Lemma: Let $x \in K$ be a non-zero element. Let $B[x]$ be the subring of K generated by B and x. Define $m[x] := m^e \subseteq B[x]$ where $m \subseteq B$ is the maximal ideal of B. Then either

$$m[x] \neq B[x] \quad \text{or} \quad m[x^{-1}] \neq B[x^{-1}]$$

Proof: Suppose both statements are false. Then have

$$① \quad u_0 + u_1 x + \cdots + u_{m'} x^{m'} = 1 \qquad (u_i \in m)$$

$$② \quad v_0 + v_1 x^{-1} + \cdots + v_n x^{-n} = 1 \qquad (v_i \in m)$$

Choose m, n as small as possible, and assume $m' \geq n$.

Multiply ② by x^n:

$$(1-v_0)x^n = v_1 x^{n-1} + \cdots + v_n$$

Now $v_0 \in m$, $1 \notin m$, so $1-v_0 \notin m$,

so $1-v_0$ is a unit since B is local.

Dividing by $1-v_0$ gives an equation

$$x^n = w_1 x^{n-1} + \cdots + w_n$$

with $w_i \in m$.

Then $x^{m'} = w_1 x^{m'-1} + \cdots + w_n x^{m'-n}$

Thus in (1) we can replace $x^{m'}$ with a smaller degree polynomial contradicting assumption that m' is a small as possible. $\blacksquare$

<u>Theorem</u>: Let (B, g) be a maximal element of Σ. Then B is a valuation ring of K.

<u>pf</u>: Let $x \neq 0$, $x \in K$. Let's

assume $\mathfrak{m}[x] \neq B[x]$ (otherwise replace x with x^{-1}.)

Then $\mathfrak{m}[x] \subseteq \mathfrak{m}' \subseteq B[x] = B'$

for some maximal ideal $\mathfrak{m}'$ of B'.

Also $\mathfrak{m}' \cap B = \mathfrak{m}$ since $\mathfrak{m}' \cap B$

is a prime ideal and contains $\mathfrak{m}$.

Thus the inclusion $B \subseteq B'$ induces a homomorphism of fields

$$k := B/\mathfrak{m} \longrightarrow B'/\mathfrak{m}' =: k'$$

and $k' = k[\bar{x}]$ where $\bar{x}$ is the image of x in $B'/\mathfrak{m}'$.

Since k' is a field, $\bar{x}$ is algebraic over k (If $\bar{x}$ were transcendental, then $k[\bar{x}]$ is a polynomial ring not a field.)

Thus k'/k is a finite field extension.

Now g induces an embedding

$$\bar{g} : B/m = k \hookrightarrow \Omega$$

We can then extend this to

$\bar{g}' : k' \longrightarrow \Omega$, which induces $g' : B' \to B'/m' = k' \xrightarrow{\bar{g}'} \Omega$, an extension of g.

By maximality of (B, g), we have $B = B'$, so $x \in B$. $\boxtimes$

<u>Cor</u>: Let $A \subseteq K$ be a subring of a field K. Then the integral closure $\bar{A}$ of A in K is the intersection of all valuation rings of K containing A.

pf: Let $B \subseteq K$ be a valuation ring of K containing A. Then as B is integrally closed in K, we have $\bar{A} \subseteq B$ since if $x \in K$ is integral over A, it is also integral over B.

Thus $\bar{A}$ is contained in the intersection of all valuation rings of K containing A.

Conversely, suppose $x \notin \bar{A}$, $x \in K$. Then $x \notin A' = A[x^{-1}]$, as otherwise we have

$$x = a_0 + a_1 x^{-1} + \dots + a_n x^{-n},$$

and multiplying this equation by x^n shows x is integral over A.

Thus x^{-1} is not a unit in A' and hence is contained in some maximal ideal $m' \subseteq A'$. Let Ω be the algebraic closure of $k' = A'/m'$, so get an inclusion

$$\bar{g}' : k' \hookrightarrow \Omega$$

and hence a composed map

$$g' : A' \longrightarrow A'/m' \xrightarrow{\bar{g}'} \Omega.$$

Thus Σ is non-empty and $\exists$ a maximal element of Σ, (B, g) with $A' \subseteq B$, $g|_{A'} = g'$.

Since $x^{-1} \in m'$, $g'(x^{-1}) = 0$, so $g(x^{-1}) = 0$. If $x \in B$, then $g(x) = g(x^{-1})^{-1}$ which isn't possible. So $x \notin B$, with B a valuation ring containing A. $\boxtimes$

<u>Goal</u>: Give a proof of Hilbert's Nullstellensatz.

<u>Prop</u>: Let $A \subseteq B$ integral domains, and B is a finitely generated A-algebra (i.e. $B \cong A[x_1, \dots, x_n]/I$ for some I). Let $v \in B$ be a non-zero. Then there exists a $u \in A$, $u \neq 0$, with the following property:

Any homomorphism

$f: A \to \Omega$ with Ω an algebraically closed field with $f(u) \neq 0$ can be extended to a homomorphism $g: B \to \Omega$

with $g(v) \neq 0$.

Proof: Go by induction on n,

the number of generators of B over

A. So we may assume B has

one generator, i.e., $B = A[x]$

(i.e., is a quotient of a polynomial

ring in one variable over A.)

Case (1): x is transcendental over A,

i.e, B is a polynomial ring over A.

Let $v = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$

and take $u = a_0$.

If $f: A \to \Omega$ has $f(u) \neq 0$,

then $\exists \xi \in \Omega$ so that

$$f(a_0)\xi^n + f(a_1)\xi^{n-1} + \cdots + f(a_n) \neq 0$$

since Ω is infinite as it is algebraically closed.

Define $g : B \to \Omega$ extending f by $g(x) = \xi$. Then $g(v) \neq 0$.

Case (2) Now suppose x is algebraic over the field of fractions of A. Then so is v^{-1} (the set of algebraic elements of $B_{(0)}$ over $A_{(0)}$ is a subfield of $B_{(0)}$.)

Thus we have equations

(1) $a_0 x^m + a_1 x^{m-1} + \cdots + a_m = 0 \quad (a_i \in A)$

(2) $a_0' v^{-n} + a_1' v^{1-n} + \cdots + a_n' = 0$

$(a_i' \in A)$

Let $u = a_0 a_0'$. Let $f : A \to \Omega$ be a homomorphism with $f(u) \neq 0$. Want to extend to $g : B \to \Omega$ with $g(v) \neq 0$.

We can extend f to

$$f_1 : \underset{\overset{\shortparallel}{A_u}}{A[u^{-1}]} \longrightarrow \Omega$$

via $f_1(u^{-1}) = f(u)^{-1}$. (This is using the universal property of localization)

We may then extend this map to $h : C \to \Omega$ where C is a valuation ring of $B_{(0)}$ and C contains $A[u^{-1}]$.

