derivations of any simple Jordan algebra $\mathfrak{H}$ can be described in intrinsic terms. Namely, for any $a \in \mathfrak{H}$ denote by L_a the operator of multiplication by a:

$$L_a x = ax. \tag{25}$$

The main identity of Jordan algebras implies that for any $a, b \in \mathfrak{H}$ the mapping

$$[L_a, L_b]\colon x \mapsto [axb] \tag{26}$$

is a derivation of $\mathfrak{H}$. This defines a linear mapping $\Lambda^2\mathfrak{H} \to \operatorname{Der}\mathfrak{H}$. If $\mathfrak{H}$ is simple, then this mapping is surjective.

In any Jordan algebra

$$[[L_a, L_b], L_c] = L_{[acb]}. \tag{27}$$

This implies the following commutation relation for derivations of the form (26):

$$[[L_a, L_b], [L_c, L_d]] = [L_{[acb]}, L_d] + [L_c, L_{[adb]}]. \tag{28}$$

We now state some results on automorphisms of complex and real exceptional Jordan algebras.

Proposition 1.4. *The group of automorphisms of any of the algebras* $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{C}))$, $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{R}))$, $\mathfrak{H}_{2,1}(\mathbb{O})$, $\mathfrak{H}_3(\mathbb{O})$ *is connected.*

Theorem 1.5. *Any matrix* $X \in \mathfrak{H}_3(\mathbb{O})$ *can, by an appropriate automorphism of the algebra* $\mathfrak{H}_3(\mathbb{O})$, *be reduced to the form* $\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, *where* $\lambda_1, \lambda_2, \lambda_3$ *are the roots of the characteristic polynomial of the matrix* X *(which are therefore always real).*

A similar reduction to diagonal form is possible for "almost any" matrix $X \in \mathfrak{H}_3(\mathfrak{A}_3(\mathbb{C}))$ (for example, for any matrix whose characteristic polynomial has no multiple roots).

For the proofs of the above statements and additional information see Chevalley and Schafer [1950], Freudenthal [1985], Braun and Koecher [1966], Jacobson [1955], Postnikov [1982].

1.5. The Algebra E_6. If one considers the special Jordan algebra $\mathfrak{H}_3(\mathbb{O})$ as an analogue of the Jordan algebra $\mathfrak{H}_3(\mathbb{R})$ of real symmetric matrices, then the group $\operatorname{Aut}\mathfrak{H}_3(\mathbb{O})$, which, as indicated above, is a compact Lie group of type F_4, is an analogue of the group SO_3 acting in $\mathfrak{H}_3(\mathbb{R})$ according to the rule $X \mapsto AXA^\top (A \in \mathrm{SO}_3)$. There is, however, another, larger group acting in $\mathfrak{H}_3(\mathbb{R})$ in a similar manner, namely the group $SL_3(\mathbb{R})$. Is there a similar group acting in $\mathfrak{H}(\mathbb{O})$? One can easily see that transformations of the form $X \mapsto AXA^\top (A \in \mathrm{SL}_3(\mathbb{R}))$ are all transformations of $\mathfrak{H}_3(\mathbb{R})$ preserving the cubic form $\det X$. We already have an analogue of the form $\det X$ for $\mathfrak{H}_3(\mathbb{O})$ — it is the norm (see formula (20)). One can therefore regard the group of linear transformations of $\mathfrak{H}_3(\mathbb{O})$ preserving the norm as an analogue of $\mathrm{SL}_3(\mathbb{R})$.

What is the structure of this group? Clearly, its tangent algebra consists of linear transformations of the space $\mathfrak{H}_3(\mathbb{O})$ annihilating the norm (in the sense of the natural action of the Lie algebra $\mathfrak{gl}(\mathfrak{H}_3(\mathbb{O}))$ in the space of cubic forms on $\mathfrak{H}_3(\mathbb{O})$). This Lie algebra can be described in a simple way.

Let K be an arbitrary field and $\mathfrak{H}$ an exceptional Jordan algebra over K. Then the algebra $\mathfrak{g}$ of linear transformations of the space $\mathfrak{H}$ annihilating the norm admits a $\mathbb{Z}_2$-grading

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_0 = \operatorname{Der}\mathfrak{H}$ and $\mathfrak{g}_1$ is the set of transformations of the form L_a (see (25)) with $T(a) = 0$. The commutation rules for elements of $\mathfrak{g}$ follow from (27) and (28). This easily implies that $\mathfrak{g}$ is a simple Lie algebra of type E_6.

We now state the result for the cases $K = \mathbb{C}$ and $\mathbb{R}$.

Theorem 1.6. *The algebra of linear transformations of the algebra $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{C}))$ annihilating the norm is a complex simple Lie algebra of type E_6. The algebras of linear transformations of the algebras $\mathfrak{H}_3(\mathfrak{A}_3(\mathbb{R}))$, $\mathfrak{H}_{2,1}(\mathbb{O})$, and $\mathfrak{H}_3(\mathbb{O})$ annihilating the norm are noncompact real simple Lie algebras of types $E\,\mathrm{I}$, $E\,\mathrm{IV}$, and $E\,\mathrm{IV}$, respectively.*

For the proof and more information see Chevalley and Schafer [1950], Freudenthal [1985], Braun and Koecher [1966], Kantor [1968], Jacobson [1955].

1.6. The Algebra E_7. Continuing the analogy cited at the beginning of the preceding subsection, one can say that there exists a group of rational transformations of the algebra $\mathfrak{H}_3(\mathbb{O})$ similar to the group of linear fractional transformations

$$X \mapsto (AX+B)(CX+D)^{-1}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_6(\mathbb{R}),$$

of the algebra $\mathfrak{H}_3(\mathbb{R})$. This group is a simple Lie group of type E_7. An explicit description of it in terms of the algebra $\mathfrak{H}_3(\mathbb{O})$ would be difficult. However, its tangent algebra (consisting of vector fields on $\mathfrak{H}_3(\mathbb{O})$) can be described simply enough.

Let K be an arbitrary field, and $\mathfrak{H}$ an exceptional Jordan algebra over K. There exists a Lie algebra $\mathfrak{g}$ of polynomial vector fields on $\mathfrak{H}$ admitting a $\mathbb{Z}$-grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_{-1}$ is the space of vector fields of zero degree

$$T_a\colon x \mapsto a \quad (a \in \mathfrak{H}), \tag{29}$$

$\mathfrak{g}_0$ is the Lie algebra of linear vector fields (i.e. linear transformations) generated by the fields of the form

$$L_a\colon x \mapsto ax \quad (a \in \mathfrak{H}), \tag{30}$$

and $\mathfrak{g}_1$ is the space of quadratic vector fields of the form

$$Q_a\colon x \mapsto 2x(xa) - x^2 a \quad (a \in \mathfrak{H}). \tag{31}$$

The algebra $\mathfrak{g}_0$ is the direct sum of a simple subalgebra of type E_6 described in Sect. 1.3 and the one-dimensional centre consisting of homotheties. The remaining commutation rules are of the following form:

$$[L_a, T_b] = T_{ab}, \quad [L_a, Q_b] = -Q_{ab}, \tag{32}$$
$$[Q_a, T_b] = 2(L_{ab} - [L_a, L_b]).$$

This easily implies that $\mathfrak{g}$ is a simple Lie algebra of type E_7.

Inversion in the algebra $\mathfrak{H}$ induces an involutory automorphism of the algebra $\mathfrak{g}$ under which T_a goes into Q_a and L_a into $-L_a$.

Let us state the result for $K = \mathbb{C}$ and $\mathbb{R}$.

Theorem 1.7. *The Lie algebra of vector fields on $\mathfrak{H}$ described above is, in the case $\mathfrak{H} = \mathfrak{H}_3(\mathfrak{A}_3(\mathbb{C}))$, the complex simple Lie algebra of type E_7, and in the cases $\mathfrak{H} = \mathfrak{H}_3(\mathfrak{A}_3(\mathbb{R}))$, $\mathfrak{H}_{2,1}(\mathbb{O})$, and $\mathfrak{H}_3(\mathbb{O})$ a noncompact real simple Lie algebra of type $E\,\mathrm{V}$, $E\,\mathrm{VII}$, and $E\,\mathrm{VII}$, respectively.*

For the proof and further results see Braun and Koecher [1966], Kantor [1968], Jacobson [1971].

1.7. Unified Construction of the Exceptional Lie Algebras. All exceptional Lie algebras except G_2 can be described from the unique standpoint suggested by Tits [1966a] (see also Schafer [1966], Jacobson [1971]). We now present another, but very similar construction proposed by E.B. Vinberg [1966] and inspired by the ideas of B.A. Rozenfel′d. One of its advantages is the symmetry, which in the Tits construction is present only in a hidden form. For the algebra F_4 both constructions coincide with that described in Sect. 1.4.

Let K be an arbitrary field, and suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two finite-dimensional composition algebras over K (see Sect. 1.1). Consider the algebra $\mathfrak{C} = \mathfrak{A} \otimes \mathfrak{B}$. Define the involution $c \mapsto \overline{c}$ in it by the formula $\overline{a \otimes b} = \overline{a} \otimes \overline{b}$. The algebra $\operatorname{Der}\mathfrak{C}$ naturally contains the subalgebra $\operatorname{Der}\mathfrak{A} \oplus \operatorname{Der}\mathfrak{B}$. For any $c_1, c_2 \in \mathfrak{C}$ we define the derivation

$$D_{c_1,c_2} \in \operatorname{Der}\mathfrak{A} \oplus \operatorname{Der}\mathfrak{B} \subset \operatorname{Der}\mathfrak{C}$$

in such a way that

$$D_{a_1\otimes b_1, a_2\otimes b_2} = (b_1, b_2) D_{a_1,a_2} + (a_1, a_2) D_{b_1,b_2}, \tag{33}$$

where $D_{a_1,a_2} \in \operatorname{Der}\mathfrak{A}$ and $D_{b_1,b_2} \in \operatorname{Der}\mathfrak{B}$ are defined by formula (12).

Denote by $\mathfrak{g}_0$ the space of skew-Hermitian matrices of order 3 with zero trace and entries in $\mathfrak{C}$, and consider the direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \operatorname{Der}\mathfrak{A} \oplus \operatorname{Der}\mathfrak{B}. \tag{34}$$

Define the commutator by the following rules:

(1) the commutator of two elements in $\operatorname{Der}\mathfrak{A} \oplus \operatorname{Der}\mathfrak{B}$ coincides with the ordinary commutator of derivations;
(2) the commutator of an element $D \in \operatorname{Der}\mathfrak{A} \oplus \operatorname{Der}\mathfrak{B}$ with a matrix $X \in \mathfrak{g}_0$ is the matrix $DX \in \mathfrak{g}_0$ obtained by applying the derivation D to all entries of the matrix X;
(3) the commutator of the matrices $X = (x_{ij}), Y = (y_{ij}) \in \mathfrak{g}_0$ is defined by the formula

$$[X, Y] = (XY - YX)_0 + \frac{1}{3} \sum_{i,j} D_{x_{ij}, y_{ij}}, \tag{35}$$

where $C_0 = C - \frac{1}{3}(\operatorname{tr} C)E$ is the projection of the matrix C onto the space of matrices with zero trace(cf. (24)).

It can be verified directly that the space $\mathfrak{g}$ equipped with this operation is a semisimple Lie algebra. The types of the resulting algebras are given in the following table:

$\dim \mathfrak{A}$ \ $\dim \mathfrak{B}$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 + A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

(36)

Embeddings of composition algebras induce embeddings of the Lie algebras listed along the rows and columns of the table, in particular, the embeddings $F_4 \subset E_6 \subset E_7 \subset E_8$.

In this construction, instead of skew-Hermitian matrices one can take pseudo-skew-Hermitian matrices X defined by the condition $X^* = -FXF$, where $F = \operatorname{diag}(1, 1, -1)$. The resulting Lie algebra is of the same type but may be (if the field K is not algebraically closed) not isomorphic to the original one.

In particular, for $K = \mathbb{R}$ we are able to choose either split or compact algebras $\mathfrak{A}$ and $\mathfrak{B}$, and take either skew-Hermitian or pseudo-skew-Hermitian matrices. This enables one to obtain all the real exceptional simple Lie algebras. Here, if at least one of the algebras $\mathfrak{A}$, $\mathfrak{B}$ is not compact, then it does not matter whether the matrices taken are skew-Hermitian or pseudo-skew-Hermitian. If both algebras $\mathfrak{A}$, $\mathfrak{B}$ are compact and one takes skew-Hermitian matrices, the resulting Lie algebra is compact; in all the other cases it is noncompact.

The following table lists the noncompact real exceptional simple Lie algebras obtained for each choice of the algebra $\mathfrak{C} = \mathfrak{A} \otimes \mathfrak{B}$. It is assumed that $\dim \mathfrak{A} = 8$, $\dim \mathfrak{B} = 2^p$ $(p = 0, 1, 2, 3)$ and in the case where both algebras $\mathfrak{A}, \mathfrak{B}$ are compact one takes pseudo-skew-Hermitian matrices (in all the other cases, as indicated above, this does not matter).

$\mathfrak{C}$ \ p	0	1	2	3
$\mathfrak{A}_3(\mathbb{R}) \otimes \mathfrak{A}_p(\mathbb{R})$		$E\,\mathrm{I}$	$E\,\mathrm{V}$	$E\,\mathrm{VIII}$
$\mathfrak{A}_3(\mathbb{R}) \otimes \mathfrak{A}_p$	$F\,\mathrm{I}$	$E\,\mathrm{II}$	$E\,\mathrm{VI}$	$E\,\mathrm{IX}$
$\mathbb{O} \otimes \mathfrak{A}_p(\mathbb{R})$		$E\,\mathrm{IV}$	$E\,\mathrm{VII}$	$E\,\mathrm{IX}$
$\mathbb{O} \otimes \mathfrak{A}_p$	$F\,\mathrm{II}$	$E\,\mathrm{III}$	$E\,\mathrm{VI}$	$E\,\mathrm{VIII}$

§ 2. Models Associated with Gradings

Some of the $\mathbb{Z}$- and $\mathbb{Z}_m$-gradings of exceptional simple Lie algebras (see Chap. 3, Sect. 3) enable one to obtain the models of these algebras described in classical terms. Such a model of the algebra G_2 was actually obtained in Sect. 1.2 (see also Chap. 3, Sect. 3.7, Example 2). The series of examples below give a few such models of the algebras E_6, E_7, and E_8. (The reader should try to construct similar models of the algebra F_4.)

Before we begin with the description of these models, some general remarks are in order. For all the gradings to be considered the subalgebra $\mathfrak{g}_0$ will be identified with some classical linear algebra acting in a "basic" vector space V (whose dimension will be given in each case separately). Each of the grading subspaces $\mathfrak{g}_p$ $(p \neq 0)$ will be identified with a space of tensors on V in such a way that the action of $\mathfrak{g}_0$ on $\mathfrak{g}_p$ is induced by the natural action of the Lie algebra $\mathfrak{gl}(V)$ in the space of all tensors of the corresponding degree. The action of $\mathfrak{g}_0$ on $\mathfrak{g}_p$ is irreducible in all cases, so the above identification is defined uniquely up to a scalar multiple.

The commutator of an element from $\mathfrak{g}_p$ and an element from $\mathfrak{g}_q$ $(p, q \neq 0)$ is a bilinear mapping $\mathfrak{g}_p \times \mathfrak{g}_q \to \mathfrak{g}_{p+q}$ commuting with the action of $\mathfrak{g}_0$. Such a mapping can be obtained as a contraction of a product of tensors belonging to $\mathfrak{g}_p$ and $\mathfrak{g}_q$, a number of Kronecker tensors δ and, if $\mathfrak{g}_0 \neq \mathfrak{gl}(V)$, the "basic tensors" of the algebra $\mathfrak{g}_0$, which are

(1) for $\mathfrak{g}_0 = \mathfrak{sl}(V)$ the skew-symmetric covariant tensor det (the volume form) of degree $n = \dim V$, and the similar skew-symmetric contravariant tensor $\det^{-1}$;
(2) for $\mathfrak{g}_0 = \mathfrak{so}(V)$ and $\mathfrak{sp}(V)$ the tensors det and $\det^{-1}$, the metric tensor g and the inverse contravariant tensor g^{-1}.

It follows from the classical theory of invariants (see, for example, Vinberg and Popov [1989], §9) that any bilinear mapping $\mathfrak{g}_p \times \mathfrak{g}_q \to \mathfrak{g}_{p+q}$ commuting with the action of $\mathfrak{g}_0$ is a linear combination of mappings obtained by the above procedure, possibly with a subsequent permutation of indices. This defines *a priori* the form of the commutator in the model of the algebra $\mathfrak{g}$ under consideration up to a few undetermined constants. An obvious variation of these constants can be achieved by varying the procedure for identifying the subspaces $\mathfrak{g}_p$ $(p \neq 0)$ with the corresponding spaces of tensors. If this does not define the result uniquely, the remaining ambiguity can be eliminated by verification of the Jacobi identity.

Now let us agree on definitions and notation pertaining to the tensor algebra. As usual, square brackets around a group of indices stand for skew-symmetrization with respect to these indices divided by $k!$, where k is the number of indices inside the brackets. The result of skew-symmetrization of a nonmixed tensor T with respect to all its indices will be denoted by $\operatorname{Alt} T$. If S and T are skew-symmetric contravariant (or covariant) tensors of ranks k and l, respectively, then their exterior product is the tensor

$$S \wedge T = \frac{(k+l)!}{k!l!} \operatorname{Alt}(S \otimes T). \tag{38}$$

(This is just one possible definition of the exterior product under which it is associative. The coefficient is chosen so that the coordinates of the tensor

$S \wedge T$ are bilinear forms of the independent coordinates of the tensors S and T with the coefficients $\pm 1, 0$.)

The coordinates of the tensors det and $\det^{-1}$ will be denoted by $\varepsilon_{i_1 \ldots i_n}$ and $\varepsilon^{i_1 \cdots i_n}$, respectively. These tensors will be assumed to be normalized in such a way that

$$\frac{1}{n!}\varepsilon_{i_1 \ldots i_n}\varepsilon^{i_1 \ldots i_n} = 1. \tag{39}$$

Let us also agree that in the examples below the grading subspaces $\mathfrak{g}_p$ are written out in the following order:

in the case of a $\mathbb{Z}_2$-grading: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$,
in the case of a $\mathbb{Z}_3$-grading: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$,
in the case of a $\mathbb{Z}$-grading: $\mathfrak{g} = \mathfrak{g}_{-d} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_d$,

where $d = \max\{p \colon \mathfrak{g}_p \neq 0\}$ is the "depth" of the grading.

Example 1. According to Chap. 3, Sect. 3.7, Example 3, the algebra E_7 admits a $\mathbb{Z}_2$-grading of the form

$$E_7 = \mathfrak{sl}(V) \oplus \wedge^4 V, \quad \dim V = 8. \tag{40}$$

The commutator in $\mathfrak{g}_1 \times \mathfrak{g}_1$ (with values in $\mathfrak{g}_0$) is defined by the formula

$$[T_1, T_2]^i_j = \frac{1}{288}\varepsilon_{i_1 i_2 i_3 i_4 j_1 j_2 j_3 j}(T_1^{i_1 i_2 i_3 i_4} T_2^{i j_1 j_2 j_3} - T_2^{i_1 i_2 i_3 i_4} T_1^{i j_1 j_2 j_3}).$$

(By modifying the procedure of identifying $\mathfrak{g}_1$ with $\wedge^4 V$ the coefficient in this formula can be made equal to an arbitrary number.) For more details on this model of E_7 see Antonyan [1981]. In terms of it one can conveniently describe the "smallest" simple E_7-module M, namely it admits a $\mathbb{Z}_2$-grading (which means that $\mathfrak{g}_i M_j \subset M_{i+j}$) of the form

$$M = \wedge^2 V^* \oplus \wedge^2 V, \tag{41}$$

where the action of $\mathfrak{g}_0$ in M_0 and M_1 coincides with the natural action of the algebra $\mathfrak{sl}(V)$ in the corresponding tensor spaces, and the action of $\mathfrak{g}_1$ (permuting M_0 and M_1) is given by the formulae

$$(TP)_{ij} = \frac{1}{48}\varepsilon_{ijk_1k_2k_3k_4l_1l_2}T^{k_1k_2k_3k_4}P^{l_1l_2} \quad (T \in \mathfrak{g}_1,\ P \in M_1),$$

$$(TQ)^{ij} = \frac{1}{2}T^{ijkl}Q_{kl} \quad (T \in \mathfrak{g}_1,\ Q \in M_0).$$

Example 2. According to Chap. 3, Sect. 3.11, Example 2, the algebra E_6 admits a $\mathbb{Z}_2$-grading of the form

$$E_6 = \mathfrak{sp}(V) \oplus \wedge^4_0 V, \quad \dim V = 8, \tag{42}$$

where $\wedge^4_0 V$ is the space of 4-vectors whose contraction with the metric tensor of the space V vanishes. Under the natural embedding of the terms of (42) in the corresponding terms of (40) the algebra E_6 becomes a subalgebra of E_7. The "smallest" E_6-module M can be identified with $\wedge^2_0 V$. Assuming that $\det = \frac{1}{24} g \wedge g \wedge g \wedge g$, the action of $\mathfrak{g}_1$ on M is given by the formula

$$(TP)^{ij} = \frac{1}{2} g_{k_1 l_1} g_{k_2 l_2} T^{ijk_1k_2} P^{l_1 l_2} \quad (T \in \mathfrak{g}_1,\ P \in M).$$

Example 3. According to Chap. 3, Sect. 3.7, Example 4, the algebra E_8 admits a $\mathbb{Z}_3$-grading of the form

$$E_8 = \wedge^3 V^* \oplus \mathfrak{sl}(V) \oplus \wedge^3 V, \quad \dim V = 9. \tag{43}$$

The commutators of the elements of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are defined by the formulae

$$[S,T]^i_j = \frac{1}{2} S^{ikl} T_{jkl} - \frac{1}{18} S^{klm} T_{klm} \delta^i_j \quad (S \in \mathfrak{g}_1,\ T \in \mathfrak{g}_{-1}),$$

$$[S_1, S_2]_{ijk} = \frac{1}{36} \varepsilon_{i_1 j_1 k_1 i_2 j_2 k_2 ijk} S_1^{i_1 j_1 k_1} S_2^{i_2 j_2 k_2} \quad (S_1,\ S_2 \in \mathfrak{g}_1),$$

$$[T^1, T^2]^{ijk} = -\frac{1}{36} \varepsilon^{i_1 j_1 k_1 i_2 j_2 k_2 ijk} T^1_{i_1 j_1 k_1} T^2_{i_2 j_2 k_2} \quad (T^1,\ T^2 \in \mathfrak{g}_{-1}).$$

For more details about this model of E_8 see Vinberg and Elashvili [1978].

Example 4. According to Chap. 3, Sect. 3.5, Example 2, the algebras E_6, E_7, E_8 admit $\mathbb{Z}$-gradings of the form

$$E_6 = \wedge^6 V^* \oplus \wedge^3 V^* \oplus \mathfrak{gl}(V) \oplus \wedge^3 V \oplus \wedge^6 V, \quad \dim V = 6, \tag{44}$$

$$E_7 = \wedge^6 V^* \oplus \wedge^3 V^* \oplus \mathfrak{gl}(V) \oplus \wedge^3 V \oplus \wedge^6 V, \quad \dim V = 7, \tag{45}$$

$$E_8 = (V^* \otimes \wedge^8 V^*) \oplus \wedge^6 V^* \oplus \wedge^3 V^* \oplus \mathfrak{gl}(V) \oplus \wedge^3 V \oplus \wedge^6 V \oplus (V \otimes \wedge^8 V),\ \dim V = 8. \tag{46}$$

There are natural embeddings $E_6 \subset E_7 \subset E_8$ compatible with these gradings. It is therefore sufficient to give commutation rules for elements of the subspaces $\mathfrak{g}_p$ $(p \neq 0)$ just for the algebra E_8. These rules are of the following form[3]:

$$[S,T]^i_j = \frac{1}{2} S^{ikl} T_{jkl} - \frac{1}{18} S^{klm} T_{klm} \delta^i_j \quad (S \in \mathfrak{g}_1,\ T \in \mathfrak{g}_{-1}),$$

$$[P,Q]^i_j = -\frac{1}{120} P^{ik_1 \dots k_5} Q_{jk_1 \dots k_5} + \frac{1}{1080} P^{k_1 \dots k_6} Q_{k_1 \dots k_6} \delta^i_j \ (P \in \mathfrak{g}_2,\ Q \in \mathfrak{g}_{-2}),$$

$$[U,W]^i_j = \frac{1}{40320} U^{ik_1 \dots k_8} W_{jk_1 \dots k_8} \quad (U \in \mathfrak{g}_3, W \in \mathfrak{g}_{-3}),$$

$$[T,P]^{ijk} = \frac{1}{6} T_{l_1 l_2 l_3} P^{l_1 l_2 l_3 ijk} \quad (T \in \mathfrak{g}_{-1},\ P \in \mathfrak{g}_2),$$

$$[Q,U]^{ijk} = -\frac{1}{120} Q_{l_1 \dots l_6} U^{l_1 \dots l_6 ijk} \quad (Q \in \mathfrak{g}_{-2},\ U \in \mathfrak{g}_3),$$

$$[T,U]^{l_1 \dots l_6} = \frac{1}{2} T_{ijk} U^{ijkl_1 \dots l_6} \quad (T \in \mathfrak{g}_{-1},\ U \in \mathfrak{g}_3),$$

$$[S_1, S_2] = S_1 \wedge S_2 \quad (S_1, S_2 \in \mathfrak{g}_1),$$

[3] The coefficients in the following formulae were computed by A.A. Katanova.

$$[S,P]^{i_1\dots i_9} = \frac{1}{1440} S^{i_1[i_2 i_3} P^{i_4\dots i_9]} \quad (S \in \mathfrak{g}_1,\ P \in \mathfrak{g}_2).$$

The remaining five formulae repeat the last five with upper indices replaced by lower ones and vice versa.

Chapter 6
Subgroups and Subalgebras of Semisimple Lie Groups and Lie Algebras

§ 1. Regular Subalgebras and Subgroups

This section considers the class of subalgebras of complex semisimple Lie algebras and the corresponding subgroups of complex semisimple Lie groups that is most amenable to analysis. The classification of regular subalgebras reduces to combinatorial problems related to root systems.

1.1. Regular Subalgebras of Complex Semisimple Lie Algebras. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. A subalgebra $\mathfrak{f} \subset \mathfrak{g}$ is said to be *regular with respect to the Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ if $\mathfrak{h}$ is contained in the normalizer $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{f})$, i.e. $[\mathfrak{h},\mathfrak{f}] \subset \mathfrak{f}$. For example, any subalgebra $\mathfrak{f}$ containing a Cartan subalgebra $\mathfrak{h}$ (a *subalgebra of maximal rank*) is regular with respect to $\mathfrak{h}$. A subalgebra $\mathfrak{f}$ is said to be *regular* if $\mathfrak{f}$ is regular with respect to some Cartan subalgebra. Since all Cartan subalgebras of $\mathfrak{g}$ are conjugate, if one studies regular subalgebras up to conjugacy it can be assumed that $\mathfrak{h}$ is fixed. A subgroup F of a connected complex semisimple Lie group G is said to be *regular* (*regular with respect to a Cartan subgroup* H) if $N_G(F)$ contains a Cartan subgroup (respectively, $H \subset N_G(F)$). In particular, if F contains a Cartan subgroup H, then F is regular with respect to H (such subgroups are called *subgroups of maximal rank*).

