Toric Varieties

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Preface

The study of toric varieties is a wonderful part of algebraic geometry. There are elegant theorems and deep connections with polytopes, polyhedra, combinatorics, commutative algebra, symplectic geometry, and topology. Toric varieties also have unexpected applications in areas as diverse as physics, coding theory, algebraic statistics, and geometric modeling. Moreover, as noted by Fulton [**105**], "toric varieties have provided a remarkably fertile testing ground for general theories." At the same time, the concreteness of toric varieties provides an excellent context for someone encountering the powerful techniques of modern algebraic geometry for the first time. Our book is an introduction to this rich subject that assumes only a modest background yet leads to the frontier of this active area of research.

Brief Summary. The text covers standard material on toric varieties, including:

(a) Convex polyhedral cones, polytopes, and fans.

(b) Affine, projective, and abstract toric varieties.

(c) Complete toric varieties and proper toric morphisms.

(d) Weil and Cartier divisors on toric varieties.

(e) Cohomology of sheaves on toric varieties.

(f) The classical theory of toric surfaces.

(g) The topology of toric varieties.

(h) Intersection theory on toric varieties.

These topics are discussed in earlier texts on the subject, such as [**93**], [**105**] and [**218**]. One difference is that we provide more details, with numerous examples, figures, and exercises to illustrate the concepts being discussed. We also provide background material when needed. In addition, we cover a large number of topics previously available only in the research literature.

The Fifteen Chapters. To give you a better idea of what is in the book, we now highlight a few topics from each chapter.

Chapters 1, 2 and 3 cover the basic material mentioned in items (a)–(c) above. The toric varieties encountered include:

- The affine toric variety $Y_{\mathscr{A}}$ of a finite set $\mathscr{A} \subseteq M \simeq \mathbb{Z}^n$ (Chapter 1).
- The affine toric variety U_σ of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ (Chapter 1).
- The projective toric variety $X_{\mathscr{A}}$ of a finite set $\mathscr{A} \subseteq M \simeq \mathbb{Z}^n$ (Chapter 2).
- The projective toric variety X_P of a lattice polytope $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ (Chapter 2).
- The abstract toric variety X_Σ of a fan Σ in $N_{\mathbb{R}} \simeq \mathbb{R}^n$ (Chapter 3).

Chapter 4 introduces Weil and Cartier divisors on toric varieties. We compute the class group and Picard group of a toric variety and define the sheaf $\mathscr{O}_{X_\Sigma}(D)$ associated to a Weil divisor D on a toric variety X_Σ.

Chapter 5 shows that the classical construction $\mathbb{P}^n = \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\mathbb{C}^*$ can be generalized to any toric variety X_Σ. The homogeneous coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$ of $\mathbb{P}^n$ also has a toric generalization, called the total coordinate ring of X_Σ.

Chapter 6 relates Cartier divisors to invertible sheaves on X_Σ. We introduce ample, basepoint free, and nef divisors and discuss their relation to convexity. The stucture of the nef cone and its dual, the Mori cone, are described in detail, as is the intersection pairing between divisors and curves.

Chapter 7 extends the relation between polytopes and projective toric varieties to a relation between polyhedra and projective toric morphisms $\phi : X_\Sigma \to U_\sigma$. We also discuss projective bundles over a toric variety and use these to classify smooth projective toric varieties of Picard number 2.

Chapter 8 relates Weil divisors to reflexive sheaves of rank one and defines Zariski p-forms. For $p = \dim X$, this gives the canonical sheaf ω_X and canonical divisor K_X. In the toric case we describe these explicitly and study the relation between reflexive polytopes and Gorenstein Fano toric varieties, meaning that $-K_{X_\Sigma}$ is ample. We find the 16 reflexive polygons in $\mathbb{R}^2$ (up to equivalence) and note the relation $|\partial P \cap M| + |\partial P^\circ \cap N| = 12$ for a reflexive polygon P and its dual P°.

Chapter 9 is about sheaf cohomology. We give two methods for computing sheaf cohomology on a toric variety and prove a dizzying array of cohomology vanishing theorems. Applications range from showing that normal toric varieties are Cohen-Macaulay to the Dehn-Sommerville equations for a simple polytope and counting lattice points in multiples of a polytope via the Ehrhart polynomial.

Chapter 10 studies toric surfaces, where we add a few twists to this classical subject. After using Hirzebruch-Jung continued fractions to compute the minimal resolution of a toric surface singularity, we discuss the toric meaning of ordinary continued fractions. We then describe unexpected connections with Gröbner fans and the McKay correspondence. Finally, we use the Riemann-Roch theorem on a smooth complete toric surface to explain the mysterious appearance of the number 12 in Chapter 8 when counting lattice points in reflexive polygons.

Chapter 11 proves the existence of toric resolutions of singularities for toric varieties of all dimensions. This is more complicated than for surfaces because of the existence of toric flips and flops. We consider simple normal crossing, crepant, log, and embedded resolutions and study how Rees algebras and multiplier ideals can be applied in the resolution problem. We also discuss toric singularities and show that a fan Σ is simplicial if and only if X_Σ has at worst finite quotient singularities and hence is rationally smooth. We also explain what canonical and terminal singularities mean in the toric context.

Chapters 12 and 13 describe the singular and equivariant cohomology of a complete simplicial toric variety X_Σ and prove the Hirzebruch-Riemann-Roch and equivariant Riemann-Roch theorems when X_Σ is smooth. We compute the fundamental group of X_Σ and study the moment map, with a brief mention of topological models of toric varieties and connections with symplectic geometry. We describe the Chow ring and intersection cohomology of a complete simplicial toric variety. After proving Riemann-Roch, we give applications to the volume polynomial and lattice point enumeration in polytopes.

Chapters 14 and 15 explore the rich connections that link geometric invariant theory, the secondary fan, the nef and moving cones, Gale duality, triangulations, wall crossings, flips, extremal contractions, and the toric minimal model program.

Appendices. The book ends with three appendices:

- Appendix A: The History of Toric Varieties.
- Appendix B: Computational Methods.
- Appendix C: Spectral Sequences.

Appendix A surveys the history of toric geometry since its origins in the early 1970s. It is fun to see how the concepts and terminology evolved. Appendix B discusses some of the software packages for toric geometry and gives examples to illustrate what they can do. Appendix C gives a brief introduction to spectral sequences and describes the spectral sequences used in Chapters 9 and 12.

Prerequisites. We assume that the reader is familiar with the material covered in basic graduate courses in algebra and topology, and to a somewhat lesser degree, complex analysis. In addition, we assume that the reader has had some previous experience with algebraic geometry, at the level of any of the following texts:

- *Ideals, Varieties and Algorithms* by Cox, Little and O'Shea [**69**].
- *Introduction to Algebraic Geometry* by Hassett [**133**].
- *Elementary Algebraic Geometry* by Hulek [**151**].
- *Undergraduate Algebraic Geometry* by Reid [**238**].

- *Computational Algebraic Geometry* by Schenck [**246**].
- *An Invitation to Algebraic Geometry* by Smith, Kahanpää, Kekäläinen and Traves [**253**].

Chapters 9 and 12 assume knowledge of some basic algebraic topology. The books by Hatcher [**135**] and Munkres [**210**] are useful references here.

Readers who have studied more sophisticated algebraic geometry texts such as Harris [**130**], Hartshorne [**131**], or Shafarevich [**245**] certainly have the background needed to read our book. For readers with a more modest background, an important prerequisite is a willingness to absorb a lot of algebraic geometry.

Background Sections. Since we do not assume a complete knowledge of algebraic geometry, Chapters 1–9 each begin with a background section that introduces the definitions and theorems from algebraic geometry that are needed to understand the chapter. References where proofs can be found are provided. The remaining chapters do not have background sections. For some of those chapters, no further background is necessary, while for others, the material is more sophisticated and the requisite background is given by careful references to the literature.

What Is Omitted. We work exclusively with varieties defined over the complex numbers $\mathbb{C}$. This means that we do not consider toric varieties over arbitrary fields (see [**92**] for a treatment of this topic), nor do we consider toric stacks (see [**39**] for an introduction). Moreover, our viewpoint is primarily algebro-geometric. Thus, while we hint at some of the connections with symplectic geometry and topology in Chapter 12, we do not do justice to this side of the story. Even within the algebraic geometry of toric varieties, there are many topics we have had to omit, though we provide some references that should help readers who want to explore these areas. We have also omitted any discussion of how toric varieties are used in physics and applied mathematics. Some pointers to the literature are given in our discussion of the recent history of toric varieties in §A.2 of Appendix A.

The Structure of the Text. We number theorems, propositions, and equations based on the chapter and the section. Thus §3.2 refers to Section 2 of Chapter 3, and Theorem 3.2.6, equation (3.2.6) and Exercise 3.2.6 all appear in this section. Definitions, theorems, propositions, lemmas, remarks, and examples are numbered together in one sequence within each section.

Some individual chapters have appendices. Within a chapter appendix the same numbering system is used, except that the section number is a capital A. This means that Theorem 3.A.3 is in the appendix to Chapter 3. On the other hand, the three appendices at the end of the book are treated in the numbering system as chapters A, B, and C. Thus Definition C.1.1 is in the first section of Appendix C.

The end (or absence) of a proof is indicated by $\square$, and the end of an example is indicated by $\Diamond$.

For the Instructor. There is much more material here than you can cover in any one-semester graduate course, probably more than you can cover in a full year in most cases. So choices will be necessary depending on the background and the interests of the student audience. We think it is reasonable to expect to cover most of Chapters 1–6, 8 and 9 in a one-semester course where the students have a minimal background in algebraic geometry. More material can be covered, of course, if the students know more algebraic geometry. If time permits, you can use toric surfaces (Chapter 10) to illustrate the power of the basic material and introduce more advanced topics such as the resolution of singularities (Chapter 11) and the Riemann-Roch theorem (Chapter 13).

Finally, we emphasize that the exercises are extremely important. We have found that when the students work in groups and present their solutions, their engagement with the material increases. We encourage instructors to consider using this strategy.

For the Student. The book assumes that you will be an active reader. This means in particular that you should do tons of exercises—this is the best way to learn about toric varieties. If you have a modest background in algebraic geometry, then reading the book requires a commitment to learn *both* toric varieties *and* algebraic geometry. It will be a lot of work but is worth the effort. This is a great subject.

Send Us Feedback. We greatly appreciate hearing from instructors, students, or general readers about what worked and what didn't. Please notify one or all of us about any typographical or mathematical errors you might find.

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Notation

The notation used in the book is organized by topic. The number in parentheses at the end of an entry indicates the chapter in which the notation first appears.

Basic Sets

$\mathbb{Z},\ \mathbb{Q},\ \mathbb{R},\ \mathbb{C}$	integers, rational numbers, real numbers, complex numbers	
$\mathbb{N}$	semigroup of nonnegative integers $\{0,1,2,\dots\}$	

The Torus

$\mathbb{C}^*$	multiplicative group of nonzero complex numbers $\mathbb{C}\setminus\{0\}$	(1)
$(\mathbb{C}^*)^n$	standard n-dimensional torus	(1)
$M,\ \chi^m$	character lattice of a torus and character of $m \in M$	(1)
$N,\ \lambda^u$	lattice of one-parameter subgroups of a torus and one-parameter subgroup of $u \in N$	(1)
T_N	torus $N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$ associated to N and M	(1)
$M_{\mathbb{R}},\ M_{\mathbb{Q}}$	vector spaces $M \otimes_{\mathbb{Z}} \mathbb{R},\ M \otimes_{\mathbb{Z}} \mathbb{Q}$ built from M	(1)
$N_{\mathbb{R}},\ N_{\mathbb{Q}}$	vector spaces $N \otimes_{\mathbb{Z}} \mathbb{R},\ N \otimes_{\mathbb{Z}} \mathbb{Q}$ built from N	(1)
$\langle m,u \rangle$	pairing of $m \in M$ or $M_{\mathbb{R}}$ with $u \in N$ or $N_{\mathbb{R}}$	(1)

Hyperplanes and Half-Spaces

H_m	hyperplane in $N_{\mathbb{R}}$ defined by $\langle m, - \rangle = 0$, $m \in M_{\mathbb{R}} \setminus \{0\}$	(1)
H_m^+	half-space in $N_{\mathbb{R}}$ defined by $\langle m, - \rangle \geq 0$, $m \in M_{\mathbb{R}} \setminus \{0\}$	(1)
$H_{u,b}$	hyperplane in $M_{\mathbb{R}}$ defined by $\langle -, u \rangle = b$, $u \in N_{\mathbb{R}} \setminus \{0\}$	(2)
$H_{u,b}^+$	half-space in $M_{\mathbb{R}}$ defined by $\langle -, u \rangle \geq b$, $u \in N_{\mathbb{R}} \setminus \{0\}$	(2)

Cones

$\mathrm{Cone}(S)$	convex cone generated by S	(1)
σ	rational convex polyhedral cone in $N_{\mathbb{R}}$	(1)
$\mathrm{Span}(\sigma)$	subspace spanned by σ	(1)
$\dim \sigma$	dimension of σ	(1)
$\sigma^\vee$	dual cone of σ	(1)
$\mathrm{Relint}(\sigma)$	relative interior of σ	(1)
$\mathrm{Int}(\sigma)$	interior of σ when $\mathrm{Span}(\sigma) = N_{\mathbb{R}}$	(1)
$\sigma^\perp$	set of $m \in M_{\mathbb{R}}$ with $\langle m, \sigma\rangle = 0$	(1)
$\tau \preceq \sigma,\ \tau \prec \sigma$	τ is a face or proper face of σ	(1)
τ^*	face of $\sigma^\vee$ dual to $\tau \subseteq \sigma$, equals $\sigma^\vee \cap \tau^\perp$	(1)

Rays

ρ	1-dimensional strongly convex cone (a ray) in $N_{\mathbb{R}}$	(1)
u_ρ	minimal generator of $\rho \cap N$, ρ a rational ray in $N_{\mathbb{R}}$	(1)
$\sigma(1)$	rays of a strongly convex cone σ in $N_{\mathbb{R}}$	(1)

Lattices

$\mathbb{Z}\mathscr{A}$	lattice generated by $\mathscr{A}$	(1)
$\mathbb{Z}'\mathscr{A}$	elements $\sum_{i=1}^s a_i m_i \in \mathbb{Z}\mathscr{A}$ with $\sum_{i=1}^s a_i = 0$	(2)
N_σ	sublattice $\mathbb{Z}(\sigma \cap N) = \mathrm{Span}(\sigma) \cap N$	(3)
$N(\sigma)$	quotient lattice N/N_σ	(3)
$M(\sigma)$	dual lattice of $N(\sigma)$, equals $\sigma^\perp \cap M$	(3)

Fans

Σ	fan in $N_{\mathbb{R}}$	(2,3)
$\Sigma(r)$	r-dimensional cones of Σ	(3)
$\Sigma_{\max}$	maximal cones of Σ	(3)
$\mathrm{Star}(\sigma)$	star of σ, a fan in $N(\sigma)$	(3)
$\Sigma^*(\sigma)$	star subdivision of Σ for $\sigma \in \Sigma$	(3)
$\Sigma^*(v)$	star subdivision of Σ for $v \in \lvert\Sigma\rvert \cap N$ primitive	(11)

Polytopes and Polyhedra

Δ_n	standard n-simplex in $\mathbb{R}^n$	(2)
$\mathrm{Conv}(S)$	convex hull of S	(1)
$\dim P$	dimension of a polyhedron P	(2)
$Q \preceq P,\ Q \prec P$	Q is a face or proper face of P	(2)
P°	dual or polar of a polytope	(2)
$A + B$	Minkowski sum	(2)
kP	multiple of a polytope or polyhedron	(2)

Cones Built From Polyhedra

C_{v}	$\mathrm{Cone}(P-\mathsf{v})$ for a vertex v of a polytope or polyhedron	(2)
σ_Q	cone of a face $Q \preceq P$ in the normal fan Σ_P	(2)
Σ_P	normal fan of a polytope or polyhedron P	(2)
$C(P)$	cone over a polytope or polyhedron	(1)
S_P	semigroup algebra of $C(P)\cap(M\times\mathbb{Z})$	(7)

Combinatorics and Lattice Points of Polytopes

f_i	number of i-dimesional faces of P	(9)
h_p	$\sum_{i=p}^{n}(-1)^{i-p}\binom{i}{p} f_i$, equals Betti number $b_{2p}(X_P)$ when P simple	(9)
$L(P)$	number of lattice points of a lattice polytope	(9)
$L^*(P)$	number of interior lattice points of a lattice polytope	(9)
$\mathrm{Ehr}_P(\ell)$	Ehrhart polynomial of a lattice polytope	(9)
$\mathrm{Ehr}_P^p(\ell)$	p-Ehrhart polynomial of a lattice polytope	(9)

Semigroups

S, $\mathbb{C}[\mathsf{S}]$	affine semigroup and its semigroup algebra	(1)
$\mathbb{N}\mathscr{A}$	affine semigroup generated by $\mathscr{A}$	(1)
$\mathsf{S}_\sigma = \mathsf{S}_{\sigma,N}$	affine semigroup $\sigma^\vee\cap M$	(1)
$\mathscr{H}$	Hilbert basis of S_σ	(1)

Rings

$R_f,\ R_S,\ R_\mathfrak{p}$	localization of R at f, a multiplicative set S, a prime ideal $\mathfrak{p}$	(1)
R'	integral closure of the integral domain R	(1)
$\widehat{R}$	completion of local ring R	(1)
$R\otimes_\mathbb{C} S$	tensor product of rings over $\mathbb{C}$	(1)
R^G	ring of invariants of G acting on R	(1,5)
$R[\mathfrak{a}]$	Rees algebra of an ideal $\mathfrak{a}\subseteq R$	(11)
$R^{[\ell]}$	Veronese subring of a graded ring R	(14)

Specific Rings

$\mathbb{C}[x_1,\dots,x_n]$	polynomial ring in n variables	(1)
$\mathbb{C}[[x_1,\dots,x_n]]$	formal power series ring in n variables	(1)
$\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$	ring of Laurent polynomials	(1)
$\mathbf{I}(V)$	ideal of an affine or projective variety	(1,2)
$\mathbb{C}[V]$	coordinate ring of an affine or projective variety	(1,2)
$\mathbb{C}[V]_d$	graded piece in degree d when V is projective	(2)
$\mathbb{C}(V)$	field of rational functions when V is irreducible	(1)
$\mathscr{O}_{V,p},\ \mathfrak{m}_{V,p}$	local ring of a variety at a point and its maximal ideal	(1)

Varieties

$\mathbf{V}(I)$	affine or projective variety of an ideal	(1,2)
V_f	subset of an affine variety V where $f \neq 0$	(1)
$\overline{S}$	Zariski closure of S in a variety	(1,3)
$T_p(X)$	Zariski tangent space of a variety at a point	(1,3)
$\dim X,\ \dim_p X$	dimension of a variety and dimension at a point	(1,3)
$\mathrm{Spec}(R)$	affine variety of coordinate ring R	(1)
$\mathrm{Proj}(S)$	projective variety of graded ring S	(7)
$X \times Y$	product of varieties	(1,3)
$X \times_S Y$	fiber product of varieties	(3)
$\widehat{X}$	affine cone of a projective variety X	(2)

Toric Varieties

$Y_{\mathscr{A}},\ X_{\mathscr{A}}$	affine and projective toric variety of $\mathscr{A} \subseteq M$	(1,2)
$U_\sigma = U_{\sigma,N}$	affine toric variety of a cone $\sigma \subseteq N_{\mathbb{R}}$	(1)
$X_\Sigma = X_{\Sigma,N}$	toric variety of a fan Σ in $N_{\mathbb{R}}$	(3)
X_P	projective toric variety of a lattice polytope or polyhedron	(2.7)
$\overline{\phi}$	lattice homomorphism of a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$	(1,3)
$\overline{\phi}_{\mathbb{R}}$	real extension of $\overline{\phi}$	(1)
γ_σ	distinguished point of U_σ	(3)
$O(\sigma)$	torus orbit corresponding to $\sigma \in \Sigma$	(3)
$V(\sigma) = \overline{O(\sigma)}$	closure of orbit of $\sigma \in \Sigma$, toric variety of $\mathrm{Star}(\sigma)$	(3)
U_P	affine toric variety of recession cone of a polyhedron	(7)
U_Σ	affine toric variety of a fan with convex support	(7)

Specific Varieties

$\mathbb{C}^n,\ \mathbb{P}^n$	affine and projective n-dimensional space	(1,2)
$\mathbb{P}(q_0,\ldots,q_n)$	weighted projective space	(2)
$\widehat{C}_d,\ C_d$	rational normal cone and curve	(1,2)
$\mathrm{Bl}_0(\mathbb{C}^n)$	blowup of $\mathbb{C}^n$ at the origin	(3)
$\mathrm{Bl}_{V(\sigma)}(X_\Sigma)$	blowup of X_Σ along $V(\sigma)$, toric variety of $\Sigma^*(\sigma)$	(3)
$\mathscr{H}_r$	Hirzebruch surface	(3)
$S_{a,b}$	rational normal scroll	(3)

Total Coordinate Ring

S	total coordinate ring of X_Σ	(5)
x_ρ	variable in S corresponding to $\rho \in \Sigma(1)$	(5)
S_β	graded piece of S in degree $\beta \in \mathrm{Cl}(X_\Sigma)$	(5)
$\deg(x^\alpha)$	degree in $\mathrm{Cl}(X_\Sigma)$ of a monomial in S	(5)

$x^{\hat{\sigma}}$	monomial $\prod_{\rho \notin \sigma(1)} x_\rho$ for $\sigma \in \Sigma$	(5)
$B(\Sigma)$	irrelevant ideal of S, generated by the $x^{\hat{\sigma}}$	(5)
$x^{\langle m \rangle}$	Laurent monomial $\prod_\rho x_\rho^{\langle m, u_\rho \rangle}$, $m \in M$	(5)
$x^{\langle m, D \rangle}$	homogenization of χ^m, $m \in P_D \cap M$	(5)
x_F	facet variable of a facet $F \prec P$	(5)
$x^{\langle m, P \rangle}$	P-monomial associated to $m \in P \cap M$	(5)
$x^{\langle \mathrm{v}, P \rangle}$	vertex monomial associated to vertex $\mathrm{v} \in P \cap M$	(5)
M	graded S-module	(5)
$\mathsf{M}(\alpha)$	shift of M by $\alpha \in \mathrm{Cl}(X_\Sigma)$	(5)

Quotient Construction

X/G	good geometric quotient	(5)
$X/\!/G$	good categorical quotient	(5)
$Z(\Sigma)$	exceptional set in quotient construction, equals $\mathbf{V}(B(\Sigma))$	(5)
G	group in quotient construction, equals $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$	(5)

Divisors

$\mathscr{O}_{X,D}$	local ring of a variety at a prime divisor	(4)
ν_D	discrete valuation of a prime divisor D	(4)
$\mathrm{div}(f)$	principal divisor of a rational function	(4)
$D \sim E$	linear equivalence of divisors	(4)
$D \geq 0$	effective divisor	(4)
$\mathrm{Div}_0(X)$	group of principal divisors on X	(4)
$\mathrm{Div}(X)$	group of Weil divisors on X	(4)
$\mathrm{CDiv}(X)$	group of Cartier divisors on X	(4)
$\mathrm{Cl}(X)$	divisor class group of a normal variety X	(4)
$\mathrm{Pic}(X)$	Picard group of a normal variety X	(4)
$\mathrm{Pic}(X)_{\mathbb{R}}$	$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$	(6)
$\mathrm{Supp}(D)$	support of a divisor	(4)
$D\|_U$	restriction of a divisor to an open set	(4)
$\{(U_i, f_i)\}$	local data of a Cartier divisor on X	(4)
$\lvert D \rvert$	complete linear system of D	(6)
$\lfloor D \rfloor$, $\lceil D \rceil$	"round down" and "round up" of a $\mathbb{Q}$-divisor	(9)

Torus-Invariant Divisors

$D_\rho = \overline{O(\rho)}$	torus-invariant prime divisor on X_Σ of ray $\rho \in \Sigma(1)$	(4)
D_F	torus-invariant prime divisor on X_P of facet $F \prec P$	(4)
$\mathrm{Div}_{T_N}(X_\Sigma)$	group of torus-invariant Weil divisors on X_Σ	(4)
$\mathrm{CDiv}_{T_N}(X_\Sigma)$	group of torus-invariant Cartier divisors on X_Σ	(4)
$\{m_\sigma\}_{\sigma \in \Sigma}$	Cartier data of a torus-invariant Cartier divisor on X_Σ	(4)

D_P	Cartier divisor of a polytope or polyhedron	(4,7)
P_D	polyhedron of a torus-invariant divisor	(4)
X_D	toric variety of a basepoint free divisor	(6)
Σ_D	fan of X_D	(6)
ϕ^*D	pullback of a Cartier divisor	(6)

Support Functions

φ_D	support function of a Cartier divisor	(4)
φ_P	support function of a polytope or polyhedron	(4)
$\mathrm{SF}(\Sigma)$	support functions for Σ	(4)
$\mathrm{SF}(\Sigma, N)$	support functions for Σ integral with respect to N	(4)

Sheaves

$\Gamma(U, \mathscr{F})$	sections of a sheaf over an open set	(4)
$\mathscr{F}\|_U$	restriction of a sheaf to an open set	(4)
$\mathscr{F}_p$	stalk of a sheaf at a point	(6)
$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$	tensor product of sheaves of $\mathscr{O}_X$-modules	(6)
$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$	sheaf of homomorphisms	(6)
$\mathscr{F}^\vee$	dual sheaf of $\mathscr{F}$, equals $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$	(6)
$f_*\mathscr{F}$	direct image sheaf	

Specific Sheaves

$\mathscr{O}_X$	structure sheaf of a variety X	(3)
$\mathscr{O}_X^*$	sheaf of invertible elements of $\mathscr{O}_X$	(4)
$\mathscr{K}_X$	constant sheaf of rational functions for X irreducible	(6)
$\mathscr{O}_X(D)$	sheaf of a Weil divisor D on X	(4)
$\mathscr{I}_Y$	ideal sheaf of a subvariety $Y \subseteq X$	(3)
$\widetilde{M}$	sheaf on $\mathrm{Spec}(R)$ of an R-module M	(4)
$\widetilde{\mathsf{M}}$	sheaf on X_Σ of the graded S-modules M	(5)
$\mathscr{O}_{X_\Sigma}(\alpha)$	sheaf of the S-module $S(\alpha)$	(5)

Vector Bundles and Locally Free Sheaves

$\mathscr{L}$, $\mathscr{E}$	invertible sheaf (line bundle) and locally free sheaf	(6)
$\pi : V \to X$	vector bundle	(6)
$\pi : V_{\mathscr{L}} \to X$	rank 1 vector bundle of an invertible sheaf $\mathscr{L}$	(6)
$f^*\mathscr{L}$	pullback of an invertible sheaf	(6)
$\phi_{\mathscr{L},W}$	map to projective space determined by $W \subseteq \Gamma(X, \mathscr{L})$	(6)
$\mathbb{P}(V)$, $\mathbb{P}(\mathscr{E})$	projective bundle of vector bundle or locally free sheaf	(7)
$\Sigma \times D$	fan for rank 1 vector bundle $V_{\mathscr{L}}$ for $\mathscr{L} = \mathscr{O}_{X_\Sigma}(D)$	(7)

Intersection Theory

$\deg(D)$	degree of a divisor on a smooth complete curve	(6)
$D \cdot C$	intersection product of Cartier divisor and complete curve	(6)
$D \equiv D'$, $C \equiv C'$	numerically equivalent Cartier divisors and complete curves	(6)
$N^1(X)$, $N_1(X)$	$(\mathrm{CDiv}(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R}$ and $(\{\text{proper 1-cycles on } X\}/\equiv) \otimes_{\mathbb{Z}} \mathbb{R}$	(6)
$\mathrm{Nef}(X)$	cone in $N^1(X)$ generated by nef divisors	(6)
$\overline{\mathrm{Mov}}(X)$	moving cone of a variety X in $N^1(X)$	(15)
$\overline{\mathrm{Eff}}(X)$	pseudoeffective cone of a variety X in $N^1(X)$	(15)
$\mathrm{NE}(X)$	cone in $N_1(X)$ generated by complete curves	(6)
$\overline{\mathrm{NE}}(X)$	Mori cone, equals the closure of $\mathrm{NE}(X)$	(6)

Differential Forms and Sheaves

$\Omega_{R/\mathbb{C}}$	module of Kähler differentials of a $\mathbb{C}$-algebra R	(8)
Ω^1_X, $\mathscr{T}_X$	cotangent and tangent sheaves of a variety X	(8)
$\mathscr{I}_Y/\mathscr{I}_Y^2$, $\mathscr{N}_{Y/X}$	conormal and normal sheaves of $Y \subseteq X$	(8)
Ω^p_X, $\widehat{\Omega}^p_X$	sheaves of p-forms and Zariski p-forms on X	(8)
K_X, ω_X	canonical divisor and canonical sheaf $\widehat{\Omega}^n_X$, $n = \dim X$	(8)
$\Omega^1_X(\log D)$	sheaf of 1-forms with logarithmic poles on D	(8)

Sheaf Cohomology

$H^0(X,\mathscr{F})$	global sections $\Gamma(X,\mathscr{F})$ of a sheaf $\mathscr{F}$ on X	(9)
$H^p(X,\mathscr{F})$	p-th sheaf cohomology group of a sheaf $\mathscr{F}$ on X	(9)
$R^p f_*\mathscr{F}$	higher direct image sheaf	(9)
$\mathrm{Ext}^p_{\mathscr{O}_X}(\mathscr{G},\mathscr{F})$	Ext groups of sheaves of $\mathscr{O}_X$-modules $\mathscr{G}$, $\mathscr{F}$	(9)
$\check{C}^\bullet(\mathscr{U},\mathscr{F})$	Čech complex for sheaf cohomology	(9)
$\chi(\mathscr{F})$	Euler characteristic of $\mathscr{F}$, equals $\sum_p (-1)^p \dim H^p(X,\mathscr{F})$	(9)

Sheaf Cohomology of a Toric Variety

$H^p(X_\Sigma,\mathscr{L})_m$	graded piece of sheaf cohomology of $\mathscr{L} = \mathscr{O}_{X_\Sigma}(D)$ for $m \in M$	(9)
$V_{D,m}$, $V^{\mathrm{supp}}_{D,m}$	subsets of $\lvert\Sigma\rvert$ used to compute $H^p(X_\Sigma,\mathscr{L})_m$	(9)

Local Cohomology

$H^p_I(\mathsf{M})$	p-th local cohomology of an R-module M for the ideal $I \subseteq R$	(9)
$\check{C}^\bullet(\mathbf{f},\mathsf{M})$	Čech complex for local cohomology when $I = \langle \mathbf{f} \rangle$	(9)
$\mathrm{Ext}^p_R(\mathsf{N},\mathsf{M})$	Ext groups of R-modules N, M	(9)

Resolution of Singularities

X_{sing}	singular locus of a variety	(11)
$\mathrm{Exc}(\phi)$	exceptional locus of a resolution of singularities	(11)

$\mathcal{J}(c\cdot\mathscr{I})$	multiplier ideal sheaf	(11)
(X,D)	log pair, $D=\sum_i a_iD_i$, $a_i\in[0,1]\cap\mathbb{Q}$	(11)

Singularities of Toric Varieties

$\mathrm{mult}(\sigma)$	multiplicity of a simplicial cone, equals $[N_\sigma : \mathbb{Z}u_1+\cdots+\mathbb{Z}u_d]$	(6,11)
P_σ	parallelotope of a simplicial cone, equals $\{\sum_i \lambda_i u_i \mid 0\le\lambda_i<1\}$	(11)
Π_σ	polytope related to canonical and terminal singularities of U_σ	(11)
Θ_σ	the polyhedron $\mathrm{Conv}(\sigma\cap N\setminus\{0\})$	$(10,11)$
Σ_{can}	fan over bounded faces of Θ_σ, reduces to canonical singularities	(11)

Topology of a Toric Variety

N_Σ	sublattice of N generated by $\lvert\Sigma\rvert\cap N$	(12)
$\pi_1(X_\Sigma)$	fundamental group of X_Σ, isomorphic to N/N_Σ	(12)
S_N	real torus $N\otimes_\mathbb{Z} S^1=\mathrm{Hom}_\mathbb{Z}(M,S^1)\simeq(S^1)^n$	(12)
$(X_\Sigma)_{\ge 0}$	nonnegative real points of a toric variety	(12)
$f,\ \mu$	algebraic and symplectic moment maps $X_P\to M_\mathbb{R}$	(12)
μ_Σ	symplectic moment map $\mathbb{C}^{\Sigma(1)}\to \mathrm{Cl}(X_\Sigma)_\mathbb{R}$	(12)

Singular Homology and Cohomology

$H^i(X,R)$	ith singular cohomology of X with coefficients in a ring R	(9)
$\widetilde{H}^i(X,R)$	i-th reduced cohomology of X	(9)
$H^i_c(X,R)$	ith cohomology of X with compact supports	(12)
$H_i(X,R)$	ith singular homology of X	(12)
$H_i^{\mathrm{BM}}(X,R)$	ith Borel-Moore homology of X	(13)
$b_i(X)$	ith Betti number of X, equals $\dim H_i(X,\mathbb{Q})$	(12)
$e(X)$	Euler characteristic of X, equals $\sum_i(-1)^i b_i(X)$	(9,10,12)
$\smile,\frown$	cap and cup products	(12)
$H^\bullet(X,R)$	cohomology ring $\bigoplus_p H^p(X,R)$ under cup product	(12)
$[W]$	cohomology class of a subvariety W in $H^{2n-2k}(X,\mathbb{Q})$	(12,13)
$[W]_r$	refined cohomology class of W in $H^{2n-2k}(X,X\setminus W,\mathbb{Q})$	(12,13)
$f_!$	generalized Gysin map	(13)
$\int_X$	integral $\int_X : H^\bullet(X,\mathbb{Q})\to\mathbb{Q}$, equals Gysin map of $X\to\{\mathrm{pt}\}$	(12,13)

Equivariant Cohomology for a Group Action

EG	a contractible space on which G acts freely	(12)
BG	the quotient EG/G	(12)
$EG\times_G X$	quotient of $EG\times X$ modulo relation $(e\cdot g,x)\sim(e,g\cdot x)$	(12)
$H_G^\bullet(X,R)$	equivariant cohomology ring, equals $H^\bullet(EG\times_G X,R)$	(12)
$\Lambda_G,\ (\Lambda_G)_\mathbb{Q}$	integral and rational equivariant cohomology ring of a point	(12)
X^G	fixed point set for action of G on X	(12)

Equivariant Cohomology for a Torus Action

$\mathrm{Sym}_{\mathbb{Z}}(M)$	symmetric algebra of M over $\mathbb{Z}$	(12)
$\mathrm{Sym}_{\mathbb{Q}}(M)$	rational symmetric algebra on M, equals $\mathrm{Sym}_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$	(12)
s	isomorphism $\mathsf{s} : \mathrm{Sym}_{\mathbb{Q}}(M) \simeq (\Lambda_T)_{\mathbb{Q}}$	(12)
$[D]_T$	equivariant cohomology class of a T-invariant divisor D	(12)
$\int_{X^{eq}}$	equivariant integral $\int_{X^{eq}} : H_T^\bullet(X,\mathbb{Q}) \to (\Lambda_T)_{\mathbb{Q}}$	(13)
$\widehat{H}_T^\bullet(X,\mathbb{Q})$	completion $\prod_{k=0}^\infty H_T^k(X,\mathbb{Q})$ of equivariant cohomology of X	(13)
$\widehat{\Lambda}$	completion of the equivariant cohomology of a point	(13)

Chow Groups and the Chow Ring

$A_k(X)$	Chow group of k-cycles modulo rational equivalence	(12)
$A^k(X)$	codimension k cycles modulo rational equivalence	(12)
$A^\bullet(X)$	integral Chow ring of X smooth and complete	(12)
$A^\bullet(X)_{\mathbb{Q}}$	rational Chow ring of X quasismooth and complete	(12)

Intersection Cohomology

$IH_i^p(X)$	ith intersection homology of X for perversity p	(12)
$IH^i(X)$	ith intersection cohomology of X for middle perversity	(12)
$IH^i(X)_{\mathbb{Q}}$	ith rational intersection cohomology of X	(12)

Cohomology Ring of a Complete Simplicial Toric Variety

$\mathscr{I}$	Stanley-Reisner ideal of the fan Σ, ideal in $\mathbb{Q}[x_1,\ldots,x_r]$	(12)
$\mathcal{J}$	ideal $\langle \sum_{i=1}^r \langle m,u_i\rangle x_i \mid m \in M\rangle \subseteq \mathbb{Q}[x_1,\ldots,x_r]$	(12)
$R_{\mathbb{Q}}(\Sigma)$	Jurkiewicz-Danilov ring $\mathbb{Q}[x_1,\ldots,x_r]/(\mathscr{I}+\mathcal{J}) \simeq H^\bullet(X_\Sigma,\mathbb{Q})$	(12)
$\mathrm{SR}_{\mathbb{Q}}(\Sigma)$	Stanley-Reisner ring $\mathbb{Q}[x_1,\ldots,x_r]/\mathscr{I} \simeq H_T^\bullet(X_\Sigma,\mathbb{Q})$	(12)

Hirzebruch-Riemann-Roch

$c_i(\mathscr{E})$	ith Chern class of a locally free sheaf $\mathscr{E}$	(13)
$\mathrm{ch}(\mathscr{L})$	Chern character of a line bundle $\mathscr{L}$	(13)
$\mathrm{Td}(X)$	Todd class of the variety X	(13)
B_k	kth Bernoulli number	(13)
$c_i = c_i(\mathscr{T}_X)$	ith Chern class of the tangent bundle	(13)
T_i	ith Todd polynomial in the c_i	(13)
$K(X)$	Grothendieck group of classes of coherent sheaves on X	(13)
$\chi^T(\mathscr{L})$	equivariant Euler characteristic	(13)
$\chi_\sigma^T(\mathscr{L})$	local contribution of $\sigma \in \Sigma(n)$ to $\chi^T(\mathscr{L})$	(13)
$\mathrm{ch}^T(\mathscr{L})$	equivariant Chern character of $\mathscr{L}$	(13)
$\mathrm{Td}^T(X)$	equivariant Todd class of X	(13)
$\mathrm{Todd}(x)$	formal Todd differential operator for the variable x	(13)

Brion's Equalities

$\mathbb{Z}[M]$	integral semigroup algebra of M	(13)
$\mathbb{Z}[[M]]$	formal semigroup module of M, formal sums $\sum_{m\in M} a_m\chi^m$	(13)
$\mathbb{Z}[[M]]_{\mathrm{Sum}}$	summable elements in $\mathbb{Z}[[M]]$	(13)
$\mathcal{S}(f)$	sum of an element $f\in\mathbb{Z}[[M]]_{\mathrm{Sum}}$	(13)
$\widetilde{\chi}(\mathscr{L})$	$\sum_{m\in M}\sum_{i=0}^n \dim H^i(X,\mathscr{L})_m\,\chi^m\in\mathbb{Z}[[M]]$	(13)

Geometric Invariant Theory

$\widehat{G}$	character group of algebraic subgroup $G\subseteq(\mathbb{C}^*)^r$	(14)
$\mathscr{L}_\chi$	sheaf of sections of rank 1 line bundle on $\mathbb{C}^r$ for character $\chi\in\widehat{G}$	(14)
$(\mathbb{C}^r)^{\mathsf{ss}}_\chi,\ (\mathbb{C}^r)^{\mathsf{s}}_\chi$	semistable and stable points for χ	(14)
R_χ	graded ring $\bigoplus_{d=0}^\infty\Gamma(\mathbb{C}^r,\mathscr{L}_{\chi^d})^G$	(14)
$\mathbb{C}^r/\!/_\chi G$	GIT quotient of $\mathbb{C}^r$ by G for χ, equals $\mathrm{Proj}(R_\chi)=(\mathbb{C}^r)^{\mathsf{ss}}_\chi/\!/G$	(14)
$B(\chi)$	irrelevant ideal of χ	(14)
$Z(\chi)$	exceptional set of χ, equals $\mathbf{V}(B(\chi))$	(14)
$P_\chi,\ P_{\mathbf{a}}$	polyhedra in $\mathbb{R}^r$ and $M_{\mathbb{R}}$ for $\chi=\chi^{\mathbf{a}}$	(14)
$F_{i,\chi},\ F_{i,\mathbf{a}}$	ith virtual facet of $P_\chi,\ P_{\mathbf{a}}$	(14)

The Secondary Fan

$\boldsymbol{\beta},\ \boldsymbol{\nu}$	lists of r vectors in $\widehat{G}_{\mathbb{R}}$ and $N_{\mathbb{R}}$	(14)
$C_{\boldsymbol{\beta}},\ C_{\boldsymbol{\nu}}$	cones generated by $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$	(14)
$\widetilde{\Gamma}_{\Sigma,I_\emptyset},\ \Gamma_{\Sigma,I_\emptyset}$	GKZ cones determined by $\Sigma, I_\emptyset$	(14)
$B(\Sigma,I_\emptyset)$	irrelevant ideal determined by $\Sigma,I_\emptyset$	(14)
Σ_{GKZ}	secondary fan of Σ	(14)
$\mathrm{Mov}_{\mathrm{GKZ}}$	moving cone of the secondary fan	(15)
P_{GKZ}	secondary polytope, normal fan is Σ_{GKZ}	(15)

Toric Minimal Model Program

$\mathcal{R}$	extremal ray of the Mori cone	(15)
$D\cdot\mathcal{R}$	intersection product $D\cdot C$ for $[C]\in\mathcal{R}\setminus\{0\}$	(15)
f_*D	birational transform of a divisor by a birational map	(15)
J_-,J_+	index sets determined by a wall relation	(15)
Σ_-,Σ_+	fans determined by a wall relation	(15)
ϕ_-,ϕ_+	toric morphisms determined by a wall relation	(15)

Miscellaneous

μ_d	multiplicative group of dth roots of unity in $\mathbb{C}$	(1)
$[a_1,\dots,a_s]$	ordinary continued fraction of a rational number	(10)
$[[b_1,\dots,b_r]]$	Hirzebruch-Jung continued fraction of a rational number	(10)

Part I. Basic Theory of Toric Varieties

Chapters 1 to 9 introduce the theory of toric varieties. This part of the book assumes only a minimal amount of algebraic geometry, at the level of *Ideals, Varieties and Algorithms* [**69**]. Each chapter begins with a background section that develops the necessary algebraic geometry.

Chapter 1

Affine Toric Varieties

§1.0. Background: Affine Varieties

We begin with the algebraic geometry needed for our study of affine toric varieties. Our discussion assumes Chapters 1–5 and 9 of [**69**].

Coordinate Rings. An ideal $I \subseteq S = \mathbb{C}[x_1,\dots,x_n]$ gives an affine variety

$$\mathbf{V}(I) = \{p \in \mathbb{C}^n \mid f(p) = 0 \text{ for all } f \in I\}$$

and an affine variety $V \subseteq \mathbb{C}^n$ gives the ideal

$$\mathbf{I}(V) = \{f \in S \mid f(p) = 0 \text{ for all } p \in V\}.$$

By the Hilbert basis theorem, an affine variety V is defined by the vanishing of finitely many polynomials in S, and for any ideal I, the Nullstellensatz tells us that $\mathbf{I}(\mathbf{V}(I)) = \sqrt{I} = \{f \in S \mid f^\ell \in I \text{ for some } \ell \geq 1\}$ since $\mathbb{C}$ is algebraically closed. The most important algebraic object associated to V is its *coordinate ring*

$$\mathbb{C}[V] = S/\mathbf{I}(V).$$

Elements of $\mathbb{C}[V]$ can be interpreted as the $\mathbb{C}$-valued polynomial functions on V. Note that $\mathbb{C}[V]$ is a $\mathbb{C}$-algebra, meaning that its vector space structure is compatible with its ring structure. Here are some basic facts about coordinate rings:

- $\mathbb{C}[V]$ is an integral domain $\Leftrightarrow$ $\mathbf{I}(V)$ is a prime ideal $\Leftrightarrow$ V is irreducible.
- Polynomial maps (also called *morphisms*) $\phi : V_1 \to V_2$ between affine varieties correspond to $\mathbb{C}$-algebra homomorphisms $\phi^* : \mathbb{C}[V_2] \to \mathbb{C}[V_1]$, where $\phi^*(g) = g \circ \phi$ for $g \in \mathbb{C}[V_2]$.
- Two affine varieties are isomorphic if and only if their coordinate rings are isomorphic $\mathbb{C}$-algebras.

- A point p of an affine variety V gives the maximal ideal
$$\{f \in \mathbb{C}[V] \mid f(p) = 0\} \subseteq \mathbb{C}[V],$$
and all maximal ideals of $\mathbb{C}[V]$ arise this way.

Coordinate rings of affine varieties can be characterized as follows (Exercise 1.0.1).

Lemma 1.0.1. *A $\mathbb{C}$-algebra R is isomorphic to the coordinate ring of an affine variety if and only if R is a finitely generated $\mathbb{C}$-algebra with no nonzero nilpotents, i.e., if $f \in R$ satisfies $f^\ell = 0$ for some $\ell \geq 1$, then $f = 0$.* □

To emphasize the close relation between V and $\mathbb{C}[V]$, we sometimes write

$$V = \mathrm{Spec}(\mathbb{C}[V]). \tag{1.0.1}$$

This can be made canonical by identifying V with the set of maximal ideals of $\mathbb{C}[V]$ via the fourth bullet above. More generally, one can take any commutative ring R and define the *affine scheme* $\mathrm{Spec}(R)$. The general definition of Spec uses all prime ideals of R, not just the maximal ideals as we have done. Thus some authors would write (1.0.1) as $V = \mathrm{Specm}(\mathbb{C}[V])$, the maximal spectrum of $\mathbb{C}[V]$. Readers wishing to learn about affine schemes should consult [**90**] and [**131**].

The Zariski Topology. An affine variety $V \subseteq \mathbb{C}^n$ has two topologies we will use. The first is the *classical topology*, induced from the usual topology on $\mathbb{C}^n$. The second is the *Zariski topology*, where the Zariski closed sets are subvarieties of V (meaning affine varieties of $\mathbb{C}^n$ contained in V) and the Zariski open sets are their complements. Since subvarieties are closed in the classical topology (polynomials are continuous), Zariski open subsets are open in the classical topology.

Given a subset $S \subseteq V$, its closure $\overline{S}$ in the Zariski topology is the smallest subvariety of V containing S. We call $\overline{S}$ the *Zariski closure* of S. It is easy to give examples where this differs from the closure in the classical topology.

Affine Open Subsets and Localization. Some Zariski open subsets of an affine variety V are themselves affine varieties. Given $f \in \mathbb{C}[V] \setminus \{0\}$, let
$$V_f = \{p \in V \mid f(p) \neq 0\} \subseteq V.$$
Then V_f is Zariski open in V and is also an affine variety, as we now explain.

Let $V \subseteq \mathbb{C}^n$ have $\mathbf{I}(V) = \langle f_1, \ldots, f_s\rangle$ and pick $g \in \mathbb{C}[x_1, \ldots, x_n]$ representing f. Then $V_f = V \setminus \mathbf{V}(g)$ is Zariski open in V. Now consider a new variable y and let $W = \mathbf{V}(f_1, \ldots, f_s, 1 - gy) \subseteq \mathbb{C}^n \times \mathbb{C}$. Since the projection map $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$ maps W bijectively onto V_f, we can identify V_f with the affine variety $W \subseteq \mathbb{C}^n \times \mathbb{C}$.

When V is irreducible, the coordinate ring of V_f is easy to describe. Let $\mathbb{C}(V)$ be the field of fractions of the integral domain $\mathbb{C}[V]$. Recall that elements of $\mathbb{C}(V)$ give rational functions on V. Then let

$$\mathbb{C}[V]_f = \{g/f^\ell \in \mathbb{C}(V) \mid g \in \mathbb{C}[V],\ \ell \geq 0\}. \tag{1.0.2}$$

In Exercise 1.0.3 you will prove that $\mathrm{Spec}(\mathbb{C}[V]_f)$ is the affine variety V_f.

Example 1.0.2. The n-dimensional torus is the affine open subset

$$(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \mathbf{V}(x_1 \cdots x_n) \subseteq \mathbb{C}^n,$$

with coordinate ring

$$\mathbb{C}[x_1, \ldots, x_n]_{x_1 \cdots x_n} = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Elements of this ring are called *Laurent polynomials*. $\Diamond$

The ring $\mathbb{C}[V]_f$ from (1.0.2) is an example of *localization*. In Exercises 1.0.2 and 1.0.3 you will show how to construct this ring for all affine varieties, not just irreducible ones. The general concept of localization is discussed in standard texts in commutative algebra such as [**10**, Ch. 3] and [**89**, Ch. 2].

Normal Affine Varieties. Let R be an integral domain with field of fractions K. Then R is *normal*, or *integrally closed*, if every element of K which is integral over R (meaning that it is a root of a monic polynomial in $R[x]$) actually lies in R. For example, any UFD is normal (Exercise 1.0.5).

Definition 1.0.3. An irreducible affine variety V is ***normal*** if its coordinate ring $\mathbb{C}[V]$ is normal.

For example, $\mathbb{C}^n$ is normal since its coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$ is a UFD and hence normal. Here is an example of a nonnormal affine variety.

Example 1.0.4. Let $C = \mathbf{V}(x^3 - y^2) \subseteq \mathbb{C}^2$. This is an irreducible plane curve with a cusp at the origin. It is easy to see that $\mathbb{C}[C] = \mathbb{C}[x,y]/\langle x^3 - y^2 \rangle$. Now let $\bar{x}$ and $\bar{y}$ be the cosets of x and y in $\mathbb{C}[C]$ respectively. This gives $\bar{y}/\bar{x} \in \mathbb{C}(C)$. A computation shows that $\bar{y}/\bar{x} \notin \mathbb{C}[C]$ and that $(\bar{y}/\bar{x})^2 = \bar{x}$. Consequently $\mathbb{C}[C]$ and hence C are not normal. We will see below that C is an affine toric variety. $\Diamond$

An irreducible affine variety V has a *normalization* defined as follows. Let

$$\mathbb{C}[V]' = \{\alpha \in \mathbb{C}(V) : \alpha \text{ is integral over } \mathbb{C}[V]\}.$$

We call $\mathbb{C}[V]'$ the *integral closure* of $\mathbb{C}[V]$. One can show that $\mathbb{C}[V]'$ is normal and (with more work) finitely generated as a $\mathbb{C}$-algebra (see [**89**, Cor. 13.13]). This gives the normal affine variety

$$V' = \mathrm{Spec}(\mathbb{C}[V]')$$

We call V' the *normalization* of V. The natural inclusion $\mathbb{C}[V] \subseteq \mathbb{C}[V]' = \mathbb{C}[V']$ corresponds to a map $V' \to V$. This is the *normalization map*.

Example 1.0.5. We saw in Example 1.0.4 that the curve $C \subseteq \mathbb{C}^2$ defined by $x^3 = y^2$ has elements $\bar{x}, \bar{y} \in \mathbb{C}[C]$ such that $\bar{y}/\bar{x} \notin \mathbb{C}[C]$ is integral over $\mathbb{C}[C]$. In Exercise 1.0.6 you will show that $\mathbb{C}[\bar{y}/\bar{x}] \subseteq \mathbb{C}(C)$ is the integral closure of $\mathbb{C}[C]$ and that the normalization map is the map $\mathbb{C} \to C$ defined by $t \mapsto (t^2, t^3)$. $\Diamond$

At first glance, the definition of normal does not seem very intuitive. Once we enter the world of toric varieties, however, we will see that normality has a very nice combinatorial interpretation and that the nicest toric varieties are the normal ones. We will also see that normality leads to a nice theory of divisors.

In Exercise 1.0.7 you will prove some properties of normal domains that will be used in §1.3 when we study normal affine toric varieties.

Smooth Points of Affine Varieties. In order to define a smooth point of an affine variety V, we first need to define *local rings* and *Zariski tangent spaces*. When V is irreducible, the *local ring* of V at p is

$$\mathscr{O}_{V,p} = \{ f/g \in \mathbb{C}(V) \mid f,g \in \mathbb{C}[V] \text{ and } g(p) \neq 0\}.$$

Thus $\mathscr{O}_{V,p}$ consists of all rational functions on V that are defined at p. Inside of $\mathscr{O}_{V,p}$ we have the maximal ideal

$$\mathfrak{m}_{V,p} = \{\phi \in \mathscr{O}_{V,p} \mid \phi(p) = 0\}.$$

In fact, $\mathfrak{m}_{V,p}$ is the unique maximal ideal of $\mathscr{O}_{V,p}$, so that $\mathscr{O}_{V,p}$ is a *local ring*. Exercises 1.0.2 and 1.0.4 explain how to define $\mathscr{O}_{V,p}$ when V is not irreducible.

The *Zariski tangent space* of V at p is defined to be

$$T_p(V) = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_{V,p}/\mathfrak{m}_{V,p}^2, \mathbb{C}).$$

In Exercise 1.0.8 you will verify that $\dim T_p(\mathbb{C}^n) = n$ for every $p \in \mathbb{C}^n$. According to [**131**, p. 32], we can compute the Zariski tangent space of a point in an affine variety as follows.

Lemma 1.0.6. *Let $V \subseteq \mathbb{C}^n$ be an affine variety and let $p \in V$. Also assume that $\mathbf{I}(V) = \langle f_1, \ldots, f_s\rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]$. For each i, let*

$$d_p(f_i) = \frac{\partial f_i}{\partial x_1}(p)\, x_1 + \cdots + \frac{\partial f_i}{\partial x_n}(p)\, x_n.$$

Then the Zariski tangent space $T_p(V)$ is isomorphic to the subspace of $\mathbb{C}^n$ defined by the equations $d_p(f_1) = \cdots = d_p(f_s) = 0$. In particular, $\dim T_p(V) \leq n$. $\square$

Definition 1.0.7. A point p of an affine variety V is ***smooth*** or ***nonsingular*** if $\dim T_p(V) = \dim_p V$, where $\dim_p V$ is the maximum of the dimensions of the irreducible components of V containing p. The point p is ***singular*** if it is not smooth. Finally, V is ***smooth*** if every point of V is smooth.

Points lying in the intersection of two or more irreducible components of V are always singular (see [**69**, Thm. 8 of Ch. 9, §6]).

Since $\dim T_p(\mathbb{C}^n) = n$ for every $p \in \mathbb{C}^n$, we see that $\mathbb{C}^n$ is smooth. For an irreducible affine variety $V \subseteq \mathbb{C}^n$ of dimension d, fix $p \in V$ and write $\mathbf{I}(V) =$

$\langle f_1,\dots,f_s\rangle$. Using Lemma 1.0.6, it is straightforward to show that V is smooth at p if and only if the Jacobian matrix

$$J_p(f_1,\dots,f_s) = \Big(\frac{\partial f_i}{\partial x_j}(p)\Big)_{1\le i\le s, 1\le j\le n} \tag{1.0.3}$$

has rank $n-d$ (Exercise 1.0.9). Here is a simple example.

Example 1.0.8. As noted in Example 1.0.4, the plane curve C defined by $x^3 = y^2$ has $\mathbf{I}(C) = \langle x^3 - y^2\rangle \subseteq \mathbb{C}[x,y]$. A point $p = (a,b) \in C$ has Jacobian

$$J_p = (3a^2, -2b),$$

so the origin is the only singular point of C. $\Diamond$

Since $T_p(V) = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_{V,p}/\mathfrak{m}_{V,p}^2, \mathbb{C})$, we see that V is smooth at p when $\dim V$ equals the dimension of $\mathfrak{m}_{V,p}/\mathfrak{m}_{V,p}^2$ as a vector space over $\mathscr{O}_{V,p}/\mathfrak{m}_{V,p}$. In terms of commutative algebra, this means that $p \in V$ is smooth if and only if $\mathscr{O}_{V,p}$ is a *regular local ring*. See [**10**, p. 123] or [**89**, 10.3].

We can relate smoothness and normality as follows.

Proposition 1.0.9. *A smooth irreducible affine variety V is normal.*

Proof. In §3.0 we will see that $\mathbb{C}[V] = \bigcap_{p\in V} \mathscr{O}_{V,p}$. By Exercise 1.0.7, $\mathbb{C}[V]$ is normal once we prove that $\mathscr{O}_{V,p}$ is normal for all $p \in V$. Hence it suffices to show that $\mathscr{O}_{V,p}$ is normal whenever p is smooth.

This follows from some powerful results in commutative algebra: $\mathscr{O}_{V,p}$ is a regular local ring when p is a smooth point of V (see above), and every regular local ring is a UFD (see [**89**, Thm. 19.19]). Then we are done since every UFD is normal. A direct proof that $\mathscr{O}_{V,p}$ is normal at a smooth point $p \in V$ is sketched in Exercise 1.0.10. $\square$

The converse of Propostion 1.0.9 can fail. We will see in §1.3 that the affine variety $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ is normal, yet $\mathbf{V}(xy - zw)$ is singular at the origin.

Products of Affine Varieties. Given affine varieties V_1 and V_2, there are several ways to show that the cartesian product $V_1 \times V_2$ is an affine variety. The most direct way is to proceed as follows. Let $V_1 \subseteq \mathbb{C}^m = \mathrm{Spec}(\mathbb{C}[x_1,\dots,x_m])$ and $V_2 \subseteq \mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[y_1,\dots,y_n])$. Take $\mathbf{I}(V_1) = \langle f_1,\dots,f_s\rangle$ and $\mathbf{I}(V_2) = \langle g_1,\dots,g_t\rangle$. Since the f_i and g_j depend on separate sets of variables, it follows that

$$V_1 \times V_2 = \mathbf{V}(f_1,\dots,f_s, g_1,\dots,g_t) \subseteq \mathbb{C}^{m+n}$$

is an affine variety.

A fancier method is to use the mapping properties of the product. This will also give an intrinsic description of its coordinate ring. Given V_1 and V_2 as above,

$V_1 \times V_2$ should be an affine variety with projections $\pi_i : V_1 \times V_2 \to V_i$ such that whenever we have a diagram

$$\begin{tikzcd} W \arrow[dr, dotted, "\nu"] \arrow[drr, bend left, "\phi_1"] \arrow[ddr, bend right, "\phi_2"'] & & \\ & V_1 \times V_2 \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & V_1 \\ & V_2 & \end{tikzcd}$$

where $\phi_i : W \to V_i$ are morphisms from an affine variety W, there should be a unique morphism ν (the dotted arrow) that makes the diagram commute, i.e., $\pi_i \circ \nu = \phi_i$. For the coordinate rings, this means that whenever we have a diagram

$$\begin{tikzcd} & \mathbb{C}[V_2] \arrow[d, "\pi_2^*"'] \arrow[ddr, bend left, "\phi_2^*"] & \\ \mathbb{C}[V_1] \arrow[r, "\pi_1^*"] \arrow[drr, bend right, "\phi_1^*"'] & \mathbb{C}[V_1 \times V_2] \arrow[dr, dotted, "\nu^*"] & \\ & & \mathbb{C}[W] \end{tikzcd}$$

with $\mathbb{C}$-algebra homomorphisms $\phi_i^* : \mathbb{C}[V_i] \to \mathbb{C}[W]$, there should be a unique $\mathbb{C}$-algebra homomorphism ν^* (the dotted arrow) that makes the diagram commute. By the universal mapping property of the *tensor product* of $\mathbb{C}$-algebras, $\mathbb{C}[V_1] \otimes_{\mathbb{C}} \mathbb{C}[V_2]$ has the mapping properties we want. Since $\mathbb{C}[V_1] \otimes_{\mathbb{C}} \mathbb{C}[V_2]$ is a finitely generated $\mathbb{C}$-algebra with no nilpotents (see the appendix to this chapter), it is the coordinate ring $\mathbb{C}[V_1 \times V_2]$. For more on tensor products, see [**10**, pp. 24–27] or [**89**, A2.2].

Example 1.0.10. Let V be an affine variety. Since $\mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[y_1, \ldots, y_n])$, the product $V \times \mathbb{C}^n$ has coordinate ring

$$\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[y_1, \ldots, y_n] = \mathbb{C}[V][y_1, \ldots, y_n].$$

If V is contained in $\mathbb{C}^m$ with $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{C}[x_1, \ldots, x_m]$, it follows that

$$\mathbf{I}(V \times \mathbb{C}^n) = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n].$$

For later purposes, we also note that the coordinate ring of $V \times (\mathbb{C}^*)^n$ is

$$\mathbb{C}[V] \otimes_{\mathbb{C}} \mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}] = \mathbb{C}[V][y_1^{\pm 1}, \ldots, y_n^{\pm 1}]. \qquad \Diamond$$

Given affine varieties V_1 and V_2, we note that the Zariski topology on $V_1 \times V_2$ is usually *not* the product of the Zariski topologies on V_1 and V_2.

Example 1.0.11. Consider $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. By definition, a basis for the product of the Zariski topologies consists of sets $U_1 \times U_2$ where U_i are Zariski open in $\mathbb{C}$. Such a set is the complement of a union of collections of "horizontal" and "vertical" lines in $\mathbb{C}^2$. This makes it easy to see that Zariski closed sets in $\mathbb{C}^2$ such as $\mathbf{V}(y-x^2)$ cannot be closed in the product topology. $\Diamond$

Exercises for §1.0.

1.0.1. Prove Lemma 1.0.1. Hint: You will need the Nullstellensatz.

1.0.2. Let R be a commutative $\mathbb{C}$-algebra. A subset $S \subseteq R$ is a *multipliciative subset* provided $1 \in S$, $0 \notin S$, and S is closed under multiplication. The *localization* R_S consists of all formal expressions g/s, $g \in R$, $s \in S$, modulo the equivalence relation

$$g/s \sim h/t \iff u(tg - sh) = 0 \text{ for some } u \in S.$$

(a) Show that the usual formulas for adding and multiplying fractions induce well-defined binary operations that make R_S into $\mathbb{C}$-algebra.

(b) If R has no nonzero nilpotents, then prove that the same is true for R_S.

For more on localization, see [**10**, Ch. 3] or [**89**, Ch. 2].

1.0.3. Let R be a finitely generated $\mathbb{C}$-algebra without nilpotents as in Lemma 1.0.1 and let $f \in R$ be nonzero. Then $S = \{1, f, f^2, \ldots\}$ is a multiplicative set. The localization R_S is denoted R_f and is called the *localization of* R *at* f.

(a) Show that R_f is a finitely generated $\mathbb{C}$-algebra without nilpotents.

(b) Show that R_f satisfies $\mathrm{Spec}(R_f) = \mathrm{Spec}(R)_f$.

(c) Show that R_f is given by (1.0.2) when R is an integral domain.

1.0.4. Let V be an affine variety with coordinate ring $\mathbb{C}[V]$. Given a point $p \in V$, let $S = \{g \in \mathbb{C}[V] \mid g(p) \neq 0\}$.

(a) Show that S is a multiplicative set. The localization $\mathbb{C}[V]_S$ is denoted $\mathscr{O}_{V,p}$ and is called the *local ring of* V *at* p.

(b) Show that every $\phi \in \mathscr{O}_{V,p}$ has a well-defined value $\phi(p)$ and that

$$\mathfrak{m}_{V,p} = \{\phi \in \mathscr{O}_{V,p} \mid \phi(p) = 0\}$$

is the unique maximal ideal of $\mathscr{O}_{V,p}$.

(c) When V is irreducible, show that $\mathscr{O}_{V,p}$ agrees with the definition given in the text.

1.0.5. Prove that a UFD is normal.

1.0.6. In the setting of Example 1.0.5, show that $\mathbb{C}[\bar{y}/\bar{x}] \subseteq \mathbb{C}(C)$ is the integral closure of $\mathbb{C}[C]$ and that the normalization $\mathbb{C} \to C$ is defined by $t \mapsto (t^2, t^3)$.

1.0.7. In this exercise, you will prove some properties of normal domains needed for §1.3.

(a) Let R be a normal domain with field of fractions K and let $S \subseteq R$ be a multiplicative subset. Prove that the localization R_S is normal.

(b) Let R_α, $\alpha \in A$, be normal domains with the same field of fractions K. Prove that the intersection $\bigcap_{\alpha\in A} R_\alpha$ is normal.

1.0.8. Prove that $\dim T_p(\mathbb{C}^n) = n$ for all $p \in \mathbb{C}^n$.

1.0.9. Use Lemma 1.0.6 to prove the claim made in the text that smoothness is determined by the rank of the Jacobian matrix (1.0.3).

1.0.10. Let V be irreducible and suppose that $p \in V$ is smooth. The goal of this exercise is to prove that $\mathscr{O}_{V,p}$ is normal using standard results from commutative algebra. Set $n = \dim V$ and consider the ring of *formal power series* $\mathbb{C}[[x_1,\dots,x_n]]$. This is a local ring with maximal ideal $\mathfrak{m} = \langle x_1,\dots,x_n\rangle$. We will use three facts:

- $\mathbb{C}[[x_1,\dots,x_n]]$ is a UFD by [**280**, p. 148] and hence normal by Exercise 1.0.5.
- Since $p \in V$ is smooth, [**207**, §1C] proves the existence of a $\mathbb{C}$-algebra homomorphism $\mathscr{O}_{V,p} \to \mathbb{C}[[x_1,\dots,x_n]]$ that induces isomorphisms
$$\mathscr{O}_{V,p}/\mathfrak{m}_{V,p}^{\ell} \simeq \mathbb{C}[[x_1,\dots,x_n]]/\mathfrak{m}^{\ell}$$
for all $\ell \geq 0$. This implies that the *completion* (see [**10**, Ch. 10])
$$\widehat{\mathscr{O}}_{V,p} = \varprojlim \mathscr{O}_{V,p}/\mathfrak{m}_{V,p}^{\ell}$$
is isomorphic to a formal power series ring, i.e., $\widehat{\mathscr{O}}_{V,p} \simeq \mathbb{C}[[x_1,\dots,x_n]]$. Such an isomorphism captures the intuitive idea that at a smooth point, functions should have power series expansions in "local coordinates" $x_1,\dots,x_n$.
- If $I \subseteq \mathscr{O}_{V,p}$ is an ideal, then
$$I = \textstyle\bigcap_{\ell=1}^{\infty}(I + \mathfrak{m}_{V,p}^{\ell}).$$
This theorem of Krull holds for any ideal I in a Noetherian local ring A and follows from [**10**, Cor. 10.19] with $M = A/I$.

Now assume that $p \in V$ is smooth.

(a) Use the third bullet to show that $\mathscr{O}_{V,p} \to \mathbb{C}[[x_1,\dots,x_n]]$ is injective.

(b) Suppose that $a,b \in \mathscr{O}_{V,p}$ satisfy $b|a$ in $\mathbb{C}[[x_1,\dots,x_n]]$. Prove that $b|a$ in $\mathscr{O}_{V,p}$. Hint: Use the second bullet to show $a \in b\mathscr{O}_{V,p} + \mathfrak{m}_{V,p}^{\ell}$ and then use the third bullet.

(c) Prove that $\mathscr{O}_{V,p}$ is normal. Hint: Use part (b) and the first bullet.

This argument can be continued to show that $\mathscr{O}_{V,p}$ is a UFD. See [**207**, (1.28)]

1.0.11. Let V and W be affine varieties and let $S \subseteq V$ be a subset. Prove that $\overline{S} \times W = \overline{S \times W}$.

1.0.12. Let V and W be irreducible affine varieties. Prove that $V \times W$ is irreducible. Hint: Suppose $V \times W = Z_1 \cup Z_2$, where Z_1, Z_2 are closed. Let $V_i = \{v \in V \mid \{v\} \times W \subseteq Z_i\}$. Prove that $V = V_1 \cup V_2$ and that V_i is closed in V. Exercise 1.0.11 will be useful.

§1.1. Introduction to Affine Toric Varieties

We first discuss what we mean by "torus" and then explore various constructions of affine toric varieties.

The Torus. The affine variety $(\mathbb{C}^*)^n$ is a group under component-wise multiplication. A *torus* T is an affine variety isomorphic to $(\mathbb{C}^*)^n$, where T inherits a group structure from the isomorphism.

The term "torus" is taken from the language of *linear algebraic groups*. We will use (without proof) basic results about tori that can be found in standard texts on algebraic groups such as [**37**], [**152**], and [**256**]. See also [**36**, Ch. 3] for a self-contained treatment of tori.

We begin with *characters* and *one-parameter subgroups*.

A *character* of a torus T is a morphism $\chi : T \to \mathbb{C}^*$ that is a group homomorphism. For example, $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ gives a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ defined by

$$\chi^m(t_1, \ldots, t_n) = t_1^{a_1} \cdots t_n^{a_n}. \tag{1.1.1}$$

One can show that *all* characters of $(\mathbb{C}^*)^n$ arise this way (see [**152**, §16]). Thus the characters of $(\mathbb{C}^*)^n$ form a group isomorphic to $\mathbb{Z}^n$.

For an arbitrary torus T, its characters form a free abelian group M of rank equal to the dimension of T. It is customary to say that $m \in M$ gives the character $\chi^m : T \to \mathbb{C}^*$.

We will need the following result concerning tori (see [**152**, §16] for a proof).

Proposition 1.1.1.

(a) *Let T_1 and T_2 be tori and let $\Phi : T_1 \to T_2$ be a morphism that is a group homomorphism. Then the image of Φ is a torus and is closed in T_2.*

(b) *Let T be a torus and let $H \subseteq T$ be an irreducible subvariety of T that is a subgroup. Then H is a torus.* □

Assume that a torus T acts linearly on a finite dimensional vector space W over $\mathbb{C}$, where the action of $t \in T$ on $w \in W$ is denoted $t \cdot w$. A basic result is that the linear maps $w \mapsto t \cdot w$ are diagonalizable and can be simultaneously diagonalized. We describe this as follows. Given $m \in M$, define the *eigenspace*

$$W_m = \{w \in W \mid t \cdot w = \chi^m(t) w \text{ for all } t \in T\}.$$

If $W_m \neq \{0\}$, then every $w \in W_m \setminus \{0\}$ is a simultaneous eigenvector for all $t \in T$, with eigenvalue given by $\chi^m(t)$. See [**256**, Thm. 3.2.3] for a proof of the following.

Proposition 1.1.2. *In the above situation, we have $W = \bigoplus_{m \in M} W_m$.* □

A *one-parameter subgroup* of a torus T is a morphism $\lambda : \mathbb{C}^* \to T$ that is a group homomorphism. For example, $u = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ gives a one-parameter subgroup $\lambda^u : \mathbb{C}^* \to (\mathbb{C}^*)^n$ defined by

$$\lambda^u(t) = (t^{b_1}, \ldots, t^{b_n}). \tag{1.1.2}$$

All one-parameter subgroups of $(\mathbb{C}^*)^n$ arise this way (see [**152**, §16]). It follows that the group of one-parameter subgroups of $(\mathbb{C}^*)^n$ is naturally isomorphic to $\mathbb{Z}^n$. For an arbitrary torus T, the one-parameter subgroups form a free abelian group N of rank equal to the dimension of T. As with the character group, an element $u \in N$ gives the one-parameter subgroup $\lambda^u : \mathbb{C}^* \to T$.

There is a natural bilinear pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$ defined as follows.

- (Intrinsic) Given a character χ^m and a one-parameter subgroup λ^u, the composition $\chi^m \circ \lambda^u : \mathbb{C}^* \to \mathbb{C}^*$ is a character of $\mathbb{C}^*$, which is given by $t \mapsto t^\ell$ for some $\ell \in \mathbb{Z}$. Then $\langle m, u \rangle = \ell$.

- (Concrete) If $T = (\mathbb{C}^*)^n$ with $m = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $u = (b_1, \ldots, b_n) \in \mathbb{Z}^n$, then one computes that

$$\langle m, u \rangle = \sum_{i=1}^{n} a_i b_i, \tag{1.1.3}$$

  i.e., the pairing is the usual dot product.

It follows that the characters and one-parameter subgroups of a torus T form free abelian groups M and N of finite rank with a pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$ that identifies N with $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and M with $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. In terms of tensor products, one obtains a canonical isomorphism $N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq T$ via $u \otimes t \mapsto \lambda^u(t)$. Hence it is customary to write a torus as T_N.

From this point of view, picking an isomorphism $T_N \simeq (\mathbb{C}^*)^n$ induces dual bases of M and N, i.e., isomorphisms $M \simeq \mathbb{Z}^n$ and $N \simeq \mathbb{Z}^n$ that turn characters into Laurent monomials (1.1.1), one-parameter subgroups into monomial curves (1.1.2), and the pairing into dot product (1.1.3).

The Definition of Affine Toric Variety. We now define the main object of study of this chapter.

Definition 1.1.3. An ***affine toric variety*** is an irreducible affine variety V containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on V. (By algebraic action, we mean an action $T_N \times V \to V$ given by a morphism.)

Obvious examples of affine toric varieties are $(\mathbb{C}^*)^n$ and $\mathbb{C}^n$. Here are some less trivial examples.

Example 1.1.4. The plane curve $C = \mathbf{V}(x^3 - y^2) \subseteq \mathbb{C}^2$ has a cusp at the origin. This is an affine toric variety with torus

$$C \setminus \{0\} = C \cap (\mathbb{C}^*)^2 = \{(t^2, t^3) \mid t \in \mathbb{C}^*\} \simeq \mathbb{C}^*,$$

where the isomorphism is $t \mapsto (t^2, t^3)$. Example 1.0.4 shows that C is a nonnormal toric variety. $\Diamond$

Example 1.1.5. The variety $V = \mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ is a toric variety with torus

$$V \cap (\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1}) \mid t_i \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^3,$$

where the isomorphism is $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$. We will see later that V is normal. $\Diamond$

Example 1.1.6. Consider the surface in $\mathbb{C}^{d+1}$ parametrized by the map

$$\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^{d+1}$$

defined by $(s, t) \mapsto (s^d, s^{d-1}t, \ldots, st^{d-1}, t^d)$. Thus Φ is defined using all degree d monomials in s, t.

Let the coordinates of $\mathbb{C}^{d+1}$ be $x_0,\ldots,x_d$ and let $I \subseteq \mathbb{C}[x_0,\ldots,x_d]$ be the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-2} & x_{d-1} \\ x_1 & x_2 & \cdots & x_{d-1} & x_d \end{pmatrix},$$

so $I = \langle x_i x_{j+1} - x_{i+1}x_j \mid 0 \le i < j \le d-1\rangle$. In Exercise 1.1.1 you will verify that $\Phi(\mathbb{C}^2) = \mathbf{V}(I)$, so that $\widehat{C}_d = \Phi(\mathbb{C}^2)$ is an affine variety. You will also prove that $\mathbf{I}(\widehat{C}_d) = I$, so that I is the ideal of all polynomials vanishing on $\widehat{C}_d$. It follows that I is prime since $\mathbf{V}(I)$ is irreducible by Proposition 1.1.8 below. The affine surface $\widehat{C}_d$ is called the *rational normal cone of degree d* and is an example of a *determinantal variety*. We will see below that I is a toric ideal.

It is straightforward to show that $\widehat{C}_d$ is a toric variety with torus

$$\Phi((\mathbb{C}^*)^2) = \widehat{C}_d \cap (\mathbb{C}^*)^{d+1} \simeq (\mathbb{C}^*)^2.$$

We will study this example from the projective point of view in Chapter 2. ◇

We next explore three equivalent ways of constructing affine toric varieties.

Lattice Points. In this book, a *lattice* is a free abelian group of finite rank. Thus a lattice of rank n is isomorphic to $\mathbb{Z}^n$. For example, a torus T_N has lattices M (of characters) and N (of one-parameter subgroups).

Given a torus T_N with character lattice M, a set $\mathscr{A} = \{m_1,\ldots,m_s\} \subseteq M$ gives characters $\chi^{m_i} : T_N \to \mathbb{C}^*$. Then consider the map

$$\Phi_{\mathscr{A}} : T_N \longrightarrow \mathbb{C}^s \tag{1.1.4}$$

defined by

$$\Phi_{\mathscr{A}}(t) = \big(\chi^{m_1}(t),\ldots,\chi^{m_s}(t)\big) \in \mathbb{C}^s.$$

Definition 1.1.7. Given a finite set $\mathscr{A} \subseteq M$, the affine toric variety $Y_{\mathscr{A}}$ is defined to be the Zariski closure of the image of the map $\Phi_{\mathscr{A}}$ from (1.1.4).

This definition is justified by the following proposition.

Proposition 1.1.8. *Given $\mathscr{A} \subseteq M$ as above, let $\mathbb{Z}\mathscr{A} \subseteq M$ be the sublattice generated by $\mathscr{A}$. Then $Y_{\mathscr{A}}$ is an affine toric variety whose torus has character lattice $\mathbb{Z}\mathscr{A}$. In particular, the dimension of $Y_{\mathscr{A}}$ is the rank of $\mathbb{Z}\mathscr{A}$.*

Proof. The map (1.1.4) can be regarded as a map

$$\Phi_{\mathscr{A}} : T_N \longrightarrow (\mathbb{C}^*)^s$$

of tori. By Proposition 1.1.1, the image $T = \Phi_{\mathscr{A}}(T_N)$ is a torus that is closed in $(\mathbb{C}^*)^s$. The latter implies that $Y_{\mathscr{A}} \cap (\mathbb{C}^*)^s = T$ since $Y_{\mathscr{A}}$ is the Zariski closure of the image. It follows that the image is Zariski open in $Y_{\mathscr{A}}$. Furthermore, T is irreducible (it is a torus), so the same is true for its Zariski closure $Y_{\mathscr{A}}$.

We next consider the action of T. Since $T \subseteq (\mathbb{C}^*)^s$, an element $t \in T$ acts on $\mathbb{C}^s$ and takes varieties to varieties. Then

$$T = t \cdot T \subseteq t \cdot Y_{\mathscr{A}}$$

shows that $t \cdot Y_{\mathscr{A}}$ is a variety containing T. Hence $Y_{\mathscr{A}} \subseteq t \cdot Y_{\mathscr{A}}$ by the definition of Zariski closure. Replacing t with t^{-1} leads to $Y_{\mathscr{A}} = t \cdot Y_{\mathscr{A}}$, so that the action of T induces an action on $Y_{\mathscr{A}}$. We conclude that $Y_{\mathscr{A}}$ is an affine toric variety.

It remains to compute the character lattice of T, which we will temporarily denote by M'. Since $T = \Phi_{\mathscr{A}}(T_N)$, the map $\Phi_{\mathscr{A}}$ gives the commutative diagram

$$\begin{tikzcd} T_N \arrow[r, "\Phi_{\mathscr{A}}"] \arrow[dr, two heads] & (\mathbb{C}^*)^s \\ & T \arrow[u, hook] \end{tikzcd}$$

where $\twoheadrightarrow$ denotes a surjective map and $\hookrightarrow$ an injective map. This diagram of tori induces a commutative diagram of character lattices

$$\begin{tikzcd} M & \mathbb{Z}^s \arrow[l, "\widehat{\Phi}_{\mathscr{A}}"'] \arrow[d, two heads] \\ & M'. \arrow[ul, hook] \end{tikzcd}$$

Since $\widehat{\Phi}_{\mathscr{A}} : \mathbb{Z}^s \to M$ takes the standard basis $e_1, \ldots, e_s$ to $m_1, \ldots, m_s$, the image of $\widehat{\Phi}_{\mathscr{A}}$ is $\mathbb{Z}\mathscr{A}$. By the diagram, we obtain $M' \simeq \mathbb{Z}\mathscr{A}$. Then we are done since the dimension of a torus equals the rank of its character lattice. $\square$

In concrete terms, fix a basis of M, so that we may assume $M = \mathbb{Z}^n$. Then the s vectors in $\mathscr{A} \subseteq \mathbb{Z}^n$ can be regarded as the columns of an $n \times s$ matrix A with integer entries. In this case, the dimension of $Y_{\mathscr{A}}$ is simply the rank of the matrix A.

We will see below that every affine toric variety is isomorphic to $Y_{\mathscr{A}}$ for some finite subset $\mathscr{A}$ of a lattice.

Toric Ideals. Let $Y_{\mathscr{A}} \subseteq \mathbb{C}^s = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_s])$ be the affine toric variety coming from a finite set $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M$. We can describe the ideal $\mathbf{I}(Y_{\mathscr{A}}) \subseteq \mathbb{C}[x_1, \ldots, x_s]$ as follows. As in the proof of Proposition 1.1.8, $\Phi_{\mathscr{A}}$ induces a map of character lattices

$$\widehat{\Phi}_{\mathscr{A}} : \mathbb{Z}^s \longrightarrow M$$

that sends the standard basis $e_1, \ldots, e_s$ to $m_1, \ldots, m_s$. Let L be the kernel of this map, so that we have an exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \longrightarrow M.$$

In down to earth terms, elements $\ell = (\ell_1, \ldots, \ell_s)$ of L satisfy $\sum_{i=1}^s \ell_i m_i = 0$ and hence record the linear relations among the m_i.

Given $\ell = (\ell_1, \dots, \ell_s) \in L$, set

$$\ell_+ = \sum_{\ell_i > 0} \ell_i e_i \quad \text{and} \quad \ell_- = -\sum_{\ell_i < 0} \ell_i e_i.$$

Note that $\ell = \ell_+ - \ell_-$ and that $\ell_+, \ell_- \in \mathbb{N}^s$. It follows easily that the binomial

$$x^{\ell_+} - x^{\ell_-} = \textstyle\prod_{\ell_i > 0} x_i^{\ell_i} - \prod_{\ell_i < 0} x_i^{-\ell_i}$$

vanishes on the image of $\Phi_{\mathscr{A}}$ and hence on $Y_{\mathscr{A}}$ since $Y_{\mathscr{A}}$ is the Zariski closure of the image.

Proposition 1.1.9. *The ideal of the affine toric variety $Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ is*

$$\mathbf{I}(Y_{\mathscr{A}}) = \big\langle x^{\ell_+} - x^{\ell_-} \mid \ell \in L \big\rangle = \big\langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L \big\rangle.$$

Proof. We leave it to the reader to prove equality of the two ideals on the right (Exercise 1.1.2). Let I_L denote this ideal and note that $I_L \subseteq \mathbf{I}(Y_{\mathscr{A}})$. We prove the opposite inclusion following [**264**, Lem. 4.1]. Pick a monomial order $>$ on $\mathbb{C}[x_1, \dots, x_s]$ and an isomorphism $T_N \simeq (\mathbb{C}^*)^n$. Thus we may assume $M = \mathbb{Z}^n$ and the map $\Phi : (\mathbb{C}^*)^n \to \mathbb{C}^s$ is given by Laurent monomials t^{m_i} in variables $t_1, \dots, t_n$. If $I_L \neq \mathbf{I}(Y_{\mathscr{A}})$, then we can pick $f \in \mathbf{I}(Y_{\mathscr{A}}) \setminus I_L$ with minimal leading monomial $x^\alpha = \prod_{i=1}^s x_i^{a_i}$. Rescaling if necessary, x^α becomes the leading term of f.

Since $f(t^{m_1}, \dots, t^{m_s})$ is identically zero as a polynomial in $t_1, \dots, t_n$, there must be cancellation involving the term coming from x^α. In other words, f must contain a monomial $x^\beta = \prod_{i=1}^s x_i^{b_i} < x^\alpha$ such that

$$\prod_{i=1}^s (t^{m_i})^{a_i} = \prod_{i=1}^s (t^{m_i})^{b_i}.$$

This implies that

$$\sum_{i=1}^s a_i m_i = \sum_{i=1}^s b_i m_i,$$

so that $\alpha - \beta = \sum_{i=1}^s (a_i - b_i) e_i \in L$. Then $x^\alpha - x^\beta \in I_L$ by the second description of I_L. It follows that $f - x^\alpha + x^\beta$ also lies in $\mathbf{I}(Y_{\mathscr{A}}) \setminus I_L$ and has strictly smaller leading term. This contradiction completes the proof. □

Given $\mathscr{A} \subseteq M$, there are several ways to compute the ideal $\mathbf{I}(Y_{\mathscr{A}}) = I_L$ of Proposition 1.1.9. In simple cases, the rational implicitization algorithm of [**69**, Ch. 3, §3] can be used. One can also compute I_L using a basis of L and ideal quotients (Exercise 1.1.3). For more on computing I_L, see [**264**, Ch. 12].

Inspired by Proposition 1.1.9, we make the following definition.

Definition 1.1.10. Let $L \subseteq \mathbb{Z}^s$ be a sublattice.

(a) The ideal $I_L = \big\langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L \big\rangle$ is called a ***lattice ideal***.

(b) A prime lattice ideal is called a ***toric ideal***.

Since toric varieties are irreducible, the ideals appearing in Proposition 1.1.9 are toric ideals. Examples of toric ideals include:

Example 1.1.4 : $\langle x^3 - y^2 \rangle \subseteq \mathbb{C}[x,y]$
Example 1.1.5 : $\langle xz - yw \rangle \subseteq \mathbb{C}[x,y,z,w]$
Example 1.1.6 : $\langle x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq d-1 \rangle \subseteq \mathbb{C}[x_0, \ldots, x_d]$.

(The latter is the ideal of the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$.) In each example, we have a prime ideal generated by binomials. As we now show, such ideals are automatically toric.

Proposition 1.1.11. *An ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_s]$ is toric if and only if it is prime and generated by binomials.*

Proof. One direction is obvious. So suppose that I is prime and generated by binomials $x^{\alpha_i} - x^{\beta_i}$. Then observe that $\mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is nonempty (it contains $(1, \ldots, 1)$) and is a subgroup of $(\mathbb{C}^*)^s$ (easy to check). Since $\mathbf{V}(I) \subseteq \mathbb{C}^s$ is irreducible, it follows that $\mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is an irreducible subvariety of $(\mathbb{C}^*)^s$ that is also a subgroup. By Proposition 1.1.1, we see that $T = \mathbf{V}(I) \cap (\mathbb{C}^*)^s$ is a torus.

Projecting on the ith coordinate of $(\mathbb{C}^*)^s$ gives a character $T \hookrightarrow (\mathbb{C}^*)^s \to \mathbb{C}^*$, which by our usual convention we write as $\chi^{m_i} : T \to \mathbb{C}^*$ for $m_i \in M$. It follows easily that $\mathbf{V}(I) = Y_{\mathscr{A}}$ for $\mathscr{A} = \{m_1, \ldots, m_s\}$, and since I is prime, we have $I = \mathbf{I}(Y_{\mathscr{A}})$ by the Nullstellensatz. Then I is toric by Proposition 1.1.9. $\square$

We will later see that all affine toric varieties arise from toric ideals. For more on toric ideals and lattice ideals, the reader should consult [**204**, Ch. 7].

Affine Semigroups. A *semigroup* is a set S with an associative binary operation and an identity element. To be an *affine semigroup*, we further require that:

- The binary operation on S is commutative. We will write the operation as $+$ and the identity element as 0. Thus a finite set $\mathscr{A} \subseteq \mathsf{S}$ gives
$$\mathbb{N}\mathscr{A} = \big\{\textstyle\sum_{m \in \mathscr{A}} a_m m \mid a_m \in \mathbb{N}\big\} \subseteq \mathsf{S}.$$
- The semigroup is finitely generated, meaning that there is a finite set $\mathscr{A} \subseteq \mathsf{S}$ such that $\mathbb{N}\mathscr{A} = \mathsf{S}$.
- The semigroup can be embedded in a lattice M.

The simplest example of an affine semigroup is $\mathbb{N}^n \subseteq \mathbb{Z}^n$. More generally, given a lattice M and a finite set $\mathscr{A} \subseteq M$, we get the affine semigroup $\mathbb{N}\mathscr{A} \subseteq M$. Up to isomorphism, all affine semigroups are of this form.

Given an affine semigroup $\mathsf{S} \subseteq M$, the *semigroup algebra* $\mathbb{C}[\mathsf{S}]$ is the vector space over $\mathbb{C}$ with S as basis and multiplication induced by the semigroup structure of S. To make this precise, we think of M as the character lattice of a torus T_N, so that $m \in M$ gives the character χ^m. Then

$$\mathbb{C}[\mathsf{S}] = \Big\{ \sum_{m \in \mathsf{S}} c_m \chi^m \mid c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m \Big\},$$

with multiplication induced by

$$\chi^m \cdot \chi^{m'} = \chi^{m+m'}.$$

If $\mathsf{S} = \mathbb{N}\mathscr{A}$ for $\mathscr{A} = \{m_1, \ldots, m_s\}$, then $\mathbb{C}[\mathsf{S}] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$.

Here are two basic examples.

Example 1.1.12. The affine semigroup $\mathbb{N}^n \subseteq \mathbb{Z}^n$ gives the polynomial ring

$$\mathbb{C}[\mathbb{N}^n] = \mathbb{C}[x_1, \ldots, x_n],$$

where $x_i = \chi^{e_i}$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$. $\Diamond$

Example 1.1.13. If $e_1, \ldots, e_n$ is a basis of a lattice M, then M is generated by $\mathscr{A} = \{\pm e_1, \ldots, \pm e_n\}$ as an affine semigroup. Setting $t_i = \chi^{e_i}$ gives the Laurent polynomial ring

$$\mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}].$$

Using Example 1.0.2, one sees that $\mathbb{C}[M]$ is the coordinate ring of the torus T_N. $\Diamond$

Affine semigroup rings give rise to affine toric varieties as follows.

Proposition 1.1.14. *Let* $\mathsf{S} \subseteq M$ *be an affine semigroup. Then:*

(a) $\mathbb{C}[\mathsf{S}]$ *is an integral domain and finitely generated as a* $\mathbb{C}$*-algebra.*

(b) $\operatorname{Spec}(\mathbb{C}[\mathsf{S}])$ *is an affine toric variety whose torus has character lattice* $\mathbb{Z}\mathsf{S}$*, and if* $\mathsf{S} = \mathbb{N}\mathscr{A}$ *for a finite set* $\mathscr{A} \subseteq M$*, then* $\operatorname{Spec}(\mathbb{C}[\mathsf{S}]) = Y_{\mathscr{A}}$.

Proof. As noted above, $\mathscr{A} = \{m_1, \ldots, m_s\}$ implies $\mathbb{C}[\mathsf{S}] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$, so $\mathbb{C}[\mathsf{S}]$ is finitely generated. Since $\mathbb{C}[\mathsf{S}] \subseteq \mathbb{C}[M]$ follows from $\mathsf{S} \subseteq M$, we see that $\mathbb{C}[\mathsf{S}]$ is an integral domain by Example 1.1.13.

Using $\mathscr{A} = \{m_1, \ldots, m_s\}$, we get the $\mathbb{C}$-algebra homomorphism

$$\pi : \mathbb{C}[x_1, \ldots, x_s] \longrightarrow \mathbb{C}[M]$$

where $x_i \mapsto \chi^{m_i} \in \mathbb{C}[M]$. This corresponds to the morphism

$$\Phi_{\mathscr{A}} : T_N \longrightarrow \mathbb{C}^s$$

from (1.1.4), i.e., $\pi = (\Phi_{\mathscr{A}})^*$ in the notation of §1.0. One checks that the kernel of π is the toric ideal $\mathbf{I}(Y_{\mathscr{A}})$ (Exercise 1.1.4). The image of π is $\mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}] = \mathbb{C}[\mathsf{S}]$, and then the coordinate ring of $Y_{\mathscr{A}}$ is

$$\begin{aligned} \mathbb{C}[Y_{\mathscr{A}}] &= \mathbb{C}[x_1, \ldots, x_s]/\mathbf{I}(Y_{\mathscr{A}}) \\ &= \mathbb{C}[x_1, \ldots, x_s]/\operatorname{Ker}(\pi) \simeq \operatorname{Im}(\pi) = \mathbb{C}[\mathsf{S}]. \end{aligned} \tag{1.1.5}$$

This proves that $\operatorname{Spec}(\mathbb{C}[\mathsf{S}]) = Y_{\mathscr{A}}$. Since $\mathsf{S} = \mathbb{N}\mathscr{A}$ implies $\mathbb{Z}\mathsf{S} = \mathbb{Z}\mathscr{A}$, the torus of $Y_{\mathscr{A}} = \operatorname{Spec}(\mathbb{C}[\mathsf{S}])$ has the desired character lattice by Proposition 1.1.8. $\square$

Here is an example of this proposition.

Example 1.1.15. Consider the affine semigroup $\mathsf{S} \subseteq \mathbb{Z}$ generated by 2 and 3, so that $\mathsf{S} = \{0, 2, 3, \dots\}$. To study the semigroup algebra $\mathbb{C}[\mathsf{S}]$, we use (1.1.5). If we set $\mathscr{A} = \{2,3\}$, then $\Phi_{\mathscr{A}}(t) = (t^2, t^3)$ and the toric ideal is $\mathbf{I}(Y_{\mathscr{A}}) = \langle x^3 - y^2 \rangle$ by Example 1.1.4. Hence

$$\mathbb{C}[\mathsf{S}] = \mathbb{C}[t^2, t^3] \simeq \mathbb{C}[x,y]/\langle x^3 - y^2 \rangle$$

and the affine toric variety $Y_{\mathscr{A}}$ is the curve $x^3 = y^2$ from Example 1.1.4. $\Diamond$

Equivalence of Constructions. Before stating our main result, we need to study the action of the torus T_N on the semigroup algebra $\mathbb{C}[M]$. The action of T_N on itself given by multiplication induces an action on $\mathbb{C}[M]$ as follows: if $t \in T_N$ and $f \in \mathbb{C}[M]$, then $t \cdot f \in \mathbb{C}[M]$ is defined by $p \mapsto f(t^{-1} \cdot p)$ for $p \in T_N$. The minus sign will be explained in §5.0.

The following lemma will be used several times in the text.

Lemma 1.1.16. *Let $A \subseteq \mathbb{C}[M]$ be a subspace stable under the action of T_N. Then*

$$A = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m.$$

Proof. Let $A' = \bigoplus_{\chi^m \in A} \mathbb{C} \cdot \chi^m$ and note that $A' \subseteq A$. For the opposite inclusion, pick $f \neq 0$ in A. Since $A \subseteq \mathbb{C}[M]$, we can write

$$f = \sum_{m \in \mathscr{B}} c_m \chi^m,$$

where $\mathscr{B} \subseteq M$ is finite and $c_m \neq 0$ for all $m \in \mathscr{B}$. Then $f \in B \cap A$, where

$$B = \operatorname{Span}(\chi^m \mid m \in \mathscr{B}) \subseteq \mathbb{C}[M].$$

An easy computation shows that $t \cdot \chi^m = \chi^m(t^{-1})\chi^m$. It follows that B and hence $B \cap A$ are stable under the action of T_N. Since $B \cap A$ is finite-dimensional, Proposition 1.1.2 implies that $B \cap A$ is spanned by simultaneous eigenvectors of T_N. This is taking place in $\mathbb{C}[M]$, where simultaneous eigenvectors are characters. It follows that $B \cap A$ is spanned by characters. Then the above expression for $f \in B \cap A$ implies that $\chi^m \in A$ for $m \in \mathscr{B}$. Hence $f \in A'$, as desired. $\square$

We can now state the main result of this section, which asserts that our various approaches to affine toric varieties all give the same class of objects.

Theorem 1.1.17. *Let V be an affine variety. The following are equivalent:*

(a) *V is an affine toric variety according to Definition 1.1.3.*

(b) *$V = Y_{\mathscr{A}}$ for a finite set $\mathscr{A}$ in a lattice.*

(c) *V is an affine variety defined by a toric ideal.*

(d) $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ *for an affine semigroup* S.

Proof. The implications (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (a) follow from Propositions 1.1.8, 1.1.9 and 1.1.14. For (a) $\Rightarrow$ (d), let V be an affine toric variety containing the torus T_N with character lattice M. Since the coordinate ring of T_N is the semigroup algebra $\mathbb{C}[M]$, the inclusion $T_N \subseteq V$ induces a map of coordinate rings

$$\mathbb{C}[V] \longrightarrow \mathbb{C}[M].$$

This map is injective since T_N is Zariski dense in V, so that we can regard $\mathbb{C}[V]$ as a subalgebra of $\mathbb{C}[M]$.

Since the action of T_N on V is given by a morphism $T_N \times V \to V$, we see that if $t \in T_N$ and $f \in \mathbb{C}[V]$, then $p \mapsto f(t^{-1} \cdot p)$ is a morphism on V. It follows that $\mathbb{C}[V] \subseteq \mathbb{C}[M]$ is stable under the action of T_N. By Lemma 1.1.16, we obtain

$$\mathbb{C}[V] = \bigoplus_{\chi^m \in \mathbb{C}[V]} \mathbb{C} \cdot \chi^m.$$

Hence $\mathbb{C}[V] = \mathbb{C}[\mathsf{S}]$ for the semigroup $\mathsf{S} = \{m \in M \mid \chi^m \in \mathbb{C}[V]\}$.

Finally, since $\mathbb{C}[V]$ is finitely generated, we can find $f_1, \ldots, f_s \in \mathbb{C}[V]$ with $\mathbb{C}[V] = \mathbb{C}[f_1, \ldots, f_s]$. Expressing each f_i in terms of characters as above gives a finite generating set of S. It follows that S is an affine semigroup. $\square$

Here is one way to think about the above proof. When an irreducible affine variety V contains a torus T_N as a Zariski open subset, we have the inclusion

$$\mathbb{C}[V] \subseteq \mathbb{C}[M].$$

Thus $\mathbb{C}[V]$ consists of those functions on the torus T_N that extend to polynomial functions on V. Then the key insight is that V is a toric variety *precisely when the functions that extend are determined by the characters that extend*.

Example 1.1.18. We have seen that $V = \mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ is a toric variety with toric ideal $\langle xy - zw \rangle \subseteq \mathbb{C}[x, y, z, w]$. Also, the torus is $(\mathbb{C}^*)^3$ via the map $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1})$. The lattice points used in this map can be represented as the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \tag{1.1.6}$$

The corresponding semigroup $\mathsf{S} \subseteq \mathbb{Z}^3$ consists of the $\mathbb{N}$-linear combinations of the column vectors. Hence the elements of S are lattice points lying in the polyhedral region in $\mathbb{R}^3$ pictured in Figure 1 on the next page. In this figure, the four vectors generating S are shown in bold, and the boundary of the polyhedral region is partially shaded. In the terminology of §1.2, this polyhedral region is a *rational*

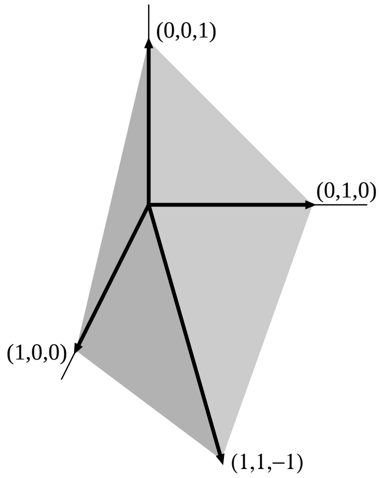


Figure 1. Cone containing the lattice points corresponding to $V = \mathbf{V}(xy - zw)$

polyhedral cone. In Exercise 1.1.5 you will show that S consists of *all* lattice points lying in the cone in Figure 1. We will use this in §1.3 to prove that V is normal. $\Diamond$

Exercises for §1.1.

1.1.1. As in Example 1.1.6, let

$$I = \langle x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq d-1 \rangle \subseteq \mathbb{C}[x_0, \dots, x_d]$$

and let $\widehat{C}_d$ be the surface parametrized by

$$\Phi(s,t) = (s^d, s^{d-1}t, \dots, st^{d-1}, t^d) \in \mathbb{C}^{d+1}.$$

(a) Prove that $\mathbf{V}(I) = \Phi(\mathbb{C}^2) \subseteq \mathbb{C}^{d+1}$. Thus $\widehat{C}_d = \mathbf{V}(I)$.

(b) Prove that $\mathbf{I}(\widehat{C}_d)$ is homogeneous.

(c) Consider lex monomial order with $x_0 > x_1 > \cdots > x_d$. Let $f \in \mathbf{I}(\widehat{C}_d)$ be homogeneous of degree ℓ and let r be the remainder of f on division by the generators of I. Prove that r can be written

$$r = h_0(x_0, x_1) + h_1(x_1, x_2) + \cdots + h_{d-1}(x_{d-1}, x_d)$$

where h_i is homogeneous of degree ℓ. Also explain why we may assume that the coefficient of x_i^ℓ in h_i is zero for $1 \leq i \leq d-1$.

(d) Use part (c) and $r(s^d, s^{d-1}t, \dots, st^{d-1}, t^d) = 0$ to show that $r = 0$.

(e) Use parts (b), (c) and (d) to prove that $I = \mathbf{I}(\widehat{C}_d)$. Also explain why the generators of I are a Gröbner basis for the above lex order.

1.1.2. Let $L \subseteq \mathbb{Z}^s$ be a sublattice. Prove that

$$\langle x^{\ell_+} - x^{\ell_-} \mid \ell \in L \rangle = \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s,\ \alpha - \beta \in L \rangle.$$

Note that when $\ell \in L$, the vectors $\ell_+, \ell_- \in \mathbb{N}^s$ have disjoint support (i.e., no coordinate is positive in both), while this may fail for arbitrary $\alpha, \beta \in \mathbb{N}^s$ with $\alpha - \beta \in L$.

1.1.3. Let I_L be a toric ideal and let $\ell^1,\dots,\ell^r$ be a basis of the sublattice $L \subseteq \mathbb{Z}^s$. Define

$$\tilde{I}_L = \langle x^{\ell^i_+} - x^{\ell^i_-} \mid i = 1,\dots,r\rangle.$$

Prove that $I_L = \tilde{I}_L : \langle x_1 \cdots x_s\rangle^\infty$. Hint: Given $\alpha,\beta \in \mathbb{N}^s$ with $\alpha - \beta \in L$, write $\alpha - \beta = \sum_{i=1}^r a_i \ell^i$, $a_i \in \mathbb{Z}$. This implies

$$x^{\alpha-\beta} - 1 = \prod_{a_i > 0} \left(\frac{x^{\ell^i_+}}{x^{\ell^i_-}}\right)^{a_i} \prod_{a_i < 0} \left(\frac{x^{\ell^i_-}}{x^{\ell^i_+}}\right)^{-a_i} - 1.$$

Show that multiplying this by $(x_1 \cdots x_s)^k$ gives an element of $\tilde{I}_L$ for $k \gg 0$. (By being more careful, one can show that this result holds for lattice ideals. See [**204**, Lem. 7.6].)

1.1.4. Fix an affine variety V. Then $f_1,\dots,f_s \in \mathbb{C}[V]$ give a polynomial map $\Phi : V \to \mathbb{C}^s$, which on coordinate rings is given by

$$\Phi^* : \mathbb{C}[x_1,\dots,x_s] \longrightarrow \mathbb{C}[V], \qquad x_i \longmapsto f_i.$$

Let $Y \subseteq \mathbb{C}^s$ be the Zariski closure of the image of Φ.

(a) Prove that $\mathbf{I}(Y) = \mathrm{Ker}(\Phi^*)$.

(b) Explain how this applies to the proof of Proposition 1.1.14.

1.1.5. Let $m_1 = (1,0,0), m_2 = (0,1,0), m_3 = (0,0,1), m_4 = (1,1,-1)$ be the columns of the matrix in Example 1.1.18 and let

$$C = \Big\{ \sum_{i=1}^4 \lambda_i m_i \mid \lambda_i \in \mathbb{R}_{\geq 0} \Big\} \subseteq \mathbb{R}^3$$

be the cone in Figure 1. Prove that $C \cap \mathbb{Z}^3$ is a semigroup generated by m_1, m_2, m_3, m_4.

1.1.6. An interesting observation is that different sets of lattice points can parametrize the same affine toric variety, even though these parametrizations behave slightly differently. In this exercise you will consider the parametrizations

$$\Phi_1(s,t) = (s^2, st, st^3) \quad \text{and} \quad \Phi_2(s,t) = (s^3, st, t^3).$$

(a) Prove that Φ_1 and Φ_2 both give the affine toric variety $Y = \mathbf{V}(xz - y^3) \subseteq \mathbb{C}^3$.

(b) We can regard Φ_1 and Φ_2 as maps

$$\Phi_1 : \mathbb{C}^2 \longrightarrow Y \quad \text{and} \quad \Phi_2 : \mathbb{C}^2 \longrightarrow Y.$$

Prove that Φ_2 is surjective and that Φ_1 is not.

In general, a finite subset $\mathscr{A} \subseteq \mathbb{Z}^n$ gives a rational map $\Phi_{\mathscr{A}} : \mathbb{C}^n \dashrightarrow Y_{\mathscr{A}}$. The image of $\Phi_{\mathscr{A}}$ in $\mathbb{C}^s$ is called a *toric set* in the literature. Thus $\Phi_1(\mathbb{C}^2)$ and $\Phi_2(\mathbb{C}^2)$ are toric sets. The papers [**169**] and [**239**] study when a toric set equals the corresponding affine toric variety.

1.1.7. In Example 1.1.6 and Exercise 1.1.1 we constructed the rational normal cone $\widehat{C}_d$ using all monomials of degree d in s,t. If we drop some of the monomials, things become more complicated. For example, consider the surface parametrized by

$$\Phi(s,t) = (s^4, s^3t, st^3, t^4) \in \mathbb{C}^4.$$

This gives a toric variety $Y \subseteq \mathbb{C}^4$. Show that the toric ideal of Y is given by

$$\mathbf{I}(Y) = \langle xw - yz, yw^2 - z^3, xz^2 - y^2w, x^2z - y^3\rangle \subseteq \mathbb{C}[x,y,z,w].$$

The ideal $\widehat{C}_4$ has quadratic generators; by removing s^2t^2, we now get cubic generators. See Example B.1.1 for a computational approach to this exercise. See also Example 2.1.10, where we will study the paramaterization Φ from the projective point of view.

1.1.8. Instead of working over $\mathbb{C}$, we will work over an algebraically closed field k of characteristic 2. Consider the affine toric variety $V \subseteq k^5$ parametrized by

$$\Phi(s,t,u) = (s^4, t^4, u^4, s^8u, t^{12}u^3) \in k^5.$$

(a) Find generators for the toric ideal $I = \mathbf{I}(V) \subseteq k[x_1,x_2,x_3,x_4,x_5]$.

(b) Show that $\dim V = 3$. You may assume that Proposition 1.1.8 holds over k.

(c) Show that $I = \sqrt{\langle x_4^4 + x_1^8x_3, x_5^4 + x_2^{12}x_3^3\rangle}$.

It follows that $V \subseteq k^5$ has codimension two and can be defined by two equations, i.e., V is a *set-theoretic complete intersection*. The paper [**12**] shows that if we replace k with an algebraically closed field of characteristic $p > 2$, then the above parametrization is *never* a set-theoretic complete intersection.

1.1.9. Prove that a lattice ideal I_L for $L \subseteq \mathbb{Z}^s$ is a toric ideal if and only if $\mathbb{Z}^s/L$ is torsion-free. Hint: When $\mathbb{Z}^s/L$ is torsion-free, it can be regarded as the character lattice of a torus. The other direction of the proof is more challenging. If you get stuck, see [**204**, Thm. 7.4].

1.1.10. Prove that $I = \langle x^2 - 1, xy - 1, yz - 1\rangle$ is the lattice ideal for the lattice

$$L = \{(a,b,c) \in \mathbb{Z}^3 \mid a+b+c \equiv 0 \bmod 2\} \subseteq \mathbb{Z}^3.$$

Also compute the primary decomposition of I to show that I is not prime.

1.1.11. Let T_N be a torus with character lattice M. Then every point $t \in T_N$ gives an evaluation map $\phi_t : M \to \mathbb{C}^*$ defined by $\phi_t(m) = \chi^m(t)$. Prove that ϕ_t is a group homomorphism and that the map $t \mapsto \phi_t$ induces a group isomorphism

$$T_N \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*).$$

1.1.12. Consider tori T_1 and T_2 with character lattices M_1 and M_2. By Example 1.1.13, the coordinate rings of T_1 and T_2 are $\mathbb{C}[M_1]$ and $\mathbb{C}[M_2]$. Let $\Phi : T_1 \to T_2$ be a morphism that is a group homomorphism. Then Φ induces maps

$$\widehat{\Phi} : M_2 \longrightarrow M_1 \quad \text{and} \quad \Phi^* : \mathbb{C}[M_2] \longrightarrow \mathbb{C}[M_1]$$

by composition. Prove that Φ^* is the map of semigroup algebras induced by the map $\widehat{\Phi}$ of affine semigroups.

1.1.13. A commutative semigroup S is *cancellative* if $u + v = u + w$ implies $v = w$ for all $u, v, w \in \mathsf{S}$ and *torsion-free* if $nu = nv$ implies $u = v$ for all $n \in \mathbb{N} \setminus \{0\}$ and $u, v \in \mathsf{S}$. Prove that S is affine if and only if it is finitely generated, cancellative, and torsion-free.

1.1.14. The requirement that an affine semigroup be finitely generated is important since lattices contain semigroups that are not finitely generated. For example, let $\tau > 0$ be irrational and consider the semigroup

$$\mathsf{S} = \{(a,b) \in \mathbb{N}^2 \mid b \geq \tau a\} \subseteq \mathbb{Z}^2.$$

Prove that S is not finitely generated. (The generators of S are related to continued fractions. For example, when $\tau = (1+\sqrt{5})/2$ is the golden ratio, the minimal generators of S are $(0,1)$ and (F_{2n}, F_{2n+1}) for $n = 1, 2, \ldots$, where F_n is the nth Fibonacci number. See [**231**] and [**259**]. Continued fractions will play an important role in Chapter 10.

1.1.15. Suppose that $\phi : M \to M$ is a group isomorphism. Fix a finite set $\mathscr{A} \subseteq M$ and let $\mathscr{B} = \phi(\mathscr{A})$. Prove that the toric varieties $Y_{\mathscr{A}}$ and $Y_{\mathscr{B}}$ are equivariantly isomorphic (meaning that the isomorphism respects the torus action).

§1.2. Cones and Affine Toric Varieties

We begin with a brief discussion of rational polyhedral cones and then explain how they relate to affine toric varieties.

Convex Polyhedral Cones. Fix a pair of dual vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$. Our discussion of cones will omit most proofs—we refer the reader to [**105**] for more details and [**218**, App. A.1] for careful statements. See also [**51, 128, 241**].

Definition 1.2.1. A ***convex polyhedral cone*** in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{Cone}(S) = \Big\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0 \Big\} \subseteq N_{\mathbb{R}},$$

where $S \subseteq N_{\mathbb{R}}$ is finite. We say that σ is ***generated*** by S. Also set $\mathrm{Cone}(\emptyset) = \{0\}$.

A convex polyhedral cone σ is in fact *convex*, meaning that $x,y \in \sigma$ implies $\lambda x + (1-\lambda)y \in \sigma$ for all $0 \leq \lambda \leq 1$, and is a *cone*, meaning that $x \in \sigma$ implies $\lambda x \in \sigma$ for all $\lambda \geq 0$. Since we will only consider convex cones, the cones satisfying Definition 1.2.1 will be called simply "polyhedral cones."

Examples of polyhedral cones include the first quadrant in $\mathbb{R}^2$ or first octant in $\mathbb{R}^3$. For another example, the cone $\mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq \mathbb{R}^3$ is pictured in Figure 2. It is also possible to have cones that contain entire lines. For example,

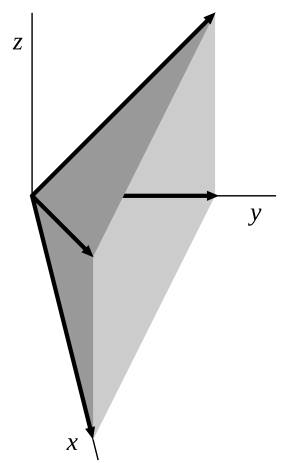


Figure 2. Cone in $\mathbb{R}^3$ generated by $e_1, e_2, e_1 + e_3, e_2 + e_3$

$\mathrm{Cone}(e_1, -e_1) \subseteq \mathbb{R}^2$ is the x-axis, while $\mathrm{Cone}(e_1, -e_1, e_2)$ is the closed upper half-plane $\{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$. As we will see below, these last two examples are not *strongly convex*.

We can also create cones using *polytopes*, which are defined as follows.

Definition 1.2.2. A ***polytope*** in $N_{\mathbb{R}}$ is a set of the form

$$P = \mathrm{Conv}(S) = \Big\{ \sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0, \ \sum_{u \in S} \lambda_u = 1 \Big\} \subseteq N_{\mathbb{R}},$$

where $S \subseteq N_{\mathbb{R}}$ is finite. We say that P is the ***convex hull*** of S.

Polytopes include all polygons in $\mathbb{R}^2$ and bounded polyhedra in $\mathbb{R}^3$. As we will see in later chapters, polytopes play a prominent role in the theory of toric varieties. Here, however, we simply observe that a polytope $P \subseteq N_{\mathbb{R}}$ gives a polyhedral cone $C(P) \subseteq N_{\mathbb{R}} \times \mathbb{R}$, called the *cone of P* and defined by

$$C(P) = \{\lambda \cdot (u, 1) \in N_{\mathbb{R}} \times \mathbb{R} \mid u \in P, \ \lambda \geq 0\}.$$

If $P = \mathrm{Conv}(S)$, then we can also describe this as $C(P) = \mathrm{Cone}(S \times \{1\})$. Figure 3 shows what this looks when P is a pentagon in the plane.

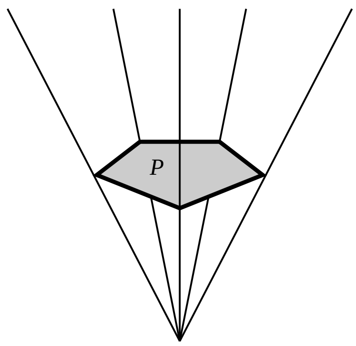


Figure 3. The cone $C(P)$ of a pentagon $P \subseteq \mathbb{R}^2$

The *dimension* $\dim \sigma$ of a polyhedral cone σ is the dimension of the smallest subspace $W = \mathrm{Span}(\sigma)$ of $N_{\mathbb{R}}$ containing σ. We call $\mathrm{Span}(\sigma)$ the *span* of σ.

Dual Cones and Faces. As usual, the pairing between $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ is denoted $\langle\ ,\ \rangle$.

Definition 1.2.3. Given a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, its ***dual cone*** is defined by

$$\sigma^\vee = \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

Duality has the following important properties.

Proposition 1.2.4. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral cone. Then $\sigma^\vee$ is a polyhedral cone in $M_{\mathbb{R}}$ and $(\sigma^\vee)^\vee = \sigma$.* $\square$

Given $m \neq 0$ in $M_{\mathbb{R}}$, we get the hyperplane

$$H_m = \{u \in N_{\mathbb{R}} \mid \langle m,u \rangle = 0\} \subseteq N_{\mathbb{R}}$$

and the closed half-space

$$H_m^+ = \{u \in N_{\mathbb{R}} \mid \langle m,u \rangle \geq 0\} \subseteq N_{\mathbb{R}}.$$

Then H_m is a *supporting hyperplane* of a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ if $\sigma \subseteq H_m^+$, and H_m^+ is a *supporting half-space*. Note that H_m is a supporting hyperplane of σ if and only if $m \in \sigma^\vee \setminus \{0\}$. Furthermore, if $m_1, \ldots, m_s$ generate $\sigma^\vee$, then it is straightforward to check that

$$\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+. \tag{1.2.1}$$

Thus every polyhedral cone is an intersection of finitely many closed half-spaces.

We can use supporting hyperplanes and half-spaces to define *faces* of a cone.

Definition 1.2.5. A ***face of a cone*** of the polyhedral cone σ is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$, written $\tau \preceq \sigma$. Using $m = 0$ shows that σ is a face of itself, i.e., $\sigma \preceq \sigma$. Faces $\tau \neq \sigma$ are called ***proper faces***, written $\tau \prec \sigma$.

The faces of a polyhedral cone have the following obvious properties.

Lemma 1.2.6. *Let $\sigma = \mathrm{Cone}(S)$ be a polyhedral cone. Then:*

(a) *Every face of σ is a polyhedral cone.*

(b) *An intersection of two faces of σ is again a face of σ.*

(c) *A face of a face of σ is again a face of σ.* □

You will prove the following useful property of faces in Exercise 1.2.1.

Lemma 1.2.7. *Let τ be a face of a polyhedral cone σ. If $v, w \in \sigma$ and $v + w \in \tau$, then $v, w \in \tau$.* □

A *facet* of σ is a face τ of codimension 1, i.e., $\dim \tau = \dim \sigma - 1$. An *edge* of σ is a face of dimension 1. In Figure 4 on the next page we illustrate a 3-dimensional cone with shaded facets and a supporting hyperplane (a plane in this case) that cuts out the vertical edge of the cone.

Here are some properties of facets.

Proposition 1.2.8. *Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a polyhedral cone. Then:*

(a) *If $\sigma = H_{m_1}^+ \cap \cdots \cap H_{m_s}^+$ for $m_i \in \sigma^\vee$, $1 \leq i \leq s$, then $\sigma^\vee = \mathrm{Cone}(m_1, \ldots, m_s)$.*

(b) *If $\dim \sigma = n$, then in (a) we can assume that the facets of σ are $\tau_i = H_{m_i} \cap \sigma$.*

(c) *Every proper face $\tau \prec \sigma$ is the intersection of the facets of σ containing τ.* □

Note how part (b) of the proposition refines (1.2.1) when $\dim \sigma = \dim N_{\mathbb{R}}$.

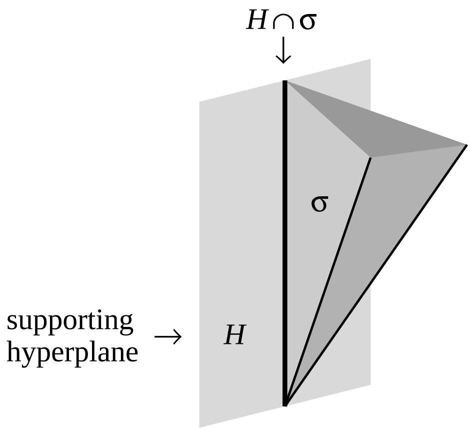


Figure 4. A cone $\sigma \subseteq \mathbb{R}^3$ and a hyperplane H supporting an edge $H \cap \sigma$

When working in $\mathbb{R}^n$, dot product allows us to identify the dual with $\mathbb{R}^n$. From this point of view, the vectors $m_1, \ldots, m_s$ in part (a) of the proposition are *facet normals*, i.e., perpendicular to the facets. This makes it easy to compute examples.

Example 1.2.9. It is easy to see that the facet normals to the cone $\sigma \subseteq \mathbb{R}^3$ in Figure 2 are $m_1 = e_1, m_2 = e_2, m_3 = e_3, m_4 = e_1 + e_2 - e_3$. Hence

$$\sigma^\vee = \mathrm{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3) \subseteq \mathbb{R}^3.$$

This is the cone of Figure 1 at the end of §1.1 whose lattice points describe the semigroup of the affine toric variety $\mathbf{V}(xy - zw)$ (see Example 1.1.18). As we will see, this is part of how cones describe normal affine toric varieties.

Now consider $\sigma^\vee$, which is the cone in Figure 1. The reader can check that the facet normals of this cone are $e_1, e_2, e_1 + e_3, e_2 + e_3$. Using duality and part (a) of Proposition 1.2.8, we obtain

$$\sigma = (\sigma^\vee)^\vee = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3).$$

Hence we recover our original description of σ. See also Example B.2.1. $\Diamond$

In this example, facets of the cone correspond to edges of its dual. More generally, given a face $\tau \preceq \sigma \subseteq N_{\mathbb{R}}$, we define

$$\begin{aligned}\tau^\perp &= \{m \in M_{\mathbb{R}} \mid \langle m, u \rangle = 0 \text{ for all } u \in \tau\}\\ \tau^* &= \{m \in \sigma^\vee \mid \langle m, u \rangle = 0 \text{ for all } u \in \tau\}\\ &= \sigma^\vee \cap \tau^\perp.\end{aligned}$$

We call τ^* the *dual face* of τ because of the following proposition.

Proposition 1.2.10. *If τ is a face of a polyhedral cone σ and $\tau^* = \sigma^\vee \cap \tau^\perp$, then:*

(a) *τ^* is a face of $\sigma^\vee$.*

(b) *The map $\tau \mapsto \tau^*$ is a bijective inclusion-reversing correspondence between the faces of σ and the faces of $\sigma^\vee$.*

(c) $\dim \tau + \dim \tau^* = n$. $\square$

Here is an example of Proposition 1.2.10 when $\dim \sigma < \dim N_{\mathbb{R}}$.

Example 1.2.11. Let $\sigma = \mathrm{Cone}(e_1, e_2) \subseteq \mathbb{R}^3$. Figure 5 shows σ and $\sigma^\vee$. You

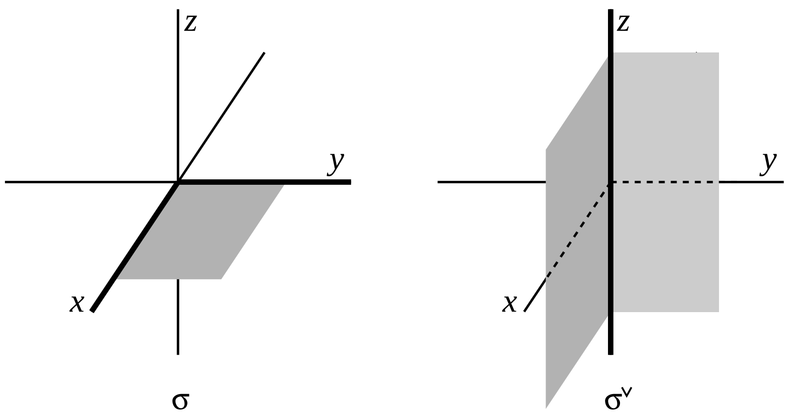


Figure 5. A 2-dimensional cone $\sigma \subseteq \mathbb{R}^3$ and its dual $\sigma^\vee \subseteq \mathbb{R}^3$

should check that the maximal face of σ, namely σ itself, gives the minimal face σ^* of $\sigma^\vee$, namely the z-axis. Note also that

$$\dim \sigma + \dim \sigma^* = 3$$

even though σ has dimension 2. $\Diamond$

Relative Interiors. As already noted, the *span* of a cone $\sigma \subseteq N_{\mathbb{R}}$ is the smallest subspace of $N_{\mathbb{R}}$ containing σ. Then the *relative interior* of σ, denoted $\mathrm{Relint}(\sigma)$, is the interior of σ in its span. Exercise 1.2.2 will characterize $\mathrm{Relint}(\sigma)$ as follows:

$$u \in \mathrm{Relint}(\sigma) \iff \langle m, u \rangle > 0 \text{ for all } m \in \sigma^\vee \setminus \sigma^\perp.$$

When the span equals $N_{\mathbb{R}}$, the relative interior is just the interior, denoted $\mathrm{Int}(\sigma)$.

For an example of how relative interiors arise naturally, suppose that $\tau \preceq \sigma$. This gives the dual face $\tau^* = \sigma^\vee \cap \tau^\perp$ of $\sigma^\vee$. Furthermore, if $m \in \sigma^\vee$, then one easily sees that

$$m \in \tau^* \iff \tau \subseteq H_m \cap \sigma.$$

In Exercise 1.2.2, you will show that if $m \in \sigma^\vee$, then

$$m \in \mathrm{Relint}(\tau^*) \iff \tau = H_m \cap \sigma.$$

Thus the relative interior $\mathrm{Relint}(\tau^*)$ tells us exactly which supporting hyperplanes of σ cut out the face τ.

Strong Convexity. Of the cones shown in Figures 1–5, all but $\sigma^\vee$ in Figure 5 have the nice property that the origin is a face. Such cones are called *strongly convex*. This condition can be stated several ways.

Proposition 1.2.12. *Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a polyhedral cone. Then:*

$$\begin{aligned}\sigma \text{ is strongly convex} &\iff \{0\} \text{ is a face of } \sigma\\ &\iff \sigma \text{ contains no positive-dimensional subspace of } N_{\mathbb{R}}\\ &\iff \sigma \cap (-\sigma) = \{0\}\\ &\iff \dim \sigma^\vee = n. \qquad \square\end{aligned}$$

You will prove Proposition 1.2.12 in Exercise 1.2.3. One corollary is that if a polyhedral cone σ is strongly convex of maximal dimension, then so is $\sigma^\vee$. The cones pictured in Figures 1–4 satisfy this condition.

In general, a polyhedral cone σ always has a minimal face that is the largest subspace W contained in σ. Furthermore:

- $W = \sigma \cap (-\sigma)$.
- $W = H_m \cap \sigma$ whenever $m \in \operatorname{Relint}(\sigma^\vee)$.
- $\overline{\sigma} = \sigma/W \subseteq N_{\mathbb{R}}/W$ is a strongly convex polyhedral cone.

See Exercise 1.2.4.

Separation. When two cones intersect in a face of each, we can separate the cones with the following result, often called the *separation lemma*.

Lemma 1.2.13 (Separation Lemma)**.** *Let σ_1, σ_2 be polyhedral cones in $N_{\mathbb{R}}$ that meet along a common face $\tau = \sigma_1 \cap \sigma_2$. Then*

$$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2$$

for any $m \in \operatorname{Relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$.

Proof. Given $A, B \subseteq N_{\mathbb{R}}$, we set $A - B = \{a - b \mid a \in A,\ b \in B\}$. A standard result from cone theory tells us that

$$\sigma_1^\vee \cap (-\sigma_2)^\vee = (\sigma_1 - \sigma_2)^\vee.$$

Now fix $m \in \operatorname{Relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$. The above equation and Exercise 1.2.4 imply that H_m cuts out the minimal face of $\sigma_1 - \sigma_2$, i.e.,

$$H_m \cap (\sigma_1 - \sigma_2) = (\sigma_1 - \sigma_2) \cap (\sigma_2 - \sigma_1).$$

However, we also have

$$(\sigma_1 - \sigma_2) \cap (\sigma_2 - \sigma_1) = \tau - \tau.$$

One inclusion is obvious since $\tau = \sigma_1 \cap \sigma_2$. For the other inclusion, write $u \in (\sigma_1 - \sigma_2) \cap (\sigma_2 - \sigma_1)$ as

$$u = a_1 - a_2 = b_2 - b_1, \quad a_1, b_1 \in \sigma_1,\ a_2, b_2 \in \sigma_2.$$

Then $a_1+b_1=a_2+b_2$ implies that this element lies in $\tau=\sigma_1\cap\sigma_2$. Since $a_1,b_1\in\sigma_1$, we have $a_1,b_1\in\tau$ by Lemma 1.2.7, and $a_2,b_2\in\tau$ follows similarly. Hence $u=a_1-a_2\in\tau-\tau$, as desired.

We conclude that $H_m\cap(\sigma_1-\sigma_2)=\tau-\tau$. Intersecting with σ_1, we obtain

$$H_m\cap\sigma_1=(\tau-\tau)\cap\sigma_1=\tau,$$

where the last equality again uses Lemma 1.2.7 (Exercise 1.2.5). If instead we intersect with $-\sigma_2$, we obtain

$$H_m\cap(-\sigma_2)=(\tau-\tau)\cap(-\sigma_2)=-\tau,$$

and $H_m\cap\sigma_2=\tau$ follows. $\square$

In the situation of Lemma 1.2.13 we call H_m a *separating hyperplane*.

Rational Polyhedral Cones. Let N and M be dual lattices with associated vector spaces $N_{\mathbb{R}}=N\otimes_{\mathbb{Z}}\mathbb{R}$ and $M_{\mathbb{R}}=M\otimes_{\mathbb{Z}}\mathbb{R}$. For $\mathbb{R}^n$ we usually use the lattice $\mathbb{Z}^n$.

Definition 1.2.14. A polyhedral cone $\sigma\subseteq N_{\mathbb{R}}$ is ***rational*** if $\sigma=\mathrm{Cone}(S)$ for some finite set $S\subseteq N$.

The cones appearing in Figures 1, 2 and 5 are rational. We note without proof that faces and duals of rational polyhedral cones are rational. Furthermore, if $\sigma=\mathrm{Cone}(S)$ for $S\subseteq N$ finite and $N_{\mathbb{Q}}=N\otimes_{\mathbb{Z}}\mathbb{Q}$, then

$$\sigma\cap N_{\mathbb{Q}}=\big\{\textstyle\sum_{u\in S}\lambda_u u\mid\lambda_u\geq 0\text{ in }\mathbb{Q}\big\}. \tag{1.2.2}$$

One new feature is that a strongly convex rational polyedral cone σ has a canonical generating set, constructed as follows. Let ρ be an edge of σ. Since σ is strongly convex, ρ is a *ray*, i.e., a half-line, and since ρ is rational, the semigroup $\rho\cap N$ is generated by a unique element $u_\rho\in\rho\cap N$. We call u_ρ the *ray generator* of ρ. Figure 6 shows the ray generator of a rational ray ρ in the plane. The dots are the lattice $N=\mathbb{Z}^2$ and the white ones are $\rho\cap N$.

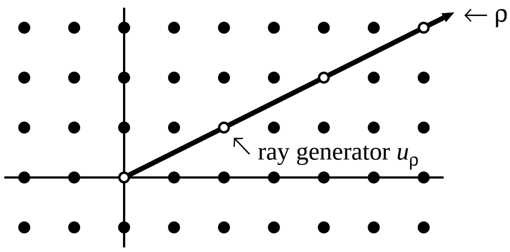


Figure 6. A rational ray $\rho\subseteq\mathbb{R}^2$ and its unique ray generator u_ρ

Lemma 1.2.15. *A strongly convex rational polyhedral cone is generated by the ray generators of its edges.* $\square$

It is customary to call the ray generators of the edges the *minimal generators* of a strongly convex rational polyhedral cone. Figures 1 and 2 show 3-dimensional strongly convex rational polyhedral cones and their ray generators.

In a similar way, a rational polyhedral cone σ of maximal dimension has unique *facet normals*, which are the ray generators of the dual $\sigma^\vee$, which is strongly convex by Proposition 1.2.12.

Here are some especially important strongly convex cones.

Definition 1.2.16. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone.

(a) σ is ***smooth*** or ***regular*** if its minimal generators form part of a $\mathbb{Z}$-basis of N,

(b) σ is ***simplicial*** if its minimal generators are linearly independent over $\mathbb{R}$.

The cone σ pictured in Figure 5 is smooth, while those in Figures 1 and 2 are not even simplicial. Note also that the dual of a smooth (resp. simplicial) cone of maximal dimension is again smooth (resp. simplicial). Later in the section we will give examples of simplicial cones that are not smooth.

Semigroup Algebras and Affine Toric Varieties. Given a rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$, the lattice points

$$\mathsf{S}_\sigma = \sigma^\vee \cap M \subseteq M$$

form a semigroup. A key fact is that this semigroup is finitely generated.

Proposition 1.2.17 (Gordan's Lemma)**.** *$\mathsf{S}_\sigma = \sigma^\vee \cap M$ is finitely generated and hence is an affine semigroup.*

Proof. Since $\sigma^\vee$ is rational polyhedral, $\sigma^\vee = \mathrm{Cone}(T)$ for a finite set $T \subseteq M$. Then $K = \{\sum_{m\in T} \delta_m m \mid 0 \le \delta_m < 1\}$ is a bounded region of $M_\mathbb{R} \simeq \mathbb{R}^n$, so that $K \cap M$ is finite since $M \simeq \mathbb{Z}^n$. Note that $T \cup (K \cap M) \subseteq \mathsf{S}_\sigma$.

We claim $T \cup (K \cap M)$ generates S_σ as a semigroup. To prove this, take $w \in \mathsf{S}_\sigma$ and write $w = \sum_{m\in T} \lambda_m m$ where $\lambda_m \ge 0$. Then $\lambda_m = \lfloor \lambda_m \rfloor + \delta_m$ with $\lfloor \lambda_m \rfloor \in \mathbb{N}$ and $0 \le \delta_m < 1$, so that

$$w = \sum_{m\in T} \lfloor \lambda_m \rfloor m + \sum_{m\in T} \delta_m m.$$

The second sum is in $K \cap M$ (remember $w \in M$). It follows that w is a nonnegative integer combination of elements of $T \cup (K \cap M)$. □

Since affine semigroups give affine toric varieties, we get the following.

Theorem 1.2.18. *Let $\sigma \subseteq N_\mathbb{R} \simeq \mathbb{R}^n$ be a rational polyhedral cone with semigroup $\mathsf{S}_\sigma = \sigma^\vee \cap M$. Then*

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_\sigma]) = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$$

is an affine toric variety. Furthermore,

$$\dim U_\sigma = n \iff \textit{the torus of } U_\sigma \textit{ is } T_N = N \otimes_\mathbb{Z} \mathbb{C}^* \iff \sigma \textit{ is strongly convex}.$$

Proof. By Gordan's Lemma and Proposition 1.1.14, U_σ is an affine toric variety whose torus has character lattice $\mathbb{Z}\mathsf{S}_\sigma \subseteq M$. To study $\mathbb{Z}\mathsf{S}_\sigma$, note that

$$\mathbb{Z}\mathsf{S}_\sigma = \mathsf{S}_\sigma - \mathsf{S}_\sigma = \{m_1 - m_2 \mid m_1, m_2 \in \mathsf{S}_\sigma\}.$$

Now suppose that $km \in \mathbb{Z}\mathsf{S}_\sigma$ for some $k > 1$ and $m \in M$. Then $km = m_1 - m_2$ for $m_1, m_2 \in \mathsf{S}_\sigma = \sigma^\vee \cap M$. Since m_1 and m_2 lie in the convex set $\sigma^\vee$, we have

$$m + m_2 = \tfrac{1}{k} m_1 + \tfrac{k-1}{k} m_2 \in \sigma^\vee.$$

It follows that $m = (m + m_2) - m_2 \in \mathbb{Z}\mathsf{S}_\sigma$, so that $M/\mathbb{Z}\mathsf{S}_\sigma$ is torsion-free. Hence

$$\text{the torus of } U_\sigma \text{ is } T_N \iff \mathbb{Z}\mathsf{S}_\sigma = M \iff \operatorname{rank} \mathbb{Z}\mathsf{S}_\sigma = n. \tag{1.2.3}$$

Since σ is strongly convex if and only if $\dim \sigma^\vee = n$ (Proposition 1.2.12), it remains to show that

$$\dim U_\sigma = n \iff \operatorname{rank} \mathbb{Z}\mathsf{S}_\sigma = n \iff \dim \sigma^\vee = n.$$

The first equivalence follows since the dimension of an affine toric variety is the dimension of its torus, which is the rank of its character lattice. We leave the proof of the second equivalence to the reader (Exercise 1.2.6). □

Remark 1.2.19.

(a) For the rest of the book, we will always assume that $\sigma \subseteq N_\mathbb{R}$ is strongly convex since we want T_N to be the torus of the affine toric variety U_σ.

(b) The reader may ask why we focus on $\sigma \subseteq N_\mathbb{R}$ since $U_\sigma = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ makes $\sigma^\vee \subseteq M_\mathbb{R}$ seem more important. The answer will become clear once we understand how normal toric varieties are constructed from affine pieces. The discussion following Proposition 1.3.16 gives a first hint of how this works.

Example 1.2.20. Let $\sigma = \operatorname{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq N_\mathbb{R} = \mathbb{R}^3$ with $N = \mathbb{Z}^3$. This is the cone pictured in Figure 2. By Example 1.2.9, $\sigma^\vee$ is the cone pictured in Figure 1, and by Example 1.1.18, the lattice points in this cone are generated by columns of matrix (1.1.6). It follows from Example 1.1.18 that U_σ is the affine toric variety $\mathbf{V}(xy - zw)$. ◊

Example 1.2.21. Fix $0 \le r \le n$ and set $\sigma = \operatorname{Cone}(e_1, \ldots, e_r) \subseteq \mathbb{R}^n$. Then

$$\sigma^\vee = \operatorname{Cone}(e_1, \ldots, e_r, \pm e_{r+1}, \ldots, \pm e_n)$$

and the corresponding affine toric variety is

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[x_1, \ldots, x_r, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}]) = \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$$

(Exercise 1.2.7). This implies that if $\sigma \subseteq N_\mathbb{R} \simeq \mathbb{R}^n$ is a smooth cone of dimension r, then $U_\sigma \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$. Figure 5 from Example 1.2.11 shows $r = 2$ and $n = 3$. ◊

Example 1.2.22. Fix a positive integer d and let $\sigma = \operatorname{Cone}(de_1 - e_2, e_2) \subseteq \mathbb{R}^2$. This has dual cone $\sigma^\vee = \operatorname{Cone}(e_1, e_1 + de_2)$. Figure 7 on the next page shows $\sigma^\vee$ when $d = 4$. The affine semigroup $\mathsf{S}_\sigma = \sigma^\vee \cap \mathbb{Z}^2$ is generated by the lattice points

$(1,i)$ for $0 \le i \le d$. When $d = 4$, these are the white dots in Figure 7. (You will prove these assertions in Exercise 1.2.8.)

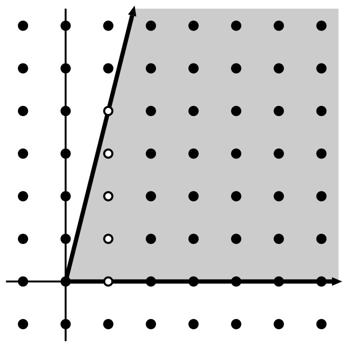


Figure 7. The cone $\sigma^\vee$ when $d = 4$

By §1.1, the affine toric variety U_σ is the Zariski closure of the image of the map $\Phi : (\mathbb{C}^*)^2 \to \mathbb{C}^{d+1}$ defined by

$$\Phi(s,t) = (s, st, st^2, \dots, st^d).$$

This map has the same image as the map $(s,t) \mapsto (s^d, s^{d-1}t, \dots, st^{d-1}, t^d)$ used in Example 1.1.6. Thus U_σ is isomorphic to the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$ whose ideal is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-2} & x_{d-1} \\ x_1 & x_2 & \cdots & x_{d-1} & x_d \end{pmatrix}.$$

Note that the cones σ and $\sigma^\vee$ are simplicial but not smooth. $\Diamond$

We will return to this example often. One thing evident in Example 1.1.6 is the difference between *cone generators* and *semigroup generators*: the cone $\sigma^\vee$ has two generators but the semigroup $\mathsf{S}_\sigma = \sigma^\vee \cap \mathbb{Z}^2$ has $d+1$.

When $\sigma \subseteq N_\mathbb{R}$ has maximal dimension, the semigroup $\mathsf{S}_\sigma = \sigma^\vee \cap M$ has a unique minimal generating set constructed as follows. Define an element $m \neq 0$ of S_σ to be *irreducible* if $m = m' + m''$ for $m', m'' \in \mathsf{S}_\sigma$ implies $m' = 0$ or $m'' = 0$.

Proposition 1.2.23. *Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone of maximal dimension and let $\mathsf{S}_\sigma = \sigma^\vee \cap M$. Then*

$$\mathscr{H} = \{m \in \mathsf{S}_\sigma \mid m \text{ is irreducible}\}$$

has the following properties:

(a) *$\mathscr{H}$ is finite and generates S_σ.*

(b) *$\mathscr{H}$ contains the ray generators of the edges of $\sigma^\vee$.*

(c) *$\mathscr{H}$ is the minimal generating set of S_σ with respect to inclusion.*

Proof. Proposition 1.2.12 implies that $\sigma^\vee$ is strongly convex, so we can find an element $u \in \sigma \cap N \setminus \{0\}$ such that $\langle m,u\rangle \in \mathbb{N}$ for all $m \in \mathsf{S}_\sigma$ and $\langle m,u\rangle = 0$ if and only if $m = 0$.

Now suppose that $m \in \mathsf{S}_\sigma$ is not irreducible. Then $m = m' + m''$ where m' and m'' are nonzero elements of S_σ. It follows that

$$\langle m,u\rangle = \langle m',u\rangle + \langle m'',u\rangle$$

with $\langle m',u\rangle, \langle m'',u\rangle \in \mathbb{N} \setminus \{0\}$, so that

$$\langle m',u\rangle < \langle m,u\rangle \quad \text{and} \quad \langle m'',u\rangle < \langle m,u\rangle.$$

Using induction on $\langle m,u\rangle$, we conclude that every element of S_σ is a sum of irreducible elements, so that $\mathscr{H}$ generates S_σ. Furthermore, using a finite generating set of S_σ, one easily sees that $\mathscr{H}$ is finite. This proves part (a). The remaining parts of the proof are covered in Exercise 1.2.9. $\square$

The set $\mathscr{H} \subseteq \mathsf{S}_\sigma$ is called the *Hilbert basis* of S_σ and its elements are the *minimal generators* of S_σ. More generally, Proposition 1.2.23 holds for any affine semigroup S satisfying $\mathsf{S} \cap (-\mathsf{S}) = \{0\}$. Algorithms for computing Hilbert bases are discussed in [**204**, 7.3] and [**264**, Ch. 13], and Hilbert bases can be computed using the computer program `Normaliz` [**57**]. See Examples B.3.1 and B.3.2.

Exercises for §1.2.

1.2.1. Prove Lemma 1.2.7. Hint: Write $\tau = H_m \cap \sigma$ for $m \in \sigma^\vee$.

1.2.2. Here are some properties of relative interiors. Let $\sigma \subseteq N_\mathbb{R}$ be a cone.

(a) Show that if $u \in \sigma$, then $u \in \mathrm{Relint}(\sigma)$ if and only if $\langle m,u\rangle > 0$ for all $m \in \sigma^\vee \setminus \sigma^\perp$.

(b) Let $\tau \preceq \sigma$ and fix $m \in \sigma^\vee$. Prove that

$$\begin{aligned} m \in \tau^* &\iff \tau \subseteq H_m \cap \sigma \\ m \in \mathrm{Relint}(\tau^*) &\iff \tau = H_m \cap \sigma. \end{aligned}$$

1.2.3. Prove Proposition 1.2.12.

1.2.4. Let $\sigma \subseteq N_\mathbb{R}$ be a polyhedral cone.

(a) Use Proposition 1.2.10 to show that σ has a unique minimal face with respect to $\preceq$. Let W denote this minimal face.

(b) Prove that $W = (\sigma^\vee)^\perp$.

(c) Prove that W is the largest subspace contained in σ.

(d) Prove that $W = \sigma \cap (-\sigma)$.

(e) Fix $m \in \sigma^\vee$. Prove that $m \in \mathrm{Relint}(\sigma^\vee)$ if and only if $W = H_m \cap \sigma$.

(f) Prove that $\overline{\sigma} = \sigma/W \subseteq N_\mathbb{R}/W$ is a strongly convex polyhedral cone.

1.2.5. Let $\tau \preceq \sigma \subseteq N_\mathbb{R}$ and let $\tau - \tau$ be defined as in the proof of Lemma 1.2.13. Prove that $\tau = (\tau - \tau) \cap \tau$. Also show that $\tau - \tau = \mathrm{Span}(\tau)$, i.e., $\tau - \tau$ is the smallest subspace of $N_\mathbb{R}$ containing τ.

1.2.6. Fix a lattice M and let $\mathrm{Span}(S)$ denote the span over $\mathbb{R}$ of a subset $S \subseteq M_{\mathbb{R}}$.

(a) Let $S \subseteq M$ be finite. Prove that $\mathrm{rank}\,\mathbb{Z}S = \dim \mathrm{Span}(S)$.

(b) Let $S \subseteq M_{\mathbb{R}}$ be finite. Prove that $\dim \mathrm{Cone}(S) = \dim \mathrm{Span}(S)$.

(c) Use parts (a) and (b) to complete the proof of Theorem 1.2.18.

1.2.7. Prove the assertions made in Example 1.2.21.

1.2.8. Prove the assertions made in Example 1.2.22. Hint: First show that when a cone is smooth, the ray generators of the cone also generate the corresponding semigroup. Then write the cone $\sigma^\vee$ of Example 1.2.22 as a union of such cones.

1.2.9. Complete the proof of Proposition 1.2.23. Hint for part (b): Show that the ray generators of the edges of $\sigma^\vee$ are irreducible in S_σ. Given an edge ρ of $\sigma^\vee$, it will help to pick $u \in \sigma \cap N \setminus \{0\}$ such that $\rho = H_u \cap \sigma^\vee$.

1.2.10. Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone generated by a set of linearly independent vectors in $N_{\mathbb{R}}$. Show that σ is strongly convex and simplicial.

1.2.11. Explain the picture illustrated in Figure 8 in terms of Proposition 1.2.8.

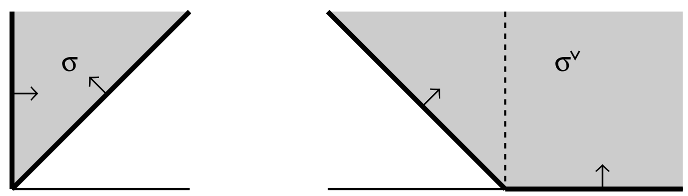


Figure 8. A cone σ in the plane and its dual

1.2.12. Let $P \subseteq N_{\mathbb{R}}$ be a polytope lying in an affine hyperplane (= translate of a hyperplane) not containing the origin. Generalize Figure 3 by showing that P gives a strongly convex polyhedral cone in $N_{\mathbb{R}}$. Draw a picture.

1.2.13. Consider the cone $\sigma = \mathrm{Cone}(3e_1 - 2e_2, e_2) \subseteq \mathbb{R}^2$.

(a) Describe $\sigma^\vee$ and find generators of $\sigma^\vee \cap \mathbb{Z}^2$. Draw a picture similar to Figure 7.

(b) Compute the toric ideal of the affine toric variety U_σ and explain how this exercise relates to Exercise 1.1.6.

1.2.14. Consider the simplicial cone $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_2 + 2e_3) \subseteq \mathbb{R}^3$.

(a) Describe $\sigma^\vee$ and find generators of $\sigma^\vee \cap \mathbb{Z}^3$.

(b) Compute the toric ideal of the affine toric variety U_σ.

1.2.15. Let σ be a strongly convex polyhedral cone of maximal dimension. Here is an example taken from [**105**, p. 132] to show that σ and $\sigma^\vee$ need not have the same number of edges. Let $\sigma \subseteq \mathbb{R}^4$ be the cone generated by $2e_i + e_j$ for all $1 \le i, j \le 4$, $i \neq j$.

(a) Show that σ has 12 edges.

(b) Show that $\sigma^\vee$ is generated by e_i and $-e_i + 2\sum_{j \neq i} e_j$, $1 \le i \le 4$ and has 8 edges.

§1.3. Properties of Affine Toric Varieties

The final task of this chapter is to explore the properties of affine toric varieties. We will also study maps between affine toric varieties.

Points of Affine Toric Varieties. We first consider various ways to describe the points of an affine toric variety.

Proposition 1.3.1. *Let $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ be the affine toric variety of the affine semigroup S. Then there are bijective correspondences between the following:*

(a) *Points $p \in V$.*

(b) *Maximal ideals $\mathfrak{m} \subseteq \mathbb{C}[\mathsf{S}]$.*

(c) *Semigroup homomorphisms $\mathsf{S} \to \mathbb{C}$, where $\mathbb{C}$ is considered as a semigroup under multiplication.*

Proof. The correspondence between (a) and (b) is standard (see [**69**, Thm. 5 of Ch. 5, §4]). The correspondence between (a) and (c) is special to the toric case.

Given a point $p \in V$, define $\mathsf{S} \to \mathbb{C}$ by sending $m \in \mathsf{S}$ to $\chi^m(p) \in \mathbb{C}$. This makes sense since $\chi^m \in \mathbb{C}[\mathsf{S}] = \mathbb{C}[V]$. One easily checks that $\mathsf{S} \to \mathbb{C}$ is a semigroup homomorphism.

Going the other way, let $\gamma : \mathsf{S} \to \mathbb{C}$ be a semigroup homomorphism. Since $\{\chi^m\}_{m\in\mathsf{S}}$ is a basis of $\mathbb{C}[\mathsf{S}]$, γ induces a surjective linear map $\mathbb{C}[\mathsf{S}] \to \mathbb{C}$ which is a $\mathbb{C}$-algebra homomorphism. The kernel of the map $\mathbb{C}[\mathsf{S}] \to \mathbb{C}$ is a maximal ideal and thus gives a point $p \in V$ by the correspondence between (a) and (b).

We construct p concretely as follows. Let $\mathscr{A} = \{m_1, \ldots, m_s\}$ generate S, so that $V = Y_{\mathscr{A}} \subseteq \mathbb{C}^s$. Let $p = (\gamma(m_1), \ldots, \gamma(m_s)) \in \mathbb{C}^s$. Let us prove that $p \in V$. By Proposition 1.1.9, it suffices to show that $x^\alpha - x^\beta$ vanishes at p for all exponent vectors $\alpha = (a_1, \ldots, a_s)$ and $\beta = (b_1, \ldots, b_s)$ satisfying

$$\sum_{i=1}^{s} a_i m_i = \sum_{i=1}^{s} b_i m_i.$$

This is easy, since γ being a semigroup homomorphism implies that

$$\prod_{i=1}^{s} \gamma(m_i)^{a_i} = \gamma\Big(\sum_{i=1}^{s} a_i m_i\Big) = \gamma\Big(\sum_{i=1}^{s} b_i m_i\Big) = \prod_{i=1}^{s} \gamma(m_i)^{b_i}.$$

It is straightforward to show that this point of V agrees with the one constructed in the previous paragraph (Exercise 1.3.1). □

As an application of this result, we describe the torus action on V. In terms of the embedding $V = Y_{\mathscr{A}} \subseteq \mathbb{C}^s$, the proof of Proposition 1.1.8 shows that the action of T_N on $Y_{\mathscr{A}}$ is induced by the usual action of $(\mathbb{C}^*)^s$ on $\mathbb{C}^s$. But how do we see the action intrinsically, without embedding into affine space? This is where semigroup homomorphisms prove their value. Fix $t \in T_N$ and $p \in V$, and let p correspond to the semigroup homorphism $m \mapsto \gamma(m)$. In Exercise 1.3.1 you will show that $t \cdot p$ is given by the semigroup homomorphism $m \mapsto \chi^m(t)\gamma(m)$. This description will prove useful in Chapter 3 when we study torus orbits.

From the point of view of group actions, the action of T_N on V is given by a map $T_N \times V \to V$. Since both sides are affine varieties, this should be a morphism, meaning that it should come from a $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[\mathsf{S}] = \mathbb{C}[V] \longrightarrow \mathbb{C}[T_N \times V] = \mathbb{C}[T_N] \otimes_{\mathbb{C}} \mathbb{C}[V] = \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[\mathsf{S}].$$

This homomorphism is given by $\chi^m \mapsto \chi^m \otimes \chi^m$ for $m \in \mathsf{S}$ (Exercise 1.3.2).

We next characterize when the torus action on an affine toric variety has a fixed point. An affine semigroup S is *pointed* if $\mathsf{S} \cap (-\mathsf{S}) = \{0\}$, i.e., if 0 is the only invertible element of S. This is the semigroup analog of strongly convex.

Proposition 1.3.2. *Let V be an affine toric variety. Then:*

(a) *If we write $V = \operatorname{Spec}(\mathbb{C}[\mathsf{S}])$, then the torus action has a fixed point if and only if S is pointed, in which case the unique fixed point is given by the semigroup homomorphism $\mathsf{S} \to \mathbb{C}$ defined by*

$$m \longmapsto \begin{cases} 1 & m = 0 \\ 0 & m \neq 0. \end{cases} \tag{1.3.1}$$

(b) *If we write $V = Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ for $\mathscr{A} \subseteq \mathsf{S} \setminus \{0\}$, then the torus action has a fixed point if and only if $0 \in Y_{\mathscr{A}}$, in which case the unique fixed point is 0.*

Proof. For part (a), let $p \in V$ be represented by the semigroup homomorphism $\gamma : \mathsf{S} \to \mathbb{C}$. Then p is fixed by the torus action if and only if $\chi^m(t)\gamma(m) = \gamma(m)$ for all $m \in \mathsf{S}$ and $t \in T_N$. This equation is satisfied for $m = 0$ since $\gamma(0) = 1$, and if $m \neq 0$, then picking t with $\chi^m(t) \neq 1$ shows that $\gamma(m) = 0$. Thus, if a fixed point exists, then it is unique and is given by (1.3.1). Then we are done since (1.3.1) is a semigroup homomorphism if and only if S is pointed.

For part (b), first assume that $V = Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ has a fixed point, in which case $\mathsf{S} = \mathbb{N}\mathscr{A}$ is pointed and the unique point p is given by (1.3.1). Then $\mathscr{A} \subseteq \mathsf{S} \setminus \{0\}$ and the proof of Proposition 1.3.1 imply that p is the origin in $\mathbb{C}^s$, so that $0 \in Y_{\mathscr{A}}$. The converse follows since $0 \in \mathbb{C}^s$ is fixed by $(\mathbb{C}^*)^s$, hence by $Y_{\mathscr{A}} \cap (\mathbb{C}^*)^s$. □

Here is a useful corollary of Proposition 1.3.2 (Exercise 1.3.3).

Corollary 1.3.3. *Let U_σ be the affine toric variety of a strongly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$. Then the torus action on U_σ has a fixed point if and only if $\dim \sigma = \dim N_{\mathbb{R}}$, in which case the fixed point is unique and is given by the maximal ideal*

$$\langle \chi^m \mid m \in \mathsf{S}_\sigma \setminus \{0\} \rangle \subseteq \mathbb{C}[\mathsf{S}_\sigma],$$

where as usual $\mathsf{S}_\sigma = \sigma^\vee \cap M$. □

We will see in Chapter 3 that this corollary is part of the correspondence between torus orbits of U_σ and faces of σ.

Normality and Saturation. We next study the question of when an affine toric variety V is normal. We need one definition before stating our normality criterion.

Definition 1.3.4. An affine semigroup $\mathsf{S} \subseteq M$ is ***saturated*** if for all $k \in \mathbb{N} \setminus \{0\}$ and $m \in M$, $km \in \mathsf{S}$ implies $m \in \mathsf{S}$.

For example, if $\sigma \subseteq N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone, then $\mathsf{S}_\sigma = \sigma^\vee \cap M$ is easily seen to be saturated (Exercise 1.3.4).

Theorem 1.3.5. *Let V be an affine toric variety with torus T_N. Then the following are equivalent:*

(a) *V is normal.*

(b) *$V = \operatorname{Spec}(\mathbb{C}[\mathsf{S}])$, where $\mathsf{S} \subseteq M$ is a saturated affine semigroup.*

(c) *$V = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_\sigma])$ $(= U_\sigma)$, where $\mathsf{S}_\sigma = \sigma^\vee \cap M$ and $\sigma \subseteq N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone.*

Proof. By Theorem 1.1.17, $V = \operatorname{Spec}(\mathbb{C}[\mathsf{S}])$ for an affine semigroup S contained in a lattice, and by Proposition 1.1.14, the torus of V has the character lattice $M = \mathbb{Z}\mathsf{S}$. Also let $n = \dim V$, so that $M \simeq \mathbb{Z}^n$.

(a) $\Rightarrow$ (b): If V is normal, then $\mathbb{C}[\mathsf{S}] = \mathbb{C}[V]$ is integrally closed in its field of fractions $\mathbb{C}(V)$. Suppose that $km \in \mathsf{S}$ for some $k \in \mathbb{N} \setminus \{0\}$ and $m \in M$. Then χ^m is a polynomial function on T_N and hence a rational function on V since $T_N \subseteq V$ is Zariski open. We also have $\chi^{km} \in \mathbb{C}[\mathsf{S}]$ since $km \in \mathsf{S}$. It follows that χ^m is a root of the monic polynomial $X^k - \chi^{km}$ with coefficients in $\mathbb{C}[\mathsf{S}]$. By the definition of normal, we obtain $\chi^m \in \mathbb{C}[\mathsf{S}]$, i.e., $m \in \mathsf{S}$. Thus S is saturated.

(b) $\Rightarrow$ (c): Let $\mathscr{A} \subseteq \mathsf{S}$ be a finite generating set of S. Then S lies in the rational polyhedral cone $\operatorname{Cone}(\mathscr{A}) \subseteq M_{\mathbb{R}}$, and $\operatorname{rank} \mathbb{Z}\mathscr{A} = n$ implies $\dim \operatorname{Cone}(\mathscr{A}) = n$ by Exercise 1.2.6. It follows that $\sigma = \operatorname{Cone}(\mathscr{A})^\vee \subseteq N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone such that $\mathsf{S} \subseteq \sigma^\vee \cap M$. In Exercise 1.3.4 you will prove that equality holds when S is saturated. Hence $\mathsf{S} = \mathsf{S}_\sigma$.

(c) $\Rightarrow$ (a): We need to show that $\mathbb{C}[\mathsf{S}_\sigma] = \mathbb{C}[\sigma^\vee \cap M]$ is normal when $\sigma \subseteq N_{\mathbb{R}}$ is a strongly convex rational polyhedral cone. Let $\rho_1, \dots, \rho_r$ be the rays of σ. Since σ is generated by its rays (Lemma 1.2.15), we have

$$\sigma^\vee = \bigcap_{i=1}^r \rho_i^\vee.$$

Intersecting with M gives $\mathsf{S}_\sigma = \bigcap_{i=1}^r \mathsf{S}_{\rho_i}$, which easily implies

$$\mathbb{C}[\mathsf{S}_\sigma] = \bigcap_{i=1}^r \mathbb{C}[\mathsf{S}_{\rho_i}].$$

By Exercise 1.0.7, $\mathbb{C}[\mathsf{S}_\sigma]$ is normal if each $\mathbb{C}[\mathsf{S}_{\rho_i}]$ is normal, so it suffices to prove that $\mathbb{C}[\mathsf{S}_\rho]$ is normal when ρ is a rational ray in $N_\mathbb{R}$. Let $u_\rho \in \rho \cap N$ be the ray generator of ρ. Since u is *primitive*, i.e, $\frac{1}{k}u_\rho \notin N$ for all $k > 1$, we can find a basis $e_1, \ldots, e_n$ of N with $u_\rho = e_1$ (Exercise 1.3.5). This allows us to assume that $\rho = \mathrm{Cone}(e_1)$, so that

$$\mathbb{C}[\mathsf{S}_\rho] = \mathbb{C}[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

by Example 1.2.21. But $\mathbb{C}[x_1, \ldots, x_n]$ is normal (it is a UFD), so its localization

$$\mathbb{C}[x_1, \ldots, x_n]_{x_2 \cdots x_n} = \mathbb{C}[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

is also normal by Exercise 1.0.7. This completes the proof. □

Example 1.3.6. We saw in Example 1.2.20 that $V = \mathbf{V}(xy - zw)$ is the affine toric variety U_σ of the cone $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$ pictured in Figure 2. Then Theorem 1.3.5 implies that V is normal, as claimed in Example 1.1.5. ◊

Example 1.3.7. By Example 1.2.22, the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$ is the affine toric variety of a strongly convex rational polyhedral cone and hence is normal by Theorem 1.3.5.

It is instructive to view this example using the parametrization

$$\Phi_{\mathscr{A}}(s,t) = (s^d, s^{d-1}t, \ldots, st^{d-1}, t^d)$$

from Example 1.1.6. Plotting the lattice points in $\mathscr{A}$ for $d = 2$ gives the white squares in Figure 9 (a) below. These generate the semigroup $\mathsf{S} = \mathbb{N}\mathscr{A}$, and the proof of Theorem 1.3.5 gives the cone $\sigma^\vee = \mathrm{Cone}(e_1, e_2)$, which is the first quadrant in the figure. At first glance, something seems wrong. The affine variety $\widehat{C}_2$ is normal, yet in Figure 9 (a) the semigroup generated by the white squares misses some lattice points in $\sigma^\vee$. This semigroup does not look saturated. How can the affine toric variety be normal?

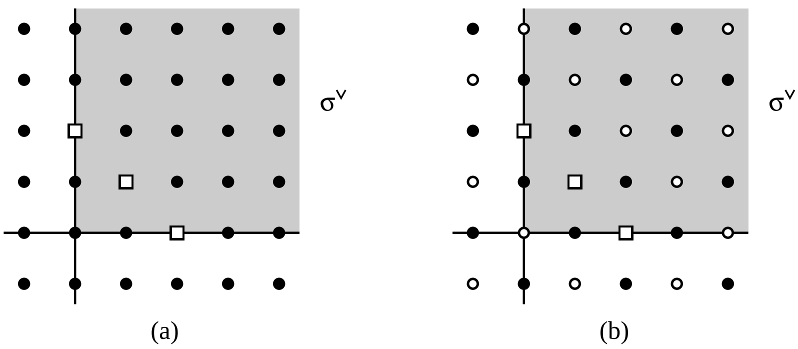


Figure 9. Lattice points for the rational normal cone $\widehat{C}_2$

The problem is that we are using the wrong lattice! Proposition 1.1.8 tells us to use the lattice $\mathbb{Z}\mathscr{A}$, which gives the white dots and squares in Figure 9 (b). This figure shows that the white squares generate the semigroup of lattice points in $\sigma^\vee$. Hence S is saturated and everything is fine. $\Diamond$

This example points out the importance of working with the correct lattice.

The Normalization of an Affine Toric Variety. The normalization of an affine toric variety is easy to describe. Let $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ for an affine semigroup S, so that the torus of V has character lattice $M = \mathbb{Z}\mathsf{S}$. Let $\mathrm{Cone}(\mathsf{S})$ denote the cone of any finite generating set of S and set $\sigma = \mathrm{Cone}(\mathsf{S})^\vee \subseteq N_\mathbb{R}$. In Exercise 1.3.6 you will prove the following.

Proposition 1.3.8. *The above cone σ is a strongly convex rational polyhedral cone in $N_\mathbb{R}$ and the inclusion $\mathbb{C}[\mathsf{S}] \subseteq \mathbb{C}[\sigma^\vee \cap M]$ induces a morphism $U_\sigma \to V$ that is the normalization map of V.* $\square$

The normalization of an affine toric variety of the form $Y_\mathscr{A}$ is constructed by applying Proposition 1.3.8 to the affine semigroup $\mathbb{N}\mathscr{A}$ and the lattice $\mathbb{Z}\mathscr{A}$.

Example 1.3.9. Let $\mathscr{A} = \{(4,0),(3,1),(1,3),(0,4)\} \subseteq \mathbb{Z}^2$. Then

$$\Phi_\mathscr{A}(s,t) = (s^4, s^3t, st^3, t^4)$$

parametrizes the surface $Y_\mathscr{A} \subseteq \mathbb{C}^4$ considered in Exercise 1.1.7. This is almost the rational normal cone $\widehat{C}_4$, except that we have omitted s^2t^2. Using $(2,2) = \frac{1}{2}\big((4,0)+(0,4)\big)$, we see that $\mathbb{N}\mathscr{A}$ is not saturated, so that $Y_\mathscr{A}$ is not normal.

Applying Proposition 1.3.8, one sees that the normalization of $Y_\mathscr{A}$ is $\widehat{C}_4$. You can check this using `Normaliz` [**57**] as explained in Example B.3.2. Note also that $\widehat{C}_4$ is an affine variety in $\mathbb{C}^5$, and the normalization map is induced by the obvious projection $\mathbb{C}^5 \to \mathbb{C}^4$. $\Diamond$

Proposition 1.3.8 tells us that $\sigma^\vee \cap M$ is the *saturation* of the semigroup S. In the appendix to Chapter 3 we will see that the normalization map $U_\sigma \to V$ constructed in Proposition 1.3.8 is onto but not necessarily one-to-one.

Smooth Affine Toric Varieties. Our next goal is to characterize when an affine toric variety is smooth. Since smooth affine varieties are normal (Proposition 1.0.9), we need only consider toric varieties U_σ coming from strongly convex rational polyhedral cones $\sigma \subseteq N_\mathbb{R}$.

We first study U_σ when σ has maximal dimension. Then $\sigma^\vee$ is strongly convex, so that $\mathsf{S}_\sigma = \sigma^\vee \cap M$ has a Hilbert basis $\mathscr{H}$. Furthermore, Corollary 1.3.3 tells us that the torus action on U_σ has a unique fixed point, denoted here by $p_\sigma \in U_\sigma$. The point p_σ and the Hilbert basis $\mathscr{H}$ are related as follows.

Lemma 1.3.10. *Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone of maximal dimension and let $T_{p_\sigma}(U_\sigma)$ be the Zariski tangent space to the affine toric variety U_σ at the above point p_σ. Then* $\dim T_{p_\sigma}(U_\sigma) = |\mathscr{H}|$.

Proof. By Corollary 1.3.3, the maximal ideal of $\mathbb{C}[\mathsf{S}_\sigma]$ corresponding to p_σ is $\mathfrak{m} = \langle \chi^m \mid m \in \mathsf{S}_\sigma \setminus \{0\}\rangle$. Since $\{\chi^m\}_{m \in \mathsf{S}_\sigma}$ is a basis of $\mathbb{C}[\mathsf{S}_\sigma]$, we obtain

$$\mathfrak{m} = \bigoplus_{m \neq 0} \mathbb{C}\chi^m = \bigoplus_{m \text{ irreducible}} \mathbb{C}\chi^m \ \oplus \bigoplus_{m \text{ reducible}} \mathbb{C}\chi^m = \Big(\bigoplus_{m \in \mathscr{H}} \mathbb{C}\chi^m\Big) \oplus \mathfrak{m}^2.$$

It follows that $\dim \mathfrak{m}/\mathfrak{m}^2 = |\mathscr{H}|$. To relate this to the maximal ideal $\mathfrak{m}_{U_\sigma, p_\sigma}$ in the local ring $\mathscr{O}_{U_\sigma, p_\sigma}$, we use the natural map

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}_{U_\sigma,p_\sigma}/\mathfrak{m}^2_{U_\sigma,p_\sigma}$$

which is always an isomorphism (Exercise 1.3.7). Since $T_{p_\sigma}(U_\sigma)$ is the dual space of $\mathfrak{m}_{U_\sigma,p_\sigma}/\mathfrak{m}^2_{U_\sigma,p_\sigma}$, we see that $\dim T_{p_\sigma}(U_\sigma) = |\mathscr{H}|$. □

The Hilbert basis $\mathscr{H}$ of S_σ gives $U_\sigma = Y_{\mathscr{H}} \subseteq \mathbb{C}^s$, where $s = |\mathscr{H}|$. This affine embedding is especially nice. Given *any* affine embedding $U_\sigma \hookrightarrow \mathbb{C}^\ell$, we have $\dim T_{p_\sigma}(U_\sigma) \leq \ell$ by Lemma 1.0.6. In other words, $\dim T_{p_\sigma}(U_\sigma)$ is a lower bound on the dimension of an affine embedding. Then Lemma 1.3.10 shows that when σ has maximal dimension, the Hilbert basis of S_σ gives the most efficient affine embedding of U_σ.

Example 1.3.11. In Example 1.2.22, we saw that the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$ is the toric variety coming from $\sigma = \mathrm{Cone}(de_1 - e_2, e_2) \subseteq \mathbb{R}^2$ and that $\mathsf{S}_\sigma = \sigma^\vee \cap \mathbb{Z}^2$ is generated by $(1,i)$ for $0 \leq i \leq d$. These generators form the Hilbert basis of S_σ, so that the Zariski tangent space of $0 \in \widehat{C}_d$ has dimension $d+1$. Hence $\mathbb{C}^{d+1}$ in the smallest affine space in which we can embed $\widehat{C}_d$. ◊

We now come to our main result about smoothness. Recall from §1.2 that a rational polyhedral cone is *smooth* if it can be generated by a subset of a basis of the lattice.

Theorem 1.3.12. *Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone. Then U_σ is smooth if and only if σ is smooth. Furthermore, all smooth affine toric varieties are of this form.*

Proof. If an affine toric variety is smooth, then it is normal by Proposition 1.0.9 and hence of the form U_σ. Also, Example 1.2.21 implies that if σ is smooth as a cone, then U_σ is smooth as a variety. It remains to prove the converse. So fix $\sigma \subseteq N_\mathbb{R}$ such that U_σ is smooth. Let $n = \dim U_\sigma = \dim N_\mathbb{R}$.

First suppose that σ has dimension n and let $p_\sigma \in U_\sigma$ be the point studied in Lemma 1.3.10. Since p_σ is smooth in U_σ, the Zariski tangent space $T_{p_\sigma}(U_\sigma)$ has dimension n by Definition 1.0.7. On the other hand, Lemma 1.3.10 implies that $\dim T_{p_\sigma}(U_\sigma)$ is the cardinality of the Hilbert basis $\mathscr{H}$ of $\mathsf{S}_\sigma = \sigma^\vee \cap M$. Thus

$$n = |\mathscr{H}| \geq |\{\text{edges } \rho \subseteq \sigma^\vee\}| \geq n,$$

where the first inequality holds by Proposition 1.2.23 (each edge $\rho \subseteq \sigma^\vee$ contributes an element of $\mathscr{H}$) and the second holds since $\dim \sigma^\vee = n$. It follows that σ has n edges and $\mathscr{H}$ consists of the ray generators of these edges. Since $M = \mathbb{Z}\mathsf{S}_\sigma$ by (1.2.3), the n edge generators of $\sigma^\vee$ generate the lattice $M \simeq \mathbb{Z}^n$ and hence form a basis of M. Thus $\sigma^\vee$ is smooth, and then $\sigma = (\sigma^\vee)^\vee$ is smooth since duality preserves smoothness.

Next suppose $\dim \sigma = r < n$. We reduce to the previous case as follows. Let $N_1 \subseteq N$ be the smallest saturated sublattice containing the generators of σ. Then N/N_1 is torsion-free, which by Exercise 1.3.5 implies the existence of a sublattice $N_2 \subseteq N$ with $N = N_1 \oplus N_2$. Note $\operatorname{rank} N_1 = r$ and $\operatorname{rank} N_2 = n - r$.

The cone σ lies in both $(N_1)_\mathbb{R}$ and $N_\mathbb{R}$. This gives affine toric varieties U_{σ,N_1} and $U_{\sigma,N}$ of dimensions r and n respectively. Furthermore, $N = N_1 \oplus N_2$ induces $M = M_1 \oplus M_2$, so that $\sigma \subseteq (N_1)_\mathbb{R}$ and $\sigma \subseteq N_\mathbb{R}$ give the affine semigroups $\mathsf{S}_{\sigma,N_1} \subseteq M_1$ and $\mathsf{S}_{\sigma,N} \subseteq M$ respectively. It is straighforward to show that

$$\mathsf{S}_{\sigma,N} = \mathsf{S}_{\sigma,N_1} \oplus M_2,$$

which in terms of semigroup algebras can be written

$$\mathbb{C}[\mathsf{S}_{\sigma,N}] \simeq \mathbb{C}[\mathsf{S}_{\sigma,N_1}] \otimes_\mathbb{C} \mathbb{C}[M_2].$$

The right-hand side is the coordinate ring of $U_{\sigma,N_1} \times T_{N_2}$. Thus

$$U_{\sigma,N} \simeq U_{\sigma,N_1} \times T_{N_2}, \tag{1.3.2}$$

which in turn implies that

$$U_{\sigma,N} \simeq U_{\sigma,N_1} \times (\mathbb{C}^*)^{n-r} \subseteq U_{\sigma,N_1} \times \mathbb{C}^{n-r}.$$

Since we are assuming that $U_{\sigma,N}$ is smooth, it follows that $U_{\sigma,N_1} \times \mathbb{C}^{n-r}$ is smooth at any point (p,q) in $U_{\sigma,N_1} \times (\mathbb{C}^*)^{n-r}$. In Exercise 1.3.8 you will show that

$$U_{\sigma,N_1} \times \mathbb{C}^{n-r} \text{ is smooth at } (p,q) \implies U_{\sigma,N_1} \text{ is smooth at } p. \tag{1.3.3}$$

Letting $p = p_\sigma \in U_{\sigma,N_1}$, the previous case implies that σ is smooth in N_1 since $\dim \sigma = \dim\, (N_1)_\mathbb{R}$. Hence σ is clearly smooth in $N = N_1 \oplus N_2$. □

Equivariant Maps between Affine Toric Varieties. We next study maps $V_1 \to V_2$ between affine toric varieties that respect the torus actions on V_1 and V_2.

Definition 1.3.13. Let $V_i = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i])$ be the affine toric varieties coming from the affine semigroups S_i, $i = 1,2$. Then a morphism $\phi : V_1 \to V_2$ is ***toric*** if the corresponding map of coordinate rings $\phi^* : \mathbb{C}[\mathsf{S}_2] \to \mathbb{C}[\mathsf{S}_1]$ is induced by a semigroup homomorphism $\widehat{\phi} : \mathsf{S}_2 \to \mathsf{S}_1$.

Here is our first result concerning toric morphisms.

Proposition 1.3.14. *Let T_{N_i} be the torus of the affine toric variety V_i, $i = 1,2$. Then:*

(a) *A morphism $\phi : V_1 \to V_2$ is toric if and only if*

$$\phi(T_{N_1}) \subseteq T_{N_2}$$

and $\phi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$ is a group homomorphism.

(b) *A toric morphism $\phi : V_1 \to V_2$ is* ***equivariant****, meaning that*

$$\phi(t \cdot p) = \phi(t) \cdot \phi(p)$$

for all $t \in T_{N_1}$ and $p \in V_1$.

Proof. Let $V_i = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_i])$, so that the character lattice of T_{N_i} is $M_i = \mathbb{Z}\mathsf{S}_i$. If ϕ comes from a semigroup homomorphism $\widehat{\phi} : \mathsf{S}_2 \to \mathsf{S}_1$, then $\widehat{\phi}$ extends to a group homomorphism $\widehat{\phi} : M_2 \to M_1$ and hence gives a commutative diagram

$$\begin{array}{ccc} \mathbb{C}[\mathsf{S}_2] & \xrightarrow{\phi^*} & \mathbb{C}[\mathsf{S}_1] \\ \downarrow & & \downarrow \\ \mathbb{C}[M_2] & \longrightarrow & \mathbb{C}[M_1]. \end{array}$$

Applying Spec, we see that $\phi(T_{N_1}) \subseteq T_{N_2}$, and $\phi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$ is a group homomorphism since $T_{N_i} = \mathrm{Hom}_{\mathbb{Z}}(M_i, \mathbb{C}^*)$ by Exercise 1.1.11. Conversely, if ϕ satisfies these conditions, then $\phi|_{T_{N_1}} : T_{N_1} \to T_{N_2}$ induces a diagram as above where the bottom map comes from a group homomorphism $\widehat{\phi} : M_2 \to M_1$. This, combined with $\phi^*(\mathbb{C}[\mathsf{S}_2]) \subseteq \mathbb{C}[\mathsf{S}_1]$, implies that $\widehat{\phi}$ induces a semigroup homomorphism $\widehat{\phi} : \mathsf{S}_2 \to \mathsf{S}_1$. This proves part (a) of the proposition.

For part (b), suppose that we have a toric map $\phi : V_1 \to V_2$. The action of T_{N_i} on V_i is given by a morphism $\Phi_i : T_{N_i} \times V_i \longrightarrow V_i$, and equivariance means that we have a commutative diagram

$$\begin{array}{ccc} T_{N_1} \times V_1 & \xrightarrow{\Phi_1} & V_1 \\ {\scriptstyle \phi|_{T_{N_1}} \times \phi}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ T_{N_2} \times V_2 & \xrightarrow{\Phi_2} & V_2. \end{array}$$

If we replace V_i with T_{N_i} in the diagram, then it certainly commutes since $\phi|_{T_{N_1}}$ is a group homomorphism. Then the whole diagram commutes since $T_{N_1} \times T_{N_1}$ is Zariski dense in $T_{N_1} \times V_1$. □

We can also characterize toric morphisms between affine toric varieties coming from strongly convex rational polyhedral cones. First note that a homomorphism $\overline{\phi} : N_1 \to N_2$ of lattices gives a group homomorphism $\phi : T_{N_1} \to T_{N_2}$ of tori. This follows from $T_{N_i} = N_i \otimes_{\mathbb{Z}} \mathbb{C}^*$, and one sees that ϕ is a morphism. Also, tensoring $\overline{\phi}$ with $\mathbb{R}$ gives $\overline{\phi}_{\mathbb{R}} : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$.

Here is the result, whose proof we leave to the reader (Exercise 1.3.9).

Proposition 1.3.15. *Suppose we have strongly convex rational polyhedral cones $\sigma_i \subseteq (N_i)_{\mathbb{R}}$ and a homomorphism $\overline{\phi} : N_1 \to N_2$. Then $\phi : T_{N_1} \to T_{N_2}$ extends to a map of affine toric varieties $\phi : U_{\sigma_1} \to U_{\sigma_2}$ if and only if $\overline{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$.* □

Faces and Affine Open Subsets. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone and let $\tau \preceq \sigma$ be a face. Then we can find $m \in \sigma^\vee \cap M$ such that $\tau = H_m \cap \sigma$. This allows us to relate the semigroup algebras of σ and τ as follows.

Proposition 1.3.16. *Let τ be a face of σ and as above write $\tau = H_m \cap \sigma$, where $m \in \sigma^\vee \cap M$. Then the semigroup algebra $\mathbb{C}[\mathsf{S}_\tau] = \mathbb{C}[\tau^\vee \cap M]$ is the localization of $\mathbb{C}[\mathsf{S}_\sigma] = \mathbb{C}[\sigma^\vee \cap M]$ at $\chi^m \in \mathbb{C}[\mathsf{S}_\sigma]$. In other words,*

$$\mathbb{C}[\mathsf{S}_\tau] = \mathbb{C}[\mathsf{S}_\sigma]_{\chi^m}.$$

Proof. The inclusion $\tau \subseteq \sigma$ implies $\mathsf{S}_\sigma \subset \mathsf{S}_\tau$, and since $\langle m, u\rangle = 0$ for all $u \in \tau$, we have $\pm m \in \tau^\vee$. It follows that

$$\mathsf{S}_\sigma + \mathbb{Z}(-m) \subseteq \mathsf{S}_\tau.$$

This inclusion is actually an equality, as we now prove. Fix a finite set $S \subseteq N$ with $\sigma = \mathrm{Cone}(S)$ and pick $m' \in \mathsf{S}_\tau$. Set

$$C = \max_{u \in S}\{|\langle m', u\rangle|\} \in \mathbb{N}.$$

It is straightforward to show that $m' + Cm \in \mathsf{S}_\sigma$. This proves that

$$\mathsf{S}_\sigma + \mathbb{Z}(-m) = \mathsf{S}_\tau,$$

from which $\mathbb{C}[\mathsf{S}_\tau] = \mathbb{C}[\mathsf{S}_\sigma]_{\chi^m}$ follows immediately. □

This interprets nicely in terms of toric morphisms. By Proposition 1.3.15, the identity map $N \to N$ and the inclusion $\tau \subseteq \sigma$ give the toric morphism $U_\tau \to U_\sigma$ that corresponds to the inclusion $\mathbb{C}[\mathsf{S}_\sigma] \subseteq \mathbb{C}[\mathsf{S}_\tau]$. By Proposition 1.3.16,

$$U_\tau = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_\tau]) = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_\sigma]_{\chi^m}) = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_\sigma])_{\chi^m} = (U_\sigma)_{\chi^m} \subseteq U_\sigma. \tag{1.3.4}$$

Thus U_τ becomes an affine open subset of U_σ when $\tau \preceq \sigma$. In particular, if two cones σ and σ' intersect in a common face $\tau = \sigma \cap \sigma'$, then we have inclusions

$$U_\sigma \supseteq U_\tau \subseteq U_{\sigma'}.$$

We will use this in Chapters 2 and 3 when we glue together affine toric varieties to create more general toric varieties.

The role of faces is the key reason why we describe affine toric varieties using $\sigma \subseteq N_{\mathbb{R}}$ rather than $\sigma^\vee \subseteq M_{\mathbb{R}}$. This answers the question raised in Remark 1.2.19.

Sublattices of Finite Index and Rings of Invariants. Another interesting class of toric morphisms arises when we keep the same cone but change the lattice. Here is an example we have already seen.

Example 1.3.17. In Example 1.3.7 the dual of $\sigma = \mathrm{Cone}(e_1, e_2) \subseteq \mathbb{R}^2$ interacts with the lattices shown in Figure 10 on the next page. To make this precise, let us name the lattices involved: the lattices

$$N' = \mathbb{Z}^2 \subseteq N = \{(a/2, b/2) \mid a, b \in \mathbb{Z},\ a + b \equiv 0 \bmod 2\}$$

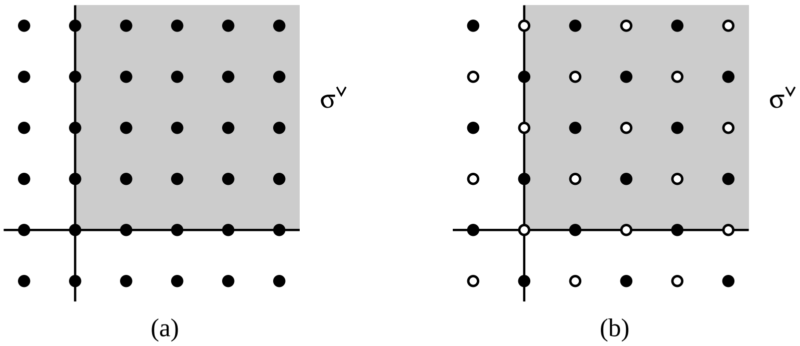


Figure 10. Lattice points of $\sigma^\vee$ relative to two lattices

have $\sigma \subseteq N'_{\mathbb{R}} \subseteq N_{\mathbb{R}}$, and the dual lattices

$$M' = \mathbb{Z}^2 \supseteq M = \{(a,b) \mid a,b \in \mathbb{Z},\ a+b \equiv 0 \bmod 2\}$$

have $\sigma^\vee \subseteq M'_{\mathbb{R}} \subseteq M_{\mathbb{R}}$. Note that duality reverses inclusions and that M and N are indeed dual under dot product. In Figure 10 (a), the black dots in the first quadrant form the semigroup $\mathsf{S}_{\sigma,N'} = \sigma^\vee \cap M'$, and in Figure 10 (b), the white dots in the first quadrant form $\mathsf{S}_{\sigma,N} = \sigma^\vee \cap M$.

This gives the affine toric varieties $U_{\sigma,N'}$ and $U_{\sigma,N}$. Clearly $U_{\sigma,N'} = \mathbb{C}^2$ since σ is smooth for N', while Example 1.3.7 shows that $U_{\sigma,N}$ is the rational normal cone $\widehat{C}_2$. The inclusion $N' \subseteq N$ gives a toric morphism

$$\mathbb{C}^2 = U_{\sigma,N'} \longrightarrow U_{\sigma,N} = \widehat{C}_2.$$

Our next task is to find a nice description of this map. $\Diamond$

In general, suppose we have lattices $N' \subseteq N$, where N' has finite index in N, and let $\sigma \subseteq N'_{\mathbb{R}} = N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. Then the inclusion $N' \subseteq N$ gives the toric morphism

$$\phi : U_{\sigma,N'} \longrightarrow U_{\sigma,N}.$$

The dual lattices satisfy $M' \supseteq M$, so that ϕ corresponds to the inclusion

$$\mathbb{C}[\sigma^\vee \cap M'] \supseteq \mathbb{C}[\sigma^\vee \cap M]$$

of semigroup algebras. The idea is to realize $\mathbb{C}[\sigma^\vee \cap M]$ as a ring of invariants of a group action on $\mathbb{C}[\sigma^\vee \cap M']$.

Proposition 1.3.18. *Let N' have finite index in N with quotient $G = N/N'$ and let $\sigma \subseteq N'_{\mathbb{R}} = N_{\mathbb{R}}$ be a strongly convex rational polyhedral cone. Then:*

(a) *There are natural isomorphisms*

$$G \simeq \operatorname{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*) = \ker(T_{N'} \to T_N).$$

(b) *G acts on $\mathbb{C}[\sigma^\vee \cap M']$ with ring of invariants*

$$\mathbb{C}[\sigma^\vee \cap M']^G = \mathbb{C}[\sigma^\vee \cap M].$$

(c) *G acts on $U_{\sigma,N'}$, and the morphism $\phi : U_{\sigma,N'} \to U_{\sigma,N}$ is constant on G-orbits and induces a bijection*

$$U_{\sigma,N'}/G \simeq U_{\sigma,N}.$$

Proof. Since $T_N = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{C}^*)$ by Exercise 1.1.11, applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'/M \longrightarrow 0$$

gives the sequence

$$1 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(M'/M,\mathbb{C}^*) \longrightarrow T_{N'} \longrightarrow T_N \longrightarrow 1.$$

This is exact since $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ is left exact and $\mathbb{C}^*$ is divisible. Note also that since N' has finite index in N, we have inclusions

$$N' \subseteq N \subseteq N_{\mathbb{Q}} \quad \text{and} \quad M \subseteq M' \subseteq M_{\mathbb{Q}}.$$

The pairing between M and N induces a pairing $M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$. Hence the map

$$M'/M \times N/N' \longrightarrow \mathbb{C}^* \qquad ([m'],[u]) \longmapsto e^{2\pi i\langle m',u\rangle}$$

is well-defined and induces $G = N/N' \simeq \mathrm{Hom}_{\mathbb{Z}}(M'/M,\mathbb{C}^*)$ (Exercise 1.3.10).

The action of $T_{N'}$ on $U_{\sigma,N'}$ induces an action of G on $U_{\sigma,N'}$ since $G \subseteq T_{N'}$. Using Exercise 1.3.1, one sees that if $g \in G$ and $\gamma \in U_{\sigma,N'}$, then $g \cdot \gamma$ is defined by the semigroup homomorphism $m' \mapsto g([m'])\gamma(m')$ for $m' \in \sigma^\vee \cap M'$. It follows that the corresponding action on the coordinate ring is given by

$$g \cdot \chi^{m'} = g([m'])^{-1}\chi^{m'}, \quad m' \in \sigma^\vee \cap M'.$$

(Exercise 5.0.1 explains why we need the inverse.) Since $m' \in M'$ lies in M if and only if $g([m']) = 1$ for all $g \in G$, the ring of invariants

$$\mathbb{C}[\sigma^\vee \cap M']^G = \{f \in \mathbb{C}[\sigma^\vee \cap M'] \mid g \cdot f = f \text{ for all } g \in G\},$$

is precisely $\mathbb{C}[\sigma^\vee \cap M]$, i.e.,

$$\mathbb{C}[\sigma^\vee \cap M']^G = \mathbb{C}[\sigma^\vee \cap M].$$

This proves part (b).

When a finite group G acts algebraically on $\mathbb{C}^n$, [**69**, Thm. 10 of Ch. 7, §4] shows that the ring of invariants $\mathbb{C}[x_1,\dots,x_n]^G \subseteq \mathbb{C}[x_1,\dots,x_n]$ gives a morphism of affine varieties

$$\mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[x_1,\dots,x_n]) \longrightarrow \mathrm{Spec}(\mathbb{C}[x_1,\dots,x_n]^G)$$

that is constant on G-orbits and induces a bijection

$$\mathbb{C}^n/G \simeq \mathrm{Spec}(\mathbb{C}[x_1,\dots,x_n]^G).$$

The proof extends without difficulty to the case when G acts algebraically on $V = \mathrm{Spec}(R)$. Here, $R^G \subseteq R$ gives a morphism of affine varieties

$$V = \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(R^G)$$

that is constant on G-orbits and induces a bijection

$$V/G \simeq \mathrm{Spec}(R^G).$$

From here, part (c) follows immediately from part (b). $\square$

We will give a careful treatment of these ideas in §5.0, where we will show that the map $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ is a *geometric quotient*.

Here are some examples of Proposition 1.3.18.

Example 1.3.19. In the situation of Example 1.3.17, one computes that G is the group $\mu_2 = \{\pm 1\}$ acting on $U_{\sigma,N'} = \mathrm{Spec}(\mathbb{C}[s,t]) \simeq \mathbb{C}^2$ by $-1 \cdot (s,t) = (-s,-t)$. Thus the rational normal cone $\widehat{C}_2$ is the quotient

$$\mathbb{C}^2/\mu_2 = U_{\sigma,N'}/\mu_2 \simeq U_{\sigma,N} = \widehat{C}_2.$$

We can see this explicitly as follows. The invariant ring is easily seen to be

$$\mathbb{C}[s,t]^{\mu_2} = \mathbb{C}[s^2,st,t^2] = \mathbb{C}[\widehat{C}_2] \simeq \mathbb{C}[x_0,x_1,x_2]/\langle x_0x_2 - x_1^2\rangle,$$

where the last isomorphism follows from Example 1.1.6. From the point of view of invariant theory, the generators s^2, st, t^2 of the ring of invariants give a morphism

$$\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^3, \quad (s,t) \to (s^2,st,t^2)$$

that is constant on μ_2-orbits. This map also separates orbits, so it induces

$$\mathbb{C}^2/\mu_2 \simeq \Phi(\mathbb{C}^2) = \widehat{C}_2,$$

where the last equality is by Example 1.1.6. But we can also think about this in terms of semigroups, where the exponent vectors of s^2, st, t^2 give the Hilbert basis of the semigroup $\mathsf{S}_{\sigma,N}$ pictured in Figure 10 (b). Everything fits together very nicely. $\Diamond$

In Exercise 1.3.11 you will generalize Example 1.3.19 to the case of the rational normal cone $\widehat{C}_d$ for arbitrary d.

Example 1.3.20. Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a simplicial cone of dimension n with ray generators $u_1,\ldots,u_n$. Then $N' = \sum_{i=1}^n \mathbb{Z}u_i$ is a sublattice of finite index in N. Furthermore, σ is smooth relative to N', so that $U_{\sigma,N'} = \mathbb{C}^n$. It follows that $G = N/N'$ acts on $\mathbb{C}^n$ with quotient

$$\mathbb{C}^n/G = U_{\sigma,N'}/G \simeq U_{\sigma,N}.$$

Hence the affine toric variety of a simplicial cone is the quotient of affine space by a finite abelian group. In the literature, varieties like $U_{\sigma,N}$ are called *orbifolds* and are said to be $\mathbb{Q}$-*factorial*. $\Diamond$

Exercises for §1.3.

1.3.1. Consider the affine toric variety $Y_{\mathscr{A}} = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$, where $\mathscr{A} = \{m_1, \ldots, m_s\}$ and $\mathsf{S} = \mathbb{N}\mathscr{A}$. Let $\gamma : \mathsf{S} \to \mathbb{C}$ be a semigroup homomorphism. In the proof of Proposition 1.3.1 we showed that $p = (\gamma(m_1), \ldots, \gamma(m_s))$ lies in $Y_{\mathscr{A}}$.

(a) Prove that the maximal ideal $\{f \in \mathbb{C}[\mathsf{S}] \mid f(p) = 0\}$ is the kernel of the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\mathsf{S}] \to \mathbb{C}$ induced by γ.

(b) The torus T_N of $Y_{\mathscr{A}}$ has character lattice $M = \mathbb{Z}\mathscr{A}$ and fix $t \in T_N$. As in the discussion following Proposition 1.3.1, $t \cdot p$ comes from the semigroup homomorphism $m \mapsto \chi^m(t)\gamma(m)$. Prove that this corresponds to the point

$$(\chi^{m_1}(t), \ldots, \chi^{m_s}(t)) \cdot (\gamma(m_1), \ldots, \gamma(m_s)) = (\chi^{m_1}(t)\gamma(m_1), \ldots, \chi^{m_s}(t)\gamma(m_s)) \in \mathbb{C}^s$$

coming from the action of $t \in T_N \subseteq (\mathbb{C}^*)^s$ on $p \in Y_{\mathscr{A}} \subseteq \mathbb{C}^s$.

1.3.2. Let $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ with $T_N = \mathrm{Spec}(\mathbb{C}[M])$, $M = \mathbb{Z}\mathsf{S}$. The action $T_N \times V \to V$ comes from a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\mathsf{S}] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[\mathsf{S}]$. Prove that this homomorphism is given by $\chi^m \mapsto \chi^m \otimes \chi^m$. Hint: Show that this formula determines the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[M] \to \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M]$ that gives the group operation $T_N \times T_N \to T_N$.

1.3.3. Prove Corollary 1.3.3.

1.3.4. Let $\mathscr{A} \subseteq M$ be a finite set.

(a) Prove that the semigroup $\mathbb{N}\mathscr{A}$ is saturated in M if and only if $\mathbb{N}\mathscr{A} = \mathrm{Cone}(\mathscr{A}) \cap M$. Hint: Apply (1.2.2) to $\mathrm{Cone}(\mathscr{A}) \subseteq M_{\mathbb{R}}$.

(b) Complete the proof of (b) $\Rightarrow$ (c) from Theorem 1.3.5.

1.3.5. Let N be a lattice.

(a) Let $N_1 \subseteq N$ be a sublattice such that N/N_1 is torsion-free. Prove that there is a sublattice $N_2 \subseteq N$ such that $N = N_1 \oplus N_2$.

(b) Let $u \in N$ be primitive as defined in the proof of Theorem 1.3.5. Prove that N has a basis $e_1, \ldots, e_n$ such that $e_1 = u$.

1.3.6. Prove Proposition 1.3.8.

1.3.7. Let p be a point of an irreducible affine variety V. Then p gives the maximal ideal $\mathfrak{m} = \{f \in \mathbb{C}[V] \mid f(p) = 0\}$ as well as the maximal ideal $\mathfrak{m}_{V,p} \subseteq \mathscr{O}_{V,p}$ defined in §1.0. Prove that the natural map $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}_{V,p}/\mathfrak{m}_{V,p}^2$ is an isomorphism of $\mathbb{C}$-vector spaces.

1.3.8. Prove (1.3.3). Hint: Use Lemma 1.0.6 and Example 1.0.10.

1.3.9. Prove Proposition 1.3.15.

1.3.10. Prove the assertions made in the proof of Proposition 1.3.18 concerning the pairing $M'/M \times N/N' \to \mathbb{C}^*$ defined by $([m'], [u]) \mapsto e^{2\pi i \langle m', u \rangle}$.

1.3.11. Let $\mu_d = \{\zeta \in \mathbb{C}^* \mid \zeta^d = 1\}$ be the group of dth roots of unity. Then μ_d acts on $\mathbb{C}^2$ by $\zeta \cdot (x, y) = (\zeta x, \zeta y)$. Adapt Example 1.3.19 to show that $\mathbb{C}^2/\mu_d \simeq \widehat{C}_d$. Hint: Use lattices $N' = \mathbb{Z}^2 \subseteq N = \{(a/d, b/d) \mid a, b \in \mathbb{Z}, a + b \equiv 0 \bmod d\}$.

1.3.12. Prove that the normalization map in Proposition 1.3.8 is a toric morphism.

1.3.13. Let $\sigma_1 \subseteq (N_1)_{\mathbb{R}}$ and $\sigma_2 \subseteq (N_2)_{\mathbb{R}}$ be strongly convex rational polyhedral cones. This gives the cone $\sigma_1 \times \sigma_2 \subseteq (N_1 \oplus N_2)_{\mathbb{R}}$. Prove that $U_{\sigma_1 \times \sigma_2} \simeq U_{\sigma_1} \times U_{\sigma_2}$. Also explain how this result applies to (1.3.2).

1.3.14. By Proposition 1.3.1, a point p of an affine toric variety $V = \mathrm{Spec}(\mathbb{C}[\mathsf{S}])$ is represented by a semigroup homomorphism $\gamma : \mathsf{S} \to \mathbb{C}$. Prove that p lies in the torus of V if and only if γ never vanishes, i.e., $\gamma(m) \neq 0$ for all $m \in \mathsf{S}$.

Appendix: Tensor Products of Coordinate Rings

In this appendix, we will prove the following result used in §1.0 in our discussion of products of affine varieties.

Proposition 1.A.1. *If R and S are finitely generated $\mathbb{C}$-algebras without nilpotents, then the same is true for $R \otimes_{\mathbb{C}} S$.*

Proof. The tensor product is clearly a finitely generated $\mathbb{C}$-algebra. Hence we need only prove that $R \otimes_{\mathbb{C}} S$ has no nilpotents. If we write $R \simeq \mathbb{C}[x_1,\ldots,x_n]/I$, then I is radical and thus has a primary decomposition $I = \bigcap_{i=1}^s P_i$, where each P_i is prime (see [**69**, Ch. 4, §7]). This gives

$$R \simeq \mathbb{C}[x_1,\ldots,x_n]/I \longrightarrow \bigoplus_{i=1}^{s} \mathbb{C}[x_1,\ldots,x_n]/P_i$$

where the map to the direct sum is injective. Each quotient $\mathbb{C}[x_1,\ldots,x_n]/P_i$ is an integral domain and hence injects into its field of fractions K_i. This yields an injection

$$R \longrightarrow \bigoplus_{i=1}^{s} K_i,$$

and since tensoring over a field preserves exactness, we get an injection

$$R \otimes_{\mathbb{C}} S \hookrightarrow \bigoplus_{i=1}^{s} K_i \otimes_{\mathbb{C}} S.$$

Hence it suffices to prove that $K \otimes_{\mathbb{C}} S$ has no nilpotents when K is a finitely generated field extension of $\mathbb{C}$. A similar argument using S then reduces us to showing that $K \otimes_{\mathbb{C}} L$ has no nilpotents when K and L are finitely generated field extensions of $\mathbb{C}$.

Since $\mathbb{C}$ has characteristic 0, the extension $\mathbb{C} \subseteq L$ has a separating transcendence basis (see [**159**, p. 519]). This means that we can find $y_1,\ldots,y_t \in L$ such that $y_1,\ldots,y_t$ are algebraically independent over $\mathbb{C}$ and $F = \mathbb{C}(y_1,\ldots,y_t) \subseteq L$ is a finite separable extension. Then

$$K \otimes_{\mathbb{C}} L \simeq K \otimes_{\mathbb{C}} (F \otimes_F L) \simeq (K \otimes_{\mathbb{C}} F) \otimes_F L.$$

But $C = K \otimes_{\mathbb{C}} F = K \otimes_{\mathbb{C}} \mathbb{C}(y_1,\ldots,y_t) = K(y_1,\ldots,y_t)$ is a field, so that we are reduced to considering

$$C \otimes_F L$$

where C and L are extensions of F and $F \subseteq L$ is finite and separable. The latter and the theorem of the primitive element imply that $L \simeq F[X]/\langle f(X) \rangle$, where $f(X)$ has distinct roots in some extension of F. Then

$$C \otimes_F L \simeq C \otimes_F F[X]/\langle f(X) \rangle \simeq C[X]/\langle f(X) \rangle.$$

Since $f(X)$ has distinct roots, this quotient ring has no nilpotents. Our result follows. $\square$

A final remark is that we can replace $\mathbb{C}$ with any perfect field since finitely generated extensions of perfect fields have separating transcendence bases (see [**159**, p. 519]).

Chapter 2

Projective Toric Varieties

§2.0. Background: Projective Varieties

Our discussion assumes that the reader is familiar with the elementary theory of projective varieties, at the level of [**69**, Ch. 8].

In Chapter 1, we introduced affine toric varieties. In general, a toric variety is an irreducible variety X over $\mathbb{C}$ containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zariski open set such that the action of $(\mathbb{C}^*)^n$ on itself extends to an action on X. We will learn in Chapter 3 that the concept of "variety" is somewhat subtle. Hence we will defer the formal definition of toric variety until then and instead concentrate on toric varieties that live in projective space $\mathbb{P}^n$, defined by

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*, \tag{2.0.1}$$

where $\mathbb{C}^*$ acts via homotheties, i.e., $\lambda \cdot (a_0, \ldots, a_n) = (\lambda a_0, \ldots, \lambda a_n)$ for $\lambda \in \mathbb{C}^*$ and $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$. Thus $(a_0, \ldots, a_n)$ are *homogeneous coordinates* of a point in $\mathbb{P}^n$ and are well-defined up to homothety.

The goal of this chapter is to use lattice points and polytopes to create toric varieties that lie in $\mathbb{P}^n$. We will use the affine semigroups and polyhedral cones introduced in Chapter 1 to describe the local structure of these varieties.

Homogeneous Coordinate Rings. A projective variety $V \subseteq \mathbb{P}^n$ is defined by the vanishing of finitely many homogeneous polynomials in the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$. The *homogeneous coordinate ring* of V is the quotient ring

$$\mathbb{C}[V] = S/\mathbf{I}(V),$$

where $\mathbf{I}(V)$ is generated by all homogeneous polynomials that vanish on V.

The polynomial ring S is graded by setting $\deg(x_i) = 1$. This gives the decomposition $S = \bigoplus_{d=0}^{\infty} S_d$, where S_d is the vector space of homogeneous polynomials of degree d. Homogeneous ideals decompose similarly, and the above coordinate ring $\mathbb{C}[V]$ inherits a grading where

$$\mathbb{C}[V]_d = S_d/\mathbf{I}(V)_d.$$

The ideal $\mathbf{I}(V) \subseteq S = \mathbb{C}[x_0,\dots,x_n]$ also defines an affine variety $\widehat{V} \subseteq \mathbb{C}^{n+1}$, called the *affine cone* of V. The variety $\widehat{V}$ satisfies

$$V = (\widehat{V} \setminus \{0\})/\mathbb{C}^*, \tag{2.0.2}$$

and its coordinate ring is the homogeneous coordinate ring of V, i.e.,

$$\mathbb{C}[\widehat{V}] = \mathbb{C}[V].$$

Example 2.0.1. In Example 1.1.6 we encountered the ideal

$$I = \langle x_i x_{j+1} - x_{i+1} x_j \mid 0 \le i < j \le d-1 \rangle \subseteq \mathbb{C}[x_0,\dots,x_d]$$

generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-2} & x_{d-1} \\ x_1 & x_2 & \cdots & x_{d-1} & x_d \end{pmatrix}.$$

Since I is homogeneous, it defines a projective variety $C_d \subseteq \mathbb{P}^d$ that is the image of the map

$$\Phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^d$$

defined in homogeneous coordinates by $(s,t) \mapsto (s^d, s^{d-1}t, \dots, st^{d-1}, t^d)$ (see Exercise 1.1.1). This shows that C_d is a curve, called the *rational normal curve* of degree d. Furthermore, the affine cone of C_d is the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$ discussed in Example 1.1.6.

We know from Chapter 1 that $\widehat{C}_d$ is an affine toric surface; we will soon see that C_d is a projective toric curve. $\Diamond$

Example 2.0.2. The affine toric variety $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ studied in Chapter 1 is the affine cone of the projective surface $V = \mathbf{V}(xy - zw) \subseteq \mathbb{P}^3$. Recall that this surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

given by $(s,t;u,v) \mapsto (su,tv,sv,tu)$. We will see below that $V \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is the projective toric variety coming from the unit square in the plane. $\Diamond$

As in the affine case, a projective variety $V \subseteq \mathbb{P}^n$ has the *classical topology*, induced from the usual topology on $\mathbb{P}^n$, and the *Zariski topology*, where the Zariski closed sets are subvarieties of V (meaning projective varieties of $\mathbb{P}^n$ contained in V) and the Zariski open sets are their complements.

Rational Functions on Irreducible Projective Varieties. A homogeneous polynomial $f \in S$ of degree $d > 0$ does not give a function on $\mathbb{P}^n$ since

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n).$$

However, the quotient of two such polynomials $f, g \in S_d$ gives the well-defined function

$$\frac{f}{g} : \mathbb{P}^n \setminus \mathbf{V}(g) \to \mathbb{C}.$$

provided $g \neq 0$. We write this as $f/g : \mathbb{P}^n \dashrightarrow \mathbb{C}$ and say that f/g is a *rational function* on $\mathbb{P}^n$.

More generally, suppose that $V \subseteq \mathbb{P}^n$ is irreducible, and let $f, g \in \mathbb{C}[V] = \mathbb{C}[\widehat{V}]$ be homogeneous of the same degree with $g \neq 0$. Then f and g give functions on the affine cone $\widehat{V}$ and hence an element $f/g \in \mathbb{C}(\widehat{V})$. By (2.0.2), this induces a rational function $f/g : V \dashrightarrow \mathbb{C}$. Thus

$$\mathbb{C}(V) = \{f/g \in \mathbb{C}(\widehat{V}) \mid f, g \in \mathbb{C}[V] \text{ homogeneous of the same degree, } g \neq 0\}$$

is the field of rational functions on V. It is customary to write the set on the left as $\mathbb{C}(\widehat{V})_0$ since it consists of the degree 0 elements of $\mathbb{C}(\widehat{V})$.

Affine Pieces of Projective Varieties. A projective variety $V \subseteq \mathbb{P}^n$ is a union of Zariski open sets that are affine. To see why, let $U_i = \mathbb{P}^n \setminus \mathbf{V}(x_i)$. Then $U_i \simeq \mathbb{C}^n$ via the map

$$(a_0, \dots, a_n) \longmapsto \left(\tfrac{a_0}{a_i}, \dots, \tfrac{a_{i-1}}{a_i}, \tfrac{a_{i+1}}{a_i}, \dots, \tfrac{a_n}{a_i}\right), \tag{2.0.3}$$

so that in the notation of Chapter 1, we have

$$U_i = \operatorname{Spec}\left(\mathbb{C}\left[\tfrac{x_0}{x_i}, \dots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \dots, \tfrac{x_n}{x_i}\right]\right).$$

Then $V \cap U_i$ is a Zariski open subset of V that maps via (2.0.3) to the affine variety in $\mathbb{C}^n$ defined by the equations

$$f\left(\tfrac{x_0}{x_i}, \dots, \tfrac{x_{i-1}}{x_i}, 1, \tfrac{x_{i+1}}{x_i}, \dots, \tfrac{x_n}{x_i}\right) = 0 \tag{2.0.4}$$

as f varies over all homogeneous polynomials in $\mathbf{I}(V)$.