By equation (1), x is integral over $A[u^{-1}]$, so $x \in C$.

So C contains B. Now take $g = h|_B$, so $g : B \to \Omega$ extends f. We need to check $g(v) \neq 0$. But equation (2) tells us that v^{-1} is integral over $A[u^{-1}]$, and hence $v^{-1} \in C$.

So $v, v^{-1} \in C$, and hence $g(v) = h(v) \neq 0$. $\blacksquare$

<u>Cor:</u> (Hilbert's Nullstellensatz)

Let k be a field and B a finitely generated k-algebra. If B is a field, then it is a finite algebraic extension of k.

Pf: Take $A = k$, $v = 1 \in B$, $\Omega = \bar{k}$ the algebraic closure of k. Then we get an extension of the inclusion $k \hookrightarrow \Omega$, so by previous proposition we get an extension $B \to \Omega$. Since B is a field, this is an inclusion, so every element of B is algebraic over k. Thus, since B is finitely generated as a k-algebra, we obtain B by adjoining a finite number of algebraic elements, hence B is a finite field extension. $\blacksquare$

Example: If A is a f.g. k-algebra, $m \subseteq A$ a maximal ideal, then A/m is a finite field extension of k.

If $A = k[x_1, \dots, x_n]$ with k algebraically closed, $m \subseteq A$ a maximal ideal, then $A/m \cong k$, being a finite field extension of an algebraically closed field.

We have an induced map

$$g : A \longrightarrow A/m \xrightarrow{\cong} k.$$

Let $g(x_i) = a_i \in k$. So $\ker g$ contains $x_1 - a_1, \dots, x_n - a_n$.

Thus $\ker g = (x_1 - a_1, \dots, x_n - a_n)$, as the latter is already a maximal ideal.

Note there is a 1-1 correspondence between maximal ideals of A and elements of k^n.

Homological techniques

Quick sketch of homological algebra.

Def: An A-module P is *projective* if for any diagram of A-modules

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd},$$

there exists a dotted arrow making the diagram commute.

Example: A free module

$$P = \bigoplus_{i \in I} A \quad \text{is always projective.}$$

Giving a map $\varphi : P \to N$ is the same as giving $\varphi(e_i)$ for each $i \in I$

where $e_i = (0, \dots, 0, 1, 0 \dots)$
$\uparrow$ i^{th} place

So in particular, given $\varphi : P \to N$, choose lifts $m_i \in M$ of $\varphi(e_i)$ and define $\varphi' : P \to M$ by

$$\varphi'(e_i) = m_i .$$

<u>**Def:**</u> An A-module I is <u>injective</u> if for any diagram of A-modules

$$\begin{tikzcd} I & \\ 0 \arrow[r] & N \arrow[u] \arrow[r] & M \arrow[ul, dotted] \end{tikzcd}$$

there exists a dotted arrow making

the diagram commutes.

Example: $\mathbb{Q}$ is an injective $\mathbb{Z}$-module,

Resolutions: Given an A-module M, there exists a projective module P_0 with $P_0 \to M \to 0$

(e.g., take a generating set $\{m_i\}_{i \in I}$ of M, and take $P_0 = \bigoplus_{i \in I} A$ and define $\varphi : P_0 \to M$ by

$$\varphi(e_i) = m_i.)$$

This map has a kernel, K_0, and repeat this process!

$$
\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & & & \\
 & & K_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[r] & P_2 \arrow[ur] \arrow[rr] & & P_1 \arrow[rr] \arrow[dr] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
K_2 \arrow[ur] & & & & K_0 \arrow[ur] \arrow[dr] & & & \\
 & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

This gives a long exact sequence

$$\cdots \longrightarrow P_1 \to P_0 \to M \to 0,$$

called a __projective resolution__ of M.

Write $P_\bullet \to M \to 0$

Less obviously, one can embed any module into an injective module and obtain __injective resolutions__

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

or $\quad 0 \to M \to I^\bullet$

Recall given a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

we have exact sequences

$$0 \to \mathrm{Hom}(M_3, N) \to \mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M_1, N)$$

$$0 \to \mathrm{Hom}(N, M_1) \to \mathrm{Hom}(N, M_2) \to \mathrm{Hom}(N, M_3)$$

$$N \otimes_A M_1 \to N \otimes_A M_2 \to N \otimes_A M_3 \to 0$$

Can we extend those sequences to long exact sequences to measure the failure of exactness?

This gives the Tor and Ext functors.

Tor We define $\mathrm{Tor}_i^A(M, N)$, $i \geq 0$

Take a projective resolution

$$P_\bullet \to M \to 0$$

$P_1 \to P_0 \to M \to 0$

$P_1 \otimes_A N \to P_0 \otimes_A N \to M \otimes_A N \to 0$

Form a complex (i.e., $d_i \circ d_{i+1} = 0 \ \forall i$)

$$\cdots \to P_3 \otimes_A N \xrightarrow{d_2} P_2 \otimes_A N \xrightarrow{d_1} P_1 \otimes_A N \xrightarrow{d_0} P_0 \otimes_A N \xrightarrow{d_{-1}} 0$$

This need not be an exact sequence.

Define

$$\mathrm{Tor}_i^A(M, N) = \frac{\ker d_{i-1}}{\mathrm{im}\, d_i}$$

(= 0 if sequence is exact.)

Remarks: One can show the following properties: (any textbook on homological algebra.)

(1) $\mathrm{Tor}_i^A(M,N)$ is independent of the choice of resolution. Further, reversing the role of M and N in the construction doesn't change the result.

(2) These are functors in both variables, i.e., given

$$M_1 \xrightarrow{f} M_2,$$ get

$$\mathrm{Tor}_i^A(M_1, N) \longrightarrow \mathrm{Tor}_i^A(M_2, N).$$

[This is done by constructing ladders

$$\begin{array}{ccccc} P_0^1 & \longrightarrow & M_1 & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \\ P_0^2 & \longrightarrow & M_2 & \longrightarrow & 0 \end{array}$$]

The same is true in the second variable.

③ Given a short exact sequence

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

$\exists$ maps $\delta : \mathrm{Tor}^A_{i+1}(M_3, N) \to \mathrm{Tor}^A_i(M_1, N)$

(called the connecting maps) $\forall i$

such that

$$\cdots \to \mathrm{Tor}_{i+1}(M_1, N) \xrightarrow{f_*} \mathrm{Tor}_{i+1}(M_2, N) \xrightarrow{g_*} \mathrm{Tor}_{i+1}(M_3, N)$$

$$\xrightarrow{\delta} \mathrm{Tor}_i(M_1, N) \xrightarrow{f_*} \mathrm{Tor}_i(M_2, N) \to \cdots$$

$$\cdots \to \mathrm{Tor}_0(M_3, N) \to 0$$

is a long exact sequence.

④ $\mathrm{Tor}^A_0(M, N) = M \otimes_A N$.