A subalgebra $\mathfrak{f}$ of $\mathfrak{g}$ is said to be *reductive* in $\mathfrak{g}$ if $\operatorname{rad}\mathfrak{f}$ consists of elements that are semisimple in $\mathfrak{g}$. This is equivalent to the fact that the adjoint representation of $\mathfrak{f}$ in $\mathfrak{g}$ is completely reducible. Virtual Lie subgroups reductive in G are defined in a similar manner.

Now we describe regular subalgebras of semisimple Lie algebras in terms of root systems. Let Δ be a root system in the sense of Chap. 3, Sect. 1.1. A subsystem $\Gamma \subset \Delta$ is said to be *closed* if

$$\alpha, \beta \in \Gamma, \quad \alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in \Gamma.$$

A subsystem $\Gamma \subset \Delta$ is said to be *symmetric* if $-\Gamma = \Gamma$, i.e.

$$\alpha \in \Gamma \Rightarrow -\alpha \in \Gamma.$$

Theorem 1.2 of Chap. 3 and Proposition 9.1 of Chap. 1 imply the following proposition.

Proposition 1.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra of it. Then all subalgebras $\mathfrak{f}$ of $\mathfrak{g}$ regular with respect to $\mathfrak{h}$ can be written in the form*

$$\mathfrak{f} = \mathfrak{f}(\mathfrak{t}, \Gamma) = \mathfrak{t} \oplus \sum_{\alpha \in \Gamma} \mathfrak{g}_\alpha,$$

where Γ is a closed subsystem in the root system $\Delta = \Delta_\mathfrak{g}(\mathfrak{h})$ and $\mathfrak{t}$ is a subspace in $\mathfrak{h}$ containing the element h_α for any $\alpha \in \Gamma \cap (-\Gamma)$. The subalgebra $\mathfrak{f}$ is algebraic if and only if so is $\mathfrak{t}$. The subalgebra $\mathfrak{f}$ is reductive in $\mathfrak{g}$ if and only if the subsystem Γ is symmetric. Finally, a reductive subalgebra $\mathfrak{f}$ is semisimple if and only if $\mathfrak{t}$ is generated by all h_α, where $\alpha \in \Gamma$. Here $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{f}$, and $\Gamma = \Delta_\mathfrak{f}(\mathfrak{t})$.

Corollary. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a decomposition into the direct sum of ideals. Then semisimple regular subalgebras of $\mathfrak{g}$ are of the form $\mathfrak{f} = \mathfrak{f}_1 \oplus \mathfrak{f}_2$, where $\mathfrak{f}_i$ $(i = 1, 2)$ is an arbitrary semisimple regular subalgebra of $\mathfrak{g}_i$.*

Example 1. The Cartan subalgebra $\mathfrak{h} = \mathfrak{f}(\mathfrak{h}, \emptyset)$ and the Borel subalgebras $\mathfrak{b}^\pm(\mathfrak{h}, \Delta^\pm)$ (see Chap. 3, Sect. 1.5) are regular with respect to $\mathfrak{h}$.

Example 2. For any $\alpha \in \Delta$ the subalgebra $\mathfrak{g}^{(\alpha)} = \langle e_\alpha, e_{-\alpha}, h_\alpha \rangle$ (see Chap. 3, Sect. 1.2) is regular and coincides with $\mathfrak{f}(\langle h_\alpha \rangle, \{\alpha, -\alpha\})$.

Let us find the Levi decomposition of a regular subalgebra. If the subalgebra $\mathfrak{f}$ is regular with respect to $\mathfrak{h}$, then the subalgebra $\operatorname{rad}\mathfrak{f}$ is clearly also regular with respect to $\mathfrak{h}$. Since $\mathfrak{h}$ consists of semisimple elements, a Levi subalgebra $\mathfrak{l} \subset \mathfrak{f}$ can also be chosen to be regular with respect to $\mathfrak{h}$. Let us see how can one describe $\mathfrak{l}$ and $\operatorname{rad}\mathfrak{f}$ in terms of roots.

Let Γ be a closed subsystem in Δ. We set $\Gamma_0 = \Gamma \cap (-\Gamma)$, $\Gamma_1 = \Gamma \setminus \Gamma_0$. Then Γ_0 and Γ_1 are closed subsystems, Γ_0 is symmetric, and

$$\alpha \in \Gamma,\ \beta \in \Gamma_1 \Rightarrow \alpha + \beta \in \Gamma_1. \tag{1}$$

Proposition 1.2. *Let $\mathfrak{t}_0$ be the linear span of the elements h_α, where $\alpha \in \Gamma_0$, and let $\mathfrak{t}_1 = \{x \in \mathfrak{t} | \alpha(x) = 0 \quad \forall \alpha \in \Gamma_0\}$. Then the subalgebra $\mathfrak{l} = \mathfrak{f}(\mathfrak{t}_0, \Gamma_0)$ is a Levi subalgebra of $\mathfrak{f} = \mathfrak{f}(\mathfrak{t}, \Gamma)$, and $\operatorname{rad}\mathfrak{f} = \mathfrak{f}(\mathfrak{t}_1, \Gamma_1)$. A subsystem Δ^+ of positive roots in Δ can be chosen in such a way that $\Gamma_1 \subset \Delta^+$.*

Proof. It follows from Proposition that $\mathfrak{l}$ is a maximal semisimple subalgebra of $\mathfrak{g}$. Now $\mathfrak{f} = \mathfrak{l} \ltimes \mathfrak{f}(\mathfrak{t}_1, \Gamma_1)$, where in view of (1) $\mathfrak{f}(\mathfrak{t}_1, \Gamma_1)$ is an ideal in $\mathfrak{f}$. If the ideal $\mathfrak{f}(\mathfrak{t}_1, \Gamma_1)$ is not solvable, one easily arrives at a contradiction with the fact that the semisimple subalgebra $\mathfrak{l}$ is maximal. Therefore $\mathfrak{f}(\mathfrak{t}_1, \Gamma_1) = \operatorname{rad}\mathfrak{f}$. The last statement is derived from the Morozov-Borel theorem (Theorem 1.6 of Chap. 3). □

Now consider the case of a subalgebra of maximal rank. As follows from Proposition 1.1 that any subalgebra of $\mathfrak{g}$ containing the Cartan subalgebra $\mathfrak{h}$

is of the form $\mathfrak{f}(\mathfrak{h}, \Gamma)$, where $\Gamma \subset \Delta$ is a closed subsystem. Conversely, any subalgebra of this form contains $\mathfrak{h}$.

Proposition 1.3. *Let Γ_1, Γ_2 be two closed subsystems of the system $\Delta = \Delta_{\mathfrak{g}}(\mathfrak{h})$. We have $\Gamma_1 \subset \Gamma_2$ if and only if $\mathfrak{f}(\mathfrak{h}, \Gamma_1) \subset \mathfrak{f}(\mathfrak{h}, \Gamma_2)$. The subalgebras $\mathfrak{f}(\mathfrak{h}, \Gamma_1)$ and $\mathfrak{f}(\mathfrak{h}, \Gamma_2)$ are conjugate in $\mathfrak{g}$ (are conjugate by means of* Aut $\mathfrak{g}$) *if and only if Γ_1 and Γ_2 are taken into each other by an element of the Weyl group $W^\vee$ of $\mathfrak{g}$ (respectively, by an element of* Aut Δ).

Proposition 1.4. *Let $\mathfrak{f}$ be a subalgebra of maximal rank of a semisimple Lie algebra $\mathfrak{g}$. Then $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{f}) = \mathfrak{f}$.*

1.2. Description of Semisimple and Reductive Regular Subalgebras. We keep the notation of Sect. 1.1. It follows from Proposition 1.1 that semisimple subalgebras of $\mathfrak{g}$ that are regular with respect to $\mathfrak{h}$ are in one-to-one correspondence with closed symmetric subsystems of the root system $\Delta = \Delta_{\mathfrak{g}}(\mathfrak{h})$. We will denote by $\mathfrak{f}(\Gamma)$ the semisimple regular subalgebra corresponding to a closed symmetric subsystem $\Gamma \subset \Delta$. A statment similar to Proposition 1.3 is the following.

Proposition 1.5. *Let Γ_1, Γ_2 be two closed symmetric subsystems in Δ. We have $\Gamma_1 \subset \Gamma_2$ if and only if $\mathfrak{f}(\Gamma_1) \subset \mathfrak{f}(\Gamma_2)$. The subalgebras $\mathfrak{f}(\Gamma_1)$ and $\mathfrak{f}(\Gamma_2)$ are conjugate in $\mathfrak{g}$ (are conjugate by means of* Aut $\mathfrak{g}$) *if and only if Γ_1 and Γ_2 are taken into each other by an element of the Weyl group $W^\vee$ of $\mathfrak{g}$ (respectively, by an element of* Aut Δ).

Thus the classification of regular semisimple subalgebras of $\mathfrak{g}$ reduces to the classification of closed symmetric subsystems $\Gamma \subset \Delta$ up to a transformation from the group $W^\vee$ or Aut Δ. We now describe the classification of the latter following E.B. Dynkin [1952b].

For any subsystem $M \subset \Delta$ we denote by $[M]$ the set of all roots in Δ that are integral linear combinations of roots in M. Any closed symmetric subsystem $\Gamma \subset \Delta$ is a root system of the subalgebra $\mathfrak{f}(\Gamma)$ and can therefore be represented in the form $\Gamma = [M]$, where M is a system of simple roots in Γ. Here the subsystems $\Gamma_1 = [M_1]$ and $\Gamma_2 = [M_2]$ are conjugate with respect to $W^\vee$ or Aut Δ if and only if M_1 and M_2 can be chosen to be conjugate. The problem is therefore reduced to the characterization and classification of subsystems M.

A subsystem $M \subset \Delta$ is called a π-system if M is linearly independent and $\alpha - \beta \notin \Delta$ for all $\alpha, \beta \in M$. It follows from Chap. 3, Proposition 1.2 that any π-system is admissible. Therefore (see Chap. 3, Theorem 1.14) any π-system $M \subset \Delta$ is a system of simple roots of the symmetric closed subsystem $\Gamma = [M]$. Conversely, any system of simple roots of a symmetric closed subsystem is a π-system in Δ. This provides the characterization of systems of simple roots of regular semisimple subalgebras.

In order to enumerate all π-systems in Δ one can use the following fact: any π-system is contained in a π-system consisting of $l = \operatorname{rk} \mathfrak{g}$ elements. Since any subset of a π-system is also a π-system, it is sufficient to find all π-systems consisting of l elements.

Let $M \subset \Delta$ be a π-system and $\widetilde{M} = M \cup \{\alpha_0\}$, where α_0 is the lowest root of the root system corresponding to an indecomposable component M_1 of the system M. Let $M' = \widetilde{M} \setminus \{\alpha\}$, where $\alpha \in M_1$. Such a transformation of a π-system M into M' is said to be *elementary*.

Theorem 1.1. *Any π-system in Δ consisting of $l = \operatorname{rk} \mathfrak{g}$ elements is obtained from a system of simple roots of Δ by a sequence of elementary transformations.*

The classification of semisimple regular subalgebras of simple Lie algebras is obtained in Dynkin [1952b] with the help of Theorem 1.1. Note that by the corollary to Proposition 1.1 the case of a semisimple Lie algebra $\mathfrak{g}$ reduces to that of a simple Lie algebra. Consider the classification of maximal semisimple regular subalgebras of simple Lie algebras.

One can easily see that the rank of any maximal closed symmetric subsystem $\Gamma \neq \Delta$ is equal to either l or $l-1$. If $\operatorname{rk}\Gamma = l$, then the corresponding π-system M is obtained from the system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ by a single elementary transformation. Thus $M = \Pi \setminus \{\alpha_i\}$, where $\widetilde{\Pi} = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ is the extended system of simple roots, $i \geq 1$ and $M \neq \Pi$. If $\operatorname{rk}\Gamma = l-1$, then the corresponding π-system is of the form $M = \Pi \setminus \{\alpha_i\}$. It turns out, however, that the removal of the root $\alpha_i \in \Pi$ from the system Π or $\widetilde{\Pi}$ does not necessarily result in a maximal closed symmetric subsystem Γ. Denote by $n_0 = 1, n_1, \ldots, n_l$ the coefficients of the linear relation introduced in Chap. 3, Sect. 3.7 (see Table 3).

Theorem 1.2 (see Goto and Grosshans [1978]) *Suppose that a reduced root system Δ is indecomposable and $M \subset \Delta$ is a π-system. A system $\Gamma = [M]$ is a maximal proper closed symmetric subsystem in Δ if and only if M can be obtained from the extended system of simple roots $\widetilde{\Pi}_N = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ in one of the following ways:*

(1) $M = \widetilde{\Pi} \setminus \{\alpha_i\}$, *where n_i is a prime number;*

(2) $M = \widetilde{\Pi} \setminus \{\alpha_0, \alpha_i\} = \Pi \setminus \{\alpha_i\}$, *where $i > 0$ and $n_i = 1$.*

Here $\operatorname{rk}\Gamma = l$ *and* $l-1$ *respectively.*

Theorem 1.2 easily implies the classification of maximal semisimple regular subalgebras of simple Lie algebras (here one has to find out which of the subsystems of M are taken into each other by an element of the group $W^\vee$ or $\operatorname{Aut}\Delta$). In particular, systems Γ of rank l correspond to maximal semisimple subalgebras of maximal rank. It is not difficult to see that these subalgebras are also maximal among all subalgebras of the algebra $\mathfrak{g}$. They are listed (up to conjugacy) in Table 5. The results of Chap. 3, Sect. 3.7 imply that all of them are of the form $\mathfrak{g}^\theta$, where $\theta \in \operatorname{Int}\mathfrak{g}$ is an automorphism of prime order.

Now consider regular subalgebras that are reductive in $\mathfrak{g}$. One can see from Proposition 1.1 that their classification reduces to that of semisimple regular subalgebras. In particular, the classification of subalgebras of maximal rank that are reductive in $\mathfrak{g}$ is equivalent to the classification of semisimple regular subalgebras (see Propositions 1.3 and 1.5). Theorem 1.2 provides the classification of maximal reductive (in $\mathfrak{g}$) subalgebras of maximal rank for simple Lie algebras $\mathfrak{g}$.

Corollary. *A maximal reductive (in $\mathfrak{g}$) subalgebra of maximal rank of a simple Lie algebra $\mathfrak{g}$ is either semisimple or has one-dimensional centre. Subalgebras of the first type are listed up to conjugacy in Table* 5, *and those of the second type in Table* 6.

Note that unlike the subalgebras in Table 5 those of Table 6 are not maximal among all subalgebras of $\mathfrak{g}$, since they are embedded in parabolic subalgebras (see Sect. 1.3). We also note the following remarkable coincidence: nonsemisimple maximal reductive subalgebras of maximal rank in simple Lie algebras are precisely subalgebras of fixed points of involutory automorphisms of type II (see Chap. 4, Sect. 1.4).

Consider one special class of reductive (in $\mathfrak{g}$) subalgebras of maximal rank. Let $\mathfrak{c}$ be a subspace of a Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$. Then $\mathfrak{z}(\mathfrak{c})$ contains $\mathfrak{h}$ and is consequently a subalgebra of maximal rank in $\mathfrak{g}$. Let G be a connected semisimple complex Lie group with tangent algebra $\mathfrak{g}$, and H its maximal torus corresponding to $\mathfrak{h}$. Then $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c})$ is the tangent algebra of the centralizer $Z_G(T)$ of the algebraic torus $T \subset H$ corresponding to the subalgebra ${}^a\mathfrak{c} \subset \mathfrak{h}$. Conversely, the tangent algebra of the centralizer of any algebraic torus in G is of the above form. It turns out that subalgebras of the form $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c})$ correspond precisely to π-systems lying in a system of simple roots $\Pi \subset \Delta$.

Theorem 1.3. *Let $M \subset \Pi$, where Π is a system of simple roots of Δ. Then the subalgebra $\mathfrak{f}(\mathfrak{h}, [M])$ is of the form $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c})$ for some subspace $\mathfrak{c}$ of $\mathfrak{g}$. Conversely, any subalgebra $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c})$, where $\mathfrak{c} \subset \mathfrak{h}$, is of the form $\mathfrak{f}(\mathfrak{h}, [M])$, where M is a subset of a system of simple roots $\Pi \subset \Delta$.*

Proof. We associate with an arbitrary subsystem $M \subset \Pi$ the element $x_M \in \mathfrak{f}(\mathbb{R})$ such that

$$\alpha(x_M) = \begin{cases} 0 & \text{if } \alpha \in M, \\ 1 & \text{if } \alpha \in \Pi \setminus M. \end{cases} \tag{2}$$

It can be verified that $\mathfrak{z}_{\mathfrak{g}}(x_M) = \mathfrak{f}(\mathfrak{h}, [M])$. This proves the first statement. Now let $\mathfrak{f} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c})$, where $\mathfrak{c} \subset \mathfrak{h}$. Then $\mathfrak{f}$ is the tangent algebra of the subgroup $F = Z_G(S)$, where S is an algebraic torus in G. It is known that there exists $y \in \mathfrak{s}(\mathbb{R}) \subset \mathfrak{f}(\mathbb{R})$ such that S coincides with the algebraic closure of the subgroup $\{\exp ty | t \in \mathbb{R}\}$. Then $F = Z_G(y)$. The element y is contained in a closed Weyl chamber $\overline{C} \subset \mathfrak{f}(\mathbb{R})$ (see Chap. 3, Sect. 1.4). If Π is the system of simple roots corresponding to C and $M = \{\alpha \in \Pi | \alpha(y) = 0\}$, then $\mathfrak{f} = \mathfrak{f}(\mathfrak{h}, [M])$. □

1.3. Parabolic Subalgebras and Subgroups. Let G be a connected complex Lie group. A Lie subgroup $F \subset G$ is said to be *parabolic* if F contains a Borel subgroup of G. Similarly, a *parabolic subalgebra* of a complex Lie algebra $\mathfrak{g}$ is a subalgebra containing a Borel subalgebra. Clearly, any parabolic subgroup (subalgebra) contains $\operatorname{Rad} G$ (respectively, $\operatorname{rad}\mathfrak{g}$). We shall assume therefore that both the Lie group G and the Lie algebra $\mathfrak{g}$ are semisimple. Clearly, parabolic subgroups and subalgebras are subgroups and subalgebras of maximal rank. Let us now describe parabolic subalgebras in the language of root systems.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, Δ^+ a fixed subsystem of positive roots in $\Delta = \Delta_{\mathfrak{g}}(\mathfrak{h})$, and Π the corresponding system of simple roots. Since all Borel subalgebras are conjugate in $\mathfrak{g}$ (Theorem 1.6 of Chap. 3), it is sufficient to consider subalgebras $\mathfrak{f} \subset \mathfrak{g}$ containing $\mathfrak{b}^+ = \mathfrak{f}(\mathfrak{h}, \Delta^+)$. By virtue of Propositions 1.1 and 1.3 such a subalgebra is of the form $\mathfrak{f} = \mathfrak{f}(\mathfrak{h}, \Gamma)$, where Γ is a closed subsystem in Δ containing Δ^+.

Proposition 1.6. *If $M \subset \Pi$, then $\Gamma_M = [M] \cup \Delta^+$ is a closed subsystem in Δ. Any closed subsystem in Δ containing Δ^+ coincides with Γ_M for some uniquely defined subsystem $M \subset \Pi$. If $M, N \subset \Pi$, and Γ_M is taken into Γ_N by an element of the group $W^\vee$, then $M = N$ and $\Gamma_M = \Gamma_N$.*

Proof. If $\Delta^+ \subset \Gamma \subset \Delta$ and Γ is a closed subsystem, we set $\Gamma_0 = \Gamma \cap (-\Gamma)$ and $M = \Gamma_0 \cap \Delta^+$. We have $\Gamma = \Gamma_0 \cup \Delta^+$. By induction on the sum of the coefficients in the simple root expansion of positive roots, we can prove that $\Gamma_0 = [M]$. If $\Gamma_N = w(\Gamma_M)$, where $w \in W^\vee$, then $[N] = w([M])$. Making use of Chap. 3, Theorem 1.9 one can assume that $N = w(M)$. Then $w(\Delta_G^+) = \Delta_G^+$ and $w(\Pi) = \Pi$, whence $w = e$ by Chap. 3, Theorem 1.9. □

For any subset $M \subset \Pi$ we set $\mathfrak{p}(M) = \mathfrak{f}(\mathfrak{h}, \Gamma_M)$, where $\Gamma_M = [M] \cup \Delta^+$. By Proposition 1.6, $\mathfrak{p}(M)$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}^+$. In particular, $\mathfrak{p}(\emptyset) = \mathfrak{b}^+$, $\mathfrak{p}(\Pi) = \mathfrak{g}$.

Propositions 1.3, 1.6, and 1.2 imply the following theorem.

Theorem 1.4. *By associating with a subsystem $M \subset \Pi$ the subalgebra $\mathfrak{p}(M) \subset \mathfrak{g}$, we obtain a one-to-one correspondence between subsets of Π and classes of conjugate parabolic subalgebras of $\mathfrak{g}$. The subalgebras $\mathfrak{p}(M)$ and $\mathfrak{p}(N)$, where $M, N \subset \Pi$, are conjugate by means of* $\operatorname{Aut}\mathfrak{g}$ *if and only if M is taken into N by an element of* $\operatorname{Aut}\Pi$. *The subalgebra $\mathfrak{f}(\mathfrak{h}, [M])$ is a reductive Levi subalgebra of $\mathfrak{p}(M)$, $\operatorname{rad}_u \mathfrak{p}(M) = \mathfrak{f}(0, \Delta^+ \setminus [M])$, $\operatorname{rad}\mathfrak{p}(M) = \mathfrak{f}(\mathfrak{z}_M, \Delta^+ \setminus [M])$, where $\mathfrak{z}_M = \{x \in \mathfrak{h} | \alpha(x) = 0 \quad \forall x \in M\}$.*

Corollary. *Maximal parabolic subalgebras of a semisimple complex Lie algebra $\mathfrak{g}$ of rank l are conjugate to subalgebras $\mathfrak{p}(M)$, where $M \subset \Pi$, $|M| = l-1$. These subalgebras are maximal among all subalgebras of $\mathfrak{g}$.*

Note that any subsystem $M \subset \Pi$ determines the following decompositions of $\mathfrak{g}$ into the direct sum of subspaces:

$$\mathfrak{g} = \mathfrak{n}_-(M) \oplus \mathfrak{f}(\mathfrak{h}, [M]) \oplus \mathfrak{n}_+(M) = \mathfrak{n}_-(M) \oplus \mathfrak{p}(M), \tag{3}$$

where $\mathfrak{n}_+(M) = \mathfrak{f}(0, \Delta^+ \setminus [M]) = \operatorname{rad}_u \mathfrak{p}(M)$, $\mathfrak{n}_-(M) = \mathfrak{f}(0, \Delta^- \setminus [M])$. For $M = \emptyset$ these decompositions have been considered in Chap. 3, Sect. 1.5.

Now consider parabolic subgroups of a connected complex Lie group G. We shall prove, in particular, that any parabolic subgroup $P \subset G$ is connected and that G/P is a simply-connected projective algebraic variety. As noted above, it is sufficient to consider the case when G is semisimple.

Fix a maximal torus H in a connected semisimple complex Lie group G and a system of positive roots $\Delta^+ \subset \Delta_G(H)$. Then any subsystem $M \subset \Pi$ corresponds to a connected algebraic subgroup $P(M) \subset G$ containing B^+. Its tangent algebra is $\mathfrak{p}(M)$. By Proposition 1.4 $(N_G(P(M)))^0 = P(M)$. By Theorem 1.4 any connected parabolic subgroup of G is conjugate to precisely one of the subgroups $P(M)$.

As we have seen in Chap. 4, Sect. 1.2, by choosing a canonical system of generators corresponding to the system Π one is able to construct a compact real form in G, which we will denote by K.

Proposition 1.7. *We have*

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{4}$$

$$G = KP(M). \tag{5}$$

Here $\mathfrak{k} \cap \mathfrak{p}(M) = \mathfrak{z}(ix_M)$, *where* $x_M \in \mathfrak{h}(\mathbb{R})$ *is the element defined by formulae* (2). *The homogeneous space* $G/P(M)$ *is compact.*

Proof. For the Lie group G considered as a real semisimple Lie group, the Iwasawa decompositions (see Theorems 4.5 and 4.6 of Chap. 4) are of the form

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}(\mathbb{R}) \oplus \mathfrak{n}^+,$$

$$G = KAN^+,$$

where $\mathfrak{n}^+ = \mathfrak{n}(\emptyset) = \operatorname{rad}_u \mathfrak{b}^+$ and $A = \exp \mathfrak{h}(\mathbb{R})$. Decompositions (4) and (5) follow from the fact that $\mathfrak{p}(M) \supset \mathfrak{b}^+ \supset \mathfrak{h}(\mathbb{R}) \oplus \mathfrak{n}^+$ and $P(M) \supset B^+ \supset AN^+$. □

Theorem 1.5. *Any parabolic subgroup P of a connected complex Lie group G is connected and coincides with its normalizer, and G/P is a simply-connected projective algebraic variety.*

Proof. One can assume that G is semisimple and $P^0 = P(M)$. Proceeding as in the proof of Theorem 1.8 of Chap. 3 and making use of decomposition (3) and Proposition 1.7, we deduce that $G/P(M)$ is a simply-connected algebraic variety and $P(M)$ coincides with its normalizer, whence $P = P(M)$. □

Homogeneous spaces G/P, where P is a parabolic subgroup in G, are called *flag homogeneous spaces* or *flag manifolds*.

1.4. Examples of Parabolic Subgroups and Flag Manifolds. The term "flag manifold" is associated with the following basic example.

Example 1. Let V be an n-dimensional vector space over the field $K = \mathbb{C}$ or $\mathbb{R}$, and $p_1, \dots, p_r$ a set of integers such that $0 < p_1 < \dots < p_r < n$. A *flag of type* $(p_1, \dots, p_r)$ in V is a set of subspaces $V_1 \subset V_2 \subset \dots \subset V_r$ in V such that $\dim V_i = p_i$ $(i = 1, \dots, r)$. A flag of type $(1, \dots, n-1)$ is called a *full flag* (or simply *flag*). Denote by $\mathcal{F}_{p_1,\dots,p_r}(V)$ the set of all flags of type $(p_1, \dots, p_r)$. In the case $r = 1$ one obtains the Grassmann manifold $\mathcal{F}_p(V) = \mathrm{Gr}_p(V)$, and in the case $r = n-1$ the manifold of (full) flags $\mathcal{F}_{12\dots n-1}(V) = \mathcal{F}(V)$.

Consider the case $K = \mathbb{C}$ and $V = \mathbb{C}^n$, $n > 1$. Clearly, the group $\mathrm{GL}_n(\mathbb{C})$ and its subgroup $\mathrm{SL}_n(\mathbb{C})$ act transitively on $\mathcal{F}_{p_1,\dots,p_r}(\mathbb{C}^n)$, and the stabilizer $P \subset \mathrm{SL}_n(\mathbb{C})$ of the flag

$$\xi = (\langle e_1, \dots, e_{p_1}\rangle, \langle e_1, \dots, e_{p_2}\rangle, \dots, \langle e_1, \dots, e_{p_r}\rangle)$$

coincides with the subgroup of all matrices of the form

$$\begin{array}{l} p_1 \\ p_2 - p_1 \\ \vdots \\ n - p_r \end{array} \begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_{r+1} \end{pmatrix}. \tag{6}$$

In the notation of Table 1 we have $P = P(M)$, where $M = \Delta_{\mathfrak{sl}_n(\mathbb{C})} \setminus \{\alpha_{p_1}, \dots, \alpha_{p_r}\}$. For the manifold of full flags we have $M = \emptyset$, i.e. P is a Borel subalgebra. By identifying the set $\mathcal{F}_{p_1,\dots,p_r}(\mathbb{C}^n)$ with $\mathrm{SL}_n(\mathbb{C})/P$ in the usual way we turn it into a flag manifold. By Theorems 1.4 and 1.5, each parabolic subgroup of $\mathrm{SL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{C})$ is conjugate to precisely one subgroup of the form (6). Maximal parabolic subgroups are obtained in the case $r = 1$, i.e. they are stabilizers of points in Grassmann manifolds.