We call $V \cap U_i$ an *affine piece* of V. These affine pieces cover V since the U_i cover $\mathbb{P}^n$. Using localization, we can describe the coordinate rings of the affine pieces as follows. The variable x_i induces an element $\bar{x}_i \in \mathbb{C}[V]$, so that we get the localization

$$\mathbb{C}[V]_{\bar{x}_i} = \{f/\bar{x}_i^k \mid f \in \mathbb{C}[V],\ k \geq 0\} \tag{2.0.5}$$

as in Exercises 1.0.2 and 1.0.3. Note that $\mathbb{C}[V]_{\bar{x}_i}$ has a well-defined $\mathbb{Z}$-grading given by $\deg(f/\bar{x}_i^k) = \deg(f) - k$ when f is homogeneous. Then

$$(\mathbb{C}[V]_{\bar{x}_i})_0 = \{f/\bar{x}_i^k \in \mathbb{C}[V]_{\bar{x}_i} \mid f \text{ is homogeneous of degree } k\} \tag{2.0.6}$$

is the subring of $\mathbb{C}[V]_{\bar{x}_i}$ consisting of all elements of degree 0. This gives an affine piece of V as follows.

Lemma 2.0.3. *The affine piece $V \cap U_i$ of V has coordinate ring*

$$\mathbb{C}[V \cap U_i] \simeq (\mathbb{C}[V]_{\bar{x}_i})_0.$$

Proof. We have an exact sequence

$$0 \longrightarrow \mathbf{I}(V) \longrightarrow \mathbb{C}[x_0,\dots,x_n] \longrightarrow \mathbb{C}[V] \longrightarrow 0.$$

If we localize at x_i, we get the exact sequence

$$0 \longrightarrow \mathbf{I}(V)_{x_i} \longrightarrow \mathbb{C}[x_0,\dots,x_n]_{x_i} \longrightarrow \mathbb{C}[V]_{\bar{x}_i} \longrightarrow 0 \tag{2.0.7}$$

since localization preserves exactness (Exercises 2.0.1 and 2.0.2). These sequences preserve degrees, so that taking elements of degree 0 gives the exact sequence

$$0 \longrightarrow (\mathbf{I}(V)_{x_i})_0 \longrightarrow (\mathbb{C}[x_0,\dots,x_n]_{x_i})_0 \longrightarrow (\mathbb{C}[V]_{\bar{x}_i})_0 \longrightarrow 0.$$

Note that $(\mathbb{C}[x_0,\dots,x_n]_{x_i})_0 = \mathbb{C}\big[\frac{x_0}{x_i},\dots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\big]$. If $f \in \mathbf{I}(V)$ is homogeneous of degree k, then

$$f/x_i^k = f\big(\tfrac{x_0}{x_i},\dots,\tfrac{x_{i-1}}{x_i},1,\tfrac{x_{i+1}}{x_i},\dots,\tfrac{x_n}{x_i}\big) \in (\mathbf{I}(V)_{x_i})_0.$$

By (2.0.4), we conclude that $(\mathbf{I}(V)_{x_i})_0$ maps to $\mathbf{I}(V \cap U_i)$. To show that this map is onto, let $g\big(\frac{x_0}{x_i},\dots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\big) \in \mathbf{I}(V \cap U_i)$. For $k \gg 0$, $x_i^k g = f(x_0,\dots,x_n)$ is homogeneous of degree k. It then follows easily that $x_i f$ vanishes on V since $g = 0$ on $V \cap U_i$ and $x_i = 0$ on the complement of U_i. Thus $x_i f \in \mathbf{I}(V)$, and then $(x_i f)/(x_i^{k+1}) \in (\mathbf{I}(V)_{x_i})_0$ maps to g. The lemma follows immediately. □

One can also explore what happens when we intersect affine pieces $V \cap U_i$ and $V \cap U_j$ for $i \neq j$. By Exercise 2.0.3, $V \cap U_i \cap U_j$ is affine with coordinate ring

$$\mathbb{C}[V \cap U_i \cap U_j] \simeq (\mathbb{C}[V]_{\bar{x}_i\bar{x}_j})_0. \tag{2.0.8}$$

We will apply this to projective toric varieties in §2.2. We will also see later in the book that Lemma 2.0.3 is related to the "Proj" construction, where Proj of a graded ring gives a projective variety, just as Spec of an ordinary ring gives an affine variety.

Products of Projective Spaces. One can study the product $\mathbb{P}^n \times \mathbb{P}^m$ of projective spaces using the bigraded ring $\mathbb{C}[x_0,\dots,x_n,y_0,\dots,y_m]$, where x_i has bidegree $(1,0)$ and y_i has bidegree $(0,1)$. Then a bihomogeneous polynomial f of bidegree (a,b) gives a well-defined equation $f = 0$ in $\mathbb{P}^n \times \mathbb{P}^m$. This allows us to define varieties in $\mathbb{P}^n \times \mathbb{P}^m$ using bihomogeneous ideals. In particular, the ideal $\mathbf{I}(V)$ of a variety $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is a bihomogeneous ideal.

Another way to study $\mathbb{P}^n \times \mathbb{P}^m$ is via the *Segre embedding*

$$\mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{nm+n+m}$$

defined by mapping $(a_0,\dots,a_n,b_0,\dots,b_m)$ to the point

$$(a_0b_0,a_0b_1,\dots,a_0b_m,a_1b_0,\dots,a_1b_m,\dots,a_nb_0,\dots,a_nb_m).$$

This map is studied in [**69**, Ex. 14 of Ch. 8, §4]. If $\mathbb{P}^{nm+n+m}$ has homogeneous coordinates x_{ij} for $0 \le i \le n, 0 \le j \le m$, then $\mathbb{P}^n \times \mathbb{P}^m \subseteq \mathbb{P}^{nm+n+m}$ is defined by the vanishing of the 2×2 minors of the $(n+1)\times(m+1)$ matrix

$$\begin{pmatrix} x_{00} & \cdots & x_{0m} \\ \vdots & & \vdots \\ x_{n0} & \cdots & x_{nm} \end{pmatrix}.$$

This follows since an $(n+1)\times(m+1)$ matrix has rank 1 if and only if it is a product $A^t B$, where A and B are nonzero row matrices of lengths $n+1$ and $m+1$.

These approaches give the same notion of what it means to be a subvariety of $\mathbb{P}^n \times \mathbb{P}^m$. A homogeneous polynomial $F(x_{ij})$ of degree d gives the bihomogeneous polynomial $F(x_i y_j)$ of bidegree (d,d). Hence any subvariety of $\mathbb{P}^{nm+n+m}$ lying in $\mathbb{P}^n \times \mathbb{P}^m$ can be defined by a bihomogeneous ideal in $\mathbb{C}[x_0,\dots,x_n,y_0,\dots,y_m]$. Going the other way takes more thought and is discussed in Exercise 2.0.5.

We also have the following useful result proved in Exercise 2.0.6.

Proposition 2.0.4. *Given subvarieties $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$, the product $V \times W$ is a subvariety of $\mathbb{P}^n \times \mathbb{P}^m$.* □

Weighted Projective Space. The graded ring associated to projective space $\mathbb{P}^n$ is $\mathbb{C}[x_0,\dots,x_n]$, where each variable x_i has degree 1. More generally, let $q_0,\dots,q_n$ be positive integers with $\gcd(q_0,\dots,q_n) = 1$ and define

$$\mathbb{P}(q_0,\dots,q_n) = (\mathbb{C}^{n+1} \setminus \{0\})/\sim,$$

where $\sim$ is the equivalence relation

$$(a_0,\dots,a_n) \sim (b_0,\dots,b_n) \iff a_i = \lambda^{q_i} b_i,\ i = 0,\dots,n \text{ for some } \lambda \in \mathbb{C}^*.$$

We call $\mathbb{P}(q_0,\dots,q_n)$ a *weighted projective space*. Note that $\mathbb{P}^n = \mathbb{P}(1,\dots,1)$.

The ring corresponding to $\mathbb{P}(q_0,\dots,q_n)$ is the graded ring $\mathbb{C}[x_0,\dots,x_n]$, where x_i now has degree q_i. A polynomial f is *weighted homogeneous* of degree d if every monomial x^α appearing in f satisfies $\alpha\cdot(q_0,\dots,q_n) = d$. The equation $f = 0$ is well-defined on $\mathbb{P}(q_0,\dots,q_n)$ when f is weighted homogeneous, and one can define varieties in $\mathbb{P}(q_0,\dots,q_n)$ using weighted homogeneous ideals of $\mathbb{C}[x_0,\dots,x_n]$.

Example 2.0.5. We can embed the weighted projective plane $\mathbb{P}(1,1,2)$ in $\mathbb{P}^3$ using the monomials $x_0^2, x_0x_1, x_1^2, x_2$ of weighted degree 2. In other words, the map

$$\mathbb{P}(1,1,2) \longrightarrow \mathbb{P}^3$$

given by

$$(a_0,a_1,a_2) \longmapsto (a_0^2, a_0a_1, a_1^2, a_2)$$

is well-defined and injective. One can check that this map induces

$$\mathbb{P}(1,1,2) \simeq \mathbf{V}(y_0y_2 - y_1^2) \subseteq \mathbb{P}^3,$$

where y_0,y_1,y_2,y_3 are homogeneous coordinates on $\mathbb{P}^3$. ◊

Later in the book we will use toric methods to construct projective embeddings of arbitrary weighted projective spaces.

Exercises for §2.0.

2.0.1. Let R be a commutative $\mathbb{C}$-algebra. Given $f \in R \setminus \{0\}$ and an exact sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$, prove that

$$0 \longrightarrow M_1 \otimes_R R_f \longrightarrow M_2 \otimes_R R_f \longrightarrow M_3 \otimes_R R_f \longrightarrow 0$$

is also exact, where R_f is the localization of R at f defined in Exercises 1.0.2 and 1.0.3.

2.0.2. Let $V \subseteq \mathbb{P}^n$ be a projective variety. If we set $S = \mathbb{C}[x_0, \ldots, x_n]$, then V has coordinate ring $\mathbb{C}[V] = S/\mathbf{I}(V)$. Let $\bar{x}_i$ be the image of x_i in $\mathbb{C}[V]$.

(a) Note that $\mathbb{C}[V]$ is an S-module. Prove that $\mathbb{C}[V]_{\bar{x}_i} \simeq \mathbb{C}[V] \otimes_S S_{x_i}$.

(b) Use part (a) and the previous exercise to prove that (2.0.7) is exact.

2.0.3. Prove the claim made in (2.0.8).

2.0.4. Let $V \subseteq \mathbb{P}^n$ be a projective variety. Take $f_0, \ldots, f_m \in S_d$ such that the intersection $V \cap \mathbf{V}(f_0, \ldots, f_m)$ is empty. Prove that the map

$$(a_0, \ldots, a_n) \longmapsto (f_0(a_0, \ldots, a_n), \ldots, f_m(a_0, \ldots, a_n))$$

induces a well-defined function $\Phi : V \to \mathbb{P}^m$.

2.0.5. Let $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be defined by $f_\ell(x_i, y_j) = 0$, where $f_\ell(x_i, y_j)$ is bihomogenous of bidegree (a_ℓ, b_ℓ), $\ell = 1, \ldots, s$. The goal of this exercise is to show that when we embed $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{nm+n+m}$ via the Segre embedding described in the text, V becomes a subvariety of $\mathbb{P}^{nm+n+m}$.

(a) For each ℓ, pick an integer $d_\ell \geq \max\{a_\ell, b_\ell\}$ and consider the polynomials $f_{\ell,\alpha,\beta} = x^\alpha y^\beta f_\ell(x_i, y_j)$ where $\ell = 1, \ldots, s$ and $|\alpha| = d_\ell - a_\ell$, $|\beta| = d_\ell - b_\ell$. Note that $f_{\ell,\alpha,\beta}$ is bihomogenous of bidegree (d_ℓ, d_ℓ). Prove that $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is defined by the vanishing of the $f_{\ell,\alpha,\beta}$.

(b) Use part (a) to show that V is a subvariety of $\mathbb{P}^{nm+n+m}$ under the Segre embedding.

2.0.6. Prove Proposition 2.0.4

2.0.7. Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. Show that after relabeling coordinates, the affine cone of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ is the variety $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ featured in many examples in Chapter 1.

§2.1. Lattice Points and Projective Toric Varieties

We first observe that $\mathbb{P}^n$ is a toric variety with torus

$$\begin{aligned} T_{\mathbb{P}^n} &= \mathbb{P}^n \setminus \mathbf{V}(x_0 \cdots x_n) = \{(a_0, \ldots, a_n) \in \mathbb{P}^n \mid a_0 \cdots a_n \neq 0\} \\ &= \{(1, t_1, \ldots, t_n) \in \mathbb{P}^n \mid t_1, \ldots, t_n \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^n. \end{aligned}$$

The action of $T_{\mathbb{P}^n}$ on itself clearly extends to an action on $\mathbb{P}^n$, making $\mathbb{P}^n$ a toric variety. To describe the lattices associated to $T_{\mathbb{P}^n}$, we use the exact sequence of tori

$$1 \longrightarrow \mathbb{C}^* \longrightarrow (\mathbb{C}^*)^{n+1} \xrightarrow{\pi} T_{\mathbb{P}^n} \longrightarrow 1$$

coming from the definition (2.0.1) of $\mathbb{P}^n$. Hence the character lattice of $T_{\mathbb{P}^n}$ is

$$\mathscr{M}_n = \{(a_0,\dots,a_n) \in \mathbb{Z}^{n+1} \mid \textstyle\sum_{i=0}^n a_i = 0\}, \tag{2.1.1}$$

and the lattice of one-parameter subgroups $\mathscr{N}_n$ is the quotient

$$\mathscr{N}_n = \mathbb{Z}^{n+1}/\mathbb{Z}(1,\dots,1).$$

Lattice Points and Projective Toric Varieties. Let T_N be a torus with lattices M and N as usual. In Chapter 1, we used a finite set of lattice points of $\mathscr{A} = \{m_1,\dots,m_s\} \subseteq M$ to create the affine toric variety $Y_\mathscr{A}$ as the Zariski closure of the image of the map

$$\Phi_\mathscr{A} : T_N \longrightarrow \mathbb{C}^s, \quad t \longmapsto (\chi^{m_1}(t),\dots,\chi^{m_s}(t)).$$

To get a projective toric variety, we regard $\Phi_\mathscr{A}$ as a map to $(\mathbb{C}^*)^s$ and compose with the homomorphism $\pi : (\mathbb{C}^*)^s \to T_{\mathbb{P}^{s-1}}$ to obtain

$$T_N \xrightarrow{\Phi_\mathscr{A}} \mathbb{C}^s \xrightarrow{\pi} T_{\mathbb{P}^{s-1}} \subseteq \mathbb{P}^{s-1}. \tag{2.1.2}$$

Definition 2.1.1. Given a finite set $\mathscr{A} \subseteq M$, the ***projective toric variety*** $X_\mathscr{A}$ is the Zariski closure in $\mathbb{P}^{s-1}$ of the image of the map $\pi \circ \Phi_\mathscr{A}$ from (2.1.2).

Proposition 2.1.2. *$X_\mathscr{A}$ is a toric variety of dimension equal to the dimension of the smallest affine subspace of $M_\mathbb{R}$ containing $\mathscr{A}$.*

Proof. The proof that $X_\mathscr{A} \subseteq \mathbb{P}^{s-1}$ is a toric variety is similar to the proof given in Proposition 1.1.8 of Chapter 1 that $Y_\mathscr{A} \subseteq \mathbb{C}^s$ is a toric variety. The assertion concerning the dimension of $X_\mathscr{A}$ will follow from Proposition 2.1.6 below. □

More concretely, $X_\mathscr{A}$ is the Zariski closure of the image of the map

$$T_N \longrightarrow \mathbb{P}^{s-1}, \quad t \longmapsto (\chi^{m_1}(t),\dots,\chi^{m_s}(t))$$

given by the characters coming from $\mathscr{A} = \{m_1,\dots,m_s\} \subseteq M$. In particular, if $M = \mathbb{Z}^n$, then χ^{m_i} is the Laurent monomial t^{m_i} and $X_\mathscr{A}$ is the Zariski closure of the image of

$$T_N \longrightarrow \mathbb{P}^{s-1}, \quad t \longmapsto (t^{m_1},\dots,t^{m_s}).$$

In the literature, $\mathscr{A} \subseteq \mathbb{Z}^n$ is often given as an $n \times s$ matrix A with integer entries, so that the elements of $\mathscr{A}$ are the columns of A.

Here is an example where the lattice points themselves are matrices.

Example 2.1.3. Let $M = \mathbb{Z}^{3\times 3}$ be the lattice of 3×3 integer matrices and let

$$\mathscr{P}_3 = \{3 \times 3 \text{ permutation matrices}\} \subseteq \mathbb{Z}^{3\times 3}.$$

Write $\mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1},\dots,t_9^{\pm 1}]$, where the variables give the generic 3×3 matrix

$$\begin{pmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \\ t_7 & t_8 & t_9 \end{pmatrix}$$

with nonzero entries. Also let $\mathbb{P}^5$ have homogeneous coordinates x_{ijk} indexed by triples such that $\left(\begin{smallmatrix}1&2&3\\i&j&k\end{smallmatrix}\right)$ is a permutation in S_3. Then $X_{\mathscr{P}_3} \subseteq \mathbb{P}^5$ is the Zariski closure of the image of the map $T_N \to \mathbb{P}^5$ given by the Laurent monomials $t_i t_j t_k$ for $\left(\begin{smallmatrix}1&2&3\\i&j&k\end{smallmatrix}\right) \in S_3$. The ideal of $X_{\mathscr{P}_3}$ is

$$\mathbf{I}(X_{\mathscr{P}_3}) = \langle x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213}\rangle \subseteq \mathbb{C}[x_{ijk}],$$

where the relation comes from the fact that the sum of the permutation matrices corresponding to $x_{123}, x_{231}, x_{312}$ equals the sum of the other three (Exercise 2.1.1). Ideals of the toric varieties arising from permutation matrices have applications to sampling problems in statistics (see [**264**, p. 148]). $\Diamond$

The Affine Cone of a Projective Toric Variety. The projective variety $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$ has an affine cone $\widehat{X}_{\mathscr{A}} \subseteq \mathbb{C}^s$. How does $\widehat{X}_{\mathscr{A}}$ relate to the affine toric variety $Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ constructed in Chapter 1?

Recall from Chapter 1 that when $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M$, the map $e_i \mapsto m_i$ induces an exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \longrightarrow M \tag{2.1.3}$$

and that the ideal of $Y_{\mathscr{A}}$ is the toric ideal

$$I_L = \big\langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}^s \text{ and } \alpha - \beta \in L \big\rangle$$

(Proposition 1.1.9). Then we have the following result.

Proposition 2.1.4. *Given $Y_{\mathscr{A}}$, $X_{\mathscr{A}}$ and I_L as above, the following are equivalent:*

(a) *$Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ is the affine cone $\widehat{X}_{\mathscr{A}}$ of $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$.*

(b) *$I_L = \mathbf{I}(X_{\mathscr{A}})$.*

(c) *I_L is homogeneous.*

(d) *There is $u \in N$ and $k > 0$ in $\mathbb{N}$ such that $\langle m_i, u\rangle = k$ for $i = 1, \ldots, s$.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from the equalities $\mathbf{I}(X_{\mathscr{A}}) = \mathbf{I}(\widehat{X}_{\mathscr{A}})$ and $I_L = \mathbf{I}(Y_{\mathscr{A}})$, and the implication (b) $\Rightarrow$ (c) is obvious.

For (c) $\Rightarrow$ (d), assume that I_L is a homogeneous ideal and take $x^\alpha - x^\beta \in I_L$ for $\alpha - \beta \in L$. If x^α and x^β had different degrees, then $x^\alpha, x^\beta \in I_L = \mathbf{I}(Y_{\mathscr{A}})$ would vanish on $Y_{\mathscr{A}}$. This is impossible since $(1, \ldots, 1) \in Y_{\mathscr{A}}$ by (2.1.2). Hence x^α and x^β have the same degree, which implies that $\ell \cdot (1, \ldots, 1) = 0$ for all $\ell \in L$. Now tensor (2.1.3) with $\mathbb{Q}$ and take duals to obtain an exact sequence

$$N_{\mathbb{Q}} \longrightarrow \mathbb{Q}^s \longrightarrow \mathrm{Hom}_{\mathbb{Q}}(L_{\mathbb{Q}}, \mathbb{Q}) \longrightarrow 0.$$

The above argument shows that $(1, \ldots, 1) \in \mathbb{Q}^s$ maps to zero in $\mathrm{Hom}_{\mathbb{Q}}(L_{\mathbb{Q}}, \mathbb{Q})$ and hence comes from an element $\tilde{u} \in N_{\mathbb{Q}}$. In other words, $\langle m_i, \tilde{u}\rangle = 1$ for all i. Clearing denominators gives the desired $u \in N$ and $k > 0$ in $\mathbb{N}$.

Finally, we prove (d) $\Rightarrow$ (a). Since $Y_{\mathscr{A}} \subseteq \widehat{X}_{\mathscr{A}}$ and $\widehat{X}_{\mathscr{A}}$ is irreducible, it suffices to show that

$$\widehat{X}_{\mathscr{A}} \cap (\mathbb{C}^*)^s \subseteq Y_{\mathscr{A}}.$$

Let $p \in \widehat{X}_{\mathscr{A}} \cap (\mathbb{C}^*)^s$. Since $X_{\mathscr{A}} \cap T_{\mathbb{P}^{s-1}}$ is the torus of $X_{\mathscr{A}}$, it follows that

$$p = \mu \cdot (\chi^{m_1}(t), \dots, \chi^{m_s}(t))$$

for some $\mu \in \mathbb{C}^*$ and $t \in T_N$. The element $u \in N$ from part (d) gives a one-parameter subgroup of T_N, which we write as $\tau \mapsto \lambda^u(\tau)$ for $\tau \in \mathbb{C}^*$. Then $\lambda^u(\tau)t \in T_N$ maps to the point $q \in Y_{\mathscr{A}}$ given by

$$q = \big(\chi^{m_1}(\lambda^u(\tau)t), \dots, \chi^{m_s}(\lambda^u(\tau)t)\big) = \big(\tau^{\langle m_1,u\rangle}\chi^{m_1}(t), \dots, \tau^{\langle m_s,u\rangle}\chi^{m_s}(t)\big),$$

since $\chi^m(\lambda^u(\tau)) = \tau^{\langle m,u\rangle}$ by the description of $\langle\ ,\ \rangle$ given in §1.1. The hypothesis of part (d) allows us to rewrite q as

$$q = \tau^k \cdot (\chi^{m_1}(t), \dots, \chi^{m_s}(t)).$$

Using $k > 0$, we can choose τ so that $p = q \in Y_{\mathscr{A}}$. This completes the proof. $\square$

The condition $\langle m_i, u\rangle = k$, $i = 1, \dots, s$, for some $u \in N$ and $k > 0$ in $\mathbb{N}$ means that $\mathscr{A}$ lies in an affine hyperplane of $M_{\mathbb{Q}}$ not containing the origin. When $M = \mathbb{Z}^n$ and $\mathscr{A}$ consists of the columns of an $n \times s$ integer matrix A, this is equivalent to $(1, \dots, 1)$ lying in the row space of A (Exercise 2.1.2).

Example 2.1.5. We will examine the rational normal curve $C_d \subseteq \mathbb{P}^d$ using two different sets of lattice points.

First let $\mathscr{A}$ consist of the columns of the $2 \times (d+1)$ matrix

$$A = \begin{pmatrix} d & d-1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & d-1 & d \end{pmatrix}.$$

The columns give the Laurent monomials defining the rational normal curve C_d in Example 2.0.1. It follows that C_d is a projective toric variety. The ideal of C_d is the homogeneous ideal given in Example 2.0.1, and the corresponding affine hyperplane of $\mathbb{Z}^2$ containing $\mathscr{A}$ (= the columns of A) consists of all points (a,b) satisfying $a + b = d$. It is equally easy to see that $(1, \dots, 1)$ is in the row space of A. In particular, we have

$$X_{\mathscr{A}} = C_d \quad \text{and} \quad Y_{\mathscr{A}} = \widehat{C}_d.$$

Now let $\mathscr{B} = \{0, 1, \dots, d-1, d\} \subseteq \mathbb{Z}$. This gives the map

$$\Phi_{\mathscr{B}} : \mathbb{C}^* \longrightarrow \mathbb{P}^d, \quad t \mapsto (1, t, \dots, t^{d-1}, t^d).$$

The resulting projective variety is the rational normal curve, i.e., $X_{\mathscr{B}} = C_d$, but the affine variety of $\mathscr{B}$ is *not* the rational normal cone, i.e., $Y_{\mathscr{B}} \neq \widehat{C}_d$. This is because $\mathbf{I}(Y_{\mathscr{B}}) \subseteq \mathbb{C}[x_0, \dots, x_d]$ is not homogeneous. For example, $x_1^2 - x_2$ vanishes at $(1, t, \dots, t^{d-1}, t^d) \in \mathbb{C}^{d+1}$ for all $t \in \mathbb{C}^*$. Thus $x_1^2 - x_2 \in \mathbf{I}(Y_{\mathscr{B}})$. $\Diamond$

Given any $\mathscr{A} \subseteq M$, there is a standard way to modify $\mathscr{A}$ so that the conditions of Proposition 2.1.4 are met: use $\mathscr{A} \times \{1\} \subseteq M \oplus \mathbb{Z}$. This lattice corresponds to the torus $T_n \times \mathbb{C}^*$, and since

$$\Phi_{\mathscr{A} \times \{1\}}(t,\mu) = (\chi^{m_1}(t)\,\mu, \ldots, \chi^{m_s}(t)\,\mu) = \mu \cdot (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)), \tag{2.1.4}$$

it follows immediately that $X_{\mathscr{A} \times \{1\}} = X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$. Since $\mathscr{A} \times \{1\}$ lies in an affine hyperplane missing the origin, Proposition 2.1.4 implies that $X_{\mathscr{A}}$ has affine cone $Y_{\mathscr{A} \times \{1\}} = \widehat{X}_{\mathscr{A}}$. When $M = \mathbb{Z}^n$ and $\mathscr{A}$ is represented by the columns of an $n \times s$ integer matrix A, we obtain $\mathscr{A} \times \{1\}$ by adding the row $(1, \ldots, 1)$ to A.

The Torus of a Projective Toric Variety. Our next task is to determine the torus of $X_{\mathscr{A}}$. We will do so by identifying its character lattice. This will also tell us the dimension of $X_{\mathscr{A}}$. Given $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M$, we set

$$\mathbb{Z}'\mathscr{A} = \big\{ \textstyle\sum_{i=1}^s a_i m_i \mid a_i \in \mathbb{Z},\ \sum_{i=1}^s a_i = 0 \big\}.$$

The rank of $\mathbb{Z}'\mathscr{A}$ is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing the set $\mathscr{A}$ (Exercise 2.1.3).

Proposition 2.1.6. *Let $X_{\mathscr{A}}$ be the projective toric variety of $\mathscr{A} \subseteq M$. Then:*

(a) *The lattice $\mathbb{Z}'\mathscr{A}$ is the character lattice of the torus of $X_{\mathscr{A}}$.*

(b) *The dimension of $X_{\mathscr{A}}$ is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing $\mathscr{A}$. In particular,*

$$\dim X_{\mathscr{A}} = \begin{cases} \operatorname{rank} \mathbb{Z}\mathscr{A} - 1 & \text{if } \mathscr{A} \text{ satisfies the conditions of Proposition 2.1.4} \\ \operatorname{rank} \mathbb{Z}\mathscr{A} & \text{otherwise.} \end{cases}$$

Proof. To prove part (a), let M' be the character lattice of the torus $T_{X_{\mathscr{A}}}$ of $X_{\mathscr{A}}$. By (2.1.2), we have the commutative diagram

$$\begin{tikzcd} T_N \arrow[r] \arrow[dr, two heads] & T_{\mathbb{P}^{s-1}} \arrow[r, hook] & \mathbb{P}^{s-1} \\ & T_{X_{\mathscr{A}}} \arrow[u, hook] & \end{tikzcd}$$

which induces the commutative diagram of character lattices

$$\begin{tikzcd} M & \mathscr{M}_{s-1} \arrow[l] \arrow[d, two heads] \\ & M' \arrow[ul, hook] \end{tikzcd}$$

since $\mathscr{M}_{s-1} = \{(a_1, \ldots, a_s) \in \mathbb{Z}^s \mid \sum_{i=0}^s a_i = 0\}$ is the character lattice of $T_{\mathbb{P}^{s-1}}$ by (2.1.1). The map $\mathscr{M}_{s-1} \to M$ is induced by the map $\mathbb{Z}^s \to M$ that sends e_i to m_i. Thus $\mathbb{Z}'\mathscr{A}$ is the image of $\mathscr{M}_{s-1} \to M$ and hence equals M' by the above diagram.

The first assertion of part (b) follows from part (a) and the observation that $\operatorname{rank} \mathbb{Z}'\mathscr{A}$ is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing $\mathscr{A}$.

Furthermore, if $Y_{\mathscr{A}}$ equals the affine cone of $X_{\mathscr{A}}$, then there is $u \in N$ with $\langle m_i, u\rangle = k > 0$ for all i by Proposition 2.1.4. This implies that $\langle \sum_{i=1}^s a_i m_i, u\rangle = k(\sum_{i=1}^s a_i)$, which gives the exact sequence

$$0 \longrightarrow \mathbb{Z}'\mathscr{A} \longrightarrow \mathbb{Z}\mathscr{A} \xrightarrow{\langle\ ,u\rangle} k\mathbb{Z} \longrightarrow 0.$$

Then $k > 0$ implies $\operatorname{rank}\mathbb{Z}\mathscr{A} - 1 = \operatorname{rank}\mathbb{Z}'\mathscr{A} = \dim X_{\mathscr{A}}$. However, if $Y_{\mathscr{A}} \neq \widehat{X}_{\mathscr{A}}$, then the ideal I_L is not homogeneous. Thus some generator $x^\alpha - y^\beta$ is not homogeneous, so that $(\alpha - \beta)\cdot(1,\ldots,1) \neq 0$. But $\alpha - \beta \in L$, where L is defined by

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^s \longrightarrow \mathbb{Z}\mathscr{A} \longrightarrow 0.$$

This implies that in the exact sequence

$$0 \longrightarrow \mathscr{M}_{s-1} \longrightarrow \mathbb{Z}^s \longrightarrow \mathbb{Z} \longrightarrow 0$$

(see (2.1.1)), the image of $L \subseteq \mathbb{Z}^s$ is $\ell\mathbb{Z} \subseteq \mathbb{Z}$ for some $\ell > 0$. This gives a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L \cap \mathscr{M}_{s-1} & \longrightarrow & L & \longrightarrow & \ell\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{M}_{s-1} & \longrightarrow & \mathbb{Z}^s & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}'\mathscr{A} & \longrightarrow & \mathbb{Z}\mathscr{A} & \rightarrow & \mathbb{Z}/\ell\mathbb{Z} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Hence $\operatorname{rank}\mathbb{Z}\mathscr{A} = \operatorname{rank}\mathbb{Z}'\mathscr{A} = \dim X_{\mathscr{A}}$. □

Example 2.1.7. Let $\mathscr{A} = \{e_1, e_2, e_1 + 2e_2, 2e_1 + e_2\} \subseteq \mathbb{Z}^2$. One computes that $\mathbb{Z}\mathscr{A} = \mathbb{Z}^2$ but $\mathbb{Z}'\mathscr{A} = \{(a,b) \in \mathbb{Z}^2 \mid a + b \equiv 0 \bmod 2\}$. Thus $\mathbb{Z}'\mathscr{A}$ has index 2 in $\mathbb{Z}\mathscr{A}$. This means that $Y_{\mathscr{A}} \neq \widehat{X}_{\mathscr{A}}$ and the map of tori

$$T_{Y_{\mathscr{A}}} \longrightarrow T_{X_{\mathscr{A}}}$$

is two-to-one, i.e., its kernel has order 2 (Exercise 2.1.4). ◊

Affine Pieces of a Projective Toric Variety. So far, our treatment of projective toric varieties has used lattice points and toric ideals. Where are the semigroups? There are actually lots of semigroups, one for each affine piece of $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$.

The affine open set $U_i = \mathbb{P}^{s-1} \setminus \mathbf{V}(x_i)$ contains the torus $T_{\mathbb{P}^{s-1}}$. Thus

$$T_{X_{\mathscr{A}}} = X_{\mathscr{A}} \cap T_{\mathbb{P}^{s-1}} \subseteq X_{\mathscr{A}} \cap U_i.$$

Since $X_{\mathscr{A}}$ is the Zariski closure of $T_{X_{\mathscr{A}}}$ in $\mathbb{P}^{s-1}$, it follows that $X_{\mathscr{A}} \cap U_i$ is the Zariski closure of $T_{X_{\mathscr{A}}}$ in $U_i \simeq \mathbb{C}^{s-1}$. Thus $X_{\mathscr{A}} \cap U_i$ is an affine toric variety.

Given $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M_{\mathbb{R}}$, the affine semigroup associated to $X_{\mathscr{A}} \cap U_i$ is easy to determine. Recall that $U_i \simeq \mathbb{C}^{s-1}$ is given by

$$(a_1, \ldots, a_s) \longmapsto (a_1/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_s/a_i).$$

Combining this and $\chi^{m_j}/\chi^{m_i} = \chi^{m_j - m_i}$ with the map (2.1.2), we see that $X_{\mathscr{A}} \cap U_i$ is the Zariski closure of the image of the map

$$T_N \longrightarrow \mathbb{C}^{s-1}$$

given by

$$t \longmapsto \big(\chi^{m_1-m_i}(t), \dots, \chi^{m_{i-1}-m_i}(t), \chi^{m_{i+1}-m_i}(t), \dots, \chi^{m_s-m_i}(t)\big). \tag{2.1.5}$$

If we set $\mathscr{A}_i = \mathscr{A} - m_i = \{m_j - m_i \mid j \neq i\}$, it follows that

$$X_{\mathscr{A}} \cap U_i = Y_{\mathscr{A}_i} = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i]),$$

where $\mathsf{S}_i = \mathbb{N}\mathscr{A}_i$ is the affine semigroup generated by $\mathscr{A}_i$. We have thus proved the following result.

Proposition 2.1.8. *Let $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$ for $\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M_{\mathbb{R}}$. Then the affine piece $X_{\mathscr{A}} \cap U_i$ is the affine toric variety*

$$X_{\mathscr{A}} \cap U_i = Y_{\mathscr{A}_i} = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i])$$

where $\mathscr{A}_i = \mathscr{A} - m_i$ and $\mathsf{S}_i = \mathbb{N}\mathscr{A}_i$. $\square$

We also note that the results of Chapter 1 imply that the torus of $Y_{\mathscr{A}_i}$ has character lattice $\mathbb{Z}\mathscr{A}_i$. Yet the torus is $T_{X_{\mathscr{A}}}$, which has character lattice $\mathbb{Z}'\mathscr{A}$ by Proposition 2.1.6. These are consistent since $\mathbb{Z}\mathscr{A}_i = \mathbb{Z}'\mathscr{A}$ for all i.

Besides describing the affine pieces $X_{\mathscr{A}} \cap U_i$ of $X_{\mathscr{A}} \subset \mathbb{P}^{s-1}$, we can also describe how they patch together. In other words, we can give a completely toric description of the inclusions

$$X_{\mathscr{A}} \cap U_i \;\supseteq\; X_{\mathscr{A}} \cap U_i \cap U_j \;\subseteq\; X_{\mathscr{A}} \cap U_j$$

when $i \neq j$. For instance, $U_i \cap U_j$ consists of all points of $X_{\mathscr{A}} \cap U_i$ where $x_j/x_i \neq 0$. In terms of $X_{\mathscr{A}} \cap U_i = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i])$, this means those points where $\chi^{m_j - m_i} \neq 0$. Thus

$$X_{\mathscr{A}} \cap U_i \cap U_j = \operatorname{Spec}\big(\mathbb{C}[\mathsf{S}_i]\big)_{\chi^{m_j-m_i}} = \operatorname{Spec}\big(\mathbb{C}[\mathsf{S}_i]_{\chi^{m_j-m_i}}\big), \tag{2.1.6}$$

so that if we set $m = m_j - m_i$, then the inclusion $X_{\mathscr{A}} \cap U_i \cap U_j \subseteq X_{\mathscr{A}} \cap U_i$ can be written

$$\operatorname{Spec}(\mathbb{C}[\mathsf{S}_i])_{\chi^m} \subseteq \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i]). \tag{2.1.7}$$

This looks very similar to the inclusion constructed in (1.3.4) using faces of cones. We will see in §2.3 that this is no accident.

We next say a few words about how the polytope $P = \operatorname{Conv}(\mathscr{A}) \subseteq M_{\mathbb{R}}$ relates to $X_{\mathscr{A}}$. As we will learn in §2.2, the dimension of P is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing P, which is the same as the smallest affine subspace of $M_{\mathbb{R}}$ containing $\mathscr{A}$. It follows from Proposition 2.1.6 that

$$\dim X_{\mathscr{A}} = \dim P.$$

Furthermore, the vertices of P give an especially efficient affine covering of $X_{\mathscr{A}}$.

Proposition 2.1.9. *Given $\mathscr{A} = \{m_1, \dots, m_s\} \subseteq M$, let $P = \mathrm{Conv}(\mathscr{A}) \subseteq M_{\mathbb{R}}$ and set $J = \big\{ j \in \{1, \dots, s\} \mid m_j$ is a vertex of $P \big\}$. Then*

$$X_{\mathscr{A}} = \bigcup_{j \in J} X_{\mathscr{A}} \cap U_j.$$

Proof. We will prove that if $i \in \{1, \dots, s\}$, then $X_{\mathscr{A}} \cap U_i \subseteq X_{\mathscr{A}} \cap U_j$ for some $j \in J$. The discussion of polytopes from §2.2 below implies that

$$P \cap M_{\mathbb{Q}} = \big\{ \textstyle\sum_{j \in J} r_j m_j \mid r_j \in \mathbb{Q}_{\geq 0},\ \sum_{j \in J} r_j = 1 \big\}.$$

Given $i \in \{1, \dots, s\}$, we have $m_i \in P \cap M$, so that m_i is a convex $\mathbb{Q}$-linear combination of the vertices m_j. Clearing denominators, we get integers $k > 0$ and $k_j \geq 0$ such that

$$k m_i = \textstyle\sum_{j \in J} k_j m_j, \quad \sum_{j \in J} k_j = k.$$

Thus $\sum_{j \in J} k_j (m_j - m_i) = 0$, which implies that $m_i - m_j \in \mathsf{S}_i$ when $k_j > 0$. Fix such a j. Then $\chi^{m_j - m_i} \in \mathbb{C}[\mathsf{S}_i]$ is invertible, so $\mathbb{C}[\mathsf{S}_i]_{\chi^{m_j - m_i}} = \mathbb{C}[\mathsf{S}_i]$. By (2.1.6), $X_{\mathscr{A}} \cap U_i \cap U_j = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_i]) = X_{\mathscr{A}} \cap U_i$, giving $X_{\mathscr{A}} \cap U_i \subseteq X_{\mathscr{A}} \cap U_j$. □

Projective Normality. An irreducible variety $V \subseteq \mathbb{P}^n$ is called *projectively normal* if its affine cone $\widehat{V} \subseteq \mathbb{C}^{n+1}$ is normal. A projectively normal variety is always normal (Exercise 2.1.5). Here is an example to show that the converse can fail.

Example 2.1.10. Let $\mathscr{A} \subseteq \mathbb{Z}^2$ consist of the columns of the matrix

$$\begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix},$$

giving the Laurent monomials $s^4, s^3 t, s t^3, t^4$. The polytope $P = \mathrm{Conv}(\mathscr{A})$ is the line segment connecting $(4,0)$ and $(0,4)$, with vertices corresponding to s^4 and t^4. The affine piece of $X_{\mathscr{A}}$ corresponding to s^4 has coordinate ring

$$\mathbb{C}[s^3t/s^4, st^3/s^4, t^4/s^4] = \mathbb{C}[t/s, (t/s)^3, (t/s)^4] = \mathbb{C}[t/s],$$

which is normal since it is a polynomial ring. Similarly, one sees that the coordinate ring corresponding to t^4 is $\mathbb{C}[s/t]$, also normal. These affine pieces cover $X_{\mathscr{A}}$ by Proposition 2.1.9, so that $X_{\mathscr{A}}$ is normal.

Since $(1,1,1,1)$ is in the row space of the matrix, $Y_{\mathscr{A}}$ is the affine cone of $X_{\mathscr{A}}$ by Proposition 2.1.4. The affine variety $Y_{\mathscr{A}}$ is not normal by Example 1.3.9, so that $X_{\mathscr{A}}$ is normal but not projectively normal. See Example B.1.2 for a sophisticated proof of this fact that uses sheaf cohomology from Chapter 9. ◇

The notion of normality used in this example is a bit ad-hoc since we have not formally defined normality for projective varieties. Once we define normality for abstract varieties in Chapter 3, we will see that Example 2.1.10 is fully rigorous.

We will say more about projective normality when we explore the connection with polytopes suggested by the above results.

Exercises for §2.1.

2.1.1. Consider the set $\mathscr{P}_3 \subseteq \mathbb{Z}^{3\times 3}$ of 3×3 permutation matrices defined in Example 2.1.3.

(a) Prove the claim made in Example 2.1.3 that three of the permutation matrices sum to the other three and use this to explain why $x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213} \in \mathbf{I}(X_{\mathscr{P}_3})$.

(b) Show that $\dim X_{\mathscr{P}_3} = 4$ by computing $\mathbb{Z}'\mathscr{P}_3$.

(c) Conclude that $\mathbf{I}(X_{\mathscr{P}_3}) = \langle x_{123}x_{231}x_{312} - x_{132}x_{321}x_{213}\rangle$.

2.1.2. Let $\mathscr{A} \subseteq \mathbb{Z}^n$ consist of the columns of an $n\times s$ matrix A with integer entries. Prove that the conditions of Proposition 2.1.4 are equivalent to the assertion that $(1,\dots,1) \in \mathbb{Z}^s$ lies in the row space of A over $\mathbb{R}$ or $\mathbb{Q}$.

2.1.3. Given a finite set $\mathscr{A} \subseteq M$, prove that the rank of $\mathbb{Z}'\mathscr{A}$ equals the dimension of the smallest affine subspace (over $\mathbb{Q}$ or $\mathbb{R}$) containing $\mathscr{A}$.

2.1.4. Verify the claims made in Example 2.1.7. Also compute $\mathbf{I}(Y_{\mathscr{A}})$ and check that it is not homogeneous.

2.1.5. Let $V \subseteq \mathbb{P}^n$ be projectively normal. Use (2.0.6) to prove that the affine pieces $V\cap U_i$ of V are normal.

2.1.6. Fix a finite subset $\mathscr{A} \subseteq M$. Given $m \in M$, let $\mathscr{A} + m = \{m' + m \mid m' \in \mathscr{A}\}$. This is the *translate* of $\mathscr{A}$ by m.

(a) Prove that $\mathscr{A}$ and its translate $\mathscr{A} + m$ give the same projective toric variety, i.e., $X_{\mathscr{A}} = X_{\mathscr{A}+m}$.

(b) Give an example to show that the affine toric varieties $Y_{\mathscr{A}}$ and $Y_{\mathscr{A}+m}$ can differ. Hint: Pick $\mathscr{A}$ so that it lies in an affine hyperplane not containing the origin. Then translate $\mathscr{A}$ to the origin.

2.1.7. In Proposition 2.1.4, give a direct proof that (d) $\Rightarrow$ (c).

2.1.8. In Example 2.1.5, the rational normal curve $C_d \subseteq \mathbb{P}^d$ was parametrized using the homogeneous monomials $s^i t^j$, $i + j = d$. Here we will consider the curve parametrized by a subset of these monomials corresponding to the exponent vectors

$$\mathscr{A} = \{(a_0,b_0),\dots,(a_n,b_n)\}$$

where $a_0 > a_1 > \cdots > a_n$ and $a_i + b_i = d$ for every i. This gives the projective curve $X_{\mathscr{A}} \subseteq \mathbb{P}^n$. We assume $n \geq 2$.

(a) If $a_0 < d$ or $a_n > 0$, explain why we can obtain the same projective curve using monomials of strictly smaller degree.

(b) Assume $a_0 = d$ and $a_n = 0$. Use Proposition 2.1.8 to show that C is smooth if and only if $a_1 = d-1$ and $a_{n-1} = 1$. Hint: For one direction, it helps to remember that smooth varieties are normal.

§2.2. Lattice Points and Polytopes

Before we can begin our exploration of the rich connections between toric varieties and polytopes, we first need to study polytopes and their lattice points.

Polytopes. Recall from Chapter 1 that a polytope $P \subseteq M_{\mathbb{R}}$ is the convex hull of a finite set $S \subseteq M_{\mathbb{R}}$, i.e., $P = \mathrm{Conv}(S)$. Similar to what we did for cones, our discussion of polytopes will omit proofs. Detailed treatments of polytopes can be found in [**51**], [**128**] and [**281**].

The *dimension* of a polytope $P \subseteq M_{\mathbb{R}}$ is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing P. Given a nonzero vector u in the dual space $N_{\mathbb{R}}$ and $b \in \mathbb{R}$, we get the *affine hyperplane* $H_{u,b}$ and *closed half-space* $H^+_{u,b}$ defined by

$$H_{u,b} = \{m \in M_{\mathbb{R}} \mid \langle m,u\rangle = b\} \quad \text{and} \quad H^+_{u,b} = \{m \in M_{\mathbb{R}} \mid \langle m,u\rangle \geq b\}.$$

A subset $Q \subseteq P$ is a *face* of P, written $Q \preceq P$, if there are $u \in N_{\mathbb{R}} \setminus \{0\}$, $b \in \mathbb{R}$ with

$$Q = H_{u,b} \cap P \ \text{ and } \ P \subseteq H^+_{u,b}.$$

We say that $H_{u,b}$ is a *supporting affine hyperplane* in this situation. Figure 1 shows a polygon with the supporting lines of its 1-dimensional faces. The arrows in the figure indicate the vectors u.

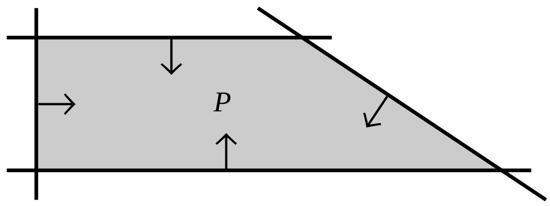


Figure 1. A polygon P and four of its supporting lines

We also regard P as a face of itself. Every face of P is again a polytope, and if $P = \mathrm{Conv}(S)$ and $Q = H_{u,b} \cap P$ as above, then $Q = \mathrm{Conv}(S \cap H_{u,b})$. Faces of P of special interest are *facets*, *edges* and *vertices*, which are faces of dimension $\dim P - 1$, 1 and 0 respectively. Facets will usually be denoted by the letter F.

Here are some properties of faces.

Proposition 2.2.1. *Let $P \subseteq M_{\mathbb{R}}$ be a polytope. Then:*

(a) *P is the convex hull of its vertices.*

(b) *If $P = \mathrm{Conv}(S)$, then every vertex of P lies in S.*

(c) *If Q is a face of P, then the faces of Q are precisely the faces of P lying in Q.*

(d) *Every proper face $Q \prec P$ is the intersection of the facets F containing Q.* □

A polytope $P \subseteq M_{\mathbb{R}}$ can also be written as a finite intersection of closed half-spaces. The converse is true provided the intersection is bounded. In other words, if an intersection

$$P = \bigcap_{i=1}^{s} H^+_{u_i,b_i}$$

is bounded, then P is a polytope. Here is a famous example.

Example 2.2.2. A $d\times d$ matrix $M\in\mathbb{R}^{d\times d}$ is *doubly-stochastic* if it has nonnegative entries and its row and column sums are all 1. If we regard $\mathbb{R}^{d\times d}$ as the affine space $\mathbb{R}^{d^2}$ with coordinates a_{ij}, then the set $\mathscr{M}_d$ of all doubly-stochastic matrices is defined by the inequalites

$$\begin{array}{ll} a_{ij}\geq 0 & (\text{all } i,j)\\ \sum_{i=1}^d a_{ij}\geq 1,\ \ \sum_{i=1}^d a_{ij}\leq 1 & (\text{all } j)\\ \sum_{j=1}^d a_{ij}\geq 1,\ \ \sum_{j=1}^d a_{ij}\leq 1 & (\text{all } i). \end{array}$$

(We use two inequalities to get one equality.) These inequalities easily imply that $0\leq a_{ij}\leq 1$ for all i,j, so that $\mathscr{M}_d$ is bounded and hence is a polytope.

Birkhoff and Von Neumann proved independently that the vertices of $\mathscr{M}_d$ are the $d!$ permutation matrices and that $\dim\mathscr{M}_d=(d-1)^2$. In the literature, $\mathscr{M}_d$ has various names, including the *Birkhoff polytope* and the *transportation polytope*. See [**281**, p. 20] for more on this interesting polytope. $\Diamond$

When P is *full dimensional*, i.e., $\dim P=\dim M_{\mathbb{R}}$, its presentation as an intersection of closed half-spaces has an especially nice form because each facet F has a *unique* supporting affine hyperplane. We write the supporting affine hyperplane and corresponding closed half-space as

$$H_F=\{m\in M_{\mathbb{R}}\mid\langle m,u_F\rangle=-a_F\}\ \text{ and }\ H_F^+=\{m\in M_{\mathbb{R}}\mid\langle m,u_F\rangle\geq -a_F\},$$

where $(u_F,a_F)\in N_{\mathbb{R}}\times\mathbb{R}$ is unique up to multiplication by a positive real number. We call u_F an *inward-pointing facet normal* of the facet F. It follows that

$$P=\bigcap_{F\text{ facet}}H_F^+=\{m\in M_{\mathbb{R}}\mid\langle m,u_F\rangle\geq -a_F\text{ for all facets }F\prec P\}. \tag{2.2.1}$$

In Figure 1, the supporting lines plus arrows determine the supporting half-planes whose intersection is the polygon P. We write (2.2.1) with minus signs in order to simplify formulas in later chapters.

Here are some important classes of polytopes.

Definition 2.2.3. Let $P\subseteq M_{\mathbb{R}}$ be a polytope of dimension d.

(a) P is a ***simplex*** or ***d-simplex*** if it has $d+1$ vertices.

(b) P is ***simplicial*** if every facet of P is a simplex.

(c) P is ***simple*** if every vertex is the intersection of precisely d facets.

Examples include the Platonic solids in $\mathbb{R}^3$:

- A tetrahedron is a 3-simplex.
- The octahedron and icosahedron are simplicial since their facets are triangles.
- The cube and dodecahedron are simple since three facets meet at every vertex.

Polytopes P_1 and P_2 are *combinatorially equivalent* if there is a bijection

$$\{\text{faces of } P_1\} \simeq \{\text{faces of } P_2\}$$

that preserves dimensions, intersections, and the face relation $\preceq$. For example, simplices of the same dimension are combinatorially equivalent, and in the plane, the same holds for polygons with the same number of vertices.

Sums, Multiples, and Duals. Given a polytope $P = \mathrm{Conv}(S)$, its multiple $rP = \mathrm{Conv}(rS)$ is again a polytope for any $r \geq 0$. If P is defined by the inequalities

$$\langle m, u_i \rangle \geq -a_i, \quad 1 \leq i \leq s,$$

then rP is given by

$$\langle m, u_i \rangle \geq -ra_i, \quad 1 \leq i \leq s.$$

In particular, when P is full dimensional, then P and rP have the same inward-pointing facet normals.

The *Minkowski sum* of subsets $A_1, A_2 \subseteq M_{\mathbb{R}}$ is

$$A_1 + A_2 = \{m_1 + m_2 \mid m_1 \in A_1, m_2 \in A_2\}.$$

Given polytopes $P_1 = \mathrm{Conv}(S_1)$ and $P_2 = \mathrm{Conv}(S_2)$, their Minkowski sum $P_1 + P_2 = \mathrm{Conv}(S_1 + S_2)$ is again a polytope. We also have the distributive law

$$rP + sP = (r+s)P.$$

When $P \subseteq M_{\mathbb{R}}$ is full dimensional and 0 is an interior point of P, we define the *dual* or *polar* polytope

$$P^\circ = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq -1 \text{ for all } m \in P\} \subseteq N_{\mathbb{R}}.$$

Figure 2 shows an example of this in the plane.

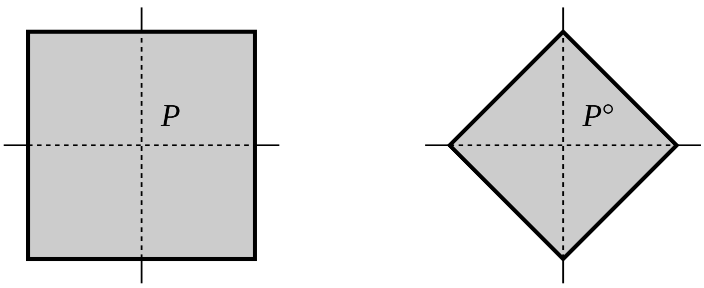


Figure 2. A polygon P and its dual P° in the plane

When we write $P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F, F \text{ facet}\}$ as in (2.2.1), we have $a_F > 0$ for all F since 0 is in the interior. Then P° is the convex hull of the vectors $(1/a_F)u_F \in N_{\mathbb{R}}$ (Exercise 2.2.1). We also have $(P^\circ)^\circ = P$ in this situation.

Lattice Polytopes. Now let M and N be dual lattices with associated vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$. A *lattice polytope* $P \subseteq M_{\mathbb{R}}$ is the convex hull of a finite set $S \subseteq M$. It follows easily that a polytope in $M_{\mathbb{R}}$ is a lattice polytope if and only if its vertices lie in M (Exercise 2.2.2).

Example 2.2.4. The *standard n-simplex* in $\mathbb{R}^n$ is

$$\Delta_n = \mathrm{Conv}(0, e_1, \ldots, e_n).$$

Another simplex in $\mathbb{R}^3$ is $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3)$, shown in Figure 3.

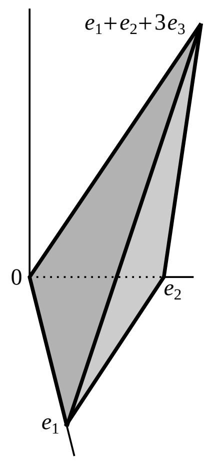


Figure 3. The simplex $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$

The lattice polytopes Δ_3 and P are combinatorially equivalent but will give very different projective toric varieties. $\Diamond$

Example 2.2.5. The Birkhoff polytope defined in Example 2.2.2 is a lattice polytope relative to the lattice of integer matrices $\mathbb{Z}^{d\times d}$ since its vertices are the permutation matrices, whose entries are all 0 or 1. $\Diamond$

One can show that faces of lattice polytopes are again lattice polytopes and that Minkowski sums and integer multiples of lattice polytopes are lattice polytopes (Exercise 2.2.2). Furthermore, every lattice polytope is an intersection of closed half-spaces defined over M, i.e., $P = \bigcap_{i=1}^{s} H^+_{u_i,b_i}$ where $u_i \in N$ and $b_i \in \mathbb{Z}$.

When a lattice polytope P is full dimensional, the facet presentation given in (2.2.1) has an especially nice form. If F is a facet of P, then the inward-pointing facet normals of F lie on a rational ray in $N_{\mathbb{R}}$. Let u_F denote the unique ray generator. The corresponding a_F is integral since $\langle m, u_F\rangle = -a_F$ when m is a vertex of F. It follows that

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -a_F \text{ for all facets } F \prec P\} \tag{2.2.2}$$

is the *unique* facet presentation of the lattice polytope P.

Example 2.2.6. Consider the square $P = \mathrm{Conv}(\pm e_1 \pm e_2) \subseteq \mathbb{R}^2$. The facet normals of P are $\pm e_1$ and $\pm e_2$ and the facet presentation of P is given by

$$\begin{aligned}\langle m, \pm e_1\rangle &\geq -1\\ \langle m, \pm e_2\rangle &\geq -1.\end{aligned}$$

Since the a_F are all equal to 1, it follows that $P^\circ = \mathrm{Conv}(\pm e_1, \pm e_2)$ is also a lattice polytope. The polytopes P and P° are pictured in Figure 2.

It is rare that the dual of a lattice polytope is a lattice polytope—this is related to the *reflexive polytopes* that will be studied later in the book.

Example 2.2.7. The 3-simplex $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$ pictured in Example 2.2.4 has facet presentation

$$\begin{aligned}\langle m, e_3\rangle &\geq 0\\ \langle m, 3e_1 - e_3\rangle &\geq 0\\ \langle m, 3e_2 - e_3\rangle &\geq 0\\ \langle m, -3e_1 - 3e_2 + e_3\rangle &\geq -3\end{aligned}$$

(Exercise 2.2.3). However, if we replace -3 with -1 in the last inequality, we get integer inequalities that define $(1/3)P$, which is *not* a lattice polytope. $\Diamond$

The combinatorial type of a polytope is an interesting object of study. This leads to the question "Is every polytope combinatorially equivalent to a lattice polytope?" If the given polytope is simplicial, the answer is "yes"—just wiggle the vertices to make them rational and clear denominators to get a lattice polytope. The same argument works for simple polytopes by wiggling the facet normals. This will enable us to prove results about arbitrary simplicial or simple polytopes using toric varieties. But in general, the answer is "no"—there exist polytopes in every dimension ≥ 8 not combinatorially equivalent to any lattice polytope. An example is described in [**281**, Ex. 6.21].

A First Guess for the Toric Variety of Polytope. A lattice polytope P gives a finite set of lattice points $P \cap M$, which in turn gives a projective toric variety $X_{P\cap M}$. This is a natural candidate for the toric variety of P. However, $X_{P\cap M}$ may fail to reflect the properties of P when there are too few lattice points.

Example 2.2.8. In Example 2.2.4, we defined the standard 3-simplex Δ_3 and the 3-simplex $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3)$. Both have only four lattice points (their vertices). Thus the toric varieties $X_{\Delta_3\cap\mathbb{Z}^3}$ and $X_{P\cap\mathbb{Z}^3}$ live in $\mathbb{P}^3$, and in fact we have $X_{\Delta_3\cap\mathbb{Z}^3} = X_{P\cap\mathbb{Z}^3} = \mathbb{P}^3$ (Exercise 2.2.3). Yet Δ_3 and P are very different lattice polytopes. For example, the nonzero vertices of Δ_3 form a basis of $\mathbb{Z}^3$, but this is not true for P. $\Diamond$

We will see below that the construction $P \mapsto P \cap M \mapsto X_{P\cap M}$ works fine when the lattice polytope P has “enough” lattice points. Hence our first task is to explore what this means.

Normal Polytopes. Here is one way to say that P has enough lattice points.

Definition 2.2.9. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is ***normal*** if

$$(kP) \cap M + (\ell P) \cap M = ((k+\ell)P) \cap M$$

for all $k, \ell \in \mathbb{N}$.

The inclusion $(kP) \cap M + (\ell P) \cap M \subseteq ((k+\ell)P) \cap M$ is automatic. Thus normality means that all lattice points of $(k+\ell)P$ come from lattice points of kP and ℓP. In particular, a lattice polytope is normal if and only if

$$\underbrace{P \cap M + \cdots + P \cap M}_{k \text{ times}} = (kP) \cap M.$$

for all integers $k \geq 1$. So normality means that P has enough lattice points to generate the lattice points in all integer multiples of P.

Lattice polytopes of dimension 1 are normal (Exercise 2.2.4). Here is another class of normal polytopes.

Definition 2.2.10. A simplex $P \subseteq M_{\mathbb{R}}$ is ***basic*** if P has a vertex m_0 such that the differences $m - m_0$, for vertices $m \neq m_0$ of P, form a subset of a $\mathbb{Z}$-basis of M.

This definition is independent of which vertex $m_0 \in P$ is chosen. The standard simplex $\Delta_n \subseteq \mathbb{R}^n$ is basic, and any basic simplex is normal (Exercise 2.2.5). More general simplices, however, need not be normal.

Example 2.2.11. We noted in Example 2.2.8 that the only lattice points of $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$ are its vertices. It follows easily that

$$e_1 + e_2 + e_3 = \tfrac{1}{6}(0) + \tfrac{1}{3}(2e_1) + \tfrac{1}{3}(2e_2) + \tfrac{1}{6}(2e_1 + 2e_2 + 6e_3) \in 2P$$

is not the sum of lattice points of P. This shows that P is not normal. In particular, P is not basic. $\lozenge$

Here is an important result on normality.

Theorem 2.2.12. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope of dimension $n \geq 2$. Then kP is normal for all $k \geq n-1$.*

Proof. This result was first explicitly stated in [**55**], though as noted in [**55**], its essential content follows from [**94**] and [**187**]. We will use ideas from [**187**] and [**223**] to show that

$$(kP) \cap M + P \cap M = ((k+1)P) \cap M \ \text{ for } \ k \in \mathbb{Z},\ k \geq n-1. \tag{2.2.3}$$

In Exercise 2.2.6 you will show that (2.2.3) implies that kP is normal for all integers $k \geq n-1$. Note also that for (2.2.3), it suffices to prove that

$$((k+1)P) \cap M \subseteq (kP) \cap M + P \cap M$$

since the other inclusion is obvious.

First consider the case where P is a simplex with no interior lattice points. Let the vertices of P be $m_0, \dots, m_n$ and take $k \geq n-1$. Then $(k+1)P$ has vertices $(k+1)m_0, \dots, (k+1)m_n$, so that a point $m \in ((k+1)P) \cap M$ is a convex combination

$$m = \textstyle\sum_{i=0}^n \mu_i (k+1) m_i, \text{ where } \mu_i \geq 0, \ \textstyle\sum_{i=0}^n \mu_i = 1.$$

If we set $\lambda_i = (k+1)\mu_i$, then

$$m = \textstyle\sum_{i=0}^n \lambda_i m_i, \text{ where } \lambda_i \geq 0, \ \textstyle\sum_{i=0}^n \lambda_i = k+1.$$

If $\lambda_i \geq 1$ for some i, then one easily sees that $m - m_i \in (kP) \cap M$. Hence $m = (m - m_i) + m_i$ is the desired decomposition. On the other hand, if $\lambda_i < 1$ for all i, then

$$n = (n-1) + 1 \leq k + 1 = \textstyle\sum_{i=0}^n \lambda_i < n+1,$$

so that $k = n-1$ and $\sum_{i=0}^n \lambda_i = n$. Now consider the lattice point

$$\widetilde{m} = \big(m_0 + \cdots + m_n\big) - m = \textstyle\sum_{i=0}^n (1 - \lambda_i) m_i.$$

The coefficients are positive since $\lambda_i < 1$ for all i, and their sum is $\sum_{i=0}^n (1-\lambda_i) = n + 1 - n = 1$. Hence $\widetilde{m}$ is a lattice point in the interior of P since $1 - \lambda_i > 0$ for all i. This contradicts our assumption on P and completes the proof when P is a lattice simplex containing no interior lattice points.

To prove (2.2.3) for the general case, it suffices to prove that P is a finite union of n-dimensional lattice simplices with no interior lattice points (Exercise 2.2.7). For this, we use Carathéodory's theorem (see [**281**, Prop. 1.15]), which asserts that for a finite set $\mathscr{A} \subseteq M_{\mathbb{R}}$, we have

$$\mathrm{Conv}(\mathscr{A}) = \bigcup \mathrm{Conv}(\mathscr{B}),$$

where the union is over all subsets $\mathscr{B} \subseteq \mathscr{A}$ consisting of $\dim \mathrm{Conv}(\mathscr{A}) + 1$ affinely independent elements. Thus each $\mathrm{Conv}(\mathscr{B})$ is a simplex. This enables us to write our lattice polytope P as a finite union of n-dimensional lattice simplices.

If an n-dimensional lattice simplex $Q = \mathrm{Conv}(w_0, \dots, w_n)$ has an interior lattice point v, then

$$Q = \bigcup_{i=0}^n Q_i, \quad Q_i = \mathrm{Conv}(w_0, \dots, \widehat{w}_i, \dots, w_n, v)$$

is a finite union of n-dimensional lattice simplices, each of which has fewer interior lattice points than Q since v becomes a vertex of each Q_i. By repeating this process on those Q_i that still have interior lattice points, we can eventually write Q and hence our original polytope P as a finite union of n-dimensional lattice simplices with no interior lattice points. You will verify the details in Exercise 2.2.7. $\square$

This theorem shows that for the nonnormal 3-simplex P of Example 2.2.11, its multiple $2P$ is normal. Here is another consequence of Theorem 2.2.12.

Corollary 2.2.13. *Every lattice polygon $P \subseteq \mathbb{R}^2$ is normal.* □

We can also interpret normality in terms of the cone of P, defined by

$$C(P) = \mathrm{Cone}(P \times \{1\}) \subseteq M_{\mathbb{R}} \times \mathbb{R}.$$

This was introduced in Figure 3 of Chapter 1. The key feature of this cone is that kP is the "slice" of $C(P)$ at height k, as illustrated in Figure 4. It follows that lattice points $m \in kP$ correspond to points $(m,k) \in C(P) \cap (M \times \mathbb{Z})$.

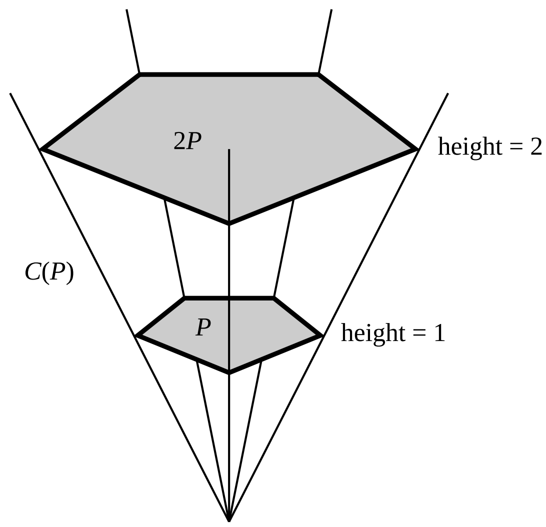


Figure 4. The cone $C(P)$ sliced at heights 1 and 2

In Exercise 2.2.8 you will show that the semigroup $C(P) \cap (M \times \mathbb{Z})$ of lattice points in $C(P)$ relates to normality as follows.

Lemma 2.2.14. *Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope. Then P is normal if and only if $(P \cap M) \times \{1\}$ generates the semigroup $C(P) \cap (M \times \mathbb{Z})$.* □

This lemma tells us that $P \subseteq M_{\mathbb{R}}$ is normal if and only if $(P \cap M) \times \{1\}$ is the Hilbert basis of $C(P) \cap (M \times \mathbb{Z})$.

Example 2.2.15. In Example 2.2.11, the simplex $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3)$ gives the cone $C(P) \subseteq \mathbb{R}^4$. The Hilbert basis of $C(P) \cap (M \times \mathbb{Z})$ is

$$(0,1), (e_1,1), (e_2,1), (e_1+e_2+3e_3,1), (e_1+e_2+e_3,2), (e_1+e_2+2e_3,2)$$

(Exercise 2.2.3). Since the Hilbert basis has generators of height greater than 1, Lemma 2.2.14 gives another proof that P is not normal. ◊

In Exercise 2.2.9, you will generalize Lemma 2.2.14 as follows.

Lemma 2.2.16. *Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a lattice polytope of dimension $n \geq 2$ and let k_0 be the maximum height of an element of the Hilbert basis of $C(P)$. Then:*

(a) $k_0 \leq n-1$.

(b) *kP is normal for any $k \geq k_0$.* □

The Hilbert basis of the simplex P of Example 2.2.15 has maximum height 2. Then Lemma 2.2.16 gives another proof that $2P$ is normal. The paper [**187**] gives a version of Lemma 2.2.16 that applies to Hilbert bases of more general cones.

Very Ample Polytopes. Here is a slightly different notion of what it means for a polytope to have enough lattice points.

Definition 2.2.17. A lattice polytope $P \subseteq M_{\mathbb{R}}$ is ***very ample*** if for every vertex $m \in P$, the semigroup $\mathsf{S}_{P,m} = \mathbb{N}(P \cap M - m)$ generated by the set $P \cap M - m = \{m' - m \mid m' \in P \cap M\}$ is saturated in M.

This definition relates to normal polytopes as follows.

Proposition 2.2.18. *A normal lattice polytope P is very ample.*

Proof. Fix a vertex $m_0 \in P$ and take $m \in M$ such that $km \in \mathsf{S}_{P,m_0}$ for some integer $k \geq 1$. To prove that $m \in \mathsf{S}_{P,m_0}$, write $km \in \mathsf{S}_{P,m_0}$ as

$$km = \textstyle\sum_{m' \in P \cap M} a_{m'}(m' - m_0), \quad a_{m'} \in \mathbb{N}.$$

Pick $d \in \mathbb{N}$ so that $kd \geq \sum_{m' \in P \cap M} a_{m'}$. Then

$$km + kdm_0 = \textstyle\sum_{m' \in P \cap M} a_{m'} m' + \big(kd - \sum_{m' \in P \cap M} a_{m'}\big) m_0 \in kdP.$$

Dividing by k gives $m + dm_0 \in dP$, which by normality implies that

$$m + dm_0 = \sum_{i=1}^{d} m_i, \quad m_i \in P \cap M \text{ for all } i.$$

We conclude that $m = \sum_{i=1}^{d}(m_i - m_0) \in \mathsf{S}_{P,m_0}$, as desired. □

Combining this with Theorem 2.2.12 and Corollary 2.2.13 gives the following.

Corollary 2.2.19. *Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a full dimensional lattice polytope. Then:*

(a) *If $\dim P \geq 2$, then kP is very ample for all $k \geq n-1$.* □

(b) *If $\dim P = 2$, then P is very ample.*

Part (a) was first proved in [**94**]. We will soon see that very ampleness is precisely the property needed to define the toric variety of a lattice polytope.

The following example taken from [**52**, Ex. 5.1] shows that very ample polytopes need not be normal, i.e., the converse of Proposition 2.2.18 is false.

Example 2.2.20. Given $1 \le i < j < k \le 6$, let $[ijk]$ denote the vector in $\mathbb{Z}^6$ with 1 in positions i, j, k and 0 elsewhere. Thus $[123] = (1,1,1,0,0,0)$. Then let

$$\mathscr{A} = \big\{[123],[124],[135],[146],[156],[236],[245],[256],[345],[346]\big\} \subseteq \mathbb{Z}^6.$$

The lattice polytope $P = \mathrm{Conv}(\mathscr{A})$ lies in the affine hyperplane of $\mathbb{R}^6$ where the coordinates sum to 3. As explained in [**52**], this configuration can be interpreted in terms of a triangulation of the real projective plane.

The points of $\mathscr{A}$ are the only lattice points of P (Exercise 2.2.10), so that $\mathscr{A}$ is the set of vertices of P. Number the points of $\mathscr{A}$ as $m_1,\dots,m_{10}$. Then

$$(1,1,1,1,1,1) = \tfrac{1}{5}\sum_{i=1}^{10} m_i = \sum_{i=1}^{10} \tfrac{1}{10}(2m_i)$$

shows that $v = (1,1,1,1,1,1) \in 2P$. Since v is not a sum of lattice points of P (when $[ijk] \in \mathscr{A}$, the vector $v - [ijk]$ is not in $\mathscr{A}$), we conclude that P is not a normal polytope.

To show that P is very ample, we first prove that $\mathscr{A} \times \{1\} \cup \{(v,2)\} \subseteq \mathbb{R}^6 \times \mathbb{R}$ is a Hilbert basis of the semigroup $C(P) \cap \mathbb{Z}^7$, where $C(P) \subseteq \mathbb{R}^6 \times \mathbb{R}$ is the cone over $P \times \{1\}$. We show how do this using `Normaliz` [**57**] in Example B.3.1. Another approach would be to use 4ti2 [**140**].

Now fix i and let S_{P,m_i} be the semigroup generated by the $m_j - m_i$. Take $m \in \mathbb{Z}^6$ such that $km \in \mathsf{S}_{P,m_i}$. As in the proof of Proposition 2.2.18, this implies that $m + dm_i \in dP$ for some $d \in \mathbb{N}$. Thus $(m + dm_i, d) \in C(P) \cap \mathbb{Z}^7$. Expressing this in terms of the above Hilbert basis easily implies that

$$m = a(v - 2m_i) + \sum_{j=1}^{10} a_j(m_j - m_i), \quad a, a_j \in \mathbb{N}.$$

If we can show that $v - 2m_i \in \mathsf{S}_{P,m_i}$, then $m \in \mathsf{S}_{P,m_i}$ follows immediately and proves that S_{P,m_i} is saturated. When $i = 1$, one can check that

$$v + [123] = [124] + [135] + [236],$$

which implies that

$$v - 2m_1 = (m_2 - m_1) + (m_3 - m_1) + (m_6 - m_1) \in \mathsf{S}_{P,m_1}.$$

One obtains similar formulas for $i = 2,\dots,10$ (Exercise 2.2.10), which completes the proof that P is very ample.

The polytope P has further interesting properties. For example, up to affine equivalence, P can be described as the convex hull of the 10 points in $\mathbb{Z}^5$ given by

$$\begin{gathered}(0,0,0,0,0),(0,0,0,0,1),(0,0,1,1,0),(0,1,0,1,1),(0,1,1,1,0)\\(1,0,1,0,1),(1,0,1,1,1),(1,1,0,0,0),(1,1,0,1,1),(1,1,1,0,0).\end{gathered}$$

Of all 5-dimensional polytopes whose vertices lie in $\{0,1\}^5$, this polytope has the most edges, namely 45 (see [**4**]). Since it has 10 vertices and $45 = \binom{10}{2}$, every pair of distinct vertices is joined by an edge. Such polytopes are 2-*neighborly*. $\Diamond$

Exercises for §2.2.

2.2.1. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional polytope with the origin as an interior point.

(a) Write $P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F$ for all facets $F\}$. Prove that $a_F > 0$ for all F and that $P^\circ = \mathrm{Conv}((1/a_F)u_F \mid F$ a facet$)$.

(b) Prove that the dual of a simplicial polytope is simple and vice versa.

(c) Prove that $(rP)^\circ = (1/r)P^\circ$ for all $r > 0$.

(d) Use part (c) to construct an example of a lattice polytope whose dual is not a lattice polytope.

2.2.2. Let $P \subseteq M_{\mathbb{R}}$ be a polytope.

(a) Prove that P is a lattice polytope if and only if the vertices of P lie in M.

(b) Prove that P is a lattice polytope if and only if P is the convex hull of its lattice points, i.e., $P = \mathrm{Conv}(P \cap M)$.

(c) Prove that every face of a lattice polytope is a lattice polytope.

(d) Prove that Minkowski sums and integer multiples of lattice polytopes are again lattice polytopes.

2.2.3. Let $P = \mathrm{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$ be the simplex studied in Examples 2.2.4, 2.2.7, 2.2.8, 2.2.11 and 2.2.15.

(a) Verify the facet presentation of P given in Example 2.2.7.

(b) Show that the only lattice points of P are its vertices.

(c) Show that the toric variety $X_{P \cap \mathbb{Z}^3}$ is $\mathbb{P}^3$, as claimed in Example 2.2.8.

(d) Show that the vectors given in Example 2.2.15 form the Hilbert basis of the semigroup $C(P) \cap (M \times \mathbb{Z})$.

2.2.4. Prove that every 1-dimensional lattice polytope is normal.

2.2.5. Recall the definition of basic simplex given in Definition 2.2.10.

(a) Show that if a simplex satisfies Definition 2.2.10 for one vertex, then it satisfies the definition for all vertices.

(b) Show that the standard simplex Δ_n is basic.

(c) Prove that a basic simplex is normal.

2.2.6. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope.

(a) Prove that (2.2.3) implies that

$$(kP) \cap M + (\ell P) \cap M = ((k+\ell)P) \cap M$$

for all integers $k \geq n-1$ and $\ell \geq 0$. Hint: When $\ell = 2$, we have

$$(kP) \cap M + P \cap M + P \cap M \subseteq (kP) \cap M + (2P) \cap M \subseteq ((k+2)P) \cap M.$$

Apply (2.2.3) twice on the right.

(b) Use part (a) to prove that kP is normal when $k \geq n-1$ and P satisifes (2.2.3).

2.2.7. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope.

(a) Follow the hints given in the text to give a careful proof that P is a finite union of n-dimensional lattice simplices with no interior lattice points.

(b) In the text, we showed that (2.2.3) holds for an n-dimensional lattice simplex with no interior lattice points. Use this and part (a) to show that (2.2.3) holds for P.

2.2.8. Prove Lemma 2.2.14.

2.2.9. In this exercise you will prove Lemma 2.2.16. As in the lemma, let k_0 be the maximum height of a generator of the Hilbert basis of $C(P) \cap (M \times \mathbb{Z})$.

(a) Adapt the proof of Gordan's Lemma (Proposition 1.2.17) to show that if $\mathscr{H}$ is the Hilbert basis of the semigroup of lattice points in a strongly convex cone $\mathrm{Cone}(\mathscr{A})$, then the lattice points in the cone can be written as the union

$$\mathbb{N}\mathscr{A} \cup \textstyle\bigcup_{m \in \mathscr{H}} \big(m + \mathbb{N}\mathscr{A}\big).$$

(b) If the Hilbert basis of $C(P) \cap (M \times \mathbb{Z})$ is $\{(m_1,h_1),\ldots,(m_s,h_s)\}$, then conclude that

$$C(P) \cap (M \times \mathbb{Z}) = \mathsf{S} \cup \textstyle\bigcup_{i=1}^{s} \big((m_i,h_i) + \mathsf{S}\big),$$

where $\mathsf{S} = \mathbb{N}((P \cap M) \times \{1\})$.

(c) Use part (b) to show that (2.2.3) holds for $k \geq k_0$.

2.2.10. Consider the polytope $P = \mathrm{Conv}(\mathscr{A})$ from Example 2.2.20.

(a) Prove that $\mathscr{A}$ is the set of lattice points of P.

(b) Complete the proof begun in the text that P is very ample.

2.2.11. Prove that every proper face of a simplicial polytope is a simplex.

2.2.12. In Corollary 2.2.19 we proved that kP is very ample for $k \geq n-1$ using Theorem 2.2.12 and Proposition 2.2.18. Give a direct proof of the weaker result that kP is very ample for k sufficiently large. Hint: A vertex $m \in P$ gives the cone $C_{P,m}$ generated by the semigroup $\mathsf{S}_{P,m}$ defined in Definition 2.2.17. The cone $C_{P,m}$ is strongly convex since m is a vertex and hence $C_{P,m} \cap M$ has a Hilbert basis. Furthermore, $C_{P,m} = C_{kP,km}$ for all $k \in \mathbb{N}$. Now argue that when k is large, $(kP) \cap M - km$ contains the Hilbert basis of $C_{P,m} \cap M$. A picture will help.

2.2.13. Fix an integer $a \geq 1$ and consider the 3-simplex $P = \mathrm{Conv}(0, ae_1, ae_2, e_3) \subseteq \mathbb{R}^3$.

(a) Work out the facet presentation of P and verify that the facet normals can be labeled so that $u_0 + u_1 + u_2 + au_3 = 0$.

(b) Show that P is normal. Hint: Show that $P \cap \mathbb{Z}^3 + (kP) \cap \mathbb{Z}^3 = ((k+1)P) \cap \mathbb{Z}^3$.

We will see later that the toric variety of P is the weighted projective space $\mathbb{P}(1,1,1,a)$.

§2.3. Polytopes and Projective Toric Varieties

Our next task is to define the toric variety of a lattice polytope. As noted in §2.2, we need to make sure that the polytope has enough lattice points. Hence we begin with very ample polytopes. Strongly convex rational polyhedral cones will play an important role in our development.

The Very Ample Case. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional very ample polytope relative to the lattice M, and let $\dim P = n$. If $P \cap M = \{m_1, \ldots, m_s\}$, then $X_{P\cap M}$ is the Zariski closure of the image of the map $T_N \to \mathbb{P}^{s-1}$ given by

$$t \longmapsto \big(\chi^{m_1}(t), \ldots, \chi^{m_s}(t)\big) \in \mathbb{P}^{s-1}.$$

Fix homogeneous coordinates $x_1, \ldots, x_s$ for $\mathbb{P}^{s-1}$.

We examine the structure of $X_{P\cap M} \subseteq \mathbb{P}^{s-1}$ using Propositions 2.1.8 and 2.1.9. For each $m_i \in P \cap M$ consider the semigroup

$$\mathsf{S}_i = \mathbb{N}(P \cap M - m_i)$$

generated by $m_j - m_i$ for $m_j \in P \cap M$. In $\mathbb{P}^{s-1}$ we have the affine open subset $U_i \simeq \mathbb{C}^{s-1}$ consisting of those points where $x_i \neq 0$. Proposition 2.1.8 showed that the affine open piece $X_{P\cap M} \cap U_i$ of $X_{P\cap M}$ is the affine toric variety

$$X_{P\cap M} \cap U_i \simeq \operatorname{Spec}(\mathbb{C}[\mathsf{S}_i]),$$

and Proposition 2.1.9 showed that

$$X_{P\cap M} = \bigcup_{m_i \text{ vertex of } P} X_{P\cap M} \cap U_i.$$

Here is our first major result about $X_{P\cap M}$.

Theorem 2.3.1. *Let $X_{P\cap M}$ be the projective toric variety of the very ample polytope $P \subseteq M_{\mathbb{R}}$, and assume that P is full dimensional with* $\dim P = n$. *Then:*

(a) *For each vertex $m_i \in P \cap M$, the affine piece $X_{P\cap M} \cap U_i$ is the affine toric variety*

$$X_{P\cap M} \cap U_i = U_{\sigma_i} = \operatorname{Spec}(\mathbb{C}[\sigma_i^\vee \cap M])$$

where $\sigma_i \subseteq N_{\mathbb{R}}$ is the strongly convex rational polyhedral cone dual to the cone $\operatorname{Cone}(P \cap M - m_i) \subseteq M_{\mathbb{R}}$. *Furthermore,* $\dim \sigma_i = n$.

(b) *The torus of $X_{P\cap M}$ has character lattice M and hence is the torus T_N.*

Proof. Let $C_i = \operatorname{Cone}(P \cap M - m_i)$. Since m_i is a vertex, it has a supporting hyperplane $H_{u,a}$ such that $P \subseteq H_{u,a}^+$ and $P \cap H_{u,a} = \{m_i\}$. It follows that $H_{u,0}$ is a supporting hyperplane of $0 \in C_i$ (Exercise 2.3.1), so that C_i is strongly convex. It is also easy to see that $\dim C_i = \dim P$ (Exercise 2.3.1). It follows that C_i and $\sigma_i = C_i^\vee$ are strongly convex rational polyhedral cones of dimension n.

We have $\mathsf{S}_i \subseteq C_i \cap M = \sigma_i^\vee \cap M$. By hypothesis, P is very ample, which means that $\mathsf{S}_i \subseteq M$ is saturated. Since S_i and $C_i = \sigma_i^\vee$ are both generated by $P \cap M - m_i$, part (a) of Exercise 1.3.4 implies $\mathsf{S}_i = \sigma_i^\vee \cap M$. (This exercise was part of the proof of the characterization of normal affine toric varieties given in Theorem 1.3.5.) Part (a) of the theorem follows immediately.

For part (b), Theorem 1.2.18 implies that T_N is the torus of U_{σ_i} since σ_i is strongly convex. Then $T_N \subseteq U_{\sigma_i} = X_{P\cap M} \cap U_i \subseteq X_{P\cap M}$ shows that T_N is also the torus of $X_{P\cap M}$. $\square$

The affine pieces $X_{P\cap M}\cap U_i$ and $X_{P\cap M}\cap U_j$ intersect in $X_{P\cap M}\cap U_i\cap U_j$. In order to describe this intersection carefully, we need to study how the cones σ_i and σ_j fit together in $N_{\mathbb{R}}$. This leads to our next topic.

The Normal Fan. The cones $\sigma_i \subseteq N_{\mathbb{R}}$ appearing in Theorem 2.3.1 fit together in a remarkably nice way, giving a structure called the *normal fan of P*.

Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope, not necessarily very ample. Faces, facets and vertices of P will be denoted by Q, F and v respectively. Hence we write the facet presentation of P as

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -a_F \text{ for all } F\}. \tag{2.3.1}$$

A vertex $\mathsf{v} \in P$ gives cones

$$C_{\mathsf{v}} = \mathrm{Cone}(P\cap M - \mathsf{v}) \subseteq M_{\mathbb{R}} \quad \text{and} \quad \sigma_{\mathsf{v}} = C_{\mathsf{v}}^{\vee} \subseteq N_{\mathbb{R}}.$$

(When $\mathsf{v} = m_i$, these are the cones C_i and σ_i studied above.) Faces $Q \subseteq P$ containing the vertex v correspond bijectively to faces $Q_{\mathsf{v}} \subseteq C_{\mathsf{v}}$ via the maps

$$\begin{aligned} Q &\longmapsto Q_{\mathsf{v}} = \mathrm{Cone}(Q\cap M - \mathsf{v})\\ Q_{\mathsf{v}} &\longmapsto Q = (Q_{\mathsf{v}} + \mathsf{v})\cap P \end{aligned} \tag{2.3.2}$$

which are inverses of each other. These maps preserve dimensions, inclusions, and intersections (Exercise 2.3.2), as illustrated in Figure 5.

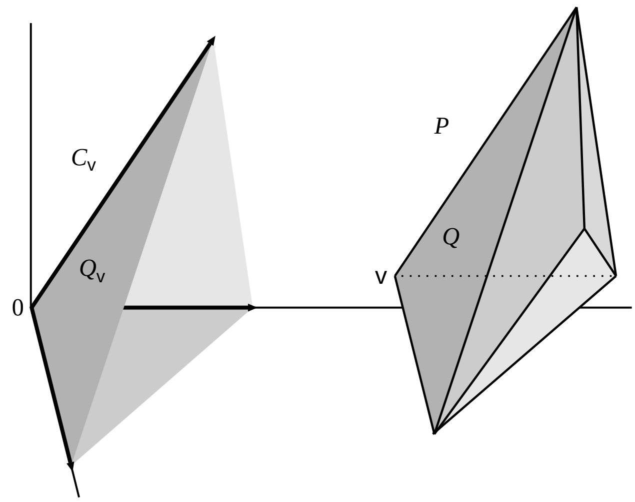


Figure 5. The cone C_{v} of a vertex $\mathsf{v} \in P$

In particular, all facets of C_{v} come from facets of P containing v, so that

$$C_{\mathsf{v}} = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq 0 \text{ for all } F \text{ containing } \mathsf{v}\}.$$

By the duality results of Chapter 1, it follows that the dual cone σ_{v} is given by

$$\sigma_{\mathsf{v}} = \mathrm{Cone}(u_F \mid F \text{ contains } \mathsf{v}).$$

This construction generalizes to arbitrary faces $Q \preceq P$ by setting

$$\sigma_Q = \mathrm{Cone}(u_F \mid F \text{ contains } Q).$$

Thus the cone σ_F is the ray generated by u_F, and $\sigma_P = \{0\}$ since $\{0\}$ is the cone generated by the empty set. Here is our main result about these cones.

Theorem 2.3.2. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope and set $\Sigma_P = \{\sigma_Q \mid Q \preceq P\}$. Then:*

(a) *For all $\sigma_Q \in \Sigma_P$, each face of σ_Q is also in Σ_P.*

(b) *The intersection $\sigma_Q \cap \sigma_{Q'}$ of any two cones in Σ_P is a face of each.*

Remark 2.3.3. A finite collection of strongly convex rational polyhedral cones that satisfies (a) and (b) of Theorem 2.3.2 is called a *fan*. General fans will be introduced in Chapter 3. Since the cones in the fan Σ_P are built from the inward-pointing normal vectors u_F, we call Σ_P the *normal fan* or *inner normal fan* of P.

Theorem 2.3.2 will follow from the results proved below. But first, we give a simple example of a polytope and its normal fan.

Example 2.3.4. The 2-simplex $\Delta_2 \subseteq \mathbb{R}^2$ has vertices $0, e_1, e_2$. Let $P = k\Delta_2$ for an integer $k > 0$. Figure 6 shows P and its normal fan Σ_P. At each vertex v_i of P, we

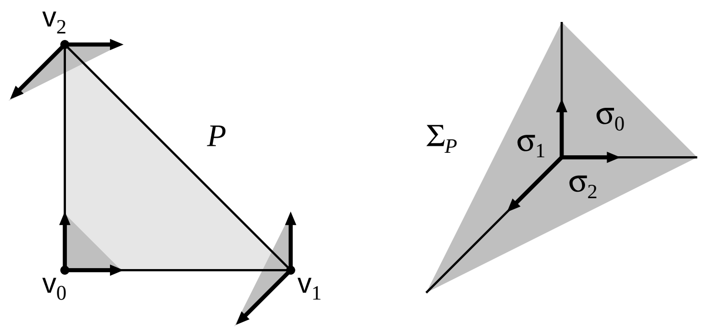


Figure 6. The triangle $P = k\Delta_2 \subseteq \mathbb{R}^2$ and its normal fan Σ_P

show the cone $\sigma_i = C_{\mathsf{v}_i}^\vee$ generated by the normal vectors of the facets containing v_i. The reassembled cones appear on the right as Σ_P.

Notice that the cones $C_{\mathsf{v}_i} \subseteq M_{\mathbb{R}}$ do not fit together nicely; rather, it is their duals $\sigma_i \subseteq N_{\mathbb{R}}$ that give the fan Σ_P. This explains why cones in $N_{\mathbb{R}}$ are the key players in toric geometry. $\Diamond$

Here is the first of the results we need to prove Theorem 2.3.2.

Lemma 2.3.5. *Let Q be a face of P and let $H_{u,b}$ be a supporting affine hyperplane of P. Then $u \in \sigma_Q$ if and only if $Q \subseteq H_{u,b} \cap P$.*

Proof. First suppose that $u \in \sigma_Q$ and write $u = \sum_{Q \subseteq F} \lambda_F u_F$, $\lambda_F \geq 0$. Then setting $b_0 = -\sum_{Q \subseteq F} \lambda_F a_F$ easily implies that $P \subseteq H^+_{u,b_0}$ and $Q \subseteq H_{u,b_0} \cap P$. Recall that the integers a_F come from the facet presentation (2.3.1). It follows that H_{u,b_0} is a supporting hyperplane of P. Since $H_{u,b}$ is a supporting hyperplane by hypothesis, we must have $b_0 = b$, hence $Q \subseteq H_{u,b} \cap P$.

Going the other way, suppose that $Q \subseteq H_{u,b} \cap P$. Take a vertex $\mathsf{v} \in Q$. Then $P \subseteq H^+_{u,b}$ and $\mathsf{v} \in H_{u,b}$ imply that $C_\mathsf{v} \subseteq H^+_{u,0}$. Hence $u \in C^\vee_\mathsf{v} = \sigma_\mathsf{v}$, so that

$$u = \textstyle\sum_{\mathsf{v} \in F} \lambda_F u_F, \quad \lambda_F \geq 0.$$

Let F_0 be a facet of P containing v but not Q, and pick $p \in Q$ with $p \notin F_0$. Then $p, \mathsf{v} \in Q \subseteq H_{u,b}$ imply that

$$\begin{aligned} b &= \langle p, u \rangle = \textstyle\sum_{\mathsf{v} \in F} \lambda_F \langle p, u_F \rangle \\ b &= \langle \mathsf{v}, u \rangle = \textstyle\sum_{\mathsf{v} \in F} \lambda_F \langle \mathsf{v}, u_F \rangle = -\textstyle\sum_{\mathsf{v} \in F} \lambda_F a_F, \end{aligned}$$

where the last equality uses $\langle \mathsf{v}, u_F \rangle = -a_F$ for $\mathsf{v} \in F$. These equations imply

$$\textstyle\sum_{\mathsf{v} \in F} \lambda_F \langle p, u_F \rangle = -\textstyle\sum_{\mathsf{v} \in F} \lambda_F a_F.$$

However, $p \notin F_0$ gives $\langle p, u_{F_0} \rangle > -a_{F_0}$, and since $\langle p, u_F \rangle \geq -a_F$ for all F, it follows that $\lambda_{F_0} = 0$ whenever $Q \not\subseteq F_0$. This gives $u \in \sigma_Q$ and completes the proof of the lemma. □

Corollary 2.3.6. *If $Q \preceq P$ and $F \prec P$ is a facet, then $u_F \in \sigma_Q$ if and only if $Q \subseteq F$.*

Proof. One direction is obvious by the definition of σ_Q, and the other direction follows from Lemma 2.3.5 since $H_{u_F, -a_F}$ is a supporting affine hyperplane of P with $H_{u_F, -a_F} \cap P = F$. □

Theorem 2.3.2 is an immediate consequence of the following proposition.

Proposition 2.3.7. *Let Q and Q' be faces of a full dimensional lattice polytope $P \subseteq M_\mathbb{R}$. Then:*

(a) *$Q \subseteq Q'$ if and only if $\sigma_{Q'} \subseteq \sigma_Q$.*

(b) *If $Q \subseteq Q'$, then $\sigma_{Q'}$ is a face of σ_Q, and all faces of σ_Q are of this form.*

(c) *$\sigma_Q \cap \sigma_{Q'} = \sigma_{Q''}$, where Q'' is the smallest face of P containing Q and Q'.*

Proof. To prove part (a), note that if $Q \subseteq Q'$, then any facet containing Q' also contains Q, which implies $\sigma_{Q'} \subseteq \sigma_Q$. The other direction follows easily from Corollary 2.3.6 since every face is the intersection of the facets containing it by Proposition 2.2.1.

For part (b), fix a vertex $\mathsf{v} \in Q$ and note that by (2.3.2), Q determines a face Q_v of C_v. Using the duality of Proposition 1.2.10, Q_v gives the dual face

$$Q_\mathsf{v}^* = C_\mathsf{v}^\vee \cap Q_\mathsf{v}^\perp = \sigma_\mathsf{v} \cap Q_\mathsf{v}^\perp$$

of the cone σ_v. Then using $\sigma_\mathsf{v} = \mathrm{Cone}(u_F \mid \mathsf{v} \in F)$ and $Q_\mathsf{v} \subseteq C_\mathsf{v} = \sigma_\mathsf{v}^\vee$, one obtains

$$Q_\mathsf{v}^* = \mathrm{Cone}(u_F \mid \mathsf{v} \in F,\ Q_\mathsf{v} \subseteq H_{u_F,0}).$$

Since $\mathsf{v} \in Q$, the inclusion $Q_\mathsf{v} \subseteq H_{u_F,0}$ is equivalent to $Q \subseteq H_{u_F,-a_F}$, which in turn is equivalent to $Q \subseteq F$. It follows that

$$Q_\mathsf{v}^* = \mathrm{Cone}(u_F \mid Q \subseteq F) = \sigma_Q, \tag{2.3.3}$$

so that σ_Q is a face of σ_v, and all faces of σ_v arise in this way.

In particular, $Q \subseteq Q'$ means that $\sigma_{Q'}$ is also a face of σ_v, and since $\sigma_{Q'} \subseteq \sigma_Q$ by part (a), we see that $\sigma_{Q'}$ is a face of σ_Q. Furthermore, every face of σ_Q is a face of σ_v by Proposition 1.2.6 and hence is of the form $\sigma_{Q'}$ for some face Q'. Using part (a) again, we see that $Q \subseteq Q'$, and part (b) follows.

For part (c), let Q'' be the smallest face of P containing Q and Q'. This exists because a face is the intersection of the facets containing it, so that Q'' is the intersection of all facets containing Q and Q' (if there are no such facets, then $Q'' = P$). By part (b) $\sigma_{Q''}$ is a face of both σ_Q and $\sigma_{Q'}$. Thus $\sigma_{Q''} \subseteq \sigma_Q \cap \sigma_{Q'}$.

It remains to prove the opposite inclusion. If $\sigma_Q \cap \sigma_{Q'} = \{0\} = \sigma_P$, then $Q'' = P$ and we are done. If $\sigma_Q \cap \sigma_{Q'} \neq \{0\}$, any nonzero u in the intersection lies in both σ_Q and $\sigma_{Q'}$. The proof of Proposition 2.3.8 given below will show that $H_{u,b}$ is a supporting affine hyperplane of P for some $b \in \mathbb{R}$. By Lemma 2.3.5, $u \in \sigma_Q$ and $u \in \sigma_{Q'}$ imply that Q and Q' lie in $H_{u,b} \cap P$. The latter is a face of P containing Q and Q', so that $Q'' \subseteq H_{u,b} \cap P$ since Q'' is the smallest such face. Applying Lemma 2.3.5 again, we see that $u \in \sigma_{Q''}$. □

Proposition 2.3.7 shows that there is a bijective correspondence between faces of P and cones of the normal fan Σ_P. Here are some further properties of this correspondence.

Proposition 2.3.8. *Let $P \subseteq M_\mathbb{R}$ be a full dimensional lattice polytope of dimension n and consider the cones σ_Q in the normal fan Σ_P of P. Then:*

(a) $\dim Q + \dim \sigma_Q = n$ *for all faces* $Q \preceq P$.

(b) $N_\mathbb{R} = \bigcup_{\mathsf{v} \text{ vertex of } P} \sigma_\mathsf{v} = \bigcup_{\sigma_Q \in \Sigma_P} \sigma_Q$.

Proof. Suppose $Q \preceq P$ and take a vertex v of Q. By (2.3.2) this gives a face Q_v of the cone C_v, which has a dual face Q_v^* of the dual cone $C_\mathsf{v}^\vee = \sigma_\mathsf{v}$. Since $Q_\mathsf{v}^* = \sigma_Q$ by (2.3.3), we have

$$\dim Q + \dim \sigma_Q = \dim Q_\mathsf{v} + \dim Q_\mathsf{v}^* = n,$$

where the first equality uses Exercise 2.3.2 and the second follows from Proposition 1.2.10. This proves part (a). For part (b), let $u \in N_{\mathbb{R}}$ be nonzero and set $b = \min\{\langle \mathsf{v}, u\rangle \mid \mathsf{v} \text{ vertex of } P\}$. Then $P \subseteq H_{u,b}^+$ and $\mathsf{v} \in H_{u,b}$ for at least one vertex of P, so that $u \in \sigma_{\mathsf{v}}$ by Lemma 2.3.5. The final equality of part (b) follows immediately. □

A fan satisfying the condition of part (b) of Proposition 2.3.8 is called *complete*. Thus the normal fan of a lattice polytope is always complete. We will learn more about complete fans in Chapter 3.

In general, multiplying a polytope by a positive integer has no effect on its normal fan, and the same is true for translations by lattice points. We record these properties in the following proposition (Exercise 2.3.3).

Proposition 2.3.9. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. Then for any lattice point $m \in M$ and any integer $k \geq 1$, the polytopes $m + P$ and kP have the same normal fan as P.* □

Examples of Normal Fans. Here are some more examples of normal fans.

Example 2.3.10. Figure 7 shows a lattice hexagon P in the plane together with its normal fan. The vertices of P are labeled $\mathsf{v}_1, \ldots, \mathsf{v}_6$, with corresponding cone $\sigma_1, \ldots, \sigma_6$ in the normal fan. In the figure, P is shown on the left, and at each vertex v_i, we have drawn the normal vectors of the facets containing v_i and shaded the cone σ_i they generate. On the right, these cones are assembled at the origin to give the normal fan.

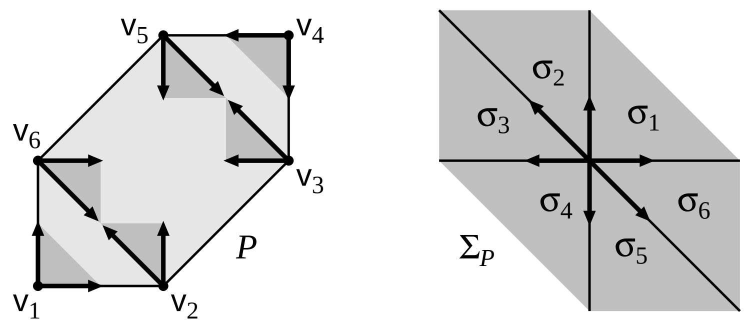


Figure 7. A lattice hexagon P and its normal fan Σ_P

Notice how one can read off the structure of P from the normal fan. For example, two cones σ_i and σ_j share a ray in Σ_P if and only if the vertices v_i and v_j lie on an edge of P.

The hexagon P is an example of a *zonotope* since it is a Minkowski sum of line segments (three in this case). Notice also that Σ_P is determined by three lines through the origin. In §6.2 we will prove that the normal fan of any zonotope is determined by an arrangement of hyperplanes containing the origin. $\Diamond$

Example 2.3.11. Consider the cube $P \subseteq \mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$. The facet normals are $\pm e_1, \pm e_2, \pm e_3$, and the facet presentation of P is

$$\langle m, \pm e_i \rangle \geq -1.$$

The origin is an interior point of P. By Exercise 2.2.1, the facet normals are the vertices of the dual polytope P°, the octahedron in Figure 8.

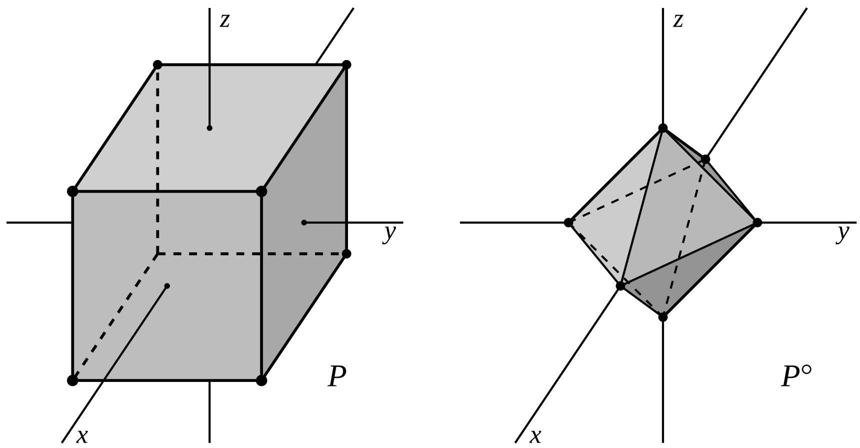


Figure 8. A cube $P \subseteq \mathbb{R}^3$ and its dual octahedron P°

However, the facet normals also give the normal fan of P, and one can check that in the above figure, the maximal cones of the normal fan are the octants of $\mathbb{R}^3$, which are just the cones over the facets of the dual polytope P°. $\Diamond$

As noted earlier, it is rare that both P and P° are lattice polytopes. However, whenever $P \subseteq M_\mathbb{R}$ is a lattice polytope containing 0 as an interior point, it is still true that maximal cones of the normal fan Σ_P are the cones over the facets of $P^\circ \subseteq N_\mathbb{R}$ (Exercise 2.3.4).

The special behavior of the polytopes P and P° discussed in Examples 2.2.6 and 2.3.11 leads to the following definition.

Definition 2.3.12. A full dimensional lattice polytope $P \subseteq M_\mathbb{R}$ is ***reflexive*** if its facet presentation is

$$P = \{m \in M_\mathbb{R} \mid \langle m, u_F \rangle \geq -1 \text{ for all facets } F\}.$$

If P is reflexive, then 0 is a lattice point of P and is the *only* interior lattice point of P (Exercise 2.3.5). Since $a_F = 1$ for all F, Exercise 2.2.1 implies that

$$P^\circ = \mathrm{Conv}(u_F \mid F \text{ facet of } P).$$

Thus P° is a lattice polytope and is in fact reflexive (Exercise 2.3.5).

We will see later that reflexive polytopes lead to some very interesting toric varieties that are important for mirror symmetry.

Intersection of Affine Pieces. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional very ample polytope and set $s = |P \cap M|$. This gives

$$X_{P\cap M} \subseteq \mathbb{P}^{s-1}.$$

If $X_{P\cap M} \cap U_{\mathsf{v}}$ is the affine piece corresponding to a vertex $\mathsf{v} \in P$, then

$$X_{P\cap M} \cap U_{\mathsf{v}} = U_{\sigma_{\mathsf{v}}} = \operatorname{Spec}(\mathbb{C}[\sigma_{\mathsf{v}}^\vee \cap M])$$

by Theorem 2.3.1. Thus the affine piece $X_{P\cap M} \cap U_{\mathsf{v}}$ is the toric variety of the cone σ_{v} in the normal fan Σ_P of P.

Our next task is to describe the intersection of two of these affine pieces.

Proposition 2.3.13. *Let $P \subseteq M_{\mathbb{R}}$ be full dimensional and very ample. If $\mathsf{v} \neq \mathsf{w}$ are vertices of P and Q is the smallest face of P containing v and w, then*

$$X_{P\cap M} \cap U_{\mathsf{v}} \cap U_{\mathsf{w}} = U_{\sigma_Q} = \operatorname{Spec}(\mathbb{C}[\sigma_Q^\vee \cap M])$$

and the inclusions

$$X_{P\cap M} \cap U_{\mathsf{v}} \supseteq X_{P\cap M} \cap U_{\mathsf{v}} \cap U_{\mathsf{w}} \subseteq X_{P\cap M} \cap U_{\mathsf{w}}$$

can be written

$$U_{\sigma_{\mathsf{v}}} \supseteq (U_{\sigma_{\mathsf{v}}})_{\chi^{\mathsf{w}-\mathsf{v}}} = U_{\sigma_Q} = (U_{\sigma_{\mathsf{w}}})_{\chi^{\mathsf{v}-\mathsf{w}}} \subseteq U_{\sigma_{\mathsf{w}}}. \tag{2.3.4}$$

Proof. We analyzed the intersection of affine pieces of $X_{P\cap M}$ in §2.1. Translated to the notation being used here, (2.1.6) and (2.1.7) imply that

$$X_{P\cap M} \cap U_{\mathsf{v}} \cap U_{\mathsf{w}} = (U_{\sigma_{\mathsf{v}}})_{\chi^{\mathsf{w}-\mathsf{v}}} = (U_{\sigma_{\mathsf{w}}})_{\chi^{\mathsf{v}-\mathsf{w}}}.$$

Thus all we need to show is that

$$(U_{\sigma_{\mathsf{v}}})_{\chi^{\mathsf{w}-\mathsf{v}}} = U_{\sigma_Q}.$$

However, we have $\mathsf{w} - \mathsf{v} \in C_{\mathsf{v}} = \sigma_{\mathsf{v}}^\vee$, so that $\tau = H_{\mathsf{w}-\mathsf{v}} \cap \sigma_{\mathsf{v}}$ is a face of σ_{v}. In this situation, Proposition 1.3.16 and equation (1.3.4) imply that

$$(U_{\sigma_{\mathsf{v}}})_{\chi^{\mathsf{w}-\mathsf{v}}} = U_\tau.$$

Thus the proposition will follow once we prove $\tau = \sigma_Q$, i.e., $H_{\mathsf{w}-\mathsf{v}} \cap \sigma_{\mathsf{v}} = \sigma_Q$. Since $\sigma_Q = \sigma_{\mathsf{v}} \cap \sigma_{\mathsf{w}}$ by Proposition 2.3.7, it suffices to prove that

$$H_{\mathsf{w}-\mathsf{v}} \cap \sigma_{\mathsf{v}} = \sigma_{\mathsf{v}} \cap \sigma_{\mathsf{w}}.$$

Let $u \in H_{\mathsf{w}-\mathsf{v}} \cap \sigma_{\mathsf{v}}$. If $u \neq 0$, there is $b \in \mathbb{R}$ such $H_{u,b}$ is a supporting affine hyperplane of P. Then $u \in \sigma_{\mathsf{v}}$ implies $\mathsf{v} \in H_{u,b}$ by Lemma 2.3.5, so that $\mathsf{w} \in H_{u,b}$ since $u \in H_{\mathsf{w}-\mathsf{v}}$. Applying Lemma 2.3.5 again, we get $u \in \sigma_{\mathsf{w}}$. Going the other way, let $u \in \sigma_{\mathsf{v}} \cap \sigma_{\mathsf{w}}$. If $u \neq 0$, pick $b \in \mathbb{R}$ as above. Then $u \in \sigma_{\mathsf{v}} \cap \sigma_{\mathsf{w}}$ and Lemma 2.3.5

imply that $\mathsf{v}, \mathsf{w} \in H_{u,b}$, from which $u \in H_{\mathsf{w}-\mathsf{v}}$ follows easily. This completes the proof. $\square$

This proposition and Theorem 2.3.1 have the remarkable result that the normal fan Σ_P completely determines the internal structure of $X_{P\cap M}$: we build $X_{P\cap M}$ from local pieces given by the affine toric varieties $U_{\sigma_{\mathsf{v}}}$, glued together via (2.3.4). We do not need the ambient projective space $\mathbb{P}^{s-1}$ for any of this—everything we need to know is contained in the normal fan.

The Toric Variety of a Polytope. We can now give the general definition of the toric variety of a polytope.

Definition 2.3.14. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. Then we define the *toric variety of P* to be

$$X_P = X_{(kP)\cap M}$$

where k is any positive integer such that kP is very ample.

Such integers k exist by Corollary 2.2.19, and if k and ℓ are two such integers, then kP and ℓP have the same normal fan by Proposition 2.3.9, namely $\Sigma_{kP} = \Sigma_{\ell P} = \Sigma_P$. It follows that while $X_{(kP)\cap M}$ and $X_{(\ell P)\cap M}$ lie in different projective spaces, they are built from the affine toric varieties $U_{\sigma_{\mathsf{v}}}$ glued together via (2.3.4). Once we develop the language of abstract varieties in Chapter 3, we will see that X_P is well-defined as an abstract variety.

We will often speak of X_P without regard to the projective embedding. When we want to use a specific embedding, we will say "X_P is embedded using kP", where we assume that kP is very ample. In Chapter 6 we will use the language of divisors and line bundles to restate this in terms of a divisor D_P on X_P such that kD_P is very ample precisely when kP is.

Here is a simple example to illustrate the difference between X_P as an abstract variety and X_P as sitting in a specific projective space.

Example 2.3.15. Consider the n-simplex $\Delta_n \subseteq \mathbb{R}^n$. We can define X_{Δ_n} using $k\Delta_n$ for any integer $k \geq 1$ since Δ_n is normal and hence very ample. The lattice points in $k\Delta_n$ correspond to the $s_k = \binom{n+k}{k}$ monomials of $\mathbb{C}[t_1, \ldots, t_n]$ of total degree $\leq k$. This gives an embedding $X_{\Delta_n} \subseteq \mathbb{P}^{s_k-1}$. When $k = 1$, $\Delta_n \cap \mathbb{Z}^n = \{0, e_1, \ldots, e_n\}$ implies that

$$X_{\Delta_n} = \mathbb{P}^n.$$

The normal fan of Δ_n is described in Exercise 2.3.6. For an arbitrary $k \geq 1$, we can regard $X_{\Delta_n} \subseteq \mathbb{P}^{s_k-1}$ as the image of the map

$$\nu_k : \mathbb{P}^n \longrightarrow \mathbb{P}^{s_k-1}$$

defined using all monomials of total degree k in $\mathbb{C}[x_0,\dots,x_n]$ (Exercise 2.3.6). It follows that this map is an embedding, usually called the *Veronese embedding*. But when we forget the embedding, the underlying toric variety is just $\mathbb{P}^n$.

The Veronese embedding allows us to construct some interesting affine open subsets of $\mathbb{P}^n$. Let $f \in \mathbb{C}[x_0,\dots,x_n]$ be nonzero and homogeneous of degree k and write $f = \sum_{|\alpha|=k} c_\alpha x^\alpha$. We write the homogeneous coordinates of $\mathbb{P}^{s-1}$ as y_α for $|\alpha| = k$. Then $L = \sum_{|\alpha|=k} c_\alpha y_\alpha$ is a nonzero linear form in the variables y_α, so that $\mathbb{P}^{s_k-1} \setminus \mathbf{V}(L)$ is a copy of $\mathbb{C}^{s_k-1}$ (Exercise 2.3.6). If follows that

$$\mathbb{P}^n \setminus \mathbf{V}(f) \simeq \nu_k(\mathbb{P}^n) \cap \big(\mathbb{P}^{s_k-1} \setminus \mathbf{V}(L)\big)$$

is an affine variety (usually not toric). This shows that $\mathbb{P}^n$ has a richer supply of affine open subsets than just the open sets $U_i = \mathbb{P}^n \setminus \mathbf{V}(x_i)$ considered earlier in the chapter. $\Diamond$

When we explain the Proj construction of $\mathbb{P}^n$ later in the book, we will see the intrinsic reason why $\mathbb{P}^n \setminus \mathbf{V}(f)$ is an affine open subset of $\mathbb{P}^n$.

Example 2.3.16. The 2-dimensional analog of the rational normal curve C_d is the *rational normal scroll* $S_{a,b}$, which is the toric variety of the polygon

$$P_{a,b} = \mathrm{Conv}(0, ae_1, e_2, be_1 + e_2) \subseteq \mathbb{R}^2,$$

where $a, b \in \mathbb{N}$ satisfy $1 \le a \le b$. The polygon $P = P_{2,4}$ and its normal fan are pictured in Figure 9.

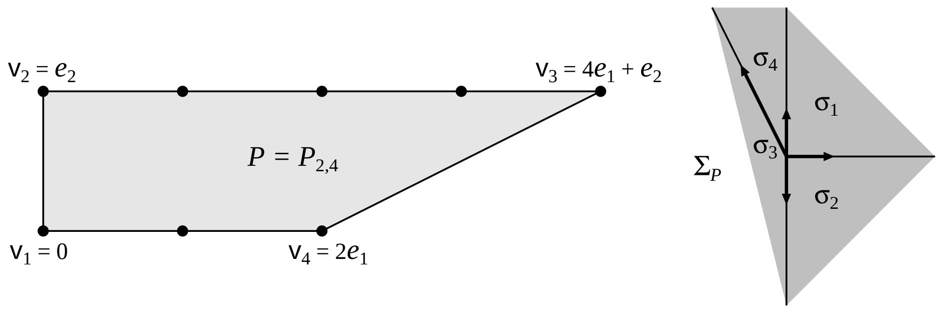


Figure 9. The polygon of a rational normal scroll and its normal fan

In general, the polygon $P_{a,b}$ has $a+b+2$ lattice points and gives the map

$$(\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^{a+b+1}, \quad (s,t) \mapsto (1, s, s^2, \dots, s^a, t, st, s^2t, \dots, s^bt)$$

such that $S_{a,b} = X_{P_{a,b}}$ is the Zariski closure of the image. To describe the image, we rewrite the map as

$$\mathbb{C} \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{a+b+1}, \quad (s,\lambda,\mu) \mapsto (\lambda, s\lambda, s^2\lambda, \dots, s^a\lambda, \mu, s\mu, s^2\mu, \dots, s^b\mu).$$

When $(\lambda,\mu)=(1,0)$, the map is $s\mapsto(1,s,s^2,\dots,s^a,0,\dots,0)$, which is the rational normal curve C_a mapped to the first $a+1$ coordinates of $\mathbb{P}^{a+b+1}$. In the same way, $(\lambda,\mu)=(0,1)$ gives the rational normal curve C_b mapped to the last $b+1$ coordinates of $\mathbb{P}^{a+b+1}$. If we think of these two curves as the "edges" of a scroll, then fixing s gives a point on each edge, and letting $(\lambda,\mu)\in\mathbb{P}^1$ vary gives the line of the scroll connecting the two points. So it really is a scroll!

An important observation is that the normal fan depends *only* on the difference $b-a$, since this determines the slope of the slanted edge of $P_{a,b}$. If we denote the difference by $r\in\mathbb{N}$, it follows that as abstract toric varieties, we have

$$X_{P_{1,r+1}}=X_{P_{2,r+2}}=X_{P_{3,r+3}}=\cdots$$

since they are all constructed from the same normal fan. In Chapter 3, we will see that this is the Hirzebruch surface $\mathscr{H}_r$.

But if we think of the projective surface $S_{a,b}\subseteq\mathbb{P}^{a+b+1}$, then a and b have a unique meaning. For example, they have a strong influence on the defining equations of $S_{a,b}$. Let the homogeneous coordinates of $\mathbb{P}^{a+b+1}$ be $x_0,\dots,x_a,y_0,\dots,y_b$ and consider the $2\times(a+b)$ matrix

$$\left(\begin{array}{cccc|cccc} x_0 & x_1 & \cdots & x_{a-1} & y_0 & y_1 & \cdots & y_{b-1}\\ x_1 & x_2 & \cdots & x_a & y_1 & y_2 & \cdots & y_b\end{array}\right).$$

One can show that $\mathbf{I}(S_{a,b})\subseteq\mathbb{C}[x_0,\dots,x_a,y_0,\dots,y_b]$ is generated by the 2×2 minors of this matrix (see [**130**, Ex. 9.11], for example). $\Diamond$

Example 2.3.16 is another example of a determinantal variety, as is the rational normal curve from Example 2.0.1. Note that the rational normal curve C_d comes from the polytope $[0,d]=d\Delta_1$, where the underlying toric variety is just $\mathbb{P}^1$.

Exercises for §2.3.

2.3.1. This exercise will use the same notation as the proof of Theorem 2.3.1.

(a) Let $H_{u,a}$ be a supporting hyperplane of a vertex $m_i\in P$. Prove that $H_{u,0}$ is a supporting hyperplane of $0\in C_i$

(b) Prove that $\dim C_i=\dim P$.

2.3.2. Consider the maps defined in (2.3.2).

(a) Show that these maps are inverses of each other and define a bijection between the faces of the cone C_v and the faces of P containing v.

(b) Prove that these maps preserve dimensions, inclusions, and intersections.

(c) Explain how this exercise relates to Exercise 2.3.1.

2.3.3. Prove Proposition 2.3.9.

2.3.4. Let $P\subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope containing 0 as an interior point, and let $P^\circ\subseteq N_{\mathbb{R}}$ be its dual polytope. Prove that the normal fan Σ_P consists of the cones over the faces of P°. Hint: Exercise 2.2.1 will be useful.

2.3.5. Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope.

(a) Prove that 0 is the only interior lattice point of P.

(b) Prove that $P^\circ \subseteq N_{\mathbb{R}}$ is reflexive.

2.3.6. This exercise is concerned with Example 2.3.15.

(a) Let $e_1,\ldots,e_n$ be the standard basis of $\mathbb{R}^n$. Prove that the normal fan of the standard n-simplex consists of the cones $\mathrm{Cone}(S)$ for all proper subsets $S \subseteq \{e_0,e_1,\ldots,e_n\}$, where $e_0 = -\sum_{i=1}^n e_i$. Draw pictures of the normal fan for $n = 1,2,3$.

(b) For an integer $k \geq 1$, show that the toric variety $X_{k\Delta_n} \subseteq \mathbb{P}^{s_k-1}$ is given by the map $\nu_k : \mathbb{P}^n \longrightarrow \mathbb{P}^{s_k-1}$defined using all monomials of total degree k in $\mathbb{C}[x_0,\ldots,x_n]$.

2.3.7. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be an n-dimensional lattice polytope and let $Q \subseteq P$ be a face. Prove the following intrinsic description of the cone $\sigma_Q \in \Sigma_P$:

$$\sigma_Q = \{u \in N_{\mathbb{R}} \mid \langle m,u\rangle \leq \langle m',u\rangle \text{ for all } m \in Q,\ m' \in P\}.$$

2.3.8. Prove that all lattice rectangles in the plane with edges parallel to the coordinate axes have the same normal fan.

§2.4. Properties of Projective Toric Varieties

We conclude this chapter by studying when the projective toric variety X_P of a polytope P is smooth or normal.

Normality. Recall from §2.1 that a projective variety is *projectively normal* if its affine cone is normal.

Theorem 2.4.1. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. Then:*

(a) *X_P is normal.*

(b) *X_P is projectively normal under the embedding given by kP if and only if kP is normal.*

Proof. Part (a) is immediate since X_P is the union of affine pieces U_{σ_v}, v a vertex of P, and U_{σ_v} is normal by Theorem 1.3.5. In Chapter 3 we will give an intrinsic definition of normality that will make this argument completely rigorous.

It remains to prove part (b). The discussion following (2.1.4) shows that the projective embedding of X_P given by $X_{(kP)\cap M}$ has $Y_{((kP)\cap M)\times\{1\}}$ as its affine cone. By Theorem 1.3.5, this is normal if and only if the semigroup $\mathbb{N}\big(((kP)\cap M)\times\{1\}\big)$ is saturated in $M \times \mathbb{Z}$. Since $((kP)\cap M)\times\{1\}$ generates the cone $C(P)$, this is equivalent to the assertion that the semigroup $C(P)\cap(M\times\mathbb{Z})$ is generated by $((kP)\cap M)\times\{1\}$. Then we are done by Lemma 2.2.14. □

Smoothness. Given the results of Chapter 1, the smoothness of X_P is equally easy to determine. We need one definition.

Definition 2.4.2. Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope.

(a) Given a vertex v of P and an edge E containing v, let w_E be the first lattice point of E different from v encountered as one tranverses E starting at v. In other words, $w_E - v$ is the ray generator of the ray $\mathrm{Cone}(E - v)$.

(b) P is ***smooth*** if for every vertex v, the vectors $w_E - v$, where E is an edge of P containing v, form a subset of a basis of M. In particular, if $\dim P = \dim M_{\mathbb{R}}$, then the vectors $w_E - v$ form a basis of M.

We can now characterize when X_P is smooth.

Theorem 2.4.3. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. Then the following are equivalent:*

(a) *X_P is a smooth projective variety.*

(b) *Σ_P is a smooth fan, meaning that every cone in Σ_P is smooth in the sense of Definition 1.2.16.*

(c) *P is a smooth polytope.*

Proof. Smoothness is a local condition, so that a variety is smooth if and only if its local pieces are smooth. Thus X_P is smooth if and only if $U_{\sigma_{\mathrm{v}}}$ is smooth for every vertex v of P, and $U_{\sigma_{\mathrm{v}}}$ is smooth if and only if σ_{v} is smooth by Theorem 1.3.12. Since faces of smooth cones are smooth and Σ_P consists of the σ_{v} and their faces, the equivalence (a) $\Leftrightarrow$ (b) follows immediately.

For (b) $\Leftrightarrow$ (c), first observe that σ_{v} is smooth if and only if its dual $C_{\mathrm{v}} = \sigma_{\mathrm{v}}^{\vee}$ is smooth. The discussion following (2.3.2) makes it easy to see that the ray generators of C_{v} are the vectors $w_E - \mathrm{v}$ from Definition 2.4.2. It follows immediately that P is smooth if and only if C_{v} is smooth for every vertex v, and we are done. □

The theorem makes it easy to check the smoothness of simple examples such as the toric variety of the hexagon in Example 2.3.10 or the rational normal scroll $S_{a,b}$ of Example 2.3.16 (Exercise 2.4.1).

We also note the following useful fact, which you will prove in Exercise 2.4.2.

Proposition 2.4.4. *Every smooth full dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$ is very ample.* □

One can also ask whether every smooth lattice polytope is normal. This is an important open problem in the study of lattice polytopes.

Here is an example of a smooth reflexive polytope whose dual is not smooth.

Example 2.4.5. Let $P = (n+1)\Delta_n - (1, \dots, 1) \subseteq \mathbb{R}^n$, where Δ_n is the standard n-simplex. Thus P is the translate of $(n+1)\Delta_n$ by $(-1, \dots, -1)$. Proposition 2.3.9 implies that P and Δ_n have the same normal fan, so that P and X_P are smooth. Note also that $X_P = X_{\Delta_n} = \mathbb{P}^n$.

The polytope P has the following interesting properties (Exercise 2.4.3). First, P has the facet presentation

$$\begin{aligned} x_i &\geq -1, \quad i = 1, \dots, n, \\ -x_1 - \cdots - x_n &\geq -1, \end{aligned}$$

so that P is reflexive with dual

$$P^\circ = \mathrm{Conv}(e_0, e_1, \dots, e_n), \quad e_0 = -e_1 - \cdots - e_n.$$

The normal fan of P° consists of cones over the faces of P. In particular, the cone of Σ_{P° corresponding to the vertex $e_0 \in P^\circ$ is the cone

$$\sigma_{e_0} = \mathrm{Conv}(v_1, \dots, v_n), \quad v_i = e_0 + (n+1)e_i.$$

Figure 10 shows P and the cone σ_{e_0} when $n = 2$.

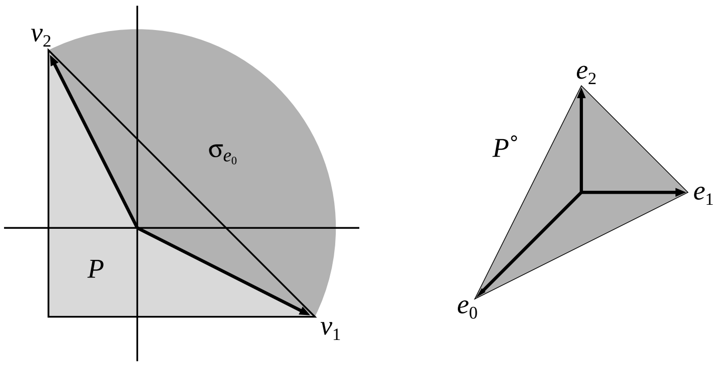


Figure 10. The cone σ_{e_0} of the normal fan of P°

For general n, observe that $v_i - v_j = (n+1)(e_i - e_j)$. This makes it easy to see that $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ has index $(n+1)^{n-1}$ in $\mathbb{Z}^n$. Thus σ_{e_0} is not smooth when $n \geq 2$. It follows that the "dual" toric variety X_{P° is singular for $n \geq 2$. Later we will construct X_{P° as the quotient of $\mathbb{P}^n$ under the action of a finite group $G \simeq (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$. $\Diamond$

Example 2.4.6. Consider $P = \mathrm{Conv}(0, 2e_1, e_2) \subseteq \mathbb{R}^2$. Since P is very ample, the lattice points $P \cap \mathbb{Z}^2 = \{0, e_1, 2e_1, e_2\}$ give the map $(\mathbb{C}^*)^2 \to \mathbb{P}^3$ defined by

$$(s,t) \longmapsto (1, s, s^2, t)$$

such that X_P is the Zariski closure of the image. If $\mathbb{P}^3$ has homogeneous coordinates y_0, y_1, y_2, y_3, then we have

$$X_P = \mathbf{V}(y_0 y_2 - y_1^2) \subseteq \mathbb{P}^3.$$

Comparing this to Example 2.0.5, we see that X_P is the weighted projective space $\mathbb{P}(1,1,2)$. Later we will learn the systematic reason why this is true.

The variety X_P is not smooth. By working on the affine piece $X_P \cap U_3$, one can check directly that $(0,0,0,1)$ is a singular point of X_P.

We can also use Theorem 2.4.3 and the normal fan of P, shown in Figure 11. One can check that the cones σ_0 and σ_1 are smooth, but σ_2 is not, so that Σ_P

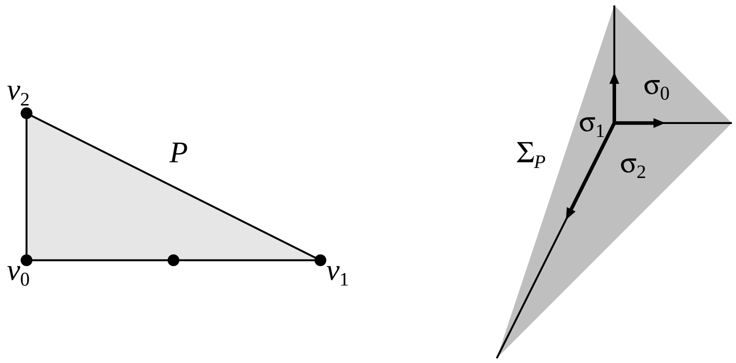


Figure 11. The polygon giving $\mathbb{P}(1,1,2)$ and its normal fan

is not a smooth fan. In terms of P, note that the vectors from v_2 to the first lattice points along the edges containing v_2 do not generate $\mathbb{Z}^2$. Either way, Theorem 2.4.3 implies that X_P is not smooth.

If you look carefully, you will see that σ_2 is the *only* nonsmooth cone of the normal fan Σ_P. Once we study the correspondence between cones and orbits in Chapter 3, we will see that the nonsmooth cone σ_2 corresponds to the singular point $(0,0,0,1)$ of X_P. $\Diamond$

Products of Projective Toric Varieties. Our final task is to understand the toric variety of a product of polytopes. Let $P_i \subseteq (M_i)_\mathbb{R} \simeq \mathbb{R}^{n_i}$ be lattice polytopes with $\dim P_i = n_i$ for $i = 1,2$. This gives a lattice polytope $P_1 \times P_2 \subseteq (M_1 \times M_2)_\mathbb{R}$ of dimension $n_1 + n_2$.

Replacing P_1 and P_2 with suitable multiples, we can assume that P_1 and P_2 are very ample. This gives projective embeddings

$$X_{P_i} \hookrightarrow \mathbb{P}^{s_i-1}, \quad s_i = |P_i \cap M_i|,$$

so that by Proposition 2.0.4, $X_{P_1} \times X_{P_2}$ is a subvariety of $\mathbb{P}^{s_1-1} \times \mathbb{P}^{s_2-1}$. Using the Segre embedding

$$\mathbb{P}^{s_1-1} \times \mathbb{P}^{s_2-1} \hookrightarrow \mathbb{P}^{s-1}, \quad s = s_1 s_2,$$

we get an embedding

$$X_{P_1} \times X_{P_2} \hookrightarrow \mathbb{P}^{s-1}. \tag{2.4.1}$$

We can understand this projective variety as follows.

Theorem 2.4.7. *If P_1 and P_2 are very ample, then:*

(a) $P_1 \times P_2 \subseteq (M_1 \times M_2)_{\mathbb{R}}$ *is a very ample polytope with lattice points*

$$(P_1 \times P_2) \cap (M_1 \times M_2) = (P_1 \cap M_1) \times (P_2 \cap M_2).$$

Thus the integer s defined above is $s = |(P_1 \times P_2) \cap (M_1 \times M_2)|$.

(b) *The image of the embedding $X_{P_1 \times P_2} \hookrightarrow \mathbb{P}^{s-1}$ determined by the very ample polytope $P_1 \times P_2$ equals the image of* (2.4.1).

(c) $X_{P_1 \times P_2} \simeq X_{P_1} \times X_{P_2}$.

Proof. For part (a), the assertions about lattice points are clear. The vertices of $P_1 \times P_2$ consist of ordered pairs (v_1, v_2) where v_i is a vertex of P_i (Exercise 2.4.4). Given such a vertex, we have

$$(P_1 \times P_2) \cap (M_1 \times M_2) - (v_1, v_2) = (P_1 \cap M_1 - v_1) \times (P_2 \cap M_2 - v_2).$$

Since P_i is very ample, we know that $\mathbb{N}(P_i \cap M_i - v_i)$ is saturated in M_i. From here, it follows easily that $P_1 \times P_2$ is very ample.

For part (b), let T_{N_i} be the torus of X_{P_i}. Since T_{N_i} is Zariski dense in X_{P_i}, it follows that $T_{N_1} \times T_{N_2}$ is Zariski dense in $X_{P_1} \times X_{P_2}$ (Exercise 2.4.4). When combined with the Segre embedding, it follows that $X_{P_1} \times X_{P_2}$ is the Zariski closure of the image of the map

$$T_{N_1} \times T_{N_2} \longrightarrow \mathbb{P}^{s_1 s_2 - 1}$$

given by the characters $\chi^m \chi^{m'}$, where m ranges over the s_1 elements of $P_1 \cap M_1$ and m' ranges over the s_2 elements of $P_2 \cap M_2$. When we identify $T_{N_1} \times T_{N_2}$ with $T_{N_1 \times N_2}$, the product $\chi^m \chi^{m'}$ becomes the character $\chi^{(m,m')}$, so that the above map coincides with the map

$$T_{N_1 \times N_2} \longrightarrow \mathbb{P}^{s-1}$$

coming from the product polytope $P_1 \times P_2 \subseteq (M_1 \times M_2)_{\mathbb{R}}$. Part (b) follows, and part (c) is an immediate consequence. $\square$

Here is an obvious example.

Example 2.4.8. Since $\mathbb{P}^n$ is the toric variety of the standard n-simplex Δ_n, it follows that $\mathbb{P}^n \times \mathbb{P}^m$ is the toric variety of $\Delta_n \times \Delta_m$.

This also works for more than two factors. Thus $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the toric variety of the cube pictured in Figure 8. $\Diamond$

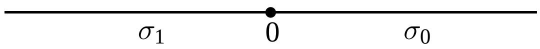
To have a complete theory of products, we need to know what happens to the normal fan. Here is the result, whose proof is left to the reader (Exercise 2.4.5).

Proposition 2.4.9. *Let $P_i \subseteq (M_i)_{\mathbb{R}}$ be full dimensional lattice polytopes for $i = 1, 2$. Then*

$$\Sigma_{P_1 \times P_2} = \Sigma_{P_1} \times \Sigma_{P_2}. \qquad \square$$

Here is an easy example.

Example 2.4.10. The normal fan of an interval $[a,b] \subseteq \mathbb{R}$, where $a < b$ in $\mathbb{Z}$, is given by



The corresponding toric variety is $\mathbb{P}^1$. The cartesian product of two such intervals is a lattice rectangle whose toric variety is $\mathbb{P}^1 \times \mathbb{P}^1$ by Theorem 2.4.7. If we set $\sigma_{ij} = \sigma_i \times \sigma_j$, then Proposition 2.4.9 gives the normal fan given in Figure 12.

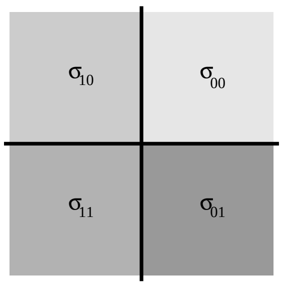


Figure 12. The normal fan of a lattice rectangle giving $\mathbb{P}^1 \times \mathbb{P}^1$

We will revisit this example in Chapter 3 when we construct toric varieties directly from fans. ◊

Proposition 2.4.9 suggests a different way to think about the product. Let v_i range over the vertices of P_i for $i = 1,2$. Then the σ_{v_i} are the maximal cones in the normal fan Σ_{P_i}, which implies that

$$X_{P_i} = \textstyle\bigcup_{v_i} U_{\sigma_{v_i}}, \quad i = 1,2. \tag{2.4.2}$$

Thus

$$\begin{aligned} X_{P_1} \times X_{P_2} &= \big(\textstyle\bigcup_{v_1} U_{\sigma_{v_1}}\big) \times \big(\textstyle\bigcup_{v_2} U_{\sigma_{v_2}}\big) \\ &= \textstyle\bigcup_{(v_1,v_2)} U_{\sigma_{v_1}} \times U_{\sigma_{v_2}} \\ &= \textstyle\bigcup_{(v_1,v_2)} U_{\sigma_{v_1} \times \sigma_{v_2}} \\ &= \textstyle\bigcup_{(v_1,v_2)} U_{\sigma_{(v_1,v_2)}} = X_{P_1 \times P_2}. \end{aligned}$$

In this sequence of equalities, the first follows from (2.4.2), the second is obvious, the third uses Exercise 1.3.13, the fourth uses Proposition 2.4.9, and the last follows since (v_1, v_2) ranges over all vertices of $P_1 \times P_2$.

This argument shows that we can construct cartesian products of varieties using affine open covers, which reduces to the cartesian product of affine varieties defined in Chapter 1. We will use this idea in Chapter 3 to define the cartesian product of abstract varieties.

Exercises for §2.4.

2.4.1. Show that the hexagon $P = \mathrm{Conv}(0, e_1, e_2, 2e_1 + e_2, e_1 + 2e_2, 2e_1 + 2e_2)$ pictured in Figure 6 and the trapezoid $P_{a,b}$ pictured in Figure 9 are smooth polygons. Also, of the polytopes shown in Figure 8, determine which ones are smooth.

2.4.2. Prove Proposition 2.4.4.

2.4.3. Consider the polytope $P = (n+1)\Delta_n - (1,\ldots,1)$ from Example 2.4.5.

(a) Verify the facet presentation of P given in the example.

(b) What is the facet presentation of P°? Hint: You know the vertices of P.

(c) Let $v_i = e_0 + (n+1)e_i$, where $i = 1,\ldots,n$ and $e_0 = -e_1 - \cdots - e_n$, and then set $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$. Use the hint given in the text to prove $\mathbb{Z}^n/L \simeq (\mathbb{Z}/(n+1)\mathbb{Z})^{n-1}$. This shows that the index of L in $\mathbb{Z}^n$ is $(n+1)^{n-1}$, as claimed in the text.

2.4.4. Let $P_i \subseteq (M_i)_{\mathbb{R}} \simeq \mathbb{R}^{n_i}$ be lattice polytopes with $\dim P_i = n_i$ for $i = 1,2$. Also let S_i be the set of vertices of P_i.

(a) Use supporting hyperplanes to prove that every element of $S_1 \times S_2$ is a vertex of $P_1 \times P_2$.

(b) Prove that $P_1 \times P_2 = \mathrm{Conv}(S_1 \times S_2)$ and conclude that $S_1 \times S_2$ is the set of vertices of $P_1 \times P_2$.

2.4.5. The goal of this exercise is to prove Proposition 2.4.9. We know from Exercise 2.4.4 that the vertices of $P_1 \times P_2$ are the ordered pairs (v_1, v_2) where v_i is a vertex of P_i.

(a) Adapt the argument of part (a) of Theorem 2.4.7 to show that $C_{(v_1,v_2)} = C_{v_1} \times C_{v_2}$. Taking duals, we see that the maximal cones of $\Sigma_{P_1 \times P_2}$ are $\sigma_{(v_1,v_2)} = \sigma_{v_1} \times \sigma_{v_2}$.

(b) Given rational polyhedral cones $\sigma_i \subseteq (N_i)_{\mathbb{R}}$ and faces $\tau_i \subseteq \sigma_i$, prove that $\tau_1 \times \tau_2$ is a face of $\sigma_1 \times \sigma_2$ and that all faces of $\sigma_1 \times \sigma_2$ arise this way.

(c) Prove that $\Sigma_{P_1 \times P_2} = \Sigma_{P_1} \times \Sigma_{P_2}$.

2.4.6. Consider positive integers $1 = q_0 \leq q_1 \leq \cdots \leq q_n$ with the property that $q_i | \sum_{j=0}^n q_j$ for $i = 0,\ldots,n$. Set $k_i = \big(\sum_{j=0}^n q_j\big)/q_i$ for $i = 1,\ldots,n$ and let

$$P_{q_0,\ldots,q_n} = \mathrm{Conv}(0, k_1e_1, k_2e_2, \ldots, k_ne_n) - (1,\ldots,1).$$

Prove that $P_{q_0,\ldots,q_n}$ is reflexive and explain how it relates to Example 2.4.5. We will prove later that the toric variety of this polytope is the weighted projective space $\mathbb{P}(q_0,\ldots,q_n)$.

2.4.7. The *Sylvester sequence* is defined by $a_0 = 2$ and $a_{k+1} = 1 + a_0a_1\cdots a_k$. It begins $2, 3, 7, 43, 1807, \ldots$ and is described in [**251**, A000058]. Now fix a positive integer $n \geq 3$ and define $q_0,\ldots,q_n$ by $q_0 = q_1 = 1$ and $q_i = 2(a_{n-1} - 1)/a_{n-i}$ for $i = 2,\ldots,n$. For $n = 3$ and 4 this gives $1,1,4,6$ and $1,1,12,28,42$. Prove that $q_0,\ldots,q_n$ satisfies the conditions of Exercise 2.4.6 and hence gives a reflexive simplex, denoted $S_{Q'_n}$ in [**215**]. This paper proves that when $n \geq 4$, $S_{Q'_n}$ has the largest volume of all n-dimensional reflexive simplices and conjectures that it also has the largest number of lattice points.

Chapter 3

Normal Toric Varieties

§3.0. Background: Abstract Varieties

The projective toric varieties studied in Chapter 2 are unions of Zariski open sets, each of which is an affine variety. We begin with a general construction of abstract varieties obtained by gluing together affine varieties in an analogous way. The resulting varieties will be *abstract* in the sense that they do not come with any given ambient affine or projective space. We will see that this is exactly the idea needed to construct a toric variety using the combinatorial data contained in a fan.

Sheaf theory, while important for later chapters, will make only a modest appearance here. For a more general approach to the concept of abstract variety, we recommend standard books such as [**90**], [**131**] or [**245**].

Regular Functions. Let $V = \mathrm{Spec}(R)$ be an affine variety. In §1.0, we defined the Zariski open subset $V_f = V \setminus \mathbf{V}(f) \subseteq V$ for $f \in R$ and showed that $V_f = \mathrm{Spec}(R_f)$, where R_f is the localization of R at f. The open sets V_f form a *basis* for the Zariski topology on V in the sense that every open set U is a (finite) union $U = \bigcup_{f\in S} V_f$ for some $S \subseteq R$ (Exercise 3.0.1).

For an affine variety, a morphism $V \to \mathbb{C}$ is called a *regular map*, so that the coordinate ring of V consists of all regular maps from V to $\mathbb{C}$. We now define what it means to be regular on an open subset of V.

Definition 3.0.1. Given an affine variety $V = \mathrm{Spec}(R)$ and a Zariski open $U \subseteq V$, we say a function $\phi : U \to \mathbb{C}$ is ***regular*** if for all $p \in U$, there exists $f_p \in R$ such that $p \in V_{f_p} \subseteq U$ and $\phi|_{V_{f_p}} \in R_{f_p}$. Then define

$$\mathscr{O}_V(U) = \{\phi : U \to \mathbb{C} \mid \phi \text{ is regular}\}.$$

The condition $p \in V_{f_p}$ means that $f_p(p) \neq 0$, and saying $\phi|_{V_{f_p}} \in R_{f_p}$ means that $\phi = a_p/f_p^{n_p}$ for some $a_p \in R$ and $n_p \geq 0$.

Here are some cases where $\mathscr{O}_V(U)$ is easy to compute.

Proposition 3.0.2. *Let $V = \operatorname{Spec}(R)$ be an affine variety. Then:*

(a) $\mathscr{O}_V(V) = R$.

(b) *If $f \in R$, then $\mathscr{O}_V(V_f) = R_f$.*

Proof. It is clear from Definition 3.0.1 that elements of R define regular functions on V, hence elements of $\mathscr{O}_V(V)$. Conversely, if $\phi \in \mathscr{O}_V(V)$, then for all $p \in V$ there is $f_p \in R$ such that $p \in V_{f_p}$ and $\phi = a_p/f_p^{n_p} \in R_{f_p}$. The ideal $I = \langle f_p^{n_p} \mid p \in V \rangle \subseteq R$ satisfies $\mathbf{V}(I) = \emptyset$ since $f_p(p) \neq 0$ for all $p \in V$. Hence the Nullstellensatz implies that $\sqrt{I} = \mathbf{I}(\mathbf{V}(I)) = R$, so there exists a finite set $S \subseteq V$ and polynomials g_p for $p \in S$ such that

$$1 = \sum_{p \in S} g_p f_p^{n_p}.$$

Hence $\phi = \sum_{p \in S} g_p f_p^{n_p} \phi = \sum_{p \in S} g_p a_p \in R$, as desired.

For part (b), let $U \subseteq V_f$ be Zariski open. Then U is Zariski open in V, and whenever $g \in R$ satisfies $V_g \subseteq U$, we have $V_g = V_{fg}$ with coordinate ring

$$R_{fg} = (R_f)_{g/f^\ell}$$

for all $\ell \geq 0$. These observations easily imply that

$$\mathscr{O}_V(U) = \mathscr{O}_{V_f}(U). \tag{3.0.1}$$

Then setting $U = V_f$ gives

$$\mathscr{O}_V(V_f) = \mathscr{O}_{V_f}(V_f) = R_f,$$

where the last equality follows by applying part (a) to $V_f = \operatorname{Spec}(R_f)$. $\square$

Local Rings. When $V = \operatorname{Spec}(R)$ is an irreducible affine variety, we can describe regular functions using the *local rings* $\mathscr{O}_{V,p}$ introduced in §1.0. A rational function in $\mathbb{C}(V)$ is contained in the local ring $\mathscr{O}_{V,p}$ precisely when it is regular in a neighborhood of p. It follows that whenever $U \subseteq V$ is open, we have

$$\bigcap_{p \in U} \mathscr{O}_{V,p} = \mathscr{O}_V(U).$$

Thus regular functions on U are rational functions on V that are defined everywhere on U. In particular, when $U = V$, Proposition 3.0.2 implies that

$$\bigcap_{p \in V} \mathscr{O}_{V,p} = \mathscr{O}_V(V) = R = \mathbb{C}[V]. \tag{3.0.2}$$

The Structure Sheaf of an Affine Variety. Given an affine variety V, the mapping

$$U \mapsto \mathscr{O}_V(U), \qquad U \subseteq V \text{ open},$$

has the following useful properties:

- When $U' \subseteq U$, Definition 3.0.1 shows that there is an obvious restriction map

$$\rho_{U,U'} : \mathscr{O}_V(U) \to \mathscr{O}_V(U')$$

  defined by $\rho_{U,U'}(\phi) = \phi|_{U'}$. It follows that $\rho_{U,U}$ is the identity map and that $\rho_{U',U''} \circ \rho_{U,U'} = \rho_{U,U''}$ whenever $U'' \subseteq U' \subseteq U$.
- If $\{U_\alpha\}$ is an open cover of $U \subseteq V$, then the sequence

$$0 \longrightarrow \mathscr{O}_V(U) \longrightarrow \prod_\alpha \mathscr{O}_V(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathscr{O}_V(U_\alpha \cap U_\beta)$$

  is exact. Here, the second arrow is defined by the restrictions ρ_{U,U_α} and the double arrow is defined by $\rho_{U_\alpha,U_\alpha\cap U_\beta}$ and $\rho_{U_\beta,U_\alpha\cap U_\beta}$. Exactness at $\mathscr{O}_V(U)$ means that regular functions are determined locally, i.e., two regular functions on U are equal if their restrictions to all U_α are equal. For the middle term, exactness means that we have an *equalizer*: an element $(f_\alpha) \in \prod_\alpha \mathscr{O}_V(U_\alpha)$ comes from $f \in \mathscr{O}_V(U)$ if and only if the restrictions $f_\alpha|_{U_\alpha\cap U_\beta} = f_\beta|_{U_\alpha\cap U_\beta}$ are equal for all α,β. This is true because regular functions on the U_α agreeing on the overlaps $U_\alpha \cap U_\beta$ patch together to give a regular function on U.

In the language of sheaf theory, these properties imply that $\mathscr{O}_V$ is a *sheaf* of $\mathbb{C}$-algebras, called the *structure sheaf* of V. We call $(V,\mathscr{O}_V)$ a *ringed space over* $\mathbb{C}$. Also, since (3.0.1) holds for all open sets $U \subseteq V_f$, we write

$$\mathscr{O}_V|_{V_f} = \mathscr{O}_{V_f}.$$

In terms of ringed spaces, this means $(V_f, \mathscr{O}_V|_{V_f}) = (V_f, \mathscr{O}_{V_f})$.

Morphisms. By §1.0, a polynomial mapping $\Phi : V_1 \to V_2$ between affine varieties corresponds to the $\mathbb{C}$-algebra homomorphism $\Phi^* : \mathbb{C}[V_2] \to \mathbb{C}[V_1]$ defined by $\Phi^*(\phi) = \phi \circ \Phi$ for $\phi \in \mathbb{C}[V_2]$. We now extend this to open sets of affine varieties.

Definition 3.0.3. Let $U_i \subseteq V_i$ be Zariski open subsets of affine varieties for $i = 1,2$. A function $\Phi : U_1 \to U_2$ is a ***morphism*** if $\phi \mapsto \phi \circ \Phi$ defines a map

$$\Phi^* : \mathscr{O}_{V_2}(U_2) \longrightarrow \mathscr{O}_{V_1}(U_1).$$

Thus $\Phi : U_1 \to U_2$ is a morphism if composing Φ with regular functions on U_2 gives regular functions on U_1. Note also that Φ^* is a $\mathbb{C}$-algebra homomorphism since it comes from composition of functions.

Example 3.0.4. Suppose that $\Phi : V_1 \to V_2$ is a morphism according to Definition 3.0.3. If $V_i = \operatorname{Spec}(R_i)$, then the above map Φ^* gives the $\mathbb{C}$-algebra homomorphism

$$R_2 = \mathscr{O}_{V_2}(V_2) \longrightarrow \mathscr{O}_{V_1}(V_1) = R_1.$$

By Chapter 1, the $\mathbb{C}$-algebra homomorphism $R_2 \to R_1$ gives a map of affine varieties $V_1 \to V_2$. In Exercise 3.0.3 you will show that this is the original map $\Phi : V_1 \to V_2$ we started with. $\Diamond$

Example 3.0.4 shows that when we apply Definition 3.0.3 to maps between affine varieties, we get the same morphisms as in Chapter 1. In Exercise 3.0.3 you will verify the following properties of morphisms:

- If U is open in an affine variety V, then
$$\mathscr{O}_V(U) = \{\phi : U \to \mathbb{C} \mid \phi \text{ is a morphism}\}.$$
Hence regular functions on U are just morphisms from U to $\mathbb{C}$.
- A composition of morphisms is a morphism.
- An inclusion of open sets $W \subseteq U$ of an affine variety V is a morphism.
- Morphisms are continuous in the Zariski topology.

We say that a morphism $\Phi : U_1 \to U_2$ is an *isomorphism* if Φ is bijective and its inverse function $\Phi^{-1} : U_2 \to U_1$ is also a morphism.

Gluing Together Affine Varieties. We now are ready to define abstract varieties by gluing together open subsets of affine varieties. The model is what happens for $\mathbb{P}^n$. Recall from §2.0 of that $\mathbb{P}^n$ is covered by open sets
$$U_i = \mathbb{P}^n \setminus \mathbf{V}(x_i) = \operatorname{Spec}\big(\mathbb{C}\big[\tfrac{x_0}{x_i}, \dots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \dots, \tfrac{x_n}{x_i}\big]\big)$$
for $i = 0, \dots, n$. Each U_i is a copy of $\mathbb{C}^n$ that uses a different set of variables. For $i \neq j$, we "glue together" these copies as follows. We have open subsets
$$(U_i)_{\frac{x_j}{x_i}} \subseteq U_i \quad \text{and} \quad (U_j)_{\frac{x_i}{x_j}} \subseteq U_j, \tag{3.0.3}$$
and we also have the isomorphism
$$g_{ji} : (U_i)_{\frac{x_j}{x_i}} \xrightarrow{\ \sim\ } (U_j)_{\frac{x_i}{x_j}} \tag{3.0.4}$$
since both give the same open set $U_i \cap U_j$ in $\mathbb{P}^n$. The notation g_{ji} was chosen so that $g_{ji}(x)$ means $x \in U_i$ since the index i is closest to x, hence $g_{ji}(x) \in U_j$. At the level of coordinate rings, g_{ji} comes from the isomorphism
$$g_{ji}^* : \mathbb{C}\big[\tfrac{x_0}{x_j}, \dots, \tfrac{x_{j-1}}{x_j}, \tfrac{x_{j+1}}{x_j}, \dots, \tfrac{x_n}{x_j}\big]_{\frac{x_i}{x_j}} \simeq \mathbb{C}\big[\tfrac{x_0}{x_i}, \dots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \dots, \tfrac{x_n}{x_i}\big]_{\frac{x_j}{x_i}}$$
defined by
$$\tfrac{x_k}{x_j} \longmapsto \tfrac{x_k}{x_i}\big/\tfrac{x_j}{x_i} \quad (k \neq j) \quad \text{and} \quad \big(\tfrac{x_i}{x_j}\big)^{-1} \longmapsto \tfrac{x_j}{x_i}.$$

We can turn this around and start from the affine varieties $U_i \simeq \mathbb{C}^n$ given above and glue together the open sets in (3.0.3) using the isomorphisms g_{ji} from (3.0.4). This gluing is consistent since $g_{ij} = g_{ji}^{-1}$ and $g_{ki} = g_{kj} \circ g_{ji}$ wherever all three maps are defined. The result of this gluing is the projective space $\mathbb{P}^n$.

To generalize this, suppose we have a finite collection $\{V_\alpha\}_\alpha$ of affine varieties and for all pairs α, β we have Zariski open sets $V_{\beta\alpha} \subseteq V_\alpha$ and isomorphisms $g_{\beta\alpha} : V_{\beta\alpha} \simeq V_{\alpha\beta}$ satisfying the following compatibility conditions:

- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ for all pairs α, β.
- $g_{\beta\alpha}(V_{\beta\alpha} \cap V_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta}$ and $g_{\gamma\alpha} = g_{\gamma\beta} \circ g_{\beta\alpha}$ on $V_{\beta\alpha} \cap V_{\gamma\alpha}$ for all α, β, γ.

The notation $g_{\beta\alpha}$ means that in the expression $g_{\beta\alpha}(x)$, the point x lies in V_α since α is the index closest to x, and the result $g_{\beta\alpha}(x)$ lies in V_β.

We are now ready to glue. Let Y be the disjoint union of the V_α and define a relation $\sim$ on Y by $a \sim b$ if and only if $a \in V_\alpha$, $b \in V_\beta$ for some α, β with $b = g_{\beta\alpha}(a)$. The first compatibility condition shows that $\sim$ is reflexive and symmetric; the second shows that it is transitive. Hence $\sim$ is an equivalence relation and we can form the quotient space $X = Y/\sim$ with the quotient topology. For each α, let

$$U_\alpha = \{[a] \in X \mid a \in V_\alpha\}.$$

Then $U_\alpha \subseteq X$ is an open set and the map $h_\alpha(a) = [a]$ defines a homeomorphism $h_\alpha : V_\alpha \simeq U_\alpha \subseteq X$. Thus X locally looks like an affine variety.

Definition 3.0.5. We call X the ***abstract variety*** determined by the above data.

An abstract variety X comes equipped with the Zariski topology whose open sets are those sets that restrict to open sets in each U_α. The Zariski closed subsets $Y \subseteq X$ are called *subvarieties* of X. We say that X is *irreducible* if it is not the union of two proper subvarieties. One can show that X is a finite union of irreducible subvarieties $X = Y_1 \cup \cdots \cup Y_s$ such that $Y_i \not\subseteq Y_j$ for $i \neq j$. We call the Y_i the *irreducible components* of X.

Here are some examples of Definition 3.0.5.

Example 3.0.6. We saw above that $\mathbb{P}^n$ can be obtained by gluing together the open sets (3.0.3) using the isomorphisms g_{ij} from (3.0.4). This shows that $\mathbb{P}^n$ is an abstract variety with affine open subsets $U_i \subseteq \mathbb{P}^n$. More generally, given a projective variety $V \subseteq \mathbb{P}^n$, we can cover V with affine open subsets $V \cap U_i$, and the gluing implicit in equation (2.0.8). We conclude that projective varieties are also abstract varieties. $\Diamond$

Example 3.0.7. In a similar way, $\mathbb{P}^n \times \mathbb{C}^m$ can be viewed as gluing affine spaces $U_i \times \mathbb{C}^m \simeq \mathbb{C}^{n+m}$ along suitable open subsets. Thus $\mathbb{P}^n \times \mathbb{C}^m$ is an abstract variety, and the same is true for subvarieties $V \subseteq \mathbb{P}^n \times \mathbb{C}^m$. $\Diamond$

Example 3.0.8. Let $V_0 = \mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[u,v])$ and $V_1 = \mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[w,z])$, with

$$\begin{aligned} V_{10} &= V_0 \setminus \mathbf{V}(v) = \mathrm{Spec}(\mathbb{C}[u,v]_v) \\ V_{01} &= V_1 \setminus \mathbf{V}(z) = \mathrm{Spec}(\mathbb{C}[w,z]_z) \end{aligned}$$

and gluing data

$$\begin{aligned} &g_{10} : V_{10} \to V_{01} \text{ coming from the } \mathbb{C}\text{-algebra homomorphism}\\ &g_{10}^* : \mathbb{C}[w,z]_z \to \mathbb{C}[u,v]_v \text{ defined by } w \mapsto uv \text{ and } z \mapsto 1/v \end{aligned}$$

and

$$\begin{aligned} &g_{01} : V_{01} \to V_{10} \text{ coming from the } \mathbb{C}\text{-algebra homomorphism}\\ &g_{01}^* : \mathbb{C}[u,v]_v \to \mathbb{C}[w,z]_z \text{ defined by } u \mapsto wz \text{ and } v \mapsto 1/z. \end{aligned}$$

One checks that $g_{01} = g_{10}^{-1}$, and the other compatibility condition is satisfied since there are only two V_i. It follows that we get an abstract variety X.

The variety X has another description. Consider the product $\mathbb{P}^1 \times \mathbb{C}^2$ with homogeneous coordinates (x_0, x_1) on $\mathbb{P}^1$ and coordinates (x, y) on $\mathbb{C}^2$. We will identify X with the subvariety $W = \mathbf{V}(x_0 y - x_1 x) \subseteq \mathbb{P}^1 \times \mathbb{C}^2$, called the *blowup of* $\mathbb{C}^2$ *at the origin*, and denoted $\mathrm{Bl}_0(\mathbb{C}^2)$. First note that $\mathbb{P}^1 \times \mathbb{C}^2$ is covered by

$$U_0 \times \mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[x_1/x_0, x, y]) \quad \text{and} \quad U_1 \times \mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[x_0/x_1, x, y]).$$

Then W is covered by $W_0 = W \cap (U_0 \times \mathbb{C}^2)$ and $W_1 = W \cap (U_1 \times \mathbb{C}^2)$. Also,

$$W_0 = \mathbf{V}(y - (x_1/x_0)x) \subseteq U_0 \times \mathbb{C}^2,$$

which gives the coordinate ring

$$\mathbb{C}[x_1/x_0, x, y]/\langle y - (x_1/x_0)x\rangle \simeq \mathbb{C}[x, x_1/x_0] \qquad \text{via } y \mapsto (x_1/x_0)x.$$

Similarly, $W_1 = \mathbf{V}(x - (x_0/x_1)y) \subseteq U_1 \times \mathbb{C}^2$ has coordinate ring

$$\mathbb{C}[x_0/x_1, x, y]/\langle x - (x_0/x_1)y\rangle \simeq \mathbb{C}[y, x_0/x_1] \qquad \text{via } x \mapsto (x_0/x_1)y.$$

You can check that these are glued together in W in exactly the same way V_0 and V_1 are glued together in X. We will generalize this example in Exercise 3.0.8. $\Diamond$

Morphisms Between Abstract Varieties. Let X and Y be abstract varieties with affine open covers $X = \bigcup_\alpha U_\alpha$ and $Y = \bigcup_\beta U'_\beta$. A *morphism* $\Phi : X \to Y$ is a Zariski continuous mapping such that the restrictions

$$\Phi|_{U_\alpha \cap \Phi^{-1}(U'_\beta)} : U_\alpha \cap \Phi^{-1}(U'_\beta) \longrightarrow U'_\beta$$

are morphisms in the sense of Definition 3.0.3.

The Structure Sheaf of an Abstract Variety. Let U be an an open subset of an abstract variety X and set $W_\alpha = h_\alpha^{-1}(U \cap U_\alpha) \subseteq V_\alpha$. Then a function $\phi : U \to \mathbb{C}$ is *regular* if

$$\phi \circ h_\alpha|_{W_\alpha} : W_\alpha \longrightarrow \mathbb{C}$$

is regular for all α. The compatibility conditions ensure that this is well-defined, so that one can define

$$\mathscr{O}_X(U) = \{\phi : U \to \mathbb{C} \mid \phi \text{ is regular}\}.$$

This gives the *structure sheaf* $\mathscr{O}_X$ of X. Thus an abstract variety is really a ringed space $(X, \mathscr{O}_X)$ with a finite open covering $\{U_\alpha\}_\alpha$ such that $(U_\alpha, \mathscr{O}_X|_{U_\alpha})$ is isomorphic to the ringed space $(V_\alpha, \mathscr{O}_{V_\alpha})$ of the affine variety V_α. (We leave the definition of isomorphism of ringed spaces to the reader.)

Open and Closed Subvarieties. Given an abstract variety X and an open subset U, we note that U has a natural structure of an abstract variety. For an affine open subset $U_\alpha \subseteq X$, $U \cap U_\alpha$ is open in U_α and hence can be written as a union $U \cap U_\alpha = \bigcup_{f \in S} (U_\alpha)_f$ for a finite subset $S \subseteq \mathbb{C}[U_\alpha]$. It follows that U is covered by finitely many affine open subsets and thus is an abstract variety. The structure sheaf $\mathscr{O}_U$ is simply the restriction of $\mathscr{O}_X$ to U, i.e., $\mathscr{O}_U = \mathscr{O}_X|_U$. Note also that a function $\phi : U \to \mathbb{C}$ is regular if and only if ϕ is a morphism as defined above.

In a similar way, a closed subset $Y \subseteq X$ also gives an abstract variety. For an affine open set $U \subseteq X$, $Y \cap U$ is closed in U and hence is an affine variety. Thus Y is covered by finitely many affine open subsets and thus is an abstract variety. This justifies the term "subvariety" for closed subsets of an abstract variety. The structure sheaf $\mathscr{O}_Y$ is related to $\mathscr{O}_X$ as follows. The inclusion $i : Y \hookrightarrow X$ is a morphism. Let $i_*\mathscr{O}_Y$ be the sheaf on X defined by $i_*\mathscr{O}_Y(U) = \mathscr{O}_Y(U \cap Y)$. Restricting functions on X to functions on Y gives a map of sheaves $\mathscr{O}_X \to i_*\mathscr{O}_Y$ whose kernel is the subsheaf $\mathscr{I}_Y \subseteq \mathscr{O}_X$ of functions vanishing on Y, meaning

$$\mathscr{I}_Y(U) = \{ f \in \mathscr{O}_X(U) \mid f(p) = 0 \text{ for all } p \in Y \cap U\}.$$

In the language of Chapter 6, we have an exact sequence of sheaves

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{O}_X \longrightarrow i_*\mathscr{O}_Y \longrightarrow 0.$$

All of the types of "variety" introduced so far can be subsumed under the concept of "abstract variety." From now on, we will usually be thinking of abstract varieties. Hence we will usually say "variety" rather than "abstract variety."

Local Rings and Rational Functions. Let p be a point of an affine variety V. Elements of the local ring $\mathscr{O}_{V,p}$ are quotients f/g in a suitable localization with $f, g \in \mathbb{C}[V]$ and $g(p) \neq 0$. It follows that V_g is a neighborhood of p in V and f/g is a regular function on V_g. In this way, we can think of elements of $\mathscr{O}_{V,p}$ as regular functions defined in a neighborhood of p.

This idea extends to the abstract case. Given a point p of a variety X and neighborhoods U_1, U_2 of p, regular functions $f_i : U_i \to \mathbb{C}$ are *equivalent at p*, written $f_1 \sim f_2$, if there is a neighbhorhood $p \in U \subseteq U_1 \cap U_2$ such that $f_1|_U = f_2|_U$.

Definition 3.0.9. Let p be a point of a variety X. Then

$$\mathscr{O}_{X,p} = \{ f : U \to \mathbb{C} \mid U \text{ is a neighborhood of } p \text{ in } X\}/\sim$$

is the ***local ring of*** X ***at*** p.

Every $\phi \in \mathscr{O}_{X,p}$ has a well-defined value $\phi(p)$. It is not difficult to see that $\mathscr{O}_{X,p}$ is a local ring with unique maximal ideal

$$\mathfrak{m}_{X,p} = \{\phi \in \mathscr{O}_{X,p} \mid \phi(p) = 0\}.$$

The local ring $\mathscr{O}_{X,p}$ can also be defined as the direct limit

$$\mathscr{O}_{X,p} = \varinjlim_{p \in U} \mathscr{O}_X(U)$$

over all neighborhoods of p in X (see Definition 6.0.1).

When X is irreducible, we can also define the field of rational functions $\mathbb{C}(X)$. A *rational function* on X is a regular function $f : U \to \mathbb{C}$ defined on a nonempty Zariski open set $U \subseteq X$, and two rational functions on X are *equivalent* if they agree on a nonempty Zariski open subset. In Exercise 3.0.4 you will show that this relation is an equivalence relation and that the set of equivalence classes is a field, called the *function field* of X, denoted $\mathbb{C}(X)$.

Normal Varieties. We return to the notion of normality introduced in Chapter 1.

Definition 3.0.10. A variety X is called ***normal*** if it is irreducible and the local rings $\mathscr{O}_{X,p}$ are normal for all $p \in X$.

At first glance, this looks different from the definition given for affine varieties in Definition 1.0.3. In fact, the two notions are equivalent in the affine case.

Proposition 3.0.11. *Let V be an irreducible affine variety. Then $\mathbb{C}[V]$ is normal if and only if the local rings $\mathscr{O}_{V,p}$ are normal for all $p \in V$.*

Proof. If $\mathscr{O}_{V,p}$ is normal for all p, then (3.0.2) shows that $\mathbb{C}[V]$ is an intersection of normal domains, all of which have the same field of fractions. Since such an intersection is normal by Exercise 1.0.7, it follows that $\mathbb{C}[V]$ is normal.

For the converse, suppose that $\mathbb{C}[V]$ is normal and let $\alpha \in \mathbb{C}(V)$ satisfy

$$\alpha^k + a_1\alpha^{k-1} + \cdots + a_k = 0, \qquad a_i \in \mathscr{O}_{V,p}.$$

Write $a_i = g_i/f_i$ with $g_i, f_i \in \mathbb{C}[V]$ and $f_i(p) \neq 0$. The product $f = f_1 \cdots f_k$ has the properties that $a_i \in \mathbb{C}[V]_f$ and $f(p) \neq 0$. The localization $\mathbb{C}[V]_f$ is normal by Exercise 1.0.7 and is contained in $\mathscr{O}_{V,p}$ since $f(p) \neq 0$. Hence $\alpha \in \mathbb{C}[V]_f \subseteq \mathscr{O}_{V,p}$. This completes the proof. □

Here is a consequence of Proposition 3.0.11 and Definition 3.0.10.

Proposition 3.0.12. *Let X be an irreducible variety with a cover consisting of affine open sets V_α. Then X is normal if and only if each V_α is normal.* □

Smooth Varieties. For an affine variety V, the definition of a *smooth point* $p \in V$ (Definition 1.0.7) used $T_p(V)$, the Zariski tangent space of V at p, and $\dim_p V$, the maximum dimension of an irreducible component of V containing p. You will show in Exercise 3.0.2 that $T_p(X)$ and $\dim_p X$ are well-defined for a point $p \in X$ of a general variety.

Definition 3.0.13. Let X be a variety. A point $p \in X$ is ***smooth*** if $\dim T_p(X) = \dim_p X$, and X is ***smooth*** if every point of X is smooth.

Products of Varieties. As another example of abstract varieties and gluing, we indicate why the product $X_1 \times X_2$ of varieties X_1 and X_2 also has the structure of a variety. In §1.0 we constructed the product of affine varieties. From here, it is relatively routine to see that if X_1 is obtained by gluing together affine varieties U_α and X_2 is obtained by gluing together affines U'_β, then $X_1 \times X_2$ is obtained by gluing together the $U_\alpha \times U'_\beta$ in the corresponding fashion. Furthermore, $X_1 \times X_2$ has the correct universal mapping property. Namely, given a diagram

$$\begin{tikzcd}
W \arrow[dr, dotted, "\nu"] \arrow[drr, bend left, "\phi_1"] \arrow[ddr, bend right, "\phi_2"'] & & \\
& X_1 \times X_2 \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & X_1 \\
& X_2 &
\end{tikzcd}$$

where $\phi_i : W \to X_i$ are morphisms, there is a unique morphism $\nu : W \to X_1 \times X_2$ (the dotted arrow) that makes the diagram commute.

Example 3.0.14. Let us construct the product $\mathbb{P}^1 \times \mathbb{C}^2$. Write $\mathbb{P}^1 = V_0 \cup V_1$ where $V_0 = \mathrm{Spec}(\mathbb{C}[u])$ and $V_1 = \mathrm{Spec}(\mathbb{C}[v])$, with the gluing given by

$$\mathbb{C}[v]_v \simeq \mathbb{C}[u]_u, \qquad v \mapsto 1/u.$$

Then $\mathbb{P}^1 \times \mathbb{C}^2$ is constructed from

$$\begin{aligned} U_0 \times \mathbb{C}^2 &= \mathrm{Spec}(\mathbb{C}[u] \otimes_{\mathbb{C}} \mathbb{C}[x,y]) \simeq \mathbb{C}^3 \\ U_1 \times \mathbb{C}^2 &= \mathrm{Spec}(\mathbb{C}[v] \otimes_{\mathbb{C}} \mathbb{C}[x,y]) \simeq \mathbb{C}^3, \end{aligned}$$

with gluing given by

$$(U_0 \times \mathbb{C}^2)_u \simeq (U_1 \times \mathbb{C}^2)_v$$

corresponding to the obvious isomorphism of coordinate rings. $\Diamond$

Separated Varieties. From the point of view of the classical topology, arbitrary gluings can lead to varieties with some strange properties.

Example 3.0.15. In Example 3.0.14 we saw how to construct $\mathbb{P}^1$ from affine varieties $V_0 = \mathrm{Spec}(\mathbb{C}[u]) \simeq \mathbb{C}$ and $V_1 = \mathrm{Spec}(\mathbb{C}[v]) \simeq \mathbb{C}$ with the gluing given by $v \mapsto 1/u$ on open sets $\mathbb{C}^* \simeq (V_0)_u \subseteq V_0$ and $\mathbb{C}^* \simeq (V_1)_v \subseteq V_1$. This expresses $\mathbb{P}^1$ as consisting of $\mathbb{C}^*$ plus two additional points. But now consider the abstract variety arising from the gluing map

$$(V_0)_u \longrightarrow (V_1)_v$$

that corresponds to the map of $\mathbb{C}$-algebras defined by $v \mapsto u$. As before, the glued variety X consists of $\mathbb{C}^*$ together with two additional points. However here we have a morphism $\pi : X \to \mathbb{C}$ whose fiber $\pi^{-1}(a)$ over $a \in \mathbb{C}^*$ contains one point, but whose fiber over 0 consists of two points, p_1 corresponding to $0 \in V_0$ and p_2 corresponding to $0 \in V_1$. If U_1, U_2 are classical open sets in X with $p_1 \in U_1$ and $p_2 \in U_2$, then $U_1 \cap U_2 \neq \emptyset$. So the *classical* topology on X is not Hausdorff. $\Diamond$

Since varieties are rarely Hausdorff in the Zariski topology (Exercise 3.0.5), we need a different way to think about Example 3.0.15. Consider the product $X \times X$ and the *diagonal mapping* $\Delta : X \to X \times X$ defined by $\Delta(p) = (p, p)$ for $p \in X$. For X from Example 3.0.15, there is a morphism $X \times X \to \mathbb{C}$ whose fiber over over 0 consists of the four points (p_i, p_j). Any Zariski closed subset of $X \times X$ containing one of these four points must contain all of them. The image of the diagonal mapping contains (p_1, p_1) and (p_2, p_2), but not the other two, so the diagonal is not Zariski closed. This example motivates the following definition.

Definition 3.0.16. We say a variety X is ***separated*** if the image of the diagonal map $\Delta : X \to X \times X$ is Zariski closed in $X \times X$.

For instance, $\mathbb{C}^n$ is separated because the image of the diagonal in $\mathbb{C}^n \times \mathbb{C}^n = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n])$ is the affine variety $\mathbf{V}(x_1 - y_1, \ldots, x_n - y_n)$. Similarly, any affine variety is separated.

The connection between failure of separatedness and failure of the Hausdorff property in the classical topology seen in Example 3.0.15 is a general phenomenon.

Theorem 3.0.17. *A variety is separated if and only if it is Hausdorff in the classical topology.* $\square$

Here are some additional properties of separated varieties (Exercise 3.0.6).

Proposition 3.0.18. *Let X be a separated variety. Then:*

(a) *If $f, g : Y \to X$ are morphisms, then $\{y \in Y \mid f(y) = g(y)\}$ is Zariski closed in Y.*

(b) *If U, V are affine open subsets of X, then $U \cap V$ is also affine.* $\square$

The requirement that X be separated is often included in the *definition* of an abstract variety. When this is done, what we have called a variety is sometimes called a *pre-variety*.

Fiber Products. Finally in this section, we will define fiber products of varieties, a construction required for the discussion of proper morphisms in §3.4. First, if we have mappings of sets $f : X \to S$ and $g : Y \to S$, then the *fiber product* $X \times_S Y$ is defined to be

$$X \times_S Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}. \tag{3.0.5}$$

The fiber product construction gives a very flexible language for describing ordinary products, intersections of subsets, fibers of mappings, the set where two mappings agree, and so forth:

- If S is a point, then $X \times_S Y$ is the ordinary product $X \times Y$.
- If X, Y are subsets of S and f, g are the inclusions, then $X \times_S Y \simeq X \cap Y$.
- If $Y = \{s\} \subseteq S$, then $X \times_S Y \simeq f^{-1}(s)$.

The third property is the reason for the name. All are easy exercises that we leave to the reader.

In analogy with the universal mapping property of the product discussed above, the fiber product has the following universal property. Whenever we have mappings $\phi_1 : W \to X$ and $\phi_2 : W \to Y$ such that $f \circ \phi_1 = g \circ \phi_2$, there is a unique $\nu : W \to X \times_S Y$ that makes the following diagram commute:

$$\begin{tikzcd}
W \arrow[dr, dotted, "\nu"] \arrow[drr, bend left, "\phi_1"] \arrow[ddr, bend right, "\phi_2"'] & & \\
& X \times_S Y \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & S.
\end{tikzcd}$$

Equation (3.0.5) defines $X \times_S Y$ as a set. To prove that $X \times_S Y$ is a variety, we assume for simplicity that S is separated. Then $f : X \to S$ and $g : Y \to S$ give a morphism $(f,g) : X \times Y \to S \times S$, and one easily checks that

$$X \times_S Y = (f,g)^{-1}(\Delta(S)),$$

where $\Delta(S) \subseteq S \times S$ is the diagonal. This is closed in $S \times S$ since S is separated, and it follows that $X \times_S Y$ is closed in $X \times Y$ and hence has a natural structure as a variety. From here, it is straightforward to show that $X \times_S Y$ has the desired universal mapping property. Proving that $X \times_S Y$ is a variety when S is not separated takes more work and will not be discussed here.

In the affine case, we can also describe the coordinate ring of $X \times_S Y$. Let $X = \operatorname{Spec}(R_1)$, $Y = \operatorname{Spec}(R_2)$, and $S = \operatorname{Spec}(R)$. The morphisms f, g correspond to ring homomorphisms $f^* : R \to R_1$, $g^* : R \to R_2$. Hence both R_1, R_2 have the structure of R-modules, and we have the tensor product $R_1 \otimes_R R_2$. This is also a finitely generated $\mathbb{C}$-algebra, though it may have nilpotents (Exercise 3.0.9). To get a coordinate ring, we need to take the quotient by the ideal N of all nilpotents.

Then one can prove that

$$X \times_S Y = \mathrm{Spec}(R_1 \otimes_R R_2/N).$$

We can avoid worrying about nilpotents by constructing $X \times_S Y$ as the *affine scheme* $\mathrm{Spec}(R_1 \otimes_R R_2)$. Interested readers can learn about the construction of fiber products as schemes in [**90**, I.3.1] and [**131**, pp. 87–89].

Exercises for §3.0.

3.0.1. Let $V = \mathrm{Spec}(R)$ be an affine variety.

(a) Show that every ideal $I \subseteq R$ can be written in the form $I = \langle f_1, \dots, f_s \rangle$, where $f_i \in R$. (This is the Hilbert basis theorem in R.)

(b) Let $W \subseteq V$ be a subvariety. Show that the complement of W in V can be written as a union of a finite collection of open affine sets of the form V_f.

(c) Deduce that every open cover of V (in the Zariski topology) has a finite subcover. (This says that affine varieties are *quasicompact* in the Zariski topology.)

3.0.2. As in the affine case, we want to say a variety X is smooth at p if $\dim T_p(X) = \dim_p X$. In this exercise, you will show that this is a well-defined notion.

(a) Show that if $p \in X$ is in the intersection of two affine open sets $V_\alpha \cap V_\beta$, then the Zariski tangent spaces $T_{V_\alpha, p}$ and $T_{V_\beta, p}$ are isomorphic as vector spaces over $\mathbb{C}$.

(b) Show that $\dim_p X$ is a well-defined integer.

(c) Deduce that the proposed notion of smoothness at p is well-defined.

3.0.3. This exercise explores some properties of the morphisms defined in Definition 3.0.3.

(a) Prove the claim made in Example 3.0.4. Hint: Take a point $p \in V_1$ and define $\mathfrak{m}_p = \{f \in R_1 \mid f(p) = 0\}$. Then describe $(\Phi^*)^{-1}(\mathfrak{m}_p)$ in terms of $\Phi(p)$.

(b) Prove the properties of morphisms listed on page 96.

3.0.4. Let X be an irreducible abstract variety.

(a) Let f, g be rational functions on X. Show that $f \sim g$ if $f|_U = g|_U$ for some nonempty open set $U \subseteq X$ is an equivalence relation.

(b) Show that the set of equivalence classes of the relation in part (a) is a field.

(c) Show that if $U \subseteq X$ is a nonempty open subset of X, then $\mathbb{C}(U) \simeq \mathbb{C}(X)$.

3.0.5. Show that a variety is Hausdorff in the Zariski topology if and only if it consists of finitely many points.

3.0.6. Consider Proposition 3.0.18.

(a) Prove part (a) of the proposition. Hint: Show first that if $F : Y \to X \times X$ is defined by $F(y) = (f(y), g(y))$, then $Z = F^{-1}(\Delta(X))$.

(b) Prove part (b) of the proposition. Hint: Show first that $U \cap V$ can be identified with $\Delta(X) \cap (U \times V) \subseteq X \times X$.

3.0.7. Let $V = \mathrm{Spec}(R)$ be an affine variety. The diagonal mapping $\Delta : V \to V \times V$ corresponds to a $\mathbb{C}$-algebra homomorphism $R \otimes_{\mathbb{C}} R \to R$. Which one? Hint: Consider the universal mapping property of $V \times V$.

3.0.8. In this exercise, we will study an important variety in $\mathbb{P}^{n-1} \times \mathbb{C}^n$, the *blowup* of $\mathbb{C}^n$ at the origin, denoted $\mathrm{Bl}_0(\mathbb{C}^n)$. This generalizes $\mathrm{Bl}_0(\mathbb{C}^2)$ from Example 3.0.8. Write the homogeneous coordinates on $\mathbb{P}^{n-1}$ as $x_0, \ldots, x_{n-1}$, and the affine coordinates on $\mathbb{C}^n$ as $y_1, \ldots, y_n$. Let

$$W = \mathrm{Bl}_0(\mathbb{C}^n) = \mathbf{V}(x_{i-1}y_j - x_{j-1}y_i \mid 1 \le i < j \le n) \subseteq \mathbb{P}^{n-1} \times \mathbb{C}^n. \tag{3.0.6}$$

Let U_{i-1}, $i = 1, \ldots, n$, be the standard affine opens in $\mathbb{P}^{n-1}$:

$$U_{i-1} = \mathbb{P}^{n-1} \setminus \mathbf{V}(x_{i-1}),$$

$i = 1, \ldots, n$ (note the slightly non-standard indexing). So the $U_{i-1} \times \mathbb{C}^n$ form a cover of $\mathbb{P}^{n-1} \times \mathbb{C}^n$.

(a) Show that for each $i = 1, \ldots, n$, $W_{i-1} = W \cap (U_{i-1} \times \mathbb{C}^n) \simeq$

$$\mathrm{Spec}\left(\mathbb{C}\left[\frac{x_0}{x_{i-1}}, \ldots, \frac{x_{i-2}}{x_{i-1}}, \frac{x_i}{x_{i-1}}, \ldots, \frac{x_{n-1}}{x_{i-1}}, y_i\right]\right)$$

using the equations (3.0.6) defining W.

(b) Give the gluing data for identifying the subsets $W_{i-1} \setminus \mathbf{V}(x_{j-1})$ and $W_{j-1} \setminus \mathbf{V}(x_{i-1})$.

3.0.9. Let $V = \mathbf{V}(y^2 - x) \subseteq \mathbb{C}^2$ and consider the morphism $\pi : V \to \mathbb{C}$ given by projection onto the x-axis. We will study the fibers of π.

(a) As noted in the text, the fiber $\pi^{-1}(0) = \{(0,0)\}$ can be represented as the fiber product $\{0\} \times_{\mathbb{C}} V$. In terms of coordinate rings, we have $\{0\} = \mathrm{Spec}(\mathbb{C}[x]/\langle x\rangle)$, $\mathbb{C} = \mathrm{Spec}(\mathbb{C}[x])$ and $V = \mathrm{Spec}(\mathbb{C}[x,y]/\langle y^2 - x\rangle$. Prove that

$$\mathbb{C}[x]/\langle x\rangle \otimes_{\mathbb{C}[x]} \mathbb{C}[x,y]/\langle y^2 - x\rangle \simeq \mathbb{C}[y]/\langle y^2\rangle.$$

Thus, the coordinate rings $\mathbb{C}[x]/\langle x\rangle$, $\mathbb{C}[x]$ and $\mathbb{C}[x,y]/\langle y^2 - x\rangle$ lead to a tensor product that has nilpotents and hence cannot be a coordinate ring.

(b) If $a \neq 0$ in $\mathbb{C}$, then $\pi^{-1}(a) = \{(a, \pm\sqrt{a})\}$. Show that the analogous tensor product is

$$\begin{aligned}\mathbb{C}[x]/\langle x - a\rangle \otimes_{\mathbb{C}[x]} \mathbb{C}[x,y]/\langle y^2 - x\rangle &\simeq \mathbb{C}[y]/\langle y^2 - a\rangle \\ &\simeq \mathbb{C}[y]/\langle y - \sqrt{a}\rangle \oplus \mathbb{C}[y]/\langle y + \sqrt{a}\rangle.\end{aligned}$$

This has no nilpotents and hence is the coordinate ring of $\pi^{-1}(a)$.

What happens in part (a) is that the two square roots coincide, so that we get only one point with "multiplicity 2." The multiplicity information is recorded in the affine scheme $\mathrm{Spec}(\mathbb{C}[y]/\langle y^2\rangle)$. This is an example of the power of schemes.

§3.1. Fans and Normal Toric Varieties

In this section we construct the toric variety X_Σ corresponding to a fan Σ. We will also relate the varieties X_Σ to many of the examples encountered previously, and we will see how properties of the fan correspond to properties such as smoothness and compactness of X_Σ.

The Toric Variety of a Fan. A toric variety continues to mean the same thing as in Chapters 1 and 2, although we now allow abstract varieties as in §3.0.

Definition 3.1.1. A ***toric variety*** is an irreducible variety X containing a torus $T_N \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to an algebraic action of T_N on X. (By algebraic action, we mean an action $T_N \times X \to X$ given by a morphism.)

The other ingredient in this section is a fan in the vector space $N_{\mathbb{R}}$.

Definition 3.1.2. A ***fan*** Σ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that:

(a) Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.

(b) For all $\sigma \in \Sigma$, each face of σ is also in Σ.

(c) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each (hence also in Σ).

Furthermore, if Σ is a fan, then:

- The ***support*** of Σ is $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_{\mathbb{R}}$.
- $\Sigma(r)$ is the set of r-dimensional cones of Σ.

We have already seen some examples of fans. Theorem 2.3.2 shows that the normal fan Σ_P of a full dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$ is a fan in the sense of Definition 3.1.2. However, there exist fans that are not equal to the normal fan of any lattice polytope. An example of such a fan will be given in Example 4.2.13.

We now show how the cones in any fan give the combinatorial data necessary to glue a collection of affine toric varieties together to yield an abstract toric variety. By Theorem 1.2.18, each cone σ in Σ gives the affine toric variety

$$U_\sigma = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_\sigma]) = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M]).$$

Recall from Definition 1.2.5 that a face $\tau \preceq \sigma$ is given by $\tau = \sigma \cap H_m$, where $m \in \sigma^\vee$ and $H_m = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0\}$ is the hyperplane defined by m. In Chapter 1, we proved two useful facts:

First, Proposition 1.3.16 used the equality

$$\mathsf{S}_\tau = \mathsf{S}_\sigma + \mathbb{Z}(-m) \tag{3.1.1}$$

to show that $\mathbb{C}[\mathsf{S}_\tau]$ is the localization $\mathbb{C}[\mathsf{S}_\sigma]_{\chi^m}$. Thus $U_\tau = (U_\sigma)_{\chi^m}$ when $\tau \preceq \sigma$.

Second, if $\tau = \sigma_1 \cap \sigma_2$, then Lemma 1.2.13 implies that

$$\sigma_1 \cap H_m = \tau = \sigma_2 \cap H_m, \tag{3.1.2}$$

for some $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$. This shows that

$$U_{\sigma_1} \supseteq (U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}} \subseteq U_{\sigma_2}. \tag{3.1.3}$$

The following proposition gives an additional property of the S_σ and their semigroup rings that we will need.

Proposition 3.1.3. *If $\sigma_1, \sigma_2 \in \Sigma$ and $\tau = \sigma_1 \cap \sigma_2$, then*
$$\mathsf{S}_\tau = \mathsf{S}_{\sigma_1} + \mathsf{S}_{\sigma_2}.$$

Proof. The inclusion $\mathsf{S}_{\sigma_1} + \mathsf{S}_{\sigma_2} \subseteq \mathsf{S}_\tau$ follows directly from the general fact that $\sigma_1^\vee + \sigma_2^\vee = (\sigma_1 \cap \sigma_2)^\vee = \tau^\vee$. For the reverse inclusion, take $p \in \mathsf{S}_\tau$ and assume that $m \in \sigma_1^\vee \cap (-\sigma_2)^\vee \cap M$ satisfies (3.1.2). Then (3.1.1) applied to σ_1 gives $p = q + \ell(-m)$ for some $q \in \mathsf{S}_{\sigma_1}$ and $\ell \in \mathbb{N}$. But $-m \in \sigma_2^\vee$ implies $-m \in \mathsf{S}_{\sigma_2}$, so that $p \in \mathsf{S}_{\sigma_1} + \mathsf{S}_{\sigma_2}$. $\square$

This result is sometimes called the *separation lemma* and is a key ingredient in showing that the toric varieties X_Σ are separated in the sense of Definition 3.0.16.

Example 3.1.4. Let $\sigma_1 = \mathrm{Cone}(e_1 + e_2, e_2)$ (as in Exercise 1.2.11), and let $\sigma_2 = \mathrm{Cone}(e_1, e_1 + e_2)$ in $N_\mathbb{R} = \mathbb{R}^2$. Then $\tau = \sigma_1 \cap \sigma_2 = \mathrm{Cone}(e_1 + e_2)$. We show the dual cones $\sigma_1^\vee = \mathrm{Cone}(e_1, -e_1 + e_2)$, $\sigma_2^\vee = \mathrm{Cone}(e_1 - e_2, e_2)$, and $\tau^\vee = \sigma_1^\vee + \sigma_2^\vee$ in Figure 1.

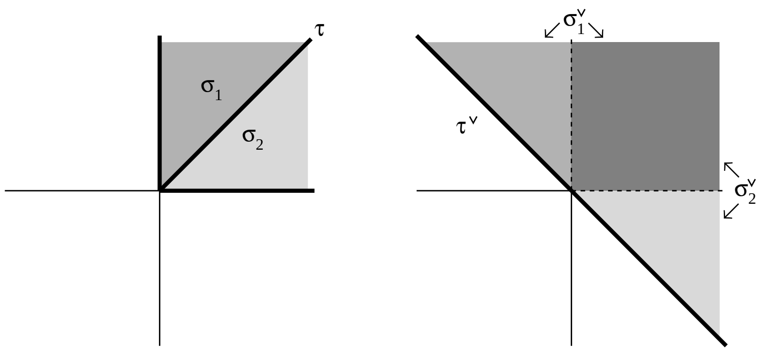


Figure 1. The cones σ_1, σ_2, τ and their duals

The dark shaded region on the right is $\sigma_1^\vee \cap \sigma_2^\vee$. Note $\tau = \sigma_1 \cap H_m = \sigma_2 \cap H_{-m}$, where $m = -e_1 + e_2 \in \sigma_1^\vee$ and $-m = e_1 - e_2 \in \sigma_2^\vee$. Since S_τ is the set of all sums $m + m'$ with $m \in \sigma_1^\vee \cap M$ and $m' \in \sigma_2^\vee \cap M$, we see that $\mathsf{S}_\tau = \mathsf{S}_{\sigma_1} + \mathsf{S}_{\sigma_2}$. $\Diamond$

Now consider the collection of affine toric varieties $U_\sigma = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_\sigma])$, where σ runs over all cones in a fan Σ. Let σ_1 and σ_2 be any two of these cones and let $\tau = \sigma_1 \cap \sigma_2$. By (3.1.3), we have an isomorphism
$$g_{\sigma_2,\sigma_1} : (U_{\sigma_1})_{\chi^m} \simeq (U_{\sigma_2})_{\chi^{-m}}$$
which is the identity on U_τ. By Exercise 3.1.1, the compatibility conditions as in §3.0 for gluing the affine varieties U_σ along the subvarieties $(U_\sigma)_{\chi^m}$ are satisfied. Hence we obtain an abstract variety X_Σ associated to the fan Σ.

Theorem 3.1.5. *Let Σ be a fan in $N_\mathbb{R}$. The variety X_Σ is a normal separated toric variety.*

Proof. Since each cone in Σ is strongly convex, $\{0\} \subseteq N$ is a face of all $\sigma \in \Sigma$. Hence we have $T_N = \operatorname{Spec}(\mathbb{C}[M]) \simeq (\mathbb{C}^*)^n \subseteq U_\sigma$ for all σ. These tori are all identified by the gluing, so we have $T_N \subseteq X_\Sigma$. We know from Chapter 1 that each U_σ has an action of T_N. The gluing isomorphism g_{σ_2,σ_1} reduces to the identity mapping on $\mathbb{C}[\mathsf{S}_{\sigma_1\cap\sigma_2}]$. Hence the actions are compatible on the intersections of every pair of sets in the open affine cover, and patch together to give an algebraic action of T_N on X_Σ.

The variety X_Σ is irreducible because all of the U_σ are irreducible affine toric varieties containing the torus T_N. Furthermore, U_σ is a normal affine variety by Theorem 1.3.5. Hence the variety X_Σ is normal by Proposition 3.0.12.

To see that X_Σ is separated it suffices to show that for each pair of cones σ_1, σ_2 in Σ, the image of the diagonal map

$$\Delta : U_\tau \to U_{\sigma_1} \times U_{\sigma_2}, \quad \tau = \sigma_1 \cap \sigma_2$$

is Zariski closed (Exercise 3.1.2). But Δ comes from the $\mathbb{C}$-algebra homomorphism

$$\Delta^* : \mathbb{C}[\mathsf{S}_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[\mathsf{S}_{\sigma_2}] \longrightarrow \mathbb{C}[\mathsf{S}_\tau]$$

defined by $\chi^m \otimes \chi^n \mapsto \chi^{m+n}$. By Proposition 3.1.3, Δ^* is surjective, so that

$$\mathbb{C}[\mathsf{S}_\tau] \simeq (\mathbb{C}[\mathsf{S}_{\sigma_1}] \otimes_{\mathbb{C}} \mathbb{C}[\mathsf{S}_{\sigma_2}])/\ker(\Delta^*).$$

Hence the image of Δ is a Zariski closed subset of $U_{\sigma_1} \times U_{\sigma_2}$. $\square$

Toric varieties were originally known as *torus embeddings*, and the variety X_Σ would be written $T_N\mathrm{emb}(\Sigma)$ in older references such as [**218**]. Other commonly used notations are $X(\Sigma)$, or $X(\Delta)$, if the fan is denoted by Δ. When we want to emphasize the dependence on the lattice N, we will write X_Σ as $X_{\Sigma,N}$.

Many of the toric varieties encountered in Chapters 1 and 2 come from fans. For example, Theorem 1.3.5 implies that a normal affine toric variety comes from a fan consisting of a single cone σ together with all of its faces. Furthermore, the projective toric variety associated to a lattice polytope in Chapter 2 comes from a fan. Here is the precise result.

Proposition 3.1.6. *Let $P \subseteq M_\mathbb{R}$ be a full dimensional lattice polytope. Then the projective toric variety $X_P \simeq X_{\Sigma_P}$, where Σ_P is the normal fan of P.*

Proof. When P is very ample, this follows immediately from the description of the intersections of the affine open pieces of X_P in Proposition 2.3.13 and the definition of the normal fan Σ_P. The general case follows since the normal fans of P and kP are the same for all positive integers k. $\square$

In general, every separated normal toric variety comes from a fan. This is a consequence of a theorem of Sumihiro from [**265**].

Theorem 3.1.7 (Sumihiro)**.** *Let the torus T_N act on a normal separated variety X. Then every point $p \in X$ has a T_N-invariant affine open neighborhood.* □

Corollary 3.1.8. *Let X be a normal separated toric variety with torus T_N. Then there exists a fan Σ in $N_{\mathbb{R}}$ such that $X \simeq X_\Sigma$.*

Proof. The proof will be sketched in Exercise 3.2.11 after we have developed the properties of T_N-orbits on toric varieties. □

Examples. We now turn to some concrete examples. Many of these are toric varieties already encountered in previous chapters.

Example 3.1.9. Consider the fan Σ in $N_{\mathbb{R}} = \mathbb{R}^2$ in Figure 2, where $N = \mathbb{Z}^2$ has standard basis e_1, e_2. This is the normal fan of the simplex Δ_2 as in Example 2.3.4. Here we show all points in the cones inside a rectangular viewing box (all figures of fans in the plane in this chapter will be drawn using the same convention.)

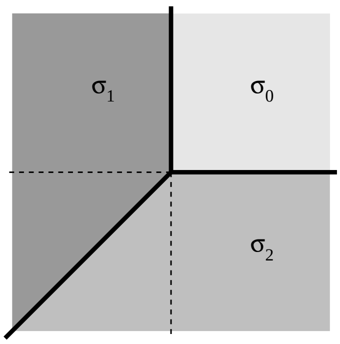


Figure 2. The fan Σ for $\mathbb{P}^2$

From the discussion in Chapter 2, we expect $X_\Sigma \simeq \mathbb{P}^2$, and we will show this in detail. The fan Σ has three 2-dimensional cones $\sigma_0 = \mathrm{Cone}(e_1, e_2)$, $\sigma_1 = \mathrm{Cone}(-e_1 - e_2, e_2)$, and $\sigma_2 = \mathrm{Cone}(e_1, -e_1 - e_2)$, together with the three rays $\tau_{ij} = \sigma_i \cap \sigma_j$ for $i \neq j$, and the origin. The toric variety X_Σ is covered by the affine opens

$$\begin{aligned} U_{\sigma_0} &= \mathrm{Spec}(\mathbb{C}[S_{\sigma_0}]) \simeq \mathrm{Spec}(\mathbb{C}[x,y]) \\ U_{\sigma_1} &= \mathrm{Spec}(\mathbb{C}[S_{\sigma_1}]) \simeq \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-1}y]) \\ U_{\sigma_2} &= \mathrm{Spec}(\mathbb{C}[S_{\sigma_2}]) \simeq \mathrm{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]). \end{aligned}$$

Moreover, by Proposition 3.1.3, the gluing data on the coordinate rings is given by

$$\begin{aligned} g_{10}^* &: \mathbb{C}[x,y]_x \simeq \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}} \\ g_{20}^* &: \mathbb{C}[x,y]_y \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}} \\ g_{21}^* &: \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}y} \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{xy^{-1}}. \end{aligned}$$

It is easy to see that if we use the usual homogeneous coordinates (x_0, x_1, x_2) on $\mathbb{P}^2$, then $x \mapsto \frac{x_1}{x_0}$ and $y \mapsto \frac{x_2}{x_0}$ identifies the standard affine open $U_i \subseteq \mathbb{P}^2$ with $U_{\sigma_i} \subseteq X_\Sigma$. Hence we have recovered $\mathbb{P}^2$ as the toric variety X_Σ. $\Diamond$

Example 3.1.10. Generalizing Example 3.1.9, let $N_\mathbb{R} = \mathbb{R}^n$, where $N = \mathbb{Z}^n$ has standard basis $e_1, \ldots, e_n$. Set

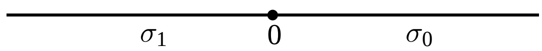
$$e_0 = -e_1 - e_2 - \cdots - e_n$$

and let Σ be the fan in $N_\mathbb{R}$ consisting of the cones generated by all proper subsets of $\{e_0, \ldots, e_n\}$. This is the normal fan of the n-simplex Δ_n, and $X_\Sigma \simeq \mathbb{P}^n$ by Example 2.3.15 and Exercise 2.3.6. You will check the details to verify that this gives the usual affine open cover of $\mathbb{P}^n$ in Exercise 3.1.3.

Example 3.1.11. We classify all 1-dimensional normal toric varieties as follows. We may assume $N = \mathbb{Z}$ and $N_\mathbb{R} = \mathbb{R}$. The only cones are the intervals $\sigma_0 = [0, \infty)$ and $\sigma_1 = (-\infty, 0]$ and the trivial cone $\tau = \{0\}$. It follows that there are only four possible fans, which gives the following list of toric varieties:

$$\begin{aligned}
&\{\tau\}, \text{ which gives } \mathbb{C}^* \\
&\{\sigma_0, \tau\} \text{ and } \{\sigma_1, \tau\}, \text{ both of which give } \mathbb{C} \\
&\{\sigma_0, \sigma_1, \tau\}, \text{ which gives } \mathbb{P}^1.
\end{aligned}$$

Here is a picture of the fan for $\mathbb{P}^1$:



This is the fan of Example 3.1.10 when $n = 1$. $\Diamond$

Example 3.1.12. By Example 2.4.8, $\mathbb{P}^n \times \mathbb{P}^m$ is the toric variety of the polytope $\Delta_n \times \Delta_m$. The normal fan of $\Delta_n \times \Delta_m$ is the product of the normal fans of each factor (Proposition 2.4.9). These normal fans are described in Example 3.1.10. It follows that the product fan Σ gives $X_\Sigma \simeq \mathbb{P}^n \times \mathbb{P}^m$.

When $n = m = 1$, we obtain the fan $\Sigma \subseteq \mathbb{R}^2 \simeq N_\mathbb{R}$ pictured in Figure 3 on the next page. Here, we can use an elementary gluing argument to show that this fan gives $\mathbb{P}^1 \times \mathbb{P}^1$. Label the 2-dimensional cones $\sigma_{ij} = \sigma_i \times \sigma'_j$ as above. Then

$$\begin{aligned}
\mathrm{Spec}(\mathbb{C}[\mathsf{S}_{\sigma_{00}}]) &\simeq \mathbb{C}[x, y] \\
\mathrm{Spec}(\mathbb{C}[\mathsf{S}_{\sigma_{10}}]) &\simeq \mathbb{C}[x^{-1}, y] \\
\mathrm{Spec}(\mathbb{C}[\mathsf{S}_{\sigma_{11}}]) &\simeq \mathbb{C}[x^{-1}, y^{-1}] \\
\mathrm{Spec}(\mathbb{C}[\mathsf{S}_{\sigma_{01}}]) &\simeq \mathbb{C}[x, y^{-1}].
\end{aligned}$$

We see that if U_0 and U_1 are the standard affine open sets in $\mathbb{P}^1$, then $U_{\sigma_{ij}} \simeq U_i \times U_j$ and it is easy to check that the gluing makes $X_\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^1$. $\Diamond$

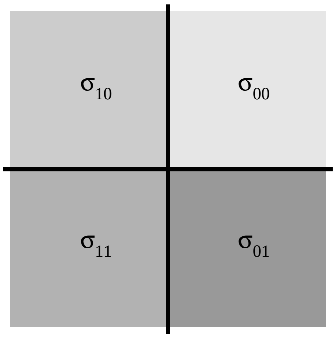


Figure 3. A fan Σ with $X_\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^1$

Example 3.1.13. Let $N = N_1 \times N_2$, with $N_1 = \mathbb{Z}^n$ and $N_2 = \mathbb{Z}^m$. Let Σ_1 in $(N_1)_\mathbb{R}$ be the fan giving $\mathbb{P}^n$, but let Σ_2 be the fan consisting of the cone $\mathrm{Cone}(e_1,\dots,e_m)$ together with all its faces. Then $\Sigma = \Sigma_1 \times \Sigma_2$ is a fan in $N_\mathbb{R}$ and the the corresponding toric variety is $X_\Sigma \simeq \mathbb{P}^n \times \mathbb{C}^m$. The case $\mathbb{P}^1 \times \mathbb{C}^2$ was studied in Example 3.0.14. ♢

Examples 3.1.12 and 3.1.13 are special cases of the following general construction, whose proof will be left to the reader (Exercise 3.1.4).

Proposition 3.1.14. *Suppose we have fans Σ_1 in $(N_1)_\mathbb{R}$ and Σ_2 in $(N_2)_\mathbb{R}$. Then*

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 \mid \sigma_i \in \Sigma_i\}$$

is a fan in $(N_1)_\mathbb{R} \times (N_2)_\mathbb{R} = (N_1 \times N_2)_\mathbb{R}$ and

$$X_{\Sigma_1 \times \Sigma_2} \simeq X_{\Sigma_1} \times X_{\Sigma_2}. \qquad \square$$

Example 3.1.15. The two cones σ_1 and σ_2 in $N_\mathbb{R} = \mathbb{R}^2$ from Example 3.1.4 (see Figure 1), together with their faces, form a fan Σ. By comparing the descriptions of the coordinate rings of V_{σ_i} given there with what we did in Example 3.0.8, it is easy to check that $X_\Sigma \simeq W$, where $W \subseteq \mathbb{P}^1 \times \mathbb{C}^2$ is the blowup of $\mathbb{C}^2$ at the origin, defined as $W = \mathbf{V}(x_0 y - x_1 x)$ (Exercise 3.1.5).

Generalizing this, let $N = \mathbb{Z}^n$ with standard basis $e_1,\dots,e_n$ and then set $e_0 = e_1 + \cdots + e_n$. Let Σ be the fan in $N_\mathbb{R}$ consisting of the cones generated by all subsets of $\{e_0,\dots,e_n\}$ not containing $\{e_1,\dots,e_n\}$. Then the toric variety X_Σ is isomorphic to the blowup of $\mathbb{C}^n$ at the origin (Exercise 3.0.8). ♢

Example 3.1.16. Let $r \in \mathbb{N}$ and consider the fan Σ_r in $N_\mathbb{R} = \mathbb{R}^2$ consisting of the four cones σ_i shown in Figure 4 on the next page, together with all of their faces.

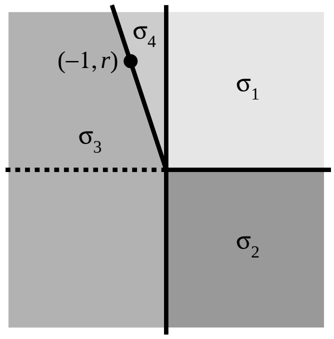


Figure 4. A fan Σ_r with $X_{\Sigma_r} \simeq \mathscr{H}_r$

The corresponding toric variety X_{Σ_r} is covered by open affine subsets,

$$\begin{aligned}
U_{\sigma_1} &= \mathrm{Spec}(\mathbb{C}[x,y]) \simeq \mathbb{C}^2\\
U_{\sigma_2} &= \mathrm{Spec}(\mathbb{C}[x,y^{-1}]) \simeq \mathbb{C}^2\\
U_{\sigma_3} &= \mathrm{Spec}(\mathbb{C}[x^{-1},x^{-r}y^{-1}]) \simeq \mathbb{C}^2\\
U_{\sigma_4} &= \mathrm{Spec}(\mathbb{C}[x^{-1},x^{r}y]) \simeq \mathbb{C}^2,
\end{aligned}$$

and glued according to (3.1.3). We call X_{Σ_r} the *Hirzebruch surface* $\mathscr{H}_r$.

Example 2.3.16 constructed the *rational normal scroll* $S_{a,b}$ using the polygon $P_{a,b}$ with $b \geq a \geq 1$. The normal fan of $P_{a,b}$ is the fan Σ_{b-a} defined above, so that as an abstract variety, $S_{a,b} \simeq \mathscr{H}_{b-a}$. Note also that $\mathscr{H}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. ◊

The Hirzebruch surfaces $\mathscr{H}_r$ will play an important role in the classification of smooth projective toric surfaces given in Chapter 10.

Example 3.1.17. Let $q_0,\dots,q_n \in \mathbb{Z}_{>0}$ satisfy $\gcd(q_0,\dots,q_n) = 1$. Consider the weighted projective space $\mathbb{P}(q_0,\dots,q_n)$ introduced in Chapter 2. Define the lattice $N = \mathbb{Z}^{n+1}/\mathbb{Z}\cdot(q_0,\dots,q_n)$ and let u_i, $i = 0,\dots,n$, be the images in N of the standard basis vectors in $\mathbb{Z}^{n+1}$, so the relation

$$q_0u_0 + \cdots + q_nu_n = 0$$

holds in N. Let Σ be the fan made up of the cones generated by all the proper subsets of $\{u_0,\dots,u_n\}$. When $q_i = 1$ for all i, we obtain $X_\Sigma \simeq \mathbb{P}^n$ by Example 3.1.10. And indeed, $X_\Sigma \simeq \mathbb{P}(q_0,\dots,q_n)$ in general. This will be proved in Chapter 5 using the toric generalization of homogeneous coordinates in $\mathbb{P}^n$.