⑤ Functoriality of δ:

Given a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & M_1 & \to & M_2 & \to & M_3 & \to & 0 \\
 & & f_1 \downarrow & & f_2 \downarrow & & \downarrow f_3 & & \\
0 & \to & M_1' & \to & M_2' & \to & M_3' & \to & 0
\end{array}$$

with exact rows, $\exists$ a commutative diagram

$$\begin{array}{ccc}
\mathrm{Tor}_{i+1}^A(M_3, N) & \xrightarrow{\delta} & \mathrm{Tor}_i^A(M_1, N) \\
f_{3*} \downarrow & & \downarrow f_{1*} \\
\mathrm{Tor}_{i+1}^A(M_3', N) & \xrightarrow[\delta]{} & \mathrm{Tor}_i^A(M_1', N)
\end{array}$$

(6) If M is projective, then

$\mathrm{Tor}_i^A(M, N) = 0 \quad \forall i > 0.$

Pf: $0 \to 0 \to 0 \to \underset{\overset{\parallel}{P_0}}{M}$ is a projective resolution of M, so so after tensoring with N get

$$0 \to 0 \to 0 \to M \otimes_A N$$

degree $\quad 3 \quad 2 \quad 1 \quad 0$

Sample easy applications

Prop: An A-module M is flat if and only if $\mathrm{Tor}_1^A(M,N)=0$ for all A-modules N.

proof: $\Longrightarrow$ Resolve N:

$$P_\bullet \to N \to 0 \quad , \qquad \text{and tensor with } M:$$

$$M \otimes_A P_\bullet \to M \otimes_A N \to 0$$

Since M is flat, this is still exact, and hence $\mathrm{Tor}_i^A(M,N)=0 \quad \forall i>0.$

$\Longleftarrow$ Given

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

get a long exact sequence

$$0=\mathrm{Tor}_1(M,N_3) \to M\otimes_A N_1 \to M\otimes_A N_2 \to M\otimes_A N_3 \to 0$$

so we get flatness

Cor: Suppose $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact with M_3 flat. Then

$$0 \to M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$$

is exact for all A-modules N.

Pf: This follows immediately from $\mathrm{Tor}_1(M_3, N) = 0$. $\boxtimes$

Analogous construction for Hom:

Given M, N A-modules, we may:

(1) Take a projective resolution of M, $P_\bullet \to M \to 0$ and apply $\mathrm{Hom}_A(\bullet, N)$, giving a complex

$$\mathrm{Hom}(P_0, N) \xrightarrow{d^0} \mathrm{Hom}(P_1, N) \xrightarrow{d^1} \mathrm{Hom}(P_2, N) \to \cdots$$

and define

$$\mathrm{Ext}^i_A(M, N) = \frac{\ker d^i}{\mathrm{Im}\, d^{i-1}}$$

(2) Take an injective resolution

$0 \to N \to I^\bullet$ and apply

$\mathrm{Hom}_A(M, \cdot)$ to get a complex

$$\mathrm{Hom}(M, I^0) \xrightarrow{d^0} \mathrm{Hom}(M, I^1) \xrightarrow{d^1} \mathrm{Hom}(M, I^2) \to \cdots$$

and define

$$\mathrm{Ext}^i_A(M, N) = \frac{\ker d^i}{\mathrm{Im}\, d^{i-1}}$$

<u>Remark</u>: The answer is independent of which method we use.

and we get the analogous properties as for Tor, the most important being

- $\mathrm{Ext}^0_A(M,N) = \mathrm{Hom}_A(M,N)$

- Given a short exact sequence

$$0 \longrightarrow M_1 \to M_2 \to M_3 \to 0,$$

we get long exact sequences

$$0 \to \mathrm{Ext}^0(M_3,N) \to \mathrm{Ext}^0(M_2,N) \to \mathrm{Ext}^0(M_1,N)$$
$$\to \mathrm{Ext}^1(M_3,N) \to \cdots$$

and

$$0 \to \mathrm{Ext}^0(N,M_1) \to \mathrm{Ext}^0(N,M_2) \to \mathrm{Ext}^0(N,M_3)$$
$$\to \mathrm{Ext}^1(N,M_1) \to \cdots$$

Def: Let A be a ring, M an A-module. We say $a \in A$ is <u>M-regular</u> if a is not a 0-divisor for M, i.e., $\nexists\, x \in M$, $x \neq 0$, $ax = 0$.

We say $a_1, \dots, a_n \in A$ is an <u>M-regular</u> sequence if a_1 is M-regular, a_2 is M/a_1M-regular, $\dots$, a_i is $M/(a_1M + \dots + a_{i-1}M)$-regular.

If $I \subseteq A$ is an ideal, the <u>I-depth of M</u>, $\operatorname{depth}_I(M)$ is the length of the longest M-regular sequence contained in I.

We write $\operatorname{depth}_I(A)$ for the depth of A as an A-module.

If (A, m) is local, Noetherian, M a f.g. A-module, we write

$$\operatorname{depth}(M) = \operatorname{depth}_m(M),$$

and we say M is <u>Cohen-Macaulay</u> if $\operatorname{depth}(M) = \dim M$.

We say (A, m) is Cohen-Macaulay if it is Cohen-Macaulay as an A-module.

An arbitrary Noetherian ring A is Cohen-Macaulay if $A_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \subseteq A$ prime.

Homological characterization of depth:

Theorem: Let A be Noetherian, M a f.g. A-module, $I \subseteq A$ an ideal of A with $IM \neq M$, $n > 0$ an integer. Then the following are equivalent:

① $\operatorname{Ext}^i_A(N, M) = 0 \quad \forall i < n$, for every f.g. A-module N with $\operatorname{Supp} N \subseteq V(I)$.

② $\operatorname{Ext}^i_A(A/I, M) = 0 \quad \forall i < n$

③ $\exists$ a f.g. A-module N with $\operatorname{Supp} N = V(I)$ such that

$$\operatorname{Ext}^i_A(N, M) = 0 \quad \forall i < n.$$

(4) $\exists$ an M-regular sequence $a_1, \dots, a_n$ of length n in I.

proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) immediate.

(3) $\Rightarrow$ (4) We have (taking $i=0$)

$$0 = \operatorname{Ext}^0(N, M) = \operatorname{Hom}(N, M)$$

If no elements of I are M-regular, then $I \subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$, the set of zero-divisors of M,

Thus $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}(M)$,

so $\exists$ an injection

$$A/\mathfrak{p} \to M.$$

Localizing at $\mathfrak{p}$ gives an injection

$$k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to M_{\mathfrak{p}}$$

so $\operatorname{Hom}_{A_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0.$

Since $\mathfrak{p} \in V(I) = \mathrm{Supp}(N)$ by assumption in (3), we have $N_\mathfrak{p} \neq 0$,

so $N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p} = N \otimes_A k(\mathfrak{p}) = N_\mathfrak{p} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) \neq 0$

by Nakayama's lemma.