Example 2. Let V be a complex n-dimensional vector space equipped with a nondegenerate bilinear form b, which is either symmetric or skew-symmetric. A flag $V_1 \subset \dots \subset V_r$ in V is said to be *isotropic* if $b|V_r = 0$. It is known that in this case $p_r = \dim V_r \leq l$, where $l = \left[\frac{n}{2}\right]$ is the Witt index of the form b. Denote by $I\mathcal{F}_{p_1,\dots,p_r}(V)$ the set of all isotropic flags of type $(p_1, \dots, p_r)$, where $0 < p_1 < \dots < p_r \leq l$. It follows from Witt's theorem (see Dieudonné (1971)) that the corresponding orthogonal group $\mathrm{O}(V)$ (if b is symmetric) or symplectic group $\mathrm{Sp}(V)$ (if b is skew-symmetric) acts transitively on $I\mathcal{F}_{p_1,\dots,p_r}(V)$. If $e_1, \dots, e_n$ is the basis in V chosen as in the Example in Chap. 1, sect. 9.5, then $\langle e_1, \dots, e_l\rangle$ is a totally isotropic subspace in V, whence $\xi = (\langle e_1, \dots, e_{p_1}\rangle, \langle e_1, \dots, e_{p_2}\rangle, \dots, \langle e_1, \dots, e_{p_r}\rangle) \in I\mathcal{F}_{p_1,\dots,p_r}(V)$. It turns out that the stabilizer G_ξ is a parabolic subgroup in $G = \mathrm{SO}_n(\mathbb{C})$ or $\mathrm{Sp}_n(\mathbb{C})$ coinciding with $P(M)$, where $M = \Pi_G \setminus \{\alpha_{p_1}, \dots, \alpha_{p_r}\}$ (in the notation of Table 1). The submanifold $I\mathcal{F}_{p_1,\dots,p_r}(V) \subset \mathcal{F}_{p_1,\dots,p_r}(V)$ is the corresponding flag manifold for the group $\mathrm{Sp}_n(\mathbb{C})$, $n = 2l$, and also for the group $\mathrm{SO}_n(\mathbb{C})$, $n = 2l+1$, which acts transitively on $I\mathcal{F}_{p_1,\dots,p_r}(V)$. In the case $n = 2l$ the group $\mathrm{SO}_n(\mathbb{C})$ acts transitively on $I\mathcal{F}_{p_1,\dots,p_r}(V)$ if $p_r < l$ and has two open orbits on $I\mathcal{F}_{p_1,\dots,p_{r-1},l}$.

1.5. Parabolic Subalgebras of Real Semisimple Lie Algebras. A subalgebra $\mathfrak{f}$ of a real semisimple Lie algebra $\mathfrak{g}$ is said to be *parabolic* if $\mathfrak{f}(\mathbb{C})$ is a parabolic subalgebra of the complex Lie algebra $\mathfrak{g}(\mathbb{C})$. As we shall see in the present subsection, parabolic subalgebras of $\mathfrak{g}$ may be described in terms of the real root decomposition of this algebra. This description is very similar to that of parabolic subalgebras of complex semisimple Lie algebras (see Sect. 1.3).

Let $\mathfrak{a}$ be a maximal $\mathbb{R}$-diagonalizable subalgebra of $\mathfrak{g}$, Σ the corresponding root system, $\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \right)$ the root decomposition (see Chap. 4, Sect. 4.1), and Θ a system of simple roots in Σ. For an arbitrary subsystem $\Psi \subset \Theta$ we set

$$\mathfrak{q}(\Psi) = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in [\Psi]} \mathfrak{g}_\lambda \right) \oplus \left(\bigoplus_{\mu \in \Sigma^+ \setminus [\Psi]} \mathfrak{g}_\mu \right), \tag{7}$$

where $[\Psi] = \langle \Psi \rangle \cap \Sigma$. One can easily see that $\mathfrak{q}(\Psi)$ is a subalgebra of $\mathfrak{g}$. We will now describe its complexification.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}(\mathbb{C})$ containing $\mathfrak{a}$, and $\Delta = \Delta_0 \cup \Delta_1$ the partition of the root system $\Delta = \Delta_{\mathfrak{g}(\mathbb{C})}(\mathfrak{h})$ defined in Chap. 4, Sect. 4.2. It is shown in Sect. 4.3 of the same chapter that the system of simple roots $\Pi \subset \Delta$ can be chosen in such a way that the restriction mapping $r \colon \Delta \to \Sigma \cup \{0\}$ takes Π onto Θ. We set $M = \Pi \cap r^{-1}(\Theta \cup \{0\})$. Then M contains $\Pi_0 = \Pi \cap \Sigma_0$ and $M \cap \Pi_1 = M \cap \Sigma_1$ is invariant under the involution ω defined in Chap. 4, Sect. 4.3. On the Satake diagram of $\mathfrak{g}$ the system M is depicted by the subdiagram containing all black vertices and together with each white vertex the vertex joined to it by an arrow. Clearly, r defines a one-to-one correspondence between subsystems of this kind in Π and arbitrary subsystems in Θ. It can be verified that $\mathfrak{q}(\Psi)(\mathbb{C})$ coincides with the parabolic subalgebra $\mathfrak{p}(M)$ of $\mathfrak{g}(\mathbb{C})$. Therefore $\mathfrak{q}(\Psi)$ is a parabolic subalgebra of $\mathfrak{g}$. We will call such parabolic subalgebras *standard*.

Theorem 1.6 (see Borel and Tits [1965]). *Any parabolic subalgebra of a real semisimple Lie algebra $\mathfrak{g}$ is conjugate to a standard parabolic subalgebra $\mathfrak{q}(\Psi)$, where $\Psi \subset \Theta$, constructed from a fixed maximal diagonalizable subalgebra and a fixed system of simple roots. If the subalgebras $\mathfrak{q}(\Psi)$ and $\mathfrak{q}(\Psi')$, where $\Psi, \Psi' \subset \Theta$, are conjugate, then $\Psi = \Psi'$.*

Consider the structure of parabolic subalgebras. A parabolic subalgebra $\mathfrak{q}(\Psi) \subset \mathfrak{g}$ is algebraic. It follows from (7) that

$$\mathfrak{q}(\Psi) = \mathfrak{l}(\Psi) \oplus \mathfrak{n}^+(\Psi), \tag{8}$$

where

$$\mathfrak{l}(\Psi) = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in [\Psi]} \mathfrak{g}_\lambda \right),$$

$$\mathfrak{n}^+(\Psi) = \bigoplus_{\lambda \in \Sigma^+ \setminus [\Psi]} \mathfrak{g}_\lambda .$$

Then $\mathfrak{n}^+(\Psi)$ is a unipotent radical, and $\mathfrak{l}(\Psi)$ is a reductive Levi subalgebra, so (8) coincides with the Chevalley decomposition of the Lie algebra $\mathfrak{q}(\Psi)$. The system $\Pi_0 \cup M$ is the system of simple roots of the reductive Lie algebra $\mathfrak{l}(\Psi)(\mathbb{C})$, while the Satake diagram of the semisimple Lie algebra $\mathfrak{s}(\Psi) = [\mathfrak{l}(\Psi), \mathfrak{l}(\Psi)]$ coincides with the corresponding subdiagram of the Satake diagram of $\mathfrak{g}$. The centre $\mathfrak{c}(\Psi)$ of the Lie algebra $\mathfrak{l}(\Psi)$ coincides with the annihilator of the system of linear forms $\Pi_0 \cup \Gamma$ in $\mathfrak{h} \cap \mathfrak{g}$. Here $\mathfrak{c}(\Psi) = \mathfrak{c}^+(\Psi) \oplus \mathfrak{c}^-(\Psi)$, where $\mathfrak{c}^-(\Psi) = \mathfrak{c}(\Psi) \cap \mathfrak{a}$, $\mathfrak{c}^+(\Psi) = \mathfrak{c}(\Psi) \cap \mathfrak{h}^+$ (see the decomposition (4.16)). We have

$$\dim \mathfrak{c}(\Psi) = \operatorname{rk} \mathfrak{g} - \operatorname{rk} \mathfrak{s}(\Psi) = |\Pi_1 \setminus \Gamma|,$$
$$\dim \mathfrak{c}^-(\Psi) = |\Theta \setminus \Psi|,$$

$\dim \mathfrak{c}^+(\Psi)$=number of pairs of distinct roots in $\Pi_1 \setminus \overline{\Psi}$ taken into each other by ω.

Let $\mathfrak{s}^+(\Psi)$ be the largest compact ideal in the Lie algebra $\mathfrak{s}(\Psi)$, and $\mathfrak{s}^-(\Psi)$ the sum of all its noncompact simple ideals. Then

$$\mathfrak{l}^+(\Psi) = \mathfrak{s}^+(\Psi) \oplus \mathfrak{c}^+(\Psi)$$

is the largest pseudocompact ideal of the Lie algebra $\mathfrak{l}(\Psi)$.

Example 1. Note that if $\Psi \subset \Psi'$, then $\mathfrak{q}(\Psi) \subset \mathfrak{q}(\Psi')$. This and Theorem 1.6 imply that all minimal parabolic Lie subalgebras of $\mathfrak{g}$ are conjugate. They correspond to the subsystem $\Psi = \emptyset$ and

$$\mathfrak{q}(\emptyset) = \mathfrak{g}_0 \oplus \left(\bigoplus_{\mu \in \Sigma^+} \mathfrak{g}_\mu \right),$$

$$\mathfrak{l}(\emptyset) = \mathfrak{g}_0, \quad \mathfrak{l}^+(\emptyset) = \mathfrak{m}, \quad \mathfrak{l}^-(\emptyset) = \mathfrak{a},$$
$$\mathfrak{n}^+(\emptyset) = \mathfrak{n}^+ = \bigoplus_{\mu \in \Sigma^+} \mathfrak{g}_\mu .$$

If $\operatorname{rk}_{\mathbb{R}} \mathfrak{g} = 1$, then all the proper parabolic subalgebras are minimal. For example, if $\mathfrak{g} = \mathfrak{so}_{1,n}$, then the proper parabolic subalgebras of $\mathfrak{g}$ are the tangent algebras of the stabilizers of various points on the sphere S^{n-1} on which the group $G = \mathrm{SO}_{1,n}$ acts as the group of conformal transformations (see Gorbatsevich and Onishchik [1988], Chap. 5, Sect. 2.4, Example 4). If $\operatorname{rk}_{\mathbb{R}} \mathfrak{g} = 0$, i.e. $\mathfrak{g}$ is compact, then there are no proper parabolic subalgebras of $\mathfrak{g}$.

Example 2. Maximal proper parabolic subalgebras of the Lie algebra $\mathfrak{g}$ correspond to arbitrary subsystems $\Psi \subset \Theta$ such that $|\Theta \setminus \Psi| = 1$. Thus number of the classes of conjugate maximal parabolic subalgebras of $\mathfrak{g}$ is equal to $\operatorname{rk}_{\mathbb{R}} \mathfrak{g}$. One can easily see that any maximal parabolic subalgebra is also maximal among all subalgebras of $\mathfrak{g}$.

Example 3. If $\mathfrak{g}$ is split, then the complexification $\mathfrak{f} \mapsto \mathfrak{f}(\mathbb{C})$ defines a one-to-one correspondence between the classes of conjugate parabolic subalgebras of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}(\mathbb{C})$. For example, for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ the classes of conjugate parabolic subalgebras can be represented by the tangent algebras of the stabilizers of standard flags in $\mathbb{R}^n$ (see Sect. 1.4, Example 1).

We conclude this subsection with the following generalization of the notion of a parabolic subgroup. A *t-subalgebra* of a real semisimple Lie algebra $\mathfrak{g}$ is any subalgebra of it containing a subalgebra that is maximal triangular in $\mathfrak{g}$. Such subalgebras play an important role in the theory of compact homogeneous spaces (see Gorbatsevich and Onishchik [1988], Chap. 5, Sect. 1). It turns out that the classification of t-subalgebras reduces to that of parabolic subalgebras of $\mathfrak{g}$, as well as to the classification of subalgebras of compact Lie algebras.

Note that any parabolic subalgebra of a Lie algebra $\mathfrak{g}$ is a t-subalgebra. Indeed, the standard parabolic subalgebra $\mathfrak{q}(\Psi)$ contains the subalgebra $\mathfrak{d} = \mathfrak{a} \oplus \mathfrak{n}^+ = \mathfrak{l}^-(\emptyset) \oplus \mathfrak{n}^+(\emptyset) \subset \mathfrak{q}(\emptyset)$, which is maximal triangular in $\mathfrak{g}$ (see Chap. 4, Sect. 4.6). To construct other t-subalgebras we write $\mathfrak{q}(\Psi)$ in the form

$$\mathfrak{q}(\Psi) = (\mathfrak{l}^+(\Psi) \oplus \mathfrak{l}^-(\Psi)) \ni \mathfrak{n}^+(\Psi),$$

where $\mathfrak{l}^-(\Psi) = \mathfrak{s}^-(\Psi) \oplus \mathfrak{c}^-(\Psi)$. Then for any subalgebra $\mathfrak{w} \subset \mathfrak{l}^+(\Psi)$ the subalgebra

$$\mathfrak{f}(\Psi, \mathfrak{w}) = (\mathfrak{w} \oplus \mathfrak{l}^-(\Psi)) \ni \mathfrak{n}^+(\Psi)$$

is a t-subalgebra.

Theorem 1.7 (see Onishchik [1967]) *Any t-subalgebra of a real semisimple Lie algebra $\mathfrak{g}$ is conjugate to a subalgebra of the form $\mathfrak{f}(\Psi, \mathfrak{w})$. If $\mathfrak{w}$, $\mathfrak{w}'$ are two subalgebras of $\mathfrak{l}^+(\Psi)$ and $\mathfrak{f}(\Psi, \mathfrak{w})$, $\mathfrak{f}(\Psi, \mathfrak{w}')$ are conjugate in $\mathfrak{g}$, then $\mathfrak{w}$ and $\mathfrak{w}'$ are taken into each other by an automorphism of $\mathfrak{g}$.*

Corollary. *If a real semisimple Lie algebra $\mathfrak{g}$ is split, then t-subalgebras of $\mathfrak{g}$ coincide with its parabolic subalgebras.*

1.6. Nonsemisimple Maximal Subalgebras. We have noted in Sect. 1.3 and 1.5 that any maximal proper parabolic subalgebra of a complex or real semisimple Lie algebra is maximal subalgebra of it. Clearly, all these maximal subalgebras are nonsemisimple (and even nonreductive). It turns out that these subalgebras exhaust all nonsemisimple maximal subalgebras of complex semisimple Lie algebras and all nonreductive maximal subalgebras of real semisimple Lie algebras.

Theorem 1.8. *Any nonsemisimple maximal subalgebra of a complex semisimple Lie algebra is parabolic.*

Proof. Let $\mathfrak{f}$ be a maximal subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$. Clearly, $\mathfrak{f}$ is an algebraic subalgebra. Suppose that $\operatorname{rad}\mathfrak{f} \neq 0$. If $\operatorname{rad}\mathfrak{f}$ consists of semisimple elements of $\mathfrak{g}$, then $\mathfrak{f}$ coincides with the centralizer of $\operatorname{rad}\mathfrak{f}$.

Hence it follows that $\mathfrak{f}$ is contained in a parabolic subalgebra (see Sect. 1.3), a contradiction. Now suppose that $\mathfrak{m} = \mathrm{rad}_u \mathfrak{f} \neq 0$. Then $\mathfrak{f}$ coincides with $\mathfrak{u}_{\mathfrak{g}}(\mathfrak{m})$. For any $x \in \mathfrak{m}$ and $y \in \mathfrak{f}$ the operator $(\mathrm{ad}\, x)(\mathrm{ad}\, y)$ is nilpotent, whence $\mathfrak{m} \subset \mathfrak{f}^{\perp}$ (the orthogonal complement with respect to the Killing form in $\mathfrak{g}$). One can actually prove that $\mathfrak{m} = \mathfrak{f}^{\perp}$, whence $\mathfrak{f} = \mathfrak{m}^{\perp}$. Clearly, $\mathfrak{m} \subset \mathfrak{n}$, where $\mathfrak{n}$ is the unipotent radical of a Borel subalgebra $\mathfrak{b}$. Then $\mathfrak{f} \supset \mathfrak{n}^{\perp} = \mathfrak{b}$, i.e. $\mathfrak{f}$ is a parabolic subalgebra. $\square$

For simple Lie algebras, Theorem 1.8 was first proved by V.V. Morozov in his doctoral dissertation "On Nonsemisimple Maximal Subgroups of Simple Groups" (Kazan, 1943) by an explicit enumeration of all possible subgroups. A formulation in terms of simple roots, equivalent to that given above, is due to F.I. Karpelevich [1951], who actually introduced the class of parabolic subalgebras. (The term "parabolic subgroup" appeared later, see Tits [1962].)

The situation in the real case is similar, the only difference being that a maximal subalgebra can be the centralizer of a nonzero pseudotoric subalgebra. The following theorem holds.

Theorem 1.9 (Mostow [1961]) *A nonsemisimple maximal subalgebra of a real semisimple Lie algebra is either parabolic or coincides with the centralizer of a pseudotoric subalgebra.*

Nonsemisimple maximal subalgebras of simple Lie algebras that are not parabolic are classified in Tao Huei-min [1966].

§ 2. Three-dimensional Simple Subalgebras and Nilpotent Elements

In this section $\mathfrak{g}$ denotes a complex semisimple Lie algebra, and G the group of its inner automorphisms, whose tangent algebra is identified with $\mathfrak{g}$. Two elements or subsets of $\mathfrak{g}$ are said to be conjugate if they are taken into each other by an inner automorphism. For any subset $M \subset \mathfrak{g}$ we denote by $Z(M)$ its centralizer in G, i.e. the group of inner automorphisms preserving each element of M.

2.1. $\mathfrak{sl}_2$-triples. By Morozov's theorem (Chap. 3, Theorem 1.3), the classification of three-dimensional simple subalgebras of $\mathfrak{g}$ is equivalent to the classification of nilpotent elements.

Let us discuss this in more detail. A triple $\{e, h, f\}$ of elements of $\mathfrak{g}$ forming a canonical basis of a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ (which means that $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$) is called an $\mathfrak{sl}_2$-*triple*. Morozov's theorem asserts that any nonzero nilpotent element e of the algebra $\mathfrak{g}$ can be included in an $\mathfrak{sl}_2$-triple. This theorem can be refined as follows.

Proposition 2.1. (1) *The element h is defined by the element e uniquely up to a transformation from $Z(e)$.*

(2) *The element f is uniquely defined by e and h.*

(3) *The element e is defined by the element h uniquely up to a transformation from $Z(h)$.*

A proof (in a more general setting) can be found, for example, in Vinberg [1979].

Corollary. *Let $\{e,h,f\}$ and $\{e',h',f'\}$ be two $\mathfrak{sl}_2$-triples. The following conditions are equivalent:*

(1) *the elements e and e' are conjugate;*
(2) *the elements h and h' are conjugate;*
(3) *the triples $\{e,h,f\}$ and $\{e',h',f'\}$ are conjugate;*
(4) *the subalgebras $\langle e,h,f\rangle$ and $\langle e',h',f'\rangle$ are conjugate.*

Thus the classification (up to conjugacy) of three-dimensional simple subalgebras of the algebra $\mathfrak{g}$ is completely equivalent to that of nonzero nilpotent elements. Both classifications are equivalent to the classification of semisimple elements h that can be included in $\mathfrak{sl}_2$-triples. This provides a convenient language for the classification based on the very effective classification up to conjugacy of all semisimple elements of $\mathfrak{g}$ (see Chap. 1, Corollary 2 to Theorem 9.8 and Chap. 3, Theorem 1.11). The main difficulty is to determine which semisimple elements can be included in an $\mathfrak{sl}_2$-triple.

The semisimple element h contained in the $\mathfrak{sl}_2$-triple $\{e,h,f\}$ is called the *characteristic* of the nilpotent element e and of the three-dimensional simple subalgebra $\langle e,h,f\rangle$. A calculation in the group $\mathrm{SL}_2(\mathbb{C})$ demonstrates that $\exp 2\pi i h$ is the unit element in the group G. One can assume that h belongs to a fixed Cartan subalgebra $\mathfrak{h}$. Then the above condition means that h is contained in the dual root lattice. *A priori* one can also prove the following proposition.

Proposition 2.2 (E.B. Dynkin [1952b]). *Let $\mathfrak{h}$ be a fixed Cartan subalgebra, $\Pi=\{\alpha_1,\dots,\alpha_l\}$ a system of simple roots of the algebra $\mathfrak{g}$ with respect to $\mathfrak{h}$, and $C\subset\mathfrak{h}$ the corresponding Weyl chamber. Then the characteristic of any nilpotent element of $\mathfrak{g}$ is conjugate to a (unique) element $h\in C$ for which all the "labels" $\alpha_i(h)$ $(i=1,\dots,l)$ are 0, 1, or 2.*

This implies, in particular, that in any semisimple Lie algebra there are only finitely many conjugacy classes of nilpotent elements. A proof of this fact based on more general ideas can be found in Vinberg and Popov [1989], Sect. 1.6 or in Steinberg [1974].

The above conditions are by no means sufficient for a semisimple element $h\in C$ to be the characteristic of a nilpotent element. Moreover, there are no (or, at least, none are known) simple conditions specifying the characteristics of nilpotent elements. The enumeration of all characteristics is a nontrivial problem, the solution of which will be considered in the following sections.

According to Proposition 2.1 all characteristics of a given nilpotent element e constitute a single $Z(e)$-orbit. One can show (see, for example, the proof of Theorem 1 in Vinberg [1979] cited above) that this orbit is a plane $\mathfrak{P}$ whose direction subspace is the tangent algebra $\mathfrak{u}$ of the unipotent radical U of the group $Z(e)$. This implies the following proposition.

Proposition 2.3. *Let $\{e, h, f\}$ be an arbitrary $\mathfrak{sl}_2$-triple. Then $Z(e, h) = Z(e, h, f)$ is a maximal reductive subgroup of the group $Z(e)$.*

Proof. Consider the action of the group $Z(e)$ on the plane $\mathfrak{P} = h + \mathfrak{u}$. Since any reductive group of affine transformations has a fixed point, any maximal reductive subgroup of $Z(e)$ has a fixed point in $\mathfrak{P}$. Therefore the stabilizer of the point $h \in \mathfrak{P}$ coinciding with $Z(e, h) = Z(e, h, f)$ contains a maximal reductive subgroup of $Z(e)$. But the group $Z(e, h, f)$ is itself reductive (Proposition 3.6, Chap. 3); hence it is a maximal reductive subgroup of $Z(e)$. □

A proof of this statement (up to connected components) is given in Springer and Steinberg [1969].

A nilpotent element e is said to be *even* if $\exp \pi i h$ (not just $\exp 2\pi i h$) is the unit element of the group G, i.e. if the corresponding three-dimensional subgroup of G is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$ (and not to $\mathrm{SL}_2(\mathbb{C})$). If $h \in C$, this means that all labels of the element h are either 0 or 2.

The following proposition provides useful information about the group $Z(e)$.

Proposition 2.4. $\dim Z(e) \le \dim Z(h)$, *where equality holds if and only if the element e is even.*

Proof. The tangent algebra of $Z(e)$ (respectively, $Z(h)$) is the centralizer $\mathfrak{z}(e)$ (respectively, $\mathfrak{z}(h)$) of the element e (respectively, h) in the algebra $\mathfrak{g}$. Consider the adjoint representation of the subalgebra $\langle e, h, f\rangle$ in $\mathfrak{g}$ and decompose it into the sum of irreducible representations. Then $\mathfrak{z}(e)$ is the linear span of the highest vectors of these representations, and $\mathfrak{z}(h)$ is the linear span of the zero weight vectors. Since the multiplicity of zero weight is equal to 1 (respectively, 0) for an odd-dimensional (respectively, even-dimensional) irreducible representation of the algebra $\mathfrak{sl}_2(\mathbb{C})$, then $\dim \mathfrak{z}(e) \le \dim \mathfrak{z}(h)$, where equality holds if and only if all irreducible representations of the decomposition are odd-dimensional. One can easily see that the latter property is equivalent to the fact that the element e is even. □

2.2. Three-dimensional Simple Subalgebras of Classical Simple Lie Algebras. Selecting (up to conjugacy) a three-dimensional simple subalgebra of the algebra $\mathfrak{sl}_n(\mathbb{C})$ is equivalent to selecting (up to isomorphism) a faithful linear representation of $\mathfrak{sl}_2(\mathbb{C})$. Any linear representation of $\mathfrak{sl}_2(\mathbb{C})$ is the sum of irreducible representations defined up to conjugacy by their dimensions (see the example in Chap. 3, Sect. 2.6). This establishes a bijection between the classes of conjugate simple three-dimensional subalgebras of $\mathfrak{sl}_n(\mathbb{C})$ and the partitions of n of the form

$$n = n_1 + \ldots + n_s, \; n_1, \ldots, n_s \in \mathbb{Z}, \; n_1 \geq \ldots \geq n_s > 0, \; n_1 > 1 \qquad (7)$$

(the partition into the sum of n units should be omitted, since it corresponds to the trivial representation). The subalgebra corresponding to the partition (7) will be denoted by $\mathfrak{a}(n_1, \ldots, n_s)$.

A nilpotent element of $\mathfrak{a}(n_1, \ldots, n_s)$ has Jordan blocks of orders $n_1, \ldots, n_s$. The uniqueness of the Jordan form now implies (independently of the general theory) a one-to-one correspondence between the conjugacy classes of simple three-dimensional subalgebras and the conjugacy classes of nonzero nilpotent elements.

Similarly, defining up to conjugacy a three-dimensional simple subalgebra of $\mathfrak{so}_n(\mathbb{C})$ (respectively, $\mathfrak{sp}_n(\mathbb{C})$) is equivalent to defining up to isomorphism by means of a transformation from the group $\mathrm{SO}_n(\mathbb{C})$ (respectively, $\mathrm{Sp}_n(\mathbb{C})$) a faithful n-dimensional orthogonal (respectively, symplectic) linear representation of $\mathfrak{sl}_2(\mathbb{C})$.

An irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ is orthogonal (respectively, symplectic) if and only if its dimension is odd (respectively, even). Here the invariant orthogonal (respectively, symplectic) metric is unique up to a scalar multiple (see Chap. 3, Sect. 2.7).

Hence it follows that an arbitrary representation of $\mathfrak{sl}_2(\mathbb{C})$ is orthogonal (respectively, symplectic) if and only if the multiplicities of its even-dimensional (respectively, odd-dimensional) irreducible components are even. Here the invariant orthogonal (respectively, symplectic) metric is unique up to a linear transformation commuting with all the operators of the representation. The latter property implies that if two orthogonal (respectively, symplectic) representations of $\mathfrak{sl}_2(\mathbb{C})$ in a space with a fixed orthogonal (respectively, symplectic) metric are isomorphic, then they are isomorphic by means of a transformation from the group $\mathrm{O}_n(\mathbb{C})$ (respectively, $\mathrm{Sp}_n(\mathbb{C})$).