Here, we will consider the special case $\mathbb{P}(1,1,2)$, where $u_0 = -u_1 - 2u_2$. The fan Σ in $N_{\mathbb{R}}$ is pictured in Figure 5 on the next page, using the plane spanned by u_1,u_2. This example is different from the ones we have seen so far. Consider $\sigma_2 = \mathrm{Cone}(u_0,u_1) = \mathrm{Cone}(-u_1-2u_2,u_1)$. Then $\sigma_2^\vee = \mathrm{Cone}(-u_2, 2u_1-u_2) \subseteq M$, so the situation is similar to the case studied in Example 1.2.22. Indeed, there is a change

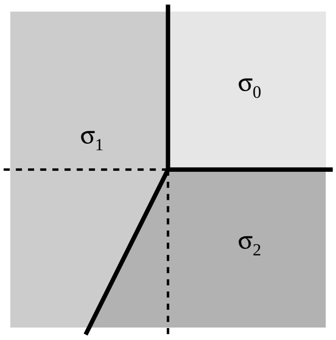


Figure 5. A fan Σ with $X_\Sigma \simeq \mathbb{P}(1,1,2)$

of coordinates defined by a matrix in $\mathrm{GL}(2,\mathbb{Z})$ that takes σ to the cone with $d = 2$ from that example. It follows that there is an isomorphism $U_{\sigma_2} \simeq \mathbf{V}(xz - y^2) \subseteq \mathbb{C}^3$ (Exercise 3.1.6). This is the rational normal cone $\widehat{C}_2$, hence has a singular point at the origin. The toric variety X_Σ is singular because of the singular point in this affine open subset.

In Example 2.4.6, we saw that the polytope $P = \mathrm{Conv}(0, 2e_1, e_2) \subseteq \mathbb{R}^2$ gives $X_P \simeq \mathbb{P}(1,1,2)$ and that the normal fan Σ_P coincides with the fan shown above. $\Diamond$

There is a dictionary between properties of X_Σ and properties of Σ that generalizes Theorem 1.3.12 and Example 1.3.20. We begin with some terminology. The first two items parallel Definition 1.2.16.

Definition 3.1.18. Let $\Sigma \subseteq N_\mathbb{R}$ be a fan.

(a) We say Σ is ***smooth*** (or ***regular***) if every cone σ in Σ is smooth (or regular).

(b) We say Σ is ***simplicial*** if every cone σ in Σ is simplicial.

(c) We say Σ is ***complete*** if its support $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$ is all of $N_\mathbb{R}$.

Theorem 3.1.19. *Let X_Σ be the toric variety defined by a fan $\Sigma \subseteq N_\mathbb{R}$. Then:*

(a) *X_Σ is a smooth variety if and only if the fan Σ is smooth.*

(b) *X_Σ is an orbifold* (*that is, X_Σ has only finite quotient singularities*) *if and only if the fan Σ is simplicial.*

(c) *X_Σ is compact in the classical topology if and only if Σ is complete.*

Proof. Part (a) follows from the corresponding statement for affine toric varieties, Theorem 1.3.12, because smoothness is a local property (Definition 3.0.13). In part (b), Example 1.3.20 gives one implication. The other implication will be proved in Chapter 11. A proof of part (c) will be given in §3.4. $\square$

The blowup of $\mathbb{C}^2$ at the origin (Example 3.1.15) is not compact, since the support of the cones in the corresponding fan is not all of $\mathbb{R}^2$. The Hirzebruch surfaces $\mathscr{H}_r$ from Example 3.1.16 are smooth and compact because every cone in the corresponding fan is smooth, and the union of the cones is $\mathbb{R}^2$. The variety $\mathbb{P}(1,1,2)$ from Example 3.1.17 is compact but not smooth. It is an orbifold (it has only finite quotient singularities) since the corresponding fan is simplicial.

Exercises for §3.1.

3.1.1. Let Σ be a fan in $N_{\mathbb{R}}$. Show that the isomorphisms g_{σ_1,σ_2} satisfy the compatibility conditions from §0 for gluing the U_σ together to create X_Σ.

3.1.2. Let X be a variety obtained by gluing affine open subsets $\{V_\alpha\}$ along open subsets $V_{\alpha\beta} \subseteq V_\alpha$ by isomorphisms $g_{\alpha\beta} : V_{\alpha\beta} \simeq V_{\beta\alpha}$. Show that X is separated when the image of $\Delta : V_{\alpha\beta} \to V_\alpha \times V_\beta$ defined by $\Delta(p) = (p, g_{\alpha\beta}(p))$ is Zariski closed for all α, β.

3.1.3. Verify that if Σ is the fan given in Example 3.1.10, then $X_\Sigma \simeq \mathbb{P}^n$.

3.1.4. Prove Proposition 3.1.14.

3.1.5. Let $N \simeq \mathbb{Z}^n$, let $e_1, \ldots, e_n \in N$ be the standard basis and let $e_0 = e_1 + \cdots + e_n$. Let Σ be the set of cones generated by all subsets of $\{e_0, \ldots, e_n\}$ not containing $\{e_1, \ldots, e_n\}$.

(a) Show that Σ is a fan in $N_{\mathbb{R}}$.

(b) Construct the affine open subsets covering the corresponding toric variety X_Σ, and give the gluing isomorphisms.

(c) Show that X_Σ is isomorphic to the blowup of $\mathbb{C}^n$ at the origin, described earlier in Exercise 3.0.8. Hint: The blowup is the subvariety of $\mathbb{P}^{n-1} \times \mathbb{C}^n$ given by $W = \mathbf{V}(x_i y_j - x_j y_i \mid 1 \leq i < j \leq n)$. Cover W by affine open subsets $W_i = W_{x_i}$ and compare those affines with your answer to part (b).

3.1.6. In this exercise, you will verify the claims made in Example 3.1.17.

(a) Show that there is a matrix $A \in \mathrm{GL}(2,\mathbb{Z})$ defining a change of coordinates that takes the cone in this example to the cone from Example 1.2.22, and find the mapping that takes $\sigma_2^\vee$ to the dual cone.

(b) Show that $\mathrm{Spec}(\mathbb{C}[\mathsf{S}_{\sigma_2}]) \simeq \mathbf{V}(xz - y^2) \subseteq \mathbb{C}^3$.

3.1.7. In $N_{\mathbb{R}} = \mathbb{R}^2$, consider the fan Σ with cones $\{0\}$, $\mathrm{Cone}(e_1)$, and $\mathrm{Cone}(-e_1)$. Show that $X_\Sigma \simeq \mathbb{P}^1 \times \mathbb{C}^*$.

§3.2. The Orbit-Cone Correspondence

In this section, we will study the orbits for the action of T_N on the toric variety X_Σ. Our main result will show that there is a bijective correspondence between cones in Σ and T_N-orbits in X_Σ. The connection comes ultimately from looking at limit points of the one-parameter subgroups of T_N defined in §1.1.

A First Example. We introduce the key features of the correspondence between orbits and cones by looking at a concrete example.

Example 3.2.1. Consider $\mathbb{P}^2 \simeq X_\Sigma$ for the fan Σ from Figure 2 of §3.1. The torus $T_N = (\mathbb{C}^*)^2 \subseteq \mathbb{P}^2$ consists of points with homogeneous coordinates $(1,s,t)$, $s,t \neq 0$. For each $u = (a,b) \in N = \mathbb{Z}^2$, we have the corresponding curve in $\mathbb{P}^2$:

$$\lambda^u(t) = (1,t^a,t^b).$$

We are abusing notation slightly; strictly speaking, the one-parameter subgroup λ^u is a curve in $(\mathbb{C}^*)^2$, but we view it as a curve in $\mathbb{P}^2$ via the inclusion $(\mathbb{C}^*)^2 \subseteq \mathbb{P}^2$.

We start by analyzing the limit of $\lambda^u(t)$ as $t \to 0$. The limit point in $\mathbb{P}^2$ depends on $u = (a,b)$. It is easy to check that the pattern is as follows:

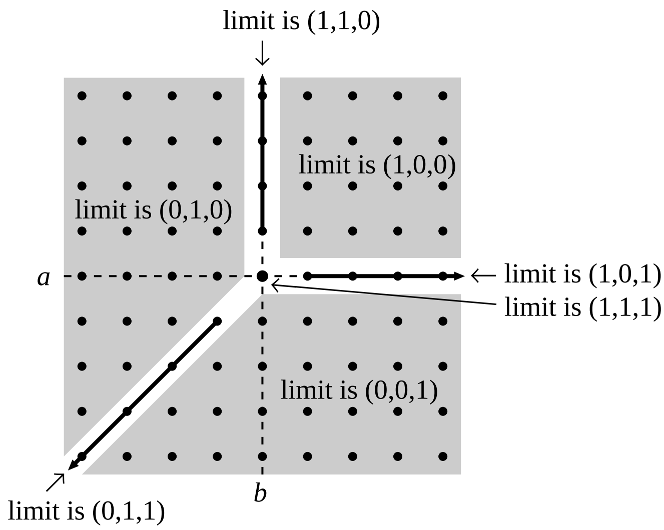


Figure 6. $\lim_{t\to 0} \lambda^u(t)$ for $u = (a,b) \in \mathbb{Z}^2$

For instance, suppose $a,b > 0$ in $\mathbb{Z}$. These points lie in the first quadrant. Here, it is obvious that $\lim_{t\to 0}(1,t^a,t^b) = (1,0,0)$. Next suppose that $a = b < 0$ in $\mathbb{Z}$, corresponding to points on the diagonal in the third quadrant. Note that

$$(1,t^a,t^b) = (1,t^a,t^a) \sim (t^{-a},1,1)$$

since we are using homogeneous coordinates in $\mathbb{P}^2$. Then $-a > 0$ implies that $\lim_{t\to 0}(t^{-a},1,1) = (0,1,1)$. You will check the remaining cases in Exercise 3.2.1.

The regions of N described in Figure 6 correspond to cones of the fan Σ. In each case, the set of u giving one of the limit points equals $N \cap \mathrm{Relint}(\sigma)$, where $\mathrm{Relint}(\sigma)$ is the *relative interior* of a cone $\sigma \in \Sigma$. In other words, we have recovered the structure of the fan Σ by considering these limits!

Now we relate this to the T_N-orbits in $\mathbb{P}^2$. By considering the description $\mathbb{P}^2 \simeq (\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^*$, you will see in Exercise 3.2.1 that there are exactly seven T_N-orbits in $\mathbb{P}^2$:

$$\begin{aligned}
O_1 &= \{(x_0,x_1,x_2) \mid x_i \neq 0 \text{ for all } i\} \ni (1,1,1)\\
O_2 &= \{(x_0,x_1,x_2) \mid x_2 = 0, \text{ and } x_0,x_1 \neq 0\} \ni (1,1,0)\\
O_3 &= \{(x_0,x_1,x_2) \mid x_1 = 0, \text{ and } x_0,x_2 \neq 0\} \ni (1,0,1)\\
O_4 &= \{(x_0,x_1,x_2) \mid x_0 = 0, \text{ and } x_1,x_2 \neq 0\} \ni (0,1,1)\\
O_5 &= \{(x_0,x_1,x_2) \mid x_1 = x_2 = 0, \text{ and } x_0 \neq 0\} = \{(1,0,0)\}\\
O_6 &= \{(x_0,x_1,x_2) \mid x_0 = x_2 = 0, \text{ and } x_1 \neq 0\} = \{(0,1,0)\}\\
O_7 &= \{(x_0,x_1,x_2) \mid x_0 = x_1 = 0, \text{ and } x_2 \neq 0\} = \{(0,0,1)\}.
\end{aligned}$$

This list shows that each orbit contains a unique limit point. Hence we obtain a correspondence between cones σ and orbits O by

$$\sigma \text{ corresponds to } O \iff \lim_{t\to 0} \lambda^u(t) \in O \text{ for all } u \in \mathrm{Relint}(\sigma).$$

We will soon see that these observations generalize to all toric varieties X_Σ. $\Diamond$

Points and Semigroup Homomorphisms. It will be convenient to use the intrinsic description of the points of an affine toric variety U_σ given in Proposition 1.3.1. We recall how this works and make some additional observations:

- Points of U_σ are in bijective correspondence with semigroup homomorphisms $\gamma : \mathsf{S}_\sigma \to \mathbb{C}$. Recall that $\mathsf{S}_\sigma = \sigma^\vee \cap M$ and $U_\sigma = \mathrm{Spec}(\mathbb{C}[\mathsf{S}_\sigma])$.
- For each cone σ we have a point of U_σ defined by
$$m \in \mathsf{S}_\sigma \longmapsto \begin{cases} 1 & m \in \mathsf{S}_\sigma \cap \sigma^\perp = \sigma^\perp \cap M \\ 0 & \text{otherwise}. \end{cases}$$
This is a semigroup homomorphism since $\sigma^\vee \cap \sigma^\perp$ is a face of $\sigma^\vee$. Thus, if $m, m' \in \mathsf{S}_\sigma$ and $m + m' \in \mathsf{S}_\sigma \cap \sigma^\perp$, then $m, m' \in \mathsf{S}_\sigma \cap \sigma^\perp$. We denote this point by γ_σ and call it the *distinguished point* corresponding to σ.
- The point γ_σ is fixed under the T_N-action if and only if $\dim \sigma = \dim N_\mathbb{R}$ (Corollary 1.3.3).
- If $\tau \preceq \sigma$ is a face, then $\gamma_\tau \in U_\sigma$. This follows since $\sigma^\perp \subseteq \tau^\perp$.

Limits of One-Parameter Subgroups. In Example 3.2.1, the limit points of one-parameter subgroups are exactly the distinguished points for the cones in the fan of $\mathbb{P}^2$ (Exercise 3.2.1). We now show that this is true for all affine toric varieties.

Proposition 3.2.2. *Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone and let $u \in N$. Then*

$$u \in \sigma \iff \lim_{t\to 0} \lambda^u(t) \text{ exists in } U_\sigma.$$

Moreover, if $u \in \mathrm{Relint}(\sigma)$, then $\lim_{t\to 0} \lambda^u(t) = \gamma_\sigma$.

Proof. Given $u \in N$, we have

$$\begin{aligned}\lim_{t\to 0}\lambda^u(t) \text{ exists in } U_\sigma &\iff \lim_{t\to 0}\chi^m(\lambda^u(t)) \text{ exists in } \mathbb{C} \text{ for all } m \in \mathsf{S}_\sigma\\ &\iff \lim_{t\to 0} t^{\langle m,u\rangle} \text{ exists in } \mathbb{C} \text{ for all } m \in \mathsf{S}_\sigma\\ &\iff \langle m,u\rangle \geq 0 \text{ for all } m \in \sigma^\vee \cap M\\ &\iff u \in (\sigma^\vee)^\vee = \sigma,\end{aligned}$$

where the first equivalence is proved in Exercise 3.2.2 and the other equivalences are clear. This proves the first assertion of the proposition.

In Exercise 3.2.2 you will also show that when $u \in \sigma \cap N$, $\lim_{t\to 0}\lambda^u(t)$ is the point corresponding to the semigroup homomorphism $\mathsf{S}_\sigma \to \mathbb{C}$ defined by

$$m \in \sigma^\vee \cap M \longmapsto \lim_{t\to 0} t^{\langle m,u\rangle}.$$

If $u \in \operatorname{Relint}(\sigma)$, then $\langle m,u\rangle > 0$ for all $m \in \mathsf{S}_\sigma \setminus \sigma^\perp$ (Exercise 1.2.2), and $\langle m,u\rangle = 0$ if $m \in \mathsf{S}_\sigma \cap \sigma^\perp$. Hence the limit point is precisely the distinguished point γ_σ. $\square$

Using this proposition, we can recover the fan Σ from X_Σ cone by cone as in Example 3.2.1. This is also the key observation needed for the proof of Corollary 3.1.8 from the previous section.

Let us apply Proposition 3.2.2 to a familiar example.

Example 3.2.3. Consider the affine toric variety $V = \mathbf{V}(xy - zw)$ studied in a number of examples from Chapter 1. For instance, in Example 1.1.18, we showed that V is the normal toric variety corresponding to a cone σ whose dual cone is

$$\sigma^\vee = \operatorname{Cone}(e_1, e_2, e_3, e_1 + e_2 - e_3), \tag{3.2.1}$$

and $V = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$.

In Example 1.1.18, we introduced the torus $T = (\mathbb{C}^*)^3$ of V as the image of

$$(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, t_1 t_2 t_3^{-1}). \tag{3.2.2}$$

Given $u = (a,b,c) \in N = \mathbb{Z}^3$, we have the one-parameter subgroup

$$\lambda^u(t) = (t^a, t^b, t^c, t^{a+b-c}) \tag{3.2.3}$$

contained in V, and we proceed to examine limit points using Proposition 3.2.2. Clearly, $\lim_{t\to 0}\lambda^u(t)$ exists in V if and only if $a,b,c \geq 0$ and $a + b \geq c$. These conditions determine the cone $\sigma \subseteq N_\mathbb{R}$ given by

$$\sigma = \operatorname{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3). \tag{3.2.4}$$

One easily checks that (3.2.1) is the dual of this cone (Exercise 3.2.3). Note also that $u \in \operatorname{Relint}(\sigma)$ means $a,b,c > 0$ and $a + b > c$, in which case the limit $\lim_{t\to 0}\lambda^u(t) = (0,0,0,0)$, which is the distinguished point γ_σ. $\Diamond$

The Torus Orbits. Now we turn to the T_N-orbits in X_Σ. We saw above that each cone $\sigma \in \Sigma$ has a distinguished point $\gamma_\sigma \in U_\sigma \subseteq X_\Sigma$. This gives the torus orbit

$$O(\sigma) = T_N \cdot \gamma_\sigma \subseteq X_\Sigma.$$

In order to determine the structure of $O(\sigma)$, we need the following lemma, which you will prove in Exercise 3.2.4.

Lemma 3.2.4. *Let σ be a strongly convex rational polyhedral cone in $N_\mathbb{R}$. Let N_σ be the sublattice of N spanned by the points in $\sigma \cap N$, and let $N(\sigma) = N/N_\sigma$.*

(a) *There is a perfect pairing*

$$\langle\ ,\ \rangle : \sigma^\perp \cap M \times N(\sigma) \to \mathbb{Z},$$

induced by the dual pairing $\langle\ ,\ \rangle : M \times N \to \mathbb{Z}$.

(b) *The pairing of part* (a) *induces a natural isomorphism*

$$\mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq T_{N(\sigma)},$$

where $T_{N(\sigma)} = N(\sigma) \otimes_\mathbb{Z} \mathbb{C}^*$ is the torus associated to $N(\sigma)$. □

To study $O(\sigma) \subseteq U_\sigma$, we recall how $t \in T_N$ acts on semigroup homomorphisms. If $p \in U_\sigma$ is represented by $\gamma : \mathsf{S}_\sigma \to \mathbb{C}$, then by Exercise 1.3.1, the point $t \cdot p$ is represented by the semigroup homomorphism

$$t \cdot \gamma : m \longmapsto \chi^m(t)\gamma(m). \tag{3.2.5}$$

Lemma 3.2.5. *Let σ be a strongly convex rational polyhedral cone in $N_\mathbb{R}$. Then*

$$\begin{aligned} O(\sigma) &= \{\gamma : \mathsf{S}_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\} \\ &\simeq \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq T_{N(\sigma)}, \end{aligned}$$

where $N(\sigma)$ is the lattice defined in Lemma 3.2.4.

Proof. The set $O' = \{\gamma : \mathsf{S}_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\}$ contains γ_σ and is invariant under the action of T_N described in (3.2.5).

Next observe that $\sigma^\perp$ is the largest vector subspace of $M_\mathbb{R}$ contained in $\sigma^\vee$. Hence $\sigma^\perp \cap M$ is a *subgroup* of $\mathsf{S}_\sigma = \sigma^\vee \cap M$. If $\gamma \in O'$, then restricting γ to $m \in \mathsf{S}_\sigma \cap \sigma^\perp = \sigma^\perp \cap M$ yields a *group homomorphism* $\widehat{\gamma} : \sigma^\perp \cap M \to \mathbb{C}^*$ (Exercise 3.2.5). Conversely, if $\widehat{\gamma} : \sigma^\perp \cap M \to \mathbb{C}^*$ is a group homomorphism, we obtain a semigroup homomorphism $\gamma \in O'$ by defining

$$\gamma(m) = \begin{cases} \widehat{\gamma}(m) & \text{if } m \in \sigma^\perp \cap M \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $O' \simeq \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$.

Now consider the exact sequence

$$0 \longrightarrow N_\sigma \longrightarrow N \longrightarrow N(\sigma) \longrightarrow 0. \tag{3.2.6}$$

Tensoring with $\mathbb{C}^*$ and using Lemma 3.2.4, we obtain a surjection

$$T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \longrightarrow T_{N(\sigma)} = N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*).$$

The bijections

$$T_{N(\sigma)} \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq O'$$

are compatible with the T_N-action, so that T_N acts transtively on O'. Then $\gamma_\sigma \in O'$ implies that $O' = T_N \cdot \gamma_\sigma = O(\sigma)$, as desired. □

The Orbit-Cone Correspondence. Our next theorem is the major result of this section. Recall that the face relation $\tau \preceq \sigma$ holds when τ is a face of σ.

Theorem 3.2.6 (Orbit-Cone Correspondence). *Let X_Σ be the toric variety of the fan Σ in $N_{\mathbb{R}}$. Then:*

(a) *There is a bijective correspondence*

$$\begin{aligned}\{\text{cones } \sigma \text{ in } \Sigma\} &\longleftrightarrow \{T_N\text{-orbits in } X_\Sigma\}\\ \sigma &\longleftrightarrow O(\sigma) \simeq \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*).\end{aligned}$$

(b) *Let $n = \dim N_{\mathbb{R}}$. For each cone $\sigma \in \Sigma$, $\dim O(\sigma) = n - \dim \sigma$.*

(c) *The affine open subset U_σ is the union of orbits*

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau).$$

(d) *$\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and*

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma),$$

where $\overline{O(\tau)}$ denotes the closure in both the classical and Zariski topologies.

For instance, Example 3.2.1 tells us that for $\mathbb{P}^2$, there are three types of cones and torus orbits:

- The trivial cone $\sigma = \{(0,0)\}$ corresponds to the orbit $O(\sigma) = T_N \subseteq \mathbb{P}^2$, which satisfies $\dim O(\sigma) = 2 = 2 - \dim \sigma$. This is a face of all the other cones in Σ, and hence all the other orbits are contained in the closure of this one by part (d). Note also that $U_\sigma = O(\sigma) \simeq (\mathbb{C}^*)^2$ by part (c), since there are no cones properly contained in σ.
- The three 1-dimensional cones τ give the torus orbits of dimension 1. Each is isomorphic to $\mathbb{C}^*$. The closures of these orbits are the coordinate axes $\mathbf{V}(x_i)$ in $\mathbb{P}^2$, each a copy of $\mathbb{P}^1$. Note that each τ is contained in two maximal cones.
- The three maximal cones σ_i in the fan Σ correspond to the three fixed points $(1,0,0), (0,1,0), (0,0,1)$ of the torus action on $\mathbb{P}^2$. There are two of these in the closure of each of the 1-dimensional torus orbits.

Proof of Theorem 3.2.6. Let O be a T_N-orbit in X_Σ. Since X_Σ is covered by the T_N-invariant affine open subsets $U_\sigma \subseteq X_\Sigma$ and $U_{\sigma_1} \cap U_{\sigma_2} = U_{\sigma_1\cap\sigma_2}$, there is a unique minimal cone $\sigma \in \Sigma$ with $O \subseteq U_\sigma$. We claim that $O = O(\sigma)$. Note that part (a) will follow immediately once we prove this claim.

To prove the claim, let $\gamma \in O$ and consider those $m \in \mathsf{S}_\sigma$ satisfying $\gamma(m) \neq 0$. In Exercise 3.2.6, you will show that these m's lie on a face of $\sigma^\vee$. But faces of $\sigma^\vee$ are all of the form $\sigma^\vee \cap \tau^\perp$ for some face $\tau \preceq \sigma$ by Proposition 1.2.10. In other words, there is a face $\tau \preceq \sigma$ such that

$$\{m \in \mathsf{S}_\sigma \mid \gamma(m) \neq 0\} = \sigma^\vee \cap \tau^\perp \cap M.$$

This easily implies $\gamma \in U_\tau$ (Exercise 3.2.6), and then $\tau = \sigma$ by the minimality of σ. Hence $\{m \in \mathsf{S}_\sigma \mid \gamma(m) \neq 0\} = \sigma^\perp \cap M$, and then $\gamma \in O(\sigma)$ by Lemma 3.2.5. This implies $O = O(\sigma)$ since two orbits are either equal or disjoint.

Part (b) follows from Lemma 3.2.5 and (3.2.6).

Next consider part (c). We know that U_σ is a union of orbits. If τ is a face of σ, then $O(\tau) \subseteq U_\tau \subseteq U_\sigma$ implies that $O(\tau)$ is an orbit contained in U_σ. Furthermore, the analysis of part (a) easily implies that any orbit contained in U_σ must equal $O(\tau)$ for some face $\tau \preceq \sigma$.

We now turn to part (d). We begin with the closure of $O(\tau)$ in the classical topology, which we denote $\overline{O(\tau)}$. This is invariant under T_N (Exercise 3.2.6) and hence is a union of orbits. Suppose that $O(\sigma) \subseteq \overline{O(\tau)}$. Then $O(\tau) \subseteq U_\sigma$, since otherwise $O(\tau) \cap U_\sigma = \emptyset$, which would imply $\overline{O(\tau)} \cap U_\sigma = \emptyset$ since U_σ is open in the classical topology. Once we have $O(\tau) \subseteq U_\sigma$, it follows that $\tau \preceq \sigma$ by part (c). Conversely, assume $\tau \preceq \sigma$. To prove that $O(\sigma) \subseteq \overline{O(\tau)}$, it suffices to show that $\overline{O(\tau)} \cap O(\sigma) \neq \emptyset$. We will do this by using limits of one-parameter subgroups as in Proposition 3.2.2.

Let γ_τ be the semigroup homomorphism corresponding to the distinguished point of U_τ, so $\gamma_\tau(m) = 1$ if $m \in \tau^\perp \cap M$, and 0 otherwise. Let $u \in \mathrm{Relint}(\sigma)$, and for $t \in \mathbb{C}^*$ define $\gamma(t) = \lambda^u(t) \cdot \gamma_\tau$. As a semigroup homomorphism, $\gamma(t)$ is

$$m \longmapsto \chi^m(\lambda^u(t))\,\gamma_\tau(m) = t^{\langle m,u\rangle}\gamma_\tau(m).$$

Note that $\gamma(t) \in O(\tau)$ for all $t \in \mathbb{C}^*$since the orbit of γ_τ is $O(\tau)$. Now let $t \to 0$. Since $u \in \mathrm{Relint}(\sigma)$, $\langle m,u\rangle > 0$ if $m \in \sigma^\vee \setminus \sigma^\perp$, and $= 0$ if $m \in \sigma^\perp$. It follows that $\gamma(0) = \lim_{t\to 0}\gamma(t)$ exists as a point in U_σ by Proposition 3.2.2, and represents a point in $O(\sigma)$. But it is also in the closure of $O(\tau)$ by construction, so that $O(\sigma) \cap \overline{O(\tau)} \neq \emptyset$. This establishes the first assertion of (d), and

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$$

follows immediately for the classical topology.

It remains to show that this set is also the Zariski closure. If we intersect $\overline{O(\tau)}$ with an affine open subset $U_{\sigma'}$, parts (c) and (d) imply that

$$\overline{O(\tau)} \cap U_{\sigma'} = \bigcup_{\tau \preceq \sigma' \preceq \sigma} O(\sigma).$$

In Exercise 3.2.6, you will show that this is the subvariety $\mathbf{V}(I) \subseteq U_{\sigma'}$ for the ideal

$$I = \langle \chi^m \mid m \in \tau^\perp \cap (\sigma')^\vee \cap M \rangle \subseteq \mathbb{C}[(\sigma')^\vee \cap M] = \mathsf{S}_{\sigma'}. \tag{3.2.7}$$

This easily implies that the classical closure $\overline{O(\tau)}$ is a subvariety of X_Σ and hence is the Zariski closure of $O(\tau)$. □

Orbit Closures as Toric Varieties. In the example of $\mathbb{P}^2$, the orbit closures $\overline{O(\tau)}$ also have the structure of toric varieties. This holds in general. We use the notation

$$V(\tau) = \overline{O(\tau)}.$$

By part (d) of Theorem 3.2.6, we have $\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq V(\tau)$, and

$$V(\tau) = \bigcup_{\tau \preceq \sigma} O(\sigma).$$

The torus $O(\tau) = T_{N(\tau)}$ is an open subset of $V(\tau)$, where $N(\tau)$ is defined in Lemma 3.2.4. We will show that $V(\tau)$ is a normal toric variety by constructing its fan. For each cone $\sigma \in \Sigma$ containing τ, let $\overline{\sigma}$ be the image cone in $N(\tau)_\mathbb{R}$ under the quotient map

$$N_\mathbb{R} \longrightarrow N(\tau)_\mathbb{R}$$

in (3.2.6). Then

$$\mathrm{Star}(\tau) = \{\overline{\sigma} \subseteq N(\tau)_\mathbb{R} \mid \tau \preceq \sigma \in \Sigma\} \tag{3.2.8}$$

is a fan in $N(\tau)_\mathbb{R}$ (Exercise 3.2.7).

Proposition 3.2.7. *For any $\tau \in \Sigma$, the orbit closure $V(\tau) = \overline{O(\tau)}$ is isomorphic to the toric variety $X_{\mathrm{Star}(\tau)}$.*

Proof. This follows from parts (a) and (d) of Theorem 3.2.6 (Exercise 3.2.7). □

Example 3.2.8. Consider the fan Σ in $N_\mathbb{R} = \mathbb{R}^3$ shown in Figure 7 on the next page. The support of Σ is the cone in Figure 2 of Chapter 1, and Σ is obtained from this cone by adding a new 1-dimensional cone τ in the center and subdividing. The orbit $O(\tau)$ has dimension 2 by Theorem 3.2.6. By Proposition 3.2.7, the orbit closure $V(\tau)$ is constructed from the cones of Σ containing τ and then collapsing τ to a point in $N(\tau)_\mathbb{R} = (N/N_\tau)_\mathbb{R} \simeq \mathbb{R}^2$. This clearly gives the fan for $\mathbb{P}^1 \times \mathbb{P}^1$, so that $V(\tau) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. ◊

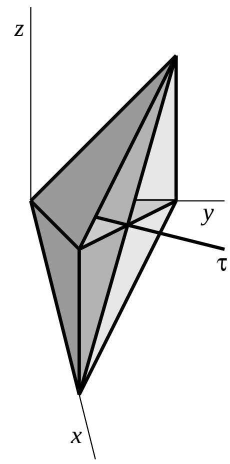


Figure 7. The fan Σ and its 1-dimensional cone τ in Example 3.2.8

A nice example of orbit closures comes from the toric variety X_P of a full dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$. Here, we use the normal fan Σ_P of P, which by Theorem 2.3.2 consists of cones

$$\sigma_Q = \mathrm{Cone}(u_F \mid F \text{ is a facet of } P \text{ containing } Q) \tag{3.2.9}$$

for each face $Q \preceq P$. Recall that u_F is the facet normal of F.

The basic idea is that the orbit closure of $V(\sigma_Q)$ is the toric variety of the lattice polytope Q. Since Q need not be full dimensional in $M_{\mathbb{R}}$, we need to be careful. The idea is to translate P by a vertex of Q so that the origin is a vertex of Q. This affects neither Σ_P nor X_P, but Q is now full dimensional in $\mathrm{Span}(Q)$ and is a lattice polytope relative to $\mathrm{Span}(Q) \cap M$. This gives the toric variety X_Q, which is easily seen to be independent of how we translate to the origin. Here is our result.

Proposition 3.2.9. *For each face $Q \preceq P$, we have $V(\sigma_Q) \simeq X_Q$.*

Proof. We sketch the proof and leave the details to reader (Exercise 3.2.8). Let

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq a_F \text{ for all facets } F \prec P\}$$

be the facet presentation of P. The facets of P containing Q also contain the origin, so that $a_F = 0$ for these facets. This implies that

$$\sigma_Q^{\perp} = \mathrm{Span}(Q),$$

and then $N(\sigma_Q)$ is dual to $\mathrm{Span}(Q) \cap M$. Note also that $N(\sigma_Q)_{\mathbb{R}} = N_{\mathbb{R}}/\mathrm{Span}(\sigma_Q)$.

To keep track of which polytope we are using, we will write the cone (3.2.9) associated to a face $Q \preceq P$ as $\sigma_{Q,P}$. Then X_P and X_Q are given by the normal fans

$$\begin{aligned}\Sigma_P &= \{\sigma_{Q',P} \subseteq N_{\mathbb{R}} \mid Q' \prec P\}\\ \Sigma_Q &= \{\sigma_{Q',Q} \subseteq N(\sigma_{Q,P})_{\mathbb{R}} \mid Q' \prec Q\}.\end{aligned}$$

By Proposition 3.2.7, the toric variety $V(\sigma_Q) = V(\sigma_{Q,P})$ is determined by the fan

$$\begin{aligned}\mathrm{Star}(\sigma_{Q,P}) &= \{\overline{\sigma} \mid \sigma_{Q,P} \prec \sigma \in \Sigma_P\}\\ &= \{\overline{\sigma_{Q',P}} \mid \sigma_{Q,P} \prec \sigma_{Q',P} \in \Sigma_P\} = \{\overline{\sigma_{Q',P}} \mid Q' \preceq Q\}.\end{aligned}$$

Then the proposition follows once one proves that $\overline{\sigma_{Q',P}} = \sigma_{Q',Q}$. □

Final Comments. The technique of using limit points of one-parameter subgroups to study a group action is also a major tool in Geometric Invariant Theory as in [**209**], where the main problem is to construct varieties (or possibly more general objects) representing orbit spaces for the actions of algebraic groups on varieties. We will apply ideas from group actions and orbit spaces to the study of toric varieties in Chapters 5 and 14.

We also note the observation made in part (d) of Theorem 3.2.6 that torus orbits have the same closure in the classical and Zariski topologies. For arbitrary subsets of a variety, these closures may differ. A torus orbit is an example of a *constructible subset*, and we will see in §3.4 that constructible subsets have the same classical and Zariski closures since we are working over $\mathbb{C}$.

Exercises for §3.2.

3.2.1. In this exercise, you will verify the claims made in Example 3.2.1 and the following discussion.

(a) Show that the remaining limits of one-parameter subgroups $\mathbb{P}^2$ are as claimed in the example.

(b) Show that the $(\mathbb{C}^*)^2$-orbits in $\mathbb{P}^2$ are as claimed in the example.

(c) Show that the limit point equals the distinguished point γ_σ of the corresponding cone in each case.

3.2.2. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone. This exercise will consider $\lim_{t\to 0} f(t)$, where $f : \mathbb{C}^* \to T_N$ is an arbitrary function.

(a) Prove that $\lim_{t\to 0} f(t)$ exists in U_σ if and only if $\lim_{t\to 0} \chi^m(f(t))$ exists in $\mathbb{C}$ for all $m \in S_\sigma$. Hint: Consider a finite set of characters $\mathscr{A}$ such that $\mathsf{S}_\sigma = \mathbb{N}\mathscr{A}$.

(b) When $\lim_{t\to 0} f(t)$ exists in U_σ, prove that the limit is given by the semigroup homomorphism that maps $m \in \mathsf{S}_\sigma$ to $\lim_{t\to 0} \chi^m(f(t))$.

3.2.3. Consider the situation of Example 3.2.3.

(a) Show that the cones in (3.2.1) and (3.2.4) are dual.

(b) Identify the limits of all one-parameter subgroups in this example, and describe the Orbit-Cone Correspondence in this case.

(c) Show that the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

defines an automorphism of $N \simeq \mathbb{Z}^3$ and the corresponding linear map on $N_\mathbb{R}$ maps the cone $\sigma^\vee$ to σ.

(d) Deduce that the affine toric varieties U_σ and $U_{\sigma^\vee}$ are isomorphic. Hint: Use Proposition 1.3.15.

3.2.4. Prove Lemma 3.2.4.

3.2.5. Let O' be as defined in the proof of Lemma 3.2.5. In this exercise, you will complete the proof that O' is a T_N-orbit in U_σ.

(a) Show that if $\gamma \in O'$, then $\widehat{\gamma} : \sigma^\perp \cap M \to \mathbb{C}^*$ is a group homomorphism.

(b) Deduce that O' has the structure of a group.

(c) Verify carefully that we have an isomorphism of groups $O' \simeq \operatorname{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*)$.

3.2.6. This exercise is concerned with the proof of Theorem 3.2.6.

(a) Let $\gamma : \mathsf{S}_\sigma \to \mathbb{C}$ be a semigroup homomorphism giving a point of U_σ. Prove that $\{m \in \mathsf{S}_\sigma \mid \gamma(m) \neq 0\} = \Gamma \cap M$ for some face $\Gamma \preceq \sigma^\vee$.

(b) Show $\overline{O(\tau)}$ is invariant under the action of T_N.

(c) Prove that $\overline{O(\tau)} \cap U_{\sigma'}$ is the variety of the ideal I defined in (3.2.7).

3.2.7. Let τ be a cone in a fan Σ, and let $\operatorname{Star}(\tau)$ be as defined in (3.2.8).

(a) Show that $\operatorname{Star}(\tau)$ is a fan in $N(\tau)_{\mathbb{R}}$.

(b) Prove Proposition 3.2.7.

3.2.8. Supply the details omitted in the proof of Proposition 3.2.9.

3.2.9. Consider the action of T_N on the affine toric variety U_σ. Use parts (c) and (d) of Theorem 3.2.6 to show that $O(\sigma)$ is the unique closed orbit of T_N acting on U_σ.

3.2.10. In Proposition 1.3.16, we saw that if τ is a face of the strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$ then $U_\tau = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_\tau])$ is an affine open subset of $U_\sigma = \operatorname{Spec}(\mathbb{C}[\mathsf{S}_\sigma])$. In this exercise, you will prove the converse, i.e., that if $\tau \subseteq \sigma$ and the induced map of affine toric varieties $\phi : U_\tau \to U_\sigma$ is an open immersion, then $\tau \preceq \sigma$, i.e., τ is a face of σ.

(a) Let $u, u' \in N \cap \sigma$, and assume $u + u' \in \tau$. Show that

$$\lim_{t \to 0} \lambda^u(t) \cdot \lim_{t \to 0} \lambda^{u'}(t) \in U_\tau.$$

(b) Show that $\lim_{t\to 0} \lambda^u(t)$ and $\lim_{t\to 0} \lambda^{u'}(t)$ are each in U_τ. Hint: Use the description of points as semigroup homomorphisms.

(c) Deduce that $u, u' \in \tau$, so τ is a face of σ.

3.2.11. In this exercise, you will use Proposition 3.2.2 and Theorem 3.2.6 to deduce Corollary 3.1.8 from Theorem 3.1.7.

(a) By Theorem 3.1.7, and the results of Chapter 1, a separated toric variety has an open cover consisting of affine toric varieties $U_i = U_{\sigma_i}$ for some collection of cones σ_i. Show that for all i, j, $U_i \cap U_j$ is also affine. Hint: Use the fact that X is separated.

(b) Show that $U_i \cap U_j$ is the affine toric variety corresponding to the cone $\tau = \sigma_i \cap \sigma_j$. Hint: Exercise 3.2.2 will be useful.

(c) If $\tau = \sigma_i \cap \sigma_j$, then show that τ is a face of both σ_i and σ_j. Hint: Use Exercise 3.2.10.

(d) Deduce that $X \simeq X_\Sigma$ for the fan consisting of the σ_i and all their faces.

§3.3. Toric Morphisms

Recall from §3.0 that if X and Y are varieties with affine open covers $X = \bigcup_\alpha U_\alpha$ and $Y = \bigcup_\beta U'_\beta$, then a morphism $\phi : X \to Y$ is a Zariski-continuous mapping such that the restrictions

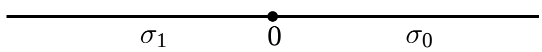
$$\phi|_{U_\alpha \cap \phi^{-1}(U'_\beta)} : U_\alpha \cap \phi^{-1}(U'_\beta) \longrightarrow U'_\beta$$

are morphisms in the sense of Definition 3.0.3 for all α, β.

In §1.3 we defined *toric morphisms* between affine toric varieties and studied their properties. When applied to arbitrary normal toric varieties, these results yield a class of morphisms whose construction comes directly from the combinatorics of the associated fans. The goal of this section is to study these special morphisms.

Definition 3.3.1. Let N_1, N_2 be two lattices with Σ_1 a fan in $(N_1)_\mathbb{R}$ and Σ_2 a fan in $(N_2)_\mathbb{R}$. A $\mathbb{Z}$-linear mapping $\overline{\phi} : N_1 \to N_2$ is ***compatible*** with the fans Σ_1 and Σ_2 if for every cone $\sigma_1 \in \Sigma_1$, there exists a cone $\sigma_2 \in \Sigma_2$ such that $\overline{\phi}_\mathbb{R}(\sigma_1) \subseteq \sigma_2$.

Example 3.3.2. Let $N_1 = \mathbb{Z}^2$ with basis e_1, e_2 and let Σ_r be the fan from Figure 4 in §3.1. By Example 3.1.16, X_{Σ_r} is the Hirzebruch surface $\mathscr{H}_r$. Also let $N_2 = \mathbb{Z}$ and consider the fan Σ giving $\mathbb{P}^1$:



as in Example 3.1.11. The mapping

$$\overline{\phi} : N_1 \longrightarrow N_2, \qquad ae_1 + be_2 \longmapsto a$$

is compatible with the fans Σ_r and Σ since each cone of Σ_r maps onto a cone of Σ. If $r \neq 0$, on the other hand, the mapping

$$\overline{\psi} : N_1 \longrightarrow N_2, \qquad ae_1 + be_2 \longmapsto b$$

is not compatible with these fans since $\sigma_3 \in \Sigma_r$ does not map into a cone of Σ. $\Diamond$

The Definition of Toric Morphism. In §1.3, we defined a toric morphism in the affine case and gave an equivalent condition in Proposition 1.3.14. For general toric varieties, it more convenient to take the result of Proposition 1.3.14 as the *definition* of toric morphism.

Definition 3.3.3. Let X_{Σ_1}, X_{Σ_2} be normal toric varieties, with Σ_1 a fan in $(N_1)_\mathbb{R}$ and Σ_2 a fan in $(N_2)_\mathbb{R}$. A morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is ***toric*** if ϕ maps the torus $T_{N_1} \subseteq X_{\Sigma_1}$ into $T_{N_2} \subseteq X_{\Sigma_2}$ and $\phi|_{T_{N_1}}$ is a group homomorphism.

The proof of part (b) of Proposition 1.3.14 generalizes easily to show that any toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is an *equivariant mapping* for the T_{N_1}- and T_{N_2}-actions. That is, we have a commutative diagram

$$\begin{array}{ccc} T_{N_1} \times X_{\Sigma_1} & \longrightarrow & X_{\Sigma_1} \\ \Big\downarrow{\scriptstyle \phi|_{T_{N_1}} \times \phi} & & \Big\downarrow{\scriptstyle \phi} \\ T_{N_2} \times X_{\Sigma_2} & \longrightarrow & X_{\Sigma_2} \end{array} \tag{3.3.1}$$

where the horizontal maps give the torus actions.

Our first result shows that toric morphisms $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ correspond to $\mathbb{Z}$-linear mappings $\overline{\phi} : N_1 \to N_2$ that are compatible with the fans Σ_1 and Σ_2.

Theorem 3.3.4. *Let N_1, N_2 be lattices and let Σ_i be a fan in $(N_i)_{\mathbb{R}}$, $i = 1, 2$.*

(a) *If $\overline{\phi} : N_1 \to N_2$ is a $\mathbb{Z}$-linear map that is compatible with Σ_1 and Σ_2, then there is a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ such that $\phi|_{T_{N_1}}$ is the map*

$$\overline{\phi} \otimes 1 : N_1 \otimes_{\mathbb{Z}} \mathbb{C}^* \longrightarrow N_2 \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

(b) *Conversely, if $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is a toric morphism, then ϕ induces a $\mathbb{Z}$-linear map $\overline{\phi} : N_1 \to N_2$ that is compatible with the fans Σ_1 and Σ_2.*

Proof. To prove part (a), let σ_1 be a cone in Σ_1. Since ϕ is compatible with Σ_1 and Σ_2, there is a cone $\sigma_2 \in \Sigma_2$ with $\overline{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$. Then Proposition 1.3.15 shows that $\overline{\phi}$ induces a toric morphism $\phi_{\sigma_1} : U_{\sigma_1} \to U_{\sigma_2}$. Using the general criterion for gluing morphisms from Exercise 3.3.1, you will show in Exercise 3.3.2 that the ϕ_{σ_1} glue together to give a morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$. Moreover, ϕ is toric because taking $\sigma_1 = \{0\}$ gives $\phi_{\{0\}} : T_{N_1} \to T_{N_2}$, which is easily seen to be the group homomorphism induced by the $\mathbb{Z}$-linear map $\overline{\phi} : N_1 \to N_2$.

For part (b), we show first that the toric morphism ϕ induces a $\mathbb{Z}$-linear map $\overline{\phi} : N_1 \to N_2$. This follows since $\phi|_{T_{N_1}}$ is a group homomorphism. Hence, given $u \in N_1$, the one-parameter subgroup $\lambda^u : \mathbb{C}^* \to T_{N_1}$ can be composed with $\phi|_{T_{N_1}}$ to give the one-parameter subgroup $\phi|_{T_{N_1}} \circ \lambda^u : \mathbb{C}^* \to T_{N_2}$. This defines an element $\overline{\phi}(u) \in N_2$. It is straightforward to show that $\overline{\phi} : N_1 \to N_2$ is $\mathbb{Z}$-linear.

It remains to show that $\overline{\phi}$ is compatible with Σ_1 and Σ_2. Take $\sigma_1 \in \Sigma_1$. By the Orbit-Cone Correspondence (Theorem 3.2.6), this gives the T_{N_1}-orbit $O_1 = O(\sigma_1) \subseteq X_{\Sigma_1}$. Because of the equivariance (3.3.1), there is a T_{N_2}-orbit $O_2 \subseteq X_{\Sigma_2}$ with $\phi(O_1) \subseteq O_2$. Using Theorem 3.2.6 again, we have $O_2 = O(\sigma_2)$ for some cone $\sigma_2 \in \Sigma_2$. Thus $\phi(O(\sigma_1)) \subseteq O(\sigma_2)$. Furthermore, if $\tau_1 \preceq \sigma_1$ is a face, then by the same reasoning, there is some cone $\tau_2 \in \Sigma_2$ such that $\phi(O(\tau_1)) \subseteq O(\tau_2)$.

We claim that in this situation τ_2 must be a face of σ_2. This follows since $O(\sigma_1) \subseteq \overline{O(\tau_1)}$ by part (d) of Theorem 3.2.6. Since ϕ is continuous in the Zariski topology, $\phi\Big(\overline{O(\tau_1)}\Big) \subseteq \overline{O(\tau_2)}$. But the only orbits contained in the closure of $O(\tau_2)$

are the orbits corresponding to cones which have τ_2 as a face. So τ_2 is a face of σ_2. It follows from part (c) of Theorem 3.2.6 that ϕ also maps the affine open subset $U_{\sigma_1} \subseteq X_{\Sigma_1}$ into $U_{\sigma_2} \subseteq X_{\Sigma_2}$, i.e.,

$$\phi(U_{\sigma_1}) \subseteq U_{\sigma_2}. \tag{3.3.2}$$

Hence ϕ induces a toric morphism $U_{\sigma_1} \to U_{\sigma_2}$, which by Proposition 1.3.15 implies that $\overline{\phi}_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$. Hence $\overline{\phi}$ is compatible with the fans Σ_1 and Σ_2. □

First Examples. Here are some examples of toric morphisms defined by mappings compatible with the corresponding fans.

Example 3.3.5. Let $N_1 = \mathbb{Z}^2$ and $N_2 = \mathbb{Z}$, and let

$$\overline{\phi} : N_1 \longrightarrow N_2, \qquad ae_1 + be_2 \longmapsto a,$$

be the first mapping in Example 3.3.2. We saw that $\overline{\phi}$ is compatible with the fans Σ_r of the Hirzebruch surface $\mathscr{H}_r$ and Σ of $\mathbb{P}^1$. Theorem 3.3.4 implies that there is a corresponding toric morphism $\phi : \mathscr{H}_r \to \mathbb{P}^1$. We will see later in the section that this mapping gives $\mathscr{H}_r$ the structure of a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. ♢

Example 3.3.6. Let $N = \mathbb{Z}^n$ and Σ be a fan in $N_{\mathbb{R}}$. For $\ell \in \mathbb{Z}_{>0}$, the multiplication map

$$\overline{\phi}_\ell : N \longrightarrow N, \qquad a \longmapsto \ell \cdot a$$

is compatible with Σ. By Theorem 3.3.4, there is a corresponding toric morphism $\phi_\ell : X_\Sigma \to X_\Sigma$ whose restriction to $T_N \subseteq X_\Sigma$ is the group endomorphism

$$\phi_\ell|_{T_N}(t_1, \ldots, t_n) = (t_1^\ell, \ldots, t_n^\ell).$$

For a concrete example, let Σ be the fan in $N_{\mathbb{R}} = \mathbb{R}^2$ from Figure 2 and take $\ell = 2$. Then we obtain the morphism $\phi_2 : \mathbb{P}^2 \to \mathbb{P}^2$ defined in homogeneous coordinates by $\phi_2(x_0, x_1, x_2) = (x_0^2, x_1^2, x_2^2)$. We will use ϕ_ℓ in Chapter 9. ♢

Sublattices of Finite Index. We get an interesting toric morphism when a lattice N' has finite index in a larger lattice N. If Σ is a fan in $N_{\mathbb{R}}$, then we can view Σ as a fan either in $N'_{\mathbb{R}}$ or in $N_{\mathbb{R}}$, and the inclusion $N' \hookrightarrow N$ is compatible with the fan Σ in $N'_{\mathbb{R}}$ and $N_{\mathbb{R}}$. As in Chapter 1, we obtain toric varieties $X_{\Sigma,N'}$ and $X_{\Sigma,N}$ depending on which lattice we consider, and the inclusion $N' \hookrightarrow N$ induces a toric morphism

$$\phi : X_{\Sigma,N'} \longrightarrow X_{\Sigma,N}.$$

Proposition 3.3.7. *Let N' be a sublattice of finite index in N and let Σ be a fan in $N_{\mathbb{R}} = N'_{\mathbb{R}}$. Let $G = N/N'$. Then*

$$\phi : X_{\Sigma,N'} \longrightarrow X_{\Sigma,N}$$

induced by the inclusion $N' \hookrightarrow N$ presents $X_{\Sigma,N}$ as the quotient $X_{\Sigma,N'}/G$.

Proof. Since N' has finite index in N, Proposition 1.3.18 shows that the finite group $G = N/N'$ is the kernel of $T_{N'} \to T_N$. It follows that G acts on $X_{\Sigma,N'}$. This action is compatible with the inclusion $U_{\sigma,N'} \subseteq X_{\Sigma,N'}$ for each cone $\sigma \in \Sigma$. Using Proposition 1.3.18 again, we see that $U_{\sigma,N'}/G \simeq U_{\sigma,N}$, which easily implies that $X_{\Sigma,N'}/G \simeq X_{\Sigma,N}$. $\square$

We will revisit Proposition 3.3.7 in Chapter 5, where we will show that the map $\phi : X_{\Sigma,N'} \to X_{\Sigma,N}$ is a *geometric quotient*.

Example 3.3.8. Let $N = \mathbb{Z}^2$, and Σ be the fan shown in Figure 5, so $X_{\Sigma,N}$ gives the weighted projective space $\mathbb{P}(1,1,2)$. Let N' be the sublattice of N given by $N' = \{(a,b) \in N \mid b \equiv 0 \bmod 2\}$, so N' has index 2 in N. Note that N' is generated by $u_1 = e_1$, $u_2 = 2e_2$ and that

$$u_0 = -e_1 - 2e_2 = -u_1 - u_2 \in N'.$$

Let $\overline{\phi} : N' \hookrightarrow N$ be the inclusion map. It is not difficult to see that with respect to the lattice N', $X_{\Sigma,N'} \simeq \mathbb{P}^2$ (Exercise 3.3.3). By Theorem 3.3.4, the $\mathbb{Z}$-linear map $\overline{\phi}$ induces a toric morphism $\phi : \mathbb{P}^2 \to \mathbb{P}(1,1,2)$, and by Proposition 3.3.7, it follows that $\mathbb{P}(1,1,2) \simeq \mathbb{P}^2/G$ for $G = N/N' \simeq \mathbb{Z}/2\mathbb{Z}$.

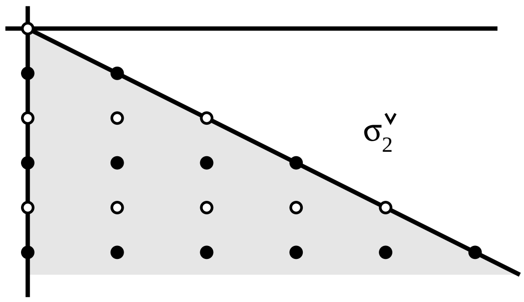


Figure 8. The semigroups $\sigma_2^\vee \cap M$ and $\sigma_2^\vee \cap M'$

The cone σ_2 from Figure 5 has the dual cone $\sigma_2^\vee$ shown in Figure 8. It is instructive to consider how $\sigma_2^\vee$ interacts with the lattice M' dual to N'. One checks that $M' \simeq \{(a,b/2) : a,b \in \mathbb{Z}\}$ and $\sigma_2^\vee = \mathrm{Cone}(2e_1 - e_2, -e_2)$. In Figure 8, the points in $\sigma_2^\vee \cap M$ are shown in white, and the points in $\sigma_2^\vee \cap M'$ not in $\sigma_2^\vee \cap M$ are shown in black. Note that the picture in $\sigma_2^\vee \cap M$ is the same (up to a change of coordinates in $\mathrm{GL}(2,\mathbb{Z})$) as Figure 10 from Chapter 1. This shows again that $\mathbb{P}(1,1,2)$ contains the affine open subset $U_{\sigma_2,N}$ isomorphic to the rational normal cone $\widehat{C}_2$. On the other hand $U_{\sigma_2,N'} \simeq \mathbb{C}^2$ is smooth. The other affine open subsets corresponding to σ_1 and σ_0 are isomorphic to $\mathbb{C}^2$ in both $\mathbb{P}^2$ and in $\mathbb{P}(1,1,2)$. $\Diamond$

Torus Factors. A toric variety X_Σ has a *torus factor* if it is equivariantly isomorphic to the product of a nontrivial torus and a toric variety of smaller dimension.

Proposition 3.3.9. *Let X_Σ be the toric variety of the fan Σ. Then the following are equivalent:*

(a) *X_Σ has a torus factor.*

(b) *There is a nonconstant morphism $X_\Sigma \to \mathbb{C}^*$.*

(c) *The u_ρ, $\rho \in \Sigma(1)$, do not span $N_\mathbb{R}$.*

Recall that $\Sigma(1)$ consists of the 1-dimensional cones of Σ, i.e., its rays, and that u_ρ is the minimal generator of a ray $\rho \in \Sigma(1)$.

Proof. If $X_\Sigma \simeq X_{\Sigma'} \times (\mathbb{C}^*)^r$ for $r > 0$ and some toric variety $X_{\Sigma'}$, then a nontrivial character of $(\mathbb{C}^*)^r$ gives a nonconstant morphism $X_\Sigma \to (\mathbb{C}^*)^r \to \mathbb{C}^*$.

If $\phi : X_\Sigma \to \mathbb{C}^*$ is a nonconstant morphism, then Exercise 3.3.4 implies that $\phi|_{T_N} = c\chi^m$ where $c \in \mathbb{C}^*$ and $m \in M \setminus \{0\}$. Multiplying by c^{-1}, we may assume that $\phi|_{T_N} = \chi^m$. Then ϕ is a toric morphism coming from a nonzero homomorphism $\overline{\phi} : N \to \mathbb{Z}$. Since $\mathbb{C}^*$ comes from the trivial fan, $\overline{\phi}_\mathbb{R}$ maps all cones of Σ to the origin. Hence $u_\rho \in \ker(\overline{\phi}_\mathbb{R})$ for all $\rho \in \Sigma(1)$, so the u_ρ do not span $N_\mathbb{R}$.

Finally, suppose that the u_ρ, $\rho \in \Sigma(1)$ span a proper subspace of $N_\mathbb{R}$. Then $N' = \mathrm{Span}(u_\rho \mid \rho \in \Sigma(1)) \cap N$ is proper sublattice of N such that N/N' is torsion-free, so N' has a complement N'' with $N = N' \times N''$. Furthermore, Σ can be regarded as a fan Σ' in $N'_\mathbb{R}$, and then Σ is the product fan $\Sigma = \Sigma' \times \Sigma''$, where Σ'' is the trivial fan in $N''_\mathbb{R}$. Then Proposition 3.1.14 gives an isomorphism

$$X_\Sigma \simeq X_{\Sigma',N'} \times T_{N''} \simeq X_{\Sigma',N'} \times (\mathbb{C}^*)^{n-k},$$

where $\dim N_\mathbb{R} = n$ and $\dim N'_\mathbb{R} = k$. □

In later chapters, toric varieties *without* torus factors will play an important role. Hence we state the following corollary of Proposition 3.3.9.

Corollary 3.3.10. *Let X_Σ be the toric variety of the fan Σ. Then the following are equivalent:*

(a) *X_Σ has no torus factors.*

(b) *Every morphism $X_\Sigma \to \mathbb{C}^*$ is constant, i.e., $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma})^* = \mathbb{C}^*$.*

(c) *The u_ρ, $\rho \in \Sigma(1)$, span $N_\mathbb{R}$.* □

We can also think about torus factors from the point of view of sublattices.

Proposition 3.3.11. *Let $N' \subseteq N$ be a sublattice with* $\dim N_\mathbb{R} = n$, $\dim N'_\mathbb{R} = k$. *Let Σ be a fan in $N'_\mathbb{R}$, which we can regard as a fan in $N_\mathbb{R}$. Then:*

(a) *If N' is spanned by a subset of a basis of N, then we have an isomorphism*

$$\phi : X_{\Sigma,N} \simeq X_{\Sigma,N'} \times T_{N/N'} \simeq X_{\Sigma,N'} \times (\mathbb{C}^*)^{n-k}.$$

(b) *In general, a basis for N' can be extended to a basis of a sublattice $N'' \subseteq N$ of finite index. Then $X_{\Sigma,N}$ is isomorphic to the quotient of*

$$X_{\Sigma,N''} \simeq X_{\Sigma,N'} \times T_{N''/N'} \simeq X_{\Sigma,N'} \times (\mathbb{C}^*)^{n-k}$$

by the finite abelian group N/N''.

Proof. Part (a) follows from the proof of Proposition 3.3.9, and part (b) follows from part (a) and Proposition 3.3.7. □

Refinements of Fans and Blowups. Given a fan Σ in $N_{\mathbb{R}}$, a fan Σ' *refines* Σ if every cone of Σ' is contained in a cone of Σ and $|\Sigma'| = |\Sigma|$. Hence every cone of Σ is a union of cones of Σ'. When Σ' refines Σ, the identity mapping on N is clearly compatible with Σ' and Σ. This yields a toric morphism $\phi : X_{\Sigma'} \to X_{\Sigma}$.

Example 3.3.12. Consider the fan Σ' in $N \simeq \mathbb{Z}^2$ pictured in Figure 1 from §3.1. This is a refinement of the fan Σ consisting of $\mathrm{Cone}(e_1, e_2)$ and its faces. The corresponding toric varieties are $X_\Sigma \simeq \mathbb{C}^2$ and $X_{\Sigma'} \simeq W = \mathbf{V}(x_0y - x_1x) \subseteq \mathbb{P}^1 \times \mathbb{C}^2$, the blowup of $\mathbb{C}^2$ at the origin (see Example 3.1.15). The identity map on N induces a toric morphism $\phi : W \to \mathbb{C}^2$. This "blowdown" morphism maps $\mathbb{P}^1 \times \{0\} \subseteq W$ to $0 \in \mathbb{C}^2$ and is injective outside of $\mathbb{P}^1 \times \{0\}$ in W. ◊

We can generalize this example and Example 3.1.5 as follows.

Definition 3.3.13. Let Σ be a fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$. Let $\sigma = \mathrm{Cone}(u_1, \ldots, u_n)$ be a smooth cone in Σ, so that $u_1, \ldots, u_n$ is a basis for N. Let $u_0 = u_1 + \cdots + u_n$ and let $\Sigma'(\sigma)$ be the set of all cones generated by subsets of $\{u_0, \ldots, u_n\}$ not containing $\{u_1, \ldots, u_n\}$. Then

$$\Sigma^*(\sigma) = (\Sigma \setminus \{\sigma\}) \cup \Sigma'(\sigma)$$

is a fan in $N_{\mathbb{R}}$ called the ***star subdivision*** of Σ along σ.

Example 3.3.14. Let $\sigma = \mathrm{Cone}(u_1, u_2, u_3) \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^3$ be a smooth cone. Figure 9 on the next page shows the star subdivision of σ into three cones

$$\mathrm{Cone}(u_0, u_1, u_2),\ \mathrm{Cone}(u_0, u_1, u_3),\ \mathrm{Cone}(u_0, u_2, u_3).$$

The fan $\Sigma^*(\sigma)$ consists of these cones, together with their faces. ◊

Proposition 3.3.15. *$\Sigma^*(\sigma)$ is a refinement of Σ, and the induced toric morphism*

$$\phi : X_{\Sigma^*(\sigma)} \longrightarrow X_\Sigma$$

makes $X_{\Sigma^*(\sigma)}$ the blowup of X_Σ at the distinguished point γ_σ corresponding to the cone σ.

Proof. Since Σ and $\Sigma^*(\sigma)$ are the same outside the cone σ, without loss of generality, we may reduce to the case that Σ is the fan consisting of σ and all of its faces, and X_Σ is the affine toric variety $U_\sigma \simeq \mathbb{C}^n$.

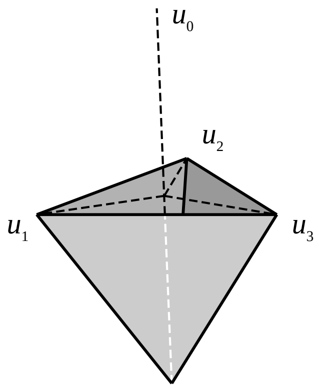


Figure 9. The star subdivision $\Sigma^*(\sigma)$ in Example 3.3.14

Under the Orbit-Cone Correspondence (Theorem 3.2.6), σ corresponds to the distinguished point γ_σ, the origin (the unique fixed point of the torus action). By Theorem 3.3.4, the identity map on N induces a toric morphism

$$\phi : X_{\Sigma^*(\sigma)} \to U_\sigma \simeq \mathbb{C}^n.$$

It is easy to check that the affine open sets covering $X_{\Sigma^*(\sigma)}$ are the same as for the blowup of $\mathbb{C}^n$ at the origin from Exercise 3.0.8, and they are glued together in the same way by Exercise 3.1.5. $\square$

The blowup X_Σ at γ_σ is sometimes denoted $\mathrm{Bl}_{\gamma_\sigma}(X_\Sigma)$. In this notation, the blowup of $\mathbb{C}^n$ at the origin is written $\mathrm{Bl}_0(\mathbb{C}^n)$.

The point blown up in Proposition 3.3.15 is a fixed point of the torus action. In some cases, torus-invariant subvarieties of larger dimension have equally nice blowups. We begin with the affine case. The standard basis $e_1,\dots,e_n$ of $\mathbb{Z}^n$ gives $\sigma = \mathrm{Cone}(e_1,\dots,e_n)$ with $U_\sigma = \mathbb{C}^n$, and the face $\tau = \mathrm{Cone}(e_1,\dots,e_r)$, $2 \le r \le n$, gives the orbit closure

$$V(\tau) = \overline{O(\tau)} = \{0\} \times \mathbb{C}^{n-r}.$$

To construct the blowup of $V(\tau)$, let $u_0 = u_1 + \cdots + u_r$ and consider the fan

$$\Sigma^*(\tau) = \{\mathrm{Cone}(A) \mid A \subseteq \{u_0,\dots,u_n\},\ \{u_1,\dots,u_r\} \not\subseteq A\}. \tag{3.3.3}$$

Example 3.3.16. Let $\sigma = \mathrm{Cone}(e_1,e_2,e_3) \subseteq N_\mathbb{R} \simeq \mathbb{R}^3$ and $\tau = \mathrm{Cone}(e_1,e_2)$. The star subdivision relative to τ subdivides σ into the cones

$$\mathrm{Cone}(e_0,e_1,e_3),\ \mathrm{Cone}(e_0,e_2,e_3),$$

as shown in Figure 10 on the next page. The fan $\Sigma^*(\tau)$ consists of these two cones, together with their faces. $\Diamond$

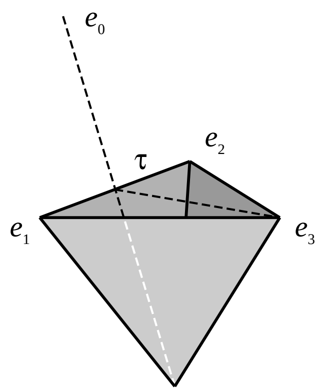


Figure 10. The star subdivision $\Sigma^*(\tau)$ in Example 3.3.16

For the fan (3.3.3), the toric variety $X_{\Sigma^*(\tau)}$ is the blowup of $\{0\} \times \mathbb{C}^{n-r} \subseteq \mathbb{C}^n$. To see why, observe that $\Sigma^*(\tau)$ is a product fan. Namely, $\mathbb{Z}^n = \mathbb{Z}^r \times \mathbb{Z}^{n-r}$, and

$$\Sigma^*(\tau) = \Sigma_1 \times \Sigma_2,$$

where Σ_1 is the fan for $\mathrm{Bl}_0(\mathbb{C}^r)$ (coming from a refinement of $\mathrm{Cone}(u_1, \dots, u_r)$) and Σ_2 is the fan for $\mathbb{C}^{n-r}$ (coming from $\mathrm{Cone}(u_{r+1}, \dots, u_n)$). It follows that

$$X_{\Sigma^*(\tau)} = \mathrm{Bl}_0(\mathbb{C}^r) \times \mathbb{C}^{n-r}.$$

Since $\mathrm{Bl}_0(\mathbb{C}^r)$ is built by replacing $0 \in \mathbb{C}^r$ with $\mathbb{P}^{r-1}$, it follows that $X_{\Sigma^*(\tau)} = \mathrm{Bl}_0(\mathbb{C}^r) \times \mathbb{C}^{n-r}$ is built by replacing $\{0\} \times \mathbb{C}^{n-r} \subseteq \mathbb{C}^n$ with $\mathbb{P}^{r-1} \times \mathbb{C}^{n-r}$. The intuitive idea is that $\mathrm{Bl}_0(\mathbb{C}^r)$ separates directions through the origin in $\mathbb{C}^r$, while the blowup $\mathrm{Bl}_{\{0\}\times\mathbb{C}^{n-r}}(\mathbb{C}^n) = X_{\Sigma^*(\tau)}$ separates *normal* directions to $\{0\} \times \mathbb{C}^{n-r}$ in $\mathbb{C}^n$. One can also study $\mathrm{Bl}_{\{0\}\times\mathbb{C}^{n-r}}(\mathbb{C}^n)$ by working on affine pieces given by the maximal cones of $\Sigma^*(\tau)$—see [**218**, Prop. 1.26].

We generalize (3.3.3) as follows.

Definition 3.3.17. Let Σ be a fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$ and assume $\tau \in \Sigma$ has the property that all cones of Σ containing τ are smooth. Let $u_\tau = \sum_{\rho\in\tau(1)} u_\rho$ and for each cone $\sigma \in \Sigma$ containing τ, set

$$\Sigma^*_\sigma(\tau) = \{\mathrm{Cone}(A) \mid A \subseteq \{u_\tau\} \cup \sigma(1),\ \tau(1) \not\subseteq A\}.$$

Then the ***star subdivision*** of Σ relative to τ is the fan

$$\Sigma^*(\tau) = \{\sigma \in \Sigma \mid \tau \not\subseteq \sigma\} \cup \bigcup_{\tau\subseteq\sigma} \Sigma^*_\sigma(\tau).$$

The fan $\Sigma^*(\tau)$ is a refinement of Σ and hence induces a toric morphism

$$\phi : X_{\Sigma^*(\tau)} \longrightarrow X_\Sigma.$$

Under the map ϕ, $X_{\Sigma^*(\tau)}$ becomes the blowup $\mathrm{Bl}_{V(\tau)}(X_\Sigma)$ of X_Σ along the orbit closure $V(\tau)$.

In Chapters 10 and 11 we will use toric morphisms coming from a generalized version of star subdivision to resolve the singularities of toric varieties.

Exact Sequences and Fibrations. Next, we consider a class of toric morphisms that have a nice local structure. To begin, consider a surjective $\mathbb{Z}$-linear mapping

$$\overline{\phi} : N \twoheadrightarrow N'.$$

If Σ in $N_\mathbb{R}$ and Σ' in $N'_\mathbb{R}$ are compatible with $\overline{\phi}$, then we have a corresponding toric morphism

$$\phi : X_\Sigma \longrightarrow X_{\Sigma'}.$$

Now let $N_0 = \ker(\overline{\phi})$, so that we have an exact sequence

$$0 \longrightarrow N_0 \longrightarrow N \xrightarrow{\overline{\phi}} N' \longrightarrow 0. \tag{3.3.4}$$

It is easy to check that

$$\Sigma_0 = \{\sigma \in \Sigma \mid \sigma \subseteq (N_0)_\mathbb{R}\}$$

is a subfan of Σ whose cones lie in $(N_0)_\mathbb{R} \subseteq N_\mathbb{R}$. By Proposition 3.3.11,

$$X_{\Sigma_0,N} \simeq X_{\Sigma_0,N_0} \times T_{N'} \tag{3.3.5}$$

since $N/N_0 \simeq N'$. Furthermore, $\overline{\phi}$ is compatible with Σ_0 in $N_\mathbb{R}$ and the trivial fan $\{0\}$ in $N'_\mathbb{R}$. This gives the toric morphism

$$\phi|_{X_{\Sigma_0,N}} : X_{\Sigma_0,N} \longrightarrow T_{N'}.$$

In fact, by the reasoning to prove Proposition 3.3.4,

$$\phi^{-1}(T_{N'}) = X_{\Sigma_0,N} \simeq X_{\Sigma_0,N_0} \times T_{N'}. \tag{3.3.6}$$

In other words, the part of X_Σ lying over $T_{N'} \subseteq X_{\Sigma'}$ is identified with the product of $T_{N'}$ and the toric variety X_{Σ_0,N_0}. We say this subset of X_Σ is a *fiber bundle* over $T_{N'}$ with fiber X_{Σ_0,N_0}.

When the fan Σ has a suitable structure relative to $\overline{\phi}$, we can make a similar statement for every torus-invariant affine open subset of $X_{\Sigma'}$.

Definition 3.3.18. In the situation of (3.3.4), we say Σ is ***split by*** Σ' and Σ_0 if there exists a subfan $\widehat{\Sigma} \subseteq \Sigma$ such that:

(a) $\overline{\phi}_\mathbb{R}$ maps each cone $\widehat{\sigma} \in \widehat{\Sigma}$ bijectively to a cone $\sigma' \in \Sigma'$ such that $\overline{\phi}(\widehat{\sigma} \cap N) = \sigma' \cap N'$. Furthermore, the map $\widehat{\sigma} \mapsto \sigma'$ defines a bijection $\widehat{\Sigma} \xrightarrow{\sim} \Sigma'$.

(b) Given cones $\widehat{\sigma} \in \widehat{\Sigma}$ and $\sigma_0 \in \Sigma_0$, the sum $\widehat{\sigma} + \sigma_0$ lies in Σ, and every cone of Σ arises this way.

Theorem 3.3.19. *If Σ is split by Σ' and Σ_0 as in Definition 3.3.18, then X_Σ is a locally trival fiber bundle over $X_{\Sigma'}$ with fiber X_{Σ_0,N_0}, i.e., $X_{\Sigma'}$ has a cover by affine open subsets U satisfying*

$$\phi^{-1}(U) \simeq X_{\Sigma_0,N_0} \times U.$$

In particular, all fibers of $X_\Sigma \to X_{\Sigma'}$ are isomorphic to X_{Σ_0,N_0}.

Proof. Fix σ' in Σ' and let $\Sigma(\sigma') = \{\sigma \in \Sigma \mid \overline{\phi}(\sigma) \subseteq \sigma'\}$. Then

$$\phi^{-1}(U_{\sigma'}) = X_{\Sigma(\sigma')}.$$

It remains to show that $X_{\Sigma(\sigma')} \simeq X_{\Sigma_0,N_0} \times U_{\sigma'}$. Since $\Sigma(\sigma')$ is split by $\Sigma_0 \cap \Sigma(\sigma')$ and $\widehat{\Sigma} \cap \Sigma(\sigma')$, we may assume $X_{\Sigma'} = U_{\sigma'}$. In other words, we are reduced to the case when Σ' consists of σ' and its proper faces.

A $\mathbb{Z}$-linear map $\overline{\nu} : N' \to N$ *splits* the exact sequence (3.3.4) provided $\overline{\phi} \circ \overline{\nu}$ is the identity on N'. A splitting induces an isomorphism

$$N_0 \times N' \simeq N.$$

By Definition 3.3.18, there is a cone $\widehat{\sigma} \in \widehat{\Sigma}$ such that $\overline{\phi}(\widehat{\sigma} \cap N) = \sigma' \cap N'$ and $\overline{\phi}_\mathbb{R}$ maps $\widehat{\sigma}$ bijectively to σ'. Using $\widehat{\sigma}$, one can find a splitting $\overline{\nu}$ with the property that $\overline{\nu}_\mathbb{R}$ maps τ' to $\widehat{\tau}$ for all $\widehat{\tau} \in \widehat{\Sigma}$ (Exercise 3.3.5). Using Definition 3.3.18 again, we see that

$$(N_0)_\mathbb{R} \times N'_\mathbb{R} \simeq N_\mathbb{R}$$

carries the product fan $(\Sigma_0, (N_0)_\mathbb{R}) \times (\Sigma', N'_\mathbb{R})$ to the fan $(\Sigma, N_\mathbb{R})$. By Proposition 3.1.14, we conclude that

$$X_\Sigma \simeq X_{\Sigma_0,N_0} \times X_{\Sigma'} \simeq X_{\Sigma_0,N_0} \times U_{\sigma'},$$

and the theorem is proved. □

Example 3.3.20. To complete the discussion begun in Examples 3.3.2 and 3.3.5, consider the toric morphism $\phi : \mathscr{H}_r \to \mathbb{P}^1$ induced by the mapping

$$\overline{\phi} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}, \qquad ae_1 + be_2 \longmapsto a.$$

The fan Σ_r of $\mathscr{H}_r$ is split by the fan of $\mathbb{P}^1$ and $\Sigma_0 = \{\sigma \in \Sigma_r \mid \overline{\phi}_\mathbb{R}(\sigma) = \{0\}\}$ because of the subfan $\widehat{\Sigma}$ of Σ_r consisting of the cones

$$\mathrm{Cone}(-e_1 + re_2), \{0\}, \mathrm{Cone}(e_1).$$

These cones are mapped bijectively to the cones in Σ' under $\overline{\phi}_\mathbb{R}$. Note also that Σ_0 consists of the cones

$$\mathrm{Cone}(e_2), \{0\}, \mathrm{Cone}(-e_2).$$

The fans $\widehat{\Sigma}$ and Σ_0 are shown in Figure 11 on the next page.

As we vary over all $\widehat{\sigma} \in \widehat{\Sigma}$ and $\sigma_0 \in \Sigma_0$, the sums $\widehat{\sigma} + \sigma_0$ give all cones of $\mathscr{H}_r$. Hence Theorem 3.3.19 shows that $\mathscr{H}_r$ is a locally trivial fibration over $\mathbb{P}^1$, with fibers isomorphic to

$$X_{\Sigma_0,N_0} \simeq \mathbb{P}^1,$$

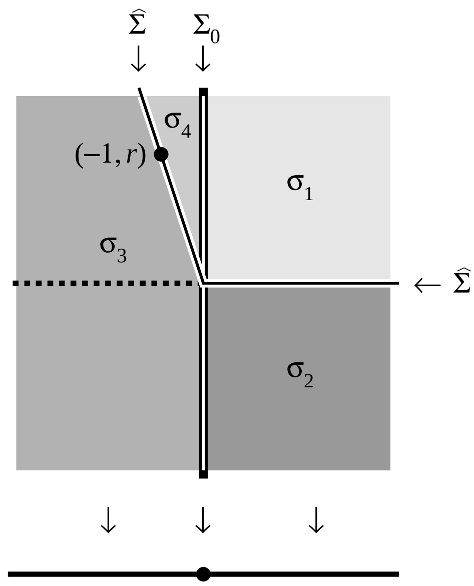


Figure 11. The Splitting of the Fan Σ_r

where $N_0 = \ker(\overline{\phi})$ gives the vertical axis in Figure 11. This fibration is not globally trivial when $r > 0$, i.e., it is not true that $\mathscr{H}_r \simeq \mathbb{P}^1 \times \mathbb{P}^1$. There is some "twisting" on the fibers involved when we try to glue together the $\phi^{-1}(U_{\sigma'}) \simeq U_{\sigma'} \times \mathbb{P}^1$ to obtain $\mathscr{H}_r$. $\Diamond$

We will give another, more precise, description of these fiber bundles and the "twisting" mentioned above using the language of sheaves in Chapter 7.

The definition of splitting fan in Definition 3.3.18 requires that $\overline{\phi}(\widehat{\sigma} \cap N) = \sigma' \cap N'$ when $\widehat{\sigma} \in \widehat{\Sigma}$ maps to $\sigma' \in \Sigma'$. Exercise 3.3.6 will give an example of how Theorem 3.3.19 can fail if this condition is not met, and Exercise 3.3.7 will explore how to modify the theorem when this happens.

Images of Distinguished Points. Each orbit $O(\sigma)$ in a toric variety X_Σ contains a distinguished point γ_σ, and each orbit closure $V(\sigma)$ is a toric variety in its own right. These structures are compatible with toric morphisms as follows.

Lemma 3.3.21. *Let $\phi : X_\Sigma \to X_{\Sigma'}$ be the toric morphism coming from a map $\overline{\phi} : N \to N'$ that is compatible with Σ and Σ'. Given $\sigma \in \Sigma$, let $\sigma' \in \Sigma'$ be the minimal cone of Σ' containing $\overline{\phi}_{\mathbb{R}}(\sigma)$. Then:*

(a) *$\phi(\gamma_\sigma) = \gamma_{\sigma'}$, where $\gamma_\sigma \in O(\sigma)$ and $\gamma_{\sigma'} \in O(\sigma')$ are the distinguished points.*

(b) *$\phi(O(\sigma)) \subseteq O(\sigma')$ and $\phi\big(V(\sigma)\big) \subseteq V(\sigma')$.*

(c) *The induced map $\phi|_{V(\sigma)} : V(\sigma) \to V(\sigma')$ is a toric morphism.*

Proof. First observe that if $\sigma_1', \sigma_2' \in \Sigma'$ contain $\overline{\phi}_{\mathbb{R}}(\sigma)$, then so does their intersection. Hence Σ' has a minimal cone containing $\overline{\phi}_{\mathbb{R}}(\sigma)$.

To prove part (a), pick $u \in \mathrm{Relint}(\sigma)$ and observe that $\overline{\phi}(u) \in \mathrm{Relint}(\sigma')$ by the minimality of σ'. Then

$$\phi(\gamma_\sigma) = \phi\big(\lim_{t\to 0} \lambda^u(t)\big) = \lim_{t\to 0} \phi(\lambda^u(t)) = \lim_{t\to 0} \lambda^{\overline{\phi}(u)}(t) = \gamma_{\sigma'},$$

where the first and last equalities use Proposition 3.2.2.

The first assertion of part (b) follows immediately from part (a) by the equivariance, and the second assertion follows by continuity (as usual, we get the same closure in the classical and Zariski topologies).

For (c), observe that $\phi|_{O(\sigma)} : O(\sigma) \to O(\sigma')$ is a morphism that is also a group homomorphism—this follows easily from equivariance. Since the orbit closures are toric varieties by Proposition 3.2.7, the map $\phi|_{V(\sigma)} : V(\sigma) \to V(\sigma')$ is a toric morphism according to Definition 3.3.3. $\square$

Exercises for §3.3.

3.3.1. Let X be a variety with an affine open cover $\{U_i\}$, and let Y be a second variety. Let $\phi_i : U_i \to Y$ be a collection of morphisms. We say that a morphism $\phi : X \to Y$ is obtained by gluing the ϕ_i if $\phi|_{U_i} = \phi_i$ for all i. Show that there exists such a $\phi : X \to Y$ if and only if for every pair i, j,

$$\phi_i|_{U_i\cap U_j} = \phi_j|_{U_i\cap U_j}.$$

3.3.2. Let N_1, N_2 be lattices, and let Σ_1 in $(N_1)_{\mathbb{R}}$, Σ_2 in $(N_2)_{\mathbb{R}}$ be fans. Let $\overline{\phi} : N_1 \to N_2$ be a $\mathbb{Z}$-linear mapping that is compatible with the corresponding fans. Using Exercise 3.3.1 above, show that the toric morphisms $\phi_{\sigma_1} : U_{\sigma_1} \to U_{\sigma_2}$ constructed in the proof of Theorem 3.3.4 glue together to form a morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$.

3.3.3. This exercise asks you to verify some of the claims made in Example 3.3.8.

(a) Verify that $X_{\Sigma,N'} \simeq \mathbb{P}^2$ with respect to the lattice N'.

(b) Verify carefully that the affine open subset $U_{\sigma_2,N} \simeq \widehat{C}_2$, where $\widehat{C}_2$ is the rational normal cone $\widehat{C}_d$ with $d = 2$.

3.3.4. A character χ^m, $m \in M$, gives a morphism $T_N \to \mathbb{C}^*$. Here you will determine *all* morphisms $T_N \to \mathbb{C}^*$.

(a) Explain why morphisms $T_N \to \mathbb{C}^*$ correspond to invertible elements in the coordinate ring of T_N.

(b) Let $c \in \mathbb{C}^*$ and $\alpha \in \mathbb{Z}^n$. Prove that ct^α is invertible in $\mathbb{C}[t_1^{\pm1}, \ldots, t_n^{\pm1}]$ and that all invertible elements of $\mathbb{C}[t_1^{\pm1}, \ldots, t_n^{\pm1}]$ are of this form.

(c) Use part (a) to show that all morphisms $T_N \to \mathbb{C}^*$ on T_N are of the form $c\chi^m$ for $c \in \mathbb{C}^*$ and $m \in M$.

3.3.5. Let $\overline{\phi} : N \to N'$ be a surjective $\mathbb{Z}$-linear mapping and let $\widehat{\sigma}$ and σ' be cones in $N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$ respectively with the property that $\overline{\phi}_{\mathbb{R}}$ maps $\widehat{\sigma}$ bijectively onto σ'. Prove that $\overline{\phi}$ has a splitting $\overline{\nu} : N' \to N$ such that $\overline{\nu}$ maps σ' to $\widehat{\sigma}$.

3.3.6. Let Σ be the fan in $\mathbb{R}^2$ with minimal generators $u_1 = (0,-1)$, $u_2 = (2,1)$ and $u_3 = (0,1)$ and maximal cones $\mathrm{Cone}(u_1,u_2)$ and $\mathrm{Cone}(u_2,u_3)$. Let $\overline{\phi} : \mathbb{Z}^2 \to \mathbb{Z}$ be projection onto the first factor and let Σ_0 be the subfan of Σ defined in the discussion following (3.3.4). Also let Σ' be the fan in $\mathbb{R}$ with maximal cone $\sigma' = \mathbb{R}_{\geq 0}$.

(a) Prove that Σ is not split by Σ' and Σ_0. Hint: Show that X_{Σ_0,N_0} and $X_{\Sigma'}$ are smooth and then show that Theorem 3.3.19 must fail.

(b) Let $\widehat{\Sigma}$ be the subfan of Σ with maximal cone $\widehat{\sigma} = \mathrm{Cone}(u_2)$. Show that this satisfies all parts of Definition 3.3.18 except for the requirement that $\overline{\phi}(\widehat{\sigma} \cap \mathbb{Z}^2) = \sigma' \cap \mathbb{Z}$. Draw a picture similar to Figure 11 in Example 3.3.20.

3.3.7. In the situation of Definition 3.3.18, we say that Σ is *weakly split by* Σ' and Σ_0 if $\widehat{\Sigma}$ satisfies Definition 3.3.18 except that we no longer require $\overline{\phi}(\widehat{\sigma} \cap N) = \sigma' \cap N'$.

(a) Explain why Exercise 3.3.6 is an example of a weak splitting that is not a splitting.

(b) In the situation of a weak splitting, prove that all fibers of $\phi : X_\Sigma \to X_{\Sigma'}$ are isomorphic to X_{Σ_0,N_0}. Hint: First assume $X_{\Sigma'} = U_{\sigma'}$. Prove that there is a sublattice of $N'' \subseteq N'$ of finite index such that Σ splits when we use the lattices $\overline{N''}$ and N''. Then show that there is a commutative diagram

$$\begin{tikzcd}
X_{\Sigma_0,N_0} \times U_{\sigma',N''} \arrow[r] \arrow[d] & X_\Sigma \arrow[d] \\
U_{\sigma',N''} \arrow[r] & U_{\sigma'}.
\end{tikzcd}$$

such that $X_{\Sigma_0,N_0} \times U_{\sigma',N''}$ is the fiber product $X_\Sigma \times_{U_{\sigma'}} U_{\sigma',N''}$ as defined in (3.0.5).

Thus, while Theorem 3.3.19 may fail for a weak splitting, at least part remains true.

3.3.8. Let Σ' be the fan obtained from the fan Σ for $\mathbb{P}^2$ in Example 3.1.9 by the following process. Subdivide the cone σ_2 into two new cones σ_{21} and σ_{22} by inserting an edge $\mathrm{Cone}(-e_2)$.

(a) Show that the resulting toric variety $X_{\Sigma'}$ is smooth.

(b) Show that $X_{\Sigma'}$ is the blowup of $\mathbb{P}^2$ at the point $V(\sigma_2)$.

(c) Show that $X_{\Sigma'}$ is isomorphic to the Hirzebruch surface $\mathscr{H}_1$.

3.3.9. Let X_Σ be the toric variety obtained from $\mathbb{P}^2$ by blowing up the points $V(\sigma_1)$ and $V(\sigma_2)$ (see Figure 2 in Example 3.1.9). Show that X_Σ is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the point $V(\sigma_{11})$ (see Figure 3 in Example 3.1.12).

3.3.10. Let Σ' be the fan obtained from the fan Σ for $\mathbb{P}(1,1,2)$ in Example 3.1.17 by the following process. Subdivide the cone σ_2 into two new cones σ_{21} and σ_{22} by inserting an edge $\mathrm{Cone}(-u_2)$.

(a) Show that the resulting toric variety $X_{\Sigma'}$ is smooth.

(b) Construct a morphism $\phi : X_{\Sigma'} \to X_\Sigma$ and determine the fiber over the unique singular point of X_Σ.

(c) One of our smooth examples is isomorphic to $X_{\Sigma'}$. Which one is it?

3.3.11. Consider the action of the group $G = \{(\zeta,\zeta^3) \mid \zeta^5 = 1\} \subseteq (\mathbb{C}^*)^2$ on $\mathbb{C}^2$. We will study the quotient $\mathbb{C}^2/G$ and its resolution of singularities using toric morphisms.

(a) Let $N' = \mathbb{Z}^2$ and $N = \{(a/5, b/5) \mid a, b \in \mathbb{Z},\ b \equiv 3a \bmod 5\}$. Also let $\zeta_5 = e^{2\pi i/5}$. Prove that the map $N \to (\mathbb{C}^*)^2$ defined by $(a/5, b/5) \mapsto (\zeta_5^a, \zeta_5^b)$ induces an exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow G \longrightarrow 0.$$

(b) Let $\sigma = \mathrm{Cone}(e_1, e_2) \subseteq N'_{\mathbb{R}} = N_{\mathbb{R}} = \mathbb{R}^2$. The inclusion $N' \to N$ induces a toric morphism $U_{\sigma,N'} \to U_{\sigma,N}$. Prove that this is the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/G$ for the above action of G on $\mathbb{C}^2$.

(c) Find the Hilbert basis (i.e., the set of irreducible elements) of the semigroup $\sigma \cap N$. Hint: The Hilbert basis has four elements.

(d) Use the Hilbert basis from part (c) to subdivide σ. This gives a fan Σ with $|\Sigma| = \sigma$. Prove that Σ is smooth relative to N and that the resulting toric morphism

$$X_{\Sigma,N} \to U_{\sigma,N} = \mathbb{C}^2/G$$

is a resolution of singularities. See Chapter 10 for more details.

(e) The group G gives the finite set $G \subseteq (\mathbb{C}^*)^2 \subseteq \mathbb{C}^2$ with ideal $\mathbf{I}(G) = \langle x^5 - 1, y - x^3 \rangle$. Read about the *Gröbner fan* in [**70**, Ch. 8, §4] and compute the Gröbner fan of $\mathbf{I}(G)$. The answer will be identical to the fan described in part (d). This is no accident, as shown in the paper [**156**] (see also §10.3). There is a lot of interesting mathematics going on here, including the McKay correspondence and the G-Hilbert scheme. See also [**206**] for the higher dimensional case.

3.3.12. Consider the fan Σ in $\mathbb{R}^3$ shown in Figure 12. This fan has five 1-dimensional

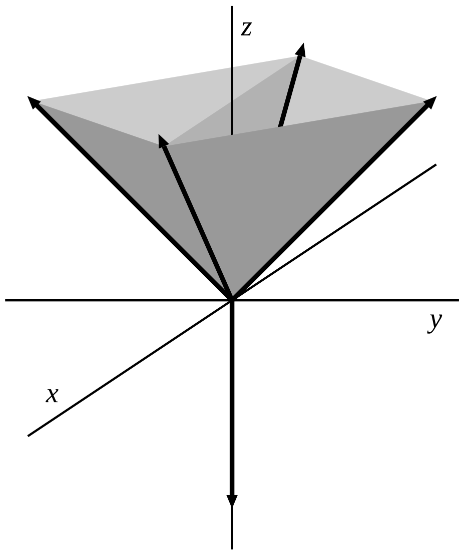


Figure 12. The fan Σ for Exercise 3.3.12

cones with four "upward" ray generators $(\pm 1, 0, 1), (0, \pm 1, 1)$ and one "downward" generator $(0, 0, -1)$. There are also nine 2-dimensional cones. Figure 12 shows five of the

2-dimensional cones; the remaining four are generated by the combining the downward generator with the four upward generators.

(a) Show that projection onto the y-axis induces a toric morphism $X_\Sigma \to \mathbb{P}^1$.

(b) Show that $X_\Sigma \to \mathbb{P}^1$ is a locally trivial fiber bundle over $\mathbb{P}^1$ with fiber $\mathbb{P}(1,1,2)$. Hint: Theorem 3.3.19 and $(1,0,1)+(-1,0,1)+2(0,0,-1)=0$. See Example 3.1.17.

(c) Explain how you can see the splitting (in the sense of Definition 3.3.18) in Figure 12. Also explain why the figure makes it clear that the fiber is $\mathbb{P}(1,1,2)$.

3.3.13. Consider the fan Σ in $\mathbb{R}^2$ with ray generators

$$u_0 = e_1 + e_2,\ u_1 = e_1,\ u_2 = e_2,\ u_3 = -e_1$$

and 1-dimensional cones $\mathrm{Cone}(u_0,u_1),\ \mathrm{Cone}(u_0,u_2),\ \mathrm{Cone}(u_2,u_3)$.

(a) Draw a picture of Σ and prove that X_Σ is the blowup of $\mathbb{P}^1 \times \mathbb{C}$ at one point.

(b) Show that the map $ae_1 + be_2 \mapsto b$ induces a toric morphism $\phi : X_\Sigma \to \mathbb{C}$ such that $\phi^{-1}(\alpha) \simeq \mathbb{P}^1$ for $\alpha \in \mathbb{C}^*$ and $\phi^{-1}(0)$ is a union of two copies of $\mathbb{P}^1$ meeting at a point. Hint: Once you understand $\phi^{-1}(0)$, show that the fan for $X_\Sigma \setminus \phi^{-1}(0)$ gives $\mathbb{P}^1 \times \mathbb{C}^*$.

(c) To get a better picture of X_Σ, consider the map $\Phi : (\mathbb{C}^*)^2 \to \mathbb{P}^3 \times \mathbb{C}$ defined by

$$\Phi(s,t) = ((s^3, s^2, st, t^2), t).$$

Let $X = \overline{\Phi((\mathbb{C}^*)^2)} \subseteq \mathbb{P}^3 \times \mathbb{C}$ be the closure of the image. Prove that $X \simeq X_\Sigma$ and that the restriction of the projection $\mathbb{P}^3 \times \mathbb{C} \to \mathbb{C}$ to X gives the toric morphism ϕ of part (b).

(d) Let x,y,z,w be coordinates on $\mathbb{P}^3$. Prove that $X \subseteq \mathbb{P}^3 \times \mathbb{C}$ is defined by the equations

$$yw - z^2 = 0,\ xz - ty^2 = 0,\ xw - tyz = 0.$$

Also use these equations to describe the fibers $\phi^{-1}(\alpha)$ for $\alpha \in \mathbb{C}$, and explain how this relates to part (b). Hint: The twisted cubic is relevant.

This is a *semi-stable degeneration of toric varieties*. See [**148**] for more details.

§3.4. Complete and Proper

The Compactness Criterion. We begin by proving part (c) of Theorem 3.1.19.

Theorem 3.4.1. *Let X_Σ be a toric variety. Then the following are equivalent:*

(a) *X_Σ is compact in the classical topology.*

(b) *The limit $\lim_{t\to 0} \lambda^u(t)$ exists in X_Σ for all $u \in N$.*

(c) *Σ is complete, i.e., $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma = N_\mathbb{R}$.*

Proof. First observe that since X_Σ is separated (Theorem 3.1.5), it is Hausdorff in the classical topology (Theorem 3.0.17). In fact, since the classical topology on each affine open set U_σ is a metric topology, X_Σ is compact if and only if every sequence of points in X_Σ has a convergent subsequence.

For (a) $\Rightarrow$ (b), assume that X_Σ is compact and fix $u \in N$. Given a sequence $t_k \in \mathbb{C}^*$ converging to 0, we get the sequence $\lambda^u(t_k) \in X_\Sigma$. By compactness, this sequence has a convergent subsequence. Passing to this subsequence, we can assume that $\lim_{k\to\infty} \lambda^u(t_k) = \gamma \in X_\Sigma$. Because X_Σ is the union of the affine open subsets U_σ for $\sigma \in \Sigma$, we may assume $\gamma \in U_\sigma$. Now take $m \in \sigma^\vee \cap M$. The character χ^m is a regular function on U_σ and hence is continuous in the classical topology. Thus

$$\chi^m(\gamma) = \lim_{k\to\infty} \chi^m(\lambda^u(t_k)) = \lim_{k\to\infty} t_k^{\langle m,u\rangle}.$$

Since $t_k \to 0$, the exponent must be nonnegative, i.e., $\langle m,u\rangle \geq 0$ for all $m \in \sigma^\vee \cap M$. This implies $\langle m,u\rangle \geq 0$ for all $m \in \sigma^\vee$, so that $u \in (\sigma^\vee)^\vee = \sigma$. Then Proposition 3.2.2 implies that $\lim_{t\to 0} \lambda^u(t)$ exists in U_σ and hence in X_Σ.

To prove (b) $\Rightarrow$ (c), take $u \in N$ and consider the limit $\lim_{t\to 0} \lambda^u(t)$. This lies in some affine open U_σ, which implies $u \in \sigma \cap N$ by Proposition 3.2.2. Thus every lattice point of $N_\mathbb{R}$ is contained in a cone of Σ. It follows that Σ is complete.

We will prove (c) $\Rightarrow$ (a) by induction on $n = \dim N_\mathbb{R}$. In the case $n = 1$, the only complete fan Σ is the fan in $\mathbb{R}$ pictured in Example 3.1.11. The corresponding toric variety is $X_\Sigma = \mathbb{P}^1$. This is homeomorphic to S^2, the 2-dimensional sphere, and hence is compact.

Now assume the statement is true for all complete fans of dimension strictly less than n, and consider a complete fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^n$. Let $\gamma_k \in X_\Sigma$ be a sequence. We will show that γ_k has a convergent subsequence.

Since X_Σ is the union of finitely many orbits $O(\tau)$, we may assume the sequence γ_k lies entirely within an orbit $O(\tau)$. If $\tau \neq \{0\}$, then the closure of $O(\tau)$ in X_Σ is the toric variety $V(\tau) = X_{\mathrm{Star}(\tau)}$ of dimension $\leq n-1$ by Proposition 3.2.7. Since Σ is complete, it is easy to check that the fan $\mathrm{Star}(\tau)$ is complete in $N(\tau)_\mathbb{R}$ (Exercise 3.4.1). Then the induction hypothesis implies that there is a convergent subsequence in $V(\tau)$. Hence, without loss of generality again, we may assume that our sequence lies entirely in the torus $T_N \subseteq X_\Sigma$.

Recall from the discussion following Lemma 3.2.5 that

$$T_N \simeq \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*).$$

Moreover, when we regard $\gamma \in T_N$ as a group homomorphism $\gamma : M \to \mathbb{C}^*$, then for any $\sigma \in \Sigma$, restriction yields a semigroup homomorphism $\sigma^\vee \cap M \to \mathbb{C}$ and hence a point γ in U_σ.

A key ingredient of the proof will be the logarithm map $L : T_N \to N_\mathbb{R}$ defined as follows. Given a point $\gamma : M \to \mathbb{C}^*$ of T_N, consider the map $M \to \mathbb{R}$ defined by the formula

$$m \longmapsto \log|\gamma(m)|.$$

This is a homomorphism and hence gives an element $L(\gamma) \in \mathrm{Hom}_\mathbb{Z}(M, \mathbb{R}) \simeq N_\mathbb{R}$. For more properties of this mapping, see Exercise 3.4.2 below.

For us, the most important property of L is the following. Suppose that a point $\gamma \in T_N$ satisfies $L(\gamma) \in -\sigma$ for some $\sigma \in \Sigma$. If $m \in \sigma^\vee \cap M$, then the definition of

L implies that

$$\log|\gamma(m)| = \langle m, L(\gamma)\rangle, \tag{3.4.1}$$

which is ≤ 0 since $m \in \sigma^\vee$ and $L(\gamma) \in -\sigma$. Hence $|\gamma(m)| \leq 1$. Thus we have proved that

$$L(\gamma) \in -\sigma \implies |\gamma(m)| \leq 1 \text{ for all } m \in \sigma^\vee \cap M. \tag{3.4.2}$$

Now apply L to our sequence, which gives a sequence $L(\gamma_k) \in N_{\mathbb{R}}$. Since Σ is complete, the same is true for the fan consisting of the cones $-\sigma$ for $\sigma \in \Sigma$. Hence, by passing to a subsequence, we may assume that there is $\sigma \in \Sigma$ such that

$$L(\gamma_k) \in -\sigma$$

for all k. By (3.4.2), we conclude that $|\gamma_k(m)| \leq 1$ for all $m \in \sigma^\vee \cap M$. It follows that the γ_k are a sequence of mappings to the closed unit disk in $\mathbb{C}$. Since the closed unit disk is compact, there is a subsequence γ_{k_ℓ} which converges to a point $\gamma \in U_\sigma$. You will check the details of this final assertion in Exercise 3.4.3. □

Complete Varieties. The compactness criterion proved in Theorem 3.4.1 uses the classical topology. It is natural to ask for an algebraic version of this theorem that uses only the Zariski topology. The crucial idea is the notion of *completeness*.

To motivate the definition of completeness, we first reformate the topological notion of compactness. You will prove the following in Exercise 3.4.4.

Proposition 3.4.2. *Let X be a locally compact Hausdorff topological space. Then the following are equivalent:*

(a) *X is compact.*

(b) *For every topological space Z, the projection map $\pi_Z : X \times Z \to Z$ is closed, i.e., $\pi_Z(W) \subseteq Z$ is closed for all closed subsets $W \subseteq X \times Z$.* □

We now define the algebraic analog of compactness.

Definition 3.4.3. A variety X is ***complete*** if for every variety Z, the projection map $\pi_Z : X \times Z \to Z$ is a closed mapping in the Zariski topology.

Here are two examples to illustrate this definition.

Example 3.4.4. Consider the affine variety $\mathbb{C}$. We claim that $\mathbb{C}$ is not complete. To see this, consider the projection map $\pi_2 : \mathbb{C} \times \mathbb{C} = \mathbb{C}^2 \to \mathbb{C}$. The closed subset $\mathbf{V}(xy-1) \subseteq \mathbb{C}^2$ does not map to a Zariski-closed subset of $\mathbb{C}$ under π_2. Hence π_2 is not a closed mapping, so that $\mathbb{C}$ is not complete. ◊

Example 3.4.5. The Projective extension theorem [**69**, Thm. 6 of Ch. 8, §5] shows that for $X = \mathbb{P}^n$, the mapping

$$\pi_{\mathbb{C}^m} : \mathbb{P}^n \times \mathbb{C}^m \to \mathbb{C}^m$$

is closed in the Zariski topology for all m. It follows that if $V \subseteq \mathbb{C}^m$ is any affine variety, the projection

$$\pi_V : \mathbb{P}^n \times V \to V$$

is a closed mapping in the Zariski topology. Then the gluing construction shows that $\pi_Z : \mathbb{P}^n \times Z \to Z$ is closed for any variety Z, so $\mathbb{P}^n$ is a complete variety. In fact, one can think of $\mathbb{P}^n$ as the prototypical complete variety. Moreover, any projective variety is complete (Exercise 3.4.5). However, there are complete varieties that are not projective—we will give a toric example in Chapter 6. $\Diamond$

Completeness is the algebraic version of compactness, and it can be shown that a variety is complete if and only if it is compact in the classical topology. This is proved in Serre's famous paper *Géométrie algébrique et géométrie analytique*, called GAGA for short. See [**248**, Prop. 6, p. 12]. As a consequence, we get the following improved version of Theorem 3.4.1.

Theorem 3.4.6. *Let X_Σ be a toric variety. Then the following are equivalent:*

(a) *X_Σ is compact in the classical topology.*

(b) *X_Σ is complete.*

(c) *The limit $\lim_{t\to 0} \lambda^u(t)$ exists in X_Σ for all $u \in N$.*

(d) *Σ is complete, i.e., $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma = N_\mathbb{R}$.* $\square$

Proper Mappings. In algebraic geometry, many concepts that apply to varieties have relative versions that apply to morphisms. To see how this works for complete varieties, we will begin in the topological category with the relative version of compactness.

Definition 3.4.7. A continuous mapping $f : X \to Y$ is ***proper*** if the inverse image $f^{-1}(T)$ is compact in X for every compact subset $T \subseteq Y$.

It is immediate that X is compact if and only if the constant mapping from X to the space $Y = \{\mathrm{pt}\}$ consisting of a single point is proper. This relative version of compactness may also be reformulated, for reasonably nice topological spaces, in the following way.

Proposition 3.4.8. *Let $f : X \to Y$ be a continuous mapping of locally compact first countable Hausdorff spaces. Then the following are equivalent:*

(a) *f is proper.*

(b) *f is a closed mapping, i.e., $f(W) \subseteq Y$ is closed for all closed subsets $W \subseteq X$, and all fibers $f^{-1}(y)$, $y \in Y$, are compact.*

(c) *Exery sequence $x_k \in X$ such that $f(x_k) \in Y$ converges in Y has a subsequence x_{k_ℓ} that converges in X.*

Proof. A proof of (a) $\Leftrightarrow$ (b) can be found in [**122**, Ch. 9, §4]. See Exercise 3.4.6 for (a) $\Leftrightarrow$ (c). $\square$

Before we can give a definition of properness that works for morphisms, we first need to reformulate the topological notion of properness. Recall from §3.0 that morphisms $f : X \to S$ and $g : Y \to S$ give the fiber product $X \times_S Y$. Fiber products can also be defined for continuous maps between topological spaces. You will prove in Exercise 3.4.6 that properness can be described using fiber products as follows.

Proposition 3.4.9. *Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. Then f is proper if and only if f is **universally closed**, meaning that for all spaces Z and all continuous mappings $g : Z \to Y$, the projection π_Z defined by the commutative diagram*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\pi_X} & X \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & Y \end{array}$$

is a closed mapping. $\square$

In algebraic geometry, we will use the following definition of properness for morphisms between varieties.

Definition 3.4.10. A morphism of varieties $\phi : X \to Y$ is ***proper*** if it is ***universally closed***, in the sense that for all varieties Z and morphisms $\psi : Z \to Y$, the projection π_Z defined by the commutative diagram

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\pi_X} & X \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle \phi} \\ Z & \xrightarrow[\psi]{} & Y \end{array}$$

is a closed mapping in the Zariski topology.

It is easy to see that a variety X is complete if and only if the constant morphism $\phi : X \to \{\mathrm{pt}\}$ is proper. Furthermore, if X is complete, then the projection map

$$\pi_Z : X \times Z \longrightarrow Z$$

is proper for any variety Z. You will prove these assertions in Exercise 3.4.7.

The Properness Criterion. Theorem 3.4.6 can be understood as a special case of the following statement for toric morphisms.

Theorem 3.4.11. *Let $\phi : X_\Sigma \to X_{\Sigma'}$ be the toric morphism corresponding to a homomorphism $\overline{\phi} : N \to N'$ that is compatible with fans Σ in $N_\mathbb{R}$ and Σ' in $N'_\mathbb{R}$. Then the following are equivalent:*

(a) *$\phi : X_\Sigma \to X_{\Sigma'}$ is proper in the classical topology (Definition 3.4.7).*

(b) *$\phi : X_\Sigma \to X_{\Sigma'}$ is a proper morphism (Definition 3.4.10).*

(c) *If $u \in N$ and $\lim_{t\to 0} \lambda^{\overline{\phi}(u)}(t)$ exists in $X_{\Sigma'}$, then $\lim_{t\to 0} \lambda^u(t)$ exists in X_Σ.*

(d) *$\overline{\phi}_\mathbb{R}^{-1}(|\Sigma'|) = |\Sigma|$.*

Proof. The proof of (a) $\Rightarrow$ (b) uses two fundamental results in algebraic geometry.

First, given any morphism of varieties $f : X \to Y$ and a Zariski closed subset $W \subseteq X$, a theorem of Chevalley tells us that the image $f(W) \subseteq Y$ is *constructible*, meaning that it can be written as a finite union $f(W) = \bigcup_i (V_i \setminus W_i)$, where V_i and W_i are Zariski closed in Y. A proof appears in [**131**, Ex. II.3.19].

Second, given any constructible subset C of a variety Y, its closure in Y in the classical topology equals its closure in the Zariski topology. When C is open in the Zariski topology, a proof is given in [**207**, Thm. (2.33)], and when C is the image of a morphism, a proof can be found in GAGA [**248**, Prop. 7, p. 12].

Now suppose that $\phi : X_\Sigma \to X_{\Sigma'}$ is proper in the classical topology and let $\psi : Z \to X_{\Sigma'}$ be a morphism. This gives the commutative diagram

$$\begin{array}{ccc} X_\Sigma \times_{X_{\Sigma'}} Z & \longrightarrow & X_\Sigma \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle \phi} \\ Z & \xrightarrow[\psi]{} & X_{\Sigma'}. \end{array}$$

Let $Y \subseteq X_\Sigma \times_{X_{\Sigma'}} Z$ be Zariski closed. We need to prove that $\pi_Z(Y)$ is Zariski closed in Z. First observe that Y is also closed in the classical topology, so that $\pi_Z(Y)$ is closed in Z in the classical topology by Proposition 3.4.9. However, $\pi_Z(Y)$ is constructible by Chevalley's theorem, and then, being classically closed, it is also Zariski closed by GAGA. Hence π_Z is a closed map in the Zariski topology for any morphism $\psi : Z \to X_{\Sigma'}$. It follows that ϕ is a proper morphism.

To prove (b) $\Rightarrow$ (c), let $u \in N$ and assume that $\gamma' = \lim_{t\to 0} \lambda^{\overline{\phi}(u)}(t)$ exists in $X_{\Sigma'}$. We first prove $\lim_{t\to 0} \lambda^u(t)$ exists in X_Σ under the extra assumption that $\overline{\phi}(u) \neq 0$. This means that $\lambda^{\overline{\phi}(u)}$ is a nontrivial one-parameter subgroup in $X_{\Sigma'}$.

Let $\overline{\lambda^u(\mathbb{C}^*)} \subseteq X_\Sigma$ be the closure of $\lambda^u(\mathbb{C}^*) \subseteq X_\Sigma$ in the classical topology. Our earlier remarks imply that this equals the Zariski closure. Since ϕ is proper, it is closed in the Zariski topology, so that $\phi\big(\overline{\lambda^u(\mathbb{C}^*)}\big)$ is closed in $X_{\Sigma'}$ in both topologies. It follows that

$$\overline{\lambda^{\overline{\phi}(u)}(\mathbb{C}^*)} \subseteq \phi\big(\overline{\lambda^u(\mathbb{C}^*)}\big).$$

Hence there is $\gamma \in \overline{\lambda^u(\mathbb{C}^*)}$ mapping to γ'. Thus there is a sequence of points $t_k \in \mathbb{C}^*$ such that $\lambda^u(t_k) \to \gamma$. Then

$$\gamma' = \phi(\gamma) = \lim_{k\to\infty} \phi(\lambda^u(t_k)) = \lim_{k\to\infty} \lambda^{\overline{\phi}(u)}(t_k).$$

This, together with $\gamma' = \lim_{t\to 0} \lambda^{\overline{\phi}(u)}(t)$ and $\overline{\phi}(u) \neq 0$, imply that $t_k \to 0$. From here, the arguments used to prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) of Theorem 3.4.1 easily imply that $\lim_{t\to 0} \lambda^u(t)$ exists in X_Σ.

For the general case when we no longer assume $\overline{\phi}(u) \neq 0$, consider the map $(\phi, 1_{\mathbb{C}}) : X_\Sigma \times \mathbb{C} \to X_{\Sigma'} \times \mathbb{C}$. This is proper since ϕ is proper (Exercise 3.4.8). Furthermore, $X_\Sigma \times \mathbb{C}$ and $X_{\Sigma'} \times \mathbb{C}$ are toric varieties by Proposition 3.1.14, and the corresponding map on lattices is $(\overline{\phi}, 1_{\mathbb{Z}}) : N \times \mathbb{Z} \to N' \times \mathbb{Z}$. Then applying the above argument to $(u,1) \in N \times \mathbb{Z}$ shows that $\lim_{t\to 0} \lambda^u(t)$ exists in X_Σ. We leave the details to the reader (Exercise 3.4.8).

For (c) $\Rightarrow$ (d), first observe that the inclusion

$$|\Sigma| \subseteq \overline{\phi}_{\mathbb{R}}^{-1}(|\Sigma'|)$$

is automatic since $\overline{\phi}$ is compatible with Σ and Σ'. For the opposite inclusion, take $u \in \overline{\phi}_{\mathbb{R}}^{-1}(|\Sigma'|) \cap N$. Then $\overline{\phi}(u) \in |\Sigma'|$, which by Proposition 3.2.2 implies that $\lim_{t\to 0} \lambda^{\overline{\phi}(u)}(t)$ exists in $X_{\Sigma'}$. By assumption, $\lim_{t\to 0} \lambda^u(t)$ exists in X_Σ. Using Proposition 3.2.2, we conclude that $u \in \sigma \cap N$ for some $\sigma \in \Sigma$. Because all the cones are rational, this immediately implies $\overline{\phi}_{\mathbb{R}}^{-1}(|\Sigma'|) \subseteq |\Sigma|$.

Finally, we prove (d) $\Rightarrow$ (a). We begin with two special cases.

Special Case 1. Suppose that a toric morphism $\phi : X_\Sigma \to T_{N'}$ satisfies (d) and has the additional property that $\overline{\phi} : N \to N'$ is onto. The fan of $T_{N'}$ consists of the trivial cone $\{0\}$, so that (d) implies

$$|\Sigma| = \overline{\phi}_{\mathbb{R}}^{-1}(0) = \ker(\overline{\phi}_{\mathbb{R}}). \tag{3.4.3}$$

When we think of Σ as a fan Σ'' in $\ker(\overline{\phi}_{\mathbb{R}}) \subseteq N_{\mathbb{R}}$, (3.3.5) implies that

$$X_\Sigma \simeq X_{\Sigma''} \times T_{N'}.$$

Then ϕ corresponds to the projection $X_{\Sigma''} \times T_{N'} \to T_{N'}$. The fan Σ'' is complete in $\ker(\overline{\phi}_{\mathbb{R}})$ by (3.4.3), so that $X_{\Sigma''}$ is compact by Theorem 3.4.1. Thus $X_{\Sigma''} \to \{\mathrm{pt}\}$ is proper, which easily implies that $X_{\Sigma''} \times T_{N'} \to T_{N'}$ is proper. We conclude that ϕ is proper in the classical topology.

Special Case 2. Suppose that a homomorphism of tori $\phi : T_N \to T_{N'}$ has the additional property that $\overline{\phi} : N \to N'$ is injective. Then (d) is obviously satisfied. An elementary proof that ϕ is proper is given in Exercise 3.4.9.

Now consider a general toric morphism $\phi : X_\Sigma \to X_{\Sigma'}$ satisfying (d). We will prove that ϕ is proper in the classical topology using part (c) of Proposition 3.4.8.

Thus assume that $\gamma_k \in X_\Sigma$ is a sequence such that $\phi(\gamma_k)$ converges in $X_{\Sigma'}$. We need to prove that a subsequence of γ_k converges in X_Σ.

Since X_Σ has only finitely many T_N-orbits, we may assume that the sequence lies in an orbit $O(\sigma)$. As in Lemma 3.3.21, let σ' be the minimal cone of Σ' containing $\overline{\phi}_{\mathbb{R}}(\sigma)$. The restriction

$$\phi|_{V(\sigma)} : V(\sigma) \to V(\sigma')$$

is a toric morphism by Lemma 3.3.21, and the fans of $V(\sigma)$ and $V(\sigma')$ are given by $\mathrm{Star}(\sigma)$ in $N(\sigma)_{\mathbb{R}}$ and $\mathrm{Star}(\sigma')$ in $N'(\sigma')_{\mathbb{R}}$ respectively. Furthermore, one can check that since Σ and Σ' satisfy (d), the same is true for the fans $\mathrm{Star}(\sigma)$ and $\mathrm{Star}(\sigma')$ (Exercise 3.4.10). Hence we may assume that $\gamma_k \in T_N$ and $\phi(\gamma_k) \in T_{N'}$ for all k.

The limit $\gamma' = \lim_{k\to\infty} \phi(\gamma_k)$ lies in an orbit $O(\tau')$ for some $\tau' \in \Sigma'$. Thus the sequence $\phi(\gamma_k)$ and its limit γ' all lie in $U_{\tau'}$. Note that $\{\sigma \in \Sigma \mid \overline{\phi}(\sigma) \subset \tau'\}$ is the fan giving $\phi^{-1}(U_{\tau'})$. Since (d) implies that

$$\overline{\phi}_{\mathbb{R}}^{-1}(\tau') = \bigcup_{\overline{\phi}_{\mathbb{R}}(\sigma) \subseteq \tau'} \sigma,$$

we can assume that $X_{\Sigma'} = U_{\tau'}$, i.e., $\phi : X_\Sigma \to U_{\tau'}$ and $\overline{\phi}^{-1}(\tau') = |\Sigma|$.

If $\tau' = \{0\}$, then $O(\tau') = U_{\tau'} = T'_N$. If we write $\overline{\phi}$ as the composition

$$N \twoheadrightarrow \overline{\phi}(N) \hookrightarrow N',$$

then $\phi : X_\Sigma \to T_{N'}$ factors as $X_\Sigma \to T_{\overline{\phi}(N)} \to T_{N'}$. Special Cases 1 and 2 imply that these maps are proper, and since the composition of proper maps is proper, we conclude that ϕ is proper.

It remains to consider the case when $\tau' \neq \{0\}$. When we think of $\gamma' \in U_{\tau'}$ as a semigroup homomorphism $\gamma' : (\tau')^\vee \cap M \to \mathbb{C}$, Lemma 3.2.5 tells us that

$$\gamma'(m') = 0 \ \text{ for all } m' \in (\tau')^\vee \cap M' \setminus (\tau')^\perp \cap M'.$$

Since the $\phi(\gamma_k) : M \to \mathbb{C}^*$ converge to γ' in $U_{\tau'}$, we see that

$$\lim_{k\to\infty} \phi(\gamma_k)(m') = 0 \ \text{ for all } m' \in (\tau')^\vee \cap M' \setminus (\tau')^\perp \cap M'.$$

Since $(\tau')^\vee \cap M'$ is finitely generated, it follows that we may pass to a subsequence and assume that

$$|\phi(\gamma_k)(m')| \le 1 \ \text{ for all } k \text{ and all } m' \in (\tau')^\vee \cap M' \setminus (\tau')^\perp \cap M'. \tag{3.4.4}$$

The logarithm map from the proof of Theorem 3.4.1 gives maps $L_N : T_N \to N_{\mathbb{R}}$ and $L_{N'} : T_{N'} \to N'_{\mathbb{R}}$ linked by a commutative diagram:

$$\begin{tikzcd} T_N \arrow[r, "L_N"] \arrow[d, "\phi|_{T_N}"'] & N_{\mathbb{R}} \arrow[d, "\overline{\phi}_{\mathbb{R}}"] \\ T_{N'} \arrow[r, "L_{N'}"'] & N'_{\mathbb{R}}. \end{tikzcd}$$

Let $\overline{\phi}^* : M' \to M$ be dual to $\overline{\phi} : N \to N'$. Then $m' \in (\tau')^\vee \cap M' \setminus (\tau')^\perp \cap M'$ implies that for all k, we have

$$\begin{aligned}\langle \overline{\phi}^*(m'), L_N(\gamma_k)\rangle &= \langle m', \overline{\phi}_{\mathbb{R}}(L_N(\gamma_k))\rangle \\ &= \langle m', L_{N'}(\phi(\gamma_k))\rangle = \log|\phi(\gamma_k)(m')| \le 0,\end{aligned} \tag{3.4.5}$$

where the first equality is standard, the second follows from the above commutative diagram, the third follows from (3.4.1), and the final inequality uses (3.4.4).

Now consider the following equivalences:

$$\begin{aligned} u \in \overline{\phi}_{\mathbb{R}}^{-1}(\tau') &\iff \overline{\phi}_{\mathbb{R}}(u) \in \tau' \\ &\iff \langle m', \overline{\phi}_{\mathbb{R}}(u)\rangle \ge 0 \ \text{ for all } m' \in (\tau')^\vee \cap M' \\ &\iff \langle \overline{\phi}^*(m'), u\rangle \ge 0 \ \text{ for all } m' \in (\tau')^\vee \cap M',\end{aligned}$$

where the first and third equivalences are obvious and the second uses $\tau' = (\tau')^{\vee\vee}$ and the rationality of τ'. But we also know that $\tau' \ne \{0\}$, which means that $(\tau')^\vee$ is a cone whose maximal subspace $(\tau')^\perp$ is a proper subset. This implies that

$$u \in \overline{\phi}_{\mathbb{R}}^{-1}(\tau') \iff \langle \overline{\phi}^*(m'), u\rangle \ge 0 \ \text{ for all } m' \in (\tau')^\vee \cap M' \setminus (\tau')^\perp \cap M'$$

(Exercise 3.4.11). Using (3.4.5), we conclude that $-L_N(\gamma_k) \in \overline{\phi}_{\mathbb{R}}^{-1}(\tau')$ for all k. But, as noted above, (d) means $\overline{\phi}^{-1}(\tau') = |\Sigma|$. It follows that

$$-L_N(\gamma_k) \in |\Sigma|$$

for all k. Passing to a subsequence, we may assume that there is $\sigma \in \Sigma$ such that

$$L_N(\gamma_k) \in -\sigma$$

for all k. From here, the proof of (c) $\Rightarrow$ (a) in Theorem 3.4.1 implies that there is a subsequence γ_{k_ℓ} which converges to a point $\gamma \in U_\sigma \subseteq X_\Sigma$. This proves that ϕ is proper in the classical topology. The proof of the theorem is now complete. $\square$

We noted earlier that a variety is complete if and only if it is compact. In a similar way, a morphism $f : X \to Y$ of varieties is a proper morphism if and only if it is proper in the classical topology. This is proved in [**126**, Prop. 3.2 of Exp. XII]. Thus the equivalence (a) $\Leftrightarrow$ (b) of Theorem 3.4.11 is a special case of this result.

Theorems 3.4.6 and 3.4.11 show that properness and completeness can be tested using one-parameter subgroups. In the case of completeness, we can formulate this as follows. Given $u \in N$, the one-parameter subgroup gives a map $\lambda^u : \mathbb{C} \setminus \{0\} \to T_N \subseteq X_\Sigma$, and saying that $\lim_{t\to 0} \lambda^u(t)$ exists in X_Σ means that λ^u extends to a morphism $\lambda_0^u : \mathbb{C} \to X_\Sigma$. In other words, whenever we have a commutative diagram with solid arrows

$$\begin{tikzcd} \mathbb{C}\setminus\{0\} \arrow[r, "\lambda^u"] \arrow[d, "i"'] & X_\Sigma \arrow[d, "\phi"] \\ \mathbb{C} \arrow[r, "\lambda_0^{\overline{\phi}(u)}"'] \arrow[ur, dashed, "\lambda_0^u"] & \{\mathrm{pt}\}, \end{tikzcd}$$

the dashed arrow λ_0^u exists so that the diagram remains commutative. The existence of λ_0^u tells us that the variety X_Σ is not missing any points, which is where the term "complete" comes from. In a similar way, the properness criterion given in part (c) of Theorem 3.4.11 can be formulated as saying that whenever $u \in N$ gives a commutative diagram,

$$\begin{tikzcd} \mathbb{C}\setminus\{0\} \arrow[r, "\lambda^u"] \arrow[d, "i"'] & X_\Sigma \arrow[d, "\phi"] \\ \mathbb{C} \arrow[r, "\lambda_0^{\overline{\phi}(u)}"'] \arrow[ur, dashed, "\lambda_0^u"] & X_{\Sigma'}, \end{tikzcd}$$

the dashed arrow λ_0^u exists so that the diagram remains commutative.

For general varieties, there are similar criteria for completeness and properness that replace $\lambda^u : \mathbb{C} \setminus \{0\} \to X_\Sigma$ and $\lambda_0^u : \mathbb{C} \to X_\Sigma$ with maps coming from *discrete valuation rings*, to be discussed in Chapter 4. An example of a discrete valuation ring is the ring of formal power series $R = \mathbb{C}[[t]]$, whose field of fractions is the field of formal Laurent series $K = \mathbb{C}((t))$. By replacing $\mathbb{C}$ with $\mathrm{Spec}(R)$ and $\mathbb{C}\setminus\{0\}$ with $\mathrm{Spec}(K)$ in the above diagrams, where R is now an arbitrary discrete valuation ring, one gets the *valuative criterion for properness* (see [**131**, Ex. II.4.11 and Thm. II.4.7]). This requires the full power of scheme theory since $\mathrm{Spec}(R)$ and $\mathrm{Spec}(K)$ are not varieties as defined in this book. Using the valuative criterion of properness, one can give a direct, purely algebraic proof of (d) $\Rightarrow$ (b) in Theorem 3.4.11 and Corollary 3.4.6 (see [**105**, Sec. 2.4] or [**218**, Sec. 1.5]).

Example 3.4.12. An important class of proper morphisms are the toric morphisms $\phi : X_{\Sigma'} \to X_\Sigma$ induced by a refinement Σ' of Σ. Condition (d) of Theorem 3.4.11 is obviously fulfilled since $\overline{\phi} : N \to N$ is the identity and every cone of Σ is a union of cones of Σ'. In particular, the blowups

$$\phi : X_{\Sigma^*(\sigma)} \to X_\Sigma$$

studied in Proposition 3.3.15 are always proper. $\Diamond$

Exercises for §3.4.

3.4.1. Let Σ be a complete fan in $N_{\mathbb{R}}$ and let τ be a cone in Σ. Show that the fan $\operatorname{Star}(\tau)$ defined in (3.2.8) is a complete fan in $N(\tau)_{\mathbb{R}}$.

3.4.2. In this exercise, you will develop some additional properties of the logarithm mapping $L : T_N \to N_{\mathbb{R}}$ defined in the proof of Theorem 3.4.1.

(a) Let S^1 be the unit circle in the complex plane, a subgroup of the multiplicative group $\mathbb{C}^*$. Show that there is an isomorphism of groups

$$\begin{aligned}\Phi : \mathbb{C}^* &\longrightarrow S^1 \times \mathbb{R}\\ z &\longmapsto (|z|, \log|z|),\end{aligned}$$

where the operation in the second factor on the right is addition.

(b) Show that the compact real n-dimensional torus $(S^1)^n$ can be viewed as a subgroup of T_N and that $L : T_N \to N_{\mathbb{R}}$ induces an isomorphism $T_N/(S^1)^n \simeq N_{\mathbb{R}}$. Hint: Use Φ from part (a).

(c) Let Σ be a fan in N. Show that the action of the compact real torus $(S^1)^n \subseteq T_N$ on T_N extends to an action on the toric variety X_Σ and that the quotient space

$$(X_\Sigma)/(S^1)^n \cong \bigcup_\sigma N(\sigma)_{\mathbb{R}},$$

where $\cong$ denotes homeomorphism of topological spaces, and the union is over all cones in the fan. Hint: Use the Orbit-Cone Correspondence (Theorem 3.2.6).

(d) Let Σ in $\mathbb{R}^2$ be the fan from Example 3.1.9, so that $X_\Sigma \simeq \mathbb{P}^2$. Show that under the action of $(S^1)^2 \subseteq (\mathbb{C}^*)^2$ as in part (c), $\mathbb{P}^2/(S^1)^2 \cong \Delta_2$, the 2-dimensional simplex.

We will say more about the topology of toric varieties in Chapter 12.

3.4.3. This exercise will complete the proof of Theorem 3.4.1. Let $\operatorname{Hom}(\sigma^\vee \cap M, \mathbb{C})$ be the set of semigroup homomorphisms $\sigma^\vee \cap M \to \mathbb{C}$. Assume that $\gamma_k \in \operatorname{Hom}(\sigma^\vee \cap M, \mathbb{C})$ is a sequence such that $|\gamma_k(m)| \leq 1$ for all $m \in \sigma^\vee \cap M$ and all k. We want to show that there is a subsequence γ_{k_ℓ} that converges to a point $\gamma \in \operatorname{Hom}(\sigma^\vee \cap M, \mathbb{C})$.

(a) The semigroup $\mathsf{S}_\sigma = \sigma^\vee \cap M$ is generated by a finite set $\{m_1, \ldots, m_s\}$. Use this fact and the compactness of $\{z \in \mathbb{C} \mid |z| \leq 1\}$ to show that there exists a subsequence γ_{k_ℓ} such that the sequences $\gamma_{k_\ell}(m_j)$ converge in $\mathbb{C}$ for all j.

(b) Deduce that the subsequence γ_{k_ℓ} converges to a $\gamma \in \operatorname{Hom}(\sigma^\vee \cap M, \mathbb{C})$.

3.4.4. Prove Proposition 3.4.2. Hint: For (b) $\Rightarrow$ (a), let Z be the one-point compactification of X and consider the projection of $\Delta = \{(x,x) \mid x \in X\} \subseteq X \times Z$.

3.4.5. Show that any projective variety is complete according to Definition 3.4.10.

3.4.6. Here you will prove some characterizations of properness stated in the text.

(a) Prove (a) $\Leftrightarrow$ (c) from Proposition 3.4.8.

(b) Prove Proposition 3.4.9. Hint: First show that if f is proper, then so is $\pi_Z : X \times_Y Z \to Z$ for any morphism $Z \to Y$. Then use (a) $\Rightarrow$ (b) of Proposition 3.4.8, which does not require first countable. If $f : X \to Y$ is universally closed, then prove that $f^{-1}(y) \to \{y\}$ is universally closed for any $y \in Y$. Then use Proposition 3.4.2 and (a) $\Rightarrow$ (b) of Proposition 3.4.8.

3.4.7. Prove that X is complete if and only $X \to \{\mathrm{pt}\}$ is proper, and that if X is complete, then $\pi_Z : X \times Z \to Z$ is proper for any variety Z.

3.4.8. Complete the proof of (b) $\Rightarrow$ (c) of Theorem 3.4.11 begun in the text.

3.4.9. Let $\phi : T_N \to T_{N'}$ be a map of tori corresponding to an injective homomorphism $\overline{\phi} : N \to N'$. Also let $\overline{\phi}^* : M' \to M$ be the dual map. Finally, let $\gamma_k \in T_N$ be a sequence such that $\phi(\gamma_k)$ converges to a point of $T_{N'}$.

(a) Prove that $\mathrm{im}(\overline{\phi}^*) \subseteq M$ has finite index. Hence we can pick an integer $d > 0$ such that $dM \subseteq \mathrm{im}(\overline{\phi}^*)$.

(b) Show that $\chi^m(\gamma_k)$ converges for all $m \in \mathrm{im}(\overline{\phi}^*)$. Conclude that $\chi^m(\gamma_k^d)$ converges for all $m \in M$, where d is as in part (a).

(c) Pick a basis of M so that $T_N \simeq (\mathbb{C}^*)^n$ and write $\gamma_k = (\gamma_{1,k}, \ldots, \gamma_{n,k}) \in (\mathbb{C}^*)^n$. Show that $(\gamma_{1,k}^d, \ldots, \gamma_{n,k}^d)$ converges to a point $(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n) \in (\mathbb{C}^*)^n$.

(d) Show that the dth roots $\tilde{\gamma}_i^{1/d}$ can be chosen so that a subsequence of the sequence $\gamma_k = (\gamma_{1,k}, \ldots, \gamma_{n,k})$ converges to a point $\gamma = (\tilde{\gamma}_1^{1/d}, \ldots, \tilde{\gamma}_n^{1/d}) \in T_N$.

(e) Explain why this implies that $T_N \to T_{N'}$ is proper in the classical topology.

3.4.10. To finish the proof of (d) $\Rightarrow$ (a) of Theorem 3.4.11, suppose we have a toric morphism $\phi : X_\Sigma \to X_{\Sigma'}$ and a cone $\sigma \in \Sigma$. Let $\sigma' \in \Sigma'$ be the smallest cone containing $\overline{\phi}_{\mathbb{R}}(\sigma)$.

(a) Prove that $\overline{\phi}$ induces a homomorphism $\overline{\phi}_\sigma : N(\sigma) \to N(\sigma')$.

(b) Assume further that $\overline{\phi}_{\mathbb{R}}^{-1}(|\Sigma'|) = |\Sigma|$. Prove that $(\overline{\phi}_\sigma)_{\mathbb{R}}^{-1}(|\mathrm{Star}(\sigma')|) = |\mathrm{Star}(\sigma)|$.

3.4.11. Let $\tau' \neq \{0\}$ be a strongly convex polyhedral cone in $N'_{\mathbb{R}}$. Prove that

$$u' \in \tau' \iff \langle m', u' \rangle \geq 0 \ \text{ for all } m' \in (\tau')^\vee \cap M \setminus (\tau')^\perp \cap M$$

and then apply this to $u' = \overline{\phi}_{\mathbb{R}}(u)$ to complete the argument in the text. Hint: To prove $\Leftarrow$, first show that the right hand side of the equivalence implies that $\langle m', u' \rangle \geq 0$ for all $m' \in (\tau')^\vee \cap M_{\mathbb{Q}} \setminus (\tau')^\perp \cap M_{\mathbb{Q}}$. Then show that $\tau' \neq \{0\}$ implies that any element of $(\tau')^\vee \cap M$ is a limit of elements in $(\tau')^\vee \cap M_{\mathbb{Q}} \setminus (\tau')^\perp \cap M_{\mathbb{Q}}$.

3.4.12. Give a second argument for the implication

$$X_\Sigma \text{ compact } \Rightarrow \Sigma \text{ complete}$$

from part (c) of Theorem 3.1.19 using induction on the dimension n of N. Hint: If Σ is not complete and $n > 1$, then there is a 1-dimensional cone τ in the boundary of the support of Σ. Consider the fan $\mathrm{Star}(\tau)$ and the corresponding toric subvariety of X_Σ.

3.4.13. Let Σ', Σ be fans in $N_{\mathbb{R}}$ compatible with the identity map $N \to N$. Prove that the toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ is proper if and only if Σ' is a refinement of Σ.

Appendix: Nonnormal Toric Varieties

In this appendix, we discuss toric varieties that are not necessarily normal. We begin with an example to show that Sumihiro's theorem (Theorem 3.1.7) on the existence of a torus-invariant affine open cover can fail in the nonnormal case.

Example 3.A.1. Consider the nodal cubic $C \subseteq \mathbb{P}^2$ defined by $y^2z = x^2(x+z)$. The only singularity of C is $p = (0,0,1)$. We claim that C is a toric variety with $C \setminus \{p\} \simeq \mathbb{C}^*$ as torus. Assuming this for the moment, consider a torus-invariant neighborhood of p. It contains p and the torus and hence is the whole curve! We conclude that p has no torus-invariant affine open neighborhood. Thus Sumihiro's theorem fails for C.

To see that C is a toric variety, we begin with the standard parametrization obtained by intersecting lines $y = tx$ with the affine curve $y^2 = x^2(x+1)$. This easily leads to the parametrization

$$x = t^2 - 1, \quad y = t(t^2-1).$$

The values $t = \pm 1$ map to the singular point p. To get a parametrization that looks more like a torus, we replace t with $\frac{t+1}{t-1}$ to obtain

$$x = \frac{4t}{(t-1)^2}, \quad y = \frac{4t(t+1)}{(t-1)^3}.$$

Then $t = 0, \infty$ map to p and $t \in \mathbb{C}^*$ maps bijectively to $C \setminus \{p\}$.

Using this parametrization, we get $\mathbb{C}^* \subseteq C$, and the action of $\mathbb{C}^*$ on itself given by multiplication extends to an action on C by making p a fixed point of the action. With some work, one can show that this action is algebraic and hence gives a toric variety. (For readers familiar with elliptic curves, the basic idea is that the description of the group law in terms of lines connecting points on the curve reduces to multiplication in $\mathbb{C}^* \subseteq C$ for our curve C.) $\Diamond$

In contrast, the projective toric varieties constructed in Chapter 2 satisfy Sumihiro's theorem by Proposition 2.1.8. Since these nonnormal toric varieties have a good local structure, it is reasonable to expect that they share some of the nice properties of normal toric varieties. In particular, they satisfy a version of the Orbit-Cone Correspondence (Theorem 3.2.6).

We begin with the affine case. Given M and a finite subset $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M$, we get the affine toric variety $Y_{\mathscr{A}} \subseteq \mathbb{C}^s$ whose torus has character group $\mathbb{Z}\mathscr{A}$ (Proposition 1.1.8). Assume $M = \mathbb{Z}\mathscr{A}$ and let $\sigma \subseteq N_{\mathbb{R}}$ be dual to $\mathrm{Cone}(\mathscr{A}) \subseteq M_{\mathbb{R}}$. By Proposition 1.3.8, the normalization of $Y_{\mathscr{A}}$ is the map

$$U_\sigma \longrightarrow Y_{\mathscr{A}}$$

induced by the inclusion of semigroup algebras

$$\mathbb{C}[\mathbb{N}\mathscr{A}] \subseteq \mathbb{C}[\sigma^\vee \cap M].$$

Recall that $\mathbb{C}[\sigma^\vee \cap M]$ is the integral closure of $\mathbb{C}[\mathbb{N}\mathscr{A}]$ in its field of fractions. We now apply standard results in commutative algebra and algebraic geometry:

- Since the integral closure $\mathbb{C}[\sigma^\vee \cap M]$ is a finitely generated $\mathbb{C}$-algebra, it is a finitely generated module over $\mathbb{C}[\mathbb{N}\mathscr{A}]$ (see [**10**, Cor. 5.8]).
- Thus the corresponding morphism $U_\sigma \to Y_{\mathscr{A}}$ is finite as defined in [**131**, p. 84].
- A finite morphism is proper with finite fibers (see [**131**, Ex. II.3.5 and II.4.1]).

Since $U_\sigma \to Y_{\mathscr{A}}$ is the identity on the torus, the image of the normalization is Zariski dense in $Y_{\mathscr{A}}$. But the image is also closed since the normalization map is proper. This proves that the normalization map is onto.

Here is an example of how the normalization map can fail to be one-to-one.

Example 3.A.2. The set $\mathscr{A} = \{e_1, e_1+e_2, 2e_2\} \subseteq \mathbb{Z}^2$ gives the parametrization $\Phi_{\mathscr{A}}(s,t) = (s, st, t^2)$, and one can check that

$$Y_{\mathscr{A}} = \mathbf{V}(y^2 - x^2 z) \subseteq \mathbb{C}^3.$$

Furthermore, $\mathbb{Z}\mathscr{A} = \mathbb{Z}^2$ and $\sigma = \mathrm{Cone}(\mathscr{A})^\vee = \mathrm{Cone}(e_1, e_2)$. It follows easily that the normalization is given by

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow Y_{\mathscr{A}} \\ (s,t) &\longmapsto (s, st, t^2). \end{aligned}$$

This map is one-to-one on the torus (the torus of $Y_{\mathscr{A}}$ is normal and hence is unchanged under normalization) but not on the t-axis, since here the map is $(0,t) \mapsto (0,0,t^2)$. We will soon see the intrinsic reason why this happens. ◊

We now determine the orbit structure of $Y_{\mathscr{A}}$.

Theorem 3.A.3. *Let $Y_{\mathscr{A}}$ be an affine toric variety with $M = \mathbb{Z}\mathscr{A}$ and let $\sigma \subseteq N_{\mathbb{R}}$ be as above. Then:*

(a) *There is a bijective correspondence*

$$\{\text{faces } \tau \text{ of } \sigma\} \longleftrightarrow \{T_N\text{-orbits in } Y_{\mathscr{A}}\}$$

such that a face of σ of dimension k corresponds to an orbit of dimension $\dim Y_{\mathscr{A}} - k$.

(b) *If $O' \subseteq Y_{\mathscr{A}}$ is the orbit corresponding to a face τ of σ, then O' is the torus with character group $\mathbb{Z}(\tau^\perp \cap \mathscr{A})$.*

(c) *The normalization $U_\sigma \to Y_{\mathscr{A}}$ induces a bijection*

$$\{T_N\text{-orbits in } U_\sigma\} \longleftrightarrow \{T_N\text{-orbits in } Y_{\mathscr{A}}\}$$

such that if $O \subseteq U_\sigma$ and $O' \subseteq Y_{\mathscr{A}}$ are the orbits corresponding to a face τ of σ, then the induced map $O \to O'$ is the map of tori corresponding to the inclusion $\mathbb{Z}(\tau^\perp \cap \mathscr{A}) \subseteq \tau^\perp \cap M$ of character groups.

Proof. We will sketch the main ideas and leave the details for the reader. The proof uses the Orbit-Cone Correspondence (Theorem 3.2.6). We regard points of U_σ and $Y_{\mathscr{A}}$ as semigroup homomorphisms, so that $\gamma : \sigma^\vee \cap M \to \mathbb{C}$ in U_σ maps to $\gamma|_{\mathbb{N}\mathscr{A}} : \mathbb{N}\mathscr{A} \to \mathbb{C}$ in $Y_{\mathscr{A}}$. Note also that $U_\sigma \to Y_{\mathscr{A}}$ is equivariant with respect to the action of T_N.

By Lemma 3.2.5, the orbit $O(\tau) \subseteq U_\sigma$ corresponding to a face τ of σ is the torus consisting of homomorphisms $\gamma : \tau^\perp \cap M \to \mathbb{C}^*$. Thus $\tau^\perp \cap M$ is the character group of $O(\tau)$. The normalization maps this orbit onto an orbit $O'(\tau) \subseteq Y_{\mathscr{A}}$, where a point γ of $O(\tau)$ maps to its restriction to $\mathbb{N}\mathscr{A}$. Since

$$(\tau^\perp \cap M) \cap \mathbb{Z}\mathscr{A} = \tau^\perp \cap \mathbb{Z}\mathscr{A} = \mathbb{Z}(\tau^\perp \cap \mathscr{A}),$$

it follows that $\mathbb{Z}(\tau^\perp \cap \mathscr{A})$ is the character group of $O'(\tau)$. This proves part (b), and the final assertion of part (c) follows easily.

Since $\sigma^\vee \cap M$ is the saturation of $\mathbb{N}\mathscr{A}$, it follows that there is an integer $d > 0$ such that $d\sigma^\vee \cap M \subseteq \mathbb{N}\mathscr{A}$. It follows easily that $\mathbb{Z}(\tau^\perp \cap \mathscr{A})$ has finite index in $\tau^\perp \cap M$, so that

$$\dim O'(\tau) = \dim O(\tau) = \dim U_\sigma - \dim \tau = \dim Y_{\mathscr{A}} - \dim \tau,$$

proving the final assertion of part (a).

Finally, every orbit in $Y_{\mathscr{A}}$ comes from an orbit in U_σ since $U_\sigma \to Y_{\mathscr{A}}$ is onto. If orbits $O(\tau_1), O(\tau_2)$ map to the same orbit of $Y_{\mathscr{A}}$, then

$$\mathbb{Z}(\tau_1^\perp \cap \mathscr{A}) = \mathbb{Z}(\tau_2^\perp \cap \mathscr{A}).$$

This implies $\tau_1^\perp = \tau_2^\perp$, so that $\tau_1 = \tau_2$. The bijections in parts (a) and (c) now follow. $\square$

We leave it to the reader to work out other aspects of the Orbit-Cone Correspondence (specifically, the analogs of parts (c) and (d) of Theorem 3.2.6) for $Y_{\mathscr{A}}$.

Let us apply Theorem 3.A.3 to our previous example.

Example 3.A.4. Let $\mathscr{A} = \{e_1, e_1+e_2, 2e_2\} \subseteq \mathbb{Z}^2$ as in Example 3.A.2. The cone $\sigma = \mathrm{Cone}(\mathscr{A})^\vee = \mathrm{Cone}(e_1, e_2)$ has a face τ such that $\tau^\perp = \mathrm{Span}(e_2)$. Thus

$$\begin{aligned}\mathbb{Z}(\tau^\perp \cap \mathscr{A}) &= \mathbb{Z}(2e_2)\\ \tau^\perp \cap M &= \mathbb{Z}e_2.\end{aligned}$$

It follows that $\mathbb{Z}(\tau^\perp \cap \mathscr{A})$ has index 2 in $\tau^\perp \cap M$, which explains why the normalization map is two-to-one on the orbit corresponding to τ. $\Diamond$

We now turn to the projective case. Here, $\mathscr{A} = \{m_1, \ldots, m_s\} \subseteq M$ gives the projective toric variety $X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1}$ whose torus has character group $\mathbb{Z}'\mathscr{A}$ (Proposition 2.1.6). Recall that $\mathbb{Z}'\mathscr{A} = \big\{\sum_{i=1}^s a_i m_i \mid a_i \in \mathbb{Z},\ \sum_{i=1}^s a_i = 0\big\}$.

One observation is that translating $\mathscr{A}$ by $m \in M$ leaves the corresponding projective variety unchanged. In other words, $X_{m+\mathscr{A}} = X_{\mathscr{A}}$ (see part (a) of Exercise 2.1.6). Thus, by translating an element of $\mathscr{A}$ to the origin, we may assume $0 \in \mathscr{A}$. Note that the torus of $X_{\mathscr{A}}$ has character lattice $\mathbb{Z}'\mathscr{A} = \mathbb{Z}\mathscr{A}$ when $0 \in \mathscr{A}$.

We defined the normalization of an affine variety in §1.0. Using a gluing construction, one can define the normalization of any variety (see [**131**, Ex. II.3.8]). We can describe the normalization of a projective toric variety $X_{\mathscr{A}}$ as follows.

Theorem 3.A.5. *Let $X_{\mathscr{A}}$ be a projective toric variety where $0 \in \mathscr{A}$ and $M = \mathbb{Z}\mathscr{A}$. If $P = \mathrm{Conv}(\mathscr{A}) \subseteq M_{\mathbb{R}}$, then the normalization of $X_{\mathscr{A}}$ is the toric variety X_{Σ_P} of the normal fan of P with respect to the lattice $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.*

Proof. Again, we sketch the proof and leave the details to the reader. We use the local description of $X_{\mathscr{A}}$ given in Propositions 2.1.8 and 2.1.9. There, we saw that $X_{\mathscr{A}}$ has an affine open covering given by the affine toric varieties $Y_{\mathscr{A}_\mathsf{v}} = \mathrm{Spec}(\mathbb{N}\mathscr{A}_\mathsf{v})$, where $\mathsf{v} \in \mathscr{A}$ is a vertex of $P = \mathrm{Conv}(\mathscr{A})$ and $\mathscr{A}_\mathsf{v} = \mathscr{A} - \mathsf{v} = \{m - \mathsf{v} \mid m \in \mathscr{A}\}$.

For the moment, assume that P is very ample. Then Theorem 2.3.1 implies that X_P has an affine open cover given by the affine toric varieties $U_{\sigma_\mathsf{v}} = \mathrm{Spec}(\sigma_\mathsf{v}^\vee \cap M)$, where $\mathsf{v} \in \mathscr{A}$ is a vertex of P and $\sigma_\mathsf{v}^\vee = \mathrm{Cone}(P \cap M - \mathsf{v})$. One can check that $\sigma_\mathsf{v}^\vee \cap M$ is the saturation of $\mathbb{N}\mathscr{A}_\mathsf{v}$, so that U_{σ_v} is the normalization of $Y_{\mathscr{A}_\mathsf{v}}$. The gluings are also compatible by equations (2.1.6), (2.1.7) and Proposition 2.3.13. It follows that we get a natural map $X_{\Sigma_P} \to X_{\mathscr{A}}$ that is the normalization of $X_{\mathscr{A}}$.

In the general case, we note that $k_0 P$ is very ample for some integer $k_0 \geq 1$ and that P and $k_0 P$ have the same normal fan. Since σ_v is a maximal cone of the normal fan, the above argument now applies in general, and the theorem is proved. $\square$

Combining this result with the Orbit-Cone Correspondence and Theorem 3.A.3 gives the following immediate corollary.

Corollary 3.A.6. *With the same hypotheses as Theorem 3.A.5, we have:*

(a) *There is a bijective correspondence*

$$\{\text{cones } \tau \text{ of } \Sigma_P\} \longleftrightarrow \{T_N\text{-orbits in } X_{\mathscr{A}}\}$$

such that a cone τ of dimension k corresponds to an orbit of dimension $\dim X_{\mathscr{A}} - k$.

(b) *If $O' \subseteq X_{\mathscr{A}}$ is the orbit corresponding to a cone τ of Σ_P, then O' is the torus with character group $\mathbb{Z}(\tau^\perp \cap \mathscr{A})$.*

(c) *The normalization $X_{\Sigma_P} \to X_{\mathscr{A}}$ induces a bijection*

$$\{T_N\text{-orbits in } X_{\Sigma_P}\} \longleftrightarrow \{T_N\text{-orbits in } X_{\mathscr{A}}\}$$

such that if $O \subseteq X_{\Sigma_P}$ and $O' \subseteq X_{\mathscr{A}}$ are the orbits corresponding to $\tau \in \Sigma_P$, then the induced map $O \to O'$ is the map of tori corresponding to the inclusion $\mathbb{Z}(\tau^\perp \cap \mathscr{A}) \subseteq \tau^\perp \cap M$ of character groups.

We leave it to the reader to work out other aspects of the Orbit-Cone Correspondence for $X_{\mathscr{A}}$. A different approach to the study of $X_{\mathscr{A}}$ appears in [**113**, Ch. 5].

Chapter 4

Divisors on Toric Varieties

§4.0. Background: Valuations, Divisors and Sheaves

Divisors are defined in terms of irreducible codimension one subvarieties. In this chapter, we will consider *Weil divisors* and *Cartier divisors*. These classes coincide on a smooth variety, but for a normal variety, the situation is more complicated. We will also study *divisor classes,* which are defined using the order of vanishing of a rational function on an irreducible divisor. We will see that normal varieties are the natural setting to develop a theory of divisors and divisor classes.

First, we give a simple motivational example.

Example 4.0.1. If $f(x) \in \mathbb{C}(x)$ is nonzero, then there is a unique $n \in \mathbb{Z}$ such that $f(x) = x^n \frac{g(x)}{h(x)}$, where $g(x)$, $h(x) \in \mathbb{C}[x]$ are not divisible by x. This works because $\mathbb{C}[x]$ is a UFD. The integer n describes the behavior of $f(x)$ at 0: if $n > 0$, $f(x)$ vanishes to order n at 0, and if $n < 0$, $f(x)$ has a pole of order $|n|$ at 0. Furthermore, the map from the multiplicative group $\mathbb{C}(x)^*$ to the additive group $\mathbb{Z}$ defined by $f(x) \mapsto n$ is easily seen to be a group homomorphism. This works in the same way if we replace 0 with any point of $\mathbb{C}$. $\Diamond$

Discrete Valuation Rings. The simple construction given in Example 4.0.1 applies in far greater generality. We begin by reviewing the algebraic machinery we will need.

Definition 4.0.2. A *discrete valuation* on a field K is a group homomorphism

$$\nu : K^* \longrightarrow \mathbb{Z}$$

that is onto and satisfies $\nu(x+y) \geq \min(\nu(x), \nu(y))$ when $x, y, x+y \in K^* = K \setminus \{0\}$. Note also that $\nu(xy) = \nu(x) + \nu(y)$. The corresponding *discrete valuation ring* is

$$R = \{x \in K^* \mid \nu(x) \geq 0\} \cup \{0\}.$$

One can check that a DVR is indeed a ring. Here are some properties of DVRs.

Proposition 4.0.3. *Let R be a DVR with valuation $\nu : K^* \to \mathbb{Z}$. Then:*

(a) *$x \in R$ is invertible in R if and only if $\nu(x) = 0$.*

(b) *R is a local ring with maximal ideal $\mathfrak{m} = \{x \in R \mid \nu(x) > 0\} \cup \{0\}$.*

(c) *R is normal.*

(d) *R is a principal ideal domain (PID).*

(e) *R is Noetherian.*

(f) *The only proper prime ideals of R are $\{0\}$ and $\mathfrak{m}$.*

Proof. First observe that since ν is a homomorphism, we have

$$\nu(x^{-1}) = -\nu(x) \tag{4.0.1}$$

for all $x \in K^*$. If $x \in R$ is a unit, then $\nu(x), \nu(x^{-1}) \geq 0$ since $x, x^{-1} \in R$. Thus $\nu(x) = 0$ by (4.0.1). Conversely, if $\nu(x) = 0$, then $\nu(x^{-1}) = 0$ by (4.0.1), so that $x^{-1} \in R$. This proves part (a).

For part (b), note that $\mathfrak{m} = \{x \in R \mid \nu(x) > 0\} \cup \{0\}$ is an ideal of R (this follows directly from Definition 4.0.2). Then part (a) easily implies that R is local with maximal ideal $\mathfrak{m}$ (Exercise 4.0.1).

To prove part (c), suppose $x \in K^* = K \setminus \{0\}$ satisfies

$$x^n + r_{n-1}x^{n-1} + \cdots + r_0 = 0,$$

with $r_i \in R$. If $x \in R$, we are done, so suppose $x \notin R$. Then $n > 1$ and $\nu(x) < 0$. Using (4.0.1) again, we see that $x^{-1} \in R$. So $x^{1-n} = (x^{-1})^{n-1} \in R$ and hence

$$x^{1-n} \cdot (x^n + r_{n-1}x^{n-1} + \cdots + r_0) = 0,$$

showing that $x = -(r_{n-1} + r_{n-2}x^{-1} + \cdots + r_0 x^{1-n}) \in R$.

Let $\pi \in R$ satisfy $\nu(\pi) = 1$ and let $I \neq \{0\}$ be an ideal of R. Pick $x \in I \setminus \{0\}$ with $k = \nu(x)$ minimal. Then $y = x\pi^{-k} \in K$ satisfies $\nu(y) = \nu(x) - k\nu(\pi) = 0$, so that y is invertible in R. From here, one proves without difficulty that $I = \langle \pi^k \rangle$. This proves part (d), and part (e) follows immediately.

For part (f), it is obvious that $\{0\}$ and the maximal ideal $\mathfrak{m}$ are prime. Note also that $\mathfrak{m} = \langle \pi \rangle$. Now let $P \neq \{0\}$ be a proper prime ideal. By the previous paragraph, $P = \langle \pi^k \rangle$ for some $k > 0$. If $k > 1$, then $\pi \cdot \pi^{k-1} \in P$ and $\pi, \pi^{k-1} \notin P$ give a contradiction. $\square$

This shows that every DVR is a Noetherian local domain of dimension one. In general, the *dimension* $\dim R$ of a Noetherian ring R is one less than the length of the longest chain $P_0 \subsetneq \cdots \subsetneq P_d$ of proper prime ideals contained in R. Among Noetherian local domains of dimension one, DVRs are characterized as follows.

Theorem 4.0.4. *If $(R,\mathfrak{m})$ is a Noetherian local domain of dimension one, then the following are equivalent:*

(a) *R is a DVR.*

(b) *R is normal.*

(c) *$\mathfrak{m}$ is principal.*

(d) *$(R,\mathfrak{m})$ is a regular local ring.*

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) follow from Proposition 4.0.3, and the equivalence (c) $\Leftrightarrow$ (d) is covered in Exercise 4.0.2. The remaining implications can be found in [**10**, Prop. 9.2]. $\square$

DVRs and Prime Divisors. DVRs have a natural geometric interpretation. Let X be an irreducible variety. A *prime divisor* $D \subseteq X$ is an irreducible subvariety of codimension one, meaning that $\dim D = \dim X - 1$. Recall from §3.0 that X has a field of rational functions $\mathbb{C}(X)$. Our goal is to define a ring $\mathscr{O}_{X,D}$ with field of fractions $\mathbb{C}(X)$ such that $\mathscr{O}_{X,D}$ is a DVR when X is normal. This will give a valuation $\nu_D : \mathbb{C}(X)^* \to \mathbb{Z}$ such that for $f \in \mathbb{C}(X)^*$, $\nu_D(f)$ gives the order of vanishing of f along D.

Definition 4.0.5. For a variety X and prime divisor $D \subseteq X$, $\mathcal{O}_{X,D}$ is the subring of $\mathbb{C}(X)$ defined by

$$\mathcal{O}_{X,D} = \{\phi \in \mathbb{C}(X) \mid \phi \text{ is defined on } U \subseteq X \text{ open with } U \cap D \neq \emptyset\}.$$

We will see below that $\mathcal{O}_{X,D}$ is a ring. Intuitively, this ring is built from rational functions on X that are defined somewhere on D (and hence defined on most of D since D is irreducible).

Since X is irreducible, Exercise 3.0.4 implies that $\mathbb{C}(X) = \mathbb{C}(U)$ whenever $U \subseteq X$ is open and nonempty. If we further assume that $U \cap D$ is nonempty, then

$$\mathscr{O}_{X,D} = \mathscr{O}_{U,U\cap D} \tag{4.0.2}$$

follows easily (Exercise 4.0.3).

Hence we can reduce to the affine case $X = \mathrm{Spec}(R)$ for an integral domain R. The *codimension* of a prime ideal $\mathfrak{p}$, also called its *height*, is defined to be $\operatorname{codim} \mathfrak{p} = \dim R - \dim \mathbf{V}(\mathfrak{p})$. It follows easily that $\mathfrak{p} \mapsto \mathbf{V}(\mathfrak{p})$ induces a bijection

$$\{\text{codimension one prime ideals of } R\} \simeq \{\text{prime divisors of } X\}.$$

Given a prime divisor $D = \mathbf{V}(\mathfrak{p})$, we can interpret $\mathscr{O}_{X,D}$ in terms of R as follows. The field of rational functions $\mathbb{C}(X)$ is the field of fractions K of R, and a rational function $\phi = f/g \in K$, $f,g \in R$, is defined somewhere on $D = \mathbf{V}(\mathfrak{p})$ precisely when $g \notin \mathbf{I}(D) = \mathfrak{p}$. It follows that

$$\mathscr{O}_{X,D} = \{f/g \in K \mid f,g \in R,\ g \notin \mathfrak{p}\},$$

which is the localization $R_{\mathfrak{p}}$ of R at the multiplicative subset $R \setminus \mathfrak{p}$ (note that $R \setminus \mathfrak{p}$ is closed under multiplication because $\mathfrak{p}$ is prime). This localization is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ (Exercise 4.0.3). It follows that

$$\mathscr{O}_{X,D} = R_{\mathfrak{p}} \tag{4.0.3}$$

when $X = \mathrm{Spec}(R)$ and $\mathfrak{p}$ is a codimension one prime ideal of R.

Example 4.0.6. In Example 4.0.1, we constructed a discrete valuation on $\mathbb{C}(x)$ by sending $f(x) \in \mathbb{C}(x)^*$ to $n \in \mathbb{Z}$, provided

$$f(x) = x^n \frac{g(x)}{h(x)}, \quad g(x), h(x) \in \mathbb{C}[x],\ g(0) \neq 0, h(0) \neq 0.$$

The corresponding DVR is the localization $\mathbb{C}[x]_{\langle x \rangle}$. It follows that the prime divisor $\{0\} = \mathbf{V}(x) \subseteq \mathbb{C} = \mathrm{Spec}(\mathbb{C}[x])$ has the local ring

$$\mathscr{O}_{\mathbb{C},\{0\}} = \mathbb{C}[x]_{\langle x \rangle}$$

which is a DVR. $\Diamond$

More generally, a normal ring or variety gives a DVR as follows.

Proposition 4.0.7.

(a) *Let R be a normal domain and $\mathfrak{p} \subseteq R$ be a codimension one prime ideal. Then the localization $R_{\mathfrak{p}}$ is a DVR.*

(b) *Let X be a normal variety and $D \subseteq X$ a prime divisor. Then the local ring $\mathscr{O}_{X,D}$ is a DVR.*

Proof. By Proposition 3.0.12, part (b) follows immediately from part (a) together with (4.0.2) and (4.0.3).

It remains to prove part (a). The maximal ideal of $R_{\mathfrak{p}}$ is the ideal $\mathfrak{m}_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ generated by $\mathfrak{p}$ in $R_{\mathfrak{p}}$. The localization of a Noetherian ring is Noetherian (Exercise 4.0.4), and the same is true for normality by Exercise 1.0.7. It follows that the local domain $(R_{\mathfrak{p}}, \mathfrak{m}_{\mathfrak{p}})$ is Noetherian and normal.

We compute the dimension of $R_{\mathfrak{p}}$ as follows. Since $\dim X = \dim R$ (see [**69**, Ex. 17 and 18 of Ch. 9, §4]), our hypothesis on $D = \mathbf{V}(\mathfrak{p})$ implies that there are no prime ideals strictly between $\{0\}$ and $\mathfrak{p}$ in R. By [**10**, Prop. 3.11], the same is true for $\{0\}$ and $\mathfrak{m}_{\mathfrak{p}}$ in $R_{\mathfrak{p}}$. It follows that $R_{\mathfrak{p}}$ has dimension one. Then $R_{\mathfrak{p}}$ is a DVR by Theorem 4.0.4. $\square$

When D is a prime divisor on a normal variety X, the DVR $\mathscr{O}_{X,D}$ means that we have a discrete valuation

$$\nu_D : \mathbb{C}(X)^* \longrightarrow \mathbb{Z},$$

where $\mathscr{O}_{X,D}$ consists of 0 and those nonzero rational functions satisfying $\nu_D(f) \geq 0$. Given $f \in \mathscr{O}_{X,D} \setminus \{0\}$, we call $\nu_D(f)$ the *order of vanishing* of f along the divisor D. Thus the maximal ideal $\mathfrak{m}_{X,D} \subseteq \mathscr{O}_{X,D}$ consists of 0 and those rational functions that vanish on D. When $f \in \mathbb{C}(X)^*$ satisfies $\nu_D(f) = \ell < 0$, we say that f has a *pole* of order $|\ell|$ along D.

Weil Divisors. Recall that a prime divisor on an irreducible variety X is an irreducible subvariety of codimension one.

Definition 4.0.8. $\mathrm{Div}(X)$ is the free abelian group generated by the prime divisors on X. A ***Weil divisor*** is an element of $\mathrm{Div}(X)$.

Thus a Weil divisor $D \in \mathrm{Div}(X)$ is a finite sum $D = \sum_i a_i D_i \in \mathrm{Div}(X)$ of prime divisors D_i with $a_i \in \mathbb{Z}$ for all i. The divisor D is *effective*, written $D \geq 0$, if the a_i are all nonnegative. The *support* of D is the union of the prime divisors appearing in D:

$$\mathrm{Supp}(D) = \bigcup_{a_i \neq 0} D_i.$$

The Divisor of a Rational Function. An important class of Weil divisors comes from rational functions. If X is normal, any prime divisor D on X corresponds to a DVR $\mathscr{O}_{X,D}$ with valuation $\nu_D : \mathbb{C}(X)^* \to \mathbb{Z}$. Given $f \in \mathbb{C}(X)^*$, the integers $\nu_D(f)$ tell us how f behaves on the prime divisors of X. Here is an important property of these integers.

Lemma 4.0.9. *If X is normal and $f \in \mathbb{C}(X)^*$, then $\nu_D(f)$ is zero for all but a finite number of prime divisors $D \subseteq X$.*

Proof. If f is constant, then it is a nonzero constant since $f \in \mathbb{C}(X)^*$. It follows that $\nu_D(f) = 0$ for all D. On the other hand, if f is nonconstant, then we can find a nonempty open subset $U \subseteq X$ such that $f : U \to \mathbb{C}$ is a nonconstant morphism. Then $V = f^{-1}(\mathbb{C}^*)$ is a nonempty open subset of X such that $f|_V : V \to \mathbb{C}^*$. The complement $X \setminus V$ is Zariski closed and hence is a union of irreducible components of dimension $< n$. Denote the irreducible components of codimension one by $D_1, \ldots, D_s$.

Now let D be prime divisor in X. If $V \cap D = \emptyset$, then $D \subseteq X \setminus V$, so that D is contained in an irreducible component of $X \setminus V$ since D is irreducible. Dimension considerations imply that $D = D_i$ for some i. On the other hand, if $V \cap D \neq \emptyset$, then f is an invertible element of $\mathscr{O}_{X,D} = \mathscr{O}_{V,V\cap D}$, which implies that $\nu_D(f) = 0$. □

Definition 4.0.10. Let X be a normal variety.

(a) The *divisor* of $f \in \mathbb{C}(X)^*$ is

$$\mathrm{div}(f) = \sum_D \nu_D(f) D,$$

where the sum is over all prime divisors $D \subseteq X$.

(b) $\mathrm{div}(f)$ is called a ***principal divisor***, and the set of all principal divisors is denoted $\mathrm{Div}_0(X)$.

(c) Divisors D and E are ***linearly equivalent***, written $D \sim E$, if their difference is a principal divisor, i.e., $D - E = \mathrm{div}(f) \in \mathrm{Div}_0(X)$ for some $f \in \mathbb{C}(X)^*$.

Lemma 4.0.9 implies that $\mathrm{div}(f) \in \mathrm{Div}(X)$. If $f, g \in \mathbb{C}(X)^*$, then $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$ and $\mathrm{div}(f^{-1}) = -\mathrm{div}(f)$ since valuations are group homomorphisms on $\mathbb{C}(X)^*$. It follows that $\mathrm{Div}_0(X)$ is a subgroup of $\mathrm{Div}(X)$.

Example 4.0.11. Let $f = c(x-a_1)^{m_1} \cdots (x-a_r)^{m_r} \in \mathbb{C}[x]$ be a polynomial of degree $m > 0$, where $c \in \mathbb{C}^*$ and $a_1, \ldots, a_r \in \mathbb{C}$ are distinct. Then:

- When $X = \mathbb{C}$, $\mathrm{div}(f) = \sum_{i=1}^r m_i \{a_i\}$.
- When $X = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, $\mathrm{div}(f) = \sum_{i=1}^r m_i \{a_i\} - m\{\infty\}$. ◊

The divisor of $f \in \mathbb{C}(X)^*$ can be written $\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f)$, where

$$\begin{aligned} \mathrm{div}_0(f) &= \sum_{\nu_D(f)>0} \nu_D(f) D \\ \mathrm{div}_\infty(f) &= \sum_{\nu_D(f)<0} -\nu_D(f) D. \end{aligned}$$

We call $\mathrm{div}_0(f)$ the *divisor of zeros* of f and $\mathrm{div}_\infty(f)$ the *divisor of poles* of f. Note that these are effective divisors.

Cartier Divisors. If $D = \sum_i a_i D_i$ is a Weil divisor on X and $U \subseteq X$ is a nonempty open subset, then

$$D|_U = \sum_{U \cap D_i \neq \emptyset} a_i U \cap D_i$$

is a Weil divisor on U called the *restriction* of D to U.

We now define a special class of Weil divisors.

Definition 4.0.12. A Weil divisor D on a normal variety X is ***Cartier*** if it is ***locally principal***, meaning that X has an open cover $\{U_i\}_{i\in I}$ such that $D|_{U_i}$ is principal in U_i for every $i \in I$. If $D|_{U_i} = \mathrm{div}(f_i)|_{U_i}$ for $i \in I$, then we call $\{(U_i, f_i)\}_{i\in I}$ the ***local data*** for D.

A principal divisor is obviously locally principal. Thus $\mathrm{div}(f)$ is Cartier for all $f \in \mathbb{C}(X)^*$. One can also show that if D and E are Cartier divisors, then $D+E$ and $-D$ are Cartier (Exercise 4.0.5). It follows that the Cartier divisors on X form a group $\mathrm{CDiv}(X)$ satisfying

$$\mathrm{Div}_0(X) \subseteq \mathrm{CDiv}(X) \subseteq \mathrm{Div}(X).$$

Divisor Classes. For Weil and Cartier divisors, linear equivalence classes form the following important groups.

Definition 4.0.13. Let X be a normal variety. Its ***class group*** is

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\mathrm{Div}_0(X),$$

and its ***Picard group*** is

$$\mathrm{Pic}(X) = \mathrm{CDiv}(X)/\mathrm{Div}_0(X).$$

We will give a more sophisticated definition of $\mathrm{Pic}(X)$ in Chapter 6. Note that since $\mathrm{CDiv}(X)$ is a subgroup of $\mathrm{Div}(X)$, we get a canonical injection

$$\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X).$$

In [**131**, II.6], Hartshorne writes "The divisor class group of a scheme is a very interesting invariant. In general it is not easy to calculate." Fortunately, divisor class groups of normal toric varieties are easy to describe, as we will see in §4.1.

More Algebra. Before we can derive further properties of divisors, we need to learn more about normal domains. Equation (3.0.2) shows that if $X = \mathrm{Spec}(R)$ is irreducible, then

$$R = \bigcap_{p \in X} \mathscr{O}_{X,p}.$$

If a point $p \in X$ corresponds to a maximal ideal $\mathfrak{m} \subseteq R$, then the local ring $\mathscr{O}_{X,p}$ is the localization $R_{\mathfrak{m}}$. Hence the above equality can be written

$$R = \bigcap_{\mathfrak{m}\ \text{maximal}} R_{\mathfrak{m}}.$$

When R is normal, we get a similar result using codimension one prime ideals.

Theorem 4.0.14. *If R is a Noetherian normal domain, then*

$$R = \bigcap_{\mathrm{codim}\, \mathfrak{p}=1} R_{\mathfrak{p}}.$$

Proof. Let K be the field of fractions of R and assume that $a/b \in K$, $a, b \in R$, lies in $R_{\mathfrak{p}}$ for all codimension one prime ideals $\mathfrak{p}$. It suffices to prove that $a \in \langle b \rangle$. This is obviously true when b is invertible in R, so we may assume that $\langle b \rangle$ is a proper ideal of R. Then we have a primary decomposition (see [**69**, Ch. 4, §7])

$$\langle b \rangle = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s, \tag{4.0.4}$$

and each prime ideal $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ is of the form $\mathfrak{p}_i = \langle b \rangle : c_i$ for some $c_i \in R$. In the terminology of [**195**, p. 38], the $\mathfrak{p}_i$ are the *prime divisors* of $\langle b \rangle$.

Since R is Noetherian and normal, the Krull principal ideal theorem states that every prime divisor of $\langle b \rangle$ has codimension one (see [**195**, Thm. 11.5] for a proof). This implies that in the primary decomposition (4.0.4), the prime divisors $\mathfrak{p}_i$ have codimension one and hence are distinct.

Note that $a/b \in R_{\mathfrak{p}_i}$ for all i by our assumption on a/b. This implies $a \in bR_{\mathfrak{p}_i}$. Since $(\mathfrak{q}_j)_{\mathfrak{p}_i} = R_{\mathfrak{p}_i}$ for $j \neq i$ (Exercise 4.0.6), localizing (4.0.4) at $\mathfrak{p}_i$ shows that for all i, we have

$$a \in bR_{\mathfrak{p}_i} = \mathfrak{q}_i R_{\mathfrak{p}_i}.$$

Since $\mathfrak{q}_i R_{\mathfrak{p}_i} \cap R = \mathfrak{q}_i$ (Exercise 4.0.6), we obtain $a \in \bigcap_{i=1}^s \mathfrak{q}_i = \langle b \rangle$. □

This result has the following useful corollary.

Corollary 4.0.15. *Let X be a normal variety and let $f : U \to \mathbb{C}$ be a morphism defined on an open set $U \subseteq X$. If $X \setminus U$ has codimension ≥ 2 in X, then f extends to a morphism defined on all of X.*

Proof. Since X has an affine open cover, we can assume that $X = \operatorname{Spec}(R)$, where R is a Noetherian normal domain. If $D \subseteq X$ is a prime divisor, then $U \cap D \neq \emptyset$ for dimension reasons. It follows that $f \in \mathscr{O}_{U,U\cap D} = \mathscr{O}_{X,D}$, so that

$$f \in \bigcap_D \mathscr{O}_{U,U\cap D} = \bigcap_D \mathscr{O}_{X,D} = \bigcap_{\operatorname{codim} \mathfrak{p}=1} R_{\mathfrak{p}} = R, \tag{4.0.5}$$

where the final equality is Theorem 4.0.14. □

These results enable us to determine when the divisor of a rational function is effective.

Proposition 4.0.16. *Let X be a normal variety. If $f \in \mathbb{C}(X)^*$, then:*

(a) $\operatorname{div}(f) \geq 0$ *if and only if* $f : X \to \mathbb{C}$ *is a morphism, i.e.,* $f \in \mathscr{O}_X(X)$.

(b) $\operatorname{div}(f) = 0$ *if and only if* $f : X \to \mathbb{C}^*$ *is a morphism, i.e.,* $f \in \mathscr{O}_X^*(X)$.

In general, $\mathscr{O}_X^*$ is the sheaf on X defined by

$$\mathscr{O}_X^*(U) = \{\text{invertible elements of } \mathscr{O}_X(U)\}.$$

This is a sheaf of abelian groups under multiplication.

Proof. If $f : X \to \mathbb{C}$ is a morphism, then $f \in \mathscr{O}_{X,D}$ for every prime divisor D, which in turn implies $\nu_D(f) \geq 0$. Hence $\operatorname{div}(f) \geq 0$. Going the other way, suppose that $\operatorname{div}(f) \geq 0$. This remains true when we restrict to an affine open subset, so we may assume that X is affine. Then $\operatorname{div}(f) \geq 0$ implies

$$f \in \bigcap_D \mathscr{O}_{X,D},$$

where the intersection is over all prime divisors. By (4.0.5), we conclude that f is defined everywhere. This proves part (a), and part (b) follows immediately since $\operatorname{div}(f) = 0$ if and only if $\operatorname{div}(f) \geq 0$ and $\operatorname{div}(f^{-1}) \geq 0$. □

Singularities and Normality. The set of singular points of a variety X is denoted

$$\mathrm{Sing}(X) \subseteq X.$$

We call $\mathrm{Sing}(X)$ the *singular locus* of X. One can show that $\mathrm{Sing}(X)$ is a proper closed subvariety of X (see [**131**, Thm. I.5.3]). When X is normal, things are even nicer.

Proposition 4.0.17. *Let X be a normal variety. Then:*

(a) $\mathrm{Sing}(X)$ *has codimension* ≥ 2 *in* X.

(b) *If X is a curve, then X is smooth.*

Proof. You will prove part (b) in Exercise 4.0.7. A proof of part (a) can be found in [**245**, Vol. 2, Thm. 3 of §II.5]. □

Computing Divisor Classes. There are two results, one algebraic and one geometric, that enable us to compute class groups in some cases.

We begin with the algebraic result.

Theorem 4.0.18. *Let R be a UFD and set $X = \mathrm{Spec}(R)$. Then:*

(a) *R is normal and every codimension one prime ideal is principal.*

(b) $\mathrm{Cl}(X) = 0$.

Proof. For part (a), we know that a UFD is normal by Exercise 1.0.5. Let $\mathfrak{p}$ be a codimension one prime ideal of R and pick $a \in \mathfrak{p} \setminus \{0\}$. Since R is a UFD,

$$a = c\prod_{i=1}^{s} p_i^{a_i},$$

with the p_i prime and c is invertible in R. Because $\mathfrak{p}$ is prime, this means some $p_i \in \mathfrak{p}$, and since $\operatorname{codim}\mathfrak{p} = 1$, this forces $\mathfrak{p} = \langle p_i \rangle$.

Turning to part (b), let $D \subseteq X$ be a prime divisor. Then $\mathfrak{p} = \mathbf{I}(D)$ is a codimension one prime ideal and hence is principal, say $\mathfrak{p} = \langle f \rangle$. Then f generates the maximal ideal of the DVR $R_\mathfrak{p}$, which implies $\nu_D(f) = 1$ (see the proof of Proposition 4.0.3). It follows easily that $\mathrm{div}(f) = D$. Then $\mathrm{Cl}(X) = 0$ since all prime divisors are linearly equivalent to 0. □

In fact, more is true: a normal Noetherian domain is a UFD if and only if every codimension one prime ideal is principal (Exercise 4.0.8).

Example 4.0.19. $\mathbb{C}[x_1, \ldots, x_n]$ is a UFD, so $\mathrm{Cl}(\mathbb{C}^n) = 0$ by Theorem 4.0.18. ◊

Before stating the geometric result, note that if $U \subseteq X$ is open and nonempty, then restriction of divisors $D \mapsto D|_U$ induces a well-defined map $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ (Exercise 4.0.9).

Theorem 4.0.20. *Let U be a nonempty open subset of a normal variety X and let $D_1,\dots,D_s$ be the irreducible components of $X \setminus U$ that are prime divisors. Then the sequence*

$$\bigoplus_{j=1}^{s} \mathbb{Z}D_j \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0$$

is exact, where the first map sends $\sum_{j=1}^{s} a_j D_j$ to its divisor class in $\mathrm{Cl}(X)$ and the second is induced by restriction to U.

Proof. Let $D' = \sum_i a_i D'_i \in \mathrm{Div}(U)$ with D'_i a prime divisor in U. Then the Zariski closure $\overline{D'_i}$ of D'_i in X is a prime divisor in X, and $D = \sum_i a_i \overline{D'_i}$ satisfies $D|_U = D'$. Hence $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is surjective.

Since each D_j restricts to 0 in $\mathrm{Div}(U)$, the composition of the two maps is trivial. To finish the proof of exactness, suppose that $[D] \in \mathrm{Cl}(X)$ restricts to 0 in $\mathrm{Cl}(U)$. This means that $D|_U$ is the divisor of some $f \in \mathbb{C}(U)^*$. Since $\mathbb{C}(U) = \mathbb{C}(X)$ and the divisor of f in $\mathrm{Div}(X)$ restricts to the divisor of f in $\mathrm{Div}(U)$, it follows that we have $f \in \mathbb{C}(X)^*$ such that

$$D|_U = \mathrm{div}(f)|_U.$$

This implies that the difference $D - \mathrm{div}(f)$ is supported on $X \setminus U$, which means that $D - \mathrm{div}(f) \in \bigoplus_{j=1}^{s} \mathbb{Z}D_j$ by the definition of the D_j. □

Example 4.0.21. Write $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ and note that $\{\infty\}$ is a prime divisor on $\mathbb{P}^1$. Then Theorem 4.0.20 and Example 4.0.19 give the exact sequence

$$\mathbb{Z}\{\infty\} \longrightarrow \mathrm{Cl}(\mathbb{P}^1) \longrightarrow \mathrm{Cl}(\mathbb{C}) = 0.$$

Hence the map $\mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^1)$ defined by $a \mapsto [a\{\infty\}]$ is surjective. This map is injective since $a\{\infty\} = \mathrm{div}(f)$ implies $\mathrm{div}(f)|_{\mathbb{C}} = 0$, so that $f \in \Gamma(\mathbb{C}, \mathscr{O}_{\mathbb{C}})^* = \mathbb{C}^*$ by Proposition 4.0.16. Hence f is constant, which forces $a = 0$. If follows that $\mathrm{Cl}(\mathbb{P}^1) \simeq \mathbb{Z}$. ◊

Later in the chapter we will use similar methods to compute the class group of an arbitrary normal toric variety.

Comparing Weil and Cartier Divisors. Once we understand Cartier divisors on normal toric varieties, it will be easy to give examples of Weil divisors that are not Cartier. On the other hand, there are varieties where *every* Weil divisor is Cartier.

Theorem 4.0.22. *Let X be a normal variety. Then:*

(a) *If the local ring $\mathscr{O}_{X,p}$ is a UFD for every $p \in X$, then every Weil divisor on X is Cartier.*

(b) *If X is smooth, then every Weil divisor on X is Cartier.*

Proof. If X is smooth, then $\mathscr{O}_{X,p}$ is a regular local ring for all $p \in X$. Since every regular local ring is a UFD (see §1.0), part (b) follows from part (a).

For part (a), it suffices to show that prime divisors are locally principal. This condition is obviously local on X, so we may assume that $X = \operatorname{Spec}(R)$ is affine. Let $D = \mathbf{V}(\mathfrak{p})$ be a prime divisor on X, where $\mathfrak{p} \subseteq R$ is a codimension one prime ideal. Note that D is obviously principal on $U = X \setminus D$ since $D|_U = 0$. It remains to show that D is locally principal in a neighborhood of a point $p \in D$.

The point p corresponds to a maximal ideal $\mathfrak{m} \subseteq R$. Thus $p \in D$ implies $\mathfrak{p} \subseteq \mathfrak{m}$. Since $\mathfrak{p} \subseteq R$ has codimension one, it follows that the prime ideal $\mathfrak{p}R_\mathfrak{m} \subseteq R_\mathfrak{m}$ also has codimension one (this follows from [**10**, Prop. 3.11]). Then Theorem 4.0.18 implies that $\mathfrak{p}R_\mathfrak{m}$ is principal since $R_\mathfrak{m}$ is a UFD by hypothesis. Thus $\mathfrak{p}R_\mathfrak{m} = (a/b)R_\mathfrak{m}$ where $a, b \in R$ and $b \notin \mathfrak{m}$. Since b is invertible in $R_\mathfrak{m}$, we in fact have $\mathfrak{p}R_\mathfrak{m} = aR_\mathfrak{m}$.

Now suppose $\mathfrak{p} = \langle a_1, \ldots, a_s \rangle \subseteq R$. Then $a_i \in \mathfrak{p}R_\mathfrak{m} = aR_\mathfrak{m}$, so that $a_i = (g_i/h_i)a$, where $g_i, h_i \in R$ and $h_i \notin \mathfrak{m}$, i.e., $h_i(p) \neq 0$. If we set $h = h_1 \cdots h_s$, then $\mathfrak{p}R_h = aR_h$ follows easily. Then $U = \operatorname{Spec}(R_h)$ is a neighborhood of p, and from here, it is straightforward to see that $D = \operatorname{div}(a)$ on U. $\square$

Example 4.0.23. Since $\mathbb{P}^1$ is smooth, Theorem 4.0.22 and Example 4.0.21 imply that $\operatorname{Pic}(\mathbb{P}^1) = \operatorname{Cl}(\mathbb{P}^1) \simeq \mathbb{Z}$. $\Diamond$

Sheaves of $\mathscr{O}_X$-modules. Weil and Cartier divisors on X lead to some important sheaves on X. Hence we need a brief excursion into sheaf theory (we will go deeper into the subject in Chapter 6). The sheaf $\mathscr{O}_X$ was defined in §3.0. The definition of a *sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules* is similar: for each open subset $U \subseteq X$, there is an $\mathscr{O}_X(U)$-module $\mathscr{F}(U)$ with the following properties:

- When $U' \subseteq U$, there is a restriction map
$$\rho_{U,U'} : \mathscr{F}(U) \to \mathscr{F}(U')$$
such that $\rho_{U,U}$ is the identity and $\rho_{U',U''} \circ \rho_{U,U'} = \rho_{U,U''}$ when $U'' \subseteq U' \subseteq U$. Furthermore, $\rho_{U,U'}$ is compatible with the restriction map $\mathscr{O}_X(U) \to \mathscr{O}_X(U')$.
- If $\{U_\alpha\}$ is an open cover of $U \subseteq X$, then the sequence
$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \prod_\alpha \mathscr{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathscr{F}(U_\alpha \cap U_\beta)$$
is exact, where the second arrow is defined by the restrictions ρ_{U,U_α} and the double arrow is defined by $\rho_{U_\alpha, U_\alpha \cap U_\beta}$ and $\rho_{U_\beta, U_\alpha \cap U_\beta}$. Exactness means the same as in §3.0.

When $U \mapsto \mathscr{F}(U)$ satisfies just the first bullet, we say that $\mathscr{F}$ is a *presheaf*.

Given a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$, elements of $\mathscr{F}(U)$ are called *sections of $\mathscr{F}$ over U*. The module of sections of $\mathscr{F}$ over $U \subseteq X$ is expressed in several ways:
$$\mathscr{F}(U) = \Gamma(U, \mathscr{F}) = H^0(U, \mathscr{F}).$$

We will use Γ in this chapter and switch to H^0 in later chapters. Traditionally, $\Gamma(X,\mathscr{F})$ is called the module of *global sections* of $\mathscr{F}$.

Example 4.0.24. Let $f : X \to Y$ be a morphism of varieties and let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules on X. The *direct image sheaf* $f_*\mathscr{F}$ on Y is defined by

$$U \longmapsto \mathscr{F}(f^{-1}(U))$$

for $U \subseteq Y$ open. Then $f_*\mathscr{F}$ is a sheaf of $\mathscr{O}_Y$-modules. For $i : Y \hookrightarrow X$, the direct image $i_*\mathscr{O}_Y$ was mentioned in §3.0. $\Diamond$

If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of $\mathscr{O}_X$-modules, then a *homomorphism of sheaves* $\phi : \mathscr{F} \to \mathscr{G}$ consists of $\mathscr{O}_X(U)$-module homomorphisms

$$\phi_U : \mathscr{F}(U) \longrightarrow \mathscr{G}(U),$$

such that the diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\rho_{U,V}"] & \mathscr{G}(U) \arrow[d, "\rho_{U,V}"] \\ \mathscr{F}(V) \arrow[r, "\phi_V"] & \mathscr{G}(V) \end{tikzcd}$$

commutes whenever $V \subseteq U$. It should be clear what it means for sheaves $\mathscr{F},\mathscr{G}$ of $\mathscr{O}_X$-modules to be isomorphic, written $\mathscr{F} \simeq \mathscr{G}$.

Example 4.0.25. Let $f : X \to Y$ be a morphism of varieties. If $U \subseteq Y$ is open, then composition with f induces a natural map

$$\mathscr{O}_Y(U) \longrightarrow \mathscr{O}_X(f^{-1}(U)) = f_*\mathscr{O}_X(U).$$

This defines a sheaf homomorphism $\mathscr{O}_Y \to f_*\mathscr{O}_X$. $\Diamond$

Over an affine variety $X = \mathrm{Spec}(R)$, there is a standard way to get sheaves of $\mathscr{O}_X$-modules. Recall that a nonzero element $f \in R$ gives the localization R_f such that $X_f = \mathrm{Spec}(R_f)$ is the open subset $X \setminus \mathbf{V}(f)$. Given an R-module M, we get the R_f-module $M_f = M \otimes_R R_f$. Then there is a unique sheaf $\widetilde{M}$ of $\mathscr{O}_X$-modules such that

$$\widetilde{M}(X_f) = M_f$$

for every nonzero $f \in R$ (see [**131**, Prop. II.5.1]). This globalizes as follows.

Definition 4.0.26. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules on a variety X.

(a) Let $U \subseteq X$ be open. Then the ***restriction*** $\mathscr{F}|_U$ is the sheaf of $\mathscr{O}_U$-modules defined by $\mathscr{F}|_U(V) = \mathscr{F}(V)$ for $V \subseteq U$ open.

(b) $\mathscr{F}$ is ***quasicoherent*** if X has an affine open cover $\{U_\alpha\}$, $U_\alpha = \mathrm{Spec}(R_\alpha)$, such that for each α, there is an R_α-module M_α satisfying $\mathscr{F}|_{U_\alpha} \simeq \widetilde{M}_\alpha$.

(c) If in addition each M_α is a finitely generated R_α-module, then we say that $\mathscr{F}$ is ***coherent***.

The Sheaf of a Weil Divisor. Let D be a Weil divisor on a normal variety X. We will show that D determines a sheaf $\mathscr{O}_X(D)$ of $\mathscr{O}_X$-modules on X. Recall that if $U \subseteq X$ is open, then $\mathscr{O}_X(U)$ consists of all morphisms $U \to \mathbb{C}$. Proposition 4.0.16 tells us that an arbitrary element $f \in \mathbb{C}(X)^*$ is a morphism on U if and only if $\mathrm{div}(f)|_U \geq 0$. It follows that the sheaf $\mathscr{O}_X$ is defined by

$$U \longmapsto \mathscr{O}_X(U) = \{f \in \mathbb{C}(X)^* \mid \mathrm{div}(f)|_U \geq 0\} \cup \{0\}.$$

In a similar way, we define the sheaf $\mathscr{O}_X(D)$ by

$$U \longmapsto \mathscr{O}_X(D)(U) = \{f \in \mathbb{C}(X)^* \mid (\mathrm{div}(f)+D)|_U \geq 0\} \cup \{0\}. \tag{4.0.6}$$

Proposition 4.0.27. *Let D be a Weil divisor on a normal variety X. Then the sheaf $\mathscr{O}_X(D)$ defined in* (4.0.6) *is a coherent sheaf of $\mathscr{O}_X$-modules on X.*

Proof. In Exercise 4.0.10 you will show that $\mathscr{O}_X(D)$ is a sheaf of $\mathscr{O}_X$-modules. The proof is a nice application of the properties of valuations.

To show that $\mathscr{O}_X(D)$ is coherent, we may assume that $X = \mathrm{Spec}(R)$. Let K be the field of fractions of R. It suffices to prove the following two assertions:

- $M = \Gamma(X, \mathscr{O}_X(D)) = \{f \in K \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\}$ is a finitely generated R-module.
- $\Gamma(X_f, \mathscr{O}_X(D)) = M_f$ for all nonzero $f \in R$.

For the first bullet, we will prove the existence of an element $h \in R \setminus \{0\}$ such that $h\Gamma(X, \mathscr{O}_X(D)) \subseteq R$. This will imply that $h\Gamma(X, \mathscr{O}_X(D))$ is an ideal of R and hence has a finite basis since R is Noetherian. It will follow immediately that $\Gamma(X, \mathscr{O}_X(D))$ is a finitely generated R-module.

Write $D = \sum_{i=1}^s a_i D_i$. Since $\mathrm{supp}(D)$ is a proper subvariety of X, we can find $g \in R \setminus \{0\}$ that vanishes on each D_i. Then $\nu_{D_i}(g) > 0$ for every i, so there is $m \in \mathbb{N}$ with $m\,\nu_{D_i}(g) > a_i$ for all i. Since $\mathrm{div}(g) \geq 0$, it follows that $m\,\mathrm{div}(g) - D \geq 0$. Now let $f \in \Gamma(X, \mathscr{O}_X(D))$. Then $\mathrm{div}(f) + D \geq 0$, so that

$$\mathrm{div}(g^m f) = m\,\mathrm{div}(g) + \mathrm{div}(f) = m\,\mathrm{div}(g) - D + \mathrm{div}(f) + D \geq 0$$

since a sum of effective divisors is effective. By Proposition 4.0.16, we conclude that $g^m f \in \mathscr{O}_X(X) = R$. Hence $h = g^m \in R$ has the desired property.

To prove the second bullet, observe that $M \subseteq K$ and $f \in R \setminus 0$ imply that

$$M_f = \Big\{\frac{g}{f^m} \mid g \in \Gamma(X, \mathscr{O}_X(D)),\ m \geq 0\Big\}.$$

It is also easy to see that $M_f \subseteq \Gamma(X_f, \mathscr{O}_X(D))$. For the opposite inclusion, let $D = \sum_{i=1}^s a_i D_i$ and write $\{1,\ldots,s\} = I \cup J$ where $D_i \cap X_f \neq \emptyset$ for $i \in I$ and $D_j \subseteq \mathbf{V}(f)$ for $j \in J$. Given $h \in \Gamma(X_f, \mathscr{O}_X(D))$, $(\mathrm{div}(h) + D)|_{X_f} \geq 0$ implies that $\nu_{D_i}(h) \geq -a_i$ for $i \in I$. There is no constraint on $\nu_{D_j}(h)$ for $j \in J$, but f vanishes on D_j for $j \in J$, so that $\nu_{D_j}(f) > 0$. Hence we can pick $m \in \mathbb{N}$ sufficiently large such that

$$m\,\nu_{D_j}(f) + \nu_{D_j}(h) > 0 \quad \text{for } j \in J.$$

Since $\operatorname{div}(f) \geq 0$, it follows easily that $\operatorname{div}(f^m h) + D \geq 0$ on X. Thus $g = f^m h \in \Gamma(X, \mathscr{O}_X(D))$, and then $h = g/f^m$ has the desired form. $\square$

The sheaves $\mathscr{O}_X(D)$ are more than just coherent; they have the additional property of being *reflexive*. Furthermore, when D is Cartier, $\mathscr{O}_X(D)$ is *invertible*. The definitions of invertible and reflexive will be given in Chapters 6 and 8 respectively.

For now, we give two results about the sheaves $\mathscr{O}_X(D)$. Here is the first.

Proposition 4.0.28. *Distinct prime divisors $D_1, \ldots, D_s$ on a normal variety X give the divisor $D = D_1 + \cdots + D_s$ and the subvariety $Y = \operatorname{Supp}(D) = D_1 \cup \cdots \cup D_s$. Then $\mathscr{O}_X(-D)$ is the ideal sheaf $\mathscr{I}_Y$ of Y, i.e.,*

$$\Gamma(U, \mathscr{O}_X(-D)) = \{ f \in \mathscr{O}_X(U) \mid f \text{ vanishes on } Y\}$$

for all open subsets $U \subseteq X$.

Proof. Since sheaves are local, we may assume that $X = \operatorname{Spec}(R)$. Then note that $f \in \Gamma(X, \mathscr{O}_X(-D))$ implies $\operatorname{div}(f) - D \geq 0$, so $\operatorname{div}(f) \geq D \geq 0$ since D is effective. Thus $f \in R$ by Proposition 4.0.16 and hence $\Gamma(X, \mathscr{O}_X(-D))$ is an ideal of R.

Let $\mathfrak{p}_i = \mathbf{I}(D_i) \subseteq R$ be the prime ideal of D_i. Then, for $f \in R$, we have

$$\nu_{D_i}(f) > 0 \iff f \in \mathfrak{p}_i R_{\mathfrak{p}_i} \iff f \in \mathfrak{p}_i,$$

where the last equivalence uses the easy equality $\mathfrak{p}_i R_{\mathfrak{p}_i} \cap R = \mathfrak{p}_i$. Hence $\operatorname{div}(f) \geq D$ if and only if f vanishes on $D_1, \ldots, D_s$, and the proposition follows. $\square$

Linear equivalence of divisors tells us the following interesting fact about the associated sheaves.

Proposition 4.0.29. *If $D \sim E$ are linearly equivalent Weil divisors, then $\mathscr{O}_X(D)$ and $\mathscr{O}_X(E)$ are isomorphic as sheaves of $\mathscr{O}_X$-modules.*

Proof. By assumption, we have $D = E + \operatorname{div}(g)$ for some $g \in \mathbb{C}(X)^*$. Then

$$\begin{aligned} f \in \Gamma(X, \mathscr{O}_X(D)) &\iff \operatorname{div}(f) + D \geq 0 \\ &\iff \operatorname{div}(f) + E + \operatorname{div}(g) \geq 0 \\ &\iff \operatorname{div}(fg) + E \geq 0 \\ &\iff fg \in \Gamma(X, \mathscr{O}_X(E)). \end{aligned}$$

Thus multiplication by g induces an isomorphism $\Gamma(X, \mathscr{O}_X(D)) \simeq \Gamma(X, \mathscr{O}_X(E))$ which is clearly an isomorphism of $\Gamma(X, \mathscr{O}_X)$-modules.

The same argument works over any Zariski open set U, and the isomorphisms are easily seen to be compatible with the restriction maps. $\square$

The converse of Proposition 4.0.29 is also true, i.e., an $\mathscr{O}_X$-module isomorphism $\mathscr{O}_X(D) \simeq \mathscr{O}_X(E)$ implies that $D \sim E$. The proof requires knowing more about the sheaves $\mathscr{O}_X(D)$ and hence will be postponed until Chapter 8.

Exercises for §4.0.

4.0.1. Complete the proof of part (b) of Proposition 4.0.3.

4.0.2. Prove (c) $\Leftrightarrow$ (d) in Theorem 4.0.4. Hint: Let $\mathfrak{m}$ be the maximal ideal of R. Since R has dimension one, it is regular if and only if $\mathfrak{m}/\mathfrak{m}^2$ has dimension one as a vector space over $R/\mathfrak{m}$. For (d) $\Rightarrow$ (c), use Nakayama's Lemma (see [**10**, Props. 2.6 and 2.8]).

4.0.3. This exercise will study the rings $\mathscr{O}_{X,D}$ and $R_\mathfrak{p}$.

(a) Prove (4.0.2).

(b) Let $\mathfrak{p}$ be a prime ideal of a ring R and let $R_\mathfrak{p}$ denote the localization of R with respect to the multiplicative subset $R \setminus \mathfrak{p}$. Prove that $R_\mathfrak{p}$ is a local ring and that its maximal ideal is the ideal $\mathfrak{p}R_\mathfrak{p} \subseteq R_\mathfrak{p}$ generated by $\mathfrak{p}$.

4.0.4. Let S be a multiplicative subset of a Noetherian ring R. Prove that the localization R_S is Noetherian.

4.0.5. Let D and E be Weil divisors on a normal variety.

(a) If D and E are Cartier, show that $D+E$ and $-D$ are also Cartier.

(b) If $D \sim E$, show that D is Cartier if and only if E is Cartier.

4.0.6. Complete the proof of Theorem 4.0.14.

4.0.7. Prove that a normal curve is smooth.

4.0.8. Let R be a Noetherian normal domain. Prove that the following are equivalent:

(a) R is a UFD.

(b) $\mathrm{Cl}(\mathrm{Spec}(R)) = 0$.

(c) Every codimension one prime ideal of R is principal.

Hint: For (b) $\Rightarrow$ (c), assume that $D = \mathrm{div}(f)$ corresponds to $\mathfrak{p}$. Use Theorem 4.0.14 to show $f \in R$ and use the Krull principal ideal theorem to show $\langle f \rangle$ is primary in R. Then $\mathfrak{p}R_\mathfrak{p} = fR_\mathfrak{p}$ and [**10**, Prop. 4.8] imply $\mathfrak{p} = \langle f \rangle$. For (c) $\Rightarrow$ (a), let $a \in R$ be noninvertible and let $D_1,\ldots,D_s$ be the codimension one irreducible components of $\mathbf{V}(a)$. If $\mathbf{I}(D_i) = \langle a_i \rangle$, compare the divisors of a and $\prod_{i=1}^s a_i^{\nu_{D_i}(a)}$ using Proposition 4.0.16.

4.0.9. Prove that the restriction map $D \mapsto D|_U$ induces a well-defined homomorphism $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$.

4.0.10. Let D be a Weil divisor on a normal variety X. Prove that (4.0.6) defines a sheaf $\mathscr{O}_X(D)$ of $\mathscr{O}_X$-modules.

4.0.11. For each of the following rings R, give a careful description of the field of fractions K and show that the ring is a DVR by constructing an appropriate discrete valuation on K.

(a) $R = \{a/b \in \mathbb{Q} \mid a,b \in \mathbb{Z}, b \neq 0, \gcd(b,p) = 1\}$, where p is a fixed prime number.

(b) $R = \mathbb{C}\{\{z\}\}$, the ring consisting of all power series in z with coefficients in $\mathbb{C}$ that have a positive radius of convergence.

4.0.12. The plane curve $\mathbf{V}(x^3 - y^2) \subseteq \mathbb{C}^2$ has coordinate ring $R = \mathbb{C}[x,y]/\langle x^3 - y^2 \rangle$. As noted in Example 1.1.15, this is the coordinate ring of the affine toric variety given by the affine semigroup $\mathsf{S} = \{0,2,3,\ldots\}$. This semigroup is not saturated, which means that $R \simeq \mathbb{C}[\mathsf{S}] = \mathbb{C}[t^2,t^3]$ is not normal by Theorem 1.3.5. It follows that R is not a DVR by

Theorem 4.0.4. Give a direct proof of this fact using only the definition of DVR. Hint: The field of fractions of $\mathbb{C}[t^2,t^3]$ is $\mathbb{C}(t)$. If $\mathbb{C}[t^2,t^3]$ comes from the discrete valuation ν, what is $\nu(t)$?

4.0.13. Let X be a normal variety. Use Proposition 4.0.16 to prove that there is an exact sequence

$$1 \longrightarrow \mathscr{O}_X(X)^* \longrightarrow \mathbb{C}(X)^* \longrightarrow \mathrm{Div}(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0,$$

where the map $\mathbb{C}(X)^* \to \mathrm{Div}(X)$ is $f \mapsto \mathrm{div}(f)$ and $\mathrm{Div}(X) \to \mathrm{Cl}(X)$ is $D \mapsto [D]$. Similarly, prove that there is an exact sequence

$$1 \longrightarrow \mathscr{O}_X(X)^* \longrightarrow \mathbb{C}(X)^* \longrightarrow \mathrm{CDiv}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

4.0.14. Let $D = \sum_{\mathrm{codim}\,\mathfrak{p}=1} a_\mathfrak{p} D_\mathfrak{p}$ be a Weil divisor on a normal affine variety $X = \mathrm{Spec}(R)$. As usual, let K be the field of fractions of R. Here you give an algebraic description of $\Gamma(X, \mathscr{O}_X(D))$ in terms of the prime ideals $\mathfrak{p}$.

(a) Let $\mathfrak{p}$ be a codimension one prime of R, so that $R_\mathfrak{p}$ is a DVR. Hence the maximal ideal $\mathfrak{p}R_\mathfrak{p}$ is principal. Use this to define $\mathfrak{p}^a R_\mathfrak{p} \subseteq K$ for all $a \in \mathbb{Z}$.

(b) Prove that

$$\Gamma(X, \mathscr{O}_X(D)) = \bigcap_{\mathrm{codim}\,\mathfrak{p}=1} \mathfrak{p}^{-a_\mathfrak{p}} R_\mathfrak{p}.$$

(c) Now assume that D is effective, i.e., $a_\mathfrak{p} \geq 0$ for all $\mathfrak{p}$. Prove that $\Gamma(X, \mathscr{O}_X(-D))$ is the ideal of R given by

$$\Gamma(X, \mathscr{O}_X(-D)) = \bigcap_{\mathrm{codim}\,\mathfrak{p}=1} \mathfrak{p}^{a_\mathfrak{p}} R_\mathfrak{p}.$$

4.0.15. Let R be an integral domain with field of fractions K. A finitely generated R-submodule of K is called a *fractional ideal*. If R is normal and D is a Weil divisor on $X = \mathrm{Spec}(R)$, explain why $\Gamma(X, \mathscr{O}_X(D)) \subseteq K$ is a fractional ideal.

§4.1. Weil Divisors on Toric Varieties

Let X_Σ be the toric variety of a fan Σ in $N_\mathbb{R}$ with $\dim N_\mathbb{R} = n$. Then X_Σ is normal of dimension n. We will use torus-invariant prime divisors and characters to give a lovely description of the class group of X_Σ.

The Divisor of a Character. The order of vanishing of a character along a torus-invariant prime divisor is determined by the polyhedral geometry of the fan.

By the Orbit-Cone Correspondence (Theorem 3.2.6), k-dimensional cones σ of Σ correspond to $(n-k)$-dimensional T_N-orbits in X_Σ. As in Chapter 3, $\Sigma(1)$ is the set of 1-dimensional cones (i.e., the rays) of Σ. Thus $\rho \in \Sigma(1)$ gives the codimension 1 orbit $O(\rho)$ whose closure $\overline{O(\rho)}$ is a T_N-invariant prime divisor on X_Σ. To emphasize that $\overline{O(\rho)}$ is a divisor we will denote it by D_ρ rather than $V(\rho)$. Then $D_\rho = \overline{O(\rho)}$ gives the DVR $\mathscr{O}_{X_\Sigma, D_\rho}$ with valuation

$$\nu_\rho = \nu_{D_\rho} : \mathbb{C}(X_\Sigma)^* \to \mathbb{Z}.$$

Recall that the ray $\rho \in \Sigma(1)$ has a minimal generator $u_\rho \in \rho \cap N$. Also note that when $m \in M$, the character $\chi^m : T_N \to \mathbb{C}^*$ is a rational function in $\mathbb{C}(X_\Sigma)^*$ since T_N is Zariski open in X_Σ.

Proposition 4.1.1. *Let X_Σ be the toric variety of a fan Σ. If the ray $\rho \in \Sigma(1)$ has minimal generator u_ρ and χ^m is character corresponding to $m \in M$, then*

$$\nu_\rho(\chi^m) = \langle m, u_\rho \rangle.$$

Proof. Since $u_\rho \in N$ is primitive, we can extend u_ρ to a basis $e_1 = u_\rho, e_2, \ldots, e_n$ of N, then we can assume $N = \mathbb{Z}^n$ and $\rho = \mathrm{Cone}(e_1) \subseteq \mathbb{R}^n$. By Example 1.2.21, the corresponding affine toric variety is

$$U_\rho = \mathrm{Spec}(\mathbb{C}[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]) = \mathbb{C} \times (\mathbb{C}^*)^{n-1}$$

and $D_\rho \cap U_\rho$ is defined by $x_1 = 0$. It follows easily that the DVR is

$$\mathscr{O}_{X_\Sigma, D_\rho} = \mathscr{O}_{U_\rho, U_\rho \cap D_\rho} = \mathbb{C}[x_1, \ldots, x_n]_{\langle x_1 \rangle}.$$

Similar to Example 4.0.6, $f \in \mathbb{C}(x_1, \ldots, x_n)^*$ has valuation $\nu_\rho(f) = \ell \in \mathbb{Z}$ when

$$f = x_1^\ell \frac{g}{h}, \quad g, h \in \mathbb{C}[x_1, \ldots, x_n] \setminus \langle x_1 \rangle.$$

To relate this to $\nu_\rho(\chi^m)$, note that $x_1, \ldots, x_n$ are the characters of the dual basis of $e_1 = u_\rho, e_2, \ldots, e_n \in N$. It follows that given any $m \in M$, we have

$$\chi^m = x_1^{\langle m, e_1 \rangle} x_2^{\langle m, e_2 \rangle} \cdots x_n^{\langle m, e_n \rangle} = x_1^{\langle m, u_\rho \rangle} x_2^{\langle m, e_2 \rangle} \cdots x_n^{\langle m, e_n \rangle}.$$

Comparing this to the previous equation implies that $\nu_\rho(\chi^m) = \langle m, u_\rho \rangle$. □

We next compute the divisor of a character. As above, a ray $\rho \in \Sigma(1)$ gives:

- A minimal generator $u_\rho \in \rho \cap N$.
- A prime T_N-invariant divisor $D_\rho = \overline{O(\rho)}$ on X_Σ.

We will use this notation for the remainder of the chapter.

Proposition 4.1.2. *For $m \in M$, the character χ^m is a rational function on X_Σ, and its divisor is given by*

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

Proof. The Orbit-Cone Correspondence (Theorem 3.2.6) implies that the D_ρ are the irreducible components of $X \setminus T_N$. Since χ^m is defined and nonzero on T_N, it follows that $\mathrm{div}(\chi^m)$ is supported on $\bigcup_{\rho \in \Sigma(1)} D_\rho$. Hence

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \nu_{D_\rho}(\chi^m) D_\rho.$$

Then we are done since $\nu_{D_\rho}(\chi^m) = \langle m, u_\rho \rangle$ by Proposition 4.1.1. □

Computing the Class Group. Divisors of the form $\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ are precisely the divisors invariant under the torus action on X_Σ (Exercise 4.1.1). Thus

$$\mathrm{Div}_{T_N}(X_\Sigma) = \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z} D_\rho \subseteq \mathrm{Div}(X_\Sigma)$$

is the group of T_N-invariant Weil divisors on X_Σ. Here is the main result of this section.