So $\mathrm{Hom}_{A_\mathfrak{p}}(N \otimes_A k(\mathfrak{p}), k(\mathfrak{p})) \neq 0$

$$\| $$

$$\mathrm{Hom}_{k(\mathfrak{p})}(N \otimes_A k(\mathfrak{p}), k(\mathfrak{p}))$$

Thus $\mathrm{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) \neq 0$, (If $I \subseteq \mathrm{Ann}(M), \mathrm{Ann}(N)$ then $\mathrm{Hom}_A(M,N) = \mathrm{Hom}_{A/I}(M,N)$)

(say take a composition

$$N_\mathfrak{p} \longrightarrow N \otimes_A k(\mathfrak{p}) \longrightarrow k(\mathfrak{p}) \longrightarrow M_\mathfrak{p})$$

$\uparrow$ quotient map; $\uparrow$ non-zero map; $\uparrow$ inclusion

Q1 of Example sheet IV tells us

that $\mathrm{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) = (\mathrm{Hom}_A(N, M))_\mathfrak{p}$

[Note this requires A Noetherian and N, M finitely generated]

Thus $\operatorname{Hom}_A(N, M) \neq 0$, which is a contradiction.

[Note: $\operatorname{Hom}_A(N, M)$ is an A-module via $(a \cdot \varphi)(n) = a\varphi(n)$ for $a \in A$, $\varphi : N \to M$.]

Thus $\exists$ an M-regular sequence $a_1 \in I$

Now consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot a_1} M \to M/a_1M \to 0.$$

$\uparrow$ Because a_1 is not a zero-divisor.

giving

$$(*) \quad \cdots \longrightarrow \operatorname{Ext}^i(N, M) \to \operatorname{Ext}^i(N, M/a_1M) \longrightarrow \operatorname{Ext}^{i+1}(N, M) \to \operatorname{Ext}^{i+1}(N, M) \to \cdots$$

Thus if $i < n-1$,

$$\mathrm{Ext}^i(N, M) \cong \mathrm{Ext}^{i+1}(N, M)$$ by hypothesis of (3), and thus

$$\mathrm{Ext}^i(N, M/a_1 M) = 0, \quad \text{for } i < n-1.$$

Now go by induction.

(4) $\Rightarrow$ (1) Induction on n, the case $n=0$ being the base case, completely vacuous.

Now assume $n > 0$ and statement is true for $n' < n$.

Put $M_1 = M/a_1 M$. M_1 has an M_1-regular sequence of length $n-1$, namely $a_2, \ldots, a_n$, so

$$\mathrm{Ext}^i(N, M_1) = 0 \quad \text{for } i < n-1.$$

Thus we have a short exact sequence

$$0 \to \operatorname{Ext}^i_A(N, M) \xrightarrow{\cdot a_1} \operatorname{Ext}^i_A(N, M)$$

$\forall\ i < n$.

<u>Hand-waving</u>: This map is multiplication by a_1.

But $\operatorname{Supp} N = V(\operatorname{Ann}(N)) \subseteq V(I)$

(by assumption of (1)) so

$$I \subseteq \sqrt{\operatorname{Ann}(N)}$$

so $a_1^r \cdot N = 0$ for some $r > 0$.

Thus a_1^r annihilates $\operatorname{Ext}^i(N, M)$.

(This is because the map

$$\operatorname{Ext}^i(N, M) \xrightarrow{\cdot a_1^r} \operatorname{Ext}^i(N, M)$$

is also induced functorially by

$N \xrightarrow{\cdot a_1^r} N$. But this is the zero map, and hence induces the zero map functorially.]

Since $\mathrm{Ext}^i(N, M) \xrightarrow{\cdot a_1} \mathrm{Ext}^i(N, M)$ is injective, we must have $\mathrm{Ext}^i(N, M) = 0$ for $i < n$. $\boxtimes$

Cor: In the situation of the theorem,

$$\mathrm{depth}_I\, M = \min\{i \mid \mathrm{Ext}^i_A(A/I, M) \neq 0\}$$

Cor: In the situation of the above theorem, suppose $a_1, \dots, a_r \in I$ is an M-regular sequence.

Then

$$\mathrm{depth}_I\left(M/(a_1M + \dots + a_rM)\right) = \mathrm{depth}_I(M) - r.$$

pf: $\operatorname{depth}_I(M/(a_1M+\dots+a_rM)) \leq$

$\operatorname{depth}_I(M) - r$

since any regular sequence for

$M/\sum a_i M$, say $b_1, \dots, b_n$, yields

a regular sequence $a_1, \dots, a_r, b_1, \dots, b_n$

for M.

To show the opposite inequality, enough to show for $r = 1$.

We have an exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot a_1} M \longrightarrow M_1 = M/a_1M \to 0$$

to see that

$$\operatorname{depth}_I M = n \implies \operatorname{Ext}^i(A/I, M) = 0 \quad \text{for } i < n$$

$$\implies \operatorname{Ext}^i(A/I, M_1) = 0 \quad \text{for } i < n-1$$

$\underset{\text{by Cor.}}{\Longrightarrow}$ $\operatorname{depth}_I M_1 \geq n-1$

$= \operatorname{depth}_I M - 1$ $\blacksquare$

<u>Remark</u>: This tells us any

M-regular sequence $a_1, \dots, a_r \in I$

can be extended to an M-regular

sequence in I of length

given by $\operatorname{depth}_I M$.

<u>Lemma</u>: Let (A, m) be a Noetherian

local ring, M, N non-zero f.g.

A-modules with $\underset{\operatorname{depth}_m M}{\underset{\|}{\operatorname{depth} M}} = k$

and $\dim N = r$. Then

$$\operatorname{Ext}^i_A(N, M) = 0 \quad \text{for } i < k - r.$$

<u>pf</u>: Induction on r.

If $r = 0$, then $\operatorname{Supp}(N) = \{m\}$

Then the result follows from Theorem.

If $r > 0$, we can make use of the filtration

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_r = N$$

with $N_i / N_{i-1} \cong A/\mathfrak{p}_i$, with $\mathfrak{p}_i$ prime. Then $\mathfrak{p}_i \in \operatorname{Supp} N$ and thus $\dim A/\mathfrak{p}_i \leq r$ for each i.

I claim now it's enough to show lemma for $N = A/\mathfrak{p}_i$. Indeed, suppose true for each $N = A/\mathfrak{p}_i$. Have

$$0 \to N_{i-1} \to N_i \to N_i / N_{i-1} = A/\mathfrak{p}_i \to 0$$

giving

$$\cdots \to \operatorname{Ext}^j(N_i/N_{i-1}, M) \to \operatorname{Ext}^j(N_i, M)$$

$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{A/P_i}$

$$\to \operatorname{Ext}^j(N_{i-1}, M) \to \operatorname{Ext}^{j+1}(N_i/N_{i-1}, M)$$

$$\to \cdots$$

While $\operatorname{Ext}^j(N_0, M) = 0 \ \forall j$

because $N_0 = 0$, and

$\operatorname{Ext}^j(N_1, M)$ injects into

$\operatorname{Ext}^j(N_0, M)$ for $j < k-r$.