Two orthogonal represention of $\mathfrak{sl}_2(\mathbb{C})$ isomorphic by means of a transformation from $\mathrm{O}_n(\mathbb{C})$ are isomorphic by means of a transformation from $\mathrm{SO}_n(\mathbb{C})$ if and only if there exists an orthogonal transformation with determinant -1 commuting with all the operators of one of these representations. It is not difficult to see that such a transformation always exists with the exception of those cases where all the irreducible components of the given representation are even-dimensional (see Proposition 3.3 below; this is possible only in the case when n is divisible by 4). In these exceptional cases the class of $\mathrm{O}_n(\mathbb{C})$-isomorphic representations splits into two classes of $\mathrm{SO}_n(\mathbb{C})$-isomorphic representations.

What we have said above results in the following description of three-dimensional simple subalgebras (and therefore nilpotent elements) of the algebras $\mathfrak{so}_n(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{C})$.

Any three-dimensional simple subalgebra of $\mathfrak{so}_n(\mathbb{C})$ (respectively, $\mathfrak{sp}_n(\mathbb{C})$) is conjugate in $\mathfrak{sl}_n(\mathbb{C})$ to the subalgebra $\mathfrak{a}(n_1, \ldots, n_s)$ defined by a partition

(7) in which each even (respectively, odd) term occurs an even number of times. The subalgebras corresponding to the same partition constitute one conjugacy class in $\mathfrak{so}_n(\mathbb{C})$ (respectively, $\mathfrak{sp}_n(\mathbb{C})$), with the exception of the subalgebras of $\mathfrak{so}_n(\mathbb{C})$ corresponding to the partition in which all the terms are even. The latter constitute two conjugacy classes in $\mathfrak{so}_n(\mathbb{C})$.

A description of the centralizers of three-dimensional simple subalgebras and nilpotent elements of classical simple Lie algebras can be found in Springer [1969].

2.3. Principal and Semiprincipal Three-dimensional Simple Subalgebras. In any semisimple Lie algebra, one class of conjugate nilpotent elements is naturally singled out. Its definition is based on the following theorem.

Theorem 2.5. *The set $\mathfrak{N}$ of all nilpotent elements of a complex semisimple Lie algebra $\mathfrak{g}$ is an irreducible algebraic variety of codimension equal to the rank of $\mathfrak{g}$.*

For the proof see, e.g. Steinberg [1974]. The most profound statement here is that of irreducibility. For a description of the situation from the general viewpoint of invariant theory see Vinberg and Popov [1989].

Corollary. *In any complex semisimple Lie algebra there is exactly one class of conjugate regular nilpotent elements.*

(For the definition of a regular element see Chap. 1, Sect. 9.3.)

Proof. The classes of conjugate nilpotent elements of the algebra $\mathfrak{g}$ are the orbits of the action of the group G on the variety $\mathfrak{N}$, and are consequently algebraic subvarieties (see, e.g. Vinberg and Popov [1989]). Since there are just finitely many such classes, exactly one of them is dense and therefore open in $\mathfrak{N}$ in the Zariski topology. Let e be a representative of this class. The comparison of dimensions yields

$$\dim \mathfrak{N} = \dim G - \dim Z(e),$$

whence $\dim Z(e) = \operatorname{rk} \mathfrak{g}$, i.e. e is a regular element. All the other classes are of lower dimension and therefore consist of singular elements. □

A regular nilpotent element can be described explicitly in terms of the root decomposition. Namely, such is the sum of (arbitrarily normalized) root vectors corresponding to the simple roots $\alpha_1, \dots, \alpha_l$:

$$e = e_{\alpha_1} + \dots + e_{\alpha_l}, \tag{8}$$

For the characteristic of this element one can take the element $h \in C$ whose labels are all equal to 2; the third element of the $\mathfrak{sl}_2$-triple is then of the form

$$f = c_1 e_{-\alpha_1} + \dots + c_l e_{-\alpha_l}, \tag{9}$$

where $c_1, \dots, c_l$ are certain appropriately chosen coefficients (Dynkin [1952b]).

Proof. The relations $[h, e] = 2e$, $[h, f] = -2f$ are obvious for the above choice of h. After an appropriate normalization of the root vectors we have

$$[e, f] = c_1 h_{\alpha_1} + \ldots + c_l h_{\alpha_l}$$

and the coefficients $c_1, \ldots, c_l$ can evidently be chosen in such a way that $[e, f] = h$. Then $\{e, h, f\}$ is an $\mathfrak{sl}_2$-triple. The form of the element h implies that e is even and therefore $\dim Z(e) = \dim Z(h)$ (Proposition 2.4). Since all the labels of h are positive, this element is regular. Therefore e is also regular. □

Theorem 2.6. *The centralizer $Z(e)$ of a regular nilpotent element e of the algebra $\mathfrak{g}$ is a connected commutative unipotent algebraic subgroup of the group G (of dimension equal to* $\operatorname{rk}\mathfrak{g}$*). The centralizer $Z(\mathfrak{a})$ of the three-dimensional simple subalgebra $\mathfrak{a}$ containing a regular nilpotent element e is trivial.*

Proof. We assume that the elements e, h, f are chosen as above. Since h is a regular semisimple element, its centralizer coincides with the Cartan subgroup H. Therefore

$$Z(\mathfrak{a}) = Z(h, e) = \{t \in H : t(e) = e\} = \{t \in H : \alpha_i(t) = 1 \quad \text{for} \quad i = 1, \ldots, l\}.$$

Since the simple roots $\alpha_1, \ldots, \alpha_l$ constitute the basis of the group of characters of the algebraic torus H, the subgroup $Z(\mathfrak{a})$ is trivial. In view of Proposition 2.3, this means that the subgroup $Z(e)$ is unipotent and therefore connected. Its commutativity is implied by the following lemma.

Lemma 2.7. *The centralizer $\mathfrak{z}(\xi)$ of any element ξ in the algebra $\mathfrak{g}$ contains a commutative subalgebra of dimension equal to* $\operatorname{rk}\mathfrak{g}$.

Proof. The element ξ can be approximated by regular semisimple elements. Thus, $\mathfrak{z}(\xi)$ contains a commutative subalgebra, which is a limit of Cartan subalgebras. □

A three-dimensional simple subalgebra of $\mathfrak{g}$ containing a regular nilpotent element is called *principal*. In the classical simple Lie algebras $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{so}_n(\mathbb{C})$, $\mathfrak{sp}_n(\mathbb{C})$, except for the algebra $\mathfrak{so}_n(\mathbb{C})$ with even n, the principal subalgebras (in the notation of Sect. 2.2) are the three-dimensional simple subalgebras $\mathfrak{a}(n)$ defined by an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. In the algebra $\mathfrak{so}_n(\mathbb{C})$ with even n the principal subalgebra is the three-dimensional simple subalgebra $\mathfrak{a}(n-1, 1)$.

According to Theorem 2.6, the centralizer of a principal three-dimensional simple subalgebra in the group G is trivial. In some semisimple Lie algebras there also exist other three-dimensional simple subalgebras possessing the same property. They are said to be *semiprincipal*, and the nilpotent elements contained in them are called *semiregular*. The latter are characterized by the fact that their centralizers are unipotent subgroups of G.

All semiprincipal three-dimensional simple subalgebras of semisimple Lie algebras were enumerated by Dynkin [1952b]. We will now state this result.

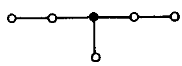
First, we note that if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, then a three-dimensional simple subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is semiprincipal if and only if its projections on $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are semiprincipal three-dimensional simple subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. It is therefore sufficient to list the semiprincipal three-dimensional simple subalgebras of simple Lie algebras.

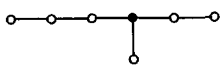
Now it is clear that if $\langle e, h, f\rangle$ is a semiprincipal three-dimensional simple subalgebra, then $\exp \pi i h$ is the unit element in the group G, i.e. e is an even nilpotent element. Therefore if $h \in C$, then all the labels of h are either 0 or 2. Let us agree to define such an element h by the Dynkin diagram of the algebra $\mathfrak{g}$ whose i-th vertex ($i = 1, \ldots, l$) is depicted by a black circle if $\alpha_i(h) = 0$, and a white one if $\alpha_i(h) = 2$ (in particular, the characteristic of a principal three-dimensional simple subalgebra is defined by the Dynkin diagram with only white circles).

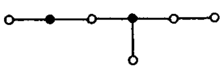
Among the classical simple Lie algebras, only the algebra $\mathfrak{so}_n(\mathbb{C})$ with even n, i.e. the algebra of type D_l (where $2l = n$) has semiprincipal but not principal three-dimensional simple subalgebras. In the notation of Sect. 2.2 they are the subalgebras $\mathfrak{a}(n-2k-1, 2k+1)$ ($k = 1, \ldots, \left[\frac{l-2}{2}\right]$). Dynkin denoted the subalgebra $\mathfrak{a}(n-2k-1, 2k+1)$ of $\mathfrak{so}_n(\mathbb{C})$ ($n = 2l$) by (a_k), and the pair $(\mathfrak{so}_n(\mathbb{C}), \mathfrak{a}(n-2k-1, 2k+1))$ by $D_l(a_k)$. The characteristic of this subalgebra is given by the diagram with k black circles:

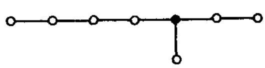
$D_l(a_k)$ 

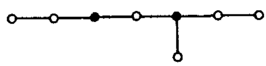
The following list gives the characteristics of all semiprincipal but not principal three-dimensional simple subalgebras $\mathfrak{a}$ of the exceptional simple Lie algebras $\mathfrak{g}$, where the pairs $(\mathfrak{g}, \mathfrak{a})$ are given in the Dynkin notation.

$E_6(a_1)$ 

$E_7(a_1)$ 

$E_7(a_2)$ 

$E_8(a_1)$ 

$E_8(a_2)$ 

2.4. Minimal Ambient Regular Subalgebras. Semiprincipal three-dimensional simple subalgebras of the algebra $\mathfrak{g}$ can be characterized by the fact that they are not contained in any proper regular subalgebra. Incidentally, this was how E.B. Dynkin originally defined them. (For the definition of regular subalgebras see Sect. 1.1).

Proof. Indeed, if a three-dimensional simple subalgebra $\mathfrak{a}$ of the algebra $\mathfrak{g}$ is not semiprincipal, then the centralizer $Z(\mathfrak{a})$, which is a reductive subgroup of the group G, contains a semisimple element s different from unity. In this case $\mathfrak{a}$ is contained in the subalgebra $\mathfrak{g}^s = \{\xi \in \mathfrak{g}: s(\xi) = \xi\}$ which is a proper (reductive) regular subalgebra of $\mathfrak{g}$.

Conversely, suppose that $\mathfrak{a}$ is contained in a proper regular subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$. Then it is contained in a maximal semisimple subalgebra $\mathfrak{g}_2$ of $\mathfrak{g}_1$, which is also a regular subalgebra of $\mathfrak{g}$. There exists a semisimple element $s \in G$ different from unity such that $\mathfrak{g}_2 \subset \mathfrak{g}^s$ (see Sect. 1.2). Clearly, $s \in Z(\mathfrak{a})$ and the subalgebra $\mathfrak{a}$ is not therefore semiprincipal. □

Hence it follows that any three-dimensional simple subalgebra of $\mathfrak{g}$ is semiprincipal in any minimal regular subalgebra to which it belongs (which is automatically semisimple). This results in the following three-step procedure for classifying all three-dimensional simple subalgebras of $\mathfrak{g}$.

Step 1. Enumerate (up to conjugacy) all semisimple regular subalgebras of $\mathfrak{g}$.

Step 2. For each of these subalgebras consider all its semiprincipal three-dimensional simple subalgebras and find their characteristics in $\mathfrak{g}$.

Step 3. By comparing the characteristics, determine which of these subalgebras are conjugate in $\mathfrak{g}$.

It was along these lines that Dynkin [1952b] obtained the classification of all three-dimensional simple subalgebras (and therefore nilpotent elements) of exceptional simple Lie algebras (for the explicit form of these subalgebras see Grélaud, Quitté, and Tauvel [1990]). The centralizers of these subalgebras, as well as of the nilpotent elements they contain, were described by Elashvili [1975] (locally), and A.V. Alekseevskij [1979] (globally).

The first step presents no difficulty, because semisimple regular subalgebras are easily described in terms of Dynkin diagrams and extended Dynkin diagrams (see Sect. 1.2). The second step involves taking the characteristics from the Weyl chamber of a semisimple regular subalgebra into the Weyl chamber of the algebra $\mathfrak{g}$ itself, which requires rather long calculations with small integers. The necessity of the third step is explained by the fact that for one and the same three-dimensional simple subalgebra of $\mathfrak{g}$ there may exist several nonconjugate minimal regular subalgebras containing it. The third step is rather difficult to carry out because the set of characteristics one has to deal with is rather large.

2.5. Minimal Ambient Complete Regular Subalgebras. An alternative variant of the procedure described in the preceding section has no Step 3.

A semisimple regular subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ will be called *complete* if its root system is the intersection of the root system of $\mathfrak{g}$ with some subspace. A system of simple roots of $\mathfrak{g}_1$ is a part of a system of simple roots of $\mathfrak{g}$ (under an appropriate choice of the latter system). By an abuse of language, complete semisimple regular subalgebras of $\mathfrak{g}$ are sometimes called Levi subalgebras since they are, in fact, Levi subalgebras (i.e. maximal semisimple subalgebras) of parabolic subalgebras of $\mathfrak{g}$.

It is easy to see that complete semisimple regular subalgebras are none other than the commutator subalgebras of (reductive) subalgebras of fixed points of algebraic tori contained in the group G. Therefore for any three-dimensional simple subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ there is a unique (up to conjugacy by means of an element from $Z(\mathfrak{a})$) complete semisimple regular subalgebra containing it. Such is the subalgebra of fixed points of (any) Cartan subgroup of $Z(\mathfrak{a})$.

For any three-dimensional simple subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ the following properties are evidently equivalent:

(1) $\mathfrak{a}$ is not contained in any proper complete semisimple regular subalgebra;
(2) $Z(\mathfrak{a})$ is finite.

Three-dimensional simple subalgebras possessing these equivalent properties will be called *pseudoprincipal*, and the nilpotent elements contained in them *pseudoregular*. The following nontrivial fact holds (see Vinberg [1979], Theorem 3, and also Kac [1980], Lemma 1.7 and Remark (c)).

Proposition 2.8. *Any pseudoregular nilpotent element is even.*

If in the classification of three-dimensional simple subalgebras (and nilpotent elements) described in Sect. 2.4 one considers only complete (instead of any semisimple regular) subalgebras, then, for the reason stated above, Step 3 of the classification is no longer necessary. The price of this is that one has to know *a priori* the classification of all pseudoprincipal (and not only semiprincipal) three-dimensional simple subalgebras.

We will now give this classification. For each such subalgebra $\mathfrak{a}$ we will give its centralizer $Z(\mathfrak{a})$ and — in order to correlate it with Dynkin's original approach – all (up to conjugacy by an element from $Z(\mathfrak{a})$) minimal ambient regular subalgebras together with the specific embedding of the subalgebra $\mathfrak{a}$ (as a semiprincipal three-dimensional simple subalgebra) in each of them.

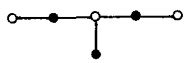
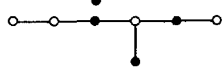
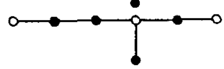
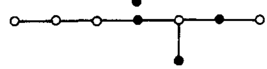
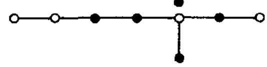
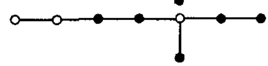
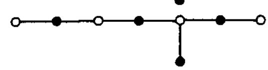
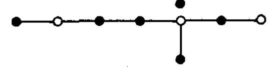
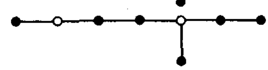
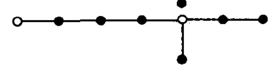
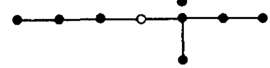
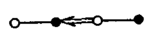
For the same reasons as for semiprincipal subalgebras, it is sufficient to enumerate pseudoprincipal three-dimensional simple subalgebras in simple Lie algebras.

In the algebra $\mathfrak{sl}_n(\mathbb{C})$ all pseudoprincipal three-dimensional simple subalgebras are principal.

In the algebra $\mathfrak{so}_n(\mathbb{C})$ the pseudoprincipal three-dimensional simple subalgebras are subalgebras of the form $\mathfrak{a}(n_1, \dots, n_s)$, where $n_1, \dots, n_s$ are distinct odd numbers (we use the notation of Sect. 2.2). The centralizer of such a subalgebra in the group $\mathrm{PSO}_n(\mathbb{C})$ for odd (respectively, even) n is isomorphic to $\mathbb{Z}_2^{s-1}$ (respectively, $\mathbb{Z}_2^{s-2}$). The minimal ambient regular subalgebras for odd (respectively, even) n are subalgebras of the form $\mathfrak{so}_{n_{i_1}+n_{i_2}}(\mathbb{C}) \oplus \dots \oplus \mathfrak{so}_{n_{i_{s-2}}+n_{i_{s-1}}}(\mathbb{C}) \oplus \mathfrak{so}_{n_{i_s}}(\mathbb{C})$ (respectively, $\mathfrak{so}_{n_{i_1}+n_{i_2}}(\mathbb{C}) \oplus \dots \oplus \mathfrak{so}_{n_{i_{s-1}}+n_{i_s}}(\mathbb{C})$), where $(i_1, \dots, i_s)$ is a permutation of the set of numbers $1, \dots, s$. Here the projection of $\mathfrak{a}(n_1, \dots, n_s)$ onto each term of the form $\mathfrak{so}_{n_i+n_j}(\mathbb{C})$ is (if one assumes, to be definite, that $i < j$) the semiprincipal three-dimensional subalgebra $\mathfrak{a}(n_i, n_j)$.

In the algebra $\mathfrak{sp}_n(\mathbb{C})$ the pseudoprincipal three-dimensional simple subalgebras are subalgebras of the form $\mathfrak{a}(n_1, \ldots, n_s)$, where $n_1, \ldots, n_s$ are distinct even numbers. The centralizer of such a subalgebra in the group $\mathrm{PSp}_n(\mathbb{C})$ is isomorphic to $\mathbb{Z}_2^{s-1}$. The minimal ambient regular subalgebra is unique: it is the subalgebra $\mathfrak{sp}_{n_1}(\mathbb{C}) \oplus \ldots \oplus \mathfrak{sp}_{n_s}(\mathbb{C})$.

The pseudoprincipal three-dimensional simple subalgebras $\mathfrak{a}$ of exceptional simple Lie algebras $\mathfrak{g}$ were actually enumerated in the work of Dynkin cited above (Dynkin [1952b], Table 23). The following table lists all such subalge-

Type of $(\mathfrak{g}, \mathfrak{a})$	Characteristic	$Z(\mathfrak{a})$	Minimal ambient regular subalgebras
$E_6(b)$		S_2	$A_5 + A_1$
$E_7(b)$		S_2	$D_6 + A_1$
$E_7(c_1)$		S_2	$D_6(a_1) + A_1$
$E_7(c_2)$		S_3	$D_6(a_2) + A_1$, $A_5 + A_2$
$E_8(b)$		S_2	$E_7 + A_1$
$E_8(c_1)$		S_2	$E_7(a_1) + A_1$
$E_8(c_2)$		S_3	$E_7(a_2) + A_1$, $E_6 + A_2$
$E_8(d_1)$		S_2	D_8
$E_8(d_2)$		S_2	$D_8(a_1)$
$E_8(d_3)$		S_3	$D_8(a_2)$, A_8
$E_8(d_4)$		S_3	$D_8(a_3)$, $E_6(a_1) + A_2$
$E_8(d_5)$		S_5	$D_4 + D_4(a_1)$, $D_5(a_1) + A_3$, $D_6(a_2) + 2A_1$, $A_5 + A_2 + A_1$, $2A_4$
$F_4(e_1)$		S_2	B_4
$F_4(e_2)$		S_2	$C_3 + A_1$
$F_4(e_3)$		S_4	$D_4(a_1)$, $A_3 + A_1$, $C_2 + 2A_1$, $2A_2$
$G_2(f)$		S_3	A_2, $2A_1$

bras except those that are semiprincipal and have already been listed above. The notation for the pairs $(\mathfrak{g}, \mathfrak{a})$ is similar to that used by Dynkin in the case of semiprincipal subalgebras. The last column lists up to conjugacy by means of an element from $Z(\mathfrak{a})$ all minimal ambient regular subalgebras $\mathfrak{g}_1$. If the notation for a simple component of $\mathfrak{g}_1$ is accompanied by no additional symbol, then the projection of $\mathfrak{a}$ onto this component is a principal three-dimensional simple subalgebra, otherwise (when it is accompanied by the symbol a_k in brackets) this projection is a semiprincipal three-dimensional simple subalgebra of the corresponding type.

The above approach is applicable (in any of its versions) to the solution of the more general problem of classifying homogeneous simple three-dimensional subalgebras (and, accordingly, homogeneous nilpotent elements) of graded semisimple Lie algebras (Vinberg [1975], Vinberg [1979]).

For another approach to these problems see Kawanaka [1987].

§ 3. Semisimple Subalgebras and Subgroups

In this section we present the classification results for connected semisimple Lie subgroups of simple complex and real Lie groups. The problem of classifying semisimple subgroups of complex classical groups was systematically considered for the first time by Malcev [1944]. The method of Malcev, based on the theory of linear representations, was subsequently improved by Dynkin [1952a, 1952b], while Karpelevich [1955] used it in order to classify semisimple subgroups of real classical groups.

3.1. Semisimple Subgroups of Complex Classical Groups. Consider the basic idea of Malcev's method quoted above. Let G be a connected complex classical group embedded in $\mathrm{GL}(V) \simeq \mathrm{GL}_n(\mathbb{C})$ in the standard way, i.e. one of the groups $\mathrm{SL}(V) \simeq \mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}(V) \simeq \mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}(V) \simeq \mathrm{Sp}_n(\mathbb{C})$, and let F be a simply-connected semisimple Lie group. Then the connected Lie subgroups of G locally isomorphic to F are the images of all possible homomorphisms $F \to G$ with discrete kernel that can be viewed as locally faithful complex linear representations of F of dimension n. Linear representations of F can be defined by the highest weights of their irreducible components (see Chap. 3, Sect. 2.2), where the dimension of the representation is computed by formula (3.7). Two questions now arise. (1) When is the image of a linear representation $R: F \to \mathrm{GL}(V)$ contained in G? (2) When are the images of linear representations $R, R': F \to \mathrm{GL}(V)$ conjugate in G or taken into each other by an automorphism of G?

Since F is semisimple, we have $(F, F) = F$, so $R(F) \subset \mathrm{SL}(V)$. A necessary and sufficient condition for the relation $R(F) \subset \mathrm{SO}(V)$ or $\mathrm{Sp}(V)$ to be satisfied for an appropriate nondegenerate bilinear form in V is that the representation R must be, respectively, orthogonal or symplectic (see Chap. 3, Sect. 2.7 for the criteria). This answers question (1).

In order to answer question (2) the following simple statement is useful.

Proposition 3.1. *Let $R, R': F \to G$ be homomorphisms with discrete kernels and $\alpha \in \operatorname{Aut} G$. Then $R'(F) = \alpha(R(F))$ if and only if $R' = \alpha R \beta$, where β is an automorphism of F.*

We will say that the homomorphism $R: F \to G$ is taken into the homomorphism $R': F \to G$ by an automorphism $\alpha \in \operatorname{Aut} G$ if $R' = \alpha R$. If, in addition, the automorphism α is inner, then the homomorphisms R and R' are said to be *conjugate*.

Corollary. *Suppose that all automorphisms of F are inner ones. The subgroups $R(F)$ and $R'(F)$ are conjugate in G or are taken into each other by an automorphism of G if and only if so are the homomorphisms R and R'.*

If F admits outer automorphisms, then in order to classify the subgroups locally isomorphic to F one first has to classify homomorphisms $F \to G$ up to conjugacy (or automorphisms of G) and then join the classes obtained from each other by multiplication on the right into an outer automorphism of F.

Let us now investigate when two automorphisms $F \to G$ can be taken into each other by an automorphism of G.

It is known (see Chap. 3, Sect. 3.1) that all automorphisms of the groups $\mathrm{SO}_{2l+1}(\mathbb{C})$, $\mathrm{Sp}_n(\mathbb{C})$ are inner ones, while any automorphism of the group $\mathrm{SO}_{2l}(\mathbb{C})$ is induced by an inner automorphism of the group $\mathrm{O}_{2l}(\mathbb{C})$. Now for $n > 2$ the group $\operatorname{Aut} \mathrm{SL}_n(\mathbb{C}) / \operatorname{Int} \mathrm{SL}_n(\mathbb{C})$ is the group of order 2 generated by the automorphism $X \mapsto (X^\top)^{-1}$. Hence it follows that if the homomorphisms $R, R': F \to G$ are conjugate in G, then $R \sim R'$ (i.e. R and R' are equivalent as linear representations). If R and R' are taken into each other by an automorphism of G, then in the case $G = \mathrm{SL}_n(\mathbb{C})$ we have either $R \sim R'$ or $R \sim R'^*$, and in the cases $G = \mathrm{SO}_n(\mathbb{C})$ and $\mathrm{Sp}_n(\mathbb{C})$ we have $R \sim R'$. The converse statement is also valid.

Proposition 3.2 (Malcev [1944]). *Let G be a classical group and $R, R': F \to G$ two homomorphisms with discrete kernels. If $R \sim R'$ and $G = \mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_{2l+1}(\mathbb{C})$, $\mathrm{Sp}_{2l}(\mathbb{C})$, then the homomorphisms R and R' are conjugate. If $R \sim R'$ and $G = \mathrm{SO}_{2l}(\mathbb{C})$, then R is taken into R' by an inner automorphism of the group $\mathrm{O}_{2l}(\mathbb{C})$. If $R \sim R'^*$ and $G = \mathrm{SL}_n(\mathbb{C})$, then R is taken into R' by an automorphism of $\mathrm{SL}_n(\mathbb{C})$.*

Proof. Let L be a compact real form of the group F. By Chap. 4, Corollary to Proposition 2.1, we can assume that $R(L)$ and $R(L')$ lie in $G \cap U_n$, i.e. in the groups SU_n, SO_n, Sp_l if $G = \mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_{2l}(\mathbb{C})$, respectively. If $R \sim R'$, then there exists $C \in \mathrm{GL}_n(\mathbb{C})$ such that $CR(g) = R'(g)C$ for all $g \in F$. By decomposing C into the product of a unitary and positive definite Hermitian matrix one can easily see that the matrix C may be assumed to be unitary. Multiplying C by a scalar, we get $\det C = 1$, which solves the problem in the case $C = \mathrm{SL}_n(\mathbb{C})$.

Now let $G = \mathrm{SO}_n(\mathbb{C})$ or $\mathrm{Sp}_{2l}(\mathbb{C})$. Then $R(g)$ and $R'(g)$ $(g \in L)$ commute with the semilinear operator J in the space $\mathbb{C}^n$ coinciding with the complex conjugation in the first case and with the quaternion structure in the second. The linear operator $B = JC^{-1}J^{-1}C \in U_n$ has the following properties: $BR(g) = R(g)B$ $(g \in L)$, $JBJ^{-1} = B^{-1}$. Let $A \in U_n$ be the operator having the same eigenspaces as B and satisfying the condition $A^2 = B$. Then A has the same properties, which implies that $C_1 = CA^{-1} \in \mathrm{O}_n$ or Sp_l, respectively, where $C_1R(g) = R'(g)C_1$ for all $g \in L$ and therefore for all $g \in F$. □

Proposition 3.3 (Malcev [1944]). *Let $R: F \to \mathrm{SO}_{2l}(\mathbb{C})$ be a linear representation and α an outer automorphism of the group $\mathrm{SO}_{2l}(\mathbb{C})$. The homomorphisms R and αR are conjugate if and only if R contains an irreducible component of odd dimension.*

These considerations, in principle, solve the classification problem for connected semisimple subgroups of complex classical groups.