Theorem 4.1.3. *We have the exact sequence*

$$M \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

where the first map is $m \mapsto \mathrm{div}(\chi^m)$ and the second sends a T_N-invariant divisor to its divisor class in $\mathrm{Cl}(X_\Sigma)$. Furthermore, we have a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

if and only if $\{u_\rho \mid \rho\in\Sigma(1)\}$ spans $N_\mathbb{R}$, i.e., X_Σ has no torus factors.

Proof. Since the D_ρ are the irreducible components of $X_\Sigma \setminus T_N$, Theorem 4.0.20 implies that we have an exact sequence

$$\mathrm{Div}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow \mathrm{Cl}(T_N) \longrightarrow 0.$$

Since $\mathbb{C}[x_1,\dots,x_n]$ is a UFD, the same is true for $\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$. This is the coordinate ring of the torus $(\mathbb{C}^*)^n$, which is isomorphic to the coordinate ring $\mathbb{C}[M]$ of the torus T_N. Hence $\mathbb{C}[M]$ is also a UFD, which implies $\mathrm{Cl}(T_N) = 0$ by Theorem 4.0.18. We conclude that $\mathrm{Div}_{T_N}(X_\Sigma) \to \mathrm{Cl}(X_\Sigma)$ is surjective.

The composition $M \to \mathrm{Div}_{T_N}(X_\Sigma) \to \mathrm{Cl}(X_\Sigma)$ is obviously zero since the first map is $m \mapsto \mathrm{div}(\chi^m)$. Now suppose that $D \in \mathrm{Div}_{T_N}(X_\Sigma)$ maps to 0 in $\mathrm{Cl}(X_\Sigma)$. Then $D = \mathrm{div}(f)$ for some $f \in \mathbb{C}(X_\Sigma)^*$. Since the support of D misses T_N, this implies that $\mathrm{div}(f)$ restricts to 0 on T_N. When regarded as an element of $\mathbb{C}(T_N)^*$, f has zero divisor on T_N, so that $f \in \mathbb{C}[M]^*$ by Proposition 4.0.16. Thus $f = c\chi^m$ for some $c \in \mathbb{C}^*$ and $m \in M$ (Exercise 3.3.4). It follows that on X_Σ,

$$D = \mathrm{div}(f) = \mathrm{div}(c\chi^m) = \mathrm{div}(\chi^m),$$

which proves exactness at $\mathrm{Div}_{T_N}(X_\Sigma)$.

Finally, suppose that $m \in M$ with $\mathrm{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle D_\rho$ is the zero divisor. Then $\langle m, u_\rho\rangle = 0$ for all $\rho \in \Sigma(1)$, which forces $m = 0$ when the u_ρ span $N_\mathbb{R}$. This gives the desired exact sequence. Conversely, if the sequence is exact, then one easily sees that the u_ρ span $N_\mathbb{R}$, which by Corollary 3.3.10 is equivalent to X_Σ having no torus factors. $\square$

In particular, we see that $\mathrm{Cl}(X_\Sigma)$ is a finitely generated abelian group.

Examples. It is easy to compute examples of class groups of toric varieties. In practice, one usually picks a basis $e_1,\dots,e_n$ of M, so that $M \simeq \mathbb{Z}^n$ and (via the dual basis) $N \simeq \mathbb{Z}^n$. Then the pairing $\langle m,u\rangle$ becomes dot product. We list the rays of Σ as $\rho_1,\dots,\rho_r$ with corresponding ray generators $u_1,\dots,u_r \in \mathbb{Z}^n$. We will think of u_i as the column vector $(\langle e_1,u_i\rangle,\dots,\langle e_n,u_i\rangle)^T$, where the superscript denotes transpose.

With this setup, the map $M \longrightarrow \mathrm{Div}_{T_N}(X_\Sigma)$ in Theorem 4.1.3 is the map

$$A : \mathbb{Z}^n \longrightarrow \mathbb{Z}^r$$

represented by the matrix whose rows are the ray generators $u_1,\dots,u_r$. In other words, $A = (u_1,\dots,u_r)^T$. By Theorem 4.1.3, the class group of X_Σ is the cokernel of this map, which is easily computed from the Smith normal form of A.

When we want to think in terms of divisors, we let D_i be the T_N-invariant prime divisor corresponding to $\rho_i \in \Sigma(1)$.

Example 4.1.4. The affine toric surface described in Example 1.2.22 comes from the cone $\sigma = \mathrm{Cone}(de_1 - e_2, e_2)$. For $d = 3$, σ is shown in Figure 1. The resulting

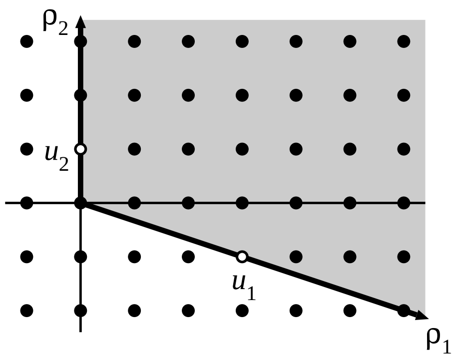


Figure 1. The cone σ when $d = 3$

toric variety U_σ is the rational normal cone $\widehat{C}_d$. Using the ray generators $u_1 = de_1 - e_2 = (d,-1)$ and $u_2 = e_2 = (0,1)$, we get the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ given by the matrix

$$A = \begin{pmatrix} d & -1 \\ 0 & 1 \end{pmatrix}.$$

This makes it easy to compute that

$$\mathrm{Cl}(\widehat{C}_d) \simeq \mathbb{Z}/d\mathbb{Z}.$$

We can also see this in terms of divisors as follows. The class group $\mathrm{Cl}(\widehat{C}_d)$ is generated by the classes of the divisors D_1, D_2 corresponding to ρ_1, ρ_2, subject to the relations coming from the exact sequence of Theorem 4.1.3:

$$\begin{aligned}0 &\sim \operatorname{div}(\chi^{e_1}) = \langle e_1, u_1\rangle D_1 + \langle e_1, u_2\rangle D_2 = d\,D_1\\ 0 &\sim \operatorname{div}(\chi^{e_2}) = \langle e_2, u_1\rangle D_1 + \langle e_2, u_2\rangle D_2 = -D_1 + D_2.\end{aligned}$$

Thus $\operatorname{Cl}(\widehat{C}_d)$ is generated by $[D_1]$ with $d[D_1] = 0$, giving $\operatorname{Cl}(\widehat{C}_d) \simeq \mathbb{Z}/d\mathbb{Z}$. $\Diamond$

Example 4.1.5. In Example 3.1.4, we saw that the blowup of $\mathbb{C}^2$ at the origin is the toric variety $\mathrm{Bl}_0(\mathbb{C}^2)$ given by the fan Σ shown in Figure 2.

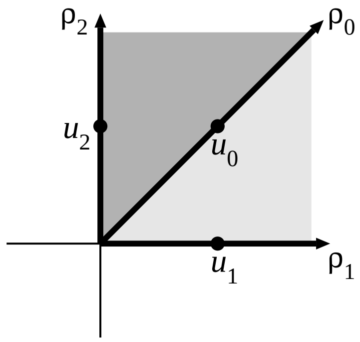


Figure 2. The fan for the blowup of $\mathbb{C}^2$ at the origin

The ray generators are $u_1 = e_1, u_2 = e_2, u_0 = e_1 + e_2$ corresponding to divisors D_1, D_2, D_0. By Theorem 4.1.3, the class group is generated by the classes of the D_i subject to the relations

$$\begin{aligned}0 &\sim \operatorname{div}(\chi^{e_1}) = D_1 + D_0\\ 0 &\sim \operatorname{div}(\chi^{e_2}) = D_2 + D_0.\end{aligned}$$

Thus $\operatorname{Cl}(\mathrm{Bl}_0(\mathbb{C}^2)) \simeq \mathbb{Z}$ with generator $[D_1] = [D_2] = -[D_0]$. This calculation can also be done using matrices as in the previous example. $\Diamond$

Example 4.1.6. The fan of $\mathbb{P}^n$ has ray generators given by $u_0 = -e_1 - \cdots - e_n$ and $u_1 = e_1, \ldots, u_n = e_n$. Thus the map $M \to \operatorname{Div}_{T_N}(\mathbb{P}^n)$ can be written as

$$\begin{aligned}\mathbb{Z}^n &\longrightarrow \mathbb{Z}^{n+1}\\ (a_1, \ldots, a_n) &\longmapsto (-a_1 - \cdots - a_n, a_1, \ldots, a_n).\end{aligned}$$

Using the map

$$\begin{aligned}\mathbb{Z}^{n+1} &\longrightarrow \mathbb{Z}\\ (b_0, \ldots, b_n) &\longmapsto b_0 + \cdots + b_n,\end{aligned}$$

one gets the exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

which proves that $\mathrm{Cl}(\mathbb{P}^n) \simeq \mathbb{Z}$, generalizing Example 4.0.21. It is easy to redo this calculation using divisors as in the previous example. $\Diamond$

Example 4.1.7. The class group $\mathrm{Cl}(\mathbb{P}^n \times \mathbb{P}^m)$ is isomorphic to $\mathbb{Z}^2$. More generally,

$$\mathrm{Cl}(X_{\Sigma_1} \times X_{\Sigma_2}) \simeq \mathrm{Cl}(X_{\Sigma_1}) \oplus \mathrm{Cl}(X_{\Sigma_2}).$$

You will prove this in Exercise 4.1.2. $\Diamond$

Example 4.1.8. The Hirzebruch surfaces $\mathscr{H}_r$ are described in Example 3.1.16. The fan for $\mathscr{H}_r$ appears in Figure 3, along with the ray generators $u_1 = -e_1 + re_2$, $u_2 = e_2$, $u_3 = e_1$, $u_4 = -e_2$.

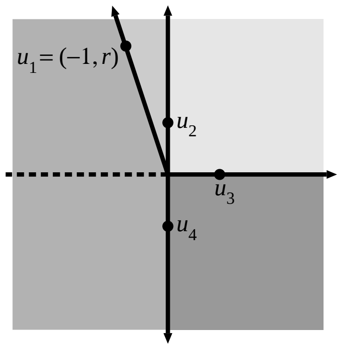


Figure 3. A fan Σ_r with $X_{\Sigma_r} \simeq \mathscr{H}_r$

The class group is generated by the classes of D_1, D_2, D_3, D_4, with relations

$$0 \sim \mathrm{div}(\chi^{e_1}) = -D_1 + D_3$$
$$0 \sim \mathrm{div}(\chi^{e_2}) = r\,D_1 + D_2 - D_4.$$

It follows that $\mathrm{Cl}(\mathscr{H}_r)$ is the free abelian group generated by $[D_1]$ and $[D_2]$. Thus

$$\mathrm{Cl}(\mathscr{H}_r) \simeq \mathbb{Z}^2.$$

In particular, $r = 0$ gives $\mathrm{Cl}(\mathscr{H}_0) = \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}^2$, which is a special case of Example 4.1.7. $\Diamond$

Exercises for §4.1.

4.1.1. This exercise will determine which divisors are invariant under the T_N-action on X_Σ. Given $t \in T_N$ and $p \in X_\Sigma$, the T_N-action gives $t \cdot p \in X_\Sigma$. If D is a prime divisor, the T_N-action gives the prime divisor $t \cdot D$. For an arbitrary Weil divisor $D = \sum_i a_i D_i$, $t \cdot D = \sum_i a_i (t \cdot D_i)$. Then D is T_N-*invariant* if $t \cdot D = D$ for all $t \in T_N$.

(a) Show that $\sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is T_N-invariant.

(b) Conversely, show that any T_N-invariant Weil divisor can be written as in part (a). Hint: Consider $\mathrm{Supp}(D)$ and use the Orbit-Cone Correspondence.

4.1.2. Given fans Σ_1 in $(N_1)_\mathbb{R}$ and Σ_2 in $(N_2)_\mathbb{R}$, we get the product fan

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 \mid \sigma_i \in \Sigma_i\},$$

which by Proposition 3.1.14 is the fan of the toric variety $X_{\Sigma_1} \times X_{\Sigma_2}$. Prove that

$$\mathrm{Cl}(X_{\Sigma_1} \times X_{\Sigma_2}) \simeq \mathrm{Cl}(X_{\Sigma_1}) \oplus \mathrm{Cl}(X_{\Sigma_2}).$$

Hint: The product fan has rays $\rho_1 \times \{0\}$ and $\{0\} \times \rho_2$ for $\rho_1 \in \Sigma_1(1)$ and $\rho_2 \in \Sigma_2(1)$.

4.1.3. Redo the divisor class group calculation given in Example 4.1.5 using matrices, and redo the calculation given in Example 4.1.6 using divisors.

4.1.4. The blowup of $\mathbb{C}^n$ at the origin is the toric variety $\mathrm{Bl}_0(\mathbb{C}^n)$ of the fan Σ described in Example 3.1.15. Prove that $\mathrm{Cl}(\mathrm{Bl}_0(\mathbb{C}^n)) \simeq \mathbb{Z}$.

4.1.5. The weighted projective space $\mathbb{P}(q_0,\dots,q_n)$, $\gcd(q_0,\dots,q_n) = 1$, is built from a fan in $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0,\dots,q_n)$. The dual lattice is

$$M = \{(a_0,\dots,a_n) \in \mathbb{Z}^{n+1} \mid a_0q_0 + \cdots + a_nq_n = 0\}.$$

Let $u_0,\dots,u_n \in N$ denote the images of the standard basis $e_0,\dots,e_n \in \mathbb{Z}^{n+1}$. The u_i are the ray generators of the fan giving $\mathbb{P}(q_0,\dots,q_n)$. Define maps

$$\begin{aligned} M &\longrightarrow \mathbb{Z}^{n+1} : m \longmapsto (\langle m,u_0\rangle,\dots,\langle m,u_n\rangle)\\ \mathbb{Z}^{n+1} &\longrightarrow \mathbb{Z} : (a_0,\dots,a_n) \longmapsto a_0q_0 + \cdots + a_nq_n. \end{aligned}$$

Show that these maps give an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} \longrightarrow 0$$

and conclude that $\mathrm{Cl}(\mathbb{P}(q_0,\dots,q_n)) \simeq \mathbb{Z}$.

§4.2. Cartier Divisors on Toric Varieties

Let X_Σ be the toric variety of a fan Σ. We will use the same notation as in §4.1, where each $\rho \in \Sigma(1)$ gives a minimal ray generator u_ρ and a T_N-invariant prime divisor $D_\rho \subseteq X_\Sigma$. In what follows, we write $\sum_\rho$ for a summation over the rays $\rho \in \Sigma(1)$ when there is no danger of confusion.

Computing the Picard Group. A Cartier divisor D on X_Σ is also a Weil divisor and hence

$$D \sim \sum_\rho a_\rho D_\rho, \quad a_\rho \in \mathbb{Z},$$

by Theorem 4.1.3. Then $\sum_\rho a_\rho D_\rho$ is Cartier since D is (Exercise 4.0.5). Let

$$\mathrm{CDiv}_{T_N}(X_\Sigma) \subseteq \mathrm{Div}_{T_N}(X_\Sigma)$$

denote the subgroup of $\mathrm{Div}_{T_N}(X_\Sigma)$ consisting of T_N-invariant Cartier divisors. Since $\mathrm{div}(\chi^m) \in \mathrm{CDiv}_{T_N}(X_\Sigma)$ for all $m \in M$, we get the following immediate corollary of Theorem 4.1.3.

Theorem 4.2.1. *We have an exact sequence*

$$M \longrightarrow \mathrm{CDiv}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0,$$

where the first map is defined above and the second sends a T_N-invariant divisor to its divisor class in $\mathrm{Pic}(X_\Sigma)$. *Furthermore, we have a short exact sequence*

$$0 \longrightarrow M \longrightarrow \mathrm{CDiv}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0$$

if and only if $\{u_\rho \mid \rho \in \Sigma(1)\}$ *spans* $N_\mathbb{R}$. □

Our next task is to determine the structure of $\mathrm{CDiv}_{T_N}(X_\Sigma)$. In other words, which T_N-invariant divisors are Cartier? We begin with the affine case.

Proposition 4.2.2. *Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex polyhedral cone. Then:*

(a) *Every T_N-invariant Cartier divisor on U_σ is the divisor of a character.*

(b) $\mathrm{Pic}(U_\sigma) = 0$.

Proof. Let $R = \mathbb{C}[\sigma^\vee \cap M]$. First suppose that $D = \sum_\rho a_\rho D_\rho$ is an effective T_N-invariant Cartier divisor. Using Proposition 4.0.16 as in the proof of Proposition 4.0.28, we see that

$$\Gamma(U_\sigma, \mathscr{O}_{U_\sigma}(-D)) = \{f \in K \mid f = 0, \text{ or } f \neq 0 \text{ and } \mathrm{div}(f) \geq D\}$$

is an ideal $I \subseteq R$. Furthermore, I is T_N-invariant since D is. Hence

$$I = \bigoplus_{\chi^m \in I} \mathbb{C} \cdot \chi^m = \bigoplus_{\mathrm{div}(\chi^m) \geq D} \mathbb{C} \cdot \chi^m \tag{4.2.1}$$

by Lemma 1.1.16.

Under the Orbit-Cone Correspondence (Theorem 3.2.6), a ray $\rho \in \sigma(1)$ gives an inclusion $O(\sigma) \subseteq \overline{O(\rho)} = D_\rho$. Thus

$$O(\sigma) \subseteq \bigcap_\rho D_\rho.$$

Now fix a point $p \in O(\sigma)$. Since D is Cartier, it is locally principal, and in particular is principal in a neighborhood U of p. Shrinking U if necessary, we may assume that $U = (U_\sigma)_h = \mathrm{Spec}(R_h)$, where $h \in R$ satisfies $h(p) \neq 0$.

Thus $D|_U = \mathrm{div}(f)|_U$ for some $f \in \mathbb{C}(U_\sigma)^*$. Since D is effective, $f \in R_h$ by Proposition 4.0.16, and since h is invertible on U, we may assume $f \in R$. Then

$$\mathrm{div}(f) = \sum_\rho \nu_{D_\rho}(f) D_\rho + \sum_{E \neq D_\rho} \nu_E(f) E \geq \sum_\rho \nu_{D_\rho}(f) D_\rho = D. \tag{4.2.2}$$

Here, $\sum_{E \neq D_\rho}$ denotes the sum over all prime divisors different from the D_ρ. The first equality is the definition of $\mathrm{div}(f)$, the second inequality follows since $f \in R$, and the final equality follows from $D|_U = \mathrm{div}(f)|_U$ since $p \in U \cap D_\rho$ for all $\rho \in \sigma(1)$. Then $f \in I$ since $\mathrm{div}(f) \geq D$ by (4.2.2).

Using (4.2.1), we can write $f = \sum_i a_i \chi^{m_i}$ with $a_i \in \mathbb{C}^*$ and $\operatorname{div}(\chi^{m_i}) \geq D$. Restricting to U, this becomes $\operatorname{div}(\chi^{m_i})|_U \geq \operatorname{div}(f)|_U$, which implies that χ^{m_i}/f is a morphism on U by Proposition 4.0.16. Then

$$1 = \frac{\sum_i a_i \chi^{m_i}}{f} = \sum_i a_i \frac{\chi^{m_i}}{f}$$

and $p \in U$ imply that $(\chi^{m_i}/f)(p) \neq 0$ for some i. Hence χ^{m_i}/f is nonvanishing in some open set V with $p \in V \subseteq U$. It follows that

$$\operatorname{div}(\chi^{m_i})|_V = \operatorname{div}(f)|_V = D|_V.$$

Since $\operatorname{div}(\chi^{m_i})$ and D have support contained in $\bigcup_\rho D_\rho$ and every D_ρ meets V (this follows from $p \in V \cap D_\rho$), we have $\operatorname{div}(\chi^{m_i}) = D$.

To finish the proof of (a), let D be an arbitrary T_N-invariant Cartier divisor on U_σ. Since $\dim \sigma^\vee = \dim M_\mathbb{R}$ (σ is strongly convex), we can find $m \in \sigma^\vee \cap M$ such that $\langle m, u_\rho \rangle > 0$ for all $\rho \in \sigma(1)$. Thus $\operatorname{div}(\chi^m)$ is a positive linear combination of the D_ρ, which implies that $D' = D + \operatorname{div}(\chi^{km}) \geq 0$ for $k \in \mathbb{N}$ sufficiently large. The above argument implies that D' is the divisor of a character, so that the same is true for D. This completes the proof of part (a), and part (b) follows immediately using Theorem 4.2.1. □

Example 4.2.3. The rational normal cone $\widehat{C}_d$ is the affine toric variety of the cone $\sigma = \operatorname{Cone}(de_1 - e_2, e_2) \subseteq \mathbb{R}^2$. We saw in Example 4.1.4 that $\operatorname{Cl}(U_\sigma) \simeq \mathbb{Z}/d\mathbb{Z}$. The edges ρ_1, ρ_2 of σ give prime divisors D_1, D_2 on $\widehat{C}_d$, and the computations of Example 4.1.4 show that $[D_1] = [D_2]$ generates $\operatorname{Cl}(U_\sigma)$. Since $\operatorname{Pic}(U_\sigma) = 0$ by Proposition 4.2.2, it follows that the Weil divisors D_1, D_2 are not Cartier if $d > 1$.

Next consider the fan Σ_0 consisting of the cones $\rho_1, \rho_2, \{0\}$. This is a subfan of the fan Σ giving $\widehat{C}_d$, and the corresponding toric variety is $X_{\Sigma_0} \simeq \widehat{C}_d \setminus \{\gamma_\sigma\}$, where γ_σ is the distinguished point that is the unique fixed point of the T_N-action on $\widehat{C}_d$. The variety X_{Σ_0} is smooth since every cone in Σ_0 is smooth (Theorem 3.1.19). Since Σ_0 and Σ have the same 1-dimensional cones, they have the same class group by Theorem 4.1.3. Thus

$$\operatorname{Pic}(X_{\Sigma_0}) = \operatorname{Cl}(X_{\Sigma_0}) = \operatorname{Cl}(X_\Sigma) = \operatorname{Cl}(\widehat{C}_d) \simeq \mathbb{Z}/d\mathbb{Z}.$$

It follows that X_{Σ_0} is a smooth toric surface whose Picard group has torsion. ◇

Example 4.2.4. One of our favorite examples is $X = \mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$, which is the toric variety of the cone $\sigma = \operatorname{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq \mathbb{R}^3$. The ray generators are

$$u_1 = e_1,\ u_2 = e_2,\ u_3 = e_1 + e_3,\ u_4 = e_2 + e_3.$$

Note that $u_1 + u_4 = u_2 + u_3$. Let $D_i \subseteq X$ be the divisor corresponding to u_i. In Exercise 4.2.1 you will verify that

$$a_1 D_1 + a_2 D_2 + a_3 D_3 + a_4 D_4 \text{ is Cartier} \iff a_1 + a_4 = a_2 + a_3$$

and that $\mathrm{Cl}(X) \simeq \mathbb{Z}$. Since $\mathrm{Pic}(X) = 0$, we see that the D_i are not Cartier, and in fact no positive multiple of D_i is Cartier. $\Diamond$

Example 4.2.3 shows that the Picard group of a normal toric variety can have torsion. However, if we assume that Σ has a cone of maximal dimension, then the torsion goes away. Here is the precise result.

Proposition 4.2.5. *Let X_Σ be the toric variety of a fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^n$. If Σ contains a cone of dimension n, then $\mathrm{Pic}(X_\Sigma)$ is a free abelian group.*

Proof. By the exact sequence in Theorem 4.2.1, it suffices to show that if D is a T_N-invariant Cartier divisor and kD is the divisor of a character for some $k > 0$, then the same is true for D. To prove this, write $D = \sum_\rho a_\rho D_\rho$ and assume that $kD = \mathrm{div}(\chi^m)$, $m \in M$.

Let σ have dimension n. Since D is Cartier, its restriction to U_σ is also Cartier. Using the Orbit-Cone Correspondence, we have

$$D|_{U_\sigma} = \sum_{\rho \in \sigma(1)} a_\rho D_\rho.$$

This is principal on U_σ by Proposition 4.2.2, so that there is $m' \in M$ such that $D|_{U_\sigma} = \mathrm{div}(\chi^{m'})|_{U_\sigma}$. This implies that

$$a_\rho = \langle m', u_\rho \rangle \quad \text{for all } \rho \in \sigma(1).$$

On the other hand, $kD = \mathrm{div}(\chi^m)$ implies that

$$k a_\rho = \langle m, u_\rho \rangle \quad \text{for all } \rho \in \Sigma(1).$$

Together, these equations imply

$$\langle km', u_\rho \rangle = k a_\rho = \langle m, u_\rho \rangle \quad \text{for all } \rho \in \sigma(1).$$

The u_ρ span $N_\mathbb{R}$ since $\dim \sigma = n$. Then the above equation forces $km' = m$, and $D = \mathrm{div}(\chi^{m'})$ follows easily. $\square$

This proposition does not contradict the torsion Picard group in Example 4.2.3 since the fan Σ_0 in that example has no maximal cone.

Comparing Weil and Cartier Divisors. Here is an application of Proposition 4.2.2.

Proposition 4.2.6. *Let X_Σ be the toric variety of the fan Σ. Then the following are equivalent:*

(a) *Every Weil divisor on X_Σ is Cartier.*

(b) $\mathrm{Pic}(X_\Sigma) = \mathrm{Cl}(X_\Sigma)$.

(c) *X_Σ is smooth.*

Proof. (a) $\Leftrightarrow$ (b) is obvious, and (c) $\Rightarrow$ (a) follows from Theorem 4.0.22. For the converse, suppose that every Weil divisor on X_Σ is Cartier and let $U_\sigma \subseteq X_\Sigma$ be the affine open subset corresponding to $\sigma \in \Sigma$. Since $\mathrm{Cl}(X_\Sigma) \to \mathrm{Cl}(U_\sigma)$ is onto by Theorem 4.0.20, it follows that every Weil divisor on U_σ is Cartier. Using $\mathrm{Pic}(U_\sigma) = 0$ from Proposition 4.2.2 and the exact sequence from Theorem 4.1.3, we conclude that $m \mapsto \mathrm{div}(\chi^m)$ induces a surjective map

$$M \longrightarrow \mathrm{Div}_{T_N}(U_\sigma) = \bigoplus_{\rho \in \sigma(1)} \mathbb{Z} D_\rho.$$

Writing $\sigma(1) = \{\rho_1, \ldots, \rho_s\}$, this map becomes

$$\begin{aligned} M &\longrightarrow \mathbb{Z}^s \\ m &\longmapsto (\langle m, u_{\rho_1} \rangle, \ldots, \langle m, u_{\rho_s} \rangle). \end{aligned} \tag{4.2.3}$$

Now define $\Phi : \mathbb{Z}^s \to N$ by $\Phi(a_1, \ldots, a_s) = \sum_{i=1}^s a_i u_{\rho_i}$. The dual map

$$\Phi^* : M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^s, \mathbb{Z}) = \mathbb{Z}^s$$

is easily seen to be (4.2.3). In Exercise 4.2.2 you will show that

$$\begin{aligned} \Phi^* \text{ is surjective} &\iff \Phi \text{ is injective and } N/\Phi(\mathbb{Z}^s) \text{ is torsion-free.} \\ &\iff u_{\rho_1}, \ldots, u_{\rho_s} \text{ can be extended to a basis of } N. \end{aligned} \tag{4.2.4}$$

The first part of the proof shows that Φ^* is surjective. Then (4.2.4) implies that the u_ρ for $\rho \in \sigma(1)$ can be extended to a basis of N, which implies that σ is smooth. Then X_Σ is smooth by Theorem 3.1.19. $\square$

Proposition 4.2.6 has a simplicial analog. Recall that X_Σ is simplicial when every $\sigma \in \Sigma$ is simplicial, meaning that the minimal generators of σ are linearly independent over $\mathbb{R}$. You will prove the following result in Exercise 4.2.2.

Proposition 4.2.7. *Let X_Σ be the toric variety of the fan Σ. Then the following are equivalent:*

(a) *Every Weil divisor on X_Σ has a positive integer multiple that is Cartier.*

(b) $\mathrm{Pic}(X_\Sigma)$ *has finite index in* $\mathrm{Cl}(X_\Sigma)$.

(c) X_Σ *is simplicial.* $\square$

In the literature, a Weil divisor is called $\mathbb{Q}$-*Cartier* if some positive integer multiple is Cartier. Thus Proposition 4.2.7 characterizes those normal toric varieties for which all Weil divisors are $\mathbb{Q}$-Cartier.

Describing Cartier Divisors. We can use Proposition 4.2.2 to characterize T_N-invariant Cartier divisors as follows. Let $\Sigma_{\max} \subseteq \Sigma$ be the set of maximal cones of Σ, meaning cones in Σ that are not proper subsets of another cone in Σ.

Theorem 4.2.8. *Let X_Σ be the toric variety of the fan Σ and let $D = \sum_\rho a_\rho D_\rho$. Then the following are equivalent:*

(a) *D is Cartier.*

(b) *D is principal on the affine open subset U_σ for all $\sigma \in \Sigma$.*

(c) *For each $\sigma \in \Sigma$, there is $m_\sigma \in M$ with $\langle m_\sigma, u_\rho\rangle = -a_\rho$ for all $\rho \in \sigma(1)$.*

(d) *For each $\sigma \in \Sigma_{\max}$, there is $m_\sigma \in M$ with $\langle m_\sigma, u_\rho\rangle = -a_\rho$ for all $\rho \in \sigma(1)$.*

Furthermore, if D is Cartier and $\{m_\sigma\}_{\sigma\in\Sigma}$ is as in part (c)*, then:*

(1) *m_σ is unique modulo $M(\sigma) = \sigma^\perp \cap M$.*

(2) *If τ is a face of σ, then $m_\sigma \equiv m_\tau \bmod M(\tau)$.*

Proof. Since $D|_{U_\sigma} = \sum_{\rho\in\sigma(1)} a_\rho D_\rho$, the equivalences (a) ⇔ (b) ⇔ (c) follow immediately from Proposition 4.2.2. The implication (c) ⇒ (d) is clear, and (d) ⇒ (c) follows because every cone in Σ is a face of some $\sigma \in \Sigma_{\max}$ and if $m_\sigma \in \Sigma_{\max}$ works for σ, it also works for all faces of σ.

For (1), suppose that $m_\sigma \in M$ satisfies $\langle m, u_\rho\rangle = -a_\rho$ for all $\rho \in \sigma(1)$. Then, given $m'_\sigma \in M$, we have

$$\begin{aligned} \langle m'_\sigma, u_\rho\rangle = -a_\rho \text{ for all } \rho\in\sigma(1) &\iff \langle m'_\sigma - m_\sigma, u_\rho\rangle = 0 \ \text{ for all } \rho \in \sigma(1)\\ &\iff \langle m'_\sigma - m_\sigma, u\rangle = 0 \text{ for all } u \in \sigma\\ &\iff m'_\sigma - m_\sigma \in \sigma^\perp \cap M = M(\sigma). \end{aligned}$$

It follows that m_σ is unique modulo $M(\sigma)$. Since m_σ works for any face τ of σ, uniqueness implies that $m_\sigma \equiv m_\tau \bmod M(\tau)$, and (2) follows. □

The m_σ of part (c) of the theorem satisfy $D|_{U_\sigma} = \mathrm{div}(\chi^{-m_\sigma})|_{U_\sigma}$ for all $\sigma \in \Sigma$. Thus $\{(U_\sigma, \chi^{-m_\sigma})\}_{\sigma\in\Sigma}$ is local data for D in the sense of Definition 4.0.12. We call $\{m_\sigma\}_{\sigma\in\Sigma}$ the *Cartier data* of D.

The minus signs in parts (c) and (d) of the theorem are related to the minus signs in the facet presentation of a lattice polytope given in (2.2.2), namely

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -a_F \text{ for all facets } F \text{ of } P\}.$$

We will say more about this below. The minus signs are also related to *support functions*, to be discussed later in the section.

When Σ is a complete fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$, part (d) of Theorem 4.2.8 can be recast as follows. Let $\Sigma(n) = \{\sigma \in \Sigma \mid \dim \sigma = n\}$. In Exercise 4.2.3 you will show that a Weil divisor $D = \sum_\rho a_\rho D_\rho$ is Cartier if and only if:

(d)′ *For each $\sigma \in \Sigma(n)$, there is $m_\sigma \in M$ with $\langle m, u_\rho\rangle = -a_\rho$ for all $\rho \in \sigma(1)$.*

Part (1) of Theorem 4.2.8 shows that these m_σ's are uniquely determined.

In general, each m_σ in Theorem 4.2.8 is only unique modulo $M(\sigma)$. Hence we can regard m_σ as a uniquely determined element of $M/M(\sigma)$. Furthermore, if τ is a face of σ, then the canonical map $M/M(\sigma) \to M/M(\tau)$ sends m_σ to m_τ.

There are two ways to turn these observations into a complete description of $\mathrm{CDiv}_{T_N}(X_\Sigma)$. For the first, write

$$\Sigma_{\max} = \{\sigma_1, \ldots, \sigma_r\}$$

and consider the map

$$\begin{aligned}\bigoplus_i M/M(\sigma_i) &\longrightarrow \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\\ (m_i)_i &\longmapsto (m_i - m_j)_{i<j}.\end{aligned}$$

In Exercise 4.2.4 you will prove the following.

Proposition 4.2.9. *There is a natural isomorphism*

$$\mathrm{CDiv}_{T_N}(X_\Sigma) \simeq \ker\big(\textstyle\bigoplus_i M/M(\sigma_i) \to \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\big). \qquad \square$$

For readers who know inverse limits (see [**10**, p. 103]), a more sophisticated description of $\mathrm{CDiv}_{T_N}(X_\Sigma)$ comes from the directed set $(\Sigma, \preceq)$, where $\preceq$ is the face relation. We get an inverse system where $\tau \preceq \sigma$ gives $M/M(\sigma) \to M/M(\tau)$, and the inverse limit gives an isomorphism

$$\mathrm{CDiv}_{T_N}(X_\Sigma) \simeq \varprojlim_{\sigma\in\Sigma} M/M(\sigma). \tag{4.2.5}$$

The Toric Variety of a Polytope. In Chapter 2, we constructed the toric variety X_P of a full dimensional lattice polytope $P \subseteq M_\mathbb{R}$. If $M_\mathbb{R} \simeq \mathbb{R}^n$, this means that $\dim P = n$. As noted above, P has a canonical presentation

$$P = \{m \in M_\mathbb{R} \mid \langle m, u_F\rangle \geq -a_F \text{ for all facets } F \text{ of } P\}, \tag{4.2.6}$$

where $a_F \in \mathbb{Z}$ and $u_F \in N$ is the inward-pointing facet normal that is the minimal generator of the ray $\rho_F = \mathrm{Cone}(u_F)$. The normal fan Σ_P consists of cones σ_Q indexed by faces $Q \preceq P$, where

$$\sigma_Q = \mathrm{Cone}(u_F \mid F \text{ contains } Q).$$

Proposition 2.3.8 implies that the fan Σ_P is complete. Furthermore, the vertices of P correspond to the maximal cones in $\Sigma_P(n)$, and the facets of P correspond to the rays in $\Sigma_P(1)$.

The ray generators of the normal fan Σ_P are the facet normals u_F. The corresponding prime divisors in X_P will be denoted D_F. Everything is now indexed by the facets F of P. The normal fan tells us the facet normals u_F in (4.2.6), but Σ_P cannot give us the integers a_F in (4.2.6). For these, we need the divisor

$$D_P = \sum_F a_F\, D_F. \tag{4.2.7}$$

As we will see in later chapters, this divisor plays a central role in the study of projective toric varieties. For now, we give the following useful result.

Proposition 4.2.10. *D_P is a Cartier divisor on X_P and $D_P \not\sim 0$.*

Proof. A vertex $\mathsf{v} \in P$ corresponds to a maximal cone σ_{v}, and a ray ρ_F lies in $\sigma_{\mathsf{v}}(1)$ if and only if $\mathsf{v} \in F$. But $\mathsf{v} \in F$ implies that $\langle \mathsf{v}, u_F \rangle = -a_F$. Note also that $\mathsf{v} \in M$ since P is a lattice polytope. Thus we have $v \in M$ such that $\langle \mathsf{v}, u_F \rangle = -a_F$ for all $\rho_F \in \sigma_{\mathsf{v}}(1)$, so that D_P is Cartier by Theorem 4.2.8. You will prove that $D_P \not\sim 0$ in Exercise 4.2.5. $\square$

In the notation of Theorem 4.2.8, $m_{\sigma_{\mathsf{v}}}$ is the vertex v. Thus the Cartier data of the Cartier divisor D_P is the set

$$\{m_{\sigma_{\mathsf{v}}}\}_{\sigma_{\mathsf{v}} \in \Sigma_P(n)} = \{\mathsf{v} \mid \mathsf{v} \text{ is a vertex of } P\}. \tag{4.2.8}$$

This is very satisfying and explains why the minus signs in (4.2.6) correspond to the minus signs in Theorem 4.2.8.

The divisor class $[D_P] \in \operatorname{Pic}(X_P)$ also has a nice interpretation. If $D \sim D_P$, then $D = D_P + \operatorname{div}(\chi^m)$ for some $m \in M$. In Proposition 2.3.9 we saw that P and its translate $P - m$ have the same normal fan and hence give the same toric variety, i.e., $X_P = X_{m+P}$. We also have

$$D = D_P + \operatorname{div}(\chi^m) = D_{P-m}$$

(Exercise 4.2.5), so that the divisor class of D_P gives all translates of P.

The divisor D_P has many more wonderful properties. We will get a glimpse of this in §4.3 and learn the full power of D_P in Chapter 6 when we study ample divisors on toric varieties.

Support Functions. The Cartier data $\{m_\sigma\}_{\sigma \in \Sigma}$ that describes a torus-invariant Cartier divisor can be cumbersome to work with. Here we introduce a more efficient computational tool. Recall that Σ has support $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_{\mathbb{R}}$.

Definition 4.2.11. Let Σ be a fan in $N_{\mathbb{R}}$.

(a) A ***support function*** is a function $\varphi : |\Sigma| \to \mathbb{R}$ that is linear on each cone of Σ. The set of all support functions is denoted $\operatorname{SF}(\Sigma)$.

(b) A support function φ is ***integral with respect to the lattice*** N if

$$\varphi(|\Sigma| \cap N) \subseteq \mathbb{Z}.$$

The set of all such support functions is denoted $\operatorname{SF}(\Sigma, N)$.

Let $D = \sum_\rho a_\rho D_\rho$ be Cartier and let $\{m_\sigma\}_{\sigma \in \Sigma}$ be the Cartier data of D as in Theorem 4.2.8. Thus

$$\langle m_\sigma, u_\rho \rangle = -a_\rho \ \text{ for all } \rho \in \sigma(1). \tag{4.2.9}$$

We now describe Cartier divisors in terms of support functions.

Theorem 4.2.12. *Let Σ be a fan in $N_{\mathbb{R}}$. Then:*

(a) *Given $D = \sum_\rho a_\rho D_\rho$ with Cartier data $\{m_\sigma\}_{\sigma\in\Sigma}$, the function*

$$\begin{aligned}\varphi_D : |\Sigma| &\longrightarrow \mathbb{R}\\ u &\longmapsto \varphi_D(u) = \langle m_\sigma, u\rangle \text{ when } u \in \sigma\end{aligned}$$

is a well-defined support function that is integral with respect to N.

(b) *$\varphi_D(u_\rho) = -a_\rho$ for all $\rho \in \Sigma(1)$, so that*

$$D = -\sum_\rho \varphi_D(u_\rho)\, D_\rho.$$

(c) *The map $D \mapsto \varphi_D$ induces an isomorphism*

$$\mathrm{CDiv}_{T_N}(X_\Sigma) \simeq \mathrm{SF}(\Sigma, N).$$

Proof. Theorem 4.2.8 tells us that each m_σ is unique modulo $\sigma^\perp \cap M$ and that $m_\sigma \equiv m_{\sigma'} \bmod (\sigma\cap\sigma')^\perp \cap M$. It follows easily that φ_D is well-defined. Also, φ_D is linear on each σ since $\varphi_D|_\sigma(u) = \langle m_\sigma, u\rangle$ for $u \in \sigma$, and it is integral with respect to N since $m_\sigma \in M$. This proves part (a), and part (b) follows from the definition of φ_D and (4.2.9).

It remains to prove part (c). First note that $\varphi_D \in \mathrm{SF}(\Sigma, N)$ by part (a). Since $D, E \in \mathrm{CDiv}_{T_N}(X_\Sigma)$ and $k \in \mathbb{Z}$ imply that

$$\begin{aligned}\varphi_{D+E} &= \varphi_D + \varphi_E\\ \varphi_{kD} &= k\,\varphi_D,\end{aligned}$$

the map $\mathrm{CDiv}_{T_N}(X_\Sigma) \to \mathrm{SF}(\Sigma, N)$ is a homomorphism, and injectivity follows from part (b). To prove surjectivity, take $\varphi \in \mathrm{SF}(\Sigma, N)$. Fix $\sigma \in \Sigma$. Since φ is integral with respect to N, it defines a $\mathbb{N}$-linear map $\varphi|_{\sigma\cap N} : \sigma \cap N \to \mathbb{Z}$, which extends to $\mathbb{N}$-linear map $\phi_\sigma : N_\sigma \to \mathbb{Z}$, where $N_\sigma = \mathrm{Span}(\sigma)\cap N$. Since

$$\mathrm{Hom}_{\mathbb{Z}}(N_\sigma, \mathbb{Z}) \simeq M/M(\sigma),$$

it follows that there is $m_\sigma \in M$ such that $\varphi|_\sigma(u) = \langle m_\sigma, u\rangle$ for $u \in \sigma$. Then $D = -\sum_\rho \varphi_D(u_\rho)\, D_\rho$ is a Cartier divisor that maps to φ. □

In terms of support functions, the exact sequence of Theorem 4.2.1 becomes

$$M \longrightarrow \mathrm{SF}(\Sigma, N) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0, \tag{4.2.10}$$

where $m \in M$ maps to the linear support function defined by $u \mapsto -\langle m, u\rangle$ and $\varphi \in \mathrm{SF}(\Sigma, N)$ maps to the divisor class $[-\sum_\rho \varphi(u_\rho)\, D_\rho] \in \mathrm{Pic}(X_\Sigma)$. Be sure you understand the minus signs.

Here is an example of how to compute with support functions.

Example 4.2.13. The eight points $\pm e_1 \pm e_2 \pm e_3$ are the vertices of a cube in $\mathbb{R}^3$. Taking the cones over the six faces gives a complete fan in $\mathbb{R}^3$. Modify this fan by replacing $e_1 + e_2 + e_3$ with $e_1 + 2e_2 + 3e_3$. The resulting fan Σ has the surprising property that $\mathrm{Pic}(X_\Sigma) = 0$. In other words, X_Σ is a complete toric variety whose Cartier divisors are all principal.

We will prove $\mathrm{Pic}(X_\Sigma) = 0$ by showing that all support functions for Σ are linear. Label the ray generators as follows, using coordinates for compactness:

$$u_1 = (1,2,3),\ u_2 = (1,-1,1),\ u_3 = (1,1,-1),\ u_4 = (-1,1,1)$$
$$u_5 = (1,-1,-1),\ u_6 = (-1,-1,1),\ u_7 = (-1,1,-1),\ u_8 = (-1,-1,-1).$$

The ray generators are shown in Figure 4. The figure also includes three maximal cones of Σ:

$$\sigma_1 = \mathrm{Cone}(u_1,u_2,u_3,u_5)$$
$$\sigma_2 = \mathrm{Cone}(u_1,u_3,u_4,u_7)$$
$$\sigma_3 = \mathrm{Cone}(u_1,u_2,u_4,u_6).$$

The shading in Figure 4 indicates $\sigma_1 \cap \sigma_2, \sigma_1 \cap \sigma_3, \sigma_2 \cap \sigma_3$. Besides $\sigma_1,\sigma_2,\sigma_3$, the fan Σ has three other maximal cones, which we call left, down, and back. Thus the cone left has ray generators u_2,u_5,u_6,u_8, and similarly for the other two.

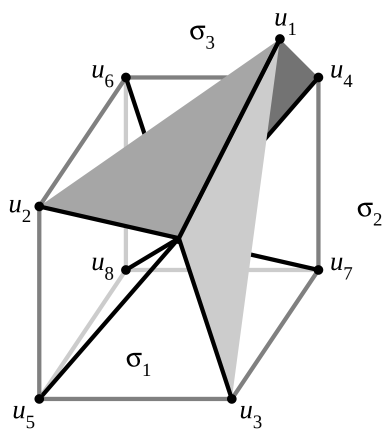


Figure 4. A fan Σ with $\mathrm{Pic}(X_\Sigma) = 0$

Take $\varphi \in \mathrm{SF}(\Sigma,\mathbb{Z}^3)$. We show that φ is linear as follows. Since $\varphi|_{\sigma_1}$ is linear, there is $m_1 \in \mathbb{Z}^3$ such that $\varphi(u) = \langle m_1,u\rangle$ for $u \in \sigma_1$. Hence the support function

$$u \longmapsto \varphi(u) - \langle m_1,u\rangle$$

vanishes identically on σ_1. Replacing φ with this support function, we may assume that $\varphi|_{\sigma_1} = 0$. Once we prove $\varphi = 0$ everywhere, it will follow that all support functions are linear, and then $\mathrm{Pic}(X_\Sigma) = 0$ by (4.2.10).

Since $u_1,u_2,u_3,u_5 \in \sigma_1$ and φ vanishes on σ_1, we have $\varphi(u_1) = \varphi(u_2) = \varphi(u_3) = \varphi(u_5) = 0$. It suffices to prove $\varphi(u_4) = \varphi(u_6) = \varphi(u_7) = \varphi(u_8) = 0$.

To do this, we use the fact that each maximal cone has four generators, which must satisfy a linear relation. Here are the cones and the corresponding relations:

cone	relation
σ_1	$2u_1+5u_5=4u_2+3u_3$
σ_2	$2u_1+4u_7=3u_3+5u_4$
σ_3	$2u_1+3u_6=4u_2+5u_4$
left	$u_2+u_8=u_5+u_6$
down	$u_3+u_8=u_5+u_7$
back	$u_4+u_8=u_6+u_7$

Since φ is linear on each cone and $\varphi(u_1)=\varphi(u_2)=\varphi(u_3)=\varphi(u_5)=0$, the second, third, fourth and fifth relations imply

$$\begin{aligned}4\varphi(u_7)&=5\varphi(u_4)\\ 3\varphi(u_6)&=5\varphi(u_4)\\ \varphi(u_8)&=\varphi(u_6)\\ \varphi(u_8)&=\varphi(u_7).\end{aligned}$$

The last two equations give $\varphi(u_6)=\varphi(u_7)$, and substituting these into the first two shows that $\varphi(u_4)=\varphi(u_6)=\varphi(u_7)=\varphi(u_8)=0$. $\Diamond$

Since the toric variety of a polytope P has the non-principal Cartier divisor D_P, its follows that the fan Σ of Example 4.2.13 is not the normal fan of *any* 3-dimensional lattice polytope. As we will see later, this implies that X_Σ is complete but not projective.

A full dimensional lattice polytope $P\subseteq M_{\mathbb{R}}$ leads to an interesting support function on the normal fan Σ_P.

Proposition 4.2.14. *Assume $P\subseteq M_{\mathbb{R}}$ is a full dimensional lattice polytope with normal fan Σ_P. Then the function $\varphi_P : N_{\mathbb{R}}\to\mathbb{R}$ defined by*

$$\varphi_P(u)=\min(\langle m,u\rangle \mid m\in P)$$

has the following properties:

(a) *φ_P is a support function for Σ_P and is integral with respect to N.*

(b) *The divisor corresponding to φ_P is the divisor D_P defined in* (4.2.7).

Proof. First note that minimum used in the definition of φ_P exists because P is compact. Now write

$$P=\{m\in M_{\mathbb{R}} \mid \langle m,u_F\rangle\geq -a_F \text{ for all facets } F \text{ of } P\}.$$

Then $D_P=\sum_F a_F\,D_F$ is Cartier by Proposition 4.2.10, and Theorem 4.2.12 shows that the corresponding support function maps u_F to $-a_F$.

It remains to show that $\varphi_P(u) \in \mathrm{SF}(\Sigma_P)$ and $\varphi_P(u_F) = -a_F$. Recall that maximal cones of Σ_P correspond to vertices of P, where the vertex v gives the maximal cone $\sigma_{\mathsf{v}} = \mathrm{Cone}(u_F \mid \mathsf{v} \in F)$. Take $u = \sum_{\mathsf{v}\in F} \lambda_F\, u_F \in \sigma_{\mathsf{v}}$, where $\lambda_F \geq 0$. Then $m \in P$ implies

$$\langle m,u\rangle = \sum_{\mathsf{v}\in F} \lambda_F \langle m,u_F\rangle \geq -\sum_{\mathsf{v}\in F} \lambda_F\, a_F. \tag{4.2.11}$$

Thus $\varphi_P(u) \geq -\sum_{\mathsf{v}\in F} \lambda_F\, a_F$. Since equality occurs in (4.2.11) when $m = \mathsf{v}$, we obtain

$$\varphi_P(u) = -\sum_{\mathsf{v}\in F} \lambda_F\, a_F = \langle \mathsf{v},u\rangle.$$

This shows that $\varphi_P \in \mathrm{SF}(\Sigma_P,N)$. Furthermore, when $\mathsf{v} \in F$, we have $\varphi_P(u_F) = \langle \mathsf{v},u_F\rangle = -a_F$, as desired. □

We will return to support functions in Chapter 6, where we will use them to give elegant criteria for a divisor to be ample or generated by its global sections.

Exercises for §4.2.

4.2.1. Prove the assertions made in Example 4.2.4.

4.2.2. Prove (4.2.4) and Proposition 4.2.7.

4.2.3. When Σ is complete, prove that $D = \sum_\rho a_\rho D_\rho$ is Cartier if and only if it satisfies condition (d)$'$ stated in the discussion following Theorem 4.2.8.

4.2.4. Prove Proposition 4.2.9.

4.2.5. A lattice polytope P gives the toric variety X_P and the divisor D_P from (4.2.7).

(a) Prove that $D_P + \mathrm{div}(\chi^m) = D_{P-m}$ for any $m \in M$.

(b) Prove that $D_P \not\sim 0$. Hint: The normal fan of P is complete.

4.2.6. Let D be a T_N-invariant Cartier divisor on X_Σ. By Theorem 4.2.8, D is determined by its Cartier data $\{m_\sigma\}_{\sigma\in\Sigma}$. Given any $m \in M$, show that $D + \mathrm{div}(\chi^m)$ has Cartier data $\{m_\sigma - m\}_{\sigma\in\Sigma}$. Be sure to explain where the minus sign comes from.

4.2.7. Let X_Σ be the toric variety of the fan Σ. Prove the following consequences of the Orbit-Cone Correspondence (Theorem 3.2.6).

(a) $O(\sigma) = \bigcap_{\rho\in\sigma(1)} D_\rho$.

(b) Rays $u_{\rho_1},\ldots,u_{\rho_r} \in \Sigma(1)$ lie in a cone of Σ if and only if $D_{\rho_1} \cap \cdots \cap D_{\rho_r} \neq \emptyset$.

4.2.8. Let Σ be a fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$ and assume that Σ has a cone of dimension n.

(a) Fix a cone $\sigma \in \Sigma$ of dimension n. Prove that

$$\mathrm{Pic}(X_\Sigma) \simeq \{\varphi \in \mathrm{SF}(\Sigma,N) \mid \varphi|_\sigma = 0\}.$$

(b) Explain how part (a) relates to Example 4.2.13.

(c) Use part (a) to give a different proof of Proposition 4.2.5.

4.2.9. Let σ be as in Example 4.2.4, but instead of using the lattice generated by e_1, e_2, e_3, instead use $N = \mathbb{Z}\cdot\frac{1}{2b}e_1 + \mathbb{Z}\cdot\frac{1}{b}e_2 + \mathbb{Z}\cdot\frac{1}{a}e_3 + \mathbb{Z}\cdot\frac{1}{2b}(e_1+e_2+e_3)$, where a, b are relatively prime positive integers with $a > 1$. Prove that no multiple of $D_1 + D_2 + D_3 + D_4$ is Cartier. Hint: The first step will be to find the minimal generators (relative to N) of the edges of σ.

4.2.10. Let X_P be the toric variety of the octahedron $P = \mathrm{Conv}(\pm e_1, \pm e_2, \pm e_3) \subseteq \mathbb{R}^3$.

(a) Show that $\mathrm{Cl}(X_P) \simeq \mathbb{Z}^5 \oplus (\mathbb{Z}/2\mathbb{Z})^2$.

(b) Use support functions and the strategy of Example 4.2.13 to show that $\mathrm{Pic}(X_P) \simeq \mathbb{Z}$.

4.2.11. In Exercise 4.1.5, you showed that the weighted projective space $\mathbb{P}(q_0,\dots,q_n)$ has class group $\mathrm{Cl}(\mathbb{P}(q_0,\dots,q_n)) \simeq \mathbb{Z}$. Prove that $\mathrm{Pic}(\mathbb{P}(q_0,\dots,q_n)) \subseteq \mathrm{Cl}(\mathbb{P}(q_0,\dots,q_n))$ maps to the subgroup $m\mathbb{Z} \subseteq \mathbb{Z}$, where $m = \mathrm{lcm}(q_0,\dots,q_n)$. Hint: Show that $\sum_{i=0}^n b_i D_i$ generates the class group, where $\sum_{i=0}^n b_i q_i = 1$. Also note that $m \in M_{\mathbb{Q}}$ lies in M if and only if $\langle m, u_i\rangle \in \mathbb{Z}$ for all i, where the u_i are from Exercise 4.1.5.

4.2.12. Let X_Σ be a smooth toric variety and let $\tau \in \Sigma$ have dimension ≥ 2. This gives the orbit closure $V(\tau) = \overline{O(\tau)} \subseteq X_\Sigma$. In §3.3 we defined the blowup $\mathrm{Bl}_{V(\tau)}(X_\Sigma)$. Prove that

$$\mathrm{Pic}(\mathrm{Bl}_{V(\tau)}(X_\Sigma)) \simeq \mathrm{Pic}(X_\Sigma) \oplus \mathbb{Z}.$$

4.2.13. A nonzero polynomial $f = \sum_{m\in\mathbb{Z}^n} c_m x^m \in \mathbb{C}[x_1,\dots,x_n]$ has *Newton polytope*

$$P(f) = \mathrm{Conv}(m \mid c_m \neq 0) \subset \mathbb{R}^n.$$

When $P(f)$ has dimension n, Proposition 4.2.14 tells us that the function $\varphi_{P(f)}(u) = \min(\langle m, u\rangle \mid m \in P(f))$ is the support function of a divisor on $X_{P(f)}$. Here we interpret $\varphi_{P(f)}$ as the *tropicalization* of f.

The *tropical semiring* $(\mathbb{R}, \oplus, \odot)$ has operations

$$\begin{aligned} a \oplus b &= \min(a,b) && \text{(tropical addition)}\\ a \odot b &= a + b && \text{(tropical multiplication)}. \end{aligned}$$

A *tropical polynomial* in real variables $x_1,\dots,x_n$ is a finite tropical sum

$$F = c_1 \odot x_1^{a_{1,1}} \odot \cdots \odot x_n^{a_{1,n}} \;\oplus\; \cdots \;\oplus\; c_r \odot x_1^{a_{r,1}} \odot \cdots \odot x_n^{a_{r,n}}$$

where $c_i \in \mathbb{R}$ and $x_i^a = x_i \odot \cdots \odot x_i$ (a times). For a more compact representation, define a tropical monomial to be $x^m = x_1^{a_1} \odot \cdots \odot x_n^{a_n}$ for $m = (a_1,\dots,a_n) \in \mathbb{N}^n$. Then, using the tropical analog of summation notation, the tropical polynomial F is

$$F = \textstyle\bigoplus_{i=1}^r c_i \odot x^{m_i}, \quad m_i = (a_{i,1},\dots,a_{i,n}).$$

(a) Show that $F = \min_{1\leq i\leq r}(c_i + a_{i,1}x_1 + \cdots + a_{i,n}x_n)$.

(b) The *tropicalization* of our original polynomial f is the tropical polynomial

$$F_f = \textstyle\bigoplus_{c_m\neq 0} 0 \odot x^m.$$

Prove that $F_f = \varphi_{P(f)}$. (The 0 is explained as follows. In tropical geometry, one often works in a larger ring where the coefficients of f are Puiseux series, and the tropicalization uses the order of vanishing of the coefficients. Here, we use a smaller ring where the coefficients of f are nonzero constants, with order of vanishing 0.)

(c) The *tropical variety* of a tropical polynomial F is the set of points in $\mathbb{R}^n$ where F is not linear. For $f = x + 2y + 3x^2 - xy^2 + 4x^2y$, compute the tropical variety of F_f and show that it consists of the rays in the normal fan of $P(f)$.

A nice introduction to tropical algebraic geometry can be found in [**240**].

§4.3. The Sheaf of a Torus-Invariant Divisor

If $D = \sum_\rho a_\rho D_\rho$ is a T_N-invariant divisor on the normal toric variety X_Σ, we get the sheaf $\mathscr{O}_{X_\Sigma}(D)$ defined in §4.0. We will study these sheaves in detail in Chapters 6 and 8. In this section we will focus primarily on global sections.

We begin with a classic example of the sheaf $\mathscr{O}_{X_\Sigma}(D)$.

Example 4.3.1. For $\mathbb{P}^n$, the divisors $D_0,\ldots,D_n$ correspond to the ray generators of the usual fan for $\mathbb{P}^n$. The computation $\mathrm{Cl}(\mathbb{P}^n) \simeq \mathbb{Z}$ from Example 4.1.6 shows that $D_0 \sim D_1 \sim \cdots \sim D_n$. These linear equivalences give isomorphisms

$$\mathscr{O}_{\mathbb{P}^n}(D_0) \simeq \mathscr{O}_{\mathbb{P}^n}(D_1) \simeq \cdots \simeq \mathscr{O}_{\mathbb{P}^n}(D_n)$$

by Proposition 4.0.29. These sheaves are denoted $\mathscr{O}_{\mathbb{P}^n}(1)$, and simiarly the sheaves $\mathscr{O}_{\mathbb{P}^n}(kD_i)$, $k \in \mathbb{Z}$, are denoted $\mathscr{O}_{\mathbb{P}^n}(k)$. We will see the intrinsic reason for this notation in Example 5.3.8. $\Diamond$

Global Sections. Let D be a T_N-invariant divisor on a toric variety X_Σ. We will give two descriptions of the global sections $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$. Here is the first.

Proposition 4.3.2. *If D is a T_N-invariant Weil divisor on X_Σ, then*

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{\mathrm{div}(\chi^m)+D\geq 0} \mathbb{C}\cdot\chi^m.$$

Proof. If $f \in \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$, then $\mathrm{div}(f) + D \geq 0$ implies $\mathrm{div}(f)|_{T_N} \geq 0$ since $D|_{T_N} = 0$. Since $\mathbb{C}[M]$ is the coordinate ring of T_N, Proposition 4.0.16 implies $f \in \mathbb{C}[M]$. Thus

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \subseteq \mathbb{C}[M].$$

Furthermore, $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$ is invariant under the T_N-action on $\mathbb{C}[M]$ since D is T_N-invariant. By Lemma 1.1.16, we obtain

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{\chi^m\in\Gamma(X_\Sigma,\mathscr{O}_{X_\Sigma}(D))} \mathbb{C}\cdot\chi^m.$$

Since $\chi^m \in \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$ if and only if $\mathrm{div}(\chi^m) + D \geq 0$, we are done. $\square$

The Polyhedron of a Divisor. For $D = \sum_\rho a_\rho D_\rho$ and $m \in M$, $\mathrm{div}(\chi^m) + D \geq 0$ is equivalent to

$$\langle m, u_\rho\rangle + a_\rho \geq 0 \quad \text{for all } \rho \in \Sigma(1),$$

which can be rewritten as

$$\langle m, u_\rho\rangle \geq -a_\rho \quad \text{for all } \rho \in \Sigma(1). \tag{4.3.1}$$

This explains the minus signs! To emphasize the underlying geometry, we define

$$P_D = \{m \in M_\mathbb{R} \mid \langle m, u_\rho\rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}. \tag{4.3.2}$$

We say that P_D is a *polyhedron* since it is an intersection of finitely many closed half spaces. This looks very similar to the canonical presentation of a polytope (see (4.2.6), for example). However, the reader should be aware that P_D need not be a polytope, and even when it is a polytope, it need not be a lattice polytope. All of this will be explained in the examples given below.

For now, we simply note that (4.3.1) is equivalent to $m \in P_D \cap M$. This gives our second description of the global sections.

Proposition 4.3.3. *If D is a T_N-invariant Weil divisor on X_Σ, then*

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m,$$

where $P_D \subseteq M_\mathbb{R}$ is the polyhedron defined in (4.3.2). $\square$

As noted above, a polyhedron is an intersection of finitely many closed half spaces. A polytope is a bounded polyhedron.

Examples. Here are some examples to illustrate the kinds of polyhedra that can occur in Proposition 4.3.3.

Example 4.3.4. The fan Σ for the blowup $\mathrm{Bl}_0(\mathbb{C}^2)$ of $\mathbb{C}^2$ at the origin has ray generators $u_0 = e_1 + e_2$, $u_1 = e_1$, $u_2 = e_2$ and corresponding divisors D_0, D_1, D_2. For the divisor $D = D_0 + D_1 + D_2$, a point $m = (x, y)$ lies in P_D if and only if

$$\begin{aligned}\langle m, u_0\rangle \geq -1 &\iff x + y \geq -1\\ \langle m, u_1\rangle \geq -1 &\iff x \geq -1\\ \langle m, u_2\rangle \geq -1 &\iff y \geq -1.\end{aligned}$$

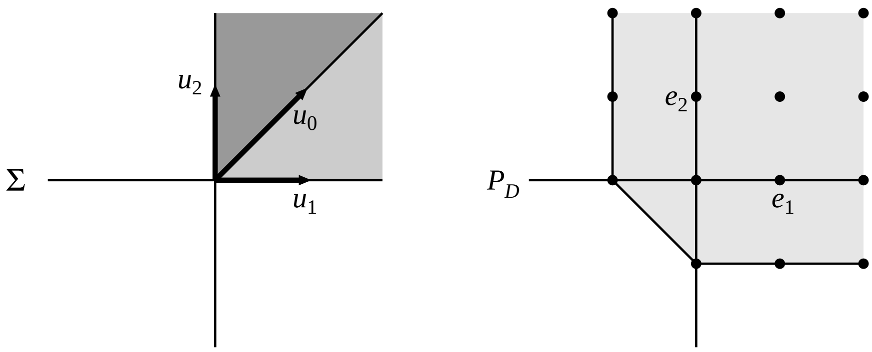


Figure 5. The fan Σ and the polyhedron P_D

The fan Σ and the polyhedron P_D are shown in Figure 5. Note that P_D is not bounded. By Proposition 4.3.3, the lattice points of P_D (the dots in Figure 5) give characters that form a basis of $\Gamma(\mathrm{Bl}_0(\mathbb{C}^2), \mathscr{O}_{\mathrm{Bl}_0(\mathbb{C}^2)}(D))$. $\Diamond$

Example 4.3.5. The fan Σ_2 for the Hirzebruch surface $\mathscr{H}_2$ has ray generators $u_1 = -e_1 + 2e_2,\ u_2 = e_2,\ u_3 = e_1,\ u_4 = -e_2$. The corresponding divisors are $D_1,\ D_2,\ D_3,\ D_4$, and Example 4.1.8 implies that the classes of D_1 and D_2 are a basis of $\mathrm{Cl}(\mathscr{H}_2) \simeq \mathbb{Z}^2$.

Consider the divisor $aD_1 + D_2$, $a \in \mathbb{Z}$, and let $P_a \subseteq \mathbb{R}^2$ be the corresponding polyhedron, which is a polytope in this case. A point $m = (x,y)$ lies in P_a if and only if

$$\begin{aligned}\langle m, u_1\rangle \geq -a &\iff y \geq \tfrac{1}{2}x - \tfrac{a}{2}\\ \langle m, u_2\rangle \geq -1 &\iff y \geq -1\\ \langle m, u_3\rangle \geq 0 &\iff x \geq 0.\\ \langle m, u_4\rangle \geq 0 &\iff y \leq 0.\end{aligned}$$

Figure 6 shows Σ_2, together with shaded areas marked A, B, C. These are related

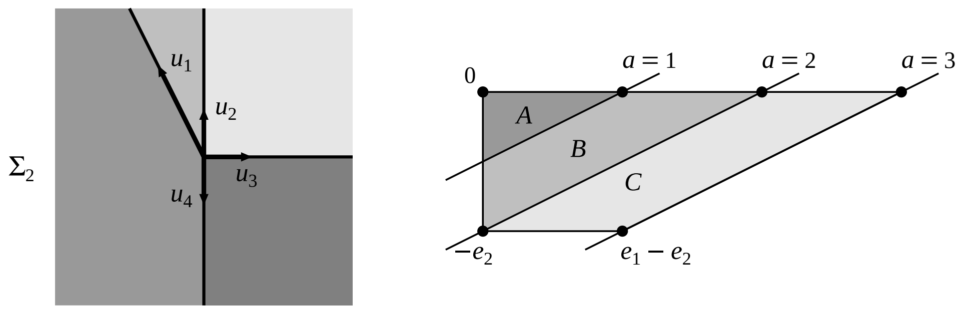


Figure 6. The fan Σ_2 and the polyhedra P_a

to the polygons P_a for $a = 1,2,3$ by the equations

$$\begin{aligned}P_1 &= A\\ P_2 &= A \cup B\\ P_3 &= A \cup B \cup C.\end{aligned}$$

Notice that as we increase a, the line $y = \frac{1}{2}x - \frac{a}{2}$ corresponding to u_1 moves to the right and makes the polytope bigger. In fact, you can see that Σ_2 is the normal fan of the lattice polytope P_a for any $a \geq 3$. For $a = 2$, we get a lattice polytope P_2, but its normal fan is not Σ_2—you can see how the "facet" with inward normal vector u_2 collapses to a point of P_2. For $a = 1$, P_1 is not a lattice polytope since $-\frac{1}{2}e_2$ is a vertex. $\Diamond$

Chapters 6 and 7 will explain how the geometry of the polyhedron P_D relates to the properties of the divisor D. In particular, we will see that the divisor $aD_1 + D_2$ from Example 4.3.5 is *ample* if and only if $a \geq 3$ since these are the only a's for which Σ_2 is the normal fan P_a.

Example 4.3.6. By Example 4.3.1, the sheaf $\mathscr{O}_{\mathbb{P}^n}(k)$ can be written $\mathscr{O}_{\mathbb{P}^n}(kD_0)$, where the divisor D_0 corresponds to the ray generator u_0 from Example 4.1.6. It is straightforward to show that the polyhedron of $D = kD_0$ is

$$P_D = \begin{cases} \emptyset & k < 0 \\ k\Delta_n & k \geq 0, \end{cases}$$

where $\Delta_n \subseteq \mathbb{R}^n$ is the standard n-simplex. We can think of characters as Laurent monomials $t^m = t_1^{a_1} \cdots t_n^{a_n}$, where $m = (a_1, \dots, a_n)$. It follows that

$$\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) \simeq \{f \in \mathbb{C}[t_1, \dots, t_n] \mid \deg(f) \leq k\}.$$

The *homogenization* of such a polynomial is

$$F = x_0^k f(x_1/x_0, \dots, x_n/x_0) \in \mathbb{C}[x_0, \dots, x_n].$$

In this way, we get an isomorphism

$$\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) \simeq \{f \in \mathbb{C}[x_0, \dots, x_n] \mid f \text{ is homogeneous with } \deg(f) = k\}.$$

The toric interpretation of homogenization will be discussed in Chapter 5. $\Diamond$

Example 4.3.7. Let X_P be the toric variety of a full dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$. The facet presentation of P gives the Cartier divisor D_P defined in (4.2.7), and one checks easily that the polyhedron P_{D_P} is the polytope P that we began with (Exercise 4.3.1). It follows from Proposition 4.3.3 that

$$\Gamma(X_P, \mathscr{O}_{X_P}(D_P)) = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m.$$

Recall from Chapter 2 that the characters χ^m for $m \in P \cap M$ give the projective toric variety $X_{P\cap M}$. The divisor kD_P gives the polytope kP (Exercise 4.3.2), so that

$$\Gamma(X_P, \mathscr{O}_{X_P}(kD_P)) = \bigoplus_{m \in (kP) \cap M} \mathbb{C} \cdot \chi^m.$$

In Chapter 2 we proved that kP is very ample for k sufficiently large, in which case $X_{(kP)\cap M}$ is the toric variety X_P. So the characters χ^m that realize X_P as a projective variety come from global sections of $\mathscr{O}_{X_P}(kD_P)$. In Chapter 6, we will pursue these ideas when we study *ample* and *very ample* Cartier divisors.

Note also that $\dim \Gamma(X_P, \mathscr{O}_{X_P}(kD_P))$ gives number of lattice points in multiples of P. This will have important consequences in later chapters. $\Diamond$

The operation sending a T_N-invariant Weil divisor $D \subseteq X_\Sigma$ to the polyhedron $P_D \subseteq M_{\mathbb{R}}$ defined in (4.3.2) has the following properties:

- $P_{kD} = kP_D$ for $k \geq 0$.
- $P_{D+\mathrm{div}(\chi^m)} = P_D - m$.
- $P_D + P_E \subseteq P_{D+E}$.

You will prove these in Exercise 4.3.2. The multiple kP_D and Minkowski sum $P_D + P_E$ are defined in §2.2, and $P - m$ is translation.

Complete Fans. When the fan Σ is complete, we have the following finiteness result that you will prove in Exercise 4.3.3.

Proposition 4.3.8. *Let X_Σ be the toric variety of a complete fan Σ in $N_\mathbb{R}$. Then:*

(a) $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}) = \mathbb{C}$*, so the only morphisms $X_\Sigma \to \mathbb{C}$ are the constant ones.*

(b) *P_D is a polytope for any T_N-invariant Weil divisor D on X_Σ.*

(c) $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$ *has finite dimension as a vector space over $\mathbb{C}$ for any Weil divisor on X_Σ.*

The assertions of parts (a) and (c) are true in greater generality: if X is *any* complete variety, then $\Gamma(X, \mathscr{O}_X) = \mathbb{C}$, and if $\mathscr{F}$ is *any* coherent sheaf on X, then $\dim \Gamma(X, \mathscr{F}) < \infty$ (see [**245**, Vol. 2, §VI.1.1 and §VI.3.4]).

Exercises for §4.3.

4.3.1. Prove the assertion $P_{D_P} = P$ made in Example 4.3.7.

4.3.2. Prove the properties of $D \mapsto P_D$ listed above.

4.3.3. Prove Proposition 4.3.8. Hint: For part (a), use completeness to show that $m = 0$ when $\langle m, u_\rho \rangle \geq 0$ for all ρ. For part (b), assume $M_\mathbb{R} = \mathbb{R}^n$ and suppose $m_i \in P_D$ satisfy $||m_i|| \to \infty$. Then consider the points $\frac{m_i}{||m_i||}$ on the sphere $S^{n-1} \subseteq \mathbb{R}^n$.

4.3.4. Let Σ be a fan in $N_\mathbb{R}$ with convex support. Then $|\Sigma| \subseteq N_\mathbb{R}$ is a convex polyhedral cone with dual $|\Sigma|^\vee \subseteq M_\mathbb{R}$.

(a) Prove that $|\Sigma|^\vee$ is the polyhedron associated to the divisor $D = 0$ on X_Σ.

(b) Conclude that $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}) = \bigoplus_{m \in |\Sigma|^\vee \cap M} \mathbb{C} \cdot \chi^m$.

(c) Use part (b) to prove part (a) of Proposition 4.3.8.

4.3.5. Example 4.3.5 studied divisors on the Hirzebruch surface $\mathscr{H}_2$. This exercise will consider the divisors $D = D_4$ and $D' = D + D_2 = D_2 + D_4$.

(a) Show that D gives the same polygon as D', i.e., $P_D = P_{D'}$.

(b) Since $\mathscr{H}_2$ is smooth, D and D' are Cartier. Compute their respective Cartier data $\{m_\sigma\}_{\sigma \in \Sigma_2(2)}$ and $\{m'_\sigma\}_{\sigma \in \Sigma_2(2)}$.

(c) Show that $P = \mathrm{Conv}(m_\sigma \mid \sigma \in \Sigma_2(2))$ and that $P \neq \mathrm{Conv}(m'_\sigma \mid \sigma \in \Sigma_2(2))$.

Thus D and D' give the same polygon but differ in how their Cartier data relates to the polygon. In Chapter 6 we will use this to prove that $\mathscr{O}_{\mathscr{H}_2}(D)$ is *generated by global sections* while $\mathscr{O}_{\mathscr{H}_2}(D')$ has *base points*.

Chapter 5

Homogeneous Coordinates on Toric Varieties

§5.0. Background: Quotients in Algebraic Geometry

Projective space $\mathbb{P}^n$ is usually defined as the quotient

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ by scalar multiplication, i.e.,

$$\lambda \cdot (a_0, \dots, a_n) = (\lambda a_0, \dots, \lambda a_n).$$

The above representation defines $\mathbb{P}^n$ as a *set*; making $\mathbb{P}^n$ into a variety requires the notion of abstract variety introduced in Chapter 3. The main goal of this chapter is to prove that every toric variety has a similar quotient construction as a variety.

Group Actions. Let G be a group acting on a variety X. We always assume that for every $g \in G$, the map $\phi_g(x) = g \cdot x$ defines a morphism $\phi_g : X \to X$. When $X = \operatorname{Spec}(R)$ is affine, $\phi_g : X \to X$ comes from a homomorphism $\phi_g^* : R \to R$. We define the *induced action* of G on R by

$$g \cdot f = \phi_{g^{-1}}^*(f)$$

for $f \in R$. In other words, $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for all $x \in X$. You will check in Exercise 5.0.1 this gives an action of G on R. Thus we have two objects:

- The set G-orbits $X/G = \{G \cdot x \mid x \in X\}$.
- The ring of invariants $R^G = \{f \in R \mid g \cdot f = f \text{ for all } g \in G\}$.

To make X/G into an affine variety, we need to define its coordinate ring, i.e., we need to determine the "polynomial" functions on X/G. A key observation is that if

$f \in R^G$, then

$$\bar{f}(G \cdot x) = f(x)$$

gives a well-defined function $\bar{f} : X/G \to \mathbb{C}$. Hence elements of G give obvious polynomial functions on X/G, which suggests that

$$\text{as an affine variety, } X/G = \operatorname{Spec}(R^G).$$

As shown by the following examples, this works in some cases but fails in others.

Example 5.0.1. Let $\mu_2 = \{\pm 1\}$ act on $\mathbb{C}^2 = \operatorname{Spec}(\mathbb{C}[s,t])$, where $-1 \in \mu_2$ acts by multiplication by -1. Note that every orbit consists of two elements, with the exception of the orbit of the origin, which is the unique fixed point of the action.

The ring of invariants $\mathbb{C}[s,t]^{\mu_2} = \mathbb{C}[s^2, st, t^2]$ is the coordinate ring of the affine toric variety $\mathbf{V}(xz - y^2)$. Hence we get a map

$$\Phi : \mathbb{C}^2/\mu_2 \longrightarrow \operatorname{Spec}(\mathbb{C}[s,t]^{\mu_2}) = \mathbf{V}(xz - y^2) \subseteq \mathbb{C}^3$$

where the orbit $\mu_2 \cdot (a,b)$ maps to (a^2, ab, b^2). This is easily seen to be a bijection, so that $\operatorname{Spec}(\mathbb{C}[s,t]^{\mu_2})$ is the perfect way to make $\mathbb{C}^2/\mu_2$ into an affine variety.

This is actually an example of the toric morphism induced by changing the lattice—see Examples 1.3.17 and 1.3.19. $\Diamond$

Example 5.0.2. Let $\mathbb{C}^*$ act on $\mathbb{C}^4 = \operatorname{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4])$, where $\lambda \in \mathbb{C}^*$ acts via

$$\lambda \cdot (a_1, a_2, a_3, a_4) = (\lambda a_1, \lambda a_2, \lambda^{-1} a_3, \lambda^{-1} a_4).$$

In this case, the ring of invariants is

$$\mathbb{C}[x_1, x_2, x_3, x_4]^{\mathbb{C}^*} = \mathbb{C}[x_1x_3, x_2x_4, x_1x_4, x_2x_3],$$

which gives the map

$$\Phi : \mathbb{C}^4/\mathbb{C}^* \longrightarrow \operatorname{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]^{\mathbb{C}^*}) = \mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$$

where the orbit $\mathbb{C}^* \cdot (a_1, a_2, a_3, a_4)$ maps to $(a_1a_3, a_2a_4, a_1a_4, a_2a_3)$. Then we have (Exercise 5.0.2):

- Φ is surjective.
- If $p \in \mathbf{V}(xy - zw) \setminus \{0\}$, then $\Phi^{-1}(p)$ consists of a single $\mathbb{C}^*$-orbit which is closed in $\mathbb{C}^4$.
- $\Phi^{-1}(0)$ consists of all $\mathbb{C}^*$-orbits contained in $\mathbb{C}^2 \times \{(0,0)\} \cup \{(0,0)\} \times \mathbb{C}^2$. Thus $\Phi^{-1}(0)$ consists of infinitely many $\mathbb{C}^*$-orbits.

This looks bad until we notice one further fact (Exercise 5.0.2):

- The fixed point $0 \in \mathbb{C}^4$ gives the unique closed orbit mapping to 0 under Φ.

If $(a,b) \neq (0,0)$, then an example of a non-closed orbit is given by

$$\mathbb{C}^* \cdot (a, b, 0, 0) = \{(\lambda a, \lambda b, 0, 0) \mid \lambda \in \mathbb{C}^*\}$$

since $\lim_{\lambda \to 0} (\lambda a, \lambda b, 0, 0) = 0$. However, restricting to closed orbits gives

$$\{\text{closed } \mathbb{C}^*\text{-orbits}\} \simeq \mathbf{V}(xy - zw).$$

We will see that this is the best we can do for this group action. ◊

Example 5.0.3. Let $\mathbb{C}^*$ act on $\mathbb{C}^{n+1} = \mathrm{Spec}(\mathbb{C}[x_0,\ldots,x_n])$ by scalar multiplication. Then the ring of invariants consists of polynomials satsifying

$$f(\lambda x_0,\ldots,\lambda x_n) = f(x_0,\ldots,x_n)$$

for all $\lambda \in \mathbb{C}^*$. Such polynomials must be constant, so that

$$\mathbb{C}[x_0,\ldots,x_n]^{\mathbb{C}^*} = \mathbb{C}.$$

It follows that the "quotient" is $\mathrm{Spec}(\mathbb{C})$, which is just a point. The reason for this is that the only closed orbit is the orbit of the fixed point $0 \in \mathbb{C}^{n+1}$. ◊

Examples 5.0.2 and 5.0.3 show what happens when there are not enough invariant functions to separate G-orbits.

The Ring of Invariants. When G acts on an affine variety $X = \mathrm{Spec}(R)$, a natural question concerns the structure of the ring of invariants. The coordinate ring R is a finitely generated $\mathbb{C}$-algebra without nilpotents. Is the same true for R^G? It clearly has no nilpotents since $R^G \subseteq R$. But is R^G finitely generated as a $\mathbb{C}$-algebra? This is related to Hilbert's Fourteenth Problem, which was settled by a famous example of Nagata that R^G *need not* be a finitely generated $\mathbb{C}$-algebra! An exposition of Hilbert's problem and Nagata's example can be found in [**83**, Ch. 4].

If we assume that R^G is finitely generated, then $\mathrm{Spec}(R^G)$ is an affine variety that is the "best" candidate for a quotient in the following sense.

Lemma 5.0.4. *Let G act on $X = \mathrm{Spec}(R)$ such that R^G is a finitely generated $\mathbb{C}$-algebra, and let $\pi : X \to Y = \mathrm{Spec}(R^G)$ be the morphism of affine varieties induced by the inclusion $R^G \subseteq R$. Then:*

(a) *Given any diagram*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & Z \\ & Y \arrow[ur, dotted, "\overline{\phi}"'] & \end{tikzcd}$$

where ϕ is a morphism of affine varieties such that $\phi(g\cdot x) = \phi(x)$ for $g \in G$ and $x \in X$, there is a unique morphism $\overline{\phi}$ making the diagram commute, i.e., $\overline{\phi}\circ\pi = \phi$.

(b) *If X is irreducible, then Y is irreducible.*

(c) *If X is normal, then Y is normal.*

Proof. Suppose that $Z = \mathrm{Spec}(S)$ and that ϕ is induced by $\phi^* : S \to R$. Then $\phi^*(S) \subseteq R^G$ follows easily from $\phi(g\cdot x) = \phi(x)$ for $g \in G, x \in X$. Thus ϕ^* factors uniquely as

$$S \xrightarrow{\overline{\phi}^*} R^G \xrightarrow{\pi^*} R.$$

The induced map $\overline{\phi} : Y \to Z$ clearly has the desired properties.

Part (b) is immediate since R^G is a subring of R. For part (c), let K be the field of fractions of R^G. If $a \in K$ is integral over R^G, then it is also integral over R and hence lies in R since R is normal. It follows that $a \in R \cap K$, which obviously equals R^G since G acts trivially on K. Thus R^G is normal. $\square$

This shows that $Y = \mathrm{Spec}(R^G)$ has some good properties when R^G is finitely generated, but there are still some unanswered questions, such as:

- Is $\pi : X \to Y$ surjective?
- Does Y have the right topology? Ideally, we would like $U \subseteq Y$ to be open if and only if $\pi^{-1}(U) \subseteq X$ is open. (Exercise 5.0.3 explores how this works for group actions on topological spaces.)
- While Y is the best affine approximation of the quotient X/G, could there be a non-affine variety that is a better approximation?

We will see that the answers to these questions are all "yes" once we work with the correct type of group action.

Good Categorical Quotients. In order to get the best properties of a quotient map, we consider the general situation where G is a group acting on a variety X and $\pi : X \to Y$ is a morphism that is constant on G-orbits. Then we have the following definition.

Definition 5.0.5. Let G act on X and let $\pi : X \to Y$ be a morphism that is constant on G-orbits. Then π is a ***good categorical quotient*** if:

(a) If $U \subseteq Y$ is open, then the natural map $\mathscr{O}_Y(U) \to \mathscr{O}_X(\pi^{-1}(U))$ induces an isomorphism
$$\mathscr{O}_Y(U) \simeq \mathscr{O}_X(\pi^{-1}(U))^G.$$

(b) If $W \subseteq X$ is closed and G-invariant, then $\pi(W) \subseteq Y$ is closed.

(c) If W_1, W_2 are closed, disjoint, and G-invariant in X, then $\pi(W_1)$ and $\pi(W_2)$ are disjoint in Y.

We often write a good categorical quotient as $\pi : X \to X/\!/G$. Here are some properties of good categorical quotients.

Theorem 5.0.6. *Let $\pi : X \to X/\!/G$ be a good categorical quotient. Then:*

(a) *Given any diagram*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & Z \\ & X/\!/G \arrow[ur, dotted, "\overline{\phi}"'] & \end{tikzcd}$$

where ϕ is a morphism such that $\phi(g \cdot x) = \phi(x)$ for $g \in G$ and $x \in X$, there is a unique morphism $\overline{\phi}$ making the diagram commute, i.e., $\overline{\phi} \circ \pi = \phi$.

(b) *π is surjective.*

(c) *A subset $U \subseteq X/\!/G$ is open if and only if $\pi^{-1}(U) \subseteq X$ is open.*

(d) *If $U \subseteq X/\!/G$ is open and nonempty, then $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a good categorical quotient.*

(e) *Given points $x, y \in X$, we have*

$$\pi(x) = \pi(y) \iff \overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset.$$

Proof. The proof of part (a) can be found in [**83**, Prop. 6.2]. The proofs of the remaining parts are left to the reader (Exercise 5.0.4). □

Algebraic Actions. So far, we have allowed G to be an arbitrary group acting on X, assuming only that for every $g \in G$, the map $x \mapsto g \cdot x$ is a morphism $\phi_g : X \to X$. We now make the further assumption that G is an affine variety. To define this carefully, we first note that the group $\mathrm{GL}_n(\mathbb{C})$ of $n \times n$ invertible matrices with entries in $\mathbb{C}$ is the affine variety

$$\mathrm{GL}_n(\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} = \mathbb{C}^{n^2} \mid \det(A) \neq 0\}.$$

A subgroup $G \subseteq \mathrm{GL}_n(\mathbb{C})$ is an *affine algebraic group* if it is also a subvariety of $\mathrm{GL}_n(\mathbb{C})$. Examples include $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $(\mathbb{C}^*)^n$, and finite groups.

If G is an affine algebraic group, then the connected component of the identity, denoted G°, has the following properties (see [**152**, 7.3]):

- G° is a normal subgroup of finite index in G.
- G° is an irreducible affine algebraic group.

An affine algebraic group G *acts algebraically* on a variety X if the G-action $(g,x) \mapsto g \cdot x$ defines a morphism

$$G \times X \to X.$$

Examples of algebraic actions include toric varieties since the torus $T_N \subseteq X$ acts algebraically on X. Examples 5.0.1, 5.0.2 and 5.0.3 are also algebraic actions.

Algebraic actions have the property that G-orbits are constructible sets in X. This has the following nice consequence for good categorical quotients.

Proposition 5.0.7. *Let an affine algebraic group G act algebraically on a variety X, and assume that a good categorical quotient $\pi : X \to X/\!/G$ exists. Then:*

(a) *If $p \in X/\!/G$, then $\pi^{-1}(p)$ contains a unique closed G-orbit.*

(b) *π induces a bijection*

$$\{\text{closed } G\text{-orbits in } X\} \simeq X/\!/G.$$

Proof. For part (a), first note that uniqueness follows immediately from part (e) of Theorem 5.0.6. To prove the existence of a closed orbit in $\pi^{-1}(p)$, let $G^\circ \subseteq G$ be the connected component of the identity. Then $\pi^{-1}(p)$ is stable under G°, so we can pick an orbit $G^\circ \cdot x \subset \pi^{-1}(p)$ such that $\overline{G^\circ \cdot x}$ has minimal dimension.

Note that $\overline{G^\circ \cdot x}$ is irreducible since G° is irreducible, and since $G^\circ \cdot x$ is constructible, there is a nonempty Zariski open subset U of $\overline{G^\circ \cdot x}$ such that $U \subseteq G^\circ \cdot x$. If $G^\circ \cdot x$ is not closed, then $\overline{G^\circ \cdot x}$ contains an orbit $G^\circ \cdot y \neq G^\circ \cdot x$. Thus

$$G^\circ \cdot y \subseteq \overline{G^\circ \cdot x} \setminus G^\circ \cdot x \subseteq \overline{G^\circ \cdot x} \setminus U.$$

However, $\overline{G^\circ \cdot x}$ is irreducible, so that

$$\dim\,(\overline{G^\circ \cdot x} \setminus U) < \dim \overline{G^\circ \cdot x}.$$

Hence $\overline{G^\circ \cdot y}$ has strictly smaller dimension, a contradiction. Thus $G^\circ \cdot x$ is closed. If $g_1, \ldots, g_t$ are coset representatives of G/G°, then

$$G \cdot x = \bigcup_{i=1}^{t} g_i\, G^\circ \cdot x$$

shows that $G \cdot x$ is also closed. This proves part (a) of the proposition, and part (b) follows immediately from part (a) and the surjectivity of π. □

For the rest of the section, we will always assume that G is an affine algebraic group acting algebraically on a variety X.

Geometric Quotients. The best quotients are those where points are orbits. For good categorical quotients, this condition is captured by requiring that orbits be closed. Here is the precise result.

Proposition 5.0.8. *Let $\pi : X \to X/\!/G$ be a good categorical quotient. Then the following are equivalent:*

(a) *All G-orbits are closed in X.*

(b) *Given points $x, y \in X$, we have*

$$\pi(x) = \pi(y) \iff x \text{ and } y \text{ lie in the same } G\text{-orbit}.$$

(c) *π induces a bijection*

$$\{G\text{-orbits in } X\} \simeq X/\!/G.$$

(d) *The image of the morphism $G \times X \to X \times X$ defined by $(g, x) \mapsto (g \cdot x, x)$ is the fiber product $X \times_{X/\!/G} X$.*

Proof. This follows easily from Theorem 5.0.6 and Proposition 5.0.7. We leave the details to the reader (Exercise 5.0.5). □

In general, a good categorical quotient is called a *geometric quotient* if it satisfies the condtions of Proposition 5.0.8. We write a geometric quotient as $\pi : X \to X/G$ since points in X/G correspond bijectively to G-orbits in X.

We have yet to give an example of a good categorical or geometric quotient. For instance, it is not clear that Examples 5.0.1, 5.0.2 and 5.0.3 satisfy Definition 5.0.5. Fortunately, once we restrict to the right kind of algebraic group, examples become abundant.

Reductive Groups. An affine algebraic group G is called *reductive* if its maximal connected solvable subgroup is a torus. Examples of reductive groups include finite groups, tori, and semisimple groups such as $\mathrm{SL}_n(\mathbb{C})$.

For us, actions by reductive groups have the following key properties.

Proposition 5.0.9. *Let G be a reductive group acting algebraically on an affine variety $X = \mathrm{Spec}(R)$. Then:*

(a) *R^G is a finitely generated $\mathbb{C}$-algebra.*

(b) *The morphism $\pi : X \to \mathrm{Spec}(R^G)$ induced by $R^G \subseteq R$ is a good categorical quotient.*

Proof. See [**83**, Prop. 3.1] for part (a) and [**83**, Thm. 6.1] for part (b). $\square$

In the situation of Proposition 5.0.9, we can write $\mathrm{Spec}(R)/\!/G = \mathrm{Spec}(R^G)$. Examples 5.0.1, 5.0.2 and 5.0.3 involve reductive groups acting on affine varieties. Hence these are good categorical quotients that have all of the properties listed in Theorem 5.0.6 and Proposition 5.0.7. Furthermore, Example 5.0.1 (the action of μ_2 on $\mathbb{C}^2$) is a geometric quotient. This last example generalizes as follows.

Example 5.0.10. Given a strongly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ and a sublattice $N' \subseteq N$ of finite index, part (b) of Proposition 1.3.18 implies that the finite group $G = N/N'$ acts on $U_{\sigma,N'}$ such that the induced map on coordinate rings is given by

$$\mathbb{C}[\sigma^\vee \cap M] \xrightarrow{\sim} \mathbb{C}[\sigma^\vee \cap M']^G \subseteq \mathbb{C}[\sigma^\vee \cap M'].$$

It follows that $\phi : U_{\sigma,N'} \to U_{\sigma,N}$ is a good categorical quotient. In fact, ϕ is a geometric quotient since the G-orbits are finite and hence closed. This completes the proof of part (c) of Proposition 1.3.18. $\Diamond$

Almost Geometric Quotients. Let us examine Examples 5.0.2 and 5.0.3 more closely. As noted above, both give good categorical quotients. However:

- (Example 5.0.3) Here we have the quotient
$$\mathbb{C}^{n+1}/\!/\mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[x_0,\dots,x_n]^{\mathbb{C}^*}) = \mathrm{Spec}(\mathbb{C}) = \{\mathrm{pt}\}.$$
So the "good" categorical quotient $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}/\!/\mathbb{C}^* = \{\mathrm{pt}\}$ is really bad.
- (Example 5.0.2) In this case, the quotient is
$$\pi : \mathbb{C}^4 \to \mathbb{C}^4/\!/\mathbb{C}^* = \mathbf{V}(xy - zw).$$
Let $U = \mathbf{V}(xy - zw) \setminus \{0\}$ and $U_0 = \pi^{-1}(U)$. Then $\pi|_{U_0} : U_0 \to U$ is a good categorical quotient by Theorem 5.0.6, and by Example 5.0.2, orbits of elements

in U_0 are closed in $\mathbb{C}^4$. Then $\pi|_{U_0}$ is a geometric quotient by Proposition 5.0.8, so that $\mathbb{C}^4/\!/\mathbb{C}^* = \mathbf{V}(xy - zw)$ is a geometric quotient outside of the origin.

The difference between these two examples is that the second is very close to being a geometric quotient. Here is a result that describes this phenomenon in general.

Proposition 5.0.11. *Let $\pi : X \to X/\!/G$ be a good categorical quotient. Then the following are equivalent:*

(a) *X has a G-invariant Zariski dense open subset U_0 such that $G\cdot x$ is closed in X for all $x \in U_0$.*

(b) *$X/\!/G$ has a Zariski dense open subset U such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a geometric quotient.*

Proof. Given U_0 satisfying (a), then $W = X \setminus U_0$ is closed and G-invariant. For $x \in U_0$, the orbit $G\cdot x \subset U_0$ is also closed and G-invariant. These are disjoint, which implies $\pi(x) \notin \pi(W)$ since π is a good categorical quotient. Since π is onto, we see that $X/\!/G = \pi(U_0) \cup \pi(W)$ is a disjoint union. If we set $U = \pi(U_0)$, then $U_0 = \pi^{-1}(U)$. Note also that U is open since $\pi(W)$ is closed and Zariski dense since U_0 is Zariski dense in X. Then $\phi|_{U_0} : U_0 \to U$ is a good categorical quotient by Theorem 5.0.6, and by assumption, orbits of elements in U_0 are closed in $\mathbb{C}^4$ and hence in U_0. It follows that $\phi|_{U_0}$ is a geometric quotient by Proposition 5.0.8.

The proof going the other way is similar and is omitted (Exercise 5.0.6). □

In general, a good categorical quotient is called an *almost geometric quotient* if it satisfies the conditions of Proposition 5.0.11. Example 5.0.2 is an almost geometric quotient while Example 5.0.3 is not.

Constructing Quotients. Now that we can handle affine quotients in the reductive case, the next step is to handle more general quotients. Here is a useful result.

Proposition 5.0.12. *Let G act on X and let $\pi : X \to Y$ be a morphism of varieties that is constant on G-orbits. If Y has an open cover $Y = \bigcup_\alpha V_\alpha$ such that*

$$\pi|_{\pi^{-1}(V_\alpha)} : \pi^{-1}(V_\alpha) \longrightarrow V_\alpha$$

is a good categorical quotient for every α, then $\pi : X \to Y$ is a good categorical quotient.

Proof. The key point is that the properties listed in Definition 5.0.5 can be checked locally. We leave the details to the reader (Exercise 5.0.7). □

Example 5.0.13. Consider a lattice N and a sublattice $N' \subseteq N$ of finite index, and let Σ be a fan in $N'_{\mathbb{R}} = N_{\mathbb{R}}$. This gives a toric morphism

$$\phi : X_{\Sigma,N'} \to X_{\Sigma,N}.$$

By Proposition 1.3.18, the finite group $G = N/N'$ is the kernel of $T_{N'} \to T_N$, so that G acts on $X_{\Sigma,N'}$. Since

$$\phi^{-1}(U_{\sigma,N}) = U_{\sigma,N'}$$

for $\sigma \in \Sigma$, Example 5.0.10 and Proposition 5.0.12 imply that ϕ is a geometric quotient. This strengthens the result proved in Proposition 3.3.7. $\Diamond$

It is sometimes possible to construct the quotient of X by G by taking rings of invariants for a suitable affine open cover. If the local quotients patch together to form a separated variety Y, then the resulting morphism $\pi : X \to Y$ is a good categorical quotient by Proposition 5.0.12. Here are two examples that illustrate this strategy.

Example 5.0.14. Let $\mathbb{C}^*$ act on $\mathbb{C}^2 \setminus \{0\}$ by scalar multiplication, where $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[x_0,x_1])$. Then $\mathbb{C}^2 \setminus \{0\} = U_0 \cup U_1$, where

$$\begin{aligned} U_0 &= \mathbb{C}^2 \setminus \mathbf{V}(x_0) = \mathrm{Spec}(\mathbb{C}[x_0^{\pm 1},x_1]) \\ U_1 &= \mathbb{C}^2 \setminus \mathbf{V}(x_1) = \mathrm{Spec}(\mathbb{C}[x_0,x_1^{\pm 1}]) \\ U_0 \cap U_1 &= \mathbb{C}^2 \setminus \mathbf{V}(x_0x_1) = \mathrm{Spec}(\mathbb{C}[x_0^{\pm 1},x_1^{\pm 1}]). \end{aligned}$$

The rings of invariants are

$$\begin{aligned} \mathbb{C}[x_0^{\pm 1},x_1]^{\mathbb{C}^*} &= \mathbb{C}[x_1/x_0] \\ \mathbb{C}[x_0,x_1^{\pm 1}]^{\mathbb{C}^*} &= \mathbb{C}[x_0/x_1] \\ \mathbb{C}[x_0^{\pm 1},x_1^{\pm 1}]^{\mathbb{C}^*} &= \mathbb{C}[(x_1/x_0)^{\pm 1}]. \end{aligned}$$

It follows that the $V_i = U_i /\!/ \mathbb{C}^*$ glue together in the usual way to create $\mathbb{P}^1$. Since $\mathbb{C}^*$-orbits are closed in $\mathbb{C}^2 \setminus \{0\}$, it follows that

$$\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$$

is a geometric quotient. $\Diamond$

This example generalizes to show that

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$$

is a good geometric quotient when $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ by scalar multiplication. At the beginning of the section, we wrote this quotient as a set-theoretic construction. It is now an algebro-geometric construction.

Our second example shows the importance of being separated.

Example 5.0.15. Let $\mathbb{C}^*$ act on $\mathbb{C}^2 \setminus \{0\}$ by $\lambda(a,b) = (\lambda a, \lambda^{-1}b)$. Then $\mathbb{C}^2 \setminus \{0\} = U_0 \cup U_1$, where U_0, U_1 and $U_0 \cap U_1$ are as in Example 5.0.14. Here, the rings of invariants are

$$\begin{aligned}\mathbb{C}[x_0^{\pm 1}, x_1]^{\mathbb{C}^*} &= \mathbb{C}[x_0x_1]\\ \mathbb{C}[x_0, x_1^{\pm 1}]^{\mathbb{C}^*} &= \mathbb{C}[x_0x_1]\\ \mathbb{C}[x_0^{\pm 1}, x_1^{\pm 1}]^{\mathbb{C}^*} &= \mathbb{C}[(x_0x_1)^{\pm 1}].\end{aligned}$$

Gluing together $V_i = U_i/\!/\mathbb{C}^*$ along $U_0 \cap U_1/\!/\mathbb{C}^*$ gives the variety obtained from two copies of $\mathbb{C}$ by identifying all nonzero points. This is the non-separated variety constructed in Example 3.0.15.

In Exercise 5.0.8 you will draw a picture of the $\mathbb{C}^*$-orbits that explains why the quotient cannot be separated in this example. $\Diamond$

In this book, we usually use the word "variety" to mean "separated variety". For example, when we say that $\pi : X \to Y$ is a good categorical or geometric quotient, we always assume that X and Y are separated. So Example 5.0.15 is *not* a good categorical quotient. In algebraic geometry, most operations on varieties preserve separatedness. Quotient constructions are one of the few exceptions where care has to be taken to check that the resulting variety is separated.

Exercises for §5.0.

5.0.1. Let G act on an affine variety $X = \mathrm{Spec}(R)$ such that $\phi_g(x) = g \cdot x$ is a morphism for all $g \in G$.

(a) Show that $g \cdot f = \phi_{g^{-1}}^*(f)$ defines an action of G on R. Be sure you understand why the inverse is necessary.

(b) The *evaluation map* $R \times X \to \mathbb{C}$ is defined by $(f, x) \mapsto f(x)$. Show that this map is invariant under the action of G on $R \times X$ given by $g \cdot (f, x) = (g \cdot f, g \cdot x)$.

5.0.2. Prove the claims made in Example 5.0.2.

5.0.3. Let G be a group acting on a Hausdorff topological space, and let X/G be the set of G-orbits. Define $\pi : X \to X/G$ by $\pi(x) = G \cdot x$. The *quotient topology* on X/G is defined by saying that $U \subseteq X/G$ is open if and only if $\pi^{-1}(U) \subseteq X$ is open.

(a) Prove that if X/G is Hausdorff, then the G-orbits are closed subsets of X.

(b) Prove that if $W \subseteq X$ is closed and G-invariant, then $\pi(W) \subseteq X/G$ is closed.

(c) Prove that if W_1, W_2 are closed, disjoint, and G-invariant in X, then $\pi(W_1)$ and $\pi(W_2)$ are disjoint in X/G.

5.0.4. Prove parts (b), (c), (d) and (e) of Theorem 5.0.6. Hint for part (b): Part (a) of Definition 5.0.5 implies that $\mathscr{O}_{X/\!/G}(U)$ injects into $\mathscr{O}_X(\pi^{-1}(U))$ for all open sets $U \subseteq X/\!/G$. Use this to prove that $\pi(X)$ is Zariski dense in $X/\!/G$. Then use part (b) of Definition 5.0.5.

5.0.5. Prove Proposition 5.0.8.

5.0.6. Complete the proof of Proposition 5.0.11.

5.0.7. Prove Proposition 5.0.12.

5.0.8. Consider the $\mathbb{C}^*$ action on $\mathbb{C}^2 \setminus \{0\}$ described in Example 5.0.15.

(a) Show that with two exceptions, the $\mathbb{C}^*$-orbits are the hyperbolas $x_1x_2 = a$, $a \neq 0$. Also describe the two remaining $\mathbb{C}^*$-orbits.

(b) Give an intuitive explanation, with picture, to show that the "limit" of the orbits $x_1x_2 = a$ as $a \to 0$ consists of two distinct orbits.

(c) Explain how part (b) relates to the non-separated quotient constructed in the example.

5.0.9. Give an example of a reductive G-action on an affine variety X such that X has a nonempty G-invariant affine open set $U \subseteq X$ with the property that the induced map $U/\!\!/G \to X/\!\!/G$ is not an inclusion.

5.0.10. Let a finite group G act on X. Then a good categorical quotient $\pi : X \to X/\!\!/G$ exists since finite groups are reductive. Explain why π is a geometric quotient.

§5.1. Quotient Constructions of Toric Varieties

Let X_Σ be the toric variety of a fan Σ in $N_\mathbb{R}$. The goal of this section is to construct X_Σ as an almost geometric quotient

$$X_\Sigma \simeq (\mathbb{C}^r \setminus Z)/\!\!/G$$

for an appropriate choice of affine space $\mathbb{C}^r$, exceptional set $Z \subseteq \mathbb{C}^r$, and reductive group G. We will use our standard notation, where each $\rho \in \Sigma(1)$ gives a minimal generator $u_\rho \in \rho \cap N$ and a T_N-invariant prime divisor $D_\rho \subseteq X_\Sigma$.

No Torus Factors. Toric varieties with no torus factors have the nicest quotient constructions. Recall from Proposition 3.3.9 that X_Σ has no torus factors when $N_\mathbb{R}$ is spanned by u_ρ, $\rho \in \Sigma(1)$, and when this happens, Theorem 4.1.3 gives the short exact sequence

$$0 \longrightarrow M \longrightarrow \textstyle\bigoplus_\rho \mathbb{Z}D_\rho \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

where $m \in M$ maps to $\mathrm{div}(\chi^m) = \sum_\rho \langle m, u_\rho \rangle D_\rho$ and $\mathrm{Cl}(X_\Sigma)$ is the class group defined in §4.0. We use the convention that in expressions such as $\bigoplus_\rho$, $\sum_\rho$ and $\prod_\rho$, the index ρ ranges over all $\rho \in \Sigma(1)$.

We write the above sequence more compactly as

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0. \tag{5.1.1}$$

Applying $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ gives

$$1 \longrightarrow \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*) \longrightarrow \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \longrightarrow \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \longrightarrow 1,$$

which remains a short exact sequence since $\mathrm{Hom}_\mathbb{Z}(-, \mathbb{C}^*)$ is left exact and $\mathbb{C}^*$ is divisible. We have natural isomorphisms

$$\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\Sigma(1)}$$
$$\mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \simeq T_N,$$

and we define the group G by

$$G = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*).$$

This gives the short exact sequence of affine algebraic groups

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow T_N \longrightarrow 1. \tag{5.1.2}$$

The Group G. The group G defined above will appear in the quotient construction of the toric variety X_Σ. For the time being, we assume that X_Σ has no torus factors.

The following result describes the structure of G and gives explicit equations for G as a subgroup of the torus $(\mathbb{C}^*)^{\Sigma(1)}$.

Lemma 5.1.1. *Let $G \subseteq (\mathbb{C}^*)^{\Sigma(1)}$ be as in* (5.1.2)*. Then:*

(a) $\mathrm{Cl}(X_\Sigma)$ *is the character group of G.*

(b) *G is isomorphic to a product of a torus and a finite abelian group. In particular, G is reductive.*

(c) *Given a basis $e_1,\dots,e_n$ of M, we have*

$$\begin{aligned} G &= \big\{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \textstyle\prod_\rho t_\rho^{\langle m,u_\rho\rangle} = 1 \text{ for all } m \in M\big\}\\ &= \big\{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \textstyle\prod_\rho t_\rho^{\langle e_i,u_\rho\rangle} = 1 \text{ for } 1 \le i \le n\big\}.\end{aligned}$$

Proof. Since $\mathrm{Cl}(X_\Sigma)$ is a finitely generated abelian group, $\mathrm{Cl}(X_\Sigma) \simeq \mathbb{Z}^\ell \times H$, where H is a finite abelian group. Then

$$G = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*) \simeq \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}^\ell \times H, \mathbb{C}^*) \simeq (\mathbb{C}^*)^\ell \times \mathrm{Hom}_\mathbb{Z}(H, \mathbb{C}^*).$$

This proves part (b) since $\mathrm{Hom}_\mathbb{Z}(H,\mathbb{C}^*)$ is a finite abelian group. For part (a), note that $\alpha \in \mathrm{Cl}(X_\Sigma)$ gives the map that sends $g \in G = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma),\mathbb{C}^*)$ to $g(\alpha) \in \mathbb{C}^*$. Thus elements of $\mathrm{Cl}(X_\Sigma)$ give characters on G, and the above isomorphisms make it easy to see that all characters of G arise this way.

For part (c), the first description of G follows from (5.1.2) since $M \to \mathbb{Z}^{\Sigma(1)}$ is defined by $m \in M \mapsto (\langle m, u_\rho\rangle) \in \mathbb{Z}^{\Sigma(1)}$, and the second description is an easy consequence of the first. □

Example 5.1.2. The ray generators of the fan for $\mathbb{P}^n$ are $u_0 = -\sum_{i=1}^n e_i, u_1 = e_1, \dots, u_n = e_n$. By Lemma 5.1.1, $(t_0,\dots,t_n) \in (\mathbb{C}^*)^{n+1}$ lies in G if and only if

$$t_0^{\langle m, -e_1 - \cdots - e_n\rangle} t_1^{\langle m,e_1\rangle} \cdots t_n^{\langle m,e_n\rangle} = 1$$

for all $m \in M = \mathbb{Z}^n$. Taking m equal to $e_1,\dots,e_n$, we see that G is defined by

$$t_0^{-1}t_1 = \cdots = t_0^{-1}t_n = 1.$$

Thus

$$G = \{(\lambda,\dots,\lambda) \mid \lambda \in \mathbb{C}^*\} \simeq \mathbb{C}^*,$$

which is the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ given by scalar multiplication. ◊

Example 5.1.3. The fan for $\mathbb{P}^1 \times \mathbb{P}^1$ has ray generators $u_1 = e_1, u_2 = -e_1, u_3 = e_2, u_4 = -e_2$ in $N = \mathbb{Z}^2$. By Lemma 5.1.1, $(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^4$ lies in G if and only if

$$t_1^{\langle m, e_1 \rangle} t_2^{\langle m, -e_1 \rangle} t_3^{\langle m, e_2 \rangle} t_4^{\langle m, -e_2 \rangle} = 1$$

for all $m \in M = \mathbb{Z}^2$. Taking m equal to e_1, e_2, we obtain

$$t_1 t_2^{-1} = t_3 t_4^{-1} = 1.$$

Thus

$$G = \{(\mu, \mu, \lambda, \lambda) \mid \mu, \lambda \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^2. \qquad \Diamond$$

Example 5.1.4. Let $\sigma = \mathrm{Cone}(de_1 - e_2, e_2) \subseteq \mathbb{R}^2$, which gives the rational normal cone $\widehat{C}_d$. Example 4.1.4 shows that $\mathrm{Cl}(\widehat{C}_d) \simeq \mathbb{Z}/d\mathbb{Z}$, so that

$$G = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/d\mathbb{Z}, \mathbb{C}^*) \simeq \mu_d,$$

where $\mu_d \subseteq \mathbb{C}^*$ is the group of dth roots of unity. To see how G acts on $\mathbb{C}^2$, one uses the ray generators $u_1 = de_1 - e_2$ and $u_2 = e_2$ to compute that

$$G = \{(\zeta, \zeta) \mid \zeta^d = 1\} \simeq \mu_d$$

(Exercise 5.1.1). This shows that G can have torsion. $\Diamond$

The Exceptional Set. For the quotient representation of X_Σ, we have the group G and the affine space $\mathbb{C}^{\Sigma(1)}$. All that is missing is the exceptional set $Z \subseteq \mathbb{C}^{\Sigma(1)}$ that we remove from $\mathbb{C}^{\Sigma(1)}$ before taking the quotient by G.

One useful observation is that G and $\mathbb{C}^{\Sigma(1)}$ depend only on $\Sigma(1)$. In order to get X_Σ, we need something that encodes the rest of the fan Σ. We will do this using a monomial ideal in the coordinate ring of $\mathbb{C}^{\Sigma(1)}$. Introduce a variable x_ρ for each $\rho \in \Sigma(1)$ and let

$$S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)].$$

Then $\mathrm{Spec}(S) = \mathbb{C}^{\Sigma(1)}$. We call S the *total coordinate ring* of X_Σ.

For each cone $\sigma \in \Sigma$, define the monomial

$$x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho.$$

Thus $x^{\hat{\sigma}}$ is the product of the variables corresponding to rays not in σ. Then define the *irrelevant ideal*

$$B(\Sigma) = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle \subseteq S.$$

A useful observation is that $x^{\hat{\tau}}$ is a multiple of $x^{\hat{\sigma}}$ whenever $\tau \preceq \sigma$. Hence, if $\Sigma_{\max}$ is the set of maximal cones of Σ, then

$$B(\Sigma) = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma_{\max} \rangle.$$

Furthermore, one sees easily that the minimal generators of $B(\Sigma)$ are precisely the $x^{\hat{\sigma}}$ for $\sigma \in \Sigma_{\max}$. Hence, once we have $\Sigma(1)$, $B(\Sigma)$ determines Σ uniquely.

Now define

$$Z(\Sigma) = \mathbf{V}(B(\Sigma)) \subseteq \mathbb{C}^{\Sigma(1)}.$$

The variety of a monomial ideal is a union of coordinate subspaces. For $B(\Sigma)$, the coordinate subspaces can be described in terms of *primitive collections*, which are defined as follows.

Definition 5.1.5. A subset $C \subseteq \Sigma(1)$ is a ***primitive collection*** if:

(a) $C \not\subseteq \sigma(1)$ for all $\sigma \in \Sigma$.

(b) For every proper subset $C' \subsetneq C$, there is $\sigma \in \Sigma$ with $C' \subseteq \sigma(1)$.

Proposition 5.1.6. *The $Z(\Sigma)$ as a union of irreducible components is given by*

$$Z(\Sigma) = \bigcup_C \mathbf{V}(x_\rho \mid \rho \in C),$$

where the union is over all primitive collections $C \subseteq \Sigma(1)$.

Proof. It suffices to determine the maximal coordinate subspaces contained in $Z(\Sigma)$. Suppose that $\mathbf{V}(x_{\rho_1},\dots,x_{\rho_s}) \subseteq Z(\Sigma)$ is such a subspace and take $\sigma \in \Sigma$. Since $x^{\hat{\sigma}}$ vanishes on $Z(\Sigma)$ and $\langle x_{\rho_1},\dots,x_{\rho_s}\rangle$ is prime, the Nullstellensatz implies $x^{\hat{\sigma}}$ is divisible by some x_{ρ_i}, i.e., $\rho_i \notin \sigma(1)$. It follows that $C = \{\rho_1,\dots,\rho_s\}$ satisfies condition (a) of Definition 5.1.5, and condition (b) follows easily from the maximality of $\mathbf{V}(x_{\rho_1},\dots,x_{\rho_s})$. Hence C is a primitive collection.

Conversely, every primitive collection C gives a maximal coordinate subspace $\mathbf{V}(x_\rho \mid \rho \in C)$ contained in $Z(\Sigma)$, and the proposition follows. □

In Exercise 5.1.2 you will show that the algebraic analog of Proposition 5.1.6 is the primary decomposition

$$B(\Sigma) = \bigcap_C \langle x_\rho \mid \rho \in C\rangle.$$

Here are some easy examples.

Example 5.1.7. The fan for $\mathbb{P}^n$ consists of cones generated by proper subsets of $\{u_0,\dots,u_n\}$, where $u_0,\dots,u_n$ are as in Example 5.1.2. Let u_i generate ρ_i, $0 \le i \le n$, and let x_i be the corresponding variable in the total coordinate ring. We compute $Z(\Sigma)$ in two ways:

- The maximal cones of the fan are given by $\sigma_i = \mathrm{Cone}(u_0,\dots,\widehat{u_i},\dots,u_n)$. Then $x^{\hat{\sigma}_i} = x_i$, so that $B(\Sigma) = \langle x_0,\dots,x_n\rangle$. Hence $Z(\Sigma) = \{0\}$.
- The only primitive collection is $\{\rho_0,\dots,\rho_n\}$, so $Z(\Sigma) = \mathbf{V}(x_0,\dots,x_n) = \{0\}$ by Proposition 5.1.6. ◊

Example 5.1.8. The fan for $\mathbb{P}^1 \times \mathbb{P}^1$ has ray generators $u_1 = e_1, u_2 = -e_1, u_3 = e_2, u_4 = -e_2$. See Example 3.1.12 for a picture of this fan. Each u_i gives a ray ρ_i and a variable x_i. We compute $Z(\Sigma)$ in two ways:

- The maximal cone $\mathrm{Cone}(u_1,u_3)$ gives the monomial x_2x_4, and similarly the other maximal cones give the monomials x_1x_4, x_1x_3, x_2x_3. Thus
$$B(\Sigma) = \langle x_2x_4, x_1x_4, x_1x_3, x_2x_3\rangle,$$
and one checks that $Z(\Sigma) = \{0\}\times\mathbb{C}^2 \cup \mathbb{C}^2\times\{0\}$.
- The only primitive collections are $\{\rho_1,\rho_2\}$ and $\{\rho_3,\rho_4\}$, so that
$$Z(\Sigma) = \mathbf{V}(x_1,x_2)\cup\mathbf{V}(x_3,x_4) = \{0\}\times\mathbb{C}^2\cup\mathbb{C}^2\times\{0\}$$
by Proposition 5.1.6. Note also that $B(\Sigma) = \langle x_1,x_2\rangle\cap\langle x_3,x_4\rangle$. $\Diamond$

A final observation is that $(\mathbb{C}^*)^{\Sigma(1)}$ acts on $\mathbb{C}^{\Sigma(1)}$ by diagonal matrices and hence induces an action on $\mathbb{C}^{\Sigma(1)}\setminus Z(\Sigma)$. It follows that $G\subseteq(\mathbb{C}^*)^{\Sigma(1)}$ also acts on $\mathbb{C}^{\Sigma(1)}\setminus Z(\Sigma)$. We are now ready to take the quotient.

The Quotient Construction. To represent X_Σ as a quotient, we first construct a toric morphism $\mathbb{C}^{\Sigma(1)}\setminus Z(\Sigma)\to X_\Sigma$. Let $\{e_\rho \mid \rho\in\Sigma(1)\}$ be the standard basis of the lattice $\mathbb{Z}^{\Sigma(1)}$. For each $\sigma\in\Sigma$, define the cone
$$\tilde{\sigma} = \mathrm{Cone}(e_\rho \mid \rho\in\sigma(1)) \subseteq \mathbb{R}^{\Sigma(1)}.$$
It is easy to see that these cones and their faces form a fan
$$\widetilde{\Sigma} = \{\tau \mid \tau\preceq\tilde{\sigma} \text{ for some } \sigma\in\Sigma\}$$
in $(\mathbb{Z}^{\Sigma(1)})_{\mathbb{R}} = \mathbb{R}^{\Sigma(1)}$. This fan has the following nice properties.

Proposition 5.1.9. *Let $\widetilde{\Sigma}$ be the fan defined above. Then:*

(a) $\mathbb{C}^{\Sigma(1)}\setminus Z(\Sigma)$ *is the toric variety of the fan* $\widetilde{\Sigma}$.

(b) *The map $e_\rho\mapsto u_\rho$ defines a map of lattices $\mathbb{Z}^{\Sigma(1)}\to N$ that is compatible with the fans $\widetilde{\Sigma}$ in $\mathbb{R}^{\Sigma(1)}$ and Σ in $N_{\mathbb{R}}$.*

(c) *The resulting toric morphism*
$$\pi : \mathbb{C}^{\Sigma(1)}\setminus Z(\Sigma)\longrightarrow X_\Sigma$$
is constant on G-orbits.

Proof. For part (a), let $\widetilde{\Sigma}_0$ be the fan consisting of $\mathrm{Cone}(e_\rho \mid \rho\in\Sigma(1))$ and its faces. Note that $\widetilde{\Sigma}$ is a subfan of $\widetilde{\Sigma}_0$. Since $\widetilde{\Sigma}_0$ is the fan of $\mathbb{C}^{\Sigma(1)}$, we get the toric variety of $\widetilde{\Sigma}$ by taking $\mathbb{C}^{\Sigma(1)}$ and then removing the orbits corresponding to all cones in $\widetilde{\Sigma}_0\setminus\widetilde{\Sigma}$. By the Orbit-Cone Correspondence (Theorem 3.2.6), this is equivalent to removing the orbit closures of the minimal elements of $\widetilde{\Sigma}_0\setminus\widetilde{\Sigma}$. But these minimal elements are precisely the primitive collections $C\subseteq\Sigma(1)$. Since the corresponding orbit closure is $\mathbf{V}(x_\rho \mid \rho\in C)$, removing these orbit closures means removing
$$Z(\Sigma) = \bigcup_C \mathbf{V}(x_\rho \mid \rho\in C).$$

For part (b), define $\overline{\pi} : \mathbb{Z}^{\Sigma(1)} \to N$ by $\overline{\pi}(e_\rho) = u_\rho$. Since the minimal generators of $\sigma \in \Sigma$ are given by u_ρ, $\rho \in \sigma(1)$, we have $\overline{\pi}_{\mathbb{R}}(\widetilde{\sigma}) = \sigma$ by the definition of $\widetilde{\sigma}$. Hence $\overline{\pi}$ is compatible with respect to the fans $\widetilde{\Sigma}$ and Σ.

The map of tori induced by $\overline{\pi}$ is the map $(\mathbb{C}^*)^{\Sigma(1)} \to T_N$ from the exact sequence (5.1.2) (you will check this in Exercise 5.1.3). Hence, if $g \in G \subseteq (\mathbb{C}^*)^{\Sigma(1)}$ and $x \in \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$, then

$$\pi(g \cdot x) = \pi(g) \cdot \pi(x) = \pi(x),$$

where the first equality holds by equivariance and the second holds since G is the kernel of $(\mathbb{C}^*)^{\Sigma(1)} \to T_N$. This proves part (c) of the proposition. $\square$

In Exercise 5.1.4 you will prove the following lemma, which will be used in the proof of the quotient construction.

Lemma 5.1.10. *Assume that V is an affine toric variety, not necessarily normal, with torus T. Given a point $\overline{p} \in V$, there is a point $q \in T$ and a one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ such that $\overline{p} = \lim_{t\to 0} \lambda(t)q$.* $\square$

We can now give the quotient construction of X_Σ.

Theorem 5.1.11. *Let X_Σ be a toric variety without torus factors and consider the toric morphism $\pi : \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \to X_\Sigma$ from Proposition 5.1.9. Then:*

(a) *π is an almost geometric quotient for the action of G on $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$. Thus*

$$X_\Sigma \simeq \big(\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)\big)/\!\!/G.$$

(b) *π is a geometric quotient if and only if Σ is simplicial.*

Proof. We begin by studying the map

$$\pi|_{\pi^{-1}(U_\sigma)} : \pi^{-1}(U_\sigma) \longrightarrow U_\sigma \tag{5.1.3}$$

for $\sigma \in \Sigma$. First observe that if $\tau, \sigma \in \Sigma$, then $\overline{\pi}_{\mathbb{R}}(\widetilde{\tau}) \subseteq \sigma$ is equivalent to $\tau \preceq \sigma$. It follows that $\pi^{-1}(U_\sigma)$ is the toric variety $U_{\widetilde{\sigma}}$ of $\widetilde{\sigma} = \mathrm{Cone}(e_\rho \mid \rho \in \sigma(1))$. This shows that (5.1.3) is the toric morphism

$$\pi_\sigma : U_{\widetilde{\sigma}} \longrightarrow U_\sigma,$$

where for simplicity we write π_σ instead of $\pi|_{\pi^{-1}(U_\sigma)}$.

Our first task is to show that π_σ is a good categorical quotient. Since G is reductive, Proposition 5.0.9 reduces this to showing that the map π_σ^* on coordinate rings induces an isomorphism

$$\mathbb{C}[U_\sigma] \simeq \mathbb{C}[U_{\widetilde{\sigma}}]^G. \tag{5.1.4}$$

The map π_σ^* can be described as follows:

- For $U_{\widetilde{\sigma}}$, the cone $\widetilde{\sigma}$ gives the semigroup
$$\widetilde{\sigma}^\vee \cap \mathbb{Z}^{\Sigma(1)} = \{(a_\rho) \in \mathbb{Z}^{\Sigma(1)} \mid a_\rho \geq 0 \text{ for all } \rho \in \sigma(1)\}.$$
Hence the coordinate ring of $U_{\widetilde{\sigma}}$ is the semigroup algebra
$$\mathbb{C}[U_{\widetilde{\sigma}}] = \mathbb{C}\big[\textstyle\prod_\rho x_\rho^{a_\rho} \mid a_\rho \geq 0 \text{ for all } \rho \in \sigma(1)\big] = S_{x^{\hat{\sigma}}},$$
where $S_{x^{\hat{\sigma}}}$ is the localization $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ at $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$.
- For U_σ, the coordinate ring is the usual semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$.
- The map $\overline{\pi} : \mathbb{Z}^{\Sigma(1)} \to N$ dualizes to the map $M \to \mathbb{Z}^{\Sigma(1)}$ sending $m \in M$ to $(\langle m, u_\rho \rangle) \in \mathbb{Z}^{\Sigma(1)}$. It follows that $\pi_\sigma^* : \mathbb{C}[\sigma^\vee \cap M] \to S_{x^{\hat{\sigma}}}$ is given by
$$\pi_\sigma^*(\chi^m) = \textstyle\prod_\rho x_\rho^{\langle m, u_\rho \rangle}.$$
Note that $\langle m, u_\rho \rangle \geq 0$ for all $\rho \in \sigma(1)$, so that the expression on the right really lies in $S_{x^{\hat{\sigma}}}$.

Thus π_σ^* can be written $\pi_\sigma^* : \mathbb{C}[\sigma^\vee \cap M] \to S_{x^{\hat{\sigma}}}$, and since π_σ is constant on G-orbits, π_σ^* factors as
$$\mathbb{C}[\sigma^\vee \cap M] \longrightarrow \big(S_{x^{\hat{\sigma}}}\big)^G \subseteq S_{x^{\hat{\sigma}}}.$$

The map π_σ has Zariski dense image in U_σ since $\pi_\sigma((\mathbb{C}^*)^{\Sigma(1)}) = T_N$ by the exact sequence (5.1.2). It follows that π_σ^* is injective. To show that its image is $(S_{x^{\hat{\sigma}}})^G$, take $f \in S_{x^{\hat{\sigma}}}$ and write it as
$$f = \sum_a c_a x^a$$
where each $x^a = \prod_\rho x_\rho^{a_\rho}$ satisfies $a_\rho \geq 0$ for all $\rho \in \sigma(1)$. Then f is G-invariant if and only if for all $t = (t_\rho) \in G$, we have
$$\sum_a c_a x^a = \sum_a c_a t^a x^a.$$
Thus f is G-invariant if and only if $t^a = 1$ for all $t \in G$ whenever $c_a \neq 0$. The map $t \mapsto t^a$ is a character on G and hence is an element of its character group $\mathrm{Cl}(X_\Sigma)$ (Lemma 5.1.1). This character is trivial when $c_a \neq 0$, so that by (5.1.1), the exponent vector $a = (a_\rho)$ must come from an element $m \in M$, i.e., $a_\rho = \langle m, u_\rho \rangle$ for all $\rho \in \Sigma(1)$. But $x^a \in S_{x^{\hat{\sigma}}}$, which implies that
$$\langle m, u_\rho \rangle = a_\rho \geq 0 \quad \text{for all } \rho \in \sigma(1).$$
Hence $m \in \sigma^\vee \cap M$, which implies that f is in the image of π_σ^*. This proves (5.1.4). We conclude that π_σ is a good categorical quotient.

We next follow ideas from [**83**, Prop. 12.1] to prove that

(5.1.5) $$\pi_\sigma : U_{\tilde{\sigma}} \to U_\sigma \text{ is a geometric quotient } \iff \sigma \text{ is simplicial.}$$

First suppose that σ is simplicial. By Proposition 5.0.8, it suffices to show that G-orbits are closed in $U_{\tilde{\sigma}}$. Let $G^\circ \subseteq G$ be the connected component of the identity. Since G° has finite index in G, it suffices to show that G°-orbits are closed in $U_{\tilde{\sigma}}$.

Take $p \in U_{\tilde{\sigma}}$ and $\overline{p} \in \overline{G^\circ \cdot p}$, where the closure is taken in $U_{\tilde{\sigma}}$. Note that $\overline{G^\circ \cdot p}$ is an affine toric variety, possibly nonnormal, with torus $T \simeq G^\circ/G^\circ_p$. By Lemma 5.1.10, there are $\lambda' : \mathbb{C}^* \to T$ and $q' \in T$ such that $\overline{p} = \lim_{t\to 0} \lambda'(t)q' \cdot p$. Lifting these to G° gives a one-parameter subgroup $\lambda : \mathbb{C}^* \to G^\circ$ and a point $q \in G^\circ$ such that

$$\overline{p} = \lim_{t\to 0} \lambda(t)q \cdot p. \tag{5.1.6}$$

Using $G^\circ \subseteq (\mathbb{C}^*)^{\Sigma(1)}$, we can write $\lambda(t) = (t^{a_\rho})$ for exponents $a_\rho \in \mathbb{Z}$. Since λ is a one-parameter subgroup of G, we have

$$\textstyle\sum_\rho a_\rho u_\rho = 0. \tag{5.1.7}$$

This follows easily from Lemma 5.1.1 (Exercise 5.1.5). Write $p = (p_\rho)$, $\overline{p} = (\overline{p}_\rho)$, and $q = (q_\rho)$. Then (5.1.6) implies

$$\overline{p}_\rho = \lim_{t\to 0} t^{a_\rho} q_\rho \cdot p_\rho.$$

Since $p, \overline{p} \in U_{\tilde{\sigma}}$ and $q \in G^\circ$, their ρth coordinates are nonzero for $\rho \notin \sigma(1)$. Then the above limit implies $a_\rho = 0$ for these ρ's, so that (5.1.7) becomes

$$\textstyle\sum_{\rho\in\sigma(1)} a_\rho u_\rho = 0.$$

But σ is simplicial, which means that the u_ρ, $\rho \in \sigma(1)$, are linearly independent. Hence $a_\rho = 0$ for all ρ, so that λ is the trivial one-parameter subgroup. Then (5.1.6) implies $\overline{p} = q \cdot p \in G^\circ \cdot p$. We conclude that $G^\circ \cdot p$ is closed in $U_{\tilde{\sigma}}$.

For the other implication of (5.1.5), suppose that $\sigma \in \Sigma$ is non-simplicial. Then there is a relation $\sum_{\rho\in\sigma(1)} a_\rho u_\rho = 0$ where $a_\rho \in \mathbb{Z}$ and $a_\rho > 0$ for at least one ρ. If we set $a_\rho = 0$ for $\rho \notin \sigma(1)$, then the one-parameter subgroup

$$\lambda(t) = (t^{a_\rho}) \in (\mathbb{C}^*)^{\Sigma(1)}$$

is a one-parameter subgroup of G by Exercise 5.1.5. Define $p = (p_\rho) \in U_{\tilde{\sigma}}$ by

$$p_\rho = \begin{cases} 1 & a_\rho \geq 0 \\ 0 & a_\rho < 0 \end{cases}$$

and consider $\lim_{t\to 0} \lambda(t) \cdot p$. The limit exists in $\mathbb{C}^{\Sigma(1)}$ since $p_\rho = 0$ for $a_\rho < 0$. Furthermore, if $\rho \notin \sigma(1)$, the ρth coordinate of $\lambda(t) \cdot p$ is 1 for all t, so that the limit $\overline{p} = \lim_{t\to 0} \lambda(t) \cdot p$ lies in $U_{\tilde{\sigma}}$. Since there is $\rho_0 \in \sigma(1)$ with $a_{\rho_0} > 0$, we have:

- Since the ρ_0th coordinate of p is nonzero, the same is true for all of $G \cdot p$.
- Since $a_{\rho_0} > 0$, the ρ_0th coordinate of $\overline{p} = \lim_{t\to 0} \lambda(t) \cdot p$ is zero.

Then $G \cdot p$ is not closed in $U_{\tilde{\sigma}}$ since its Zariski closure contains $\overline{p} \in U_{\tilde{\sigma}} \setminus G \cdot p$. This shows that π_σ is not a geometric quotient and completes the proof of (5.1.5).

We can now prove the theorem. Since the maps (5.1.3) are good categorical quotients, the same is true for $\pi : \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \to X_\Sigma$ by Proposition 5.0.12. To prove part (a), let $\Sigma' \subseteq \Sigma$ be the subfan of simplicial cones of Σ. Then $X_{\Sigma'}$ is open in X_Σ, and since $\Sigma'(1) = \Sigma(1)$, $X_{\Sigma'}$ and X_Σ have the same total coordinate ring S and same group G. In Exercise 5.1.5, you will show that

$$\pi^{-1}(X_{\Sigma'}) = \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma') = \bigcup_{\sigma \in \Sigma'} U_{\widetilde{\sigma}}. \tag{5.1.8}$$

As above, $\pi|_{\pi^{-1}(X_{\Sigma'})} : \pi^{-1}(X_{\Sigma'}) \to X_{\Sigma'}$ is a good categorical quotient, and π_σ is a geometric quotient for each $\sigma \in \Sigma'$ by (5.1.5). It follows easily that $\pi|_{\pi^{-1}(X_{\Sigma'})}$ is a geometric quotient, and then Proposition 5.0.11 implies that π is an almost geometric quotient. This argument also implies that π is a geometric quotient when Σ is simplicial, which proves half of part (b). The other half of part (b) follows from (5.1.5). The proof of the theorem is now complete. $\square$

One nice feature of the quotient $X_\Sigma = (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))/\!\!/G$ is that it is compatible with the tori, meaning that we have a commutative diagram

$$\begin{array}{ccc} X_\Sigma & \simeq & (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))/\!\!/G \\ \uparrow & & \uparrow \\ T_N & \simeq & (\mathbb{C}^*)^{\Sigma(1)}/G \end{array}$$

where the isomorphism on the bottom comes from (5.1.2) and the vertical arrows are inclusions.

Examples. Here are some examples of the quotient construction.

Example 5.1.12. By Examples 5.1.2 and 5.1.7, $\mathbb{P}^n$ has quotient representation

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts by scalar multiplication. This is a geometric quotient since Σ is smooth and hence simplicial. $\Diamond$

Example 5.1.13. By Examples 5.1.3 and 5.1.8, $\mathbb{P}^1 \times \mathbb{P}^1$ has quotient representation

$$\mathbb{P}^1 \times \mathbb{P}^1 = \big(\mathbb{C}^4 \setminus (\{0\} \times \mathbb{C}^2 \cup \mathbb{C}^2 \times \{0\})\big)/(\mathbb{C}^*)^2,$$

where $(\mathbb{C}^*)^2$ acts via $(\mu,\lambda)\cdot(a,b,c,d) = (\mu a, \mu b, \lambda c, \lambda d)$. This is again a geometric quotient. $\Diamond$

Example 5.1.14. Fix positive integers $q_0,\dots,q_n$ with $\gcd(q_0,\dots,q_n) = 1$ and let N be the lattice $\mathbb{Z}^{n+1}/\mathbb{Z}(q_0,\dots,q_n)$. The images of the standard basis in $\mathbb{Z}^{n+1}$ give primitive elements $u_0,\dots,u_n \in N$ satisfying $q_0u_0 + \cdots + q_nu_n = 0$. Let Σ be the fan consisting of all cones generated by proper subsets of $\{u_0,\dots,u_n\}$.

As in Example 3.1.17, the corresponding toric variety is denoted $\mathbb{P}(q_0,\dots,q_n)$. Using the quotient construction, we can now explain why this is called a weighted projective space.

We have $\mathbb{C}^{\Sigma(1)} = \mathbb{C}^{n+1}$ since Σ has $n+1$ rays, and $Z(\Sigma) = \{0\}$ by the argument used in Example 5.1.7. It remains to compute $G \subseteq (\mathbb{C}^*)^{n+1}$. In Exercise 4.1.5, you showed that the maps $m \in M \mapsto (\langle m, u_0\rangle, \dots, \langle m, u_n\rangle) \in \mathbb{Z}^{n+1}$ and $(a_0, \dots, a_n) \in \mathbb{Z}^{n+1} \mapsto a_0q_0 + \cdots + a_nq_n \in \mathbb{Z}$ give the short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{5.1.9}$$

This shows that the class group is $\mathbb{Z}$. Note also that $e_i \in \mathbb{Z}^{n+1}$ maps to $q_i \in \mathbb{Z}$. In Exercise 5.1.6 you will compute that

$$G = \{(t^{q_0}, \dots, t^{q_n}) \mid t \in \mathbb{C}^*\} \simeq \mathbb{C}^*.$$

This is the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ given by

$$t \cdot (u_0, \dots, u_n) = (t^{q_0} u_0, \dots, t^{q_n} u_n).$$

Since Σ is simplicial (every proper subset of $\{u_0, \dots, u_n\}$ is linearly independent in $N_\mathbb{R}$), we get the geometric quotient

$$\mathbb{P}(q_0, \dots, q_n) = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*.$$

This gives the set-theoretic definition of $\mathbb{P}(q_0, \dots, q_n)$ from §2.0 and also gives its structure as a variety since we have a geometric quotient. $\Diamond$

Example 5.1.15. Consider the cone $\sigma = \mathrm{Cone}(e_1, e_2, e_1+e_3, e_2+e_3) \subseteq \mathbb{R}^3$. To find the quotient representation of U_σ, we label the ray generators as

$$u_1 = e_1,\ u_2 = e_2 + e_3,\ u_3 = e_2,\ u_4 = e_1 + e_3.$$

Then $\mathbb{C}^{\Sigma(1)} = \mathbb{C}^4$ and $Z(\Sigma) = \emptyset$ since $x^{\hat{\sigma}} = 1$. To determine the group $G \subseteq (\mathbb{C}^*)^4$, note that the exact sequence (5.1.1) becomes

$$0 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 \mapsto a_1 + a_2 - a_3 - a_4 \in \mathbb{Z}$. This makes it straightforward to show that

$$G = \{(\lambda, \lambda, \lambda^{-1}, \lambda^{-1}) \mid \lambda \in \mathbb{C}^*\} \simeq \mathbb{C}^*.$$

Hence we get the quotient presentation

$$U_\sigma = \mathbb{C}^4/\!/\mathbb{C}^*.$$

In Example 5.0.2, we gave a naive argument that the quotient was $\mathbf{V}(xy - zw)$. We now see that the intrinsic meaning of Example 5.0.2 is the quotient construction of U_σ given by Theorem 5.1.11. This example is not a geometric quotient since σ is not simplicial. $\Diamond$

Example 5.1.16. Let $\mathrm{Bl}_0(\mathbb{C}^2)$ be the blowup of $\mathbb{C}^2$ at the origin, whose fan Σ is shown in Figure 1 on the next page. By Example 4.1.5, $\mathrm{Cl}(\mathrm{Bl}_0(\mathbb{C}^2)) \simeq \mathbb{Z}$ with generator $[D_1] = [D_2] = -[D_0]$. Hence $G = \mathbb{C}^*$ and the irrelevant ideal is $B(\Sigma) = \langle x, y\rangle$. This gives the geometric quotient

$$\mathrm{Bl}_0(\mathbb{C}^2) \simeq \big(\mathbb{C}^3 \setminus (\mathbb{C} \times \{0,0\})\big)/\mathbb{C}^*,$$

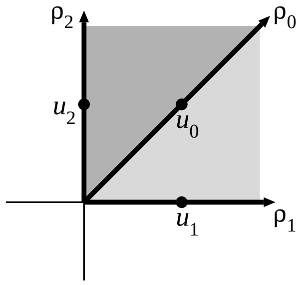


Figure 1. The fan Σ for the blowup of $\mathbb{C}^2$ at the origin

where the $\mathbb{C}^*$-action is given by $\lambda \cdot (t,x,y) = (\lambda^{-1}t, \lambda x, \lambda y)$.

We also have $\mathbb{C}[t,x,y]^{\mathbb{C}^*} = \mathbb{C}[tx,ty]$. Then the inclusion

$$\mathbb{C}^3 \setminus (\mathbb{C} \times \{0,0\}) \subseteq \mathbb{C}^3$$

induces the map on quotients

$$\phi : \mathrm{Bl}_0(\mathbb{C}^2) \simeq \big(\mathbb{C}^3 \setminus (\mathbb{C} \times \{0,0\})\big)/\mathbb{C}^* \longrightarrow \mathbb{C}^3/\!/\mathbb{C}^* \simeq \mathbb{C}^2,$$

where the final isomorphism uses

$$\mathbb{C}^3/\!/\mathbb{C}^* = \mathrm{Spec}(\mathbb{C}[t,x,y]^{\mathbb{C}^*}) = \mathrm{Spec}(\mathbb{C}[tx,ty]).$$

In terms of homogeneous coordinates, $\phi(t,x,y) = (tx,ty)$. This map is the toric morphism $\mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2$ induced by the refinement of $\mathrm{Cone}(u_1,u_2)$ given by Σ.

The quotient representation makes it easy to see why $\mathrm{Bl}_0(\mathbb{C}^2)$ is the blowup of $\mathbb{C}^2$ at the origin. Given a point of $\mathrm{Bl}_0(\mathbb{C}^2)$ with homogeneous coordinates (t,x,y), there are two possibilities:

- $t \neq 0$, in which case $t \cdot (t,x,y) = (1,tx,ty)$. This maps to $(tx,ty) \in \mathbb{C}^2$ and is nonzero since x,y cannot both be zero. It follows that the part of $\mathrm{Bl}_0(\mathbb{C}^2)$ where $t \neq 0$ looks like $\mathbb{C}^2 \setminus \{0,0\}$.
- $t = 0$, in which case $(0,x,y)$ maps to the origin in $\mathbb{C}^2$. Since $\lambda \cdot (0,x,y) = (\lambda x, \lambda y)$ and x,y cannot both be zero, it follows that the part of $\mathrm{Bl}_0(\mathbb{C}^2)$ where $t = 0$ looks like $\mathbb{P}^1$.

This shows that $\mathrm{Bl}_0(\mathbb{C}^2)$ is a built from $\mathbb{C}^2$ by replacing the origin with a copy of $\mathbb{P}^1$, which is called the *exceptional locus* E. Since $E = \phi^{-1}(0,0)$, we see that $\phi : X_\Sigma \to \mathbb{C}^2$ induces an isomorphism

$$\mathrm{Bl}_0(\mathbb{C}^2) \setminus E \simeq \mathbb{C}^2 \setminus \{(0,0)\}.$$

Note also that E is the divisor D_0 corresponding to the ray ρ_0. You should be able to look at Figure 1 and see instantly that $D_0 \simeq \mathbb{P}^1$.

We can also check that lines through the origin behave properly. Consider the line L defined by $ax + by = 0$, where $(a,b) \neq (0,0)$. When we pull this back to

$\mathrm{Bl}_0(\mathbb{C}^2)$, we get the subvariety defined by

$$a(tx)+b(ty)=0.$$

This is the *total transform* of L. It factors as $t(ax+by)=0$. Note that $t=0$ defines the exceptional locus, so that once we remove this, we get the curve in $\mathrm{Bl}_0(\mathbb{C}^2)$ defined by $ax+by=0$. This is the *proper transform* of L, which meets the exceptional locus E at the point with homogeneous coordinates $(0,-b,a)$, corresponding to $(-b,a)\in\mathbb{P}^1$. In this way, we see how blowing up separates tangent directions through the origin. $\Diamond$

The General Case. So far, we have assumed that X_Σ has no torus factors. When torus factors are present, X_Σ still has a quotient construction, though it is no longer canonical.

Let X_Σ be a toric variety with a torus factor. By Proposition 3.3.9, the ray generators u_ρ, $\rho\in\Sigma(1)$, span a proper subspace of $N_\mathbb{R}$. Let N' be the intersection of this subspace with N, and pick a complement N'' so that $N=N'\oplus N''$. The cones of Σ all lie in $N'_\mathbb{R}$ and hence give a fan Σ' in $N'_\mathbb{R}$. As in the proof of Proposition 3.3.9. we obtain

$$X_\Sigma \simeq X_{\Sigma',N'}\times(\mathbb{C}^*)^r$$

where $N''\simeq\mathbb{Z}^r$. Theorem 5.1.11 applies to $X_{\Sigma',N'}$ since u_ρ, $\rho\in\Sigma'(1)=\Sigma(1)$, span $N'_\mathbb{R}$ by construction. Note also that $B(\Sigma')=B(\Sigma)$ and $Z(\Sigma')=Z(\Sigma)$. Hence

$$X_{\Sigma',N'}\simeq(\mathbb{C}^{\Sigma(1)}\setminus Z(\Sigma))/\!/G.$$

It follows that

$$\begin{aligned} X_\Sigma &\simeq X_{\Sigma',N'}\times(\mathbb{C}^*)^r\\ &\simeq \big((\mathbb{C}^{\Sigma(1)}\setminus Z(\Sigma))/\!/G\big)\times(\mathbb{C}^*)^r\\ &\simeq \big((\mathbb{C}^{\Sigma(1)}\times(\mathbb{C}^*)^r)\setminus(Z(\Sigma)\times(\mathbb{C}^*)^r)\big)/\!/G, \end{aligned}\tag{5.1.10}$$

In the last line, we use the trivial action of G on $(\mathbb{C}^*)^r$. You will verify the last isomorphism in Exercise 5.1.7.

We can rewrite (5.1.10) as follows. Using $(\mathbb{C}^*)^r=\mathbb{C}^r\setminus\mathbf{V}(x_1\cdots x_r)$, we obtain

$$(\mathbb{C}^{\Sigma(1)}\times(\mathbb{C}^*)^r)\setminus(Z(\Sigma)\times(\mathbb{C}^*)^r)=\mathbb{C}^{\Sigma(1)+r}\setminus Z'(\Sigma),$$

where $\mathbb{C}^{\Sigma(1)+r}=\mathbb{C}^{\Sigma(1)}\times\mathbb{C}^r$ and $Z'(\Sigma)=(Z(\Sigma)\times\mathbb{C}^r)\cup(\mathbb{C}^{\Sigma(1)}\times\mathbf{V}(x_1\cdots x_r))$. Hence the quotient presentation of X_Σ is the almost geometric quotient

$$X_\Sigma\simeq\big(\mathbb{C}^{\Sigma(1)+r}\setminus Z'(\Sigma)\big)/\!/G.\tag{5.1.11}$$

This differs from Theorem 5.1.11 in two ways:

- The representation (5.1.11) is non-canonical since it depends on the choice of the complement N''.

- $Z'(\Sigma)$ contains $\mathbf{V}(x_1\cdots x_r)\times\mathbb{C}^{\Sigma(1)}$ and hence has codimension 1 in $\mathbb{C}^{\Sigma(1)+r}$. In constrast, $Z(\Sigma)$ always has codimension ≥ 2 in $\mathbb{C}^{\Sigma(1)}$ (this follows from Proposition 5.1.6 since every primitive collection has at least two elements).

In practice, (5.1.11) is rarely used, while Theorem 5.1.11 is a common tool in toric geometry.

Exercises for §5.1.

5.1.1. In Example 5.1.4, verify carefully that $G=\{(\zeta,\zeta)\mid\zeta\in\mu_d\}$.

5.1.2. Prove that $B(\Sigma)=\bigcap_C\langle x_\rho\mid\rho\in C\rangle$, where the intersection ranges over all primitive collections $C\subseteq\Sigma(1)$.

5.1.3. In Proposition 5.1.9, we defined $\overline{\pi}:\mathbb{Z}^{\Sigma(1)}\to N$, and in the proof we use the map of tori $(\mathbb{C}^*)^{\Sigma(1)}\to T_N$ induced by $\overline{\pi}$. Show that this is the map appearing in (5.1.2).

5.1.4. This exercise will prove Lemma 5.1.10. In parts (a) and (b), we consider a normal affine toric variety U_σ and a point $\overline{p}\in U_\sigma$. By Theorem 3.2.6, there is a face $\tau\preceq\sigma$ such that $p\in O(\tau)\subseteq U_\tau\subseteq U_\sigma$. Also take $u\in\mathrm{Relint}(\tau)\cap N$.

(a) Prove $\lim_{t\to 0}\lambda^u(t)=\gamma_\tau$, where $\gamma_\tau\in O(\tau)$ is the distinguished point defined in §3.2. Hint: Proposition 3.2.2.

(b) Find $q\in T_N$ such that $\lim_{t\to 0}\lambda^u(t)q=\overline{p}$. Hint: T_N acts transitively on $O(\tau)$.

(c) Prove Lemma 5.1.10. Hint: Let $U_\sigma\to V$ be the normalization map. Then apply Theorem 3.A.3.

5.1.5. This exercise is concerned with the proof of Theorem 5.1.11.

(a) Prove that a one-parameter subgroup $\lambda(t)=(t^{a_\rho})\in(\mathbb{C}^*)^{\Sigma(1)}$ takes values in G if and only if $\sum_\rho a_\rho u_\rho=0$. Hint: Use Lemma 5.1.1. You can give a more conceptual proof by taking the dual of (5.1.1).

(b) Prove (5.1.8) and conclude that the quotient construction of $X_{\Sigma'}$ is the map $\pi|_{\pi^{-1}(X_{\Sigma'})}:\pi^{-1}(X_{\Sigma'})\to X_{\Sigma'}$ used in the proof of Theorem 5.1.11.

5.1.6. Show that the group G in Example 5.1.14 is given by $G=\{(t^{q_0},\dots,t^{q_n})\mid t\in\mathbb{C}^*\}$. Hint: Pick integers b_i such that $\sum_{i=0}^n b_iq_i=1$. Given $(t_0,\dots,t_n)\in G$, set $t=\prod_{i=0}^n t_i^{a_i}$. Also note that if $e_0,\dots,e_n$ is the standard basis of $\mathbb{Z}^{n+1}$, then $q_ie_j-q_je_i\in\mathbb{Z}^{n+1}$ maps to $0\in\mathbb{Z}$ in (5.1.9).

5.1.7. Let X be a variety with trivial G action. Prove that $(X\times U_{\widetilde{\sigma}})/\!/G\simeq X\times U_\sigma$ and use this to verify the final line of (5.1.10).

5.1.8. Consider the usual fan Σ for $\mathbb{P}^2$ with the lattice $N=\{(a,b)\in\mathbb{Z}^2\mid a+b\equiv 0 \bmod d\}$, where d is a positive integer.

(a) Prove that the ray generators are $u_1=(d,0)$, $u_2=(0,d)$ and
$$u_0=\begin{cases}(-d,-d) & d \text{ odd}\\ (-d/2,-d/2) & d \text{ even.}\end{cases}$$

(b) Prove that the dual lattice is $M=\{(a/d,b/d)\mid a,b\in\mathbb{Z},\ a-b\equiv 0 \bmod d\}$.

(c) Prove that $\mathrm{Cl}(X_\Sigma)=\mathbb{Z}\oplus\mathbb{Z}/d\mathbb{Z}$ (d odd) or $\mathbb{Z}\oplus\mathbb{Z}/\frac{d}{2}\mathbb{Z}$ (d even).

(d) Compute the quotient representation of X_Σ.

5.1.9. Find the quotient representation of the Hirzeburch surface $\mathscr{H}_r$ in Example 3.1.16.

5.1.10. Prove that G acts freely on $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$ when the fan Σ is smooth. Hint: Let $\sigma \in \Sigma$ and suppose that $g = (t_\rho) \in G$ fixes $u = (u_\rho) \in U_{\widetilde{\sigma}}$. Show that $t_\rho = 1$ for $\rho \notin \sigma$ and then use Lemma 5.1.1 to show that $t_\rho = 1$ for all ρ.

5.1.11. Prove that G acts with finite isotropy subgroups on $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$ when the fan Σ is simplicial. Hint: Adapt the argument used in Exercise 5.1.10.

5.1.12. Prove that $2 \leq \operatorname{codim}(Z(\Sigma)) \leq |\Sigma(1)|$. When Σ is a complete simplicial fan, a stronger result states that either

(a) $2 \leq \operatorname{codim}(Z(\Sigma)) \leq \lfloor \frac{1}{2}\dim X_\Sigma \rfloor + 1$, or

(b) $|\Sigma(1)| = \dim X_\Sigma + 1$ and $Z(\Sigma) = \{0\}$.

This is proved in [**19**, Prop. 2.8]. See the next exercise for more on part (b).

5.1.13. Let Σ be a complete fan such that $|\Sigma(1)| = n+1$, where $n = \dim X_\Sigma$. Prove that there is a weighted projective space $\mathbb{P}(q_0,\dots,q_n)$ and a finite abelian group H acting on $\mathbb{P}(q_0,\dots,q_n)$ such that
$$X_\Sigma \simeq \mathbb{P}(q_0,\dots,q_n)/H.$$
These are called *fake weighted projective spaces* in [**60**] and [**167**]. Also prove that the following are equivalent:

(a) X_Σ is a weighted projective space.

(b) $\operatorname{Cl}(X_\Sigma) \simeq \mathbb{Z}$.

(c) N is generated by u_ρ, $\rho \in \Sigma(1)$.

Hint: Label the ray generators $u_0,\dots,u_n$. First show that Σ is simplicial and that there are positive integers $q_0,\dots,q_n$ satisfying $\sum_{i=0}^n q_i u_i = 0$ and $\gcd(q_0,\dots,q_n) = 1$. Then consider the sublattice of N generated by the u_i and use Example 5.1.14. You will also need Proposition 3.3.7. If you get stuck, see [**19**, Lem. 2.11].

5.1.14. In the proof of Theorem 5.1.11, we showed that a non-simplicial cone leads to a non-closed G-orbit. Show that the non-closed G-orbit exhibited in Example 5.0.2 is an example of this construction. See also Example 5.1.15.

5.1.15. Example 5.1.16 gave the quotient construction of the blowup of $0 \in \mathbb{C}^2$ and used the quotient construction to describe the properties of the blowup. Give a similar treatment for the blowup of $\mathbb{C}^r \subseteq \mathbb{C}^n$ using the star subdivision described in §3.3.

§5.2. The Total Coordinate Ring

In this section we assume that X_Σ is a toric variety without torus factors. Its *total coordinate ring*
$$S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$$
was defined in §5.1. This ring gives $\mathbb{C}^{\Sigma(1)} = \operatorname{Spec}(S)$ and contains the irrelevant ideal
$$B(\Sigma) = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma \rangle$$
used in the quotient construction of X_Σ. In this section we will explore how this ring relates to the algebra and geometry of X_Σ.

The Grading. An important feature of the total coordinate ring S is its grading by the class group $\mathrm{Cl}(X_\Sigma)$. We have the exact sequence (5.1.1)

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

where $a = (a_\rho) \in \mathbb{Z}^{\Sigma(1)}$ maps to the divisor class $\big[\sum_\rho a_\rho D_\rho\big] \in \mathrm{Cl}(X_\Sigma)$. Given a monomial $x^a = \prod_\rho x_\rho^{a_\rho} \in S$, define its degree to be

$$\deg(x^a) = \big[\textstyle\sum_\rho a_\rho D_\rho\big] \in \mathrm{Cl}(X_\Sigma).$$

For $\beta \in \mathrm{Cl}(X_\Sigma)$, we let S_β denote the corresponding graded piece of S.

The grading on S is closely related to the group $G = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$. Recall that $\mathrm{Cl}(X_\Sigma)$ is the character group of G, where as usual $\beta \in \mathrm{Cl}(X_\Sigma)$ gives the character $\chi^\beta : G \to \mathbb{C}^*$. The action of G on $\mathbb{C}^{\Sigma(1)}$ induces an action on S with the property that given $f \in S$, we have

$$\begin{aligned} f \in S_\beta &\iff g\cdot f = \chi^\beta(g^{-1})\, f \text{ for all } g \in G \\ &\iff f(g\cdot x) = \chi^\beta(g)\, f(x) \text{ for all } g \in G,\ x \in \mathbb{C}^{\Sigma(1)} \end{aligned} \tag{5.2.1}$$

(Exercise 5.2.1). Thus the graded pieces of S are the eigenspaces of the action of G on S. We say that $f \in S_\beta$ is *homogeneous* of degree β.

Example 5.2.1. The total coordinate ring of $\mathbb{P}^n$ is $\mathbb{C}[x_0,\dots,x_n]$. By Example 4.1.6, the map $\mathbb{Z}^{n+1} \to \mathbb{Z} = \mathrm{Cl}(\mathbb{P}^n)$ is $(a_0,\dots,a_n) \mapsto a_0 + \cdots + a_n$. This gives the grading on $\mathbb{C}[x_0,\dots,x_n]$ where each variable x_i has degree 1, so that "homogeneous polynomial" has the usual meaning.

In Exercise 5.2.2 you will generalize this by showing that the total coordinate ring of the weighted projective space $\mathbb{P}(q_0,\dots,q_n)$ is $\mathbb{C}[x_0,\dots,x_n]$, where the variable x_i now has degree q_i. Here, "homogeneous polynomial" means weighted homogeneous polynomial. $\Diamond$

Example 5.2.2. The fan for $\mathbb{P}^n \times \mathbb{P}^m$ is the product of the fans of $\mathbb{P}^n$ and $\mathbb{P}^m$, and by Example 4.1.7, the class group is

$$\mathrm{Cl}(\mathbb{P}^n \times \mathbb{P}^m) \simeq \mathrm{Cl}(\mathbb{P}^n) \times \mathrm{Cl}(\mathbb{P}^m) \simeq \mathbb{Z}^2.$$

The total coordinate ring is $\mathbb{C}[x_0,\dots,x_n,y_0,\dots,y_m]$, where

$$\deg(x_i) = (1,0) \quad \deg(y_i) = (0,1)$$

(Exercise 5.2.3). For this ring, "homogeneous polynomial" means bihomogeneous polynomial. $\Diamond$

Example 5.2.3. Example 5.1.16 gave the quotient representation of the blowup $\mathrm{Bl}_0(\mathbb{C}^2)$ of $\mathbb{C}^2$ at the origin. The fan Σ of $\mathrm{Bl}_0(\mathbb{C}^2)$ is shown in Example 5.1.16 and has ray generators u_0, u_1, u_2, corresponding to variables t, x, y in the total coordinate ring $S = \mathbb{C}[t,x,y]$. Since $\mathrm{Cl}(\mathrm{Bl}_0(\mathbb{C}^2)) \simeq \mathbb{Z}$, one can check that the grading on S is given by

$$\deg(t) = -1 \text{ and } \deg(x) = \deg(y) = 1$$

(Exercise 5.2.4). Thus total coordinate rings can have some elements of positive degree and other elements of negative degree. $\Diamond$

The Toric Ideal-Variety Correspondence. For n-dimensional projective space $\mathbb{P}^n$, a homogeneous ideal $I \subseteq \mathbb{C}[x_0,\dots,x_n]$ defines a projective variety $\mathbf{V}(I) \subseteq \mathbb{P}^n$. This generalizes to more general toric varieties X_Σ as follows.

We first assume that Σ is simplicial, so that we have a geometric quotient

$$\pi : \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \longrightarrow X_\Sigma$$

by Theorem 5.1.11. Given $p \in X_\Sigma$, we say a point $x \in \pi^{-1}(p)$ gives *homogeneous coordinates* for p. Since π is a geometric quotient, we have $\pi^{-1}(p) = G \cdot x$. Thus *all* homogeneous coordinates for p are of the form $g \cdot x$ for some $g \in G$.

Now let S be the total coordinate ring of X_Σ and let $f \in S$ be homogeneous for the $\mathrm{Cl}(X_\Sigma)$-grading on S, say $f \in S_\beta$. Then

$$f(g \cdot x) = \chi^\beta(g)\, f(x)$$

by (5.2.1), so that $f(x) = 0$ for *one* choice of homogeneous coordinates of $p \in X_\Sigma$ if and only if $f(x) = 0$ for *all* homogeneous coordinates of p. It follows that the equation $f = 0$ is well-defined in X_Σ. We can use this to define subvarieties of X_Σ as follows.

Proposition 5.2.4. *Let S be the total coordinate ring of the simplicial toric variety X_Σ. Then:*

(a) *If $I \subseteq S$ is a homogeneous ideal, then*

$$\mathbf{V}(I) = \{\pi(x) \in X_\Sigma \mid f(x) = 0 \text{ for all } f \in I\}$$

is a closed subvariety of X_Σ.

(b) *All closed subvarieties of X_Σ arise this way.*

Proof. Given $I \subseteq S$ as in part (a), notice that

$$W = \{x \in \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \mid f(x) = 0 \text{ for all } f \in I\}$$

is a closed G-invariant subset of $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$. By part (b) of the definition of good categorical quotient (Definition 5.0.5), $\mathbf{V}(I) = \pi(W)$ is closed in X_Σ.

Conversely, given a closed subset $Y \subseteq X_\Sigma$, its inverse image

$$\pi^{-1}(Y) \subseteq \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$$

is closed and G-invariant. Then the same is true for the Zariski closure

$$\overline{\pi^{-1}(Y)} \subseteq \mathbb{C}^{\Sigma(1)}.$$

It follows without difficulty that $I = \mathbf{I}(\overline{\pi^{-1}(Y)}) \subseteq S$ is a homogeneous ideal satisfying $\mathbf{V}(I) = Y$. $\square$

Example 5.2.5. The equation $x_\rho = 0$ defines the T_N-invariant closed subvariety $\mathbf{V}(x_\rho) \subseteq X_\Sigma$ which is easily seen to be the prime divisor D_ρ. This shows that D_ρ always has a global equation, though it fails to have local equations when D_ρ is not Cartier (see Example 4.2.3). $\Diamond$

Classically, the Weak Nullstellensatz gives a necessary and sufficient condition for the variety of an ideal to be empty. This applies to $\mathbb{C}^n$ and $\mathbb{P}^n$ as follows:

- For $\mathbb{C}^n$: Given an ideal $I \subseteq \mathbb{C}[x_1,\ldots,x_n]$,
$$\mathbf{V}(I) = \emptyset \text{ in } \mathbb{C}^n \iff 1 \in I.$$
- For $\mathbb{P}^n$: Given a homogeneous ideal $I \subseteq \mathbb{C}[x_0,\ldots,x_n]$,
$$\mathbf{V}(I) = \emptyset \text{ in } \mathbb{P}^n \iff \langle x_0,\ldots,x_n\rangle^\ell \subseteq I \text{ for some } \ell \geq 0.$$

For a toric version of the weak Nullstellensatz, we use the irrelevant ideal $B(\Sigma) = \langle x^{\hat\sigma} \mid \sigma \in \Sigma\rangle \subseteq S$.

Proposition 5.2.6 (The Toric Weak Nullstellensatz)**.** *Let X_Σ be a simplicial toric variety with total coordinate ring S and irrelevant ideal $B(\Sigma) \subseteq S$. If $I \subseteq S$ is a homogeneous ideal, then*
$$\mathbf{V}(I) = \emptyset \textit{ in } X_\Sigma \iff B(\Sigma)^\ell \subseteq I \textit{ for some } \ell \geq 0.$$

Proof. Let $\mathbf{V}_a(I) \subseteq \mathbb{C}^{\Sigma(1)}$ denote the affine variety defined by $I \subseteq S$. Then:
$$\begin{aligned}\mathbf{V}(I) = \emptyset \text{ in } X_\Sigma &\iff \mathbf{V}_a(I) \cap \big(\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)\big) = \emptyset\\ &\iff \mathbf{V}_a(I) \subseteq Z(\Sigma) = \mathbf{V}_a(B(\Sigma))\\ &\iff B(\Sigma)^\ell \subseteq I \text{ for some } \ell \geq 0,\end{aligned}$$
where the last equivalence uses the Nullstellensatz in $\mathbb{C}^{\Sigma(1)}$. $\square$

For $\mathbb{C}^n$ and $\mathbb{P}^n$, the irrelevant ideal is $\langle 1\rangle \subseteq \mathbb{C}[x_1,\ldots,x_n]$ and $\langle x_0,\ldots,x_n\rangle \subseteq \mathbb{C}[x_0,\ldots,x_n]$ respectively. For $\mathbb{C}^n$, the grading on $\mathbb{C}[x_1,\ldots,x_n]$ by $\mathrm{Cl}(\mathbb{C}^n) = \{0\}$ is trivial, so that every ideal is homogeneous. Thus the toric weak Nullstellensatz implies the classical version of the weak Nullstellensatz for both $\mathbb{C}^n$ and $\mathbb{P}^n$.

The relation between ideals and varieties is not perfect because different ideals can define the same subvariety. In $\mathbb{C}^n$ and $\mathbb{P}^n$, we avoid this by using radical ideals:

- For $\mathbb{C}^n$: There is a bijective correspondence
$$\{\text{closed subvarieties of } \mathbb{C}^n\} \longleftrightarrow \{\text{radical ideals } I \subseteq \mathbb{C}[x_1,\ldots,x_n]\}.$$
- For $\mathbb{P}^n$: There is a bijective correspondence
$$\{\text{closed subvarieties of } \mathbb{P}^n\} \longleftrightarrow \left\{\begin{array}{c}\text{radical homogeneous ideals}\\ I \subseteq \langle x_0,\ldots,x_n\rangle \subseteq \mathbb{C}[x_0,\ldots,x_n]\end{array}\right\}.$$

Here is the toric version of this correspondence.

Proposition 5.2.7 (The Toric Ideal-Variety Correspondence). *Let X_Σ be a simplicial toric variety. Then there is a bijective correspondence*

$$\{\text{closed subvarieties of } X_\Sigma\} \longleftrightarrow \left\{\begin{array}{c}\text{radical homogeneous}\\ \text{ideals } I \subseteq B(\Sigma) \subseteq S\end{array}\right\}.$$

Proof. Given a closed subvariety $Y \subseteq X_\Sigma$, we can find a homogeneous ideal $I \subseteq S$ with $\mathbf{V}(I) = Y$ by Proposition 5.2.4. Then $\sqrt{I}$ is also homogeneous and satisfies $\mathbf{V}(\sqrt{I}) = \mathbf{V}(I) = Y$, so we may assume that I is radical. Since

$$\mathbf{V}_a(I \cap B(\Sigma)) = \mathbf{V}_a(I) \cup \mathbf{V}_a(B(\Sigma)) = \mathbf{V}_a(I) \cup Z(\Sigma)$$

in $\mathbb{C}^{\Sigma(1)}$, we see that $I \cap B(\Sigma) \subseteq B(\Sigma)$ is a radical homogeneous ideal satisfying $\mathbf{V}(I \cap B(\Sigma)) = Y$. This proves surjectivity.

To prove injectivity, suppose that $I, J \subseteq B(\Sigma)$ are radical homogeneous ideals with $\mathbf{V}(I) = \mathbf{V}(J)$ in X_Σ. Then

$$\mathbf{V}_a(I) \cap \big(\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)\big) = \mathbf{V}_a(J) \cap \big(\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)\big).$$

However, $I, J \subseteq B(\Sigma)$ implies that $Z(\Sigma)$ is contained in $\mathbf{V}_a(I)$ and $\mathbf{V}_a(J)$. Hence the above equality implies

$$\mathbf{V}_a(I) = \mathbf{V}_a(J),$$

so that $I = J$ by the Nullstellensatz since I and J are radical. □

For general ideals, another way to recover injectivity is to work with closed subschemes rather than closed subvarieties. We will say more about this in the appendix to Chapter 6.

When X_Σ is not simplicial, there is still a relation between ideals in the total coordinate ring and closed subvarieties of X_Σ.

Proposition 5.2.8. *Let S be the total coordinate ring of the toric variety X_Σ. Then:*

(a) *If $I \subseteq S$ is a homogeneous ideal, then*

$$\mathbf{V}(I) = \{p \in X_\Sigma \mid \textit{there is } x \in \pi^{-1}(p) \textit{ with } f(x) = 0 \text{ for all } f \in I\}$$

is a closed subvariety of X_Σ.

(b) *All closed subvarieties of X_Σ arise this way.*

Proof. The proof is identical to the proof of Proposition 5.2.4. □

The main difference between Propositions 5.2.4 and 5.2.8 is the phrase "there is $x \in \pi^{-1}(p)$". In the simplicial case, all such x are related by the group G, while this may fail in the non-simplicial case. One consequence is that the ideal-variety correspondence of Proposition 5.2.7 breaks down in the nonsimplicial case. Here is a simple example.

Example 5.2.9. In Example 5.1.15 we described the quotient representation of $U_\sigma = \mathbb{C}^4/\!/\mathbb{C}^*$ for the cone $\sigma = \mathrm{Cone}(e_1, e_2, e_1+e_3, e_2+e_3) \subseteq \mathbb{R}^3$, and in Example 5.0.2 we saw that the quotient map

$$\pi : \mathbb{C}^4 \longrightarrow U_\sigma = \mathbf{V}(xy-zw) \subseteq \mathbb{C}^4$$

is given by $\pi(a_1,a_2,a_3,a_4) = (a_1a_3, a_2a_4, a_1a_4, a_2a_3)$. Note that the irrelevant ideal is $B(\Sigma) = \mathbb{C}[x_1,x_2,x_3,x_4]$.

The ideals $I_1 = \langle x_1, x_2\rangle$ and $I_2 = \langle x_3, x_4\rangle$ are distinct radical homogeneous ideals contained in $B(\Sigma)$ that give the same subvariety in U_σ:

$$\mathbf{V}(I_1) = \pi(\mathbf{V}_a(I_1)) = \pi(\mathbb{C}^2 \times \{0\}) = \{0\} \in U_\sigma$$
$$\mathbf{V}(I_2) = \pi(\mathbf{V}_a(I_2)) = \pi(\{0\} \times \mathbb{C}^2) = \{0\} \in U_\sigma.$$

Thus Proposition 5.2.7 fails to hold for this toric variety. $\Diamond$

Local Coordinates. Let X_Σ be an n-dimensional toric variety. When Σ contains a smooth cone σ of dimension n, we get an affine open set

$$U_\sigma \subseteq X_\Sigma \quad \text{with} \quad U_\sigma \simeq \mathbb{C}^n$$

The usual coordinates for $\mathbb{C}^n$ are compatible with the homogeneous coordinates for X_Σ in the following sense. The cone σ gives the map $\phi_\sigma : \mathbb{C}^{\sigma(1)} \to \mathbb{C}^{\Sigma(1)}$ that sends $(a_\rho)_{\rho\in\sigma(1)}$ to the point $(b_\rho)_{\rho\in\Sigma(1)}$ defined by

$$b_\rho = \begin{cases} a_\rho & \rho \in \sigma(1) \\ 1 & \text{otherwise}. \end{cases}$$

Proposition 5.2.10. *Let $\sigma \in \Sigma$ be a smooth cone of dimension $n = \dim X_\Sigma$ and let $\phi_\sigma : \mathbb{C}^{\sigma(1)} \to \mathbb{C}^{\Sigma(1)}$ be defined as above. Then we have a commutative diagram*

$$\begin{tikzcd} \mathbb{C}^{\sigma(1)} \arrow[r, hook, "\phi_\sigma"] \arrow[d] & \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \arrow[d] \\ U_\sigma \arrow[r, hook] & X_\Sigma, \end{tikzcd}$$

where the vertical maps are the quotient maps from Theorem 5.1.11. Furthermore, the vertical map on the left is an isomorphism.

Proof. We first show commutativity. In the proof of Theorem 5.1.11 we saw that $\pi^{-1}(U_\sigma) = U_{\widetilde{\sigma}}$. Since the image of ϕ_σ lies in $U_{\widetilde{\sigma}}$, we are reduced to the diagram

$$\begin{tikzcd} \mathbb{C}^{\sigma(1)} \arrow[rr, hook, "\phi_\sigma"] \arrow[dr] & & U_{\widetilde{\sigma}} \arrow[dl] \\ & U_\sigma. & \end{tikzcd}$$

Since everything is affine, we can consider the corresponding diagram of coordinate rings

$$\begin{tikzcd} \mathbb{C}[x_\rho \mid \rho \in \sigma(1)] & & \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]_{x^{\hat\sigma}} \arrow[ll, "\phi_\sigma^*"'] \\ & \mathbb{C}[\sigma^\vee \cap M], \arrow[ul, "\alpha^*"] \arrow[ur, "\beta^*"'] & \end{tikzcd}$$

where $\alpha^*(\chi^m) = \prod_{\rho\in\sigma(1)} x_\rho^{\langle m,u_\rho\rangle}$ and $\beta^*(\chi^m) = \prod_{\rho\in\Sigma(1)} x_\rho^{\langle m,u_\rho\rangle}$ for $m \in \sigma^\vee \cap M$. It is clear that $\phi_\sigma^* \circ \beta^* = \alpha^*$, and commutativity follows.

For the final assertion, note that α^* is an isomorphism since the u_ρ, $\rho \in \sigma(1)$, form a basis of N by our assumption on σ. This completes the proof. $\square$

It follows that if a closed subvariety $Y \subseteq X_\Sigma$ is defined by an ideal $I \subseteq S$, then the affine piece $Y \cap U_\sigma \subseteq U_\sigma \simeq \mathbb{C}^{\sigma(1)}$ is defined by the dehomogenized ideal $\tilde{I} \subseteq \mathbb{C}[x_\rho \mid \rho \in \sigma(1)]$ obtained by setting $x_\rho = 1$, $\rho \notin \sigma(1)$, in all polynomials of I. We will give examples of this below, and in §5.4, we will explore the corresponding notion of homogenization.

Proposition 5.2.10 can be generalized to any cone $\sigma \in \Sigma$ satisfying $\dim \sigma = \dim X_\Sigma$ (Exercise 5.2.5).

Example 5.2.11. In Example 5.1.16 we described the quotient construction of the blowup of $\mathbb{C}^2$ at the origin. This variety can be expressed as the union $\mathrm{Bl}_0(\mathbb{C}^2) = U_{\sigma_1} \cup U_{\sigma_2}$, where $\sigma_1, \sigma_2 \in \Sigma$ are as in Example 5.1.16.

The map $\mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2$ is given by $(t,x,y) \mapsto (tx,ty)$ in homogeneous coordinates. Combining this with the local coordinate maps from Proposition 5.2.10, we obtain

$$\begin{aligned} U_{\sigma_1} \subseteq X_\Sigma \to \mathbb{C}^2 &: (t,x) \mapsto (t,x,1) \mapsto (tx,t) \\ U_{\sigma_2} \subseteq X_\Sigma \to \mathbb{C}^2 &: (t,y) \mapsto (t,1,y) \mapsto (t,ty). \end{aligned}$$

Consider the curve $f(x,y) = 0$ in the plane $\mathbb{C}^2$, where $f(x,y) = x^3 - y^2$. We study this on the blowup $\mathrm{Bl}_0(\mathbb{C}^2)$ using local coordinates as follows:

- On U_{σ_1}, we get $f(tx,t) = 0$, i.e., $(tx)^3 - t^2 = t^2(tx^3 - 1) = 0$. Since $t = 0$ defines the exceptional locus, we get the proper transform $tx^3 - 1 = 0$.
- On U_{σ_2}, we get $f(t,ty) = 0$, i.e., $t^3 - (ty)^2 = t^2(t - y^2) = 0$, with proper transform $t - y^2 = 0$.

Hence the proper transform is a smooth curve in $\mathrm{Bl}_0(\mathbb{C}^2)$. This method of studying the blowup of a curve is explained in many elementary texts on algebraic geometry, such as [**236**, p. 100].

We relate this to the homogeneous coordinates of $\mathrm{Bl}_0(\mathbb{C}^2)$ as follows. Using the above map $X_\Sigma \to \mathbb{C}^2$, we get the curve in X_Σ defined by $f(tx,ty) = 0$, i.e., $(tx)^3 - (ty)^2 = t^2(tx^3 - y^2) = 0$. Hence the proper transform is $tx^3 - y^2 = 0$. Then:

- Setting $y=1$ gives the proper transform $tx^3-1=0$ on U_{σ_1}.
- Setting $x=1$ gives the proper transform $t-y^2=0$ on U_{σ_2}.

Hence the "local" proper transforms computed above are obtained from the homogeneous proper transform by setting appropriate coordinates equal to 1. $\Diamond$

Exercises for §5.2.

5.2.1. Prove (5.2.1).

5.2.2. Show that the total coordinate ring of the weighted projective space $\mathbb{P}(q_0,\dots,q_n)$ is $\mathbb{C}[x_0,\dots,x_n]$ where $\deg(x_i)=q_i$. Hint: See Example 5.1.14.

5.2.3. Prove the claims made about the total coordinate ring of the product $\mathbb{P}^n\times\mathbb{P}^m$ made in Example 5.2.2.

5.2.4. Prove the claims made about the class group and the total coordinate ring of the blowup of $\mathbb{P}^2$ at the origin made in Example 5.2.3.

5.2.5. Let X_Σ be the toric variety of the fan Σ and assume as usual that X_Σ has no torus factors. A subfan $\Sigma'\subseteq\Sigma$ is *full* if $\Sigma'=\{\sigma\in\Sigma\mid\sigma(1)\subseteq\Sigma'(1)\}$. Consider a full subfan $\Sigma'\subseteq\Sigma$ with the property that $X_{\Sigma'}$ has no torus factors.

(a) Define the map $\phi:\mathbb{C}^{\Sigma'(1)}\to\mathbb{C}^{\Sigma(1)}$ by sending $(a_\rho)_{\rho\in\Sigma'(1)}$ to the point $(b_\rho)_{\rho\in\Sigma(1)}$ given by
$$b_\rho=\begin{cases}a_\rho & \rho\in\Sigma'(1)\\ 1 & \text{otherwise.}\end{cases}$$
Prove that there is a commutative diagram
$$\begin{tikzcd}\mathbb{C}^{\Sigma'(1)}\setminus Z(\Sigma') \arrow[r, hook, "\phi"] \arrow[d] & \mathbb{C}^{\Sigma(1)}\setminus Z(\Sigma) \arrow[d]\\ X_{\Sigma'} \arrow[r, hook] & X_\Sigma,\end{tikzcd}$$
where the vertical maps are the quotient maps from Theorem 5.1.11.

(b) Explain how part (a) generalizes Proposition 5.2.10.

(c) Use part (a) to give a version of Proposition 5.2.10 that applies to any cone $\sigma\in\Sigma$ satisfying $\dim\sigma=\dim X_\Sigma$.

5.2.6. The quintic $y^2=x^5$ in $\mathbb{C}^2$ has a unique singular point at the origin. We will resolve the singularity using successive blowups.

(a) Show that the proper transform of this curve in $\mathrm{Bl}_0(\mathbb{C}^2)$ is defined by $y^2-t^3x^5=0$. This uses the homogeneous coordinates t,x,y from Example 5.2.3.

(b) Show that the proper transform is smooth on U_{σ_1} but singular on U_{σ_2}.

(c) Subdivide σ_2 to obtain a smooth fan Σ'. The toric variety $X_{\Sigma'}$ has variables u,t,x,y, where u corresponds to the ray that subdivides σ_2. Show that $\mathrm{Cl}(X_{\Sigma'})\simeq\mathbb{Z}^2$ with
$$\deg(u)=(0,-1),\ \deg(t)=(-1,0),\ \deg(x)=(1,1),\ \deg(y)=(1,2).$$

(d) Show that $(u,t,x,y)\mapsto(utx,u^2ty)$ defines a toric morphism $X_{\Sigma'}\to\mathbb{C}^2$ and use this to show that the proper transform of the quintic in $\mathbb{C}^2$ is defined by $y^2-ut^3x^5=0$.

(e) Show that the proper transform is smooth by inspecting it in local coordinates.

5.2.7. Adapt the method Exercise 5.2.6 to desingularize $y^2 = x^{2n+1}$, $n \geq 1$ an integer.

5.2.8. Given an ideal I in a commutative ring R, its *Rees algebra* is the graded ring

$$R[I] = \bigoplus_{i=0}^{\infty} I^i t^i \subseteq R[t],$$

where t is a new variable and $I^0 = R$. There is also the *extended Rees algebra*

$$R[I, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} I^i t^i \subseteq R[t, t^{-1}],$$

where $I^i = R$ for $i \leq 0$. These rings are graded by letting $\deg(t) = 1$, so that elements of R have degree 0. See [**56**, 4.4] and §11.3 for more about Rees algebras.

(a) When $I = \langle x, y \rangle \subseteq R = \mathbb{C}[x, y]$, prove that the extended Rees algebra $R[I, t^{-1}]$ is the polynomial ring $\mathbb{C}[xt, yt, t^{-1}]$

(b) Prove that the ring of part (a) is isomorphic to the total coordinate ring of the blowup of $\mathbb{C}^2$ at the origin.

(c) Generalize parts (a) and (b) to the case of $I = \langle x_1, \ldots, x_n \rangle \subseteq R = \mathbb{C}[x_1, \ldots, x_n]$.

§5.3. Sheaves on Toric Varieties

Given a toric variety X_Σ, we show that graded modules over the total coordinate ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ give quasicoherent sheaves on X_Σ. We continue to assume that X_Σ has no torus factors.

Graded Modules. The grading on S gives a direct sum decomposition

$$S = \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} S_\alpha$$

such that $S_\alpha \cdot S_\beta \subseteq S_{\alpha+\beta}$ for all $\alpha, \beta \in \mathrm{Cl}(X_\Sigma)$.

Definition 5.3.1. An S-module M is ***graded*** if it has a decomposition

$$\mathsf{M} = \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} \mathsf{M}_\alpha$$

such that $S_\alpha \cdot \mathsf{M}_\beta \subseteq \mathsf{M}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathrm{Cl}(X_\Sigma)$. Given $\alpha \in \mathrm{Cl}(X_\Sigma)$, the ***shift*** $\mathsf{M}(\alpha)$ is the graded S-module satisfying

$$\mathsf{M}(\alpha)_\beta = \mathsf{M}_{\alpha+\beta}$$

for all $\beta \in \mathrm{Cl}(X_\Sigma)$.

The passage from a graded S-module to a quasicoherent sheaf on X_Σ requires some tools from the proof of Theorem 5.1.11. A cone $\sigma \in \Sigma$ gives the monomial $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho \in S$, and by (5.1.4), the map $\chi^m \mapsto x^{\langle m \rangle} = \prod_\rho x_\rho^{\langle m, u_\rho \rangle}$ induces an isomorphism

$$\pi_\sigma^* : \mathbb{C}[\sigma^\vee \cap M] \xrightarrow{\sim} (S_{x^{\hat{\sigma}}})^G \subseteq S_{x^{\hat{\sigma}}},$$

where $S_{x^{\hat{\sigma}}}$ is the localization of S at $x^{\hat{\sigma}}$. Since monomials are homogeneous, $S_{x^{\hat{\sigma}}}$ is also graded by $\mathrm{Cl}(X_\Sigma)$, and its elements of degree 0 are precisely its G-invariants (Exercise 5.3.1), i.e., $(S_{x^{\hat{\sigma}}})_0 = (S_{x^{\hat{\sigma}}})^G$. Hence the above isomorphism becomes

$$\pi_\sigma^* : \mathbb{C}[\sigma^\vee \cap M] \xrightarrow{\sim} (S_{x^{\hat{\sigma}}})_0. \tag{5.3.1}$$

These isomorphisms glue together just as we would hope.

Lemma 5.3.2. *Let $\tau = \sigma \cap m^\perp$ be a face of σ. Then $(S_{x^{\hat{\tau}}})_0 = ((S_{x^{\hat{\sigma}}})_0)_{\pi_\sigma^*(\chi^m)}$, and there is a commutative diagram of isomorphisms*

$$\begin{tikzcd}
(S_{x^{\hat{\sigma}}})_0 \arrow[r] \arrow[d] & ((S_{x^{\hat{\tau}}})_0)_{\pi_\sigma^*(\chi^m)} \arrow[d] \\
\mathbb{C}[\sigma^\vee \cap M] \arrow[r] & \mathbb{C}[\tau^\vee \cap M]_{\chi^m}.
\end{tikzcd}$$

Proof. Since $\tau = \sigma \cap m^\perp$, we have $\langle m, u_\rho \rangle = 0$ when $\rho \in \tau(1)$ and $\langle m, u_\rho \rangle > 0$ when $\rho \in \sigma(1) \setminus \tau(1)$. This means that $S_{x^{\hat{\tau}}} = (S_{x^{\hat{\sigma}}})_{\pi_\sigma^*(\chi^m)}$. Taking elements of degree zero commutes with localization, hence $(S_{x^{\hat{\tau}}})_0 = ((S_{x^{\hat{\sigma}}})_0)_{\pi_\sigma^*(\chi^m)}$. The vertical maps in the diagram come from (5.3.1), and the horizontal maps are localization. In Exercise 5.3.2 you will chase the diagram to show that it commutes. □

From Modules to Sheaves. We now construct the sheaf of a graded module.

Proposition 5.3.3. *Let M be a graded S-module. Then there is a quasicoherent sheaf $\widetilde{\mathsf{M}}$ on X_Σ such that for every $\sigma \in \Sigma$, the sections of $\widetilde{\mathsf{M}}$ over $U_\sigma \subseteq X_\Sigma$ are*

$$\Gamma(U_\sigma, \widetilde{\mathsf{M}}) = (\mathsf{M}_{x^{\hat{\sigma}}})_0.$$

Proof. Since M is a graded S-module, it is immediate that $\mathsf{M}_{x^{\hat{\sigma}}}$ is a graded $S_{x^{\hat{\sigma}}}$-module. Hence $(\mathsf{M}_{x^{\hat{\sigma}}})_0$ is an $(S_{x^{\hat{\sigma}}})_0$-module, which induces a sheaf $(\mathsf{M}_{x^{\hat{\sigma}}})_0^{\sim}$ on $U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M]) = \mathrm{Spec}((S_{x^{\hat{\sigma}}})_0)$. The argument of Lemma 5.3.2 applies verbatim to show that

$$(\mathsf{M}_{x^{\hat{\tau}}})_0 = ((\mathsf{M}_{x^{\hat{\sigma}}})_0)_{\pi_\sigma^*(\chi^m)}.$$

Thus the sheaves $(\mathsf{M}_{x^{\hat{\sigma}}})_0^{\sim}$ patch to give a sheaf $\widetilde{\mathsf{M}}$ on X_Σ which is quasicoherent by construction. □

Example 5.3.4. The total coordinate ring of $\mathbb{P}^n$ is $S = \mathbb{C}[x_0, \ldots, x_n]$ with the standard grading where every variable has degree 1. The quasicoherent sheaf on $\mathbb{P}^n$ associated to a graded S-module was first described by Serre in his foundational paper *Faisceaux algébriques cohérents* [**247**], called FAC for short. ◇

An important special case is when M is a finitely generated graded S-module. We will need the following finiteness result to understand the sheaf $\widetilde{\mathsf{M}}$.

Lemma 5.3.5. *$(S_{x^{\hat{\sigma}}})_\alpha$ is finitely generated as a $(S_{x^{\hat{\sigma}}})_0$-module for all $\sigma \in \Sigma$ and $\alpha \in \mathrm{Cl}(X_\Sigma)$.*

Proof. Write $\alpha = [\sum_\rho a_\rho D_\rho]$ and consider the rational polyhedral cone

$$\widehat{\sigma} = \{(m,\lambda) \in M_{\mathbb{R}} \times \mathbb{R} \mid \lambda \geq 0,\ \langle m, u_\rho\rangle \geq -\lambda a_\rho \text{ for all } \rho\} \subseteq M_{\mathbb{R}} \times \mathbb{R}.$$

By Gordan's Lemma, $\widehat{\sigma} \cap (M \times \mathbb{Z})$ is a finitely generated semigroup. Let the generators with last coordinate equal to 1 be $(m_1,1),\ldots,(m_r,1)$. Then you will prove in Exercise 5.3.3 that the monomials $\prod_\rho x_\rho^{\langle m_i,u_\rho\rangle + a_\rho}$, $i = 1,\ldots,r$, generate $(S_{x^{\hat{\sigma}}})_\alpha$ as a $(S_{x^{\hat{\sigma}}})_0$-module. □

Here are some coherent sheaves on X_Σ.

Proposition 5.3.6. *The sheaf* $\widetilde{\mathsf{M}}$ *on* X_Σ *is coherent when* M *is a finitely generated graded* S*-module.*

Proof. Because M is graded, we may assume its generators are homogeneous of degrees $\alpha_1,\ldots,\alpha_r$. Given $\sigma \in \Sigma$, it follows immediately that $\mathsf{M}_{x^{\hat{\sigma}}}$ is finitely generated over $S_{x^{\hat{\sigma}}}$ with generators in the same degrees. However, we need to be careful when taking elements of degree 0. Multiply a generator of degree α_i by the $(S_{x^{\hat{\sigma}}})_0$-module generators of $(S_{x^{\hat{\sigma}}})_{-\alpha_i}$ (finitely many by the previous lemma). Doing this for all i gives finitely many elements in $(\mathsf{M}_{x^{\hat{\sigma}}})_0$ that generate $(\mathsf{M}_{x^{\hat{\sigma}}})_0$ as an $(S_{x^{\hat{\sigma}}})_0$-module (Exercise 5.3.3). It follows that $\widetilde{\mathsf{M}}$ is coherent. □

Given $\alpha \in \mathrm{Cl}(X_\Sigma)$, the shifted S-module $S(\alpha)$ gives a coherent sheaf on X_Σ denoted $\mathscr{O}_{X_\Sigma}(\alpha)$. This is a sheaf we already know.

Proposition 5.3.7. *Fix* $\alpha \in \mathrm{Cl}(X_\Sigma)$. *Then:*

(a) *There is a natural isomorphism* $S_\alpha \simeq \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha))$.

(b) *If* $D = \sum_\rho a_\rho D_\rho$ *is a Weil divisor satisfying* $\alpha = [D]$, *then*

$$\mathscr{O}_{X_\Sigma}(D) \simeq \mathscr{O}_{X_\Sigma}(\alpha).$$

Proof. By definition, the sections of $\mathscr{O}_{X_\Sigma}(\alpha)$ over U_σ are

$$\Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(\alpha)) = (S(\alpha)_{x^{\hat{\sigma}}})_0 = (S_{x^{\hat{\sigma}}})_\alpha$$

for $\sigma \in \Sigma$. Since the open cover $\{U_\sigma\}_{\sigma\in\Sigma}$ of X_Σ satisfies $U_\sigma \cap U_\tau = U_{\sigma\cap\tau}$, the sheaf axiom gives the exact sequence

$$0 \longrightarrow \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha)) \longrightarrow \prod_\sigma (S_{x^{\hat{\sigma}}})_\alpha \rightrightarrows \prod_{\sigma,\tau} (S_{x^{\widehat{\sigma\cap\tau}}})_\alpha.$$

The localization $(S_{x^{\hat{\sigma}}})_\alpha$ has a basis consisting of all Laurent monomials $\prod_\rho x_\rho^{b_\rho}$ of degree α such that $b_\rho \geq 0$ for all $\rho \in \sigma(1)$. Then the exact sequence implies that $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha))$ has a basis consisting of all Laurent monomials $\prod_\rho x_\rho^{b_\rho}$ of degree α such that $b_\rho \geq 0$ for all $\rho \in \Sigma(1)$. These are precisely the monomials in S of degree α, which gives the desired isomorphism $S_\alpha \simeq \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha))$.

We turn to part (b). Given a Weil divisor $D = \sum_\rho a_\rho D_\rho$ with $\alpha = [D]$, we need to construct a sheaf isomorphism $\mathscr{O}_{X_\Sigma}(D) \simeq \mathscr{O}_{X_\Sigma}(\alpha)$. By the above description of the sections over U_σ, it suffices to prove that for every $\sigma \in \Sigma$, we have isomorphisms

$$\Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(D)) \simeq (S_{x^{\hat{\sigma}}})_\alpha. \tag{5.3.2}$$

compatible with inclusions $U_\tau \subseteq U_\sigma$ induced by $\tau \preceq \sigma$ in Σ.

To construct this isomorphism, we apply Proposition 4.3.3 to U_σ to obtain

$$\Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{\substack{m\in M \\ \langle m,u_\rho\rangle \geq -a_\rho,\, \rho\in\sigma(1)}} \mathbb{C}\cdot\chi^m.$$

A lattice point $m \in M$ gives the Laurent monomial

$$x^{\langle m,D\rangle} = \prod_\rho x_\rho^{\langle m,u_\rho\rangle + a_\rho}. \tag{5.3.3}$$

When $\langle m,u_\rho\rangle \geq -a_\rho$ for $\rho \in \sigma(1)$, this lies in $S_{x^{\hat{\sigma}}}$, and in fact $x^{\langle m,D\rangle} \in (S_{x^{\hat{\sigma}}})_\alpha$ since

$$\deg(x^{\langle m,D\rangle}) = \big[\textstyle\sum_\rho(\langle m,u_\rho\rangle + a_\rho)D_\rho\big] = \big[\mathrm{div}(\chi^m) + D\big] = [D] = \alpha.$$

We claim that map $\chi^m \mapsto x^{\langle m,D\rangle}$ induces the desired isomorphism (5.3.2).

Suppose that $\chi^m, \chi^{m'}$ map to the same monomial. Then $\langle m,u_\rho\rangle = \langle m',u_\rho\rangle$ for all ρ. This implies $m = m'$ since X_Σ has no torus factors. Furthermore, if $x^b = \prod_\rho x_\rho^{b_\rho} \in (S_{x^{\hat{\sigma}}})_\alpha$, then $[\sum_\rho b_\rho D_\rho] = \alpha = [\sum_\rho a_\rho D_\rho]$, so that there is $m \in M$ such that $b_\rho = \langle m,u_\rho\rangle + a_\rho$ for all ρ. Since x^b is a monomial in $S_{x^{\hat{\sigma}}}$, $b_\rho \geq 0$ for $\rho \in \sigma(1)$, hence $\langle m,u_\rho\rangle \geq -a_\rho$ for $\rho \in \sigma(1)$. Then $\chi^m \in \Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(D))$ maps to x^b. This defines an isomorphism (5.3.2) which is easily seen to be compatible with the inclusion of faces. □

Example 5.3.8. For $\mathbb{P}^n$ we have $S = \mathbb{C}[x_0,\ldots,x_n]$ with the standard grading by $\mathbb{Z} = \mathrm{Cl}(\mathbb{P}^n)$. Then $\mathscr{O}_{\mathbb{P}^n}(k)$ is the sheaf associated to $S(k)$ for $k \in \mathbb{Z}$. The classes of the toric divisors $D_0 \sim \cdots \sim D_n$ correspond to $1 \in \mathbb{Z}$, so that

$$\mathscr{O}_{\mathbb{P}^n}(k) \simeq \mathscr{O}_{\mathbb{P}^n}(kD_0) \simeq \cdots \simeq \mathscr{O}_{\mathbb{P}^n}(kD_n).$$

Thus $\mathscr{O}_{\mathbb{P}^n}(k)$ is a canonical model for the sheaf $\mathscr{O}_{\mathbb{P}^n}(kD_i)$. This justifies what we did in Example 4.3.1.

Also note that when $k \geq 0$, we have

$$\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(k)) = S_k.$$

Hence global sections of $\mathscr{O}_{\mathbb{P}^n}(k)$ are homogeneous polynomials in $x_0,\ldots,x_n$ of degree k, which agrees with what we computed in Example 4.3.6. ◊

Sheaves versus Modules. An important result is that *all* quasicoherent sheaves on X_Σ come from graded modules.

Proposition 5.3.9. *Let $\mathscr{F}$ be a quasicoherent sheaf on X_Σ. Then:*

(a) *There is a graded S-module M such that $\widetilde{\mathsf{M}} \simeq \mathscr{F}$.*

(b) *If $\mathscr{F}$ is coherent, then M can be chosen to be finitely generated over S.*

The proof will be given in the appendix to Chapter 6 since it involves tensor products of sheaves from §6.0.

Although the map $\mathsf{M} \mapsto \widetilde{\mathsf{M}}$ is surjective (up to isomorphism), it is far from injective. In particular, there are nontrivial graded modules that give the trivial sheaf. This phenomenon is well-known for $\mathbb{P}^n$, where a finitely generated graded module M over $S = \mathbb{C}[x_0,\ldots,x_n]$ gives the trivial sheaf on $\mathbb{P}^n$ if and only if $\mathsf{M}_\ell = 0$ for $\ell \gg 0$ (see [**131**, Ex. II.5.9]). This is equivalent to

$$\langle x_0,\ldots,x_n\rangle^\ell\,\mathsf{M} = 0$$

for $\ell \gg 0$ (Exercise 5.3.4). Since $\langle x_0,\ldots,x_n\rangle$ is the irrelevant ideal for $\mathbb{P}^n$, this suggests a toric generalization. In the smooth case, we have the following result.

Proposition 5.3.10. *Let $B(\Sigma) \subseteq S$ be the irrelevant ideal of S for a smooth toric variety X_Σ, and let M be a finitely generated graded S-module. Then $\widetilde{\mathsf{M}} = 0$ if and only if $B(\Sigma)^\ell\,\mathsf{M} = 0$ for $\ell \gg 0$.*

Proof. First observe that $\widetilde{\mathsf{M}} = 0$ if and only if it vanishes on each affine open subset $U_\sigma \subseteq X_\Sigma$. But on an affine variety, the correspondence between quasicoherent sheaves and modules is bijective (see [**131**, Cor. II.5.5]). Hence $\widetilde{\mathsf{M}} = 0$ if and only if $(\mathsf{M}_{x^{\hat\sigma}})_0 = 0$ for all $\sigma \in \Sigma$.

Next suppose that $B(\Sigma)^\ell\,\mathsf{M} = 0$ for some $\ell \ge 0$. Then $(x^{\hat\sigma})^\ell\mathsf{M} = 0$, which easily implies that $\mathsf{M}_{x^{\hat\sigma}} = 0$. Then $\widetilde{\mathsf{M}} = 0$ follows from the previous paragraph. This part of the argument works for any toric variety.

For the converse, we have $(\mathsf{M}_{x^{\hat\sigma}})_0 = 0$ for all $\sigma \in \Sigma$. Given $h \in \mathsf{M}_\alpha$, we will show that $(x^{\hat\sigma})^\ell h = 0$ for some $\ell \ge 0$, which will imply $B(\Sigma)^\ell\,\mathsf{M} = 0$ for $\ell \gg 0$ since M is finitely generated. Let $\alpha = [D]$, where $D = \sum_\rho a_\rho D_\rho$. Since σ is smooth, D is Cartier, so there is $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho\rangle = -a_\rho$ for all $\rho \in \sigma(1)$ (this is part of the Cartier data for D). Replacing D with $D + \mathrm{div}(\chi^{m_\sigma})$, we may assume that $D = \sum_{\rho\notin\sigma(1)} a_\rho D_\rho$. Now set $k = \max(0, a_\rho \mid \rho \notin \sigma(1))$ and observe that

$$x^b = (x^{\hat\sigma})^k \prod_{\rho\notin\sigma(1)} x_\rho^{-a_\rho} = \prod_{\rho\notin\sigma(1)} x_\rho^{k-a_\rho} \in S.$$

Furthermore, $x^b h/(x^{\hat\sigma})^k \in \mathsf{M}_{x^{\hat\sigma}}$ has degree 0. Hence $x^b h/(x^{\hat\sigma})^k = 0$ in $\mathsf{M}_{x^{\hat\sigma}}$, which by the definition of localization implies that there is $s \ge 0$ with

$$(x^{\hat\sigma})^s \cdot x^b h = 0 \text{ in } \mathsf{M}.$$

Since x^b involves only x_ρ for $\rho \notin \sigma(1)$, we can find $x^a \in S$ such that $x^a \cdot x^b$ is a power of $x^{\hat{\sigma}}$. Hence multiplying the above equation by x^a implies $(x^{\hat{\sigma}})^\ell h = 0$ for some $\ell \geq 0$, as desired. $\square$

Unfortunately, the situation is more complicated when X_Σ is not smooth. Here is an example to show what can go wrong when X_Σ is simplicial.

Example 5.3.11. The weighted projective space $\mathbb{P}(1,1,2)$ has total coordinate ring $S = \mathbb{C}[x,y,z]$, where x,y have degree 1 and z has degree 2, and the irrelevant ideal is $B(\Sigma) = \langle x,y,z \rangle$. The graded S-module $\mathsf{M} = S(1)/(xS(1)+yS(1))$ has only elements of odd degree. Then $(\mathsf{M}_z)_0 = 0$ since z has degree 2, and it is clear that $(\mathsf{M}_x)_0 = (\mathsf{M}_y)_0 = 0$. It follows that $\widetilde{\mathsf{M}} = 0$, yet one easily checks that $B(\Sigma)^\ell \mathsf{M} = z^\ell \mathsf{M} \neq 0$ for all $\ell \geq 0$. Thus Proposition 5.3.10 fails for $\mathbb{P}(1,1,2)$. $\Diamond$

Exercise 5.3.5 explores a version of Proposition 5.3.10 that applies to simplicial toric varieties. The condition that $B(\Sigma)^\ell\,\mathsf{M} = 0$ is replaced with the weaker condition that $B(\Sigma)^\ell\,\mathsf{M}_\alpha = 0$ for all $\alpha \in \mathrm{Pic}(X_\Sigma)$.

We will say more about the relation between quasicoherent sheaves and graded S-modules in the appendix to Chapter 6.

Exercises for §5.3.

5.3.1. As described in §5.0, the action of G on $\mathbb{C}^{\Sigma(1)}$ induces an action of G on the total coordinate ring S. Also recall that $g \in G$ is a homomorphism $g : \mathrm{Cl}(X_\Sigma) \to \mathbb{C}^*$.

(a) Given $x^a \in S$ and $g \in G$, show that $g \cdot x^a = g^{-1}(\alpha)\,x^a$, where $\alpha = \deg(x^a)$.

(b) Show that $S^G = S_0$ and that a similar result holds for the localization S_{x^b}.

5.3.2. Complete the proof of Lemma 5.3.2.

5.3.3. Complete the proofs of Lemma 5.3.5 and Proposition 5.3.6.

5.3.4. Let $S = \mathbb{C}[x_0,\dots,x_n]$ where $\deg(x_i) = 1$ for all i, and let M be a finitely generated graded S-module. Show that $\mathsf{M}_\ell = 0$ for $\ell \gg 0$ if and only if $\langle x_0,\dots,x_n\rangle^\ell\,\mathsf{M} = 0$ for $\ell \gg 0$.

5.3.5. Let X_Σ be a simplicial toric variety and let M be a finitely generated graded S-module. Prove that $\widetilde{\mathsf{M}} = 0$ if and only if $B(\Sigma)^\ell\,\mathsf{M}_\alpha = 0$ for all $\ell \gg 0$ and $\alpha \in \mathrm{Pic}(X_\Sigma)$.

5.3.6. Let X_Σ be a smooth toric variety. State and prove a version of Proposition 5.3.10 that applies to arbitrary graded S-modules M. Also explain what happens when X_Σ is simplicial, as in Exercise 5.3.5.

§5.4. Homogenization and Polytopes

The final section of the chapter will explore the relation between torus-invariant divisors on a toric variety X_Σ and its total coordinate ring. We will also see that when X_Σ comes from a polytope P, the quotient construction of X_Σ relates nicely to the definition of projective toric variety given in Chapter 2.

Homogenization. When working with affine and projective space, one often needs to homogenize polynomials. This process generalizes nicely to the toric context. The full story involves characters, polyhedra, divisors, sheaves, and graded pieces of the total coordinate ring.

A Weil divisor $D = \sum_\rho a_\rho D_\rho$ on X_Σ gives the polyhedron

$$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}.$$

Proposition 4.3.3 tells us that the global sections of the sheaf $\mathscr{O}_{X_\Sigma}(D)$ are spanned by characters coming from lattice points of P_D, i.e.,

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m.$$

This relates to the total coordinate ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ as follows. Given $m \in P_D \cap M$, the *D-homogenization* of χ^m is the monomial

$$x^{\langle m, D \rangle} = \prod_\rho x_\rho^{\langle m, u_\rho \rangle + a_\rho}$$

defined in (5.3.3). The inequalities defining P_D guarantee that $x^{\langle m, D \rangle}$ lies in S. Here are the basic properties of these monomials.

Proposition 5.4.1. *Assume that X_Σ has no torus factors. If D and P_D are as above and $\alpha = [D] \in \mathrm{Cl}(X_\Sigma)$ is the divisor class of D, then:*

(a) *For each $m \in P_D \cap M$, the monomial $x^{\langle m, D \rangle}$ lies in S_α.*

(b) *The map sending the character χ^m of $m \in P_D \cap M$ to the monomial $x^{\langle m, D \rangle}$ induces an isomorphism*

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \simeq S_\alpha.$$

Proof. Part (a) follows from the proof of Proposition 5.3.7. As for part (b), we use the same proposition to conclude that

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \simeq \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha)) \simeq S_\alpha.$$

One easily sees that this isomorphism is given by $\chi^m \mapsto x^{\langle m, D \rangle}$. $\square$

Here are some examples of homogenization.

Example 5.4.2. The fan for $\mathbb{P}^n$ has ray generators $u_0 = -\sum_{i=1}^n e_i$ and $u_i = e_i$ for $i = 1, \ldots, n$. This gives variables x_i and divisors D_i for $i = 0, \ldots, n$. Since $M = \mathbb{Z}^n$, the character of $m = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ is the Laurent monomial $t^m = \prod_{i=1}^n t_i^{b_i}$.

For a positive integer d, the divisor $D = d\,D_0$ has polyhedron $P_D = d\Delta_n$, where Δ_n is the standard n-simplex. Given $m = (b_1,\dots,b_n) \in d\Delta_n$, its homogenization is

$$\begin{aligned} x^{\langle m,D\rangle} &= x_0^{\langle m,u_0\rangle+d} x_1^{\langle m,u_1\rangle+0} \cdots x_n^{\langle m,u_n\rangle+0} \\ &= x_0^{-b_1-\cdots-b_n+d} x_1^{b_1} \cdots x_n^{b_n} \\ &= x_0^d \left(\frac{x_1}{x_0}\right)^{b_1} \cdots \left(\frac{x_n}{x_0}\right)^{b_n}, \end{aligned}$$

which is the usual way to homogenize $t^m = \prod_{i=1}^n t_i^{b_i}$ with respect to x_0.

This monomial has degree $d = [dD_0] \in \mathrm{Cl}(\mathbb{P}^n) = \mathbb{Z}$, in agreement with Proposition 5.4.1. The proposition also implies the standard fact that monomials of degree d in $x_0,\dots,x_n$ correspond to lattice points in $d\Delta_n$. $\Diamond$

Example 5.4.3. For $\mathbb{P}^1 \times \mathbb{P}^1$, we have ray generators $u_1 = e_1, u_2 = -e_1, u_3 = e_2, u_4 = -e_2$ with corresponding variables x_i and divisors D_i. Given nonnegative integers k,ℓ, we get the divisor $D = kD_2 + \ell D_4$. The polyhedron P_D is the rectangle with vertices $(0,0),(k,0),(0,\ell),(k,\ell)$, and given $(a,b) \in P_D \cap \mathbb{Z}^2$, the Laurent monomial $t_1^a t_2^b$ homogenizes to

$$x_1^a x_2^{k-a} x_3^b x_4^{\ell-b} = x_2^k x_4^\ell \left(\frac{x_1}{x_2}\right)^a \left(\frac{x_3}{x_4}\right)^b,$$

which is the usual way of turning a two-variable monomial into a bihomogeneous monomial of degree (k,ℓ) (remember that $\deg(x_1) = \deg(x_2) = (1,0)$ and $\deg(x_3) = \deg(x_4) = (0,1)$). Thus monomials of degree (k,ℓ) correspond to lattice points in the rectangle P_D. $\Diamond$

Example 5.4.4. The fan for $\mathrm{Bl}_0(\mathbb{C}^2)$ is shown in Example 5.1.16, and its total coordinate ring $S = \mathbb{C}[t,x,y]$ is described in Example 5.2.3. If we pick $D = 0$, then the polyhedron $P_D \subset \mathbb{R}^2$ is defined by the inequalities

$$\langle m, u_i\rangle \geq 0, \quad i = 0,1,2.$$

Since u_1,u_2 form a basis of $N = \mathbb{Z}^2$ and $u_0 = u_1 + u_2$, P_D is the first quadrant in $\mathbb{R}^2$. Given $m = (a,b) \in P_D \cap \mathbb{Z}^2$, the monomial $t_1^a t_2^b$ homogenizes to

$$t^{\langle m,u_0\rangle} x^{\langle m,u_1\rangle} y^{\langle m,u_2\rangle} = t^{a+b} x^a y^b = (tx)^a (ty)^b.$$

where the ray generators u_0,u_1,u_2 correspond to the variables t,x,y.