Thus $\operatorname{Ext}^j(N_1, M) = 0$ for $j < k-r$

Repeating, we get

$$0 = \operatorname{Ext}^j(N_1, M) = \operatorname{Ext}^j(N_2, M)$$

$$= \cdots = \operatorname{Ext}^j(N, M) \quad \forall j < k-r.$$

So now assume $N = A/P$

with $\dim N = r$.

Pick $x \in m \setminus \mathfrak{p}$, so

$$0 \to N \xrightarrow{\cdot x} N \to N' = A/(\mathfrak{p}+(x)) \to 0$$

is exact. Further,

$$\dim A/(\mathfrak{p}+(x)) \leq \dim A/\mathfrak{p} - 1$$
$$= r-1.$$

By induction hypothesis, we have

$$0 = \operatorname{Ext}^i(N', M) \to \operatorname{Ext}^i(N, M) \xrightarrow{\cdot x} \operatorname{Ext}^i(N, M)$$
$$\to \underline{\operatorname{Ext}^{i+1}(N', M) = 0}$$

for $i < k-r$

Note this vanishing holds for $i = k-r-1$ because $\dim N' \leq r-1$ and the induction hypothesis.

Thus $x \cdot \operatorname{Ext}^i(N, M) = \operatorname{Ext}^i(N, M)$ for $i < k-r$, so $\operatorname{Ext}^i(N, M) = 0$ by Nakayama's lemma.

[Note: to apply Nakayama, we need $\operatorname{Ext}^i(N, M)$ finitely generated.] $\boxtimes$

<u>Theorem</u>: Let $(A, \mathfrak{m})$ be a Noetherian local ring, M a f.g. A-module. Then

$$\operatorname{depth}(M) \le \dim A/\mathfrak{p} \quad \text{for}$$

every $\mathfrak{p} \in \operatorname{Ass}(M)$. In particular,

$\operatorname{depth}(M) \le \dim M$

<u>Pf</u>: If $\mathfrak{p} \in \operatorname{Ass}(M)$, then

$\operatorname{Hom}_A(A/\mathfrak{p}, M) \neq 0$, so

$\operatorname{depth} M \le \dim A/\mathfrak{p}$ by the lemma. $\boxtimes$

Lemma: Let (A, m) be a Noetherian Local ring, M a f.g. A-module, $a_1, \dots, a_r \in m$ an M-regular sequence.

Then

$$\dim M \Big/ \left(\sum_{i=1}^{r} a_i M\right) = \dim M - r.$$

proof: By the dimension theorem

($d(M)$ = smallest # of generators for an ideal of def'n for $A/\mathrm{Ann}(M)$,

= $\dim M$)

$$\dim M / \sum a_i M \geq \dim M - r,$$

since if

$$\ell\left(M \Big/ \left(\sum_{i=1}^{r} a_i M + \sum_{j=1}^{d} b_j M\right)\right) < +\infty,$$

with minimal choice of length of

$b_1, \dots, b_d$, so that $d = \dim M / \sum a_i M$,

we see $\dim M \leq d + 1$.

Now if $a \in A$ is an M-regular element, we have

$$\operatorname{Supp}(M/aM) = \operatorname{Supp}(M \otimes_A A/(a))$$

(Example Sheet II)

$$= \operatorname{Supp} M \cap \operatorname{Supp}(A/(a))$$

$$= \operatorname{Supp} M \cap V((a))$$

Also, a is not in any minimal element of $\operatorname{Supp}(M)$, as those are associated primes, and the set of 0-divisors for M is the union of associated primes.

Thus $\dim M/aM < \dim M$.

Thus $\dim M/aM \leq \dim M - 1$

and inductively

$$\dim M / \sum_{i=1}^{r} a_i M \leq \dim M - r. \qquad \boxtimes$$

Theorem: (A, m) a Noetherian local ring, M a f.g. A-module. Then

① If M is Cohen-Macaulay (i.e., depth $M = \dim M$) then

depth $M = \dim A/p \quad \forall p \in \mathrm{Ass}(M)$.

In particular, M has no embedded primes.

② If $a_1, \dots, a_r \in m$ is an M-regular sequence, and

$M' = M / \sum_{i=1}^{r} a_i M$, then $M/(a_1 \dots a_r)M$

M is Cohen-Macaulay $\iff$ M' is Cohen-Macaulay.

(3) If M is Cohen-Macaulay, then for every $\mathfrak{p} \in \operatorname{Spec} A$, the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is Cohen-Macaulay, and if $M_{\mathfrak{p}} \neq 0$, then $\operatorname{depth}_{\mathfrak{p}}(M) = \operatorname{depth}_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}})$.

Proof: (1) Assume $M \neq 0$, so depth M = dim M. Since $\mathfrak{p} \in \operatorname{Ass}(M)$ satisfies $\mathfrak{p} \in \operatorname{Supp}(M)$,

dim $M \geq$ dim $A/\mathfrak{p}$ and

depth $(M) \leq$ dim $(A/\mathfrak{p})$ by

previous theorem. Thus depth M = dim $A/\mathfrak{p}$.

(3) By Nakayama's lemma, $M = 0$ iff $M' = 0$. If $M \neq 0$, then

$$\begin{array}{ccc} \dim M' & = & \dim M - r \quad \text{by lemma} \\ \vee\!\mid & & \shortparallel \\ \operatorname{depth} M' & & \operatorname{depth} M - r \quad \text{since} \\ & & M \text{ is C-M} \end{array}$$

But $\operatorname{depth} M' = \operatorname{depth} M - r$ by corollary, so $\operatorname{depth} M' = \dim M'$, so M' is C-M.

(3) Let $\mathfrak{p} \subseteq A$ be prime, and assume $M_\mathfrak{p} \neq 0$, so $\mathfrak{p} \in \operatorname{Supp} M$ so $\mathfrak{p} \supseteq \operatorname{Ann} M$. We have

$(*)$ $$\dim M_\mathfrak{p} \geq \operatorname{depth}_{\mathfrak{p} A_\mathfrak{p}} M_\mathfrak{p} \geq \operatorname{depth}_\mathfrak{p} M$$

For the last inequality, if $a_1, \ldots, a_r \in \mathfrak{p}$ an M-regular sequence.

Then $\frac{a_1}{1}, \dots, \frac{a_r}{1} \in \mathfrak{p}A_\mathfrak{p}$ is an $M_\mathfrak{p}$-regular sequence, by exactness of localization, i.e., if

$$M/(a_1, \dots, a_{i-1})M \xrightarrow{\cdot a_i} M/(a_1, \dots, a_{i-1})M$$

is injective, so is the localization.