3.2. Maximal Connected Subgroups of Complex Classical Groups. In this section we present the results of Dynkin [1952a] listing all maximal connected Lie subgroups (i.e. connected Lie subgroups that are maximal among connected subgroups) of the complex classical groups. This is equivalent to the classification of maximal semisimple subalgebras of complex classical Lie algebras. The following theorems classify the maximal connected subgroups into three categories — reducible (as linear groups), irreducible nonsimple, and irreducible simple. We remind the reader that any irreducible connected Lie subgroup in $\mathrm{SL}(V)$ is semisimple (Chap. 1, Corollary 3 to Theorem 1.1).

Theorem 3.1. *Let F be a maximal connected Lie subgroup of a complex classical group G. If F is reducible, then the following three cases are possible:*

(a) *F is a maximal parabolic subgroup of G;*

(b) *$G = \mathrm{SO}_n(\mathbb{C})$, F is conjugate to the subgroup $\mathrm{SO}_k(\mathbb{C}) \times \mathrm{SO}_{n-k}(\mathbb{C})$ $(0 < k < n)$;*

(c) *$G = \mathrm{Sp}_n(\mathbb{C})$, F is conjugate to the subgroup $\mathrm{Sp}_k(\mathbb{C}) \times \mathrm{Sp}_{n-k}(\mathbb{C})$ (k and n are even, $0 < k < n$).*

Conversely, all subgroups of the form (a), (b), (c) *are maximal connected Lie subgroups.*

Theorem 3.2. *Let F be a nonsimple irreducible maximal connected Lie subgroup of a complex classical Lie group G. Then the following cases are possible:*

(a) *$G = \mathrm{SL}_n(\mathbb{C})$, F is conjugate to the subgroup $\mathrm{SO}_s(\mathbb{C}) \otimes \mathrm{SO}_t(\mathbb{C})$, where $n = st$, $s, t \geq 2$;*

(b) $G = \mathrm{SO}_n(\mathbb{C})$, *$F$ is conjugate either to the subgroup* $\mathrm{SO}_s(\mathbb{C}) \otimes \mathrm{SO}_t(\mathbb{C})$, *where* $n = st$, $s, t \geq 3$, $s, t \neq 4$ *or to the subgroup* $\mathrm{Sp}_s(\mathbb{C})$, *where* $n = st$, $s, t \geq 2$;

(c) $G = \mathrm{Sp}_n(\mathbb{C})$, *$F$ is conjugate to the subgroup* $\mathrm{Sp}_s(\mathbb{C}) \otimes \mathrm{SO}_t(\mathbb{C})$, *where* $n = st$, $s \geq 2$, $t \geq 3$, $t \neq 4$, *or* $s = 2$, $t = 4$.

Conversely, all subgroups of the form (a), (b), (c) *are maximal connected Lie subgroups.*

Theorem 3.3. *Let* $R: F \to \mathrm{GL}(V)$ *be a nontrivial irreducible linear representation of a simply-connected simple complex Lie group F. If there are no nondegenerate bilinear forms in V invariant under R, then $R(F)$ is a maximal connected subgroup of* $\mathrm{SL}(V)$, *and if R is orthogonal or sympletic, then R is a maximal connected subgroup of* $\mathrm{SO}(V)$ *or* $\mathrm{Sp}(V)$, *respectively. The only exceptions are the representations listed in Table* 7.

Note that Dynkin [1952a] also lists proper subgroups of the groups $\mathrm{SL}(V)$, $\mathrm{SO}(V)$, $\mathrm{Sp}(V)$ containing $R(F)$, where R is a representation from Table 7.

3.3. Semisimple Subalgebras of Exceptional Complex Lie Algebras. This section contains a brief review of the results of Dynkin [1952b] on semisimple subalgebras of exceptional Lie algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra. The subalgebras $\mathfrak{f}, \mathfrak{f}' \subset \mathfrak{g}$ are said to be *L-conjugate* if for any linear representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ the subalgebras $\rho(\mathfrak{f})$ and $\rho(\mathfrak{f}')$ are conjugate in $\mathfrak{gl}(V)$. The main result of Dynkin [1952b] is the classification of semisimple subalgebras of exceptional complex Lie algebras up to L-conjugacy.

Note that if $\mathfrak{g}$ is a simple Lie algebra other than $\mathfrak{so}_{2l}$, then the subalgebras $\mathfrak{f}, \mathfrak{f}'$ are L-conjugate in $\mathfrak{g}$ if the subalgebras $\rho(\mathfrak{f})$, $\rho(\mathfrak{f}')$ of $\mathfrak{gl}(V)$ are conjugate for just one representation $\rho = \rho_0 = \rho(\pi_1)$ having the least possible dimension (for $\mathfrak{g} = \mathfrak{so}_{2l}(\mathbb{C})$ this condition must also be satisfied for $\rho(\pi_{l-1})$ or $\rho(\pi_l)$). Therefore one can define (up to L-conjugacy) semisimple subalgebras $\mathfrak{f}$ of the exceptional Lie algebras $\mathfrak{g}$ with the help of linear representations $\rho_0|\mathfrak{f}$. Now Proposition 1.2 implies that for the classical Lie algebras other than $\mathfrak{so}_{2l}(\mathbb{C})$ the L-conjugacy of two subalgebras implies their conjugacy in $\mathfrak{g}$, while for $\mathfrak{g} = \mathfrak{so}_{2l}(\mathbb{C})$ this is not the case. Similarly, for the exceptional Lie algebras, in most cases the L-conjugacy of subalgebras implies their conjugacy. However, there are examples of subalgebras isomorphic to $\mathfrak{sl}_3(\mathbb{C})$ and G_2 in E_6 that are L-conjugate but not conjugate.

A proper subalgebra $\mathfrak{f}$ of a simple Lie algebra $\mathfrak{g}$ is said to be an *R-subalgebra* if $\mathfrak{f}$ is contained in a proper regular subalgebra of $\mathfrak{g}$, and an *S-subalgebra* otherwise. For any faithful linear representation of the algebra $\mathfrak{g}$, R-subalgebras are represented by reducible linear Lie algebras. The properties of S-subalgebras are similar to those of irreducible subalgebras of classical Lie algebras. In particular, any S-subalgebra is semisimple. Since regular semisimple subalgebras are easily classified in terms of root systems (see Sect. 1.2), the most difficult part is the classification of S-subalgebras. We now state two theorems from Dynkin [1952b] relating to S-subalgebras.

Theorem 3.4. *The simple S-subalgebras of rank > 1 of the exceptional Lie algebras are classified as follows:*

(a) *in G_2 and F_4 there are no such subalgebras;*

(b) *in E_6 there are two classes of conjugate S-subalgebras of type A_2 and two classes of type G_2, taken into each other by an outer automorphism of the algebra E_6, and one class of conjugate subalgebras of type C_4 and one of type F_4;*

(c) *in E_7 there is one class of conjugate subalgebras of type A_2;*

(d) *in E_8 there is one class of conjugate subalgebras of type B_2.*

Theorem 3.5. *The classes of L-conjugate maximal S-subalgebras $\mathfrak{f}$ in the exceptional Lie algebras $\mathfrak{g}$ are described by the following table (the upper index denotes the index of the subalgebra in the sense of Chap.* 1, Sect. 2.9):

$\mathfrak{g}$	$\mathfrak{f}$
G_2	$\mathfrak{sl}_2(\mathbb{C})^{28}$
F_4	$\mathfrak{sl}_2(\mathbb{C})^{156}$, $G_2^1 \oplus \mathfrak{sl}_2(\mathbb{C})^8$
E_6	$\mathfrak{sl}_2(\mathbb{C})^9$, G_2^3, $\mathfrak{sp}_8(\mathbb{C})^1$, $G_2^1 \oplus \mathfrak{sl}_3(\mathbb{C})^2$, F_4^1
E_7	$\mathfrak{sl}_2(\mathbb{C})^{399}$, $\mathfrak{sl}_2(\mathbb{C})^{231}$, $\mathfrak{sl}_3(\mathbb{C})^{21}$, $G_2^1 \oplus \mathfrak{sp}_6(\mathbb{C})^1$, $F_4^1 \oplus \mathfrak{sl}_2(\mathbb{C})^3$, $G_2^2 \oplus \mathfrak{sl}_2(\mathbb{C})^7$, $\mathfrak{sl}_2(\mathbb{C})^{24} \oplus \mathfrak{sl}_2(\mathbb{C})^{15}$
E_8	$\mathfrak{sl}_2(\mathbb{C})^{1240}$, $\mathfrak{sl}_2(\mathbb{C})^{760}$, $\mathfrak{sl}_2(\mathbb{C})^{520}$, $G_2^1 \oplus F_4^1$, $\mathfrak{sl}_3(\mathbb{C})^6 \oplus \mathfrak{sl}_2(\mathbb{C})^{16}$, $\mathfrak{so}_5(\mathbb{C})^{12}$

3.4. Semisimple Subalgebras of Real Semisimple Lie Algebras. Let $\mathfrak{g}$ be a real semisimple Lie algebra,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{12}$$

its Cartan decomposition, and θ the corresponding involutory automorphism. A subalgebra $\mathfrak{f} \subset \mathfrak{g}$ is said to be *canonically embedded* in $\mathfrak{g}$ with respect to the decomposition (12) if $\theta(\mathfrak{f}) = \mathfrak{f}$ or, equivalently, if

$$\mathfrak{f} = (\mathfrak{f} \cap \mathfrak{k}) \oplus (\mathfrak{f} \cap \mathfrak{p}). \tag{13}$$

An algebraic subalgebra $\mathfrak{f} \subset \mathfrak{g}$ is said to be *reductive* if $\operatorname{ad}\mathfrak{f}$ is the tangent algebra of a reductive algebraic subgroup in $\mathrm{GL}(\mathfrak{g})$, where ad is the adjoint representation of the algebra $\mathfrak{g}$.

Theorem 3.6. *An algebraic subalgebra of a real semisimple Lie algebra $\mathfrak{g}$ is reductive if and only if it is canonically embedded in $\mathfrak{g}$ with respect to a Cartan decomposition.*

Proof. If $\mathfrak{f}$ is canonically embedded in $\mathfrak{g}$, then the form $k_{\mathfrak{g}}$ is nondegenerate on $\mathfrak{f}(\mathbb{C})$ and therefore $k_{\mathfrak{g}(\mathbb{C})}$ is nondegenerate on $\mathfrak{f}(\mathbb{C})$. The reductivity follows from Chap. 1, Proposition 6.2.

Conversely, let $\mathfrak{f}$ be a reductive algebraic subalgebra of $\mathfrak{g}$. Consider the Lie algebra $\mathfrak{g}(\mathbb{C})$ and its complex conjugation σ with respect to $\mathfrak{g}$. It follows from Chap. 4, Corollary 3 to Theorem 3.3 that there exists a real structure τ in $\mathfrak{g}(\mathbb{C})$ such that the subalgebra $\mathfrak{u} = \mathfrak{g}(\mathbb{C})^\tau$ is compact, $\sigma\tau = \tau\sigma$ and $\tau(\mathfrak{f}(\mathbb{C})) = \mathfrak{f}(\mathbb{C})$. Then $\mathfrak{f}$ is canonically embedded in $\mathfrak{g}$ with respect to the Cartan decomposition $\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{u}) \oplus (\mathfrak{g} \cap (i\mathfrak{u}))$. □

Note that if a canonically embedded subalgebra $\mathfrak{f}$ is semisimple, then the decomposition (13) is its Cartan decomposition. Theorem 3.6 was proved independently by Karpelevich [1953] and Mostow [1955]. In his work, Karpelevich also established the connection between semisimple subalgebras of a real semisimple algebra and totally geodesic submanifolds of the Riemannian symmetric space corresponding to it (see also Onishchik [1980]).

Now consider the problem of classifying semisimple subalgebras of real semisimple Lie algebras. As in the complex case (see Sect 3.1), one can consider instead the problem of classifying injective homomorphisms of semisimple real Lie algebras $\varphi\colon \mathfrak{f} \to \mathfrak{g}$ up to inner automorphisms or up to automorphisms of $\mathfrak{g}$. Such a homomorphism admits a complexification $\varphi(\mathbb{C})\colon \mathfrak{f}(\mathbb{C}) \to \mathfrak{g}(\mathbb{C})$, which leads to the following essential question. Let $\psi\colon \mathfrak{a} \to \mathfrak{b}$ be a homomorphism of complex semisimple Lie algebras, and let $\mathfrak{f}, \mathfrak{g}$ be real forms of the algebras $\mathfrak{a}, \mathfrak{b}$ respectively. When is $\psi(\mathfrak{f})$ contained in $\mathfrak{g}$? Theorem 3.6 implies the following statement.

Theorem 3.7 (Karpelevich [1955]). *Let $\psi\colon \mathfrak{a} \to \mathfrak{b}$ be a homomorphism of complex semisimple Lie groups, and $\mathfrak{f}, \mathfrak{g}$ real forms in $\mathfrak{a}$ and $\mathfrak{b}$, respectively. The relation $\psi(\mathfrak{f}) \subset \mathfrak{g}$ is satisfied if and only if $\theta\psi = \psi\eta$, where η, θ are involutory automorphisms of the algebras $\mathfrak{a}, \mathfrak{b}$ belonging to the classes of conjugate automorphisms corresponding to the forms $\mathfrak{f}, \mathfrak{g}$ (see Theorem 1.3 of Chap. 4).*

In the case where $\mathfrak{b}$ is a classical complex Lie algebra, ψ is a linear representation of the algebra $\mathfrak{a}$, and the condition of Theorem 3.7 can be written as the condition on the highest weights of the representation ψ. In these terms, Karpelevich [1955] gives an explicit description of linear representations taking a given real form of a simple Lie algebra $\mathfrak{a}$ into a given real form of a classical Lie algebra $\mathfrak{b}$. The case of a reductive Lie algebra $\mathfrak{a}$ was studied in Komrakov [1989].

The results of Karpelevich [1955] together with the results of Sect. 3.2 make it also possible to classify maximal subalgebras of real classical Lie algebras (see, for example, Taufik [1979], Komrakov [1990], [1991]).

A wider class of primitive subalgebras connected with primitive actions of connected Lie groups (see Gorbatsevich and Onishchik [1988], Chap. 2, Sect. 1.4) has also been investigated. Their classification reduces to the case when the ambient Lie algebra $\mathfrak{g}$ is simple, and the subalgebra is reductive in $\mathfrak{g}$. This case was completely analyzed by Komrakov [1991]. He proved, in particular, that the only non-maximal primitive simple irreducible subalgebra of a complex classical simple Lie algebra is that of type $\mathcal{D}_6$ in $\mathfrak{so}_{495}(\mathbb{C})$ determined by the representation with highest weight π_4 (see Table 7).

Chapter 7
On the Classification of Arbitrary Lie Groups and Lie Algebras of a Given Dimension

§ 1. Classification of Lie Groups and Lie Algebras of Small Dimension

1.1. Lie Algebras of Small Dimension. At the time of writing there is a classification of all Lie algebras of dimension ≤ 6 over the fields $\mathbb{C}$ and $\mathbb{R}$ (see Mubarakzianov [1963], Yamaguchi [1981]). In larger dimensions, only partial results are known. For example, there is a classification of all complex nilpotent Lie algebras of dimension 7. For any fixed dimension one can (with the help of a computer if the dimension is large) enumerate all semisimple Lie algebras making use of the classification of simple Lie algebras (see Chap. 4, Sect. 1.5). At the time of writing there is no unified approach to the classification of arbitrary Lie algebras. In this section we consider individual classification results. Let us agree to define Lie algebras by the commutation relations $[X_i, X_j] = C_{ij}^k \cdot X_k$, writing them just for $i \leq j$, and only if they are nonzero ones.

Theorem 1.1. *Let* $\mathfrak{g}$ *be a complex Lie algebra of dimension* ≤ 4. *Then* $\mathfrak{g}$ *is isomorphic to one (and just one) of the following Lie algebras*:

(i) $\dim \mathfrak{g} = 1 \quad \mathfrak{g} \simeq \mathbb{C}$;
(ii) $\dim \mathfrak{g} = 2 \quad \mathfrak{g} \simeq \mathbb{C}^2, \mathfrak{r}_2(\mathbb{C})$;
(iii) $\dim \mathfrak{g} = 3 \quad \mathfrak{g} \simeq \mathbb{C}^3$, $\mathfrak{n}_3(\mathbb{C})$, $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{r}_3(\mathbb{C})$, $\mathfrak{r}_{3,\lambda}(\mathbb{C})$, $\mathfrak{sl}_2(\mathbb{C})$;
(iv) $\dim \mathfrak{g} = 4 \quad \mathfrak{g} \simeq \mathbb{C}^4$, $\mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^2$, $\mathfrak{r}_3(\mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$, $\mathfrak{n}_4(\mathbb{C})$ *or one of the (solvable) Lie algebras* $\mathfrak{g}_{4,i}$, $1 \leq i \leq 8$ *given in the basis* X_1, X_2, X_3, X_4 *by the relations*

$\mathfrak{g}_{4,1}$: $[X_1, X_4] = X_2$, $[X_3, X_4] = X_1$;
$\mathfrak{g}_{4,2}$: $[X_1, X_4] = X_1$, $[X_2, X_4] = X_2$;
$\mathfrak{g}_{4,3}$: $[X_1, X_4] = X_1$, $[X_2, X_4] = X_1 + X_2$, $[X_3, X_4] = X_2 + X_3$;
$\mathfrak{g}_{4,4}$: $[X_1, X_4] = cX_1$, $[X_2, X_4] = (1+c)X_2$, $[X_3, X_4] = X_1 + cX_3$, $c \in \mathbb{C}$;
$\mathfrak{g}_{4,5}$: $[X_1, X_4] = X_1$, $[X_2, X_4] = X_2$;
$\mathfrak{g}_{4,6}$: $[X_2, X_3] = X_1$, $[X_1, X_4] = X_1$, $[X_2, X_4] = X_2$;
$\mathfrak{g}_{4,7}$: $[X_2, X_3] = X_1$, $[X_1, X_4] = 2X_2$, $[X_2, X_4] = X_2$, $[X_3, X_4] = X_2 + X_3$;
$\mathfrak{g}_{4,8}$: $[X_2, X_3] = X_1$, $[X_1, X_4] = cX_1$, $[X_2, X_4] = X_2$, $[X_3, X_4] = (c-1)X_3$, $c \neq 1$, $c \in \mathbb{C}$.

Let us now describe the Lie algebras listed in (i)–(iii) above (with the exception of the obvious cases $\mathbb{C}^k$ and $\mathfrak{sl}_2(\mathbb{C})$):

$\mathfrak{r}_2(\mathbb{C})$: $[X_1, X_2] = X_2$, solvable;
$\mathfrak{n}_3(\mathbb{C})$: $[X_1, X_2] = X_3$, nilpotent;

$\mathfrak{r}_3(\mathbb{C})$: $[X_1, X_2] = X_2$, $[X_1, X_3] = X_2 + X_3$, solvable;
$\mathfrak{r}_{3,\lambda}(\mathbb{C})$: , $[X_1, X_2] = X_2$, $[X_1, X_3] = \lambda \cdot X_3$, $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$, solvable;
$\mathfrak{n}_4(\mathbb{C})$: $[X_1, X_i] = X_{i+1}$, $i = 2, 3$, nilpotent.

Proof. Let us give an outline of the proof of the theorem for $\dim \mathfrak{g} \leq 3$. For $\dim \mathfrak{g} = 1, 2$ the statement of the theorem is almost obvious. Consider the case $\dim \mathfrak{g} = 3$. If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{C})$ (see Chap. 3, Sect. 1.2). If $\mathfrak{g}$ is not semisimple, then it is solvable of the form $\mathbb{C} \ni_\varphi \mathbb{C}^2$, where $\varphi\colon \mathbb{C} \to \mathfrak{gl}_2(\mathbb{C})$ is a homomorphism (which is uniquely defined by the matrix $A = \varphi(1)$). The classification of the Lie algebras of the form $\mathbb{C} \ni_\varphi \mathbb{C}^2$ up to isomorphism is equivalent to the classification of the matrices A up to similarity and multiplication by a nonzero scalar multiplier. By means of such transformations the matrix A can be reduced to one of the following forms:

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_1(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}, \quad \text{where} \quad |\lambda| \leq 1,$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

These matrices correspond to the Lie algebras $\mathbb{C}^3$, $\mathfrak{r}_{3,\lambda}(\mathbb{C})$, $\mathfrak{n}_3(\mathbb{C})$, and $\mathfrak{r}_3(\mathbb{C})$, respectively. □

Theorem 1.2 (see Morozov [1958]). *Let $\mathfrak{g}$ be a nilpotent Lie algebra over the field k of characteristic 0 and $\dim \mathfrak{g} \leq 6$. Then $\mathfrak{g}$ either decomposes into the direct sum of Lie algebras of lower dimension or is isomorphic to one (and only one) of the following Lie algebras:*

(a) $\dim \mathfrak{g} = 1$
  $\mathfrak{g}_{1,1} = k$;
(b) $\dim \mathfrak{g} = 3$
  $\mathfrak{g}_{3,1}$: $[X_1, X_2] = X_3$ $(\mathfrak{g}_{3,1} \simeq \mathfrak{n}_2(k))$;
(c) $\dim \mathfrak{g} = 4$
  $\mathfrak{g}_{4,1}$: $[X_1, X_i] = X_{i+1}$, $i = 2, 3$ $(\mathfrak{g}_{4,1} \simeq \mathfrak{n}_4(k))$;
(d) $\dim \mathfrak{g} = 5$
  $\mathfrak{g}_{5,1}$: $[X_1, X_2] = X_5$, $[X_3, X_4] = X_5$;
  $\mathfrak{g}_{5,2}$: $[X_1, X_2] = X_4$, $[X_1, X_3] = X_5$;
  $\mathfrak{g}_{5,3}$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_2, X_5] = X_4$;
  $\mathfrak{g}_{5,4}$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_2, X_3] = X_5$;
  $\mathfrak{g}_{5,5}$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_1, X_4] = X_5$;
  $\mathfrak{g}_{5,6}$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_1, X_4] = X_5$, $[X_2, X_3] = X_5$;
(e) $\dim \mathfrak{g} = 6$
  $\mathfrak{g}_{6,1}$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_1, X_5] = X_6$;

$\mathfrak{g}_{6,2}$: $[X_1,X_2]=X_3$, $[X_1,X_3]=X_4$, $[X_1,X_4]=X_5$, $[X_1,X_5]=X_6$;
$\mathfrak{g}_{6,3}$: $[X_1,X_2]=X_6$, $[X_1,X_3]=X_4$, $[X_2,X_3]=X_5$;
$\mathfrak{g}_{6,4}$: $[X_1,X_2]=X_5$, $[X_1,X_3]=X_6$, $[X_2,X_4]=X_6$;
$\mathfrak{g}_{6,5}$: $[X_1,X_3]=X_5$, $[X_1,X_4]=X_6$, $[X_2,X_4]=X_5$, $[X_2,X_3]=\gamma X_6$, $\gamma \in k^\times \bmod k^{\times 2}$;
$\mathfrak{g}_{6,6}$: $[X_1,X_2]=X_6$, $[X_1,X_3]=X_4$, $[X_1,X_4]=X_5$, $[X_2,X_3]=X_5$;
$\mathfrak{g}_{6,7}$: $[X_1,X_3]=X_4$, $[X_1,X_4]=X_5$, $[X_2,X_3]=X_6$;
$\mathfrak{g}_{6,8}$: $[X_1,X_2]=X_3+X_5$, $[X_1,X_3]=X_4$, $[X_2,X_5]=X_6$;
$\mathfrak{g}_{6,9}$: $[X_1,X_2]=X_3$, $[X_1,X_3]=X_4$, $[X_1,X_5]=X_6$, $[X_2,X_3]=X_6$;
$\mathfrak{g}_{6,10}$: $[X_1,X_2]=X_3$, $[X_1,X_3]=X_5$, $[X_1,X_4]=X_6$, $[X_2,X_4]=X_5$, $[X_2,X_3]=\gamma X_6$, $\gamma \in k^\times \bmod k^{\times 2}$;
$\mathfrak{g}_{6,11}$: $[X_1,X_2]=X_3$, $[X_1,X_3]=X_4$, $[X_1,X_4]=X_5$, $[X_2,X_3]=X_6$;
$\mathfrak{g}_{6,12}$: $[X_1,X_3]=X_4$, $[X_1,X_4]=X_6$, $[X_2,X_5]=X_6$;
$\mathfrak{g}_{6,13}$: $[X_1,X_2]=X_5$, $[X_1,X_3]=X_4$, $[X_1,X_4]=X_6$, $[X_2,X_5]=X_6$;
$\mathfrak{g}_{6,14}$: $[X_1,X_3]=X_4$, $[X_1,X_4]=X_6$, $[X_2,X_3]=X_5$, $[X_2,X_5]=cX_6$, $c\in k$, $c\neq 0$;
$\mathfrak{g}_{6,15}$: $[X_1,X_2]=X_3+X_5$, $[X_1,X_3]=X_4$, $[X_1,X_4]=X_6$, $[X_2,X_5]=X_6$;
$\mathfrak{g}_{6,16}$: $[X_1,X_3]=X_4$, $[X_1,X_4]=X_5$, $[X_1,X_5]=X_6$, $[X_2,X_3]=X_5$, $[X_2,X_4]=X_6$;
$\mathfrak{g}_{6,17}$: $[X_1,X_2]=X_3$, $[X_1,X_3]=X_4$, $[X_1,X_4]=X_6$, $[X_2,X_5]=X_6$;
$\mathfrak{g}_{6,18}$: $[X_1,X_2]=X_3$, $[X_1,X_3]=X_4$, $[X_1,X_4]=X_6$, $[X_2,X_3]=X_5$, $[X_2,X_5]=cX_6$, $c\in k$, $c\neq 0$;
$\mathfrak{g}_{6,19}$: $[X_1,X_2]=X_3$, $[X_1,X_3]=X_4$, $[X_1,X_4]=X_5$, $[X_1,X_5]=X_6$, $[X_2,X_3]=X_6$;
$\mathfrak{g}_{6,20}$: $[X_1,X_2]=X_3$, $[X_1,X_3]=X_4$, $[X_1,X_4]=X_5$, $[X_1,X_5]=X_6$, $[X_2,X_3]=X_5$, $[X_2,X_4]=X_6$;
$\mathfrak{g}_{6,21}$: $[X_1,X_2]=X_3$, $[X_1,X_5]=X_6$, $[X_2,X_3]=X_4$, $[X_2,X_4]=X_5$, $[X_3,X_4]=X_6$;
$\mathfrak{g}_{6,22}$: $[X_1,X_2]=X_3$, $[X_1,X_3]=X_5$, $[X_1,X_5]=X_6$, $[X_2,X_3]=X_4$, $[X_2,X_4]=X_5$, $[X_3,X_4]=X_6$.

Note that if the field k is algebraically closed, then the Lie algebras $\mathfrak{g}_{6,5}$ and $\mathfrak{g}_{6,10}$ must be excluded from the list, because they are isomorphic to $\mathfrak{g}_{3,1}\oplus\mathfrak{g}_{3,1}$ and $\mathfrak{g}_{6,8}$, respectively.