For example, the singular cubic $t_1^3 - t_2^2 = 0$ homogenizes to $(tx)^3 - (ty)^2 = 0$, which is the equation enountered in Example 5.2.11 when resolving the singularity of this curve. $\Diamond$

One thing to keep in mind when doing toric homogenization is that characters χ^m (in general) or Laurent monomials t^m (in specific examples) are intrinsically defined on the torus T_N or $(\mathbb{C}^*)^n$. The homogenization process produces a "global object" $x^{\langle m,D\rangle}$ relative to a divisor D that lives in the total coordinate ring or, via Proposition 5.4.1, in the global sections of $\mathscr{O}_{X_\Sigma}(D)$.

We next study the isomorphisms $S_\alpha \simeq \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$ from Proposition 5.4.1. We will see that they are compatible with linear equivalence and multiplication.

First suppose that D and E are linearly equivalent torus-invariant divisors. This means that $D = E + \mathrm{div}(\chi^m)$ for some $m \in M$. Proposition 4.0.29 implies that $f \mapsto f\chi^m$ induces an isomorphism

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \simeq \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(E)). \tag{5.4.1}$$

Turning to the associated polyhedra, we proved $P_E = P_D + m$ in Exercise 4.3.2. An easy calculation shows that if $m' \in P_D$, then

$$x^{\langle m', D\rangle} = x^{\langle m'+m, E\rangle}$$

(Exercise 5.4.1). Hence (5.4.1) fits into a commutative diagram of isomorphisms

$$\begin{tikzcd}
\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(E)) \arrow[dl, "\sim"] \\
& S_\alpha. &
\end{tikzcd} \tag{5.4.2}$$

Here, $\alpha = [D] = [E] \in \mathrm{Cl}(X_\Sigma)$ and the "diagonal" maps are the isomorphisms from Proposition 5.4.1. You will verify these claims in Exercise 5.4.1.

It follows that S_α gives a "canonical model" for $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$, since the latter depends on the particular choice of divisor D in the class α. It is also possible to give a "canonical model" for the polyhedron P_D (Exercise 5.4.2).

Next consider multiplication. Let D and E be torus-invariant divisors on X_Σ and set $\alpha = [D]$, $\beta = [E]$ in $\mathrm{Cl}(X_\Sigma)$. Then $f \otimes g \mapsto fg$ induces a $\mathbb{C}$-linear map

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \otimes_{\mathbb{C}} \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(E)) \longrightarrow \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D+E))$$

such that the isomorphisms of Proposition 5.4.1 give a commutative diagram

$$\begin{tikzcd}
\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \otimes_{\mathbb{C}} \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(E)) \arrow[r] \arrow[d] & \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D+E)) \arrow[d] \\
S_\alpha \otimes_{\mathbb{C}} S_\beta \arrow[r] & S_{\alpha+\beta}
\end{tikzcd} \tag{5.4.3}$$

where the bottom map is multiplication in the total coordinate ring (Exercise 5.4.3). Thus homogenization turns multiplication of sections into ordinary multiplication.

Polytopes. A full dimensional lattice polytope $P \subseteq M_\mathbb{R}$ gives a toric variety X_P. Recall that X_P can be constructed in two ways:

- As the toric variety X_{Σ_P} of the normal fan Σ_P of P (Chapter 3).
- As the projective toric variety $X_{(kP)\cap M}$ of the set of characters $(kP) \cap M$ for $k \gg 0$ (Chapter 2).

We will see that both descriptions relate nicely to homogeneous coordinates and the total coordinate ring.

Given P as above, set $n = \dim P$ and let $P(i)$ denote the set of i-dimensional faces of P. Thus $P(0)$ consists of vertices and $P(n-1)$ consists of facets. The facet presentation of P given in equation (2.2.2) can be written as

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \text{ for all } F \in P(n-1)\}. \tag{5.4.4}$$

In terms of the normal fan Σ_P, we have bijections

$$\begin{aligned} P(0) &\longleftrightarrow \Sigma_P(n) \quad (\text{vertices} \longleftrightarrow \text{maximal cones}) \\ P(n-1) &\longleftrightarrow \Sigma_P(1) \quad (\text{facets} \longleftrightarrow \text{rays}). \end{aligned}$$

When dealing with polytopes we index everything by facets rather than rays. Thus each facet $F \in P(n-1)$ gives:

- The facet normal u_F, which is the ray generator of the corresponding cone.
- The torus-invariant prime divisor $D_F \subseteq X_P$.
- The variable x_F in the total coordinate ring S. We call x_F a *facet variable*.

We also have the divisor

$$D_P = \sum_F a_F\, D_F$$

from (4.2.7). The polytope P_{D_P} of this divisor is the polytope P we began with (Exercise 4.3.1). Hence, if we set $\alpha = [D_P] \in \mathrm{Cl}(X_P)$, then we get isomorphisms

$$S_\alpha \simeq \Gamma(X_P, \mathscr{O}_{X_P}(D_P)) \simeq \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m.$$

In this situation, we write the D_P-homogenization of χ^m as

$$x^{\langle m,P \rangle} = \prod_F x_F^{\langle m, u_F \rangle + a_F}.$$

We call $x^{\langle m,P \rangle}$ a *P-monomial*.

The exponent of the variable x_F in $x^{\langle m,P \rangle}$ gives the *lattice distance* from m to the facet F. To see this, note that F lies in the supporting hyperplane defined by $\langle m, u_F \rangle + a_F = 0$. If the exponent of x_F is $a \geq 0$, then to get from the supporting hyperplane to m, we must pass through the a parallel hyperplanes, namely $\langle m, u_F \rangle + a_F = j$ for $j = 1, \ldots, a$. Here is an example.

Example 5.4.5. Consider the toric variety X_P of the polygon $P \subset \mathbb{R}^2$ with vertices $(1,1), (-1,1), (-1,0), (0,-1), (1,-1)$, shown in Figure 2 on the next page. In terms of (5.4.4), we have $a_1 = \cdots = a_5 = 1$, where the indices correspond to the facet variables $x_1, \ldots, x_5$ indicated in Figure 2. The 8 points of $P \cap \mathbb{Z}^2$ give the P-monomials

$$\begin{array}{ccc} x_2 x_3^2 x_4^2 & x_1 x_2^2 x_3^2 x_4 & x_1^2 x_2^3 x_3^2 \\ x_3 x_4^2 x_5 & x_1 x_2 x_3 x_4 x_5 & x_1^2 x_2^2 x_3 x_5 \\ & x_1 x_4 x_5^2 & x_1^2 x_2 x_5^2, \end{array}$$

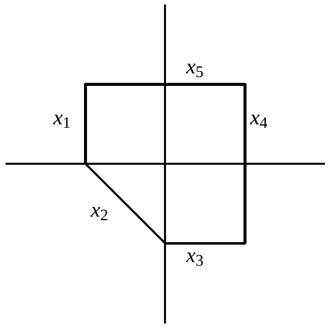


Figure 2. A polygon with facets labeled by variables

where the position of each P-monomial $x^{\langle m,P\rangle}$ corresponds to the position of the lattice point $m \in P \cap \mathbb{Z}^2$. The exponents are easy to understand if you think in terms of lattice distances to facets. $\Diamond$

The lattice-distance interpretation of the exponents in $x^{\langle m,P\rangle}$ shows that lattice points in the interior $\mathrm{int}(P)$ of P correspond to those P-monomials divisible by $\prod_F x_F$. For example, the only P-monomial in Example 5.4.5 divisible by $x_1 \cdots x_5$ corresponds to the unique interior lattice point.

We next relate the constructions of toric varieties given in Chapter 2 and in §5.1. In Chapter 2, we wrote the lattice points of P as $P \cap M = \{m_1, \ldots, m_s\}$ and considered the map

$$\Phi : T_N \longrightarrow \mathbb{P}^{s-1}, \quad t \longmapsto (\chi^{m_1}(t), \ldots, \chi^{m_s}(t)). \tag{5.4.5}$$

The projective (possibly non-normal) toric variety $X_{P\cap M}$ is the Zariski closure of the image of Φ.

On the other hand, we have the quotient construction of X_P

$$X_P \simeq \big(\mathbb{C}^r \setminus Z(\Sigma_P)\big)/\!/G,$$

where we write $\mathbb{C}^r = \mathbb{C}^{\Sigma_P(1)}$. Also, the exceptional set $Z(\Sigma_P)$ can be described in terms of the P-monomials coming from the vertices of the polytope.

Lemma 5.4.6. *The **vertex monomials** $x^{\langle \mathrm{v},P\rangle}$, v a vertex of P, have the following properties:*

(a) $\sqrt{\langle x^{\langle \mathrm{v},P\rangle} \mid \mathrm{v} \in P(0)\rangle} = B(\Sigma_P)$, *where* $B(\Sigma_P) = \langle x^{\hat\sigma} \mid \sigma \in \Sigma(n)\rangle$ *is the irrelevant ideal of* S.

(b) $Z(\Sigma_P) = \mathbf{V}(x^{\langle \mathrm{v},P\rangle} \mid \mathrm{v} \in P(0))$.

Proof. We saw above that vertices $v \in P(0)$ correspond bijectively to cones $\sigma_{\mathrm{v}} = \mathrm{Cone}(u_F \mid \mathrm{v} \in F) \in \Sigma_P(n)$. Then the lattice-distance interpretation of $x^{\langle \mathrm{v},P\rangle}$ shows the facet variables x_F appearing in $x^{\langle \mathrm{v},P\rangle}$ are precisely the variables appearing in $x^{\hat\sigma_{\mathrm{v}}}$. This implies part (a), and part (b) follows immediately. $\square$

If we set $\alpha = [D_P]$ as above, then the P-monomials $x^{\langle m_i,P\rangle}$, $i = 1,\dots,s$, form a basis of S_α and give a map

$$\Psi : \mathbb{C}^r \setminus Z(\Sigma_P) \longrightarrow \mathbb{P}^{s-1} \quad p \longmapsto (p^{\langle m_1,P\rangle},\dots,p^{\langle m_s,P\rangle}), \tag{5.4.6}$$

where $p^{\langle m_i,P\rangle}$ is the evaluation of the monomial $x^{\langle m_i,P\rangle}$ at the point $p \in \mathbb{C}^r \setminus Z(\Sigma_P)$. This map is well-defined since for each $p \in \mathbb{C}^r \setminus Z(\Sigma_P)$, Lemma 5.4.6 implies that at least one P-monomial (in fact, at least one vertex monomial) must be nonzero.

The maps (5.4.5) and (5.4.6) fit into a diagram

$$\begin{tikzcd}
(\mathbb{C}^*)^r \arrow[r, hook] \arrow[d] & \mathbb{C}^r \setminus Z(\Sigma_P) \arrow[d, "\pi"] \arrow[ddr, "\Psi"] & \\
T_N \arrow[r, hook] \arrow[drr, "\Phi"'] & X_P \arrow[dr, dotted, "\phi"] & \\
 & & \mathbb{P}^{s-1}.
\end{tikzcd}$$

Here, the map $(\mathbb{C}^*)^r \to T_N$ is from (5.1.2) and $\pi : \mathbb{C}^r \setminus Z(\Sigma_P) \to X_P$ is the quotient map. This diagram has the following properties.

Proposition 5.4.7. *There is a morphism $\phi : X_P \to \mathbb{P}^{s-1}$ represented by the dotted arrow in the above diagram that makes the entire diagram commute. Furthermore, the image of ϕ is precisely the projective toric variety $X_{P\cap M}$.*

Proof. When we regard the x_F as characters on $(\mathbb{C}^*)^r = (\mathbb{C}^*)^{\Sigma_P(1)}$, the exact sequence (5.1.1) tells us that

$$\chi^m = \prod_F x_F^{\langle m,u_F\rangle} \tag{5.4.7}$$

for $m \in M$. Multiplying each side by $\prod_F x_F^{a_F}$, we obtain

$$\Big(\prod_F x_F^{a_F}\Big)\chi^m = x^{\langle m,D\rangle}.$$

If we let $m = m_i$, $i = 1,\dots,s$ and apply this to a point in $p \in (\mathbb{C}^*)^r$, we see that $\Psi(p)$ and $\Phi(p)$ give the same point in projective space since the vector for $\Psi(p)$ equals $\prod_F p_F^{a_F}$ times the vector for $\Phi(p)$. It follows that, ignoring ϕ for the moment, the rest of the above diagram commutes.

We next show that Ψ is constant on G-orbits. This holds since P-monomials are homogeneous of the same degree. In more detail, fix points $t = (t_F) \in G$, $p = (p_F) \in \mathbb{C}^r \setminus Z(\Sigma_P)$ and a P-monomial $x^{\langle m,D\rangle} = \prod_F x_F^{\langle m,u_F\rangle + a_F}$. Then evaluating

$x^{\langle m,D\rangle}$ at $t\cdot p$ gives

$$\begin{aligned}(t\cdot p)^{\langle m,D\rangle} &= \prod_F (t_F p_F)^{\langle m,u_F\rangle + a_F}\\ &= \Big(\prod_F t_F^{\langle m,u_F\rangle}\Big)\Big(\prod_F t_F^{a_F}\Big)p^{\langle m,D\rangle} = \Big(\prod_F t_F^{a_F}\Big)p^{\langle m,D\rangle},\end{aligned}$$

where the last equality follows from the description of G given in Lemma 5.1.1. Arguing as in the previous paragraph, it follows that $\Psi(t\cdot p)$ and $\Psi(p)$ give the same point in $\mathbb{P}^{s-1}$. This proves the existence of ϕ since π is a good categorical quotient, and this choice of ϕ makes the entire diagram commute.

The final step is to show that the image of $\phi : X_P \to \mathbb{P}^{s-1}$ is the Zariski closure $X_{P\cap M}$ of the image of $\Phi : T_N \to \mathbb{P}^{s-1}$. First observe that

$$\phi(X_P) = \phi(\overline{T_N}) \subseteq \overline{\phi(T_N)} = \overline{\Phi(T_N)} = X_{P\cap M}$$

since ϕ is continuous in the Zariski topology and $\phi|_{T_N} = \Phi$ by commutativity of the diagram. However, $\phi(X_P)$ is Zariski closed in $\mathbb{P}^{s-1}$ since X_P is projective. You will give two proofs of this in Exercise 5.4.4, one topological (using constructible sets and compactness) and one algebraic (using completeness and properness). Once we know that $\phi(X_P)$ is Zariski closed, $\Phi(T_N) \subseteq \phi(X_P)$ implies

$$X_{P\cap M} = \overline{\Phi(T_N)} \subseteq \phi(X_P),$$

and $\phi(X_P) = X_{P\cap M}$ follows. □

In Chapter 2, we used the map Φ, constructed from characters, to parametrize a big chunk of the projective toric variety $X_{P\cap M}$. In contrast, Proposition 5.4.7 uses the map Ψ, constructed from P-monomials, to parametrize *all* of $X_{P\cap M}$.

If the lattice polytope P is very ample, then the results of Chapter 2 imply that $X_{P\cap M}$ is the toric variety X_P. So in the very ample case, the P-monomials give an explicit construction of the quotient $\big(\mathbb{C}^r \setminus Z(\Sigma_P)\big)/\!/G$ by mapping $\mathbb{C}^r \setminus Z(\Sigma_P)$ to projective space via the P-monomials. It follows that we have two ways to take the quotient of $\mathbb{C}^r$ by G:

- At the beginning of the chapter, we took G-invariant polynomials—elements of S_0—to construct an affine quotient.
- Here, we use P-monomials—elements of S_α—to construct a projective quotient, after removing a set $Z(\Sigma_P)$ of "bad" points.

The P-monomials are not G-invariant but instead transform the *same* way under G. This is why we map to projective space rather than affine space. We will explore these ideas further in Chapter 14 when we discuss *geometric invariant theory*.

When P is very ample, we have a projective embedding $X_P \subseteq \mathbb{P}^{s-1}$ given by the P-monomials in S_α. If $y_1,\dots,y_s$ are homogeneous coordinates of $\mathbb{P}^{s-1}$, then the *homogeneous coordinate ring* of $X_P \subseteq \mathbb{P}^{s-1}$ is

$$\mathbb{C}[X_P] = \mathbb{C}[y_1,\dots,y_s]/\mathbf{I}(X_P)$$

as in §2.0. We also have the affine cone $\widehat{X}_P \subseteq \mathbb{C}^s$ of X_P, and $\mathbb{C}[X_P]$ is the ordinary coordinate ring of $\widehat{X}_P$, i.e.,

$$\mathbb{C}[X_P] = \mathbb{C}[\widehat{X}_P].$$

Recall that $\mathbb{C}[X_P]$ is an $\mathbb{N}$-graded ring since $\mathbf{I}(X_P)$ is a homogeneous ideal.

Another $\mathbb{N}$-graded ring is $\bigoplus_{k=0}^\infty S_{k\alpha}$. This relates to $\mathbb{C}[X_P]$ as follows.

Theorem 5.4.8. *Let P be a very ample lattice polytope with $\alpha = [D_P] \in \mathrm{Cl}(X_P)$. Then:*

(a) $\bigoplus_{k=0}^\infty S_{k\alpha}$ *is normal.*

(b) *There is a natural inclusion $\mathbb{C}[X_P] \subseteq \bigoplus_{k=0}^\infty S_{k\alpha}$ such that $\bigoplus_{k=0}^\infty S_{k\alpha}$ is the normalization of $\mathbb{C}[X_P]$.*

(c) *The following are equivalent:*

  (1) $X_P \subseteq \mathbb{P}^{s-1}$ *is projectively normal.*

  (2) *P is normal.*

  (3) $\bigoplus_{k=0}^\infty S_{k\alpha} = \mathbb{C}[X_P]$.

  (4) $\bigoplus_{k=0}^\infty S_{k\alpha}$ *is generated as a $\mathbb{C}$-algebra by its elements of degree* 1.

Proof. Consider the cone

$$C(P) = \mathrm{Cone}(P \times \{1\}) \subseteq M_\mathbb{R} \times \mathbb{R}.$$

This cone is pictured in Figure 4 of §2.2. Recall that kP is the "slice" of $C(P)$ at height k. Since the divisor D_{kP} associated to kP is kD_P, homogenization with respect to kP induces an isomorphism

$$S_{k\alpha} \simeq \Gamma(X_P, \mathscr{O}_{X_P}(kD_P)) \simeq \bigoplus_{m \in (kP)\cap M} \mathbb{C}\cdot\chi^m.$$

Now consider the dual cone $\sigma_P = C(P)^\vee \subseteq N_\mathbb{R} \times \mathbb{R}$. The semigroup algebra $\mathbb{C}[C(P)\cap(M\times\mathbb{Z})]$ is the coordinate ring of the affine toric variety U_{σ_P}. Given $(m,k) \in C(P)\cap(M\times\mathbb{Z})$, we write the corresponding character as $\chi^m t^k$.

The algebra $\mathbb{C}[C(P)\cap(M\times\mathbb{Z})]$ is graded using the last coordinate, the "height." Since $(m,k) \in C(P)\cap(M\times\mathbb{Z})$ if and only if $m \in kP$ (this is the "slice" observation made above), we have

$$\mathbb{C}[C(P)\cap(M\times\mathbb{Z})]_k = \bigoplus_{m\in(kP)\cap M} \mathbb{C}\cdot\chi^m t^k.$$

Using (5.4.3), we obtain a graded $\mathbb{C}$-algebra isomorphism

$$\bigoplus_{k=0}^{\infty} S_{k\alpha} \simeq \mathbb{C}[C(P)\cap(M\times\mathbb{Z})].$$

This proves that $\bigoplus_{k=0}^{\infty} S_{k\alpha}$ is normal.

We next claim that U_{σ_P} is the normalization of the affine cone $\widehat{X}_P$. For this, we let $\mathscr{A} = (P\cap M)\times\{1\} \subseteq M\times\mathbb{Z}$. As noted in the proof of Theorem 2.4.1, the affine cone of $X_P = X_{P\cap M}$ is $\widehat{X}_P = Y_{\mathscr{A}}$. Since P is very ample, one easily checks that $\mathscr{A}$ generates $M\times\mathbb{Z}$, i.e., $\mathbb{Z}\mathscr{A} = M\times\mathbb{Z}$ (Exercise 5.4.5). It is also clear that $\mathscr{A}$ generates the cone $C(P) = \sigma_P^{\vee}$. Hence U_{σ_P} is the normalization of $\widehat{X}_P$ by Proposition 1.3.8. This immediately implies part (b).

For part (c), we observe that (1) $\Leftrightarrow$ (2) follows from Theorem 2.4.1, and (1) $\Leftrightarrow$ (3) follows from parts (a) and (b) since the projective normality of $X_P \subseteq \mathbb{P}^{s-1}$ is equivalent to the normality of $\mathbb{C}[X_P]$. Also (3) $\Rightarrow$ (4) is obvious since $\mathbb{C}[X_P]$ is generated by the images of $y_1,\ldots,y_s$, which have degree 1. Finally, you will show in Exercise 5.4.6 that (4) $\Rightarrow$ (2), completing the proof. $\square$

Further Examples. We begin with an example of that illustrates how there can be many different polytopes that give the same toric variety.

Example 5.4.9. The toric surface in Example 5.4.5 was defined using the polygon shown in Figure 2. In Figure 3 we see four polygons $A, A\cup B, A\cup C, A\cup D$, all

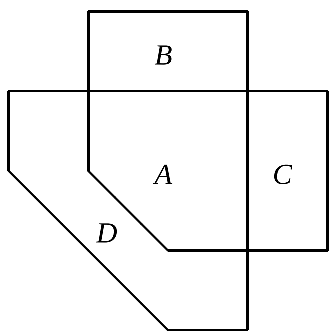


Figure 3. Four polygons $A, A\cup B, A\cup C, A\cup D$ with the same normal fan

of which have the same normal fan and hence give the same toric variety. Since we are in dimension 2, these polygons are very ample (in fact, normal), so that Theorem 5.4.8 applies.

These four polygons give four different projective embeddings, each of which has its own homogeneous coordinate ring as a projective variety. By Theorem 5.4.8, these homogeneous coordinate rings all live in the total coordinate ring S. This explains the “total” in “total coordinate ring.” $\Diamond$

Our next example involves torsion in the grading of the total coordinate ring.

Example 5.4.10. The fan Σ for $\mathbb{P}^4$ has ray generators $u_0 = -\sum_{i=1}^4 e_i$ and $u_i = e_i$ for $i = 1,\dots,4$ in $N = \mathbb{Z}^4$ and is the normal fan of the standard simplex $\Delta_4 \subseteq \mathbb{R}^4$. Another polytope with the same normal fan is

$$P = 5\Delta_4 - (1,1,1,1) \subseteq M_{\mathbb{R}} = \mathbb{R}^4,$$

so that $X_P = \mathbb{P}^4$. We saw that P is reflexive in Example 2.4.5. One checks that $D_P = D_0 + \cdots + D_4$ has degree $5 \in \mathbb{Z} \simeq \mathrm{Cl}(\mathbb{P}^4)$. Since P is a translate of $5\Delta_4$, (5.4.2) implies that the P-monomials for $m \in P \cap \mathbb{Z}^4$ coincide with the homogenizations coming from $5\Delta_4$, which are homogeneous polynomials of degree 5 in $S = \mathbb{C}[x_0,\dots,x_4]$.

Since P is reflexive, its dual P° is also a lattice polytope. Furthermore,

$$P^\circ = \mathrm{Conv}(u_0,\dots,u_4) \subseteq N_{\mathbb{R}} = \mathbb{R}^4$$

since the ray generators of the normal fan of P° are the *vertices* of P by duality for reflexive polytopes (be sure you understand this—Exercise 5.4.7). The vertices of P are

$$\begin{aligned} &\mathsf{v}_0 = (-1,-1,-1,-1),\ \mathsf{v}_1 = (4,-1,-1,-1),\ \mathsf{v}_2 = (-1,4,-1,-1)\\ &\mathsf{v}_3 = (-1,-1,4,-1),\ \mathsf{v}_4 = (-1,-1,-1,4). \end{aligned} \tag{5.4.8}$$

The v_i generate a sublattice $M_1 \subseteq M = \mathbb{Z}^4$. In Exercise 5.4.7 you will show that the map $M \to \mathbb{Z}^5$ defined by

$$m \in M \longmapsto (\langle m,u_0\rangle,\dots,\langle m,u_4\rangle) \in \mathbb{Z}^5$$

induces an isomorphism

$$M/M_1 \simeq \big\{(a_0,a_1,a_2,a_3,a_4) \in (\mathbb{Z}/5\mathbb{Z})^5 : \textstyle\sum_{i=0}^4 a_i = 0\big\}/(\mathbb{Z}/5\mathbb{Z}) \tag{5.4.9}$$

where $\mathbb{Z}/5\mathbb{Z} \subseteq (\mathbb{Z}/5\mathbb{Z})^5$ is the diagonal subgroup. Then $M/M_1 \simeq (\mathbb{Z}/5\mathbb{Z})^3$, so that M_1 is a lattice of index 125 in M.

The dual toric variety X_{P° is determined by the normal fan Σ° of P°. The ray generators of Σ° are the vectors $\mathsf{v}_0,\dots,\mathsf{v}_4$ from (5.4.8). The only possible complete fan in $\mathbb{R}^4$ with these ray generators is the fan whose cones are generated by all proper subsets of $\{\mathsf{v}_0,\dots,\mathsf{v}_4\}$. Since $\mathsf{v}_0 + \cdots + \mathsf{v}_4 = 0$ and the v_i generate M_1, the toric variety of Σ° relative to M_1 is $\mathbb{P}^4$, i.e., $X_{\Sigma^\circ,M_1} = \mathbb{P}^4$. (Remember that Σ° is a fan in $(M_1)_{\mathbb{R}} = M_{\mathbb{R}}$.) Since $M_1 \subseteq M$ has index 125, Proposition 3.3.7 implies

$$X_{P^\circ} = X_{P^\circ,M} \simeq X_{P^\circ,M_1}/(M/M_1) = \mathbb{P}^4/(M/M_1).$$

Hence the dual toric variety X_{P° is the quotient of $\mathbb{P}^4$ by a group of order 125.

The total coordinate ring S° is the polynomial ring $\mathbb{C}[y_0,\dots,y_4]$, graded by $\mathrm{Cl}(X_{P^\circ})$. The notation is challenging, since by duality N is the character lattice of the torus of X_{P°. Thus (5.1.1) becomes the short exact sequence

$$0 \longrightarrow N \longrightarrow \mathbb{Z}^5 \longrightarrow \mathrm{Cl}(X_{P^\circ}) \longrightarrow 0,$$

where $N \to \mathbb{Z}^5$ is $u \mapsto (\langle \mathsf{v}_0, u\rangle, \ldots, \langle \mathsf{v}_4, u\rangle)$. If we let $N_1 = \mathrm{Hom}_{\mathbb{Z}}(M_1, \mathbb{Z})$, then $M_1 \subseteq M$ dualizes to $N \subseteq N_1$ of index 125. Now consider the diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & N & \longrightarrow & \mathbb{Z}^5 & \longrightarrow & \mathrm{Cl}(X_{P^\circ}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & \mathbb{Z}^5 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & N_1/N & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

with exact rows and columns. In the middle row, we use $\mathrm{Cl}(X_{\Sigma^\circ, M_1}) = \mathrm{Cl}(\mathbb{P}^4) = \mathbb{Z}$. By the snake lemma, we obtain the exact sequence

$$0 \longrightarrow N_1/N \longrightarrow \mathrm{Cl}(X_{P^\circ}) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

so $\mathrm{Cl}(X_{P^\circ}) \simeq \mathbb{Z} \oplus N/N_1$. Thus the class group has torsion.

The polytope P° has only six lattice points in N: the vertices $u_0, \ldots, u_4$ and the origin (Exercise 5.4.7). When we homogenize these, we get six P°-monomials

$$y^{\langle 0, D\rangle} = \prod_{j=0}^{4} y_j^{\langle \mathsf{v}_j, 0\rangle + 1} = y_0 \cdots y_4$$
$$y^{\langle u_i, D\rangle} = \prod_{j=0}^{4} y_j^{\langle \mathsf{v}_j, u_i\rangle + 1} = y_i^5, \quad i = 0, \ldots, 4$$

since $\langle \mathsf{v}_j, u_i\rangle = 5\delta_{ij} - 1$ (Exercise 5.4.7). ◊

The equation

$$c_0 y_0^5 + \cdots + c_4 y_4^5 + c_5 y_0 \cdots y_4 = 0$$

defines a hypersurface $Y \subseteq X_{P^\circ}$ since it is built from P°-monomials. If we want an irreducible hypersurface, we must have $c_0, \ldots, c_4 \neq 0$, in which case Y is isomorphic (via the torus action) to a hypersurface of the form

$$y_0^5 + \cdots + y_4^5 + \lambda y_0 \cdots y_4 = 0.$$

This is the *quintic mirror family*, which played a crucial role in the development of mirror symmetry. See [**68**] for an introduction to this astonishing subject.

Exercises for §5.4.

5.4.1. Let D, E be linearly equivalent torus-invariant divisors with $D = \mathrm{div}(\chi^m) + E$.

(a) If $m' \in P_D \cap M$, then prove that $x^{\langle m', D\rangle} = x^{\langle m'+m, E\rangle}$.

(b) Prove (5.4.2).

5.4.2. Fix a torus-invariant divisor $D = \sum_\rho a_\rho D_\rho$ and consider its associated polyhedron $P_D = \{m \in M_\mathbb{R} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho\}$. Define

$$\phi_D : M_\mathbb{R} \longrightarrow \mathbb{R}^{\Sigma(1)}$$

by $\phi_D(m) = (\langle m, u_\rho \rangle + a_\rho) \in \mathbb{R}^{\Sigma(1)}$.

(a) Prove that ϕ_D embeds $M_\mathbb{R}$ as an affine subspace of $\mathbb{R}^{\Sigma(1)}$. Hint: Remember that X_Σ has no torus factors.

(b) Prove that ϕ_D induces a bijection

$$\phi_D|_{P_D} : P_D \simeq \phi_D(M_\mathbb{R}) \cap \mathbb{R}^{\Sigma(1)}_{\geq 0}.$$

This realizes P_D as the polyhedron obtained by intersecting the positive orthant $\mathbb{R}^{\Sigma(1)}_{\geq 0}$ of $\mathbb{R}^{\Sigma(1)}$ with an affine subspace.

(c) Let $D = \mathrm{div}(\chi^m) + E$. Prove that $\phi_D(P_D) = \phi_E(P_E)$. Thus the polyhedron in $\mathbb{R}^{\Sigma(1)}$ constructed in part (b) depends only on the divisor class of D. This is the "canonical model" of P_D.

5.4.3. Prove that the diagram (5.4.3) is commutative.

5.4.4. The proof of Proposition 5.4.7 claimed that the image of $\phi : X_P \to \mathbb{P}^{s-1}$ was Zariski closed. This follows from the general fact that if $\phi : X \to Y$ is a morphism of varieties and X is complete, then $\phi(X)$ is Zariski closed in Y. You will prove this two ways.

(a) Give a topological proof that uses constructible sets and compactness. Hint: Remember that projective space is compact.

(b) Give an algebraic proof that uses completeness and properness from §3.4. Hint: Show that $X \times Y \to Y$ is proper and use the graph of ϕ.

5.4.5. Let $P \subseteq M_\mathbb{R}$ be a very ample lattice polytope and let $\mathscr{A} = (P \cap M) \times \{1\} \subseteq M \times \mathbb{Z}$. Prove that $\mathbb{Z}\mathscr{A} = M \times \mathbb{Z}$. Hint: First show that $\mathbb{Z}'\mathscr{A} = M \times \{0\}$, where $\mathbb{Z}'\mathscr{A}$ is defined in the discussion preceding Proposition 2.1.6.

5.4.6. Prove of (4) $\Rightarrow$ (2) in part (c) of Theorem 5.4.8. Hint: (4) implies that the map $S_\alpha \otimes_\mathbb{C} S_{k\alpha} \to S_{(k+1)\alpha}$ is onto for all $k \geq 0$.

5.4.7. This exercise is concerned with Example 5.4.10.

(a) Prove that if $P \subseteq \mathbb{R}^n$ is reflexive, then the vertices of P are the ray generators of the normal fan of P°.

(b) Prove (5.4.9).

(c) Prove $\langle \mathrm{v}_j, u_i \rangle = 5\delta_{ij} - 1$, where v_j, u_i are defined in Example 5.4.10.

(d) Let $G = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_{P^\circ}), \mathbb{C}^*) \subseteq (\mathbb{C}^*)^5$. Use Proposition 1.3.18 to prove

$$G = \{(\lambda\zeta_0, \dots, \lambda\zeta_4) \mid \lambda \in \mathbb{C}^*,\ \zeta_i \in \mu_5,\ \zeta_0 \cdots \zeta_4 = 1\} \simeq \mathbb{C}^* \oplus M/M_1.$$

(e) Use part (e) and the quotient construction of X_{P° to give another proof that $X_{P^\circ} = \mathbb{P}^4/(M/M_1)$. Also give an explicit description of the action of M/M_1 on $\mathbb{P}^4$.

5.4.8. This exercise will give another way to think about homogenization. Let $e_1, \dots, e_n$ be a basis of M, so that $t_i = \chi^{e_i}$, $i = 1, \dots, n$, are coordinates for the torus T_N.

(a) Adapt the proof of (5.4.7) to show that $t_i = \prod_\rho x_\rho^{\langle e_i, u_\rho \rangle}$ when we think of the x_ρ as characters on $(\mathbb{C}^*)^{\Sigma(1)}$.

(b) Given $m \in P_D \cap M$, part (a) tells us that the Laurent monomial t^m can be regarded as a Laurent monomial in the x_ρ. Show that we can "clear denominators" by multiplying by $\prod_\rho x_\rho^{a_\rho}$ to obtain a monomial in the polynomial ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$.

(c) Show that this monomial obtained in part (b) is the homogenization $x^{\langle m,D\rangle}$.

5.4.9. Consider the toric variety X_P of Example 5.4.5.

(a) Compute $\mathrm{Cl}(X_P)$ and find the classes of the four polygons appearing in Figure 3.

(b) Show that X_P is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point.

5.4.10. Consider the reflexive polytope $P = 4\Delta_3 - (1,1,1) \subseteq \mathbb{R}^3$. Work out the analog of Example 5.4.10 for P.

5.4.11. Fix an integer $a \geq 1$ and consider the 3-simplex $P = \mathrm{Conv}(0, ae_1, ae_2, e_3) \subseteq \mathbb{R}^3$. In Exercise 2.2.13, we claimed that the toric variety of P is the weighted projective space $\mathbb{P}(1,1,1,a)$. Prove this.

5.4.12. Consider positive integers $1 = q_0 \leq q_1 \leq \cdots \leq q_n$ with the property that $q_i \mid \sum_{j=0}^n q_j$ for $i = 0,\ldots,n$. Set $k_i = \big(\sum_{j=0}^n q_j\big)/q_i$ for $i = 1,\ldots,n$ and let

$$P_{q_0,\ldots,q_n} = \mathrm{Conv}(0, k_1e_1, k_2e_2, \ldots, k_ne_n) - (1,\ldots,1) \subseteq \mathbb{R}^n.$$

This lattice polytope is reflexive by Exercise 2.4.6. Prove that the associated toric variety is the weighted projective space $\mathbb{P}(q_0, q_2, \ldots, q_n)$.

Chapter 6

Line Bundles on Toric Varieties

§6.0. Background: Sheaves and Line Bundles

Sheaves of $\mathscr{O}_X$-modules on a variety X were introduced in §4.0. Recall that for an affine variety $V = \mathrm{Spec}(R)$, an R-module M gives a sheaf $\widetilde{M}$ on V such that $\widetilde{M}(V_f) = M_f$ for all $f \neq 0$ in R. Globalizing this leads to quasicoherent sheaves on X. These include coherent sheaves, which locally come from finitely generated modules. In this section we develop the language of sheaf theory and discuss vector bundles and line bundles.

The Stalk of a Sheaf at a Point. Since sheaves are local in nature, we need a method for inspecting a sheaf at a point $p \in X$. This is provided by the notion of *direct limit* over a *directed set*.

Definition 6.0.1. A partially ordered set $(I, \preceq)$ is a ***directed set*** if

$$\text{for all } i, j \in I, \text{ there exists } k \in I \text{ such that } i \preceq k \text{ and } j \preceq k.$$

If $\{R_i\}$ is a family of rings indexed by a directed set $(I, \preceq)$ such that whenever $i \preceq j$ there is a homomorphism

$$\mu_{ji} : R_i \longrightarrow R_j$$

satisfying $\mu_{ii} = 1_{R_i}$ and $\mu_{kj} \circ \mu_{ji} = \mu_{ki}$, then the R_i form a ***directed system***. Let S be the submodule of $\bigoplus_{i \in I} R_i$ generated by the relations $r_j - \mu_{ji}(r_i)$, for $r_i \in R_i$ and $i \preceq j$. Then the ***direct limit*** is defined as

$$\varinjlim_{i \in I} R_i = \big(\textstyle\bigoplus_{i \in I} R_i\big)/S.$$

For simplicity, we often write the direct limit as $\varinjlim R_i$. Note also that references such as [**10**] write μ_{ij} instead of μ_{ji}.

For every $i \in I$, there is a natural map $R_i \to \varinjlim R_i$ such that whenever $i \preceq j$, the elements $r \in R_i$ and $\mu_{ji}(r) \in R_j$ have the same image in $\varinjlim R_i$. More generally, two elements $r_i \in R_i$ and $r_j \in R_j$ are identified in $\varinjlim R_i$ if there is a diagram

$$\begin{tikzcd} R_i \arrow[dr, "\mu_{ki}"] & \\ & R_k \\ R_j \arrow[ur, "\mu_{kj}"'] & \end{tikzcd}$$

such that $\mu_{ki}(r_i) = \mu_{kj}(r_j)$.

Example 6.0.2. Given $p \in X$, the definition of sheaf shows that the rings $\mathscr{O}_X(U)$, indexed by neighborhoods U of p, form a directed system under inclusion, so that the μ_{ji} are the restriction maps $\rho_{U,U'}$ for $p \in U' \subseteq U$. The direct limit is the local ring $\mathscr{O}_{X,p}$. For a quasicoherent sheaf $\mathscr{F}$, take an affine open subset $V = \mathrm{Spec}(R)$ containing p so that $\mathscr{F}(V) = M$, where M is an R-module. If $\mathfrak{m}_p = \mathbf{I}(p) \subseteq R$ is the corresponding maximal ideal, then $\mathscr{O}_{X,p}$ is the localization $R_{\mathfrak{m}_p}$, and

$$\varinjlim_{p\in U} \mathscr{F}(U) = M_{\mathfrak{m}_p},$$

where $M_{\mathfrak{m}_p}$ is the localization of M at the maximal ideal $\mathfrak{m}_p$. $\Diamond$

The term *sheaf* has agrarian origins: farmers harvesting their wheat tied a rope around a big bundle, and left it standing to dry. Think of the footprint of the bundle as an open set, so that increasingly smaller neighborhoods around a point on the ground pick out smaller and smaller bits of the bundle, narrowing to a single stalk.

Definition 6.0.3. The ***stalk*** of a sheaf $\mathscr{F}$ at a point $p \in X$ is $\mathscr{F}_p = \varinjlim_{p\in U} \mathscr{F}(U)$.

Injective and Surjective. A homomorphism $\phi : \mathscr{F} \to \mathscr{G}$ of $\mathscr{O}_X$-modules was defined in §4.0. We can also define what it means for ϕ to be injective or surjective. The definition is a bit unexpected, since we need to take into account the fact that sheaves are built to convey local data.

Definition 6.0.4. A sheaf homomorphism

$$\phi : \mathscr{F} \longrightarrow \mathscr{G}$$

is ***injective*** if for any point $p \in X$ and open subset $U \subseteq X$ containing p, there exists an open subset $V \subseteq U$ containing p, with ϕ_V injective. Also, ϕ is ***surjective*** if for any point p and open subset U containing p and any $g \in \mathscr{G}(U)$, there is an open subset $V \subseteq U$ containing p and $f \in \mathscr{F}(V)$ such that $\phi_V(f) = \rho_{U,V}(g)$.

In Exercise 6.0.1 you will prove that for a sheaf homomorphism $\phi : \mathscr{F} \to \mathscr{G}$,

$$U \longmapsto \ker(\phi_U : \mathscr{F}(U) \to \mathscr{G}(U))$$

defines a sheaf denoted $\ker(\phi)$. You will also show that ϕ is injective exactly when the "naive" idea works, i.e., $\ker(\phi) = 0$. On the other hand, surjectivity of a sheaf homomorphism need not mean that the maps ϕ_U are surjective for all U. Here is an example.

Example 6.0.5. On $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, consider the Weil divisor $D = \{0\} \subseteq \mathbb{C} \subseteq \mathbb{P}^1$. If we write of $\mathbb{P}^1 = U_0 \cup U_1$ with $U_0 = \mathrm{Spec}(\mathbb{C}[t])$ and $U_1 = \mathrm{Spec}(\mathbb{C}[t^{-1}])$, then $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t)$. Since

$$\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(D)) = \{ f \in \mathbb{C}(t)^* \mid \mathrm{div}(f) + D \geq 0\} \cup \{0\},$$

it follows easily that we have global sections

$$1, t^{-1} \in \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(D)).$$

For any $f \in \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(D))$, multiplication by f gives a sheaf homomorphism $\mathscr{O}_{\mathbb{P}^1}(-D) \to \mathscr{O}_{\mathbb{P}^1}$. Doing this for $1, t^{-1} \in \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(D))$ gives

$$\mathscr{O}_{\mathbb{P}^1}(-D) \oplus \mathscr{O}_{\mathbb{P}^1}(-D) \longrightarrow \mathscr{O}_{\mathbb{P}^1}.$$

(Direct sums of sheaves will be defined below.) In Exercise 6.0.2 you will check that this sheaf homomorphism is surjective. However, taking global sections gives

$$0 \oplus 0 = \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-D)) \oplus \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-D)) \longrightarrow \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) = \mathbb{C},$$

which is clearly not surjective. $\Diamond$

There is an additional point to make here. Given $\phi : \mathscr{F} \to \mathscr{G}$, the presheaf

$$U \longmapsto \mathrm{im}(\phi_U : \mathscr{F}(U) \to \mathscr{G}(U))$$

need not be a sheaf. Fortunately, this can be rectified. Given a presheaf $\mathscr{F}$, there is an associated sheaf $\mathscr{F}^+$, the *sheafification* of $\mathscr{F}$, which is defined by

$$\begin{aligned}\mathscr{F}^+(U) = \{ f : U \to \textstyle\prod_{p \in U} \mathscr{F}_p \mid &\text{ for all } p \in U, f(p) \in \mathscr{F}_p \text{ and there is}\\ &p \in V_p \subseteq U \text{ and } t \in \mathscr{F}(V_p) \text{ with } f(x) = t_p \text{ for all } x \in V_p\}.\end{aligned}$$

See [**131**, II.1] for a proof that $\mathscr{F}^+$ is a sheaf with the same stalks as $\mathscr{F}_p$. Hence

$$U \longmapsto \mathrm{im}(\phi_U)$$

has a natural sheaf associated to it, denoted $\mathrm{im}(\phi)$.

Exactness. We define exact sequences of sheaves as follows.

Definition 6.0.6. A sequence of sheaves

$$\mathscr{F}^{i-1} \xrightarrow{d^{i-1}} \mathscr{F}^{i} \xrightarrow{d^{i}} \mathscr{F}^{i+1}$$

is ***exact*** at $\mathscr{F}^i$ if there is an equality of sheaves

$$\ker(d^i) = \operatorname{im}(d^{i-1}).$$

The local nature of sheaves is again highlighted by the following result, whose proof may be found in [**131**, II.1].

Proposition 6.0.7. *The sequence in Definition 6.0.6 is exact if and only if*

$$\mathscr{F}_p^{i-1} \xrightarrow{d_p^{i-1}} \mathscr{F}_p^{i} \xrightarrow{d_p^{i}} \mathscr{F}_p^{i+1}$$

is exact for all $p \in X$. □

It follows from Example 6.0.5 that if

$$0 \longrightarrow \mathscr{F}^1 \xrightarrow{d^1} \mathscr{F}^2 \xrightarrow{d^2} \mathscr{F}^3 \longrightarrow 0 \tag{6.0.1}$$

is a short exact sequence of sheaves, the corresponding sequence of global sections may fail to be exact. However, we always have the following partial exactness, which you will prove in Exercise 6.0.3.

Proposition 6.0.8. *Given a short exact sequence of sheaves* (6.0.1)*, taking global sections gives the exact sequence*

$$0 \longrightarrow \Gamma(X,\mathscr{F}^1) \xrightarrow{d^1} \Gamma(X,\mathscr{F}^2) \xrightarrow{d^2} \Gamma(X,\mathscr{F}^3).$$

In Chapter 9 we will use *sheaf cohomology* to extend this exact sequence.

Example 6.0.9. For an affine variety $V = \operatorname{Spec}(R)$, an R-module M gives a quasicoherent sheaf $\widetilde{M}$ on V. This operation preserves exactness, i.e., an exact sequence of R-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

gives an exact sequence of sheaves

$$0 \longrightarrow \widetilde{M}_1 \longrightarrow \widetilde{M}_2 \longrightarrow \widetilde{M}_3 \longrightarrow 0$$

(see [**131**, Prop. II.5.2]). ◊

Here is a toric generalization of this example.

Example 6.0.10. Let $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ be the total coordinate ring of a toric variety X_Σ without torus factors. We saw in §5.3 that a graded S-module M gives the quasicoherent sheaf $\widetilde{\mathsf{M}}$ on X.

Then an exact sequence $0 \to \mathsf{M}_1 \to \mathsf{M}_2 \to \mathsf{M}_3 \to 0$ of graded S-modules gives an exact sequence

$$0 \longrightarrow \widetilde{\mathsf{M}}_1 \longrightarrow \widetilde{\mathsf{M}}_2 \longrightarrow \widetilde{\mathsf{M}}_3 \longrightarrow 0$$

on X_Σ. To see why, note that for $\sigma \in \Sigma$, the restriction of $\widetilde{\mathsf{M}}_i$ to $U_\sigma \subseteq X_\Sigma$ is the sheaf associated to $((\mathsf{M}_i)_{x^{\hat{\sigma}}})_0$, the elements of degree 0 in the localization of M_i at $x^{\hat{\sigma}} \in S$. Localization preserves exactness, as does taking elements of degree 0. The desired exactness then follows from Example 6.0.9. $\Diamond$

Another example is the following exact sequence of sheaves from §3.0.

Example 6.0.11. A closed subvariety $i : Y \hookrightarrow X$ gives two sheaves:

- The sheaf $\mathscr{I}_Y$, defined by $\mathscr{I}_Y(U) = \{f \in \mathscr{O}_X(U) \mid f(p) = 0 \text{ for } p \in Y \cap U\}$.
- The direct image sheaf $i_*\mathscr{O}_Y$, defined by $i_*\mathscr{O}_Y(U) = \mathscr{O}_Y(Y \cap U)$.

These are coherent sheaves on X and are related by the exact sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{O}_X \longrightarrow i_*\mathscr{O}_Y \longrightarrow 0. \qquad \Diamond$$

Operations on Quasicoherent Sheaves of $\mathscr{O}_X$. Operations on modules over a ring have natural analogs for quasicoherent sheaves. In particular, given quasicoherent sheaves $\mathscr{F}, \mathscr{G}$, it is easy to show that $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ defines the quasicoherent sheaf $\mathscr{F} \oplus \mathscr{G}$. We can also define $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ via

$$U \longmapsto \mathrm{Hom}_{\mathscr{O}_X(U)}(\mathscr{F}(U), \mathscr{G}(U)).$$

In Exercise 6.0.4 you will show that $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is a quasicoherent sheaf.

On the other hand, $U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$ is only a presheaf, so the tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is defined to be the sheaf associated to this presheaf. This sheaf is again quasicoherent and satisfies

$$\Gamma(U, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) = \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$$

whenever $U \subseteq X$ is an affine open set (see [**131**, Prop. II.5.2]).

Global Generation. For a module M over a ring, there is always a surjection from a free module onto M. This is true for a sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules when $\Gamma(X, \mathscr{F})$ is, in a certain sense, large enough.

Definition 6.0.12. A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is ***generated by global sections*** if there exists a set $\{s_i\} \subseteq \Gamma(X, \mathscr{F})$ such that at any point $p \in X$, the images of the s_i generate the stalk $\mathscr{F}_p$.

Any global section $s \in \Gamma(X, \mathscr{F})$ gives a sheaf homomorphism $\mathscr{O}_X \to \mathscr{F}$. It follows that if $\mathscr{F}$ is generated by $\{s_i\}_{i \in I}$, there is a surjection of sheaves

$$\bigoplus_{i \in I} \mathscr{O}_X \longrightarrow \mathscr{F}.$$

In the next section we will see that when X is toric, there is a particularly nice way of determining when the sheaves $\mathscr{O}_X(D)$ are generated by global sections.

Locally Free Sheaves and Vector Bundles. We begin with locally free sheaves.

Definition 6.0.13. A sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is ***locally free of rank r*** if there exists an open cover $\{U_\alpha\}$ of X such that for all α, $\mathscr{F}|_{U_\alpha} \simeq \mathscr{O}_{U_\alpha}^r$.

Locally free sheaves are closely related to vector bundles.

Definition 6.0.14. A variety V is a ***vector bundle of rank*** r over a variety X if there is a morphism

$$\pi : V \longrightarrow X$$

and an open cover $\{U_i\}$ of X such that:

(a) For every i, there is an isomorphism

$$\phi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}^r$$

such that ϕ_i followed by projection onto U_i is $\pi|_{\pi^{-1}(U_i)}$.

(b) For every pair i, j, there is $g_{ij} \in \mathrm{GL}_r(\Gamma(U_i \cap U_j, \mathscr{O}_X))$ such that the diagram

$$\begin{tikzcd} & U_i \cap U_j \times \mathbb{C}^r \\ \pi^{-1}(U_i \cap U_j) \arrow[ur, "\phi_i|_{\pi^{-1}(U_i\cap U_j)}"] \arrow[dr, "\phi_j|_{\pi^{-1}(U_i\cap U_j)}"'] & \\ & U_i \cap U_j \times \mathbb{C}^r \arrow[uu, "1\times g_{ij}"'] \end{tikzcd}$$

commutes.

Data $\{(U_i, \phi_i)\}$ satisfying properties (a) and (b) is called a *trivialization*. The map $\phi_i : \pi^{-1}(U_i) \simeq U_i \times \mathbb{C}^r$ gives a *chart*, where $\pi^{-1}(p) \simeq \mathbb{C}^r$ for $p \in U_i$. We call $\pi^{-1}(p)$ the *fiber* over p. See Figure 1 on the next page.

For $p \in U_i \cap U_j$, the isomorphisms

$$\mathbb{C}^r \simeq \{p\} \times \mathbb{C}^r \xleftarrow{\sim} \pi^{-1}(p) \xrightarrow{\sim} \{p\} \times \mathbb{C}^r \simeq \mathbb{C}^r$$

given by ϕ_i and ϕ_j are related by the linear map $g_{ij}(p)$. Hence the fiber $\phi^{-1}(p)$ has a well-defined vector space structure. This shows that a vector bundle really is a "bundle" of vector spaces.

On a vector bundle, the g_{ij} are called *transition functions* and can be regarded as a *family* of transition matrices that vary as $p \in U_i \cap U_j$ varies. Just as there is no preferred basis for a vector space, there is no canonical choice of basis for a particular fiber. Note also that the transition functions satisfy the compatibility conditions

$$\begin{aligned} g_{ik} &= g_{ij} \circ g_{jk} \text{ on } U_i \cap U_j \cap U_k \\ g_{ij} &= g_{ji}^{-1} \qquad \text{on } U_i \cap U_j. \end{aligned} \tag{6.0.2}$$

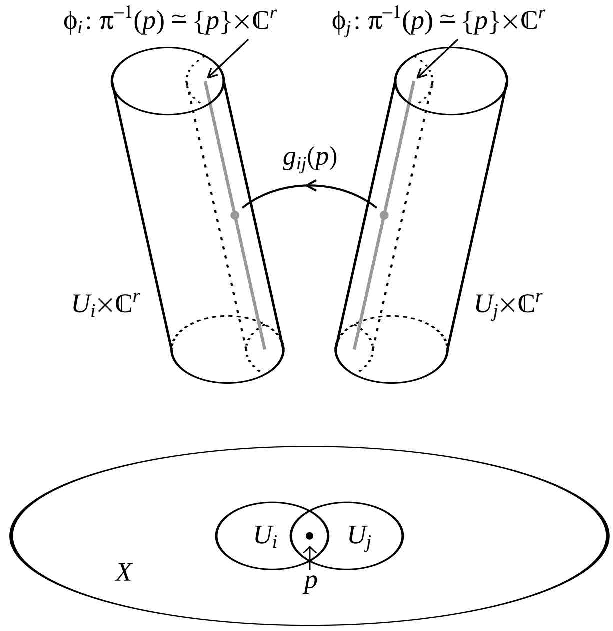


Figure 1. Visualizing a vector bundle

Definition 6.0.15. A ***section*** of a vector bundle V over $U \subseteq X$ open is a morphism

$$s : U \longrightarrow V$$

such that $(\pi \circ s)(p) = p$ for all $p \in U$. A section $s : X \to V$ is a ***global section***.

A section s picks out a point $s(p)$ in each fiber $\pi^{-1}(p)$, as shown in Figure 2.

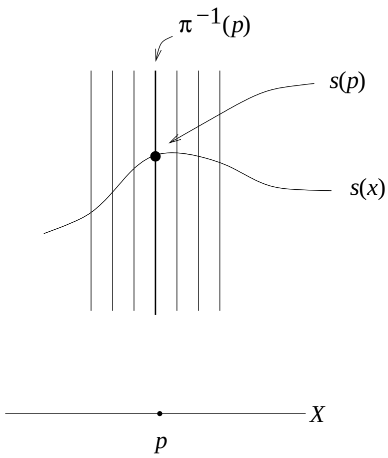


Figure 2. For a section s, $s(p) \in \pi^{-1}(p)$

We can describe a vector bundle and its global sections purely in terms of the transition functions g_{ij} as follows.

Proposition 6.0.16. *Let X be a variety with an affine open cover $\{U_i\}$, and assume that for every i,j, we have $g_{ij} \in \mathrm{GL}_r(\Gamma(U_i \cap U_j, \mathscr{O}_X))$ satisfying the compatibility conditions* (6.0.2). *Then:*

(a) *There is a vector bundle $\pi : V \to X$ of rank r, unique up to isomorphism, whose transition functions are the g_{ij}.*

(b) *A global section $s : X \to V$ is uniquely determined by a collection of r-tuples $s_i \in \mathscr{O}_X^r$ such that for all i,j,*

$$s_i|_{U_i \cap U_j} = g_{ij} s_j|_{U_i \cap U_j}.$$

Proof. One easily checks that the g_{ij}^{-1} satisfy the gluing conditions from §3.0. It follows that the affine varieties $U_i \times \mathbb{C}^r$ glue together to give a variety V. Furthermore, the projection maps $U_i \times \mathbb{C}^r \to U_i$ glue together to give a morphism $\pi : V \to X$. It follows easily that the open set of V corresponding to $U_i \times \mathbb{C}^r$ is $\pi^{-1}(U_i)$, which gives an isomorphism $\phi_i : \pi^{-1}(U_i) \simeq U_i \times \mathbb{C}^r$. Hence V is a vector bundle with transition functions g_{ij}.

Given a section $s : X \to V$, $\phi_i \circ s|_{U_i}$ is a section of $U_i \times \mathbb{C}^r \to U_i$. Thus

$$\phi_i \circ s|_{U_i}(p) = (p, s_i(p)) \in U_i \times \mathbb{C}^r,$$

where $s_i \in \mathscr{O}_X(U_i)^r$. By Definition 6.0.14, the s_i satisfy the desired compatibility condition, and since every global section arises this way, we are done. $\square$

Let $\mathscr{F}(U)$ denote the set of all sections of V over U. One easily sees that $\mathscr{F}$ is a sheaf on X and in fact is a sheaf of $\mathscr{O}_X$-modules since the fibers are vector spaces. In fact, $\mathscr{F}$ is an especially nice sheaf.

Proposition 6.0.17. *The sheaf of sections of a vector bundle is locally free.*

Proof. For a trivial vector bundle $U \times \mathbb{C}^r \to U$, the proof of Proposition 6.0.16 shows that a section is determined by a morphism $U \to \mathbb{C}^r$, i.e., an element of $\mathscr{O}_U(U)^r$. Thus the sheaf associated to a trivial vector bundle over U is $\mathscr{O}_U^r$.

For a general vector bundle $\pi : V \to X$ with trivialization $\{(U_i, \phi_i)\}$, each U_i gives an isomorphism of vector bundles

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\phi_i"] \arrow[dr, "\pi|_{\pi^{-1}(U_i)}"'] & & U_i \times \mathbb{C}^r \arrow[dl] \\ & U_i. & \end{tikzcd}$$

Since isomorphic vector bundles have isomorphic sheaves of sections, it follows that if $\mathscr{F}$ is the sheaf of sections of $\pi : V \to X$, then $\mathscr{F}|_{U_i} \simeq \mathscr{O}_{U_i}^r$. $\square$

Line Bundles and Cartier Divisors. Since a vector space of dimension one is a line, a vector bundle of rank 1 is called a *line bundle*. Despite the new terminology, line bundles are actually familiar objects when X is normal.

Theorem 6.0.18. *The sheaf $\mathscr{L} = \mathscr{O}_X(D)$ of a Cartier divisor D on a normal variety X is the sheaf of sections of a line bundle $V_{\mathscr{L}} \to X$.*

Proof. Recall from Chapter 4 that a Cartier divisor is locally principal, so that X has an affine open cover $\{U_i\}_{i\in I}$ with $D|_{U_i} = \operatorname{div}(f_i)|_{U_i}$, $f_i \in \mathbb{C}(X)^*$. Thus $\{(U_i, f_i)\}_{i\in I}$ is local data for D. Note also that

$$\operatorname{div}(f_i)|_{U_i\cap U_j} = \operatorname{div}(f_j)|_{U_i\cap U_j},$$

which implies $f_i/f_j \in \mathscr{O}_X(U_i \cap U_j)^*$ by Proposition 4.0.16.

We use this data to construct a line bundle as follows. Since

$$\operatorname{GL}_1(O_X(U_i\cap U_j)) = O_X(U_i\cap U_j)^*,$$

the quotients $g_{ij} = f_i/f_j$ may be regarded as transition functions. These satisfy the hypotheses of Proposition 6.0.16 and hence give a line bundle $\pi : V_{\mathscr{L}} \to X$.

A global section $f \in \Gamma(X, \mathscr{O}_X(D))$ satisfies $\operatorname{div}(f) + D \geq 0$, so that on U_i,

$$\operatorname{div}(f f_i)|_{U_i} = \operatorname{div}(f)|_{U_i} + \operatorname{div}(f_i)|_{U_i} = (\operatorname{div}(f) + D)|_{U_i} \geq 0.$$

This shows that $s_i = f_i f \in \mathscr{O}_X(D)(U_i)$. Then

$$g_{ij}s_j = f_i/f_j \cdot f_j f = f_i f = s_i,$$

which by part (b) of Proposition 6.0.16 gives a global section of $\pi : V_{\mathscr{L}} \to X$. Conversely, the proposition shows that a global section of $V_{\mathscr{L}} \to X$ gives functions $s_i \in \mathscr{O}_X(D)(U_i)$ such that $g_{ij}s_j = s_i$. It follows that $f = s_i/f_i \in \mathbb{C}(X)$ is independent of i. One easily checks that $f \in \Gamma(X, \mathscr{O}_X(D))$. The same argument works when we restrict to any open subset of X. It follows that $\mathscr{L} = \mathscr{O}_X(D)$ is the sheaf of sections of $\pi : V_{\mathscr{L}} \to X$. □

We will see shortly that this process is reversible, i.e., there is a one-to-one correspondence between line bundles and sheaves coming from Cartier divisors. First, we give an important example.

Example 6.0.19. When we regard $\mathbb{P}^n$ as the set of lines through the origin in $\mathbb{C}^{n+1}$, each point $p \in \mathbb{P}^n$ corresponds to a line $\ell_p \subseteq \mathbb{C}^{n+1}$. We assemble these lines into a line bundle as follows. Let $x_0,\ldots,x_n$ be homogeneous coordinates on $\mathbb{P}^n$ and $y_0,\ldots,y_n$ be coordinates on $\mathbb{C}^{n+1}$. Define

$$V \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}$$

as the locus where the matrix

$$\begin{pmatrix} x_0 & \cdots & x_n \\ y_0 & \cdots & y_n \end{pmatrix}$$

has rank one. Thus V is defined by the vanishing of $x_iy_j - x_jy_i$. Then define the map $\pi : V \to \mathbb{P}^n$ to be projection on the first factor of $\mathbb{P}^n \times \mathbb{C}^{n+1}$. To see that V is a line bundle, consider the open subset $\mathbb{C}^n \simeq U_i \subseteq \mathbb{P}^n$ where x_i is invertible. On $\pi^{-1}(U_i)$ the equations defining V become

$$\frac{x_j}{x_i}y_i = y_j, \quad \text{for all } j \neq i.$$

Thus $(x_0,\dots,x_n,y_0,\dots,y_n) \mapsto (x_0,\dots,,x_n,y_i)$ defines an isomorphism

$$\phi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}.$$

In other words, y_i is a local coordinate for the line $\mathbb{C}$ over U_i. Switching to the coordinate system over U_j, we have the local coordinate y_j, which over $U_i \cap U_j$ is related to y_i via

$$\frac{x_i}{x_j}y_j = y_i.$$

Hence the the transition function from $U_i \cap U_j \times \mathbb{C}$ to $U_i \cap U_j \times \mathbb{C}$ is given by

$$g_{ij} = \frac{x_i}{x_j} \in \mathscr{O}_{\mathbb{P}^n}(U_i \cap U_j)^*.$$

This bundle is called the *tautological bundle* on $\mathbb{P}^n$. In Example 6.0.21 below, we will describe the sheaf of sections of this bundle. ◊

Projective spaces are the simplest type of Grassmannian, and just as in this example, the construction of the Grassmannian shows that it comes equipped with a tautological vector bundle. In Exercise 6.0.5 you will determine the transition functions for the Grassmannian $\mathbb{G}(1,3)$.

Invertible Sheaves and the Picard Group. Propositions 6.0.17 and 6.0.18 imply that the sheaf $\mathscr{O}_X(D)$ of a Cartier divisor is locally free of rank 1. In general, a locally free sheaf of rank 1 is called an *invertible sheaf*.

The relation between Cartier divisors, line bundles and invertible sheaves is described in the following theorem.

Theorem 6.0.20. *Let $\mathscr{L}$ be an invertible sheaf on a normal variety X. Then:*

(a) *There is a Cartier divisor D on X such that $\mathscr{L} \simeq \mathscr{O}_X(D)$.*

(b) *There is a line bundle $V_{\mathscr{L}} \to X$ whose sheaf of sections is isomorphic to $\mathscr{L}$.*

Proof. The part (b) of the theorem follows from part (a) and Proposition 6.0.18. It remains to prove part (a).

Since X is irreducible, any nonempty open $U \subseteq X$ gives a domain $\mathscr{O}_X(U)$ with field of fractions $\mathbb{C}(U)$. By Exercise 3.0.4, $\mathbb{C}(U) = \mathbb{C}(X)$, so that $U \mapsto \mathbb{C}(U)$ defines a constant sheaf on X, denoted $\mathscr{K}_X$. This sheaf is relevant since $\mathscr{O}_X(D)$ is defined as a subsheaf of $\mathscr{K}_X$.

First assume that $\mathscr{L}$ is a subsheaf of $\mathscr{K}_X$. Pick an open cover $\{U_i\}$ of X such that $\mathscr{L}|_{U_i} \simeq \mathscr{O}_X|_{U_i}$ for every i. Over U_i, this gives homomorphisms

$$\mathscr{O}_X(U_i) \simeq \mathscr{L}(U_i) \hookrightarrow \mathbb{C}(X).$$

Let $f_i^{-1} \in \mathbb{C}(X)$ be the image of $1 \in \mathscr{O}_X(U_i)$. One can show without difficulty that $f_i/f_j \in \mathscr{O}_X(U_i \cap U_j)^*$. Then $\{(U_i, f_i)\}$ is local data for a Cartier divisor D on X satisfying $\mathscr{L} = \mathscr{O}_X(D)$.

For the general case, observe that on an irreducible variety, every locally constant sheaf is globally constant (Exercise 6.0.6). Now let $\mathscr{L}$ be any invertible sheaf on X. On a small enough open set U, $\mathscr{L}(U) \simeq \mathscr{O}_X(U)$, so that

$$\mathscr{L}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{K}_X(U) \simeq \mathscr{O}_X(U) \otimes_{\mathscr{O}_X(U)} \mathscr{K}_X(U) \simeq \mathscr{K}_X(U) = \mathbb{C}(X).$$

Thus $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is locally constant and hence constant. This easily implies that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{K}_X \simeq \mathscr{K}_X$, and composing this with the inclusion

$$\mathscr{L} \longrightarrow \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{K}_X$$

expresses $\mathscr{L}$ as a subsheaf of $\mathscr{K}_X$. □

We note without proof that the line bundle corresponding to an invertible sheaf is unique up to isomorphism. Because of this result, algebraic geometers tend to use the terms *line bundle* and *invertible sheaf* interchangeably, even though strictly speaking the latter is the sheaf of sections of the former.

We next discuss some properties of invertible sheaves coming from Cartier divisors. A first result is that if D and E are Cartier divisors on X, then

$$\mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{O}_X(E) \simeq \mathscr{O}_X(D+E). \tag{6.0.3}$$

This follows because $f \otimes g \mapsto fg$ induces a sheaf homomorphism

$$\mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{O}_X(E) \longrightarrow \mathscr{O}_X(D+E)$$

which is clearly an isomorphism on any open set where $\mathscr{O}_X(D)$ is trivial.

By standard properties of tensor product, the isomorphism (6.0.3) induces an isomorphism

$$\mathscr{O}_X(E) \simeq \mathcal{H}om_{\mathscr{O}_X}(\mathscr{O}_X(D), \mathscr{O}_X(D+E)).$$

In particular, when $E = -D$, we obtain

$$\mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-D) \simeq \mathscr{O}_X \quad \text{and} \quad \mathscr{O}_X(-D) \simeq \mathscr{O}_X(D)^\vee,$$

where $\mathscr{O}_X(D)^\vee = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{O}_X(D), \mathscr{O}_X)$ is the *dual* of $\mathscr{O}_X(D)$.

More generally, the tensor product of invertible sheaves is again invertible, and if $\mathscr{L}$ is invertible, then $\mathscr{L}^\vee = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$ is invertible and

$$\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}^\vee \simeq \mathscr{O}_X.$$

This explains why locally free sheaves of rank 1 are called invertible.

Example 6.0.21. There is a nice relation between the tautological bundle on $\mathbb{P}^n$ and the invertible sheaf $\mathscr{O}_{\mathbb{P}^n}(1)$ introduced in Example 4.3.1. Recall that the T_N-invariant divisors $D_0, \dots, D_n$ on $\mathbb{P}^n$ are all linearly equivalent, and so define isomorphic sheaves, usually denoted $\mathscr{O}_{\mathbb{P}^n}(1)$. The local data for the Cartier divisor D_0 is easily seen to be $\{(U_i, \frac{x_0}{x_i})\}$, where $U_i \subseteq \mathbb{P}^n$ is the open set where $x_i \neq 0$. Thus the transition functions for $\mathscr{O}_X(D_0)$ are given by

$$g_{ij} = \frac{\frac{x_0}{x_i}}{\frac{x_0}{x_j}} = \frac{x_j}{x_i}.$$

These are the inverses of the transition functions for the tautological bundle from Example 6.0.19. It follows that the sheaf of sections of the tautological bundle is $\mathscr{O}_{\mathbb{P}^n}(1)^\vee = \mathscr{O}_{\mathbb{P}^n}(-1)$. $\Diamond$

We can also explain when Cartier divisors give isomorphic invertible sheaves.

Proposition 6.0.22. *Two Cartier divisors D, E give isomorphic invertible sheaves $\mathscr{O}_X(D) \simeq \mathscr{O}_X(E)$ if and only if $D \sim E$.*

Proof. By Proposition 4.0.29, linearly equivalent Cartier divisors give isomorphic sheaves. For the converse, we first prove that $\mathscr{O}_X(D) = \mathscr{O}_X$ implies $D = 0$.

Assume $\mathscr{O}_X(D) = \mathscr{O}_X$. Then $1 \in \Gamma(X, \mathscr{O}_X) = \Gamma(X, \mathscr{O}_X(D))$, so $D \geq 0$. If $D \neq 0$, then we can pick an irreducible divisor D_0 that appears in D with positive coefficient. The local ring $\mathscr{O}_{X,D_0}$ is a DVR, so we can find $h \in \mathscr{O}_{X,D_0}$ with $\nu_{D_0}(h) = 1$. Set $U = X \setminus W$, where W is the union of all irreducible divisors $D' \neq Z_0$ with $\nu_{D'}(h) \neq 0$. There are only finitely many such divisors, so that U is a nonempty open subset of X with $U \cap D_0 \neq \emptyset$. Then $h \in \Gamma(U, \mathscr{O}_X)$, and $h^{-1} \notin \Gamma(U, \mathscr{O}_X)$ since h vanishes on $U \cap D_0$. However,

$$(D + \mathrm{div}(h^{-1}))|_U = (D - \mathrm{div}(h))|_U = (D - D_0)|_U \geq 0,$$

so that $h^{-1} \in \Gamma(U, \mathscr{O}_X(D)) = \Gamma(U, \mathscr{O}_X)$. This contradiction proves $D = 0$.

Now suppose that Cartier divisors D, E satisfy $\mathscr{O}_X(D) \simeq \mathscr{O}_X(E)$. Tensoring each side with $\mathscr{O}_X(-E)$ and applying (6.0.3), we see that $\mathscr{O}_X(D-E) \simeq \mathscr{O}_X$. If $1 \in \Gamma(X, \mathscr{O}_X)$ maps to $g \in \Gamma(X, \mathscr{O}_X(D-E))$ via this isomorphism, then

$$g\mathscr{O}_X = \mathscr{O}_X(D-E)$$

as subsheaves of $\mathscr{K}_X$. Thus

$$\mathscr{O}_X = g^{-1}\mathscr{O}_X(D-E) = \mathscr{O}(D - E + \mathrm{div}(g)),$$

where the last equality follows from the proof of Proposition 4.0.29. By the previous paragraph, we have $D - E + \mathrm{div}(g) = 0$, which implies that $D \sim E$. $\square$

In Chapter 4, the Picard group was defined as the quotient

$$\mathrm{Pic}(X) = \mathrm{CDiv}(X)/\mathrm{Div}_0(X).$$

We can interpret this in terms of invertible sheaves as follows. Given $\mathscr{L}$ invertible, Theorem 6.0.20 tells us that $\mathscr{L} \simeq \mathscr{O}_X(D)$ for some Cartier divisor D, which is unique up to linear equivalence by Proposition 6.0.22. Hence we have a bijection

$$\mathrm{Pic}(X) \simeq \{\text{isomorphism classes of invertible sheaves on } X\}.$$

The right-hand side has a group structure coming from tensor product of invertible sheaves. By (6.0.3), the above bijection is a group isomorphism.

In more sophisticated treatments of algebraic geometry, the Picard group of an arbitrary variety is defined using invertible sheaves. Also, Cartier divisors can be defined on an irreducible variety in terms of local data, without assuming normality (see [**131**, II.6]), though one loses the connection with Weil divisors. Since most of our applications involve toric varieties coming from fans, we will continue to assume normality when discussing Cartier divisors.

Stalks, Fibers, and Sections. From here on, we will think of a line bundle $\mathscr{L}$ on X as the sheaf of sections of a rank 1 vector bundle $\pi : V_{\mathscr{L}} \to X$. Given a section $s \in \mathscr{L}(U)$ and $p \in U$, we get the following:

- Since $V_{\mathscr{L}}$ is a vector bundle of rank 1, we have the *fiber* $\pi^{-1}(p) \simeq \mathbb{C}$. Then $s : U \to V_{\mathscr{L}}$ gives $s(p) \in \pi^{-1}(p)$.
- Since $\mathscr{L}$ is a locally free sheaf of rank 1, we have the *stalk* $\mathscr{L}_p \simeq \mathscr{O}_{X,p}$. Then $s \in \mathscr{L}(U)$ gives $s_p \in \mathscr{L}_p$.

In Exercise 6.0.7 you will show that these are related via the equivalences

$$\begin{aligned} s(p) \neq 0 \text{ in } \pi^{-1}(p) &\iff s_p \notin \mathfrak{m}_p\mathscr{L}_p \\ &\iff s_p \text{ generates } \mathscr{L}_p \text{ as an } \mathscr{O}_{X,p}\text{-module} \end{aligned} \tag{6.0.4}$$

A section s *vanishes* at $p \in X$ if $s(p) = 0$ in $\pi^{-1}(p)$, i.e., if $s_p \in \mathfrak{m}_p\mathscr{L}_p$.

Basepoints. It can happen that many sections of a line bundle vanish at a point p. This leads to the following definition.

Definition 6.0.23. A subspace $W \subseteq \Gamma(X, \mathscr{L})$ ***has no basepoints*** or ***is basepoint free*** if for every $p \in X$, there is $s \in W$ with $s(p) \neq 0$.

As noted earlier, a global section $s \in \Gamma(X, \mathscr{L})$ gives a sheaf homomorphism $\mathscr{O}_X \to \mathscr{L}$. Thus a subspace $W \subseteq \Gamma(X, \mathscr{L})$ gives

$$W \otimes_{\mathbb{C}} \mathscr{O}_X \longrightarrow \mathscr{L}$$

defined by $s \otimes h \mapsto hs$. Then (6.0.4) and Proposition 6.0.7 imply the following.

Proposition 6.0.24. *A subspace $W \subseteq \Gamma(X, \mathscr{L})$ has no basepoints if and only if $W \otimes_{\mathbb{C}} \mathscr{O}_X \to \mathscr{L}$ is surjective.* □

For a line bundle $\mathscr{L} = \mathscr{O}_X(D)$ of a Cartier divisor D on a normal variety, the vanishing locus of a global section has an especially nice interpretation. The local data $\{(U_i, f_i)\}$ of D gives the rank 1 vector bundle $\pi : V_{\mathscr{L}} \to X$ with transition functions $g_{ij} = f_i/f_j$. Hence we can think of a nonzero global section of $\mathscr{O}_X(D)$ in two ways:

- A rational function $f \in \mathbb{C}(X)^*$ satisfying $D + \mathrm{div}(f) \geq 0$.
- A morphism $s : X \to V_{\mathscr{L}}$ whose composition with π is the identity on X.

The relation between s and f is given in the proof of Theorem 6.0.18: over U_i, the section s looks like $(p, s_i(p))$ for $s_i = f_i f \in \mathscr{O}_X(U_i)$. It follows that $s = 0$ exactly when $s_i = 0$. Since $D|_{U_i} = \mathrm{div}(f_i)|_{U_i}$, the divisor of s_i on U_i is given by

$$\mathrm{div}(f_i f)|_{U_i} = (D + \mathrm{div}(f))|_{U_i}.$$

These patch together in the obvious way, so that the *divisor of zeros* of s is

$$\mathrm{div}_0(s) = D + \mathrm{div}(f).$$

Thus the divisor of zeros of a global section is an effective divisor that is linearly equivalent to D. It is also easy to see that *any* effective divisor linearly equivalent to D is the divisor of zeros of a global section of $\mathscr{O}_X(D)$ (Exercise 6.0.8).

In terms of Cartier divisors, Proposition 6.0.24 has the following corollary.

Corollary 6.0.25. *The following are equivalent for a Cartier divisor D:*

(a) *$\mathscr{O}_X(D)$ is generated by global sections in the sense of Definition 6.0.12.*

(b) *D is **basepoint free**, meaning that $\Gamma(X, \mathscr{O}_X(D))$ is basepoint free.*

(c) *For every $p \in X$ there is $s \in \Gamma(X, \mathscr{O}_X(D))$ with $p \notin \mathrm{Supp}(\mathrm{div}_0(s))$.* $\square$

The Pullback of a Line Bundle. Let $\mathscr{L}$ be a line bundle on X and $V_{\mathscr{L}} \to X$ the associated rank 1 vector bundle. A morphism $f : Z \to X$ gives the fibered product $f^*V_{\mathscr{L}} = V_{\mathscr{L}} \times_X Z$ from §3.0 that fits into the commutative diagram

$$\begin{tikzcd} f^*V_{\mathscr{L}} \arrow[r] \arrow[d] & V_{\mathscr{L}} \arrow[d, "\pi"] \\ Z \arrow[r, "f"'] & X. \end{tikzcd}$$

It is easy to see that $f^*V_{\mathscr{L}}$ is a rank 1 vector bundle over Z.

Definition 6.0.26. The ***pullback*** $f^*\mathscr{L}$ of the sheaf $\mathscr{L}$ is the sheaf of sections of the rank 1 vector bundle $f^*V_{\mathscr{L}}$ defined above.

Thus the pullback of a line bundle is again a line bundle. Furthermore, there is a natural map on global sections

$$f^* : \Gamma(X, \mathscr{L}) \longrightarrow \Gamma(Z, f^*\mathscr{L})$$

defined as follows. A global section $s : X \to V_{\mathscr{L}}$ gives the commutative diagram:

$$\begin{tikzcd}
Z \arrow[rr, "f"] \arrow[dr, dotted, "f^*(s)"] \arrow[ddr, bend right, "1_Z"'] & & X \arrow[d, "s"] \\
& f^*V_{\mathscr{L}} \arrow[r] \arrow[d] & V_{\mathscr{L}} \arrow[d, "\pi"] \\
& Z \arrow[r, "f"'] & X.
\end{tikzcd}$$

The universal property of fibered products guarantees the existence and uniqueness of the dotted arrow $f^*(s) : Z \to f^*V_{\mathscr{L}}$ that makes the diagram commute. It follows that $f^*(s) \in \Gamma(Z, f^*\mathscr{L})$.

Example 6.0.27. Let $X \subseteq \mathbb{P}^n$ be a projective variety. If we write the inclusion as $i : X \hookrightarrow \mathbb{P}^n$, then the line bundle $\mathscr{O}_{\mathbb{P}^n}(1)$ gives the line bundle $i^*\mathscr{O}_{\mathbb{P}^n}(1)$ on X. When the projective embedding of X is fixed, this line bundle is often denoted $\mathscr{O}_X(1)$.

Thus a projective variety always comes equipped with a line bundle. However, it is not unique, since the same variety may have many projective embeddings. You will work out an example of this in Exercise 6.0.9. $\Diamond$

In general, given a sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules on X and a morphism $f : Z \to X$, one gets a sheaf $f^*\mathscr{F}$ of $\mathscr{O}_Z$-modules on Z. The definition is more complicated, so we refer the reader to [**131**, II.5] for the details.

Line Bundles and Maps to Projective Space. We now reverse Example 6.0.27 by using a line bundle $\mathscr{L}$ on X to create a map to projective space.

Fix a finite-dimensional subspace $W \subseteq \Gamma(X, \mathscr{L})$ with no basepoints and let $W^\vee = \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$ be its dual. The projective space of $W^\vee$ is

$$\mathbb{P}(W^\vee) = (W^\vee \setminus \{0\})/\mathbb{C}^*.$$

We define a map $\phi_{\mathscr{L},W} : X \to \mathbb{P}(W^\vee)$ as follows. Fix $p \in X$ and pick a nonzero element $v_p \in \pi^{-1}(p) \simeq \mathbb{C}$, where $\pi : V_{\mathscr{L}} \to X$ is the rank 1 vector bundle associated to $\mathscr{L}$. For each $s \in W$, there is $\lambda_s \in \mathbb{C}$ such that $s(p) = \lambda_s v_p$. Then the map defined by $\ell_p(s) = \lambda_s$ is linear and nonzero since W has no basepoints. Thus $\ell_p \in W^\vee$, and since v_p is unique up to an element of $\mathbb{C}^*$, the same is true for ℓ_p. Then

$$\phi_{\mathscr{L},W}(p) = \ell_p$$

defines the desired map $\phi_{\mathscr{L},W} : X \to \mathbb{P}(W^\vee)$.

Lemma 6.0.28. *The map $\phi_{\mathscr{L},W} : X \to \mathbb{P}(W^\vee)$ is a morphism.*

Proof. Let $s_0, \ldots, s_m$ be a basis of W and let $U_i = \{p \in X \mid s_i(p) \neq 0\}$. These open sets cover X since W has no basepoints. Furthermore, the natural map

$$U_i \times \mathbb{C} \longrightarrow \pi^{-1}(U_i), \quad (p, \lambda) \longmapsto \lambda s_i(p)$$

is easily seen to be an isomorphism. Since all sections of $U_i \times \mathbb{C} \to \mathbb{C}$ are of the form $p \mapsto (p, h(p))$ for $h \in \mathscr{O}_X(U_i)$, it follows that for all $0 \le j \le m$, we can write $s_j|_{U_i} = h_{ij} s_i|_{U_i}$, $h_{ij} \in \mathscr{O}_X(U_i)$.

The definition of $\phi_{\mathscr{L},W}$ uses a nonzero vector $v_p \in \pi^{-1}(p)$. Over U_i, we can use $s_i(p) \in \pi^{-1}(p)$. Then $s_j(p) = h_{ij}(p) s_i(p)$ implies $\ell_p(s_j(p)) = h_{ij}(p)$. Since $\ell \mapsto (\ell(s_0), \dots, \ell(s_m))$ gives an isomorphism $\mathbb{P}(W^\vee) \simeq \mathbb{P}^m$, $\phi_{\mathscr{L},W}|_{U_i}$ can be written

$$U_i \longrightarrow \mathbb{P}^m, \quad p \longmapsto (h_{i0}(p), \dots, h_{im}(p)), \tag{6.0.5}$$

which is a morphism since $h_{ii} = 1$. □

When W has no basepoints and $s_0, \dots, s_m$ span W, $\phi_{\mathscr{L},W}$ is often written

$$X \longrightarrow \mathbb{P}^m, \quad p \longrightarrow (s_0(p), \dots, s_m(p)) \in \mathbb{P}^m \tag{6.0.6}$$

with the understanding that this means (6.0.5) on $U_i = \{p \in X \mid s_i(p) \neq 0\}$.

Furthermore, when $\mathscr{L} = \mathscr{O}_X(D)$, we can think of the global sections s_i as rational functions g_i such that $D + \operatorname{div}(g_i) \ge 0$. Then $\phi_{\mathscr{L},W}$ can be written

$$X \longrightarrow \mathbb{P}^m, \quad p \longrightarrow (g_0(p), \dots, g_m(p)) \in \mathbb{P}^m. \tag{6.0.7}$$

Since $g_i(p)$ may be undefined, this needs explanation. The local data $\{(U_j, f_j)\}$ of D implies that $f_j g_0, \dots, f_j g_m \in \mathscr{O}_X(U_j)$. Then (6.0.7) means that $\phi_{\mathscr{L},W}|_{U_j}$ is

$$U_j \longrightarrow \mathbb{P}^m, \quad p \longrightarrow (f_j g_0(p), \dots, f_j g_m(p)) \in \mathbb{P}^m.$$

This is a morphism on U_j since the global sections corresponding to $g_0, \dots, g_m$ have no base points.

Exercises for §6.0.

6.0.1. For a sheaf homomorphism $\phi : \mathscr{F} \to \mathscr{G}$, show that

$$U \longmapsto \ker(\phi_U)$$

defines a sheaf. Also prove that the following are equivalent:

(a) The kernel sheaf is identically zero.

(b) ϕ_U is injective for every open subset U.

(c) ϕ is injective as defined in Definition 6.0.4.

6.0.2. In Example 6.0.5, prove that $\mathscr{O}_{\mathbb{P}^1}(-D) \oplus \mathscr{O}_{\mathbb{P}^1}(-D) \to \mathscr{O}_{\mathbb{P}^1}$ is surjective.

6.0.3. Prove Proposition 6.0.8.

6.0.4. Let $\mathscr{F}, \mathscr{G}$ be quasicoherent sheaves on X. Prove that $U \mapsto \operatorname{Hom}_{\mathscr{O}_X(U)}(\mathscr{F}(U), \mathscr{G}(U))$ defines a quasicoherent sheaf $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$.

6.0.5. The Grassmannian $\mathbb{G}(1,3)$ is defined as the space of lines in $\mathbb{P}^3$, or equivalently, of 2-dimensional subspaces of $V = \mathbb{C}^4$. This exercise will construct the *tautological bundle* on $\mathbb{G}(1,3)$, which assembles these 2-dimensional subspaces into a rank 2 vector bundle over $\mathbb{G}(1,3)$. A point of $\mathbb{G}(1,3)$ corresponds to a full rank matrix

$$p = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix}$$

up to left multiplication by elements of $\mathrm{GL}_2(\mathbb{C})$. Then define

$$V \subseteq \mathbb{G}(1,3) \times \mathbb{C}^4$$

to consist of all pairs $(\binom{\alpha}{\beta}, v)$ such that $v \in \mathrm{Span}(\alpha, \beta)$.

(a) A pair $(\binom{\alpha}{\beta}, v)$ gives the 3×4 matrix

$$A = \begin{pmatrix} v \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} v_0 & v_1 & v_2 & v_3 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix}.$$

Prove that $(\binom{\alpha}{\beta}, v)$ is a point of V if and only if the maximal minors of A vanish. This shows that $V \subseteq \mathbb{G}(1,3) \times \mathbb{C}^4$ is a closed subvariety.

(b) Projection onto the first factor gives a morphism $\pi : V \to \mathbb{G}(1,3)$. Explain why the fiber over $p \in \mathbb{G}(1,3)$ is the 2-dimensional subspace of $\mathbb{C}^4$ corresponding to p.

(c) Given $0 \le i < j \le 3$, define

$$U_{ij} = \{\binom{\alpha}{\beta} \in \mathbb{G}(1,3) \mid \alpha_i\beta_j - \alpha_j\beta_i \neq 0\}.$$

Prove that $U_{ij} \simeq \mathbb{C}^4$ and that the U_{ij} give an affine open cover of $\mathbb{G}(1,3)$.

(d) Given $0 \le i < j \le 3$, pick $k < l$ such that $\{i, j, k, l\} = \{0,1,2,3\}$. Prove that the map $(p, v) \mapsto (p, v_k, v_l)$ gives an isomorphism

$$\pi^{-1}(U_{ij}) \xrightarrow{\sim} U_{ij} \times \mathbb{C}^2.$$

(e) By part (d), V is a vector bundle over $\mathbb{G}(1,3)$. Determine its transition functions.

6.0.6. Prove that a locally constant sheaf on an irreducible variety is constant.

6.0.7. Prove (6.0.4).

6.0.8. Prove that an effective divisor linearly equivalent to a Cartier divisor D is the divisor of zeros of a global section of $\mathscr{O}_X(D)$.

6.0.9. Let $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ be the Veronese mapping defined in Example 2.3.15. Prove that $\nu_d^* \mathscr{O}_{\mathbb{P}^d}(1) = \mathscr{O}_{\mathbb{P}^1}(d)$.

6.0.10. Let $f : Z \to X$ be a morphism and let $\mathscr{L}$ be a line bundle on X that is generated by global sections. Prove that the pullback line bundle $f^*\mathscr{L}$ is generated by global sections.

6.0.11. Let D be a Cartier divisor on a complete normal variety X.

(a) $f, g \in \Gamma(X, \mathscr{O}_X(D)) \setminus \{0\}$ give effective divisors $D + \mathrm{div}(f), D + \mathrm{div}(g)$ on X. Prove that these divisors are equal if and only if $f = \lambda g$, $\lambda \in \mathbb{C}^*$.

(b) The *complete linear system* of D is defined to be

$$|D| = \{E \in \mathrm{CDiv}(X) \mid E \sim D,\ E \ge 0\}.$$

Thus the complete linear system of D consists of all effective Cartier divisors on X linearly equivalent to D. Use part (a) to show that $|D|$ can be identified with the projective space of $\Gamma(X, \mathscr{O}_X(D))$, i.e., there is a natural bijection

$$|D| = \mathbb{P}(\Gamma(X, \mathscr{O}_X(D))) = (\Gamma(X, \mathscr{O}_X(D)) \setminus \{0\})/\mathbb{C}^*.$$

(c) Assume that D has no basepoints and set $W = \Gamma(X, \mathscr{O}_X(D))$. Then we can identify $\mathbb{P}(W^\vee)$ with the set of hyperplanes in $\mathbb{P}(W) = |D|$. Prove that the morphism $\phi_{\mathscr{O}_X(D),W} : X \to \mathbb{P}(W^\vee)$ is given by

$$\phi_{\mathscr{O}_X(D),W} = \{E \in |D| \mid p \in \mathrm{Supp}(E)\} \subseteq |D|.$$

§6.1. Ample and Basepoint Free Divisors on Complete Toric Varieties

In this section we will study two special classes of Cartier divisors on complete toric varieties. We begin with the basepoint free case.

Basepoint Free Divisors. Consider the toric variety X_Σ of a fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^n$ and let $D = \sum_\rho a_\rho D_\rho$ be a torus-invariant Cartier divisor on X_Σ. By Propositions 4.3.3 and 4.3.8, we have the global sections

$$\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m,$$

where $P_D \subseteq M_\mathbb{R}$ is the polyhedron defined by

(6.1.1) $$P_D = \{m \in M_\mathbb{R} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}.$$

Since $D = \sum_\rho a_\rho D_\rho$ is Cartier, there are $m_\sigma \in M$ for $\sigma \in \Sigma$ such that

(6.1.2) $$\langle m_\sigma, u_\rho \rangle = -a_\rho, \quad \rho \in \sigma(1).$$

Furthermore, when $\Sigma_{\max} = \Sigma(n)$, D is uniquely determined by the Cartier data $\{m_\sigma\}_{\sigma \in \Sigma(n)}$. Then we have the following preliminary result.

Proposition 6.1.1. *If $\Sigma_{\max} = \Sigma(n)$, then the following are equivalent:*

(a) *D has no basepoints, i.e., $\mathscr{O}_{X_\Sigma}(D)$ is generated by global sections.*

(b) *$m_\sigma \in P_D$ for all $\sigma \in \Sigma(n)$.*

Proof. First suppose that D is generated by global sections and take $\sigma \in \Sigma(n)$. The T_N-orbit corresponding to σ is a fixed point p of the T_N-action, and by the Orbit-Cone Correspondence,

$$\{p\} = \bigcap_{\rho \in \sigma(1)} D_\rho.$$

By Corollary 6.0.25, there is a global section s such that p is not in the support of the divisor of zeros $\mathrm{div}_0(s)$ of s. Since $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$ is spanned by χ^m for $m \in P_D \cap M$, we can assume that s is given by χ^m for some $m \in P_D \cap M$. The discussion preceding Corollary 6.0.25 shows that the divisor of zeros of s is

$$\mathrm{div}_0(s) = D + \mathrm{div}(\chi^m) = \sum_\rho (a_\rho + \langle m, u_\rho \rangle) D_\rho.$$

The point p is not in the support of $\mathrm{div}_0(s)$ yet lies in D_ρ for every $\rho \in \sigma(1)$. This forces $a_\rho + \langle m, u_\rho \rangle = 0$ for $\rho \in \sigma(1)$. Since σ is n-dimensional, we conclude that $m_\sigma = m \in P_D$.

For the converse, take $\sigma \in \Sigma(n)$. Since $m_\sigma \in P_D$, the character χ^{m_σ} gives a global section s whose divisor of zeros is $\mathrm{div}_0(s) = D + \mathrm{div}(\chi^{m_\sigma})$. Using (6.1.2), one sees that the support of $\mathrm{div}_0(s)$ misses U_σ, so that s is nonvanishing on U_σ. Then we are done since the U_σ cover X_Σ. □

Here is an example to illustrate Proposition 6.1.1.

Example 6.1.2. The fan for the Hirzebruch surface $\mathscr{H}_2$ is shown in Figure 3. Let

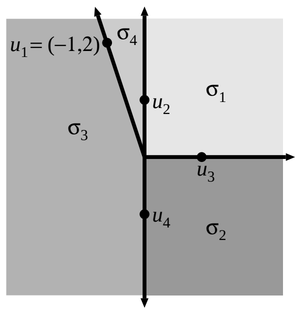


Figure 3. A fan Σ_2 with $X_{\Sigma_2} = \mathscr{H}_2$

D_i be the divisor corresponding to u_i. We will study the divisors

$$D = D_4 \quad \text{and} \quad D' = D_2 + D_4.$$

Write the Cartier data for D and D' with respect to $\sigma_1, \ldots, \sigma_4$ as $\{m_i\}$ and $\{m_i'\}$ respectively. Figure 4 shows P_D and m_i (left) and $P_{D'}$ and m_i' (right) (see also

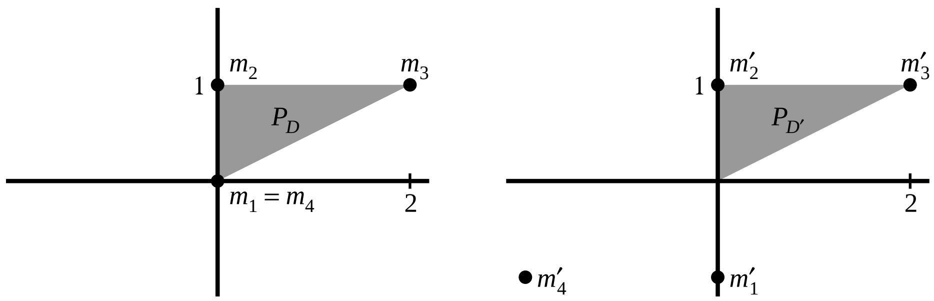


Figure 4. P_D and m_i (left) and $P_{D'}$ and m_i' (right)

Exercise 4.3.5). This figure and Proposition 6.1.1 make it clear that D is basepoint free while D' is not. $\Diamond$

Support Functions and Their Graphs. Let $D = \sum_\rho a_\rho D_\rho$ be a Cartier divisor on a toric variety X_Σ. As in Chapter 4, its *support function* $\varphi_D : |\Sigma| \to \mathbb{R}$ is determined by the following properties:

- φ_D is linear on each cone $\sigma \in \Sigma$.
- $\varphi_D(u_\rho) = -a_\rho$ for all $\rho \in \Sigma(1)$.

This is where the $\{m_\sigma\}_{\sigma\in\Sigma}$ from (6.1.2) appear naturally, since the explicit formula for $\varphi_D|_\sigma$ is given by $\varphi_D(u) = \langle m_\sigma, u\rangle$ for all $u \in \sigma$.

When $M = \mathbb{Z}^2$ and Σ is complete, it is easy to visualize the graph of φ_D in $M_\mathbb{R} \times \mathbb{R} = \mathbb{R}^3$: imagine a tent, with centerpole extending from $(0,0,0)$ down the z-axis, and tent stakes placed at positions $(u_\rho, -a_\rho)$. Here is an example.

Example 6.1.3. Take $\mathbb{P}^1 \times \mathbb{P}^1$ and consider the divisor $D = D_1 + D_2 + D_3 + D_4$. This gives the support function where $\varphi_D(u_i) = -1$ for the four ray generators u_1, u_2, u_3, u_4 of the fan of $\mathbb{P}^1 \times \mathbb{P}^1$. The graph of φ_D is shown in Figure 5. This

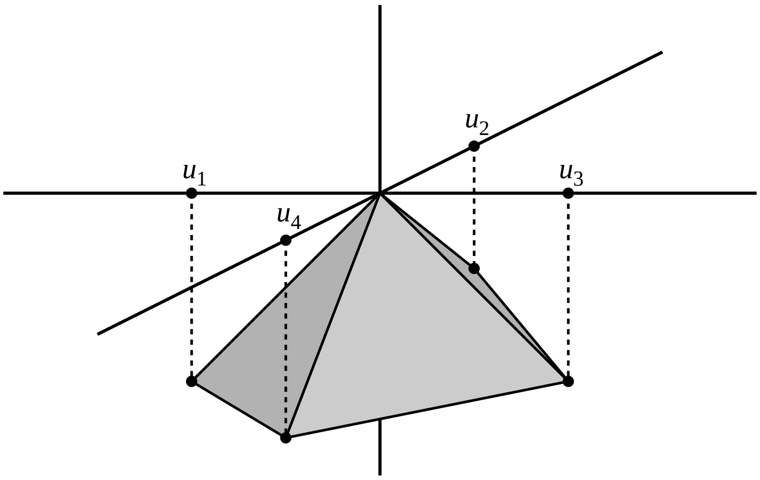


Figure 5. The graph of φ_D

should be visualized as an infinite Egyptian pyramid, with apex at the origin and edges going through $(u_i, -1)$ for $1 \le i \le 4$. $\Diamond$

Convex Functions. We now introduce the key concept of convexity.

Definition 6.1.4. Let $S \subseteq N_\mathbb{R}$ be convex. A function $\varphi : S \to \mathbb{R}$ is ***convex*** if

$$\varphi(tu + (1-t)v) \ge t\varphi(u) + (1-t)\varphi(v),$$

for all $u, v \in S$ and $t \in [0,1]$.

We caution the reader that some books define convexity with the inequality going the other way.

Continuing with the tent analogy, a support function φ_D is convex exactly if there are unimpeded lines of sight inside the tent. It is clear that for Example 6.1.3, the support function is convex.

Full Dimensional Convex Support. In this chapter, our main focus is on complete fans. However, the natural setting for convexity is the class of fans Σ in $N_{\mathbb{R}}$ which satisfy the following two conditions:

- $|\Sigma| \subseteq N_{\mathbb{R}}$ is convex.
- $\dim |\Sigma| = n = \dim N_{\mathbb{R}}$.

We say that Σ has *convex support of full dimension*. Such fans satisfy

$$|\Sigma| = \mathrm{Cone}(u_\rho \mid \rho \in \Sigma(1)) = \bigcup_{\sigma \in \Sigma(n)} \sigma. \tag{6.1.3}$$

In particular, the maximal cones of Σ have dimension n, so that we can focus on the cones $\sigma \in \Sigma(n)$, just as in the complete case.

Support Functions and Convexity. The following lemma describes when a support function is convex. Given a fan Σ in $N_{\mathbb{R}} \simeq \mathbb{R}^n$, a cone $\tau \in \Sigma(n-1)$ is called a *wall* when it is the intersection of two n-dimensional cones $\sigma, \sigma' \in \Sigma(n)$, i.e, when $\tau = \sigma \cap \sigma'$ forms the wall separating σ and σ'. If Σ is complete, every $\tau \in \Sigma(n-1)$ is a wall.

Lemma 6.1.5. *Let D be a Cartier divisor on a toric variety whose fan Σ has convex support of full dimension. Then the following are equivalent:*

(a) *The support function $\varphi_D : |\Sigma| \to \mathbb{R}$ is convex.*

(b) *$\varphi_D(u) \le \langle m_\sigma, u\rangle$ for all $u \in |\Sigma|$ and $\sigma \in \Sigma(n)$.*

(c) *$\varphi_D(u) = \min_{\sigma \in \Sigma(n)} \langle m_\sigma, u\rangle$ for all $u \in |\Sigma|$.*

(d) *For every wall $\tau = \sigma \cap \sigma'$, there is $u_0 \in \sigma' \setminus \sigma$ with $\varphi_D(u_0) \le \langle m_\sigma, u_0\rangle$.*

Proof. First assume (a) and fix v in the interior of $\sigma \in \Sigma(n)$. Given $u \in |\Sigma|$, we can find $t \in (0,1)$ with $tu + (1-t)v \in \sigma$. By convexity, we have

$$\begin{aligned}\langle m_\sigma, tu + (1-t)v\rangle &= \varphi_D(tu + (1-t)v)\\ &\ge t\varphi_D(u) + (1-t)\varphi_D(v) = t\varphi_D(u) + (1-t)\langle m_\sigma, v\rangle.\end{aligned}$$

This easily implies $\langle m_\sigma, u\rangle \ge \varphi_D(u)$, proving (b). The implication (b) $\Rightarrow$ (c) is immediate since $\varphi_D(u) = \langle m_\sigma, u\rangle$ for $u \in \sigma$, and (c) $\Rightarrow$ (a) follows because the minimum of a finite set of linear functions is always convex (Exercise 6.1.1).

Since (b) $\Rightarrow$ (d) is obvious, it remains to prove the converse. Assume (d) and fix a wall $\tau = \sigma \cap \sigma'$. Then σ' lies on one side of the wall. We claim that

$$\langle m_{\sigma'}, u\rangle \le \langle m_\sigma, u\rangle, \quad \text{when } u, \sigma' \text{ are on the same side of } \tau. \tag{6.1.4}$$

This is easy. The wall is defined by $\langle m_\sigma - m_{\sigma'}, u\rangle = 0$. Then (d) implies that the halfspace containing σ' is defined by $\langle m_\sigma - m_{\sigma'}, u\rangle \ge 0$, and (6.1.4) follows.

Now take $u \in |\Sigma|$ and $\sigma \in \Sigma(n)$. Since $|\Sigma|$ is convex, we can pick v in the interior of σ so that the line segment $\overline{uv}$ intersects every wall of Σ in a single point, as shown in Figure 6 on the next page. Using (6.1.4) repeatedly, we obtain

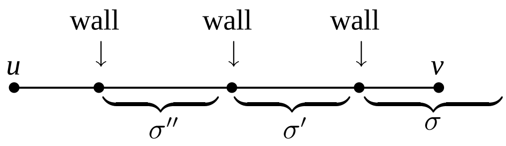


Figure 6. Crossing walls from u to v along $\overline{uv}$

$$\langle m_\sigma, u\rangle \geq \langle m_{\sigma'}, u\rangle \geq \langle m_{\sigma''}, u\rangle \geq \cdots.$$

When we arrive at the cone containing u, the pairing becomes $\varphi_D(u)$, so that $\langle m_\sigma, u\rangle \geq \varphi_D(u)$. This proves (b). □

In terms of the tent analogy, part (b) of the lemma means that if we have a convex support function and extend one side of the tent in all directions, the rest of the tent lies below the resulting hyperplane. Then part (d) means that it suffices to check this locally where two sides of the tent meet.

The proof of our main result about convexity will use the following lemma that describes the polyhedron of a Cartier divisor in terms of its support function.

Lemma 6.1.6. *Let Σ be a fan and $D = \sum_\rho a_\rho D_\rho$ be a Cartier divisor on X_Σ. Then*

$$P_D = \{m \in M_{\mathbb{R}} \mid \varphi_D(u) \leq \langle m, u\rangle \text{ for all } u \in |\Sigma|\}.$$

Proof. Assume $\varphi_D(u) \leq \langle m, u\rangle$ for all $u \in |\Sigma|$. Applying this with $u = u_\rho$ gives

$$-a_\rho = \varphi_D(u_\rho) \leq \langle m, u_\rho\rangle,$$

so that $m \in P_D$ by the definition of P_D. For the opposite inclusion, take $m \in P_D$ and $u \in |\Sigma|$. Thus $u \in \sigma \in \Sigma$, so that $u = \sum_{\rho\in\sigma(1)} \lambda_\rho u_\rho$, $\lambda_\rho \geq 0$. Then

$$\begin{aligned}\langle m, u\rangle = \textstyle\sum_{\rho\in\sigma(1)} \lambda_\rho \langle m, u_\rho\rangle &\geq \textstyle\sum_{\rho\in\sigma(1)} \lambda_\rho(-a_\rho)\\ &= \textstyle\sum_{\rho\in\sigma(1)} \lambda_\rho \varphi_D(u_\rho) = \varphi_D(u),\end{aligned}$$

where the inequality follows from $m \in P_D$, and the last two equalities follow from the defining properties of φ_D. □

We now expand Proposition 6.1.1 to give a more complete characterization of when a divisor is basepoint free. Recall that P_D is a polytope when Σ is complete.

Theorem 6.1.7. *Assume $|\Sigma|$ is convex of full dimension n and let φ_D be the support function of a Cartier divisor D on X_Σ. Then the following are equivalent:*

(a) *D is basepoint free.*

(b) *$m_\sigma \in P_D$ for all $\sigma \in \Sigma(n)$.*

(c) *$\varphi_D(u) = \min_{\sigma\in\Sigma(n)} \langle m_\sigma, u\rangle$ for all $u \in |\Sigma|$.*

(d) *$\varphi_D : |\Sigma| \to \mathbb{R}$ is convex.*

If addition Σ is complete, then (a)–(d) *are equivalent to the following:*

(e) $P_D = \mathrm{Conv}(m_\sigma \mid \sigma \in \Sigma(n))$.

(f) $\{m_\sigma \mid \sigma \in \Sigma(n)\}$ *is the set of vertices of* P_D.

(g) $\varphi_D(u) = \min_{m\in P_D}\langle m,u\rangle$ *for all* $u \in N_{\mathbb{R}}$.

Proof. The equivalences (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d) were proved in Proposition 6.1.1 and Lemma 6.1.5. Furthermore, Lemmas 6.1.5 and 6.1.6 imply that

$$\begin{aligned}\varphi_D \text{ is convex} &\iff \varphi_D(u) \le \langle m_\sigma, u\rangle \text{ for all } \sigma \in \Sigma(n),\ u \in |\Sigma|\\ &\iff m_\sigma \in P_D \text{ for all } \sigma \in \Sigma(n).\end{aligned}$$

This proves (d) $\Leftrightarrow$ (b), so that (a), (b), (c) and (d) are equivalent.

Assume (b) and note that P_D is a polytope since Σ is complete. Then $m_\sigma \in P_D$ and $\varphi_D(u) = \min_{\sigma\in\Sigma(n)}\langle m_\sigma, u\rangle$. Combining these with Lemma 6.1.6, we obtain

$$\varphi_D(u) \le \min_{m\in P_D}\langle m,u\rangle \le \min_{\sigma\in\Sigma(n)}\langle m_\sigma,u\rangle = \varphi_D(u),$$

proving (g). The implication (g) $\Rightarrow$ (d) follows since the minimum of a compact set of linear functions is convex (Exercise 6.1.1). So (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (g).

The implications (f) $\Rightarrow$ (e) $\Rightarrow$ (b) are clear. It remains to prove (b) $\Rightarrow$ (f). Take $\sigma \in \Sigma(n)$. Let u be in the interior of σ and set $a = \varphi_D(u)$. By Exercise 6.1.2, $H_{u,a} = \{m \in M_{\mathbb{R}} \mid \langle m,u\rangle = a\}$ is a supporting hyperplane of P_D and

$$H_{u,a} \cap P_D = \{m_\sigma\}. \tag{6.1.5}$$

This implies that m_σ is a vertex of P_D. Conversely, let $H_{u,a}$ be a supporting hyperplane of a vertex $\mathsf{v} \in P_D$. Thus $\langle m,u\rangle \ge a$ for all $m \in P_D$, with equality if and only if $m = \mathsf{v}$. Since (b) holds, we also have (c) and (g). By (g), $\varphi_D(u) = \min_{m\in P_D}\langle m,u\rangle = \langle m,\mathsf{v}\rangle = a$. Combining this with (c), we obtain

$$\varphi_D(u) = \min_{\sigma\in\Sigma(n)}\langle m_\sigma,u\rangle = a.$$

Hence $\langle m_\sigma, u\rangle = a$ must occur for some $\sigma \in \Sigma(n)$, which forces $\mathsf{v} = m_\sigma$. $\square$

Example 6.1.8. In Example 6.1.2 we showed that on the Hirzebruch surface $\mathscr{H}_2$, $D = D_4$ is basepoint free while $D' = D_2 + D_4$ is not. Theorem 6.1.7 gives a different proof using support functions. Figure 7 on the next page shows the graph of the support function φ_D. Notice that the portion of the "roof" containing the points u_1, u_2, u_3 and the origin lies in the plane $z = 0$, and it is clear that for φ_D, there are unimpeded lines of sight within the tent. In other words, φ_D is convex.

The support function $\varphi_{D'}$ is shown in Figure 8 on the next page. Here, the line of sight from u_1 to u_3 lies in the plane $z = 0$, yet the ridgeline going from the origin to the point $(u_2, -1)$ on the tent lies below the plane $z = 0$. Hence this line of sight does not lie inside the tent, so that $\varphi_{D'}$ is not convex. $\Diamond$

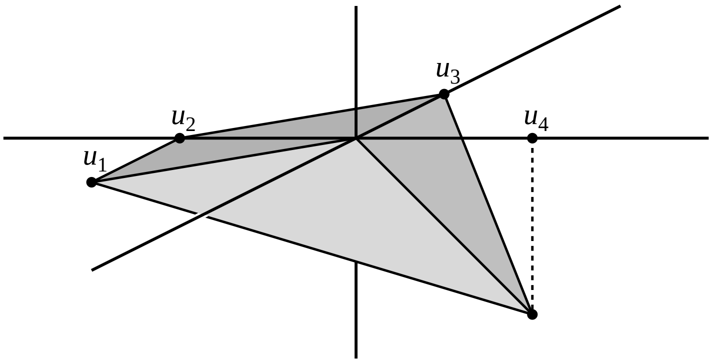


Figure 7. The graph of $\varphi_D = \varphi_{D_4}$ in Example 6.1.8

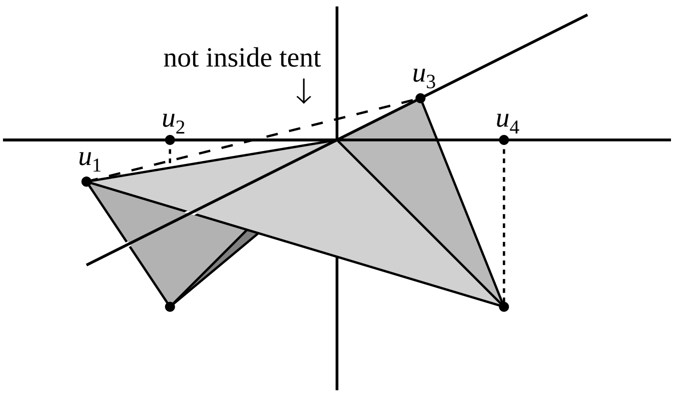


Figure 8. The graph of $\varphi_{D'} = \varphi_{D_2+D_4}$ in Example 6.1.8

When D is basepoint free, Theorem 6.1.7 implies that the vertices of P_D are the lattice points m_σ, $\sigma \in \Sigma(n)$. One caution is that in general, the m_σ need not be distinct, i.e., $\sigma \neq \sigma'$ can have $m_\sigma = m_{\sigma'}$. An example is given by the divisor $D = D_4$ considered in Example 6.1.2—see Figure 4. As we will see later, this behavior illustrates the difference between basepoint free and ample.

It can also happen that P_D has strictly smaller dimension than the dimension of X_Σ. You will work out a simple example of this in Exercise 6.1.3.

Ample Divisors. We now introduce the second key concept of this section.

Definition 6.1.9. Let D be a Cartier divisor on a complete normal variety X. As we noted in §4.3, $W = \Gamma(X, \mathscr{O}_X(D))$ is finite-dimensional.

(a) The divisor D and the line bundle $\mathscr{O}_X(D)$ are ***very ample*** when D has no basepoints and $\phi_D = \phi_{\mathscr{O}_X(D),W} : X \to \mathbb{P}(W^\vee)$ is a closed embedding.

(b) D and $\mathscr{O}_X(D)$ are ***ample*** when kD is very ample for some integer $k > 0$.

We will see that support functions give a simple, elegant characterization of when a torus-invariant Cartier divisor is ample. But first, we explore how the very ample polytopes from Definition 2.2.17 relate to Definition 6.1.9.

Very Ample Polytopes. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ a full dimensional lattice polytope with facet presentation

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \text{ for all facets } F\}.$$

This gives the complete normal fan Σ_P and the toric variety X_P. Write

$$P \cap M = \{m_1, \ldots, m_s\}.$$

A vertex $m_i \in P$ corresponds to a maximal cone

$$\sigma_i = \mathrm{Cone}(P \cap M - m_i)^{\vee} \in \Sigma_P(n). \tag{6.1.6}$$

Proposition 4.2.10 implies that $D_P = \sum_F a_F D_F$ is Cartier since $\langle m_i, u_F \rangle = -a_F$ when $m_i \in F$.

Recall from Definition 2.2.17 that P is *very ample* if for every vertex $m_i \in P$, the semigroup $\mathbb{N}(P \cap M - m_i)$ is saturated in M. The definition of X_P given in Chapter 2 used very ample polytopes. This is no accident.

Proposition 6.1.10. *Let X_P and D_P be as above. Then:*

(a) *D_P is ample and basepoint free.*

(b) *If $n \geq 2$, then kD_P is very ample for every $k \geq n-1$.*

(c) *D_P is very ample if and only if P is a very ample polytope.*

Proof. First observe that the polytope of the divisor D_P is the polytope P we began with, i.e., $P_{D_P} = P$. This has two consequences:

- D_P is basepoint free by Proposition 6.1.1, which proves the final assertion of part (a).
- If $P \cap M = \{m_1, \ldots, m_s\}$, then the characters χ^{m_i} span $W = \Gamma(X_P, \mathscr{O}_{X_P}(D_P))$.

Since D_P is basepoint free, these global sections give the morphism

$$\phi_{D_P} = \phi_{\mathscr{O}_{X_P}(D_P), W} \longrightarrow \mathbb{P}^{s-1}$$

by Lemma 6.0.28. As explained in (6.0.7), ϕ_D can be written

$$\phi_{D_P}(p) = (\chi^{m_1}(p), \ldots, \chi^{m_s}(p)). \tag{6.1.7}$$

It follows that ϕ_{D_P} factors as

$$X_P \to X_{P\cap M} \subseteq \mathbb{P}^{s-1},$$

where $X_{P\cap M}$ is the projective toric variety of $P \cap M \subseteq M$ from §2.1. We need to understand when $X_P \to X_{P\cap M}$ is an isomorphism.

Fix coordinates $x_1,\ldots,x_s$ of $\mathbb{P}^{s-1}$ and let $I \subseteq \{1,\ldots,s\}$ be the set of indices such that m_i is a vertex of P. Hence each $i \in I$ gives a vertex m_i and a corresponding maximal cone σ_i in the normal fan of P.

If $i \in I$, then $\langle m_i, u_F\rangle = -a_F$ for every facet F containing m_i. For all other facets F, $\langle m_i, u_F\rangle > -a_F$. Hence, if s_i is the global section corresponding to χ^{m_i}, then the support of $\operatorname{div}(s_i)_0 = D_P + \operatorname{div}(\chi^{m_i})$ consists of those divisors missing the affine open toric variety $U_{\sigma_i} \subseteq X_P$ of σ_i. It follows that U_{σ_i} is the nonvanishing locus of s_i.

Under ϕ_{D_P}, this nonvanishing locus maps to the affine open subset $U_i \subseteq \mathbb{P}^{s-1}$ where $x_i \neq 0$. Since $X_P = \bigcup_{i\in I} U_{\sigma_i}$, and $X_{P\cap M} \subseteq \bigcup_{i\in I} U_i$ by Proposition 2.1.9, it suffices to study the maps

$$U_{\sigma_i} \longrightarrow X_{P\cap M} \cap U_i$$

of affine toric varieties. By Proposition 2.1.8,

$$X_{P\cap M} \cap U_i = \operatorname{Spec}(\mathbb{C}[\mathbb{N}(P\cap M - m_i)]).$$

Since $\sigma_i^\vee = \operatorname{Cone}(P\cap M - m_i)$ by (6.1.6), we have an inclusion of semigroups

$$\mathbb{N}(P\cap M - m_i) \subseteq \sigma_i^\vee \cap M.$$

This is an equality precisely when $\mathbb{N}(P\cap M - m_i)$ is saturated in M. Since $U_{\sigma_i} = \operatorname{Spec}(\mathbb{C}[\sigma_i^\vee \cap M])$, we obtain the equivalences:

$$\begin{aligned}
D_P \text{ is very ample} &\iff X_P \to X_{P\cap M} \text{ is an isomorphism}\\
&\iff U_{\sigma_i} \to X_{P\cap M} \cap U_i \text{ is an isomorphism for all } i \in I\\
&\iff \mathbb{C}[\mathbb{N}(P\cap M - m_i)] \to \mathbb{C}[\sigma^\vee \cap M] \text{ is an}\\
&\qquad \text{isomorphism for all } i \in I\\
&\iff \mathbb{N}(P\cap M - m_i) \text{ is saturated for all } i \in I\\
&\iff P \text{ is very ample.}
\end{aligned}$$

This proves part (c) of the proposition. For part (b), recall that if $n \geq 2$ and P is arbitrary, then kP is very ample when $k \geq n-1$ by Corollary 2.2.19. Hence $kD_P = D_{kP}$ is very ample. This implies that D_P is ample (the case $n = 1$ is trivial), which completes the proof of part (a). $\square$

Example 6.1.11. In Example 2.2.11, we showed that the simplex

$$P = \operatorname{Conv}(0, e_1, e_2, e_1 + e_2 + 3e_3) \subseteq \mathbb{R}^3$$

is not normal. We show that P is not very ample as follows. From Chapter 2 we know that the only lattice points of P are its vertices, so that $\phi_{D_P} : X_P \to \mathbb{P}^3$. Since X_P is singular (Exercise 6.1.4) of dimension 3, it follows that ϕ_{D_P} cannot be a closed embedding. Hence P and D_P are not very ample. However, $2P$ and $2D_P$ are very ample by Proposition 6.1.10. $\lozenge$

Ampleness and Strict Convexity. We next determine when a Cartier divisor $D = \sum_\rho a_\rho D_\rho$ on X_Σ is ample. Our criterion will involve the support function φ_D of D. Recall that the Cartier data $\{m_\sigma\}_{\sigma\in\Sigma(n)}$ of D satisfies

$$\langle m_\sigma, u\rangle = \varphi_D(u), \quad \text{for all } u \in \sigma.$$

Definition 6.1.12. Assume that Σ has full dimensional convex support. Then the support function φ_D of a Cartier divisor D on X_Σ is ***strictly convex*** if it is convex and for every $\sigma \in \Sigma(n)$ satisfies

$$\langle m_\sigma, u\rangle = \varphi_D(u) \iff u \in \sigma.$$

The following lemma, which you will prove in Exercise 6.1.5, shows that there are many ways to think about strict convexity.

Lemma 6.1.13. *Let D Cartier divisor on a toric variety whose fan Σ has convex support of full dimension. Then the following are equivalent:*

(a) *The support function $\varphi_D : |\Sigma| \to \mathbb{R}$ is strictly convex.*

(b) *$\varphi_D(u) < \langle m_\sigma, u\rangle$ for all $u \in |\Sigma| \setminus \sigma$ and $\sigma \in \Sigma(n)$.*

(c) *For every wall $\tau = \sigma\cap\sigma'$, there is $u_0 \in \sigma'\setminus\sigma$ with $\varphi_D(u_0) < \langle m_\sigma, u_0\rangle$.*

(d) *φ_D is convex and $m_\sigma \neq m_{\sigma'}$ when $\sigma \neq \sigma'$ in $\Sigma(n)$ and $\sigma\cap\sigma'$ is a wall.*

(e) *φ_D is convex and $m_\sigma \neq m_{\sigma'}$ when $\sigma \neq \sigma'$ in $\Sigma(n)$.*

(f) *$\langle m_\sigma, u_\rho\rangle > -a_\rho$ for all $\rho \in \Sigma(1)\setminus\sigma(1)$ and $\sigma \in \Sigma(n)$.*

(g) *$\varphi_D(u+v) > \varphi_D(u) + \varphi_D(v)$ for all $u, v \in |\Sigma|$ not in the same cone of Σ.* □

We now relate strict convexity to ampleness.

Theorem 6.1.14. *Assume that φ_D is the support function of a Cartier divisor $D = \sum_\rho a_\rho D_\rho$ on a complete toric variety X_Σ. Then*

$$D \text{ is ample} \iff \varphi_D \text{ is strictly convex.}$$

Furthermore, if $n \geq 2$ and D is ample, then kD is very ample for all $k \geq n-1$.

Proof. First suppose that D is very ample. Very ample divisors have no basepoints, so φ_D is convex by Theorem 6.1.7. If strict convexity fails, then Lemma 6.1.13 implies that Σ has a wall $\tau = \sigma\cap\sigma'$ with $m_\sigma = m_{\sigma'}$. Let $V(\tau) = \overline{O(\tau)} \subseteq X_\Sigma$.

Let P_D be the polyhedron of D from (6.1.1), which is a polytope since Σ is complete. Let $P_D \cap M = \{m_1, \ldots, m_s\}$, so that $\phi_D : X_\Sigma \to \mathbb{P}^{s-1}$ can be written

$$\phi_D(p) = (\chi^{m_1}(p), \ldots, \chi^{m_s}(p))$$

as in (6.1.7). In this enumeration, $m_\sigma = m_{\sigma'} = m_{i_0}$ for some i_0. We will study ϕ_D on the open subset $U_\sigma \cup U_{\sigma'} \subseteq X_\Sigma$.

First consider U_σ. Theorem 6.1.7 implies that $m_\sigma \in P_D$, so that the section corresponding to χ^{m_σ} is nonvanishing on U_σ by the proof of Proposition 6.1.1. It follows that on U_σ, ϕ_D is given by

$$\phi_D(p) = (\chi^{m_1 - m_\sigma}(p), \ldots, \chi^{m_s - m_\sigma}(p)) \in U_{i_0} \simeq \mathbb{C}^{s-1},$$

where $U_{i_0} \subseteq \mathbb{P}^{s-1}$ is the open subset where $x_{i_0} \neq 0$.

Since $m_\sigma = m_{\sigma'}$, the same argument works on $U_{\sigma'}$. This gives a morphism

$$\phi_D|_{U_\sigma \cup U_{\sigma'}} : U_\sigma \cup U_{\sigma'} \longrightarrow U_{i_0} \simeq \mathbb{C}^{s-1}.$$

The only n-dimensional cones of Σ containing τ are σ, σ' since τ is a wall. Hence

$$V(\tau) \subset U_\sigma \cup U_{\sigma'}$$

by the Orbit-Cone Correspondence. Note also $V(\tau) \simeq \mathbb{P}^1$ since τ is a wall. Since $\mathbb{P}^1$ is complete, Proposition 4.3.8 implies that all morphisms from $\mathbb{P}^1$ to affine space are constant. Thus ϕ_D maps $V(\tau)$ to a point, which is impossible since D is very ample. Hence φ_D is strictly convex when D is very ample.

If D is ample, then kD is very ample for $k \gg 0$. Thus $\varphi_{kD} = k\varphi_D$ must be strictly convex, which implies that φ_D is strictly convex.

For the converse, assume φ_D is strictly convex. Let $\{m_\sigma\}_{\sigma \in \Sigma(n)}$ be the Cartier data of D. Since φ is convex, Theorem 6.1.7 shows that the m_σ are the vertices of P_D. Hence P_D is a lattice polytope.

If P_D is not full dimensional, then there are $u \neq 0$ in $N_\mathbb{R}$ and $k \in \mathbb{R}$ such that $\langle m_\sigma, u\rangle = k$ for all $\sigma \in \Sigma(n)$. Then Theorem 6.1.7 implies

$$\varphi_D(u) = \langle m_\sigma, u\rangle = k$$

for all $\sigma \in \Sigma(n)$. Using strict convexity and Definition 6.1.12, we conclude that $u \in \sigma$ for all $\sigma \in \Sigma(n)$. Hence $u = 0$ since Σ is complete. This contradicts $u \neq 0$ and proves that P_D is full dimensional.

Hence P_D gives the toric variety X_{P_D} with normal fan Σ_{P_D}. Furthermore, X_{P_D} has the ample divisor D_{P_D} from Proposition 6.1.10. We studied the support function of this divisor in Proposition 4.2.14, where we showed that it is the function

$$\varphi_{P_D}(u) = \min_{m \in P_D} \langle m, u\rangle.$$

However, this is precisely φ_D by Theorem 6.1.7. Hence $\varphi_{P_D} = \varphi_D$ is strictly convex with respect to Σ (by hypothesis) and Σ_{P_D} (by the first part of the proof).

Definition 6.1.13 implies that the maximal cones of the fan are the maximal subsets of $N_\mathbb{R}$ on which a strictly convex support function is linear. This, combined with the previous paragraph, implies that $\Sigma = \Sigma_{P_D}$. Thus

$$X_\Sigma = X_{P_D}. \tag{6.1.8}$$

Furthermore, we also have

$$D = D_{P_D} \tag{6.1.9}$$

since the divisors have the same support function. Since D_{P_D} is an ample divisor by Proposition 6.1.10, it follows that D is also ample.

The final assertion of the theorem follows from Proposition 6.1.10. □

The relation between polytopes and ample divisors given by (6.1.8) and (6.1.9) will be explored in §6.2. These facts also give the following nice result.

Theorem 6.1.15. *On a smooth complete toric variety X_Σ, a divisor D is ample if and only if it is very ample.*

Proof. If D is ample, then X_Σ is the toric variety of P_D by (6.1.8). Since X_Σ is smooth, P_D is very ample by Theorem 2.4.3 and Proposition 2.4.4. Since D is the divisor of P_D by (6.1.9), D is very ample by Proposition 6.1.10. □

Computing Ample Divisors. Given a wall $\tau \in \Sigma(n-1)$, write $\tau = \sigma \cap \sigma'$ and pick $\rho' \in \sigma'(1) \setminus \sigma(1)$. Then a Cartier divisor $D = \sum_\rho a_\rho D_\rho$ gives the *wall inequality*

$$\langle m_\sigma, u_{\rho'} \rangle > -a_{\rho'}. \tag{6.1.10}$$

Lemma 6.1.13 and Theorem 6.1.14 imply that D is ample if and only if it satisfies the wall inequality (6.1.10) for every wall of Σ.

In terms of divisor classes, recall the map $\mathrm{CDiv}_T(X_\Sigma) \to \mathrm{Pic}(X_\Sigma)$ whose kernel consists of divisors of characters. If we fix $\sigma_0 \in \Sigma(n)$, then we have an isomorphism

$$\big\{D = \textstyle\sum_\rho a_\rho D_\rho \in \mathrm{CDiv}_T(X_\Sigma) \mid a_\rho = 0 \text{ for all } \rho \in \sigma_0(1)\big\} \simeq \mathrm{Pic}(X_\Sigma) \tag{6.1.11}$$

(Exercise 6.1.6). Then (6.1.10) gives inequalities for determining when a divisor class is ample. Here is a classic example.

Example 6.1.16. Let us determine the ample divisors on the Hirzebruch surface $\mathscr{H}_r$. The fan for $\mathscr{H}_2$ is shown in Figure 3 of Example 6.1.2, and this becomes the fan for $\mathscr{H}_r$ by redefining u_1 to be $u_1 = (-1, r)$. Hence we have ray generators u_1, u_2, u_3, u_4 and maximal cones $\sigma_1, \sigma_2, \sigma_3, \sigma_4$.

In Examples 4.3.5 and 4.1.8, we used D_1 and D_2 to give a basis of $\mathrm{Pic}(\mathscr{H}_r) = \mathrm{Cl}(\mathscr{H}_r)$. Here, it is more convenient to use D_3 and D_4. More precisely, applying (6.1.11) for the cone σ_4, we obtain

$$\mathrm{Pic}(\mathscr{H}_r) \simeq \{aD_3 + bD_4 \mid a, b \in \mathbb{Z}\}.$$

To determine when $aD_3 + bD_4$ is ample, we compute $m_i = m_{\sigma_i}$ to be

$$m_1 = (-a, 0),\ m_2 = (-a, b),\ m_3 = (rb, b),\ m_4 = (0, 0).$$

Then (6.1.10) gives four wall inequalities which reduce to $a, b > 0$. Thus

$$aD_3 + bD_4 \text{ is ample} \iff a, b > 0. \tag{6.1.12}$$

For an arbitrary divisor $D = \sum_{i=1}^4 a_i D_i$, the relations

$$0 \sim \mathrm{div}(\chi^{e_1}) = -D_1 + D_3$$
$$0 \sim \mathrm{div}(\chi^{e_2}) = r D_1 + D_2 - D_4$$

show that $D \sim (a_1 - ra_2 + a_3)D_3 + (a_2 + a_4)D_4$. Hence

$$\textstyle\sum_{i=1}^4 a_i D_i \text{ is ample } \iff a_1 + a_3 > ra_2,\ a_2 + a_4 > 0.$$

Sometimes ampleness is easier to check if we think geometrically in terms of support functions. For $D = aD_3 + bD_4$, look back at Figure 7 and imagine moving the vertex at u_3 downwards. This gives the graph of φ_D, which is strictly convex when $a, b > 0$. ♢

Here is an example of how to determine ampleness using support functions.

Example 6.1.17. The fan for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ has the eight orthants of $\mathbb{R}^3$ as its maximal cones, and the ray generators are $\pm e_1, \pm e_2, \pm e_3$. Take the positive orthant $\mathbb{R}^3_{\geq 0}$ and subdivide further by adding the new ray generators

$$a = (2,1,1),\ b = (1,2,1),\ c = (1,1,2),\ d = (1,1,1).$$

We obtain a complete fan Σ by filling the first orthant with the cones in Figure 9, which shows the intersection of $\mathbb{R}^3_{\geq 0}$ with the plane $x + y + z = 1$. You will check that Σ is smooth in Exercise 6.1.7.

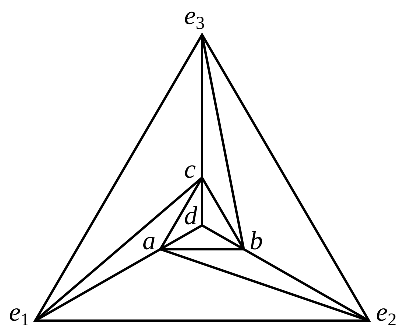


Figure 9. Cones of Σ lying in $\mathbb{R}^3_{\geq 0}$

Let $D = \sum_\rho a_\rho D_\rho$ be a Cartier divisor on X_Σ. Replacing D with $D + \mathrm{div}(\chi^m)$ for $m = (-a_{e_1}, -a_{e_2}, -a_{e_3})$, we can assume that φ_D satisfies

$$\varphi_D(e_1) = \varphi_D(e_2) = \varphi_D(e_3) = 0.$$

Now observe that $e_1 + b = (2,2,1) = e_2 + a$. Since e_1 and b do not lie in a cone of Σ , part (g) of Lemma 6.1.13 implies that

$$\varphi_D(e_1 + b) > \varphi_D(e_1) + \varphi_D(b) = \varphi_D(b).$$

However, e_2 and a generate a cone of Σ, so that

$$\varphi_D(a) = \varphi_D(e_2) + \varphi_D(a) = \varphi_D(e_2 + a) = \varphi_D(e_1 + b).$$

Together, these imply $\varphi_D(a) > \varphi_D(b)$. By symmetry, we obtain

$$\varphi_D(a) > \varphi_D(b) > \varphi_D(c) > \varphi_D(a),$$

an impossibility. Hence Σ has no strictly convex support functions, which shows that X_Σ is a smooth complete nonprojective variety. See also Example B.2.2 for a computational approach using the `Polyhedra` package of `Macaulay2` [**123**]. $\Diamond$

We will say more computing ample divisors later in the chapter.

The Toric Chow Lemma. Recall from Chapter 3 that a refinement Σ' of Σ gives a proper birational toric morphism $X_{\Sigma'} \to X_\Sigma$. We will now use the methods of this section to prove the *toric Chow lemma*, which asserts that any complete fan has a refinement that gives a projective toric variety. Here is the precise result.

Theorem 6.1.18. *A complete fan Σ has a refinement Σ' such that $X_{\Sigma'}$ is projective.*

Proof. Suppose Σ is a fan in $N_\mathbb{R} \simeq \mathbb{R}^n$. Let Σ' be obtained from Σ by considering the complete fan obtained from

$$\bigcup_{\tau \in \Sigma(n-1)} \mathrm{Span}(\tau).$$

So for each wall τ, we take the entire hyperplane spanned by the wall. This yields a subdivison Σ' with the property that

$$\bigcup_{\tau' \in \Sigma'(n-1)} \tau' = \bigcup_{\tau \in \Sigma(n-1)} \mathrm{Span}(\tau),$$

i.e., each hyperplane $\mathrm{Span}(\tau)$ is a union of walls of Σ', and all walls of Σ' arise this way.

Choosing $m_\tau \in M$ so that

$$\{u \in N_\mathbb{R} \mid \langle m_\tau, u\rangle = 0\} = \mathrm{Span}(\tau),$$

define the map $\varphi : N_\mathbb{R} \to \mathbb{R}$ by

$$\varphi(u) = -\sum_{\tau \in \Sigma(n-1)} |\langle m_\tau, u\rangle|.$$

Note that φ takes integer values on N and is convex by the triangle inequality (this explains the minus sign).

Let us show that φ is piecewise linear with respect to Σ'. Fix $\tau \in \Sigma(n-1)$ and note that each cone of Σ' is contained in one of the closed half-spaces bounded by $\mathrm{Span}(\tau)$. This implies that $u \mapsto |\langle m_\tau, u\rangle|$ is linear on each cone of Σ'. Hence the same is true for φ.

Finally, we prove that φ is strictly convex. Suppose that $\tau' = \sigma_1' \cap \sigma_2'$ is a wall of Σ'. Then $\tau' \subseteq \mathrm{Span}(\tau_0)$, $\tau_0 \in \Sigma(n-1)$. We label σ_1' and σ_2' so that

$$\begin{aligned}
\varphi|_{\sigma_1'}(u) &= -\langle m_{\tau_0}, u\rangle - \textstyle\sum_{\tau\neq\tau_0 \text{ in } \Sigma(n-1)} |\langle m_\tau, u\rangle|, \ u \in \sigma_1'\\
\varphi|_{\sigma_2'}(u) &= \quad \langle m_{\tau_0}, u\rangle - \textstyle\sum_{\tau\neq\tau_0 \text{ in } \Sigma(n-1)} |\langle m_\tau, u\rangle|, \ u \in \sigma_2'.
\end{aligned}$$

The sum $\sum_{\tau\neq\tau_0 \text{ in } \Sigma(n-1)} |\langle m_\tau, u\rangle|$ is linear on $\sigma_1' \cup \sigma_2'$, so φ is represented by different linear functions on each side of the wall τ'. Since φ is convex, it is strictly convex by Lemma 6.1.13. Then $X_{\Sigma'}$ is projective since $D' = -\sum_{\rho'} \varphi(u_{\rho'}) D_{\rho'}$ is ample by Theorem 6.1.14. $\square$

Using the results of Chapter 11, one can improve this result by showing that $X_{\Sigma'}$ can be chosen to be smooth and projective.

Exercises for §6.1.

6.1.1. Let $S \subseteq M_\mathbb{R}$ be a compact set and define $\phi : N_\mathbb{R} \to \mathbb{R}$ by $\phi(u) = \min_{m\in S}\langle m, u\rangle$. Explain carefully why the minimum exists and prove that ϕ is convex.

6.1.2. Let $H_{u,a}$ be as in the proof of (b) $\Rightarrow$ (d) of Theorem 6.1.7. Prove that $H_{u,a}$ is a supporting hyperplane of P_D that satisfies (6.1.5). Hint: Write $u = \sum_{\rho\in\sigma(1)} \lambda_\rho u_\rho$, $\lambda_\rho > 0$. Then show $m \in P_D$ implies $\langle m, u\rangle = \sum_{\rho\in\sigma(1)} \lambda_\rho \langle m, u_\rho\rangle \geq \varphi_D(u)$.

6.1.3. As noted in the text, the polytope P_D of a basepoint free Cartier divisor on a complete toric variety X_Σ can have dimension strictly less than $\dim X_\Sigma$. Here are some examples.

(a) Let D be one of the four torus-invariant prime divisors on $\mathbb{P}^1 \times \mathbb{P}^1$. Show that P_D is a line segment.

(b) Consider $(\mathbb{P}^1)^n$ and fix an integer d with $0 < d < n$. Find a basepoint free divisor D on $(\mathbb{P}^1)^n$ such that $\dim P_D = d$. Hint: See Exercise 6.1.9 below.

6.1.4. Show that the toric variety X_P of the polytope P in Example 6.1.11 is singular.

6.1.5. This exercise is devoted to proving that the statements (a)–(g) of Lemma 6.1.13 are equivalent. Many of the implications use Lemma 6.1.5.

(a) Prove (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (d).

(b) Prove (b) $\Rightarrow$ (e) and (b) $\Rightarrow$ (f) $\Rightarrow$ (c).

(c) Prove (c) $\Rightarrow$ (b) by adapting the proof of (d) $\Rightarrow$ (b) from Lemma 6.1.5.

(d) Prove (b) $\Leftrightarrow$ (g) and use the obvious implication (e) $\Rightarrow$ (d) to complete the proof of the lemma.

6.1.6. Let X_Σ be the toric variety of a fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^n$ and fix $\sigma_0 \in \Sigma(n)$. Prove that the natural map $\mathrm{CDiv}_T(X_\Sigma) \to \mathrm{Pic}(X_\Sigma)$ induces an isomorphism

$$\big\{D = \textstyle\sum_\rho a_\rho D_\rho \in \mathrm{CDiv}_T(X_\Sigma) \mid a_\rho = 0 \text{ for all } \rho \in \sigma_0(1)\big\} \simeq \mathrm{Pic}(X_\Sigma).$$

6.1.7. Prove that the toric variety X_Σ of Example 6.1.17 is smooth.

6.1.8. For the following toric varieties X_Σ, compute $\mathrm{Pic}(X_\Sigma)$ and describe which torus-invariant divisors are ample and which are basepoint free.

(a) X_Σ is the toric variety of the smooth complete fan Σ in $\mathbb{R}^2$ with
$$\Sigma(1) = \{\pm e_1, \pm e_2, e_1 + e_2\}.$$
(b) X_Σ is the blowup $\mathrm{Bl}_p(\mathbb{P}^n)$ of $\mathbb{P}^n$ at a fixed point p of the torus action.

(c) X_Σ is the toric variety of the fan Σ from Exercise 3.3.12. See Figure 12 from Chapter 3.

(d) X_Σ is the toric variety of the fan obtained from the fan of Figure 12 from Chapter 3 by combining the two upward pointing cones.

6.1.9. The toric variety $(\mathbb{P}^1)^n$ has ray generators $\pm e_1, \ldots, \pm e_n$. Let $D_1^\pm, \ldots, D_n^\pm$ denote the corresponding torus-invariant divisors. Consider $D = \sum_{i=1}^n (a_i^+ D_i^+ + a_i^- D_i^-)$.

(a) Show that D is basepoint free if and only if $a_i^+ + a_i^- \geq 0$ for all i.

(b) Show that D is ample if and only if $a_i^+ + a_i^- > 0$ for all i.

6.1.10. Let $D = \sum_\rho a_\rho D_\rho$ be an ample divisor on a complete toric variety X_Σ. Define
$$\sigma = \mathrm{Cone}((u_\rho, -a_\rho) \mid \rho \in \Sigma(1)) \subseteq N_\mathbb{R} \times \mathbb{R}.$$
(a) Prove that σ is strongly convex.

(b) Prove that the boundary of σ is the graph of the support function φ_D.

(c) Prove that Σ is the set of cones obtained by projecting proper faces of σ onto $M_\mathbb{R}$.

6.1.11. Let Σ be the fan from Example 4.2.13. Prove the X_Σ is not projective.

§6.2. Polytopes and Projective Toric Varieties

We begin with the set of polytopes
$$\big\{P \subseteq M_\mathbb{R} \mid P \text{ is a full dimensional lattice polytope}\big\}$$
and the set of pairs
$$\big\{(X_\Sigma, D) \mid \Sigma \text{ a complete fan in } N_\mathbb{R},\ D \text{ a torus-invariant ample divisor on } X_\Sigma\big\}.$$
These sets are related as follows.

Theorem 6.2.1. *The maps $P \mapsto (X_P, D_P)$ and $(X_\Sigma, D) \mapsto P_D$ define bijections between the above sets that are inverses of each other.*

Proof. The map $P \mapsto (X_P, D_P)$ comes from Proposition 6.1.10, where we showed that D_P is an ample divisor on X_P. Also recall from Proposition 3.1.6 that X_P is the toric variety of the normal fan Σ_P, which is a fan in $N_\mathbb{R}$. For $(X_\Sigma, D) \mapsto P_D$, we showed that $P_D \subseteq M_\mathbb{R}$ is a full dimensional lattice polytope in the proof of Theorem 6.1.14.

It remains to prove that these maps are inverses of each other. One direction is easy, since $P \mapsto (X_P, D_P) \mapsto P_{D_P} = P$, where the equality is Exercise 4.3.1. Going the other way, we have $(X_\Sigma, D) \mapsto P_D \mapsto (X_{P_D}, D_{P_D}) = (X_\Sigma, D)$, where the equality follows from (6.1.8) and (6.1.9) in the proof of Theorem 6.1.14. $\square$

The goal of this section is to look more deeply into the above relationship. In particular, we are interested in the following questions:

- Suppose P and Q are full dimensional lattice polytopes with $X_P = X_Q$. How are P and Q related?
- Suppose D is a torus-invariant Cartier divisor on X_P that is basepoint free. How are P and P_D related?

The answers to these questions will involve generalized fans, pullbacks of divisors, and Minkowski sums of polytopes.

Generalized Fans. The polytope P_D of a basepoint free Cartier divisor D is a lattice polytope by Theorem 6.1.7, but need not be full dimensional (see Exercise 6.1.3). If we want P_D to have a "normal fan," we need to allow for more general fans. Here is the definition we will use.

Definition 6.2.2. A ***generalized fan*** Σ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subseteq N_{\mathbb{R}}$ such that:

(a) Every $\sigma \in \Sigma$ is a rational polyhedral cone.

(b) For all $\sigma \in \Sigma$, each face of σ is also in Σ.

(c) For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each (hence also in Σ).

This agrees with the definition of fan given in Definition 3.1.2, except that the cones are no longer required be strongly convex. The definitions of *support* and *complete* extend to generalized fans in the obvious way. A generalized fan Σ that is a ordinary fan is called *nondegenerate*; otherwise Σ is *degenerate*. Generalized fans will play an important role in Chapters 14 and 15.

Let Σ be a generalized fan. Then $\sigma_0 = \bigcap_{\sigma \in \Sigma} \sigma$ is the minimal cone in Σ. It has no proper faces and hence must be a subspace of $N_{\mathbb{R}}$. Let $\overline{N} = N/(\sigma_0 \cap N)$ with quotient map $\pi : N \to \overline{N}$. You will prove the following in Exercise 6.2.1:

- Σ is a fan if and only if $\sigma_0 = \{0\}$.
- For $\sigma \in \Sigma$, $\overline{\sigma} = \sigma/\sigma_0 \subseteq N_{\mathbb{R}}/\sigma_0 = \overline{N}_{\mathbb{R}}$ is a strongly convex rational polyhedral cone such that $\sigma = \pi_{\mathbb{R}}^{-1}(\overline{\sigma})$.
- $\overline{\Sigma} = \{\overline{\sigma} \mid \sigma \in \Sigma\}$ is a fan in $\overline{N}_{\mathbb{R}}$.

The toric variety X_Σ of the generalized fan Σ is defined to be the toric variety of the usual fan $\overline{\Sigma}$, i.e., $X_\Sigma = X_{\overline{\Sigma}}$.

The Normal Fan of a Lattice Polytope. Some of most interesting generalized fans come from polyhedra. Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope. We do not assume that P is full dimensional. A vertex $\mathsf{v} \in P$ gives the cone

$$C_{\mathsf{v}} = \mathrm{Cone}(P \cap M - \mathsf{v}) \subseteq M_{\mathbb{R}}.$$

Similar to §2.3, the dual cone $\sigma_{\mathsf{v}} = C_{\mathsf{v}}^{\vee} \subseteq N_{\mathbb{R}}$ is a rational polyhedral cone, and these cones give a generalized fan as follows (Exercise 6.2.2).

Proposition 6.2.3. *Given a lattice polytope $P \subseteq M_{\mathbb{R}}$, the set*

$$\Sigma_P = \{\sigma \mid \sigma \preceq \sigma_{\mathsf{v}},\ \mathsf{v} \text{ is a vertex of } P\}$$

is a complete generalized fan in $N_{\mathbb{R}}$. Furthermore:

(a) *The minimal cone of Σ_P is the dual of* $\mathrm{Span}(m - m' \mid m, m' \in P \cap M) \subseteq M_{\mathbb{R}}$.

(b) *Σ_P is a fan if and only if $P \subseteq M_{\mathbb{R}}$ is full dimensional.* □

We call Σ_P the *normal fan* of P. The toric variety X_P is then defined to be the toric variety of the generalized fan X_{Σ_P}, i.e., $X_P = X_{\Sigma_P}$.

Example 6.2.4. Let $P \subseteq M_{\mathbb{R}}$ be a line segment whose vertices are lattice points. The cone C_{v} at each vertex is a ray, so that the normal fan Σ_P consists of two closed half-spaces and the hyperplane where they intersect. Taking the quotient by this hyperplane gives the usual fan for $\mathbb{P}^1$, so that $X_P = \mathbb{P}^1$. ◊

The Normal Fan of a Basepoint Free Divisor. If Σ is a complete fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$ and $D = \sum_\rho a_\rho D_\rho$ has no basepoints and Cartier data $\{m_\sigma\}_{\sigma \in \Sigma(n)}$, then P_D is a lattice polytope with the m_σ as vertices. We can describe the normal fan Σ_{P_D} of P_D as follows.

Proposition 6.2.5. *Let $D = \sum_\rho a_\rho D_\rho$ be a basepoint free Cartier divisor on X_Σ with polytope P_D. Then:*

(a) *If $\mathsf{v} \in P_D$ is a vertex, then the corresponding cone $\sigma_{\mathsf{v}} = C_{\mathsf{v}}^\vee$ in the normal fan Σ_{P_D} is the union*

$$\sigma_{\mathsf{v}} = \bigcup_{\substack{\sigma \in \Sigma(n) \\ m_\sigma = \mathsf{v}}} \sigma.$$

(b) *Σ is a refinement Σ_{P_D}.*

Proof. Part (b) follows immediately from part (a). Let $\mathsf{v} \in P_D$ be a vertex. Since $C_{\mathsf{v}} = \mathrm{Cone}(P_D \cap M - \mathsf{v})$ is strongly convex, its dual $\sigma_{\mathsf{v}} = C_{\mathsf{v}}^\vee$ has dimension n in $N_{\mathbb{R}}$. It follows that part (a) is equivalent to the assertion

$$\text{for all } \sigma \in \Sigma(n),\ \mathrm{Int}(\sigma) \cap \mathrm{Int}(\sigma_{\mathsf{v}}) \neq \emptyset \text{ implies } m_\sigma = \mathsf{v}, \tag{6.2.1}$$

where "Int" denotes the interior (Exercise 6.2.3). Also note that any $u \in \sigma_{\mathsf{v}}$ satisfies

$$\langle m - \mathsf{v}, u\rangle \geq 0, \quad \text{for all } m \in P_D \cap M.$$

In particular, $m_\sigma \in P_D$ for $\sigma \in \Sigma(n)$ since D is basepoint free, so that

$$\langle m_\sigma, u\rangle \geq \langle \mathsf{v}, u\rangle, \quad \text{for all } \sigma \in \Sigma(n). \tag{6.2.2}$$

We now prove (6.2.1). Assume $\mathrm{Int}(\sigma) \cap \mathrm{Int}(\sigma_{\mathsf{v}}) \neq \emptyset$ and let u be an element of the intersection. Since $\mathsf{v} = m_{\sigma'}$ for some $\sigma' \in \Sigma(n)$, we have

$$\langle \mathsf{v}, u\rangle \geq \varphi_D(u) = \langle m_\sigma, u\rangle$$

by convexity and part (b) of Lemma 6.1.5. Combining this with (6.2.2), we see that

$$\langle m_\sigma, u\rangle = \langle \mathsf{v}, u\rangle, \quad \text{for all } u \in \mathrm{Int}(\sigma)\cap\mathrm{Int}(\sigma_{\mathsf{v}}).$$

Since $\mathrm{Int}(\sigma)\cap\mathrm{Int}(\sigma_{\mathsf{v}})$ is open, this forces $m = m_\sigma$, proving (6.2.1). $\square$

This proposition gives a nice way to think about the normal fan Σ_{P_D}. One begins with the Cartier data $\{m_\sigma\}_{\sigma\in\Sigma(n)}$ of D and then combines all cones $\sigma \in \Sigma(n)$ whose m_σ's give the same vertex of P_D. These combined cones and their faces satisfy the conditions for being a fan, except that strong convexity fails when P_D is not full dimensional. Here is an example of how this works.

Example 6.2.6. For the Hirzebruch surface $\mathscr{H}_2$, consider the divisors $D = D_4$ and $D' = D_1$. The polytope P_D from Figure 4 of Example 6.1.2 is shown on the left in Figure 10 on the next page. By Proposition 6.2.5, $m_1 = m_4$ tells us to combine σ_1 and σ_4, as shown on the right in Figure 10. Thus the normal fan of P_D is a fan with three maximal cones.

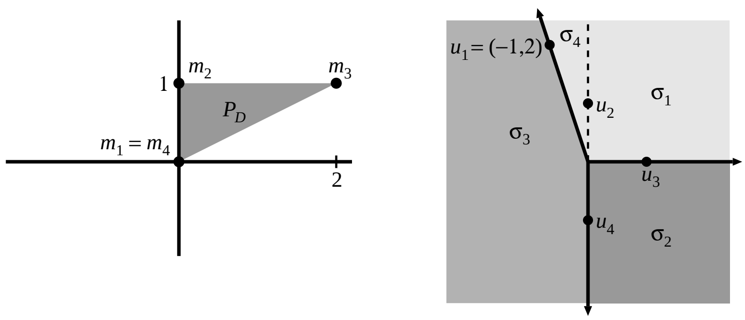


Figure 10. P_D (left) and its normal fan (right)

The polytope $P_{D'}$ is the line segment shown on the left in Figure 11. Here, we combine σ_1 and σ_2 (since $m_1 = m_2$) and also combine σ_3 and σ_4 (since $m_3 = m_4$). This gives the degenerate normal fan shown on the right in Figure 11. Thus the toric variety of $P_{D'}$ is $\mathbb{P}^1$. $\Diamond$

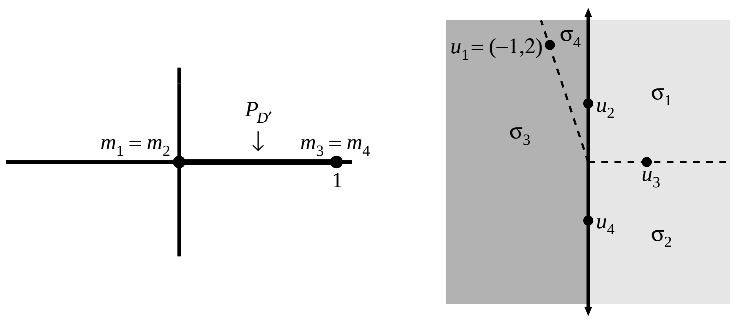


Figure 11. $P_{D'}$ (left) and its degenerate normal fan (right)

Pulling Back via Toric Morphisms. In order to understand the full implications of Proposition 6.2.5, we need the following description of pullbacks of torus-invariant Cartier divisors by toric morphisms.

Proposition 6.2.7. *Assume that $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is the toric morphism induced by $\overline{\phi} : N_1 \to N_2$, and let D_2 be a torus-invariant Cartier divisor with support function $\varphi_{D_2} : |\Sigma_2| \to \mathbb{R}$. Then there is a unique torus-invariant Cartier divisor D_1 on X_{Σ_1} with the following properties:*

(a) $\mathscr{O}_{X_{\Sigma_1}}(D_1) \simeq \phi^* \mathscr{O}_{X_{\Sigma_2}}(D_2)$.

(b) *The support function φ_{D_1} is the composition*

$$|\Sigma_1| \xrightarrow{\overline{\phi}} |\Sigma_2| \xrightarrow{\varphi_{D_2}} \mathbb{R}.$$

Proof. Let the local data of D_2 be $\{(U_\sigma, \chi^{-m_\sigma})\}_{\sigma \in \Sigma_2}$, where σ now refers to an arbitrary cone of Σ_2. Recall that the minus sign comes from $\langle m_\sigma, u_\rho \rangle = -a_\rho$ when $\rho \in \sigma(1)$. Then the proof of Theorem 6.0.18 shows that $\mathscr{O}_{X_{\Sigma_2}}(D_2)$ is the sheaf of sections of a rank 1 vector bundle $V \to X_{\Sigma_2}$ with transition functions

$$g_{\sigma\tau} = \chi^{m_\tau - m_\sigma}.$$

Now take $\sigma' \in \Sigma_1$ and let $\sigma \in \Sigma_2$ be the smallest cone satisfying $\overline{\phi}_{\mathbb{R}}(\sigma') \subseteq \sigma$. Using the dual map $\overline{\phi}^* : M_2 \to M_1$, we set

$$m_{\sigma'} = \overline{\phi}^*(m_\sigma).$$

Since $\phi(U_{\sigma'}) \subseteq U_\sigma$, one can show without difficulty that

$$g_{\sigma'\tau'} = \chi^{m_{\tau'} - m_{\sigma'}} \in \mathscr{O}_{X_{\Sigma_1}}(U_{\sigma'} \cap U_{\tau'})^*.$$

Then $\{(U_{\sigma'}, \chi^{-m_{\sigma'}})\}_{\sigma' \in \Sigma_1}$ is the local data for a Cartier divisor D_1 on X_{Σ_1}. It is straightforward to verify that D_1 has the required properties (Exercise 6.2.4). □

In the situation of Proposition 6.2.7, we call D_1 is the *pullback* of D_2 via ϕ since $\mathscr{O}_{X_{\Sigma_1}}(D_1)$ is the pullback of $\mathscr{O}_{X_{\Sigma_2}}(D_2)$ via ϕ. We denote this by $D_1 = \phi^* D_2$.

The Structure of Basepoint Free Divisors. Proposition 6.2.5 shows that Σ refines the normal fan Σ_{P_D}. Hence we should have a toric morphism $X_\Sigma \to X_{P_D}$. This is certainly true when Σ_{P_D} is nondegenerate, and as we will see below, it remains true when Σ_{P_D} is degenerate. More importantly, D is (up to linear equivalence) the pullback of an ample divisor on X_{P_D} via this morphism.

Theorem 6.2.8. *Let D be a basepoint free Cartier divisor on a complete toric variety, and let X_D be the toric variety of the polytope $P_D \subseteq M_{\mathbb{R}}$. Then the refinement Σ of Σ_{P_D} induces a proper toric morphism*

$$\phi : X_\Sigma \longrightarrow X_{P_D}.$$

Furthermore, D is linearly equivalent to the pullback via ϕ of the ample divisor on X_{P_D} coming from P_D.

Proof. The minimal cone σ_0 of Σ_{P_D} is a subspace of $N_{\mathbb{R}}$. Let $\overline{N} = N/(\sigma_0 \cap N)$, with quotient map $\overline{\phi} : N \to \overline{N}$. Since Σ refines Σ_{P_D} and Σ_{P_D} projects to a genuine fan in $\overline{N}_{\mathbb{R}}$, it follows that $\overline{\phi}$ induces a toric morphism as claimed. Note also that ϕ is proper since X_Σ and X_{P_D} are complete.

Let $\overline{M} \subseteq M$ be dual to $\overline{\phi} : N \to \overline{N}$. Part (a) of Proposition 6.2.3 implies that $\overline{M}_{\mathbb{R}} = \mathrm{Span}(m - m' \mid m, m' \in P_D \cap M\}$. Translating P_D by a lattice point, we may assume that $P_D \subseteq \overline{M}_{\mathbb{R}}$. This changes our original divisor D by a linear equivalence.

The polytope P_D gives the ample divisor $\overline{D} = D_{P_D}$ on X_{P_D}. Since D is basepoint free, Theorem 6.1.7 implies that

$$\varphi_D(u) = \min_{m \in P_D} \langle m, u \rangle.$$

Using $P_D \subseteq \overline{M}_{\mathbb{R}}$, one sees that φ_D factors through $\overline{\phi} : N \to \overline{N}$, and in fact,

$$\varphi_D = \varphi_{\overline{D}} \circ \overline{\phi}_{\mathbb{R}}$$

(Exercise 6.2.5). By Proposition 6.2.7, D is the pullback of $\overline{D} = D_{P_D}$ via ϕ. $\square$

Theorem 6.2.8 implies that a Cartier divisor without basepoints on a complete toric variety has a very nice structure: it is linearly equivalent to the pullback (via a toric morphism) of an ample divisor on a projective toric variety of possibly smaller dimension. This will be useful when we study the geometric invariant theory of toric varieties in Chapters 14 and 15.

Here are two examples to illustrate what can happen in Theorem 6.2.8.

Example 6.2.9. The toric variety X_Σ of Example 6.1.17 has no ample divisors, but it does have nontrivial basepoint free divisors. The ray generators of Σ are

$$\pm e_1, \pm e_2, \pm e_3, a, b, c, d,$$

with corresponding toric divisors

$$D_1^{\pm}, D_2^{\pm}, D_3^{\pm}, D_a, D_b, D_c, D_d.$$

Then one can show that

$$D = 2D_1^- + 2D_2^- + 2D_3^- - D_a - D_b - D_c - D_d$$

is basepoint free (Exercise 6.2.6). Thus the support function φ_D is convex.

Figure 9 in Example 6.1.17 shows that $\mathrm{Cone}(e_1, e_2, d)$ is a union of three cones of Σ. Using $\varphi_D(e_1) = \varphi_D(e_2) = 0$ and $\varphi_D(a) = \varphi_D(b) = \varphi_D(d) = 1$, one sees that these three cones all have $m_\sigma = e_3$ (Exercise 6.2.6). Hence we should combine these three cones. The same thing happens in $\mathrm{Cone}(e_1, e_3, d)$ and $\mathrm{Cone}(e_2, e_3, d)$.

In the first orthant, the fan of X_{P_D} looks like Figure 12 on the next page when intersected with $x + y + z = 1$. Hence X_{P_D} is the blowup of $(\mathbb{P}^1)^3$ at the point corresponding to the first orthant (Exercise 6.2.6). Also, $\phi : X_\Sigma \to X_{P_D}$ is a proper birational toric morphism since Σ refines the (nondegenerate) normal fan Σ_{P_D}. $\Diamond$

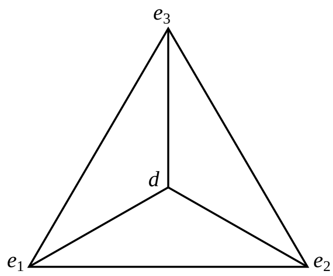


Figure 12. Combined cones of Σ lying in $\mathbb{R}^3_{\geq 0}$ in Example 6.2.9

Example 6.2.10. Consider the divisor $D = 3D_1 + D_2 - D_4 = D_1 + \mathrm{div}(\chi^{e_2})$ on the Hirzebruch surface $\mathscr{H}_2$. Then $P_D = \mathrm{Conv}(-e_2, e_1 - e_2) = \mathrm{Conv}(0, e_1) - e_2$. This gives the degenerate normal fan shown in Figure 11 of Example 6.2.6, and $\phi : X_\Sigma \to X_{P_D} = \mathbb{P}^1$ is the toric morphism from Example 3.3.5. Then $D \sim D_1$, which is the pullback of an ample divisor on $\mathbb{P}^1$. $\Diamond$

$\mathbb{N}$-***Minkowski Summands.*** We now return to the questions asked at the beginning of the section. In terms of normal fans, the answers are easy to give:

- Full dimensional lattice polytopes P and Q in $M_\mathbb{R}$ give the same toric variety if and only if they have the same normal fan.
- If D is a torus-invariant basepoint free Cartier divisor on X_P, then the normal fan of P refines the normal fan of P_D by Proposition 6.2.5.

By rephrasing this in terms of Minkowski sums, we can state both of these purely in the language of polytopes. Here is the definition.

Definition 6.2.11. Given lattice polytopes P and Q in $M_\mathbb{R}$, Q is an $\mathbb{N}$-***Minkowski summand*** of P if

$$Q + Q' = kP,$$

where $k \in \mathbb{N}$ is positive and $Q' \subseteq M_\mathbb{R}$ is a lattice polytope.

Example 6.2.12. The rectangle $Q = \mathrm{Conv}(0, 2e_1, e_2, 2e_1 + e_2)$ is an $\mathbb{N}$-Minkowski summand of the hexagon $P = \mathrm{Conv}(0, e_1, e_2, 2e_1 + e_2, e_1 + 2e_2, 2e_1 + 2e_2)$, as shown by Figure 13. $\Diamond$

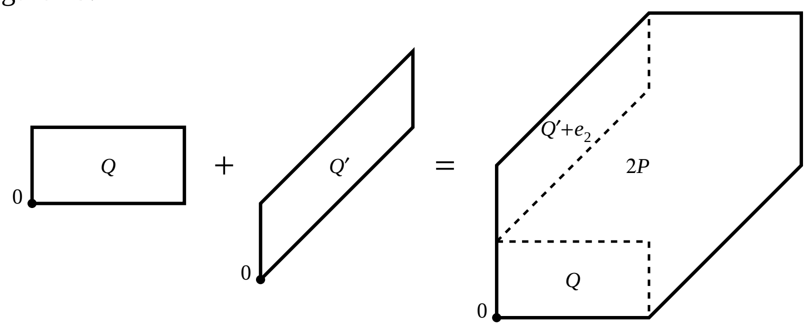


Figure 13. Q is an $\mathbb{N}$-Minkowski summand of P since $Q + Q' = 2P$

Minkowski sums are related to normal fans as follows [**28**, Prop. 1.2].

Proposition 6.2.13. *Let P and Q be lattice polytopes in $M_{\mathbb{R}}$. Then:*

(a) *Q is an $\mathbb{N}$-Minkowski summand of P if and only if Σ_P refines Σ_Q.*

(b) *Σ_{P+Q} is the coarsest common refinement of Σ_P and Σ_Q, i.e., any fan that refines Σ_P and Σ_Q also refines Σ_{P+Q}.* □

Proposition 6.2.13 does not assume that the lattice polytopes P and Q have full dimension, so the normal fans Σ_P and Σ_Q in the proposition may be degenerate. Also note that Σ_{P+Q} is common refinement of Σ_P and Σ_Q by part (a). So the point of part (b) is that Σ_{P+Q} is the most efficient common refinement.

We can now describe when two polytopes give the same toric variety.

Corollary 6.2.14. *Full dimensional lattice polytopes in $M_{\mathbb{R}}$ give the same toric variety if and only if each is an $\mathbb{N}$-Minkowski summand of the other.* □

Proof. This follows immediately from Proposition 6.2.13 since two fans are equal if and only if each refines the other. □

We also have the following lovely result about basepoint free divisors.

Corollary 6.2.15. *Let P be a full dimensional lattice polytope in $M_{\mathbb{R}}$. Then a polytope $Q \subseteq M_{\mathbb{R}}$ is an $\mathbb{N}$-Minkowski summand of P if and only if there is a torus-invariant basepoint free Cartier divisor D on X_P such that $Q = P_D$.*

Proof. If D is basepoint free on X_P, then Propositions 6.2.5 and 6.2.13 imply that P_D is an $\mathbb{N}$-Minkowski summand of P. For the converse, suppose that Q is an $\mathbb{N}$-Minkowski summand of P. Then Σ_P refines Σ_Q. We will write the maximal cones of Σ_Q as σ_{v} for $\mathsf{v} \in Q$ a vertex. Also let $n = \dim X_P$. We define D as follows. Each $\sigma \in \Sigma_P(n)$ is contained in σ_{v} for some vertex $\mathsf{v} \in Q$. Then D is the Cartier divisor on X_P whose Cartier data $\{m_\sigma\}_{\sigma\in\Sigma_P(n)}$ is defined by $m_\sigma = \mathsf{v}$ when $\sigma \subseteq \sigma_{\mathsf{v}}$.

Thus $D = \sum_{\rho\in\Sigma_P(1)} a_\rho D_\rho$, where $a_\rho = -\langle v, u_\rho\rangle$ when $u_\rho \in \sigma_{\mathsf{v}}$. To prove that $P_D = Q$, take $m \in P_D$, so that $\langle m, u_\rho\rangle \geq -a_\rho$ for all $\rho \in \Sigma_P(1)$. This implies

$$\langle m - \mathsf{v}, u_\rho\rangle = \langle m, u_\rho\rangle + a_\rho \geq 0 \text{ for all } u_\rho \in \sigma_{\mathsf{v}}.$$

These u_ρ's generate σ_{v} since Σ_P refines Σ_Q, so that $m - \mathsf{v} \in \sigma_{\mathsf{v}}^\vee = C_{\mathsf{v}}$. Hence

$$m \in \textstyle\bigcap_{\mathsf{v} \text{ is a vertex of } Q} (C_{\mathsf{v}} + \mathsf{v}) = Q,$$

where the equality follows from Exercise 6.2.7. The opposite inclusion $Q \subseteq P_D$ is straightforward and hence is left to the reader. This proves $P_D = Q$, and then D is basepoint free by Proposition 6.1.1. □

Example 6.2.16. Consider the rectangle Q and the hexagon P defined in Example 6.2.12. Since Q is an $\mathbb{N}$-Minkowski summand of P, it gives a basepoint free divisor D on X_P. Let the ray generators $u_1, \ldots, u_6$ of Σ_P be arranged clockwise around the origin, starting with $u_1 = e_2$. Then the recipe for D given in the proof of Corollary 6.2.15 makes it easy to show that

$$D = D_3 + D_4 + 2D_5 + D_6,$$

where D_i is the toric divisor corresponding to u_i (Exercise 6.2.8). $\Diamond$

Zonotopes. Recall from Example 2.3.10 that a *zonotope* is a Minkowski sum of line segments. Here we show that zonotopes have especially nice normal fans. A *central hyperplane arrangement* in $N_{\mathbb{R}}$ consists of finitely many rational hyperplanes $H \subseteq N_{\mathbb{R}}$ whose intersection is the origin. This determines a fan in $N_{\mathbb{R}}$ whose maximal cones are the closures of the connected components of the complement of the arrangement.

Example 6.2.17. The hexagon P from Example 6.2.12 is a zonotope since $P = \mathrm{Conv}(0,e_1) + \mathrm{Conv}(0,e_2) + \mathrm{Conv}(0,e_1+e_2)$. Figure 14 reproduces Figure 7 from

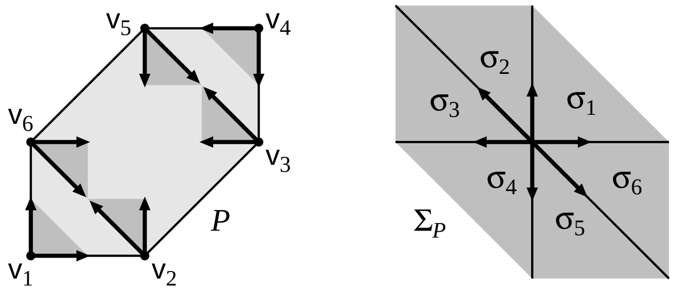


Figure 14. A zonotope P and its normal fan Σ_P

Example 2.3.10. As you can see, the normal fan of P comes from an arrangement of three lines through the origin in $\mathbb{R}^2$. $\Diamond$

Proposition 6.2.18. *The normal fan of a full dimensional lattice zonotope P comes from a central hyperplane arrangement.*

Proof. First note that a Minkowski sum of parallel line segements is again a line segment. Thus we can write $P = L_1 + \cdots + L_s$ as a Minkowski sum of line segments where no two segments are parallel. Each normal fan Σ_{L_i} is determined by the hyperplane normal H_i to L_i, as explained in Example 6.2.4. By Proposition 6.2.13, $\Sigma_P = \Sigma_{L_1+\cdots+L_s}$ is the coarsest common refinement of $\Sigma_{L_1},\ldots,\Sigma_{L_s}$. This is clearly the fan determined by the central hyperplane arrangement $H_1,\ldots,H_s$. Note that that $H_1 \cap \cdots \cap H_s = \{0\}$ since Σ_P is a nondegenerate fan. $\square$

See [**281**, Thm. 7.16] for a different proof of Proposition 6.2.18 that uses linear programming.

Exercises for §6.2.

6.2.1. Prove the properties of generalized fans stated in three bullets in the discussion following Definition 6.2.2.

6.2.2. Prove Proposition 6.2.3.

6.2.3. Prove that part (a) of Propostion 6.2.5 follows from (6.2.1).

6.2.4. Complete the proof of Proposition 6.2.7.

6.2.5. Complete the proof of Theorem 6.2.8.

6.2.6. This exercise deals with Example 6.2.9.

(a) Let $D = 2D_1^- + 2D_2^- + 2D_3^- - D_a - D_b - D_c - D_d$ be the divisor from Example 6.2.9. Prove that P_D is the polytope with 10 vertices
$$e_1, e_2, e_3, 2e_1, 2e_2, 2e_3, 2e_1+2e_2, 2e_1+2e_3, 2e_2+2e_3, 2e_1+2e_2+2e_3$$
and conclude that D is basepoint free.

(b) In Example 6.2.9, we asserted that certain maximal cones of Σ must be combined to get the maximal cones of Σ_{P_D}. Prove that this is correct.

(c) Show that X_{P_D} is the blowup of $(\mathbb{P}^1)^3$ at the point corresponding to the first orthant.

6.2.7. This exercise is concerned with the proof of Corollary 6.2.15.

(a) Given a lattice polytope $Q \subseteq M_{\mathbb{R}}$, let $C_{\mathsf{v}} = \mathrm{Cone}(Q \cap M - \mathsf{v})$ for $\mathsf{v} \in Q$ a vertex. Prove that $\bigcap_{\mathsf{v} \text{ is a vertex of } Q}(C_{\mathsf{v}} + \mathsf{v}) = Q$.

(b) Complete the proof of the corollary by showing $Q \subseteq P_D$.

6.2.8. In Example 6.2.16, the rectangle Q is an $\mathbb{N}$-Minkowski summand of the hexagon P.

(a) In the example, we claimed that $D = D_3 + D_4 + 2D_5 + D_6$. Prove this.

(b) Let $Q' = \mathrm{Conv}(0, e_2, 2e_1 + 2e_2, 2e_1 + 3e_2)$. Prove carefully that $Q + Q' = 2P$ and compute the basepoint free divisor D' determined by Q'.

6.2.9. Suppose that full dimensional lattice polytopes $P, Q \subseteq M_{\mathbb{R}}$ give the same toric variety X_Σ. Prove that $P + Q$ also gives X_Σ.

6.2.10. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope. A face $Q \preceq P$ determines a cone σ_Q in the normal fan of P. This gives the orbit closure $V(\sigma_Q) \subseteq X_P$, and $V(\sigma_Q) \simeq X_Q$ by Proposition 3.2.9. This gives an inclusion $i : X_Q \to X_P$ which is not a toric morphism when $Q \prec P$. Prove that $i^*\mathscr{O}_{X_P}(D_P) \simeq \mathscr{O}_{X_Q}(D_Q)$.

§6.3. The Nef and Mori Cones

In §6.1, we gave some nice criteria for when a Cartier divisor D is basepoint free or ample. We now study the structure of the set of basepoint free divisors and the set of ample divisors inside $\mathrm{Pic}(X_\Sigma)_{\mathbb{R}} = \mathrm{Pic}(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{R}$.

The main concept of this section is that of *numerical effectivity*. Roughly speaking, the goal is to define a pairing between divisors and curves, such that for a Cartier divisor D and complete curve C on a variety X, the number $D \cdot C$ counts the number of points of $D \cap C$, with appropriate multiplicity.

Example 6.3.1. Suppose $X = \mathbb{P}^2$ with homogeneous coordinates x, y, z, and let $D = \mathbf{V}(y)$ and $C = \mathbf{V}(zy - x^2)$. Then D and C meet at the single point $p = (0,0,1)$, where they share a common tangent. If we replace D with the linearly equivalent divisor $E = \mathbf{V}(y - z)$, then clearly E and C meet in two points. This suggests that the point $\{p\} = D \cap C$ should be counted twice, since it is a tangent point. Hence we should have $D \cdot C = 2$. $\Diamond$

Despite this encouraging example, there are several technical hurdles to overcome in order to make this precise in a general setting. Note that in $\mathbb{C}^2$, two lines may or may not meet, so to get a reasonable theory, we will work with *complete curves* C on a normal variety X. We also need to restrict to *Cartier divisors* D on X. With these assumptions, the intersection product $D \cdot C$ should possess the following properties:

- $(D + E) \cdot C = D \cdot C + E \cdot C$.
- $D \cdot C = E \cdot C$ when $D \sim E$.
- Let D be a prime divisor on X such that $D \cap C$ is finite. Assume each $p \in D \cap C$ is smooth in C, D, X and that the tangent spaces $T_p(C) \subseteq T_p(X)$ and $T_p(D) \subseteq T_p(X)$ meet transversely. Then $D \cdot C = |D \cap C|$.

Using these properties, one can give a rigorous proof of the computation $D \cdot C = 2$ from Example 6.3.1.

The Degree of a Line Bundle. The key tool we will use is the notion of the *degree* of a divisor on an irreducible smooth complete curve C. Such a divisor can be written as a finite sum $D = \sum_i a_i\, p_i$ where $a_i \in \mathbb{Z}$ and $p_i \in C$.

Definition 6.3.2. Let $D = \sum_i a_i\, p_i$ be a divisor on an irreducible smooth complete curve C. Then the ***degree*** of D is the integer

$$\deg(D) = \sum_i a_i \in \mathbb{Z}.$$

Note the obvious property $\deg(D + E) = \deg(D) + \deg(E)$. The following key result is proved in [**131**, Cor. II.6.10].

Theorem 6.3.3. *Every principal divisor on an irreducible smooth complete curve has degree zero.* $\square$

In other words, $\deg(\mathrm{div}(f)) = 0$ for all nonzero rational functions f on an irreducible smooth complete curve C. Thus

$$\deg(D) = \deg(E) \text{ when } D \sim E \text{ on } C,$$

and the degree map induces a surjective homomorphism

$$\deg : \mathrm{Pic}(C) \longrightarrow \mathbb{Z}.$$

Note that all Weil divisors are Cartier since C is smooth.

In §6.0 we showed that $\mathrm{Pic}(C)$ is the set of isomorphism classes of line bundles on C. Hence we can define the degree $\deg(\mathscr{L})$ of a line bundle $\mathscr{L}$ on C. This leads immediately to the following result.

Proposition 6.3.4. *Let C be an irreducible smooth complete curve. Then a line bundle $\mathscr{L}$ has a **degree** $\deg(\mathscr{L})$ such that $\mathscr{L} \mapsto \deg(\mathscr{L})$ has the following properties:*

(a) $\deg(\mathscr{L} \otimes \mathscr{L}') = \deg(\mathscr{L}) + \deg(\mathscr{L}')$.

(b) $\deg(\mathscr{L}) = \deg(\mathscr{L}')$ *when* $\mathscr{L} \simeq \mathscr{L}'$.

(c) $\deg(\mathscr{L}) = \deg(D)$ *when* $\mathscr{L} \simeq \mathscr{O}_C(D)$. □

The Normalization of a Curve. We defined the normalization of an affine variety in §1.0, and by gluing together the normalizations of affine pieces, one can define the normalization of any variety (see [**131**, Ex. II.3.8]). In particular, an irreducible curve C has a normalization map

$$\phi : \overline{C} \longrightarrow C,$$

where $\overline{C}$ is an normal variety. Here are the key propeties of $\overline{C}$.

Proposition 6.3.5. *Let $\overline{C}$ be the normalization of an irreducible curve C. Then:*

(a) *$\overline{C}$ is smooth.*

(b) *$\overline{C}$ is complete whenever C is complete.*

Proof. Since C is a curve, Proposition 4.0.17 implies that C is smooth. Part (b) is covered by [**131**, Ex. II.5.8]. □

One can prove that every irreducible smooth complete curve is projective. See [**131**, Ex. II.5.8].

The Intersection Product. We now have the tools needed to define the intersection product. Let X be a normal variety. Given a Cartier divisor D on X and an irreducible complete curve $C \subseteq X$, we have

- The line bundle $\mathscr{O}_X(D)$ on X.
- The normalization $\phi : \overline{C} \longrightarrow C$.

Then $\phi^*\mathscr{O}_X(D)$ is a line bundle on the irreducible smooth complete curve $\overline{C}$.

Definition 6.3.6. The ***intersection product*** of D and C is $D \cdot C = \deg(\phi^*\mathscr{O}_X(D))$.

Here are some properties of the intersection product.

Proposition 6.3.7. *Let C be an irreducible complete curve and D, E Cartier divisors on a normal variety X. Then:*

(a) $(D+E)\cdot C = D\cdot C + E\cdot C$.

(b) $D\cdot C = E\cdot C$ *when* $D \sim E$.

Proof. The pullback of line bundles is compatible with tensor product, so that part (a) follows from (6.0.3) and Proposition 6.3.4. Part (b) is an easy consequence of Propositions 6.0.22 and 6.3.4. □

The intersection product extends to $\mathbb{Q}$-Cartier divisors as follows. Recall from Chapter 4 that a Weil divisor D is $\mathbb{Q}$-Cartier if ℓD is Cartier for some integer $\ell > 0$. Given an irreducible complete curve $C \subseteq X$, let

$$D \cdot C = \frac{1}{\ell}(\ell D) \cdot C \in \mathbb{Q}. \tag{6.3.1}$$

In Exercise 6.3.1 you will show that this intersection product is well-defined and satisfies Propostion 6.3.7.

Intersection Products on Toric Varieties. In the toric case, $D \cdot C$ is easy to compute when D and C are torus-invariant in X_Σ. In order for C to be torus-invariant and complete, we must have $C = V(\tau) = \overline{O(\tau)}$, where $\tau = \sigma \cap \sigma' \in \Sigma(n-1)$ is the wall separating cones $\sigma, \sigma' \in \Sigma(n)$, $n = \dim X_\Sigma$. We do not assume Σ is complete.

In this situation, we have the sublattice $N_\tau = \mathrm{Span}(\tau) \cap N \subseteq N$ and the quotient $N(\tau) = N/N_\tau$. Let $\overline{\sigma}$ and $\overline{\sigma}'$ be the images of σ and σ' in $N(\tau)_\mathbb{R}$. Since τ is a wall, $N(\tau) \simeq \mathbb{Z}$ and $\overline{\sigma}$, $\overline{\sigma}'$ are rays that correspond to the rays in the usual fan for $\mathbb{P}^1$. In particular, $V(\tau) \simeq \mathbb{P}^1$ is smooth, so no normalization is needed when computing the intersection product.

Proposition 6.3.8. *Let $C = V(\tau)$ be the complete torus-invariant curve in X_Σ coming from the wall $\tau = \sigma \cap \sigma'$. Let D be a Cartier divisor with Cartier data $m_\sigma, m_{\sigma'} \in M$ corresponding to $\sigma, \sigma' \in \Sigma(n)$. Also pick $u \in \sigma' \cap N$ that maps to the minimal generator of $\overline{\sigma}' \subseteq N(\tau)_\mathbb{R}$. Then*

$$D \cdot C = \langle m_\sigma - m_{\sigma'}, u\rangle \in \mathbb{Z}.$$

Proof. Since $V(\tau) \subseteq U_\sigma \cup U_{\sigma'}$, we can assume $X_\Sigma = U_\sigma \cup U_{\sigma'}$ and Σ is the fan consisting of σ, σ' and their faces. We also have

$$D|_{U_\sigma} = \mathrm{div}(\chi^{-m_\sigma})|_{U_\sigma},\ D|_{U_{\sigma'}} = \mathrm{div}(\chi^{-m_{\sigma'}})|_{U_{\sigma'}}.$$

The proof of Proposition 6.2.7 shows that the line bundle $\mathscr{O}_{X_\Sigma}(D)$ is determined by the transition function $g_{\sigma'\sigma} = \chi^{m_\sigma - m_{\sigma'}}$. Thus

$$D \cdot C = \deg(i^*\mathscr{O}_{X_\Sigma}(D)),$$

where $i : V(\tau) \hookrightarrow X_\Sigma$ is the inclusion map. The pullback bundle is determined by the restriction of $g_{\sigma'\sigma}$ to

$$V(\tau) \cap U_\sigma \cap U_{\sigma'} = V(\tau) \cap U_\tau = O(\tau),$$

where $O(\tau)$ is the T_N-orbit corresponding to τ. This is also the torus of the toric variety $V(\tau) = \overline{O(\tau)}$. In Lemma 3.2.5, we showed that $\tau^\perp \cap M$ is the dual of $N(\tau)$ and that

$$O(\tau) \simeq \mathrm{Hom}_\mathbb{Z}(M \cap \tau^\perp, \mathbb{C}^*) \simeq T_{N(\tau)}.$$

Now comes the key observation: since the linear functions given by $m_\sigma, m_{\sigma'}$ agree on τ, we have $m_\sigma - m_{\sigma'} \in \tau^\perp \cap M$. Thus $i^*\mathscr{O}_{X_\Sigma}(D)$ is the line bundle on $V(\tau)$ whose transition function is $g_{\sigma'\sigma} = \chi^{m_\sigma - m_{\sigma'}}$ for $m_\sigma - m_{\sigma'} \in \tau^\perp \cap M$.

It follows that $i^*\mathscr{O}_{X_\Sigma}(D) \simeq \mathscr{O}_{V(\tau)}(\overline{D})$, where $\overline{D}$ is the divisor on $V(\tau)$ given by the Cartier data

$$m_{\overline{\sigma}} = 0,\ \ m_{\overline{\sigma}'} = m_{\sigma'} - m_\sigma.$$

Let $p_\sigma, p_{\sigma'}$ be the torus fixed points corresponding to σ, σ'. Since $u \in \sigma' \cap N$ maps to the minimal generator $\overline{u} \in \overline{\sigma}' \cap N(\tau)$, we have

$$\overline{D} = \langle -m_{\overline{\sigma}}, -\overline{u}\rangle\, p_\sigma + \langle -m_{\overline{\sigma}'}, \overline{u}\rangle\, p_{\sigma'} = \langle m_\sigma - m_{\sigma'}, u\rangle\, p_{\sigma'},$$

where the second equality follows from $m_{\overline{\sigma}'} = m_{\sigma'} - m_\sigma \in \tau^\perp \cap M$. Hence

$$D \cdot C = \deg(i^*\mathscr{O}_{X_\Sigma}(D)) = \deg(\overline{D}) = \langle m_\sigma - m_{\sigma'}, u\rangle. \qquad \square$$

Example 6.3.9. Consider the toric surface whose fan Σ in $\mathbb{R}^2$ has ray generators

$$u_1 = e_1,\ u_2 = e_2,\ u_0 = 2e_1 + 3e_2$$

and maximal cones

$$\sigma = \mathrm{Cone}(u_1, u_0),\ \sigma' = \mathrm{Cone}(u_2, u_0).$$

The support of Σ is the first quadrant and $\tau = \sigma \cap \sigma' = \mathrm{Cone}(u_0)$ gives the complete torus-invariant curve $C = V(\tau) \subseteq X_\Sigma$.

If D_1, D_2, D_0 are the divisors corresponding to u_1, u_2, u_0, then

$$D = aD_1 + bD_2 + cD_0 \text{ is Cartier} \iff 2a + 3b \equiv c \bmod 6.$$

When this condition is satisfied, we have

$$m_\sigma = -ae_1 + \frac{2a-c}{3}e_2,\ m_{\sigma'} = \frac{3b-c}{2}e_1 - be_2.$$

Also, $u = e_1 + 2e_2 \in \sigma'$ maps to the minimal generator of $\overline{\sigma}'$ since u, u_0 form a basis of $\mathbb{Z}^2$. (You will check these assertions in Exercise 6.3.2.) Thus

$$D \cdot C = \langle m_\sigma - m_{\sigma'}, u\rangle = \frac{2a + 3b - c}{6}$$

by Proposition 6.3.8. Note that D is $\mathbb{Q}$-Cartier since Σ is simplicial. Then (6.3.1) shows that the formula for $D \cdot C$ holds for arbitrary integers a, b, c. In particular,

$$D_1 \cdot C = \frac{1}{3},\ D_2 \cdot C = \frac{1}{2},\ D_0 \cdot C = -\frac{1}{6}.$$

In the next section we will see that these intersection products follow directly from the relation $-u_0 + 2u_1 + 3u_2 = 0$ and the fact that $\mathbb{Z}u_0 = \mathrm{Span}(\tau) \cap \mathbb{Z}^2$. $\Diamond$

Nef Divisors. We now define an important class of Cartier divisors.

Definition 6.3.10. Let X be a normal variety. Then a Cartier divisor D on X is ***nef*** (short for ***numerically effective***) if

$$D \cdot C \geq 0$$

for every irreducible complete curve $C \subseteq X$.

A divisor linearly equivalent to a nef divisor is nef. Here is another result.

Proposition 6.3.11. *Every basepoint free divisor is nef.*

Proof. The pullback of a line bundle generated by global sections is generated by global sections (Exercise 6.0.10). Thus, given $\phi : \overline{C} \longrightarrow C$ and D basepoint free, the line bundle $\mathscr{L} = \phi^*(\mathscr{O}_X(D))$ is generated by global sections. This allows us to write $\mathscr{L} = \mathscr{O}_{\overline{C}}(D')$ for a basepoint free divisor D' on $\overline{C}$. A nonzero global section of $\mathscr{O}_{\overline{C}}(D')$ gives an effective divisor E' linearly equivalent to D'. Then

$$D \cdot C = \deg(\phi^*(\mathscr{O}_X(D))) = \deg(\mathscr{O}_{\overline{C}}(D')) = \deg(D') = \deg(E') \geq 0,$$

where the last inequality follows since E' is effective. □

In the toric case, nef divisors are especially easy to understand.

Theorem 6.3.12. *Let D be a Cartier divisor on a toric variety X_Σ whose fan Σ has convex support of full dimension. The following are equivalent:*

(a) *D is basepoint free, i.e., $\mathscr{O}_X(D)$ is generated by global sections.*

(b) *D is nef.*

(c) *$D \cdot C \geq 0$ for all torus-invariant irreducible complete curves $C \subseteq X$.*

Proof. The first item implies the second by Proposition 6.3.11, and the second item implies the third by the definition of nef. So suppose that $D \cdot C \geq 0$ for all torus-invariant irreducible curves C. We can replace D with a linearly equivalent torus-invariant divisor. Then, by Theorem 6.1.7, it suffices to show that φ_D is convex.

Take a wall $\tau = \sigma \cap \sigma'$ of Σ and set $C = V(\tau)$. If we pick $u \in \sigma' \cap N$ as in Proposition 6.3.8, then

$$\langle m_\sigma - m_{\sigma'}, u \rangle = D \cdot C \geq 0,$$

so that

$$\langle m_\sigma, u \rangle \geq \langle m_{\sigma'}, u \rangle = \varphi_D(u).$$

Note that $u \notin \sigma$ since the image of u is nonzero in $N(\tau) = N/(\mathrm{Span}(\tau) \cap N)$. Then Lemma 6.1.5 implies that φ_D is convex. □

A variant of the above proof leads to the following ampleness criterion, which you will prove in Exercise 6.3.3.

Theorem 6.3.13 (Toric Kleiman Criterion). *Let D be a Cartier divisor on a complete toric variety X_Σ. Then D is ample if and only if $D \cdot C > 0$ for all torus-invariant irreducible curves $C \subseteq X_\Sigma$.* □

Note that one direction of the proof follows from the general fact that on any complete normal variety, an ample divisor D satisfies $D \cdot C > 0$ for all irreducible curves $C \subseteq X$ (Exercise 6.3.4).

Theorems 6.3.12 and 6.3.13 were well-known in the smooth case and proved more recently (and independently) in [**185**], [**197**] and [**212**] in the complete case.

Numerical Equivalence of Divisors. The intersection product leads to an important equivalence relation on Cartier divisors.

Definition 6.3.14. Let X be a normal variety.

(a) A Cartier divisor D on X is ***numerically equivalent to zero*** if $D \cdot C = 0$ for all irreducible complete curves $C \subseteq X$.

(b) Cartier divisors D and E are ***numerically equivalent***, written $D \equiv E$, if $D - E$ is numerically equivalent to zero.

What does this say in the toric case?

Proposition 6.3.15. *Let D be a Cartier divisor on a toric variety X_Σ whose fan Σ has convex support of full dimension. Then $D \sim 0$ if and only if $D \equiv 0$.*

Proof. Clearly if D is principal then D is numerically equivalent to zero. For the converse, assume $D \equiv 0$ and let $\tau = \sigma \cap \sigma'$ be a wall of Σ. If we pick $u \in \sigma'$ as in Proposition 6.3.8, then

$$0 = D \cdot C = \langle m_\sigma - m_{\sigma'}, u \rangle$$

for $C = V(\tau)$. This forces $m_\sigma = m_{\sigma'}$ since $m_\sigma - m_{\sigma'} \in \tau^\perp$ and $u \notin \sigma$. From here, one sees that $m_\sigma = m_{\sigma'}$ for all $\sigma, \sigma' \in \Sigma(n)$, and it follows that D is principal. □

Numerical Equivalence of Curves. We also get an interesting equivalence relation on curves. Let $Z_1(X)$ be the free abelian group generated by irreducible complete curves $C \subseteq X$. An element of $Z_1(X)$ is called a *proper* 1-*cycle*.

Definition 6.3.16. Let X be a normal variety.

(a) A proper 1-cycle C on X is ***numerically equivalent to zero*** if $D \cdot C = 0$ for all Cartier divisors D on X.

(b) Proper 1-cycles C and C' are ***numerically equivalent***, written $C \equiv C'$, if $C - C'$ is numerically equivalent to zero.

The intersection product $(D, C) \mapsto D \cdot C$ extends naturally to a pairing

$$\mathrm{CDiv}(X) \times Z_1(X) \longrightarrow \mathbb{Z}.$$

between Cartier divisors and 1-cycles. In order to get a nondegenerate pairing, we work over $\mathbb{R}$ and mod out by numerical equivalence.

Definition 6.3.17. For a normal variety X, define

$$N^1(X) = (\mathrm{CDiv}(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad N_1(X) = (Z_1(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R}.$$

It follows easily that we get a well-defined nondegenerate bilinear pairing

$$N^1(X) \times N_1(X) \longrightarrow \mathbb{R}.$$

A deeper fact is that $N^1(X)$ and $N_1(X)$ have finite dimension over $\mathbb{R}$. Thus $N^1(X)$ and $N_1(X)$ are dual vector spaces via intersection product.

The Nef and Mori Cones. The vector spaces $N^1(X)$ and $N_1(X)$ contain some interesting cones.

Definition 6.3.18. Let X be a normal variety.

(a) $\mathrm{Nef}(X)$ is the cone in $N^1(X)$ generated by classes of nef Cartier divisors. We call $\mathrm{Nef}(X)$ the ***nef cone***.

(b) $\mathrm{NE}(X)$ is the cone in $N_1(X)$ generated by classes of irreducible complete curves.

(c) $\overline{\mathrm{NE}}(X)$ is the closure of $\mathrm{NE}(X)$ in $N_1(X)$. We call $\overline{\mathrm{NE}}(X)$ the ***Mori cone***.

Here are some easy observations about the nef and Mori cones.

Lemma 6.3.19.

(a) $\mathrm{Nef}(X)$ *and* $\overline{\mathrm{NE}}(X)$ *are closed convex cones and are dual to each other, i.e.,*

$$\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)^\vee \quad \textit{and} \quad \overline{\mathrm{NE}}(X) = \mathrm{Nef}(X)^\vee.$$

(b) $\mathrm{NE}(X)$ *has full dimension in* $N_1(X)$.

(c) $\mathrm{Nef}(X)$ *is strongly convex in* $N^1(X)$.

Proof. It is obvious that $\mathrm{Nef}(X)$, $\mathrm{NE}(X)$ and $\overline{\mathrm{NE}}(X)$ are convex cones, and $\mathrm{Nef}(X)$ is closed since it is defined by inequalities of the form $D \cdot C \geq 0$. In fact,

$$\mathrm{Nef}(X) = \mathrm{NE}(X)^\vee$$

by the definition of nef. Then $\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)^\vee$ follows easily. In general, $\mathrm{NE}(X)$ need not be closed. However, since the closure of a convex cone is its double dual, we have

$$\overline{\mathrm{NE}}(X) = \mathrm{NE}(X)^{\vee\vee} = \mathrm{Nef}(X)^\vee.$$

Note that the cone $\mathrm{NE}(X)$ has full dimension since $N_1(X)$ is spanned by the classes of irreducible complete curves. Hence the same is true for its closure $\overline{\mathrm{NE}}(X)$. Then $\mathrm{Nef}(X)$ is strongly convex since its the dual has full dimension. $\square$

Let X_Σ be a toric variety whose fan Σ has convex support of full dimension and set $\operatorname{Pic}(X_\Sigma)_\mathbb{R} = \operatorname{Pic}(X_\Sigma) \otimes_\mathbb{Z} \mathbb{R}$. We have an inclusion

$$\operatorname{Pic}(X_\Sigma) \subseteq \operatorname{Pic}(X_\Sigma)_\mathbb{R}$$

since $\operatorname{Pic}(X_\Sigma)$ is torsion-free by Proposition 4.2.5. Thus $\operatorname{Pic}(X_\Sigma)$ is a lattice in the vector space $\operatorname{Pic}(X_\Sigma)_\mathbb{R}$. Note also that

$$\operatorname{Pic}(X_\Sigma)_\mathbb{R} = N^1(X_\Sigma) \tag{6.3.2}$$

since numerical and linear equivalence coincide by Proposition 6.3.15. In this case, we will write $\operatorname{Pic}(X_\Sigma)_\mathbb{R}$ instead of $N^1(X_\Sigma)$.

Theorem 6.3.20. *Let X_Σ be a toric variety whose fan Σ has convex support of full dimension. Then:*

(a) $\operatorname{Nef}(X_\Sigma)$ *is a rational polyhedral cone in* $\operatorname{Pic}(X_\Sigma)_\mathbb{R}$.

(b) $\overline{\mathrm{NE}}(X_\Sigma) = \mathrm{NE}(X_\Sigma)$ *is a rational polyhedral cone in* $N_1(X_\Sigma)$. *Furthermore,*

$$\overline{\mathrm{NE}}(X_\Sigma) = \sum_{\tau \text{ a wall of } \Sigma} \mathbb{R}_{\geq 0}[V(\tau)],$$

where $[V(\tau)] \in N_1(X_\Sigma)$ *is the class of* $V(\tau)$.

Proof. Part (a) is an immediate consequence of part (b). For part (b), let $\Gamma = \sum_{\tau \text{ a wall of } \Sigma} \mathbb{R}_{\geq 0}[V(\tau)]$ and note that Γ is a rational polyhedral cone contained in $\mathrm{NE}(X_\Sigma)$. Furthermore, Theorem 6.3.12 easily implies $\operatorname{Nef}(X_\Sigma) = \Gamma^\vee$. Then

$$\overline{\mathrm{NE}}(X_\Sigma) = \operatorname{Nef}(X_\Sigma)^\vee = \Gamma^{\vee\vee} = \Gamma \subseteq \mathrm{NE}(X_\Sigma) \subseteq \overline{\mathrm{NE}}(X_\Sigma),$$

where the third equality follows since Γ is polyhedral. $\square$

The formula from part (b) of Theorem 6.3.20, namely

$$\overline{\mathrm{NE}}(X_\Sigma) = \sum_{\tau \text{ a wall of } \Sigma} \mathbb{R}_{\geq 0}[V(\tau)],$$

is called the *Toric cone theorem*. Although the Mori cone equals $\mathrm{NE}(X_\Sigma)$ in this case, we will continue to write $\overline{\mathrm{NE}}(X_\Sigma)$ for consistency with the literature. For the same reason we write $\mathbb{R}_{\geq 0}[V(\tau)]$ instead of $\operatorname{Cone}([V(\tau)])$.

The Mori cone of an arbitrary normal variety can have a complicated structure. The *cone theorem* shows that some parts of the Mori cone are locally polyhedral. See [**179**, Ch. 3] and [**194**, Ch. 7] for a discussion of this important result.

Since every irreducible complete curve $C \subseteq X_\Sigma$ gives a class in $\overline{\mathrm{NE}}(X_\Sigma)$, we get the following corollary of the toric cone theorem.

Corollary 6.3.21. *Assume the fan Σ has convex support of full dimension. Then any irreducible complete curve on X_Σ is numerically equivalent to a non-negative linear combination of torus-invariant complete curves.* $\square$

When X_Σ is projective we can say more about the nef and Mori cones.

Theorem 6.3.22. *Let X_Σ be a projective toric variety. Then:*

(a) $\mathrm{Nef}(X_\Sigma)$ *and* $\overline{\mathrm{NE}}(X_\Sigma)$ *are dual strongly convex rational polyhedral cones of full dimension.*

(b) *A Cartier divisor D is ample if and only if its class in* $\mathrm{Pic}(X_\Sigma)_{\mathbb{R}}$ *lies in the interior of* $\mathrm{Nef}(X_\Sigma)$.

Proof. By hypothesis, X_Σ has an ample divisor D. Then $D \cdot C > 0$ for every irreducible curve in X_Σ. This easily implies that the class of D lies in the interior of $\mathrm{Nef}(X_\Sigma)$. Thus $\mathrm{Nef}(X_\Sigma)$ has full dimension and hence its dual $\overline{\mathrm{NE}}(X_\Sigma)$ is strongly convex. When combined with Lemma 6.3.19, part (a) follows easily.

The strict inequality $D \cdot C > 0$ also shows that every irreducible curve gives a nonzero class in $N_1(X_\Sigma)$. Now suppose that the class of D is in the interior of the nef cone. Then $[D]$ defines a supporting hyperplane of the origin of the dual cone $\overline{\mathrm{NE}}(X_\Sigma)$. Since $0 \neq [C] \in \overline{\mathrm{NE}}(X_\Sigma)$ for every irreducible curve $C \subseteq X_\Sigma$, we have $D \cdot C > 0$ for all such C. Hence D is ample by Theorem 6.3.13. □

It follows that $\overline{\mathrm{NE}}(X_\Sigma)$ is strongly convex in the projective case. The rays of $\overline{\mathrm{NE}}(X_\Sigma)$ are called *extremal rays*, which by the toric cone theorem are of the form $\mathbb{R}_{\geq 0}[V(\tau)]$. The corresponding walls τ are called *extremal walls*.

Here is an example of the cones $\mathrm{Nef}(X_\Sigma)$ and $\overline{\mathrm{NE}}(X_\Sigma)$.

Example 6.3.23. For the Hirzebruch surface $\mathscr{H}_r$, we showed in Example 6.1.16 that $\mathrm{Pic}(\mathscr{H}_r) = \{a[D_3] + b[D_4] \mid a, b \in \mathbb{Z}\}$. Figure 15 shows $\mathrm{Nef}(\mathscr{H}_r)$ and $\overline{\mathrm{NE}}(\mathscr{H}_r)$.

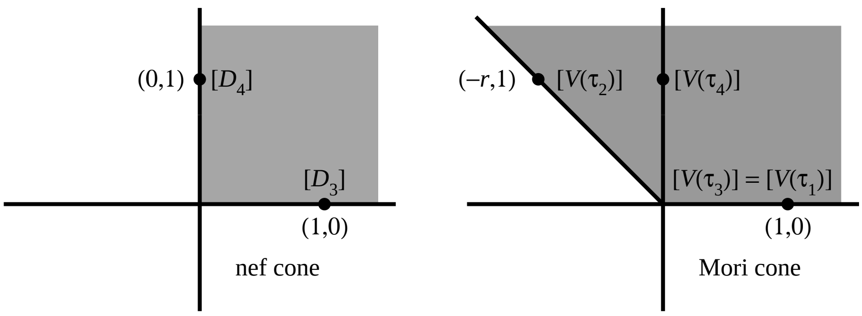


Figure 15. The nef and Mori cones of $\mathscr{H}_r$

Here, $\tau_i = \mathrm{Cone}(u_i)$, so that $D_i = V(\tau_i)$. Using both notations helps distinguish between $\mathrm{Nef}(\mathscr{H}_r)$ (built from divisors) and $\overline{\mathrm{NE}}(\mathscr{H}_r)$ (built from curves).

The description of the nef cone follows from the characterization of ample divisors on $\mathscr{H}_r$ given in Example 6.1.16. The Mori cone is generated by the classes of the $V(\tau_i)$ by the toric cone theorem. Using the the basis given by $D_3 = V(\tau_3)$,

$D_4 = V(\tau_4)$ and the linear equivalences

$$D_1 \sim D_3, \quad D_2 \sim -rD_3 + D_4$$

from Example 6.1.16, we get the picture of $\overline{\mathrm{NE}}(\mathscr{H}_r)$ shown in Figure 15. It follows that $[V(\tau_2)]$ and $[V(\tau_3)] = [V(\tau_1)]$ generate extremal rays, while $[V(\tau_4)]$ does not. Thus τ_1, τ_2, τ_3 are extremal walls.

The explicit duality between the cones $\mathrm{Nef}(X_\Sigma)$ and $\overline{\mathrm{NE}}(X_\Sigma)$ in Figure 15 will be described in the next section.

Theorem 6.3.22 tells us that ample divisors correspond to lattice points in the interior of $\mathrm{Nef}(\mathscr{H}_r)$. Thus lattice points on the boundary correspond to divisors that are basepoint free but not ample. We can see this vividly by looking at the polytopes P_D associated to divisors D whose classes lie in $\mathrm{Nef}(\mathscr{H}_r)$.

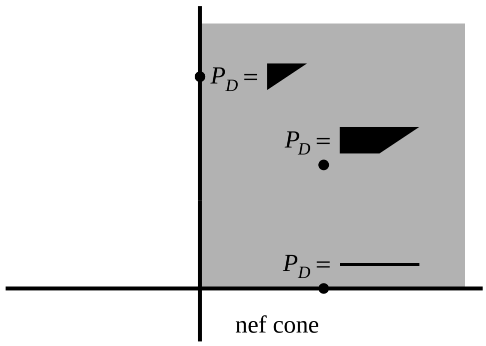


Figure 16. Polytopes P_D associated to divisors D in nef cone of $\mathscr{H}_r$

Figure 16 shows that when D is in the interior of the nef cone, P_D is a polygon whose normal fan is the fan of $\mathscr{H}_r$. On the boundary of the nef cone, however, things are different: P_D is a triangle on the vertical ray and and a line segment on the horizontal ray. This follows from Figures 10 and 11 in Example 6.2.6. $\Diamond$

The Simplicial Case. When X_Σ is complete and simplicial, a result to be proved in §6.4 gives the following criterion for X_Σ to be projective.

Proposition 6.3.24. *A complete simplicial toric variety X_Σ is projective if and only if its nef cone $\mathrm{Nef}(X_\Sigma) \subseteq \mathrm{Pic}(X_\Sigma)_\mathbb{R}$ has full dimension in $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$.*

Proof. One direction is an immediate consequence of Theorem 6.3.22. For the converse, suppose that $\mathrm{Nef}(X_\Sigma)$ has full dimension. Then we can find a Cartier divisor D whose class lies in the interior of $\mathrm{Nef}(X_\Sigma)$. Since $\mathrm{Nef}(X_\Sigma) = \overline{\mathrm{NE}}(X_\Sigma)^\vee$, it follows that $D \cdot C > 0$ for every curve C whose class in $\overline{\mathrm{NE}}(X_\Sigma)$ is nonzero. Hence, if we can show that the torus-invariant curves $V(\tau)$, $\tau \in \Sigma$ a wall, give nonzero classes in $\overline{\mathrm{NE}}(X_\Sigma)$, then Theorem 6.3.13 will imply that D is ample, proving that X_Σ is projective.

A wall $\tau \in \Sigma$ is a facet of some maximal cone $\sigma \in \Sigma$, and since Σ is simplicial, there is $\rho \in \Sigma(1)$ such that $\sigma(1) = \tau(1) \cup \{\rho\}$. Then Lemma 6.4.2 implies that $D_\rho \cdot V(\tau) > 0$. Hence the class of $V(\tau)$ in $\overline{\mathrm{NE}}(X_\Sigma)$ is nonzero. □

Here is a nice application of this result.

Proposition 6.3.25. *A complete toric surface X_Σ is projective.*

Proof. Picking a basis of N, we may assume $N = \mathbb{Z}^2$ and $N_\mathbb{R} = \mathbb{R}^2$. Let $\Sigma(1) = \{\rho_1, \ldots, \rho_r\}$ and pick $\nu_i \in \rho_i$ with $||\nu_i|| = 1$. Then define $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that φ is linear on the cones of Σ and satisfies $\varphi(\nu_i) = -1$ for all i. The tent analogy (see Figure 5 in Example 6.1.3) shows that φ is strictly convex with respect Σ.

Let $D = \sum_{i=1}^r -\varphi(u_i) D_i = \sum_{i=1}^r ||u_i|| D_i$, where $u_i \in \mathbb{Z}^2$ is the minimal generator of ρ_i. Note that $[D] \in \mathrm{Pic}(X_\Sigma)_\mathbb{R}$ since Σ is simplicial. Strict convexity and the proof of Theorem 6.3.12 imply that $D \cdot C > 0$ for every torus-invariant curve $C \subseteq X_\Sigma$, so that $[D] \in \mathrm{Nef}(X_\Sigma)$. The strict inequalities show that $[D]$ is an interior point, so that X_Σ is projective by Proposition 6.3.24. □

When X_Σ is not simplicial, the criterion given in Proposition 6.3.24 can fail. Here is an example due to Fujino [**100**].