We go by induction on $\mathrm{depth}_\mathfrak{p}(M)$.

If $\mathrm{depth}_\mathfrak{p}(M) = 0$, then

$$\mathfrak{p} \subseteq \bigcup_{\mathfrak{p}' \in \mathrm{Ass}(M)} \mathfrak{p}' \quad (\text{the set of zero-divisors for } M)$$

so $\mathfrak{p} \subseteq \mathfrak{p}'$ for some $\mathfrak{p}' \in \mathrm{Ass}(M)$.

But $\mathrm{Ann}(M) \subseteq \mathfrak{p} \subseteq \mathfrak{p}'$

and by ①, the elements

of $\mathrm{Ass}(M)$ are the minimal primes over $\mathrm{Ann}(M)$, and thus $\mathfrak{p} = \mathfrak{p}'$ is minimal over $\mathrm{Ann}\, M$ and $\dim M_{\mathfrak{p}} = 0$.

Thus $\mathrm{depth}_{\mathfrak{p}}(M) = \dim M_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ is C-M by $(*)$.

Now $\mathrm{depth}_{\mathfrak{p}}(M) > 0$, and let $a \in \mathfrak{p}$ be M-regular, $M_1 = M/aM$

$\frac{a}{1}$ is $M_{\mathfrak{p}}$-regular (since localization preserves exact sequences). Thus

$$\dim (M_1)_{\mathfrak{p}} = \dim M_{\mathfrak{p}}/a M_{\mathfrak{p}}$$
$$= \dim M_{\mathfrak{p}} - 1$$
(by lemma)

and $\operatorname{depth}_{\mathfrak{p}}(M_1) = \operatorname{depth}_{\mathfrak{p}}(M) - 1$

(by corollary)

Since M_1 is C-M by ②,

by the induction hypothesis,

$$\begin{array}{ccc} \dim (M_1)_{\mathfrak{p}} & = & \operatorname{depth}_{\mathfrak{p}} (M_1) \\ \parallel & & \parallel \\ \dim M_{\mathfrak{p}} - 1 & & \operatorname{depth}_{\mathfrak{p}} M - 1 \end{array}$$

$\therefore \quad \dim M_{\mathfrak{p}} = \operatorname{depth}_{\mathfrak{p}} M \underset{\text{by } (*)}{=} \operatorname{depth}_{\mathfrak{p}A_{\mathfrak{p}}} M_{\mathfrak{p}}$

$\therefore \quad M_{\mathfrak{p}}$ is C-M. $\square$

<u>Theorem</u>: Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring. Then

① For any sequence $a_1, \dots, a_r \in \mathfrak{m}$, the following conditions are equivalent:

ⓐ $a_1, \dots, a_r$ is A-regular

(b) $\operatorname{ht}(a_1, \dots, a_i) = i \quad (1 \le i \le r)$

(c) $\operatorname{ht}(a_1, \dots, a_r) = r.$

(d) $\exists\, a_{r+1}, \dots, a_n \quad (n = \dim A)$

such that $a_1, \dots, a_n$ is a system of parameters (i.e., generate an ideal of def'n)

(2) For every proper ideal $I \subset A$,

$\operatorname{ht} I = \operatorname{depth}_I A$ and

$\operatorname{ht} I + \dim A/I = \dim A$

($\le$ is always true)

(Such a ring is called catenary)

pf: (1) (a) $\Rightarrow$ (b)

We showed in general that every minimal prime over $(a_1, \dots, a_i)$ has height $\le i$, so $ht(a_1, \dots, a_i) \le i$.

On the other hand

$$0 < ht(a_1) < ht(a_1, a_2) < ht(a_1, a_2, a_3)$$

inductively

as a_i is not contained in any minimal prime over (0)

Thus $ht(a_1, \dots, a_i) \ge i$.

(b) $\Rightarrow$ (c) Trivial.

(c) $\Rightarrow$ (d). Trivial if $\dim A = r$, as then $a_1, \dots, a_r$ are already a system of parameters.

If $\dim A > r$, then m is not a minimal prime over $(a_1, \dots, a_r)$ (if not, $\dim A/(a_1, \dots, a_r) = 0$) and can take $a_{r+1} \in m$ which is not in any minimal prime over $(a_1, \dots, a_r)$. (If such doesn't exist, then m would be contained in the union of such minimal primes and hence would be equal to a minimal prime.) Thus $\operatorname{ht}(a_1, \dots, a_{r+1}) = r+1$, and continue,

Once we get $a_1, \dots, a_n$ with $n = \dim A$, we have

$$\dim A/(a_1, \dots, a_n) = 0,$$

so $a_1, \dots, a_n$ is a system of parameters.

[Note we didn't use A C-M, only $\operatorname{ht}(a_1, \dots, a_r) = r$.]

(d) $\Rightarrow$ (a) It suffices to show that any system of parameters

$x_1, \ldots, x_n$ is A-regular.

[Note we will need the C-M hypothesis.]

If $\mathfrak{p} \in \mathrm{Ass}(A)$, then $\dim A/\mathfrak{p} = n$

since A is C-M. So $x_1 \notin \mathfrak{p}$

(otherwise can't be part of a system of parameters)

Thus x_1 is A-regular

Put $A' = A/(x_1)$. Then A'

is C-M of dimension $n-1$,

and the images of $x_2, \ldots, x_n$

form a system of parameters of A'.

Thus inductively, $x_2, \ldots, x_n$ are

A'-regular.

(2) Let $\mathrm{ht}(I) = r$. Then

can choose $a_1, \ldots, a_r \in I$

such that $ht(a_1, \dots, a_i) = i$ $\forall i \le r$

[Choose a_1 not contained in a minimal prime contained in I: if such doesn't exist, then

$$\underline{I} \subseteq \bigcup_{\substack{p \supseteq \underline{I} \\ p \text{ minimal}}} p, \qquad \text{so } \underline{I} \subseteq p$$

for some p minimal and $ht(\underline{I}) = 0$

Similarly, given $a_1, \dots, a_{i-1}$, can choose $a_i \in \underline{I}$ not contained in a minimal prime over $(a_1, \dots, a_{i-1})$. If such doesn't exist, then $\exists$ such a minimal prime containing $\underline{I}$, so $ht\, \underline{I} \le r-1$.]

Thus by part ① of this theorem, $a_1, \dots, a_r$ are an A-regular sequence. So

$$\operatorname{ht} I \leq \operatorname{depth}_I A.$$

Conversely, if $b_1, \ldots, b_s \in I$

is an A-regular sequence, then by (I)

$$\operatorname{ht}(b_1, \ldots, b_s) = s \leq \operatorname{ht}(I) = r$$

Thus $\operatorname{depth}_I A \leq \operatorname{ht} I$.