Theorem 1.3. *Let* $\mathfrak{g}$ *be a Lie algebra over the field* $\mathbb{C}$ *and* $\dim\mathfrak{g}=5$ *or* 6. *Then if the Lie algebra* $\mathfrak{g}$ *is not solvable, it is isomorphic to one* (*and only one*) *of the following Lie algebras.*

(i) $\dim\mathfrak{g}=5$: $\mathfrak{g}\simeq\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}^2$, $\mathfrak{sl}_2(\mathbb{C}) \in_{\varphi}\mathbb{C}^2$, where $\varphi\colon\mathfrak{sl}_2(\mathbb{C})\to\mathfrak{gl}_2(\mathbb{C})$ is a tautological embedding;

(ii) $\dim\mathfrak{g}=6$: , $\mathfrak{g}\simeq\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{sl}_2(\mathbb{C}) \in_{\mathrm{ad}}\mathbb{C}^3$ (which corresponds to the adjoint representation), or $\mathfrak{sl}_2(\mathbb{C}) \in_{\varphi}\mathfrak{n}_3(\mathbb{C})$, where the homomorphism $\varphi\colon\mathfrak{sl}_2(\mathbb{C})\to\operatorname{Der}\mathfrak{n}_3(\mathbb{C})$ is described as follows: if $\mathfrak{n}_3(\mathbb{C})$ is generated by elements X_1, X_2 such that $[X_1,X_2]=X_3$ is a generator in $Z(\mathfrak{n}_3(\mathbb{C}))$, then

$\mathfrak{sl}_2(\mathbb{C})$ acts in a standard way on the linear span of X_1, X_2. This generates the action on $\mathfrak{n}_3(\mathbb{C})$.

The most difficult part of the classification is the enumeration of all solvable Lie algebras. The solvable Lie algebras of dimension 5 (over the fields $\mathbb{C}$ and $\mathbb{R}$) are listed in Mubarkzianov [1963]. The 6-dimensional Lie algebras are classified in Yamaguchi [1981].

We now present some results on the classification of real Lie algebras.

Theorem 1.4. *Let* $\mathfrak{g}$ *be a real Lie algebra,* $\dim \mathfrak{g} \leq 3$. *Then* $\mathfrak{g}$ *is isomorphic to one (and only one) of the following Lie algebras:*

(i) $\dim \mathfrak{g} = 1 \quad \mathfrak{g} = \mathbb{R}$;

(ii) $\dim \mathfrak{g} = 2 \quad \mathfrak{g} = \mathbb{R}^2,\ \mathfrak{r}_2(\mathbb{R})$;

(iii) $\dim \mathfrak{g} = 3 \quad \mathfrak{g} = \mathbb{R}^3,\ \mathfrak{n}_3(\mathbb{R}),\ \mathfrak{r}_2(\mathbb{R}) \oplus \mathbb{R},\ \mathfrak{r}_3(\mathbb{R}),\ \mathfrak{r}_{3,\lambda}(\mathbb{R})$ $(\lambda \in \mathbb{R},\ 0 < |\lambda| \leq 1)$, $\mathfrak{r}'_{3,\lambda}(\mathbb{R})$ $(\lambda \in \mathbb{R},\ \lambda \neq 0)$, $\mathfrak{n}(2)$, $\mathfrak{su}_2$, $\mathfrak{sl}_2(\mathbb{R})$.

The Lie algebras $\mathfrak{r}_3(\mathbb{R}),\ \mathfrak{r}_{3,\lambda}(\mathbb{R}),\ \mathfrak{r}'_{3,\lambda}(\mathbb{R}),\ \mathfrak{n}(2)$ *are of the form* $\mathbb{R} \ni_{\varphi} \mathbb{R}^2$, *where the matrix* $A = \varphi(1)$ *is, respectively,* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

With the growth of dimension, the classification of Lie algebras becomes disastrously more complicated. Multiparameter families of pairwise nonisomorphic Lie algebras arise, growing in number. One of the known approaches to the classification consists in analyzing the properties of the space $\mathcal{L}_n$ of structural tensors of n-dimensional Lie algebras.

1.2. Connected Lie Groups of Dimension ≤ 3. Let G be a connected Lie group and $\pi\colon \widetilde{G} \to G$ the universal covering. Then $G = \widetilde{G}/Z$, where $Z = \operatorname{Ker} \pi$ is a discrete central subgroup. Therefore the classification of connected Lie groups of a given dimension reduces to the classification of simply-connected Lie groups of the same dimension, and to the enumeration of all possible discrete subgroups of $Z(\widetilde{G})$. The classification of simply-connected Lie groups is equivalent to the classification of the corresponding Lie algebras, for which, in the case of smaller dimensions, see Sect. 1.1. It is easy to see that the discrete subgroups of $Z(\widetilde{G})$ must be classified up to conjugacy by elements from $\operatorname{Aut} \widetilde{G}$. This results in a rather complicated classification problem which at the time of writing is solved only for $\dim G \leq 3$. Now we present the corresponding results for real Lie groups.

(A) If $\dim G = 1$, then G is isomorphic to either $\mathbb{T} \simeq \mathrm{SO}_2$ or $\mathbb{R}^1$.

(B) If $\dim G = 2$, then either G is abelian and then it is isomorphic to $\mathbb{R}^2$, $\mathbb{T} \times \mathbb{R}$, or $\mathbb{T}^2$, or G is nonabelian and in this case it is isomorphic to the group $\operatorname{Aff}^0 \mathbb{R}^1$ of proper affine transformations of the straight line (this group is simply-connected and has trivial centre).

(C) If $\dim G = 3$, then the results of Sect. 1.1 imply that the simply-connected covering $\widetilde{G}$ of G is isomorphic to $\mathbb{R}^3$, $N_3(\mathbb{R})$, SU_2, $\mathcal{A} = \widetilde{\mathrm{SL}_2}(\mathbb{R})$,

$\widetilde{E}_0^0(2)$, the universal covering of $E^0(2)$, $\mathbb{R}_3$, $R_{3,\lambda}$, $R'_{3,\lambda}$, where the last three Lie groups correspond to the Lie algebras $\mathfrak{r}_3$, $\mathfrak{r}_{3,\lambda}(\mathbb{R})$, $\mathfrak{r}'_{3,\lambda}(\mathbb{R})$.

(i) If $\widetilde{G} \simeq \mathbb{R}^3$, then G is isomorphic to $\mathbb{R}^3$, $\mathbb{R}^2 \times \mathbb{T}$, $\mathbb{R} \times \mathbb{T}^2$, or $\mathbb{T}^3$.

(ii) If $\widetilde{G} \simeq N_3(\mathbb{R})$, then G is isomorphic to the Lie group $N_3(\mathbb{R})$ or the Lie group $N_3^* = N_3(\mathbb{R})/Z(N_3(\mathbb{Z}))$, where $Z(N_3(\mathbb{Z}))$ is the group of integer points of the centre $Z(N_3(\mathbb{R})) \simeq \mathbb{R}$ of the group $N_3(\mathbb{R})$.

(iii) If $\widetilde{G} \simeq \mathrm{SU}_2$, then G is isomorphic to either SU_2 or SO_3.

(iv) If $\widetilde{G} \simeq \mathcal{A}$, then G is isomorphic to either $\mathcal{A}$ or the Lie group $A_1(m)$, which is the m-fold covering of the Lie group $A_1(1) = \mathrm{PSL}_2(\mathbb{R})$, $m \in \mathbb{N}$.

(v) If $\widetilde{G} \simeq \widetilde{E}^0(2)$, then G is isomorphic to either the Lie group $\widetilde{E}^0(2)$ or the group E_k, which is the k-fold covering of the Lie group $E_1 = E^0(2)$, $k \in \mathbb{N}$.

(vi) If $\widetilde{G} \simeq R_3$, $R_{3,\lambda}$, or $R'_{3,\lambda}$ $(\lambda \neq 0)$, then G is isomorphic to R_3, $R_{3,\lambda}$, or $R'_{3,\lambda}$ respectively, because the centres of the simply-connected Lie groups R_3, $R_{3,\lambda}$, and $R'_{3,\lambda}$ (for $\lambda \neq 0$) are trivial.

(vii) If $\widetilde{G} \simeq R_{3,0}$, then G is isomorphic to either $R_2 \times \mathbb{R}$ or $R_2 \times \mathbb{T}$, where $R_2 = \mathrm{Aff}^{\,0}\mathbb{R}$.

This proves the following theorem.

Theorem 1.5. *An arbitrary three-dimensional connected real Lie group is isomorphic to one of the following pairwise nonisomorphic Lie groups*:

$$\mathbb{R}^3, \mathbb{R}^2 \times \mathbb{T}, \mathbb{R} \times \mathbb{T}^2, \mathbb{T}^3, N_3(\mathbb{R}), N_3^*, R_2 \times \mathbb{R}, R_2 \times \mathbb{T}, R_3, R_{3,\lambda}\ (\lambda \neq 0),$$
$$R'_{3,\lambda}\ (\lambda = 0), E^0(2), E_k\ (k \in \mathbb{N}), \mathrm{SU}_2, \mathrm{SO}_3, \mathcal{A}, A_1(m)\ (m \in \mathbb{N}).$$

§ 2. The Space of Lie Algebras. Deformations and Contractions

2.1. The Space of Lie Algebras. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an arbitrary field k. Choose a basis $\{e_j\}$, $1 \leq i \leq n = \dim \mathfrak{g}$, in $\mathfrak{g}$ and set $[e_i, e_j] = C_{ij}^k \cdot e_k$. The coefficients C_{ij}^k are called structural constants. They form a *structural* tensor, which is an element of the space $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$ identified with the bilinear skew-symmetric mapping $\mu: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ defining the commutator on $\mathfrak{g}$. The fact that $\mathfrak{g}$ is a Lie algebra is equivalent to the following relations between the structural constants:

$$C_{ij}^k + C_{ji}^k = 0 \quad \text{for} \quad 1 \leq i, j, k \leq n;$$
$$C_{ij}^l \cdot C_{lk}^m + C_{jk}^l \cdot C_{li}^m + C_{ki}^l \cdot C_{lj}^m = 0 \quad \text{for} \quad 1 \leq i, j, k, l, m \leq n. \qquad (*)$$

For convenience of notation, the structural constants C_{ij}^k are usually defined only for $i < j$ (since they are skew-symmetric in i, j). By ordering them in some way, the sets of structural constants can be viewed as points in k^N, where $N = n^3$.

Denote by $\mathcal{L}_n(k)$ (or $\mathcal{L}_n$ if there can be no misunderstanding about the field k) the set of structural tensors $\{C^k_{ij}\}$ corresponding to all possible n-dimensional Lie algebras over k, and all possible bases in them. By virtue of relation $(*)$, the space $\mathcal{L}_n(k)$, viewed as a subset of k^N, is defined by $\frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} = \frac{n(n-1)(n+1)}{6}$ algebraic (mainly, quadratic) equations. Therefore $\mathcal{L}_n(k)$ is an algebraic subset in k^N (defined over a prime subfield of k, i.e. over either $\mathbb{Q}$ or $\mathbb{F}_p$). It can be regarded as a scheme defined by the ideal corresponding to the relations $(*)$. In the general case the algebraic subset $\mathcal{L}_n(k)$ is reducible and nonreduced (see Rauch [1972]).

Example 1. The space $\mathcal{L}_1(k)$ consists of a single point (C^1_{11}) corresponding to the abelian Lie algebra.

Example 2. The space $\mathcal{L}_2(k)$ is isomorphic to k^2, since for $n = 2$ relations $(*)$ are of the form

$$C^k_{12} = C^k_{22} = 0,$$

$$C^k_{12} + C^k_{21} = 0 \quad \text{for} \quad k = 1, 2.$$

Therefore a two-dimensional Lie algebra is determined by two arbitrary constants C^1_{12} and C^2_{12}. The point for which $C^1_{12} = C^2_{12} = 0$ correspond to the abelian Lie algebra k^2, and all other points of $\mathcal{L}_2(k)$ correspond to pairwise isomorphic Lie algebras all of which are isomorphic to the Lie algebra $\mathfrak{r}_2(k)$ defined, in an appropriate basis e_1, e_2, by the relation $[e_1, e_2] = e_2$.

For $n \geq 3$, with the growth of n the description of the spaces $\mathcal{L}_n(k)$ becomes much more complicated (see Sect. 2.6 below).

If $k = \mathbb{C}$ or $\mathbb{R}$, then the space $\mathcal{L}_n(k)$ is an analytic (even algebraic) subset of $\mathbb{C}^N$ or $\mathbb{R}^N$, respectively, and therefore $\mathcal{L}_n(k)$ admits a triangulation. In particular, as a topological space, $\mathcal{L}_n(k)$ is locally piecewise connected. This enables one to study the spaces $\mathcal{L}_n(\mathbb{C})$, $\mathcal{L}_n(\mathbb{C})$ with the help of deformations (i.e. curves that are continuous, piecewise smooth, or even piecewise analytic) in $\mathcal{L}_n$; for more details see Sect. 2.3 below.

Let k be a field of characteristic 0. The algebraic set $\mathcal{L}_n(k)$ decomposes into the union of finitely many irreducible components $\mathcal{L}^\alpha_n(k)$, $\alpha = 1, 2, \ldots, a_n$. Each irreducible component is a cone in k^N with vertex at the point $0 \in k^N$. For $k = \mathbb{C}$, only $a_1, \ldots, a_7$ have been computed at the time of writing (see Sect. 2.6); for $n \geq 8$ the number a_n of irreducible components of $\mathcal{L}_n(\mathbb{C})$ has only almost trivial upper bounds and some lower ones (see Sect. 2.6). For the field $k = \mathbb{R}$ the situation is even more unclear.

2.2. Orbits of the Action of the Group $\mathrm{GL}_n(k)$ **on** $\mathcal{L}_n(k)$. Let V be a vector space over an arbitrary field k. The natural action of the group $\mathrm{GL}(V)$ induces a linear action of this group on the space $V^* \otimes V^* \otimes V$, which may be viewed as the space of structural tensors $\mu\colon V \otimes V \to V$ of all possible algebra structures on V. For $\mu \in V^* \otimes V^* \otimes V$ and $g \in G$ we have

$$(g * \mu)(X, Y) = g \cdot \mu(g^{-1}X, g^{-1}Y), \quad \text{where} \quad X, Y \in V.$$

The set of structural tensors μ corresponding to Lie algebras is invariant under this action of the group $\mathrm{GL}(V)$. Now we will consider the coordinate notation.

The natural action of the group $\mathrm{GL}_n(k)$ on k^n induces the linear action of the group $\mathrm{GL}_n(k)$ on the space of all tensors of type $(2,1)$ on k^n, and, in particular, on its subspace $\mathcal{L}_n(k)$. If $g = [g_i^j] \in \mathrm{GL}_n(k)$ and $\mu = \{C_{ij}^k\} \in \mathcal{L}_n(k)$, then the action of the element g on the structural tensor μ is described as follows:

$$(g * \mu)_{ij}^k = g_i^p \cdot g_j^q \cdot \overset{(-1)}{g_r^k} \cdot C_{pq}^r,$$

where $\overset{(-1)}{g_r^k}$ are the entries of the matrix g^{-1} inverse to g. The orbits of this $\mathrm{GL}_n(k)$ action on $\mathcal{L}_n(k)$ are precisely the sets of structural tensors corresponding to isomorphic Lie algebras. Denote by $O_{\mathfrak{g}}$ the orbit in $\mathcal{L}_n(k)$ corresponding to an n-dimensional Lie algebra $\mathfrak{g}$. The isotropy subgroup of any point in $O_{\mathfrak{g}}$ is isomorphic to the group $\mathrm{Aut}\,\mathfrak{g}$. The orbit $O_{\mathfrak{g}}$ can therefore be identified with the homogeneous space $\mathrm{GL}_n(k)/\mathrm{Aut}\,\mathfrak{g}$.

Consider, for example, the orbits of the group $\mathrm{GL}_2(k)$ on $\mathcal{L}_2(k)$. The space $\mathcal{L}_2(k)$ is isomorphic to k^2, and the action of $\mathrm{GL}_2(k)$ on $\mathcal{L}_2(k)$ corresponds to the natural action of the group $\mathrm{GL}_2(k)$ on k^2. One of these orbits is the point $0 \in k^2$ corresponding to the abelian Lie algebra $\mathfrak{g} \simeq k^2$. The complement $k^2 \setminus \{0\}$ of this point is the second (open) orbit of the group $\mathrm{GL}_2(k)$. This orbit corresponds to the solvable Lie algebra $\mathfrak{r}_2(k)$. The group $\mathrm{Aut}\,\mathfrak{r}_2(k)$ is isomorphic to the group $\mathrm{ST}_2(k) = T_2(k) \cap \mathrm{SL}_2(k)$, so the orbit $O_{\mathfrak{r}_2}(k)$ is identified with $\mathrm{GL}_2(k)/\mathrm{ST}_2(k)$.

Consider an arbitrary orbit $O_{\mathfrak{g}}$ in $\mathcal{L}_n(k)$. Its algebraic closure ${}^aO_{\mathfrak{g}}$ contains $O_{\mathfrak{g}}$ as a Zariski open set (see Onishchik and Vinberg [1990], Chap. 3, Sect. 1.5). Therefore $\overline{O}_{\mathfrak{g}} \setminus O_{\mathfrak{g}}$ consists of orbits of lower dimension that $O_{\mathfrak{g}}$ (there may be infinitely many such orbits). If k is either $\mathbb{C}$ or $\mathbb{R}$, then $O_{\mathfrak{g}}$ is a smooth submanifold of k^N. Denote by $\overline{O}_{\mathfrak{g}}$ the closure of the orbit $O_{\mathfrak{g}}$ in the Euclidean topology. For $k = \mathbb{C}$ the set $\overline{O}_{\mathfrak{g}}$ coincides with ${}^aO_{\mathfrak{g}}$.

Consider the point $o_n \in \mathcal{L}_n(k)$ corresponding to the abelian Lie algebra $\mathfrak{g} \simeq k^n$ and let us show that $o_n \in \overline{O}_{\mathfrak{g}}$ for any Lie algebra $\mathfrak{g} \in \mathcal{L}_n(k)$. With that in mind consider the action of the scalar linear operator $t \cdot E$ (which is nondegenerate for $t \neq 0$) on $\mathcal{L}_n(k)$:

$$(tE) * \{C_{ij}^k\} = \{t \cdot C_{ij}^k\}.$$

Hence it follows, in particular, that an arbitrary orbit $O_{\mathfrak{g}}$ is a cone without the point o_n. As $t \to 0$ we have $(tE) * \{C_{ij}^k\} \to 0$, whence $o_n \in \overline{O}_{\mathfrak{g}}$. Using the terms which will be introduced in Sect. 2.3, this means that any Lie algebra can be contracted into an abelian one. We can easily see that $o_n \in {}^aO_{\mathfrak{g}}$ for Lie algebras $\mathfrak{g}$ over an arbitrary field k.

The above considerations imply that there is only one closed (even in the Euclidean topology if $k = \mathbb{C}$ or $\mathbb{R}$) orbit in $\mathcal{L}_n(k)$, namely the point

$o_n \in \mathcal{L}_n(k)$. For other Lie algebras $\mathfrak{g}$ the space ${}^aO_{\mathfrak{g}}$ is more or less different from $O_{\mathfrak{g}}$.

Example 3. Consider the Lie algebra $\mathfrak{g} = \mathfrak{n}_3(k) \oplus k^{n-3} \in \mathcal{L}_n(k)$, $n \geq 3$. We can show that ${}^aO_{\mathfrak{g}} = O_{\mathfrak{g}} \cup \{o_n\}$, i.e. ${}^aO_{\mathfrak{g}}$ differs from $O_{\mathfrak{g}}$ in the least possible way — by the point $\{0\}$ only.

There are other Lie algebras $\mathfrak{g}$ for which ${}^aO_{\mathfrak{g}} = O_{\mathfrak{g}} \cup \{0\}$. For example, such is the Lie algebra $\mathfrak{r}_{3,+1} \in \mathcal{L}_n(\mathbb{C})$ described in Sect. 1.

Consider the space $Z^2(\mathfrak{g}, \mathfrak{g})$ of 2-cocycles of the Lie algebra $\mathfrak{g}$ acting on itself by means of the adjoint representation, and the subspace of coboundaries $B^2(\mathfrak{g}, \mathfrak{g})$ in it.

Theorem 2.1 (see Rauch [1972]). *Let μ be a point of the scheme $\mathcal{L}_n(k)$ corresponding to the Lie algebra $\mathfrak{g}$. Then*

(i) *the tangent Zariski space to the scheme $\mathcal{L}_n(k)$ at the point μ coincides with $Z^2(\mathfrak{g}, \mathfrak{g})$;*

(ii) *the tangent space to the orbit $O_{\mathfrak{g}}$ at the point μ coincides with $B^2(\mathfrak{g}, \mathfrak{g})$.*

2.3. Deformations of Lie Algebras. In this section we consider Lie algebras defined over the fields $k = \mathbb{R}$ and $\mathbb{C}$. A *deformation* in $\mathcal{L}_n(k)$ is a continuous curve $c: [0, \varepsilon] \to \mathcal{L}_n(k)$, where $\varepsilon > 0$ (of course, the closed interval $[0, \varepsilon]$ may be replaced by any other closed interval $[a, b]$). If this curve is (piecewise) smooth, analytic, etc., then the deformation is said to be (piecewise) smooth, analytic, etc., respectively. For $t \in [0, \varepsilon]$ denote by $\mathfrak{g}(t)$ the Lie algebra corresponding to the structural tensor $c(t) \in \mathcal{L}_n(k)$.

A deformation c is said to be *trivial* if all $c(t)$, $t \in [0, \varepsilon]$, belong to the same orbit of the group $\mathrm{GL}_n(k)$, i.e. if all the Lie algebras $\mathfrak{g}(t)$ are isomorphic.

Two continuous deformations c and c' are said to be *locally equivalent* if there exists a continuous mapping $g: [0, \varepsilon^*] \to \mathrm{GL}_n(k)$, where $0 < \varepsilon^* \leq \min(\varepsilon, \varepsilon')$, such that $c'(t) = g(t) * c(t)$ for all $t \in [0, \varepsilon^*]$. Clearly, if a deformation C is trivial, then it is locally equivalent to the identity, i.e. the deformation $c_0(t)$ such that $c_0(t) = c_0(0)$ for all $t \in [0, \varepsilon_0]$. The notions of local equivalence for (piecewise) smooth, analytic, etc. deformations are defined in a similar manner.

An example of a deformation is the line $(tE)*\{C_{ij}^k\} = \{tC_{ij}^k\}$ (see Sect. 2.2). For $t \neq 0$ all the corresponding Lie algebras $\mathfrak{g}(t)$ are isomorphic, while for $t = 0$ we obtain the abelian Lie algebra k^n. Now let us consider more complicated examples.

Example 4. Let $\mathfrak{r} = \mathbb{R} \Subset {}_{\varphi}\mathbb{R}^2$ be the semidirect sum corresponding to the homomorphism $\varphi: \mathbb{R} \to \mathfrak{gl}_2(\mathbb{R})$ such that $\varphi(1) = \operatorname{diag}(\lambda, \mu)$, where $\lambda, \mu > 0$. In the Lie algebras of that form we will always choose a basis $\{e_i\}$ in the standard way, i.e. as the union of the standard bases in $\mathbb{R}^k$, $k = 1, 2$.

Let $\varphi_t(1) = \operatorname{diag}((1+t)\lambda, (1+t)\mu)$, $\mathfrak{r}(t) = \mathbb{R} \ltimes_{\varphi_t} \mathbb{R}^2$. It is clear that $\mathfrak{r}(0) \simeq \mathfrak{r}$ and for all $t \neq -1$ the Lie algebras $\mathfrak{r}(t)$ are isomorphic. By choosing standard bases in $\mathfrak{r}(t)$, we obtain the deformation $c\colon [0,1] \to \mathcal{L}_3(\mathbb{R})$, which is a trivial one.

Now let $\mathfrak{r}'(t) = \mathbb{R} \ltimes_{\varphi_t} \mathbb{R}^2$, where $\varphi_t(1) = \operatorname{diag}((1+t)\lambda, (1+t+\alpha t^2)\mu)$. Then if $\lambda, \mu \neq 0$ and $\alpha > 0$, we can easily check that all the Lie algebras $\mathfrak{r}(t)$ are pairwise nonisomorphic. As a result, we obtain an analytic deformation in $\mathcal{L}_n(k)$, which is nontrivial on any closed interval $[a,b]$, where $0 < a < b$.

Denote by $\mathcal{N}_n(k)$ and $\mathcal{R}_n(k)$ the subschemes in $\mathcal{L}_n(k)$ consisting of structural tensors of nilpotent and solvable Lie algebras respectively. Clearly, $\mathcal{N}_n(k)$, $\mathcal{R}_n(k)$ are algebraic subvarieties in $\mathcal{L}_n(k)$ invariant under the action of the group $\mathrm{GL}_n(k)$. The tangent Zariski space for points in $\mathcal{R}_n(k)$ and $\mathcal{N}_n(k)$ is described in Rauch [1972] (see also Vergne [1970]).

Example 5. Consider the following deformation in $\mathcal{N}_6(\mathbb{R})$ (which is, of course, also a deformation in $\mathcal{L}_6(\mathbb{R})$):

$$C_{14}^6 = C_{24}^5 = C_{12}^3 = C_{13}^5 = 1,$$

$$C_{23}^6 = t,$$

and the remaining $C_{ij}^k = 0$ for $1 \leq i < j < 6$, $t \in]0,1]$. It follows from Morozov [1958] that some of the corresponding Lie algebras $\mathfrak{g}(t)$ are nonisomorphic over $\mathbb{Q}$, but this deformation in $\mathcal{N}_6(\mathbb{R})$ is trivial.

If $c\colon [0,\varepsilon] \to \mathcal{L}_n(k)$ is an analytic deformation, then we have the convergent series

$$c(t) = F_0 + F_1 \cdot t + \ldots + F_m t^m + \ldots,$$

where $F_m = \{F_m\}_{ij}^k = \frac{c^{(m)}(0)}{m!} \in k^N$, $N = n^3$. It is sometimes convenient to consider (instead of analytic) *formal deformations*, i.e. formal series $c(t) = \sum_{p=0}^{\infty} F_p \cdot t^p$, where $F_p = \{F_{p,i,j}^k\}$ are tensors of type $(2,1)$, such that the components $C_{ij}^k(t)$ of the tensor $C(t)$ (formally) satisfy conditions $(*)$. The notions of triviality and equivalence for formal deformations are introduced in the natural manner. In particular, two formal deformations c and c' are said to be formally equivalent if there exists a formal series $g(t) = \sum_{p=0}^{\infty} G_p t^p$ (where $G_p \in \mathfrak{gl}_n(k)$ and the matrix G_0 is nondegenerate, which ensures that the series $g(t)$ has an inverse) such that $C_{ij}'^k(t) = g_i^p(t) \cdot g_j^q(t) \cdot \overset{(-1)}{g_r^k}(t) \cdot C_{pq}^r(t)$ (here $\overset{(-1)}{g_r^k}$ are terms of the formal series $g(t)^{-1}$).

Any analytic deformation generates a formal one, no any formal deformation is even equivalent to an analytic one.