Example 6.3.26. Consider the complete fan in $\mathbb{R}^3$ with six minimal generators

$$\begin{aligned} &u_1 = (1,0,1), \quad u_2 = (0,1,1), \quad u_3 = (-1,-1,1) \\ &u_4 = (1,0,-1),\ u_5 = (0,1,-1),\ u_6 = (-1,-1,-1) \end{aligned}$$

and six maximal cones

$$\begin{aligned} &\mathrm{Cone}(u_1,u_2,u_3),\ \mathrm{Cone}(u_1,u_2,u_4),\ \mathrm{Cone}(u_2,u_4,u_5) \\ &\mathrm{Cone}(u_1,u_3,u_4,u_6),\ \mathrm{Cone}(u_2,u_3,u_5,u_6),\ \mathrm{Cone}(u_4,u_5,u_6). \end{aligned}$$

You will draw a picture of this fan in Exercise 6.3.5 and show that the resulting complete toric variety satisfies

$$\mathrm{Pic}(X_\Sigma) \simeq \{a(D_1 + D_4) \mid a \in 3\mathbb{Z}\} \simeq \mathbb{Z}.$$

The cones $\sigma = \mathrm{Cone}(u_1,u_2,u_4)$ and $\sigma' = \mathrm{Cone}(u_2,u_4,u_5)$ meet along the wall

$$\tau = \sigma \cap \sigma' = \mathrm{Cone}(u_2,u_4).$$

However, any Cartier divisor $D = \sum_{i=1}^6 a_i D_i$ satisfies $m_\sigma = m_{\sigma'}$ (Exercise 6.3.5), so that the irreducible complete curve $C = V(\tau)$ satisfies

$$D \cdot C = 0$$

by Proposition 6.3.8. This holds for all Cartier divisors on X_Σ, so $C \equiv 0$. Then X_Σ has no ample divisors by the toric Kleiman criterion, so that X_Σ is nonprojective.

By Exercise 6.3.5, the nef cone of X_Σ is the half-line

$$\mathrm{Nef}(X_\Sigma) = \{a[D_1 + D_4] \mid a \geq 0\}.$$

This has full dimension in $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$, yet X_Σ is not projective. Note also that the Cartier divisor $D = 3(D_1+D_4)$ gives a class in the interior of the nef cone, yet D is not ample. Hence part (b) of Theorem 6.3.22 also fails for X_Σ. The failure is due to the existence of irreducible curves in X_Σ that are numerically equivalent to zero. This shows that numerical equivalence of curves can be badly behaved in complete toric varieties that are neither projective nor simplicial. $\Diamond$

Exercises for §6.3.

6.3.1. Let X be a normal variety. Prove that (6.3.1) gives a well-defined pairing between $\mathbb{Q}$-Cartier divisors and irreducible complete curves. Also show that this pairing satisfies Propostion 6.3.7.

6.3.2. Derive the formulas for m_σ and $m_{\sigma'}$ given in Example 6.3.9.

6.3.3. Prove Theorem 6.3.13.

6.3.4. Prove that on a complete normal variety, an ample divisor D satisfies $D\cdot C > 0$ for all irreducible curves $C \subseteq X$.

6.3.5. Consider the fan Σ from Example 6.3.26.

(a) Draw a picture of this fan in $\mathbb{R}^3$.

(b) Prove that $\mathrm{Pic}(X_\Sigma) \simeq \{a(D_1+D_4) \mid a \in 3\mathbb{Z}\}$.

(c) Prove that $3(D_1+D_4)$ is nef.

§6.4. The Simplicial Case

Here we assume that Σ is a simplicial fan in $N_\mathbb{R} \simeq \mathbb{R}^n$. Then Proposition 4.2.7 implies that every Weil divisor is $\mathbb{Q}$-Cartier. Since we will be working in $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$, it follows that we can drop the adjective "Cartier" when discussing divisors.

Relations Among Minimal Generators. We begin our discussion of the simplicial case with another way to think of elements of $N_1(X_\Sigma)$. Recall from Theorem 4.1.3 that we have an exact sequence

$$M \xrightarrow{\alpha} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\beta} \mathrm{Cl}(X_\Sigma) \longrightarrow 0 \tag{6.4.1}$$

where $\alpha(m) = (\langle m, u_\rho\rangle)_{\rho\in\Sigma(1)}$ and β sends the standard basis element $e_\rho \in \mathbb{Z}^{\Sigma(1)}$ to $[D_\rho] \in \mathrm{Cl}(X_\Sigma)$.

Proposition 6.4.1. *Let Σ be a simplicial fan in $N_\mathbb{R}$ with convex support of full dimension. Then there are dual exact sequences*

$$0 \longrightarrow M_\mathbb{R} \xrightarrow{\alpha} \mathbb{R}^{\Sigma(1)} \xrightarrow{\beta} \mathrm{Pic}(X_\Sigma)_\mathbb{R} \longrightarrow 0$$

and

$$0 \longrightarrow N_1(X_\Sigma) \xrightarrow{\beta^*} \mathbb{R}^{\Sigma(1)} \xrightarrow{\alpha^*} N_\mathbb{R} \longrightarrow 0,$$

where

$$\begin{aligned} \alpha^*(e_\rho) &= u_\rho, && e_\rho \text{ a standard basis vector of } \mathbb{R}^{\Sigma(1)} \\ \beta^*([C]) &= (D_\rho \cdot C)_{\rho\in\Sigma(1)}, && C \subseteq X_\Sigma \text{ an irreducible complete curve}. \end{aligned}$$

Thus $N_1(X_\Sigma)$ can be interpreted as the space of linear relations among the minimal generators of Σ. Furthermore, given $D = \sum_\rho a_\rho D_\rho$ and a relation $\sum_\rho b_\rho u_\rho = 0$, the intersection pairing of $[D] \in \mathrm{Pic}(X_\Sigma)_\mathbb{R}$ and $R = (b_\rho)_{\rho\in\Sigma(1)} \in N_1(X_\Sigma)$ is

$$[D]\cdot R = \sum_\rho a_\rho b_\rho.$$

Proof. Since Σ is simplicial, all Weil divisors are $\mathbb{Q}$-Cartier. Hence

$$\mathrm{Pic}(X_\Sigma)_\mathbb{R} = \mathrm{Pic}(X_\Sigma)\otimes_\mathbb{Z}\mathbb{R} = \mathrm{Cl}(X_\Sigma)\otimes_\mathbb{Z}\mathbb{R}.$$

Tensoring with $\mathbb{R}$ preserves exactness, so exactness of the first sequence follows from (6.4.1). Note also that $\mathrm{Pic}(X_\Sigma)_\mathbb{R} = N^1(X_\Sigma)$ by (6.3.2).

The dual of an exact sequence of finite-dimensional vector spaces is still exact. Then the perfect pairings

$$\begin{gathered} M_\mathbb{R}\times N_\mathbb{R} \to \mathbb{R} : (m,u) \mapsto \langle m,u\rangle \\ \mathrm{Pic}(X_\Sigma)_\mathbb{R}\times N_1(X_\Sigma) \to \mathbb{R} : ([D],[C]) \mapsto D\cdot C \end{gathered}$$

easily imply that for $m \in M_\mathbb{R}$ and $[C] \in N_1(X_\Sigma)$, we have

$$\alpha(m) = (\langle m,u_\rho\rangle)_{\rho\in\Sigma(1)} \implies \alpha^*(e_\rho) = u_\rho$$

and

$$\beta(e_\rho) = [D_\rho] \implies \beta^*([C]) = (D_\rho\cdot C)_{\rho\in\Sigma(1)}.$$

Finally, the duality between the two exact sequences reduces to dot product on the middle terms $\mathbb{R}^{\Sigma(1)}$. This proves the final assertion of the proposition. □

The map $\beta^* : N_1(X_\Sigma) \to \mathbb{R}^{\Sigma(1)}$ in Proposition 6.4.1 implies that an irreducible complete curve $C \subseteq X_\Sigma$ gives the relation

$$\textstyle\sum_\rho (D_\rho\cdot C)\, u_\rho = 0 \text{ in } N_\mathbb{R}.$$

This can be proved directly as follows. First observe that $m \in M$ gives

$$\textstyle\sum_\rho \langle m,u_\rho\rangle D_\rho = \mathrm{div}(\chi^m) \sim 0.$$

Taking the intersection product with C, we see that

$$\textstyle\sum_\rho \langle m,u_\rho\rangle (D_\rho\cdot C) = 0$$

holds for all $m \in M_\mathbb{R}$. Writing this as $\big\langle m, \sum_\rho (D_\rho\cdot C)\, u_\rho\big\rangle = 0$, we obtain

$$\textstyle\sum_\rho (D_\rho\cdot C)\, u_\rho = 0 \text{ in } N_\mathbb{R}. \tag{6.4.2}$$

This argument shows that (6.4.2) holds for *any* simplicial toric variety.

Intersection Products. Our next task is to compute $D_\rho \cdot C$ when C is a torus-invariant complete curve in X_Σ. This means $C = V(\tau)$, where $\tau \in \Sigma(n-1)$ is a wall, i.e., the intersection of two cones in $\Sigma(n)$. Here, we only assume that Σ is a simplicial fan in $N_\mathbb{R} \simeq \mathbb{R}^n$, with no hypotheses on its support.

We begin with an easy case. Fix a wall

$$\tau = \sigma \cap \sigma'.$$

Since Σ is simplicial, we can label the minimal generators of σ so that

$$\begin{aligned}\sigma &= \mathrm{Cone}(u_{\rho_1}, u_{\rho_2}, \dots, u_{\rho_n})\\ \tau &= \mathrm{Cone}(u_{\rho_2}, \dots, u_{\rho_n}).\end{aligned}$$

Thus τ is the facet of σ "opposite" to ρ_1. We will compute the intersection product $D_{\rho_1} \cdot V(\tau)$ in terms of the *multiplicity* (also called the *index*) of a simplicial cone. This is defined as follows. If γ is a simplicial cone with minimal generators $u_1, \dots, u_k$, then $\mathrm{mult}(\gamma)$ is the index of the sublattice

$$\mathbb{Z}u_1 + \cdots + \mathbb{Z}u_k \subseteq N_\gamma = \mathrm{Span}(\gamma) \cap N = (\mathbb{R}u_1 + \cdots + \mathbb{R}u_k) \cap N.$$

Lemma 6.4.2. *If τ, σ and ρ_1 are as above, then*

$$D_{\rho_1} \cdot V(\tau) = \frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma)}.$$

Proof. Since $\{u_{\rho_1}, \dots, u_{\rho_n}\}$ is a basis of $N_\mathbb{Q}$, we can find $m \in M_\mathbb{Q}$ such that

$$\langle m, u_{\rho_i} \rangle = \begin{cases} -1 & i = 1 \\ 0 & i = 2, \dots, n. \end{cases}$$

Pick a positive integer ℓ such that $\ell m \in M$. On $U_\sigma \cup U_{\sigma'}$, ℓD_{ρ_1} is the Cartier divisor determined by $m_\sigma = \ell m$ and $m_{\sigma'} = 0$. By (6.3.1) and Proposition 6.3.8,

$$D_{\rho_1} \cdot V(\tau) = \frac{1}{\ell}(\ell D_{\rho_1}) \cdot V(\tau) = \frac{1}{\ell}\langle \ell m, u \rangle = \langle m, u \rangle,$$

where $u \in \sigma'$ maps to a generator of $\overline{\sigma}' \cap N(\tau)$. Recall that $N(\tau) = N/N_\tau$.

When we combine u with a basis of N_τ, we get a basis of N. Thus there is a positive integer β such that $u_{\rho_1} = -\beta u + v$, $v \in N_\tau$. The minus sign appears because u and u_{ρ_1} lie on opposite sides of τ. By considering the sublattices

$$\mathbb{Z}u_{\rho_1} + \mathbb{Z}u_{\rho_2} + \cdots + \mathbb{Z}u_{\rho_n} \subseteq \mathbb{Z}u_{\rho_1} + N_\tau \subseteq \mathbb{Z}u + N_\tau = N,$$

one sees that $\beta = \mathrm{mult}(\sigma)/\mathrm{mult}(\tau)$. Thus

$$u = -\frac{1}{\beta}\big(u_{\rho_1} - v\big) = -\frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma)}\big(u_{\rho_1} - v\big).$$

Since $m \in \tau^\perp$, it follows that

$$D_{\rho_1} \cdot V(\tau) = \langle m, u \rangle = -\frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma)}\langle m, u_{\rho_1} \rangle = \frac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma)}. \qquad \square$$

Corollary 6.4.3. *Let Σ be a simplicial fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$ and $\tau \in \Sigma(n-1)$ be a wall. If $\rho \in \Sigma(1)$ and τ generate a smooth cone of $\Sigma(n)$, then*

$$D_\rho \cdot V(\tau) = 1 \qquad \square$$

Given a wall $\tau \in \Sigma(n-1)$, our next task is to compute $D_\rho \cdot V(\tau)$ for the other rays $\rho \in \Sigma(1)$. Let $\tau = \sigma \cap \sigma'$ and write

$$\begin{aligned} \sigma &= \mathrm{Cone}(u_{\rho_1}, \ldots, u_{\rho_n}) \\ \sigma' &= \mathrm{Cone}(u_{\rho_2}, \ldots, u_{\rho_{n+1}}) \\ \tau &= \mathrm{Cone}(u_{\rho_2}, \ldots, u_{\rho_n}). \end{aligned} \tag{6.4.3}$$

This situation is pictured in Figure 17.

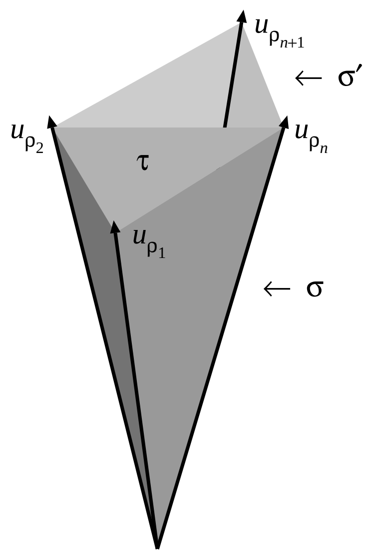


Figure 17. $\tau = \sigma \cap \sigma'$

The $n+1$ minimal generators $u_{\rho_1}, \ldots, u_{\rho_{n+1}}$ are linearly dependent. Hence they satisfy a linear relation, which we write as

$$\alpha u_{\rho_1} + \sum_{i=2}^{n} b_i u_{\rho_i} + \beta u_{\rho_{n+1}} = 0. \tag{6.4.4}$$

We may assume $\alpha, \beta > 0$ since u_{ρ_1} and $u_{\rho_{n+1}}$ lie on opposite sides of the wall τ. Then (6.4.4) is unique up to multiplication by a positive constant since $u_{\rho_1}, \ldots, u_{\rho_n}$ are linearly independent. We call (6.4.4) a *wall relation*.

On the other hand, setting $C = V(\tau)$ in (6.4.2) gives the linear relation

$$\sum_\rho (D_\rho \cdot V(\tau))\, u_\rho = 0 \tag{6.4.5}$$

As we now prove, the two relations are the same up to a positive constant.

Proposition 6.4.4. *The relations* (6.4.4) *and* (6.4.5) *are equal after multiplication by a positive constant. Furthermore:*

(a) $D_\rho \cdot V(\tau) = 0$ *for all* $\rho \notin \{\rho_1, \dots, \rho_{n+1}\}$.

(b) $D_{\rho_1} \cdot V(\tau) = \dfrac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma)}$ *and* $D_{\rho_{n+1}} \cdot V(\tau) = \dfrac{\mathrm{mult}(\tau)}{\mathrm{mult}(\sigma')}$.

(c) $D_{\rho_i} \cdot V(\tau) = \dfrac{b_i\,\mathrm{mult}(\tau)}{\alpha\,\mathrm{mult}(\sigma)} = \dfrac{b_i\,\mathrm{mult}(\tau)}{\beta\,\mathrm{mult}(\sigma')}$ *for* $i = 2, \dots, n$.

Proof. Part (b) follows immediately from Lemma 6.4.2. Also observe that when $\rho \notin \{\rho_1, \dots, \rho_{n+1}\}$, ρ and τ never lie in the same cone of Σ, so $D_\rho \cap V(\tau) = \emptyset$ by the Orbit-Cone Correspondence. This easily implies $D_\rho \cdot V(\tau) = 0$ (Exercise 6.4.1). This proves part (a) and implies that (6.4.5) reduces to the equation

$$(D_{\rho_1} \cdot V(\tau))\, u_{\rho_1} \;+\; \sum_{i=2}^{n} (D_{\rho_i} \cdot V(\tau))\, u_{\rho_i} \;+\; (D_{\rho_{n+1}} \cdot V(\tau))\, u_{\rho_{n+1}} = 0.$$

The coefficients of u_{ρ_1} and $u_{\rho_{n+1}}$ are positive by part (b), so up to a positive constant, this must be the wall relation (6.4.4). The first assertion of the lemma follows.

Since the above relation equals (6.4.4) up to a nonzero constant, we obtain

$$b_i(D_{\rho_1} \cdot V(\tau)) = \alpha(D_{\rho_i} \cdot V(\tau)), \quad b_i(D_{\rho_{n+1}} \cdot V(\tau)) = \beta(D_{\rho_i} \cdot V(\tau)),$$

for $i = 2, \dots, n$. Then the formulas for $D_{\rho_i} \cdot V(\tau)$ in part (c) follow from part (b). $\square$

For a simplicial toric variety, Proposition 6.4.4 provides everything we need to compute $D \cdot V(\tau)$ when τ is a wall of Σ.

Example 6.4.5. Consider the fan Σ in $\mathbb{R}^2$ from Example 6.3.9. We have the wall

$$\tau = \mathrm{Cone}(u_0) = \sigma \cap \sigma' = \mathrm{Cone}(u_1, u_0) \cap \mathrm{Cone}(u_2, u_0),$$

where $u_1 = e_1$, $u_2 = e_2$ and $u_0 = 2e_1 + 3e_2$. Computing multiplicities gives

$$\mathrm{mult}(\tau) = 1,\ \mathrm{mult}(\sigma) = 3,\ \mathrm{mult}(\sigma') = 2.$$

Then part (b) of Proposition 6.4.4 implies

$$D_1 \cdot V(\tau) = \frac{1}{3},\ \ D_2 \cdot V(\tau) = \frac{1}{2},$$

and the relation

$$2 \cdot u_1 + (-1) \cdot u_0 + 3 \cdot u_2 = 0$$

implies

$$D_0 \cdot V(\tau) = \frac{-1 \cdot 1}{2 \cdot 3} = \frac{-1 \cdot 1}{3 \cdot 2} = -\frac{1}{6}$$

by part (c) of the proposition. Hence we recover the intersection products computed in Example 6.3.9. $\Diamond$

When X_Σ is smooth, all multiplicities are 1. Hence the wall relation (6.4.4) can be written uniquely as

$$u_{\rho_1} + \sum_{i=2}^{n} b_i u_{\rho_i} + u_{\rho_{n+1}} = 0, \quad b_i \in \mathbb{Z}, \tag{6.4.6}$$

and then the intersection formulas of Proposition 6.4.4 reduce to

$$D_{\rho_1} \cdot V(\tau) = D_{\rho_{n+1}} \cdot V(\tau) = 1, \quad D_{\rho_i} \cdot V(\tau) = b_i,\ i = 2, \dots, n. \tag{6.4.7}$$

Example 6.4.6. For the Hirzebruch surface $\mathscr{H}_r$, the four curves coming from walls are also divisors. Recall that the minimal generators are

$$u_1 = -e_1 + re_2,\ u_2 = e_2,\ u_3 = e_1, u_4 = -e_2,$$

arranged clockwise around the origin (see Figure 3 from Example 6.1.2 for the case $r = 2$). Hence the wall generated by u_1 gives the relation

$$u_2 - 0 \cdot u_3 + u_4 = 0,$$

which implies

$$D_1 \cdot D_1 = 0$$

by (6.4.7). On the other hand, the wall generated by u_2 gives the relation

$$u_1 - r \cdot u_2 + u_3 = 0.$$

Then (6.4.7) implies

$$D_2 \cdot D_2 = -r.$$

Similarly, one can check that

$$D_3 \cdot D_3 = 0,\ D_4 \cdot D_4 = r,$$

and by Corollary 6.4.3 or (6.4.7) we also have

$$D_1 \cdot D_2 = D_2 \cdot D_3 = D_3 \cdot D_4 = D_4 \cdot D_1 = 1.$$

These computations give an explicit description of the duality between the nef and Mori cones shown in Figure 15 of Example 6.3.23 (Exercise 6.4.2). ◊

In general, a $\mathbb{Q}$-Cartier divisor D on a complete surface has *self-intersection* $D \cdot D = D^2$. Self-intersections will play a prominent role in Chapter 10 when we study toric surfaces.

Example 6.4.7. Let X_Σ be a complete toric surface. Write $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$, where the ρ_i are arranged clockwise around the origin in $N_\mathbb{R} \simeq \mathbb{R}^2$. Each ρ_i gives a minimal generator u_i and a toric divisor D_i. Note also that $\mathrm{Pic}(X_\Sigma)_\mathbb{R} \simeq \mathbb{R}^{r-2}$.

Proposition 6.3.25 tells us that X_Σ is projective, so that its Mori cone $\overline{\mathrm{NE}}(X_\Sigma)$ is strongly convex of dimension $r-2$. Hence a minimal generating set has at least $r-2$ elements. Since the r classes $[D_i] = [V(\rho_i)]$ generate by the toric cone theorem, we almost know the minimal generators.

Now suppose that X_Σ is smooth. Then the wall relation for $D_i = V(\rho_i)$ is $u_{i-1} + b_i u_i + u_{i+1} = 0$ by (6.4.6), where $b_i = D_i^2$ by (6.4.7). Given a divisor $D = \sum_{i=1}^r a_i D_i$, Proposition 6.4.1 implies that

$$D \cdot D_i = a_{i-1} + b_i a_i + a_{i+1},$$

so that D is nef (resp. ample) if and only if

$$a_{i-1} + b_i a_i + a_{i+1} \geq 0 \text{ (resp. } > 0)$$

for $i = 1, \dots, r$. This makes it easy to study nef and ample divisors on X_Σ. ◊

Primitive Collections. In the projective case, there is a beautiful criterion for a Cartier divisor to be nef or ample in terms of the *primitive collections* introduced in Definition 5.1.5. Recall that

$$P = \{\rho_1, \dots, \rho_k\} \subseteq \Sigma(1)$$

is a primitive collection if P is not contained in $\sigma(1)$ for all $\sigma \in \Sigma$ but any proper subset is. Since Σ is simplicial, primitive means that P does not generate a cone of Σ but every proper subset does. This is the definition given by Batyrev in [**14**].

Example 6.4.8. Consider the complete fan Σ in $\mathbb{R}^3$ shown in Figure 18. One can check that

$$\{\rho_1, \rho_3\},\ \{\rho_0, \rho_2, \rho_4\}$$

are the only primitive collections of Σ. ◊

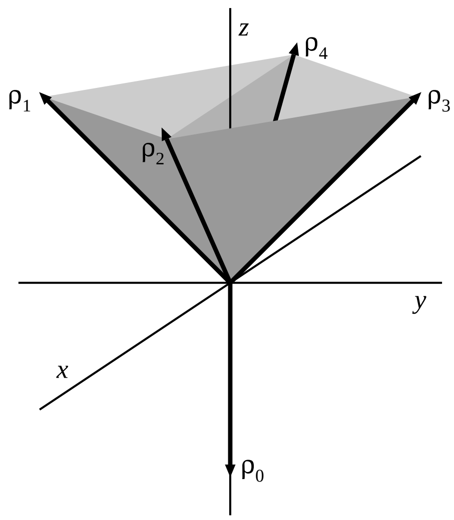


Figure 18. A complete fan in $\mathbb{R}^3$ with two primitive collections

Here is the promised characterization, due to Batyrev [**14**] in the smooth case.

Theorem 6.4.9. *Let X_Σ be a projective simplicial toric variety. Then:*

(a) *A Cartier divisor D is nef if and only if its support function φ_D satisfies*

$$\varphi_D(u_{\rho_1}+\cdots+u_{\rho_k}) \geq \varphi_D(u_{\rho_1})+\cdots+\varphi_D(u_{\rho_k})$$

for all primitive collections $P=\{\rho_1,\ldots,\rho_k\}$ of Σ.

(b) *A Cartier divisor D is ample if and only if its support function φ_D satisfies*

$$\varphi_D(u_{\rho_1}+\cdots+u_{\rho_k}) > \varphi_D(u_{\rho_1})+\cdots+\varphi_D(u_{\rho_k})$$

for all primitive collections $P=\{\rho_1,\ldots,\rho_k\}$ of Σ.

Before we discuss the proof of Theorem 6.4.9, we need to study the relations that come from primitive collections.

Definition 6.4.10. Let $P=\{\rho_1,\ldots,\rho_k\}\subseteq\Sigma(1)$ be a primitive collection for the complete simplicial fan Σ. Hence $\sum_{i=1}^k u_{\rho_i}$ lies in the relative interior of a cone $\gamma\in\Sigma$. Thus there is a unique expression

$$u_{\rho_1}+\cdots+u_{\rho_k}=\textstyle\sum_{\rho\in\gamma(1)} c_\rho u_\rho, \quad c_\rho\in\mathbb{Q}_{>0}.$$

Then $u_{\rho_1}+\cdots+u_{\rho_k}-\sum_{\rho\in\gamma(1)} c_\rho u_\rho=0$ is the ***primitive relation*** of P.

The coefficient vector of this relation is $r(P)=(b_\rho)_{\rho\in\Sigma(1)}\in\mathbb{R}^{\Sigma(1)}$, where

$$b_\rho=\begin{cases} 1 & \rho\in P,\ \rho\notin\gamma(1)\\ 1-c_\rho & \rho\in P\cap\gamma(1)\\ -c_\rho & \rho\in\gamma(1),\ \rho\notin P\\ 0 & \text{otherwise}.\end{cases} \tag{6.4.8}$$

Then $\sum_\rho b_\rho u_\rho=0$, so that $r(P)$ gives an element of $N_1(X_\Sigma)$ by Proposition 6.4.1. In Exercise 6.4.3, you will prove that $c_\rho<1$ when $\rho\in P\cap\gamma(1)$. This means that P is determined by the positive entries in the coefficient vector $r(P)$.

The Mori cone for X_Σ has a nice description in terms of primitive relations.

Theorem 6.4.11. *If X_Σ is a projective simplicial toric variety, then*

$$\overline{\mathrm{NE}}(X_\Sigma)=\sum_P \mathbb{R}_{\geq 0}r(P),$$

where the sum is over all primitive collections P of Σ.

Proof. Given a Cartier divisor $D=\sum_\rho a_\rho D_\rho$, Proposition 6.4.1 shows that

$$[D]\cdot r(P)=\sum_\rho a_\rho b_\rho=\sum_{i=1}^k a_{\rho_i}-\sum_{\rho\in\gamma(1)} a_\rho c_\rho.$$

Since the support function of D satisfies $\varphi_D(u_\rho) = -a_\rho$ and is linear on γ, we can rewrite this as

$$[D]\cdot r(P) = -\varphi_D(u_{\rho_1}) - \cdots - \varphi_D(u_{\rho_k}) + \varphi_D(u_{\rho_1} + \cdots + u_{\rho_k}). \tag{6.4.9}$$

If D is nef, then it is basepoint free (Theorem 6.3.12), so φ_D is convex. It follows that $[D]\cdot r(P) \geq 0$, which proves $r(P) \in \overline{\mathrm{NE}}(X_\Sigma)$. Note also that $r(P)$ is nonzero.

To finish the proof, we need to show that $\overline{\mathrm{NE}}(X_\Sigma)$ is generated by the $r(P)$. In the discussion following the proof of Theorem 6.3.22, we noted that $\overline{\mathrm{NE}}(X_\Sigma)$ is generated by its extremal rays, each of which is of the form $\mathbb{R}_{\geq 0}[V(\tau)]$ for an extremal wall τ. It suffices to show that $[V(\tau)]$ is a positive multiple of $r(P)$ for some primitive collection P.

We first make a useful observation about nef divisors. Given a cone $\sigma \in \Sigma$, we claim that any nef divisor is linearly equivalent to a divisor of the form

$$D = \textstyle\sum_\rho a_\rho D_\rho, \quad a_\rho = 0,\ \rho \in \sigma(1) \text{ and } a_\rho \geq 0,\ \rho \notin \sigma(1). \tag{6.4.10}$$

To prove this, first recall that any nef divisor is linearly equivalent to a torus-invariant nef divisor $D = \sum_\rho a_\rho D_\rho$. Then we have $m_\sigma \in M$ with $\langle m_\sigma, u_\rho\rangle = -a_\rho$ for $\rho \in \sigma(1)$. Since D is nef, it is also basepoint free, so that

$$\langle m_\sigma, u_\rho\rangle \geq \varphi_D(u_\rho) = -a_\rho, \quad \rho \in \Sigma(1),$$

by Theorem 6.1.7. Replacing D with $D + \mathrm{div}(\chi^{m_\sigma})$, we obtain (6.4.10).

Now assume we have an extremal wall τ and let $C = V(\tau)$. Consider the set

$$P = \{\rho \mid D_\rho \cdot C > 0\}.$$

We will prove that P is a primitive collection whose primitive relation is the class of C, up to a positive constant. Our argument is taken from [**71**], which is based on ideas of Kresch [**181**].

We first prove by contradiction that $P \not\subseteq \sigma(1)$ for all $\sigma \in \Sigma$. Suppose $P \subseteq \sigma(1)$ and take an ample divisor D (remember that X_Σ is projective). Then in particular D is nef, so we may assume that D is of the form (6.4.10). Since $a_\rho = 0$ for $\rho \in \Sigma(1)$, we have

$$D\cdot C = \textstyle\sum_{\rho\notin\sigma(1)} a_\rho D_\rho \cdot C.$$

However, $a_\rho \geq 0$ by (6.4.10), and $P \subseteq \sigma(1)$ implies $D_\rho \cdot C \leq 0$ for $\rho \notin \sigma(1)$. It follows that $D\cdot C \leq 0$, which is impossible since D is ample. Thus no cone of Σ contains all rays in P.

It follows that some subset $Q \subseteq P$ is a primitive collection. This gives the primitive relation with coefficient vector $r(Q) \in N_1(X_\Sigma)$, and we also have the class $[C] \in N_1(X_\Sigma)$. Let

$$\beta = [C] - \lambda r(Q) \in N_1(X_\Sigma),$$

where $\lambda > 0$. We claim that if λ is sufficiently small, then

$$\{\rho \mid [D_\rho]\cdot\beta < 0\} \subseteq \{\rho \mid D_\rho \cdot C < 0\}. \tag{6.4.11}$$

To prove this, first observe that the definition of β implies

$$D_\rho \cdot C = \lambda[D_\rho] \cdot r(Q) + [D_\rho] \cdot \beta.$$

Suppose $[D_\rho] \cdot \beta < 0$ and $D_\rho \cdot C \geq 0$. This forces $[D_\rho] \cdot r(Q) > 0$. Proposition 6.4.1 implies that $[D_\rho] \cdot r(Q)$ is the coefficent of u_ρ in the primitive relation of Q, which by (6.4.8) is positive only when $\rho \in Q$. Then $Q \subseteq P$ implies $D_\rho \cdot C > 0$ by the definition of P. But we can clearly choose λ sufficiently small so that

$$D_\rho \cdot C > \lambda[D_\rho] \cdot r(Q) \quad \text{whenever } D_\rho \cdot C > 0.$$

This inequality and the above equation imply $[D_\rho] \cdot \beta > 0$, which is a contradiction.

We next claim that $\beta \in \overline{\mathrm{NE}}(X_\Sigma)$. By (6.4.11), we have

$$\{\rho \mid [D_\rho] \cdot \beta < 0\} \subseteq \{\rho \mid D_\rho \cdot C < 0\} \subseteq \tau(1),$$

where the second inclusion follows from $C = V(\tau)$ and Proposition 6.4.4. Now let D be nef, and by (6.4.10) with $\sigma = \tau$, we may assume that

$$D = \sum_\rho a_\rho D_\rho, \quad a_\rho = 0,\ \rho \in \tau(1) \ \text{ and } \ a_\rho \geq 0,\ \rho \notin \tau(1).$$

Then

$$[D] \cdot \beta = \sum_{\rho \notin \tau(1)} a_\rho [D_\rho] \cdot \beta \geq 0,$$

where the final inequality follows since $a_\rho \geq 0$ and $[D_\rho] \cdot \beta < 0$ can happen only when $\rho \in \tau(1)$. This proves that $\beta \in \overline{\mathrm{NE}}(X_\Sigma)$.

We showed earlier that $r(Q) \in \overline{\mathrm{NE}}(X_\Sigma)$. Thus the equation

$$[C] = \lambda r(Q) + \beta$$

expresses $[C]$ as a sum of elements of $\overline{\mathrm{NE}}(X_\Sigma)$. But $[C]$ is extremal, i.e., it lies in a 1-dimensional face of $\overline{\mathrm{NE}}(X_\Sigma)$. By Lemma 1.2.7, this forces $r(Q)$ and β to lie in the face. Since $r(Q)$ is nonzero, it generates the face, so that $[C]$ is a positive multiple of $r(Q)$.

The relation corresponding to C has coefficients $(D_\rho \cdot C)_{\rho \in \Sigma(1)}$, and P is the set of ρ's where $D_\rho \cdot C > 0$. But this relation is a positive multiple of $r(Q)$, whose positive entries correspond to Q. Thus $P = Q$ and the proof is complete. □

It is now straightforward to prove Theorem 6.4.9 using Theorem 6.4.11 and (6.4.9) (Exercise 6.4.4). We should also mention that these results hold more generally for any projective toric variety (see [**71**]).

Example 6.4.12. Let Σ be the fan shown in Figure 18 of Example 6.4.8. The minimal generators of $\rho_0, \ldots, \rho_4$ are

$$u_0 = (0,0,-1), u_1 = (0,-1,1), u_2 = (1,0,1), u_3 = (0,1,1), u_4 = (-1,0,1).$$

The computations you did for part (c) of Exercise 6.1.8 imply that X_Σ is projective. Since the only primitive collections are $\{\rho_1,\rho_3\}$ and $\{\rho_0,\rho_2,\rho_4\}$, Theorem 6.4.9 implies that a Cartier divisor D is nef if and only if

$$\varphi_D(u_1+u_3) \geq \varphi_D(u_1)+\varphi_D(u_3)$$
$$\varphi_D(u_0+u_2+u_4) \geq \varphi_D(u_0)+\varphi_D(u_2)+\varphi_D(u_4)$$

and ample if and only if these inequalities are strict. One can also check that

$$\mathrm{Pic}(X_\Sigma) \simeq \{a[D_1]+b[D_2] \mid a,b \in 2\mathbb{Z}\}$$

and aD_1+bD_2 is nef (resp. ample) if and only if $a \geq b \geq 0$ (resp. $a > b > 0$). Exercise 6.4.5 will relate this example to the proof of Theorem 6.4.11.

Besides Σ, the minimal generators $u_0,\ldots,u_4$ support two other complete fans in $\mathbb{R}^3$: first, the fan Σ' obtained by replacing $\mathrm{Cone}(u_2,u_3)$ with $\mathrm{Cone}(u_1,u_3)$ in Figure 18, and second, the fan Σ_0 obtained by removing this wall to create the cone $\mathrm{Cone}(u_1,u_2,u_3,u_4)$. Since $\Sigma(1)=\Sigma'(1)=\Sigma_0(1)$, the toric varieties $X_\Sigma,X_{\Sigma'},X_{\Sigma_0}$ have the same class group, though X_{Σ_0} has strictly smaller Picard group since it is not simplicial. Thus

$$\mathrm{Pic}(X_{\Sigma_0})_\mathbb{R} \subseteq \mathrm{Pic}(X_\Sigma)_\mathbb{R} = \mathrm{Pic}(X_{\Sigma'})_\mathbb{R} \simeq \mathbb{R}^2.$$

This allows us to draw all three nef cones in the same copy of $\mathbb{R}^2$. In Exercise 6.4.5 you show that we get the cones shown in Figure 19. The ideas behind this figure

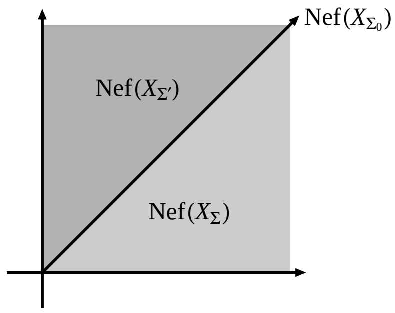


Figure 19. The nef cones of $X_\Sigma,X_{\Sigma'},X_{\Sigma_0}$

will be developed in Chapters 14 and 15 when we study geometric invariant theory and the minimal model program for toric varieties. $\Diamond$

Exercises for §6.4.

6.4.1. This exercise will describe a situation where $D\cdot C$ is guaranteed to be zero.

(a) Let X be normal and assume that C is a complete irreducible curve disjoint from the support of a Cartier divisor D. Prove that $D\cdot C=0$. Hint: Use $U = X\setminus \mathrm{Supp}(D)$.

(b) Let τ be a wall of a fan Σ and pick $\rho \in \Sigma(1)$ such that ρ and τ do not lie in the same cone of Σ. Use the Cone-Orbit Correspondence to prove that $D_\rho \cap V(\tau) = \emptyset$, and conclude that $D_\rho \cdot V(\tau) = 0$.

6.4.2. As in Example 6.4.2, the classes $[D_3], [D_4]$ give a basis of $\mathrm{Pic}(\mathscr{H}_r)_{\mathbb{R}}$. Since $\mathscr{H}_r$ is a smooth complete surface, the intersection product $D_i \cdot V(\tau_j)$ is written $D_i \cdot D_j$.

(a) Give an explicit formula for $(a[D_3] + b[D_4]) \cdot (a[D_3] + b[D_4])$ using the computations of Example 6.4.2.

(b) Use part (a) to show that the cones shown in Figure 15 in Example 6.3.23 are dual to each other.

6.4.3. In the primitive relation defined in Definition 6.4.10, prove $c_\rho < 1$ when $\rho \in P \cap \gamma(1)$. Hint: If $\rho_1 \in \gamma(1)$ and $c_{\rho_1} \geq 1$, then cancel u_{ρ_1} and show that $u_{\rho_2}, \ldots, u_{\rho_k} \in \gamma$.

6.4.4. Prove Theorem 6.4.9 using Theorem 6.4.11 and (6.4.9).

6.4.5. Consider the fan Σ from Examples 6.4.8 and 6.4.12. Every wall of Σ is of the form $\tau_{ij} = \mathrm{Cone}(u_i, u_j)$ for suitable $i < j$. Let $r(\tau_{ij}) \in \mathbb{R}^5$ denote the wall relation of τ_{ij}. Normalize by a positive constant so that the entries of $r(\tau_{ij})$ are integers with $\gcd = 1$.

(a) Show the nine walls give the three distinct wall relations $r(\tau_{02}), r(\tau_{03}), r(\tau_{24})$.

(b) Show that $r(\tau_{03}) + r(\tau_{24}) = r(\tau_{02})$ and conclude that τ_{03} and τ_{24} are extremal walls whose classes generate the Mori cone of X_Σ.

(c) For each extremal wall of part (b), determine the corresponding primitive collection. You should be able to read the primitive collection directly from the wall relation.

(d) Show that the nef cones of $X_\Sigma, X_{\Sigma'}, X_{\Sigma_0}$ give the cones shown in Figure 19.

6.4.6. Let X_Σ be the blowup of $\mathbb{P}^n$ at a fixed point of the torus action. Thus $\mathrm{Pic}(X_\Sigma) \simeq \mathbb{Z}^2$.

(a) Compute the nef and Mori cones of X_Σ and draw pictures similar to Figure 15 in Example 6.3.23.

(b) Determine the primitive relations and extremal walls of X_Σ.

6.4.7. Let $\mathcal{P}_r$ be the toric surface obtained by changing the ray u_1 in the fan of the Hirzebruch surface $\mathscr{H}_r$ from $(-1, r)$ to $(-r, 1)$. Assume $r > 1$.

(a) Prove that $\mathcal{P}_r$ is singular. How many singular points are there?

(b) Determine which divisors $a_1D_1 + a_2D_2 + a_3D_3 + a_4D_4$ are Cartier and compute $D_i \cdot D_j$ for all i, j.

(c) Determine the primitive relations and extremal walls of $\mathcal{P}_r$.

Appendix: Quasicoherent Sheaves on Toric Varieties

Now that we know more about sheaves (specifically, tensor products and exactness), we can complete the discussion of quasicoherent sheaves on toric varieties begun in §5.3. In this appendix, X_Σ will denote a toric variety with no torus factors. The total coordinate ring of X_Σ is $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$, which is graded by $\mathrm{Cl}(X_\Sigma)$.

Recall from §5.3 that for $\alpha \in \mathrm{Cl}(X_\Sigma)$, the shifted S-module $S(\alpha)$ gives the sheaf $\mathscr{O}_{X_\Sigma}(\alpha)$ satisfying $\mathscr{O}_{X_\Sigma}(\alpha) \simeq \mathscr{O}_{X_\Sigma}(D)$ for any Weil divisor with $\alpha = [D]$. In §6.0 we constructed a sheaf homomorphism $\mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{O}_X(E) \to \mathscr{O}_X(D+E)$. In a similar way, one can define

$$\mathscr{O}_{X_\Sigma}(\alpha) \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{O}_{X_\Sigma}(\beta) \longrightarrow \mathscr{O}_{X_\Sigma}(\alpha+\beta). \tag{6.A.1}$$

for $\alpha, \beta \in \mathrm{Cl}(X_\Sigma)$ such that if $\alpha = [D]$ and $\beta = [E]$, then the diagram

$$\begin{array}{ccc} \mathscr{O}_{X_\Sigma}(D)) \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{O}_{X_\Sigma}(E)) & \longrightarrow & \mathscr{O}_{X_\Sigma}(D+E)) \\ \downarrow & & \downarrow \\ \mathscr{O}_{X_\Sigma}(\alpha) \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{O}_{X_\Sigma}(\beta) & \longrightarrow & \mathscr{O}_{X_\Sigma}(\alpha+\beta) \end{array}$$

commutes, where the vertical maps are isomorphisms.

From Sheaves to Modules. The main construction of §5.3 takes a graded S-module M and produces a quasicoherent sheaf $\widetilde{\mathsf{M}}$ on X_Σ. We now go in the reverse direction and show that *every* quasicoherent sheaf on X_Σ arises in this way. We will use the following construction.

Definition 6.A.1. For a sheaf $\mathscr{F}$ of $\mathscr{O}_{X_\Sigma}$-modules on X_Σ and $\alpha \in \mathrm{Cl}(X_\Sigma)$, define

$$\mathscr{F}(\alpha) = \mathscr{F} \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{O}_{X_\Sigma}(\alpha)$$

and then set

$$\Gamma_*(\mathscr{F}) = \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} \Gamma(X_\Sigma, \mathscr{F}(\alpha)).$$

For example, $\Gamma_*(\mathscr{O}_{X_\Sigma}) = S$ since $\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha)) \simeq S_\alpha$ by Proposition 5.3.7. Using this and (6.A.1), we see that $\Gamma_*(\mathscr{F})$ is a graded S-module.

We want to show that $\mathscr{F}$ is isomorphic to the sheaf associated to $\Gamma_*(\mathscr{F})$ when $\mathscr{F}$ is quasicoherent. We will need the following lemma due to Mustaţă [**212**]. Recall that for $\sigma \in \Sigma$, we have the monomial $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho \in S$. Let $\alpha_\sigma = \deg(x^{\hat{\sigma}}) \in \mathrm{Cl}(X_\Sigma)$.

Lemma 6.A.2. *Let $\mathscr{F}$ be a quasicoherent sheaf on X_Σ. Then:*

(a) *If $v \in \Gamma(U_\sigma, \mathscr{F})$, then there are $\ell \geq 0$ and $u \in \Gamma(X_\Sigma, \mathscr{F}(\ell\alpha_\sigma))$ such that u restricts to $(x^{\hat{\sigma}})^\ell v \in \Gamma(U_\sigma, \mathscr{F}(\ell\alpha_\sigma))$.*

(b) *If $u \in \Gamma(X_\Sigma, \mathscr{F})$ restricts to 0 in $\Gamma(U_\sigma, \mathscr{F})$, then there is $\ell \geq 0$ such that $(x^{\hat{\sigma}})^\ell u = 0$ in $\Gamma(X_\Sigma, \mathscr{F}(\ell\alpha_\sigma))$.*

Proof. For part (a), fix $\sigma \in \Sigma$ and take $v \in \Gamma(U_\sigma, \mathscr{F})$. Given $\tau \in \Sigma$, let v_τ be the restriction of v to $U_\sigma \cap U_\tau$. By (3.1.3), we can find $m \in (-\sigma^\vee) \cap \tau^\vee \cap M$ such that $U_\sigma \cap U_\tau = (U_\tau)_{\chi^m} = \mathrm{Spec}(\mathbb{C}[\tau^\vee \cap M]_{\chi^m})$. In terms of the total coordinate ring S, we have $\mathbb{C}[\tau^\vee \cap M] \simeq (S_{x^{\hat{\tau}}})_0$ by (5.3.1). Hence the coordinate ring of $U_\sigma \cap U_\tau$ is the localization

$$\big((S_{x^{\hat{\tau}}})_0\big)_{x^{\langle m \rangle}},$$

where $x^{\langle m \rangle} = \prod_\rho x_\rho^{\langle m, u_\rho \rangle} \in (S_{x^{\hat{\tau}}})_0$ since $m \in \tau^\vee \cap M$. This enables us to write

$$U_\sigma \cap U_\tau = (U_\tau)_{x^{\langle m \rangle}}.$$

Since $\mathscr{F}$ is quasicoherent, $\mathscr{F}|_{U_\tau}$ is determined by its sections $G = \Gamma(U_\tau, \mathscr{F})$, and then $\Gamma(U_\sigma \cap U_\tau, \mathscr{F})$ is the localization $G_{x^{\langle m \rangle}}$.

It follows that $v_\tau \in \Gamma(U_\sigma, \mathscr{F})$ equals $\tilde{u}_\tau/(x^{\langle m \rangle})^k$, where $k \geq 0$ and $\tilde{u}_\tau \in \Gamma(U_\tau, \mathscr{F})$. Hence $\tilde{u}_\tau$ restricts to $(x^{\langle m \rangle})^k v \in \Gamma(U_\sigma, \mathscr{F})$. Since $m \in (-\sigma^\vee)$, we see that

$$x^a = (x^{\hat{\sigma}})^\ell (x^{\langle m \rangle})^{-k} \in S \tag{6.A.2}$$

for $\ell \gg 0$. This monomial has degree $\ell\alpha_\sigma$. Then $u_\tau = x^a\tilde{u}_\tau \in \Gamma(U_\tau, \mathscr{F}(\ell\alpha_\sigma))$ restricts to $(x^{\hat{\sigma}})^\ell v_\tau \in \Gamma(U_\sigma \cap U_\tau, \mathscr{F}(\ell\alpha_\sigma))$. By making ℓ sufficiently large, we can find one ℓ that works for all $\tau \in \Sigma$.

To study whether the u_τ patch to give a global section of $\mathscr{F}(\ell\alpha_\sigma)$, take $\tau_1, \tau_2 \in \Sigma$ and set $\gamma = \tau_1 \cap \tau_2$. Thus $U_\gamma = U_{\tau_1} \cap U_{\tau_2}$, and

$$w = u_{\tau_1}|_{U_\gamma} - u_{\tau_2}|_{U_\gamma} \in \Gamma(U_\gamma, \mathscr{F}(\ell\alpha_\sigma)) \tag{6.A.3}$$

restricts to $0 \in \Gamma(U_\sigma \cap U_\gamma, \mathscr{F}(\ell\alpha_\sigma))$. Arguing as above, this group of sections is the localization $\Gamma(U_\gamma, \mathscr{F}(\ell\alpha_\sigma))_{x^{\langle m\rangle}}$, where $m \in \gamma^\vee \cap (-\sigma^\vee) \cap M$ such that $U_\sigma \cap U_\gamma = (U_\gamma)_{\chi^m}$. Since w gives the zero element in this localization, there is $k \geq 0$ with $(x^{\langle m\rangle})^k w = 0$ in $\Gamma(U_\gamma, \mathscr{F}(\ell\alpha_\sigma))$. If we multiply by $x^b = (x^{\hat{\sigma}})^{\ell'}(x^{\langle m\rangle})^{-k}$ for $\ell' \gg 0$, we obtain $(x^{\hat{\sigma}})^{\ell'} w = 0$ in $\Gamma(U_\gamma, \mathscr{F}((\ell'+\ell)\alpha_\sigma))$. Another way to think of this is that if we made ℓ in (6.A.2) big enough to begin with, then in fact $w = 0$ in $\Gamma(U_\gamma, \mathscr{F}(\ell\alpha_\sigma))$ for all τ, τ'. Given the definition (6.A.3) of w, it follows that the u_τ patch to give a global section $u \in \Gamma(X_\Sigma, \mathscr{F}(\ell\alpha_\sigma))$ with the desired properties.

The proof of part (b) is similar and is left to the reader. $\square$

Proposition 6.A.3. *Let $\mathscr{F}$ be a quasicoherent sheaf on X_Σ. Then $\mathscr{F}$ is isomorphic to the sheaf associated to the graded S-module $\Gamma_*(\mathscr{F})$.*

Proof. Let $\mathsf{M} = \Gamma_*(\mathscr{F})$ and recall from §5.3 that for every $\sigma \in \Sigma$, the restriction of $\widetilde{\mathsf{M}}$ to U_σ is the sheaf associated to the $(S_{x^{\hat{\sigma}}})_0$-module $(\mathsf{M}_{x^{\hat{\sigma}}})_0$.

We first construct a sheaf homomorphism $\widetilde{\mathsf{M}} \to \mathscr{F}$. Elements of $(\mathsf{M}_{x^{\hat{\sigma}}})_0$ are $u/(x^{\hat{\sigma}})^\ell$ for $u \in \Gamma(X_\Sigma, \mathscr{F}(\ell\alpha_\sigma))$. Since $(x^{\hat{\sigma}})^{-\ell}$ is a section of $\mathscr{O}_{X_\Sigma}(-\ell\alpha_\sigma)$ over U_σ, the map

$$\Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(-\ell\alpha_\sigma)) \otimes_{\mathbb{C}} \Gamma(U_\sigma, \mathscr{F}(\alpha_\sigma)) \longrightarrow \Gamma(U_\sigma, \mathscr{F})$$

induces a homomorphism of $(S_{x^{\hat{\sigma}}})_0$-modules

$$(\mathsf{M}_{x^{\hat{\sigma}}})_0 \longrightarrow \Gamma(U_\sigma, \mathscr{F}). \tag{6.A.4}$$

This gives compatible sheaf homomorphisms $\widetilde{\mathsf{M}}|_{U_\sigma} \to \mathscr{F}|_{U_\sigma}$ that patch to give $\widetilde{\mathsf{M}} \to \mathscr{F}$.

Since $\mathscr{F}$ is quasicoherent, it suffices to show that (6.A.4) is an isomorphism for every $\sigma \in \Sigma$. First suppose that $u/(x^{\hat{\sigma}})^k \in (\mathsf{M}_{x^{\hat{\sigma}}})_0$ maps to $0 \in \Gamma(U_\sigma, \mathscr{F})$. It follows easily that u restricts to zero in $\Gamma(U_\sigma, \mathscr{F}(k\alpha_\sigma))$. By Lemma 6.A.2 applied to $\mathscr{F}(k\alpha_\sigma)$, there is $\ell \geq 0$ such that $(x^{\hat{\sigma}})^\ell u = 0$ in $\Gamma(X_\Sigma, \mathscr{F}((\ell+k)\alpha_\sigma))$. Then

$$\frac{u}{(x^{\hat{\sigma}})^k} = \frac{(x^{\hat{\sigma}})^\ell u}{(x^{\hat{\sigma}})^{\ell+k}} = 0 \quad \text{in } (\mathsf{M}_{x^{\hat{\sigma}}})_0,$$

which shows that (6.A.4) is injective. To prove surjectivity, take $v \in \Gamma(U_\sigma, \mathscr{F})$ and apply Lemma 6.A.2 to find $\ell \geq 0$ and $u \in \Gamma(X_\Sigma, \mathscr{F}(\ell\alpha_\sigma))$ such that u restricts to $(x^{\hat{\sigma}})^\ell v$. It follows immediately that $u/(x^{\hat{\sigma}})^\ell \in (\mathsf{M}_{x^{\hat{\sigma}}})_0$ maps to v. $\square$

This result proves part (a) of Proposition 5.3.9. We now turn our attention to part (b) of the proposition, which applies to coherent sheaves.

Proposition 6.A.4. *Every coherent sheaf $\mathscr{F}$ on X_Σ is isomorphic to the sheaf associated to a finitely generated graded S-module.*

Proof. On the affine open subset U_σ, we can find finitely many sections $f_{i,\sigma} \in \Gamma(U_\sigma, \mathscr{F})$ which generate $\mathscr{F}$ over U_σ. By Lemma 6.A.2, we can find $\ell \geq 0$ such that $(x^{\hat{\sigma}})^\ell f_{i,\sigma}$ comes from a global section $g_{i,\sigma}$ of $\mathscr{F}(\ell\alpha_\sigma)$. Now consider the graded S-module $\mathsf{M} \subseteq \Gamma_*(\mathscr{F})$ generated by the $g_{i,\sigma}$. Proposition 6.A.3 gives an isomorphism

$$\widetilde{\Gamma_*(\mathscr{F})} \simeq \mathscr{F}.$$

Hence $\mathsf{M} \subseteq \Gamma_*(\mathscr{F})$ gives a sheaf homomorphism $\widetilde{\mathsf{M}} \to \mathscr{F}$ which is injective by the exactness proved in Example 6.0.10. Over U_σ, we have $f_{i,\sigma} = g_{i,\sigma}/(x^{\hat{\sigma}})^\ell \in (\mathsf{M}_{x^{\hat{\sigma}}})_0$, and since these sections generate $\mathscr{F}$ over U_σ, it follows that $\widetilde{\mathsf{M}} \simeq \mathscr{F}$. Then we are done since M is clearly finitely generated. $\square$

The proof of Proposition 6.A.4 used a submodule of $\Gamma_*(\mathscr{F})$ because the full module need not be finitely generated when $\mathscr{F}$ is coherent. Here is an easy example.

Example 6.A.5. A point $p \in \mathbb{P}^n$ gives a subvariety $i : \{p\} \hookrightarrow \mathbb{P}^n$. The sheaf $\mathscr{F} = i_*\mathscr{O}_{\{p\}}$ can be thought of as a copy of $\mathbb{C}$ sitting over the point p. The line bundle $\mathscr{O}_{\mathbb{P}^n}(a)$ is free in a neighborhood of p, so that $\mathscr{F}(a) \simeq \mathscr{F}$ for all $a \in \mathbb{Z}$. Thus

$$\Gamma_*(\mathscr{F}) = \bigoplus_{a\in\mathbb{Z}} \Gamma(\mathbb{P}^n, \mathscr{F}(a)) = \bigoplus_{a\in\mathbb{Z}} \mathbb{C}.$$

This module is not finitely generated over S since it is nonzero in all negative degrees. $\Diamond$

Subschemes and Homogeneous Ideals. For readers who know about schemes, we can apply the above results to describe subschemes of a toric variety X_Σ with no torus factors.

Let $I \subseteq S$ be a homogeneous ideal. By Proposition 6.0.10, this gives a sheaf of ideals $\mathscr{I} \subseteq \mathscr{O}_{X_\Sigma}$ whose quotient is the structure sheaf $\mathscr{O}_Y$ of a closed subscheme $Y \subseteq X_\Sigma$. This differs from the subvarieties considered in the rest of the book since the structure sheaf $\mathscr{O}_Y$ may have nilpotents.

Proposition 6.A.6. *Every subscheme $Y \subseteq X_\Sigma$ is defined by a homogeneous ideal $I \subseteq S$.*

Proof. Given an ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_{X_\Sigma}$, we get a homomorphism of S-modules

$$\Gamma_*(\mathscr{I}) \longrightarrow \Gamma_*(\mathscr{O}_{X_\Sigma}) = S.$$

If $I \subseteq S$ is the image of this map, then the map factors $\Gamma_*(\mathscr{I}) \twoheadrightarrow I \hookrightarrow S$, where the first arrow is surjective and the second injective. By Example 6.0.10 and Proposition 6.A.3, the inclusion $\mathscr{I} \subseteq \mathscr{O}_{X_\Sigma}$ factors as $\mathscr{I} \twoheadrightarrow \tilde{I} \hookrightarrow \mathscr{O}_{X_\Sigma}$. It follows immediately that $\mathscr{I} = \tilde{I}$. $\square$

In the case of $\mathbb{P}^n$, it is well-known that different graded ideals can give the same ideal sheaf. The same happens in the toric situation, and as in §5.3, we get the best answer in the smooth case. Not surprisingly, the irrelevant ideal $B(\Sigma) \subseteq S$ plays a key role in the following result from [**65**, Cor. 3.8].

Proposition 6.A.7. *Homogeneous ideals $I, J \subseteq S$ in the total coordinate of a smooth toric variety X_Σ give the same ideal sheaf of $\mathscr{O}_{X_\Sigma}$ if and only if $I : B(\Sigma)^\infty = J : B(\Sigma)^\infty$.* $\square$

There is a less elegant version of this result that applies to simplicial toric varieties. See [**65**] for a proof and more details about the relation between graded modules and sheaves. See also [**204**] for more on multigraded commutative algebra.

Chapter 7

Projective Toric Morphisms

§7.0. Background: Quasiprojective Varieties and Projective Morphisms

Many results of Chapter 6 can be generalized, but in order to do so, we need to learn about *quasiprojective varieties* and *projective morphisms*.

Quasiprojective Varieties. Besides affine and projective varieties, we also have the following important class of varieties.

Definition 7.0.1. A variety is ***quasiprojective*** if it is isomorphic to an open subset of a projective variety.

Here are some easy properties of quasiprojective varieties.

Proposition 7.0.2.

(a) *Affine varieties and projective varieties are quasiprojective.*

(b) *Every closed subvariety of a quasiprojective variety is quasiprojective.*

(c) *A product of quasiprojective varieties is quasiprojective.*

Proof. You will prove this in Exercise 7.0.1. □

Projective Morphisms. In algebraic geometry, concepts that apply to varieties sometimes have relative versions that apply to morphisms between varieties. For example, in §3.4, we defined *completeness* and *properness*, where the former applies to varieties and the latter applies to morphisms. Sometimes we say that the *relative version* of a complete variety is a proper morphism. In the same way, the relative version of a *projective variety* is a *projective morphism*.

We begin with a special case. Let $f : X \to Y$ be a morphism and $\mathscr{L}$ a line bundle on X with a basepoint free finite-dimensional subspace $W \subseteq \Gamma(X, \mathscr{L})$. Then combining $f : X \to Y$ with the morphism $\phi_{\mathscr{L},W} : X \to \mathbb{P}(W^\vee)$ from §6.0 gives a morphism $X \to Y \times \mathbb{P}(W^\vee)$ that fits into a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f\times\phi_{\mathscr{L},W}"] \arrow[dr, "f"'] & Y \times \mathbb{P}(W^\vee) \arrow[d, "p_1"] \\ & Y. \end{tikzcd} \tag{7.0.1}$$

If $f \times \phi_{\mathscr{L},W}$ is a *closed embedding* (meaning that its image $Z \subseteq Y \times \mathbb{P}(W^\vee)$ is closed and the induced map $X \to Z$ is an isomorphism), then you will show in Exercise 7.0.2 that f has the following nice properties:

- f is proper.
- For every $p \in Y$, the fiber $f^{-1}(p)$ is isomorphic to a closed subvariety of $\mathbb{P}(W^\vee)$ and hence is projective.

The general definition of projective morphism must include this special case. In fact, going from the special case to the general case is not that hard.

Definition 7.0.3. A morphism $f : X \to Y$ is ***projective*** if there is a line bundle $\mathscr{L}$ on X and an affine open cover $\{U_i\}$ of Y with the property that for each i, there is a basepoint free finite-dimensional subspace $W_i \subseteq \Gamma(f^{-1}(U_i), \mathscr{L})$ such that

$$f^{-1}(U_i) \xrightarrow{f_i\times\phi_{\mathscr{L}_i,W_i}} U_i \times \mathbb{P}(W_i^\vee)$$

is a closed embedding, where $f_i = f|_{f^{-1}(U_i)}$ and $\mathscr{L}_i = \mathscr{L}|_{f^{-1}(U_i)}$. We say that $f : X \to Y$ is ***projective with respect to*** $\mathscr{L}$.

The general case has the properties noted above in the special case.

Proposition 7.0.4. *Let $f : X \to Y$ be projective. Then:*

(a) *f is proper.*

(b) *For every $p \in Y$, the fiber $f^{-1}(p)$ is a projective variety.* □

Here are some further properties.

Proposition 7.0.5.

(a) *The composition of projective morphisms is projective.*

(b) *A closed embedding is a projective morphism.*

(c) *A variety X is projective if and only if $X \to \{\text{pt}\}$ is a projective morphism.*

Proof. Parts (a) and (b) are proved in [**127**, (5.5.5)]. For part (c), one direction follows immediately from the previous proposition. Conversely, let $i : X \hookrightarrow \mathbb{P}^n$ be projective, and assume that X is *nondegenerate*, meaning that X is not contained in any hyperplane of $\mathbb{P}^n$. Now let $\mathscr{L} = \mathscr{O}_X(1) = i^*\mathscr{O}_{\mathbb{P}^n}(1)$. Then

$$i^* : \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) \longrightarrow \Gamma(X, \mathscr{L})$$

is injective since X is nondegenerate. In Exercise 7.0.3 you will show that

$$\Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) = \mathrm{Span}(x_0, \ldots, x_n)$$

and that if $W \subseteq \Gamma(X, \mathscr{L})$ is the image of i^*, then $\phi_{\mathscr{L},W}$ is the embedding we began with. Hence Definition 7.0.3 is satisfied for $X \to \{\mathrm{pt}\}$ and $\mathscr{L}$. $\square$

When the domain is quasiprojective, the relation between proper and projective is especially easy to understand.

Proposition 7.0.6. *Let $f : X \to Y$ be a morphism where X is quasiprojective. Then:*

$$f \text{ is proper} \iff f \text{ is projective}.$$

Proof. One direction is obvious since projective implies proper. For the opposite direction, X is quasiprojective, which implies that there is a morphism

$$g : X \longrightarrow Z$$

such that Z is projective, $g(X) \subseteq Z$ is open, and $X \simeq g(X)$ via g. Then one can prove without difficulty that the product map

$$f \times g : X \longrightarrow Y \times Z \tag{7.0.2}$$

induces an isomorphism $X \simeq (f \times g)(X)$.

Since $f : X \to Y$ is proper, $f \times g : X \to Y \times Z$ is also proper (Exercise 7.0.4). Hence the image of $f \times g$ is closed in $X \times Z$ since proper morphisms are universally closed. Thus $X \simeq (f \times g)(X)$ and $(f \times g)(X)$ is closed in $Y \times Z$. This proves that (7.0.2) is a closed embedding.

Now take a closed embedding $Z \hookrightarrow \mathbb{P}^s$. Arguing as above, we get a closed embedding of X into $Y \times \mathbb{P}^s$. From here, it is straightforward to show that f is projective (Exercise 7.0.4). $\square$

To complicate matters, there are two definitions of projective morphism used in the literature. In [**131**, II.4], a projective morphism is defined as the special case considered in (7.0.1), while [**127**, (5.5.2)] and [**273**, 5.3] give a much more general definition. Theorem 7.A.4 of the appendix to this chapter shows that the more general definition is equivalent to Definition 7.0.3.

Projective Bundles. Vector bundles give rise to an interesting class of projective morphisms.

Let $\pi : V \to X$ be a vector bundle of rank $n \geq 1$. Recall from §6.0 that V has a trivialization $\{(U_i, \phi_i)\}$ with $\phi_i : \pi^{-1}(U_i) \simeq U_i \times \mathbb{C}^n$. Furthermore, the transition functions $g_{ij} \in \mathrm{GL}_n(\Gamma(U_i \cap U_j, \mathscr{O}_X))$ make the diagram

$$\begin{tikzcd} & U_i \cap U_j \times \mathbb{C}^n \\ \pi^{-1}(U_i \cap U_j) \arrow[ur, "\phi_i|_{\pi^{-1}(U_i\cap U_j)}"] \arrow[dr, "\phi_j|_{\pi^{-1}(U_i\cap U_j)}"'] & \\ & U_i \cap U_j \times \mathbb{C}^n \arrow[uu, "1\times g_{ij}"'] \end{tikzcd}$$

commute. Note that $1 \times g_{ij}$ induces an isomorphism

$$1 \times \overline{g}_{ij} : U_i \cap U_j \times \mathbb{P}^{n-1} \simeq U_i \cap U_j \times \mathbb{P}^{n-1}.$$

This gives gluing data for a variety $\mathbb{P}(V)$. It is clear that π induces a morphism $\overline{\pi} : \mathbb{P}(V) \to X$ and that ϕ_i induces the trivialization

$$\overline{\phi}_i : \overline{\pi}^{-1}(U_i) \simeq U_i \times \mathbb{P}^{n-1}.$$

The discussion following Theorem 7.A.4 in the appendix to this chapter shows that $\overline{\pi} : \mathbb{P}(V) \to X$ is a projective morphism. We call $\mathbb{P}(V)$ the *projective bundle* of V.

Example 7.0.7. Let W be a finite-dimensional vector space over $\mathbb{C}$ of positive dimension. Then, for any variety X, the trivial bundle $X \times W \to X$ gives the trivial projective bundle $X \times \mathbb{P}(W) \to X$. $\Diamond$

There is also a version of this construction for locally free sheaves. If $\mathscr{E}$ is locally free of rank n, then $\mathscr{E}$ is the sheaf of sections of a vector bundle $V_{\mathscr{E}} \to X$ of rank n. When $n = 1$, we proved this in Theorem 6.0.20. Then define

$$\mathbb{P}(\mathscr{E}) = \mathbb{P}(V_{\mathscr{E}}^{\vee}), \tag{7.0.3}$$

where $V_{\mathscr{E}}^{\vee}$ is the dual vector bundle of $V_{\mathscr{E}}$. Here are some properties of $\mathbb{P}(\mathscr{E})$.

Lemma 7.0.8.

(a) $\mathbb{P}(\mathscr{L}) = X$ *when* $\mathscr{L}$ *is locally free of rank* 1.

(b) $\mathbb{P}(\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L}) = \mathbb{P}(\mathscr{E})$ *when* $\mathscr{E}$ *is locally free and* $\mathscr{L}$ *is a line bundle.*

(c) *If a homomorphism* $\mathscr{E} \to \mathscr{F}$ *of locally free sheaves is surjective, then the induced map* $\mathbb{P}(\mathscr{F}) \to \mathbb{P}(\mathscr{E})$ *of projective bundles is injective.*

Proof. You will prove this in Exercise 7.0.5. The dual in (7.0.3) is the reason why $\mathscr{E} \to \mathscr{F}$ gives $\mathbb{P}(\mathscr{F}) \to \mathbb{P}(\mathscr{E})$. $\square$

The appearance of the dual in (7.0.3) can be explained as follows. Let $\mathscr{L}$ be a line bundle with $W \subseteq \Gamma(X, \mathscr{L})$ basepoint free of finite dimension. As in §6.0, this gives a morphism

$$\phi_{\mathscr{L},W} : X \longrightarrow \mathbb{P}(W^{\vee}).$$

Let $\mathscr{E} = W \otimes_{\mathbb{C}} \mathscr{O}_X$. The corresponding vector bundle is $V_{\mathscr{E}} = X \times W$, so

$$\mathbb{P}(\mathscr{E}) = \mathbb{P}(V_{\mathscr{E}}^{\vee}) = X \times \mathbb{P}(W^{\vee}). \tag{7.0.4}$$

By Proposition 6.0.24, the natural map $\mathscr{E} \to \mathscr{L}$ is surjective since W has no basepoints. By Lemma 7.0.8, we get an injection of projective bundles

$$\mathbb{P}(\mathscr{L}) \longrightarrow \mathbb{P}(\mathscr{E}).$$

The lemma also implies $\mathbb{P}(\mathscr{L}) = X$. Using this and (7.0.4), we get an injection

$$X \longrightarrow X \times \mathbb{P}(W^{\vee}).$$

Projection onto the second factor gives a morphism $X \to \mathbb{P}(W^{\vee})$, which is the morphism $\phi_{\mathscr{L},W}$ from §6.0 (Exercise 7.0.6).

Proj of a Graded Ring. As described in [**90**, III.2] and [**131**, II.2], a graded ring

$$S = \bigoplus_{d=0}^{\infty} S_d$$

gives the scheme $\mathrm{Proj}(S)$ such that for every non-nilpotent $f \in S_d$, $d > 0$, we have the affine open subset $D_+(f) \subseteq \mathrm{Proj}(S)$ with

$$D_+(f) \simeq \mathrm{Spec}(S_{(f)}),$$

where $S_{(f)}$ is the homogeneous localization of S at f, i.e.,

$$S_{(f)} = \Big\{ \frac{g}{f^{\ell}} \mid g \in S_{\ell d},\ \ell \in \mathbb{N} \Big\}.$$

Furthermore, if homogeneous elements $f_1, \ldots, f_s \in S$ satisfy

$$\sqrt{\langle f_1, \ldots, f_s \rangle} = S_+ = \bigoplus_{d>0} S_d,$$

then the affine open subsets $D_+(f_1), \ldots, D_+(f_s)$ cover $\mathrm{Proj}(S)$. Thus we construct $\mathrm{Proj}(S)$ by gluing together the affine varieties $D_+(f_i)$, just as we construct $\mathbb{P}^n$ by gluing together copies of $\mathbb{C}^n$.

The inclusions $S_0 \subseteq S_{(f)}$ for all f give a natural morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$. We have the following important result from [**131**, Prop. II.7.10].

Theorem 7.0.9. *The morphism* $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0)$ *is projective.* □

Example 7.0.10. Let $U = \mathrm{Spec}(R)$ and consider the graded ring

$$S = R[x_0, \ldots, x_n]$$

such that each x_i has degree 1. Then

$$\mathrm{Proj}(S) = U \times \mathbb{P}^n,$$

where $\mathrm{Proj}(S) \to \mathrm{Spec}(S_0) = \mathrm{Spec}(R) = U$ is projection onto the first factor. ◊

In [**131**, II.7], the projective bundle $\mathbb{P}(\mathscr{E})$ of a locally free sheaf $\mathscr{E}$ on X is constructed via a relative version of the Proj construction. More generally, one can define the "projective bundle" $\mathbb{P}(\mathscr{E})$ for any coherent sheaf $\mathscr{E}$ on X.

Exercises for §7.0.

7.0.1. Prove Proposition 7.0.2.

7.0.2. Prove Proposition 7.0.4. Hint: First prove the special case given by (7.0.1). Recall from §3.4 that $\mathbb{P}^n$ is complete, so that $\mathbb{P}^n \to \{\text{pt}\}$ is proper.

7.0.3. Complete the proof of Proposition 7.0.5.

7.0.4. Let $\alpha : X \to Y$ and $\beta : Y \to Z$ be morphisms such that $\beta \circ \alpha : X \to Z$ is proper. Prove that $\alpha : X \to Y$ is also proper. Hint: As noted in the comments following Corollary 3.4.6, being proper is equivalent to being topologically proper (Definition 3.4.7). Also, $T \subseteq Y$ implies $\alpha^{-1}(T) \subseteq (\beta \circ \alpha)^{-1}(\beta(T))$.

7.0.5. Prove Lemma 7.0.8. Hint: Work on an open cover where the bundles are trivial.

7.0.6. In the discussion following (7.0.4), we constructed a morphism $X \to \mathbb{P}(W^\vee)$ using the surjection $\mathscr{E} = W \otimes_{\mathbb{C}} \mathscr{O}_X \to \mathscr{L}$. Prove that this coincides with the morphism $\phi_{\mathscr{L},W}$.

7.0.7. Show that $\mathbb{C}^2 \setminus \{(0,0)\}$ is quasiprojective but neither affine nor projective.

§7.1. Polyhedra and Toric Varieties

This section and the next will study quasiprojective toric varieties and projective toric morphisms. Our starting point is the observation that just as polytopes give projective toric varieties, polyhedra give projective toric morphisms.

Polyhedra. Recall that a polyhedron $P \subseteq M_{\mathbb{R}}$ is the intersection of finitely many closed half-spaces

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_i \rangle \geq -a_i,\ i = 1, \ldots, s\}.$$

A basic structure theorem tells us that P is a Minkowski sum

$$P = Q + C,$$

where Q is a polytope and C is a polyhedral cone (see [**281**, Thm. 1.2]). If P is presented as above, then the cone part of P is

$$C = \{m \in M_{\mathbb{R}} \mid \langle m, u_i \rangle \geq 0,\ i = 1, \ldots, s\}. \tag{7.1.1}$$

(Exercise 7.1.1). Following [**281**], we call C the *recession cone* of P.

Similar to polytopes, polyhedra have supporting hyperplanes, faces, facets, vertices, edges, etc. One difference is that some polyhedra have no vertices.

Lemma 7.1.1. *Let $P \subseteq M_{\mathbb{R}}$ be a polyhedron with recession cone C.*

(a) *The set* $V = \{\mathsf{v} \in P \mid \mathsf{v} \text{ is a vertex}\}$ *is finite and is nonempty if and only if C is strongly convex.*

(b) *If C is strongly convex, then* $P = \operatorname{Conv}(V) + C$.

Proof. You will prove this in Exercises 7.1.2–7.1.5. □

Example 7.1.2. The polyhedron $P = \{(a_1,\dots,a_n) \in \mathbb{R}^n \mid a_i \geq 0, \sum_{i=1}^n a_i \geq 1\}$ has vertices $e_1,\dots,e_n$ and recession cone $C = \mathrm{Cone}(e_1,\dots,e_n)$. ◊

Lattice Polyhedra. We now generalize the notion of lattice polytope.

Definition 7.1.3. A polyhedron $P \subseteq M_\mathbb{R}$ is a ***lattice polyhedron*** with respect to the lattice $M \subseteq M_\mathbb{R}$ if

(a) The recession cone of P is a strongly convex rational polyhedral cone.

(b) The vertices of P lie in the lattice M.

A full dimensional lattice polyhedron has a unique facet presentation

$$P = \{m \in M_\mathbb{R} \mid \langle m, u_F\rangle \geq -a_F \text{ for all facets } F\}, \tag{7.1.2}$$

where $u_F \in N$ is a primitive inward pointing facet normal. This was defined in Chapter 2 for full dimensional lattice polytopes but applies equally well to full dimensional lattice polyhedra. Then the *cone of P* is the cone $C(P) \subseteq M_\mathbb{R} \times \mathbb{R}$ by

$$C(P) = \{(m,\lambda) \in M_\mathbb{R} \times \mathbb{R} \mid \langle m, u_F\rangle \geq -\lambda a_F \text{ for all } F,\ \lambda \geq 0\}.$$

When P is a polytope, this reduces to the cone $C(P) = \mathrm{Cone}(P \times \{1\})$ considered in §2.2.

Example 7.1.4. The blowup of $\mathbb{C}^2$ at the origin is given by the fan Σ in $\mathbb{R}^2$ with minimal generators $u_0 = e_1 + e_2, u_1 = e_1, u_2 = e_2$ and maximal cones $\mathrm{Cone}(u_0,u_1)$, $\mathrm{Cone}(u_0,u_2)$. For the divisor $D = D_0 + D_1 + D_2$, we computed in Figure 5 from Example 4.3.4 that the polyhedron P_D is a 2-dimensional lattice polyhedron whose recession cone C is the first quadrant.

Figure 1 on the next page shows the 3-dimensional cone $C(P_D)$ with P_D at height 1. Notice how the cone C of P_D appears naturally at height 0 in Figure 1. ◊

Some of the properties suggested by Figure 1 hold in general.

Lemma 7.1.5. *Let P be a full dimensional lattice polyhedron in $M_\mathbb{R}$ with recession cone C. Then $C(P)$ is a strongly convex cone in $M_\mathbb{R} \times \mathbb{R}$ and*

$$C(P) \cap (M_\mathbb{R} \times \{0\}) = C.$$

Proof. The final assertion of the lemma follows from (7.1.1) and the definition of $C(P)$. For strong convexity, note that $C(P) \subseteq M_\mathbb{R} \times \mathbb{R}_{\geq 0}$ implies

$$C(P) \cap (-C(P)) \subseteq M_\mathbb{R} \times \{0\}.$$

Then we are done since C is strongly convex. □

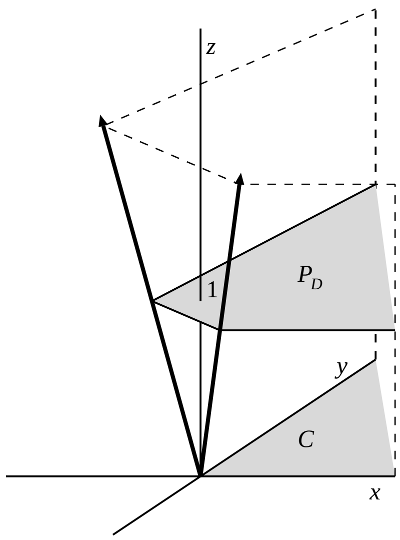


Figure 1. The cone $C(P_D)$ in Example 7.1.4

We say that a point $(m,\lambda) \in C(P)$ has *height* λ. Furthermore, when $\lambda > 0$, the "slice" of $C(P)$ at height λ is λP. If we write $P = Q + C$, where Q is a polytope, then for $\lambda > 0$,

$$\lambda P = \lambda Q + C$$

since C is a cone. It follows that as $\lambda \to 0$, the polytope shrinks to a point so that at height 0, only the cone C remains, as in Lemma 7.1.5. You can see how this works in Figure 1.

The Toric Variety of a Polyhedron. In Chapter 2, we constructed the normal fan of a full dimensional lattice polytope. We now do the same for a full dimensional lattice polyhedron P. Given a vertex $\mathsf{v} \in P$, we get the cone

$$C_{\mathsf{v}} = \mathrm{Cone}(P \cap M - \mathsf{v}) \subseteq M_{\mathbb{R}}.$$

Note that $\mathsf{v} \in M$ since P is a lattice polyhedron. It follows easily that C_{v} is a strongly convex rational polyhedral cone of maximal dimension, so that the same is true for its dual

$$\sigma_{\mathsf{v}} = C_{\mathsf{v}}^{\vee} = \mathrm{Cone}(P \cap M - \mathsf{v})^{\vee} \subseteq N_{\mathbb{R}}.$$

These cones fit together nicely.

Theorem 7.1.6. *Given a full dimensional lattice polyhedron $P \subseteq M_{\mathbb{R}}$ with recession cone C, the set*

$$\Sigma_P = \{\sigma \mid \sigma \preceq \sigma_{\mathsf{v}},\ \mathsf{v} \textit{ is a vertex of } P\}$$

is a fan in $N_{\mathbb{R}}$. Furthermore:

(a) *The support of* Σ_P *is* $|\Sigma_P| = C^\vee$.

(b) Σ_P *has full dimensional convex support in* $N_{\mathbb{R}}$.

Proof. The proof that we get a fan is similar to the proof for the polytope case (see §2.3) and hence is omitted. To prove part (a), we need to show

$$\bigcup_{\mathsf{v}\in V} \sigma_{\mathsf{v}} = C^\vee,$$

where V is the set of vertices of P. Now take $\mathsf{v} \in V$ and $m \in C \cap M$. Then $m = (\mathsf{v}+m) - \mathsf{v} \in P \cap M - \mathsf{v}$, which easily implies $C \subseteq \mathrm{Cone}(P \cap M - \mathsf{v})$. Taking duals, we obtain $\sigma_{\mathsf{v}} \subseteq C^\vee$. For the opposite inclusion, take $u \in C^\vee$ and pick $\mathsf{v} \in V$ such that $\langle \mathsf{v}, u\rangle \le \langle \mathsf{w}, u\rangle$ for all $\mathsf{w} \in V$. We show $u \in \sigma_{\mathsf{v}}$ as follows. Any $m \in P \cap M$ can be written $m = \sum_{\mathsf{w}\in V} \lambda_{\mathsf{w}} \mathsf{w} + m'$ where $\lambda_{\mathsf{w}} \ge 0$, $\sum_{\mathsf{w}\in V} \lambda_{\mathsf{w}} = 1$ and $m' \in C$. Then

$$\langle m, u\rangle = \textstyle\sum_{\mathsf{w}\in V} \lambda_{\mathsf{w}} \langle \mathsf{w}, u\rangle + \langle m', u\rangle \ge \sum_{\mathsf{w}\in V} \lambda_{\mathsf{w}} \langle \mathsf{v}, u\rangle = \langle \mathsf{v}, u\rangle.$$

Thus $\langle m - \mathsf{v}, u\rangle \ge 0$ for all $m - \mathsf{v} \in P \cap M - \mathsf{v}$, which proves $u \in \sigma_{\mathsf{v}}$.

Part (b) now follows since $C^\vee$ is clearly convex, and has full dimension since C is strongly convex. $\square$

The fan Σ_P of Theorem 7.1.6 is the *normal fan* of P. We define X_P to be the toric variety X_{Σ_P} of Σ_P. Here is an example.

Example 7.1.7. The polyhedron $P = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_i \ge 0, \sum_{i=1}^n a_i \ge 1\}$ of Example 7.1.2 has vertices $e_1, \ldots, e_n$. The facet of P defined by $\sum_{i=1}^n a_i = 1$ has $e_1 + \cdots + e_n$ as inward normal. Then the vertex e_i gives the cone

$$\sigma_{e_i} = \mathrm{Cone}(e_1 + \cdots + e_n, e_1, \ldots, \widehat{e_i}, \ldots, e_n).$$

These cones form the fan of the blowup of $\mathbb{C}^n$ at the origin, so $X_P = \mathrm{Bl}_0(\mathbb{C}^n)$. $\Diamond$

Note that X_P is not complete in this example. In general, the normal fan has support $|\Sigma_P| = C^\vee$. We measure the deviation from completeness as follows.

The support $|\Sigma_P|$ is a rational polyhedral cone but need not be strongly convex. Recall that $W = |\Sigma_P| \cap (-|\Sigma_P|)$ is the largest subspace contained in $|\Sigma_P|$. Hence $|\Sigma_P|$ gives the following:

- The sublattice $W \cap N \subseteq N$ and the quotient lattice $N_P = N/(W \cap N)$.
- The strongly convex cone $\sigma_P = |\Sigma_P|/W \subseteq N_{\mathbb{R}}/W = (N_P)_{\mathbb{R}}$.
- The affine toric variety U_P of σ_P.

The projection map $\overline{\phi} : N \to N_P$ is compatible with the fans of X_P and U_P since $\overline{\phi}_{\mathbb{R}}(|\Sigma_P|) = \sigma_P$. Hence we get a toric morphism

$$\phi : X_P \longrightarrow U_P.$$

Since $|\Sigma_P| = \overline{\phi}_{\mathbb{R}}^{-1}(\sigma_P)$ (Exercise 7.1.6), Theorem 3.4.11 implies that ϕ is proper.

The key result of this section is that $\phi : X_P \to U_P$ is a projective morphism. We first give an elementary proof in Theorem 7.1.10. We will also give a more sophisticated proof that applies Proj construction from §7.0 to the semigroup algebra

$$S_P = \mathbb{C}[C(P) \cap (M \times \mathbb{Z})]. \tag{7.1.3}$$

The character associated to $(m,k) \in C(P) \cap (M \times \mathbb{Z})$ is written $\chi^m t^k$, and S_P is graded by height, i.e, $\deg(\chi^m t^k) = k$. In Proposition 7.1.13 we will prove that

$$X_P \simeq \mathrm{Proj}(S_P).$$

Then standard properties of Proj will imply that $\phi : X_P \to U_P$ is projective.

The Divisor of a Polyhedron. Let P be a full dimensional lattice polyhedron in $M_{\mathbb{R}}$. As in the polytope case, facets of P correspond to rays in the normal fan Σ_P, so that each facet F gives a prime torus-invariant divisor $D_F \subseteq X_P$. Thus the facet presentation (7.1.2) of P gives the divisor

$$D_P = \textstyle\sum_F a_F D_F, \tag{7.1.4}$$

where the sum is over all facets of P. Results from Chapter 4 (Proposition 4.2.10 and Example 4.3.7) easily adapt to the polyhedral case to show that D_P is Cartier (with $m_{\sigma_{\mathsf{v}}} = \mathsf{v}$ for every vertex v) and the polyhedron of D_P is P, i.e., $P = P_{D_P}$. Then Proposition 4.3.3 implies that

$$\Gamma(X_P, \mathscr{O}_{X_P}(D_P)) = \textstyle\bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m. \tag{7.1.5}$$

The definition of projective morphism given in §7.0 involves a line bundle $\mathscr{L}$ and a finite-dimensional subspace W of global sections. The line bundle will be $\mathscr{O}_{X_P}(kD_P)$ for a suitably chosen integer $k \geq 1$ and W will be determined by certain carefully chosen lattice points of kP. The reason we need a multiple is that P might not have enough lattice points.

Normal and Very Ample Polyhedra. In Chapter 2, we defined normal and very ample polytopes, which are different ways of saying that there are enough lattice points. For a lattice polyhedron P, the definitions are the same.

Definition 7.1.8. Let $P \subseteq M_{\mathbb{R}}$ be a lattice polyhedron. Then:

(a) P is ***normal*** if for all integers $k \geq 1$, every lattice point of kP is a sum of k lattice points of P.

(b) P is ***very ample*** if for every vertex $\mathsf{v} \in P$, the semigroup $\mathbb{N}(P \cap M - \mathsf{v})$ generated by $P \cap M - \mathsf{v}$ is saturated in M.

We have the following result about normal and very ample polyhedra.

Proposition 7.1.9. *Let $P \subseteq M_{\mathbb{R}}$ be a lattice polyhedron. Then:*

(a) *If P is normal, then P is very ample.*

(b) *If* $\dim P = n \geq 2$, *then kP is normal and hence very ample for all $k \geq n-1$.*

Proof. Part (a) follows from the proof of Proposition 2.2.18. For part (b), write $P = Q + C$, where Q is a lattice polytope and C is the recession cone of P. Let $C = \operatorname{Cone}(m_1, \dots, m_s)$, $m_i \in M$. In Exercise 7.1.7 you will show that

$$C = \bigcup_{m \in C \cap M} \operatorname{Conv}(0, sm_1, \dots, sm_s) + m. \tag{7.1.6}$$

Since $P = Q + C$ is a full dimensional polyhedron, $Q + \operatorname{Conv}(0, sm_1, \dots, sm_s)$ is a full dimensional lattice polytope. It follows that $P = Q + C$ is a union of full dimensional lattice polytopes. Then part (b) follows by applying Theorem 2.2.12 to each of these polytopes. □

The Projective Morphism. Let P be a full dimensional lattice polyhedron in $M_{\mathbb{R}}$. Assume P is very ample and pick a finite set $\mathscr{A} \subseteq P \cap M$ such that:

- $\mathscr{A}$ contains the vertices of P.
- For every vertex $\mathsf{v} \in P$, $\mathscr{A} - \mathsf{v}$ generates $\operatorname{Cone}(P \cap M - \mathsf{v}) \cap M = \sigma_{\mathsf{v}}^{\vee} \cap M$.

We can always satsify the first condition, and the second is possible since P is very ample. Using (7.1.5), we get the subspace

$$W = \operatorname{Span}(\chi^m \mid m \in \mathscr{A}) \subseteq \Gamma(X_P, \mathscr{O}_{X_P}(D_P)).$$

We claim that W has no basepoints since $\mathscr{A}$ contains the vertices of P. To prove this, let v be a vertex. Recall that $D_P + \operatorname{div}(\chi^{\mathsf{v}})$ is the divisor of zeros of the global section given by χ^{v}. One computes that

$$D_P + \operatorname{div}(\chi^{\mathsf{v}}) = \textstyle\sum_F (a_F + \langle \mathsf{v}, u_F \rangle) D_F.$$

Since $\langle \mathsf{v}, u_F \rangle = -a_F$ for all facets containing v and $\langle \mathsf{v}, u_F \rangle > -a_F$ for all other facets, the support of $D_P + \operatorname{div}(\chi^{\mathsf{v}})$ is the complement of the affine open subset $U_{\sigma_{\mathsf{v}}} \subseteq X_P$, i.e., the nonvanishing set of the section is precisely $U_{\sigma_{\mathsf{v}}}$. Then we are done since the $U_{\sigma_{\mathsf{v}}}$ cover X_P.

It follows that we get a morphism

$$\phi_{\mathscr{L},W} : X_P \longrightarrow \mathbb{P}(W^{\vee})$$

for $\mathscr{L} = \mathscr{O}_{X_P}(D_P)$. Here is our result.

Theorem 7.1.10. *Let P be a full dimensional lattice polyhedron. Then:*

(a) *The toric variety X_P is quasiprojective.*

(b) *$\phi : X_P \to U_P$ is a projective morphism.*

Proof. First suppose that P is very ample. The proof of part (a) is similar to the proof of Proposition 6.1.10. Let $\mathscr{L}$, W and $\mathscr{A}$ be as above and write $\mathscr{A} = \{m_1, \dots, m_s\}$. Consider the projective toric variety

$$X_{\mathscr{A}} \subseteq \mathbb{P}^{s-1} = \mathbb{P}(W^{\vee}).$$

Let $I \subseteq \{1,\dots,s\}$ be the set of indices corresponding to vertices of P. So $i \in I$ gives a vertex m_i and a corresponding cone $\sigma_i = \sigma_{m_i}$ in Σ_P. Also let $U_i \subseteq \mathbb{P}^{s-1}$ be the affine open subset where the ith coordinate is nonzero. By our choice of $\mathscr{A}$, the proof of Proposition 6.1.10 shows that $\phi_{\mathscr{L},W}$ induces an isomorphism

$$U_{\sigma_i} \simeq X_{\mathscr{A}} \cap U_i.$$

Since X_P is the union of the U_{σ_i} for $i \in I$, it follows that

$$\phi_{\mathscr{L},W} : X_P \xrightarrow{\sim} X_{\mathscr{A}} \cap \textstyle\bigcup_{i\in I} U_i. \tag{7.1.7}$$

Since $X_{\mathscr{A}}$ is projective, this shows that X_P is quasiprojective. Part (b) now follows immediately from Proposition 7.0.6 since $\phi : X_P \to U_P$ is proper.

When P is not very ample, we know that a positive multiple kP is. Since P and kP have the same normal fan and same recession cone, the maps $X_P \to U_P$ and $X_{kP} \to U_{kP}$ are identical. Hence the general case follows immediately from the very ample case. □

Here are two examples to illustrate Theorem 7.1.10.

Example 7.1.11. The polyhedron P from Example 7.1.7 is very ample (in fact, it is normal), and the set $\mathscr{A}$ used in the proof of Theorem 7.1.10 can be chosen to be $\mathscr{A} = \{e_1,\dots,e_n,2e_1,\dots,2e_n\}$ (Exercise 7.1.8). This gives $X_{\mathscr{A}} \subseteq \mathbb{P}^{2n-1}$, where $\mathbb{P}^{2n-1}$ has variables $x_1,\dots,x_n,w_1,\dots,w_n$ corresponding to the elements $e_1,\dots,e_n$, $2e_1,\dots,2e_n$ of $\mathscr{A}$. Then $X_{\mathscr{A}} \subseteq \mathbb{P}^{2n-1}$ is defined by the equations $x_i^2 w_j = x_j^2 w_i$ for $1 \le i < j \le n$ (Exercise 7.1.8). Since $X_P = \mathrm{Bl}_0(\mathbb{C}^n)$ by Example 7.1.7, the isomorphism (7.1.7) implies

$$\begin{aligned}\mathrm{Bl}_0(\mathbb{C}^n) \simeq \{(x_1,\dots,x_n,w_1,\dots,w_n) \in \mathbb{P}^{2n-1} \mid (x_1,\dots,x_n) \neq (0,\dots,0)\\ \text{and } x_i^2 w_j = x_j^2 w_i \text{ for } 1 \le i < j \le n\}.\end{aligned}$$

We get a better description of $\mathrm{Bl}_0(\mathbb{C}^n)$ using the vertices $\mathscr{B} = \{e_1,\dots,e_n\}$ of P. This gives a map $X_P \longrightarrow \mathbb{P}^{n-1}$ which, when combined with $X_P \to U_P = \mathbb{C}^n$, gives a morphism

$$\Phi : X_P \longrightarrow \mathbb{P}^{n-1} \times \mathbb{C}^n.$$

Let $\mathbb{P}^{n-1}$ and $\mathbb{C}^n$ have variables $x_1,\dots,x_n$ and $y_1,\dots,y_n$ respectively. Then Φ is an embedding onto the subvariety of $\mathbb{P}^{n-1} \times \mathbb{C}^n$ defined by $x_i y_j = x_j y_i$ for $1 \le i < j \le n$ (Exercise 7.1.8). Hence

$$\mathrm{Bl}_0(\mathbb{C}^n) \simeq \{(x_1,\dots,x_n,y_1,\dots,y_n) \in \mathbb{P}^{n-1} \times \mathbb{C}^n \mid x_i y_j = x_j y_i, 1 \le i < j \le n\}.$$

This description of the blowup $\mathrm{Bl}_0(\mathbb{C}^n)$ can be found in many books on algebraic geometry and appeared earlier in this book as Exercise 3.0.8. Note also that the projective morphism of Theorem 7.1.10 is the blowdown map $\mathrm{Bl}_0(\mathbb{C}^n) \to \mathbb{C}^n$. ◊

Example 7.1.12. Consider the full dimensional lattice polyhedron $P \subseteq \mathbb{R}^2$ defined by the inequalities

$$x \le 2,\ 0 \le y \le 2, y \ge x+1.$$

This polyhedron has vertices $v_1 = (1,0), v_2 = (2,1), v_3 = (2,2)$ shown in Figure 2. The left side of the figure also shows the recession cone C and the decompostion $P = Q + C$, where Q is the convex hull of the vertices.

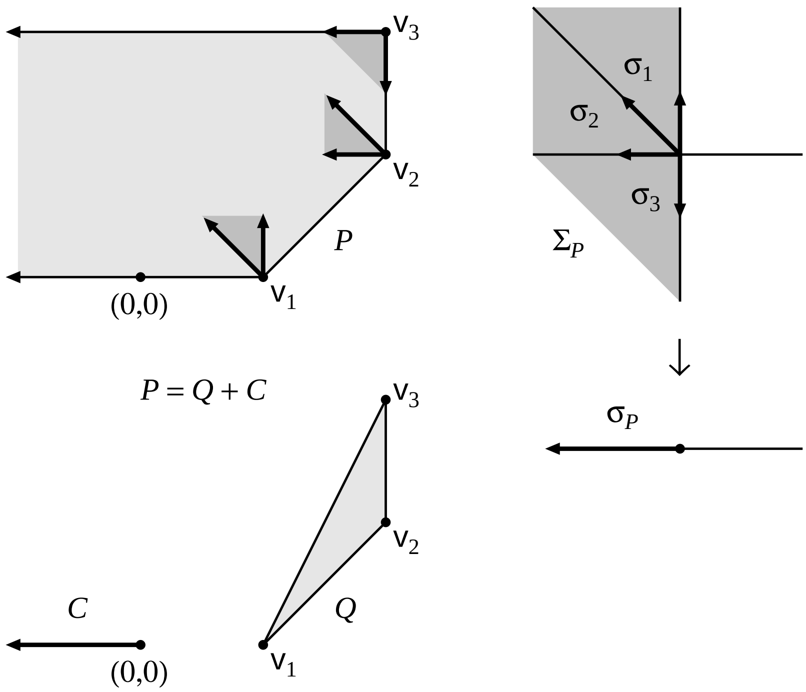


Figure 2. The polyhedron $P = Q + C$, the normal fan Σ_P, and the cone σ_P

The normal vectors at each vertex v_i are reassembled on the right to give the maximal cones σ_i of the normal fan Σ_P. Note also that $|\Sigma_P|$ is not strictly convex, so we mod out by its maximal subspace to get the strictly convex cone σ_P. The projection map on the right of Figure 2 gives the projective morphism $X_P \to U_P$, where $U_P \simeq \mathbb{C}$ is the toric variety of σ_P. $\Diamond$

The Proj Construction. We conclude this section by explaining how construct X_P using Proj. Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polyhedron. By (7.1.3), the cone $C(P) \subseteq M_{\mathbb{R}} \times \mathbb{R}$ gives the semigroup algebra

$$S_P = \mathbb{C}[C(P) \cap (M \times \mathbb{Z})].$$

We use the height grading defined by $\deg(\chi^m t^k) = k$. Then we can relate $\mathrm{Proj}(S_P)$ to X_P as follows.

Theorem 7.1.13. *There is a natural isomorphism $X_P \simeq \mathrm{Proj}(S_P)$.*

Proof. Let V be the set of vertices of P. In Exercise 7.1.9 you will prove:

- $\sqrt{\langle \chi^{\mathsf{v}} t \mid \mathsf{v} \in V\rangle} = (S_P)_+ = \bigoplus_{d>0}(S_P)_d$.
- If $\mathsf{v} \in V$, then $(S_P)_{(\chi^{\mathsf{v}} t)} = \mathbb{C}[\sigma_{\mathsf{v}}^\vee \cap M]$, where $\sigma_{\mathsf{v}} = \mathrm{Cone}(P \cap M - \mathsf{v})^\vee$.

By the first bullet, $\mathrm{Proj}(S_P)$ is covered by the affine open subsets $\mathrm{Spec}((S_P)_{(\chi^{\mathsf{v}} t)})$, and the second shows that $\mathrm{Spec}((S_P)_{(\chi^{\mathsf{v}} t)})$ is the affine toric variety of the cone σ_{v}. These patch together in the correct way to give $X_P \simeq \mathrm{Proj}(S_P)$. □

We can also interpret the morphism $\phi : X_P \to U_P$ in terms of Proj. The idea is to compute $(S_P)_0$, the degree 0 part of the graded ring $S_P = \mathbb{C}[C(P) \cap (M \times \mathbb{Z})]$. The slice of $C(P)$ at height 0 is the recession cone C of P. Recall that $N_P = N/(W \cap N)$, where $W \subseteq C^\vee$ is the largest subspace contained in $C^\vee$ and that U_P is the affine toric variety of σ_P, which is the image of $C^\vee$ in $(N_P)_\mathbb{R}$. Then the inclusion $M_P \subseteq M$ dual to $N \to N_P$ gives

$$\sigma_P^\vee = C \subseteq (M_P)_\mathbb{R} \subseteq M_\mathbb{R}.$$

It follows that $(S_P)_0 = \mathbb{C}[C \cap M] = \mathbb{C}[\sigma_P^\vee \cap M]$. This implies $\mathrm{Spec}((S_P)_0) = U_P$, so that $\mathrm{Proj}(S_P) \to \mathrm{Spec}((S_P)_0)$ becomes $\phi : X_P \to U_P$. It follows that ϕ is projective by Proposition 7.0.9. This gives a second proof of Theorem 7.1.10.

It is also possible to prove directly that $\mathrm{Proj}(S_P) \to \mathrm{Spec}((S_P)_0)$ is projective without using Proposition 7.0.9. See Exercise 7.1.10.

Exercises for §7.1.

7.1.1. Prove (7.1.1). Hint: Fix $m_0 \in P$ and take any $m \in C$. Show that $m_0 + \lambda m \in P$ for $\lambda > 0$, so $\langle m_0 + \lambda m, u_i\rangle \geq -a_i$. Then divide by λ and let $\lambda \to \infty$.

7.1.2. Let $P = Q + C$ be a polyhedron in $M_\mathbb{R}$ where Q is a polytope and C is a polyhedral cone. Define $\varphi_P(u) = \min_{m \in P}\langle m, u\rangle$ for $u \in C^\vee$.

(a) Show that $\varphi_P(u) = \min_{m \in Q}\langle m, u\rangle$ for $u \in C^\vee$ and conclude that $\varphi_P : C^\vee \to \mathbb{R}$ is well-defined.

(b) Show that $\varphi_P(u) = \min_{\mathsf{v} \in V_Q}\langle \mathsf{v}, u\rangle$ for $u \in C^\vee$, where V_Q is the set of vertices of Q.

(c) Show that $P = \{m \in M_\mathbb{R} \mid \varphi_P(u) \leq \langle m, u\rangle \text{ for all } u \in C^\vee\}$. Hint: For the non-obvious direction, represent P as the intersection of closed half-spaces coming from supporting hyperplanes.

7.1.3. Let P be a polyhedron in $M_\mathbb{R}$ with recession cone C and let $W = C \cap (-C)$ be the largest subspace contained in C. Prove that every face of P contains a translate of W and conclude that P has no vertices when C is not strongly convex.

7.1.4. Let $P = Q + C$ be a polyhedron in $M_\mathbb{R}$ where Q is a polytope and C is a strongly convex polyhedral cone. Let V_Q be the set of vertices of Q. Assume that there is $\mathsf{v} \in V_Q$ and u in the interior of $C^\vee$ such that $\langle \mathsf{v}, u\rangle < \langle \mathsf{w}, u\rangle$ for all $\mathsf{w} \neq \mathsf{v}$ in V_Q. Prove that v is a vertex of P. Hint: Show that $H_{u,a}$, $a = \langle \mathsf{v}, u\rangle$, is a supporting hyperplane of P such that $H_{u,a} \cap P = \mathsf{v}$. Also show if v and u satisfy the hypothesis of the problem, then so do v and u' for any u' sufficiently close to u. Finally, Exercise 7.1.2 will be useful.

7.1.5. Let $P = Q + C$ be a polyhedron in $M_\mathbb{R}$ where Q is a polytope and C be a strongly convex polyhedral cone. Let V_Q be the set of vertices of Q and let

$$U_0 = \{u \in \mathrm{Int}(C^\vee) \mid \langle \mathsf{v}, u\rangle \neq \langle \mathsf{w}, u\rangle \text{ whenever } \mathsf{v} \neq \mathsf{w} \text{ in } V_Q\}.$$

(a) Show that U_0 is open and dense in $C^\vee$. Hint: $\dim C^\vee = \dim N_\mathbb{R}$.

(b) Use Exercise 7.1.4 to show that for every $u \in U_0$, there is a vertex v of P such that $\varphi_P(u) = \langle \mathsf{v}, u\rangle$. Conclude that the set V_P of vertices of P is nonempty and finite.

(c) Show that $\varphi_P(u) = \min_{\mathsf{v}\in V_P} \langle \mathsf{v}, u\rangle$ for $u \in C^\vee$.

(d) Conclude that $P = \mathrm{Conv}(V_P) + C$. Hint: The first step is to show that $\varphi_P = \varphi_{P'}$, where $P' = \mathrm{Conv}(V_P) + C$. Then use part (c) of Exercise 7.1.2.

7.1.6. Let $C \subseteq N_\mathbb{R}$ be a polyhedral cone. Set $W = C \cap (-C)$ and let $\sigma = \gamma(C) \subset N_\mathbb{R}/W$ for the quotient map $\gamma : N_\mathbb{R} \to N_\mathbb{R}/W$. Show that σ is strongly convex and $C = \gamma^{-1}(\sigma)$.

7.1.7. Prove (7.1.6). Hint: Given $\sum_{i=1}^s \lambda_i m_i \in C$, let $m = \sum_{i=1}^s \lfloor \lambda_i \rfloor m_i \in C \cap M$.

7.1.8. Prove the claims made in Example 7.1.11.

7.1.9. Supply proofs of the two bullets from the proof of Theorem 7.1.13.

7.1.10. Here you will give an elementary proof that $\mathrm{Proj}(S_P) \to \mathrm{Spec}((S_P)_0)$ is projective, where S_P is the graded semigroup algebra from (7.1.3).

(a) Explain why we can assume that P is normal.

(b) Show that $C(P) \cap (M \times \mathbb{Z})$ is generated by its elements of height ≤ 1 when P is normal.

(c) Assume P is normal and let $\mathscr{H}$ be a Hilbert basis of $C(P) \cap (M \times \mathbb{Z})$. Then $\mathscr{H} = \mathscr{H}_0 \cup \mathscr{H}_1$, where elements of $\mathscr{H}_i$ have height i and write $\mathscr{H}_1 = \{(m_1, 1), \ldots, (m_s, 1)\}$. Prove that S_P is generated as an $(S_P)_0$-algebra by $\chi^{m_1} t, \ldots, \chi^{m_s} t$ and conclude that there is a surjective homomorphism of graded rings

$$(S_P)_0[x_1, \ldots, x_s] \longrightarrow S_P, \qquad x_i \longmapsto \chi^{m_i} t.$$

(d) Prove that when P is normal, there is a commutative diagram

$$\begin{tikzcd} X_P \arrow[r, "\alpha"] \arrow[dr, "\phi"'] & U_P \times \mathbb{P}^{s-1} \arrow[d, "p_1"] \\ & U_P \end{tikzcd}$$

such that α is a closed embedding and ϕ is a projective morphism.

7.1.11. In this exercise, you will prove a stronger version of part (b) of Theorem 7.1.10. Let $X_\mathscr{A}$ and W be as in the proof of the theorem. Prove that there is a commutative diagram

$$\begin{tikzcd} X_P \arrow[r, "\phi \times \phi_{\mathscr{L},W}"] \arrow[dr, "\phi"'] & U_P \times \mathbb{P}(W^\vee) \arrow[d, "p_1"] \\ & U_P \end{tikzcd} \tag{7.1.8}$$

such that $\phi \times \phi_{\mathscr{L},W} : X_P \to U_P \times \mathbb{P}(W^\vee)$ is a closed embedding. Hint: Proposition 7.0.6.

7.1.12. Let $\sigma \subseteq N_\mathbb{R}$ be a strongly convex rational polyhedral cone. This gives the semigroup algebra $\mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap M]$. Given a monomial ideal $\mathfrak{a} = \langle \chi^{m_1}, \ldots, \chi^{m_s}\rangle \subseteq \mathbb{C}[S_\sigma]$, we get the polyhedron

$$P = \mathrm{Conv}(m \in M \mid \chi^m \in \mathfrak{a}),$$

Prove that $P = \mathrm{Conv}(m_1, \ldots, m_s) + \sigma^\vee$.

§7.2. Projective Morphisms and Toric Varieties

We begin our study of projective toric morphisms with a toric variety X_Σ whose fan Σ has full dimensional convex support. We construct an affine toric variety from $|\Sigma|$ as follows. The largest subspace contained in $|\Sigma|$ is $W = |\Sigma| \cap (-|\Sigma|)$. Similar to §7.1, we have:

- The sublattice $W \cap N \subseteq N$ and the quotient lattice $N_\Sigma = N/(W \cap N)$.
- The strongly convex cone $\sigma_\Sigma = |\Sigma|/W \subseteq N_\mathbb{R}/W = (N_\Sigma)_\mathbb{R}$.
- The affine toric variety $U_\Sigma = U_{\sigma_\Sigma}$.

The projection map $\overline{\phi} : N \to N_\Sigma$ is compatible with the fans of X_Σ and U_Σ since $\overline{\phi}_\mathbb{R}(|\Sigma|) = \sigma_\Sigma$. This gives a toric morphism

$$\phi : X_\Sigma \longrightarrow U_\Sigma. \tag{7.2.1}$$

which as in §7.1 is easily seen to be proper. The difference between here and §7.1 is that $\phi : X_\Sigma \to U_\Sigma$ may fail to be projective. Our first goal is to understand when this happens. As you might suspect, the answer involves polyhedra, support functions, and convexity.

The Polyhedron of a Divisor. A Weil divisor $D = \sum_\rho a_\rho D_\rho$ on X_Σ gives the polyhedron

$$P_D = \{m \in M_\mathbb{R} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho\}.$$

When Σ is complete, this is a polytope, but in general we have

$$P_D = Q + C,$$

where Q is a polytope and C is the recession cone of P_D.

Lemma 7.2.1. *Assume $|\Sigma|$ is convex of full dimension and let $D = \sum_\rho a_\rho D_\rho$ be a Weil divisor on X_Σ. If $P_D \neq \emptyset$, then:*

(a) *The recession cone of P_D is $|\Sigma|^\vee$.*

(b) *The set $V = \{\mathsf{v} \in P_D \mid \mathsf{v} \text{ is a vertex}\}$ is nonempty and finite.*

(c) $P_D = \mathrm{Conv}(V) + |\Sigma|^\vee$.

Proof. Combining (7.1.1) with the definition of P_D, we see that the recession cone of P_D is

$$\{m \in M_\mathbb{R} \mid \langle m, u_\rho \rangle \geq 0 \text{ for all } \rho\} = |\Sigma|^\vee$$

since $|\Sigma| = \mathrm{Cone}(u_\rho \mid \rho \in \Sigma(1))$ by (6.1.3). This proves part (a). The recession cone is strongly convex since $|\Sigma|$ has full dimension, so that parts (b) and (c) follow from Lemma 7.1.1. $\square$

Divisors and Convexity. Now that we know about recession cones, the convexity result proved in Theorem 6.1.7 can be improved as follows.

Theorem 7.2.2. *Assume $|\Sigma|$ is convex of full dimension n and let φ_D be the support function of a Cartier divisor D on X_Σ. Then the following are equivalent:*

(a) *D is basepoint free.*

(b) *$m_\sigma \in P_D$ for all $\sigma \in \Sigma(n)$.*

(c) *$\varphi_D(u) = \min_{\sigma\in\Sigma(n)}\langle m_\sigma, u\rangle$ for all $u \in |\Sigma|$.*

(d) *$\varphi_D : |\Sigma| \to \mathbb{R}$ is convex.*

(e) *$P_D = \mathrm{Conv}(m_\sigma \mid \sigma \in \Sigma(n)) + |\Sigma|^\vee$.*

(f) *$\{m_\sigma \mid \sigma \in \Sigma(n)\}$ is the set of vertices of P_D.*

(g) *$\varphi_D(u) = \min_{m\in P_D}\langle m, u\rangle$ for all $u \in |\Sigma|$.*

In particular, P_D is a lattice polyhedron when D is basepoint free.

Proof. Parts (a)–(d) are equivalent by Theorem 6.1.7. Furthermore, (b) $\Rightarrow$ (f) and (b) $\Rightarrow$ (g) follow as in the proof of Theorem 6.1.7, and (f) $\Rightarrow$ (e) follows from Lemma 7.2.1. Also, (e) $\Rightarrow$ (b) is obvious. Finally, (g) $\Rightarrow$ (d) follows from part (b) of Exercise 7.1.2. $\square$

Strict Convexity. Our next task is to show that $\phi : X_\Sigma \to U_\Sigma$ is projective if and only if X_Σ has a Cartier divisor D with strictly convex support function. We continue to assume that Σ has full dimensional convex support. As in §6.1, φ_D is strictly convex if it is convex and for each $\sigma \in \Sigma(n)$, the equation $\varphi_D(u) = \langle m_\sigma, u\rangle$ holds only on σ. The strict convexity criteria from Lemma 6.1.13 apply to this situation.

When φ_D is strictly convex, the polyhedron P_D has an especially nice relation to the fan Σ.

Proposition 7.2.3. *Assume that $|\Sigma|$ is convex of full dimension and $D = \sum_\rho a_\rho D_\rho$ has a strictly convex support function. Then:*

(a) *P_D is a full dimensional lattice polyhedron.*

(b) *Σ is the normal fan of P_D.*

Proof. Theorem 7.2.2 and Lemma 6.1.13 imply that the m_σ, $\sigma \in \Sigma(n)$, are distinct and give the vertices of the polyhedron. As in §7.1, a vertex $m_\sigma \in P_D$ gives the cone $C_{m_\sigma} = \mathrm{Cone}(P_D \cap M - m_\sigma)$. We claim that

$$\sigma = C_{m_\sigma}^\vee.$$

This easily implies that P_D has full dimension and that Σ is the normal fan of P_D.

Fix $\sigma \in \Sigma(n)$ and let $\rho \in \sigma(1)$. Then $m \in P_D \cap M$ implies

$$\langle m, u_\rho\rangle \geq \varphi_D(u_\rho) = \langle m_\sigma, u_\rho\rangle, \tag{7.2.2}$$

where the inequality holds by Lemma 6.1.6 and the equality holds since $u_\rho \in \sigma$. Thus $\langle m - m_\sigma, u_\rho\rangle \geq 0$ for all $m \in P_D \cap M$, so that $u_\rho \in C_{m_\sigma}^\vee$ for all $\rho \in \sigma(1)$. Hence

$$\sigma \subseteq C_{m_\sigma}^\vee.$$

Since $|\Sigma|^\vee$ is the recession cone of P_D, the proof of Theorem 7.1.6 implies

$$C_{m_\sigma}^\vee \subseteq |\Sigma| = \textstyle\bigcup_{\sigma \in \Sigma(n)} \sigma.$$

Now take $u \in \mathrm{Int}(C_{m_\sigma}^\vee)$. Hence $u \in \sigma'$ for some $\sigma' \in \Sigma(n)$. Then $u \in C_{m_\sigma}^\vee$ and $m_{\sigma'} - m_\sigma \in C_{m_\sigma}$ imply

$$\langle m_{\sigma'} - m_\sigma, u\rangle \geq 0, \text{ so } \langle m_{\sigma'}, u\rangle \geq \langle m_\sigma, u\rangle.$$

On the other hand, if we apply (7.2.2) to the cone σ' and $m = m_\sigma$, we obtain $\langle m_\sigma, u_\rho\rangle \geq \langle m_{\sigma'}, u_\rho\rangle$. We conclude that

$$\langle m_\sigma, u\rangle = \langle m_{\sigma'}, u\rangle,$$

and the same equality holds for all elements of $\mathrm{Int}(C_{m_\sigma}^\vee) \cap \sigma'$. This easily implies that $m_\sigma = m_{\sigma'}$. Then $\sigma = \sigma'$ by strict convexity, so that $u \in \sigma$. $\square$

Here is the first major result of this section.

Theorem 7.2.4. *Let $\phi : X_\Sigma \to U_\sigma$ be the proper toric morphism where U_σ is affine. Then $|\Sigma|$ is convex. Furthermore, the following are equivalent:*

(a) *X_Σ is quasiprojective.*

(b) *ϕ is a projective morphism.*

(c) *X_Σ has a torus-invariant Cartier divisor with strictly convex support function.*

Proof. Since ϕ is proper, Theorem 3.4.11 implies that $|\Sigma| = \overline{\phi}_{\mathbb{R}}^{-1}(\sigma)$. Thus $|\Sigma|$ is convex. To prove (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c), first assume that $|\Sigma|$ has full dimension.

If (c) is true, then Σ is the normal fan of the full dimensional lattice polyhedron P_D by Proposition 7.2.3. It follows that $X_\Sigma = X_{P_D}$, which is quasiprojective by Theorem 7.1.10, proving (a). Furthermore, (a) $\Rightarrow$ (b) by Proposition 7.0.6.

If (b) is true, we will use the theory of ampleness developed in [**127**]. The essential facts we need are summarized in the appendix to this chapter. Since ϕ is projective, there is a line bundle $\mathscr{L}$ on X_Σ that satisfies Definition 7.0.3. Then, since U_σ is affine, Theorem 7.A.4 and Proposition 7.A.6 imply that

- $\mathscr{L}^{\otimes k} = \mathscr{L} \otimes_{\mathscr{O}_{X_\Sigma}} \cdots \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{L}$ (k times) is generated by global sections for some integer $k > 0$.
- The nonvanishing set of a global section of $\mathscr{L}$ is an affine open subset of X_Σ.

By §7.0, $\mathscr{L} \simeq \mathscr{O}_{X_\Sigma}(D)$ for some Cartier divisor on X, and since linearly equivalent Cartier divisors give isomorphic line bundles, we may assume that D is torus-invariant (this follows from Theorem 4.2.1). Then $\mathscr{O}_{X_\Sigma}(kD)$ is generated by global sections for some $k > 0$. This implies that $\varphi_{kD} = k\varphi_D$ is convex by Theorem 7.2.2, so that φ_D is convex as well. We show that φ_D is strictly convex by contradiction.

If strict convexity fails, then Lemma 6.1.13 implies that there is a wall $\tau = \sigma \cap \sigma'$ in Σ with $m_\sigma = m'_\sigma$. Then $m = m_\sigma = m'_\sigma$ corresponds to a global section s, which by the proof of Proposition 6.1.1 is nonvanishing on U_σ (since $m = m_\sigma$) and on $U_{\sigma'}$ (since $m = m_{\sigma'}$). Thus the nonvanishing set contains $U_\sigma \cup U_{\sigma'}$, which contains the complete curve $V(\tau) \subseteq U_\sigma \cup U_{\sigma'}$. But being affine, the nonvanishing set cannot contain a complete curve (Exercise 7.2.1). This completes the proof of the theorem when $|\Sigma|$ has full dimension.

Finally, suppose that $|\Sigma|$ fails to have full dimension. Let $N_1 = \mathrm{Span}(|\Sigma|) \cap N$ and pick $N_0 \subseteq N$ such that $N = N_0 \oplus N_1$. The cones of Σ lie in $(N_1)_\mathbb{R}$ and hence give a fan Σ_1 in $(N_1)_\mathbb{R}$. If N_0 has rank r, then Proposition 3.3.11 implies that

$$X_\Sigma \simeq (\mathbb{C}^*)^r \times X_{\Sigma_1}. \tag{7.2.3}$$

It follows that $\varphi_D : |\Sigma| = |\Sigma_1| \to \mathbb{R}$ is the support function of a Cartier divisor D_1 on X_{Σ_1}. Note also that $|\Sigma_1|$ is convex of full dimension in $(N_1)_\mathbb{R}$. Since $(\mathbb{C}^*)^r$ is quasiprojective, this allows us to reduce to the case of full dimensional support. You will supply the omitted details in Exercise 7.2.2. □

f-Ample and f-Very Ample Divisors. The definitions of ample and very ample from §6.1 generalize to the relative setting as follows. Recall from Definition 7.0.3 that a morphism $f : X \to Y$ is projective with respect to the line bundle $\mathscr{L}$ when for a suitable open cover $\{U_i\}$ of Y, we can find global sections $s_0, \ldots, s_{k_i}$ of $\mathscr{L}$ over $f^{-1}(U_i)$ that give a closed embedding

$$f^{-1}(U_i) \longrightarrow U_i \times \mathbb{P}^{k_i}.$$

Then we have the following definition.

Definition 7.2.5. Let D be a Cartier divisor on X and $f : X \to Y$ be proper.

(a) The divisor D and the line bundle $\mathscr{O}_X(D)$ are ***f-very ample*** if f is projective with respect to the line bundle $\mathscr{L} = \mathscr{O}_X(D)$.

(b) D and $\mathscr{O}_X(D)$ are ***f-ample*** when kD is f-very ample for some integer $k > 0$.

Hence $f : X \to Y$ is projective if and only if X has an f-ample line bundle. In the toric case, Proposition 7.1.9 and Theorem 7.2.4 give the following result.

Theorem 7.2.6. *Let $\phi : X_\Sigma \to U_\sigma$ be a proper toric morphism where U_σ is affine, and let $D = \sum_\rho a_\rho D_\rho$ be a Cartier divisor on X_Σ. Also let $n = \dim X_\Sigma$. Then:*

(a) *D is ϕ-ample if and only if φ_D is strictly convex.*

(b) *If $n \geq 2$ and D is ϕ-ample, then kD is ϕ-very ample for all $k \geq n$.* □

Here is are two examples of Theorem 7.2.6.

Example 7.2.7. Consider the blowdown morphism $\phi : \mathrm{Bl}_0(\mathbb{C}^n) \to \mathbb{C}^n$. The fan for $\mathrm{Bl}_0(\mathbb{C}^n)$ has minimal generators $u_0 = e_1 + \cdots + e_n$ and $u_i = e_i$ for $1 \leq i \leq n$. Let D_0 be the divisor corresponding to u_0. The support function φ_{-D_0} of $-D_0$ is easily seen to be strictly convex (Exercise 7.2.4). Thus $-D_0$ is ϕ-ample by Theorem 7.2.6. Note also that the polyhedron P_{-D_0} is the polyhedron P from Example 7.1.7. $\Diamond$

Example 7.2.8. The P be a full dimensional lattice polyhedron in $M_{\mathbb{R}}$. The map $\phi : X_P \to U_P$ is projective by Theorem 7.1.10. We also have the Cartier divisor D_P on X_P defined in (7.1.4). As noted in the discussion following (7.1.4), $P = P_{D_P}$ and the vertices of P give the Cartier data of D_P, so that φ_{D_P} is strictly convex by Theorem 7.2.2 and Lemma 6.1.13. Hence D_P is ϕ-ample by Theorem 7.2.6. $\Diamond$

Semiprojective Toric Varieties. Following [**137**], we say that X_Σ is *semiprojective* if the natural map $\phi : X_\Sigma \to \mathrm{Spec}(\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}))$ is projective and X_Σ has a torus fixed point. We can characterize semiprojective toric varieties as follows.

Proposition 7.2.9. *Given a toric variety X_Σ, the following are equivalent:*

(a) *X_Σ is semiprojective.*

(b) *X_Σ is quasiprojective and Σ has full dimensional convex support in $N_{\mathbb{R}}$.*

(c) *$X_\Sigma = X_P$ is the toric variety of a full dimensional lattice polyhedron $P \subseteq M_{\mathbb{R}}$.*

Proof. By the Orbit-Cone Correspondence (Theorem 3.2.6), X_Σ has a torus fixed point if and only if Σ has a full dimensional cone, which is equivalent to Σ having full dimensional support. In Exercise 7.2.3 you will show that $\mathrm{Spec}(\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}))$ is a normal affine toric variety U_σ. Then (a) $\Leftrightarrow$ (b) follows from Theorem 7.2.4.

The equivalence (b) $\Leftrightarrow$ (c) follows from Proposition 7.2.3 and Theorem 7.2.4. This completes the proof. $\square$

A semiprojective toric X_Σ variety comes equipped with a projective morphism $\phi : X_\Sigma \to \mathrm{Spec}(\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}))$, and a full dimensional lattice polyhedron P comes with a projective morphism $\phi : X_P \to U_P$ by Theorem 7.1.10. These maps are the same by Exercise 7.2.3.

We can also extend the relation between polytopes and ample divisors on complete toric varieties described in §6.2. Consider the set of polyhedra

$$\big\{P \subseteq M_{\mathbb{R}} \mid P \text{ is a full dimensional lattice polytope}\big\}$$

and the set of pairs

$$\begin{aligned}\big\{(X_\Sigma, D) \mid \Sigma \text{ is a fan in } N_{\mathbb{R}},\ X_\Sigma \text{ is semiprojective, and}\\ D \text{ is a torus-invariant } \phi\text{-ample divisor on } X_\Sigma\big\}.\end{aligned}$$

These sets are related as follows.

Theorem 7.2.10. *The maps $P \mapsto (X_P, D_P)$ and $(X_\Sigma, D) \mapsto P_D$ define bijections between the above sets that are inverses of each other.*

Proof. First note that X_P is semiprojective by Proposition 7.2.9, and D_P is ϕ-ample by Example 7.2.8. Going the other way, suppose that X_Σ is semiprojective and D is ϕ-ample. Then Theorem 7.2.6 and Proposition 7.2.9 imply that D_P has a strictly convex support function, so that P_D is a full dimensional lattice polyhedron by Proposition 7.2.3.

Using Proposition 7.2.3 and $P = P_{D_P}$, it is easy to see that the two maps are inverses of each other. □

Projective Toric Morphisms. Suppose we have fans Σ in $N_\mathbb{R}$ and Σ' in $N'_\mathbb{R}$. Recall from §3.3 that a toric morphism

$$\phi : X_\Sigma \longrightarrow X_{\Sigma'}$$

is induced from a map of lattices

$$\overline{\phi} : N \longrightarrow N'$$

compatible with Σ and Σ', i.e., for each $\sigma \in \Sigma$ there is $\sigma' \in \Sigma'$ with $\overline{\phi}_\mathbb{R}(\sigma) \subseteq \sigma'$.

We first determine when a torus-invariant Cartier divisor on X_Σ is ϕ-ample. Since projective morphisms are proper, we can assume that ϕ is proper, which by Theorem 3.4.11 is equivalent to

$$|\Sigma| = \overline{\phi}_\mathbb{R}^{-1}(|\Sigma'|). \tag{7.2.4}$$

Here is our result.

Theorem 7.2.11. *Let $\phi : X_\Sigma \to X_{\Sigma'}$ be a proper toric morphism and let $D = \sum_\rho a_\rho D_\rho$ be a Cartier divisor on X_Σ. Also let $n = \dim X_\Sigma$. Then:*

(a) *D is ϕ-ample if and only if for every $\sigma' \in \Sigma'$, φ_D is strictly convex on $\overline{\phi}_\mathbb{R}^{-1}(\sigma')$.*

(b) *If $n \geq 2$ and D is ϕ-ample, then kD is ϕ-very ample for all $k \geq n-1$.*

Proof. The idea is to study what happens over the affine open subsets $U_{\sigma'} \subseteq X_{\Sigma'}$ for $\sigma' \in \Sigma'$. Observe that $\phi^{-1}(U_{\sigma'})$ is the toric variety corresponding to the fan

$$\Sigma_{\sigma'} = \{\sigma \in \Sigma \mid \overline{\phi}_\mathbb{R}(\sigma) \subseteq \sigma'\}.$$

Thus $\phi^{-1}(U_{\sigma'}) = X_{\Sigma_{\sigma'}}$. Let $\phi_{\sigma'} = \phi|_{\phi^{-1}(U_{\sigma'})}$ and consider the diagram

$$\begin{array}{ccc}
X_\Sigma & \xrightarrow{\ \phi\ } & X_{\Sigma'} \\
\uparrow & & \uparrow \\
\phi^{-1}(U_{\sigma'}) & \xrightarrow{\ \phi_{\sigma'}\ } & U_{\sigma'} \\
\| & & \| \\
X_{\Sigma_{\sigma'}} & \xrightarrow{\ \phi_{\sigma'}\ } & U_{\sigma'}.
\end{array}$$

Also let $D_{\sigma'}$ be the restriction of D to $\phi^{-1}(U_{\sigma'}) = X_{\Sigma_{\sigma'}}$.

By Proposition 7.A.5, D is ϕ-ample if and only if the restriction $D|_{\phi^{-1}(U_{\sigma'})}$ is $\phi|_{\phi^{-1}(U_{\sigma'})}$-ample for all $\sigma' \in \Sigma'$. Using the above notation, this becomes

$$D \text{ is } \phi\text{-ample} \iff D_{\sigma'} \text{ is } \phi_{\sigma'}\text{-ample for all } \sigma' \in \Sigma'.$$

However, Theorem 7.2.6 implies that

$$D_{\sigma'} \text{ is } \phi_{\sigma'}\text{-ample} \iff \varphi_{D_{\sigma'}} \text{ is strictly convex}.$$

This completes the proof of the theorem. □

It is now easy to characterize when a toric morphism is projective.

Theorem 7.2.12. *Let $\phi : X_\Sigma \to X_{\Sigma'}$ be a toric morphism. Then the following are equivalent:*

(a) *ϕ is projective.*

(b) *ϕ is proper and X_Σ has a torus-invariant Cartier divisor D whose support function φ_D is strictly convex on $\overline{\phi}_{\mathbb{R}}^{-1}(\sigma')$ for all $\sigma' \in \Sigma'$.* □

You will prove Theorem 7.2.12 in Exercise 7.2.5. The first proof of this theorem was given in [**172**, Thm. 13 of Ch. I]. In Chapter 11 we will use this result to construct interesting examples of projective toric morphisms.

Exercises for §7.2.

7.2.1. Prove that an affine variety cannot contain a complete variety of positive dimension. Hint: If X is complete and irreducible, then $\Gamma(X, \mathscr{O}_X) = \mathbb{C}$.

7.2.2. This exercise will complete the proof of Theorem 7.2.4. Let $\phi : X_\Sigma \to U_\sigma$ satisfy the hypothesis of the theorem and write X_Σ as in (7.2.3). We also have the Cartier divisors D on X_Σ and D_1 on X_{Σ_1} as in the proof of the theorem.

(a) Assume that ϕ is projective. Prove that X_Σ is quasiprojective and conclude that X_{Σ_1} is quasiprojective. Now use the first part of the proof to show that φ_D is strictly convex. Hint: See Exercise 7.0.1.

(b) Assume that φ_D is strictly convex. Prove that X_{Σ_1} is quasiprojective and conclude that X_Σ is quasiprojective. Then use Proposition 7.0.6.

7.2.3. Given a toric variety X_Σ, let $C = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq 0 \text{ for all } \rho \in \Sigma(1)\}$, and let σ be the strongly convex cone obtained by taking the quotient of $C^\vee$ by its minimal face. Prove that $U_\sigma \simeq \operatorname{Spec}(\Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}))$.

7.2.4. Prove that the support function φ_{-D_0} in Example 7.2.7 is strictly convex. We will generalize this result considerably in Chapter 11.

7.2.5. Prove Theorem 7.2.12.

§7.3. Projective Bundles and Toric Varieties

Given a vector bundle or projective bundle over a toric variety, the nicest case is when the bundle is also a toric variety. This will lead to some lovely examples of toric varieties.

Toric Vector Bundles and Cartier Divisors. A Cartier divisor $D = \sum_\rho a_\rho D_\rho$ on a toric variety X_Σ gives the line bundle $\mathscr{L} = \mathscr{O}_{X_\Sigma}(D)$, which is the sheaf of sections of the rank 1 vector bundle $\pi : V_\mathscr{L} \to X_\Sigma$.

We will show that $V_\mathscr{L}$ is a toric variety and π is a toric morphism by constructing the fan of $V_\mathscr{L}$ in terms of Σ and D. To motivate our construction, recall that for $m \in M$, we have

$$\begin{aligned} \chi^m \in \Gamma(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) &\iff m \in P_D \\ &\iff \langle m,u\rangle \geq \varphi_D(u) \text{ for all } u \in |\Sigma| \\ &\iff \text{the graph of } u \mapsto \langle m,u\rangle \text{ lies} \\ &\qquad \textit{above} \text{ the graph of } \varphi_D. \end{aligned}$$

The first equivalence follows from Proposition 4.3.3 and the second from Proposition 6.1.6. The key word is "above": it tells us to focus on the part of $N_\mathbb{R} \times \mathbb{R}$ that lies above the graph of φ_D.

We define the fan $\Sigma \times D$ in $N_\mathbb{R} \times \mathbb{R}$ as follows. Given $\sigma \in \Sigma$, set

$$\begin{aligned} \widetilde{\sigma} &= \{(u,\lambda) \mid u \in \sigma,\ \lambda \geq \varphi_D(u)\} \\ &= \mathrm{Cone}((0,1),(u_\rho,-a_\rho) \mid \rho \in \sigma(1)), \end{aligned}$$

where the second equality follows since $\varphi_D(u_\rho) = -a_\rho$ and φ_D is linear on σ. Note that $\widetilde{\sigma}$ is a strongly convex rational polyhedral cone in $N_\mathbb{R} \times \mathbb{R}$. Then let $\Sigma \times D$ be the set consisting of the cones $\widetilde{\sigma}$ for $\sigma \in \Sigma$ and their faces. This is a fan in $N_\mathbb{R} \times \mathbb{R}$, and the projection $\overline{\pi} : N \times \mathbb{Z} \to N$ is clearly compatible with $\Sigma \times D$ and Σ. Hence we get a toric morphism

$$\pi : X_{\Sigma\times D} \longrightarrow X_\Sigma.$$

Proposition 7.3.1. *$\pi : X_{\Sigma\times D} \to X_\Sigma$ is a rank 1 vector bundle whose sheaf of sections is $\mathscr{O}_{X_\Sigma}(D)$.*

Proof. We first show that π is a toric fibration as in Theorem 3.3.19. The kernel of $\overline{\pi} : N \times \mathbb{Z} \to N$ is $N_0 = \{0\} \times \mathbb{Z}$, and the fan $\Sigma_0 = \{\sigma \in \Sigma \times D \mid \sigma \subseteq (N_0)_\mathbb{R}\}$ has $\sigma_0 = \mathrm{Cone}((0,1))$ as its unique maximal cone. Also, for $\sigma \in \Sigma$ let

$$\widehat{\sigma} = \mathrm{Cone}((u_\rho,-a_\rho) \mid \rho \in \sigma(1)).$$

This is the face of $\widetilde{\sigma}$ consisting of points (u,λ) where $\varphi_D(u) = \lambda$. Thus $\widehat{\sigma} \in \Sigma \times D$ and in fact $\widehat{\Sigma} = \{\widehat{\sigma} \mid \sigma \in \Sigma\}$ is a subfan of $\Sigma \times D$. Since $\widetilde{\sigma} = \widehat{\sigma} + \sigma_0$ and $\overline{\pi}_\mathbb{R}$

maps $\widehat{\sigma}$ bijectively to σ, we see that $\Sigma \times D$ is split by Σ and Σ_0 in the sense of Definition 3.3.18. Since $X_{\Sigma_0,N_0} = \mathbb{C}$, Theorem 3.3.19 implies that

$$\pi^{-1}(U_\sigma) \simeq U_\sigma \times \mathbb{C}.$$

To see that this gives the desired vector bundle, we study the transition functions. First note that $\pi^{-1}(U_\sigma) = U_{\widetilde{\sigma}}$, so that the above isomorphism is

$$U_{\widetilde{\sigma}} \simeq U_\sigma \times \mathbb{C},$$

which by projection induces a map $U_{\widetilde{\sigma}} \to \mathbb{C}$. It is easy to check that this map is $\chi^{(-m_\sigma,1)}$, where $\varphi_D(u) = \langle m_\sigma, u\rangle$ for $u \in \sigma$ (Exercise 7.3.1). Note that

$$(-m_\sigma, 1) \in \widetilde{\sigma}^\vee \cap (M \times \mathbb{Z}),$$

follows directly from the definition of $\widetilde{\sigma}$. Then, given another cone $\tau \in \Sigma$, the transition map from $U_{\sigma\cap\tau} \times \mathbb{C} \subseteq U_\tau \times \mathbb{C}$ to $U_{\sigma\cap\tau} \times \mathbb{C} \subseteq U_\sigma \times \mathbb{C}$ is given by $(u,t) \mapsto (u, g_{\sigma\tau}(u)t)$, where $g_{\sigma\tau} = \chi^{m_\tau - m_\sigma}$ (Exercise 7.3.1).

We are now done, since the proof of Proposition 6.2.7 shows that $\mathscr{O}_{X_\Sigma}(D)$ is the sheaf of sections of a rank 1 vector bundle over X_Σ whose transition functions are $g_{\sigma\tau} = \chi^{m_\tau - m_\sigma}$. $\square$

This construction is easy but leads to some surprisingly rich examples.

Example 7.3.2. Consider $\mathbb{P}^n$ with its usual fan and let D_0 correspond to the minimal generator $u_0 = -e_1 - \cdots - e_n$. Recall that $\mathscr{O}_{\mathbb{P}^n}(-D_0)$ is denoted $\mathscr{O}_{\mathbb{P}^n}(-1)$. This gives the rank 1 vector bundle $V \to \mathbb{P}^n$ described in Proposition 7.3.1 whose fan Σ in $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ has minimal generators

$$e_1, \ldots, e_{n+1}, -e_1 - \cdots - e_n + e_{n+1}.$$

You will check this in Exercise 7.3.2.

We can also describe this vector bundle geometrically as follows. Consider the lattice polyhedron in $\mathbb{R}^{n+1}$ given by

$$P = \mathrm{Conv}(0, e_1, \ldots, e_n) + \mathrm{Cone}(e_{n+1}, e_1 + e_{n+1}, \ldots, e_n + e_{n+1}).$$

The normal fan of P is the fan Σ (Exercise 7.3.2), so that X_P is the above vector bundle V. Note also that $|\Sigma|$ is dual to the recession cone of P.

It is easy to see that $|\Sigma|$ is a smooth cone of dimension $n+1$, so that the projective toric morphism $X_P \to U_P$ constructed in §7.1 becomes $X_P \to \mathbb{C}^{n+1}$. When combined with the vector bundle map $X_P = V \to \mathbb{P}^n$, we get a morphism

$$X_P \longrightarrow \mathbb{P}^n \times \mathbb{C}^{n+1}.$$

When the coordinates of $\mathbb{P}^n$ and $\mathbb{C}^{n+1}$ are ordered correctly, the image is precisely the variety defined by $x_i y_j = x_j y_i$ (Exercise 7.3.2). In this way, we recover the description of $V \to \mathbb{P}^n$ given in Example 6.0.19. $\Diamond$

Proposition 7.3.1 extends easily to decomposable toric vector bundles. Suppose we have r Cartier divisors $D_i = \sum_\rho a_{i\rho} D_\rho$, $i = 1, \dots, r$. This gives the locally free sheaf

$$\mathscr{O}_{X_\Sigma}(D_1) \oplus \cdots \oplus \mathscr{O}_{X_\Sigma}(D_r) \tag{7.3.1}$$

of rank r. To construct the fan of the corresponding vector bundle, we work in $N_\mathbb{R} \times \mathbb{R}^r$. Let $e_1, \dots, e_r$ be the standard basis of $\mathbb{R}^r$ and write elements of $N_\mathbb{R} \times \mathbb{R}^r$ as $u + \lambda_1 e_1 + \cdots + \lambda_r e_r$. Then, given $\sigma \in \Sigma$, we get the cone

$$\begin{aligned}\widetilde{\sigma} &= \{u + \lambda_1 e_1 + \cdots + \lambda_r e_r \mid u \in \sigma,\ \lambda_i \geq \varphi_{D_i}(u) \text{ for } i = 1, \dots, r\} \\ &= \operatorname{Cone}(u_\rho - a_{1\rho} e_1 - \cdots - a_{r\rho} e_r \mid \rho \in \sigma(1)) + \operatorname{Cone}(e_1, \dots, e_r).\end{aligned}$$

One can show without difficulty that the set consisting of the cones $\widetilde{\sigma}$ for $\sigma \in \Sigma$ and their faces is a fan in $N_\mathbb{R} \times \mathbb{R}^r$ such that the toric variety of this fan is the vector bundle over X_Σ whose sheaf of sections is (7.3.1) (Exercise 7.3.3).

Besides decomposable vector bundles, one can also define a *toric vector bundle* $\pi : V \to X_\Sigma$. Here, rather than assume that V is a toric variety, one makes the weaker assumption the torus of X_Σ acts on V such that the action is linear on the fibers and π is equivariant. Toric vector bundles have been classified by Klyachko [**178**] and others—see [**225**] for the historical background. Oda noted in [**217**, p. 41] that if a toric vector bundle is a toric variety in its own right, then the bundle is a direct sum of line bundles, as above. This can be proved using Klyachko's results.

Toric Projective Bundles. The decomposable toric vector bundles have associated toric projective bundles. Cartier divisors $D_0, \dots, D_r$ give the locally free sheaf

$$\mathscr{E} = \mathscr{O}_{X_\Sigma}(D_0) \oplus \cdots \oplus \mathscr{O}_{X_\Sigma}(D_r),$$

of rank $r+1$. Then $\mathbb{P}(\mathscr{E}) \to X_\Sigma$ is a projective bundle whose fibers look like $\mathbb{P}^r$.

To describe the fan of $\mathbb{P}(\mathscr{E})$, we first give a new description of the fan of $\mathbb{P}^r$. In $\mathbb{R}^{r+1}$, we use the standard basis $e_0, \dots, e_r$. The "first orthant" $\operatorname{Cone}(e_0, \dots, e_r)$ has $r+1$ facets

$$F_i = \operatorname{Cone}(e_0, \dots, \hat{e}_i, \dots, e_r), \quad i = 0, \dots, r.$$

Now set $\overline{N} = \mathbb{Z}^{r+1}/\mathbb{Z}(e_0 + \cdots + e_r)$. Then the images $\overline{e}_i$ of e_i sum to zero in $\overline{N}$ and the images $\overline{F}_i$ of F_i give the fan for $\mathbb{P}^r$ in $\overline{N}_\mathbb{R}$.

The construction of $\mathbb{P}(\mathscr{E})$ given in §7.0 involves taking the dual vector bundle. Thus $\mathbb{P}(\mathscr{E}) = \mathbb{P}(V_\mathscr{E})$, where $V_\mathscr{E}$ is the vector bundle whose sheaf of sections is

$$\mathscr{O}_{X_\Sigma}(-D_0) \oplus \cdots \oplus \mathscr{O}_{X_\Sigma}(-D_r).$$

The fan of $V_\mathscr{E}$ is built from cones

$$\operatorname{Cone}(u_\rho + a_{0\rho} e_0 + \cdots + a_{r\rho} e_r \mid \rho \in \sigma(1)) + \operatorname{Cone}(e_0, \dots, e_r)$$

and their faces, as σ ranges over the cones $\sigma \in \Sigma$. To get the fan for $\mathbb{P}(\mathscr{E}) = \mathbb{P}(V_{\mathscr{E}})$, take $\sigma \in \Sigma$ and let F_i be a facet of $\mathrm{Cone}(e_0, \dots, e_r)$. This gives the cone

$$\mathrm{Cone}(u_\rho + a_{0\rho}e_0 + \cdots + a_{r\rho}e_r \mid \rho \in \sigma(1)) + F_i \subseteq N_{\mathbb{R}} \times \mathbb{R}^{r+1},$$

and one sees that $\sigma_i \subseteq N_{\mathbb{R}} \times \overline{N}_{\mathbb{R}}$ is the image of this cone under the projection map $N_{\mathbb{R}} \times \mathbb{R}^{r+1} \to N_{\mathbb{R}} \times \overline{N}_{\mathbb{R}}$.

Proposition 7.3.3. *The cones $\{\sigma_i \mid \sigma \in \Sigma, i = 0, \dots, r\}$ and their faces form a fan $\Sigma_{\mathscr{E}}$ in $N_{\mathbb{R}} \times \overline{N}_{\mathbb{R}}$ whose toric variety $X_{\mathscr{E}}$ is the projective bundle $\mathbb{P}(\mathscr{E})$.*

Proof. Consider the fan Σ_0 in $\overline{N}_{\mathbb{R}}$ given by the $\overline{F}_i$ and their faces. Also, for $\sigma \in \Sigma$, let $\widehat{\sigma}$ be the image of $\mathrm{Cone}(u_\rho + a_{0\rho}e_0 + \cdots + a_{r\rho}e_r \mid \rho \in \sigma(1))$ in $N_{\mathbb{R}} \times \overline{N}_{\mathbb{R}}$. Then one easily adapts the proof of Proposition 7.3.1 to show that the toric variety $X_{\mathscr{E}}$ of $\Sigma_{\mathscr{E}}$ is a fibration over X_Σ with fiber $\mathbb{P}^r$. Furthermore, working over an affine open subset of X_Σ, one sees that $X_{\mathscr{E}}$ is obtained from $V_{\mathscr{E}}$ by the process described in §7.0. We leave the details as Exercise 7.3.4. $\square$

In practice, one usually replaces $\overline{N} = \mathbb{Z}^{r+1}/\mathbb{Z}(e_0 + \cdots + e_r)$ with $\mathbb{Z}^r$ and the basis $\mathbf{e}_1, \dots, \mathbf{e}_r$. Then set $\mathbf{e}_0 = -\mathbf{e}_1 - \cdots - \mathbf{e}_r$ and we redefine F_i as

$$F_i = \mathrm{Cone}(\mathbf{e}_0, \dots, \hat{\mathbf{e}}_i, \dots, \mathbf{e}_r) \subseteq \mathbb{R}^r \tag{7.3.2}$$

and for a cone $\sigma \in \Sigma$, redefine σ_i as

$$\sigma_i = \mathrm{Cone}(u_\rho + (a_{1\rho} - a_{0\rho})\mathbf{e}_1 + \cdots + (a_{r\rho} - a_{0\rho})\mathbf{e}_r \,|\, \rho \in \sigma(1)) + F_i \tag{7.3.3}$$

in $N_{\mathbb{R}} \times \mathbb{R}^r$. This way, $\Sigma_{\mathscr{E}}$ is a fan in $N_{\mathbb{R}} \times \mathbb{R}^r$. Here is a classic example.

Example 7.3.4. The fan for $\mathbb{P}^1$ has minimal generators u_1 and $u_0 = -u_1$. Also let $\mathscr{O}_{\mathbb{P}^1}(1) = \mathscr{O}_{\mathbb{P}^1}(D_0)$, where D_0 is the divisor corresponding to u_0. Fix an integer $a \geq 0$ and consider

$$\mathscr{E} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(a).$$

As above, we get a fan $\Sigma_{\mathscr{E}}$ in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$. The minimal generators u_0, u_1 live in the first factor. In the second factor, the vectors $\mathbf{e}_0 = -\sum_{i=1}^r \mathbf{e}_i, \mathbf{e}_1, \dots, \mathbf{e}_r$ in the above construction reduce to $\mathbf{e}_0 = -\mathbf{e}_1, \mathbf{e}_1$. Thus $F_0 = \mathrm{Cone}(\mathbf{e}_1)$ and $F_1 = \mathrm{Cone}(\mathbf{e}_0)$. We will use $u_1, \mathbf{e}_1$ as the basis of $\mathbb{R}^2$.

The maximal cones for the fan of $\mathbb{P}^1$ are $\sigma = \mathrm{Cone}(u_1)$ and $\sigma' = \mathrm{Cone}(u_0)$. Then $\Sigma_{\mathscr{E}}$ has four cones:

$$\begin{aligned}
\widetilde{\sigma}_0 &= \mathrm{Cone}(u_1 + (0-0)\mathbf{e}_1) + F_0 = \mathrm{Cone}(u_1, \mathbf{e}_1)\\
\widetilde{\sigma}_1 &= \mathrm{Cone}(u_1 + (0-0)\mathbf{e}_1) + F_1 = \mathrm{Cone}(u_1, -\mathbf{e}_1)\\
\widetilde{\sigma}'_0 &= \mathrm{Cone}(u_0 + (a-0)\mathbf{e}_1) + F_0 = \mathrm{Cone}(-u_1 + a\mathbf{e}_1, \mathbf{e}_1)\\
\widetilde{\sigma}'_1 &= \mathrm{Cone}(u_0 + (a-0)\mathbf{e}_1) + F_1 = \mathrm{Cone}(-u_1 + a\mathbf{e}_1, -\mathbf{e}_1).
\end{aligned}$$

This is the fan for the Hirzebruch surface $\mathscr{H}_a$. Thus

$$\mathscr{H}_a = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(a)).$$

Note also that the toric morphism $\mathscr{H}_a \to \mathbb{P}^1$ constructed earlier is the projection map for the projective bundle. $\Diamond$

This example generalizes as follows.

Example 7.3.5. Given integers $s, r \geq 1$ and $0 \leq a_1 \leq \cdots \leq a_r$, consider the projective bundle

$$\mathbb{P}(\mathscr{E}) = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^s}(a_r)).$$

The fan $\Sigma_{\mathscr{E}}$ of $\mathbb{P}(\mathscr{E})$ has a nice description. We will work in $\mathbb{R}^s \times \mathbb{R}^r$, where $\mathbb{R}^s$ has basis $u_1, \ldots, u_s$ and $\mathbb{R}^r$ has basis $\mathbf{e}_1, \ldots, \mathbf{e}_r$. Also set $u_0 = -\sum_{j=1}^s u_j$ and $\mathbf{e}_0 = -\sum_{i=1}^s \mathbf{e}_i$. As usual, u_0 corresponds to the divisor D_0 of $\mathbb{P}^s$ such that $\mathscr{O}_{\mathbb{P}^s}(a_i) = \mathscr{O}_{\mathbb{P}^s}(a_i D_0)$.

The description (7.3.3) of the cones in Σ uses generators of the form

$$u_\rho + (a_{1\rho} - a_{0\rho})\mathbf{e}_1 + \cdots + (a_{r\rho} - a_{0\rho})\mathbf{e}_r, \tag{7.3.4}$$

where the u_ρ are minimal generators of the fan of the base of the projective bundle. Here, the u_ρ's are $u_0, \ldots, u_s$. Since we are using the divisors $0, a_1 D_0, \ldots, a_r D_0$, the formula (7.3.4) simplifies dramatically, giving minimal generators

$$\begin{aligned} u_\rho = u_0 &: \ \mathbf{v}_0 = u_0 + a_1\mathbf{e}_1 + \cdots + a_r\mathbf{e}_r \\ u_\rho = u_j &: \ \mathbf{v}_j = u_j, \ j = 1, \ldots, s. \end{aligned}$$

Since the maximal cones of $\mathbb{P}^s$ are $\mathrm{Cone}(u_0, \ldots, \hat{u}_j, \ldots, u_s)$, (7.3.2) and (7.3.3) imply that the maximal cones of Σ are

$$\mathrm{Cone}(\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_s) + \mathrm{Cone}(\mathbf{e}_0, \ldots, \hat{\mathbf{e}}_i, \ldots, \mathbf{e}_r)$$

for all $j = 0, \ldots, s$ and $i = 0, \ldots, r$. It is also easy to see that the minimal generators $\mathbf{v}_0, \ldots, \mathbf{v}_s, \mathbf{e}_0, \ldots, \mathbf{e}_r$ have the following properties:

- $\mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{e}_1, \ldots, \mathbf{e}_r$ form a basis of $\mathbb{Z}^s \times \mathbb{Z}^r$.
- $\mathbf{e}_0 + \cdots + \mathbf{e}_r = 0$.
- $\mathbf{v}_0 + \cdots + \mathbf{v}_s = a_1\mathbf{e}_1 + \cdots + a_r\mathbf{e}_r$.

The first two bullets are clear, and the third follows from $\sum_{j=0}^s u_j = 0$ and the definition of the $\mathbf{v}_j$.

One also sees that $X_{\mathscr{E}} = \mathbb{P}(\mathscr{E})$ is smooth of dimension $s + r$. Since $\Sigma_{\mathscr{E}}$ has $(s+1) + (r+1) = s + r + 2$ minimal generators, the description of the Picard group given in §4.2 implies that

$$\mathrm{Pic}(\mathbb{P}(\mathscr{E})) \simeq \mathbb{Z}^2.$$

(Exercise 7.3.5). Also observe that $\{\mathbf{v}_0, \ldots, \mathbf{v}_s\}$ and $\{\mathbf{e}_0, \ldots, \mathbf{e}_r\}$ give primitive collections of $\Sigma_{\mathscr{E}}$. We will see below that these are the only primitive collections of $\Sigma_{\mathscr{E}}$. Furthermore, they are extremal in the sense of §6.4 and their primitive relations generate the Mori cone of $\mathbb{P}(\mathscr{E})$.

This is a very rich example! $\Diamond$

A Classification Theorem. Kleinschmidt [**177**] classified all smooth projective toric varieties with Picard number 2, i.e., with $\mathrm{Pic}(X_\Sigma) \simeq \mathbb{Z}^2$. The rough idea is that they are the toric projective bundles described in Example 7.3.5. Following ideas of Batyrev [**14**], we will use primitive collections to obtain the classification.

We begin with some results from [**14**] about primitive collections. Recall from §6.4 that a primitive collection $P = \{\rho_1, \dots, \rho_k\} \subseteq \Sigma(1)$ gives the primitive relation

$$u_{\rho_1} + \cdots + u_{\rho_k} - \textstyle\sum_{\rho \in \gamma(1)} c_\rho u_\rho = 0, \quad c_\rho \in \mathbb{Q}_{>0}, \tag{7.3.5}$$

where $\gamma \in \Sigma$ is the minimal cone containing $u_{\rho_1} + \cdots + u_{\rho_k}$. When X_Σ is smooth and projective, these primitive relations have some nice properties.

Proposition 7.3.6. *Let X_Σ be a smooth projective toric variety. Then:*

(a) *In the primitive relation* (7.3.5), *$P \cap \gamma(1) = \emptyset$ and $c_\rho \in \mathbb{Z}_{>0}$ for all $\rho \in \sigma(1)$.*

(b) *There is a primitive collection P with primitive relation $u_{\rho_1} + \cdots + u_{\rho_k} = 0$.*

Proof. The c_ρ are integral since Σ is smooth. Let the minimal generators of γ be $u_1, \dots, u_\ell$, so the primitive relation becomes

$$u_{\rho_1} + \cdots + u_{\rho_k} = c_1 u_1 + \cdots + c_\ell u_\ell.$$

To prove part (a), suppose for example that $u_{\rho_1} = u_1$. Then

$$u_{\rho_2} + \cdots + u_{\rho_k} = (c_1 - 1)u_1 + c_2 u_2 \cdots + c_\ell u_\ell.$$

Note that $u_{\rho_2}, \dots, u_{\rho_k}$ generate a cone of Σ since P is a primitive collection. So the above equation expresses an element of a cone of Σ in terms of minimal generators in two different ways. Since Σ is smooth, these must coincide. To see what this means, we consider two cases:

- $c_1 > 1$. Then $\{u_{\rho_2}, \dots, u_{\rho_k}\} = \{u_1, u_2, \dots, u_\ell\}$, so that $u_{\rho_i} = u_1$ for some $i > 1$. This is impossible since $u_{\rho_1} = u_1$.
- $c_1 = 1$. Then $\{u_{\rho_2}, \dots, u_{\rho_k}\} = \{u_2, \dots, u_\ell\}$. Since $u_{\rho_1} = u_1$, we obtain $P \subseteq \gamma(1)$, which is impossible since P is a primitive collection.

Since c_1 must be positive, we conclude that $u_{\rho_1} = u_1$ leads to a contradiction. From here, it is easy to see that $P \cap \gamma(1) = \emptyset$.

Turning to part (b), let φ be the support function of an ample divisor on X_Σ. Thus φ is strictly convex. Since Σ is complete, we can find an expression

$$b_1 u_{\rho_1} + \cdots + b_s u_{\rho_s} = 0 \tag{7.3.6}$$

such that $b_1, \dots, b_s$ are positive integers. Note that $u_{\rho_1}, \dots, u_{\rho_s}$ cannot lie in a cone of Σ. By strict convexity and Lemma 6.1.13, it follows that

$$0 = \varphi(0) = \varphi(b_1 u_{\rho_1} + \cdots + b_s u_{\rho_s}) > b_1 \varphi(u_{\rho_1}) + \cdots + b_s \varphi(u_{\rho_s}). \tag{7.3.7}$$

Pick a relation (7.3.6) so that the right-hand side is as big as possible.

The set $\{u_{\rho_1},\dots,u_{\rho_s}\}$ is not contained in a cone of Σ and hence has a subset that is a primitive collection. By relabeling, we may assume that $\{u_{\rho_1},\dots,u_{\rho_k}\}$, $k \le s$, is a primitive collection. Using (7.3.6) and the primitive relation (7.3.5), we obtain the nonnegative relation

$$\sum_{\rho\in\gamma(1)} c_\rho u_\rho + \sum_{i=1}^{k}(b_i-1)u_{\rho_i} + \sum_{i=k+1}^{s} b_i u_{\rho_i} = 0.$$

Since φ is linear on γ and strictly convex,

$$\sum_{\rho\in\gamma(1)} c_\rho\varphi(u_\rho) = \varphi\Big(\sum_{\rho\in\gamma(1)} c_\rho u_\rho\Big) = \varphi\Big(\sum_{i=1}^{k} u_{\rho_i}\Big) > \sum_{i=1}^{k}\varphi(u_{\rho_i}),$$

which implies that

$$\begin{aligned}&\sum_{\rho\in\gamma(1)} c_\rho\varphi(u_\rho) + \sum_{i=1}^{k}(b_i-1)\varphi(u_{\rho_i}) + \sum_{i=k+1}^{s} b_i\varphi(u_{\rho_i})\\ >\ & \sum_{i=1}^{k}\varphi(u_{\rho_i}) \ + \ \sum_{i=1}^{k}(b_i-1)\varphi(u_{\rho_i}) + \sum_{i=k+1}^{s} b_i\varphi(u_{\rho_i}) = \sum_{i=1}^{s} b_i\varphi(u_{\rho_i}).\end{aligned}$$

This contradicts the maximality of the right-hand side of (7.3.7), unless $k = s$ and $b_1 = \cdots = b_k = 1$, in which case we get the desired primitive collection. $\square$

We now prove Kleinschmidt's classification theorem.

Theorem 7.3.7. *Let X_Σ be a smooth projective toric variety with $\mathrm{Pic}(X_\Sigma) \simeq \mathbb{Z}^2$. Then there are integers $s, r \ge 1$, $s + r = \dim X_\Sigma$ and $0 \le a_1 \le \cdots \le a_r$ with*

$$X_\Sigma \simeq \mathbb{P}\big(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^s}(a_r)\big).$$

Proof. Let $n = \dim X_\Sigma$. Then $\mathrm{Pic}(X_\Sigma) \simeq \mathbb{Z}^2$ and Theorem 4.2.1 imply that $\Sigma(1)$ has $n+2$ elements. We recall two facts about divisors D on X_Σ:

- If D is nef and $\sigma \in \Sigma(n)$, then $D \sim \sum_\rho a_\rho D_\rho$ where $a_\rho = 0$ for $\rho \in \sigma(1)$ and $a_\rho \ge 0$ for $\rho \notin \sigma(1)$.
- If $D \ge 0$ and $D \sim 0$, then $D = 0$ since X_Σ is complete.

The first bullet was proved in (6.4.10), and the second is an easy consequence of Propositions 4.0.16 and 4.3.8.

By assumption, X_Σ has an ample divisor D which lies in the interior of the nef cone $\mathrm{Nef}(X_\Sigma)$. Changing D if necessary, we can assume that D is effective and $[D] \in \mathrm{Pic}(X_\Sigma)_\mathbb{R}$ is not a scalar multiple of any $[D_\rho]$ for $\rho \in \Sigma(1)$. The line determined by $[D]$ divides $\mathrm{Pic}(X_\Sigma)_\mathbb{R} \simeq \mathbb{R}^2$ into closed half-planes H^+ and H^-. Then define

$$\begin{aligned}P &= \{\rho \in \Sigma(1) \mid [D_\rho] \in H^+\}\\ Q &= \{\rho \in \Sigma(1) \mid [D_\rho] \in H^-\}.\end{aligned}$$

Note that $P \cup Q = \Sigma(1)$, and $P \cap Q = \emptyset$ by our choice of D. We claim that

$$\text{(7.3.8)} \qquad \begin{aligned} &\Sigma(n) = \{\sigma_{\rho,\rho'} \mid \rho \in P, \rho' \in Q\}, \text{ where} \\ &\sigma_{\rho,\rho'} = \mathrm{Cone}(u_{\hat{\rho}} \mid \hat{\rho} \in \Sigma(1) \setminus \{\rho,\rho'\}). \end{aligned}$$

To prove this, first take $\sigma \in \Sigma(n)$. Since $|\sigma(1)| = n$ and $|\Sigma(1)| = n+2$, we have

$$\text{(7.3.9)} \qquad \Sigma(1) = \sigma(1) \cup \{\rho,\rho'\}.$$

Applying the first bullet above to D and σ, we get $[D] = a[D_\rho] + b[D_{\rho'}]$ where $a,b > 0$ since $[D]$ is a multiple of neither $[D_\rho]$ nor $[D_{\rho'}]$. It follows that $[D_\rho]$ and $[D_{\rho'}]$ lie on opposite sides of the line determined by $[D]$. We can relabel so that $\rho \in P$ and $\rho' \in Q$, and then σ has the desired form by (7.3.9).

For the converse, take $\rho \in P$ and $\rho' \in Q$. Since $\mathrm{Pic}(X_\Sigma)_{\mathbb{R}} \simeq \mathbb{R}^2$, we can find a linear dependence

$$a_0[D_\rho] + b_0[D_{\rho'}] + c_0[D] = 0, \quad a_0, b_0, c_0 \in \mathbb{Z} \text{ not all } 0.$$

We can assume that $a_0, b_0 \geq 0$ since $[D_\rho]$ and $[D_{\rho'}]$ lie on opposite sides of the line determined by $[D]$. Note also that $c_0 < 0$ by the second bullet above, and then $a_0, b_0 > 0$ by our choice of D. It follows that $D' = a_0 D_\rho + b_0 D_{\rho'}$ is ample. In Exercise 7.3.6 you will show that

$$X_\Sigma \setminus \mathrm{Supp}(D') = X_\Sigma \setminus (D_\rho \cup D_{\rho'})$$

is the nonvanishing set of a global section of $\mathscr{O}_{X_\Sigma}(D')$ and hence is affine. This set is also torus-invariant and hence is an affine toric variety. Thus it must be U_σ for some $\sigma \in \Sigma$. In other words,

$$X_\Sigma = U_\sigma \cup D_\rho \cup D_{\rho'}.$$

Since $U_\sigma \cap (D_\rho \cup D_{\rho'}) = \emptyset$, the Orbit-Cone correspondence (Theorem 3.2.6) implies that σ satsisfies (7.3.9) and hence gives an element of $\Sigma(n)$. This completes the proof of (7.3.8).

An immediate consequence of this description of $\Sigma(n)$ is that P and Q are primitive collections. Be sure you understand why. It is also true that P and Q are the *only* primitive collections of Σ. To prove this, suppose that we had a third primitive collection R. Then $P \not\subseteq R$, so there is $\rho \in P \setminus R$, and similarly there is $\rho' \in Q \setminus R$ since $Q \not\subseteq P$. By (7.3.8), the rays of R all lie in $\sigma_{\rho,\rho'} \in \Sigma(n)$, which contradicts the definition of primitive collection.

Since X_Σ is projective and smooth, Proposition 7.3.6 guarantees that Σ has a primitive collection whose elements sum to zero. We may assume that P is this primitive collection. Let $|P| = r+1$ and $|Q| = s+1$, so $r,s \geq 1$ since primitive collections have at least two elements, and $r+s = n$ since $|P|+|Q| = n+2$.

Now rename the minimal generators of the rays in P as $\mathbf{e}_0, \ldots, \mathbf{e}_r$. Thus

$$\mathbf{e}_0 + \cdots + \mathbf{e}_r = 0.$$

The next step is to rename the minimal generators of the rays in P as $\mathbf{v}_0,\dots,\mathbf{v}_s$. Proposition 7.3.6 implies that $\sum_{j=0}^{s}\mathbf{v}_j$ lies in a cone $\gamma\in\Sigma$ whose rays lie in the complement of Q, which is P. Since P is a primitive collection, γ must omit at least one element of P, which we may assume to be the ray generated by $\mathbf{e_0}$. Then the primitive relation of Q can be written

$$\mathbf{v}_0+\cdots+\mathbf{v}_s = a_1\mathbf{e}_1+\cdots+a_r\mathbf{e}_r,$$

and by further relabeling, we may assume $0\le a_1\le\cdots\le a_r$. Finally, observe that $\mathbf{v}_1,\dots,\mathbf{v}_s,\mathbf{e}_1,\dots,\mathbf{e}_r$ generate a maximal cone of Σ by (7.3.8). Since Σ is smooth, it follows that these $r+s$ vectors form a basis of N. Comparing all of this to Example 7.3.5, we conclude that the toric variety of Σ is the projective bundle $\mathbb{P}\big(\mathscr{O}_{\mathbb{P}^s}\oplus\mathscr{O}_{\mathbb{P}^s}(a_1)\oplus\cdots\oplus\mathscr{O}_{\mathbb{P}^s}(a_r)\big)$. $\square$

The classification result proved in [**177**] is more general than the one given in Theorem 7.3.7. By using a result from [**191**] on sphere triangulations with few vertices, Kleinschmidt does not need to assume that X_Σ is projective. Another proof of Theorem 7.3.7 that does not assume projective can be found in [**14**, Thm. 4.3]. We should also mention that (7.3.8) can be proved using the *Gale transforms* discussed in [**93**, II.4–6] and [**281**, Ch. 6]. We will explain this is §15.2.

Exercises for §7.3.

7.3.1. Here you will supply some details needed to prove Theorem 7.3.1.

(a) In the proof we constructed a map $U_{\widetilde{\sigma}}\to\mathbb{C}$. Show that this map is $\chi^{(-m_\sigma,1)}$, where $\varphi_D(u)=\langle m_\sigma,u\rangle$ for $u\in\sigma$.

(b) Given cones $\sigma,\tau\in\Sigma$, the transition map from $U_{\sigma\cap\tau}\times\mathbb{C}\subseteq U_\tau\times\mathbb{C}$ to $U_{\sigma\cap\tau}\times\mathbb{C}\subseteq U_\sigma\times\mathbb{C}$ is given by $(u,t)\mapsto(u,g_{\sigma\tau}(u)t)$. Prove that $g_{\sigma\tau}=\chi^{m_\tau-m_\sigma}$.

7.3.2. In Example 7.3.2, we study the rank 1 vector bundle $V\to\mathbb{P}^n$ whose sheaf of sections is $\mathscr{O}_{\mathbb{P}^n}(-1)$. Let Σ be the fan of V in $\mathbb{R}^{n+1}$.

(a) Prove that $e_1,\dots,e_{n+1},-e_1-\cdots-e_n+e_{n+1}$ are the minimal generators of Σ.

(b) Prove that Σ is the normal fan of

$$P=\mathrm{Conv}(0,e_1,\dots,e_n)+\mathrm{Cone}(e_{n+1},e_1+e_{n+1},\dots,e_n+e_{n+1}).$$

(c) The example constructs a morphism $V\to\mathbb{P}^n\times\mathbb{C}^{n+1}$. Prove that the image of this map is defined by $x_iy_j=x_jy_i$ and explain how this relates to Example 6.0.19.

7.3.3. Consider the locally free sheaf (7.3.1) and the cones $\widetilde{\sigma}\subseteq N_\mathbb{R}\times\mathbb{R}^r$ defined in the discussion following (7.3.1). Prove that these cones and their faces give a fan in $N_\mathbb{R}\times\mathbb{R}^r$ whose toric variety is the vector bundle with (7.3.1) as sheaf of sections.

7.3.4. Complete the proof of Proposition 7.3.3.

7.3.5. Let $\mathbb{P}(\mathscr{E})\to\mathbb{P}^s$ be the toric projective bundle constructed in Example 7.3.5. Prove that $\mathrm{Pic}(\mathbb{P}(\mathscr{E}))\simeq\mathbb{Z}^2$.

7.3.6. Let D be an ample effective divisor on a complete normal variety X. The goal of this exercise is to prove that $X\setminus\mathrm{Supp}(D)$ is affine.

(a) Assume that D is very ample. Let $s \in \Gamma(X, \mathscr{O}_X(D))$ be nonzero and consider the *nonvanishing set* of s defined by $U = \{s \in X \mid s(x) \neq 0\}$. Prove that U is affine. Hint: Show that a basis $s = s_0, s_1, \ldots, s_m$ of $\Gamma(X, \mathscr{O}_X(D))$ gives a closed embedding $X \to \mathbb{P}^m$. Let $\mathbb{P}^m$ have homogeneous coordinates $x_0, \ldots, x_m$ and regard X as a subset of $\mathbb{P}^m$. Prove that $U = X \cap U_0$, where $U_0 \subseteq \mathbb{P}^m$ is where $x_0 \neq 0$.

(b) Explain why part (a) remains true when D is ample but not necessarily very ample. Hint: $s^k \in \Gamma(X, \mathscr{O}_X(kD))$.

(c) Since D is effective, $1 \in \Gamma(X, \mathscr{O}_X(D))$ is a global section. Prove that the nonvanishing set of this global section is $X \setminus \mathrm{Supp}(D)$. Hint: For $s \in \Gamma(X, \mathscr{O}_X(D))$, recall the definition of $\mathrm{div}_0(s)$ given in §4.0.

Parts (b) and (c) imply that $X \setminus \mathrm{Supp}(D)$ is affine when D is ample, as desired. Note also that part (b) is a special case of Proposition 7.A.6.

7.3.7. By Example 2.3.16, the *rational normal scroll* $S_{a,b}$ is the toric variety of

$$P_{a,b} = \mathrm{Conv}(0, ae_1, e_2, be_1 + e_2) \subseteq \mathbb{R}^2,$$

where $a, b \in \mathbb{N}$ satisfy $1 \leq a \leq b$, and $S_{a,b} \simeq \mathscr{H}_{b-a}$ by Example 3.1.16. Thus rational normals scroll are Hirzebruch surfaces. Here you will explore an n-dimensional analog.

Take integers $1 \leq d_0 \leq d_1 \leq \cdots \leq d_{n-1}$. Then $P_{d_0,\ldots,d_{n-1}}$ is the lattice polytope in $\mathbb{R}^n$ having the $2n$ lattice points

$$0,\ d_0e_1,\ e_2,\ e_2 + d_1e_1,\ e_3,\ e_3 + d_2e_1, \ldots, e_n,\ e_n + d_{n-1}e_1$$

as vertices. The toric variety of $P_{d_0,\ldots,d_{n-1}}$ is denoted $S_{d_0,\ldots,d_{n-1}}$.

(a) Explain why $P_{d_0,\ldots,d_{n-1}}$ is a "truncated prism" whose base in $\{0\} \times \mathbb{R}^{n-1}$ is the standard simplex Δ_{n-1}, and above the vertices of Δ_{n-1} there are edges of lengths $d_0, \ldots, d_{n-1}$. Here, "above" means the e_1 direction. Draw a picture when $n = 3$.

(b) Prove that $S_{d_0,\ldots,d_{n-1}} \simeq \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(d_0) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(d_{n-1}))$.

(c) $S_{d_0,\ldots,d_{n-1}}$ is smooth by part (b), so that $P_{d_0,\ldots,d_{n-1}}$ is very ample and hence gives a projective embedding of $S_{d_0,\ldots,d_{n-1}}$. Explain how this embedding consists of n embeddings of $\mathbb{P}^1$ such that for each point $p \in \mathbb{P}^1$, the resulting n points in projective space are connected by an $(n-1)$-dimensional plane that lies in $S_{d_0,\ldots,d_{n-1}}$.

(d) Explain how part (c) relates to the scroll discussion in Example 2.3.16.

(e) Show that the $(n-1)$-dimensional plane associated to $p \in \mathbb{P}^1$ in part (c) is the fiber of the projective bundle $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(d_0) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^1}(d_{n-1})) \to \mathbb{P}^1$.

7.3.8. Consider the toric variety $\mathbb{P}(\mathscr{E})$ constructed in Example 7.3.5.

(a) Prove that $\mathbb{P}(\mathscr{E})$ is projective. Hint: Proposition 7.0.5.

(b) Show that $\mathbb{P}(\mathscr{E}) \simeq \mathbb{P}(\mathscr{O}_{\mathbb{P}^s}(1) \oplus \mathscr{O}_{\mathbb{P}^s}(a_1 + 1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^s}(a_r + 1))$. Hint: Part (b) of Lemma 7.0.8.

(c) Find a lattice polytope in $\mathbb{R}^s \times \mathbb{R}^r$ whose toric variety is $\mathbb{P}(\mathscr{E})$. Hint: In the polytope of Exercise 7.3.7, each vertex of $\{0\} \times \Delta_{n-1} \subseteq \mathbb{R} \times \mathbb{R}^{n-1}$ is attached to a line segment in the normal direction. Also observe that a line segment is a multiple of Δ_1. Adapt this by using $\{0\} \times \Delta_r \subseteq \mathbb{R}^s \times \mathbb{R}^r$ as "base" and then, at each vertex of Δ_r, attach a positive mutliple of Δ_s in the normal direction.

7.3.9. Let X_Σ be a projective toric variety and let $D_0, \dots, D_r$ be torus-invariant ample divisors on X_Σ. Each D_i gives a lattice polytope $P_i = D_{P_i}$ whose normal fan is Σ. Prove that the projective bundle $\mathbb{P}(\mathscr{O}_{X_\Sigma}(D_0) \oplus \cdots \oplus \mathscr{O}_{X_\Sigma}(D_r))$ is the toric variety of the polytope in $N_\mathbb{R} \times \mathbb{R}$

$$\mathrm{Conv}(P_0 \times \{0\} \cup P_1 \times \{e_1\} \cup \cdots \cup P_r \times \{e_r\}).$$

Hint: If you get stuck, see [**61**, Sec. 3]. Do you see how this relates to Exercise 7.3.8?

7.3.10. Use primitive collections to show that $\mathbb{P}^n$ is the only smooth projective toric variety with Picard number 1.

Appendix: More on Projective Morphisms

In this appendix, we discuss some technical details related to projective morphisms.

Ampleness. A comprehensive treatment of ampleness appears in Volume II of *Éléments de géométrie algébrique* (EGA) by Grothendieck and Dieudonné [**127**]. The results we need from EGA are spread out over several sections. Here we collect the definitions and theorems we will use in our discusion of ampleness.[1]

Definition 7.A.1. A line bundle $\mathscr{L}$ on a variety X is ***absolutely ample*** if for every coherent sheaf $\mathscr{F}$ on X, there is an integer k_0 such that $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes k}$ is generated by global sections for all $k \geq k_0$.

By [**127**, (4.5.5)], this is equivalent to what EGA calls "ample" in [**127**, (4.5.3)]. We use the name "absolutely ample" to prevent confusion with Definition 6.1.9, where "ample" is defined for line bundles on complete varieties. Here is another definition from EGA.

Definition 7.A.2. Let $f : X \to Y$ be a morphism. A line bundle $\mathscr{L}$ on X is ***relatively ample with respect to*** f if Y has an affine open cover $\{U_i\}$ such that for every i, $\mathscr{L}|_{f^{-1}(U_i)}$ is absolutely ample on $f^{-1}(U_i)$.

This is [**127**, (4.6.1)]. When mapping to an affine variety, relatively ample and absolutely ample coincide. More precisely, we have the following result from [**127**, (4.6.6)].

Proposition 7.A.3. *Let $f : X \to Y$ be a morphism, where Y is affine, and let $\mathscr{L}$ be a line bundle on X. Then:*

$$\mathscr{L} \text{ is relatively ample with respect to } f \iff \mathscr{L} \text{ is absolutely ample.} \qquad \square$$

The reader should be warned that in EGA, "relatively ample with respect to f" and "f-ample" are synonyms. In this text, they are slightly different, since "relatively ample with respect to f" refers to Definition 7.A.2 while "f-ample" refers to Definition 7.2.5. Fortunately, they coincide when the map f is proper.

Theorem 7.A.4. *Let $f : X \to Y$ be a proper morphism and $\mathscr{L}$ a line bundle on X. Then the following are equivalent:*

(a) *$\mathscr{L}$ is relatively ample with respect to f in the sense of Definition 7.A.2.*

[1] The theory developed in EGA applies to very general schemes. The varieties and morphisms appearing in this book are nicely behaved—the varieties are quasi-compact and noetherian, the morphisms are of finite type, and coherent is equivalent to quasicoherent of finite type. Hence most of the special hypotheses needed for some of the results in [**127**] are automatically true in our situation.

(b) *$\mathscr{L}$ is f-ample in the sense of Definition 7.2.5.*

(c) *There is an integer $k > 0$ such that f is projective with respect to $\mathscr{L}^{\otimes k}$ in the sense of Definition 7.0.3.*

Proof. First observe that (b) and (c) are equivalent by Definition 7.2.5. Now suppose that f is projective with respect to $\mathscr{L}^{\otimes k}$. Then there is an affine open covering $\{U_i\}$ of Y such that for each i, there is a finite-dimensional subspace $W \subseteq \Gamma(U_i, \mathscr{L}^{\otimes k})$ that gives a closed embedding of $f^{-1}(U_i)$ into $U_i \times \mathbb{P}(W^\vee)$ for each i.

The locally free sheaf $\mathscr{E} = W^\vee \otimes_{\mathbb{C}} \mathscr{O}_{U_i}$ is the sheaf of sections of the trivial vector bundle $U_i \times W^\vee \to U_i$. This gives the projective bundle $\mathbb{P}(\mathscr{E}) = U_i \times \mathbb{P}(W^\vee)$, so that we have a closed embedding

$$f^{-1}(U_i) \longrightarrow \mathbb{P}(\mathscr{E}).$$

By definition [**127**, (4.4.2)], $\mathscr{L}^{\otimes k}|_{f^{-1}(U_i)}$ is very ample for $f|_{f^{-1}(U_i)}$. Then [**127**, (4.6.9)] implies that $\mathscr{L}|_{f^{-1}(U_i)}$ is relatively ample with respect to $f|_{f^{-1}(U_i)}$, and hence absolutely ample by Proposition 7.A.3. Then $\mathscr{L}$ is relatively ample with respect to f by Definition 7.A.2.

Finally, suppose that $\mathscr{L}$ is relatively ample with respect to f and let $\{U_i\}$ be an affine open covering of Y. Then [**127**, (4.6.4)] implies that $\mathscr{L}|_{f^{-1}(U_i)}$ is relatively ample with respect to $f|_{f^{-1}(U_i)}$. By [**127**, (4.6.9)], $\mathscr{L}^{\otimes k}|_{f^{-1}(U_i)}$ is very ample for $f|_{f^{-1}(U_i)}$, which by definition [**127**, (4.4.2)] means that $f^{-1}(U_i)$ can be embedded in $\mathbb{P}(\mathscr{E})$ for a coherent sheaf $\mathscr{E}$ on U_i. Then the proof of [**273**, Thm. 5.44] shows how to find finitely many sections of $\mathscr{L}^{\otimes k}$ over $f^{-1}(U_i)$ that give a suitable embedding of $f^{-1}(U_i)$ into $U_i \times \mathbb{P}(W^\vee)$. $\square$

In EGA [**127**, (5.5.2)], the definition of when a morphism $f : X \to Y$ is projective involves two equivalent conditions stated in [**127**, (5.5.1)]. The first condition uses the projective bundle $\mathbb{P}(\mathscr{E})$ of a coherent sheaf $\mathscr{E}$ on Y, and the second uses $\mathrm{Proj}(\mathscr{S})$, where $\mathscr{S}$ is a quasicoherent graded $\mathscr{O}_Y$-algebra such that $\mathscr{S}_1$ is coherent and generates $\mathscr{S}$. By [**127**, (5.5.3)], projective is equivalent to proper and quasiprojective, and by the defintion of quasiprojective [**127**, (5.5.1)], this means that X has a line bundle relatively ample with respect to f. Hence Theorem 7.A.4 shows that the definition of projective morphism given in EGA is equivalent to Definition 7.0.3.

We close with two further results about projective morphisms. Proofs can be found in [**127**, (4.6.4)] and [**127**, (5.5.7)] respectively.

Proposition 7.A.5. *Let $f : X \to Y$ be a proper morphism and $\mathscr{L}$ a line bundle on X. Given an affine open cover $\{U_i\}$ of Y, the following are equivalent:*

(a) *$\mathscr{L}$ is f-ample.*

(b) *For every i, $\mathscr{L}|_{f^{-1}(U_i)}$ is $f|_{f^{-1}(U_i)}$-ample.* $\square$

Proposition 7.A.6. *Let $f : X \to Y$ be a projective morphism with Y affine and let $\mathscr{L}$ be an f-ample line bundle on X. Then:*

(a) *Given a global section $s \in \Gamma(X, \mathscr{L})$, let $X_s \subseteq X$ be the open subset where s is nonvanishing. Then X_s is an affine open subset of X.*

(b) *There is an integer k_0 such that $\mathscr{L}^{\otimes k}$ is generated by global sections for all $k \geq k_0$.* $\square$

Chapter 8

The Canonical Divisor of a Toric Variety

§8.0. Background: Reflexive Sheaves and Differential Forms

This chapter will study the canonical divisor of a toric variety. The theory developed in Chapters 6 and 7 dealt with Cartier divisors and line bundles. As we will see, the canonical divisor of a normal toric variety is a Weil divisor that is not necessarily Cartier. We will also study the sheaves associated to Weil divisors.

Reflexive Sheaves. A Weil divisor D on a normal variety X gives the sheaf $\mathscr{O}_X(D)$ defined by

$$\Gamma(U, \mathscr{O}_X(D)) = \{f \in \mathbb{C}(X)^* \mid (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\}.$$

Our first task is to characterize these sheaves.

Recall that the dual of a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is $\mathscr{F}^\vee = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$. We say that $\mathscr{F}$ is *reflexive* if the natural map

$$\mathscr{F} \longrightarrow \mathscr{F}^{\vee\vee}$$

is an isomorphism. It is easy to see that locally free sheaves are reflexive. Here are some properties of reflexive sheaves.

Proposition 8.0.1. *Let $\mathscr{F}$ be a coherent sheaf on a normal variety X and consider the inclusion $j : U_0 \hookrightarrow X$ where U_0 is open with* $\mathrm{codim}(X \setminus U_0) \geq 2$. *Then:*

(a) *$\mathscr{F}^\vee$ and hence $\mathscr{F}^{\vee\vee}$ are reflexive.*

(b) *If $\mathscr{F}$ is reflexive, then $\mathscr{F} \simeq j_*(\mathscr{F}|_{U_0})$.*

(c) *If $\mathscr{F}|_{U_0}$ is locally free, then $\mathscr{F}^{\vee\vee} \simeq j_*(\mathscr{F}|_{U_0})$.*

Proof. Recall from §4.0 that the direct image $j_*\mathscr{G}$ of a sheaf $\mathscr{G}$ on U_0 is defined by $\Gamma(U, j_*\mathscr{G}) = \Gamma(U \cap U_0, \mathscr{G})$ for $U \subseteq X$ open.

Parts (a) and (b) of the proposition are proved in [**132**, Cor. 1.2 and Prop. 1.6]. For part (c), we first observe that restriction is compatible with taking the dual, i.e., $(\mathscr{G}^\vee)|_{U_0} = (\mathscr{G}|_{U_0})^\vee$ for any coherent sheaf $\mathscr{G}$ on X. Then

$$\mathscr{F}^{\vee\vee} \simeq j_*((\mathscr{F}^{\vee\vee})|_{U_0}) = j_*((\mathscr{F}|_{U_0})^{\vee\vee}) \simeq j_*(\mathscr{F}|_{U_0}),$$

where the first isomorphism follows from parts (a) and (b), and the last follows since $\mathscr{F}|_{U_0}$ is locally free and hence reflexive. □

Later in the section we will study the sheaf Ω_X^p of p-forms on X. This sheaf is locally free when X is smooth. For X normal, however, Ω_X^p may be badly behaved, though it is locally free on the smooth locus of X. Hence we can use part (c) of Proposition 8.0.1 to create a reflexive version of Ω_X^p.

For more on reflexive sheaves, the reader should consult [**132**] and [**235**].

Reflexive Sheaves of Rank One. We first define the rank of a coherent sheaf on an irreducible variety X. Recall that $\mathscr{K}_X$ is the constant sheaf on X given by $\mathbb{C}(X)$.

Definition 8.0.2. Given a $\mathscr{F}$ coherent sheaf on irreducible variety X, the global sections of $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ form a finite-dimensional vector space over $\mathbb{C}(X)$ whose dimension is the ***rank*** of $\mathscr{F}$.

For a locally free sheaf, the rank is just the rank of the associated vector bundle. Other properties of the rank will be studied in Exercise 8.0.1.

In the smooth case, reflexive sheaves of rank 1 are easy to understand.

Proposition 8.0.3. *On a smooth variety, a coherent sheaf of rank* 1 *is reflexive if and only if it is a line bundle.* □

This is proved in [**132**, Prop. 1.9]. We now have all of the tools needed to characterize which coherent sheaves on a normal variety come from Weil divisors.

Theorem 8.0.4. *Let $\mathscr{L}$ be a coherent sheaf on a normal variety X. Then the following are equivalent:*

(a) *$\mathscr{L}$ is reflexive of rank* 1.

(b) *There is an open subset $j : U_0 \hookrightarrow X$ such that* $\operatorname{codim}(X \setminus U_0) \geq 2$, *$\mathscr{L}|_{U_0}$ is a line bundle on U_0, and $\mathscr{L} \simeq j_*(\mathscr{L}|_{U_0})$.*

(c) *$\mathscr{L} \simeq \mathscr{O}_X(D)$ for some Weil divisor D on X.*

Proof. (a) $\Rightarrow$ (b) Since X is normal, its singular locus $Y = \operatorname{Sing}(X)$ has codimension at least two in X by Proposition 4.0.17. Then $U_0 = X \setminus Y$ is smooth, which implies that $\mathscr{L}|_{U_0}$ is a line bundle by Proposition 8.0.3. Hence $\mathscr{L} \simeq j_*(\mathscr{L}|_{U_0})$ by Proposition 8.0.1.

(b) $\Rightarrow$ (c) The line bundle $\mathscr{L}|_{U_0}$ can be written as $\mathscr{O}_{U_0}(E)$ for some Cartier divisor $E = \sum_i a_i E_i$ on U_0. Consider the Weil divisor $D = \sum_i a_i D_i$, where D_i is the Zariski closure of E_i in X. Given $f \in \mathbb{C}(X)^*$, note that

$$\mathrm{div}(f) + D \geq 0 \iff (\mathrm{div}(f) + D)|_{U_0} \geq 0$$

since $\mathrm{codim}(X \setminus U_0) \geq 2$, and the same holds over any open set of X. Combining this with $E = D|_{U_0}$, we obtain

$$\mathscr{O}_X(D) \simeq j_* \mathscr{O}_{U_0}(E) = j_*(\mathscr{L}|_{U_0}) \simeq \mathscr{L}.$$

(c) $\Rightarrow$ (a) The proof of (b) $\Rightarrow$ (c) shows that $\mathscr{O}_X(D) \simeq j_*(\mathscr{O}_X(D)|_{U_0})$. But $\mathrm{codim}(X \setminus U_0) \geq 2$, and $\mathscr{O}_X(D)|_{U_0} = \mathscr{O}_{U_0}(D|_{U_0})$ is locally free since U_0 is smooth. Thus $j_*(\mathscr{O}_X(D)|_{U_0}) \simeq \mathscr{O}_X(D)^{\vee\vee}$ by Proposition 8.0.1, so $\mathscr{O}_X(D) \simeq \mathscr{O}_X(D)^{\vee\vee}$ is reflexive and has rank 1 since it is a line bundle on U_0. $\square$

Tensor Products and Duals. Given Weil divisors D, E on a normal variety X, the map $f \otimes g \mapsto fg$ defines a sheaf homomorphism

$$\mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{O}_X(E) \longrightarrow \mathscr{O}_X(D+E). \tag{8.0.1}$$

This is an isomorphism when D or E is Cartier but may fail to be an isomorphism in general.

Example 8.0.5. Consider the affine quadric cone $X = \mathbf{V}(y^2 - xz) \subseteq \mathbb{C}^3$. From examples in previous chapters, we know that this is a normal toric surface. The line $L = \mathbf{V}(y,z)$ gives a Weil divisor that is not Cartier, though $2L$ is Cartier (this follows from Example 4.2.3). The coordinate ring of X is $R = \mathbb{C}[x,y,z]/\langle y^2 - xz\rangle$. Let x, y, z denote the images of the variables in R. In Exercise 8.0.2 you will show the following:

- $\Gamma(X, \mathscr{O}_X(-L))$ is the ideal $\langle y,z\rangle \subseteq R$.
- $\Gamma(X, \mathscr{O}_X(-2L))$ is the ideal $\langle z\rangle \subseteq R$ (principal since $-2L$ is Cartier).
- On global sections, the image of the map

  $$\mathscr{O}_X(-L) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-L) \longrightarrow \mathscr{O}_X(-2L)$$

  is $\langle y,z\rangle^2$, which is a proper subset of $\Gamma(X, \mathscr{O}_X(-2L)) = \langle z\rangle$.

It follows that $\mathscr{O}_X(-L) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-L) \not\simeq \mathscr{O}_X(-2L)$. $\Diamond$

If we apply (8.0.1) when $E = -D$, we get a map

$$\mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-D) \longrightarrow \mathscr{O}_X,$$

which in turn induces a map

$$\mathscr{O}_X(-D) \longrightarrow \mathscr{O}_X(D)^\vee. \tag{8.0.2}$$

As noted in §6.0, this is an isomorphism when D is Cartier. In general, we have the following result about the maps (8.0.1) and (8.0.2).

Proposition 8.0.6. *Let D,E be Weil divisors on a normal variety X. Then* (8.0.1) *induces an isomorphism*

$$(\mathscr{O}_X(D)\otimes_{\mathscr{O}_X}\mathscr{O}_X(E))^{\vee\vee}\simeq\mathscr{O}_X(D+E).$$

Furthermore, (8.0.2) *is an isomorphism, i.e.,*

$$\mathscr{O}_X(-D)\simeq\mathscr{O}_X(D)^{\vee}.$$

Proof. The first isomorphism follows from Proposition 8.0.1 since $\mathscr{O}_X(D+E)$ is reflexive and (8.0.1) is an isomorphism on the smooth locus of X. The second isomorphism follows similarly since both sheaves are reflexive and (8.0.2) is an isomorphism on the smooth locus. □

***Divisor Classes*.** Recall that Weil divisors D and E on X are linearly equivalent, written $D\sim E$, if $D=E+\mathrm{div}(f)$ for some $f\in\mathbb{C}(X)^*$.

Proposition 8.0.7. *Let X be a normal variety.*

(a) *If D and E are Weil divisors on X, then*

$$\mathscr{O}_X(D)\simeq\mathscr{O}_X(E)\iff D\sim E.$$

(b) *If D is a Weil divisor on X, then*

$$D \textit{ is Cartier } \iff \mathscr{O}_X(D) \textit{ is a line bundle}.$$

Proof. Linearly equivalent divisors give isomorphic sheaves by Proposition 4.0.29. Conversely, $\mathscr{O}_X(D)\simeq\mathscr{O}_X(E)$ implies

$$\mathscr{O}_X(D)\otimes_{\mathscr{O}_X}\mathscr{O}_X(-E)\simeq\mathscr{O}_X(E)\otimes_{\mathscr{O}_X}\mathscr{O}_X(-E).$$

Taking the double dual and using Proposition 8.0.6, we get $\mathscr{O}_X(D-E)\simeq\mathscr{O}_X$. From here, showing that $D\sim E$ follows exactly as in the proof of Proposition 6.0.22.

One direction of part (b) was proved in Chapter 6 (see Proposition 6.0.17 and Theorem 6.0.18). Conversely, if $\mathscr{O}_X(D)$ is a line bundle on X, then Theorem 6.0.20 shows that $\mathscr{O}_X(D)\simeq\mathscr{O}_X(E)$ for some Cartier divisor E. Thus $D\sim E$ by part (a). Then we are done since any divisor linearly equivalent to a Cartier divisor is Cartier by Exercise 4.0.5. □

In Chapter 4 we defined the class group $\mathrm{Cl}(X)$ and Picard group $\mathrm{Pic}(X)$ in terms of Weil and Cartier divisors. Then, in Chapter 6, we reinterpreted $\mathrm{Pic}(X)$ as the group of isomorphism classes of line bundles, where the group operation was tensor product and the inverse was the dual. We can now reinterpret $\mathrm{Cl}(X)$ as the group of isomorphism classes of reflexive sheaves of rank 1, where the group operation is the double dual of the tensor product and the inverse is the dual. This follows immediately from Propositions 8.0.6 and 8.0.7.

Kähler Differentials. In order to give an algebraic definition of differential forms on a variety, we begin with the case of a $\mathbb{C}$-algebra.

Definition 8.0.8. Let R be a $\mathbb{C}$-algebra. The ***module of Kähler differentials*** of R over $\mathbb{C}$, denoted $\Omega_{R/\mathbb{C}}$, is the R-module generated by the formal symbols df for $f \in R$, modulo the relations

(a) $d(cf+g) = c\,df + dg$ for all $c \in \mathbb{C}$, $f, g \in R$.

(b) $d(fg) = f\,dg + g\,df$ for all $f, g \in R$.

Example 8.0.9. If $R = \mathbb{C}[x_1, \dots, x_n]$, then

$$\Omega_{R/\mathbb{C}} \simeq \bigoplus_{i=1}^{n} R\,dx_i.$$

This follows because the relations defining $\Omega_{R/\mathbb{C}}$ imply $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$ for all $f \in R$ (Exercise 8.0.3). $\Diamond$

A $\mathbb{C}$-algebra homomorphism $R \to S$ induces a natural homomorphism

$$\Omega_{R/\mathbb{C}} \longrightarrow \Omega_{S/\mathbb{C}}.$$

When we regard $\Omega_{S/\mathbb{C}}$ as an R-module, we obtain a homomorphism of S-modules

$$S \otimes_R \Omega_{R/\mathbb{C}} \longrightarrow \Omega_{S/\mathbb{C}}.$$

Here is a case when this map is easy to understand. See [**195**, Thm. 25.2] for a proof.

Proposition 8.0.10. *Let $R \to S$ be a surjection of $\mathbb{C}$-algebras with kernel I. Then there is an exact sequence of S-modules*

$$I/I^2 \longrightarrow S \otimes_R \Omega_{R/\mathbb{C}} \longrightarrow \Omega_{S/\mathbb{C}} \longrightarrow 0,$$

where $[f] \in I/I^2$ maps to $1 \otimes df \in S \otimes_R \Omega_{R/\mathbb{C}}$. $\square$

Example 8.0.11. Let $R = \mathbb{C}[x_1, \dots, x_n]$ and $S = R/I$, where $I = \langle f_1, \dots, f_s \rangle$. The generators of I give a surjection $R^s \to I$ and hence a surjection $S^s \to I/I^2$. Combining this with Proposition 8.0.10 and Example 8.0.9, we obtain an exact sequence

$$S^s \xrightarrow{\alpha} S^n \longrightarrow \Omega_{S/\mathbb{C}} \longrightarrow 0,$$

where α is given by the reduction of the $n \times s$ Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_s}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_s}{\partial x_n} \end{pmatrix} \tag{8.0.3}$$

modulo the ideal I (Exercise 8.0.4). This presentation of $\Omega_{S/\mathbb{C}}$ is very useful for computing examples. $\Diamond$

Kähler differentials also behave well under localization, as you will prove in Exercise 8.0.5.

Proposition 8.0.12. *Let R_f be the localization of R at a non-nilpotent element $f \in R$. Then $\Omega_{R_f/\mathbb{C}} \simeq \Omega_{R/\mathbb{C}} \otimes R_f$.* □

Cotangent and Tangent Sheaves. Now we globalize Definition 8.0.8.

Definition 8.0.13. Let X be a variety. The ***cotangent sheaf*** Ω^1_X is the sheaf of $\mathscr{O}_X$-modules defined via

$$\Omega^1_X(U) = \Omega_{\mathscr{O}_X(U)/\mathbb{C}}$$

on affine open sets U. The ***tangent sheaf*** $\mathscr{T}_X$ is the dual sheaf

$$\mathscr{T}_X = (\Omega^1_X)^\vee = \mathcal{H}om_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X).$$

The reason for the superscript in the notation for the cotangent sheaf will become clear later in this chapter. In Exercise 8.0.6 you will use Example 8.0.11 and Proposition 8.0.12 to show that Ω^1_X is a coherent sheaf. See [**131**, II.8] for a slightly different approach to defining the sheaf Ω^1_X, and [**131**, II.8, Comment 8.9.2] for the connection between these methods.

When $U = \operatorname{Spec}(R)$ is an affine open of X, the definition of the tangent sheaf implies that

$$\mathscr{T}_X(U) = \operatorname{Hom}_R(\Omega_{R/\mathbb{C}}, R).$$

This can also be described in terms of derivations—see Exercises 8.0.7 and 8.0.8.

When X is smooth, these sheaves are nicely behaved, as shown by the following result from [**131**, Thm. II.8.15].

Theorem 8.0.14. *A variety X is smooth if and only if Ω^1_X is locally free. When this happens, Ω^1_X and $\mathscr{T}_X$ are locally free sheaves of rank n, $n = \dim X$.* □

In the smooth toric case, it is easy to see that the cotangent sheaf is locally free.

Example 8.0.15. A smooth cone $\sigma \subseteq N_\mathbb{R} \simeq \mathbb{R}^n$ of dimension r gives the affine toric variety

$$U_\sigma \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{n-r} \subseteq \mathbb{C}^n.$$

Then Example 8.0.9 and Proposition 8.0.12 imply that $\Omega^1_{U_\sigma}$ is locally free of rank n. It follows immediately that $\Omega^1_{X_\Sigma}$ is locally free for any smooth toric variety X_Σ. ◊

We know from Chapter 6 that a locally free sheaf is the sheaf of sections of a vector bundle. When X is smooth, the vector bundles corresponding to Ω^1_X and $\mathscr{T}_X$ are called the *cotangent bundle* and *tangent bundle* respectively.

Example 8.0.16. We construct the cotangent bundle for $\mathbb{P}^2$. Recall that $\mathbb{P}^2$ has a covering by the affine open sets

$$\begin{aligned}
U_{\sigma_0} &\simeq \operatorname{Spec}(\mathbb{C}[x,y])\\
U_{\sigma_1} &\simeq \operatorname{Spec}(\mathbb{C}[yx^{-1},x^{-1}])\\
U_{\sigma_2} &\simeq \operatorname{Spec}(\mathbb{C}[xy^{-1},y^{-1}]).
\end{aligned}$$

where $\sigma_0,\sigma_1,\sigma_2$ are the maximal cones in the usual fan for $\mathbb{P}^2$.

Let $\mathbb{C}^2 = \operatorname{Spec}(R)$ for $R = \mathbb{C}[x,y]$. The module $\Omega_{R/\mathbb{C}}$ is a free R-module of rank 2 with generators dx, dy by Example 8.0.9. Thus a 1-form on $\mathbb{C}^2$ may be written uniquely as $f_1\,dx + f_2\,dy$, where $f_i \in R$. To generalize this to $\mathbb{P}^2$, we require that after changing coordinates, dx and dy transform via the Jacobian matrix described in Example 8.0.11. More precisely, the matrix for the transition function ϕ_{ij} will be the Jacobian of the map $U_{\sigma_j} \to U_{\sigma_i}$.

On U_{σ_2}, the coordinates (a_1,a_2) are represented in terms of the (x,y) coordinates on U_{σ_0} as (xy^{-1},y^{-1}), yielding

$$\phi_{20} = \begin{pmatrix} 1/y & -x/y^2\\ 0 & -1/y^2\end{pmatrix}.$$

Next, we compute ϕ_{12}. Things get messy if we keep everything in (x,y) coordinates, so we first translate to coordinates (a_1,a_2) on U_{σ_2}, and then translate back. On U_{σ_2} we identify (a_1,a_2) with (xy^{-1},y^{-1}). Then U_{σ_1} has coordinates

$$(yx^{-1},x^{-1}) = \left(\frac{1}{a_1},\frac{a_2}{a_1}\right).$$

So in terms of (a_1,a_2), we have

$$\phi_{12} = \begin{pmatrix} -1/a_1^2 & 0\\ -a_2/a_1^2 & 1/a_1\end{pmatrix}.$$

Rewriting this in terms of (x,y) yields

$$\phi_{12} = \begin{pmatrix} -y^2/x^2 & 0\\ -y/x^2 & y/x\end{pmatrix}.$$

Finally, computing ϕ_{10} directly, we obtain

$$\phi_{10} = \begin{pmatrix} -y/x^2 & 1/x\\ -1/x^2 & 0\end{pmatrix}.$$

A check shows that $\phi_{10} = \phi_{12}\circ\phi_{20}$. Similar computations show that the compatibility criteria are satisfied for all i,j,k, i.e.,

$$\phi_{ik} = \phi_{ij}\circ\phi_{jk}.$$

Since $\det(\phi_{ij})$ is invertible on $U_{\sigma_i}\cap U_{\sigma_j}$, the same is true for ϕ_{ij}. Hence we obtain a rank 2 vector bundle on $\mathbb{P}^2$ whose sheaf of sections is $\Omega^1_{\mathbb{P}^2}$. $\Diamond$

Relation with the Zariski Tangent Space. The definition of the tangent sheaf $\mathscr{T}_X$ seems far removed from the definition of the Zariski tangent space $T_p(X) = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2, \mathbb{C})$ given in Chapter 1. Here we explain (without proof) the connection.

The stalk $(\mathscr{T}_X)_p$ of the tangent sheaf at $p \in X$ can be described as follows. The stalk of Ω_X^1 at p is the module of Kähler differentials

$$(\Omega_X^1)_p = \Omega_{\mathscr{O}_{X,p}/\mathbb{C}},$$

where $\mathscr{O}_{X,p}$ is the local ring of X at p. Since $\mathscr{O}_{X,p}/\mathfrak{m}_{X,p} \simeq \mathbb{C}$ and $\Omega_{\mathbb{C}/\mathbb{C}} = 0$ (easy to check), the exact sequence of Proposition 8.0.10 gives a surjection

$$\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2 \longrightarrow \Omega_{\mathscr{O}_{X,p}/\mathbb{C}} \otimes_{\mathscr{O}_{X,p}} \mathbb{C}$$

which is an isomorphism of vector spaces over $\mathbb{C}$ by [**131**, Prop. II.8.7]. Since $\mathscr{T}_X$ is dual to Ω_X^1, taking the dual of the above isomorphism gives

$$(\mathscr{T}_X)_p \otimes_{\mathscr{O}_{X,p}} \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2, \mathbb{C}) = T_p(X). \tag{8.0.4}$$

This omits many details but should help you understand why $\mathscr{T}_X$ is the correct definition of tangent sheaf.

Example 8.0.17. Let $V \subseteq \mathbb{C}^n$ be defined by $I = \mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]$. The coordinate ring of V is $S = \mathbb{C}[x_1, \ldots, x_n]/I$, so that Example 8.0.11 gives the exact sequence

$$S^s \longrightarrow S^n \longrightarrow \Omega_{S/\mathbb{C}} \longrightarrow 0.$$

Now take $p \in V$ and tensor with $\mathscr{O}_{V,p}$ to obtain the exact sequence

$$\mathscr{O}_{V,p}^s \longrightarrow \mathscr{O}_{V,p}^n \longrightarrow \Omega_{\mathscr{O}_{V,p}/\mathbb{C}} \longrightarrow 0$$

(Exercise 8.0.9). If we tensor this with $\mathbb{C}$ and dualize, (8.0.4) and the isomorphism $\Omega_{\mathscr{O}_{X,p}/\mathbb{C}} \otimes_{\mathscr{O}_{X,p}} \mathbb{C} \simeq \mathfrak{m}_{X,p}/\mathfrak{m}_{X,p}^2$ give the exact sequence

$$0 \longrightarrow T_p(V) \longrightarrow \mathbb{C}^n \xrightarrow{\ \delta\ } \mathbb{C}^s,$$

where δ comes from the $s \times n$ Jacobian matrix $\big(\frac{\partial f_i}{\partial x_j}(p)\big)$ (Exercise 8.0.9). This explains the description of $T_p(V)$ given in Lemma 1.0.6. $\Diamond$

Conormal and Normal Sheaves. Given a closed subvariety $i : Y \hookrightarrow X$, it is natural to ask how their cotangent sheaves relate. We begin with the exact sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{O}_X \longrightarrow i_*\mathscr{O}_Y \longrightarrow 0,$$

which we write more informally as

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_Y \longrightarrow 0.$$

The quotient sheaf $\mathscr{I}_Y/\mathscr{I}_Y^2$ has a natural structure as a sheaf of $\mathscr{O}_Y$-modules, as does $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_Y$ for any sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules. The following basic result is proved in [**131**, Prop. II.8.12 and Thm. II.8.17].

Theorem 8.0.18. *Let Y be a closed subvariety of a variety X. Then:*

(a) *There is an exact sequence of $\mathscr{O}_Y$-modules*

$$\mathscr{I}_Y/\mathscr{I}_Y^2 \longrightarrow \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{O}_Y \longrightarrow \Omega_Y^1 \longrightarrow 0.$$

(b) *If X and Y are smooth, then this sequence is also exact on the left and $\mathscr{I}_Y/\mathscr{I}_Y^2$ is locally free of rank equal to the codimension of Y.* □

Note that part (a) of this theorem is a global version of Proposition 8.0.10. We call $\mathscr{I}_Y/\mathscr{I}_Y^2$ the *conormal sheaf* of Y in X and call its dual

$$\mathscr{N}_{Y/X} = (\mathscr{I}_Y/\mathscr{I}_Y^2)^\vee = \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathscr{O}_Y)$$

the *normal sheaf* of Y in X. When X and Y are smooth, we can dualize the sequence appearing in Theorem 8.0.18 to obtain the exact sequence

$$0 \longrightarrow \mathscr{T}_Y \longrightarrow \mathscr{T}_X \otimes_{\mathscr{O}_X} \mathscr{O}_Y \longrightarrow \mathscr{N}_{Y/X} \longrightarrow 0.$$

The vector bundle associated to $\mathscr{N}_{Y/X}$ is the *normal bundle* of Y in X. Then the above sequence says that when the tangent bundle of X is restricted to the subvariety Y, it contains the tangent bundle of Y with quotient given by the normal bundle. This is the algebraic analog of what happens in differential geometry, where the normal bundle is the orthogonal complement of the tangent bundle of Y.

Differential Forms. We call Ω_X^1 the *sheaf of* 1*-forms*, and we define the *sheaf of p-forms* to be the wedge product

$$\Omega_X^p = \textstyle\bigwedge^p \Omega_X^1.$$

For any sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules, the exterior power $\bigwedge^p \mathscr{F}$ is the sheaf associated to the presheaf which to each open set U assigns the $\mathscr{O}_X(U)$-module $\bigwedge^p \mathscr{F}(U)$.