$\therefore$ $\operatorname{depth}_I A = \operatorname{ht} I$.

Now $\operatorname{ht}(I) = \inf\{\operatorname{ht} \mathfrak{p} \mid \mathfrak{p} \in V(I)\}$

$$\dim A/I = \sup\{\dim A/\mathfrak{p} \mid \mathfrak{p} \in V(I)\}$$

So if $\operatorname{ht}(\mathfrak{p}) = \dim A - \dim A/\mathfrak{p}$

for all primes $\mathfrak{p}$, the equality

$$\operatorname{ht}(I) = \dim A - \dim A/I \quad \text{follows}$$

in general.

So let $\mathfrak{p}$ be a prime,

$n = \dim A = \operatorname{depth} A$, $\operatorname{ht} \mathfrak{p} = r$.

Then $A_\mathfrak{p}$ is C-M, and

$$\operatorname{ht}(\mathfrak{p}) = \dim A_\mathfrak{p} = \operatorname{depth}_\mathfrak{p}(A).$$

So we can find an A-regular sequence $a_1, \dots, a_r \in \mathfrak{p}$

Then $A/(a_1, \dots, a_r)$ is C-M of dimension $n-r$ and $\mathfrak{p}$ is a minimal prime over $(a_1, \dots, a_r)$ (as $\operatorname{ht}(a_1, \dots, a_r) = r$) Thus

$\dim A/\mathfrak{p} = n-r$ by part (1) of the previous theorem. $\blacksquare$

<u>Def</u>: Let A be Noetherian, $I \subseteq A$ an ideal, $\operatorname{Ass}_A(A/I) = \{\mathfrak{p}_1, \dots \mathfrak{p}_s\}$.

We say I is <u>unmixed</u> if

$ht(P_i) = ht(I) \ \forall i$.

We say the <u>unmixedness theorem holds</u> in A if whenever

$I = (a_1, \dots, a_r)$ is of height r,

I is unmixed.

[Note this is equivalent to A/I having no embedded primes, as all minimal primes over I have height $\geq r$.]

[Algebraic geometry: this says $V(I)$ is equi-dimensional.]

<u>Theorem</u>: Let A be Noetherian. Then A is Cohen-Macaulay if and only if the unmixedness

theorem holds in A.

pf: $\Leftarrow$ Suppose unmixedness holds.

Let $\mathfrak{p} \subseteq A$ be a prime of height r. We want to show $A_\mathfrak{p}$ is C-M, i.e., depth $A_\mathfrak{p} = \dim A_\mathfrak{p} = r$

We can find $a_1, \dots, a_r \in \mathfrak{p}$ so that $ht(a_1, \dots, a_i) = i$ for $1 \leq i \leq r$

[See proof of part ② of previous theorem, the bit in red.]

Then $(a_1, \dots, a_i)$ is unmixed by assumption, so a_{i+1} lies in no associated prime of $A/(a_1, \dots, a_i)$

so a_{i+1} is not a ~~zero~~-divisor in $A/(a_1, \dots, a_i)$.

Thus $a_1, \dots, a_r$ is a regular sequence in $\mathfrak{p}$, so

$$r \leq \operatorname{depth}_{\mathfrak{p}} A \leq \operatorname{depth}(A_{\mathfrak{p}}) \leq \dim A_{\mathfrak{p}} = r$$

(as an A-regular sequence in $\mathfrak{p}$ gives an $A_{\mathfrak{p}}$-regular sequence)

$\therefore$ $\operatorname{depth} A_{\mathfrak{p}} = r$. Thus $A_{\mathfrak{p}}$ is C-M.

$\Rightarrow$ Suppose A is C-M. To show A satisfies the unmixedness theorem, enough to show $A_{\mathfrak{m}}$ satisfies unmixedness for each maximal ideal $\mathfrak{m} \subseteq A$.

Pf: If $I = (a_1, \dots, a_r)$ is of height r, and $\mathfrak{p} \in \operatorname{Ass}_A(A/I)$.

then choose a maximal ideal m containing p. Then p^e is an associated prime of

$$(A/I)_m = A_m/I^e = A_m/(a_1, \dots, a_r).$$

So enough to show p^e is a minimal prime of $A_m/(a_1, \dots, a_r)$. $\boxtimes$

Thus we may assume A is a C-M local ring. We know (0) is unmixed. Let $(a_1, \dots, a_r)$ be of height $r > 0$. Then $a_1, \dots, a_r$ is a regular sequence by the previous theorem, and hence $A/(a_1, \dots, a_r)$ is also C-M, so $(a_1, \dots, a_r)$ is unmixed. $\boxtimes$

Theorem: Let A be C-M. Then $A[x_1, \dots, x_n]$ is also C-M.

Proof: Can assume $n=1$, so we'll show $B = A[x]$ is C-M.

Let $\mathfrak{p} \subseteq B$ be a prime, $q = \mathfrak{p} \cap A$

Want to show $B_{\mathfrak{p}}$ is C-M.

Now $B_{\mathfrak{p}}$ is a localization of $B_q = A_q[x]$. Thus we can assume, by replacing A with A_q, that A is a local ring and q is its maximal ideal.

Thus $B/qB = k[x]$, $k = A/q$ a field.

Recall the primes of B/qB

are in 1-1 correspondence with the primes $\mathfrak{p}$ of B with $\mathfrak{p} \cap B = \mathfrak{q}$,

i.e., $\exists$ a bijection

$$\operatorname{Spec} B/\mathfrak{q}B \longrightarrow f^{-1}(\mathfrak{q})$$

where $f: \operatorname{Spec} B \to \operatorname{Spec} A$ is induced by the inclusion $A \hookrightarrow B$.

Thus if $\mathfrak{p} \in f^{-1}(\mathfrak{q})$, there are two cases:

Case (1): $\mathfrak{p}$ corresponds to a maximal ideal in $k[x]$, so

$$\mathfrak{p} = \mathfrak{q}[x] + (f), \quad \text{where } f \in B$$

is a monic polynomial of positive degree, and whose image in $k[x]$ is irreducible.

Case 2 $\;\; \mathfrak{p} = \mathfrak{q}[x]$.