Let $c(t)$ be a formal deformation, and consider the corresponding Lie algebra with the commutation law

$$\Phi_t(a,b) = [a,b]_t = F_0(a,b) + F_1(a,b) \cdot t + \ldots,$$

where $\Phi_0(a,b) = [a,b]$ is the commutator in the Lie algebra $\mathfrak{g} = \mathfrak{g}(0)$, $a, b \in k^n$. The Jacobi identity for $[\,,\,]_t$ takes the form

$$0 = \{[[a,b]_t, c]_t\} = \{F_0([a,b],c)\} + \{F_1([a,b],c)\}t + \{[F_1(a,b),c]\} \cdot t + \dots,$$

where $\{\ \}$ denotes the sum of all terms obtained by cyclic permutation of the arguments. As a result, the Jacobi identity generates a sequence of relations between the tensors F_p. The first nontrivial relation among them is of the form

$$\begin{aligned} &F_1([a,b],c) + F_1([b,c],a) + F_1([c,a],b) \\ &+ [F_1(a,b),c] + [F_1(b,c),a] + [F_1(c,a),b] = 0 \end{aligned}$$

This condition means that F_1 is a 2-cocycle of the Lie algebra $\mathfrak{g}_0 = \mathfrak{g}(0)$ acting by the adjoint representation on itself, i.e. $F_1 \in Z^2(\mathfrak{g},\mathfrak{g})$. Thus a formal deformation $c(t)$ corresponds to an element from $Z^2(\mathfrak{g},\mathfrak{g})$.

Similarly, if $c(t)$ is a smooth deformation, we set $F_1 = \frac{dc}{dt}\big|_{t=0}$. It is easy to verify that $F_1 \in Z^2(\mathfrak{g},\mathfrak{g})$. Therefore the vectors tangent to the curves in $\mathcal{L}_n(k)$ ($k = \mathbb{R}, \mathbb{C}$) lie in the space $Z^2(\mathfrak{g},\mathfrak{g})$, which by Theorem 2.1 is identified with the tangent space to the scheme $\mathcal{L}_n(k)$ at the point $\{C_{ij}^k(0)\}$ corresponding to the Lie algebra $\mathfrak{g}_0$.

In the general case some elements of $Z^2(\mathfrak{g},\mathfrak{g})$ do not correspond to smooth deformations in $\mathcal{L}_n(k)$, since $\mathcal{L}_n(k)$ has (for example, if $n \geq 13$, see Sect. 2.4) singular points for which there are vectors in the tangent Zariski space that are not tangent to $\mathcal{L}_n(k)$ in the sense of differential geometry.

The elements of $Z^2(\mathfrak{g},\mathfrak{g})$ are called *infinitesimal deformations*, and those corresponding to certain deformations (smooth, analytic, or formal) are said to be *integrable*. The existence of a nonintegrable infinitesimal deformation is demonstrated in Example 2.6 (see Sect. 2.4).

The condition that the tensor F_1 is a cocycle is only the first nontrivial condition imposed by the Jacobi identity on $\{F_p\}$. The subsequent conditions may be interpreted as the requirement that the consecutive Massey powers of F_1 are cocycles (see Feigin and Fuks [1988]). For analytic integrability there is an additional requirement, namely that the series corresponding to the formal deformation must converge.

Theorem 2.2 (see Rauch [1972]). *Let $\mathfrak{g}$ be an n-dimensional Lie algebra over the field $k = \mathbb{R}$ or $\mathbb{C}$. If $H^3(\mathfrak{g},\mathfrak{g}) = 0$, then the point $\{C_{ij}^k\} \in \mathcal{L}_n(k)$ corresponding to the Lie algebra $\mathfrak{g}$ is nonsingular in $\mathcal{L}_n(k)$, and any infinitesimal deformation of it is analytically integrable.*

Unfortunately, the class of Lie algebras to which Theorem 2.2 is applicable is rather narrow.

The following theorem demonstrates that in the analysis of deformations it is not the cocycle F_1 that is important but the corresponding cohomology class in $H^2(\mathfrak{g},\mathfrak{g})$.

Theorem 2.3 (see Feigin and Fuks [1988]). *Let c, c' be two equivalent formal (or locally equivalent smooth) deformations of the point $\{C_{ij}^k\} \in \mathcal{L}_n(k)$*

(where $k = \mathbb{R}$ or $\mathbb{C}$). Then the cocycles in $Z^2(\mathfrak{g}, \mathfrak{g})$ corresponding to these deformations are cohomological.

Proof. Let c, c' be locally equivalent smooth deformations whose equivalence is given by the smooth mapping $g: [0, \varepsilon^*] \to \mathrm{GL}_n(k)$. A direct computation shows that

$$F_1'(a,b) = F_1(a,b) + [a, G(b)] + [G(a), b] - G([a,b]),$$

where $G = \left.\frac{dg}{dt}\right|_{t=0} \in \mathfrak{gl}_n(k)$, $a, b \in \mathfrak{g}_0$. The resulting equality means that F_1 and F_1' are cohomological. □

Besides deformations of Lie algebras, one can also consider deformations of their subalgebras, as well as deformations of homomorphisms of Lie algebras. They are described on similar lines and closely related to the study of the corresponding cohomology space of Lie algebras (see Nijenhuis [1968]).

In other words, deformations are one-parameter families of Lie algebras (or, to be more precise, structural tensors). One can also consider multiparameter families whose analysis reduces, to a large extent, to that of deformations. We also note that the important notion of a versal family of Lie algebras naturally arises and is actively used in the analysis of multiparameter families of Lie algebras.

2.4. Rigid Lie Algebras. A Lie algebra $\mathfrak{g}$ is said to be *rigid* if any analytic (occasionally one also considers any smooth) deformation of the corresponding point in $\mathcal{L}_n(k)$ is locally trivial ($k = \mathbb{R}$ or $\mathbb{C}$).

A Lie algebra is said to be formally rigid if any formal deformation of the corresponding point in $\mathcal{L}_n(k)$ is trivial (i.e. formally equivalent to the identity).

The rigidity of a Lie algebra $\mathfrak{g}$ means that all the Lie algebras (more precisely, their structural tensors) close to $\mathfrak{g}$ are isomorphic to it, since any Lie algebra that is close to it can be obtained from $\mathfrak{g}$ by an analytic deformation.

Rigid algebras are also characterized by the fact that for them the corresponding orbit $O_{\mathfrak{g}}$ is open in $\mathcal{L}_n(k)$ in the Zariski topology (this is equivalent to the fact that it is open in the Euclidean topology). If $\mathfrak{g}$ is a Lie algebra over an arbitrary field, then it is said to be rigid if the orbit $O_{\mathfrak{g}}$ is Zariski open in $\mathcal{L}_n(k)$. Theorem 2.1 implies the following statement.

Theorem 2.4 (see Richardson [1967]). *Let $\mathfrak{g}$ be an n-dimensional Lie algebra over an algebraically closed field k of characteristic 0 or over the field $k = \mathbb{R}$. If $H^2(\mathfrak{g}, \mathfrak{g}) = 0$, then $\mathfrak{g}$ is a rigid Lie algebra.*

Conversely, if $\mathfrak{g}$ is a rigid Lie algebra and the point $\{C_{ij}^k\}$ corresponding to it in $\mathcal{L}_n(k)$ is nonsingular, then $H^2(\mathfrak{g}, \mathfrak{g}) = 0$.

The condition that the point $\{C_{ij}^k\} \in \mathcal{L}_n(k)$ is nonsingular included in the last statement of Theorem 2.4 is essential, as the following example shows.

Example 6 (see Richardson [1967]). Suppose that the Lie algebra $\mathfrak{g}_n$ is the semidirect sum $\mathfrak{sl}_2(\mathbb{C}) \in_{\varphi_n} \mathbb{C}^n$ corresponding to the irreducible representation $\varphi_n: \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$. Then, if $n \equiv 1 \mod 4$ and $n \geq 13$, the algebra $\mathfrak{g}_n$ is rigid although $H^2(\mathfrak{g}, \mathfrak{g}) \neq \{0\}$.

By Theorem 2.4, the points in $\mathcal{L}_n$ corresponding to such Lie algebras $\mathfrak{g}_n$ are singular ones. In addition, $Z^2(\mathfrak{g}_n, \mathfrak{g}_n) \neq B^2(\mathfrak{g}_n, \mathfrak{g}_n)$ and there exist nonintegrable infinitesimal deformations of the Lie algebra $\mathfrak{g}_n$.

Theorems 2.4 and 2.2 from Chap. 2 imply the following corollary.

Corollary. *An arbitrary semisimple Lie algebra (over the field $\mathbb{R}$ or $\mathbb{C}$) is rigid.*

A more general statement is contained in the following theorem.

Theorem 2.5 (Tolpygo [1972]), Leger and Luks [1972]). *Let $\mathfrak{g}$ be an arbitrary parabolic subalgebra of a semisimple Lie algebra. Then $H^p(\mathfrak{g}, \mathfrak{g}) = 0$ for all $p \geq 0$.*

In particular, any such Lie algebra $\mathfrak{g}$ is rigid.

By virtue of Theorem 2.5, any Borel subalgebra of an arbitrary complex semisimple Lie algebra is rigid and thus we obtain a family of rigid solvable Lie algebras. There is a classification of all rigid complex solvable Lie algebras of dimension ≤ 8 (Goze and Ancochea-Bermudez [1985]).

The structure of arbitrary rigid Lie algebras was studied in Carles [1984]. In particular, any complex rigid Lie algebra is an algebraic (linear) Lie algebra.

Note that the number of rigid Lie algebras (considered up to isomorphism) in $\mathcal{L}_n(k)$ is finite since for them the closure of the orbit $O_{\mathfrak{g}}$ is an irreducible component and the number of the latter in $\mathcal{L}_n(k)$ is finite.

Theorem 2.6 (see Goze and Ancochea-Bermudez [1985]).

(i) *The number of pairwise nonisomorphic rigid Lie algebras in $\mathcal{L}(\mathbb{C})$ is not less than $\exp\left(\frac{\alpha n}{\ln n}\right)$, where $\alpha = \frac{(\ln 2)^2}{2} \approx 0.24$ (for sufficiently large n).*

(ii) *The number of pairwise nonisomorphic solvable Lie algebras in $\mathcal{L}_n(k)$ for $n \geq 81$ is not less than $\Gamma(\sqrt{n})$, where $\Gamma(x)$ is Euler's Γ-function.*

The statements of Theorem 2.6 yield, in particular, lower bounds on the number of irreducible components in the spaces $\mathcal{L}_n(\mathbb{C})$ and, by virtue of (ii), in $\mathcal{R}_n(\mathbb{C})$.

2.5. Contractions of Lie Algebras. The operation of contraction of a Lie algebra is, in a sense, opposite to that of deformation.

Let $C = \{c_{ij}^k\} \in \mathcal{L}_n(k)$ be a structural tensor ($k = \mathbb{R}$ or $\mathbb{C}$). Consider a mapping (continuous, smooth, analytic, etc.) $g: [0, \varepsilon] \to \mathrm{GL}_n(k)$ and set

$$C_{ij}^k(t) = (g * c)_{ij}^k = g_i^p \cdot g_j^q \cdot \overset{(-1)}{g_k^r} \cdot C_{pq}^r.$$

Suppose that for all $1 \leq i, j, k \leq n$ there exist the limits $\lim_{t \to +0} C_{ij}^k(t)$, which we will denote by $C_{ij}^k(0)$. The subspace $\mathcal{L}_n(k)$ is closed in k^N, $N = n^3$, and therefore the point $\{C_{ij}^k(0)\}$ belongs to $\mathcal{L}_n(k)$. Moreover, this point lies in the closure (relative to the Euclidean topology) $\overline{O}_{\mathfrak{g}}$ of the orbit $O_{\mathfrak{g}}$ of the corresponding Lie algebra $\mathfrak{g}$. The limit point $\{C_{ij}^k(0)\}$ and the corresponding Lie algebra $\mathfrak{g}(0)$ are said to be *contractions* of the point $C = \{C_{ij}^k\}$ and of the Lie algebra $\mathfrak{g}$ corresponding to it.

Any Lie algebra can be contracted into an abelian one (see Sect. 2.2 above). Consider other examples of contractions of Lie algebras.

Example 7. IW-contractions (Inönü-Wigner contractions, see Lykhmus [1969]).

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subalgebra of it. Consider a subspace $\mathfrak{p} \subset \mathfrak{g}$ complementary to $\mathfrak{h}$ and denote by $p_{\mathfrak{h}}$ and $p_{\mathfrak{p}}$ the projections onto $\mathfrak{h}$ and $\mathfrak{p}$ parallel to $\mathfrak{p}$ and $\mathfrak{h}$ respectively. Set

$$g(t) = p_{\mathfrak{p}} + t \cdot p_{\mathfrak{h}}.$$

Clearly, $g(t) \in \mathrm{GL}_n(k)$ for $t \neq 0$. A direct calculation shows that there exists the limit $\{C_{ij}^k(0)\}$ and that in the corresponding Lie algebra $\mathfrak{g}(0)$ the subspace $\mathfrak{p}$ is an ideal, i.e. $\mathfrak{g}(0) = \mathfrak{h} + \mathfrak{p}$ is the semidirect sum. The action of $\mathfrak{h}$ on $\mathfrak{p}$ is induced by the adjoint action of $\mathfrak{h}$ on the quotient space $\mathfrak{g}/\mathfrak{h}$.

Consider some special cases.

(a) $\mathfrak{g} = \mathfrak{so}_3 \supset \mathfrak{h} = \mathfrak{so}_2$. Here the IW-contraction yields the Lie algebra $\mathfrak{g}(0) \simeq \mathfrak{e}(2)$, which is the tangent algebra of the Lie group $E(2)$ of motions of the Euclidean plane.

(b) $\mathfrak{g} = \mathfrak{so}_{1,3} \supset \mathfrak{h} = \mathfrak{so}_3$. Here the IW-contraction yields the Galileo algebra $\mathfrak{so}_3 \Subset \mathbb{R}^3$, which is the tangent algebra of the group $E(3)$ of motions of the Euclidean space E^3. This contraction corresponds to the limit transition from the special theory of relativity to the Galilean mechanics (as the velocity of light $c \to \infty$).

(c) Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of an arbitrary real Lie algebra $\mathfrak{g}$ (see Chap. 3, Sect. 7.1). Taking for $\mathfrak{h}$ a maximal compact subalgebra $\mathfrak{k}$, we obtain an IW-contraction of the Lie algebra $\mathfrak{g}$, which turns it into a Lie algebra of the form $\mathfrak{k} \Subset {}_{\varphi}V$, where V is an abelian ideal, and the action φ of the subalgebra k on V is induced by the adjoint representation. Contractions of this kind admit a geometric interpretation (see Rosenfel'd [1969]). Note that for $\mathfrak{g}_{1,3}$ we obtain the case (b).

(d) The following diagram shows all possible IW-contractions of three-dimensional Lie algebras (see Conaster [1972]); only the contractions of all Lie algebras (except $\mathfrak{n}_3$) in $\mathbb{R}^3$ (corresponding to the subalgebra $\mathfrak{h} = \{0\}$) are not listed:

$$\begin{tikzcd}
 & & & \mathfrak{sl}_2(\mathbb{R}) \arrow[d] \arrow[dr] & \mathfrak{su}_2 \arrow[d] & \\
\mathfrak{r}_2 \oplus \mathbb{R} \arrow[drr] & \mathfrak{r}_{3,1} \arrow[dr] & \mathfrak{r}_{3,\lambda}\ \lambda \neq \pm 1 \arrow[d] & \mathfrak{r}_{3,-1} \arrow[dl] & \mathfrak{e}(2) & \mathfrak{r}'_{3,\lambda} \arrow[dlll] \\
 & & \mathfrak{n}_3 \arrow[d] & & & \\
 & & \mathbb{R}^3 & & &
\end{tikzcd}$$

Let us describe the Lie algebras appearing in the diagram. The Lie algebra $\mathfrak{r}_{3,\lambda}$ is of the form $\mathbb{R} +_{\varphi} \mathbb{R}^2$, where $\varphi(1) = \operatorname{diag}(1, \lambda)$. The Lie algebra $\mathfrak{r}'_{3,\lambda}$ is of the same form $\mathbb{R} +_{\psi} \mathbb{R}^2$ but here $\psi(1) = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}$. The notation for the rest of the algebras in the diagram is standard.

One can consider contractions of a more general type than the IW ones. For example, the Saletan contractions are of the form $g(t) = A + t \cdot B$, where $A, B \in \mathfrak{gl}_n(\mathbb{R})$ and the operator $A + B$ is nonsingular (see Saletan [1961]). However, no simple class of contractions has yet been found that might enable one to obtain an arbitrary point in the closure $\overline{O}_{\mathfrak{g}}$ of the orbit $O_{\mathfrak{g}} \subset \mathcal{L}_n(k)$.

Contractions first appeared from physical considerations; besides the ones mentioned above they have many other physical interpretations (see, e.g., Barut and Rączka [1977]). Contractions also arise naturally in the study of the Lie algebras of symmetries of differential equations (they correspond to the "exceptional" values of the parameters involved in these equations, see Ovsyannikov [1978]). Other geometrical interpretations of contractions are also known (see, e.g., Rosenfel'd and Karpova [1966]).

Besides contractions of Lie algebras, one can also consider contractions of Lie groups, as well as homomorphisms, representations etc.

2.6. Spaces $\mathcal{L}_n(k)$ for Small n. The spaces $\mathcal{L}_n(k)$ for $n = 1, 2$ were described in Sect. 1. At the time of writing the description of irreducible components of the spaces $\mathcal{L}_n$ for $n \leq 7$ is known (see Kirillov and Neretin [1984], Carles and Diakité [1984]). In what follows we give the results for the cases $n = 3, 4, 5$.

We begin with the following construction. Let $\mathfrak{n}$ be a nilpotent Lie algebra. Denote by $\mathcal{R}_n(\mathfrak{n}, k)$ the subset of $\mathcal{R}_n(k)$ consisting of solvable Lie algebras $\mathfrak{g}$ whose nilradical is isomorphic to $\mathfrak{n}$. Denote by $\mathcal{R}_n^0(\mathfrak{n}, k)$ the subset of $\mathcal{R}_n(\mathfrak{n}, k)$ consisting of Lie algebras admitting a decomposition into the semidirect sum $\mathfrak{g} = \mathfrak{a} \in \mathfrak{n}$, where $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ (which is evidently abelian). It turns out that for $n \leq 7$ the closure ${}^a\mathcal{R}_n^0(\mathfrak{n}, k)$ of the subset $\mathcal{R}_n^0(\mathfrak{n}, k)$ coincides with $\mathcal{R}_n(\mathfrak{n}, k)$. Moreover, for $n \leq 7$ the spaces ${}^a\mathcal{R}_n^0(\mathfrak{n}, k)$ are irreducible components of the space $\mathcal{L}_n(k)$ as well.

The following is the list of all irreducible components in $\mathcal{L}_n(k)$ for $n = 3, 4, 5$ (the dimensions are given in parenthesis), Carles and Diakité [1984].

$\mathcal{L}_3(\mathbb{C})$: ${}^a\mathcal{R}_3^0(\mathbb{C}^2, \mathbb{C})$ (6), ${}^aO_{\mathfrak{sl}_2(\mathbb{C})}$ (6);

$\mathcal{L}_4(\mathbb{C})$: ${}^a\mathcal{R}_4^0(\mathbb{C}^2,\mathbb{C})$ (12), ${}^a\mathcal{R}_4^0(\mathbb{C}^3,\mathbb{C})$ (12), ${}^a\mathcal{R}_4^0(\mathfrak{n}_3(\mathbb{C}),\mathbb{C})$ (12), ${}^aO_{\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}}$ (12);

$\mathcal{L}_5(\mathbb{C})$: ${}^a\mathcal{R}_5^0(\mathbb{C}^3,\mathbb{C})$ (21), ${}^a\mathcal{R}_5^0(\mathfrak{n}_3(\mathbb{C}),\mathbb{C})$ (20), ${}^a\mathcal{R}_5^0(\mathbb{C}^4,\mathbb{C})$ (20), ${}^a\mathcal{R}_5^0(\mathfrak{n}_4(\mathbb{C})\oplus\mathbb{C},\mathbb{C})$ (20), ${}^a\mathcal{R}_5^0(\mathfrak{n}_4(\mathbb{C}),\mathbb{C})$ (20), ${}^aO_{\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{r}_2(\mathbb{C})}$ (20), ${}^aO_{\mathfrak{sl}_2(\mathbb{C})\in\mathbb{C}^2}$ (19).

Here the following notation is used: $\mathfrak{n}_4(\mathbb{C})$ is the nilpotent Lie algebra described in Sect. 10.1, and $\mathfrak{sl}_2(\mathbb{C}) \in \mathbb{C}^2$ is the semidirect sum corresponding to the natural action of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}^2$.

The next table provides information on the number of irreducible components in $\mathcal{L}_n(\mathbb{C})$ and $\mathcal{R}_n(\mathbb{C})$ and on the number of rigid Lie algebras (or, which is the same, the number of open orbits in $\mathcal{L}_n(\mathbb{C})$ and $\mathcal{R}_n(\mathbb{C})$) for $n \le 7$ (see Goze and Ancochea-Bermudez [1985]).

	1	2	3	4	5	6	7
Number of components in $\mathcal{R}_n(\mathbb{C})$	1	1	1	3	5	13	40
Number of rigid Lie algebras in $\mathcal{R}_n(\mathbb{C})$	1	1	0	1	1	3	8
Number of components in $\mathcal{L}_n(\mathbb{C})$	1	1	2	4	7	17	49
Number of rigid Lie algebras in $\mathcal{L}_n(\mathbb{C})$	1	1	1	2	3	6	14

The space $\mathcal{N}_n(k)$ consisting of structural tensors of nilpotent Lie algebras is also of interest. Some general results on the space $\mathcal{N}_n(\mathbb{C})$ can be found in Vergne [1970]. For more details on the spaces $\mathcal{N}_n(\mathbb{C})$ for $n \le 7$ see Magnin [1986], Carles [1989]. Hakimjanov [1989] studied deformations of nilradicals of parabolic subalgebras of semisimple Lie algebras. This provides information on some components of the spaces $\mathcal{N}_n(\mathbb{C})$.

Tables

Table 1. The weights of the groups B_l, C_l, D_l, and F_4 are expressed in the table in terms of an orthonormal basis $(\varepsilon_1, \ldots, \varepsilon_l)$ of the space $\mathfrak{h}(\mathbb{R})^*$. The weights of the groups A_l, E_7, E_8, and G_2 are expressed in terms of the vectors $\varepsilon_1, \ldots, \varepsilon_{l+1}$ satisfying the condition $\sum \varepsilon_i = 0$. For these vectors

$$(\varepsilon_i, \varepsilon_i) = \frac{l}{l+1}, \quad (\varepsilon_i, \varepsilon_j) = -\frac{1}{l+1} \quad \text{when} \quad i \neq j.$$

The weights of the group E_6 are expressed in terms of the vectors $\varepsilon_1, \ldots, \varepsilon_6 \in f(\mathbb{R})^*$ constructed as for A_5 and of the vector $\varepsilon \in \mathfrak{h}(\mathbb{R})^*$ which is orthogonal to them and satisfies the condition $(\varepsilon, \varepsilon) = \frac{1}{2}$.

The indices $i, j, \ldots$ in the expression of any weight are assumed to be different.

In the column "Dynkin diagram" the numbering of simple roots adopted in all tables is given.

The column "Simple roots" also gives the highest root δ, and the column "Fundamental weights" also indicates their sum γ (equal to half the sum of the positive roots).

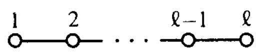
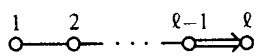
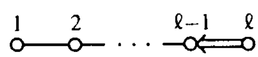
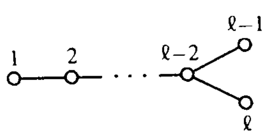
type of G	Dynkin diagrams	dim G	Roots and simple roots
A_l ($l \geqslant 1$)		$l^2 + 2l$	$\varepsilon_i - \varepsilon_j$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $\delta = \varepsilon_1 - \varepsilon_{l+1} = \pi_1 + \pi_l$
B_l ($l \geqslant 2$)		$2l^2 + l$	$\pm\varepsilon_i \pm \varepsilon_j$, $\pm\varepsilon_i$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i < l)$, $\alpha_l = \varepsilon_l$, $\delta = \varepsilon_1 + \varepsilon_2 = \pi_2$
C_l ($l \geqslant 2$)		$2l^2 + l$	$\pm\varepsilon_i \pm \varepsilon_j$, $\pm 2\varepsilon_i$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i < l)$, $\alpha_l = 2\varepsilon_l$, $\delta = 2\varepsilon_1 = 2\pi_1$
D_l ($l \geqslant 3$)		$2l^2 - l$	$\pm\varepsilon_i \pm \varepsilon_j$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i < l)$, $\alpha_l = \varepsilon_{l-1} + \varepsilon_l$, $\delta = \varepsilon_1 + \varepsilon_2 = \begin{cases} \pi_2 \text{ for } l \geqslant 4, \\ \pi_2 + \pi_3 \text{ for } l = 3 \end{cases}$

Table 1 (cont.)

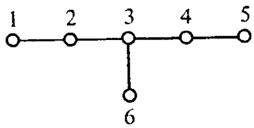
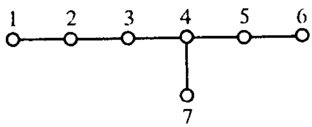
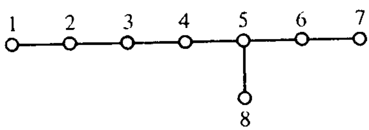
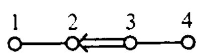
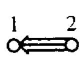
type of G	Dynkin diagrams	dim G	Roots and simple roots
E_6		78	$\varepsilon_i - \varepsilon_j,\ \pm 2\varepsilon,$ $\varepsilon_i + \varepsilon_j + \varepsilon_k \pm \varepsilon$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}\ (i < 6),$ $\alpha_6 = \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon,$ $\delta = 2\varepsilon = \pi_6$
E_7		133	$\varepsilon_i - \varepsilon_j,$ $\varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}\ (i < 7),$ $\alpha_7 = \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8,$ $\delta = -\varepsilon_7 + \varepsilon_8 = \pi_6$
E_8		248	$\varepsilon_i - \varepsilon_j,\ \pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)$ $\alpha_i = \varepsilon_i - \varepsilon_{i+1}\ (i < 8),$ $\alpha_8 = \varepsilon_6 + \varepsilon_7 + \varepsilon_8,$ $\delta = \varepsilon_1 - \varepsilon_9 = \pi_1$
F_4		52	$\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i$ $(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)/2$ $\alpha_1 = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2,$ $\alpha_2 = \varepsilon_4,$ $\alpha_3 = \varepsilon_3 - \varepsilon_4,$ $\alpha_4 = \varepsilon_2 - \varepsilon_3,$ $\delta = \varepsilon_1 + \varepsilon_2 = \pi_4$
G_2		14	$\varepsilon_i - \varepsilon_j,\ \pm\varepsilon_i$ $\alpha_1 = -\varepsilon_2,$ $\alpha_2 = \varepsilon_2 - \varepsilon_3$ $\delta = \varepsilon_1 - \varepsilon_3 = \pi_2$

Table 1 (cont.)