Example 8.0.19. For $\mathbb{C}^n$, $\Omega_{\mathbb{C}^n}^1$ is the sheaf associated to the free R-module $\Omega_{R/\mathbb{C}} = \bigoplus_{i=1}^n R\,dx_i$, $R = \mathbb{C}[x_1,\dots,x_n]$. Then $\Omega_{\mathbb{C}^n}^p$ is the sheaf associated to

$$\textstyle\bigwedge^p \Omega_{R/\mathbb{C}} = \displaystyle\bigoplus_{1\le i_1<\cdots<i_p\le n} R\,dx_{i_1}\wedge\cdots\wedge dx_{i_p}.$$

It follows that $\Omega_{\mathbb{C}^n}^p$ is free of rank $\binom{n}{p}$. ◊

More generally, Theorem 8.0.14 implies that Ω_X^p is locally free of rank $\binom{n}{p}$ when X is smooth of dimension n. In particular, Ω_X^n is a line bundle in this case.

Zariski p-Forms and the Canonical Sheaf. For a normal variety X, the sheaf of p-forms Ω_X^p may fail to be locally free. However, this sheaf is locally free on the smooth locus of X, and the complement of the smooth locus has codimension ≥ 2 since X is normal. Hence we can use Proposition 8.0.1 to define the *sheaf of Zariski p-forms*

$$\widehat{\Omega}_X^p = (\Omega_X^p)^{\vee\vee} = j_*\Omega_{U_0}^p, \tag{8.0.5}$$

where $j : U_0 \hookrightarrow X$ is the inclusion of the smooth locus of X. It follows that $\widehat{\Omega}^p_X$ is a reflexive sheaf of rank $\binom{n}{p}$, where $n = \dim X$.

For later purposes, we note that by Proposition 8.0.1, (8.0.5) is valid for *any* smooth open subset $U_0 \subseteq X$ whose complement has codimension ≥ 2.

The case $p = n$ is especially important.

Definition 8.0.20. The ***canonical sheaf*** of a normal variety X is

$$\omega_X = \widehat{\Omega}^n_X,$$

where n is the dimension of X. This is a reflexive sheaf of rank 1, so that

$$\omega_X \simeq \mathscr{O}_X(D)$$

for some Weil divisor D on X. We call this divisor a ***canonical divisor*** of X, often denoted K_X.

Proposition 8.0.7 shows that the canonical divisor K_X is well-defined up to linear equivalence and hence gives a unique divisor class in $\mathrm{Cl}(X)$, known as the *canonical class* of X. In the toric case, we will see later in the chapter that there is a natural choice for the canonical divisor.

When X is smooth, we call ω_X the *canonical bundle* since it is a line bundle. In this case, the canonical divisor is Cartier. There are also singular varieties whose canonical divisors are Cartier—these are the *Gorenstein varieties* to be studied later in the chapter.

While it often suffices to know ω_X up to isomorphism, there are situations where a unique model of ω_X is required. One such construction uses $\bigwedge^n \Omega_{\mathbb{C}(X)/\mathbb{C}}$, where $\mathbb{C}(X)$ is the field of rational functions on X. We can regard $\bigwedge^n \Omega_{\mathbb{C}(X)/\mathbb{C}}$ as the constant sheaf of rational n-forms on X, similar to the way that $\mathbb{C}(X)$ gives the constant sheaf $\mathscr{K}_X$ of rational functions on X. There is an obvious sheaf map

$$\Omega^n_X \longrightarrow \textstyle\bigwedge^n \Omega_{\mathbb{C}(X)/\mathbb{C}}.$$

You will prove the following result in Exercises 8.0.10 and 8.0.11.

Proposition 8.0.21. *When X is normal, image of the map $\Omega^n_X \to \bigwedge^n \Omega_{\mathbb{C}(X)/\mathbb{C}}$ is*

$$\omega_X \subseteq \textstyle\bigwedge^n \Omega_{\mathbb{C}(X)/\mathbb{C}}. \qquad \square$$

The canonical sheaf can be defined for any irreducible variety X as a subsheaf of $\bigwedge^n \Omega_{\mathbb{C}(X)/\mathbb{C}}$, though the definition is more sophisticated (see [**184**, §9]). When X is projective, another approach is given in [**131**, III.7], where ω_X is called the *dualizing sheaf*. We will see in Chapter 9 that ω_X plays a key role in Serre duality.

Exercises for §8.0.

8.0.1. The rank of a coherent sheaf on an irreducible variety was defined in Definition 8.0.2. Here are some properties of the rank.

(a) Let an irreducible affine variety have coordinate ring R with field of fractions K. Let M be a finitely generated R-module. Show that $M \otimes_R K$ is a finite-dimensional vector space over K whose dimension equals the rank of the coherent sheaf $\widetilde{M}$ on $\mathrm{Spec}(R)$.

(b) Let $\mathscr{F}$ be a coherent sheaf on X let $U \subseteq X$ be an nonempty open subset. Prove that $\mathscr{F}$ and $\mathscr{F}|_U$ have the same rank.

(c) Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to$ be an exact sequence of sheaves on X. Prove that $\mathrm{rank}(\mathscr{G}) = \mathrm{rank}(\mathscr{F}) + \mathrm{rank}(\mathscr{H})$.

8.0.2. Prove the claims made in Example 8.0.5.

8.0.3. Let $R = \mathbb{C}[x_1,\dots,x_n]$. In Example 8.0.9 we claimed that $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i$ in $\Omega_{R/\mathbb{C}}$ for all $f \in R$. Prove this.

8.0.4. Prove that the map α in the exact sequence from Example 8.0.11 comes from the Jacobian matrix (8.0.3).

8.0.5. Prove Proposition 8.0.12.

8.0.6. Prove that the cotangent sheaf Ω^1_X defined in Definition 8.0.13 is a coherent sheaf.

8.0.7. Given a $\mathbb{C}$-algebra R and an R-module M, a $\mathbb{C}$-*derivation* $\delta : R \to M$ is a $\mathbb{C}$-linear map that satisfies the Leibniz rule, i.e., $\delta(fg) = f\delta(g) + g\delta(f)$ for all $f,g \in R$.

(a) Show that $f \to df$ defines a $\mathbb{C}$-derivation $d : R \to \Omega_{R/\mathbb{C}}$.

(b) More generally, show that if $\phi : \Omega_{R/\mathbb{C}} \to M$ is an R-module homomorphism, then $\phi \circ d : R \to M$ is a $\mathbb{C}$-derivation.

8.0.8. Continuing Exercise 8.0.7, we let $\mathrm{Der}_{\mathbb{C}}(R,M)$ denote the set of all $\mathbb{C}$-derivations $\delta : R \to M$. This is an R-module where $(r\delta)(f) = r\delta(f)$.

(a) Use part (b) of Exercise 8.0.7 to construct an R-module isomorphism $\mathrm{Der}_{\mathbb{C}}(R,M) \simeq \mathrm{Hom}_R(\Omega_{R/\mathbb{C}},M)$. Explain why $d : R \to \Omega_{R/\mathbb{C}}$ is called the *universal derivation*.

(b) Let $\mathscr{T}_X$ be the tangent sheaf of a variety X and let $U = \mathrm{Spec}(R)$ be an affine open subset of X. Prove that $\mathscr{T}_X(U) = \mathrm{Der}_{\mathbb{C}}(R,R)$.

8.0.9. Fill in the details omitted in Example 8.0.17.

8.0.10. Let $j : U \hookrightarrow X$ be the inclusion of a nonempty open subset of a variety X.

(a) Show that there is a sheaf map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ for any sheaf $\mathscr{F}$ on X.

(b) Show that the map of part (a) is an isomorphism when X is irreducible and $\mathscr{F}$ is a constant sheaf.

8.0.11. Prove Proposition 8.0.21. Hint: Ω^1_X is locally free when restricted to the smooth locus of X. Exercise 8.0.10 will be useful.

8.0.12. Let $1 \in T_N$ be the identity element of the torus T_N and let $\mathfrak{m} \subseteq \mathbb{C}[M]$ be the corresponding maximal ideal. Set $N_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}$ and $M_{\mathbb{C}} = M \otimes_{\mathbb{Z}} \mathbb{C}$.

(a) Let $f = \sum_m c_m \chi^m \in \mathbb{C}[M]$. Show that $f \in \mathfrak{m}^2$ if and only if $\sum_m c_m = 0$ in $\mathbb{C}$ and $\sum_m c_m m = 0$ in $M_{\mathbb{C}}$. Hint: Pick a basis $e_1,\dots,e_n$ of M and set $t_i = \chi^{e_i}$, so that f is a Laurent monomial in $t_1,\dots,t_n$. Then show that $f \in \mathfrak{m}^2$ if and only if $f \in \mathfrak{m}$ and $\frac{\partial f}{\partial t_i}(1) = 0$ for all i.

(b) Use part (a) to construct an isomorphism $\mathfrak{m}/\mathfrak{m}^2 \simeq M_{\mathbb{C}}$, and conclude that the Zariski tangent space of T_N at the identity is naturally isomorphic to $N_{\mathbb{C}}$.

8.0.13. Let F be a free module of rank n over a ring R and fix $0 \le p \le n$. Recall that wedge product induces an isomorphism $\bigwedge^{n-p} F \simeq \mathrm{Hom}_R(\bigwedge^p F, \bigwedge^n F)$.

(a) If X is smooth of dimension n, then show that $\Omega_X^{n-p} \simeq \mathscr{H}om_{\mathscr{O}_X}(\Omega_X^p, \Omega_X^n)$.

(b) Show that if X is a normal variety of dimension n, then there is an isomorphism $\widehat{\Omega}_X^{n-p} \simeq \mathscr{H}om_{\mathscr{O}_X}(\widehat{\Omega}_X^p, \omega_X)$. Hint: If you get stuck, see [**76**, Prop. 4.7].

(c) In a similar vein, show that the tangent sheaf $\mathscr{T}_X$ of a normal variety X satisfies $\mathscr{T}_X \simeq \mathscr{H}om_{\mathscr{O}_X}(\widehat{\Omega}_X^1, \mathscr{O}_X)$.

§8.1. One-Forms on Toric Varieties

In this section we will describe two interesting exact sequences that involve the sheaves $\omega^1_{X_\Sigma}$ and $\widehat{\Omega}^1_{X_\Sigma}$ on a normal toric variety X_Σ.

The Torus. The coordinate ring of the torus T_N is the semigroup algebra $\mathbb{C}[M]$. Then the map

$$\Omega_{\mathbb{C}[M]/\mathbb{C}} \longrightarrow M \otimes_{\mathbb{Z}} \mathbb{C}[M]$$

defined by $d\chi^m \mapsto m \otimes \chi^m$ is easily seen to be an isomorphism. It follows that

$$\Omega^1_{T_N} \simeq M \otimes_{\mathbb{Z}} \mathscr{O}_{T_N}, \tag{8.1.1}$$

and dualizing, we obtain

$$\mathscr{T}_{T_N} \simeq N \otimes_{\mathbb{Z}} \mathscr{O}_{T_N} = N_{\mathbb{C}} \otimes_{\mathbb{C}} \mathscr{O}_{T_N}.$$

This makes intuitive sense since $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ as a complex Lie group. Thus its tangent space at the identity is $N \otimes_{\mathbb{Z}} \mathbb{C} = N_{\mathbb{C}}$ via the exponential map. This is also true algebraically, as shown in Exercise 8.0.12. The group action transports the tangent space $N_{\mathbb{C}}$ over the whole torus, which explains the above trivialization of the tangent bundle $\mathscr{T}_{T_N}$.

As a consequence, the 1-form $\frac{d\chi^m}{\chi^m}$ is a global section of $\Omega^1_{T_N}$ that maps to $m \otimes 1$ in (8.1.1) and hence is invariant under the action of T_N. See Exercise 8.1.1 for more on invariant 1-forms on the torus.

The First Exact Sequence. Now consider the toric variety X_Σ of the fan Σ. For $\rho \in \Sigma(1)$, the inclusion $i : D_\rho \hookrightarrow X_\Sigma$ gives the sheaf $i_*\mathscr{O}_{D_\rho}$ on X_Σ, which following §8.0 we write as $\mathscr{O}_{D_\rho}$. Using the map $M \to \mathbb{Z}$ given by $m \mapsto \langle m, u_\rho \rangle$, we obtain the composition

$$M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} = \mathscr{O}_{X_\Sigma} \longrightarrow \mathscr{O}_{D_\rho}.$$

This gives a natural map

$$\beta : M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \longrightarrow \textstyle\bigoplus_\rho \mathscr{O}_{D_\rho}, \tag{8.1.2}$$

where the direct sum is over all $\rho \in \Sigma(1)$. We also have a canonical map

$$\alpha : \Omega^1_{X_\Sigma} \longrightarrow M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \tag{8.1.3}$$

constructed as follows. On the affine piece $U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$, α is defined by

$$d\chi^m \in \Omega_{\mathbb{C}[\sigma^\vee\cap M]/\mathbb{C}} \longmapsto m \otimes \chi^m \in M \otimes_{\mathbb{Z}} \mathbb{C}[\sigma^\vee \cap M].$$

These $\mathbb{C}[\sigma^\vee \cap M]$-module homomorphisms $\Omega_{\mathbb{C}[\sigma^\vee\cap M]/\mathbb{C}} \to M \otimes_{\mathbb{Z}} \mathbb{C}[\sigma^\vee \cap M]$ patch to give the desired map (8.1.3) (Exercise 8.1.2). Note that over the torus T_N, the map α of (8.1.3) reduces to the isomorphism (8.1.1).

Theorem 8.1.1. *For a smooth toric variety X_Σ, the sequence*

$$0 \longrightarrow \Omega^1_{X_\Sigma} \xrightarrow{\ \alpha\ } M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \xrightarrow{\ \beta\ } \bigoplus\nolimits_\rho \mathscr{O}_{D_\rho} \longrightarrow 0$$

formed using (8.1.2) *and* (8.1.3) *is exact.*

Proof. We first verify that $\beta\circ\alpha$ is the zero map. On the affine piece $U_\sigma \subseteq X_\Sigma$, the subvariety $D_\rho \cap U_\sigma \subseteq U_\sigma$ is defined by the ideal $I_\rho = \mathbf{I}(D_\rho \cap U_\sigma) \subseteq \mathbb{C}[\sigma^\vee \cap M]$. By Propositions 4.0.28 and 4.3.2, we have

$$\text{(8.1.4)}\quad I_\rho = \Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(-D_\rho)) = \bigoplus_{\mathrm{div}(\chi^m)|_{U_\sigma} \ge D_\rho|_{U_\sigma}} \mathbb{C}\cdot\chi^m = \bigoplus_{m\in\sigma^\vee\cap M,\, \langle m,u_\rho\rangle>0} \mathbb{C}\cdot\chi^m.$$

Over U_σ, the composition $\Omega^1_{X_\Sigma} \to M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \to \mathscr{O}_{D_\rho}$ takes a 1-form $d\chi^m$, $m \in \sigma^\vee \cap M$, to $\langle m, u_\rho\rangle\, \overline{\chi^m} \in \mathbb{C}[\sigma^\vee \cap M]/I_\rho$. This is obviously zero if $\langle m,u_\rho\rangle = 0$, and if $\langle m, u_\rho\rangle \neq 0$, it vanishes since $\chi^m \in I_\rho$ in this case.

We now verify that the sequence is exact over U_σ. Since σ is smooth, we may assume $\sigma = \mathrm{Cone}(e_1,\dots,e_r)$, where $r \le n$ and $e_1,\dots,e_n$ is a basis of N. Then $U_\sigma = \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$. Let $x_1,\dots,x_n$ denote the characters of the corresponding dual basis of M, also denoted $e_1,\dots,e_n$. The coordinate ring of U_σ is $R = \mathbb{C}[x_1,\dots,x_r,x_{r+1}^{\pm1},\dots,x_n^{\pm1}]$, and the 1-forms on U_σ form the free R-module $\Omega_{R/\mathbb{C}} = \bigoplus_{i=1}^n R\,dx_i$ by Example 8.0.9 and Proposition 8.0.12. Since α takes dx_i to $e_i \otimes x_i$, we see that α can be regarded as the map

$$\Omega_{R/\mathbb{C}} = \bigoplus_{i=1}^n R\,dx_i \longrightarrow M \otimes_{\mathbb{R}} R = \bigoplus_{i=1}^n R$$

that sends $\sum_{i=1}^n f_i\,dx_i$ to $(f_1x_1,\dots,f_nx_n)$. This gives the exact sequence

$$0 \longrightarrow \Omega_{R/\mathbb{C}} \longrightarrow \bigoplus_{i=1}^n R \longrightarrow \bigoplus_{i=1}^r R/\langle x_i\rangle \longrightarrow 0$$

since $x_{r+1},\dots,x_n$ are units in R, and the theorem follows. $\square$

Logarithmic Forms. The exact sequence of Theorem 8.1.1 has a lovely interpretation in terms of residues of logarithmic 1-forms. The idea is that $M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}$ can be thought of as the sheaf $\omega^1_{X_\Sigma}(\log D)$ of 1-forms on X_Σ with logarithmic poles along $D = \sum_\rho D_\rho$. We begin with an example.

Example 8.1.2. The coordinate ring of $\mathbb{C}^n$ is $R = \mathbb{C}[x_1,\dots,x_n]$, and the divisor D is the sum of the coordinate hyperplanes $D_i = \mathbf{V}(x_i)$. As above, $\Omega_{R/\mathbb{C}} = \bigoplus_{i=1}^n R\,dx_i$. Now introduce some denominators: a rational 1-form ω has *logarithmic poles along D* if

$$\omega = \sum_{i=1}^n f_i \frac{dx_i}{x_i}, \quad f_i \in R.$$

These form the free R-module $\bigoplus_{i=1}^n R\,\frac{dx_i}{x_i}$, and the corresponding sheaf is defined to be $\Omega^1_{\mathbb{C}^n}(\log D)$. The formal calculation

$$\frac{d\chi^m}{\chi^m} = \sum_{i=1}^n \langle m, e_i\rangle \frac{dx_i}{x_i}$$

shows that the map $\frac{d\chi^m}{\chi^m} \mapsto m \otimes 1$ induces an isomorphism of sheaves

$$\Omega^1_{\mathbb{C}^n}(\log D) \simeq M \otimes_{\mathbb{C}^n} \mathscr{O}_{\mathbb{C}^n}$$

such that the map $\alpha : \Omega^1_{\mathbb{C}^n} \to M \otimes_{\mathbb{C}^n} \mathscr{O}_{\mathbb{C}^n}$ of (8.1.3) is induced by the inclusion of 1-forms $\Omega^1_{\mathbb{C}^n} \hookrightarrow \Omega^1_{\mathbb{C}^n}(\log D)$. $\Diamond$

This construction works for any smooth affine toric variety U_σ, and the sheaves of logarithmic 1-forms on U_σ patch to give the sheaf $\omega^1_{X_\Sigma}(\log D)$ for any smooth toric variety X_Σ. Furthermore, we have a canonical isomorphism

$$\omega^1_{X_\Sigma}(\log D) \simeq M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \tag{8.1.5}$$

such that the map α of (8.1.3) comes from the inclusion of 1-forms.

The construction of $\omega^1_{X_\Sigma}(\log D)$ can be done more generally. Let X be a smooth variety. A divisor $D = \sum_i D_i$ on X has *simple normal crossings* if every D_i is smooth and irreducible, and for every $p \in X$, the divisors containing p meet nicely. More precisely, if $I_p = \{i \mid p \in D_i\}$, we require that the tangent spaces $T_p(D_i) \subseteq T_p(X)$ meet transversely, i.e.,

$$\operatorname{codim}\Big(\bigcap_{i\in I_p} T_p(D_i)\Big) = |I_p|.$$

For example, the divisor $D = \sum_\rho D_\rho$ is a simple normal crossing divisor on any smooth toric variety X_Σ. A nice discussion of $\Omega^1_X(\log D)$ for complex manifolds can be found in [**125**, p. 449].

The Poincaré Residue Map. Let $f(z)$ be an analytic (also called holomorphic) function defined in a punctured neighborhood of a point $p \in \mathbb{C}$. Take a counterclockwise loop C around p on X such that $f(z)$ has no other poles inside C. Then the *residue* of the 1-form $\omega = f(z)\,dz$ at p is the contour integral

$$\operatorname{res}_p(\omega) = \frac{1}{2\pi i}\int_C \omega.$$

In particular, if $p = 0$ and $f(z) = \frac{g(z)}{z}$, with $g(z)$ analytic at zero, then the residue theorem tells us that $\mathrm{res}_p(\omega)$ is the coefficient of $\frac{1}{z}$ in the Laurent series for $f(z)$ at 0, and is equal to $g(0)$. Note that the 1-form $\omega = f(z)\,dz = g(z)\frac{dz}{z}$ has a logarithmic pole at $p = 0$.

When there are several variables, we can do the same construction by working one variable at a time. Here is an example.

Example 8.1.3. Given $f = f(x_1,\dots,x_n) \in R = \mathbb{C}[x_1,\dots,x_n]$, we get the logarithmic 1-form $\omega = f\frac{dx_1}{x_1}$. In terms of the above discussion of residues, we can regard $f(0,x_2,\dots,x_n)$ as the "residue" of ω at $\mathbf{V}(x_1)$. Note also that $f(0,x_2,\dots,x_n)$ represents the class of f in $R/\langle x_1\rangle$. Doing this for every variable shows that the map

$$\Omega^1_{\mathbb{C}^n}(\log D) \simeq M \otimes_{\mathbb{Z}} \mathscr{O}_{\mathbb{C}^n} \xrightarrow{\beta} \bigoplus_{i=1}^{n} \mathscr{O}_{D_i}$$

can be interpreted as a sum of "residue" maps. $\Diamond$

More generally, if X is a smooth variety and $D = \sum_i D_i$ a simple normal crossing divisor, one can define the ***Poincaré residue map***

$$P_r : \Omega^1_X(\log D) \longrightarrow \bigoplus_i \mathscr{O}_{D_i}$$

(see [**227**, p. 254]) such that we have an exact sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log D) \xrightarrow{P_r} \bigoplus_i \mathscr{O}_{D_i} \longrightarrow 0. \tag{8.1.6}$$

When applied to a smooth toric variety X_Σ and the divisor $D = \sum_\rho D_\rho$, this gives the exact sequence of Theorem 8.1.1 via the isomorphism (8.1.5).

The Normal Case. When X_Σ is normal, we get an analog of Theorem 8.1.1 that uses the sheaf $\widehat{\Omega}^1_{X_\Sigma}$ of Zariski 1-forms in place of $\Omega^1_{X_\Sigma}$. Since $M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}$ is reflexive, taking the double dual of (8.1.3) gives a map

$$\widehat{\Omega}^1_{X_\Sigma} \longrightarrow M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}.$$

Theorem 8.1.4. *Let X_Σ be a normal toric variety. Then:*

(a) *The sequence*

$$0 \longrightarrow \widehat{\Omega}^1_{X_\Sigma} \longrightarrow M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \longrightarrow \textstyle\bigoplus_\rho \mathscr{O}_{D_\rho}$$

is exact.

(b) *If X_Σ is simplicial, then the map on the right is surjective, so that*

$$0 \longrightarrow \widehat{\Omega}^1_{X_\Sigma} \longrightarrow M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \longrightarrow \textstyle\bigoplus_\rho \mathscr{O}_{D_\rho} \longrightarrow 0$$

is exact.

Proof. Let $j : U_0 \subseteq X_\Sigma$ be the inclusion map for $U_0 = \bigcup_\rho U_\rho$. Note that U_0 is a smooth toric variety whose fan has the same 1-dimensional cones as Σ, and $\operatorname{codim}(X \setminus U_0) \geq 2$ by the Orbit-Cone Correspondence. By Theorem 8.1.1, we have an exact sequence

$$0 \longrightarrow \Omega^1_{U_0} \longrightarrow M \otimes_{\mathbb{Z}} \mathscr{O}_{U_0} \longrightarrow \bigoplus_\rho \mathscr{O}_{D_\rho \cap U_0} \longrightarrow 0,$$

so that applying j_* gives the exact sequence

$$0 \longrightarrow j_*\Omega^1_{U_0} \longrightarrow j_*(M \otimes_{\mathbb{Z}} \mathscr{O}_{U_0}) \longrightarrow \bigoplus_\rho j_*\mathscr{O}_{D_\rho \cap U_0}$$

since j_* is left exact (Exercise 8.1.3). However, $j_*\Omega^1_{U_0} = \widehat{\Omega}^1_{X_\Sigma}$ by the remarks following (8.0.5), and $j_*(M \otimes_{\mathbb{Z}} \mathscr{O}_{U_0}) = M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}$ by Proposition 8.0.1. Hence we get an exact sequence

$$0 \longrightarrow \widehat{\Omega}^1_{X_\Sigma} \longrightarrow M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \longrightarrow \bigoplus_\rho j_*\mathscr{O}_{D_\rho \cap U_0}.$$

In Exercise 8.1.4 you will show that the maps $M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \to j_*\mathscr{O}_{D_\rho \cap U_0}$ factor as

$$M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \longrightarrow \mathscr{O}_{D_\rho} \longrightarrow j_*\mathscr{O}_{D_\rho \cap U_0},$$

where $\mathscr{O}_{D_\rho} \to j_*\mathscr{O}_{D_\rho \cap U_0}$ is injective. The exact sequence of part (a) then follows immediately.

It remains to show that $M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \to \bigoplus_\rho \mathscr{O}_{D_\rho}$ is surjective when X_Σ is simplicial. Given $\sigma \in \Sigma$, we need to show that

$$M \otimes_{\mathbb{Z}} \mathscr{O}_{U_\sigma} \longrightarrow \bigoplus_{\rho \in \sigma(1)} \mathscr{O}_{D_\rho \cap U_\sigma}$$

is surjective. Fix $\rho \in \sigma(1)$ and pick $m \in M$ such that $\langle m, u_\rho \rangle \neq 0$ and $\langle m, u_{\rho'} \rangle = 0$ for all $\rho' \neq \rho$ in $\sigma(1)$. Such an m exists since σ is simplicial. Then $m \otimes 1$ maps to a nonzero constant function on $\mathscr{O}_{D_\rho \cap U_\sigma}$ and to the zero function on $\mathscr{O}_{D_{\rho'} \cap U_\sigma}$ for $\rho' \neq \rho$. The desired surjectivity now follows easily. □

When X_Σ has no torus factors, we learned in §5.3 that graded modules over the total coordinate ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ give quasicoherent sheaves on X_Σ. It is easy to describe a graded S-module that gives $\widehat{\Omega}^1_{X_\Sigma}$. For each ρ, there are two maps

$$M \otimes_{\mathbb{Z}} S \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}} S = S \longrightarrow S/\langle x_\rho \rangle,$$

where the first map comes from $m \mapsto \langle m, u_\rho \rangle$ and the second map is obvious. This gives a homomorphism $M \otimes_{\mathbb{Z}} S \to \bigoplus_\rho S/\langle x_\rho \rangle$, and we define $\widehat{\Omega}^1_S$ to be the kernel of this map. Hence we have an exact sequence of graded S-modules

$$0 \longrightarrow \widehat{\Omega}^1_S \longrightarrow M \otimes S \longrightarrow \bigoplus_\rho S/\langle x_\rho \rangle. \tag{8.1.7}$$

Using Example 6.0.10 and Theorem 8.1.4, we obtain the following result.

Corollary 8.1.5. *When X_Σ has no torus factors, $\widehat{\Omega}^1_{X_\Sigma}$ is the sheaf associated to the graded S-module $\widehat{\Omega}^1_S$.* □

The Euler Sequence. In [**131**, Thm. II.8.13], Hartshorne constructs an exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-1)^{n+1} \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow 0, \tag{8.1.8}$$

called the *Euler sequence* of $\mathbb{P}^n$. He goes on to say "This is a fundamental result, upon which we will base all future calculations involving differentials on projective varieties." Of course, $\mathbb{P}^n$ is toric, and there is a toric generalization of this result, due to Batyrev and Mel′nikov [**20**] and Jaczewski [**160**] in the smooth case and Batyrev and Cox [**19**, Thm. 12.1] in the simplicial case.

Theorem 8.1.6. *Let X_Σ be a simplicial toric variety with no torus factors, i.e., $\{u_\rho \mid \rho \in \Sigma(1)\}$ spans $N_\mathbb{R}$. Then there is an exact sequence*

$$0 \longrightarrow \widehat{\Omega}^1_{X_\Sigma} \longrightarrow \textstyle\bigoplus_\rho \mathscr{O}_{X_\Sigma}(-D_\rho) \longrightarrow \mathrm{Cl}(X_\Sigma) \otimes_\mathbb{Z} \mathscr{O}_{X_\Sigma} \longrightarrow 0.$$

Furthermore, if X_Σ is smooth, then the sequence can be written

$$0 \longrightarrow \Omega^1_{X_\Sigma} \longrightarrow \textstyle\bigoplus_\rho \mathscr{O}_{X_\Sigma}(-D_\rho) \longrightarrow \mathrm{Pic}(X_\Sigma) \otimes_\mathbb{Z} \mathscr{O}_{X_\Sigma} \longrightarrow 0.$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d, dotted] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{\Omega}^1_{X_\Sigma} \arrow[r] \arrow[d, dotted] & M \otimes_\mathbb{Z} \mathscr{O}_{X_\Sigma} \arrow[r] \arrow[d] & \bigoplus_\rho \mathscr{O}_{D_\rho} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_\rho \mathscr{O}_{X_\Sigma}(-D_\rho) \arrow[r] \arrow[d, dotted] & \bigoplus_\rho \mathscr{O}_{X_\Sigma} \arrow[r] \arrow[d] & \bigoplus_\rho \mathscr{O}_{D_\rho} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Cl}(X_\Sigma) \otimes_\mathbb{Z} \mathscr{O}_{X_\Sigma} \arrow[r] \arrow[d, dotted] & \mathrm{Cl}(X_\Sigma) \otimes_\mathbb{Z} \mathscr{O}_{X_\Sigma} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The top row is from Theorem 8.1.4 and is exact since X_Σ is simplicial. Also, by Proposition 4.0.28, each $\rho \in \Sigma(1)$ gives an exact sequence

$$0 \longrightarrow \mathscr{O}_{X_\Sigma}(-D_\rho) \longrightarrow \mathscr{O}_{X_\Sigma} \longrightarrow \mathscr{O}_{D_\rho} \longrightarrow 0,$$

and the middle row is the direct sum of these exact sequences. The third row is the obvious exact sequence that uses the identity map on $\mathrm{Cl}(X_\Sigma) \otimes_\mathbb{Z} \mathscr{O}_{X_\Sigma}$.

Since X_Σ has no torus factors, we have the exact sequence

$$0 \longrightarrow M \longrightarrow \textstyle\bigoplus_\rho \mathbb{Z} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

from Theorem 4.1.3, and tensoring this with $\mathscr{O}_{X_\Sigma}$ gives the middle column. The column on the right is the another obvious exact sequence, and one can check without difficulty that the solid arrows in the diagram commute (Exercise 8.1.5).

Then commutativity and exactness imply the existence of the dotted arrows in the diagram, which give the desired exact sequence by a standard diagram chase. $\square$

The exact sequence of sheaves in Theorem 8.1.6 is the (generalized) *Euler sequence* of the toric variety X_Σ. We will use it in the next section to determine the canonical sheaf of X_Σ. The Euler sequence also encodes relations generalizing the classical Euler relation for homogeneous polynomials (see Exercise 8.1.8). Note also that in [**160**], Jaczewski shows that smooth toric varieties can be characterized as smooth varieties which admit a generalized Euler sequence.

Exercises for §8.1.

8.1.1. We will study invariant 1-forms and derivations on the torus. Since the torus $T_N = \operatorname{Spec}(\mathbb{C}[M])$ is affine, we know from Exercise 8.0.8 that the derivations $\operatorname{Der}_{\mathbb{C}}(\mathbb{C}[M],\mathbb{C}[M])$ give the global sections of the tangent sheaf $\mathscr{T}_{T_N}$.

(a) For $u \in N$, define $\partial_u : \mathbb{C}[M] \to \mathbb{C}[M]$ by
$$\partial_u(\chi^m) = \langle m,u\rangle \chi^m.$$
Prove that $\partial_u \in \operatorname{Der}_{\mathbb{C}}(\mathbb{C}[M],\mathbb{C}[M])$

(b) Let $x_1,\ldots,x_n$ be the characters corresponding to the elements of M dual to some particular basis $e_1,\ldots,e_n$ of N. Thus $\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1},\ldots,x_n^{\pm 1}]$. Prove that $\partial_{e_i} = x_i \frac{\partial}{\partial x_i}$ and that $\Gamma(T_N,\mathscr{T}_{T_N})$ is the free $\mathbb{C}[M]$-module generated by $x_1\frac{\partial}{\partial x_1},\ldots,x_n\frac{\partial}{\partial x_n}$.

(c) Dualizing, conclude that $\Gamma(T_N,\Omega^1_{T_N})$ is the free $\mathbb{C}[M]$-module generated by the T_N-invariant differentials $\frac{dx_1}{x_1},\ldots,\frac{dx_n}{x_n}$.

8.1.2. Consider the affine toric variety $U_\sigma = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$.

(a) Prove that the map
$$d\chi^m \in \Omega_{\mathbb{C}[\sigma^\vee\cap M]/\mathbb{C}} \longmapsto m\otimes\chi^m \in M\otimes_{\mathbb{Z}}\mathbb{C}[\sigma^\vee\cap M]$$
defines a $\mathbb{C}[\sigma^\vee \cap M]$-module homomorphism.

(b) For a toric variety X_Σ, prove that these homomorphisms patch together to give the map $\alpha : \Omega^1_{X_\Sigma} \longrightarrow M\otimes_{\mathbb{Z}}\mathscr{O}_{X_\Sigma}$ in (8.1.3).

8.1.3. Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of sheaves on X and let $f : X \to Y$ be a morphism.

(a) Prove that $0 \to f_*\mathscr{F} \to f_*\mathscr{G} \to f_*\mathscr{H}$ is exact on Y.

(b) Suppose that $Y = \{\mathrm{pt}\}$ and $f : X \to Y$ is the obvious map. Use part (a) to give a new proof of Proposition 6.0.8.

8.1.4. In the proof of Theorem 8.1.4, show that the map $M\otimes_{\mathbb{Z}}\mathscr{O}_{X_\Sigma} \to j_*\mathscr{O}_{D_\rho\cap U_0}$ factors as
$$M\otimes_{\mathbb{Z}}\mathscr{O}_{X_\Sigma} \longrightarrow \mathscr{O}_{D_\rho} \longrightarrow j_*\mathscr{O}_{D_\rho\cap U_0},$$
where $\mathscr{O}_{D_\rho} \to j_*\mathscr{O}_{D_\rho\cap U_0}$ is injective.

8.1.5. The proof of Theorem 8.1.6 contains the square

$$\begin{array}{ccc} M\otimes_{\mathbb{Z}}\mathscr{O}_{X_\Sigma} & \longrightarrow & \bigoplus_\rho \mathscr{O}_{D_\rho} \\ \big\downarrow & & \big\downarrow \\ \bigoplus_\rho \mathscr{O}_{X_\Sigma} & \longrightarrow & \bigoplus_\rho \mathscr{O}_{D_\rho}. \end{array}$$

Describe the maps in this square carefully and prove that it commutes.

8.1.6. Show that the Euler sequence from Theorem 8.1.6 reduces to (8.1.8) when $X = \mathbb{P}^n$.

8.1.7. Sometimes the name *Euler sequence* is used to refer to an exact sequence for the tangent sheaf $\mathscr{T}_{X_\Sigma}$ of a smooth toric variety.

(a) Show that for $\mathbb{P}^n$, we have an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow \mathscr{O}_{\mathbb{P}^n}(1)^{n+1} \longrightarrow \mathscr{T}_{\mathbb{P}^n} \longrightarrow 0.$$

Hint: Use (8.1.8).

(b) What is the corresponding sequence for a general smooth toric variety X_Σ for Σ as in Theorem 8.1.6?

8.1.8. Let f be a homogeneous polynomial of degree d in $\mathbb{C}[x_1,\dots,x_n]$. The classical *Euler relation* is the equation

$$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = d\cdot f. \tag{8.1.9}$$

In this exercise, you will prove this relation and consider generalizations encoded by the generalized Euler sequence from a toric variety.

(a) Prove (8.1.9). Hint: Differentiate the equation

$$f(tx_1,\dots,tx_d) = t^d f(x_1,\dots,x_n)$$

with respect to t.

(b) To see how the classical Euler relation generalizes, recall from Chapter 5 that given a toric variety X_Σ with no torus factors (i.e., $\{u_\rho \mid \rho\in\Sigma(1)\}$ spans $N_\mathbb{R}$), we have the total coordinate ring

$$S = \mathbb{C}[x_\rho \mid \rho\in\Sigma(1)],$$

graded by $\mathrm{Cl}(X_\Sigma)$. The graded pieces S_β for $\beta\in\mathrm{Cl}(X_\Sigma)$ consist of homogeneous polynomials as described by (5.2.1) from Chapter 5. If $\phi\in\mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma),\mathbb{Z})$ and $f\in S_\beta$ show that we have a generalized Euler relation

$$\sum_{\rho\in\Sigma(1)} \phi([D_\rho])\, x_\rho \frac{\partial f}{\partial x_\rho} = \phi(\beta)\cdot f.$$

Hint: Follow what you did for part a, which is the case $X = \mathbb{P}^{n-1}$.

(c) When $\mathrm{Cl}(X_\Sigma)$ has rank greater than 1, there will be several distinct generalized Euler relations on homogeneous elements of S. Compute the Euler relations on $X = \mathbb{P}^1\times\mathbb{P}^1$.

§8.2. Differential Forms on Toric Varieties

For a toric variety X_Σ, we have the sheaf of p-forms $\Omega^p_{X_\Sigma}$ and the sheaf of Zariski p-forms $\widehat{\Omega}^p_{X_\Sigma}$. By §8.0, the canonical sheaf of X_Σ is $\omega_{X_\Sigma} = \widehat{\Omega}^n_{X_\Sigma}$, $n = \dim X_\Sigma$.

Properties of Wedge Products. We will need the following properties of wedge products of free R-modules.

Proposition 8.2.1. *Let F,G,H be free R-modules of finite rank.*

(a) *An R-module homomorphism $\phi : F \to G$ induces a homomorphism*

$$\textstyle\bigwedge^p \phi : \bigwedge^p F \longrightarrow \bigwedge^p G.$$

(b) *Let $0 \to F \to G \to H \to 0$ be an exact sequence with* $\operatorname{rank} F = m$ *and* $\operatorname{rank} H = n$*. Then* $\operatorname{rank} G = m+n$ *and there is a natural isomorphism*

$$\textstyle\bigwedge^{m+n} G \simeq \bigwedge^m F \otimes_R \bigwedge^n H.$$

Proof. Part (a) is straightforward (see Exercise 8.2.1 for an explicit description of $\bigwedge^p \phi$), as is the rank assertion in part (b). For the isomorphism of part (b), we assume $n > 0$ and define a map $\bigwedge^m F \otimes_R \bigwedge^n H \to \bigwedge^{m+n} G$ as follows. If the maps in the exact sequence are $\alpha : F \to G$ and $\beta : G \to H$, then one checks that $\bigwedge^m \alpha : \bigwedge^m F \to \bigwedge^m G$ is injective and $\bigwedge^n \beta : \bigwedge^n G \to \bigwedge^n H$ is surjective. Then $\mu \otimes \nu \in \bigwedge^m F \otimes_R \bigwedge^n H$ maps to $\bigwedge^m \alpha(\mu) \wedge \nu' \in \bigwedge^{m+n} G$, where $\bigwedge^n \nu' = \nu$. This map is well-defined and gives the desired isomorphism (Exercise 8.2.1). $\square$

A corollary of this proposition is that if $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of locally free sheaves on a variety X with $\operatorname{rank} \mathscr{F} = m$ and $\operatorname{rank} \mathscr{H} = n$, then $\operatorname{rank} \mathscr{G} = m+n$ and there is a natural isomorphism

$$\textstyle\bigwedge^{m+n} \mathscr{G} \simeq \bigwedge^m \mathscr{F} \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{H}. \tag{8.2.1}$$

Example 8.2.2. Suppose that $Y \subseteq X$ is a smooth subvariety of a smooth variety, and let $n = \dim X$, $m = \dim Y$. Then we have the exact sequence

$$0 \longrightarrow \mathscr{I}_Y/\mathscr{I}_Y^2 \longrightarrow \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{O}_Y \longrightarrow \Omega_Y^1 \longrightarrow 0$$

from Theorem 8.0.18, where $\mathscr{I}_Y \subseteq \mathscr{O}_X$ is the ideal sheaf of Y. By (8.2.1), we obtain

$$\textstyle\bigwedge^n (\Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{O}_Y) \simeq \bigwedge^{n-m} (\mathscr{I}_Y/\mathscr{I}_Y^2) \otimes_{\mathscr{O}_Y} \bigwedge^m \Omega_Y^1.$$

One can check that $\bigwedge^n (\Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{O}_Y) \simeq (\bigwedge^n \Omega_X^1) \otimes_{\mathscr{O}_X} \mathscr{O}_Y$. Now recall that $\omega_X = \bigwedge^n \Omega_X^1$ and $\omega_Y = \bigwedge^m \Omega_Y^1$ and that the normal sheaf of $Y \subseteq X$ is $\mathscr{N}_{Y/X} = (\mathscr{I}_Y/\mathscr{I}_Y^2)^\vee$. Hence the above isomorphism implies

$$\textstyle\omega_Y \simeq \omega_X \otimes_{\mathscr{O}_X} \bigwedge^{n-m} \mathscr{N}_{Y/X}.$$

This isomorphism is called the *adjunction formula*. $\Diamond$

The Canonical Sheaf of a Toric Variety. Our first major result gives a formula for the canonical sheaf of a toric variety.

Theorem 8.2.3. *For a toric variety X_Σ, the canonical sheaf ω_{X_Σ} is given by*

$$\omega_{X_\Sigma} \simeq \mathscr{O}_{X_\Sigma}\big(-\textstyle\sum_\rho D_\rho\big).$$

Thus $K_{X_\Sigma} = -\sum_\rho D_\rho$ is a torus-invariant canonical divisor on X_Σ.

Proof. We first assume that X_Σ is smooth with no torus factors. Then we have the Euler sequence

$$0 \longrightarrow \Omega^1_{X_\Sigma} \longrightarrow \bigoplus_\rho \mathscr{O}_{X_\Sigma}(-D_\rho) \longrightarrow \mathrm{Pic}(X_\Sigma) \otimes_\mathbb{Z} \mathscr{O}_{X_\Sigma} \longrightarrow 0$$

from Theorem 8.1.6. Each $\mathscr{O}_{X_\Sigma}(-D_\rho)$ is a line bundle since X_Σ is smooth, and if we set $r = |\Sigma(1)|$, then one sees easily that $\mathrm{Pic}(X_\Sigma) \otimes_\mathbb{Z} \mathscr{O}_{X_\Sigma} \simeq \mathscr{O}^{r-n}_{X_\Sigma}$. Hence we can apply part (b) of Proposition 8.2.1 to obtain

$$\bigwedge^n \Omega^1_{X_\Sigma} \otimes_{\mathscr{O}_{X_\Sigma}} \bigwedge^{r-n} \mathscr{O}^{r-n}_{X_\Sigma} \simeq \bigwedge^r \big(\bigoplus_\rho \mathscr{O}_{X_\Sigma}(-D_\rho)\big). \tag{8.2.2}$$

It follows by induction from Proposition 8.2.1 that the right-hand side of (8.2.2) is isomorphic to

$$\bigotimes_\rho \mathscr{O}_{X_\Sigma}(-D_\rho) \simeq \mathscr{O}_{X_\Sigma}\big(-\textstyle\sum_\rho D_\rho\big).$$

Turning to the left-hand side of (8.2.2), note that $\bigwedge^{r-n} \mathscr{O}^{r-n}_{X_\Sigma} \simeq \mathscr{O}_{X_\Sigma}$, so that the left-hand side is isomorphic to

$$\bigwedge^n \Omega^1_{X_\Sigma} = \Omega^n_{X_\Sigma} = \omega_{X_\Sigma}$$

since X_Σ is smooth. This proves the result when X_Σ is smooth without torus factors. In Exercise 8.2.2 you will deduce the result for an arbitrary smooth toric variety.

Now suppose that X_Σ is normal but not necessarily smooth. Let $j : U_0 \subseteq X_\Sigma$ be the inclusion map for $U_0 = \bigcup_\rho U_\rho$. We saw in the proof of Theorem 8.1.4 that U_0 is a smooth toric variety satsifying $\mathrm{codim}(X \setminus U_0) \geq 2$. Now consider ω_{X_Σ} and $\mathscr{O}_{X_\Sigma}(-\sum_\rho D_\rho)$. Since the fans for U_0 and X_Σ have the same 1-dimensional cones, these sheaves become isomorphic over U_0 by the smooth case. Since these sheaves are reflexive and $\mathrm{codim}(X \setminus U_0) \geq 2$, we conclude that $\omega_{X_\Sigma} \simeq \mathscr{O}_{X_\Sigma}(-\sum_\rho D_\rho)$ by Proposition 8.0.1. □

Here are some examples.

Example 8.2.4. Theorem 8.2.3 implies that the canonical bundle of $\mathbb{P}^n$ is

$$\omega_{\mathbb{P}^n} \simeq \mathscr{O}_{\mathbb{P}^n}(-n-1)$$

for all $n \geq 1$ since $\mathrm{Cl}(\mathbb{P}^n) \simeq \mathbb{Z}$ and $D_0 \sim D_1 \sim \cdots \sim D_n$. In Exercise 8.2.3, you will see another way to understand and derive this isomorphism. ♢

Example 8.2.5. The previous example shows that $\omega_{\mathbb{P}^2} \simeq \mathscr{O}_{\mathbb{P}^2}(-3)$. We will compute this directly using

$$\omega_{\mathbb{P}^2} = \Omega^2_{\mathbb{P}^2} = \bigwedge^2 \Omega^1_{\mathbb{P}^2}$$

and the description of $\Omega^1_{\mathbb{P}^2}$ as a rank 2 vector bundle given in Example 8.0.16. Recall that the transition functions for this bundle are given by:

$$\phi_{20} = \begin{pmatrix} 1/y & -x/y^2 \\ 0 & -1/y^2 \end{pmatrix}, \quad \phi_{12} = \begin{pmatrix} -y^2/x^2 & 0 \\ -y/x^2 & y/x \end{pmatrix}, \quad \phi_{10} = \begin{pmatrix} -y/x^2 & 1/x \\ -1/x^2 & 0 \end{pmatrix}.$$

By Exercise 8.2.1, the corresponding maps on $\bigwedge^2$ are given by the determinants of these 2×2 matrices:

$$\textstyle\bigwedge^2\phi_{20} = \frac{-1}{y^3},\ \ \bigwedge^2\phi_{12} = \frac{-y^3}{x^3},\ \ \bigwedge^2\phi_{10} = \frac{1}{x^3}.$$

Note that each is a cube. It is also evident that

$$\textstyle\bigwedge^2\phi_{10} = \bigwedge^2\phi_{12}\cdot\bigwedge^2\phi_{20},$$

so that these give the transition functions for a line bundle on $\mathbb{P}^2$.

On the other hand, $\bigwedge^2\Omega^1_{\mathbb{P}^2} \simeq \mathscr{O}_{\mathbb{P}^2}(-3)$ says the canonical bundle of $\mathbb{P}^2$ is the third tensor power of the tautological bundle described in Examples 6.0.19 and 6.0.21. To see this directly, we first need to calibrate the coordinate systems. Example 8.0.16 used coordinates x,y from U_{σ_0}, and Example 6.0.19 used homogeneous coordinates x_0,x_1,x_2 for $\mathbb{P}^2$, with the standard open cover $U_i = \mathbb{P}^2\setminus\mathbf{V}(x_i)$.

Letting $x = x_0/x_2$ and $y = x_1/x_2$ gives an isomorphism $U_{\sigma_0} = U_2$. Translating coordinates for U_{σ_1}, we have

$$\big(1/x, y/x\big) = \big(1/(x_0/x_2), (x_1/x_2)/(x_0/x_2)\big) = \big(x_2/x_0, x_1/x_0\big),$$

hence $U_{\sigma_1} = U_0$. A similar computation shows $U_{\sigma_2} = U_1$. We are now set for the final calculation. Keep in mind that the ϕ_{ij} are in the coordinate system with charts U_{σ_i}. We will use θ_{ij} to denote the same transition function, but using the U_i charts. Thus we have

$$\begin{array}{rclclcl}
\bigwedge^2\phi_{20} &=& -1/y^3 &=& (-x_2/x_1)^3 &=& \theta_{12}\\
\bigwedge^2\phi_{12} &=& -y^3/x^3 &=& (-x_1/x_0)^3 &=& \theta_{01}\\
\bigwedge^2\phi_{10} &=& 1/x^3 &=& (x_2/x_0)^3 &=& \theta_{02}.
\end{array}$$

Up to a sign, these are indeed the cubes of the transition functions that we computed for the tautological bundle $\mathscr{O}_{\mathbb{P}^n}(-1)$ in Example 6.0.19. In Exercise 8.2.4, you will work through the definition of the canonical bundle directly to find the transition functions given in Example 8.0.16. ◊

Example 8.2.6. When we computed the class group of the Hirzebruch surface $\mathscr{H}_r$ in Example 4.1.8, we wrote the divisors D_ρ as D_1,D_2,D_3,D_4 and showed that

$$\begin{aligned} D_3 &\sim D_1\\ D_4 &\sim rD_1 + D_2.\end{aligned}$$

Thus $\mathrm{Cl}(\mathscr{H}_r) = \mathrm{Pic}(\mathscr{H}_r) \simeq \mathbb{Z}^2$ is freely generated by the classes of D_1 and D_2. It follows that the canonical bundle can be written

$$\omega_{\mathscr{H}_r} \simeq \mathscr{O}_{\mathscr{H}_r}(-D_1-D_2-D_3-D_4) \simeq \mathscr{O}_{\mathscr{H}_r}(-(r+2)D_1-2D_2)$$

by Theorem 8.2.3. ◊

The Canonical Module. For a toric variety X_Σ without torus factors, the canonical sheaf ω_{X_Σ} comes from a graded S-module, where $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ is the total coordinate ring of X_Σ. This module is easy to describe explicitly.

Each variable $x_\rho \in S$ has degree $\deg(x_\rho) = [D_\rho] \in \mathrm{Cl}(X_\Sigma)$. Define

$$\beta_0 = \deg\big(\textstyle\prod_\rho x_\rho\big) = \big[\textstyle\sum_\rho D_\rho\big] \in \mathrm{Cl}(X_\Sigma).$$

Then $S(-\beta_0)$ is the graded S-module where $S(-\beta_0)_\alpha = S_{\alpha-\beta_0}$ for $\alpha \in \mathrm{Cl}(X_\Sigma)$. As in §5.3, the coherent sheaf associated to $S(-\beta_0)$ is denoted $\mathscr{O}_{X_\Sigma}(-\beta_0)$. We have the following result.

Proposition 8.2.7. $\mathscr{O}_{X_\Sigma}(-\beta_0) \simeq \omega_{X_\Sigma}$.

Proof. According to Proposition 5.3.7, $\mathscr{O}_{X_\Sigma}(-\beta_0) \simeq \mathscr{O}_{X_\Sigma}(D)$ for any Weil divisor with $-\beta_0 = [D] \in \mathrm{Cl}(X_\Sigma)$. The definition of β_0 allows us to pick $D = -\sum_\rho D_\rho$. Then Theorem 8.2.3 implies

$$\mathscr{O}_{X_\Sigma}(-\beta_0) \simeq \mathscr{O}_{X_\Sigma}\big(-\textstyle\sum_\rho D_\rho\big) \simeq \omega_{X_\Sigma}. \qquad \square$$

We call $S(-\beta_0)$ the *canonical module* of S.

Corollary 8.2.8. *For any normal toric variety X_Σ we have an exact sequence*

$$0 \longrightarrow \omega_{X_\Sigma} \longrightarrow \mathscr{O}_{X_\Sigma} \longrightarrow \textstyle\bigoplus_\rho \mathscr{O}_{D_\rho}.$$

Proof. First suppose that X_Σ has no torus factors. Multiplication by $\prod_\rho x_\rho$ induces an exact sequence of graded S-modules

$$0 \longrightarrow S(-\beta_0) \longrightarrow S \longrightarrow \textstyle\bigoplus_\rho S/\langle x_\rho\rangle$$

since $\beta_0 = \deg(\prod_\rho x_\rho)$. The sheaf associated to $S/\langle x_\rho\rangle$ is $\mathscr{O}_{D_\rho}$ (Exercise 8.2.5), and then we are done by Example 6.0.10 and Proposition 8.2.7. When X_Σ has a torus factor, the result follows using the strategy described in Exercise 8.2.2. $\square$

We can also describe ω_{X_Σ} in terms of n-forms in the x_ρ. Fix a basis $e_1,\dots,e_n$ of M. For each n-element subset $I = \{\rho_1,\dots,\rho_n\} \subseteq \Sigma(1)$, we get the $n\times n$ determinant

$$\det(u_I) = \det(\langle e_i, u_{\rho_j}\rangle).$$

This depends on the ordering of the ρ_i, as does the n-form $dx_{\rho_1}\wedge\cdots\wedge dx_{\rho_n}$, though the product

$$\det(u_I)\, dx_{\rho_1}\wedge\cdots\wedge dx_{\rho_n}$$

depends only on $e_1,\dots,e_n$. It follows that the n-form

$$\Omega_0 = \sum_{|I|=n} \det(u_I)\, \big(\textstyle\prod_{\rho\notin I} x_\rho\big)\, dx_{\rho_1}\wedge\cdots\wedge dx_{\rho_n} \tag{8.2.3}$$

is well-defined up to ± 1. If we set $\deg(dx_\rho) = \deg(x_\rho)$, then $\Omega_0 \in \bigwedge^n \Omega_{S/\mathbb{C}}$ is homogeneous of degree β_0. This gives the submodule

$$S\,\Omega_0 \subseteq \textstyle\bigwedge^n \Omega_{S/\mathbb{C}},$$

which is isomorphic to $S(-\beta_0)$ since $\deg(\Omega_0) = \beta_0$.

To see where Ω_0 comes from, let $L = \mathbb{C}(x_\rho \mid \rho \in \Sigma(1))$ be the field of fractions of S. Then L is the function field of $\mathbb{C}^{\Sigma(1)}$, and the surjective toric morphism

$$\pi : \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \longrightarrow X_\Sigma$$

from Proposition 5.1.9 induces an injection on function fields

$$\pi^* : \mathbb{C}(X_\Sigma) \longrightarrow L.$$

Furthermore, the proof of Theorem 5.1.11 implies that for any $m \in M$, the character $\chi^m \in \mathbb{C}(X_\Sigma)$ maps to

$$\pi^*(\chi^m) = \textstyle\prod_\rho x_\rho^{\langle m,u_\rho\rangle}.$$

We regard $\mathbb{C}(X_\Sigma)$ as a subfield of L via π^*, so that $\mathbb{C}(X_\Sigma) \subseteq L$ and $\chi^m = \prod_\rho x_\rho^{\langle m,u_\rho\rangle}$. This induces an inclusion of Kähler n-forms

$$\textstyle\bigwedge^n \Omega_{\mathbb{C}(X_\Sigma)/\mathbb{C}} \subseteq \bigwedge^n \Omega_{L/\mathbb{C}}. \tag{8.2.4}$$

A basis $e_1, \ldots, e_n$ of M gives coordinates $t_i = \chi^{e_i}$ for the torus T_N, and we know from §8.1 that $\frac{dt_i}{t_i}$ is a T_N-invariant section of $\Omega^1_{T_N}$. Then

$$\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$$

is a T_N-invariant section of $\Omega^n_{T_N}$ which pulls back to a rational n-form in $\bigwedge^n \Omega_{L/\mathbb{C}}$ via (8.2.4). Note that

$$\frac{dt_i}{t_i} = \sum_\rho \langle e_i, u_\rho\rangle \frac{dx_\rho}{x_\rho}$$

since $t_i = \prod_\rho x_\rho^{\langle e_i,u_\rho\rangle}$. When we mutiply by $\prod_\rho x_\rho$ to clear denominators, we obtain

$$\begin{aligned}\textstyle\prod_\rho x_\rho \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} &= \textstyle\prod_\rho x_\rho \Big(\sum_\rho \langle e_1, u_\rho\rangle \frac{dx_\rho}{x_\rho}\Big) \wedge \cdots \wedge \Big(\sum_\rho \langle e_n, u_\rho\rangle \frac{dx_\rho}{x_\rho}\Big)\\ &= \sum_{|I|=n} \det(u_I) \big(\textstyle\prod_{\rho\notin I} x_\rho\big)\, dx_{\rho_1} \wedge \cdots \wedge dx_{\rho_n} = \Omega_0.\end{aligned}$$

Hence Ω_0 arises in a completely natural way.

This can be interpreted in terms of sheaves as follows. We regard (8.2.4) as an inclusion of constant sheaves on X_Σ. Then the inclusion

$$\omega_{X_\Sigma} \subseteq \textstyle\bigwedge^n \Omega_{\mathbb{C}(X_\Sigma)/\mathbb{C}}$$

from Proposition 8.0.21 induces an inclusion

$$\omega_{X_\Sigma} \subseteq \textstyle\bigwedge^n \Omega_{L/\mathbb{C}}.$$

On the other hand, it is easy to see that $S\,\Omega_0 \subseteq \bigwedge^n \Omega_{S/\mathbb{C}}$ induces an inclusion

$$\widetilde{S\Omega_0} \subseteq \textstyle\bigwedge^n \Omega_{L/\mathbb{C}}.$$

Using the above derivation of Ω_0, one can prove that

$$\widetilde{S\Omega_0} = \omega_{X_\Sigma}$$

as subsheaves of the constant sheaf $\bigwedge^n \Omega_{L/C}$—see [**184**, Prop. 14.14]. Note that this is an *equality* of sheaves, not just an isomorphism. This shows that from the point of view of differential forms, $S\Omega_0$ deserves to be called the canonical module. It is isomorphic to the earlier version $S(-\beta_0)$ of the canonical module via the map that takes $\Omega_0 \in S\Omega_0$ to $1 \in S(-\beta_0)$.

The Affine Case. The canonical sheaf ω_{U_σ} of a normal affine toric variety U_σ is determined by its module of global sections, the *canonical module*. When we think of ω_{U_σ} as the ideal sheaf $\mathscr{O}_{U_\sigma}(-\sum_\rho D_\rho)$, we get the ideal

$$\Gamma(U_\sigma, \omega_{U_\sigma}) \subseteq \Gamma(U_\sigma, \mathscr{O}_{U_\sigma}) = \mathbb{C}[\sigma^\vee \cap M].$$

Proposition 8.2.9. $\Gamma(U_\sigma, \omega_{U_\sigma}) \subseteq \mathbb{C}[\sigma^\vee \cap M]$ *is the ideal generated by the characters* χ^m *for all* $m \in M$ *in the interior of* $\sigma^\vee$.

Proof. Corollary 8.2.8 gives the exact sequence

$$0 \longrightarrow \Gamma(U_\sigma, \omega_{U_\sigma}) \longrightarrow \mathbb{C}[\sigma^\vee \cap M] \longrightarrow \textstyle\bigoplus_\rho \mathbb{C}[\sigma^\vee \cap M]/I_\rho,$$

where $I_\rho = \mathbf{I}(D_\rho \cap U_\sigma)$ is the ideal of $D_\rho \cap U_\sigma \subset U_\sigma$. Since

$$I_\rho = \bigoplus_{m \in \sigma^\vee \cap M, \langle m, u_\rho\rangle > 0} \mathbb{C}\cdot\chi^m \subseteq \bigoplus_{m\in\sigma^\vee\cap M} \mathbb{C}\cdot\chi^m = \mathbb{C}[\sigma^\vee \cap M]$$

by (8.1.4), it follows that $\Gamma(U_\sigma, \omega_{X_\Sigma})$ is the direct sum of $\mathbb{C}\cdot\chi^m$ for all $m \in M$ such that $\langle m, u_\rho\rangle > 0$ for all ρ. Since the u_ρ generate σ, this is equivalent to saying that m is in the interior of $\sigma^\vee$. □

The Projective Case. A full dimensional lattice polytope $P \subseteq M_\mathbb{R}$ gives two interesting graded rings, each with its own canonical module. For the first, we use an auxiliary variable t so that for every $k \in \mathbb{N}$, a lattice point $m \in (kP) \cap M$ gives a character $\chi^m t^k$ on $T_N \times \mathbb{C}^*$. These characters span the semigroup algebra

$$S_P = \mathbb{C}[C(P) \cap (M \times \mathbb{Z})] = \bigoplus_{m \in (kP)\cap M} \mathbb{C}\cdot\chi^m t^k, \tag{8.2.5}$$

where $C(P) = \mathrm{Cone}(P \times \{1\}) \subseteq M_\mathbb{R} \times \mathbb{R}$ is the cone of P. Recall that the "slice" of $C(P)$ at height k is kP (see Figure 4 is §2.2). The ring S_P is graded by setting $\deg(\chi^m t^k) = k$. Then Proposition 8.2.9 tells us that the canonical sheaf of the associated affine toric variety comes from the ideal

$$I_P = \bigoplus_{m \in \mathrm{Int}(kP)\cap M} \mathbb{C}\cdot\chi^m t^k \subseteq S_P.$$

This is the *canonical module* of S_P.

The second ring associated to P uses the total coordinate ring S of the projective toric variety X_P and the corresponding ample divisor D_P. Let $\alpha = [D_P] \in \mathrm{Cl}(X_P)$ be the divisor class of D_P. The graded pieces $S_{k\alpha} \subseteq S$, $k \in \mathbb{N}$, form the graded ring

$$S_{\bullet\alpha} = \bigoplus_{k=0}^{\infty} S_{k\alpha}.$$

The canonical module of S is $S(-\beta_0)$, which is isomorphic to the ideal $\big\langle \prod_\rho x_\rho \big\rangle \subseteq S$ via multiplication by $\prod_\rho x_\rho$ since $\beta_0 = \deg\big(\prod_\rho x_\rho\big)$. When restricted to $S_{\bullet\alpha}$, this gives the ideal

$$I_{\bullet\alpha} = \bigoplus_{k=0}^{\infty} I_{k\alpha} \subseteq S_{\bullet\alpha},$$

where $I_{k\alpha} \subset S_{k\alpha}$ is generated by all monomials of degree $k\alpha$ in which every variable appears to a positive power.

It follows that the polytope P gives graded rings S_P and $S_{\bullet\alpha}$, each of which has an ideal representing the canonical module. These are related as follows.

Theorem 8.2.10. *The graded rings S_P and $S_{\bullet\alpha}$ are naturally isomorphic via an isomorphism that takes I_P to $I_{\bullet\alpha}$.*

Proof. The proof of Theorem 5.4.8 used homogenization to construct an isomorphism $S_P \simeq S_{\bullet\alpha}$. It is straightforward to see that this isomorphism carries the ideal $I_P \subseteq S_P$ to the ideal $I_{\bullet\alpha} \subseteq S_{\bullet\alpha}$ (Exercise 8.2.6). □

Recall from Theorem 7.1.13 that $X_P \simeq \mathrm{Proj}(S_P)$ via the Proj construction from §7.0. Furthemore, graded S_P-modules give quasicoherent sheaves on $\mathrm{Proj}(S_P)$, as described in [**131**, II.5] (this is similar to the construction given in §5.3). Then the following is true (Exercise 8.2.7).

Proposition 8.2.11. *The sheaf on X_P associated to the ideal $I_P \subseteq S_P$ by the* Proj *construction is the canonical sheaf of X_P.* □

The situation becomes even nicer when P is a normal polytope. The ample divisor D_P is very ample and hence embeds X_P into a projective space, which gives the homogeneous coordinate ring $\mathbb{C}[X_P]$. As we learned in §2.0, $\mathbb{C}[X_P]$ is also the ordinary coordinate ring of its affine cone $\widehat{X}_P$, i.e.,

$$\mathbb{C}[X_P] = \mathbb{C}[\widehat{X}_P].$$

Furthermore, since P is normal, Theorem 5.4.8 gives isomorphisms

$$S_P \simeq S_{\bullet\alpha} \simeq \mathbb{C}[X_P]$$

and implies that $\widehat{X}_P$ is the normal affine toric variety given by

$$\widehat{X}_P = \mathrm{Spec}(S_P) = \mathrm{Spec}(S_{\bullet\alpha}).$$

It follows that the canonical sheaf of the affine cone of X_P comes from the ideal $I_P \subseteq S_P$, and when we use the grading on S_P and I_P, they give X_P and its canonical sheaf via $X_P \simeq \operatorname{Proj}(S_P)$. Everything fits together very nicely.

There is a more general notion of canonical module that applies to any graded Cohen-Macaulay ring $S_\bullet = \bigoplus_{k=0}^\infty S_k$ where S_0 is a field—see [**56**, Sec. 3.6]. The canonical modules of $S_P \simeq S_{\bullet\alpha}$ constructed above are canonical in this sense.

When the Canonical Divisor is Cartier. For a toric variety X_Σ, the Weil divisor $K_{X_\Sigma} = -\sum_\rho D_\rho$ is called the canonical divisor. Note that K_{X_Σ} need not be a Cartier divisor if X_Σ is not smooth. In fact, from Theorem 4.2.8, we have the following characterization of the cases when the canonical divisor is Cartier.

Proposition 8.2.12. *Let X_Σ be a normal toric variety. Then K_{X_Σ} is Cartier if and only if for each maximal cone $\sigma \in \Sigma$, there exists $m_\sigma \in M$ such that*

$$\langle m_\sigma, u_\rho\rangle = 1 \text{ for all } \rho \in \sigma(1). \qquad \square$$

Similarly, K is $\mathbb{Q}$-Cartier if and only if for each maximal $\sigma \in \Sigma$, there exists $m_\sigma \in M_\mathbb{Q}$ such that $\langle m_\sigma, u_\rho\rangle = 1$ for all $\rho \in \sigma(1)$.

Example 8.2.13. Let $\sigma = \operatorname{Cone}(de_1 - e_2, e_2) \subseteq \mathbb{R}^2$. The affine toric variety U_σ is the rational normal cone $\widehat{C}_d$. We computed $\operatorname{Cl}(\widehat{C}_d) \simeq \mathbb{Z}/d\mathbb{Z}$ in Example 4.1.4, where the Weil divisors D_1, D_2 coming from the rays satisfy $D_2 \sim D_1, dD_1 \sim 0$.

The canonical divisor $K_{\widehat{C}_d} = -D_1 - D_2$ has divisor class corresponding to $[-2] \in \mathbb{Z}/d\mathbb{Z}$. Since the Picard group of a normal affine toric variety is trivial (Proposition 4.2.2), it follows that $K_{\widehat{C}_d}$ is Cartier if and only if $d \leq 2$.

Another way to see this is via Proposition 8.2.12, where one easily computes that $m_\sigma = (2/d)e_1 + e_2$ satisfies $\langle m_\sigma, de_1 - e_2\rangle = \langle m_\sigma, e_2\rangle = 1$. This lies in $M = \mathbb{Z}^2$ if and only if $d \leq 2$. $\Diamond$

We will use the following terminology.

Definition 8.2.14. Let K_X be a canonical divisor on a normal variety X. We say that X is ***Gorenstein*** if K_X is a Cartier divisor.

Since the canonical sheaf $\omega_X = \mathscr{O}_X(K_X)$ is reflexive, we have

$$X \text{ is Gorenstein} \iff K_X \text{ is Cartier} \iff \omega_X \text{ is a line bundle}$$

by Proposition 8.0.7. All smooth varieties are Gorenstein, of course. We will study further examples of singular Gorenstein varieties in the next section of the chapter.

Refinements. A refinement Σ' of a fan Σ induces a proper birational toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$. The canonical sheaves of $X_{\Sigma'}$ and X_Σ are related as follows.

Theorem 8.2.15. *Let $\phi : X_{\Sigma'} \to X_\Sigma$ be the toric morphism induced by a refinement Σ' of a fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^n$. Then*

$$\phi_*\omega_{X_{\Sigma'}} \simeq \omega_{X_\Sigma}.$$

In particular, if Σ' is a smooth refinement of Σ, then

$$\phi_*\Omega^n_{X_{\Sigma'}} \simeq \omega_{X_\Sigma}.$$

Proof. First assume $X_\Sigma = U_\sigma$, so that Σ' refines σ. Since $K_{\Sigma'} = -\sum_{\rho' \in \Sigma'(1)} D_{\rho'}$, the description of global sections given in Proposition 4.3.3 implies that

$$\Gamma(X_{\Sigma'}, \omega_{X_{\Sigma'}}) = \bigoplus_{m \in P' \cap M} \mathbb{C} \cdot \chi^m,$$

where

$$P' = \{m \in M_\mathbb{R} \mid \langle m, u_{\rho'} \rangle \geq 1 \text{ for all } \rho' \in \Sigma'(1)\}.$$

We clearly have $P' \cap M \subseteq \operatorname{Int}(\sigma^\vee) \cap M$ since $\sigma(1) \subseteq \Sigma'(1)$. The opposite inclusion also holds, as we now prove. Given $m \in \operatorname{Int}(\sigma^\vee) \cap M$, we have $\langle m, u \rangle > 0$ for all $u \neq 0$ in σ. In particular, $\langle m, u_{\rho'} \rangle > 0$ for all $\rho' \in \Sigma'(1)$. This is an integer, so that $\langle m, u_{\rho'} \rangle \geq 1$, which implies $m \in P' \cap M$, as desired. Since

$$\Gamma(U_\sigma, \omega_{U_\sigma}) = \bigoplus_{m \in \operatorname{Int}(\sigma^\vee) \cap M} \mathbb{C} \cdot \chi^m$$

by Proposition 8.2.9, we conclude that $\Gamma(X_{\Sigma'}, \omega_{X_{\Sigma'}}) = \Gamma(U_\sigma, \omega_{U_\sigma})$. Then we have $\phi_*\omega_{X_{\Sigma'}} \simeq \omega_{U_\sigma}$ since U_σ is affine.

In the general case, X_Σ is covered by affine open subsets U_σ for $\sigma \in \Sigma$, and $\phi^{-1}(U_\sigma)$ is the toric variety of the refinement of σ induced by Σ'. The above paragraph gives isomorphisms

$$\phi_*\omega_{X_{\Sigma'}}|_{U_\sigma} \simeq \omega_{X_\Sigma}|_{U_\sigma}$$

which are compatible with the inclusion $U_\tau \subseteq U_\sigma$ when τ is a face of σ. Hence these isomorphisms patch to give the desired isomorphism $\phi_*\omega_{X_{\Sigma'}} \simeq \omega_{X_\Sigma}$. □

In Chapter 11, we will prove the existence of smooth refinements. It follows that for any n-dimensional toric variety X_Σ, there are two ways of constructing the canonical sheaf ω_{X_Σ} from the sheaf of n-forms of a smooth toric variety:

- (Internal) $\omega_{X_\Sigma} = j_*\Omega^n_{U_0}$, where $j : U_0 \subseteq X_\Sigma$ is the inclusion of the smooth locus of X_Σ.
- (External) $\omega_{X_\Sigma} = \phi_*\Omega^n_{X_{\Sigma'}}$, where $\phi : X_{\Sigma'} \to X_\Sigma$ is the toric morphism coming from a smooth refinement Σ' of Σ.

Sheaves of p-forms. The sheaves $\widehat{\Omega}^p_{X_\Sigma}$ for $1 < p < n = \dim(X_\Sigma)$ are also important for the geometry of X_Σ. We construct a sequence

$$0 \longrightarrow \widehat{\Omega}^p_{X_\Sigma} \xrightarrow{\alpha_p} \bigwedge^p M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \xrightarrow{\beta_p} \bigoplus_\rho \bigwedge^{p-1}(\rho^\perp \cap M) \otimes_{\mathbb{Z}} \mathscr{O}_{D_\rho} \tag{8.2.6}$$

as follows. The map $\alpha : \Omega^1_{X_\Sigma} \to M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}$ from (8.1.3) induces

$$\bigwedge^p \alpha : \Omega^p_{X_\Sigma} = \bigwedge^p \Omega^1_{X_\Sigma} \longrightarrow \bigwedge^p M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma},$$

and then $\alpha_p = \big(\bigwedge^p \alpha\big)^{\vee\vee}$. To define β_p, recall that for $u \in N$, the *contraction map* $i_u : \bigwedge^p M \to \bigwedge^{p-1} M$ has the property that

$$i_u(m_1 \wedge \cdots \wedge m_p) = \sum_{i=1}^p (-1)^{i-1} \langle m_i, u\rangle\, m_1 \wedge \cdots \wedge \widehat{m}_i \wedge \cdots \wedge m_p \tag{8.2.7}$$

when $m_1, \ldots, m_p \in M$. Note also that $\mathrm{im}(i_u) \subseteq \bigwedge^{p-1}(u^\perp \cap M)$ (Exercise 8.2.8).

An element $\rho \in \Sigma(1)$ gives u_ρ as usual. Using $i_{u_\rho} : \bigwedge^p M \to \bigwedge^{p-1}(\rho^\perp \cap M)$ and $\mathscr{O}_{X_\Sigma} \to \mathscr{O}_{D_\rho}$, we obtain the composition

$$\bigwedge^p M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \longrightarrow \bigwedge^{p-1}(\rho^\perp \cap M) \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \longrightarrow \bigwedge^{p-1}(\rho^\perp \cap M) \otimes_{\mathbb{Z}} \mathscr{O}_{D_\rho}.$$

This gives a natural map

$$\beta_p : \bigwedge^p M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \longrightarrow \bigoplus_\rho \bigwedge^{p-1}(\rho^\perp \cap M) \otimes_{\mathbb{Z}} \mathscr{O}_{D_\rho}.$$

Theorem 8.2.16. *The sequence* (8.2.6) *is exact for any normal toric variety X_Σ.*

Proof. We begin with the smooth case. Since exactness of a sheaf sequence is local, we can work over an affine toric variety $U_\sigma = \mathbb{C}^r \times (\mathbb{C}^*)^{n-r}$, where σ is generated by the first r elements of a basis of $e_1, \ldots, e_n$ of N. By abuse of notation, $e_1, \ldots, e_n$ will denote the corresponding dual basis of M. Setting $x_i = \chi^{e_i}$, we get

$$U_\sigma = \mathrm{Spec}(R), \quad R = \mathbb{C}[x_1, \ldots, x_r, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Then (8.2.6) comes from the exact sequence of R-modules

$$0 \longrightarrow \bigwedge^p \Omega_{R/\mathbb{C}} \xrightarrow{\alpha_p} \bigwedge^p M \otimes_{\mathbb{Z}} R \xrightarrow{\beta_p} \bigoplus_{i=1}^r \bigwedge^{p-1}(e_i^\perp \cap M) \otimes_{\mathbb{Z}} R/\langle x_i\rangle,$$

where α_p maps $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ to $e_{i_1} \wedge \cdots \wedge e_{i_p} \otimes x_{i_1} \cdots x_{i_p}$ by the description of α given in (8.1.3).

It is thus obvious that $\beta_p \circ \alpha_p = 0$. To prove exactness, we regard $\bigwedge^p M \otimes_{\mathbb{Z}} R$ as $\bigwedge^p F$, where F is the free R module with basis $e_1, \ldots, e_n$. Now suppose that $\beta_p(\omega) = 0$ for some $\omega \in \bigwedge^p F$. We can write ω uniquely as

$$\omega = \sum_{i_1 < \cdots < i_p} f_{i_1 \ldots i_p}\, e_{i_1} \wedge \cdots \wedge e_{i_p}.$$

Exactness will follow once we prove $x_{i_1} \cdots x_{i_p}$ divides $f_{i_1 \ldots i_p}$ for all $i_1 < \cdots < i_p$.

If an index $i > r$ appears in $f_{i_1\ldots i_p}$, then x_i automatically divides $f_{i_1\ldots i_p}$ since x_i is invertible in R. Now suppose that an index $i \le r$ appears in $f_{i_1\ldots i_p}$. We can write $\omega = e_i \wedge \omega_1 + \omega_2$, where e_i does not appear in ω_2 and all $f_{i_1\ldots i_p}$ involving the index i appear in ω_1. Since $i_{e_1}(\omega) = \omega_1$, the only way for $\beta_p(\omega)$ to vanish is for $f_{i_1\ldots i_p}$ to be zero in $R/\langle x_i\rangle$, i.e., for x_i to divide $f_{i_1\ldots i_p}$. This completes the proof of exactness in the smooth case.

The proof for the general case follows from the smooth case by an argument similar to what we did in the proof of Theorem 8.1.4. □

Note that when $p = 1$, the exact sequence (8.2.6) reduces to the sequence appearing in Theorem 8.1.4, and when $p = n$, we get the sequence in Corollary 8.2.8 since $\omega_{X_\Sigma} = \widehat{\Omega}^n_{X_\Sigma}$.

When X_Σ has no torus factors, it is easy to find a graded S-module whose associated sheaf is $\widehat{\Omega}^p_{X_\Sigma}$. Adapting the definition of β_p in (8.2.6) to the module case, we get a homomorphism

$$\textstyle\bigwedge^p M \otimes_{\mathbb{Z}} S \longrightarrow \bigoplus_\rho \bigwedge^{p-1}(\rho^\perp \cap M) \otimes_{\mathbb{Z}} S/\langle x_\rho\rangle$$

of graded S-modules whose kernel we denote $\widehat{\Omega}^p_S$. This gives an exact sequence of graded S-modules

$$0 \longrightarrow \widehat{\Omega}^p_S \longrightarrow \textstyle\bigwedge^p M \otimes_{\mathbb{Z}} S \longrightarrow \bigoplus_\rho \bigwedge^{p-1}(\rho^\perp \cap M) \otimes_{\mathbb{Z}} S/\langle x_\rho\rangle.$$

Using Example 6.0.10, we obtain the following corollary of Theorem 8.2.16.

Corollary 8.2.17. *If X_Σ has no torus factors, then $\widehat{\Omega}^p_{X_\Sigma}$ is the sheaf associated to the graded S-module $\widehat{\Omega}^p_S$.* □

The Affine Case. For an affine toric variety U_σ, the sheaf $\widehat{\Omega}^p_{U_\sigma}$ is determined by its global sections, which can be described using Theorem 8.2.16. We will need the following notation. If $m \in M$ and $\sigma \in \Sigma$, let

$$V_\sigma(m) = \mathrm{Span}_{\mathbb{C}}(m_0 \in M \mid m_0 \in \text{the minimal face of } \sigma^\vee \text{ containing } m) \subseteq M_{\mathbb{C}},$$

where $M_{\mathbb{C}} = M \otimes_{\mathbb{Z}} \mathbb{C}$. Here is a result due to Danilov [**76**, Prop. 4.3].

Proposition 8.2.18. *Let U_σ be the toric variety of the cone σ. Then we have an isomorphism*

$$\Gamma(U_\sigma, \widehat{\Omega}^p_{U_\sigma}) \simeq \bigoplus_{m \in \sigma^\vee \cap M} \textstyle\bigwedge^p V_\sigma(m) \cdot \chi^m.$$

Proof. Using the inclusion $\rho^\perp \cap M \subseteq M$ and the exact sequence of Theorem 8.2.16 over U_σ, we obtain the exact sequence

$$0 \longrightarrow \Gamma(U_\sigma, \widehat{\Omega}^p_{U_\sigma}) \longrightarrow \textstyle\bigwedge^p M \otimes_{\mathbb{Z}} \Gamma(U_\sigma, \mathscr{O}_{U_\sigma}) \xrightarrow{\beta_p} \bigoplus_\rho \bigwedge^{p-1} M \otimes_{\mathbb{Z}} \Gamma(U_\sigma, \mathscr{O}_{D_\rho})$$

of modules over $\Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}) = \mathbb{C}[\sigma^\vee \cap M] = \bigoplus_{m\in\sigma^\vee\cap M} \mathbb{C}\cdot\chi^m$. Also note that

$$\Gamma(U_\sigma, \mathscr{O}_{D_\rho}) = \bigoplus_{m\in\sigma^\vee\cap M, \langle m,u_\rho\rangle=0} \mathbb{C}\cdot\chi^m \tag{8.2.8}$$

by the analysis given in the proof of Theorem 8.1.1. It follows that we have a direct sum decomposition $\Gamma(U_\sigma, \widehat{\Omega}^p_{U_\sigma}) = \bigoplus_{m\in\sigma^\vee\cap M} \Gamma(U_\sigma, \widehat{\Omega}^p_{U_\sigma})_m$, where

$$0 \longrightarrow \Gamma(U_\sigma, \widehat{\Omega}^p_{U_\sigma})_m \longrightarrow \textstyle\bigwedge^p M_{\mathbb{C}} \xrightarrow{\beta_p} \bigoplus_{\langle m,u_\rho\rangle=0} \bigwedge^{p-1} M_{\mathbb{C}} \tag{8.2.9}$$

is exact for $m \in \sigma^\vee \cap M$ and β_p is the sum of the contraction maps i_{u_ρ} for all ρ satisfying $\langle m, u_\rho\rangle = 0$.

Thus $\omega \in \bigwedge^p M_{\mathbb{C}}$ is in the kernel of β_p if and only if $i_{u_\rho}(\omega) = 0$ for all ρ with $\langle m, u_\rho\rangle = 0$. You will show in Exercise 8.2.8 that $i_{u_\rho}(\omega) = 0$ if and only if $\omega \in \bigwedge^p(\rho^\perp)_{\mathbb{C}}$. It follows that the kernel of β_p in (8.2.9) is the intersection

$$\bigcap_{\langle m,u_\rho\rangle=0} \textstyle\bigwedge^p(\rho^\perp)_{\mathbb{C}} = \bigwedge^p\Big(\displaystyle\bigcap_{\langle m,u_\rho\rangle=0} (\rho^\perp)_{\mathbb{C}}\Big). \tag{8.2.10}$$

However, the intersection

$$F = \bigcap_{\langle m,u_\rho\rangle=0} \rho^\perp \cap \sigma^\vee$$

is the minimal face of $\sigma^\vee$ containing m, and one sees easily that

$$\mathrm{Span}_{\mathbb{R}}(m_0 \in M \mid m_0 \in F) = \textstyle\bigcap_{\langle m,u_\rho\rangle=0} \rho^\perp.$$

It follows that $V_\sigma(m) = \bigcap_{\langle m,u_\rho\rangle=0} (\rho^\perp)_{\mathbb{C}}$. This plus (8.2.10) imply that the kernel of β_p in (8.2.9) is $\bigwedge^p V_\sigma(m)$, as claimed. □

The Simplicial Case. When X_Σ is simplicial, the sequence (8.2.6) is exact on the right when $p = 1$ by part (b) of Theorem 8.1.4. For $p > 1$, similar though longer exact sequences exist in the simplicial case, as we now describe without proof.

Given a fan Σ, consider the sheaf on X_Σ defined by

$$K^j(\Sigma, p) = \bigoplus_{\sigma\in\Sigma(j)} \textstyle\bigwedge^{p-j}(\sigma^\perp \cap M) \otimes_{\mathbb{Z}} \mathscr{O}_{V(\sigma)},$$

where $V(\sigma) = \overline{O(\sigma)} \subseteq X_\Sigma$ is the orbit closure corresponding to $\sigma \in \Sigma$. Thus

$$K^0(\Sigma, p) = \textstyle\bigwedge^p M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}$$

$$K^1(\Sigma, p) = \bigoplus_{\rho\in\Sigma(1)} \textstyle\bigwedge^{p-1}(\rho^\perp \cap M) \otimes_{\mathbb{Z}} \mathscr{O}_{V(\rho)} = \displaystyle\bigoplus_{\rho\in\Sigma(1)} \textstyle\bigwedge^{p-1}(\rho^\perp \cap M) \otimes_{\mathbb{Z}} \mathscr{O}_{D_\rho},$$

and the exact sequence (8.2.6) can be written

$$0 \longrightarrow \widehat{\Omega}^p_{X_\Sigma} \longrightarrow K^0(\Sigma, p) \longrightarrow K^1(\Sigma, p).$$

When X_Σ is simplicial, one can extend this exact sequence as follows.

Theorem 8.2.19. *When X_Σ is simplicial, there is an exact sequence*

$$0 \longrightarrow \widehat{\Omega}^p_{X_\Sigma} \longrightarrow K^0(\Sigma,p) \longrightarrow K^1(\Sigma,p) \longrightarrow \cdots \longrightarrow K^p(\Sigma,p) \longrightarrow 0. \qquad \square$$

A proof of this result can be found in [**218**, Sec. 3.2], where $K^\bullet(\Sigma,p)$ is called the *pth Ishida complex*. We will use this exact sequence in Chapter 9 to prove a vanishing theorem for sheaf cohomology on simplicial toric varieties.

Exercises for §8.2.

8.2.1. Given a map of free R-modules $\phi : F \to G$, we can pick bases of F,G and represent ϕ by a $n \times m$ matrix with entries in R, where $m = \mathrm{rank}(F), n = \mathrm{rank}(G)$. These bases induce bases of the wedge products $\bigwedge^p F$ and $\bigwedge^p G$. Then prove that the induced map $\bigwedge^p \phi : \bigwedge^p F \to \bigwedge^p G$ is given by the $p \times p$ minors of ϕ (with appropriate signs).

8.2.2. Complete the proof of Theorem 8.2.3 in the smooth case by showing how to reduce to the case when X_Σ is smooth with no torus factors. Hint: First prove that any smooth toric variety is equivariantly isomorphic to a product $X_\Sigma \times (\mathbb{C}^*)^r$, where X_Σ is a smooth toric variety with no torus factors. Then consider $X_\Sigma \times \mathbb{C}^r$.

8.2.3. Let T_N be the torus of $\mathbb{P}^n$. If $x_0,\ldots,x_n$ are the usual homogeneous coordinates, then $y_i = x_i/x_0$ for $i = 1,\ldots,n$ are affine coordinates for the open subset where $x_0 \neq 0$. The differential n-form $\omega = \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_n}{y_n}$ spans $\Gamma(T_N, \Omega^n_{T_N}) = \Gamma(T_N, \bigwedge^n \Omega^1_{T_N})$ and has poles of order 1 along each $D_i = \mathbf{V}(x_i)$ for $i = 1,\ldots,n$. Show that if we write $z_j = x_j/x_i$ for the affine coordinates on the complement of $\mathbf{V}(x_i)$, and change coordinates to the z_j, $j \neq i$, then we can see ω also has a pole of order 1 along $\mathbf{V}(x_0)$. Hence on $\mathbb{P}^n$, ω defines a section of $\mathscr{O}_{\mathbb{P}^n}(-\sum_{i=0}^n D_i)$.

8.2.4. Using the open subsets $U_{\sigma_i} \subseteq \mathbb{P}^2$ (see Figure 2 in Example 3.1.9), compute $\Omega^1_{\mathbb{P}^2}(U_{\sigma_i})$, and write down the transition functions on $U_{\sigma_i} \cap U_{\sigma_j}$. Compare to Example 8.0.16. If the result of your computation differs, describe how you can explain this via a change of basis.

8.2.5. Let S be the total coordinate ring of a toric variety X_Σ without torus factors. Prove that $\mathscr{O}_{D_\rho}$ is the sheaf associated to the graded S-module $S/\langle x_\rho \rangle$. Hint: Consider the exact sequence $0 \to \langle x_\rho \rangle \to S \to S/\langle x_\rho \rangle \to 0$ and apply Proposition 5.3.7 and Example 6.0.10.

8.2.6. Prove Theorem 8.2.10.

8.2.7. Prove that the sheaf associated to the ideal $I_P \subseteq S_P$ from Theorem 8.2.10 is the canonical sheaf of $X_P = \mathrm{Proj}(S_P)$. Hint: Let S be the total coordinate ring of X_P. We saw in §5.3 that a graded S-module such as $S(-\beta_0)$ gives a sheaf on X_P. Compare this to how $I_{\bullet\alpha}$ gives a sheaf on $\mathrm{Proj}(S_{\bullet\alpha})$. See the proof of Theorem 7.1.13.

8.2.8. Let F be a free module of finite rank over a domain R. Given $u \in F^\vee$, we get the kernel $u^\perp \subseteq F$ and the contraction map $i_u : \bigwedge^p F \to \bigwedge^{p-1} F$. Assume $u = re_1$ where $r \in R$ and $e_1,\ldots,e_n$ is a basis of $F^\vee$.

(a) Prove that the image of i_u is contained in $\bigwedge^{p-1} u^\perp$.

(b) Given $\omega \in \bigwedge^p F$, prove that $i_u(\omega) = 0$ if and only if $\omega \in \bigwedge^p u^\perp$.

8.2.9. Assume that X_Σ has no torus factors. For $\beta \in \mathrm{Cl}(X_\Sigma)$, define $\widehat{\Omega}^p_{X_\Sigma}(\beta)$ to be the sheaf associated to the graded S-module $\widehat{\Omega}^p_S(\beta)$. Prove that $\Gamma(X_\Sigma, \widehat{\Omega}^p_{X_\Sigma}(\beta)) \simeq (\widehat{\Omega}^p_S)_\beta$. Hint: If you get stuck, see [**19**, Prop. 8.5].

8.2.10. Compute the n-form Ω_0 defined in (8.2.3) for $\mathbb{P}^n$, $\mathbb{P}(q_0,\dots,q_n)$, and $\mathbb{P}^1\times\mathbb{P}^1$.

8.2.11. Prove that our favorite affine toric variety $V=\mathbf{V}(xy-zw)\subseteq\mathbb{C}^4$ is Gorenstein. Hint: Example 1.2.20.

8.2.12. Prove that a product of two Gorenstein toric varieties is again Gorenstein. Hint: Use Propositions 3.1.14 and 8.2.12.

8.2.13. We will see in Chapter 10 that every 2-dimensional rational cone in $\mathbb{R}^2$ is lattice equivalent to a cone of the form $\sigma=\mathrm{Cone}(e_2,de_1-ke_2)$ for integers $d>0$ and $0\le k<d$ with $\gcd(d,k)=1$. Prove that U_σ is Gorenstein if and only if $d=k+1$.

8.2.14. A strongly convex rational polyhedral cone is *Gorenstein* if its associated affine toric variety is Gorenstein. In the discussion of the algebra (8.2.5), we saw that a full dimensional lattice polytope $P\subseteq M_{\mathbb{R}}$ gives the cone

$$C(P)=\mathrm{Cone}(P\times\{1\})\subseteq M_{\mathbb{R}}\times\mathbb{R}.$$

(a) Prove that $C(P)$ is Gorenstein.

(b) Prove that the dual cone $C(P)^\vee\subseteq N_{\mathbb{R}}\times\mathbb{R}$ is Gorenstein if and only if P is reflexive.

In general, a cone is *reflexive Gorenstein* if both the cone and its dual are Gorenstein. Reflexive Gorenstein cones play an important role in mirror symmetry—see [**18, 204**].

8.2.15. Explain why $S_{\bullet\alpha-\beta_0}=\bigoplus_{k=0}^\infty S_{k\alpha-\beta_0}$ gives another model for the canonical module of the graded ring $S_{\bullet\alpha}$ discussed in the text.

§8.3. Fano Toric Varieties

We finish this chapter with a discussion of an interesting class of projective toric varieties and their corresponding polytopes.

Definition 8.3.1. A complete normal variety X is said to be a ***Gorenstein Fano variety*** if the anticanonical divisor $-K_X$ is Cartier and ample.

Thus Gorenstein Fano varieties are projective. When X is smooth, we will simply say that X is Fano.

Example 8.3.2. Example 8.2.5 shows that $\mathscr{O}_{\mathbb{P}^2}(-K_{\mathbb{P}^2})\simeq\mathscr{O}_{\mathbb{P}^2}(3)$ is ample, so $\mathbb{P}^2$ is a Fano variety. We continue this example to introduce the key ideas that will lead to a classification of 2-dimensional Gorenstein Fano toric varieties.

The standard fan for $\mathbb{P}^2=X_\Sigma$ has minimal generators $u_0=-e_1-e_2$, $u_1=e_1$ and $u_2=e_2$. The polytope corresponding to the anticanonical divisor of $\mathbb{P}^2$ is

$$P=\{m\in\mathbb{R}^2\mid\langle m,u_i\rangle\ge-1, i=0,1,2\}.$$

In Exercise 8.3.1 you will check that

$$P=\mathrm{Conv}(-e_1-e_2,2e_1-e_2,-e_1+2e_2).$$

This lattice polygon is shown in Figure 1 on the next page. The open circle in the figure represents the origin, which is the unique interior lattice point of P. Also,

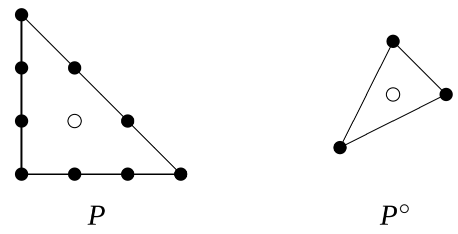


Figure 1. The anticanonical polytope P and its dual P° for $\mathbb{P}^2$

by Exercise 8.3.1, the dual polytope $P^\circ = \{u \in N_{\mathbb{R}} \mid \langle m,u\rangle \geq -1 \text{ for all } m \in P\}$ is given by

$$P^\circ = \mathrm{Conv}(e_1, e_2, -e_1 - e_2),$$

which is the polytope generated by the ray generators of the fan of $\mathbb{P}^2$. $\Diamond$

Example 8.3.3. In Exercise 8.3.2, you will generalize Example 8.3.2 by showing that the weighted projective space $\mathbb{P}(q_0,\ldots,q_n)$ is Gorenstein Fano if and only if $q_i \,|\, q_0 + \cdots + q_n$ for all $0 \leq i \leq n$. $\Diamond$

The special features of Example 8.3.2 involve an unexpected relation between Fano toric varieties and the *reflexive polytopes* introduced in Chapter 2.

Fano Toric Varieties and Reflexive Polytopes. Recall from Definition 2.3.12 that a lattice polytope in $M_{\mathbb{R}}$ is *reflexive* if its facet presentation is

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -1 \text{ for all facets } F\}. \tag{8.3.1}$$

It follows that if P is reflexive, the origin is the unique interior lattice point of P (Exercise 2.3.5). Since $a_F = 1$ for all facets F, the dual polytope is

$$P^\circ = \mathrm{Conv}(u_F \mid F \text{ is a facet of } P) \tag{8.3.2}$$

(Exercise 2.2.1). Finally, P° is a lattice polytope and is reflexive (Exercise 2.3.5). The polytopes pictured in Figure 1 are reflexive.

The following result gives the connection between projective Gorenstein Fano toric varieties and reflexive polytopes generalizing what we saw above for $\mathbb{P}^2$.

Theorem 8.3.4. *Let X_Σ be a normal toric variety. If X_Σ is a projective Gorenstein Fano variety, then the polytope associated to the anticanonical divisor $-K_{X_\Sigma} = \sum_\rho D_\rho$ is reflexive. Conversely, if X_P is the projective toric variety associated to a reflexive polytope P, then X_P is a Gorenstein Fano variety.*

Proof. If X_Σ is Gorenstein Fano, then $-K_{X_\Sigma}$ is Cartier and ample. This implies that polytope associated to $-K_{X_\Sigma} = \sum_\rho D_\rho$ has facet presentation

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho\rangle \geq -1 \text{ for all } \rho \in \Sigma(1)\}.$$

and hence is reflexive by (8.3.1). Conversely, let P be a reflexive polytope in $M_{\mathbb{R}}$. The normal fan Σ_P of P defines the variety $X_P = X_{\Sigma_P}$. The facet presentation (8.3.1) of P has $a_F = 1$ for every facet F of P. Hence the Cartier divisor corresponding to P is $D_P = \sum_F D_F = -K_{X_P}$. We proved that D_P is ample in Proposition 6.1.10, so that $-K_{X_P}$ is ample. Hence X_P is Gorenstein Fano. □

Classification. By Theorem 8.3.4, classifying toric Gorenstein Fano varieties is equivalent to classifying reflexive polytopes P in $M_{\mathbb{R}}$. Since reflexive polytopes contain the origin as an interior point, "classify" means up to invertible linear maps of $M_{\mathbb{R}}$ induced by isomorphisms of M. This is called *lattice equivalence*. The first interesting case is in dimension two, where toric Gorenstein Fano surfaces correspond to 16 equivalence classes of reflexive polygons.

We begin with some general facts about a reflexive polytope P. First note that the lattice points of P are the origin and the lattice points lying on the boundary. Furthermore, any boundary lattice point is primitive. We will also need two less obvious results from [**214**].

The first result concerns projections of reflexive polytopes. Given a reflexive polygon P and a primitive element $m \in M$, there is a projection map

$$\pi_m : M_{\mathbb{R}} \longrightarrow M_{\mathbb{R}}/\mathbb{R}m$$

whose image is a polytope whose vertices lie in $M/\mathbb{Z}m$.

Lemma 8.3.5. *Let P be a reflexive polytope in $M_{\mathbb{R}} \simeq \mathbb{R}^n$ and let m be a lattice point m in the boundary of P. Then $\pi_m(P)$ is a lattice polytope in $M_{\mathbb{R}}/\mathbb{R}m$ containing the origin as an interior lattice point, and*

$$\pi_m(P) = \pi_m\Big(\bigcup\nolimits_{m\in F \text{ facet of } P} F\Big).$$

Proof. For each $p \in \pi_m(P)$, $\pi_m^{-1}(p)$ is a line parallel to the line $\mathbb{R}m$. By taking the point in P that maps to p and is farthest along this line in the direction of m, we see that the points in $\pi_m(P)$ are in bijective correspondence with the points in

$$U = \{x \in P \mid x + \lambda m \notin P \text{ for all } \lambda > 0\}.$$

If F is a facet of P containing m, one easily sees that $F \subseteq U$. Conversely, let $x \in U$. One can show without difficulty there exists some facet F of P, not parallel to m, containing x and such that $\langle u_F, m\rangle < 0$. Since P is reflexive and $m \in M$, $\langle u_F, m\rangle$ is an integer ≥ -1. Hence $\langle u_F, m\rangle = -1$, so that $x, m \in F$. Thus $U \subseteq \bigcup_{m\in F} F$, and from here the lemma follows easily. □

The second result is the following fact about pairs of lattice points on the boundary of a reflexive polytope. The vertices are primitive vectors in M, but we can have other lattice points on the boundary of P as well.

Lemma 8.3.6. *Let m, m' be distinct lattice points on the boundary of a reflexive polytope P. Then exactly one of the following holds:*

(a) *m and m' lie in a common edge of P,*

(b) *$m + m' = 0$, or*

(c) *$m + m'$ is also on the boundary of P.*

The proof is left to the reader as Exercise 8.3.3.

The 2-Dimensional Case. The following theorem classifies reflexive polygons in the plane $M_{\mathbb{R}} \simeq \mathbb{R}^2$, up to lattice equivalence. Figure 2 shows 16 lattice polygons, where the open circle in the center of each polygon is the origin and the solid circles are the lattice points on the boundary. The numbers in the labels give the number of boundary lattice points. Polygons 3 and 9 are the dual pair from Example 8.3.2.

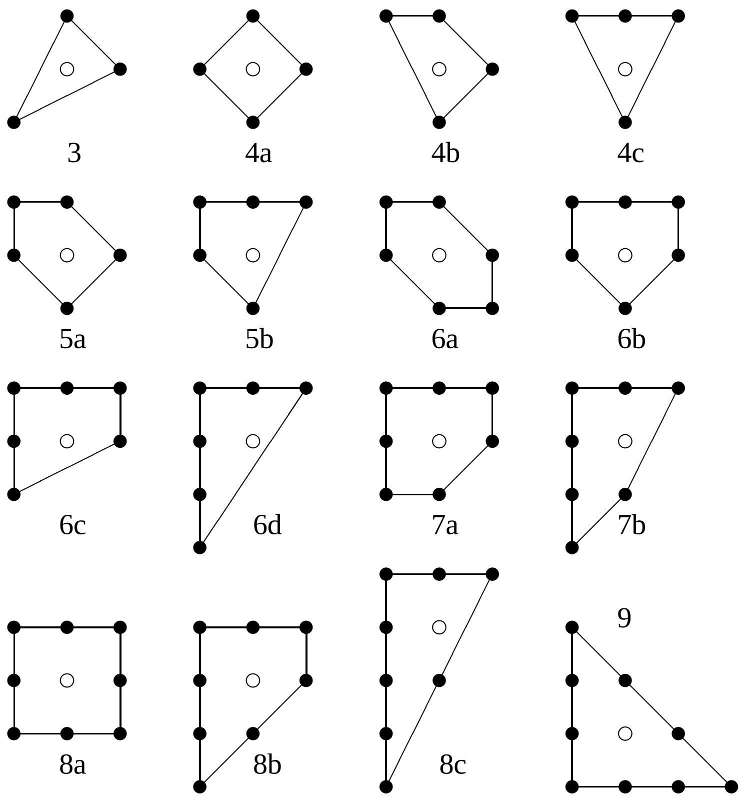


Figure 2. The 16 equivalence classes of reflexive lattice polygons in $\mathbb{R}^2$

Theorem 8.3.7. *There are exactly 16 equivalence classes of reflexive polygons in the plane, shown in Figure 2.*

Proof. We will sketch the proof following [**215**, Proposition 4.1], and leave the details for the reader to verify (Exercise 8.3.4). We consider several cases.

Case A. First assume that P is a reflexive polygon such that each edge contains exactly two lattice points, and fix one such pair. These lattice points must form a basis for M since the triangle formed by these two vertices and the origin has lattice points only at the vertices (part (a) of Exercise 8.3.4). Hence we can use a lattice equivalence to place the two of them at e_1 and e_2.

Subcase A.1. If P has exactly three vertices, the third is located at $ae_1 + be_2$ for some $a, b \in \mathbb{Z}$. Since 0 is the only lattice point in the interior, we must have $a = b = -1$ (part (b) of Exercise 8.3.4). This gives the polygon of type 3.

Subcase A.2. Next, still in Case A, assume that there are three distinct vertices m, m', m'' such that $m + m' = m''$. There must be edges of P containing the pairs m', m'' and m', m'' (part (c) of Exercise 8.3.4). Hence each pair must yield a basis for M and we can place $m = e_1, m_2 = -e_1 + e_2$, and then $m'' = e_2$. Project P from $m'' = e_2$ using Lemma 8.3.5. It follows that P can only contain points $-e_1, 0, e_1$ on the line $y = 0$ and $-e_2, e_1 - e_2$ on the line $y = -1$. Moreover, P cannot contain any points with $y \leq -2$. It follows that the only other possible polygons in this case are 4b, 5a, 6a (part (d) of Exercise 8.3.4) up to lattice equivalence.

Subcase A.3. Finally, in Case A, if we are not in subcases A.1 or A.2, then by Lemma 8.3.6, if m and m' are not in the same edge, we must have $m + m' = 0$. The only possibility here is clearly polygon 4a.

Case B. Assume there are exactly three lattice points on some edge of P. By an isomorphism of M, we can place these at $-e_1 + e_2, e_2, e_1 + e_2$. Projecting from $m = e_2$ and using Lemma 8.3.5, we see that P must be contained in the strip $-1 \leq x \leq 1$. Moreover, P cannot contain any points with $y < -3$, or else $-e_2 \neq 0$ would be an interior lattice point. Up to lattice equivalence, the possibilities are 4c, 5b, 6b, 6c, 6d, 7a, 7b, 8a, 8b, and 8c (part (e) of Exercise 8.3.4).

Case C. Assume no edge of P has exactly three lattice points and some edge has four or more. Place the vertex of this edge at $-e_1 - e_2$ so that $-e_1$, $-e_1 + e_2$ and $-e_1 + 2e_2$ also lie on this edge. Projecting from $-e_1$ via Lemma 8.3.5 shows that P lies above the line $y = -1$, and since the origin is an interior point, there must be a lattice point $m = ae_1 + be_2$ with $a > 0$ and $b \geq -1$.

Subcase C.1 If $b = -1$, then $2e_1 - e_2$ must lie in P. If $-e_1 + 2e_2$ and $2e_1 - e_2$ do not lie on an edge of P, their sum $e_1 + e_2$ would lie in P by Lemma 8.3.6 and force e_1, e_2 to be interior points. This is impossible, so $-e_1 + 2e_2$ and $2e_1 - e_2$ lie on an edge of P. Hence P has type 9.

Subcase C.2 If $b = 0$, then $m = e_1$, for otherwise e_1 would be an interior point. Applying Lemma 8.3.6 to e_1 and $-e_1 + e_2$ shows that $e_2 \in P$. If no edge connects

e_1, e_2, then Lemma 8.3.6 would imply $e_1 + e_2 \in P$. This point and $-e_1 + 2e_2$ would force e_2 to be interior, again impossible. Hence e_1, e_2 lie on an edge, which forces P to be contained in the polygon of type 9. Then our hypotheses on P force equality.

Subcase C.3 If $b \geq 1$, then e_2 becomes an interior point of P (part (f) of Exercise 8.3.4). Hence this subcase cannot occur. $\square$

There are many other proofs of the classification given in Theorem 8.3.7, including [**74, 214, 230**]. References to further proofs are given in [**74**, Thm. 6.10].

The collection of 2-dimensional reflexive polygons also exhibits some very interesting symmetries and regularities. For instance, we know that if P is reflexive, then its dual is also reflexive. In Exercise 8.3.5 you will determine which are the dual pairs in the list above. There is also a very interesting relation between the numbers of boundary lattice points in P and P°: in all cases, we have

$$|\partial P \cap M| + |\partial P^\circ \cap N| = 12. \tag{8.3.3}$$

We will see in Chapter 10 that there is an explanation for this coincidence coming from the cohomology theory of sheaves on surfaces.

Higher Dimensions. Reflexive polytopes of dimension greater than two and their associated toric varieties are being actively studied. See for instance [**215**] and the references therein. One reason for the interest in these varieties is the relation with mirror symmetry. It is known that there are a finite number of equivalence classes of reflexive polytopes in all dimensions. Using their computer program PALP, Kreuzer and Skarke [**183**] determined that there are 4319 classes of 3-dimensional reflexive polytopes and 473800776 classes of 4-dimensional reflexive polytopes. Since these numbers grow so quickly, most more recent work has focused on subclasses, for instance the polytopes giving smooth Fano toric varieties. These polytopes have a deceptively simple description (Exercise 8.3.6). There are 5 types of such polytopes in dimension 2 (Exercise 8.3.7), and Batyrev [**17**] and Sato [**244**] have shown that there are 18 types in dimension 3 and 124 types in dimension 4. See [**216**] for a classification algorithm and further references.

Exercises for §8.3.

8.3.1. Verify the claims about the polygons P and P° defined in Example 8.3.2.

8.3.2. Prove the claim about $\mathbb{P}(q_0, \ldots, q_n)$ made in Example 8.3.3. Hint: Exercise 4.2.11.

8.3.3. Prove Lemma 8.3.6. Hint: Assume that (a) and (b) do not hold. Show that for any facet F of P, $\langle u_F, m\rangle + \langle u_F, m'\rangle > -2$, and use this to conclude that (c) must hold.

8.3.4. In this exercise, you will supply the details for the proof of Theorem 8.3.7.

(a) Let $m, m' \in M \simeq \mathbb{Z}^2$ and assume that $\mathrm{Conv}(0, m, m')$ has no lattice points other than the vertices. Prove that m, m' form a basis of M.

(b) Show that if P is a reflexive triangle with vertices at e_1, e_2, then the third vertex must be $-e_1 - e_2$. Hint: One way to show this succinctly is to use the projections from the vertices e_1 and e_2 as in Lemma 8.3.5.

(c) In the case that each facet contains exactly two vertices, show that if there are vertices m, m', m'' with $m + m' = m''$, then the pairs m, m'' and m', m'' must lie in edges.

(d) Complete the proof that the polygons of types 4b, 5a, 6a are the only possibilities in Subcase A.2. Hint: Show that $\mathrm{Conv}(\pm e_2, \pm(e_2 - e_1), e_1)$ is lattice equivalent to 5a.

(e) Show that every reflexive polygon in Case B is equivalent to one of type 4c, 5b, 6b, 6c, 6d, 7a, 7b, 8a, 8b, or 8c.

(f) Prove the claim made in Subcase C.3.

8.3.5. Consider the 16 reflexive polygons in Figure 2. Since each polygon in the figure is reflexive, its dual must also appear in the figure, up to lattice equivalence.

(a) For each polygon in the Figure 2, determine its dual polygon and where the dual fits in the classification. In some cases, you will need to find an isomorphism of M that takes the dual to a polygon in the figure. Also, some of the polygon are self-dual, up to lattice equivalence.

(b) Show that the relation (8.3.3) holds for each polar pair.

8.3.6. A lattice polytope is called *Fano* if the origin is an interior point and the vertices of every facet form a basis of the lattice. Let $Q \subseteq N_{\mathbb{R}}$ be a Fano polytope.

(a) Prove that Q is reflexive.

(b) Part (a) implies that $Q^\circ \subseteq M_{\mathbb{R}}$ is a lattice polytope. Prove that X_{Q° is a smooth Fano toric variety whose normal fan is formed by taking cones over proper faces of Q.

(c) Prove that every smooth Fano toric variety arises this way.

8.3.7. Of the 16 Gorenstein Fano varieties classified in Theorem 8.3.7, exactly five are smooth. Find them.

8.3.8. Some of the polygons in Figure 2 give well-known toric varieties. For example, type 9 gives $\mathbb{P}^2$ and type 8a gives $\mathbb{P}^1 \times \mathbb{P}^1$. This follows easily by computing the normal fan. Here you will describe the toric surfaces coming from some of the other polygons in Figure 2. This exercise is based on [**74**, Rem. 6.11].

(a) Show that type 8b corresponds to the blow-up of $\mathbb{P}^2$ at one of the torus fixed points. Also show that this is the Hirzebruch surface $\mathscr{H}_1$. Hint: Remember the description of blowing up given in §2.3.

(b) Similarly show that type 7a (resp. 6a) corresponds to the blow-up of $\mathbb{P}^2$ at two (resp. three) torus fixed points.

(c) Show that type 3 corresponds to a quotient $\mathbb{P}^2/(\mathbb{Z}/3\mathbb{Z})$ and is isomorphic to the surface in $\mathbb{P}^3$ defined by $w^3 = xyz$. Hint: Compute the normal fan and show that its minimal generators span a sublattice of index 3 in $\mathbb{Z}^2$. Proposition 3.3.7 will be helpful. For the final assertion, use the characters coming from the lattice points of the polygon.

(d) Show that type 8c gives the weighted projective space $\mathbb{P}(1,1,2)$ and type 4c gives the quotient $\mathbb{P}(1,1,2)/(\mathbb{Z}/2\mathbb{Z})$. Hint: See Example 3.1.17.

(e) Show that type 4a gives the quotient $\mathbb{P}^1 \times \mathbb{P}^1/(\mathbb{Z}/2\mathbb{Z})$.

(f) Show that type 6d gives the weighted projective space $\mathbb{P}(1,2,3)$.

(g) Show that type 7b (resp. 6b) corresponds to the blow-up of $\mathbb{P}(1,1,2)$ at one (resp. two) smooth torus fixed points.

(h) Show that type 5b gives the blow-up of $\mathbb{P}(1,2,3)$ at its unique smooth torus fixed point.

8.3.9. In this exercise you will consider a toric surface that is not quite Fano. The lattice polygon $P = \mathrm{Conv}(\pm 3e_1, \pm 3e_2, \pm 2e_1 \pm 2e_2) \subseteq M_{\mathbb{R}} = \mathbb{R}^2$ gives a toric surface X_P. Let $D = \sum_\rho D_\rho$ be the anticanonical divisor of X_P.

(a) Prove that the ample Cartier divisor D_P associated to P is given by $D_P = 6D$. Conclude that D is not Cartier and that $6D$ is the smallest integer multiple of D that is Cartier.

(b) Show that the normal fan of P has minimal generators $\pm e_1 \pm 2e_2, \pm 2e_1 \pm e_2$ and that the minimal generators are the vertices of $Q = \mathrm{Conv}(\pm e_1 \pm 2e_2, \pm 2e_1 \pm e_2) \subseteq N_{\mathbb{R}}$.

(c) Show that the dual of Q is $Q^\circ = \frac{1}{6}P$ and conclude that 6 is the smallest integer multiple of Q° that is a lattice polytope.

8.3.10. A complete toric surface X_Σ is *log del Pezzo* if some integer multiple of its anti-canonical divisor $-K_{X_\Sigma}$ is an ample Cartier divisor. The *index* of X_Σ is the smallest positive integer ℓ such that such that $-\ell K_{X_\Sigma}$ is Cartier. This exercise and the next will consider this interesting class of toric surfaces.

(a) Prove that a complete toric surface is log del Pezzo of index 1 if and only if it is Gorenstein Fano.

(b) Prove that the toric surface of Exercise 8.3.9 is log del Pezzo of index 6.

8.3.11. A lattice polygon $Q \subseteq N_{\mathbb{R}}$ is called *LDP* if the origin is an interior point of Q and the vertices of Q are primitive vectors in N. The dual $Q^\circ \subseteq M_{\mathbb{R}}$ of a LDP polygon contains the origin as an interior point but may fail to be a lattice polygon. The *index* of Q is the smallest positive integer ℓ such that such that ℓQ° is a lattice polygon. This exercise will explore the relation between LDP polygons and toric log del Pezzo surfaces.

(a) Show that the polygon Q of Exercise 8.3.9 is LDP of index 6.

(b) An LDP polygon $Q \subseteq N_{\mathbb{R}}$ gives a fan Σ in $N_{\mathbb{R}}$ by taking the cones over the faces of Q. Show that the minimal generators of Σ are the vertices of Q and that the toric surface X_Σ is log del Pezzo of index equal to the index of Q.

(c) Conversely, let X_Σ be a toric log del Pezzo surface and let Q be the convex hull of the minimal generators of Σ. Note that Q is LDP and that Σ is the fan obtained by taking the cones over the faces of Q. Hint: The key point is to show that every minimal generator of Σ is a vertex of Q. This can be proved using the strict convexity of the support function of $-\ell K_{X_\Sigma}$.

This exercise shows that classifying toric log del Pezzo surfaces up to isomorphism is equivalent to classifying LDP polygons up to lattice equivalence. This is an active area of research—see [**168**].

8.3.12. Let X_Σ be a smooth projective toric variety. As in Definition 6.4.10, a primitive colection $P = \{\rho_1, \ldots, \rho_k\} \subseteq \Sigma(1)$ gives a primitive relation

$$u_{\rho_1} + \cdots + u_{\rho_k} = \textstyle\sum_{\rho \in \gamma(1)} c_\rho u_\rho, \quad c_\rho \in \mathbb{Z}_{>0},$$

where $\gamma \in \Sigma$ is the minimal cone containing $\sum_{i=1}^k u_{\rho_i}$. Following [**17**], the *degree* of P is $\Delta(P) = k - \sum_{\rho \in \gamma(1)} c_\rho$. Prove that X_Σ is Fano if and only if $\Delta(P) > 0$ for all primitive collections P. This is [**17**, Prop. 2.3.6]. Hint: Use (6.4.9) and Theorem 6.4.9.

Chapter 9

Sheaf Cohomology of Toric Varieties

§9.0. Background: Sheaf Cohomology

By Proposition 6.0.8, a short exact sequence of sheaves on X

$$0 \to \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0 \tag{9.0.1}$$

gives rise to an exact sequence of global sections

$$0 \longrightarrow \Gamma(X,\mathscr{F}) \longrightarrow \Gamma(X,\mathscr{G}) \longrightarrow \Gamma(X,\mathscr{H}). \tag{9.0.2}$$

The failure of $\Gamma(X,\mathscr{G}) \to \Gamma(X,\mathscr{H})$ to be surjective is measured by a sheaf cohomology group. The main goal of this chapter is to understand sheaf cohomology on a toric variety.

Sheaves and Cohomology. A sheaf $\mathscr{F}$ on a variety X has *sheaf cohomology groups* $H^p(X,\mathscr{F})$. The abstract definition uses an exact sequence of sheaves

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^0 \xrightarrow{d^0} \mathscr{I}^1 \xrightarrow{d^1} \mathscr{I}^2 \xrightarrow{d^2} \cdots,$$

where $\mathscr{I}^0, \mathscr{I}^1, \ldots$ are *injective*. This term is from homological algebra: a sheaf $\mathscr{I}$ is injective if given a sheaf homomorphism $\mathscr{H} \xrightarrow{\alpha} \mathscr{I}$ and an injection $\beta : \mathscr{H} \to \mathscr{G}$, there exists a sheaf homomorphism θ making the diagram below commute:

$$\begin{tikzcd} & \mathscr{I} & \\ 0 \arrow[r] & \mathscr{H} \arrow[u, "\alpha"] \arrow[r, "\beta"] & \mathscr{G}. \arrow[ul, dotted, "\theta"'] \end{tikzcd}$$

We say that $\mathscr{I}^\bullet$ is an *injective resolution* of $\mathscr{F}$. From $\mathscr{I}^\bullet$ we get the complex of global sections

$$\Gamma(X,\mathscr{I}^\bullet):\ \Gamma(X,\mathscr{I}^0) \xrightarrow{d^0} \Gamma(X,\mathscr{I}^1) \xrightarrow{d^1} \Gamma(X,\mathscr{I}^2) \xrightarrow{d^2} \cdots .$$

The term *complex* refers to the fact that $d^{p+1}\circ d^p = 0$ for all $p \geq 0$. Then the pth *sheaf cohomology group* of $\mathscr{F}$ is defined to be

$$H^p(X,\mathscr{F}) = H^p(\Gamma(X,\mathscr{I}^\bullet)) = \ker(d^p)/\mathrm{im}(d^{p-1}), \tag{9.0.3}$$

where for $p = 0$, we define d^{-1} to be the zero map $0 \to \Gamma(X,\mathscr{I}^0)$. One can prove that injective resolutions always exist and that two different injective resolutions of $\mathscr{F}$ give the same sheaf cohomology groups.

This definition of $H^p(X,\mathscr{F})$ has some very nice properties, including:

- $H^0(X,\mathscr{F}) = \Gamma(X,\mathscr{F})$.
- A sheaf homomorphism $\mathscr{F} \to \mathscr{G}$ induces a homomorphism of cohomology groups $H^i(X,\mathscr{F}) \to H^i(X,\mathscr{G})$ that is compatible with composition and takes the identity to the identity, i.e., $\mathscr{F} \mapsto H^i(X,\mathscr{F})$ is a functor.
- A short exact sequence of sheaves (9.0.1) gives a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X,\mathscr{F}) \longrightarrow H^0(X,\mathscr{G}) \longrightarrow H^0(X,\mathscr{H}) &\xrightarrow{\partial_0} \\ H^1(X,\mathscr{F}) \longrightarrow H^1(X,\mathscr{G}) \longrightarrow H^1(X,\mathscr{H}) &\xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{p-1}} \\ H^p(X,\mathscr{F}) \longrightarrow H^p(X,\mathscr{G}) \longrightarrow H^p(X,\mathscr{H}) &\xrightarrow{\partial_p} \cdots . \end{aligned}$$

We call $\partial_p : H^p(X,\mathscr{H}) \to H^{p+1}(X,\mathscr{F})$ a *connecting homomorphism*.

You will prove the first bullet in Exercise 9.0.1. For the second bullet, the key step is to show that given a sheaf homomorphism $\alpha : \mathscr{F} \to \mathscr{G}$ and injective resolutions $\mathscr{F} \to \mathscr{A}^\bullet, \mathscr{G} \to \mathscr{B}^\bullet$, there are sheaf homomorphisms $\alpha^p : \mathscr{A}^p \to \mathscr{B}^p$ such that the diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d, "\alpha"'] & \mathscr{A}^0 \arrow[r, "d^0"] \arrow[d, "\alpha^0"'] & \mathscr{A}^1 \arrow[r, "d^1"] \arrow[d, "\alpha^1"'] & \cdots \\ \mathscr{G} \arrow[r] & \mathscr{A}^0 \arrow[r, "d^0"] & \mathscr{A}^1 \arrow[r, "d^1"] & \cdots \end{tikzcd}$$

commutes. We say that $\alpha^\bullet : \mathscr{A}^\bullet \to \mathscr{B}^\bullet$ is a map of complexes. Then α^p induces the desired map $H^p(X,\mathscr{F}) \to H^p(X,\mathscr{G})$. Finally, for the last bullet, an exact sequence (9.0.1) lifts to an exact sequence of injective resolutions

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{A}^\bullet \arrow[r] & \mathscr{B}^\bullet \arrow[r] & \mathscr{C}^\bullet \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{F} \arrow[r] \arrow[u] & \mathscr{G} \arrow[r] \arrow[u] & \mathscr{H} \arrow[r] \arrow[u] & 0, \end{tikzcd}$$

and one can show that taking global sections gives an exact sequence of complexes

$$0 \longrightarrow \Gamma(X,\mathscr{A}^\bullet) \longrightarrow \Gamma(X,\mathscr{B}^\bullet) \longrightarrow \Gamma(X,\mathscr{C}^\bullet) \longrightarrow 0. \tag{9.0.4}$$

It is a general fact in homological algebra that *any* exact sequence of complexes

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

gives a long exact sequence

$$\begin{aligned} &0 \longrightarrow H^0(A^\bullet) \longrightarrow H^0(B^\bullet) \longrightarrow H^0(C^\bullet) \xrightarrow{\partial_0} \\ &H^1(A^\bullet) \longrightarrow H^1(B^\bullet) \longrightarrow H^1(C^\bullet) \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{p-1}} \\ &H^p(A^\bullet) \longrightarrow H^p(B^\bullet) \longrightarrow H^p(C^\bullet) \xrightarrow{\partial_p} \cdots . \end{aligned} \tag{9.0.5}$$

Applied to (9.0.4), we get the desired long exact sequence in sheaf cohomology.

In the language of homological algebra, Γ is *left exact* since (9.0.2) is exact. Then the sheaf cohomology groups are the *derived functors* of Γ. The texts [**125, 131, 273**] discuss sheaf cohomology and homological algebra in more detail. We especially recommend Appendix 3 of [**89**] and Chapters 2 and 3 of [**158**].

Čech Cohomology. While the abstract definition of sheaf cohomology has nice properties, it is not useful for explicit computations. Fortunately, there is a down-to-earth way of viewing sheaf cohomology, in terms of the Čech complex, which we now describe.

Let $\mathscr{U} = \{U_i\}_{i=1}^{\ell}$ be an open cover of X. The definition of a sheaf $\mathscr{F}$ on X shows that $H^0(X,\mathscr{F}) = \Gamma(X,\mathscr{F})$ is the kernel of the map

$$\prod_{1\le i\le \ell} \mathscr{F}(U_i) \xrightarrow{d^0} \prod_{1\le i<j\le \ell} \mathscr{F}(U_i\cap U_j), \tag{9.0.6}$$

where d^0 is defined as follows: if $\alpha = (\alpha_i) \in \prod_{1\le i\le \ell}\mathscr{F}(U_i)$, then the component of $d^0(\alpha)$ in $\mathscr{F}(U_i\cap U_j)$ for $i<j$ is given by $\alpha_j|_{U_i\cap U_j} - \alpha_i|_{U_i\cap U_j}$. Here, $\alpha_j \mapsto \alpha_j|_{U_i\cap U_j}$ is the restriction map $\mathscr{F}(U_j) \to \mathscr{F}(U_i\cap U_j)$, and similarly for $\alpha_i \mapsto \alpha_i|_{U_i\cap U_j}$.

To get "higher" information about how sections of $\mathscr{F}$ fit together, we extend (9.0.6) to the *Čech complex*. We will use the following notation. Let $[\ell] = \{1,\ldots,\ell\}$ be the index set for the open cover and let $[\ell]_p$ denote the set of all $(p+1)$-tuples $(i_0,\ldots,i_p)$ of elements of I satisfying $i_0 < \cdots < i_p$.

Definition 9.0.1. The group of pth ***Čech cochains*** is

$$\check{C}^p(\mathscr{U},\mathscr{F}) = \bigoplus_{(i_0,\ldots,i_p)\in[\ell]_p} \mathscr{F}(U_{i_0}\cap\cdots\cap U_{i_p}).$$

One can think of an element of $\check{C}^p(\mathscr{U},\mathscr{F})$ as a function α that assigns an element of $\mathscr{F}(U_{i_0}\cap\cdots\cap U_{i_p})$ to each $(i_0,\ldots,i_p)\in[\ell]_p$. Then we define a differential

$$\check{C}^p(\mathscr{U},\mathscr{F}) \xrightarrow{d^p} \check{C}^{p+1}(\mathscr{U},\mathscr{F})$$

by describing how $d^p(\alpha)$ operates on elements of $[\ell]_{p+1}$:

$$d^p(\alpha)(i_0,\ldots,i_{p+1}) = \sum_{k=0}^{p+1}(-1)^k\alpha(i_0,\ldots,\widehat{i_k},\ldots,i_{p+1})|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}.$$

As above, $\alpha(i_0,\ldots,\widehat{i_k},\ldots,i_{p+1})|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}$ is obtained by applying the restriction map

$$\mathscr{F}(U_{i_0}\cap\cdots\cap\widehat{U}_{i_k}\cap\cdots\cap U_{i_{p+1}}) \longrightarrow \mathscr{F}(U_{i_0}\cap\cdots\cap U_{i_{p+1}})$$

to $\alpha(i_0,\ldots,\widehat{i_k},\ldots,i_{p+1})$. In Exercise 9.0.2 you will verify that $d^p\circ d^{p-1} = 0$.

Definition 9.0.2. Given a sheaf $\mathscr{F}$ on X and an open cover $\mathscr{U} = \{U_i\}_{i\in I}$ of X, the ***Čech complex*** is

$$\check{C}^\bullet(\mathscr{U},\mathscr{F}):\ 0 \longrightarrow \check{C}^0(\mathscr{U},\mathscr{F}) \xrightarrow{d^0} \check{C}^1(\mathscr{U},\mathscr{F}) \xrightarrow{d^1} \check{C}^2(\mathscr{U},\mathscr{F}) \xrightarrow{d^2} \cdots,$$

and the pth ***Čech cohomology group*** is

$$\check{H}^p(\mathscr{U},\mathscr{F}) = H^p(\check{C}^\bullet(\mathscr{U},\mathscr{F})) = \ker(d^p)/\mathrm{im}(d^{p-1}).$$

Notice that $\check{H}^0(\mathscr{U},\mathscr{F}) = H^0(X,\mathscr{F}) = \Gamma(X,\mathscr{F})$ since $\mathscr{F}$ is a sheaf. However, $\check{H}^p(\mathscr{U},\mathscr{F})$ need not equal $H^p(X,\mathscr{F})$ for $p>0$. We will soon see that there is a nice case where equality occurs for all p.

***Cohomology of a Quasicoherent Sheaf*.** To compute the sheaf cohomology of a quasicoherent sheaf $\mathscr{F}$ on a variety X using Čech cohomology, the idea is to use an open cover $\mathscr{U}$ of X such that $\check{H}^p(\mathscr{U},\mathscr{F})$ equals $H^p(X,\mathscr{F})$ for all p. The following vanishing theorem of Serre is very useful in this regard. Proofs can be found in [**125**], [**131**] and [**273**].

Theorem 9.0.3 (Serre Vanishing for Affine Varieties)**.** *Let $\mathscr{F}$ be a quasicoherent sheaf on an affine variety U. Then $H^p(U,\mathscr{F}) = 0$ for all $p>0$.* □

By Theorem 9.0.3, we can compute the cohomology of a quasicoherent sheaf $\mathscr{F}$ using *any* affine open cover. Here is the rough intuition:

- Consider an arbitrary open cover $\mathscr{U} = \{U_i\}$ of X. Since we can construct X by "gluing together" the U_i, the cohomology of X should be obtained from the cohomologies of the U_i and their various intersections. In other words, $H^\bullet(X,\mathscr{F})$ should be determined by the cohomology groups $H^\bullet(U_{i_0}\cap\cdots\cap U_{i_{p+1}},\mathscr{F})$ as we vary over all p.
- Now suppose that $\mathscr{U} = \{U_i\}$ is an *affine* open cover. Then $U_{i_0}\cap\cdots\cap U_{i_{p+1}}$ is affine and hence has vanishing higher cohomology by Serre's theorem. So all that is left is $H^0(U_{i_0}\cap\cdots\cap U_{i_{p+1}},\mathscr{F})$. This gives the Čech complex, which thus computes the sheaf cohomology of $\mathscr{F}$.

This suggests the following result.

Theorem 9.0.4. *Let $\mathscr{U} = \{U_i\}$ be an affine open cover of a variety X and let $\mathscr{F}$ be a quasicoherent sheaf on X. Then there are natural isomorphisms*

$$\check{H}^p(\mathscr{U},\mathscr{F}) \simeq H^p(X,\mathscr{F})$$

for all $p \geq 0$.

The proof will be given later in the section after we introduce a spectral sequence that makes the above intuition rigorous. A more elementary proof that does not use spectral sequences can be found in [**131**, Thm. III.4.5].

Here is an application of Theorem 9.0.4.

Example 9.0.5. We compute the cohomology of $\mathscr{O}_{\mathbb{P}^1}$, $\mathscr{O}_{\mathbb{P}^1}(-1)$ and $\Omega^1_{\mathbb{P}^1}$ on $\mathbb{P}^1$. Consider the affine open cover $\mathscr{U} = \{U_0, U_1\}$ of $\mathbb{P}^1$, where

$$U_0 = \operatorname{Spec}(\mathbb{C}[x]) \quad \text{and} \quad U_1 = \operatorname{Spec}(\mathbb{C}[x^{-1}]).$$

Note also that

$$U_0 \cap U_1 = \operatorname{Spec}(\mathbb{C}[x,x^{-1}]).$$

For any quasicoherent sheaf $\mathscr{F}$ on $\mathbb{P}^1$, the Čech complex is

$$\mathscr{F}(U_0) \oplus \mathscr{F}(U_1) \xrightarrow{d^0} \mathscr{F}(U_0 \cap U_1).$$

It follows that $H^p(\mathbb{P}^1,\mathscr{F}) = 0$ for $p \geq 2$. Hence we need only consider $H^0(\mathbb{P}^1,\mathscr{F})$ and $H^1(\mathbb{P}^1,\mathscr{F})$. For simplicity, we write these sheaf cohomology groups as $H^0(\mathscr{F})$ and $H^1(\mathscr{F})$ respectively.

For $\mathscr{F} = \mathscr{O}_{\mathbb{P}^1}$, the Čech complex becomes

$$\mathbb{C}[x] \oplus \mathbb{C}[x^{-1}] \xrightarrow{d^0} \mathbb{C}[x,x^{-1}] \tag{9.0.7}$$

where $d^0(f(x), g(x^{-1})) = f(x) - g(x^{-1})$. Then

$$H^0(\mathscr{O}_{\mathbb{P}^1}) = \ker(d^0) = \mathbb{C}$$
$$H^1(\mathscr{O}_{\mathbb{P}^1}) = \operatorname{coker}(d^0) = 0.$$

The assertion for H^0 is clear since $f(x) - g(x^{-1}) = 0$ implies that f and g are the same constant, and the assertion for H^1 follows since an element of $\mathbb{C}[x,x^{-1}]$ can be written

$$\underbrace{a_{-m}x^{-m} + \cdots + a_{-1}x^{-1}}_{-g(x^{-1})} + \underbrace{a_0 + a_1x + \cdots + a_nx^n}_{f(x)}.$$

We can represent $\mathscr{O}_{\mathbb{P}^1}(-1)$ as the line bundle $\mathscr{O}_{\mathbb{P}^1}(-D)$, where the divisor D is one of the fixed points of the torus action on $\mathbb{P}^1$. It follows that we have an exact sequence of sheaves

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathscr{O}_{\mathbb{P}^1} \longrightarrow \mathscr{O}_D \longrightarrow 0.$$

The long exact sequence in sheaf cohomology gives

$$0 \to H^0(\mathscr{O}_{\mathbb{P}^1}(-1)) \to H^0(\mathscr{O}_{\mathbb{P}^1}) \to H^0(\mathscr{O}_D) \to H^1(\mathscr{O}_{\mathbb{P}^1}(-1)) \to H^1(\mathscr{O}_{\mathbb{P}^1}) \to \cdots.$$

The map $H^0(\mathscr{O}_{\mathbb{P}^1}) \to H^0(\mathscr{O}_D)$ is the isomorphism that sends a constant function on $\mathbb{P}^1$ to the same constant function on D. Since $H^1(\mathscr{O}_{\mathbb{P}^1}) = 0$, the long exact sequence implies that

$$H^0(\mathscr{O}_{\mathbb{P}^1}(-1)) = H^1(\mathscr{O}_{\mathbb{P}^1}(-1)) = 0. \tag{9.0.8}$$

Finally, for $\Omega^1_{\mathbb{P}^1}$, we use the Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^1} \longrightarrow \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathscr{O}_{\mathbb{P}^1} \longrightarrow 0$$

from (8.1.8). Since $H^p(X, \mathscr{F} \oplus \mathscr{G}) \simeq H^p(X, \mathscr{F}) \oplus H^p(X, \mathscr{G})$ (Exercise 9.0.3), the vanishing (9.0.8) and the long exact sequence for cohomology imply that

$$\begin{aligned} H^0(\Omega^1_{\mathbb{P}^1}) &= 0 \\ H^1(\Omega^1_{\mathbb{P}^1}) &\simeq H^0(\mathscr{O}_{\mathbb{P}^1}) = \mathbb{C}. \end{aligned}$$

Earlier, in Example 6.0.5, we showed that the surjective sheaf homomorphism

$$\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1) \to \mathscr{O}_{\mathbb{P}^1}$$

is not surjective on global sections. This is what forces $H^1(\Omega^1_{\mathbb{P}^1})$ to be nonzero. $\Diamond$

A key part of Example 9.0.5 was the surjectivity of (9.0.7). This is the algebraic analog of a *Cousin problem* in complex analysis. A discussion of Cousin problems and their relation to sheaves and cohomology can be found in [**180**, Ch. 13].

Serre Vanishing for Projective Varieties. The Serre vanishing theorem for affine varieties (Theorem 9.0.3) has a projective version.

Theorem 9.0.6 (Serre Vanishing for Projective Varieties)**.** *Let $\mathscr{L}$ be an ample line bundle on a projective variety X. Then for any coherent sheaf $\mathscr{F}$ on X, we have*

$$H^p(X, \mathscr{F} \otimes_X \mathscr{L}^{\otimes \ell}) = 0$$

for all $p > 0$ and $\ell \gg 0$. □

A proof can be found in [**131**, Prop. II.5.3]. We will use this result later in the chapter to prove some interesting vanishing theorems for toric varieties.

Higher Direct Images. Given a morphism $f : X \to Y$ of varieties and a sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules on X, the *direct image* is the sheaf $f_*\mathscr{F}$ on Y defined by

$$U \longmapsto \mathscr{F}(f^{-1}(U))$$

for $U \subseteq Y$ open. We noted in Example 4.0.24 that $f_*\mathscr{F}$ is a sheaf of $\mathscr{O}_Y$-modules.

The definition of $f_*\mathscr{F}$ implies in particular that

$$H^0(Y, f_*\mathscr{F}) = H^0(X, \mathscr{F})$$

since $f^{-1}(Y) = X$. More generally, there are homomorphisms

$$H^p(Y, f_*\mathscr{F}) \longrightarrow H^p(X, \mathscr{F}), \tag{9.0.9}$$

which need not be isomorphisms for $p > 0$.

In this situation, we also get the *higher direct image* $R^p f_* \mathscr{F}$, which is the sheaf on Y associated to the presheaf defined by

$$U \longmapsto H^p(f^{-1}(U), \mathscr{F}).$$

Proposition 9.0.7. *Let $f : X \to Y$ be a morphism and $\mathscr{F}$ be a quasicoherent sheaf on X. Then:*

(a) *The higher direct images $R^p f_* \mathscr{F}$ are quasicoherent sheaves on Y.*

(b) *If $U \subseteq Y$ is affine, then $R^p f_* \mathscr{F}|_U$ is the sheaf associated to the $\mathscr{O}_Y(U)$-module $H^p(f^{-1}(U), \mathscr{F})$.* □

A proof can be found in [**131**, Propositions II.5.8 and III.8.5]. One especially nice case is when the higher direct images $R^p f_* \mathscr{F}$ vanish for $p > 0$. We will see below that the maps (9.0.9) are isomorphisms when this happens. The proof involves our next topic, spectral sequences.

Spectral Sequences. Readers not familar with spectral sequences should glance at Appendix C before proceeding farther. Here we discuss two spectral sequences relevant to this section.

First suppose that $f : X \to Y$ is a morphism and $\mathscr{F}$ is a quasicoherent sheaf on X. As above, we get the higher direct images $R^p f_* \mathscr{F}$, which are sheaves on Y. Proposition 9.0.7 shows that these sheaves compute the cohomology of $\mathscr{F}$ over certain open subsets of X. So when we "put these together," i.e., compute $H^p(Y, R^q f_* \mathscr{F})$, we should get the cohomology of $\mathscr{F}$ on all of X. The precise form of this intuition is the *Leray spectral sequence*:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathscr{F}) \Rightarrow H^{p+q}(X, \mathscr{F}).$$

Furthermore, the map $H^p(Y, f_* \mathscr{F}) \to H^p(X, \mathscr{F})$ from (9.0.9) is the edge homomorphism $E_2^{p,0} \to H^p(X, \mathscr{F})$ (see Definition C.1.6). This spectral sequence is discussed in Theorem C.2.1 and in more detail in [**115**, II.4.17] and [**125**, p. 463].

Now assume $R^q f_* \mathscr{F} = 0$ for $q > 0$. Then

$$E_2^{p,q} = \begin{cases} H^p(Y, f_* \mathscr{F}) & q = 0 \\ 0 & q > 0. \end{cases}$$

Then Proposition C.1.7 implies that (9.0.9) is an isomorphism. Thus we have proved the following.

Proposition 9.0.8. *Suppose $f : X \to Y$ is a morphism and $\mathscr{F}$ is a quasicoherent sheaf on X such that $R^q f_* \mathscr{F} = 0$ for $q > 0$. Then the map* (9.0.9) *is an isomorphism $H^p(Y, f_* \mathscr{F}) \simeq H^p(X, \mathscr{F})$.* □

For our second spectral sequence, let $\mathscr{U} = \{U_i\}$ be an open cover of X and $\mathscr{F}$ be a sheaf on X. In the discussion leading up to Theorem 9.0.4, we asserted that

$H^\bullet(X,\mathscr{F})$ is determined by $H^\bullet(U_{i_0}\cap\cdots\cap U_{i_{p+1}},\mathscr{F})$ as we vary over all p. The precise meaning is given by the E_1 spectral sequence

$$E_1^{p,q} = \bigoplus_{(i_0,\dots,i_p)\in[\ell]_p} H^q(U_{i_0}\cap\cdots\cap U_{i_p},\mathscr{F}) \Rightarrow H^{p+q}(X,\mathscr{F}), \tag{9.0.10}$$

where the differential $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ is induced by inclusion, with signs similar to the differential in the Čech complex. This spectral sequence is constructed in [**115**, II.5.4]. See also Theorem C.2.2.

We now have the tools needed to prove Theorem 9.0.4.

Proof of Theorem 9.0.4. We are assuming that $\mathscr{U}=\{U_i\}$ is an affine open cover of X. First observe that the $q=0$ terms of (9.0.10) are given by

$$E_1^{p,0} = \bigoplus_{(i_0,\dots,i_p)\in[\ell]_p} H^0(U_{i_0}\cap\cdots\cap U_{i_p},\mathscr{F}) = \check{C}^p(\mathscr{U},\mathscr{F}),$$

and the differentials $d_1^{p,0}$ are the differentials in the Čech complex. Hence

$$\begin{aligned} E_2^{p,0} &= \ker(d_1 : E_1^{p,0} \to E_1^{p+1,0})\big/\mathrm{im}(d_1 : E_1^{p-1,0} \to E_1^{p,0}) \\ &= H^p(\check{C}^\bullet(\mathscr{U},\mathscr{F})) = \check{H}^p(\mathscr{U},\mathscr{F}). \end{aligned}$$

Since $\mathscr{F}$ is quasicoherent and the intersections $U_{i_0}\cap\cdots\cap U_{i_p}$ are affine for all $p\geq 0$, it follows from Theorem 9.0.3 that $E_1^{p,q}=0$ for $q>0$. This implies that $E_2^{p,q}=0$ for $q>0$. Using Proposition C.1.7 again, we conclude that the edge homomorphism

$$E_2^{p,0} = \check{H}^p(\mathscr{U},\mathscr{F}) \longrightarrow H^p(X,\mathscr{F})$$

is an isomorphism for all $p\geq 0$. □

Cohen-Macaulay Varieties. We next discuss a class of varieties that play a crucial role in duality theory and are interesting in their own right.

We first define what it means for a local ring $(R,\mathfrak{m})$ to be Cohen-Macaulay. Elements $f_1,\dots,f_s\in\mathfrak{m}$ form a *regular sequence* if f_i is not a zero divisor in $R/\langle f_1,\dots,f_{i-1}\rangle$ for all i, and the *depth* of R is the maximal length of a regular sequence. Then R is *Cohen-Macaulay* if its depth equals its dimension. Examples include regular local rings. A nice discussion of what Cohen-Macaulay means can be found in [**246**, 10.2]. See also [**179**, pp. 153–155].

A variety X is *Cohen-Macaulay* if its local rings $(\mathscr{O}_{X,p},\mathfrak{m}_{X,p})$ are Cohen-Macaulay for all $p\in X$. Thus smooth varieties are Cohen-Macaulay. Later in the chapter we will prove that normal toric varieties are Cohen-Macaulay.

Serre Duality. Cohen-Macaulay varieties provide the natural setting for a basic duality theorem of Serre. Here is the simplest version.

Theorem 9.0.9 (Serre Duality I)**.** *Let ω_X be the canonical sheaf of a complete normal Cohen-Macaulay variety X of dimension n. Then for every locally free sheaf $\mathscr{F}$ of finite rank on X, there are natural isomorphisms*

$$H^p(X,\mathscr{F})^\vee \simeq H^{n-p}(X,\omega_X \otimes_{\mathscr{O}_X} \mathscr{F}^\vee).$$

In particular, when D is a Cartier divisor on X and K_X is a canonical divisor, we have isomorphisms

$$H^p(X,\mathscr{O}_X(D))^\vee \simeq H^{n-p}(X,\mathscr{O}_X(K_X - D)). \qquad \square$$

A proof for the projective case can be found in [**131**, Thm. III.7.6]. The assertion for divisors follows from

$$\omega_X \otimes_{\mathscr{O}_X} \mathscr{O}_X(D)^\vee \simeq \mathscr{O}_X(K_X) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-D) \simeq \mathscr{O}_X(K_X - D),$$

where the last isomorphism holds since D is Cartier.

There is also a more general version of Serre duality that applies when $\mathscr{F}$ is coherent but not necessarily locally free. The cohomology groups $H^p(X,\mathscr{F})$ are the derived functors of the global section functor $\mathscr{F} \mapsto \Gamma(X,\mathscr{F})$, and in the same way, the *Ext groups* $\mathrm{Ext}^p_{\mathscr{O}_X}(\mathscr{G},\mathscr{F})$ are the derived functors of the Hom functor $\mathscr{F} \mapsto \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathscr{F})$ for fixed $\mathscr{G}$. Then we have the following result.

Theorem 9.0.10 (Serre Duality II)**.** *Let ω_X be the canonical sheaf of a complete normal Cohen-Macaulay variety X of dimension n. Then for every coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms*

$$H^p(X,\mathscr{F})^\vee \simeq \mathrm{Ext}^{n-p}_{\mathscr{O}_X}(\mathscr{F},\omega_X). \qquad \square$$

When X is projective, this is proved in [**131**, Thm. III.7.6]. We will give a version of Serre duality especially adapted to the toric case in §9.2.

When X fails to be Cohen-Macaulay, there is a more general duality theorem where canonical sheaf ω_X is replaced with the *dualizing complex* $\omega_X^\bullet$. A discussion of this version of duality can be found in [**218**, §3.2].

Singular Cohomology. Our discussion of the sheaf cohomology of a toric variety in §9.1 will use some algebraic topology. Here we review the topological invariants we will need, beginning with the *singular cohomology groups*

$$H^p(Z,R),$$

where Z is a topological space and R is a commutative ring, usually $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. These are defined using continuous maps

$$\gamma : \Delta_n \longrightarrow Z,$$

where Δ_n is the standard n-simplex. There are several good introductions to singular cohomology, including [**124**], [**135**] and [**210**].

Here are some important properties of singular cohomology:

- A continuous map $f : Z \to W$ induces $f^* : H^p(W,R) \to H^p(Z,R)$ such that homotopic maps induce the same map on cohomology and the identity map induces the identity on cohomology.
- If $i : A \hookrightarrow Z$ is a deformation retract (i.e., there is a continuous map $r : Z \to A$ such that $r \circ i = 1_A$ and $i \circ r$ is homotopic to 1_Z), then $i^* : H^p(Z,R) \to H^p(A,R)$ is an isomorphism.
- If Z is contractible (i.e, there is $z \in Z$ such that $\{z\} \hookrightarrow Z$ is a deformation retract), then
$$H^p(Z,R) = \begin{cases} R & p = 0 \\ 0 & \text{otherwise.} \end{cases}$$
- For $n > 1$, the singular cohomology of the $(n-1)$-sphere $S^{n-1} \subseteq \mathbb{R}^n$ is
$$H^p(S^{n-1},R) = \begin{cases} R & p = 0, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

We will always assume that Z is a locally contractible metric space. This allows us to interpret the singular cohomology of Z in terms of sheaf cohomology as follows. The ring R defines a presheaf on Z where R is the group of sections over every nonempty open $U \subset Z$. The corresponding sheaf is the *constant sheaf* of R. By [**45**, III.1], the sheaf cohomology of the constant sheaf of R on Z is the singular cohomology $H^p(Z,R)$.

The Cohomology Ring. For a commutative ring R, $H^\bullet(Z,R)$ is a graded R-algebra with multiplication given by cup product. A continuous map induces a ring homomorphism on cohomology, so in particular, a deformation retract $i : A \hookrightarrow Z$ induces a ring isomorphism $i^* : H^\bullet(Z,R) \simeq H^\bullet(A,R)$.

Here are some examples.

Example 9.0.11. The real torus $(S^1)^n$ has cohomology ring
$$H^\bullet((S^1)^n,R) = R[\alpha_1,\ldots,\alpha_n]$$
where the α_i lie in $H^1((S^1)^n,R)$ and satisfy the relations $\alpha_i\alpha_j = -\alpha_j\alpha_i$ and $\alpha_i^2 = 0$ (see Examples 3.11 and 3.15 of [**135**]). Thus
$$H^\bullet((S^1)^n,R) \simeq \textstyle\bigwedge^\bullet R^n,$$
i.e., the cohomology ring is the exterior algebra of a free R-module. $\Diamond$

Example 9.0.12. The torus $(\mathbb{C}^*)^n$ contains the real torus $(S^1)^n$ as a deformation retract via $(t_1,\ldots,t_n) \mapsto (t_1/|t_1|,\ldots,t_n/|t_n|)$. Hence
$$H^\bullet((\mathbb{C}^*)^n,R) \simeq H^\bullet((S^1)^n,R) \simeq \textstyle\bigwedge^\bullet R^n.$$
More canonically, the torus $T_N = N \otimes \mathbb{C}^*$ has cohomology ring
$$H^\bullet(T_N,R) \simeq \textstyle\bigwedge^\bullet M \otimes_{\mathbb{Z}} R,$$

where M is the dual of N. Thus a lattice homomorphism $N \to N'$ gives a map $T_N \to T_{N'}$ of tori, and the induced map $H^\bullet(T_{N'}, R) \to H^\bullet(T_N, R)$ is the map

$$\bigwedge^\bullet M' \otimes_{\mathbb{Z}} R \longrightarrow \bigwedge^\bullet M \otimes_{\mathbb{Z}} R$$

determined by the dual homomorphism $M' \to M$. $\Diamond$

Example 9.0.13. The cohomology ring of $\mathbb{P}^n$ over $\mathbb{Z}$ is

$$H^\bullet(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}[\alpha]/\langle \alpha^{n+1} \rangle,$$

where $\alpha \in H^2(\mathbb{P}^n, \mathbb{Z})$ (see [**135**, Thm. 3.12]). In Theorem 12.4.4, we will give a similar quotient description of the cohomology ring of any smooth complete toric variety. $\Diamond$

Reduced Cohomology. Given a topological space Z, the canonical map $Z \to \{\text{pt}\}$ sends all elements of Z to the single point denoted pt. This induces the map

$$R = H^0(\{\text{pt}\}, R) \longrightarrow H^0(Z, R)$$

whose cokernel is denoted $\widetilde{H}^0(Z, R)$. Thus, assuming $R \neq 0$,

$$\widetilde{H}^0(Z, R) = 0 \iff Z \text{ is path connected}.$$

The *reduced cohomology* of Z with coefficients in R is defined for $p \geq 0$ by

$$\widetilde{H}^p(Z, R) = \begin{cases} \widetilde{H}^0(Z, R) & p = 0 \\ H^p(Z, R) & p > 0, \end{cases}$$

and for $p = -1$ by

$$\widetilde{H}^{-1}(Z, R) = \begin{cases} 0 & Z \neq \emptyset \\ R & Z = \emptyset. \end{cases}$$

The definition of $\widetilde{H}^{-1}$ may look strange but means that the sequence

$$(9.0.11) \qquad 0 \longrightarrow \widetilde{H}^{-1}(Z, R) \longrightarrow R \longrightarrow H^0(Z, R) \longrightarrow \widetilde{H}^0(Z, R) \longrightarrow 0$$

is exact for all Z, including $Z = \emptyset$ (Exercise 9.0.4). We will use $\widetilde{H}^{-1}$ and (9.0.11) in §9.1 when we compute sheaf cohomology on a toric variety.

Exercises for §9.0.

9.0.1. Use (9.0.3) to show that $H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

9.0.2. Check that the map d^p defined on the Čech cochains satisfies $d^p \circ d^{p-1} = 0$.

9.0.3. Let $\mathscr{F}, \mathscr{G}$ be sheaves on X. Prove $H^p(X, \mathscr{F} \oplus \mathscr{G}) \simeq H^p(X, \mathscr{F}) \oplus H^p(X, \mathscr{G})$.

9.0.4. Prove the exactness of (9.0.11).

9.0.5. A morphism $f : X \to Y$ is *affine* if Y has an affine open cover $\{U_i\}$ such that $f^{-1}(U_i)$ is affine for all i. Now assume that $f : X \to Y$ is affine and let $\mathscr{F}$ be a quasicoherent sheaf on X. Use Theorem 9.0.3, Proposition 9.0.7 and Theorem 9.0.8 to prove that $H^p(Y, f_*\mathscr{F}) \simeq H^p(X, \mathscr{F})$ for all $p \geq 0$.

9.0.6. Use Exercise 9.0.5 to prove the following isomorphisms in cohomology:

(a) $H^p(X, i_*\mathscr{F}) \simeq H^p(Y, \mathscr{F})$ when $i : Y \hookrightarrow X$ is closed in X and $\mathscr{F}$ is quasicoherent.

(b) $H^p(X, \pi_*\mathscr{F}) \simeq H^p(V, \mathscr{F})$ when $\pi : V \to X$ is a vector bundle and $\mathscr{F}$ is quasicoherent.

9.0.7. Let X be a variety that is a union of affine open subsets $X = U_1 \cup \cdots \cup U_s$.

(a) Let $\mathscr{F}$ be quasicoherent on X. Prove that $H^p(X, \mathscr{F}) = 0$ for $p > s-1$.

(b) Let $\mathscr{F}$ be quasicoherent on $\mathbb{P}^n$. Prove that $H^p(\mathbb{P}^n, \mathscr{F}) = 0$ for $p > n$.

9.0.8. Consider the $(n-1)$-sphere $S^{n-1} \subseteq \mathbb{R}^n$. Show that

$$\widetilde{H}^p(S^{n-1}, R) \simeq \begin{cases} R & p = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Hint: Remember that S^0 consists of two points.

§9.1. Cohomology of Toric Divisors

Demazure [**82**] gave a concrete description of the sheaf cohomology of the sheaf $\mathscr{O}_{X_\Sigma}(D)$ of a torus-invariant Cartier divisor D on a toric variety X_Σ. We generalize this to torus-invariant $\mathbb{Q}$-Cartier divisors, inspired by papers of Eisenbud, Mustaţă and Stillman [**91**], Hering, Küronya and Payne [**141**], and Perling [**226**].

The Toric Čech Complex. When we compute sheaf cohomology using Čech cohomology, the obvious choice of open cover is

$$\mathscr{U} = \{U_\sigma\}_{\sigma \in \Sigma_{\max}},$$

where $\Sigma_{\max}$ is the set of maximal cones in Σ. We write these as σ_i and order them according to their indices.

Given a torus-invariant Cartier divisor $D = \sum_\rho a_\rho D_\rho$ on X_Σ, the Čech complex is given by

$$\check{C}^p(\mathscr{U}, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{\gamma = (i_0,\dots,i_p) \in [\ell]_p} H^0(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}, \mathscr{O}_{X_\Sigma}(D)).$$

If we set $\sigma_\gamma = \sigma_{i_0} \cap \cdots \cap \sigma_{i_p} \in \Sigma$ for $\gamma = (i_0, \dots, i_p) \in [\ell]_p$, then we rewrite this as

$$\check{C}^p(\mathscr{U}, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{\gamma \in [\ell]_p} H^0(U_{\sigma_\gamma}, \mathscr{O}_{X_\Sigma}(D)). \tag{9.1.1}$$

The Grading on Cohomology. For an affine open subset U_σ, Proposition 4.3.3 implies that the sections of $\mathscr{O}_{X_\Sigma}(D)$ over U_σ can be written

$$H^0(U_\sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_m \mathbb{C} \cdot \chi^m,$$

where the direct sum is over all $m \in M$ such that $\langle m, u_\rho \rangle \geq -a_\rho$ for all $\rho \in \sigma(1)$. We can write this as

$$H^0(U_\sigma, \mathscr{O}_{X_\Sigma}(D)) = \bigoplus_{m \in M} H^0(U_\sigma, \mathscr{O}_{X_\Sigma}(D))_m,$$

where for $m \in M$,

$$H^0(U_\sigma, \mathscr{O}_{X_\Sigma}(D))_m = \begin{cases} \mathbb{C} \cdot \chi^m & \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \sigma(1) \\ 0 & \text{otherwise}. \end{cases} \tag{9.1.2}$$

This induces a grading of the Čech complex (9.1.1). The Čech differential is built from the restriction maps and hence respects the grading of the complex. Since $H^p(X_\Sigma, \mathscr{O}_X(D)) = \check{H}^p(\mathscr{U}, \mathscr{O}_X(D))$, we obtain a natural decomposition of sheaf cohomology

$$H^p(X_\Sigma, \mathscr{O}_X(D)) = \bigoplus_{m \in M} H^p(X_\Sigma, \mathscr{O}_X(D))_m.$$

When decomposing cohomology this way, we often refer to the $m \in M$ as *weights*. Here is an example of how weights can be used to compute sheaf cohomology.

Example 9.1.1. On $\mathbb{P}^2$, label the rays as usual: $u_0 = -e_1 - e_2$, $u_1 = e_1$, $u_2 = e_2$, and maximal cones σ_i, starting with σ_0 in the first quadrant and going counterclockwise. We will compute $H^p(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a))$ for $a \in \mathbb{Z}$, where $\mathscr{O}_{\mathbb{P}^2}(a) = \mathscr{O}_{\mathbb{P}^2}(aD_0)$ for the divisor D_0 corresponding to u_0.

Let U_i be the affine open corresponding to σ_i. Then U_{ij} is $U_i \cap U_j$ and U_{012} is the triple intersection. This allows us to write the Čech complex as

$$0 \longrightarrow \check{C}^0(a) \xrightarrow{\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}} \check{C}^1(a) \xrightarrow{\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}} \check{C}^2(a) \longrightarrow 0, \tag{9.1.3}$$

where

$$\begin{aligned} \check{C}^0(a) &= \bigoplus_{i=0}^{2} H^0(U_i, \mathscr{O}_{\mathbb{P}^2}(aD_0)) \\ \check{C}^1(a) &= \bigoplus_{i<j} H^0(U_{ij}, \mathscr{O}_{\mathbb{P}^2}(aD_0)) \\ \check{C}^2(a) &= H^0(U_{012}, \mathscr{O}_{\mathbb{P}^2}(aD_0)). \end{aligned}$$

We will compute the cohomology of this complex using the graded pieces for $m \in M = \mathbb{Z}^2$. For this purpose, let l_0, l_1, l_2 be the lines in $M_\mathbb{R} = \mathbb{R}^2$ defined by $\langle m, u_0 \rangle = -a, \langle m, u_1 \rangle = 0, \langle m, u_2 \rangle = 0$. These lines divide the plane into various regions called *chambers*. To get a disjoint decomposition, we define

$$C_{-++} = \{m \in M_\mathbb{R} \mid \langle m, u_0 \rangle < -a, \langle m, u_1 \rangle \geq 0, \langle m, u_2 \rangle \geq 0\}.$$

Note that a minus sign corresponds to strict inequality (< 0) while a plus sign corresponds to a weak inequality (≥ 0). The other chambers C_{+-+}, C_{+--}, etc. are defined similarly.

Suppose $a < 0$. The corresponding chamber decomposition of $M_\mathbb{R}$ is shown in Figure 1 on the next page. The labels l_i are placed on the plus side of the lines, i.e., where $\langle m, u_0 \rangle \geq -a, \langle m, u_1 \rangle \geq 0, \langle m, u_2 \rangle \geq 0$. Each chamber is labeled with its sign pattern, and the shading indicates which chamber the points on the lines belong to.

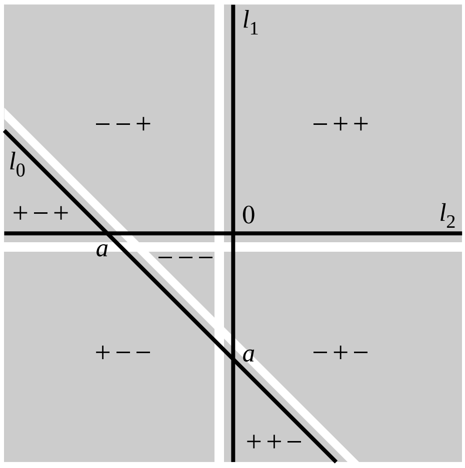


Figure 1. The chamber decomposition for $a < 0$

The cone σ_i is generated by u_j, u_k, where $\{i, j, k\} = \{0, 1, 2\}$. Thus

$$\begin{aligned} H^0(U_0, \mathscr{O}_{\mathbb{P}^2}(a))_m \neq 0 &\iff m \in C_{-++} \cap M \\ H^0(U_1, \mathscr{O}_{\mathbb{P}^2}(a))_m \neq 0 &\iff m \in C_{+-+} \cap M \\ H^0(U_2, \mathscr{O}_{\mathbb{P}^2}(a))_m \neq 0 &\iff m \in C_{++-} \cap M. \end{aligned} \tag{9.1.4}$$

This follows from (9.1.2) (Exercise 9.1.1). Hence we know when $\check{C}^0(a)_m \neq 0$.

The cone $\sigma_1 \cap \sigma_2$ is generated by u_0, so its dual is the half-plane of $M_{\mathbb{R}}$ where $\langle m, u_0 \rangle \geq a$, i.e., on the plus side of l_0. Using Figure 1 and (9.1.2) again, we get

$$H^0(U_{12}, \mathscr{O}_{\mathbb{P}^2}(a))_m \neq 0 \iff m \in (C_{+-+} \cup C_{+--} \cup C_{++-}) \cap M.$$

Similar results hold for U_{01} and U_{02}, so we know when $\check{C}^1(a)_m \neq 0$. Finally, since U_{012} is the torus of $\mathbb{P}^2$, we have

$$\check{C}^2(a)_m = H^0(U_{012}, \mathscr{O}_{\mathbb{P}^2}(a))_m \neq 0 \text{ for all } m \in M.$$

Putting everything together, we get Table 1 on the next page, which shows the dimension of $\check{C}^p(a)$ for $m \in M$ and $a < 0$. For example, $\dim \check{C}^1(a)_m = 2$ when $m \in C_{-++}$ since both U_{01} and U_{02} contribute 1-dimensional subspaces.

When $a < 0$, it is now easy to understand the Čech complex

$$0 \longrightarrow \check{C}^0(a)_m \longrightarrow \check{C}^1(a)_m \longrightarrow \check{C}^2(a)_m \longrightarrow 0. \tag{9.1.5}$$

The first line of Table 1 corresponds to $m \in C_{-++} \cup C_{+-+} \cup C_{++-}$. One can check without difficulty that for these m's, the complex (9.1.5) is exact, and the same is true for the second row as well.

$m \in M$ is in	$\dim \check{C}^0(a)_m$	$\dim \check{C}^1(a)_m$	$\dim \check{C}^2(a)_m$
$C_{-++} \cup C_{+-+} \cup C_{++-}$	1	2	1
$C_{--+} \cup C_{-+-} \cup C_{+--}$	0	1	1
C_{---}	0	0	1

Table 1. Dimension of $\check{C}^p(a)_m$ for $a < 0$

These remarks imply that if $m \in M$ is *not* in the interior of the triangle

$$a\Delta_2 = \mathrm{Conv}\{(0,0),(a,0),(0,a)\},$$

then

$$H^p(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a))_m = 0 \text{ for all } p.$$

However, if m is a lattice point in the interior of $a\Delta_2$, then $H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a))_m$ is nonzero. Summing up, we have:

$$\dim H^p(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a)) = \begin{cases} 0 & p \neq 2 \\ \big|\mathrm{Int}(a\Delta_2) \cap M\big| = \binom{-a-1}{2} & p = 2 \end{cases} \tag{9.1.6}$$

when $a < 0$. In what follows, we write $h^p(a) = \dim H^p(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a))$ for $a \in \mathbb{Z}$.

In Exercise 9.1.2 you will adapt the above methods to show that

$$h^p(a) = \begin{cases} \big|a\Delta_2 \cap M\big| = \binom{a+2}{2} & p = 0 \\ 0 & p > 0 \end{cases} \tag{9.1.7}$$

when $a \geq 0$. Thus, for any line bundle on $\mathbb{P}^2$, we have *completely determined* the dimensions of the cohomology groups $H^p(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a))$. The values for $-7 \leq a \leq 4$ are depicted in Table 2. Note the symmetry in the table. In Exercise 9.1.3 you will explore how this symmetry relates to Serre duality. ◊

a	$h^0(a)$	$h^1(a)$	$h^2(a)$
-7	0	0	15
-6	0	0	10
-5	0	0	6
-4	0	0	3
-3	0	0	1
-2	0	0	0
-1	0	0	0
0	1	0	0
1	3	0	0
2	6	0	0
3	10	0	0
4	15	0	0

Table 2. Sheaf cohomology of $\mathscr{O}_{\mathbb{P}^2}(a)$ on $\mathbb{P}^2$

Computing Cohomology. Given a divisor $D = \sum_\rho a_\rho D_\rho$ on X_Σ and $m \in M$, define two subsets of $|\Sigma|$:

- (When Σ is general) $V_{D,m} = \bigcup_{\sigma\in\Sigma} \mathrm{Conv}(u_\rho \mid \rho \in \sigma(1),\ \langle m, u_\rho\rangle < -a_\rho)$.
- (When D is $\mathbb{Q}$-Cartier) $V_{D,m}^{\mathrm{supp}} = \{u \in |\Sigma| \mid \langle m, u\rangle < \varphi_D(u)\}$, where φ_D is the support function of D (see Exercise 9.1.4).

Here is an example of these sets.

Example 9.1.2. Let X_Σ be the toric surface obtained by blowing up $\mathbb{P}^2$ at the three fixed points of the torus action. The minimal generators $u_1, \ldots, u_6$ of the fan Σ give divisors $D_1, \ldots, D_6$ on X_Σ. Let $D = -D_3 + 2D_5 - D_6$ and $m = e_1$. Since

$$\langle m, u_1\rangle \geq 0,\ \langle m, u_2\rangle \geq 0, \langle m, u_3\rangle < 1,\ \langle m, u_4\rangle < 0, \langle m, u_5\rangle \geq -2,\ \langle m, u_6\rangle < 1,$$

we get the sets $V_{D,m}$ and $V_{D,m}^{\mathrm{supp}}$ shown in Figure 2 (Exercise 9.1.5). The open circle

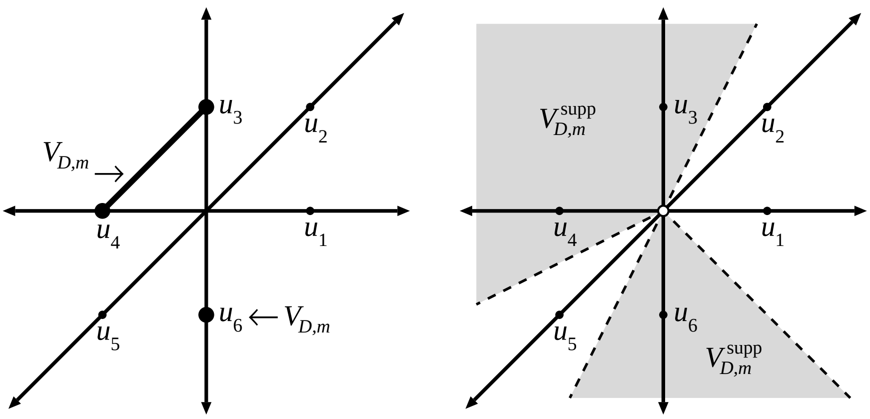


Figure 2. $V_{D,m}$ and $V_{D,m}^{\mathrm{supp}}$ for $D = -D_3 + 2D_5 - D_6$ and $m = e_1$

at the origin on the right side of the figure is a reminder that $V_{D,m}^{\mathrm{supp}}$ does not contain the origin. ◊

The sets $V_{D,m}$ and $V_{D,m}^{\mathrm{supp}}$ enable us to compute sheaf cohomology as follows.

Theorem 9.1.3. *Let $D = \sum_\rho a_\rho D_\rho$ be a Weil divisor on X_Σ. Fix $m \in M$ and $p \geq 0$.*

(a) $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \simeq \widetilde{H}^{p-1}(V_{D,m}, \mathbb{C})$.

(b) *If D is $\mathbb{Q}$-Cartier, then* $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \simeq \widetilde{H}^{p-1}(V_{D,m}^{\mathrm{supp}}, \mathbb{C})$.

Proof. For part (a), first note that if $\sigma \in \Sigma$, then

$$V_{D,m} \cap \sigma = \mathrm{Conv}(u_\rho \mid \rho \in \sigma(1),\ \langle m, u_\rho\rangle < -a_\rho). \tag{9.1.8}$$

One inclusion is trivial; the other follows easily using Lemma 1.2.7. Hence, for all $\sigma \in \Sigma$, we have

$$\begin{aligned} H^0(U_\sigma, \mathscr{O}_{X_\Sigma}(D))_m \neq \{0\} &\iff \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \sigma(1) \\ &\iff V_{D,m} \cap \sigma = \emptyset, \end{aligned} \tag{9.1.9}$$

where the first equivalence uses (9.1.2) and the second follows from (9.1.8).

The equation (9.1.8) shows that $V_{D,m} \cap \sigma$ is convex and hence connected when it is nonempty. It follows that $H^0(V_{D,m} \cap \sigma, \mathbb{C}) = \mathbb{C}$ when $V_{D,m} \cap \sigma \neq \emptyset$. Since $H^0(U_\sigma, \mathscr{O}_{X_\Sigma}(D))_m = \mathbb{C} \cdot \chi^m$ when it is nonzero, the equivalences (9.1.9) give a canonical exact sequence

$$0 \longrightarrow H^0(U_\sigma, \mathscr{O}_{X_\Sigma}(D))_m \longrightarrow \mathbb{C} \longrightarrow H^0(V_{D,m} \cap \sigma, \mathbb{C}) \longrightarrow 0 \tag{9.1.10}$$

for all $\sigma \in \Sigma$. It follows that for every $p \geq 0$, we get an exact sequence

$$0 \longrightarrow \bigoplus_{\gamma \in [\ell]_p} H^0(U_{\sigma_\gamma}, \mathscr{O}_{X_\Sigma}(D))_m \longrightarrow \bigoplus_{\gamma \in [\ell]_p} \mathbb{C} \longrightarrow \bigoplus_{\gamma \in [\ell]_p} H^0(V_{D,m} \cap \sigma_\gamma, \mathbb{C}) \longrightarrow 0,$$

which we write as

$$0 \longrightarrow \check{C}^p(\mathscr{U}, \mathscr{O}_{X_\Sigma}(D))_m \longrightarrow B^p \longrightarrow C^p \longrightarrow 0.$$

The formula for the differential in the Čech complex can be used to define differentials $B^p \to B^{p+1}$ and $C^p \to C^{p+1}$. Then we get an exact sequence of complexes

$$0 \longrightarrow \check{C}^\bullet(\mathscr{U}, \mathscr{O}_{X_\Sigma}(D))_m \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0.$$

Since $\check{H}^p(\mathscr{U}, \mathscr{O}_{X_\Sigma}(D)) \simeq H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$, the long exact sequence from (9.0.5) becomes

$$0 \to H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \to H^0(B^\bullet) \to H^0(C^\bullet) \to H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \to \cdots .$$

In Exercises 9.1.6 and 9.1.7 you will use the theory of Koszul complexes to show that the complex $B^\bullet$ has very simple cohomology:

$$H^p(B^\bullet) = \begin{cases} \mathbb{C} & p = 0 \\ 0 & p > 0. \end{cases} \tag{9.1.11}$$

Thus our long exact sequence breaks up into an exact sequence

$$0 \to H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \to \mathbb{C} \to H^0(C^\bullet) \to H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \to 0$$

and isomorphisms

$$H^{p-1}(C^\bullet) \simeq H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m, \quad p \geq 2.$$

We will show below that

$$H^p(C^\bullet) \simeq H^p(V_{D,m}, \mathbb{C}), \quad p \geq 0. \tag{9.1.12}$$

For $p \geq 2$, this gives the desired isomorphism

$$\widetilde{H}^{p-1}(V_{D,m}, \mathbb{C}) = H^{p-1}(V_{D,m}, \mathbb{C}) \simeq H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m.$$

When $p=1$, we obtain the exact sequence

$$0 \to H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \to \mathbb{C} \to H^0(V_{D,m}, \mathbb{C}) \to H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \to 0.$$

Since $H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \neq 0$ implies $V_{D,m} = \emptyset$ (Exercise 9.1.8), we obtain

$$\begin{aligned} \widetilde{H}^0(V_{D,m}, \mathbb{C}) &\simeq H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \\ \widetilde{H}^{-1}(V_{D,m}, \mathbb{C}) &\simeq H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m, \end{aligned} \tag{9.1.13}$$

where the last line uses the exact sequence (9.0.11).

It remains to prove (9.1.12). Since $|\Sigma|$ is the union of the maximal cones and $V_{D,m} \subseteq |\Sigma|$, we get the closed cover $\mathscr{C} = \{V_{D,m} \cap \sigma_i\}$ of $V_{D,m}$, where the σ_i are the maximal cones of Σ. Furthermore, $C^\bullet$ is the "Čech" complex $\check{C}^\bullet(\mathscr{C}, \mathbb{C})$ for the constant sheaf $\mathbb{C}$ on $V_{D,m}$, where we use quotation marks since $\mathscr{C}$ is a closed cover rather than an open cover.

Similar to the spectral sequence of an open covering discussed in §9.0, there is a spectral sequence for the closed covering $\mathscr{C} = \{V_{D,m} \cap \sigma_i\}$ of $V_{D,m}$. This E_1 spectral sequence is

$$E_1^{p,q} = \bigoplus_{\gamma \in [\ell]_p} H^q(V_{D,m} \cap \sigma_\gamma, \mathbb{C}) \Rightarrow H^{p+q}(V_{D,m}, \mathbb{C}).$$

As explained in Theorem C.2.3 in Appendix C or [**115**, II.5.2], the differential $d_1^{p,q}$ is determined by the Čech differential. In particular,

$$E_1^{p,0} = \check{C}^p(\mathscr{C}, \mathbb{C}) = C^p.$$

Also, we noted earlier in the proof that each $V_{D,m} \cap \sigma_\gamma$ is convex. It follows that $V_{D,m} \cap \sigma_\gamma$ is contractible, so that

$$E_1^{p,q} = \bigoplus_{\gamma \in [\ell]_p} H^q(V_{D,m} \cap \sigma_\gamma, \mathbb{C}) = 0, \quad q > 0.$$

As in the proof of Theorem 9.0.4, this implies that

$$E_2^{p,q} = \begin{cases} H^p(\check{C}^\bullet(\mathscr{C}, \mathbb{C})) = H^p(C^\bullet) & q = 0 \\ 0 & q > 0. \end{cases}$$

Similar to proof of Theorem 9.0.4, Proposition C.1.3 of Appendix C implies that the edge homomorphisms are isomorphisms, i.e.,

$$H^p(C^\bullet) \simeq H^p(V_{D,m}, \mathbb{C}).$$

The proves (9.1.12) and completes the proof of part (a).

For part (b), we assume that D is $\mathbb{Q}$-Cartier with support function φ_D. Then one can modify the proof of (9.1.9) to obtain the equivalence

$$H^0(U_\sigma, \mathscr{O}_{X_\Sigma}(D))_m \neq \{0\} \iff V_{D,m}^{\text{supp}} \cap \sigma = \emptyset$$

(Exercise 9.1.9). This allows us to replace (9.1.10) with the exact sequence

$$0 \longrightarrow H^0(U_\sigma, \mathscr{O}_{X_\Sigma}(D))_m \longrightarrow \mathbb{C} \longrightarrow H^0(V_{D,m}^{\mathrm{supp}} \cap \sigma, \mathbb{C}) \longrightarrow 0.$$

From here, the proof is identical to what we did in part (a) (Exercise 9.1.9). □

Example 9.1.4. Consider the divisor $D = -D_3 + 2D_5 - D_6$ on the surface X_Σ from Example 9.1.2. Let us compute $H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m$ for $m = e_1$. The sets $V_{D,m}$ and $V_{D,m}^{\mathrm{supp}}$ are shown in Figure 2 of Example 9.1.2. Since both sets have two connected components, Theorem 9.1.3 implies that

$$H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \simeq \widetilde{H}^0(V_{D,m}, \mathbb{C}) \simeq \widetilde{H}^0(V_{D,m}^{\mathrm{supp}}, \mathbb{C}) \simeq \mathbb{C}. \qquad \Diamond$$

Each representation $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m$ in Theorem 9.1.3 has its advantages. For example, the vanishing theorems in §9.2 use $V_{D,m}^{\mathrm{supp}}$ because of its relation to convexity, while for surfaces, we will see below that $V_{D,m}$ is easier to work with. Other treatments of $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m$ can be found in [**91**], [**141**] and [**226**].

Also note that Theorem 9.1.3 can be stated using relative cohomology instead of reduced cohomology. Consider, for example, the case when D is $\mathbb{Q}$-Cartier. Then the inclusion $V_{D,m}^{\mathrm{supp}} \subseteq |\Sigma|$ gives the long exact sequence

$$\cdots \longrightarrow H^p(|\Sigma|, V_{D,m}^{\mathrm{supp}}, \mathbb{C}) \longrightarrow H^p(|\Sigma|, \mathbb{C}) \longrightarrow H^p(V_{D,m}^{\mathrm{supp}}, \mathbb{C}) \longrightarrow \cdots.$$

Since $|\Sigma|$ is contractible (it is a cone), this sequence and Theorem 9.1.3 imply

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \simeq H^p(|\Sigma|, V_{D,m}^{\mathrm{supp}}, \mathbb{C}), \quad p \geq 0 \tag{9.1.14}$$

(Exercise 9.1.11). Since $V_{D,m}^{\mathrm{supp}}$ is open in $|\Sigma|$, algebraic geometers often write this relative cohomology group as $H^p_{Z_{D,m}}(|\Sigma|, \mathbb{C})$, where $Z_{D,m}$ is the closed set

$$Z_{D,m} = |\Sigma| \setminus V_{D,m}^{\mathrm{supp}} = \{u \in |\Sigma| \mid \langle m, u\rangle \geq \varphi_D(u)\}.$$

Then part (b) of Theorem 9.1.3 can be restated as

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \simeq H^p_{Z_{D,m}}(|\Sigma|, \mathbb{C}).$$

This is the version that appears in [**105**, p. 74].

The Surface Case. When X_Σ is a complete toric surface, the set $V_{D,m}$ appearing in Theorem 9.1.3 has an especially simple topology that can be encoded by a sign sequence. We illustrate this with an example.

Example 9.1.5. We return to the situation of Example 9.1.4. Write the divisor $D = -D_3 + 2D_5 - D_6$ as $D = \sum_i a_i D_i$, and label the ray generated by u_i with $+$ if $\langle m, u_i\rangle \geq -a_i$ and with $-$ if $\langle m, u_i\rangle < -a_i$. Then the picture of $V_{D,m}$ from Figure 2 of Example 9.1.2 gives the sign pattern shown in Figure 3 on the next page.

If we start at u_1 and go counterclockwise around the origin, we get the sign pattern $++--+-$, which we regard as cyclic. The positions of the $-$'s determine $V_{D,m}$, so that connected components of $V_{D,m}$ correspond to strings of consecutive $-$'s in the sign pattern. $\Diamond$

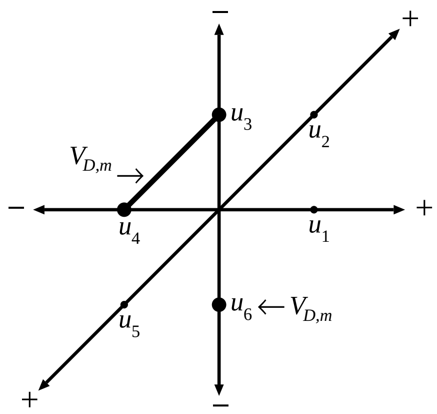


Figure 3. The sign pattern for $D = -D_3 + 2D_5 - D_6$ and $m = e_1$ in Example 9.1.5

These observations hold in general. Let X_Σ be a complete toric surface with minimal generators $u_1, \dots, u_r$ arranged counterclockwise around the origin. Given a torus-invariant Weil divisor $D = \sum_i a_i D_i$ on X_Σ and $m \in M \simeq \mathbb{Z}^2$, the *sign pattern* $\mathrm{sign}_D(m)$ is the string of length r whose ith entry is $+$ if $\langle m, u_i \rangle \geq -a_i$ and $-$ if $\langle m, u_i \rangle < -a_i$. We regard $\mathrm{sign}_D(m)$ as cyclic. Thus, for example, $--+++++-$ has only one string of consecutive $-$'s.

We have the following result from [**142**] and [**154**] (Exercise 9.1.12).

Proposition 9.1.6. *Given a Weil divisor $D = \sum_i a_i D_i$ on a complete toric surface X_Σ, the dimension of $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m$ is given by*

$$\dim H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m = \begin{cases} 1 & \text{if } \mathrm{sign}_D(m) = + \cdots + \ (\Leftrightarrow V_{D,m} = \emptyset) \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \dim H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m &= \max(0, \#\text{connected components of } V_{D,m} - 1) \\ &= \max(0, \#\text{strings of consecutive } -\text{'s in } \mathrm{sign}_D(m) - 1) \end{aligned}$$

$$\dim H^2(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m = \begin{cases} 1 & \text{if } \mathrm{sign}_D(m) = - \cdots - \ (\Leftrightarrow V_{D,m} \text{ is a cycle}) \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Example 9.1.7. Let us revisit Example 9.1.1. For each $m \in M = \mathbb{Z}^2$, the minimal generators u_0, u_1, u_2 give a sign pattern for the line bundle $\mathscr{O}_{\mathbb{P}^2}(a) = \mathscr{O}_{\mathbb{P}^2}(aD_0)$. When $a < 0$, all possible sign patterns are recorded in Figure 1 of Example 9.1.1. Using Proposition 9.1.6, we obtain:

- $H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a)) = 0$ since $+++$ does not appear.
- $H^1(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a)) = 0$ since all patterns have one string of consecutive $-$'s.
- $H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a)) = \bigoplus_{m \in \mathrm{Int}(a\Delta_2)} \mathbb{C} \cdot \chi^m$ since $---$ labels the interior of $a\Delta_2$.

Thus the computation of Example 9.1.1 follows immediately from Figure 1. $\Diamond$

Example 9.1.8. Consider the surface X_Σ and divisor $D = -D_3 + 2D_5 - D_6$ from Example 9.1.4. The sign patterns $\text{sign}_D(m)$ for all $m \in M \simeq \mathbb{Z}^2$ give the chamber decomposition shown in Figure 4 (Exercise 9.1.10).

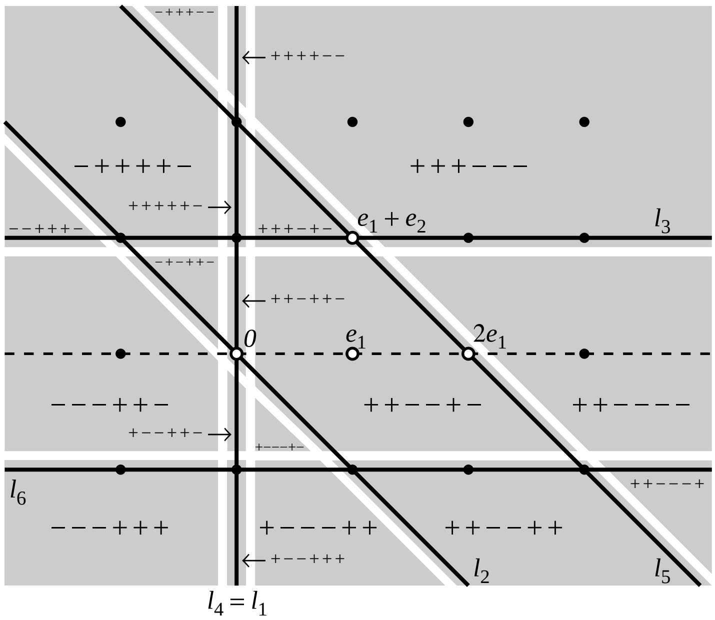


Figure 4. The chamber decompostion for $D = -D_3 + 2D_5 - D_6$

As in Figure 1, we have lines l_i defined by $\langle m, u_i\rangle = -a_i$, where $D = \sum_i a_i D_i$. We put the label l_i on the side where $\langle m, u_i\rangle \geq -a_i$, and the shading indicates where the boundary points lie. Each chamber is labeled with its sign pattern (some labels are rather small), and we get five 1-dimensional chambers since $l_1 = l_4$.

We now compute cohomology using Proposition 9.1.6. First,

$$H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = H^2(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = 0$$

since neither $++++++$ nor $------$ appear anywhere. Furthermore, of the six chambers with sign patterns having more than one string of consecutive $-$'s, three of them ($-+-++-$, $+----+-$, plus the 1-dimensional $+--++-$) have no lattice points. Of the remaining three, the lattice points are:

- $m = 0$ from $++-++-$ (1-dimensional).
- $m = e_1, 2e_1$ from $++--+-$.
- $m = e_1 + e_2$ from $+++-+-$.

These lattice points are indicated with white dots in Figure 4. Hence

$$H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \simeq \mathbb{C}\cdot\chi^0 \oplus \mathbb{C}\cdot\chi^{e_1} \oplus \mathbb{C}\cdot\chi^{2e_1} \oplus \mathbb{C}\cdot\chi^{e_1+e_2}.$$

In particular, $\dim H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = 4$. Exercise B.8.2 explains how to check this computation using `Macaulay2` [**123**]. $\Diamond$

The paper [**142**] applies these methods to classify line bundles with vanishing higher cohomology on the toric surface coming from three consecutive blowups of the Hirzebruch surface $\mathscr{H}_2$.

Exercises for §9.1.

9.1.1. In Example 9.1.1, verify the sign patterns in Figure 1.

9.1.2. Use the methods of Example 9.1.1 to prove (9.1.7).

9.1.3. The canonical bundle of $\mathbb{P}^2$ is $\omega_{\mathbb{P}^2} = \mathscr{O}_{\mathbb{P}^2}(-3)$. Use this and Serre duality to explain the symmetry in Table 2 in Example 9.1.1.

9.1.4. Recall that a Weil divisor $D = \sum_\rho a_\rho D_\rho$ on X_Σ is $\mathbb{Q}$-Cartier if some positive integer multiple of ℓD is Cartier. Let $\varphi_{\ell D}$ be the support function of ℓD as in Theorem 4.2.12.

(a) Show that $(1/\ell)\varphi_{\ell D} : |\Sigma| \to \mathbb{R}$ is a support function (Definition 4.2.11) and depends only on D. We define the *support function of D* to be $\varphi_D = (1/\ell)\varphi_{\ell D}$.

(b) Show that D is $\mathbb{Q}$-Cartier if and only if for every $\sigma \in \Sigma$, there is $m_\sigma \in M_\mathbb{Q}$ such that $\langle m_\sigma, u_\rho\rangle = -a_\rho$ for all $\rho \in \sigma(1)$. When this is satisfied, show that $\varphi_D(u) = \langle m_\sigma, u\rangle$ for all $u \in \sigma$.

9.1.5. Verify Figure 2 in Example 9.1.2

9.1.6. Given a ring R and elements $f_1,\ldots,f_\ell \in R$, define

$$d^p : \textstyle\bigwedge^p R^\ell \longrightarrow \bigwedge^{p+1} R^\ell$$

by $d^p(\alpha) = (\sum_{i=1}^\ell f_i e_i)\wedge\alpha$, where $e_1,\ldots,e_\ell$ are the standard basis of R^ℓ. Setting $K^p = \bigwedge^p R^\ell$, we get the complex

$$K^\bullet : \ K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{\ell-2}} K^{\ell-1} \xrightarrow{d^{\ell-1}} K^\ell,$$

and for an R-module M, we get the *Koszul complex*

$$K^\bullet(f_1,\ldots,f_\ell;\mathsf{M}) = K^\bullet \otimes_R \mathsf{M}.$$

Thus $K^\bullet = K^\bullet(f_1,\ldots,f_\ell;R)$.

(a) Given $\varphi : R^\ell \to R$, show that there are maps $s^p : K^p \to K^{p-1}$ such that $s^1 = \varphi$ and $s^{p+q}(\alpha\wedge\beta) = s^p(\alpha)\wedge\beta + (-1)^p\alpha\wedge s^q(\beta)$ for all $\alpha \in K_p$ and $\beta \in K_q$.

(b) Show that for all $\alpha \in K^p$, the maps s^p satisfy

$$d^{p-1}(s^p(\alpha)) + s^{p+1}(d^p(\alpha)) = \varphi(\textstyle\sum_{i=1}^\ell f_i e_i)\,\alpha.$$

(c) Now assume that $\langle f_1,\ldots,f_\ell\rangle = R$. Prove that $K^\bullet(f_1,\ldots,f_\ell;\mathsf{M})$ is exact for every R-module M. Hint: Define $\varphi(e_i) = g_i$ where $\sum_{i=1}^\ell g_i f_i = 1$.

9.1.7. When $f_1 = \cdots = f_\ell = 1$ and M is an R-module, the Koszul complex $K^\bullet(1,\ldots,1;\mathsf{M})$ of Exercise 9.1.6 becomes

$$\mathsf{M} \longrightarrow \bigoplus_{1\le i\le\ell}\mathsf{M} \longrightarrow \bigoplus_{1\le i<j\le\ell}\mathsf{M} \longrightarrow \bigoplus_{1\le i<j<k\le\ell}\mathsf{M} \longrightarrow \cdots, \tag{9.1.15}$$

where the entries of the matrices representing the differentials are 0 or ±1.

(a) Use the previous exercise to prove that (9.1.15) is exact.

(b) Prove that the complex $B^\bullet$ defined in the proof of Theorem 9.1.3 satisfies (9.1.11). Hint: Show $B^\bullet$ is the Koszul complex (9.1.15) with $\mathsf{M} = \mathbb{C}$, minus its first term.

9.1.8. Prove that $H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \neq 0$ if and only if $V_{m,D} = \emptyset$. Then prove (9.1.13).

9.1.9. Complete the proof of part (b) of Theorem 9.1.3.

9.1.10. Verify Figure 4 in Example 9.1.8.

9.1.11. Prove (9.1.14).

9.1.12. Prove Proposition 9.1.6.

9.1.13. Prove that $H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \neq 0$ implies that $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m = 0$ for all $p > 0$. Hint: Exercise 9.1.8.

§9.2. Vanishing Theorems I

In this section we prove two basic vanishing theorems for $\mathbb{Q}$-Cartier divisors on a toric variety X_Σ. We then apply these results to show that normal toric varieties are Cohen-Macaulay and hence satisfy Serre duality. We also give a version of Serre duality that is special to the toric case.

Nef $\mathbb{Q}$-Cartier Divisors. A Weil divisor D on a normal variety X is $\mathbb{Q}$-Cartier if some positive integer multiple is Cartier. Also recall that the intersection product $D \cdot C$ is defined whenever D is $\mathbb{Q}$-Cartier and C is a complete irreducible curve in X. Then D is *nef* if $D \cdot C \geq 0$ for all complete irreducible curves $C \subseteq X$. This generalizes the definition of nef Cartier divisor given in §6.3.

In the toric case, we characterize $\mathbb{Q}$-Cartier nef divisors as follows.

Lemma 9.2.1. *Let $D = \sum_\rho a_\rho D_\rho$ be a $\mathbb{Q}$-Cartier divisor on X_Σ. If Σ has convex support, then the following are equivalent:*

(a) *D is nef.*

(b) *For some integer $\ell > 0$, ℓD is Cartier and basepoint free, i.e., $\mathscr{O}_{X_\Sigma}(\ell D)$ is a line bundle generated by its global sections.*

(c) *The support function $\varphi_D : |\Sigma| \to \mathbb{R}$ is convex.*

Proof. The proof combines ideas from Theorems 6.1.7, 6.3.12 and 7.2.2. We leave the details to the reader (Exercise 9.2.1). □

In the Cartier case, we know that D is nef if and only if $\mathscr{O}_{X_\Sigma}(D)$ is generated by global sections (Theorems 6.1.7 and 6.3.12). This fails in the $\mathbb{Q}$-Cartier case, as shown by the following example.

Example 9.2.2. Let $X_\Sigma = \mathbb{P}(2,3,5)$ and let D_0, D_1, D_2 be the divisors given by the minimal generators u_0, u_1, u_2. Note that $2u_0 + 3u_1 + 5u_2 = 0$. Then $\mathrm{Cl}(X_\Sigma) \simeq \mathbb{Z}$, where the classes of D_0, D_1, D_2 map to $2, 3, 5$ respectively. Let $D = D_1 - D_0$ and note that $D_0 \sim 2D, D_1 \sim 3D, D_2 \sim 5D$. It also easy to see that D is $\mathbb{Q}$-Cartier and nef. However, you will check in Exercise 9.2.2 that $H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = 0$. Thus $\mathscr{O}_{X_\Sigma}(D)$ is not generated by its global sections. $\Diamond$

Demazure Vanishing. The most basic vanishing theorem for toric varieties is the following result, first proved by Demazure [**82**] for Cartier divisors.

Theorem 9.2.3 (Demazure Vanishing)**.** *Let D be a $\mathbb{Q}$-Cartier divisor on X_Σ. If $|\Sigma|$ is convex and D is nef, then*

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = 0 \textit{ for all } p > 0.$$

Proof. The support function $\varphi_D : |\Sigma| \to \mathbb{R}$ is convex by Lemma 9.2.1. Fix $m \in M$ and let $V_{D,m}^{\mathrm{supp}} = \{u \in |\Sigma| \mid \langle m,u\rangle < \varphi_D(u)\}$ as in Theorem 9.1.3. Let $u,v \in V_{D,m}^{\mathrm{supp}}$, so that $\langle m,u\rangle < \varphi_D(u)$ and $\langle m,v\rangle < \varphi_D(u)$. Then, for $0 \le t \le 1$, we have

$$\begin{aligned}\langle m,(1-t)u+tv\rangle &= (1-t)\langle m,u\rangle + t\langle m,v\rangle\\ &< (1-t)\varphi_D(u) + t\varphi_D(v)\\ &\le \varphi_D((1-t)u+tv).\end{aligned}$$

The last line follows since φ_D is convex. This implies that $V_{D,m}^{\mathrm{supp}}$ is convex and hence contractible. Combining this with Theorem 9.1.3, we obtain

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \simeq \widetilde{H}^{p-1}(V_{D,m}^{\mathrm{supp}}, \mathbb{C}) = 0 \text{ for all } p > 0. \qquad \square$$

In particular, if D is a basepoint free Cartier divisor on X_Σ, then D is nef, so that Theorem 9.2.3 implies

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = 0 \text{ for all } p > 0.$$

Example 9.2.4. For $\mathbb{P}^n$, aD_0 is basepoint free for $a \ge 0$. Hence the higher cohomology of $\mathscr{O}_{\mathbb{P}^n}(a) = \mathscr{O}_{\mathbb{P}^n}(aD_0)$ vanishes by Theorem 9.2.3, which agrees with what we found in Example 9.1.1 when $n = 2$. $\Diamond$

Vanishing of Higher Direct Images. Here is an easy application of Demazure vanishing.

Theorem 9.2.5. *Let $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ be a proper toric morphism and let $\mathscr{L}$ be a line bundle on X_{Σ_2} Then:*

(a) $R^p\phi_*\phi^*\mathscr{L} = 0$ *for all* $p > 0$.

(b) *If the map $\overline{\phi} : N_1 \to N_2$ on the lattices of one-parameter subgroups is surjective, then $\phi_*\phi^*\mathscr{L} \simeq \mathscr{L}$.*

Proof. First assume $\mathscr{L} = \mathscr{O}_{X_{\Sigma_2}}$, so that $\phi^*\mathscr{L} = \mathscr{O}_{X_{\Sigma_1}}$. Let $U_\sigma \subseteq X_{\Sigma_2}$ be the affine open corresponding to $\sigma \in \Sigma_2$. By Proposition 9.0.7, $R^p\phi_*\mathscr{O}_{X_{\Sigma_1}}|_{U_\sigma}$ is the sheaf associated to the module $H^p(\phi^{-1}(U_\sigma), \mathscr{O}_{X_{\Sigma_1}})$. Since ϕ is proper, Theorem 3.4.11 tells us that $\phi^{-1}(U_\sigma)$ is a toric variety whose fan is supported on the convex set

$\overline{\phi}_{\mathbb{R}}^{-1}(\sigma)$. Thus $H^p(\phi^{-1}(U_\sigma), \mathscr{O}_{X_{\Sigma_1}}) = 0$ for $p > 0$ by Theorem 9.2.3. It follows that $R^p\phi_*\mathscr{O}_{X_{\Sigma_1}} = 0$ for $p > 0$ by Proposition 9.0.7.

The morphism ϕ induces a homomorphism of sheaves $\mathscr{O}_{X_{\Sigma_2}} \to \phi_*\mathscr{O}_{X_{\Sigma_1}}$ on X_{Σ_2} (Example 4.0.25). It suffices to show that this map is an isomorphism on each affine open $U_\sigma \subseteq X_{\Sigma_2}$, $\sigma \in \Sigma_2$. Note also that the restriction $\phi^{-1}(U_\sigma) \to U_\sigma$ is again a proper toric morphism. Hence we may assume that $X_{\Sigma_2} = U_\sigma$. Then $|\Sigma_1| = \overline{\phi}_{\mathbb{R}}^{-1}(\sigma)$ is a convex cone, so that

$$H^0(X_{\Sigma_1}, \mathscr{O}_{X_{\Sigma_1}}) = \bigoplus_{m \in |\Sigma_1|^\vee \cap M_1} \mathbb{C} \cdot \chi^m$$

by Exercise 4.3.4. We analyze $|\Sigma_1|^\vee \cap M_1$ as follows.

The surjection $N_1 \to N_2$ gives an injection $M_2 \subseteq M_1$ on character lattices. Then $\sigma \subseteq (N_2)_{\mathbb{R}}$ has dual $\sigma^\vee \subseteq (M_2)_{\mathbb{R}} \subseteq (M_1)_{\mathbb{R}}$ and $|\Sigma_1| \subseteq (N_1)_{\mathbb{R}}$ has dual $|\Sigma_1|^\vee \subseteq (M_1)_{\mathbb{R}}$. Also, $|\Sigma_1| = \overline{\phi}_{\mathbb{R}}^{-1}(\sigma)$ implies $\sigma^\vee = |\Sigma_1|^\vee$. It follows that

$$|\Sigma_1|^\vee \cap M_1 = \sigma^\vee \cap M_1 = \sigma^\vee \cap (M_2)_{\mathbb{R}} \cap M_1 = \sigma^\vee \cap M_2,$$

where the last equality follows since M_1 is saturated in M_2 by the surjectivity of $N_1 \to N_2$. Hence

$$H^0(X_{\Sigma_1}, \mathscr{O}_{X_{\Sigma_1}}) = \bigoplus_{m \in \sigma^\vee \cap M_2} \mathbb{C} \cdot \chi^m = H^0(U_\sigma, \mathscr{O}_{U_\sigma}).$$

This shows that $\mathscr{O}_{U_\sigma} \to \phi_*\mathscr{O}_{X_{\Sigma_1}}$ induces an isomorphism on global sections, which gives the desired sheaf isomorphism since we are working with quasicoherent sheaves over an affine variety.

For a line bundle $\mathscr{L}$ on X_{Σ_2}, take $U_\sigma \subseteq X_{\Sigma_2}$. Then $\mathscr{L}|_{U_\sigma} \simeq \mathscr{O}_{X_{\Sigma_2}}|_{U_\sigma}$, so that

$$R^p\phi_*\phi^*\mathscr{L}|_{U_\sigma} \simeq R^p\phi_*\phi^*\mathscr{O}_{X_{\Sigma_2}}|_{U_\sigma} \simeq R^p\phi_*\mathscr{O}_{X_{\Sigma_1}}|_{U_\sigma} = 0$$

for $p > 0$. Furthermore, when $\overline{\phi} : N_1 \to N_2$ is surjective, we have

$$\phi_*\phi^*\mathscr{L} \simeq \phi_*\mathscr{O}_{X_{\Sigma_1}} \otimes_{\mathscr{O}_{X_{\Sigma_2}}} \mathscr{L} \simeq \mathscr{L},$$

where the first isomorphism is part (a) of Exercise 9.2.3 and the second follows from $\phi_*\mathscr{O}_{X_{\Sigma_1}} \simeq \mathscr{O}_{X_{\Sigma_2}}$. □

The Injectivity Lemma. We introduce a method that will be used several times in this section and the next. The general framework uses a morphism $f : Y \to X$ and coherent sheaves $\mathscr{G}$ on Y and $\mathscr{F}$ on X such that:

- f is an affine morphism, meaning $f^{-1}(U) \subseteq X$ is affine when $U \subseteq Y$ is affine.
- There is a split injection $i : \mathscr{F} \to f_*\mathscr{G}$ of $\mathscr{O}_X$-modules. This means that there is a homomorphism $r : f_*\mathscr{G} \to \mathscr{F}$ such that $r \circ i = 1_{\mathscr{F}}$.

By functoriality, a split injection $i : \mathscr{F} \to f_*\mathscr{G}$ induces a split injection

$$H^p(X, \mathscr{F}) \longrightarrow H^p(X, f_*\mathscr{G}).$$

However, since f is affine, we also have

$$H^p(X, f_*\mathscr{G}) \simeq H^p(Y,\mathscr{G})$$

by Exercise 9.0.5. Hence we get an injection

$$H^p(X,\mathscr{F}) \hookrightarrow H^p(Y,\mathscr{G}).$$

In particular, $H^p(Y,\mathscr{G}) = 0$ implies $H^p(X,\mathscr{F}) = 0$. This method is used in the proofs of many vanishing theorems—see [**186**, Ch. 4].

In the toric context, Fujino [**101**] identified the best choice for the morphism f. Given a fan Σ in $N_{\mathbb{R}}$ and a positive integer ℓ, let $\overline{\phi}_\ell : N \to N$ be multiplication by ℓ. This maps Σ to itself and hence induces a toric morphism $\phi_\ell : X_\Sigma \to X_\Sigma$. The dual of $\overline{\phi}_\ell$ is multiplication by ℓ on M, and the restriction of ϕ_ℓ to $T_N = M \otimes_{\mathbb{Z}} \mathbb{C}^*$ is the group homomorphism given by raising to the ℓth power. Furthermore, given any $\sigma \in \Sigma$, we have $\phi_\ell^{-1}(U_\sigma) = U_\sigma$ since $(\overline{\phi}_\ell)_{\mathbb{R}}^{-1}(\sigma) = \sigma$. This shows that ϕ_ℓ is an affine morphism.

Here is our first injectivity lemma.

Lemma 9.2.6. *Let D be a Weil divisor on X_Σ and let ℓ be a positive integer. Then there is an injection*

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \hookrightarrow H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D)) \ \textit{for all } p \geq 0.$$

Proof. We will construct a split injection $\mathscr{O}_{X_\Sigma}(D) \hookrightarrow \phi_{\ell *}\mathscr{O}_{X_\Sigma}(\ell D)$. The lemma then follows immediately from the above discussion.

For $D = \sum_\rho a_\rho D_\rho$ and $\sigma \in \Sigma$, let $P = \{m \in M_{\mathbb{R}} \mid \langle m,u\rangle \geq -a_\rho \text{ for all } \rho \in \sigma(1)\}$. Over U_σ, the sheaves $\mathscr{O}_{X_\Sigma}(D)$ and $\mathscr{O}_{X_\Sigma}(\ell D)$ come from the $\mathbb{C}[\sigma^\vee \cap M]$-modules

$$\bigoplus_{m \in P\cap M} \mathbb{C}\cdot\chi^m \quad \text{and} \quad \bigoplus_{m \in (\ell P)\cap M} \mathbb{C}\cdot\chi^m.$$

Since $\phi_\ell^{-1}(U_\sigma) = U_\sigma$, $\phi_{\ell *}\mathscr{O}_{X_\Sigma}(\ell D)$ is determined by $\bigoplus_{m \in (\ell P)\cap M} \mathbb{C}\cdot\chi^m$ with the module structure given by $\chi^m \cdot a = \chi^{\ell m} a$. This implies that the map

$$\bigoplus_{m \in P\cap M} \mathbb{C}\cdot\chi^m \longrightarrow \bigoplus_{m \in (\ell P)\cap M} \mathbb{C}\cdot\chi^m \tag{9.2.1}$$

that sends χ^m to $\chi^{\ell m}$ is a homomorphism of $\mathbb{C}[\sigma^\vee \cap M]$-modules (Exercise 9.2.4). Furthermore, one can show that the map given by

$$r(\chi^m) = \begin{cases} 0 & m \notin \ell M \\ \chi^{m'} & m = \ell m',\ m' \in M \end{cases} \tag{9.2.2}$$

defines a $\mathbb{C}[\sigma^\vee \cap M]$-module splitting of (9.2.1) (Exercise 9.2.4).

The map (9.2.1) and the splitting r are easily seen to be compatible with the inclusion $U_\tau \subseteq U_\sigma$ when τ is a face of σ. It follows that the maps (9.2.1) patch to give the split injection $\mathscr{O}_{X_\Sigma}(D) \hookrightarrow \phi_{\ell *}\mathscr{O}_{X_\Sigma}(\ell D)$. $\square$

Batyrev-Borisov Vanishing. In Example 9.1.1, we saw that $\mathscr{O}_{\mathbb{P}^2}(a)$ has nontrivial H^2 when $a \leq -3$. If we write $a = -b$ for $b > 0$, we can rewrite (9.1.6) as

$$\dim H^p(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-b)) = \begin{cases} 0 & p \neq 2 \\ \big|\mathrm{Int}(b\Delta_2) \cap M\big| & p = 2. \end{cases}$$

This has been generalized by Batyrev and Borisov [**18**]. Recall that a Weil divisor $D = \sum_\rho a_\rho D_\rho$ gives the polyhedron

$$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\},$$

which is a polytope when Σ is complete (Proposition 4.3.8).

Theorem 9.2.7 (Batyrev-Borisov Vanishing)**.** *Let $D = \sum_\rho a_\rho D_\rho$ be a $\mathbb{Q}$-Cartier divisor on a complete toric variety X_Σ. If D is nef, then*

(a) $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D)) = 0$ *for all* $p \neq \dim P_D$.

(b) *When* $p = \dim P_D$, $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D)) \simeq \bigoplus_{m \in \mathrm{Relint}(P_D) \cap M} \mathbb{C} \cdot \chi^{-m}$.

Proof. First assume that D is Cartier and $\dim P_D = \dim N_{\mathbb{R}} = \dim X_\Sigma$. Then D is basepoint free since it is nef (Theorem 6.3.12). By Proposition 6.2.5 and Theorem 6.2.8, combining cones of Σ that share the same linear functional relative to φ_D gives a fan Σ_D such that Σ refines Σ_D and φ_D is strictly convex relative to Σ_D.

For $m \in M$, $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D))_m \simeq \widetilde{H}^{p-1}(V^{\mathrm{supp}}_{-D,m}, \mathbb{C})$ by Theorem 9.1.3. Set

$$Z_{-D,m} = N_{\mathbb{R}} \setminus V^{\mathrm{supp}}_{-D,m} = \{u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq \varphi_{-D}(u)\}.$$

Since φ_D is strictly convex relative to Σ_D, the proof of Theorem 9.2.3 implies that $Z_{-D,m}$ is convex. In fact, it is strongly convex. To see why, note that for any nonzero $u \in N_{\mathbb{R}}$, the strict convexity of φ_D implies $0 = \varphi_D(u + (-u)) > \varphi_D(u) + \varphi_D(-u)$ since u and $-u$ do not lie in the same cone of Σ_D. Thus

$$\varphi_{-D}(-u) = -\varphi_D(-u) > \varphi_D(u) \quad \text{for all } u \neq 0 \text{ in } N_{\mathbb{R}}.$$

Now assume $u \in Z_{-D,m} \setminus \{0\}$. Then $\langle m, u \rangle \geq \varphi_{-D}(u)$, so that

$$\varphi_D(u) \geq \langle m, -u \rangle.$$

Combining the displayed inequalities gives $\varphi_{-D}(-u) > \langle m, -u \rangle$, which implies that $-u \in V^{\mathrm{supp}}_{-D,m} = N_{\mathbb{R}} \setminus Z_{-D,m}$. Hence $Z_{-D,m}$ is strongly convex.