As B is flat over A, $B_{\mathfrak{p}}$ is also flat over A. Thus any A-regular sequence $a_1, \ldots, a_r$ in $\mathfrak{q}$ $(r = \dim A)$ is also $B_{\mathfrak{p}}$-regular

$\left(0 \to A/(a_1, \ldots, a_i) \xrightarrow{\cdot a_{i+1}} A/(a_1, \ldots, a_i) \right.$ can be tensored with $B_{\mathfrak{p}}$ to get injective map, showing $a_1, \ldots, a_r$ is a $B_{\mathfrak{p}}$-regular sequence.$\left. \right)$

Recall also we showed for a flat morphism $\phi: A \to B$, $\mathfrak{p} \in \operatorname{Spec} B$, $\mathfrak{q} = \phi^{-1}(\mathfrak{p}) = \phi^{*}(\mathfrak{p})$, then

$$ht(\mathfrak{p}) = ht(\mathfrak{q}) + ht(\mathfrak{p}/\mathfrak{q}^e)$$

Thus if $\mathfrak{p} = \mathfrak{q}[x] = \mathfrak{q}^e$, then

$ht(\mathfrak{p}) = ht(\mathfrak{q})$ (Case 2)

and if $\mathfrak{p} = \mathfrak{q}[x] + (f)$, we have

$ht(\mathfrak{p}) = ht(\mathfrak{q}) + 1$ (Case 1)

In Case 2, $\dim B_{\mathfrak{p}} = ht(\mathfrak{p}) = ht(\mathfrak{q}) = \dim A$

and we've seen $\operatorname{depth} B_{\mathfrak{p}} \geq \operatorname{depth} A$,

$\therefore \operatorname{depth} B_{\mathfrak{p}} = \dim B_{\mathfrak{p}}$, so C-M

In Case 1, $\dim B_{\mathfrak{p}} = ht(\mathfrak{p}) = ht(\mathfrak{q}) + 1 = \dim A + 1$

But $a_1, \ldots, a_r, f$ form a regular

sequence for $B_{\mathfrak{p}}$ since f is

a non-zero divisor in

$$(A/(a_1, \ldots, a_r))[x],$$

being a monic polynomial.

Thus depth $B_\mathfrak{p} \geq r+1$, so depth $B_\mathfrak{p} = \dim B_\mathfrak{p}$, hence $B_\mathfrak{p}$ is C-M. $\blacksquare$

Goal: We will show regular local rings are C-M.

Theorem: Let $(A, \mathfrak{m})$ be a regular local ring, $x_1, \ldots, x_i \in \mathfrak{m}$.
Then the following are equivalent:

① $x_1, \ldots, x_i$ is a subset of a system of parameters for A generating $\mathfrak{m}$.

② The images of $x_1, \ldots, x_i$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $k = A/\mathfrak{m}$

③ $A/(x_1, \dots, x_i)$ is an $\dim A - i$ dimensional regular local ring.

Pf: ① $\Rightarrow$ ② If $x_1, \dots, x_i, x_{i+1}, \dots, x_n$ is a system of parameters generating m, $n = \dim A$, then their images $\bar{x}_1, \dots, \bar{x}_n \in m/m^2$ span m/m^2.

But $\dim_k m/m^2 = n = \dim A$, since A regular, so $\bar{x}_1, \dots, \bar{x}_n$ are linearly independent.

① $\Rightarrow$ ③ $\dim A/(x_1, \dots, x_i) = \dim A - i$ since dividing out by a sequence of elements of a system of parameters decrease the dimension by at most one at each step, and $A/(x_1, \dots, x_n) = A/m = k$ is dim. 0.

Also, $x_{i+1}, \ldots, x_n$ generate the maximal ideal of $A/(x_1, \ldots, x_i)$, hence the latter ring is also regular.

(3) $\Rightarrow$ (1) If the maximal ideal $m/(x_1, \ldots, x_i)$ of $A/(x_1, \ldots, x_i)$ is generated by images of $y_1, \ldots, y_{n-i} \in m$, then $x_1, \ldots, x_i, y_1, \ldots, y_{n-i}$ generate m, and hence are a system of parameters.

(2) $\Rightarrow$ (1) Using $\dim_k m/m^2 = n$, we can choose $x_{i+1}, \ldots, x_n \in m$ so that $\bar{x}_1, \ldots, \bar{x}_n \in m/m^2$ form a basis for m/m^2. Then by Nakayama's lemma, $x_1, \ldots, x_n$ generate m.

Thus $x_1, \ldots, x_i$ are a subset of a system of parameters. $\boxed{/}$

<u>Theorem</u>: A regular local ring is an integral domain.

<u>pf</u>: Let (A, m) be regular local ring of dimension n. Proof by induction on n.

If $n=0$, then m is generated by zero elements, i.e., $m = (0)$, so A is a field.

If $n=1$, then $m=(x)$ is principal, $ht(m)=1$, so $\exists$ a prime ideal $P \subsetneq m$. If $y \in P$, we can write $y = ax$ for some $a \in A$

and since $x \notin \mathfrak{p}$, $a \in \mathfrak{p}$. Thus

$\mathfrak{p} = x\mathfrak{p}$. (Every element of $\mathfrak{p}$ is an element of $\mathfrak{p}$ times x.)

Hence by Nakayama's lemma $\mathfrak{p} = 0$.

Thus A is an integral domain.

If $n > 1$, let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal primes of A. Since $\mathfrak{m} \not\subseteq \mathfrak{m}^2$ and $\mathfrak{m} \not\subseteq \mathfrak{p}_i$, there exists an element $x \in \mathfrak{m}$ not lying in $\mathfrak{p}_1, \dots, \mathfrak{p}_r, \mathfrak{m}^2$

Thus the image of x in $\mathfrak{m}/\mathfrak{m}^2$ is non-zero, so $A/(x)$ is a regular local ring of dimension $n-1$ by previous theorem. Thus by induction this is an integral domain.

Thus (x) is a prime ideal.

Thus $\mathfrak{P}_i \subsetneq (x)$ for some i

The same argument as in $n=1$ case

implies $\mathfrak{P}_i = 0$. Thus A is an

integral domain. $\boxtimes$

<u>Cor</u>: A regular local ring is

Cohen-Macaulay.

<u>Pf</u>: A system of parameters $x_1, \dots, x_n$

generating $\mathfrak{m} \subseteq A$ is a regular

sequence since

$$A, \quad A/(x_1), \quad \dots, \quad A/(x_1, \dots, x_n)$$

are regular local rings, hence integral

domains. $\boxtimes$

Natural theorems to look at next:

Serre's characterization of regular local rings.

For M an A-module, we define

$$\text{Proj. Dim } M = \inf \{ n \mid 0 \to P_n \to \cdots \to P_0 \to M \to 0 \text{ is a projective resolution of } M \}$$

The global dimension of a ring A is

$$\text{gl. dim } A = \sup \{ \text{proj. dim } M \mid M \text{ an } A\text{-module} \}$$

Theorem (Serre) Let A be Noetherian, local. Then

$$A \text{ is regular} \iff \text{gl. dim. } A = \dim A$$

$$\iff \text{gl. dim. } A < \infty.$$

This is used to prove:

Theorem. (Serre) If A is a regular local ring, $\mathfrak{p} \subseteq A$ prime, then $A_{\mathfrak{p}}$ is also a regular local ring.

Theorem: (Auslander-Buchsbaum) Regular local rings are UFDs.

Auslander - Buchsbaum