Type of G	Fundamental weights	dim $R(\pi_1)$	Weights of $R(\pi_1)$
A_l ($l \geqslant 1$)	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i$, $\gamma = l\varepsilon_1 + (l-1)\varepsilon_2 + \cdots + \varepsilon_l$	$l+1$	ε_i
B_l ($l \geqslant 2$)	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (i < l)$, $\pi_l = (\varepsilon_1 + \cdots + \varepsilon_l)/2$ $\gamma = [(2l-1)\varepsilon_1 + (2l-3)\varepsilon_2 + \cdots + \varepsilon_l]/2$	$2l+1$	$\pm\varepsilon_i, 0$
C_l ($l \geqslant 2$)	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i$, $\gamma = l\varepsilon_1 + (l-1)\varepsilon_2 + \cdots + \varepsilon_l$	$2l$	$\pm\varepsilon_i$
D_l ($l \geqslant 3$)	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (i < l-1)$, $\pi_{l-1} = (\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l)/2$ $\pi_l = (\varepsilon_1 + \cdots + \varepsilon_{l-1} + \varepsilon_l)/2$ $\gamma = (l-1)\varepsilon_1 + (l-2)\varepsilon_2 + \cdots + \varepsilon_{l-1}$	$2l$	$\pm\varepsilon_i$
E_6	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i + \min\{i, 6-i\}\cdot\varepsilon \quad (i < 6)$, $\pi_6 = 2\varepsilon$, $\gamma = 5\varepsilon_1 + 4\varepsilon_2 + \cdots + \varepsilon_5 + 11\varepsilon$	27	$\varepsilon_i \pm \varepsilon$, $-\varepsilon_i - \varepsilon_j$
E_7	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i + \min\{i, 8-i\}\cdot\varepsilon_8 \quad (i < 7)$, $\pi_7 = 2\varepsilon_8$, $\gamma = 6\varepsilon_1 + 5\varepsilon_2 + \cdots + \varepsilon_6 + 17\varepsilon_8$	56	$\pm(\varepsilon_i + \varepsilon_j)$
E_8	$\pi_i = \varepsilon_1 + \cdots + \varepsilon_i - \min\{i, 15-2i\}\cdot\varepsilon_9 \quad (i < 8)$, $\pi_8 = -3\varepsilon_9$, $\gamma = 7\varepsilon_1 + 6\varepsilon_2 + \cdots + \varepsilon_7 - 22\varepsilon_9$	248	$\varepsilon_i - \varepsilon_j$, $\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)$, 0 (of multiplicity 8)
F_4	$\pi_1 = \varepsilon_1$, $\pi_2 = (3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2$ $\pi_3 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$, $\pi_4 = \varepsilon_1 + \varepsilon_2$, $\gamma = (11\varepsilon_1 + 5\varepsilon_2 + 3\varepsilon_3 + \varepsilon_4)/2$	26	$\pm\varepsilon_i$, $(\pm\varepsilon_1 \pm \varepsilon_2$ $\pm \varepsilon_3 \pm \varepsilon_4)/2$ 0 (of multiplicity 2)
G_2	$\pi_1 = \varepsilon_1$, $\pi_2 = \varepsilon_1 - \varepsilon_3$, $\gamma = 2\varepsilon_1 - \varepsilon_3$	7	$\pm\varepsilon_i, 0$

Table 2. The table lists the centres and groups of outer automorphisms of simply-connected simple complex Lie groups. The fifth column contains the order of the canonical involution ν of the system of simple roots. The last column indicates the element $b \in P^\nu$ determining the element $z_0 \in Z(G)$ which defines whether the bilinear invariant of the self-adjoint irreducible representation is symmetric or skew-symmetric (see Chap. 3, Sect. 2.7).

For the groups E_8, F_4, and G_2 not mentioned in the table the centres and the groups of outer automorphisms are trivial and any linear representation is orthogonal.

Type of G	$Z(G) \simeq P^\vee/Q^\vee$	The generators of $P^\vee/Q^\vee$	$\operatorname{Aut} G/\operatorname{Int} G$	$\lvert\nu\rvert$	b
A_l $(l > 1)$	$\mathbb{Z}_{l+1}$	$(h_1 + 2h_2 + \cdots + lh_l)/(l+1)$	$\mathbb{Z}_2$	2	$(h_1 + h_3 + \cdots + h_l)/2$ for $l = 2q+1$, 0 otherwise
A_1	$\mathbb{Z}_2$	$h_1/2$	$\{e\}$	1	$h_1/2$
B_l	$\mathbb{Z}_2$	$h_l/2$	$\{e\}$	1	$h_l/2$ for $l = 4q+1, 4q+2$, 0 otherwise
C_l	$\mathbb{Z}_2$	$(h_1 + h_3 + h_5 + \cdots)/2$	$\{e\}$	1	$(h_1 + h_3 + h_5 + \cdots)/2$
D_l $(l$ odd$)$	$\mathbb{Z}_4$	$(h_1 + h_3 + \cdots + h_{l-2})/2 + (h_{l-1} - h_l)/4$	$\mathbb{Z}_2$	2	$(h_{l-1} + h_l)/2$ for $l = 4q+3$, 0 otherwise
D_l $(l$ even$)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(h_1 + h_3 + \cdots + h_{l-1})/2$, $(h_{l-1} + h_l)/2$	$\mathbb{Z}_2$ for $l > 4$, S_3 for $l = 4$	1	$(h_{l-1} + h_l)/2$ for $l = 4q+2$, 0 otherwise
E_6	$\mathbb{Z}_3$	$(h_1 - h_2 + h_4 - h_5)/3$	$\mathbb{Z}_2$	2	0
E_7	$\mathbb{Z}_2$	$(h_1 + h_3 + h_7)/2$	$\{e\}$	1	$(h_1 + h_3 + h_7)/2$

Table 3. The table lists Dynkin diagrams of linearly dependent admissible systems of vectors. In the notation of Sect. 3.8 and 3.9 of Chap. 3 the diagram of type $L_n^{(q)}$ is none other than the Dynkin diagram of the extended system of simple σ-roots of the simple Lie algebra $\mathfrak{g}$ of type L_n, where σ is the automorphism of order q of the system of simple roots of $\mathfrak{g}$. The numerical labels on the diagrams coincide with the numbers $\frac{n_i}{q}$, where n_i are the coefficients of the linear relation (32) of Chap. 3.

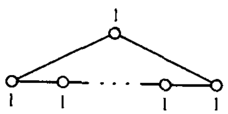
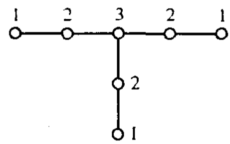
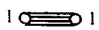
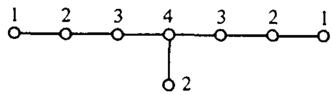
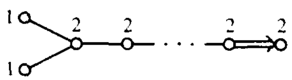
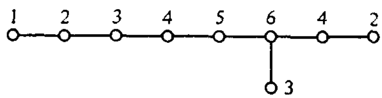
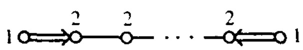
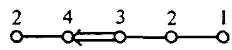
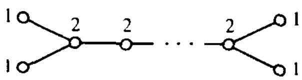
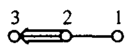
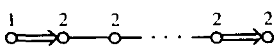
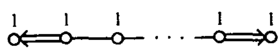
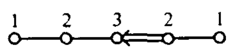
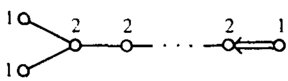
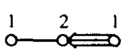
Type	Affine diagram	Type	Affine diagram
$A_l^{(1)}$ $(l \geqslant 2)$		$E_6^{(1)}$	
$A_1^{(1)}$		$E_7^{(1)}$	
$B_l^{(1)}$ $(l \geqslant 3)$		$E_8^{(1)}$	
$C_l^{(1)}$ $(l \geqslant 2)$		$F_4^{(1)}$	
$D_l^{(1)}$ $(l \geqslant 4)$		$G_2^{(1)}$	
$A_{2l}^{(2)}$ $(l \geqslant 2)$		$D_{l+1}^{(2)}$ $(l \geqslant 2)$	
$A_2^{(2)}$		$E_6^{(2)}$	
$A_{2l-1}^{(2)}$ $(l \geqslant 3)$			
$D_4^{(3)}$			

Table 4. The table lists noncompact real Lie algebras $\mathfrak{g}$ that do not admit a complex structure, i.e. the real form of complex simple Lie algebras $\mathfrak{g}(\mathbb{C})$. The table also indicates the type of the system Σ of real roots and the restriction map $r\colon \Pi_1 \to \Theta$, where Θ is the system of simple roots of Σ. The simple roots from Π are denoted by α_j, those from Θ by λ_j; the numbering in both these systems is the same as in Table 1.

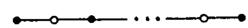
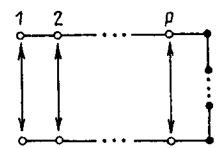
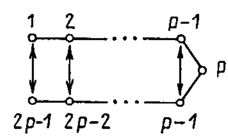
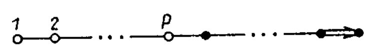
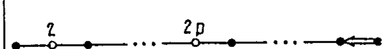
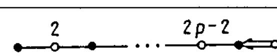
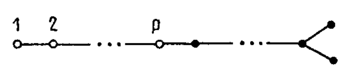
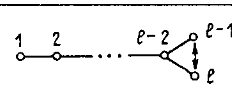
$\mathfrak{g}(\mathbb{C})$	$\mathfrak{g}$	$\mathfrak{k}$	$\dim\mathfrak{k}$	$\dim\mathfrak{p}$	$\mathrm{rk}_{\mathbb{R}}\mathfrak{g}$	Satake diagram
$\mathfrak{sl}_{l+1}(\mathbb{C})$ $(l\geqslant 1)$	$\mathfrak{sl}_{l+1}(\mathbb{R})$	$\mathfrak{so}_{l+1}$	$\frac{1}{2}l(l+1)$	$\frac{1}{2}l(l+3)$	l	
	$\mathfrak{sl}_{l+1}(\mathbb{H})$ $(l=2p+1,$ $p\geqslant 1)$	$\mathfrak{sp}_{p+1}$	$(p+1)$ $\times(2p+3)$	$p(2p+3)$	p	
	$\mathfrak{su}_{p,l+1-p}$ $(1\leq p\leq \frac{l}{2})$	$\mathfrak{su}_p$ $\oplus\mathfrak{u}_{l+1-p}$	p^2+ $(l+1-$ $-p)^2-1$	$2p\times$ $(l+1-$ $-p)$	p	
	$\mathfrak{su}_{p,p}$ $(l=2p-1,$ $p\geqslant 2)$	$\mathfrak{su}_p\oplus\mathfrak{u}_p$	$2p^2-1$	$2p^2$	p	
$\mathfrak{so}_{2l+1}(\mathbb{C})$ $(l\geqslant 1)$	$\mathfrak{so}_{p,2l+1-p}$ $(1\leq p\leq l)$	$\mathfrak{so}_p$ $\oplus\mathfrak{so}_{2l+1-p}$	$p(2p+1)-$ $-(2l+1$ $-p)(4l+$ $+3-2p)$	$p(2l+1-$ $-p)$	p	
$\mathfrak{sp}_{2l}(\mathbb{C})$	$\mathfrak{sp}_{2l}(\mathbb{R})$	$\mathfrak{u}_l$	l^2	$l(l+1)$	l	
	$\mathfrak{sp}_{p,l-p}$ $(1\leq p$ $\leq\frac{1}{2}(l-1))$	$\mathfrak{sp}_p\oplus\mathfrak{sp}_{l-p}$	$p(2p+1)+$ $+(l-p)\times$ $\times(2l-2p$ $+1)$	$4p(l-p)$	p	
	$\mathfrak{sp}_{p,p}$ $(l=2p)$	$\mathfrak{sp}_p\oplus\mathfrak{sp}_p$	$2p(2p+1)$	$4p^2$	p	
$\mathfrak{so}_{2l}(\mathbb{C})$ $(l\geqslant 4)$	$\mathfrak{so}_{p,2l-p}$ $(1\leq p$ $\leq l-2)$	$\mathfrak{so}_p$ $\times\mathfrak{so}_{2l-p}$	$\frac{1}{2}p(p-1)+$ $\frac{1}{2}(2l-p)\times$ $(2l-p+1)$	$p(2l-p)$	p	
	$\mathfrak{so}_{l-1,l+1}$	$\mathfrak{so}_{l-1}$ $\times\mathfrak{so}_{l+1}$	$\frac{1}{2}(l-1)\times$ $(l-2)+$ $\frac{1}{2}l(l+1)$	l^2-1	$l-1$	

Table 4 (cont.)

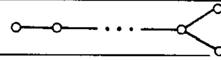
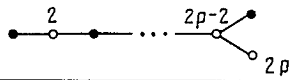
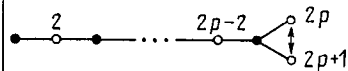
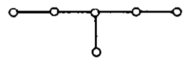
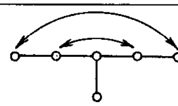
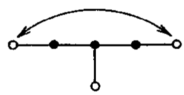
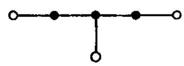
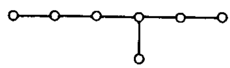
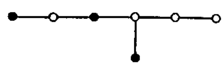
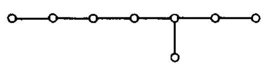
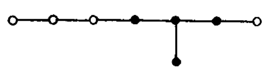
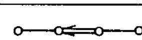
$\mathfrak{g}(\mathbb{C})$	$\mathfrak{g}$	$\mathfrak{k}$	$\dim \mathfrak{k}$	$\dim \mathfrak{p}$	$\mathrm{rk}_{\mathbb{R}}\mathfrak{g}$	Satake diagram
$\mathfrak{so}_{2l}(\mathbb{C})$ $(l \geqslant 4)$	$\mathfrak{so}_{l,l}$	$\mathfrak{so}_l \times \mathfrak{so}_l$	$l(l-1)$	l^2	l	
	$\mathfrak{u}^*_{2p}(\mathbb{H})$ $(l = 2p)$	$\mathfrak{u}_{2p}$	$4p^2$	$2p(2p-1)$	p	
	$\mathfrak{u}^*_{2p+1}(\mathbb{H})$ $(l = 2p+1)$	$\mathfrak{u}_{2p+1}$	$(2p+1)^2$	$2p(2p+1)$	p	
E_6	E I	$\mathfrak{sp}_4$	36	42	6	
	E II	$\mathfrak{su}_2 \oplus \mathfrak{su}_8$	38	40	4	
	E III	$\mathfrak{so}_{10} \oplus \mathbb{R}$	46	32	2	
	E IV	F_4	52	26	2	
E_7	E V	$\mathfrak{su}_8$	63	70	7	
	E VI	$\mathfrak{su}_2 \oplus \mathfrak{so}_{12}$	69	64	4	
	E VII	$E_6 \oplus \mathbb{R}$	79	54	3	
E_8	E VIII	$\mathfrak{so}_{16}$	120	128	8	
	E IX	$\mathfrak{su}_2 \oplus E_7$	136	112	4	
F_4	F I	$\mathfrak{su}_2 \oplus \mathfrak{sp}_3$	24	28	4	
	F II	$\mathfrak{so}_9$	36	16	4	
G_2	G	$\mathfrak{so}_3 \oplus \mathfrak{so}_3$	6	8	2	

Table 4 (cont.)

$\mathfrak{g}(\mathbb{C})$	$\mathfrak{g}$	Σ	r	$\dim \mathfrak{g}_{\lambda_j}$	$\dim \mathfrak{g}_{2\lambda}$
$\mathfrak{sl}_{l+1}(\mathbb{C})$ $(l \geqslant 1)$	$\mathfrak{sl}_{l+1}(\mathbb{R})$	A_l	$r(\alpha_j) = \lambda_j$ $(1 \leq j \leq l)$	1	0
	$\mathfrak{sl}_{p+1}(\mathbb{H})$ $(l = 2p+1, p \geqslant 1)$	A_p	$r(\alpha_{2j}) = \lambda_j$ $(1 \leq j \leq p)$	4	0
	$\mathfrak{su}_{p,l+1-p}$ $(1 \leq p \leq \frac{l}{2}$	BC_p	$r(\alpha_j) =$ $= r(\alpha_{l+1-j})$ $= \lambda_j$ $(1 \leq j \leq p)$	2 $(j \leq p+1)$	0
				$2(l+1-2p)$ $(j = p)$	1
	$\mathfrak{su}_{p,p}$ $(l = 2p-1, p \geqslant 2)$	C_p	$r(\alpha_j) =$ $= r(\alpha_{2p-j})$ $= \lambda_j$ $(1 \leq j \leq p)$	2 $(j \leq p-1)$ 1 $(j = p)$	0
$\mathfrak{so}_{2l+1}(\mathbb{C})$ $(l \geqslant 1)$	$\mathfrak{so}_{p,2l+1-p}$ $(1 \leqslant p \leqslant l$	B_p	$r(\alpha_j) = \lambda_j$ $(1 \leqslant j \leqslant p)$	1 $(j \leq p-1)$ $2(l-p)+1$ $(j = p)$	0
$\mathfrak{sp}_{2l}(\mathbb{C})$	$\mathfrak{sp}_{2l}(\mathbb{R})$	C_l	$r(\alpha_j) = \lambda_j$ $(1 \leqslant j \leqslant l)$	1	0
$\mathfrak{sp}_{2l}(\mathbb{C})$	$\mathfrak{sp}_{p,l-p}$ $(1 \leqslant p \leqslant \frac{1}{2}(l-1))$	BC_p	$r(\alpha_{2j}) = \lambda_j$ $(1 \leqslant j \leqslant p)$	4 $(j \leqslant p-1)$	0
				$4(l-2p)$ $(j = p)$	3
	$\mathfrak{sp}_{p,p}$ $(l = 2p)$	C_p	$r(\alpha_{2j}) = \lambda_j$ $(1 \leqslant j \leqslant p)$	4 $(j \leqslant p-1)$ 3 $(j = p)$	0

Table 4 (cont.)

$\mathfrak{g}(\mathbb{C})$	$\mathfrak{g}$	Σ	r	$\dim \mathfrak{g}_{\lambda_j}$	$\dim \mathfrak{g}_{2\lambda}$
$\mathfrak{so}_{2l}(\mathbb{C})$ $l \geqslant 4$	$\mathfrak{so}_{p,2l-p}$ $(1 \leqslant p \leqslant l-2)$	B_p	$r(\alpha_j) = \lambda_j$ $(1 \leqslant j \leqslant p)$	1 $(j \leqslant p-1)$ $2(l-p)$ $(j = p)$	0
	$\mathfrak{so}_{l-1,l+1}$	B_{l-1}	$r(\alpha_j) = \lambda_j$ $(1 \leqslant j \leqslant l-1)$ $r(\alpha_l) = \lambda_{l-1}$	1 $(j \leqslant l-2)$ 2 $(j = l-1)$	0
	$\mathfrak{so}_{l,l}$	D_l	$r(\alpha_j) = \lambda_j$ $(1 \leqslant j \leqslant l)$	1	0
$\mathfrak{so}_{2l}(\mathbb{C})$ $(l \geqslant 4)$	$\mathfrak{u}^*_{2p}(\mathbb{H})$ $(l = 2p)$	C_p	$r(\alpha_{2j}) = \lambda_j$ $(1 \leqslant j \leqslant p)$	4 $(j \leqslant p-1)$ 1 $(j = p)$	0
	$\mathfrak{u}^*_{2p+1}(\mathbb{H})$ $(l = 2p+1)$	BC_p	$r(\alpha_{2j}) = \lambda_j$ $(1 \leqslant j \leqslant p)$ $r(\alpha_{2p+1}) = \lambda_p$	4	0 $(j \leqslant p-1)$ 1 $(j = p)$
E_6	E I	E_6	$r(\alpha_j) = \lambda_j$ $(1 \leqslant j \leqslant 6)$	1	0
	E II	F_4	$r(\alpha_1) = r(\alpha_5) = \lambda_1$ $r(\alpha_2) = r(\alpha_4) = \lambda_2$ $r(\alpha_3) = \lambda_3$ $r(\alpha_6) = \lambda_4$	2 $(j = 1, 2)$ 1 $(j = 3, 4)$	0
	E III	BC_2	$r(\alpha_1) = r(\alpha_5) = \lambda_2$ $r(\alpha_6) = \lambda_1$	6 $(j = 1)$	0
				8 $(j = 2)$	1
	E IV	A_2	$r(\alpha_1) = \lambda_1$	8	0

Table 4 (cont.)

$\mathfrak{g}(\mathbb{C})$	$\mathfrak{g}$	Σ	r	$\dim \mathfrak{g}_{\lambda_j}$	$\dim \mathfrak{g}_{2\lambda}$
E_7	E V	E_7	$r(\alpha_j) = \lambda_j$ $(1 \leqslant j \leqslant 7)$	1	0
	E VI	F_4	$r(\alpha_2) = \lambda_1$ $r(\alpha_1) = \lambda_2$ $r(\alpha_5) = \lambda_3$ $r(\alpha_6) = \lambda_4$	4 $(j = 1, 2)$ 1 $(j = 3, 4)$	0
	E VII	C_3	$r(\alpha_6) = \lambda_1$ $r(\alpha_2) = \lambda_2$ $r(\alpha_1) = \lambda_3$	8 $(j = 1, 2)$ 1 $(j = 3)$	0
E_8	E VIII	F_4	$r(\alpha_j) = \lambda_j$ $(1 \leqslant j \leqslant 8)$	1	0
E_8	E IX	E_8	$r(\alpha_7) = \lambda_1$ $r(\alpha_3) = \lambda_2$ $r(\alpha_2) = \lambda_3$ $r(\alpha_1) = \lambda_4$	8 $(j = 1, 2)$ 1 $(j = 3, 4)$	0
F_4	F I	F_4	$r(\alpha_j) = \lambda_j$ $(1 \leqslant j \leqslant 4)$	1	0
	F II	BC_1	$r(\alpha_1) = \lambda_1$	8	7
G_2	G	G_2	$r(\alpha_j) = \lambda_j$ $(j = 1, 2)$	1	0

Table 5. The table lists all semisimple maximal subalgebras $\mathfrak{f}$ of maximal rank in simple complex Lie algebras $\mathfrak{g}$ (up to conjugacy in $\mathfrak{g}$). For each subalgebra $\mathfrak{f}$ the system of its simple roots M is given, as well as the order p of the inner automorphism θ of the algebra $\mathfrak{g}$ for which $\mathfrak{g}^\theta = \mathfrak{f}$.

$\mathfrak{g}$	$\mathfrak{f}$	M	p
$\mathfrak{so}_{2l+1}(\mathbb{C})$, $l \geqslant 2$	$\mathfrak{so}_{2k}(\mathbb{C}) \oplus \mathfrak{so}_{2(l-k)+1}(\mathbb{C})$, $2 \leqslant k \leqslant l$	$\widetilde{\Pi} \setminus \{\alpha_k\}$	2
$\mathfrak{sp}_{2l}(\mathbb{C})$, $l \leqslant 3$	$\mathfrak{sp}_{2k}(\mathbb{C}) \oplus \mathfrak{sp}_{2(l-k)+1}(\mathbb{C})$, $1 \leqslant k \leqslant \left[\frac{l}{2}\right]$	$\widetilde{\Pi} \setminus \{\alpha_k\}$	2
$\mathfrak{so}_{2l}(\mathbb{C})$, $l \geqslant 4$	$\mathfrak{so}_{2k}(\mathbb{C}) \oplus \mathfrak{so}_{2(l-k)}(\mathbb{C})$, $2 \leqslant k \leqslant \frac{l+1}{2}$	$\widetilde{\Pi} \setminus \{\alpha_k\}$	2
E_6	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}_6(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_2\}$	2
	$\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_3\}$	3
E_7	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}_{12}(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_2\}$	2
	$\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_6(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_3\}$	3
	$\mathfrak{sl}_8(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_7\}$	2
E_8	$\mathfrak{sl}_2(\mathbb{C}) \oplus E_7$	$\widetilde{\Pi} \setminus \{\alpha_1\}$	2
	$\mathfrak{sl}_3(\mathbb{C}) \oplus E_6$	$\widetilde{\Pi} \setminus \{\alpha_2\}$	3
	$\mathfrak{sl}_5(\mathbb{C}) \oplus \mathfrak{sl}_5(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_4\}$	5
	$\mathfrak{so}_{16}(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_7\}$	2
	$\mathfrak{sl}_9(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_8\}$	3
F_4	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sp}_6(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_1\}$	2
	$\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_2\}$	3
	$\mathfrak{so}_9(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_4\}$	2
G_2	$\mathfrak{sl}_3(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_1\}$	3
	$\mathfrak{so}_4(\mathbb{C})$	$\widetilde{\Pi} \setminus \{\alpha_2\}$	2

Table 6. The table lists all nonsemisimple maximal reductive subalgebras $\mathfrak{f}$ of maximal rank of simple complex Lie algebras $\mathfrak{g}$ (up to automorphisms of the algebra $\mathfrak{g}$). For each subalgebra $\mathfrak{f}$ the system M of its simple roots is given.

$\mathfrak{g}$	$\mathfrak{f}$	M
$\mathfrak{sl}_n(\mathbb{C})$ $n \geqslant 2$	$\mathfrak{sl}_k(\mathbb{C}) \oplus \mathfrak{sl}_{n-k}(\mathbb{C}) \oplus \mathbb{C}$ $1 \leqslant k \leqslant \left[\frac{n}{2}\right]$	$\Pi \setminus \{\alpha_k\}$
$\mathfrak{so}_{2l+2}(\mathbb{C})$ $l \geqslant 2$	$\mathfrak{so}_2(\mathbb{C}) \oplus \mathfrak{so}_{2l-1}(\mathbb{C})$	$\Pi \setminus \{\alpha_1\}$
$\mathfrak{sp}_{2l}(\mathbb{C})$ $l \geqslant 3$	$\mathfrak{gl}_l(\mathbb{C})$	$\Pi \setminus \{\alpha_l\}$
$\mathfrak{so}_{2l}(\mathbb{C})$ $l \geqslant 4$	$\mathfrak{so}_2(\mathbb{C}) \oplus \mathfrak{so}_{2l-2}(\mathbb{C})$ $\mathfrak{gl}_l(\mathbb{C})$	$\Pi \setminus \{\alpha_1\}$ $\Pi \setminus \{\alpha_l\}$
E_6	$\mathfrak{so}_{10}(\mathbb{C}) \oplus \mathbb{C}$	$\Pi \setminus \{\alpha_1\}$
E_7	$E_6 \oplus \mathbb{C}$	$\Pi \setminus \{\alpha_1\}$

Table 7. The table lists the highest weights and dimensions of irreducible representations R of simple complex Lie groups F for which the group $R(F)$ is not maximal among connected Lie subgroups of the group $\mathrm{SL}(V)$ (if R has no bilinear invariants), $\mathrm{SO}(V)$ (if R if orthogonal), and $\mathrm{Sp}(V)$ (if R is symplectic). The three cases are characterized by the values $\varepsilon = 0, 1, -1$, respectively.

F	Highest weight of the representation R	$\dim R$	ε
$\mathrm{SL}_{l+1}(\mathbb{C})$	$\pi_1 + \pi_3$, $l \geqslant 3$	$3\binom{l+2}{4}$	0
	$2\pi_1 + \pi_2$, $l \geqslant 2$	$3\binom{l+3}{4}$	0
$\mathrm{SL}_2(\mathbb{C})$	$6\pi_1$	7	1
$\mathrm{SL}_6(\mathbb{C})$	$\pi_2 + \pi_4$	189	1
$\mathrm{Spin}_{4k+3}(\mathbb{C})$, $k \geqslant 1$	$m\pi_{2k+1}$, $m \geqslant 1$	$\prod_{s=1}^{2k} \frac{\binom{m+2s-1}{m}}{\binom{m+s}{m}}$	$(-1)^{(k+1)m}$
$\mathrm{Spin}_9(\mathbb{C})$	$\pi_1 + \pi_4$	128	1
$\mathrm{Sp}_6(\mathbb{C})$	$2\pi_2$	90	1
	$2\pi_2 + \pi_3$	350	-1
$\mathrm{Spin}_{10}(\mathbb{C})$	$\pi_2 + \pi_4$	560	0
$\mathrm{Spin}_{12}(\mathbb{C})$	π_4	495	1
	$\pi_3 + \pi_5$	4928	-1
E_6	π_2	351	0
	$\pi_4 + \pi_6$	17550	0
E_7	π_2	1539	1
	π_3	27664	-1
	π_4	365750	1
	$\pi_5 + \pi_7$	3792096	-1
G_2	$m\pi_1$, $m \geqslant 2$	$\frac{2m+5}{5}\binom{m+4}{4}$	1

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