We next prove that

$$Z_{-D,m} = \{0\} \iff m \in -\mathrm{Int}(P_D) \cap M. \tag{9.2.3}$$

If $m \in -\mathrm{Int}(P_D) \cap M$, then $-m \in \mathrm{Int}(P_D)$, so that

$$\langle -m, u_\rho \rangle > -a_\rho = \varphi_D(u_\rho) \quad \text{for all } \rho \in \Sigma(1).$$

Thus

$$\langle m, u_\rho \rangle < \varphi_{-D}(u_\rho) \quad \text{for all } \rho \in \Sigma(1), \tag{9.2.4}$$

which easily implies that $Z_{-D,m} = \{0\}$ (Exercise 9.2.5). On the other hand, if we have $m \notin -\mathrm{Int}(P_D) \cap M$, then one of the inequalities of (9.2.4) must fail, i.e.,

$$\langle m, u_{\rho_0} \rangle \geq \varphi_{-D}(u_{\rho_0}) \quad \text{for some } \rho_0 \in \Sigma(1).$$

Then $u_{\rho_0} \in Z_{-D,m}$, so that $Z_{-D,m} \neq \{0\}$. This proves (9.2.3).

We are now ready to compute $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D))_m$. If $m \in -\mathrm{Int}(P_D) \cap M$, then $V^{\mathrm{supp}}_{-D,m} = N_\mathbb{R} \setminus \{0\}$ is homotopic to S^{n-1}. Hence

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D))_m \simeq \widetilde{H}^{p-1}(V^{\mathrm{supp}}_{-D,m}, \mathbb{C}) \simeq \widetilde{H}^{p-1}(S^{n-1}, \mathbb{C}) = \begin{cases} 0 & p \neq n \\ \mathbb{C} & p = n. \end{cases}$$

If $m \notin -\mathrm{Int}(P_D) \cap M$, then $V^{\mathrm{supp}}_{-D,m} = \mathbb{R}^n \setminus Z_{-D,m}$, where $Z_{-D,m}$ is a closed strongly convex cone of positive dimension. If $u \neq 0$ in $Z_{-D,m}$, we showed above that $-u \in V^{\mathrm{supp}}_{-D,m}$. It is easy to see that $V^{\mathrm{supp}}_{D,m}$ contracts to $-u$ (Exercise 9.2.6). Hence

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D))_m \simeq \widetilde{H}^{p-1}(V^{\mathrm{supp}}_{-D,m}, \mathbb{C}) = 0 \quad \text{for all } p \geq 0.$$

This completes the proof when D is Cartier and P_D has dimension n.

Now suppose D is Cartier but $\dim P_D < n$. We will use Proposition 6.2.5 and Theorem 6.2.8. After translation, P_D spans $(M_D)_\mathbb{R} \subseteq M_\mathbb{R}$, where $M_D \subseteq M$ is dual to a surjection $\overline{\phi} : N \to N_D$, and the normal fan of P_D lies in $(N_D)_\mathbb{R}$. If X_D is the toric variety of $P_D \subseteq (M_D)_\mathbb{R}$, then Theorem 6.2.8 shows that $\overline{\phi}$ induces a toric morphism $\phi : X_\Sigma \to X_D$ such that D is the pullback of the ample divisor D' on X_D determined by P_D. In particular, $\mathscr{O}_{X_\Sigma}(-D) \simeq \phi^*\mathscr{O}_{X_D}(-D')$.

Since ϕ is proper and $\overline{\phi} : N \to N_D$ is surjective, we have

$$\begin{aligned} R^p\phi_*\mathscr{O}_{X_\Sigma}(-D) &= 0,\ p > 0 \\ \phi_*\mathscr{O}_{X_\Sigma}(-D) &= \mathscr{O}_{X_D}(-D') \end{aligned}$$

by Theorem 9.2.5. Then Proposition 9.0.8 implies that

$$H^p(X_D, \mathscr{O}_{X_D}(-D')) \simeq H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D)). \tag{9.2.5}$$

This proves the theorem, as we now explain. Since M_D is saturated in M by the surjectivity of $N \to N_D$, we have

$$P_D \cap M = P_D \cap (M_D)_\mathbb{R} \cap M = P_{D'} \cap M_D,$$

Since $P_D = P_{D'}$ is full dimensional in $(M_D)_\mathbb{R}$, we are done by (9.2.5) and the full dimensional case considered above.

Finally, we need to consider what happens when D is $\mathbb{Q}$-Cartier. Pick an integer $\ell > 0$ such that ℓD is Cartier. By Lemma 9.2.6, we have an injection

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D)) \hookrightarrow H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-\ell D)).$$

Since ℓD is Cartier and nef with $P_{\ell D} = \ell P_D$, the Cartier case proved above implies that $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D)) = 0$ for $p \neq \dim P_D$. Furthermore, when $p = \dim P_D$,

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-\ell D)) = \bigoplus_{m \in M} \widetilde{H}^{p-1}(V^{\mathrm{supp}}_{-\ell D,m}, \mathbb{C}) \cdot \chi^m = \bigoplus_{m \in \mathrm{Relint}(\ell P_D) \cap M} \mathbb{C} \cdot \chi^{-m},$$

from which we conclude

$$\widetilde{H}^{p-1}(V^{\mathrm{supp}}_{-\ell D,m}, \mathbb{C}) = \begin{cases} \mathbb{C} & m \in -\mathrm{Relint}(\ell P_D) \cap M \\ 0 & \text{otherwise}. \end{cases}$$

Since $V^{\mathrm{supp}}_{-D,m} = V^{\mathrm{supp}}_{-\ell D,\ell m}$ and $\ell m \in -\mathrm{Relint}(\ell P_D) \cap M \Leftrightarrow m \in -\mathrm{Relint}(P_D) \cap M$,

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D)) = \bigoplus_{m \in M} \widetilde{H}^{p-1}(V^{\mathrm{supp}}_{-D,m}, \mathbb{C}) \cdot \chi^m = \bigoplus_{m \in \mathrm{Relint}(P_D) \cap M} \mathbb{C} \cdot \chi^{-m}. \qquad \square$$

Example 9.2.8. We saw in Example 9.2.4 that the higher cohomology of $\mathscr{O}_{\mathbb{P}^n}(a)$ vanishes when $a \geq 0$. When $a < 0$, $\mathscr{O}_{\mathbb{P}^n}(a) = \mathscr{O}_{\mathbb{P}^n}(-|a|D_0)$. Since $|a|D_0$ is ample with polytope $|a|\Delta_n$, Theorem 9.2.7 implies

$$\dim H^p(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(a)) = \begin{cases} 0 & p \neq n \\ \big|\mathrm{Int}(a\Delta_n) \cap M\big| & p = n \end{cases}$$

when $a < 0$. This generalizes (9.1.6) from Example 9.1.1. $\Diamond$

The Cohen-Macaulay Property. Here is a surprising consequence of Batyrev-Borisov vanishing.

Theorem 9.2.9. *A normal toric variety is Cohen-Macaulay.*

Proof. First suppose that X_Σ is projective and let D be a torus-invariant ample divisor on X_Σ. Corollary 5.72 of [**179**, p. 182] implies that X_Σ is Cohen-Macaulay if and only if

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-\ell D)) = 0 \text{ for all } \ell \gg 0,\ p < n. \tag{9.2.6}$$

Now take any integer $\ell > 0$. Since D is ample, the divisor ℓD is nef and its polytope $P_{\ell D}$ has dimension $n = \dim X_\Sigma$. Then (9.2.6) follows from Theorem 9.2.7. We conclude that X_Σ is Cohen-Macaulay.

Next consider an affine toric variety U_σ. We can find a projective toric variety that contains U_σ as an affine open subset (Exercise 9.2.7). Since being Cohen-Macaulay is a local property, it follows that U_σ is Cohen-Macaulay, and then any normal toric variety X_Σ is Cohen-Macaulay. $\square$

This result was originally proved by Hochster [**146**]. Another proof can be found in [**76**, Thm. 3.4]. A projective variety is called *arithmetically Cohen-Macaulay* (*aCM* for short) if its affine cone is Cohen-Macaulay. In Exercise 9.2.8 you will show that normal lattice polytopes give aCM toric varieties.

Serre Duality. Theorem 9.2.9 shows that Serre duality holds for any normal toric variety. However, the version stated in Theorem 9.0.9 only applies to locally free sheaves, while the more general version Theorem 9.0.10 uses Ext groups. Many of the sheaves we deal with, such as $\mathscr{O}_{X_\Sigma}(D)$ and $\widehat{\Omega}^p_{X_\Sigma}$, fail to be locally free when X_Σ is not smooth. Fortunately, there is an "Ext-free" version of Serre duality that holds for these sheaves.

Theorem 9.2.10 (Toric Serre Duality)**.** *Assume that X_Σ is a complete toric variety of dimension n.*

(a) *If D is a $\mathbb{Q}$-Cartier divisor and K_{X_Σ} is the canonical divisor, then we have natural isomorphisms*

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))^\vee \simeq H^{n-p}(X_\Sigma, \mathscr{O}_{X_\Sigma}(K_{X_\Sigma} - D)).$$

(b) *If X_Σ is simplicial and $\mathscr{F}$ is locally free, then there are natural isomorphisms*

$$H^p(X_\Sigma, \widehat{\Omega}^q_{X_\Sigma} \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{F})^\vee \simeq H^{n-p}(X_\Sigma, \widehat{\Omega}^{n-q}_{X_\Sigma} \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{F}^\vee).$$

Remark 9.2.11. Here are two interesting aspects of Theorem 9.2.10:

(a) $\mathscr{O}_{X_\Sigma}(K_{X_\Sigma} - D)$ need not be isomorphic to

$$\omega_{X_\Sigma} \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{O}_{X_\Sigma}(D)^\vee = \mathscr{O}_{X_\Sigma}(K_{X_\Sigma}) \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{O}_{X_\Sigma}(-D)$$

when D is $\mathbb{Q}$-Cartier. So Theorem 9.2.10 does not follow from Theorem 9.0.9.

(b) Every Weil divisor is $\mathbb{Q}$-Cartier on a simplicial toric variety. Theorem 9.2.10 holds for all Weil divisors in this case.

Before beginning the proof of Theorem 9.2.10, we introduce some tools from commutative algebra. This is typical of algebraic geometry—once a variety ceases to be smooth, the theory becomes more technically demanding.

In §9.0 we defined the depth of a local ring $(R, \mathfrak{m})$. More generally, the *depth* of a finitely generated R-module F is the maximal length of a sequence $f_1, \ldots, f_s \in \mathfrak{m}$ such that f_i is not a zero divisor in $F/\langle f_1, \ldots, f_{i-1}\rangle F$ for all i, and the *dimension* of F is $\dim R/\mathrm{Ann}(F)$, where $\mathrm{Ann}(F) = \{f \in R \mid f \cdot F = 0\}$.

Then we define a coherent sheaf $\mathscr{F}$ on a variety X to be *maximal Cohen-Macaulay* (*MCM* for short) if for every point $x \in X$, the stalk $\mathscr{F}_x$ is an $\mathscr{O}_{X,x}$-module whose depth and dimension both equal $\dim X$. For example, if X is Cohen-Macaulay, then every locally free sheaf on X is MCM.

The following result from [**226**, Prop. 4.24] shows that MCM sheaves satisfy a nice version of Serre duality. See also [**179**, Thm. 5.71].

Theorem 9.2.12 (Serre Duality III)**.** *Let $\mathscr{F}$ be a MCM sheaf on a complete normal Cohen-Macaulay variety X of dimension n. There there are natural isomorphisms*

$$H^p(X, \mathscr{F})^\vee \simeq H^{n-p}(X, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \omega_X)).$$

Proof. We will use the sheaf version of Ext, where the sheaves $\mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{F},\omega_X)$ are the derived functors of $\mathscr{F} \mapsto \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\omega_X)$. The stalk at $x \in X$ is

$$\mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{F},\omega_X)_x = \mathrm{Ext}^q_{\mathscr{O}_{X,x}}(\mathscr{F}_x,(\omega_X)_x). \tag{9.2.7}$$

The relation between the Ext groups $\mathrm{Ext}^q_{\mathscr{O}_X}(\mathscr{F},\omega_X)$ and Ext sheaves $\mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{F},\omega_X)$ is described by the spectral sequence

$$E_2^{p,q} = H^p(X,\mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{F},\omega_X)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(\mathscr{F},\omega_X)$$

from Theorem C.2.4 of Appendix C.

Since $\mathscr{F}$ is MCM, [**56**, Cor. 3.5.11] implies that $\mathrm{Ext}^q_{\mathscr{O}_{X,x}}(\mathscr{F}_x,(\omega_X)_x) = 0$ for $q > 0$ and $x \in X$. By (9.2.7), we obtain $\mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{F},\omega_X) = 0$ for $q > 0$. Hence

$$H^p(X,\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\omega_X)) \simeq \mathrm{Ext}^p_{\mathscr{O}_X}(\mathscr{F},\omega_X)$$

by Proposition C.1.7 of Appendix C. Since $H^p(X,\mathscr{F})^\vee \simeq \mathrm{Ext}^{n-p}_{\mathscr{O}_X}(\mathscr{F},\omega_X)$ by the version of Serre duality given in Theorem 9.0.10, we are done. □

We can now prove the toric version of Serre duality.

Proof of Theorem 9.2.10. We first show that $\mathscr{O}_{X_\Sigma}(D)$ is MCM when D is $\mathbb{Q}$-Cartier. Pick $\ell > 0$ such that ℓD is Cartier. We will use splitting methods, though for clarity we replace $\overline{\phi}_\ell : N \to N$ with the sublattice $\ell N \subseteq N$. The dual lattice $\ell^{-1}M \supseteq M$ gives semigroup algebras $\mathsf{S}_{\sigma,N} = \mathbb{C}[\sigma^\vee \cap M] \subseteq \mathsf{S}_{\sigma,\ell N} = \mathbb{C}[\sigma^\vee \cap \ell^{-1}M]$.

Over an affine open subset U_σ, we can pick $m_\sigma \in \ell^{-1}M$ such that $\varphi_D(u) = \langle m_\sigma, u\rangle$ for $u \in \sigma$. Now consider the $S_{\sigma,N}$-module $A = \Gamma(U_\sigma,\mathscr{O}_{X_\Sigma}(D))$. If we set $P = m_\sigma + \sigma^\vee$, then one easily sees that

$$A = \bigoplus_{m \in P\cap M} \mathbb{C}\cdot\chi^m \subseteq B = \bigoplus_{m\in P\cap \ell^{-1}M} \mathbb{C}\cdot\chi^m = \chi^{m_\sigma}\mathsf{S}_{\sigma,\ell N}, \tag{9.2.8}$$

where the last equality uses $m_\sigma \in \ell^{-1}M$. Note that $B = \chi^{m_\sigma}\mathsf{S}_{\sigma,\ell N}$ is free over $\mathsf{S}_{\sigma,\ell N}$ and hence is MCM over $\mathsf{S}_{\sigma,\ell N}$. However, $\mathsf{S}_{\sigma,\ell N}$ is finitely generated as a $\mathsf{S}_{\sigma,N}$-module (Exercise 9.2.9), so that B is MCM over $S_{\sigma,N}$ by [**56**, Ex. 1.2.26].

Another property of MCM modules is that any nontrivial direct summand of an MCM module is again MCM. This follows from [**56**, Thm. 3.5.7]. Hence it suffices to split (9.2.8). In this situation, we use the map $r : B \to A$ that sends $\chi^m \in B$ to

$$r(\chi^m) = \begin{cases} \chi^m & \text{if } \chi^m \in A \\ 0 & \text{otherwise}. \end{cases}$$

Similar to what we did in Lemma 9.2.6, r is a homomorphism of $S_{\sigma,N}$-modules. From here, it follows easily that $\mathscr{O}_{X_\Sigma}(D)$ is MCM when D is $\mathbb{Q}$-Cartier.

Now we prove duality. Since $\mathscr{O}_{X_\Sigma}(D)$ is MCM, Theorem 9.2.12 implies that

$$\begin{aligned}H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))^\vee &\simeq H^{n-p}(X_\Sigma, \mathscr{H}om_{\mathscr{O}_{X_\Sigma}}(\mathscr{O}_{X_\Sigma}(D), \omega_{X_\Sigma}))\\ &= H^{n-p}(X_\Sigma, \mathscr{H}om_{\mathscr{O}_{X_\Sigma}}(\mathscr{O}_{X_\Sigma}(D), \mathscr{O}_{X_\Sigma}(K_{X_\Sigma}))).\end{aligned}$$

However, $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X(D), \mathscr{O}_X(E)) \simeq \mathscr{O}_X(E-D)$ for any Weil divisors D, E on a normal variety X (Exercise 9.2.10). This gives the desired duality for $\mathscr{O}_{X_\Sigma}(D)$.

For part(b), we first show that $\widehat{\Omega}^q_{X_\Sigma}$ is MCM when X_Σ is simplicial. Since Σ is complete, we can work locally over U_σ, $\dim \sigma = n$. The minimal generators of σ form a basis of a sublattice $N' \subset N$ of finite index such that σ is smooth relative to N'. Let M' be the dual of N'. As above, we get semigroup algebras $\mathsf{S}_{\sigma,N} \subseteq \mathsf{S}_{\sigma,N'}$.

By Proposition 8.2.18, the restriction of $\widehat{\Omega}^q_{X_\Sigma}$ to $U_\sigma = U_{\sigma,N}$ is determined by the $\mathsf{S}_{\sigma,N}$-module

$$A = \bigoplus_{m \in \sigma^\vee \cap M} \textstyle\bigwedge^q V_\sigma(m) \cdot \chi^m,$$

where $V_\sigma(m) = \mathrm{Span}_{\mathbb{C}}(m_0 \in M \mid m_0 \in \text{the minimal face of } \sigma^\vee \text{ containing } m)$. Now consider the larger $\mathsf{S}_{\sigma,N}$-module defined by

$$B = \bigoplus_{m \in \sigma^\vee \cap M'} \textstyle\bigwedge^q V_\sigma(m) \cdot \chi^m.$$

Since $V_\sigma(m)$ is unaffected when M is replaced by M', we see that as a $\mathsf{S}_{\sigma,N'}$-module,

$$B = \Gamma(U_{\sigma,N'}, \widehat{\Omega}^q_{U_{\sigma,N'}}).$$

However, $\widehat{\Omega}^q_{U_{\sigma,N'}}$ is locally free since $U_{\sigma,N'}$ is smooth by our choice of N'. Hence B is MCM over $\mathsf{S}_{\sigma,N'}$. Arguing as in part (a), B is MCM over $\mathsf{S}_{\sigma,N}$ and we have a splitting map r defined by

$$r(\chi^m) = \begin{cases} \chi^m & \text{if } m \in \sigma^\vee \cap M\\ 0 & \text{otherwise}.\end{cases}$$

Thus $\widehat{\Omega}^q_{X_\Sigma}$ is MCM, and then $\widehat{\Omega}^q_{X_\Sigma} \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{F}$ is MCM since $\mathscr{F}$ is locally free.

The proof is now easy to finish, since Theorem 9.2.12 implies

$$H^p(X_\Sigma, \widehat{\Omega}^q_{X_\Sigma} \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{F})^\vee \simeq H^{n-p}(X_\Sigma, \mathscr{H}om_{\mathscr{O}_{X_\Sigma}}(\widehat{\Omega}^q_{X_\Sigma} \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{F}, \omega_{X_\Sigma}))$$

However, using the local freeness of $\mathscr{F}$ and Exercise 8.0.13, we have

$$\mathscr{H}om_{\mathscr{O}_{X_\Sigma}}(\widehat{\Omega}^q_{X_\Sigma} \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{F}, \omega_{X_\Sigma}) \simeq \mathscr{H}om_{\mathscr{O}_{X_\Sigma}}(\widehat{\Omega}^q_{X_\Sigma}, \omega_{X_\Sigma}) \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{F}^\vee \simeq \widehat{\Omega}^{n-q}_{X_\Sigma} \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{F}^\vee.$$

From here, the theorem follows easily. □

The proof of part (a) of the theorem was inspired by [**76**, Lem. 3.4.2]. Other proofs that $\mathbb{Q}$-Cartier divisors give MCM sheaves can be found in [**54**, Cor. 4.2.2] and [**226**, Prop. 4.22]. In part (b) we followed [**76**, Prop. 4.8].

We should also mention that besides $\mathbb{Q}$-Cartier divisors, there can be other Weil divisors that give MCM sheaves. For example, ω_X is MCM on any normal Cohen-Macaulay variety (see [**56**, Def. 3.3.1]), so that ω_{X_Σ} is MCM on any toric variety X_Σ, even when the canonical class K_{X_Σ} fails to be $\mathbb{Q}$-Cartier. You will give a toric proof of this in Exercise 9.2.11.

In Exercise 9.2.12 you will use Alexander duality to give a purely toric proof of Serre duality for simplicial toric varieties.

Exercises for §9.2.

9.2.1. Prove Proposition 9.2.1.

9.2.2. Verify the claims made in Example 9.2.2.

9.2.3. We saw in Example 4.0.25 a morphism of varieties $f : X \to Y$ induces a homomorphism $\mathscr{O}_Y \to f_*\mathscr{O}_X$ of $\mathscr{O}_Y$-modules.

(a) Let $\mathscr{L}$ be a line bundle on Y. Construct an isomorphism $f_*f^*\mathscr{L} \simeq (f_*\mathscr{O}_X) \otimes_{\mathscr{O}_Y} \mathscr{L}$ of $\mathscr{O}_Y$-modules. Hint: Construct a homomorphism and then study the homomorphism over open subsets of Y where $\mathscr{L}$ is trivial.

(b) Generalize part (b) by showing that for any sheaf $\mathscr{G}$ of $\mathscr{O}_X$-modules on X and any line bundle $\mathscr{L}$ on Y, there is an isomorphism $f_*(\mathscr{G} \otimes_{\mathscr{O}_X} f^*\mathscr{L}) \simeq f_*\mathscr{G} \otimes_{\mathscr{O}_Y} \mathscr{L}$. This is the *projection formula*.

9.2.4. Consider the map $\bigoplus_{m\in P\cap M} \mathbb{C}\cdot\chi^m \longrightarrow \bigoplus_{m\in(\ell P)\cap M} \mathbb{C}\cdot\chi^m$ from (9.2.1), where the $\mathbb{C}[\sigma^\vee \cap M]$-module structure on $\bigoplus_{m\in(\ell P)\cap M} \mathbb{C}\cdot\chi^m$ is given by $\chi^m \cdot a = \chi^{\ell m} a$. Prove that (9.2.1) and (9.2.2) are $\mathbb{C}[\sigma^\vee \cap M]$-module homomorphisms.

9.2.5. Prove that (9.2.4) implies $Z_{-D,m} = \{0\}$, as claimed in the proof of Theorem 9.2.7.

9.2.6. As in the proof of Theorem 9.2.7, assume that $Z_{-D,m}$ is strongly convex and that $u \in Z_{-D,m}$ is nonzero. Thus $-u \in V^{\text{supp}}_{-D,m}$. Prove that for every $v \in V^{\text{supp}}_{-D,m}$, the line segment $\overline{uv}$ is contained in $V^{\text{supp}}_{-D,m}$ (this means that $V^{\text{supp}}_{-D,m}$ is *star shaped* with respect to $-u$). Conclude that the constant map $\gamma : V^{\text{supp}}_{-D,m} \to \{-u\}$ is a contraction.

9.2.7. Prove that every affine toric variety U_σ is contained in a projective toric variety X_Σ.

9.2.8. The lattice points of a normal lattice polytope P give a projective embedding of the toric variety X_P. Prove that X_P is aCM in this embedding.

9.2.9. Assume that $N_1 \subseteq N_2$ has finite index, with dual $M_2 \subseteq M_1$, and let σ be a cone in $(N_1)_\mathbb{R} = (N_2)_\mathbb{R}$. This gives semigroup algebras $\mathsf{S}_{\sigma,N_2} = \mathbb{C}[\sigma^\vee \cap M_2] \subseteq \mathsf{S}_{\sigma,N_1} = \mathbb{C}[\sigma^\vee \cap M_1]$. Prove that S_{σ,N_1} is finitely generated as a module over S_{σ,N_2}. Hint: Let $m_1,\dots,m_r \in M_2$ generate $\sigma^\vee$ and consider $\{\sum_{i=1}^r \delta_i m_i \mid 0 \le \delta_i < 1\} \cap M_1$.

9.2.10. Let D,E be Weil divisors on a normal variety X and let $U \subseteq X$ be the smooth locus. Then $\Gamma(X,\mathscr{O}_X(D)) \simeq \Gamma(U,\mathscr{O}_X(D)|_U)$, and the same holds for E.

(a) Construct a natural map $\Phi_X : \Gamma(X,\mathscr{O}_X(E-D)) \to \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X(D),\mathscr{O}_X(E))$ and prove that Φ_X is an isomorphism when X is smooth.

(b) Prove that $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X(D),\mathscr{O}_X(E))$ is reflexive. Hint: Let U is the smooth locus and study the restriction map $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X(D),\mathscr{O}_X(E)) \to \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_X(D)|_U,\mathscr{O}_X(D)|_U)$.

(c) Show that Φ_X is an isomorphism and that $\mathscr{O}_X(E-D) \simeq \mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X(D), \mathscr{O}_X(E))$.

9.2.11. Following [**76**, Cor. 3.5], you will show that ω_{X_Σ} is MCM on a normal toric variety. Fix a cone $\sigma \subseteq N_\mathbb{R}$ and pick a basis $e_1, \ldots, e_n$ of M such that e_n is in the interior of $\sigma^\vee$. Let $\ell = \mathrm{lcm}(\langle e_n, u_\rho\rangle \mid \rho \in \sigma(1))$ and let $\overline{M}$ be the lattice with basis $e_1, \ldots, e_{n-1}, \ell^{-1}e_n$, with dual $\overline{N} \subseteq N$. By Proposition 8.2.9, ω_{U_σ} comes from the ideal $\bigoplus_{m \in \mathrm{Int}(\sigma^\vee)\cap M} \mathbb{C}\cdot\chi^m \subseteq \mathbb{C}[\sigma^\vee \cap M]$.

(a) Prove that $\overline{u}_\rho = (\ell/\langle e_n, u_\rho\rangle)u_\rho$ lies in $\overline{N}$. Then use $\langle \ell^{-1}e_n, \overline{u}_\rho\rangle = 1$ to show that $\overline{u}_\rho$ is the minimal generator of ρ with respect to $\overline{N}$.

(b) Conclude that the canonical divisor of $U_{\sigma,\overline{N}}$ is Cartier.

(c) Construct a splitting of $\bigoplus_{m \in \mathrm{Int}(\sigma^\vee)\cap M} \mathbb{C}\cdot\chi^m \subseteq \bigoplus_{m \in \mathrm{Int}(\sigma^\vee)\cap \overline{M}} \mathbb{C}\cdot\chi^m$ and conclude that the canonical sheaf of $U_{\sigma,N}$ is MCM.

9.2.12. Alexander duality (see [**210**, §71]) states that if $A \subseteq S^{n-1}$ is a closed subset such that the pair (S^{n-1}, A) is triangulable (see [**210**, p. 150]), then

$$\widetilde{H}^{p-1}(A, \mathbb{C})^\vee \simeq \widetilde{H}^{n-p-1}(S^{n-1} \setminus A, \mathbb{C}).$$

You will prove Serre duality when $D = \sum_\rho a_\rho D_\rho$ on a complete simplicial toric variety X_Σ of dimension n. Let $K = K_{X_\Sigma}$ be the canonical divisor of X_Σ. Theorem 9.1.3 implies

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \simeq \widetilde{H}^{p-1}(V_{D,m}, \mathbb{C}),\ m \in M.$$

Set $A_{D,m} = V_{D,m}$ and $\Delta_\sigma = \mathrm{Conv}(u_\rho \mid \rho \in \sigma(1))$. Your goal is to prove that

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m^\vee \simeq H^{n-p}(X_\Sigma, \mathscr{O}_{X_\Sigma}(K-D))_{-m}. \tag{9.2.9}$$

(a) Explain why $S = \bigcup_{\sigma\in\Sigma} \Delta_\sigma$ is homeomorphic to S^{n-1}. Note that $A_{D,m}, A_{K-D,-m} \subseteq S$.

(b) Prove that $A_{D,m} \cap A_{K-D,-m} = \emptyset$ and that for $\sigma \in \Sigma$, σ is contained in neither $A_{D,m}$ nor $A_{K-D,-m}$ if and only if there are $\rho_1, \rho_2 \in \sigma(1)$ such that $\langle m, u_{\rho_1}\rangle < -a_{\rho_1}$ and $\langle m, u_{\rho_2}\rangle \geq -a_{\rho_2}$. We say that σ is *intermediate* when this happens.

(c) Fix an intermediate cone $\sigma \in \Sigma$. Let Δ_σ^- be the face of Δ_σ generated by u_ρ's with $\langle m, u_\rho\rangle < -a_\rho$ and Δ_σ^+ be the face generated by u_ρ's with $\langle m, u_\rho\rangle \geq -a_\rho$. Show that every $u \in \Delta_\sigma$ can be written uniquely as $u = (1-t)u^+ + tu^-$ where $u^+ \in \Delta_\sigma^+$, $u^- \in \Delta_\sigma^-$ and $0 \leq t \leq 1$. Then show that $\Delta_\sigma^+ \subseteq \Delta_\sigma \setminus \Delta_\sigma^-$ is a deformation retract.

(d) Prove that $A_{K-D,-m} \subseteq S \setminus A_{D,m}$ is a deformation retract.

(e) Finally, prove (9.2.9) by applying Alexander duality to $A_{D,m} \subseteq S$.

This exercise was inspired by [**76**, Prop. 7.7.1] and [**105**, Sec. 4.4].

§9.3. Vanishing Theorems II

Vanishing theorems play an important role in algebraic geometry. A glance at the index of Lazarsfeld's two-volume treatise [**186**] lists vanishing theorems due to

> Bogomolov, Demailly, Fujita, Grauert-Riemenschneider,
> Griffiths, Kawamata-Viehweg, Kodaira, Kollár, Le Potier,
> Manivel, Miyoka, Nadel, Nakano, and Serre.

We will explore toric versions of several of these results. In some cases, the toric version is stronger, which is to be expected since toric varieties are so special.

Twisting. If D is a Cartier divisor and $\mathscr{F}$ a coherent sheaf on a normal variety X, then the *twist* of $\mathscr{F}$ by D is the sheaf

$$\mathscr{F}(D) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(D).$$

For example, if K_X is a canonical divisor on X, then

$$\omega_X(D) = \omega_D \otimes_{\mathscr{O}_X} \mathscr{O}_X(D) \simeq \mathscr{O}_X(K_X) \otimes_{\mathscr{O}_X} \mathscr{O}_X(D) \simeq \mathscr{O}_X(K_X + D).$$

This notation will be used some of the vanishing theorems stated below. However, when we start working with non-Cartier divisors, we will drop the twist notation.

Kodaira and Nakano. Two of the earliest vanishing theorems are due to Kodaira and Nakano. For an ample divisor D on a smooth projective variety X of dimension n, Kodaira vanishing asserts that

$$H^p(X, \omega_X(D)) = 0, \ p > 0,$$

and Nakano vanishing states that

$$H^p(X, \Omega_X^q(D)) = 0, \ p + q > n.$$

Nakano's theorem generalizes Kodaira's since $\omega_X = \Omega_X^n$ in the smooth case.

In the toric case, we get more vanishing, in what has become known as the Bott-Steenbrink-Danilov vanishing theorem.

Theorem 9.3.1 (Bott-Steenbrink-Danilov Vanishing)**.** *Let D be an ample divisor on a projective toric variety X_Σ. Then for all $p > 0$ and $q \geq 0$, we have*

$$H^p(X_\Sigma, \widehat{\Omega}^q_{X_\Sigma}(D)) = 0.$$

Proof. We give a proof due to Fujino [**101**] that uses the morphism $\phi_\ell : X_\Sigma \to X_\Sigma$ from Lemma 9.2.6. We can assume that D is torus-invariant with support function $\varphi_D : N_\mathbb{R} \to \mathbb{R}$. Let $\mathscr{L} = \mathscr{O}_{X_\Sigma}(D)$ be the associated line bundle.

The morphism φ_ℓ has two key properties. The first concerns the pullback $\phi_\ell^* \mathscr{O}_{X_\Sigma}(D)$. Proposition 6.2.7 implies that $\phi_\ell^* \mathscr{O}_{X_\Sigma}(D)$ comes from a divisor whose support function is $\varphi_D \circ \overline{\phi}_\ell$. Since $\overline{\phi}_\ell$ is multiplication by ℓ, $\varphi_D \circ \overline{\phi}_\ell$ is the support function of ℓD. Hence

$$\phi_\ell^* \mathscr{L} = \phi_\ell^* \mathscr{O}_{X_\Sigma}(D) \simeq \mathscr{O}_{X_\Sigma}(\ell D) \simeq \mathscr{O}_{X_\Sigma}(D)^{\otimes \ell} = \mathscr{L}^{\otimes \ell}.$$

The second key property of ϕ_ℓ is that there is a split injection

$$\widehat{\Omega}^q_{X_\Sigma} \hookrightarrow \phi_{\ell *} \widehat{\Omega}^q_{X_\Sigma}. \tag{9.3.1}$$

We will assume this for now.

Given these properties of ϕ_ℓ, the theorem follows easily. Tensoring (9.3.1) with $\mathscr{L}$ gives a split injection

$$\widehat{\Omega}^q_{X_\Sigma} \otimes_{X_\Sigma} \mathscr{L} \hookrightarrow \phi_{\ell *} \widehat{\Omega}^q_{X_\Sigma} \otimes_{X_\Sigma} \mathscr{L}.$$

Since $\phi_{\ell*}\widehat{\Omega}^q_{X_\Sigma} \otimes_{X_\Sigma} \mathscr{L} \simeq \phi_{\ell*}(\widehat{\Omega}^q_{X_\Sigma} \otimes_{X_\Sigma} \phi_\ell^*\mathscr{L})$ (this is the projection formula from Exercise 9.2.3) and $\phi_\ell^*\mathscr{L} \simeq \mathscr{L}^{\otimes \ell}$, we obtain a split injection

$$\widehat{\Omega}^q_{X_\Sigma} \otimes_{X_\Sigma} \mathscr{L} \hookrightarrow \phi_{\ell*}(\widehat{\Omega}^q_{X_\Sigma} \otimes_{X_\Sigma} \mathscr{L}^{\otimes \ell}).$$

As in the discussion leading up to Lemma 9.2.6, this gives the injection

$$H^p(X_\Sigma, \widehat{\Omega}^q_{X_\Sigma} \otimes_{X_\Sigma} \mathscr{L}) \hookrightarrow H^p(X_\Sigma, \widehat{\Omega}^q_{X_\Sigma} \otimes_{X_\Sigma} \mathscr{L}^{\otimes \ell}).$$

When $p > 0$, the right-hand side vanishes for ℓ sufficiently large by Serre vanishing (Theorem 9.0.6). Hence the left-hand side also vanishes for $p > 0$, which is what we want.

It remains to prove (9.3.1). If we set $D = 0$ in the proof of Lemma 9.2.6, we get a split injection $i : \mathscr{O}_{X_\Sigma} \to \phi_{\ell*}\mathscr{O}_{X_\Sigma}$. Recall that locally,

- $\phi_{\ell*}\mathscr{O}_{X_\Sigma}$ looks like $\mathscr{O}_{X_\Sigma}$, with module structure is given by $\chi^m \cdot a = \chi^{\ell m} a$.
- i sends χ^m to $\chi^{\ell m}$.
- The splitting $r : \phi_{\ell*}\mathscr{O}_{X_\Sigma} \to \mathscr{O}_{X_\Sigma}$ is defined by

$$r(\chi^m) = \begin{cases} 0 & m \notin \ell M \\ \chi^{m'} & m = \ell m', m' \in M. \end{cases}$$

If we tensor $i : \mathscr{O}_{X_\Sigma} \to \phi_{\ell*}\mathscr{O}_{X_\Sigma}$ with $\bigwedge^q M$, we get a sheaf homomorphism

$$\bigwedge^q M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \hookrightarrow \phi_{\ell*}\big(\bigwedge^q M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}\big) \tag{9.3.2}$$

together with a splitting map

$$\phi_{\ell*}\big(\bigwedge^q M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}\big) \longrightarrow \bigwedge^q M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}. \tag{9.3.3}$$

Thus (9.3.2) is a split injection.

By Theorem 8.2.16, $\widehat{\Omega}^q_{X_\Sigma}$ sits inside $\bigwedge^q M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}$, so that $\phi_{\ell*}\widehat{\Omega}^q_{X_\Sigma}$ is a subsheaf of $\phi_{\ell*}\big(\bigwedge^q M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}\big)$. These subsheaves relate to (9.3.2) and (9.3.3) as follows. Over U_σ, Theorem 8.2.18 implies

$$\Gamma(U_\sigma, \widehat{\Omega}^q_{X_\Sigma}) \simeq \bigoplus_{m \in \sigma^\vee \cap M} \bigwedge^q V_\sigma(m) \cdot \chi^m \subseteq \bigoplus_{m \in \sigma^\vee \cap M} \bigwedge^q M_{\mathbb{C}} \cdot \chi^m = \Gamma(U_\sigma, \bigwedge^q M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma}),$$

where $V_\sigma(m) \subseteq M_{\mathbb{C}}$ is spanned by the lattice points lying in the same minimal face of $\sigma^\vee$ as m. Then over U_σ, we have:

- $\phi_{\ell*}\widehat{\Omega}^q_{X_\Sigma}$ looks like $\widehat{\Omega}^q_{X_\Sigma}$ with module structure $\chi^m \cdot a = \chi^{\ell m} a$.
- The map (9.3.2) takes $\bigwedge^q V_\sigma(m) \cdot \chi^m$ to

$$\bigwedge^q V_\sigma(m) \cdot \chi^{\ell m} = \bigwedge^q V_\sigma(\ell m) \cdot \chi^{\ell m} \subseteq \Gamma(U_\sigma, \widehat{\Omega}^q_{X_\Sigma})$$

  since $V_\sigma(m) = V_\sigma(\ell m)$. Thus (9.3.2) induces a map (9.3.1).

- The map (9.3.2) takes $\bigwedge^q V_\sigma(m)\cdot\chi^m$ to 0 or, when $m=\ell m'$, to
$$\textstyle\bigwedge^q V_\sigma(m)\cdot\chi^{m'}=\bigwedge^q V_\sigma(m')\cdot\chi^{\ell m'}\subseteq\Gamma(U_\sigma,\widehat{\Omega}^q_{X_\Sigma})$$
since $V_\sigma(m)=V_\sigma(m')$. Thus (9.3.3) induces a map $\phi_{\ell*}\widehat{\Omega}^q_{X_\Sigma}\to\widehat{\Omega}^q_{X_\Sigma}$.

This gives the desired split injection (9.3.1). □

Although Theorem 9.3.1 generalizes the vanishing theorems of Kodaira and Nakano, its name "Bott-Steenbrink-Danilov" reflects the more special vanishing that happens for projective spaces (Bott [**40**]), weighted projective spaces (Steenbrink [**260**]), and projective toric varieties (Danilov [**76**], stated without proof). Theorem 9.3.1 was first proved by Batyrev and Cox [**19**] in the simplicial case and by Buch, Thomsen, Lauritzen and Mehta [**58**] in general. Further proofs have been given by Fujino [**101**] (noted above) and Mustaţă [**212**].

The Simplicial Case. When X_Σ is simplicial, there is another vanishing theorem involving the sheaves $\widehat{\Omega}^q_{X_\Sigma}$.

Theorem 9.3.2. *If X_Σ is a complete simplicial toric variety, then*
$$H^p(X_\Sigma,\widehat{\Omega}^q_{X_\Sigma})=0 \textit{ for all } p\neq q.$$

Proof. Since X_Σ is simplicial, we have the Ishida complex

$$0\longrightarrow\widehat{\Omega}^q_{X_\Sigma}\longrightarrow K^0(\Sigma,q)\longrightarrow K^1(\Sigma,q)\longrightarrow\cdots\longrightarrow K^q(\Sigma,q)\longrightarrow 0 \tag{9.3.4}$$

which is exact by Theorem 8.2.19. Recall that
$$K^j(\Sigma,q)=\bigoplus_{\sigma\in\Sigma(j)}\textstyle\bigwedge^{q-j}(\sigma^\perp\cap M)\otimes_{\mathbb{Z}}\mathscr{O}_{V(\sigma)}$$

and $V(\sigma)=\overline{O(\sigma)}\subseteq X_\Sigma$ is the orbit closure corresponding to $\sigma\in\Sigma$. Since $V(\sigma)$ is a toric variety, its structure sheaf has vanishing higher cohomology by Demazure vanishing, so that $H^p(X_\Sigma,K^j(\Sigma,q))=0$ for $p>0$. Since the above sequence has $q+1$ terms with vanishing higher cohomology, an easy argument using the long exact sequence in cohomology implies $H^p(X_\Sigma,\widehat{\Omega}^q_{X_\Sigma})=0$ for $p>q$ (Exercise 9.3.1).

Now suppose $p<q$. Since X_Σ is simplicial, the version of Serre duality given in Theorem 9.2.10 implies
$$H^p(X_\Sigma,\widehat{\Omega}^q_{X_\Sigma})^\vee\simeq H^{n-p}(X_\Sigma,\widehat{\Omega}^{n-q}_{X_\Sigma})$$
The right-hand side vanishes since $n-p>n-q$, and the result follows. □

When X_Σ complete but not necessarily simplicial, Danilov [**76**, Cor. 12.7] proves that $H^p(X_\Sigma,\widehat{\Omega}^q_{X_\Sigma})=0$ when $q>p$.

In [**198**], Mavlyutov discusses a vanishing theorem for $\widehat{\Omega}^q_{X_\Sigma}(D)$, where D is a nef Cartier divisor. His result goes as follows.

Theorem 9.3.3. *Let X_Σ be a complete simplicial toric variety. If D is a nef Cartier divisor on X_Σ, then*

$$H^p(X_\Sigma, \widehat{\Omega}^q_{X_\Sigma}(D)) = 0$$

whenever $p > q$ or $q > p + \dim P_D$. □

The proof for $p > q$ is relatively easy (Exercise 9.3.2). Note that Theorem 9.3.2 is the case $D = 0$ of Theorem 9.3.3. The paper [**198**] computes $H^p(X_\Sigma, \widehat{\Omega}^q_{X_\Sigma}(D))$ explicitly for all p, q.

$\mathbb{Q}$-***Weil Divisors.*** A $\mathbb{Q}$-*Weil divisor* or D on a normal variety X is a formal $\mathbb{Q}$-linear combination of prime divisors. Thus a positive integer multiple of D is an ordinary Weil divisor, often called *integral* in this context. A $\mathbb{Q}$-Weil divisor is $\mathbb{Q}$-*Cartier* if some positive multiple is integral and Cartier. In the literature (see [**186**, 1.1.4]), $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisors are often called $\mathbb{Q}$-*divisors*. These divisors and their close cousins, $\mathbb{R}$-divisors, are essential tools in modern algebraic geometry.

For $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the greatest integer $\leq x$ and $\lceil x \rceil$ is the least integer $\geq x$. Then, given a $\mathbb{Q}$-Weil divisor $D = \sum_i a_i D_i$, we get integral Weil divisors

$$\begin{aligned} \lfloor D \rfloor &= \textstyle\sum_i \lfloor a_i \rfloor D_i \quad \text{(the "round down" of } D) \\ \lceil D \rceil &= \textstyle\sum_i \lceil a_i \rceil D_i \quad \text{(the "round up" of } D). \end{aligned}$$

We now prove an injectivity lemma for $\mathbb{Q}$-Weil divisors due to Fujino [**101**].

Lemma 9.3.4. *Let D be a $\mathbb{Q}$-Weil divisor on a toric variety X_Σ and let $\ell > 0$ be an integer such that ℓD is integral. Then for all $p \geq 0$ there is an injection*

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\lfloor D \rfloor)) \hookrightarrow H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D)).$$

Proof. We adapt the proof of Lemma 9.2.6 to our situation. Write $D = \sum_\rho a_\rho D_\rho$, where $a_\rho = b_\rho + \varepsilon_\rho$ for $b_\rho \in \mathbb{Z}$ and $0 \leq \varepsilon_\rho < 1$. Thus $\lfloor D \rfloor = \sum_\rho b_\rho D_\rho$. With ℓ as above, we claim that $\phi_\ell : X_\Sigma \to X_\Sigma$ from Lemma 9.2.6 gives a split injection

$$\mathscr{O}_{X_\Sigma}(\lfloor D \rfloor) \hookrightarrow \phi_{\ell *}\mathscr{O}_{X_\Sigma}(\ell D). \tag{9.3.5}$$

Assuming (9.3.5), the lemma follows from the remarks leading up to Lemma 9.2.6.

It remains to prove the existence of a split injection (9.3.5). Take an affine open subset $U_\sigma \subseteq X_\Sigma$ and consider

$$P_\ell = \{m \in M_\mathbb{R} \mid \langle m, u_\rho \rangle \geq -\ell a_\rho \text{ for all } \rho \in \sigma(1)\}.$$

Then

$$\Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(\lfloor D \rfloor)) = \bigoplus_{m \in P_1 \cap M} \mathbb{C} \cdot \chi^m, \ \ \Gamma(U_\sigma, \mathscr{O}_{X_\Sigma}(\ell D)) = \bigoplus_{m \in P_\ell \cap M} \mathbb{C} \cdot \chi^m,$$

where the first equality uses $a_\rho = b_\rho + \varepsilon_\rho$, $0 \le \varepsilon_\rho < 1$. Since $P_\ell \cap \ell M = \ell(P_1 \cap M)$, the map $m \mapsto \ell m$ induces an inclusion

$$\bigoplus_{m\in P_1\cap M} \mathbb{C}\cdot\chi^m \hookrightarrow \bigoplus_{m\in P_\ell\cap M} \mathbb{C}\cdot\chi^m.$$

This is a $\mathbb{C}[\sigma^\vee \cap M]$-module homomorphism, provided that the right-hand side has module structure $\chi^m \cdot a = \chi^{\ell m}a$, and the usual formula for the splitting map r is also a $\mathbb{C}[\sigma^\vee \cap M]$-module homomorphism. From here, we get the required split injection (9.3.5) without difficulty. $\square$

Here are $\mathbb{Q}$-divisor versions of Demazure and Batyrev-Borisov vanishing.

Theorem 9.3.5. *Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on a toric variety X_Σ.*

(a) *If $|\Sigma|$ is convex and D is nef, then*

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\lfloor D\rfloor)) = 0 \text{ for all } p > 0.$$

(b) *If Σ is complete and D is nef, then*

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-\lceil D\rceil)) = 0 \text{ for all } p \ne \dim P_D.$$

Proof. Pick $\ell > 0$ with ℓD Cartier. For part (a), $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D)) = 0$ for $p > 0$ by Theorem 9.2.3. The desired vanishing follows immediately from Lemma 9.3.4. For part (b), replacing D with $-D$ in Lemma 9.3.4 gives

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-\lceil D\rceil)) = H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\lfloor -D\rfloor)) \hookrightarrow H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-\ell D)),$$

and then we are done by Theorem 9.2.7. $\square$

Here is an example that uses part (a) of Theorem 9.3.5 to extend Demazure vanishing beyond nef $\mathbb{Q}$-Cartier divisors.

Example 9.3.6. The fan for the Hirzebruch surface $\mathscr{H}_r$ has minimal generators $u_1 = (-1,r), u_2 = (0,1), u_3 = (1,0), u_4 = (0,-1)$, giving divisors $D_1,\ldots,D_4$. By Example 6.1.16,

$$\mathrm{Pic}(\mathscr{H}_r)_\mathbb{R} \simeq \{aD_3 + bD_4 \mid a,b \in \mathbb{R}\}$$

and the nef cone is generated by D_3 and D_4. Since $D_1 \sim D_3$ and $D_2 \sim D_4 - rD_3$, it follows that a $\mathbb{Q}$-Weil divisor $D = a_1D_1 + \cdots + a_4D_4$ is nef if and only

$$a_1 + a_3 \ge ra_2 \quad \text{and} \quad a_2 + a_4 \ge 0. \tag{9.3.6}$$

We will show that the divisors $aD_3 + bD_4$, $a,b \ge -1$, have vanishing higher cohomology when $r > 0$. Given such a divisor, pick a rational number $0 < \varepsilon < \frac{1}{2}$ and consider the $\mathbb{Q}$-Weil divisor

$$D = 2\varepsilon D_1 + \tfrac{\varepsilon}{r}D_2 + (a+1-\varepsilon)D_3 + (b+1-\tfrac{\varepsilon}{r})D_4.$$

This satisfies (9.3.6) and hence is nef. Then $\lfloor D\rfloor = aD_3 + bD_4$ has vanishing higher cohomology by Theorem 9.3.5. Taking $a = -1$ or $b = -1$, we get non-nef divisors whose higher cohomology vanishes. $\Diamond$

Lemma 9.3.4 also leads to the following result due to Mustaţă [**212**].

Theorem 9.3.7. *Let X_Σ be a projective toric variety and let $\rho_1,\dots,\rho_r \in \Sigma(1)$ be distinct. Then for $p > 0$ and any ample Cartier divisor D on X_Σ, we have*

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D - D_{\rho_1} - \cdots - D_{\rho_r})) = 0.$$

Proof. Let $B = D_{\rho_1} + \cdots + D_{\rho_r}$. Since D is ample, Serre vanishing (Theorem 9.0.6) implies that we can find $\ell > 0$ such that $H^p(X_\Sigma, X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D - B)) = 0$ for $p > 0$. Now let $E = D - \ell^{-1}B$. Since $\lfloor E \rfloor = D - B$, the inclusion

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\lfloor E \rfloor)) \hookrightarrow H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell E))$$

from Lemma 9.3.4 implies

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D - B)) \hookrightarrow H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D - B)) = 0,\ p > 0. \qquad \square$$

Here is a non-$\mathbb{Q}$-Cartier divisor with vanishing higher cohomology.

Example 9.3.8. Consider the complete fan Σ in $\mathbb{R}^3$ shown in Figure 5. It has

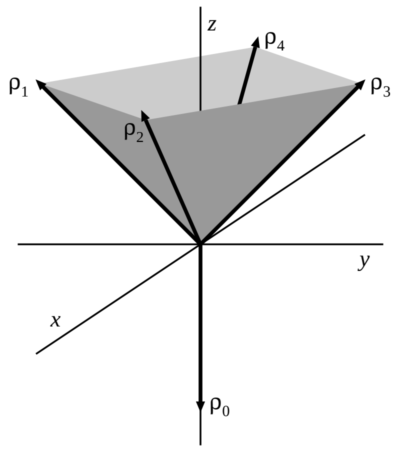


Figure 5. A complete fan in $\mathbb{R}^3$

divisors $D_0,\dots,D_4$ corresponding to $\rho_0,\dots,\rho_4$. In Exercise 9.3.3 you will show that for a $\mathbb{Q}$-Weil divisor $D = a_0 D_0 + \cdots + a_4 D_4$, we have:

- D is $\mathbb{Q}$-Cartier if and only $a_1 + a_3 = a_2 + a_4$.
- D is $\mathbb{Q}$-Cartier and nef if and only $a_1 + a_3 = a_2 + a_4 \geq 0$.

In particular, $D = D_3 + D_4$ is $\mathbb{Q}$-Cartier and nef, so that $D_4 = D - D_3$ has vanishing higher cohomology by Theorem 9.3.7. Yet D_4 is not $\mathbb{Q}$-Cartier. $\Diamond$

Iitaka Dimension and Big Divisors. Given a nef Cartier divisor D on a complete toric variety X_Σ and an integer $\ell > 0$, the global sections $W_\ell = H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D))$ give a morphism $\phi_{W_\ell} : X_\Sigma \to \mathbb{P}(W_\ell^\vee)$ as in Lemma 6.0.28. Also recall that since D is nef, its polytope P_D is a lattice polytope. The map ϕ_{W_ℓ} and the polytope ℓP_D are related as follows.

Lemma 9.3.9. *For $\ell \gg 0$, the image of ϕ_{W_ℓ} is isomorphic to the toric variety of ℓP_D. In particular,* $\dim \phi_{W_\ell}(X_\Sigma) = \dim P_D$ *for $\ell \gg 0$.*

Proof. This follows from Proposition 6.2.8 (Exercise 9.3.4). □

This situation is a special case of the definition of the *Iitaka dimension* $\kappa(X,D)$ (see [**186**, Def. 2.1.3]) of a Cartier divisor D on a complete irreducible variety X. In this terminology, Lemma 9.3.9 implies that a nef Cartier divisor D on a complete toric variety X_Σ has Iitaka dimension

$$\kappa(X_\Sigma, D) = \dim P_D.$$

In general, a Cartier divisor is *big* if it has maximal Iitaka dimension, which for a nef Cartier divisor D on a complete toric variety X_Σ means

$$D \text{ is big} \iff \dim P_D = \dim X_\Sigma.$$

It should be clear what it means for a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor to be big and nef.

Kawamata-Viehweg. The classic version of Kodaira vanishing can be stated as

$$H^p(X, \mathscr{O}_X(K_X + D)) = 0 \text{ for all } p > 0$$

when D is an ample line bundle on a smooth projective variety X. In the 1982, Kawamata and Viehweg independently weakened the hypotheses on D. Here is the toric version of their result.

Theorem 9.3.10 (Toric Kawamata-Viehweg)**.** *Let X_Σ be a complete toric variety and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on X_Σ that is big and nef. Then*

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(K_{X_\Sigma} + \lceil D \rceil)) = 0 \textit{ for all } p > 0.$$

Proof. Let $n = \dim X_\Sigma$. When D is Cartier, Serre duality implies

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(K_{X_\Sigma} + D))^\vee \simeq H^{n-p}(X_\Sigma, \mathscr{O}_{X_\Sigma}(-D)).$$

Since D is nef, the latter vanishes for $n - p \neq \dim P_D$ by Theorem 9.2.7, and since D is big, the condition on p becomes $n - p \neq n$, i.e, $p \neq 0$.

In general, write $D = \sum_\rho a_\rho D_\rho$ where $a_\rho = b_\rho - \varepsilon_\rho$ for $b_\rho \in \mathbb{Z}$ and $0 \leq \varepsilon_\rho < 1$. Pick ℓ such that ℓD is Cartier and $0 < \varepsilon_\rho + \ell^{-1} < 1$ for all ρ. Let $E = D + \ell^{-1} K_{X_\Sigma}$. Since $\lfloor b_\rho - \varepsilon_\rho - \ell^{-1} \rfloor = b_\rho - 1$, we have $\lfloor E \rfloor = \lceil D \rceil + K_{X_\Sigma}$. Thus the inclusion

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\lfloor E \rfloor)) \hookrightarrow H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell E))$$

from Lemma 9.3.4 implies
$$H^p(X_\Sigma,\mathscr{O}_{X_\Sigma}(K_{X_\Sigma}+\lceil D\rceil))\hookrightarrow H^p(X_\Sigma,\mathscr{O}_{X_\Sigma}(K_{X_\Sigma}+\ell D)).$$
Then we are done by the Cartier case already proved. $\square$

Another approach to Theorem 9.3.10 is to prove a version of Serre duality that implies $H^p(X_\Sigma,\mathscr{O}_{X_\Sigma}(K_{X_\Sigma}+\lceil D\rceil))^\vee \simeq H^{n-p}(X_\Sigma,\mathscr{O}_{X_\Sigma}(-\lceil D\rceil))$ (Exercise 9.3.5).

Grauert-Riemenschneider. Our next vanishing result features the higher direct images of the canonical sheaf. We will need the following preliminary result.

Proposition 9.3.11. *Let Σ be a simplicial fan whose support $|\Sigma|$ is strongly convex. Then $H^p(X_\Sigma,\omega_{X_\Sigma})=0$ for all $p>0$.*

Proof. Fix $p\geq 1$ and $m\in M$. Theorem 9.1.3 with $D=K_{X_\Sigma}=-\sum_\rho D_\rho$ implies
$$H^p(X_\Sigma,\omega_{X_\Sigma})_m = H^p(X_\Sigma,\mathscr{O}_{X_\Sigma}(D))_m \simeq \widetilde{H}^{p-1}(V_{D,m},\mathbb{C}),$$
where
$$\begin{aligned}V_{D,m} &= \textstyle\bigcup_{\sigma\in\Sigma}\mathrm{Conv}(u_\rho \mid \rho\in\sigma(1),\ \langle m,u_\rho\rangle<1)\\ &= \textstyle\bigcup_{\sigma\in\Sigma}\mathrm{Conv}(u_\rho \mid \rho\in\sigma(1),\ \langle m,u_\rho\rangle\leq 0).\end{aligned}$$
The last equality follows since $\langle m,u_\rho\rangle\in\mathbb{Z}$. Now consider the set
$$W=\{u\in|\Sigma|\setminus\{0\}\mid \langle m,u\rangle\leq 0\}=(|\Sigma|\setminus\{0\})\cap\{u\in N_\mathbb{R}\mid\langle m,u\rangle\leq 0\},$$
and observe that W is convex since $|\Sigma|$ is strongly convex. Note also that
$$V_{D,m}\subseteq W. \tag{9.3.7}$$
The proposition will follow once we prove that (9.3.7) is a deformation retract.

Given $\sigma\in\Sigma$ with $W\cap\sigma\neq\emptyset$, we construct $r_\sigma : W\cap\sigma\to V_{D,m}\cap\sigma$ as follows. The assumption $W\cap\sigma\neq\emptyset$ implies that
$$A=\{\rho\in\sigma(1)\mid\langle m,u_\rho\rangle\leq 0\}\neq\emptyset.$$
Since σ is simplicial, $u\in\sigma$ can be uniquely written $u=\sum_{\rho\in\sigma(1)}\lambda_\rho u_\rho$, $\lambda_\rho\geq 0$, and when $u\in W\cap\sigma$, define $r_\sigma(u)=(\sum_{\rho\in A}\lambda_\rho)^{-1}\sum_{\rho\in A}\lambda_\rho u_\rho$. The sum $\sum_{\rho\in A}\lambda_\rho$ is nonzero since $u\in W\cap\sigma$, so that $r_\sigma(u)\in\mathrm{Conv}(u_\rho\mid\rho\in A)\subseteq V_{D,m}$. It is also easy to see that r_σ is compatible with r_τ whenever τ is a face of σ. Thus we get a map $r: W\to V_{D,m}$, which is the desired retraction by Exercise 9.3.6. $\square$

Here is a toric version of Grauert-Riemenschneider vanishing.

Theorem 9.3.12 (Toric Grauert-Riemenschneider Vanishing)**.** *Let $\phi: X_\Sigma\to X_{\Sigma'}$ be a surjective proper toric morphism between toric varieties of the same dimension. If X_Σ is simplicial, then*
$$R^p\phi_*\omega_{X_\Sigma}=0 \text{ for all } p>0.$$
If in addition ϕ is birational, then $\phi_*\omega_{X_\Sigma}\simeq\omega_{X_{\Sigma'}}$.

Proof. Our hypothesis on ϕ implies that the associated lattice map $\overline{\phi} : N \to N'$ induces an isomorphism $\overline{\phi}_{\mathbb{R}} : N_{\mathbb{R}} \simeq N'_{\mathbb{R}}$ such that Σ is the inverse image of Σ'. Thus, given $\sigma \in \Sigma'$, we see that $\overline{\phi}_{\mathbb{R}}^{-1}(\sigma)$ is strongly convex. This is the support of the fan of $\phi^{-1}(U_\sigma)$, so that

$$H^p(\phi^{-1}(U_\sigma), \omega_{X_\Sigma}) = 0$$

for $p > 0$ by Proposition 9.3.11. Then $R^p\phi_*\omega_{X_\Sigma} = 0$ for $p > 0$ by Proposition 9.0.7.

The final assertion of the theorem is an easy consequence of Theorem 8.2.15 (Exercise 9.3.7). □

Most versions of Theorem 9.3.12 in the literature assume that X_Σ is smooth.

Other Vanishing Theorems. There are many more toric vanishing theorems. Both Fujino [**101**] and Mustaţă [**212**] state general vanishing theorems that imply versions of Theorems 9.3.1, 9.3.5, 9.3.7 and 9.3.10. Further vanishing results can be found in [**102**] and [**226**].

Exercises for §9.3.

9.3.1. A sheaf $\mathscr{G}$ on a variety X is *acyclic* if $H^p(X, \mathscr{G}) = 0$ for all $p > 0$. Now suppose that we have an exact sequence of sheaves

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}_0 \longrightarrow \mathscr{G}_1 \longrightarrow \cdots \longrightarrow \mathscr{G}_r \longrightarrow 0,$$

where $\mathscr{G}_0, \dots, \mathscr{G}_r$ are acyclic. Prove $H^p(X, \mathscr{F}) = 0$ for all $p > r$ by induction on r.

9.3.2. Use (9.3.4) and the previous exercise to prove Theorem 9.3.3 when $p > q$.

9.3.3. Let $D_0, \dots, D_4$ be the divisors from Example 9.3.8 and consider a $\mathbb{Q}$-Weil divisor $D = a_0D_0 + \cdots + a_4D_4$.

(a) Show that $D_0 \sim 2D_3 + 2D_4, D_1 \sim D_3, D_2 \sim D_4$.

(b) Show that D is $\mathbb{Q}$-Cartier if and only $a_1 + a_3 = a_2 + a_4$.

(c) Show that D is $\mathbb{Q}$-Cartier and nef if and only $a_1 + a_3 = a_2 + a_4 \geq 0$.

9.3.4. Complete the proof of Lemma 9.3.9.

9.3.5. Let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor on a complete toric variety X_Σ.

(a) Pick $\ell > 0$ such that ℓD is Cartier and consider $\mathscr{O}_{X_\Sigma}(\lfloor D \rfloor) \hookrightarrow \phi_{\ell *}\mathscr{O}_{X_\Sigma}(\ell D)$ from (9.3.5). Use this to prove that $\mathscr{O}_{X_\Sigma}(\lfloor D \rfloor)$ is MCM. Hint: Replace $\overline{\phi}_\ell : N \to N$ with $\ell N \subseteq N$ as in the proof of Theorem 9.2.10.

(b) Adapt the proof of Theorem 9.2.10 to show that

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\lfloor D \rfloor))^\vee \simeq H^{n-p}(X_\Sigma, \mathscr{O}_{X_\Sigma}(K_{X_\Sigma} - \lfloor D \rfloor)).$$

(c) Prove $H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(-\lceil D \rceil))^\vee \simeq H^{n-p}(X_\Sigma, \mathscr{O}_{X_\Sigma}(K_{X_\Sigma} + \lceil D \rceil))$. Hint: $\lfloor -D \rfloor = -\lceil D \rceil$.

(d) Prove Theorem 9.3.10 using part (c) and Theorem 9.3.5.

9.3.6. Consider the map $r_\sigma : W \cap \sigma \to V_{D,m} \cap \sigma$ defined in the proof of Proposition 9.3.11.

(a) Prove that these maps patch to give a retraction $r : W \to V_{D,m}$.

(b) Prove that when regarded as a map from $W \cap \sigma$ to itself, r_σ is homotopic to the identity. Then formulate and prove a similar result for r.

9.3.7. Let $\phi : X_\Sigma \to X_{\Sigma'}$ be a proper birational toric morphism.

(a) Prove that $\overline{\phi} : N \to N'$ is an isomorphism.

(b) Let Σ_0 be the fan in $N_\mathbb{R}$ consisting of the cones $\overline{\phi}_\mathbb{R}^{-1}(\sigma)$ for $\sigma \in \Sigma'$. Prove that Σ is a refinement of Σ_0.

(c) Complete the proof of Theorem 9.3.12.

9.3.8. Given a toric variety X_Σ, let D be a Weil divisor and $E = \sum_\rho a_\rho D_\rho$ a $\mathbb{Q}$-Weil divisor such that $0 \leq a_\rho \leq 1$ for all ρ and ℓE is integral for some integer $\ell > 0$. Prove that there is an injection

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \hookrightarrow H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(D + \ell(D+E))) \text{ for all } p \geq 0.$$

This is a strengthened version of Theorem 0.1 from [**212**]. Hint: As suggested by [**101**], apply Lemma 9.3.4 to the $\mathbb{Q}$-Weil divisor $D + \frac{\ell}{\ell+1}E$.

§9.4. Applications to Lattice Polytopes

In this section we use vanishing theorems to study lattice polytopes.

The Euler Characteristic of a Sheaf. Let $\mathscr{F}$ be a coherent sheaf on a complete variety X. Its *Euler characteristic* $\chi(\mathscr{F})$ is defined to be the alternating sum

$$\chi(\mathscr{F}) = \sum_{p \geq 0} (-1)^p \dim H^p(X, \mathscr{F}).$$

Our hypotheses on X and $\mathscr{F}$ guarantee that $\dim H^p(X, \mathscr{F}) < \infty$ for all p, and $H^p(X, \mathscr{F}) = 0$ for $p > \dim X$ by [**131**, Thm. III.2.7]. Hence $\chi(\mathscr{F})$ is a well-defined integer. The Euler characteristic satisfies

$$\chi(\mathscr{G}) = \chi(\mathscr{F}) + \chi(\mathscr{H}) \tag{9.4.1}$$

whenever we have an exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ of coherent sheaves. This follows from the long exact sequence in cohomology (Exercise 9.4.1).

Given a line bundle $\mathscr{L}$ on X, define

$$\mathscr{L}^{\otimes \ell} = \begin{cases} \underbrace{\mathscr{L} \otimes_{\mathscr{O}_X} \cdots \otimes_{\mathscr{O}_X} \mathscr{L}}_{\ell \text{ times}} & \ell > 0 \\ (\mathscr{L}^\vee)^{\otimes(-\ell)} & \ell < 0 \\ \mathscr{O}_X & \ell = 0, \end{cases}$$

where as usual $\mathscr{L}^\vee = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$. A basic result [**176**] states that

$$\chi(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes \ell}), \quad \ell \in \mathbb{Z},$$

is a polynomial in ℓ, called the *Hilbert polynomial*. Exercises 9.4.2 and 9.4.3 sketch a proof in the ample case. When $\mathscr{L} = \mathscr{O}(D)$ for a Cartier divisor D, the Hilbert polynomial is written

$$\chi(\mathscr{F}(\ell D)), \quad \ell \in \mathbb{Z},$$

via the twisting convention introduced in §9.2.

Example 9.4.1. We will compute $\chi(\mathscr{O}_{\mathbb{P}^n}(\ell))$. When $\ell \geq 0$, Example 9.2.4 implies that $H^p(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\ell)) = 0$ for $p > 0$, so that

$$\chi(\mathscr{O}_{\mathbb{P}^n}(\ell)) = \dim H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\ell)) = |\ell\Delta_n \cap \mathbb{Z}^n| = \tbinom{\ell+n}{n}. \tag{9.4.2}$$

The second equality follows from Proposition 4.3.3 since the polytope associated to ℓD_0 is $\ell\Delta_n$, where Δ_n is the standard n-simplex from Example 4.3.6. You will prove the last equality in Exercise 9.4.4. When $\ell < 0$, Example 9.2.8 implies

$$\begin{aligned} \chi(\mathscr{O}_{\mathbb{P}^n}(\ell)) = (-1)^n \dim H^n(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\ell)) &= (-1)^n |\mathrm{Int}(\ell\Delta_n) \cap \mathbb{Z}^n| \\ &= (-1)^n \tbinom{-\ell-1}{n}, \end{aligned} \tag{9.4.3}$$

where the last equality uses Exercise 9.4.4.

Now consider the polynomial

$$p(x) = \frac{(x+n)(x+n-1)\cdots(x+1)}{n!} \in \mathbb{Q}[x]$$

and observe that $p(\ell) = \binom{\ell+n}{n}$ when $\ell \in \mathbb{N}$.

We claim that $p(\ell) = \chi(\mathscr{O}_{\mathbb{P}^n}(\ell))$ for all $\ell \in \mathbb{Z}$. For $\ell \geq 0$ this follows easily from (9.4.2). When $\ell < 0$, note that

$$p(\ell) = (-1)^n \tbinom{-\ell-1}{n}$$

(Exercise 9.4.4). Then $p(\ell) = \chi(\mathscr{O}_{\mathbb{P}^n}(\ell))$ for $\ell < 0$ by (9.4.3).

Note also that when $\ell > 0$, (9.4.2) and (9.4.3) imply that

$$\begin{aligned} p(\ell) &= |\ell\Delta_n \cap \mathbb{Z}^n| \\ p(-\ell) &= (-1)^n |\mathrm{Int}(\ell\Delta_n) \cap \mathbb{Z}^n|. \end{aligned}$$

We will see below that this is a special case of *Ehrhart reciprocity*. ◊

The Ehrhart Polynomial. Given a full dimensional lattice polytope $P \subseteq M_{\mathbb{R}}$, the functions

$$\begin{aligned} L(P) &= |P \cap M| \\ L^*(P) &= |\mathrm{Int}(P) \cap M| \end{aligned}$$

count lattice points in P or in its interior. For example, if $\ell > 0$ is an integer, then (9.4.2) and (9.4.3) imply that $L(\ell\Delta_n) = \binom{\ell+n}{n}$ and $L^*(\ell\Delta_n) = \binom{\ell-1}{n}$.

Theorem 9.4.2 (Ehrhart Reciprocity)**.** *Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a full dimensional lattice polytope. There is a polynomial* $\mathrm{Ehr}_P(x) \in \mathbb{Q}[x]$ *such that if $\ell \in \mathbb{N}$, then*

$$\mathrm{Ehr}_P(\ell) = L(\ell P).$$

Furthermore, if $\ell \in \mathbb{N}$ is positive, then

$$\mathrm{Ehr}_P(-\ell) = (-1)^n L^*(\ell P).$$

Proof. Let X_P be the toric variety of P and D_P the associated ample divisor. We will show that the Hilbert polynomial

$$\mathrm{Ehr}_P(\ell) = \chi(\mathscr{O}_{X_P}(\ell D_P))$$

has the desired properties. The case when $\ell \geq 0$ is easy since ℓD_P is basepoint free by Proposition 6.1.10. Thus

$$\chi(\mathscr{O}_{X_P}(\ell D_P)) = \dim H^0(X_P, \mathscr{O}_{X_P}(\ell D_P)) = |\ell P \cap M| = L(\ell P)$$

by Demazure vanishing (Theorem 9.2.3) and Example 4.3.7.

Now assume $\ell > 0$. Then $P_{\ell D_P} = \ell P$ is full dimensional, hence

$$\begin{aligned}\chi(\mathscr{O}_{X_P}(-\ell D_P)) = (-1)^n \dim H^n(X_P, \mathscr{O}_{X_P}(-\ell D_P)) &= (-1)^n |\mathrm{Int}(\ell P) \cap M| \\ &= (-1)^n L^*(\ell P)\end{aligned}$$

by Batyrev-Borisov vanishing (Theorem 9.2.7). □

The polynomial Ehr_P in Theorem 9.4.2 is called the *Ehrhart polynomial* of P. An elementary approach to Ehrhart polynomials and Ehrhart Reciprocity can be found in [**22**].

When P is very ample, the Ehrhart polynomial has a nice interpretation. The very ample divisor D_P on X_P gives a projective embedding $i : X_P \hookrightarrow \mathbb{P}^{s-1}$, where $s = |P \cap M|$. Its homogeneous ideal $\mathbf{I}(X_P) \subseteq \mathbb{C}[x_1, \ldots, x_s]$ gives the homogeneous coordinate ring

$$\mathbb{C}[X_P] = \mathbb{C}[x_1, \ldots, x_s]/\mathbf{I}(X_P).$$

This graded ring has a Hilbert function

$$\ell \longmapsto \dim \mathbb{C}[X_P]_\ell, \quad \ell \geq 0,$$

which is a polynomial for $\ell \gg 0$ (see [**70**, Ch. 6, §4]). This is the *Hilbert polynomial* of $X_P \subseteq \mathbb{P}^{s-1}$,

Proposition 9.4.3. *If P is very ample, then the Ehrhart polynomial* Ehr_P *equals the Hilbert polynomial of the toric variety X_P under the projective embedding given by the very ample divisor D_P.*

Proof. Consider the exact sequence

$$0 \longrightarrow \mathscr{I}_{X_P} \longrightarrow \mathscr{O}_{\mathbb{P}^{s-1}} \longrightarrow \mathscr{O}_{X_P} \longrightarrow 0.$$

Since $\mathscr{O}_{X_P}(D_P)$ is the restriction of $\mathscr{O}_{\mathbb{P}^{s-1}}(1)$ to X_P, tensoring with $\mathscr{O}_{\mathbb{P}^{s-1}}(\ell)$ gives an exact sequence

$$0 \longrightarrow \mathscr{I}_{X_P}(\ell) \longrightarrow \mathscr{O}_{\mathbb{P}^{s-1}}(\ell) \longrightarrow \mathscr{O}_{X_P}(\ell D_P) \longrightarrow 0.$$

In Exercise 9.4.5 you will show that the resulting long exact sequence is

$$0 \to \mathbf{I}(X_P)_\ell \to \mathbb{C}[x_1,\dots,x_s]_\ell \to H^0(X_P,\mathscr{O}_{X_P}(\ell D_P)) \to H^1(\mathbb{P}^{s-1},\mathscr{I}_{X_P}(\ell)) \to \cdots .$$

The H^1 term vanishes for $\ell \gg 0$ by Serre vanishing (Theorem 9.0.6). Hence for ℓ large, we get an isomorphism

$$\mathbb{C}[X_P]_\ell \simeq H^0(X_P,\mathscr{O}_{X_P}(\ell D_P)).$$

This implies that the Hilbert polynomial of X_P is the Ehrhart polynomial of P. $\square$

We can describe the degree and leading coefficient of the Ehrhart polynomial. For the leading coefficient, we use the *normalized volume* in $M_\mathbb{R}$. Let $e_1,\dots,e_n$ be a basis of M and consider the simplex $\Delta_n = \mathrm{Conv}(0,e_1,\dots,e_n) \subseteq M_\mathbb{R}$. Then the normalized volume is the usual n-dimensional Lebesgue measure, scaled so that Δ_n has volume equal to 1. Thus the "unit cube" $\{\sum_{i=1}^n \lambda_i e_i \mid 0 \le \lambda_i \le 1\}$ has normalized volume $n!$.

Given a full dimensional lattice polytope $P \subseteq M_\mathbb{R}$, its normalized volume is denoted $\mathrm{Vol}(P)$ and is computed by the limit

$$\frac{\mathrm{Vol}(P)}{n!} = \lim_{\ell\to\infty} \frac{L(\ell P)}{\ell^n}. \tag{9.4.4}$$

This is proved in many places, including [**22**, Lem. 3.19]. Since $L(\ell P)$ is given by the polynomial $\mathrm{Ehr}_P(\ell)$, it follows easily that Ehr_P has degree n and its leading coefficient $\mathrm{Vol}(P)/n!$ (Exercise 9.4.7).

Here is a classic application of these ideas.

Example 9.4.4. Let $P \subset \mathbb{R}^2$ be a lattice polygon with Ehrhart polynomial Ehr_P. The leading coefficient of Ehr_P is $\frac{1}{2}\mathrm{Vol}(P)$, which is the usual Euclidean area $\mathrm{Area}(P)$ since the normalized volume of the unit square is 2. The constant term is also easy to compute, since $\mathrm{Ehr}_P(0) = L(0\cdot P) = 1$ by Theorem 9.4.2. Thus

$$\mathrm{Ehr}_P(x) = \mathrm{Area}(P)\,x^2 + \tfrac{1}{2}Bx + 1,$$

where B is yet to be determined. The reason for the $\frac{1}{2}$ will soon become clear.

By Ehrhart reciprocity, we have

$$\begin{aligned} &\mathrm{Area}(P) + \tfrac{1}{2}B + 1 = \mathrm{Ehr}_P(1) = L(P)\\ &\mathrm{Area}(P) - \tfrac{1}{2}B + 1 = \mathrm{Ehr}_P(-1) = (-1)^2L^*(P) = L^*(P). \end{aligned} \tag{9.4.5}$$

Solving for B gives $B = L(P) - L^*(P) = |\partial P \cap M|$, where ∂P is the boundary of P. Thus the Ehrhart polynomial of P is

$$\mathrm{Ehr}_P(x) = \mathrm{Area}(P)\,x^2 + \tfrac{1}{2}|\partial P \cap M|\,x + 1.$$

Furthermore, solving the bottom equation of (9.4.5) for the area gives

$$\mathrm{Area}(P) = L^*(P) + \tfrac{1}{2}|\partial P \cap M| - 1.$$

This equation, called *Pick's formula*, shows how to compute the area of a lattice polygon in terms of its lattice points. $\Diamond$

Example 9.4.5. Let $P = \mathrm{Conv}\{(0,0,0),(1,0,0),(0,1,0),(1,1,3)\}$ be the lattice simplex shown in Figure 3 of Example 2.2.4. Since the normalized volume is 3 and $\mathrm{Ehr}_P(0) = L(0\cdot P) = 1$, the Ehrhart polynomial is $\mathrm{Ehr}_P(x) = \frac{1}{2}x^3 + ax^2 + bx + 1$. We noted in Example 2.2.8 that the only lattice points of P are its vertices. By Theorem 9.4.2, we obtain

$$\begin{aligned} \mathrm{Ehr}_P(1) &= \tfrac{1}{2} + a + b + 1 = L(P) = 4 && \text{(4 lattice points)}\\ \mathrm{Ehr}_P(-1) &= -\tfrac{1}{2} + a - b + 1 = (-1)^3 L^*(P) = 0 && \text{(no interior lattice points)} \end{aligned}$$

Hence $a = 1$, $b = \frac{3}{2}$, so that $\mathrm{Ehr}_P(x) = \frac{1}{2}x^3 + x^2 + \frac{3}{2}x + 1$. In Example B.3.3 we will explain how to compute $\mathrm{Ehr}_P(x)$ using `Normaliz` [**57**]. $\Diamond$

Examples 9.4.4 and 9.4.5 used $\mathrm{Ehr}_P(0) = L(0\cdot P) = 1$. The latter equality is obvious since $0\cdot P = \{0\}$. However, the equality $\mathrm{Ehr}_P(0) = L(0\cdot P)$ (proved in Theorem 9.4.2) is more subtle since it can fail when we replace the polytope P with a *polytopal complex* (a collection of polytopes where the intersection of any two is a face of each). Here is a simple example.

Example 9.4.6. Consider the boundary $\partial\Delta_2$ of the 2-simplex $\Delta_2 \subseteq \mathbb{R}^2$. Thus $\partial\Delta_2$ consists of three line segments. One easily computes that

$$L(\ell\,\partial\Delta_2) = |(\ell\,\partial\Delta_2) \cap \mathbb{Z}^2| = 3\ell$$

when $\ell > 0$ is an integer. This gives the "Ehrhart polynomial" $\mathrm{Ehr}_{\partial\Delta_2}(x) = 3x$ with the property that $\mathrm{Ehr}_{\partial\Delta_2}(\ell) = L(\ell\,\partial\Delta_2)$ for integers $\ell > 0$. However,

$$\mathrm{Ehr}_{\partial\Delta_2}(0) = 3\cdot 0 = 0 \neq L(0\cdot\partial\Delta_2) = L(\{0\}) = 1.$$

Further examples (and an explanation) will be given in Execise 9.4.8. $\Diamond$

The p-Ehrhart Polynomials. Following Materov [**193**], we use the sheaves $\widehat{\Omega}^p_{X_P}$ to generalize the Ehrhart polynomial when the lattice polytope P is simple. Recall that a full dimensional lattice polytope $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ is *simple* if every vertex is the intersection of exactly n facets. Hence each vertex has n facet normals that generate the corresponding cone in the normal fan Σ_P. It follows that Σ_P is simplicial whenever P is simple. Hence the toric variety X_P is simplicial.

We proved earlier that the Ehrhart polynomial $\mathrm{Ehr}_P(x)$ of P satisfies

$$\mathrm{Ehr}_P(\ell) = \chi(\mathscr{O}_{X_P}(\ell D_P)),\ \ell \in \mathbb{Z},$$

where D_P is the ample divisor coming from P. More generally, given an integer $0 \leq p \leq n$, the Euler characteristic $\chi(\widehat{\Omega}^p_{X_P}(\ell D_P))$ is a polynomial function of ℓ. Then the *p-Ehrhart polynomial of P*, denoted $\mathrm{Ehr}^p_P(x)$, is the unique polynomial in $\mathbb{Q}[x]$ that satisfies

$$\mathrm{Ehr}^p_P(\ell) = \chi(\widehat{\Omega}^p_{X_P}(\ell D_P)),\ \ell \in \mathbb{Z}.$$

Note that $\mathrm{Ehr}^0_P(x)$ is the ordinary Ehrhart polynomial $\mathrm{Ehr}_P(x)$. To state the properties of $\mathrm{Ehr}^p_P(x)$, we will use the following notation:

$$\begin{aligned} P(i) &= \{Q \mid Q \text{ is an } i\text{-dimensional face of } P\} \\ f_i &= |P(i)| = \#\, i\text{-dimensional faces of } P \\ h_p &= \sum_{i=p}^{n} (-1)^{i-p} \binom{i}{p} f_i. \end{aligned}$$

The f_i are the *face numbers* of P. Furthermore, if Q is a face of P, we define

$$\begin{aligned} L(Q) &= |Q \cap M| \\ L^*(Q) &= |\mathrm{Relint}(Q) \cap M|. \end{aligned}$$

These invariants are related to the p-Ehrhart polynomials by the following result of Materov [**193**].

Theorem 9.4.7. *Let X_P be the toric variety of a full dimensional simple lattice polytope $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$.*

(a) *If $\ell > 0$ is an integer, then*

$$\mathrm{Ehr}^p_P(\ell) = \sum_{i=p}^{n} \binom{i}{p} \sum_{Q \in P(i)} L^*(\ell Q).$$

(b) *If $\ell \geq 0$ is an integer, then*

$$\mathrm{Ehr}^p_P(\ell) = \sum_{i=0}^{p} (-1)^i \binom{n-i}{n-p} \sum_{Q \in P(n-i)} L(\ell Q).$$

(c) *If $0 \leq p \leq n$, then*

$$\mathrm{Ehr}^p_P(-x) = (-1)^n \,\mathrm{Ehr}^{n-p}_P(x).$$

(d) *If $0 \leq p \leq n$, then*

$$\mathrm{Ehr}^p_P(0) = (-1)^p h_p.$$

We will defer the proof for now. Theorem 9.4.7 has some nice consequences. First, setting $x = 0$ in part (c) and using part (d), we see that

$$(-1)^p h_p = (-1)^n (-1)^{n-p} h_{n-p}.$$

This proves that

$$h_p = h_{n-p} \tag{9.4.6}$$

for $0 \le p \le n$. These are called the *Dehn-Sommerville equations*.

Also, if we write $\mathrm{Ehr}_P^p(\ell)$ using part (b) of the theorem and $\mathrm{Ehr}_P^{n-p}(\ell)$ using part (a), then part (c) gives the following formulas for $\ell > 0$:

$$\begin{aligned}
\mathrm{Ehr}_P^p(\ell) &= \sum_{i=0}^{p} (-1)^i \binom{n-i}{n-p} \sum_{Q \in P(n-i)} L(\ell Q),\\
\mathrm{Ehr}_P^p(-\ell) = (-1)^n \,\mathrm{Ehr}_P^{n-p}(\ell) &= (-1)^n \sum_{i=n-p}^{n} \binom{i}{n-p} \sum_{Q \in P(i)} L^*(\ell Q).
\end{aligned}$$

For $p = 0$ and $\ell > 0$, these equations reduce to Ehrhart reciprocity:

$$\begin{aligned}
\mathrm{Ehr}_P^0(\ell) &= L(\ell P)\\
\mathrm{Ehr}_P^0(-\ell) &= (-1)^n L^*(\ell P).
\end{aligned}$$

Thus Theorem 9.4.7 simultaneously generalizes the Dehn-Sommerville equations and Ehrhart reciprocity. The proof of the theorem will use the following lemma.

Lemma 9.4.8. *Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a full dimensional lattice polytope with toric variety X_P and ample divisor D_P. Given an integer $\ell > 0$ and $m \in (\ell P) \cap M$, let $V_{\ell P}(m)$ be the subspace of $M_{\mathbb{C}}$ such that $V_{\ell P}(m) + m$ is the smallest affine subspace of $M_{\mathbb{C}}$ containing the minimal face of ℓP containing m. Then*

$$H^0(X_P, \widehat{\Omega}^p_{X_P}(\ell D_P)) = \bigoplus_{m \in (\ell P) \cap M} \textstyle\bigwedge^p V_{\ell P}(m) \cdot \chi^m.$$

Proof. We adapt the strategy used in the proof of Proposition 8.2.18. To simplify notation, set $D = \ell D_P$. Using $\rho^\perp \cap M \subseteq M$ and tensoring (8.2.6) with $\mathscr{O}_{X_P}(D)$, we obtain the exact sequence

$$0 \longrightarrow \widehat{\Omega}^p_{X_P}(D) \longrightarrow \textstyle\bigwedge^p M \otimes_{\mathbb{Z}} \mathscr{O}_{X_P}(D) \longrightarrow \bigoplus_\rho \bigwedge^{p-1} M \otimes_{\mathbb{Z}} \mathscr{O}_{D_\rho}(D).$$

Take global sections over X_P and consider the graded piece for $m \in M$. Since $H^0(X_P, \mathscr{O}_{X_P}(D)) = \bigoplus_{m \in (\ell P) \cap M} \mathbb{C} \cdot \chi^m$, we get the exact sequence

$$0 \longrightarrow H^0(X_P, \widehat{\Omega}^p_{X_P}(D))_m \longrightarrow \textstyle\bigwedge^p M_{\mathbb{C}} \longrightarrow \bigoplus_\rho \bigwedge^{p-1} M \otimes_{\mathbb{Z}} H^0(X_P, \mathscr{O}_{D_\rho}(D))_m$$

when $m \in (\ell P) \cap M$. To compute $H^0(X_P, \mathscr{O}_{D_\rho}(D))_m$, recall from Proposition 4.0.28 that we have an exact sequence

$$0 \longrightarrow \mathscr{O}_{X_P}(-D_\rho) \longrightarrow \mathscr{O}_{X_P} \longrightarrow \mathscr{O}_{D_\rho} \longrightarrow 0.$$

Tensor this with $\mathscr{O}_{X_P}(D)$ and take global sections to obtain

$$0 \longrightarrow H^0(X_P, \mathscr{O}_{X_P}(D-D_\rho)) \longrightarrow H^0(X_P, \mathscr{O}_{X_P}(D)) \longrightarrow H^0(X_P, \mathscr{O}_{D_\rho}(D)) \longrightarrow 0,$$

where the exactness on right follows from $H^1(X_P, \mathscr{O}_{X_P}(D-D_\rho)) = 0$ courtesy of Theorem 9.3.7. Comparing the polytopes of $D-D_\rho$ and $D = \sum_\rho a_\rho D_\rho$ shows that

$$H^0(X_P, \mathscr{O}_{D_\rho}(D))_m = \begin{cases} \mathbb{C} & \langle m, u_\rho\rangle = -\ell a_\rho \\ 0 & \text{otherwise}. \end{cases}$$

Let F_ρ be the facet of ℓP corresponding to ρ. For $m \in \ell P$, $\langle m, u_\rho\rangle = -\ell a_\rho$ if and only if $m \in F_\rho$. Hence we get an exact sequence

$$0 \longrightarrow H^0(X_P, \widehat{\Omega}^p_{X_P}(D))_m \longrightarrow \textstyle\bigwedge^p M_{\mathbb{C}} \xrightarrow{\beta_p} \bigoplus_{m\in F_\rho} \bigwedge^{p-1} M_{\mathbb{C}},$$

where β_p is a sum of contraction maps i_{u_ρ} defined in (8.2.7).

Thus $\omega \in \bigwedge^p M_{\mathbb{C}}$ is in the kernel of β_p if and only if $i_{u_\rho}(\omega) = 0$ for all ρ with $m \in F_\rho$. We know from Exercise 8.2.8 that $i_{u_\rho}(\omega) = 0$ if and only if $\omega \in \bigwedge^p(\rho^\perp)_{\mathbb{C}}$. It follows that the kernel of β_p in (8.2.9) is the intersection

$$\bigcap_{m\in F_\rho} \textstyle\bigwedge^p(\rho^\perp)_{\mathbb{C}} = \bigwedge^p\Big(\bigcap_{m\in F_\rho}(\rho^\perp)_{\mathbb{C}}\Big). \tag{9.4.7}$$

Since $F = \bigcap_{m\in F_\rho} F_\rho$ is the minimal face of ℓP containing m, one sees easily that

$$\mathrm{Span}_{\mathbb{R}}(m_0 - m \in M \mid m_0 \in F) = \textstyle\bigcap_{m\in F_\rho} \rho^\perp.$$

Thus $V_{\ell P}(m) = \bigcap_{m\in F_\rho}(\rho^\perp)_{\mathbb{C}}$. This and (9.4.7) imply $\ker(\beta_p) = \bigwedge^p V_{\ell P}(m)$. $\square$

We are now ready to prove our main result.

Proof of Theorem 9.4.7. We begin with part (a). Lemma 9.4.8 implies that

$$\dim H^0(X_P, \widehat{\Omega}^p_{X_P}(\ell D_P)) = \sum_{m\in(\ell P)\cap M} \binom{\dim V_{\ell P}(m)}{p}$$

when $\ell > 0$. Given a face Q of P and $m \in (\ell P)\cap M$, note that

$$m \in \mathrm{Relint}(\ell Q)\cap M \iff Q \text{ is the minimal face of } P \text{ containing } m.$$

When this happens, we have $\dim Q = \dim V_{\ell P}(m)$. Since the i-dimensional faces of ℓP are ℓQ for $Q \in P(i)$, we obtain

$$\dim H^0(X_P, \widehat{\Omega}^p_{X_P}(\ell D_P)) = \sum_{i\ge 0}\binom{i}{p}\sum_{Q\in P(i)} L^*(\ell Q) = \sum_{i=p}^n \binom{i}{p}\sum_{Q\in P(i)} L^*(\ell Q).$$

We also have $H^q(X_P, \widehat{\Omega}^p_{X_P}(\ell D_P)) = 0$ for $q > 0$ by Theorem 9.3.1. Hence

$$\mathrm{Ehr}^p_P(\ell) = \chi(\widehat{\Omega}^p_{X_P}(\ell D_P)) = \dim H^0(X_P, \widehat{\Omega}^p_{X_P}(\ell D_P)) = \sum_{i=p}^n \binom{i}{p}\sum_{Q\in P(i)} L^*(\ell Q),$$

which proves part (a).

Now consider $\operatorname{Ehr}_P^p(0)$. If $\ell > 0$ and $Q \in P(i)$, then $L^*(\ell Q) = (-1)^i \operatorname{Ehr}_Q(-\ell)$ by Ehrhart reciprocity. Since the previous display holds for all $\ell > 0$, we obtain the polynomial identity

$$\operatorname{Ehr}_P^p(x) = \sum_{i=p}^{n} \binom{i}{p} \sum_{Q\in P(i)} (-1)^i \operatorname{Ehr}_Q(-x). \tag{9.4.8}$$

Setting $x = 0$ gives

$$\operatorname{Ehr}_P^p(0) = \sum_{i=p}^{n} (-1)^i \binom{i}{p} \sum_{Q\in P(i)} \operatorname{Ehr}_Q(0) = (-1)^p h_p$$

by the definition of h_p, and part (d) follows.

For part (c), we use Serre duality as given in Theorem 9.2.10, which implies

$$H^q(X_\Sigma, \widehat{\Omega}^p_{X_P}(\ell D_P))^\vee \simeq H^{n-q}(X_\Sigma, \widehat{\Omega}^{n-p}_{X_P}(-\ell D_P)) \tag{9.4.9}$$

since ℓD_P is Cartier for any $\ell \in \mathbb{Z}$. This easily implies

$$\chi(\widehat{\Omega}^p_{X_P}(\ell D_P)) = (-1)^n \chi(\widehat{\Omega}^{n-p}_{X_P}(-\ell D_P))$$

(Exercise 9.4.9). Thus $\operatorname{Ehr}_P^p(\ell) = (-1)^n \operatorname{Ehr}_P^{n-p}(-\ell)$ for $\ell \in \mathbb{Z}$, proving part (d).

Finally, for part (b), take $\ell \geq 0$ and consider

$$\operatorname{Ehr}_P^p(\ell) = (-1)^n \operatorname{Ehr}_P^{n-p}(-\ell) = (-1)^n \sum_{i=n-p}^{n} \binom{i}{n-p} \sum_{Q\in P(i)} (-1)^i \operatorname{Ehr}_Q(\ell),$$

where the first equality uses part (d) and the second uses (9.4.8) with p replaced by $n-p$. Since $\ell \geq 0$ implies $\operatorname{Ehr}_Q(\ell Q) = L(\ell Q)$, we obtain

$$\operatorname{Ehr}_P^p(\ell) = \sum_{i=n-p}^{n} (-1)^{n-i} \binom{i}{n-p} \sum_{Q\in P(i)} L(\ell Q) = \sum_{j=0}^{p} (-1)^j \binom{n-j}{n-p} \sum_{Q\in P(i)} L(\ell Q),$$

where the last equality follows by setting $j = n-i$. This completes the proof. □

Identities equivalent to Theorem 9.4.7 were discovered by McMullen in 1977 in a very different context. See [**22**, Ch. 5] for details and references, including a non-toric version of Theorem 9.4.7 in [**22**, Ex. 5.8–5.10].

Examples. We first give a classic example of the Dehn-Sommerville equations.

Example 9.4.9. Let P be a simple 3-dimensional lattice polytope in $\mathbb{R}^3$. The Dehn-Sommerville equations can be written

$$\begin{aligned} h_3 = h_0, &\text{ so } f_3 = f_0 - f_1 + f_2 - f_3 \\ h_2 = h_1, &\text{ so } f_2 - 3f_3 = f_1 - 2f_2 + 3f_3. \end{aligned}$$

Since $f_3 = 1$ (P is the only 3-dimensional face of itself), we obtain

$$\begin{aligned} f_0 - f_1 + f_2 &= 2 \\ f_1 &= 3f_2 - 6. \end{aligned}$$

The first equation Euler's celebrated formula $V - E + F = 2$, which holds for any 3-dimensional polytope. The second seems more mysterious, but when combined with the first reduces to $2f_1 = 3f_0$. This holds because is P is simple—every vertex meets three edges and every edge meets two vertices. $\Diamond$

There is a version of the Dehn-Sommerville equations for the dual polytope $P^\circ \subseteq N_{\mathbb{R}}$, which is simplicial since P is simple (Exercise 9.4.10). More on the Dehn-Sommerville equations and toric varieties can be found in [**105**, Sec. 5.6]. We also recommend [**22**, Ch. 5] and [**281**, Sec. 8.3].

We next give an application of Theorem 9.4.7.

Example 9.4.10. The standard n-simplex Δ_n has $\mathbb{P}^n$ as its associated toric variety. Given $\ell > 0$, note that

$$\sum_{Q \in \Delta_n(i)} L^*(\ell\Delta_n) = \binom{n+1}{n-i}\binom{\ell-1}{i}$$

because

- An n-simplex has $\binom{n+1}{i+1} = \binom{n+1}{n-i}$ faces of dimension i.
- Each $Q \in \Delta_n(i)$ is lattice isomorphic to the standard i-simplex and hence has $\binom{\ell-1}{i}$ interior lattice points by (9.4.3).

Then part (a) of Theorem 9.4.7 gives the formula

$$\mathrm{Ehr}^p_{\Delta_n}(\ell) = \sum_{i=p}^{n} \binom{i}{p}\binom{n+1}{n-i}\binom{\ell-1}{i}.$$

Since $\binom{i}{p}\binom{\ell-1}{i} = \binom{\ell-1}{p}\binom{\ell-p-1}{i-p}$, this becomes

$$\mathrm{Ehr}^p_{\Delta_n}(\ell) = \binom{\ell-1}{p}\sum_{i\geq 0}\binom{n+1}{n-i}\binom{\ell-p-1}{i-p} = \binom{\ell-1}{p}\binom{n+\ell-p}{\ell},$$

where the last equality uses the Vandermonde identity discussed in Exercise 9.4.11. Hence

$$\dim H^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(\ell)) = \binom{\ell-1}{p}\binom{n+\ell-p}{\ell}.$$

This formula was first proved by Bott [**40**] in 1957. $\Diamond$

Cohomology of p-Forms. Given a simple polytope P as above, the sheaf $\widehat{\Omega}^p_{X_P}$ has Euler characteristic

$$\chi(\widehat{\Omega}^p_{X_P}) = \mathrm{Ehr}^p_P(0) = (-1)^p h_p \tag{9.4.10}$$

by Theorem 9.4.7. The factor of $(-1)^p$ is explained as follows.

Theorem 9.4.11. *Let X_P be the toric variety of a full dimensional simple lattice polytope in $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$. Then*

$$\dim\, H^q(X_P, \widehat{\Omega}^p_{X_P}) = \begin{cases} h_p & q = p \\ 0 & q \neq p. \end{cases}$$

Proof. The toric variety X_P is simplicial, so that Theorem 9.3.2 applies to X_P. This shows that $H^q(X_P, \widehat{\Omega}^p_{X_P}) = 0$ for $q \neq p$. It follows that

$$\chi(\widehat{\Omega}^p_{X_P}) = (-1)^p \dim\, H^p(X_P, \widehat{\Omega}^p_{X_P}).$$

The theorem follows by comparing this with (9.4.10). □

A different proof of $\dim\, H^p(X_P, \widehat{\Omega}^p_{X_P}) = h_p$ will be sketched in Exercise 9.4.12. Theorem 9.4.11 has a nice application to the singular cohomology of X_P because of the the *Hodge decomposition*

$$H^k(X_P, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(X_P, \widehat{\Omega}^p_{X_P}). \tag{9.4.11}$$

When X_P is smooth, this is a classical fact—see, for example, [**125**, p. 116]. In the simplicial case, see [**261**] for a proof. Combined with Theorem 9.4.11, we obtain

$$\dim H^k(X_P, \mathbb{C}) = \begin{cases} h_p & k = 2p \\ 0 & k \text{ odd}. \end{cases} \tag{9.4.12}$$

We will give a topological proof of this formula in §12.4.

Exercises for §9.4.

9.4.1. Prove (9.4.1).

9.4.2. Here are some cases where it is easy to show that Euler characteristics give Hilbert polynomials.

(a) Let F be a finitely generated graded module over the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_n]$. By [**70**, Ch. 6, Thm. (3.8)], there is an exact sequence

$$0 \longrightarrow F_r \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F \longrightarrow 0,$$

where each F_i is a finite direct sum of modules for the form $S(a)$. Now let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n$. Prove that there is an exact sequence of sheaves

$$0 \longrightarrow \mathscr{F}_r \longrightarrow \cdots \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_0 \longrightarrow \mathscr{F} \longrightarrow 0,$$

where each $\mathscr{F}_i$ is a finite direct sum of sheaves of the form $\mathscr{O}_{\mathbb{P}^n}(a)$. Hint: Apply Example 6.0.10 with $X_\Sigma = \mathbb{P}^n$.

(b) Use part (a) together with (9.4.1) and Example 9.4.1 to show that $\chi(\mathscr{F}(\ell))$ is a polynomial in $\ell \in \mathbb{Z}$, where $\mathscr{F}(\ell) = \mathscr{F} \otimes_{\mathscr{O}_{\mathbb{P}^n}} \mathscr{O}_{\mathbb{P}^n}(\ell)$.

(c) Let $\mathscr{F}$ be a coherent sheaf on a complete variety X and let D be a very ample divisor on X. Use part (b) and Exercise 9.0.6 to show that $\chi(\mathscr{F}(\ell D))$ is a polynomial in $\ell \in \mathbb{Z}$.

9.4.3. Let $\mathscr{F}$ be a coherent sheaf on a projective variety X, and let D be an ample divisor on X. Thus there is $k_0 > 0$ such that kD is very ample for all $k \geq k_0$.

(a) Let a, k be integers with $k \geq k_0$. Use Exercise 9.4.2 to show that there is a polynomial $p_{a,k}(x) \in \mathbb{Q}[x]$ such that $p_{a,k}(\ell) = \chi(\mathscr{F}((a+\ell k)D))$ for all $\ell \in \mathbb{Z}$.

(b) Show that the polynomials $p_{a,k}(x/k)$ and $p_{a,m}(x/m)$ are equal when $k, m \geq k_0$. Conclude that there is a polynomial $p_a(x)$ such that $p_a(\ell) = \chi(\mathscr{F}((a+\ell)D))$ for all $\ell \geq k_0$.

(c) Show that the polynomials $p_a(x-a)$ and $p_b(x-b)$ are equal for any $a, b \in \mathbb{Z}$ and conclude that there is a polynomial $p(x) \in \mathbb{Q}[x]$ such that $p(\ell) = \chi(\mathscr{F}(\ell D))$ for all $\ell \in \mathbb{Z}$.

9.4.4. In this exercise ℓ will always denote an integer.

(a) Prove that $|(\ell\Delta_n) \cap \mathbb{Z}^n| = \binom{\ell+n}{n}$ for ℓ nonnegative. Hint: Lattice points in $\ell\Delta_n$ give monomials of degree ℓ in $x_0, \ldots, x_n$ by Exercise 4.3.6. Here is a combinatorial proof. Begin with a list of 1's of length $\ell+n$. In n of the positions, convert the 1 to a 0. This divides the remaining 1's into $n+1$ groups. The number of elements in each group gives the exponents of $x_0, \ldots, x_n$. See also [**69**, Ex. 13 of Ch. 9, §3].

(b) Prove that $|\mathrm{Int}(\ell\Delta_n) \cap \mathbb{Z}^n| = \binom{\ell-1}{n}$ for ℓ positive. Hint: Show that shifting the interior lattice points by $(1, \ldots, 1)$ gives the lattice points in $(\ell-n-1)\Delta_n$.

(c) Prove that $p(x) = (x+n)(x+n-1)\cdots(x+1)/n!$ satisfies $p(\ell) = (-1)^n\binom{-\ell-1}{n}$ for ℓ negative.

9.4.5. A projective variety $X \subseteq \mathbb{P}^n$ gives $0 \to \mathscr{I}_X \to \mathscr{O}_{\mathbb{P}^n} \to \mathscr{O}_X \to 0$. After tensoring with $\mathscr{O}_{\mathbb{P}^n}(\ell)$, we have an exact sequence $0 \to \mathscr{I}_X(\ell) \to \mathscr{O}_{\mathbb{P}^n}(\ell) \to \mathscr{O}_X(\ell) \to 0$ whose long exact sequence in cohomology begins

$$0 \to H^0(\mathbb{P}^n, \mathscr{I}_X(\ell)) \to H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\ell)) \to H^0(\mathbb{P}^n, \mathscr{O}_X(\ell)) \to H^1(\mathbb{P}^n, \mathscr{I}_X(\ell)) \to \cdots .$$

We know from Example 4.3.6 that $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\ell)) = S_\ell$, where $S = \mathbb{C}[x_0, \ldots, x_n]$ with the standard grading. Also recall that as a sheaf on $\mathbb{P}^n$, $\mathscr{O}_X(\ell)$ stands for $i_*\mathscr{O}_X(\ell)$, where $i : X \hookrightarrow \mathbb{P}^n$ is the inclusion map. It follows that $H^0(\mathbb{P}^n, \mathscr{O}_X(\ell)) = H^0(X, \mathscr{O}_X(\ell))$.

(a) Show that $H^0(\mathbb{P}^n, \mathscr{I}_X(\ell)) = \mathbf{I}(X)_\ell$, where $\mathbf{I}(X) \subseteq S$ is the ideal of X.

(b) Show that the Hilbert polynomial of the coordinate ring $\mathbb{C}[X] = S/\mathbf{I}(X)$ is the Euler characteristic $\chi(\mathscr{O}_X(\ell))$.

9.4.6. Recall that $X \subseteq \mathbb{P}^n$ is projectively normal if and only if its affine cone is normal. By [**131**, Ex. II.5.14], this is equivalent to saying that X is normal and $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\ell)) \to H^0(X, \mathscr{O}_X(\ell))$ is surjective for all $\ell \geq 0$. Combine this with the previous exercise to show that X is projectively normal if and only if X is normal and $H^1(\mathbb{P}^n, \mathscr{I}_X(\ell)) = 0$ for all $\ell \geq 0$. See Example B.1.2 for a computational example.

9.4.7. Let $p(x)$ be a polynomial such that $\lim_{\ell\to\infty} p(\ell)/\ell^n$ exists and is nonzero.

(a) Prove that $n = \deg(p(x))$ and that the above limit is the leading coefficient of $p(x)$.

(b) Use part (a) to prove the assertion made following (9.4.4) concerning the degree and leading coefficient of the Ehrhart polynomial.

9.4.8. Here we compute the "Ehrhart polynomials" of some simple polytopal complexes.

(a) As in Example 9.4.6, let $\partial\Delta_2$ be the boundary of the standard simplex $\Delta_2 \subseteq \mathbb{R}^2$. Prove that $|(\ell\cdot\partial\Delta_2)\cap\mathbb{Z}^2| = 3\ell$, so that $\mathrm{Ehr}_{\partial\Delta_2}(x) = 3x$, with constant term 0.

(b) Consider the "butterfly" $B = \mathrm{Conv}(0, e_1 \pm e_2) \cup \mathrm{Conv}(0, -e_1 \pm e_2)$ with boundary ∂B. Prove that $|(\ell\cdot\partial B)\cap\mathbb{Z}^2| = 8\ell - 1$, so that $\mathrm{Ehr}_{\partial B}(x) = 8x - 1$, with constant term -1.

(c) The *Euler characteristic* of a well-behaved topological space Z is defined to be $e(Z) = \sum_p (-1)^p \operatorname{rank} H^p(Z,\mathbb{Z})$. Show that $e(\partial\Delta_2) = 0$ and $e(\partial B) = -1$.

The answer to part (c) is not a coincidence—given any polytopal complex $\mathscr{P}$ whose vertices are lattice points, the constant term of its Ehrhart polynomial $\mathrm{Ehr}_{\mathscr{P}}$ is $e(\mathscr{P})$ (see [**189**]). Since a polytope P is contractible, this gives a different way of seeing that the constant term of Ehr_P is 1.

9.4.9. Use (9.4.9) to prove that $\chi(\widehat{\Omega}^p_{X_P}(\ell D_P)) = (-1)^n \chi(\widehat{\Omega}^{n-p}_{X_P}(-\ell D_P))$.

9.4.10. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a full dimensional simplicial lattice polytope and define

$$h_p^{\mathrm{simp}} = \sum_{i=p}^{n} (-1)^{i-p} \binom{i}{p} f_{n-i-1}.$$

Rescaling and translating P does not affect the face numbers f_i. So we may that assume the origin is an interior point of P. Let $P^\circ \subseteq N_{\mathbb{R}}$ be the dual polytope of P. Use Exercise 2.3.4 and Proposition 2.3.8 to prove the following:

(a) P° is simple.

(b) The face numbers f_i^P of P and $f_i^{P^\circ}$ of P° are related by $f_{n-i-1}^P = f_i^{P^\circ}$.

(c) $h_p^{\mathrm{simp}} = h_{n-p}^{\mathrm{simp}}$.

9.4.11. The goal of this exercise is to prove the identity used in Example 9.4.10.

(a) Prove the *Vandermonde identity*, which states that $\sum_{i+j=k}\binom{a}{i}\binom{b}{j} = \binom{a+b}{k}$ for $a, b \in \mathbb{N}$. Hint: $(1+x)^a(1+x)^b = (1+x)^{a+b}$.

(b) Use part (a) to show that $\sum_{i\geq 0}\binom{n+1}{n-i}\binom{\ell-p-1}{i-p} = \binom{n+\ell-p}{\ell}$, as claimed in Example 9.4.10.

9.4.12. For a full dimensional simple polytope $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$, Theorem 8.2.19 gives

$$0 \longrightarrow \widehat{\Omega}^p_{X_P} \longrightarrow K^0(\Sigma_P, p) \longrightarrow K^1(\Sigma_P, p) \longrightarrow \cdots \longrightarrow K^p(\Sigma_P, p) \longrightarrow 0,$$

where $K^i(\Sigma_P, p) = \bigoplus_{\sigma\in\Sigma_P(i)} \bigwedge^{p-i}(\sigma^\perp\cap M)\otimes_{\mathbb{Z}} \mathscr{O}_{V(\sigma)}$ and $V(\sigma)\subseteq X_P$ is the closure of the orbit corresponding to $\sigma\in\Sigma_P$. You will use this to compute $\dim H^p(X_P, \widehat{\Omega}^p_{X_P})$.

(a) Show that $\chi(\widehat{\Omega}^p_{X_P}) = \sum_{j=0}^p (-1)^j \chi(K^j(\Sigma_P, p))$.

(b) Use Theorem 9.2.3 to show that $\chi(K^j(\Sigma_P, p)) = \dim H^0(X_P, K^j(\Sigma_P, p))$.

(c) Use Proposition 2.3.8 to show that $\dim H^0(X_P, K^j(\Sigma_P, p)) = \binom{n-j}{n-p} f_{n-j}$. Hint: $\sigma^\perp\cap M$ has rank $n-j$ for $\sigma\in\Sigma_P(j)$.

(d) Use Theorem 9.3.2 to conclude that $\dim H^p(X_P, \widehat{\Omega}^p_{X_P}) = h_{n-p}$. Hint: Set $j = n-i$.

9.4.13. When a polytope P has rational but not integral coordinates, the counting function $L(\ell P)$ is almost a polynomial. Here you will study $P = \mathrm{Conv}(0, e_1, \frac{1}{2}e_2) \subseteq \mathbb{R}^2$.

(a) Given $\ell \in \mathbb{N}$, prove that

$$L(\ell P) = \begin{cases} \frac{1}{4}\ell^2 + \ell + 1 & \ell \text{ even} \\ \frac{1}{4}\ell^2 + \ell + \frac{3}{4} & \ell \text{ odd.} \end{cases}$$

This is an example of a *quasipolynomial*. See [**22**, Sec. 3.7] for a discussion of the Ehrhart quasipolynomial of a rational polytope.

(b) The weighted projective plane $\mathbb{P}(1,1,2)$ is given by the fan in $\mathbb{R}^2$ with minimal generators $u_0 = -e_1 - 2e_2, u_1 = e_1, u_2 = e_2$. Show that $X_{2P} = \mathbb{P}(1,1,2)$ with $D_{2P} = 2D_0$, where D_0 is the divisor corresponding to u_0. Also show that

$$\chi(\mathscr{O}_{\mathbb{P}(1,1,2)}(\ell D_0)) = L(\ell P).$$

The Euler characteristic is *not* a polynomial in ℓ. The reason is that D_0 is not Cartier.

9.4.14. Let $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^n$ be a full dimensional lattice poltyope, not necessarily simple. Let $\mathrm{Ehr}_P^p(x)$ be the unique polynomial satisfying $\mathrm{Ehr}_P^p(\ell) = \chi(\widehat{\Omega}^p_{X_P}(\ell D_P))$ for all $\ell \in \mathbb{Z}$.

(a) Prove that if $\ell > 0$ in an integer, then

$$\mathrm{Ehr}_P^p(\ell) = \sum_{i=p}^{n} \binom{i}{p} \sum_{Q \in P(i)} L^*(\ell Q).$$

Hint: Look at the hypotheses of Theorem 9.3.1 and Lemma 9.4.8.

(b) Prove that $\mathrm{Ehr}_P^0(-\ell) = (-1)^n \mathrm{Ehr}_P^n(\ell)$. Hint: Serre duality.

(c) Prove that $\mathrm{Ehr}_P^p(0) = (-1)^p h_p$. Hint: Follow the proof of Theorem 9.4.7.

This shows that some parts of Theorem 9.4.7 hold for arbitrary lattice polytopes. In the next exercise you will see that other parts can fail.

9.4.15. This exercise will show how things can go wrong for a non-simple polytope. Let $P = \mathrm{Conv}(\pm e_1, \pm e_2, e_3) \subseteq \mathbb{R}^3$. Note that P is not simple, so that X_P is not simplicial.

(a) Show that $h_1 \neq h_2$.

(b) Conclude that for this polytope, (9.4.9) cannot hold for all p, q, ℓ. So the version of Serre duality stated in part (b) of Theorem 9.2.10 can fail for non-simplicial toric varieties. Hint: The Dehn-Sommerville equations follow from Theorem 9.4.7.

9.4.16. Suppose that $P \subseteq \mathbb{R}^3$ is a lattice simplex whose only lattice points are its vertices, and let k be the normalized volume of P. Prove that $2P$ has $k-1$ interior lattice points. Hint: Adapt Exercise 9.4.5.

§9.5. Local Cohomology and the Total Coordinate Ring

In this section, we use local cohomology and Ext to compute the cohomology of a coherent sheaf on a toric variety X_Σ. Our treatment is based on [**91**].

We first review the basics of local cohomology.

Local Cohomology. Let R be a finitely generated $\mathbb{C}$-algebra, $I \subseteq R$ an ideal, and M an R-module. First define

$$\Gamma_I(\mathsf{M}) = \{a \in \mathsf{M} \mid I^k \cdot a = 0 \text{ for some } k \in \mathbb{N}\}.$$

This is the *I-torsion submodule* of M. One checks easily that $\mathsf{M} \mapsto \Gamma_I(\mathsf{M})$ is left exact. Hence, just as we did for global sections of sheaves in §9.0, we get the derived functors $\mathsf{M} \mapsto H^p_I(\mathsf{M})$ such that

- $H^0_I(\mathsf{M}) = \Gamma_I(\mathsf{M})$.
- A short exact sequence $0 \to \mathsf{M} \to \mathsf{N} \to \mathsf{P} \to 0$ gives a long exact sequence
$$0 \to H^0_I(\mathsf{M}) \to H^0_I(\mathsf{N}) \to H^0_I(\mathsf{P}) \to H^1_I(\mathsf{M}) \to H^1_I(\mathsf{N}) \to H^1_I(\mathsf{P}) \to \cdots .$$

The Local Čech Complex. As with sheaf cohomology, there is a Čech complex for local cohomology. The sheaf case used an affine open cover; here we use generators of the ideal. More precisely, if $I = \langle f_1, \ldots, f_\ell \rangle$, let $[\ell] = \{1, \ldots, \ell\}$ be the index set and as in §9.0, let $[\ell]_p$ denote the set of all $(p+1)$-tuples $(i_0, \ldots, i_p)$ of elements of I satisfying $i_0 < \cdots < i_p$. Also set $\mathbf{f} = (f_1, \ldots, f_\ell)$

Given an R-module M, define

$$\check{C}^p(\mathbf{f}, \mathsf{M}) \ = \bigoplus_{(i_0,\ldots,i_{p-1}) \in [\ell]_{p-1}} \mathsf{M}_{f_{i_0} \cdots f_{i_{p-1}}},$$

where $\mathsf{M}_{f_{i_0} \cdots f_{i_{p-1}}}$ is the localization of M at $f_{i_0} \cdots f_{i_{p-1}}$. An element of $\check{C}^p(\mathbf{f}, \mathsf{M})$ is a function α that assigns an element of $\mathsf{M}_{f_{i_0} \cdots f_{i_{p-1}}}$ to each $(i_0, \ldots, i_{p-1}) \in [\ell]_{p-1}$. Then define a differential

$$d^p : \check{C}^p(\mathbf{f}, \mathsf{M}) \longrightarrow \check{C}^{p+1}(\mathbf{f}, \mathsf{M})$$

by

$$d^p(\alpha)(i_0, \ldots, i_p) = \sum_{k=0}^{p} (-1)^k \alpha(i_0, \ldots, \widehat{i_k}, \ldots, i_p),$$

where we regard $\alpha(i_0, \ldots, \widehat{i_k}, \ldots, i_p)$ as an element of $\mathsf{M}_{f_{i_0} \cdots f_{i_p}}$ via the map

$$\mathsf{M}_{f_{i_0} \cdots \widehat{f_{i_k}} \cdots f_{i_p}} \to \mathsf{M}_{f_{i_0} \cdots f_{i_p}}$$

given by localization at f_{i_k}. Similar to Exercise 9.0.2, we have $d^p \circ d^{p-1} = 0$.

Definition 9.5.1. Given an R-module M and generators $\mathbf{f} = (f_1, \ldots, f_\ell)$ of I, the ***local Čech complex*** is

$$\check{C}^\bullet(\mathbf{f}, \mathsf{M}) : \ 0 \longrightarrow \check{C}^0(\mathbf{f}, \mathsf{M}) \xrightarrow{d^0} \check{C}^1(\mathbf{f}, \mathsf{M}) \xrightarrow{d^1} \check{C}^2(\mathbf{f}, \mathsf{M}) \xrightarrow{d^2} \cdots .$$

Just as the Čech complex computes sheaf cohomology, the local Čech complex computes local cohomology. See [**158**, Thm. 7.13] for a proof of the following.

Theorem 9.5.2. *Given an R-module M and generators $\mathbf{f} = (f_1, \dots, f_\ell)$ of I, there are natural isomorphisms*

$$H^p_I(\mathsf{M}) \simeq H^p(\check{C}^\bullet(\mathbf{f}, \mathsf{M}))$$

for all $p \geq 0$. □

Example 9.5.3. For $I = \langle x, y\rangle \subseteq S = \mathbb{C}[x,y]$, the local Čech complex is

$$0 \longrightarrow S \longrightarrow S_x \oplus S_y \longrightarrow S_{xy} \longrightarrow 0.$$

Theorem 9.5.2 implies that $H^2_I(S)$ is the cokernel of $S_x \oplus S_y \to S_{xy}$. Consider $x^{-1}y^{-1}\mathbb{C}[x^{-1}, y^{-1}] = \mathrm{Span}_{\mathbb{C}}(x^{-a}y^{-b} \mid a, b > 0)$ with S-module structure given by

$$x^i y^j \cdot x^{-a}y^{-b} = \begin{cases} x^{-(a-i)}y^{-(b-j)} & i < a, j < b \\ 0 & \text{otherwise}. \end{cases}$$

Then the map $S_{xy} \to x^{-1}y^{-1}\mathbb{C}[x^{-1}, y^{-1}]$ defined by $f(x,y)/(xy)^k \mapsto f(x,y) \cdot x^{-k}y^{-k}$ gives an exact sequence

$$S_x \oplus S_y \longrightarrow S_{xy} \longrightarrow x^{-1}y^{-1}\mathbb{C}[x^{-1}, y^{-1}] \longrightarrow 0 \tag{9.5.1}$$

(Exercise 9.5.1). Thus $H^2_I(S) \simeq x^{-1}y^{-1}\mathbb{C}[x^{-1}, y^{-1}]$.

It follows that if x and y have degree 1, then $H^2_I(S)$ is graded with

$$H^2_I(S) = \cdots \oplus H^2_I(S)_{-4} \oplus H^2_I(S)_{-3} \oplus H^2_I(S)_{-2} \oplus 0 \oplus \cdots$$

and $\dim H^2_I(S)_{-\ell} = \ell - 1$ for $\ell > 0$ (Exercise 9.5.1). ◊

The number $\ell - 1$ just computed is also the dimension of $H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-\ell))$ (see Example 9.4.1). This is no accident, since we will prove below that

$$H^2_I(S)_a \simeq H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(a))$$

for all $a \in \mathbb{Z}$.

Relation with Ext. For us, an especially useful aspect of local cohomology is its relation to Ext. In §9.0 we introduced Ext in the context of sheaves. There is also a module version, where for a fixed R-module N, the derived functors of $\mathsf{M} \mapsto \mathrm{Hom}_R(\mathsf{N}, \mathsf{M})$ are denoted $\mathrm{Ext}^p_R(\mathsf{N}, \mathsf{M})$. They have the expected properties, and more importantly, are easy to compute by computer algebra systems such as `Macaulay2`.

To see the relation between Ext and local cohomology, we begin with the observation that an R-module homomorphism $\phi : R/I^k \to \mathsf{M}$ is determined by $\phi(1)$, and choosing any $a \in \mathsf{M}$ gives a homomorphism, provided that $I^k \cdot a = 0$. It follows that $\mathrm{Hom}_R(R/I^k, \mathsf{M})$ consists of those elements of M annihilated by I^k. Comparing this to the definition of $\Gamma_I(\mathsf{M})$, we obtain

$$\Gamma_I(\mathsf{M}) = \varinjlim_k \mathrm{Hom}_R(R/I^k, \mathsf{M}).$$

Since direct limit is an exact functor (Exercise 9.5.2), it follows without difficulty that the derived functors are related the same way (see [**158**, Thm. 7.8] for a proof).

Theorem 9.5.4. $H^p_I(\mathsf{M}) = \varinjlim_k \operatorname{Ext}^p_R(R/I^k, \mathsf{M})$. □

We will need a variant of this result. For an ideal $I = \langle f_1, \ldots, f_\ell \rangle$, let

$$I^{[k]} = \langle f_1^k, \ldots, f_\ell^k \rangle.$$

Theorem 9.5.5. $H^p_I(\mathsf{M}) = \varinjlim_k \operatorname{Ext}^p_R(R/I^{[k]}, \mathsf{M})$.

Proof. This follows from [**158**, Rem. 7.9] since $I^{[k]} \subseteq I^k$ and $I^{\ell k} \subseteq I^{[k]}$. □

Here is a simple example.

Example 9.5.6. Let $I = \langle x, y \rangle \subseteq S = \mathbb{C}[x,y]$. We know by Example 9.5.3 that $H^2_I(S)_{-5}$ has dimension 4. To compute this using Ext, let $A = S/I^{[k]} = S/\langle x^k, y^k \rangle$. Since A and S are graded S-modules, the Ext group $\operatorname{Ext}^2_S(A,S)$ is also graded. As described in Example B.4.1, we can compute the graded piece $\operatorname{Ext}^2_S(A,S)_{-5}$ for various values of k using `Macaulay2`. When $k=3$, we find that

$$\dim \operatorname{Ext}^2_S(S/I^{[3]}, S)_{-5} = 2.$$

Since $\dim H^2_I(S)_{-5} = 4$, we see that $k=3$ is not big enough. If we switch to $k \geq 4$, then the answer stabilizes at the number 4. ◊

In this example, $\operatorname{Ext}^2_S(S/I^{[k]}, S)_{-5} = H^2_I(S)_{-5}$ for k sufficiently large. We will prove below that for any degree $a \in \mathbb{Z}$, we have

$$\operatorname{Ext}^2_S(S/I^{[k]}, S)_a = H^2_I(S)_a$$

provided that k is sufficiently large. The problem is that $\operatorname{Ext}^2_S(S/I^{[k]}, S)$ is finitely generated over S but $H^2_I(S)$ is not (Exercise 9.5.4). So we cannot use the same k for all degrees a. Explicit bounds on k in terms of a are needed in order to turn the method of Example 9.5.6 into an algorithm.

This concludes our overview of local cohomology; for more, see [**89**, App. 4] or [**158**]. Our next task to is apply local cohomology to toric varieties.

The Toric Case. The total coordinate ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ of a toric variety X_Σ is graded by the class group $\operatorname{Cl}(X_\Sigma)$, where a monomial $\prod_\rho x_\rho^{a_\rho}$ has degree $[\sum_\rho a_\rho D_\rho] \in \operatorname{Cl}(X_\Sigma)$. We also have the irrelevant ideal

$$B(\Sigma) = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma_{\max} \rangle, \quad x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho,$$

introduced in Chapter 5. We assume that X_Σ has no torus factors.

We proved in Chapter 5 that a finitely generated graded S-module M and a divisor class $\alpha \in \operatorname{Cl}(X_\Sigma)$ give the following:

- The coherent sheaf $\mathscr{F} = \widetilde{\mathsf{M}}$ on X_Σ. Every coherent sheaf on X_Σ arises in this way (Proposition 5.3.9).
- The shifted module $\mathsf{M}(\alpha)$, where $\mathsf{M}(\alpha)_\beta = \mathsf{M}_{\alpha+\beta}$ for $\beta \in \mathrm{Cl}(X_\Sigma)$.

The sheaf associated to $S(\alpha)$ is denoted $\mathscr{O}_{X_\Sigma}(\alpha)$, and Proposition 5.3.7 tells us that $\mathscr{O}_{X_\Sigma}(\alpha) \simeq \mathscr{O}_{X_\Sigma}(D)$ when D is a Weil divisor with divisor class α. More generally, if $\mathscr{F} = \widetilde{\mathsf{M}}$, then $\mathscr{F}(\alpha)$ will denote the sheaf associated to $\mathsf{M}(\alpha)$. When α is the class of a Cartier divisor, one can prove that

$$\mathscr{F}(\alpha) \simeq \mathscr{F} \otimes_{X_\Sigma} \mathscr{O}_{X_\Sigma}(\alpha) \tag{9.5.2}$$

(Exercise 9.5.3). Our first main result is that the local cohomology for the irrelevant ideal $B(\Sigma)$ computes the sheaf cohomology of *all* twists of a coherent sheaf on X_Σ.

Theorem 9.5.7. *Let M be a finitely generated graded S-module with associated coherent sheaf $\mathscr{F} = \widetilde{\mathsf{M}}$ on X_Σ. If $p \geq 2$, then*

$$H^p_{B(\Sigma)}(\mathsf{M}) \simeq \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} H^{p-1}(X_\Sigma, \mathscr{F}(\alpha)).$$

Furthermore, we have an exact sequence

$$0 \longrightarrow H^0_{B(\Sigma)}(\mathsf{M}) \longrightarrow \mathsf{M} \longrightarrow \bigoplus_{\alpha \in \mathrm{Cl}(X_\Sigma)} H^0(X_\Sigma, \mathscr{F}(\alpha)) \longrightarrow H^1_{B(\Sigma)}(\mathsf{M}) \longrightarrow 0.$$

Proof. Let $\Sigma_{\max} = \{\sigma_1, \ldots, \sigma_\ell\}$, so that $B(\Sigma)$ is generated by the monomials $f_i = x^{\hat{\sigma}_i}$, $i \in [\ell] = \{1, \ldots, \ell\}$. Then the terms of the local Čech complex $\check{C}^\bullet(\mathbf{f}, \mathsf{M})$ are direct sums of localizations M at products of various f_i's. These localizations are $\mathrm{Cl}(X_\Sigma)$-graded since M is graded and the f_i are monomials. The differentials also preserve the grading, which by Theorem 9.5.2 implies that $H^p_{B(\Sigma)}(\mathsf{M})$ has a natural $\mathrm{Cl}(X_\Sigma)$-grading such that for all $\alpha \in \mathrm{Cl}(X_\Sigma)$, we have

$$H^p_{B(\Sigma)}(\mathsf{M})_\alpha = H^p(\check{C}^\bullet(\mathbf{f}, \mathsf{M})_\alpha).$$

We will relate $\check{C}^p(\mathbf{f}, \mathsf{M})_\alpha$ to the Čech complex for $\mathscr{F}(\alpha)$ given in (9.1.1).

To compute $\check{C}^p(\mathbf{f}, \mathsf{M})_\alpha$, first observe that

$$\check{C}^p(\mathbf{f}, \mathsf{M})_\alpha = \bigoplus_{(i_0,\ldots,i_{p-1}) \in [\ell]_{p-1}} \big(\mathsf{M}_{f_{i_0}\cdots f_{i_{p-1}}}\big)_\alpha = \bigoplus_{(i_0,\ldots,i_{p-1}) \in [\ell]_{p-1}} \big(\mathsf{M}(\alpha)_{f_{i_0}\cdots f_{i_{p-1}}}\big)_0$$

since shifting commutes with localization. For $\gamma = (i_0, \ldots, i_{p-1}) \in [\ell]_{p-1}$, note that

$$f_{i_0} \cdots f_{i_{p-1}} = x^{\hat{\sigma}_{i_0}} \cdots x^{\hat{\sigma}_{i_{p-1}}} = \prod_{\rho \notin \sigma(1)} x_\rho^{a_\rho}, \tag{9.5.3}$$

where $a_\rho = |\{i_j \mid \rho \notin \sigma_{i_j}\}| > 0$. If we set $\sigma_\gamma = \sigma_{i_0} \cap \cdots \cap \sigma_{i_{p-1}}$, then $x^{\hat{\sigma}_\gamma}$ gives the same localization as (9.5.3). Since $\mathscr{F}(\alpha)$ is the sheaf associated to $\mathsf{M}(\alpha)$, Proposition 5.3.3 implies that

$$\big(\mathsf{M}(\alpha)_{f_{i_0}\cdots f_{i_{p-1}}}\big)_0 = \big(\mathsf{M}(\alpha)_{x^{\hat{\sigma}_\gamma}}\big)_0 \simeq \Gamma(U_{\sigma_\gamma}, \mathscr{F}(\alpha)).$$

Comparing this with the Čech complex (9.1.1) for the open cover $\mathscr{U} = \{U_{\sigma_i}\}_{i\in[\ell]}$, we obtain

$$\check{C}^p(\mathbf{f},\mathsf{M})_\alpha = \bigoplus_{\gamma\in[\ell]_{p-1}} \Gamma(U_{\sigma_\gamma},\mathscr{F}(\alpha)) \simeq \check{C}^{p-1}(\mathscr{U},\mathscr{F}(\alpha)).$$

This isomorphism is compatible with the differentials. It follows that the local Čech complex $\check{C}^\bullet(\mathbf{f},\mathsf{M})_\alpha$ is obtained from the Čech complex $\check{C}^\bullet(\mathscr{U},\mathscr{F}(\alpha))$ by deleting the first term and shifting the remaining terms. When $p \geq 2$, this implies

$$H^p_{B(\Sigma)}(\mathsf{M})_\alpha \simeq H^{p-1}(X_\Sigma,\mathscr{F}(\alpha)),$$

and with a little work (Exercise 9.5.5) we also get an exact sequence

$$0 \longrightarrow H^0_{B(\Sigma)}(\mathsf{M})_\alpha \longrightarrow \mathsf{M}_\alpha \longrightarrow H^0(X_\Sigma,\mathscr{F}(\alpha)) \longrightarrow H^1_{B(\Sigma)}(\mathsf{M})_\alpha \longrightarrow 0. \qquad \square$$

Theorems 9.5.5 and 9.5.7 imply that when $p \geq 1$,

$$H^p(X_\Sigma,\mathscr{F}(\alpha)) \simeq \varinjlim_k \operatorname{Ext}^{p+1}_S(S/B(\Sigma)^{[k]},\mathsf{M})_\alpha$$

when $\mathscr{F} = \widetilde{\mathsf{M}}$ and $\alpha \in \operatorname{Cl}(X_\Sigma)$. We can compute $\operatorname{Ext}^{p+1}_S(S/B(\Sigma)^{[k]},\mathsf{M})_\alpha$ by the methods of Example 9.5.6. The problem is the direct limit. We tackle this next.

Stabilization of Ext. In the toric case, Ext has a nice relation to local cohomology.

Lemma 9.5.8. *Let M be a finitely generated* $\operatorname{Cl}(X_\Sigma)$*-graded S-module and fix* $\alpha \in \operatorname{Cl}(X_\Sigma)$. *If Σ is a complete fan, then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, the natural map $S/B(\Sigma)^{[k+1]} \to S/B(\Sigma)^{[k]}$ induces an isomorphism*

$$\operatorname{Ext}^p_S(S/B(\Sigma)^{[k]},\mathsf{M})_\alpha \simeq \operatorname{Ext}^p_S(S/B(\Sigma)^{[k+1]},\mathsf{M})_\alpha.$$

In particular, $k \geq k_0$ implies

$$\operatorname{Ext}^p_S(S/B(\Sigma)^{[k]},\mathsf{M})_\alpha \simeq H^p_{B(\Sigma)}(\mathsf{M})_\alpha.$$

Proof. We give a proof only for $\mathsf{M} = S$ following Mustaţă [**211**, Thm. 1.1]. For the general case, one replaces M with a free resolution and uses a spectral sequence argument. See [**91**, Prop. 4.1] for the details.

Earlier we described Ext using the derived functors of $\mathsf{M} \mapsto \operatorname{Hom}_R(\mathsf{N},\mathsf{M})$ for fixed N. Ext also comes from the derived functors of $\mathsf{N} \mapsto \operatorname{Hom}_R(\mathsf{N},\mathsf{M})$ for fixed M, where one uses a projective resolution of N instead of an injective resolution of M (see, for example, [**89**, A3.11]). In particular, we can compute $\operatorname{Ext}^p_S(S/B(\Sigma)^{[k]},S)$ using any free resolution of $S/B(\Sigma)^{[k]}$. An especially nice resolution is given by the *Taylor resolution*, which is described as follows.

We begin with $S/B(\Sigma)$. The minimal generators of $B(\Sigma)$ are $f_i = x^{\hat{\sigma}_i}$ for $i \in [\ell]$. Let F_s be a free S module with basis e_I for all $I \subseteq [\ell]$ with $|I| = s$. To define the differential $d_s : F_s \to F_{s-1}$, take $I,J \subseteq [\ell]$ with $|I| = |J|+1 = s$ and list the elements of I as $i_1 < \cdots < i_s$. Then define

$$c_{IJ} = \begin{cases} 0 & J \not\subseteq I \\ (-1)^r f_I/f_J & I = J \cup \{i_r\}, \end{cases}$$

where $f_I = \operatorname{lcm}(f_i \mid i \in I)$ and similarly for f_J. Then

$$d_s(e_I) = \textstyle\sum_J c_{IJ} e_J.$$

Since $F_0 = S$, we have an obvious map $F_0 \to S/B(\Sigma)$. One can prove that

$$0 \longrightarrow F_\ell \xrightarrow{d_\ell} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow S/B(\Sigma) \longrightarrow 0$$

is exact (see [**89**, Ex. 17.11]). This is the *Taylor resolution* $(F_\bullet, d_\bullet)$ of $S/B(\Sigma)$.

This construction applies to any monomial ideal. In particular, it works for $B(\Sigma)^{[k]} = \langle f_1^k, \ldots, f_\ell^k \rangle$. Let $(F_\bullet^{[k]}, d_\bullet^{[k]})$ denote the Taylor resolution of $S/B(\Sigma)^{[k]}$. It has the same modules $F_s^{[k]} = F_s$, and since the f_i are square-free, we have $f_I^k = \operatorname{lcm}(f_i^k \mid i \in I)$. Thus the differentials $d_s^{[k]}$ in the Taylor resolution of $S/B(\Sigma)^{[k]}$ are given by

$$d_s^{[k]}(e_I) = \textstyle\sum_J c_{IJ}^k e_J,\ c_{IJ} \text{ as above.}$$

We now compare the Taylor resolutions of $S/B(\Sigma)^{[k]}$ and $S/B(\Sigma)^{[k+1]}$. The maps $\phi_s : F_s^{[k+1]} \to F_s^{[k]}$ defined by $\phi_s(e_I) = f_I e_I$ induce a commutative diagram

$$\begin{tikzcd} F_s^{[k+1]} \arrow[r, "\phi_s"] \arrow[d, "d_s^{[k+1]}"'] & F_s^{[k]} \arrow[d, "d_s^{[k]}"] \\ F_{s-1}^{[k+1]} \arrow[r, "\phi_{s-1}"] & F_{s-1}^{[k]}. \end{tikzcd}$$

These maps are compatible with the surjection $S/B(\Sigma)^{[k+1]} \to S/B(\Sigma)^{[k]}$. Hence

$$H^p(\operatorname{Hom}_S(F_\bullet^{[k+1]}, S)) \to H^p(\operatorname{Hom}_S(F_\bullet^{[k]}, S)) \tag{9.5.4}$$

induced by the ϕ_s can be identified with the canonical map

$$\operatorname{Ext}_S^p(S/B(\Sigma)^{[k]}, S) \to \operatorname{Ext}_S^p(S/B(\Sigma)^{[k+1]}, S).$$

The next key idea involves the use of a finer grading than the $\operatorname{Cl}(X_\Sigma)$-grading used so far. Recall that this grading is induced by the map

$$\mathbb{Z}^{\Sigma(1)} \longrightarrow \operatorname{Cl}(X_\Sigma) \tag{9.5.5}$$

where $x^{\mathbf{a}} = \prod_\rho x_\rho^{a_\rho}$ has degree $[\sum_\rho a_\rho D_\rho]$. The ring S also has a $\mathbb{Z}^{\Sigma(1)}$-grading where $\deg(x^{\mathbf{a}}) = \mathbf{a}$. Since $B(\Sigma)^{[k]}$ is a monomial ideal, the quotient ring $S/B(\Sigma)^{[k]}$ is $\mathbb{Z}^{\Sigma(1)}$-graded. Then $\operatorname{Ext}_S^i(S/B(\Sigma)^{[k]}, S)$ inherits a natural $\mathbb{Z}^{\Sigma(1)}$-grading.

Now grade the Taylor resolution $(F_\bullet^{[k]}, d_\bullet^{[k]})$ by setting $\deg(e_I) = k\deg(f_I)$. This guarantees two things:

- The differential $d_s^{[k]}$ has degree 0, so the isomorphism $\mathrm{Ext}_S^p(S/B(\Sigma)^{[k]},S) \simeq H^p(\mathrm{Hom}_S(F_\bullet^{[k]},S)$ is $\mathbb{Z}^{\Sigma(1)}$-graded.
- The map ϕ_s has degree 0, so (9.5.4) is $\mathbb{Z}^{\Sigma(1)}$-graded.

It follows that $(F_s^{[k]})^\vee = \mathrm{Hom}_S(F_s^{[k]},S)$ has dual basis e_I^* with $\deg(e_I^*) = -k\deg(f_I)$.

Given $\mathbf{a},\mathbf{b} \in \mathbb{Z}^{\Sigma(1)}$, define

$$\mathbf{a} \geq \mathbf{b} \text{ if and only if } a_\rho \geq b_\rho \text{ for all } \rho \in \Sigma(1).$$

We claim that if $\mathbf{a} \geq (-k,\dots,-k)$, then

$$(\phi_s^\vee)_{\mathbf{a}} : (F_s^{[k]})_{\mathbf{a}}^\vee \longrightarrow (F_s^{[k+1]})_{\mathbf{a}}^\vee \tag{9.5.6}$$

is an isomorphism for all s.

To prove this, first observe that $\phi_s^\vee(e_I^*) = f_I e_I^*$. It follows that $(\phi_s^\vee)_{\mathbf{a}}$ is injective. Now take $x^{\mathbf{b}} e_I^* \in (F_s^{[k+1]})_{\mathbf{a}}^\vee$. Then $\mathbf{b} - (k+1)\deg(f_I) = \mathbf{a}$, so

$$\mathbf{b} = (k+1)\deg(f_I) + \mathbf{a} \geq (k+1)\deg(f_I) + (-k,\dots,-k).$$

Since f_I is square-free, $(k+1)\deg(f_I)$ is a vector with whose ρth entry is $(k+1)$ if x_ρ divides f_I and 0 otherwise. Then the above inequality implies that f_I divides $x^{\mathbf{b}}$, so that $x^{\mathbf{b}} e_I^*$ is in the image of $(\phi_s^\vee)_{\mathbf{a}}$. It follows that (9.5.6) is an isomorphism.

The local cohomology $H^p_{B(\Sigma)}(S)$ has a $\mathbb{Z}^{\Sigma(1)}$-grading which is compatible with its $\mathrm{Cl}(X_\Sigma)$-grading via (9.5.5). This follows easily from Theorem 9.5.5 and the Taylor resolution. If $\alpha \in \mathrm{Cl}(X_\Sigma)$ and $p \geq 2$, then

$$H^p_{B(\Sigma)}(S)_\alpha \simeq H^{p-1}(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha)).$$

by Theorem 9.5.7. The right-hand side is finite-dimensional since Σ is complete. This implies that when we decompose $H^p_{B(\Sigma)}(S)$ into its nonzero $\mathbb{Z}^{\Sigma(1)}$-graded pieces $H^p_{B(\Sigma)}(S)_{\mathbf{a}}$, only finitely many can appear in $H^p_{B(\Sigma)}(S)_\alpha$. For these finitely many $\mathbf{a}$'s, pick k_0 such that they all satisfy $\mathbf{a} \geq (-k_0,\dots,-k_0)$. This k_0 has the required properties. The argument for $p = 0,1$ is covered in Exercise 9.5.6. $\square$

Theorem 9.5.7 and Lemma 9.5.8 imply that for $p \geq 1$ and $k \gg 0$,

$$H^p(X_\Sigma, \mathscr{O}_{X_\Sigma}(\alpha)) \simeq H^{p+1}_{B(\Sigma)}(S)_\alpha \simeq \mathrm{Ext}_S^{p+1}(S/B(\Sigma)^{[k]},S).$$

Notice how these isomorphisms generalize Examples 9.5.3 and 9.5.6. Our final task is to give an explicit method for deciding when k is big enough.

Bounds. For a complete fan Σ, graded S-module M, and divisor class $\alpha \in \mathrm{Cl}(X_\Sigma)$, the paper [**91**] gives an explicit value for the number k_0 appearing in Lemma 9.5.8. We will state the results of [**91**] without proof, though the following example suggests some of the ideas involved.

Example 9.5.9. Let Σ be a complete fan in $M_{\mathbb{R}} \simeq \mathbb{R}^2$ with minimal generators $u_1,\dots,u_r$ arranged counterclockwise around the origin. Suppose we have a divisor $D = \sum_i a_i D_i$ on X_Σ and $m \in M$ such that $H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \neq 0$. It follows from Proposition 9.1.6 that the sign pattern of m (determined by $\langle m,u_i\rangle + a_i \geq 0$ or < 0) has at least two strings of consecutive $-$'s. The set $I = \{i \in [r] \mid \langle m,u_i\rangle + a_i < 0\}$ records the locations of the $-$'s in the sign pattern of m.

Since $H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \neq 0$ is finite-dimensional, there are only finitely many m's with the same sign pattern. In other words, the inequalities

$$\begin{aligned} \langle m,u_i\rangle + a_i &< 0, \ i \in I\\ \langle m,u_i\rangle + a_i &\geq 0, \ i \in [r]\setminus I \end{aligned}$$

have only finitely many integer solutions, which means that the region of the plane defined by these inequalities is bounded. You can see and example of this in Figure 4 from Example 9.1.8. Since this region determined by the u_i and a_i, it should be possible to bound the size of the m's that appear in terms of the u_i and a_i.

To relate this to Lemma 9.5.8, let $\mathbf{b} = (\langle m,u_1\rangle + a_1,\dots,\langle m,u_r\rangle + a_r)$. Then it is easy to see that

$$H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \simeq H^2_{B(\Sigma)}(S)_{\mathbf{b}}.$$

Thus bounding the m's with $H^1(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))_m \neq 0$ is equivalent to bounding the $\mathbf{b}$'s with $H^2_{B(\Sigma)}(S)_{\mathbf{b}} \neq 0$. And once we find this bound, we know how to pick k_0 so that $\mathbf{b} \geq (-k_0,\dots,-k_0)$ for all such $\mathbf{b}$'s. The proof of Lemma 9.5.8 shows that this k_0 works. $\Diamond$

In the general case, we proceed as follows. Pick a basis $e_1,\dots,e_n$ of M and let A be the matrix whose rows give the coefficients of the u_ρ's with respect to the chosen basis. Then A is an $r\times n$ matrix, where $r = |\Sigma(1)|$. For this matrix, define

$$\begin{aligned} q_n &= \min(|\text{nonzero } n\times n \text{ minors of } A|)\\ Q_1 &= \max(|\text{entries of } A|)\\ Q_{n-1} &= \max(|(n-1)\times(n-1) \text{ minors of } A|). \end{aligned} \tag{9.5.7}$$

The following bound is proved in [**91**, Cor. 3.3].

Theorem 9.5.10. *Given a complete fan Σ in $N_R \simeq \mathbb{R}^n$ and a divisor class $\alpha = [\sum_\rho a_\rho D_\rho] \in \mathrm{Cl}(X_\Sigma)$, we have*

$$\mathrm{Ext}^{p+1}_S(S/B(\Sigma)^{[k]}, S)_\alpha \simeq H^p_{B(\Sigma)}(S)_\alpha$$

for all $k \geq k_0$, where

$$k_0 = n^2 \max_\rho(|a_\rho|)\frac{Q_1 Q_{n-1}}{q_n}. \qquad \square$$

For a finitely generated graded S-module M, the formula for k_0 involves the minimal free resolution of M. Write the resolution as

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathsf{M} \longrightarrow 0$$

and write

$$F_j = \sum_{\beta\in \mathrm{Cl}(X_\Sigma)} S(-\beta)^{r_{j,\beta}}.$$

Finally, for each β with $r_{j,\beta} \neq 0$, write $\beta = [\sum_\rho a_{\beta,\rho} D_\rho] \in \mathrm{Cl}(X_\Sigma)$. Then [**91**, Prop. 4.1] gives the following bound.

Theorem 9.5.11. *Let Σ and α be as in Theorem 9.5.10, and let M be a finitely generated graded S-module. Then*

$$\mathrm{Ext}_S^{p+1}(S/B(\Sigma)^{[k]}, M)_\alpha \simeq H^p_{B(\Sigma)}(M)_\alpha$$

for all $k \geq k_0$, where

$$k_0 = n^2 \max_{\rho,j,r_{j,\beta}\neq 0}(|a_\rho - a_{\beta,\rho}|)\frac{Q_1 Q_{n-1}}{q_n}. \qquad \square$$

The Cotangent Bundle. We end this section by using our methods to calculate the cohomology of the cotangent bundle of a smooth complete toric variety X_Σ. The first step is to find a graded S-module M with $\Omega^1_{X_\Sigma} \simeq \widetilde{\mathsf{M}}$.

In (8.1.7) we constructed an exact sequence

$$0 \longrightarrow \Omega^1_S \longrightarrow M \otimes_{\mathbb{Z}} S \to \bigoplus_\rho S/\langle x_\rho\rangle$$

where $M \otimes_{\mathbb{Z}} S \to \bigoplus_\rho S/\langle x_\rho\rangle$ is defined by $m \otimes f \mapsto \sum_\rho \langle m, u_\rho\rangle [f]$, and in Corollary 8.1.5 we showed that $\Omega^1_{X_\Sigma}$ is the sheaf of Ω^1_S. However, we need a description of Ω^1_S that is easier to implement on a computer. We do this as follows.

Pick bases of M and $\mathrm{Pic}(X_\Sigma)$ and order the elements of $\Sigma(1)$ as $\rho_1,\ldots,\rho_r$. Then the basic exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0$$

can be written

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^r \xrightarrow{B} \mathbb{Z}^{r-n} \longrightarrow 0, \tag{9.5.8}$$

where A is an $r \times n$ matrix whose ith row consists of the coefficients of u_{ρ_i} in the basis of M. This is the same matrix A that appears in Theorem 9.5.10. The $(r-n)\times r$ matrix B is called the *Gale dual* of A.

Lemma 9.5.12. *The $(r-n)\times r$ matrix*

$$\theta = B \cdot \begin{pmatrix} x_{\rho_1} & 0 & \cdots & 0 \\ 0 & x_{\rho_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{\rho_r} \end{pmatrix}$$

induces a graded homomorphism $\theta : \bigoplus_{i=1}^r S(-\deg(x_{\rho_i})) \to S^{r-n}$ *of degree* 0 *such that* $\Omega^1_{X_\Sigma}$ *is the sheaf associated to* $\ker(\theta)$.

Proof. Let $\mathbf{x}$ be the $r \times r$ diagonal matrix of variables appearing in the statement of the lemma. Similar to the proof of Theorem 8.1.6, we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d, dotted] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^1_S \arrow[r] \arrow[d, dotted] & \mathbb{Z}^n \otimes_{\mathbb{Z}} S \arrow[r] \arrow[d, "A"] & \bigoplus_{i=1}^r S/\langle x_{\rho_i}\rangle \arrow[d] & \\
0 \arrow[r] & \bigoplus_{i=1}^r S(-\deg(x_{\rho_i})) \arrow[r, "\mathbf{x}"] \arrow[d, dotted] & \mathbb{Z}^r \otimes_{\mathbb{Z}} S \arrow[r] \arrow[d, "B"] & \bigoplus_{i=1}^r S/\langle x_{\rho_i}\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}^{r-n} \otimes_{\mathbb{Z}} S \arrow[r] & \mathbb{Z}^{r-n} \otimes_{\mathbb{Z}} S \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

The rows are exact, as are the center and right columns. By the diagram chase from the proof of Theorem 8.1.6, the dotted arrows are exact, and by commutivity, the dotted arrow from $\bigoplus_{i=1}^r S(-\deg(x_{\rho_i}))$ to $\mathbb{Z}^{r-n} \otimes_{\mathbb{Z}} S$ is given by $\theta = B \cdot \mathbf{x}$. □

Example 9.5.13. For $\mathbb{P}^2$, the matrices A and B of (9.5.8) are given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

Thus $\theta = (x\ y\ z)$, which gives the exact sequence

$$0 \longrightarrow \Omega^1_S \longrightarrow S(-1)^3 \xrightarrow{(x\ y\ z)} S.$$

Using `Macaulay2` as in Example B.4.2, one computes the free resolution

$$0 \longrightarrow S(-3) \xrightarrow{\begin{pmatrix} z \\ x \\ -y \end{pmatrix}} S(-2)^3 \longrightarrow \Omega^1_S \longrightarrow 0. \tag{9.5.9}$$

The numbers from (9.5.7) are $q_2 = Q_1 = 1$, so that for $a \in \mathbb{Z}$, the formula for k_0 from Theorem 9.5.11 is

$$k_0 = 4\max(|a-2|, |a-3|).$$

For a in the range $-4 \le a \le 4$, we can use $k_0 = 28$. This implies that for $p \ge 1$,

$$H^p(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a)) \simeq \operatorname{Ext}^{p+1}_S(S/\langle x^{28}, y^{28}, z^{28}\rangle, \Omega^1_S)_a$$

when $-4 \le a \le 4$. This can be computed by the methods of Example 9.5.6. We can also compute $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a))$ directly from Ω^1_S (Exercise 9.5.7). The results are shown in Table 3 on the next page.

$a\backslash p$	0	1	2
-4	0	0	15
-3	0	0	8
-2	0	0	3
-1	0	0	0
0	0	1	0
1	0	0	0
2	3	0	0
3	8	0	0
4	15	0	0

Table 3. $\dim H^p(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a))$ for $-4 \leq a \leq 4$

In Exercise 9.5.8 you will calculate this table by hand. Notice also that the symmetry of the table comes from Serre duality. See Example B.5.2 and Exercise B.5.1 for different way to compute Table 3. $\Diamond$

Here is a slightly more complicated example.

Example 9.5.14. Let us compute the first cohomology group of various twists of the cotangent sheaf of the Hirzebruch surface $\mathscr{H}_2$. We will use the notation of Example 9.3.6. The goal is to compute the cohomology groups of $\Omega^1_{\mathscr{H}_2}(a,b) = \Omega^1_{\mathscr{H}_2}(aD_3 + bD_4)$ for all a, b.

One way to proceed would be to follow the approach of the previous example using `Macaulay2`. This would involve the following steps:

- Construct the total coordinate ring $S = \mathbb{Q}[x_1, x_2, x_3, x_4]$ of $\mathscr{H}_2$ is with its $\mathbb{Z}^2$-grading and irrelevant ideal $B = \langle x_1x_3, x_2x_4 \rangle$.
- Use the 2×4 matrix θ to define the S-module Ω^1_S as a kernel as in Lemma 9.5.8.
- Use a free resolution of Ω^1_S to compute the bound k_0 from Theorem 9.5.11.
- Compute various graded pieces of $\mathrm{Ext}^{p+1}_S(S/B^{[k_0]}, \Omega^1_S)$ as explained in §B.4 of Appendix B.

Alternatively, we could use the `Macaulay2` package `NormalToricVarieties` described in Examples B.5.1 and B.5.2. This package automates most of the above steps and makes it easy to calculate $H^p(\mathscr{H}_2, \Omega^1_{\mathscr{H}_2}(a,b))$ directly and efficiently. Table 4 records some computations of H^1 done this way (see Example B.5.2 for the details). There are many things we can see from this table, including:

- Theorem 9.4.11 implies that $\dim H^1(\mathscr{H}_2, \Omega^1_{\mathscr{H}_2}) = h_1 = f_1 - 2f_2 = 4 - 2\cdot 1 = 2$ since $\mathscr{H}_2$ comes from a quadrilateral (see Examples 2.3.16 and 3.1.16). This explains the 2 when $(a,b) = (0,0)$.
- Serre duality implies that $H^1(\mathscr{H}_2, \Omega^1_{\mathscr{H}_2}(a,b))^\vee \simeq H^1(\mathscr{H}_2, \Omega^1_{\mathscr{H}_2}(-a,-b))$. This explains the symmetry in the table.

$a\backslash b$	–2	–1	0	1	2
–2	0	0	3	4	3
–1	0	0	2	2	2
0	1	1	2	1	1
1	2	2	2	0	0
2	3	4	3	0	0

Table 4. $\dim H^1(\mathscr{H}_2, \Omega^1_{\mathscr{H}_2}(a,b))$ for $-2 \leq a,b \leq 2$

- Bott-Danilov-Steenbrink vanishing implies that $H^1(\mathscr{H}_2, \Omega^1_{\mathscr{H}_2}(a,b)) = 0$ for $a,b > 0$. This explains the 0s in the lower right corner of the table. $\Diamond$

See [**91**, Sec. 5] for an algorithmic approach to these calculations. Using [**190**, Cor. 3.4], one can describe an especially efficient method for computing the sheaf cohomology of a toric variety. This method is implemented in Greg Smith's `Macaulay2` package `NormalToricVarieites` [**252**]. A equivalent method was conjectured independently [**35**].

Exercises for §9.5.

9.5.1. Prove the exactness of (9.5.1).

9.5.2. Suppose that (A_i, ϕ_i), (A_i', ϕ_i'), (A_i'', ϕ_i'') are directed systems, and that for each i, there exist d_i and δ_i commuting with the ϕ's, such that we have an exact sequence

$$0 \longrightarrow A_i \xrightarrow{d_i} A_i' \xrightarrow{\delta_i} A_i'' \longrightarrow 0.$$

Prove the exactness of the sequence

$$0 \longrightarrow \varinjlim_i A_i \longrightarrow \varinjlim_i A_i' \longrightarrow \varinjlim_i A_i'' \longrightarrow 0.$$

9.5.3. Prove (9.5.2) when $\alpha \in \mathrm{Cl}(X_\Sigma)$ is the class of a Cartier divisor. Hint: Look carefully at Proposition 5.3.3 and remember that the restriction of $\mathscr{O}_{X_\Sigma}(\alpha)$ to U_σ is trivial.

9.5.4. For $I = \langle x,y\rangle \subseteq S = \mathbb{C}[x,y]$, prove that $H^2_I(S)$ is not finitely generated as an S-module. Hint: Use Example 9.5.3 to show that $H^2_I(S)_a \neq 0$ for all $a \leq -2$.

9.5.5. Complete the proof of Theorem 9.5.7 by proving that there is an exact sequence

$$0 \longrightarrow H^0_{B(\Sigma)}(\mathsf{M})_\alpha \longrightarrow \mathsf{M}_\alpha \longrightarrow H^0(X_\Sigma, \mathscr{F}(\alpha)) \longrightarrow H^1_{B(\Sigma)}(\mathsf{M})_\alpha \longrightarrow 0$$

for all $\alpha \in \mathrm{Cl}(X_\Sigma)$.

9.5.6. Prove that $H^0_{B(\Sigma)}(S) = H^1_{B(\Sigma)}(S) = 0$. Hint: Proposition 5.3.7.

9.5.7. Here are some details to check from Example 9.5.13.

(a) Use the exact sequence from Lemma 9.5.12 to show that $(\Omega^1_S)_\alpha \simeq H^0(X_\Sigma, \Omega^1_{X_\Sigma}(\alpha))$ for all $\alpha \in \mathrm{Pic}(X_\Sigma)$ when X_Σ is smooth and complete.

(b) Verify the first column of Table 3.

(c) Show that the first column agrees with the Bott formula from Example 9.4.10.

9.5.8. We can check Table 3 of Example 9.5.13 with a "barehanded" approach. Observe that we have exact sequences involving $\Omega^1_{\mathbb{P}^2}$:

$$0 \longrightarrow \Omega^1_{\mathbb{P}^2} \longrightarrow \mathscr{O}^3_{\mathbb{P}^2}(-1) \longrightarrow \mathscr{O}_{\mathbb{P}^2} \longrightarrow 0$$
$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-3) \longrightarrow \mathscr{O}^3_{\mathbb{P}^2}(-2) \longrightarrow \Omega^1_{\mathbb{P}^2} \longrightarrow 0.$$

The first comes from Lemma 9.5.12, and the second comes by sheafifying the free resolution of Ω^1_S computed in Example 9.5.13. Twist these sequences by $a \in \mathbb{Z}$ and consider the resulting long exact sequences in cohomology. Using the vanishing theorems we know, conclude that the nonzero values for $H^p(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a))$ are:

$$\begin{aligned} \dim H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a)) &= a^2 - 1 && \text{if } a \geq 2 \\ \dim H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a)) &= 1 && \text{if } a = 1 \\ \dim H^2(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(a)) &= a^2 - 1 && \text{if } a \leq -2. \end{aligned}$$

These formulas give the numbers in Table 3.

Part II. Topics in Toric Geometry

Chapters 10 to 15 explore further topics in theory of toric varieties. This part of the book assumes a wider knowledge of algebraic geometry, though we give careful references for all of the tools used in our study of toric geometry. The topics presented here are just of few of many rich areas of inquiry encountered in the study of toric varieties.

Chapter 10

Toric Surfaces

In this chapter, we will apply the theory developed so far to study the structure of 2-dimensional normal toric varieties (toric surfaces). We will describe their singularities, introduce the idea of a resolution of singularities, and also classify smooth complete toric surfaces. Along the way, we will encounter two types of continued fractions, Hilbert bases, the Gröbner fan, the McKay correspondence, the Riemann-Roch theorem, the sectional genus, and the number 12.

§10.1. Singularities of Toric Surfaces and Their Resolutions

Singular Points of Toric Surfaces. If X_Σ is the toric surface of a fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^2$, then minimal generators of the rays $\rho \in \Sigma(1)$ are primitive and hence extend to a basis of N. Then Theorem 3.1.19 implies that the toric surface obtained by removing the fixed points of the torus action (i.e., the points corresponding to the 2-dimensional cones under the Orbit-Cone Correspondence) is smooth. There are only finitely many such points, so X_Σ has at most finitely many singular points. Moreover, 2-dimensional cones are always simplicial, so from Example 1.3.20, each of these singular points is a finite abelian quotient singularity (isomorphic to the image of the origin in the quotient $\mathbb{C}^2/G$ where G is a finite abelian group).

All cones are assumed to be rational and polyhedral. A 2-dimensional strongly convex cone in $N_\mathbb{R} \simeq \mathbb{R}^2$ has the following normal form that will facilitate our study of the singularities of toric surfaces.

Proposition 10.1.1. *Let $\sigma \subseteq N_\mathbb{R} \simeq \mathbb{R}^2$ be a 2-dimensional strongly convex cone. Then there exists a basis e_1, e_2 for N such that*

$$\sigma = \mathrm{Cone}(e_2, de_1 - ke_2),$$

where $d > 0$, $0 \leq k < d$, and $\gcd(d,k) = 1$.

Proof. We will need the following modified division algorithm here and at several other points in this chapter (Exercise 10.1.1):

(10.1.1) Given integers l and $d > 0$, there are unique integers s and k such that $l = sd - k$ and $0 \leq k < d$.

Say $\sigma = \text{Cone}(u_1, u_2)$, where u_i are primitive vectors. Since u_1 is primitive, we can take it as part of a basis of N, and we let $e_2 = u_1$. Since σ is strongly convex, for any basis e_1', e_2 for N, it will be true that

$$u_2 = de_1' + le_2$$

for some $d \neq 0$. By replacing e_1' by $-e_1'$ if necessary, we can assume $d > 0$. By (10.1.1), there are integers s, k such that $l = sd - k$, where $0 \leq k < d$. Using this integer s, let $e_1 = e_1' + se_2$. Then e_1, e_2 is also a basis for N and

$$u_2 = de_1 + (l - sd)e_2 = de_1 - ke_2.$$

Hence $\sigma = \text{Cone}(e_2, de_1 - ke_2)$ as claimed, and $\gcd(d, k) = 1$ follows since u_2 is primitive. □

We will call the integers d, k in this statement the *parameters* of the cone σ, and $\{e_1, e_2\}$ is called a *normalized basis* for N relative to σ. The uniqueness of d, k will be studied in Proposition 10.1.3 below.

Using the normal form, we next describe the local structure of the point p_σ in the affine toric variety U_σ. Recall from Example 1.3.20 that if $N' \subseteq N$ is the sublattice generated by the ray generators of σ, then $U_\sigma \simeq \mathbb{C}^2/G$, where $G = N/N'$. In our situation, $N = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and

$$N' = \mathbb{Z}e_2 \oplus \mathbb{Z}(de_1 - ke_2) = d\mathbb{Z}e_1 \oplus \mathbb{Z}e_2,$$

so it follows easily that

$$G = N/N' \simeq \mathbb{Z}/d\mathbb{Z}. \tag{10.1.2}$$

In particular, for singularities of toric surfaces, the finite group G is always cyclic.

The action of G on $\mathbb{C}^2$ is determined by the integers d, k as follows. We write

$$\mu_d = \{\zeta \in \mathbb{C} \mid \zeta^d = 1\}$$

for the group of dth roots of unity in $\mathbb{C}$. Then a choice of a primitive dth root of unity defines an isomorphism of groups $\mu_d \simeq \mathbb{Z}/d\mathbb{Z}$.

Proposition 10.1.2. *Let M' be the dual lattice of N' and let $m_1, m_2 \in M'$ be dual to u_1, u_2 in N'. Using the coordinates $x = \chi^{m_1}$ and $y = \chi^{m_2}$ of $\mathbb{C}^2$, the action of $\zeta \in \mu_d \simeq N/N'$ on $\mathbb{C}^2$ is given by*

$$\zeta \cdot (x, y) = (\zeta x, \zeta^k y).$$

Furthermore, $U_\sigma \simeq \mathbb{C}^2/\mu_d$ with respect to this action.

Proof. The general discussion in §1.3 shows that the quotient $N/N' \simeq \mathbb{Z}/d\mathbb{Z}$ acts on the coordinate ring of $\mathbb{C}^2$ via

$$(u+N')\cdot\chi^{m'} = e^{2\pi i\langle m',u\rangle}\chi^{m'}, \tag{10.1.3}$$

where $m' \in \sigma^\vee \cap M'$ and $u = je_1$ for $0 \le j \le d-1$.

An easy calculation shows that $\langle m_1, e_1\rangle = 1/d$ and $\langle m_2, e_1\rangle = k/d$. Hence if we set up the isomorphism $\mu_d \simeq N/N'$ by mapping $e^{2\pi ij/d} \mapsto je_1 + N'$, then for all $\zeta = e^{2\pi ij/d} \in \mu_d$, we have

$$\zeta\cdot(x,y) = (e^{2\pi ij/d}x, e^{2\pi ijk/d}y) = (\zeta x, \zeta^k y)$$

by (10.1.3). This is what we wanted to show. $\square$

We next describe the slight but manageable ambiguity in the normal form for 2-dimensional cones. Two cones are *lattice equivalent* if there is a bijective $\mathbb{Z}$-linear mapping $\varphi : N \to N$ taking one cone to the other. After choice of basis for N, such mappings are defined by matrices in $\mathrm{GL}(2,\mathbb{Z})$.

Proposition 10.1.3. *Let* $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ *and* $\widetilde{\sigma} = \mathrm{Cone}(e_2', \widetilde{d}e_1' - \widetilde{k}e_2')$ *be cones in normal form that are lattice equivalent. Then* $\widetilde{d} = d$ *and either* $\widetilde{k} = k$ *or* $k\widetilde{k} \equiv 1 \bmod d$.

Proof. Since the cones are lattice equivalent, writing N' and $\widetilde{N}'$ for the sublattices as in (10.1.2), there is a bijective $\mathbb{Z}$-linear mapping $\varphi : N \to N$ such that $\varphi(N') = \widetilde{N}'$. Hence $N/\widetilde{N}' \simeq N/N'$, so $\widetilde{d} = d$. The statement about k and $\widetilde{k}$ is left to the reader in Exercise 10.1.2. $\square$

Here are two examples to illustrate Proposition 10.1.2.

Example 10.1.4. First consider a cone

$$\sigma = \mathrm{Cone}(e_2, de_1 - e_2)$$

with parameters $d > 1$ (so the cone is not smooth) and $k = 1$. This is precisely the cone considered in Example 1.2.22. The corresponding toric surface U_σ is the rational normal cone $\widehat{C}_d \subseteq \mathbb{C}^{d+1}$. The quotient $\widehat{C}_d \simeq \mathbb{C}^2/\mu_d$ was studied in the special case $d = 2$ in Example 1.3.19, and the general case was described in Exercise 1.3.11. With the notation of Proposition 10.1.2, $\zeta \in \mu_d$ acts on $(x,y) \in \mathbb{C}^2$ via $\zeta\cdot(x,y) = (\zeta x, \zeta y)$ and the ring of invariants is

$$\mathbb{C}[x,y]^{\mu_d} = \mathbb{C}[x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d],$$

so

$$U_\sigma \simeq \mathbb{C}^2/\mu_d \simeq \mathrm{Spec}(\mathbb{C}[x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d]).$$

On the other hand, from Example 1.2.22, we also have the description

$$U_\sigma \simeq \mathrm{Spec}(\mathbb{C}[s, st, st^2, \ldots, st^d]).$$

Exercise 10.1.3 studies the relation between these representations of the coordinate ring of U_σ. $\Diamond$

Example 10.1.5. Next consider a cone σ with parameters d and $k = d-1$, so $d = k+1$. We will express everything in terms of the parameter k in the following. Unlike the previous example, this is a case we have not encountered previously. Note that $k \equiv -1 \bmod d$. Hence by Proposition 10.1.2, the action of $G = N/N'$ on $\mathbb{C}^2$ is given by

$$\zeta \cdot (x,y) = (\zeta x, \zeta^{-1} y).$$

It is easy to check that the ring of invariants here is

$$\mathbb{C}[x,y]^{\mu_{k+1}} = \mathbb{C}[x^{k+1}, y^{k+1}, xy].$$

Moreover we have an isomorphism of rings

$$\begin{aligned}\varphi : \mathbb{C}[X,Y,Z]/\langle Z^{k+1} - XY\rangle &\simeq \mathbb{C}[x^{k+1}, y^{k+1}, xy]\\ X &\mapsto x^{k+1}\\ Y &\mapsto y^{k+1}\\ Z &\mapsto xy,\end{aligned}$$

so we may identify the toric surface U_σ with the variety $\mathbf{V}(Z^{k+1} - XY) \subseteq \mathbb{C}^3$. $\Diamond$

The origin is the unique singular point of the affine variety of Example 10.1.5 and is called a *rational double point* (or *Du Val singularity*) of type A_k. Another standard form of these singularities is given in Exercise 10.1.4. They are called *double* points because the lowest degree nonzero term in the defining equation has degree two (i.e., the *multiplicity* of the singularity is two). The *rational* double points are the simplest singularities from a certain point of view. The exact definition, which we will give in §10.4, depends on the notion of a resolution of singularities, which will be introduced shortly. All rational double points appear as singularities of quotient surfaces $\mathbb{C}^2/G$ where G is a finite subgroup of $\mathrm{SU}(2,\mathbb{C})$. There is a complete classification of such points in terms of the *Dynkin diagrams* of types A_k, D_k, E_6, E_7, and E_8. The groups corresponding to the diagrams D_k, E_6, E_7, E_8 are not abelian, so by the comment after (10.1.2), such points do not appear on toric surfaces. We will see one way that the Dynkin diagram A_k appears from the geometry of the toric surface U_σ in Exercise 10.1.5, and we will return to this example in §10.4. More details on these singularities can be found in [**245**, Ch. VI] and in the article [**85**].

Here is another interesting aspect of Example 10.1.5. Recall that a normal variety X is *Gorenstein* if its canonical divisor is Cartier (Definition 8.2.14). The following result was proved in Exercise 8.2.13 of Chapter 8.

Proposition 10.1.6. *For a cone $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ in normal form, the affine toric surface U_σ is Gorenstein if and only if $k = d-1$.* $\square$

Toric Resolution of Singularities. Let X be a normal toric surface, and denote by X_{sing} the finite set of singular points of X (possibly empty).

Definition 10.1.7. A proper morphism $\varphi : Y \to X$ is a ***resolution of singularities*** of X if Y is a smooth surface and φ induces an isomorphism of varieties

$$Y \setminus \varphi^{-1}(X_{\text{sing}}) \simeq X \setminus X_{\text{sing}}. \tag{10.1.4}$$

Such a mapping modifies X to produce a smooth variety without changing the smooth locus $X \setminus X_{\text{sing}}$. One of the most appealing aspects of toric varieties is the way that many questions that are difficult for general varieties admit simple and concrete solutions in the toric case. The problem of finding resolutions of singularities is a perfect example. We illustrate this by constructing explicit resolutions of singularities of the toric surfaces from Examples 10.1.4 and 10.1.5.

Example 10.1.8. Consider the rational normal cone of degree d, the affine toric surface U_σ for $\sigma = \text{Cone}(e_2, de_1 - e_2)$ studied in Example 10.1.4. Let Σ be the fan in Figure 1 obtained by inserting a new ray $\tau = \text{Cone}(e_1)$ subdividing σ into two 2-dimensional cones:

$$\begin{aligned}\sigma_1 &= \text{Cone}(e_2, e_1)\\ \sigma_2 &= \text{Cone}(e_1, de_1 - e_2).\end{aligned}$$

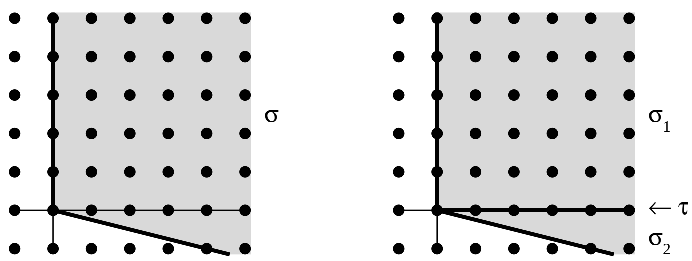


Figure 1. The cone σ and the refinement given by σ_1, σ_2, τ

We now use some results from Chapter 3. The identity mapping on the lattice N is compatible with the fans Σ and σ as in Definition 3.3.1. By Theorem 3.3.4, we have a corresponding toric blowup morphism

$$\phi : X_\Sigma \longrightarrow U_\sigma. \tag{10.1.5}$$

Note that both σ_1 and σ_2 (as well as all of their faces) are smooth cones. Hence Theorem 3.1.19 implies that X_Σ is a smooth surface. In addition, the toric morphism ϕ is proper by Theorem 3.4.11 since Σ is a refinement of σ. Finally, we claim that ϕ satisfies (10.1.4). This follows from the Orbit-Cone Correspondence on the two surfaces: if p_σ is the distinguished point corresponding to the 2-dimensional cone σ (the singular point of U_σ at the origin), then ϕ restricts to an isomorphism

$$X_\Sigma \setminus \phi^{-1}(p_\sigma) \simeq U_\sigma \setminus \{p_\sigma\} = (U_\sigma)_{\text{smooth}}.$$

The inverse image $E = \phi^{-1}(p_\sigma)$ is the curve on X_Σ given by the closure of the T_N-orbit $O(\tau)$ corresponding to the ray τ. That is, the singular point "blows up" to $E \simeq \mathbb{P}^1$ on the smooth surface. It follows that X_Σ and the morphism (10.1.5) give a toric resolution of singularities of the rational normal cone. We call E the *exceptional divisor* on the smooth surface. We will say more about how E sits inside the surface X_Σ in §10.4. $\Diamond$

Example 10.1.9. We consider the case $d = 4$ of Example 10.1.5, for which the surface U_σ has a rational double point of type A_3. We will leave the details, as well as the generalization to all $d \geq 2$, to the reader (Exercise 10.1.5). It is easy to find subdivisions of

$$\sigma = \text{Cone}(e_2, 4e_1 - 3e_2)$$

yielding collections of smooth cones. The most economical way to do this is to insert three new rays $\rho_1 = \text{Cone}(e_1)$, $\rho_2 = \text{Cone}(2e_1 - e_2)$, $\rho_3 = \text{Cone}(3e_1 - 2e_2)$ to obtain a fan Σ consisting of four 2-dimensional cones and their faces.

The fan produced by this subdivision is somewhat easier to visualize if we draw the cones relative to a different basis u_1, u_2 for N. For $u_1 = e_2$ and $u_2 = e_1 - e_2$, the cone $\sigma = \text{Cone}(u_1, u_1 + 4u_2)$ and the fan Σ with maximal cones

$$\begin{aligned}
\sigma_1 &= \text{Cone}(u_1, u_1 + u_2)\\
\sigma_2 &= \text{Cone}(u_1 + u_2, u_1 + 2u_2)\\
\sigma_3 &= \text{Cone}(u_1 + 2u_2, u_1 + 3u_2)\\
\sigma_4 &= \text{Cone}(u_1 + 3u_2, u_1 + 4u_2)
\end{aligned} \tag{10.1.6}$$

appear in Figure 2.

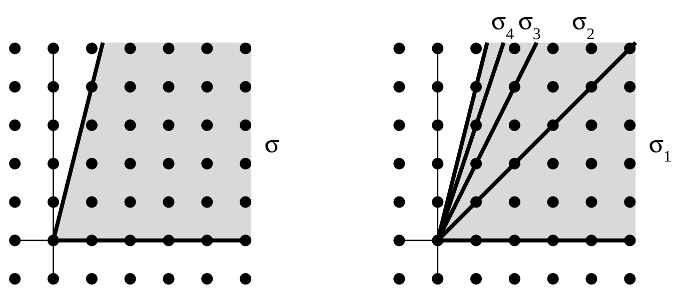


Figure 2. The cone σ and the refinement Σ

You will check that each of these cones is smooth. Hence X_Σ is a smooth surface. Since Σ is a refinement of σ, we have a proper toric morphism

$$\phi : X_\Sigma \to U_\sigma.$$

As in the previous example, ϕ restricts to an isomorphism from $X_\Sigma \setminus \phi^{-1}(p_\sigma)$ to $X_\Sigma \setminus \{p_\sigma\}$. In this case, the exceptional divisor $E = \phi^{-1}(p_\sigma)$ is the union

$$E = V(\tau_1) \cup V(\tau_2) \cup V(\tau_3)$$

on X_Σ. The curves $V(\tau_i)$ are isomorphic to $\mathbb{P}^1$. The first two intersect transversely at the fixed point of the T_N-action on X_Σ corresponding to the cone σ_2, while the second two intersect transversely at the fixed point corresponding to σ_3. $\Diamond$

In these examples, we constructed toric resolutions of affine toric surfaces with just one singular point. The same techniques can be applied to any normal toric surface X_Σ.

Theorem 10.1.10. *Let X_Σ be a normal toric surface. There exists a smooth fan Σ' refining Σ such that the associated toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ is a toric resolution of singularities.*

Proof. It suffices to show the existence of the smooth fan Σ' refining Σ. The reasoning given in Example 10.1.8 applies to show that the corresponding toric morphism ϕ is proper and birational, hence a resolution of singularities of X_Σ.

We will prove this by induction on an integer invariant of fans that measures the complexity of the singularities on the corresponding surfaces. Let $\sigma_1, \ldots, \sigma_\ell$ denote the 2-dimensional cones in a fan Σ. For each i, we will write N_i for the sublattice of N generated by the ray generators of σ_i. Then we define

$$s(\Sigma) = \sum_{i=1}^{\ell} (\operatorname{mult}(\sigma_i) - 1),$$

where $\operatorname{mult}(\sigma_i) = [N : N_i]$ as in §6.3. If $s(\Sigma) = 0$, then $\ell = 0$ or $\operatorname{mult}(\sigma_i) = 1$ for all i. It is easy to see that this implies that Σ is a smooth fan. Hence X_Σ is already smooth and we take this as the base case for our induction.

For the induction step, we assume that the existence of smooth refinements has been established for all fans Σ with $s(\Sigma) < s$, and consider a fan Σ with $s(\Sigma) = s$. If $s \geq 1$, then there exists some nonsmooth cone σ_i in Σ. By Proposition 10.1.1, there is a basis e_1, e_2 for N such that $\sigma_i = \operatorname{Cone}(e_2, de_1 - ke_2)$ with parameters $d > 0$, $0 \leq k < d$, and $\gcd(d,k) = 1$. Consider the refinement Σ' of Σ obtained by subdividing the cone σ_i into two new cones

$$\sigma_i' = \operatorname{Cone}(e_2, e_1)$$
$$\sigma_i'' = \operatorname{Cone}(e_1, de_1 - ke_2)$$

with a new 1-dimensional cone $\rho = \text{Cone}(e_1)$. We must show that $s(\Sigma') < s(\Sigma)$ to invoke the induction hypothesis and conclude the proof.

In $s(\Sigma)$, the terms corresponding to the other cones σ_j for $j \neq i$ are unchanged. The cone σ_i' is smooth since e_1, e_2 is the normalized basis of N relative to σ_i. So it contributes a zero term in $s(\Sigma')$. Now consider the cone σ_i''. In order to compute its contribution to $s(\Sigma')$, we must determine the parameters of σ_i''.

In terms of the basis e_1, e_2 for N, the $\mathbb{Z}$-linear mapping defined by the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(a "90-degree rotation") takes σ_i'' to $\text{Cone}(e_2, ke_1 + de_2)$. Since $A \in \text{GL}(2, \mathbb{Z})$, it defines an automorphism of N, and hence σ_i'' will have the same parameters as $\text{Cone}(e_2, ke_1 + de_2)$. But now we apply (10.1.1) to write

$$d = sk - l \tag{10.1.7}$$

where $0 \leq l < k$. Since $\gcd(d, k) = 1$, we have $\gcd(k, l) = 1$ as well. Hence the cone σ_i'' has parameters k and l obtained from (10.1.7). Since $k < d$, if N_i'' is the sublattice generated by the ray generators of σ_i'', then by (10.1.2),

$$[N : N_i''] = k < [N : N_i] = d.$$

It follows that $s(\Sigma') < s(\Sigma)$, and the proof is complete by induction. □

We will see in the next section that in the affine case, the refinement that gives the resolution of singularities of U_σ has a very nice description. As a preview, notice that in Examples 10.1.8 and 10.1.9, the refinement of the given cone σ was produced by subdividing along the rays through the Hilbert basis (the irreducible elements) of the semigroup $\sigma \cap N$.

A resolution of a non-normal toric surface singularity can be constructed by first saturating the associated semigroup as in Theorem 1.3.5, then applying the results of this section. Toric resolutions of singularities for toric varieties of dimension three and larger also exist. However, we postpone the higher-dimensional case until Chapter 11.

Exercises for §10.1.

10.1.1. Adapt the usual proof of the integer division algorithm to prove (10.1.1).

10.1.2. In this exercise, you will develop further properties of the parameters d, k in the normal form for cones from Proposition 10.1.1 and prove part of Proposition 10.1.3.

(a) Show that if $\widetilde{\sigma}$ is obtained from a cone σ by parameters d, k by a $\mathbb{Z}$-linear mapping of N defined by a matrix in $\text{GL}(2, \mathbb{Z})$, then the parameter $\widetilde{k}$ of $\widetilde{\sigma}$ satisfies either $\widetilde{k} = k$, or $k\widetilde{k} \equiv 1 \bmod d$. Hint: There is a choice of *orientation* to be made in the normalization process. Recall that $\gcd(d, k) = 1$, so there are integers $\widetilde{d}, \widetilde{k}$ such that $d\widetilde{d} + k\widetilde{k} = 1$.

(b) Show that if σ is a cone with parameters d,k, then the dual cone $\sigma^\vee \subseteq M_\mathbb{R}$ has parameters $d,d-k$. Hint: Use the normal form for σ, write down $\sigma^\vee$ in the corresponding dual basis in M, then change bases in M to normalize $\sigma^\vee$.

10.1.3. With the notation in Example 10.1.4, show that

$$\mathbb{C}[s,st,st^2,\dots,st^d] \simeq \mathbb{C}[x^d,x^{d-1}y,\dots,xy^{d-1},y^d]$$

under $s \mapsto x^d$ and $t \mapsto y/x$, and use Proposition 10.1.2 to explain where these identifications come from in terms of the semigroup S_σ. Hint: We have $s = \chi^{m_1}$ and $t = \chi^{m_2}$ where e_1,e_2 is the normalized basis for N and m_1,m_2 is the dual basis for M.

10.1.4. In Example 10.1.5, we gave one form of the rational double point of type A_k, namely the singular point at $(0,0,0)$ on the surface $V = \mathbf{V}(Z^{k+1}-XY) \subseteq \mathbb{C}^3$. Another commonly used normal form for this type of singularity is the singular point at $(0,0,0)$ on the surface $W = \mathbf{V}(X^{k+1}+Y^2+Z^2)$. Show that V and W are isomorphic as affine varieties, hence the singularities at the origin are analytically equivalent. Hint: There is a linear change of coordinates in $\mathbb{C}^3$ that does this.

10.1.5. In this exercise, you will check the claims made in Example 10.1.9 and show how to extend the results there to the case $\sigma = \mathrm{Cone}(e_2, de_1-(d-1)e_2)$ for general d.

(a) Check that each of the four cones in (10.1.6) is smooth, so that the toric surface X_Σ is smooth by Theorem 3.1.19.

(b) For general d, show how to insert new rays ρ_i to subdivide σ and obtain a fan Σ whose associated toric surface is smooth. Try to do this with as few new rays as possible. Hence we obtain toric resolutions of singularities $\phi : X_\Sigma \to U_\sigma$ for all d.

(c) Identify the inverse image $C = \phi^{-1}(p_\sigma)$ in general. For instance, how many irreducible components does C have? How are they connected? Hint: One way to represent the structure is to draw a graph with vertices corresponding to the components and connect two vertices by an edge if and only if the components intersect on X_Σ. Do you notice a relation between this graph and the Dynkin diagram $A_k = A_{d-1}$ mentioned before? We will discuss the relation in detail in §10.4.

§10.2. Continued Fractions and Toric Surfaces

To relate continued fractions to toric surfaces, we begin with the affine toric surface U_σ of a cone $\sigma \subseteq N_\mathbb{R} \simeq \mathbb{R}^2$ in normal form with parameters d,k. We will always assume $d > k > 0$, so that U_σ has a unique singular point.

Hirzebruch-Jung Continued Fractions. When we construct a resolution of singularities of U_σ by following the proof of Theorem 10.1.10, the first step is to refine the cone $\sigma = \mathrm{Cone}(e_2, de_1-ke_2)$ to a fan containing the 2-dimensional cones

$$\sigma' = \mathrm{Cone}(e_2,e_1) \quad \text{and} \quad \sigma'' = \mathrm{Cone}(e_1, de_1-ke_2).$$

The first is smooth, but the second may not be. However, we saw in the proof of Theorem 10.1.10 that the cone σ'' has parameters k,k_1 satsifying

$$d = b_1k - k_1,$$

where $b_1 \geq 2$, $0 \leq k_1 < k$ as in (10.1.1). We used slightly different notation before, writing s rather than b_1 and l rather than k_1; the new notation will help us keep track of what happens as we continue the process and refine the cone σ''.

Using the normalized basis for N relative to σ'', we insert a new ray and obtain a new smooth cone and a second, possibly nonsmooth cone with parameters k_1, k_2, where

$$k = b_2 k_1 - k_2$$

using (10.1.1). Doing this repeatedly yields a *modified Euclidean algorithm*

$$\begin{aligned} d &= b_1 k - k_1 \\ k &= b_2 k_1 - k_2 \\ &\vdots \\ k_{r-3} &= b_{r-1} k_{r-2} - k_{r-1} \\ k_{r-2} &= b_r k_{r-1} \end{aligned} \tag{10.2.1}$$

that computes the parameters of the new cones produced as we successively subdivide to produce the fan giving the resolution of singularities. The process terminates with $k_r = 0$ for some r as shown, since as in the usual Euclidean algorithm, the k_i are a strictly decreasing sequence of nonnegative numbers. Also, by (10.1.1), we have $b_i \geq 2$ for all i.

The equations (10.2.1) can be rearranged:

$$\begin{aligned} d/k &= b_1 - k_1/k \\ k/k_1 &= b_2 - k_2/k_1 \\ &\vdots \\ k_{r-3}/k_{r-2} &= b_{r-1} - k_{r-1}/k_{r-2} \\ k_{r-2}/k_{r-1} &= b_r \end{aligned} \tag{10.2.2}$$

and spliced together to give a type of continued fraction expansion for the rational number d/k, with minus signs:

$$d/k = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_r}}}. \tag{10.2.3}$$

This is the *Hirzebruch-Jung continued fraction expansion* of d/k. For obvious typographical reasons, it is desirable to have a more compact way to represent these expressions. We will use the notation

$$d/k = [[b_1, b_2, \ldots, b_r]].$$

The integers b_i are the *partial quotients* of the Hirzebruch-Jung continued fraction, and the truncated Hirzebruch-Jung continued fractions

$$[[b_1, b_2, \dots, b_i]], \quad 1 \le i \le r,$$

are the *convergents*.

Example 10.2.1. Consider the rational number $17/11$. The Hirzebruch-Jung continued fraction expansion is

$$17/11 = [[2,3,2,2,2,2]],$$

as may be verified directly using the modified Euclidean algorithm (10.2.1). $\Diamond$

Proposition 10.2.2. *Let $d > k > 0$ be integers with $\gcd(d,k) = 1$ and let $d/k = [[b_1, \dots, b_r]]$. Define sequences P_i and Q_i recursively as follows. Set*

$$\begin{aligned} P_0 &= 1, & Q_0 &= 0 \\ P_1 &= b_1, & Q_1 &= 1, \end{aligned} \tag{10.2.4}$$

and for all $2 \le i \le r$, let

$$\begin{aligned} P_i &= b_i P_{i-1} - P_{i-2} \\ Q_i &= b_i Q_{i-1} - Q_{i-2}. \end{aligned} \tag{10.2.5}$$

Then the P_i, Q_i satisfy:

(a) *The P_i and Q_i are increasing sequences of integers.*

(b) $[[b_1, \dots, b_i]] = P_i/Q_i$ *for all* $1 \le i \le r$.

(c) $P_{i-1}Q_i - P_iQ_{i-1} = 1$ *for all* $1 \le i \le r$.

(d) *The convergents form a strictly decreasing sequence:*

$$\frac{d}{k} = \frac{P_r}{Q_r} < \frac{P_{r-1}}{Q_{r-1}} < \cdots < \frac{P_1}{Q_1}.$$

Proof. The proof of part (a) is left to the reader (Exercise 10.2.1).

To prove part (b), first observe that the expression on the right side of (10.2.3) makes sense when the b_j are any rational numbers (not just integers) such that all denominators in (10.2.3) are nonzero. We will show that the sequences defined by (10.2.5) satisfy

$$[[b_1, \dots, b_s]] = \frac{P_s}{Q_s}$$

for all such lists $b_1, \dots, b_s$. The proof is by induction on the length s of the list. When $s = 1$, we have $[[b_1]] = b_1 = \frac{P_1}{Q_1}$ by (10.2.4). Now assume that the result has been proved for all lists of of length t and consider the expression

$$[[b_1, \dots, b_{t+1}]] = [[b_1, \dots, b_t - \frac{1}{b_{t+1}}]],$$

where the right side comes from a list of length t. By the induction hypothesis, this equals

$$\frac{\left(b_t - \frac{1}{b_{t+1}}\right)P_{t-1} - P_{t-2}}{\left(b_t - \frac{1}{b_{t+1}}\right)Q_{t-1} - Q_{t-2}}.$$

By the recurrences (10.2.5), this equals

$$\frac{P_t - \frac{1}{b_{t+1}}P_{t-1}}{Q_t - \frac{1}{b_{t+1}}Q_{t-1}} = \frac{b_{t+1}P_t - P_{t-1}}{b_{t+1}Q_t - Q_{t-1}} = \frac{P_{t+1}}{Q_{t+1}},$$

which is what we wanted to show.

Part (c) will be proved by induction on i. The base case $i = 1$ follows directly from (10.2.4). Now assume that the result has been proved for $i \le s$, and consider $i = s+1$. Using the recurrences (10.2.5), we have

$$\begin{aligned} P_sQ_{s+1} - P_{s+1}Q_s &= P_s(b_sQ_s - Q_{s-1}) - (b_sP_s - P_{s-1})Q_s \\ &= P_{s-1}Q_s - P_sQ_{s-1} \\ &= 1 \end{aligned}$$

by the induction hypothesis.

Finally, from part (b), for each $1 \le i \le r-1$, we have

$$\frac{P_{i-1}}{Q_{i-1}} = \frac{P_i}{Q_i} + \frac{1}{Q_{i-1}Q_i}.$$

Hence

$$\frac{P_i}{Q_i} < \frac{P_{i-1}}{Q_{i-1}}$$

since $Q_{i-1}Q_i > 0$ by part (a). Hence part (d) follows. □

Hirzebruch-Jung Continued Fractions and Resolutions. When σ is a cone with parameters $d > k > 0$, the process of computing the Hirzebruch-Jung continued fraction of d/k yields a convenient method for finding a refinement Σ of σ such that $\phi : X_\Sigma \to U_\sigma$ is a toric resolution of singularities.

Theorem 10.2.3. *Let $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ be in normal form. Let $u_0 = e_2$ and use the integers P_i and Q_i from Proposition 10.2.2 to construct vectors*

$$u_i = P_{i-1}e_1 - Q_{i-1}e_2, \quad 1 \le i \le r+1.$$

Then the cones

$$\sigma_i = \mathrm{Cone}(u_{i-1}, u_i), \quad 1 \le i \le r+1,$$

have the following properties:

(a) *Each σ_i is a smooth cone and u_{i-1}, u_i are its ray generators.*

(b) *For each i, $\sigma_{i+1} \cap \sigma_i = \mathrm{Cone}(u_i)$.*

(c) *$\sigma_1 \cup \cdots \cup \sigma_{r+1} = \sigma$, so the fan Σ consisting of the σ_i and their faces gives a smooth refinement of σ.*

(d) *The toric morphism $\phi : X_\Sigma \to U_\sigma$ is a resolution of singularities.*

Proof. Both statements in part (a) follow easily from part (c) of Proposition 10.2.2.

For part (b), we note that the ratio $-Q_{i-1}/P_{i-1}$ represents the *slope* of the line through u_i in the coordinate system relative to the normalized basis e_1, e_2 for σ. By part (d) of Proposition 10.2.2, these slopes form a strictly decreasing sequence for $i \geq 0$, which implies the statement in part (b).

Part (c) follows from part (b) by noting that $u_0 = e_2$ and $P_r/Q_r = d/k$, so $u_{r+1} = de_1 - ke_2$. Hence the cones σ_i fill out σ.

Part (d) now follows by the reasoning used in Examples 10.1.8 and 10.1.5. $\square$

Example 10.2.4. Consider the cone $\sigma = \mathrm{Cone}(e_2, 7e_1 - 5e_2)$ in normal form. To construct the resolution of singularities of the affine toric surface U_σ, we simply compute the Hirzebruch-Jung continued fraction expansion of the rational number $d/k = 7/5$ using the modified Euclidean algorithm:

$$
\begin{aligned}
7 &= 2 \cdot 5 - 3\\
5 &= 2 \cdot 3 - 1\\
3 &= 3 \cdot 1.
\end{aligned}
$$

Hence $b_0 = b_1 = 2, b_2 = 3$, and

$$7/5 = [[2,2,3]]. \tag{10.2.6}$$

Then from Proposition 10.2.2 we have

$$
\begin{aligned}
&P_0 = 1, && Q_0 = 0\\
&P_1 = 2, && Q_1 = 1\\
&P_2 = b_2P_1 - P_0 = 3, && Q_2 = b_2Q_1 - Q_0 = 2\\
&P_3 = b_3P_2 - P_1 = 7, && Q_3 = b_3Q_2 - Q_1 = 5.
\end{aligned}
$$

Theorem 10.2.3 gives the vectors

$$u_0 = e_2,\ u_1 = e_1,\ u_2 = 2e_1 - e_2,\ u_3 = 3e_1 - 2e_2,\ u_4 = 7e_1 - 5e_2$$

and the cones

$$
\begin{aligned}
\sigma_1 &= \mathrm{Cone}(e_2, e_1)\\
\sigma_2 &= \mathrm{Cone}(e_1, 2e_1 - e_2)\\
\sigma_3 &= \mathrm{Cone}(2e_1 - e_2, 3e_1 - 2e_2)\\
\sigma_4 &= \mathrm{Cone}(3e_1 - 2e_2, 7e_1 - 5e_2)
\end{aligned}
\tag{10.2.7}
$$

shown in Figure 3 on the next page. The cones σ_i give a smooth refinement of σ. You will see another example of this process in Exercise 10.2.2. $\Diamond$

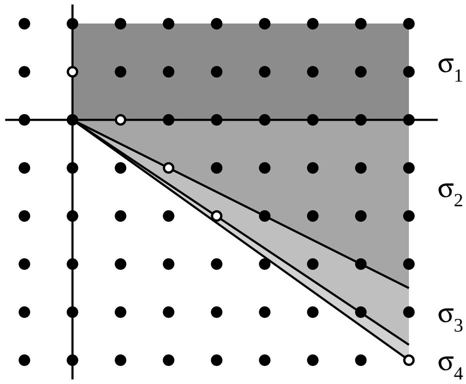


Figure 3. The refinement Σ with open circles at $u_i = P_{i-1}e_1 - Q_{i-1}e_2$ in Example 10.2.4

Next we show that the vectors u_i from Theorem 10.2.3 determine the partial quotients in the Hirzebruch-Jung continued fraction expansion of d/k.

Theorem 10.2.5. *Let $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ be in normal form, and let*

$$d/k = [[b_1, b_2, \dots, b_r]].$$

Then the vectors $u_0, u_1, \dots, u_{r+1}$ constructed in Theorem 10.2.3 satisfy

$$u_{i-1} + u_{i+1} = b_i u_i, \quad b_i \geq 2, \tag{10.2.8}$$

for $1 \leq i \leq r$.

Proof. By the recurrences (10.2.5),

$$\begin{aligned} u_{i-1} + u_{i+1} &= (P_{i-2}e_1 - Q_{i-2}e_2) + (P_i e_1 - Q_i e_2) \\ &= (P_{i-2} + P_i)e_1 - (Q_{i-2} + Q_i)e_2 \\ &= b_i(P_{i-1}e_1 - Q_{i-1}e_2) = b_i u_i. \end{aligned}$$ □

Later in this chapter we will see several important consequences of (10.2.8) connected with the geometry of smooth toric surfaces.

The nonuniqueness of Proposition 10.1.3 has a nice relation to Theorem 10.2.3. For instance, Example 10.2.4 used the Hirzebruch-Jung expansion $7/5 = [[2,2,3]]$. Since $5 \cdot 3 \equiv 1 \bmod 7$, the cone of Example 10.2.4 also has parameters $d = 7$, $k = 3$. We leave it to the reader to check that

$$7/3 = [[3,2,2]],$$

with the partial quotients the same as those in (10.2.6), but listed in reverse order. This pattern holds for all Hirzebruch-Jung continued fractions. We give a proof that uses the properties of the associated toric surfaces.

Proposition 10.2.6. *Let* $0 < k, \widetilde{k} < d$ *and assume* $k\widetilde{k} \equiv 1 \bmod d$. *If the Hirzebruch-Jung continued fraction expansion of* d/k *is*

$$d/k = [[b_1, b_2, \ldots, b_r]],$$

then the Hirzebruch-Jung continued fraction expansion of $d/\widetilde{k}$ *is*

$$d/\widetilde{k} = [[b_r, b_{r-1}, \ldots, b_1]].$$

Proof. Let $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ and $\widetilde{\sigma} = \mathrm{Cone}(e_2, de_1 - \widetilde{k}e_2)$ be the corresponding cones in normal form. Since $k\widetilde{k} \equiv 1 \bmod m$, there is an integer $\widetilde{d}$ such that $d\widetilde{d} + k\widetilde{k} = 1$. The $\mathbb{Z}$-linear mapping $\varphi : N \to N$ defined with respect to the basis e_1, e_2 by the matrix

$$A = \begin{pmatrix} \widetilde{k} & d \\ \widetilde{d} & -k \end{pmatrix}$$

is bijective, maps $\widetilde{\sigma}$ to σ, and is orientation-reversing. Thus $\varphi(de_1 - \widetilde{k}e_2) = e_2$ and $\varphi(e_2) = de_1 - ke_2$. If we apply Theorem 10.2.3 to σ, then we obtain vectors u_i satisfying the equations

$$u_{i-1} + u_{i+1} = b_i u_i$$

for all $1 \le i \le r$. We claim that when we apply the mapping φ^{-1} defined by the inverse of the matrix A above, then the vectors u_i are taken to corresponding vectors $\widetilde{u}_i$ for the cone $\widetilde{\sigma}$. But since φ and φ^{-1} are orientation-reversing, the partial quotients in the Hirzebruch-Jung continued fraction will be listed in the opposite order. You will complete the proof of this assertion in Exercise 10.2.3. □

Hilbert Bases and Convex Hulls. Our next result gives two alternative ways to understand the vectors u_i in Theorem 10.2.3. The idea is that σ gives two objects:

- The semigroup $\sigma \cap N$. Since σ is strongly convex, its irreducible elements form the unique minimal generating set called the *Hilbert basis* of $\sigma \cap N$. (See Proposition 1.2.23.)
- The convex hull $\Theta_\sigma = \mathrm{Conv}(\sigma \cap (N \setminus \{0\}))$. This is an unbounded polygon in the plane whose bounded edges contain finitely many lattice points.

Example 10.2.7. Consider the cone $\sigma = \mathrm{Cone}(e_2, 7e_1 - 5e_2)$ from Example 10.2.4. Figure 4 on the next page shows the convex hull Θ_σ, where the white circles represent the lattice points on the bounded edges. We will see below that these lattice points give the Hilbert basis of $\sigma \cap N$. ◊

Here is the general result suggested by Example 10.2.7.

Theorem 10.2.8. *Let* $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ *be in normal form and let*

$$S = \{u_0, u_1, \ldots, u_{r+1}\}$$

be the set of vectors constructed in Theorem 10.2.3. Then:

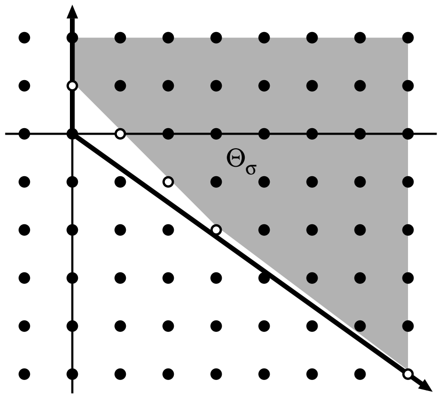


Figure 4. Convex hull Θ_σ and lattice points on bounded edges in Example 10.2.7

(a) *S is the Hilbert basis of the semigroup $\sigma \cap N$.*

(b) *S is the set of lattice points on the bounded edges of Θ_σ.*

Proof. For part (a), we use the notation of Theorem 10.2.3, where the cone σ_i is generated by u_{i-1}, u_i. Then $\sigma_i \cap N$ is generated as a semigroup by u_{i-1}, u_i since σ_i is smooth. Using $\sigma = \sigma_1 \cup \cdots \cup \sigma_{r+1}$, one sees easily that S generates $\sigma \cap N$.

We claim next that all the u_i are irreducible elements of $\sigma \cap N$. This is clear for $u_0 = e_2$ and $u_{r+1} = de_1 - ke_2$ since they are the ray generators for σ. If $1 \le i \le r$ and u_i is not irreducible, then u_i would have to be a linear combination of the vectors in $S \setminus \{u_i\}$ with nonnegative integer coefficients, i.e.,

$$u_i = P_{i-1}e_1 - Q_{i-1}e_2 = \sum_{j \neq i} c_j u_j = \Big(\sum_{j \neq i} c_j P_{j-1}\Big)e_1 - \Big(\sum_{j \neq i} c_j Q_{j-1}\Big)e_2$$

with $c_j \ge 0$ in $\mathbb{Z}$. Hence

$$P_{i-1} = \sum_{j \neq i} c_j P_{j-1}, \qquad Q_{i-1} = \sum_{j \neq i} c_j Q_{j-1}.$$

Since the P_i and Q_i are strictly increasing by part (a) of Proposition 10.2.2, we must have $c_j = 0$ for all $j > i$. But this would imply that u_i is a linear combination with nonnegative integer coefficients of the vectors in $\{u_0, \ldots, u_{i-1}\}$. This contradicts the observation made in the proof of Theorem 10.2.3 that the slopes of the u_i are strictly decreasing. It follows that the u_i are irreducible elements of $\sigma \cap N$.

Finally, we must show that there are no other irreducible elements in $\sigma \cap N$. But this follows from what we have already said. Since $\sigma = \sigma_1 \cup \cdots \cup \sigma_{r+1}$, if u is irreducible, then $u \in \sigma_i \cap N$ for some i. But then $u = c_{i-1}u_{i-1} + c_i u_i$ for some $c_{i-1}, c_i \ge 0$ in $\mathbb{Z}$. Thus u is irreducible only if $u = u_{i-1}$ or u_i.

For part (b), first observe that by Proposition 10.2.2, we have

$$\begin{aligned} P_{i-2}Q_i - P_iQ_{i-2} &= P_{i-2}(b_iQ_{i-1} - Q_{i-2}) - (b_iP_{i-1} - P_{i-2})Q_{i-2} \\ &= b_i(P_{i-2}Q_{i-1} - P_{i-1}Q_{i-2}) = b_i \geq 2. \end{aligned}$$

Combining this with part (c) of Proposition 10.2.2, one obtains the inequality

$$\frac{-(Q_{i-1} - Q_{i-2})}{P_{i-1} - P_{i-2}} \leq \frac{-(Q_i - Q_{i-1})}{P_i - P_{i-1}}.$$

Since $u_i = P_{i-1}e_1 - Q_{i-1}e_2$, this inequality tells us that the slopes of the line segments $\overline{u_{i-1}u_i}$ and $\overline{u_iu_{i+1}}$ are related by

$$\text{slope of } \overline{u_{i-1}u_i} \leq \text{slope of } \overline{u_iu_{i+1}}.$$

This implies that these line segments lie on boundary of Θ_σ. From here, it is easy to see that the u_i are the lattice points of the bounded edges of Θ_σ. □

Ordinary Continued Fractions. The Hirzebruch-Jung continued fractions studied above are less familiar than ordinary continued fraction expansions in which the minus signs are replaced by plus signs. If $d > k > 0$ are integers, then the ordinary continued fraction expansion of d/k may be obtained by performing the same sequence of integer divisions used in the usual Euclidean algorithm for the gcd. Starting with $k_{-1} = d$ and $k_0 = k$, we write a_i for the quotient and k_i for the remainder at each step, so that the ith division is given by

$$k_{i-2} = a_ik_{i-1} + k_i, \tag{10.2.9}$$

where $0 \leq k_i < k_{i-1}$.

Let k_{s-1} be the final nonzero remainder (which equals $\gcd(d,k)$). The resulting equations splice together to form the *ordinary continued fraction*

$$d/k = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_s}}}.$$

To distinguish these from Hirzebruch-Jung continued fractions, we will use the notation

$$d/k = [a_1, a_2, \ldots, a_s] \tag{10.2.10}$$

for the ordinary continued fraction. The a_i are the (ordinary) *partial quotients* of d/k, and the truncated continued fractions

$$[a_1, a_2, \ldots, a_i], \quad 1 \leq i \leq s,$$

are the (ordinary) *convergents* of d/k.

Example 10.2.9. By Example 10.2.1, the Hirzebruch-Jung continued fraction of $17/11$ is

$$17/11 = [[2,3,2,2,2,2]],$$

and the ordinary continued fraction is

$$17/11 = [1,1,1,5].$$

The partial quotients and the lengths are different. However, each expansion determines the other, and there are methods for computing the Hirzebruch-Jung partial quotients b_j in terms of the ordinary partial quotients a_i and vice versa. See [**75**, Prop. 3.6], [**145**, p. 257], [**231**, Prop. 2.3], and Exercise 10.2.4. $\Diamond$

The following result is mostly parallel to Proposition 10.2.2, but shows that ordinary continued fractions are slightly *more* complicated than Hirzebruch-Jung continued fractions. The proof is left to the reader (Exercise 10.2.5).

Proposition 10.2.10. *Let $d > k > 0$ be integers with $\gcd(d,k) = 1$, and let $d/k = [a_1,\ldots,a_s]$. Define sequences p_i and q_i recursively as follows. First set*

$$\begin{aligned} p_0 &= 1, & q_0 &= 0 \\ p_1 &= a_1, & q_1 &= 1, \end{aligned} \tag{10.2.11}$$

and for all $2 \le i \le s$, let

$$\begin{aligned} p_i &= a_i p_{i-1} + p_{i-2} \\ q_i &= a_i q_{i-1} + q_{i-2}. \end{aligned} \tag{10.2.12}$$

Then the p_i, q_i satisfy:

(a) $[a_1,\ldots,a_i] = p_i/q_i$ *for all* $1 \le i \le s$.

(b) $p_i q_{i-1} - p_{i-1} q_i = (-1)^i$ *for all* $1 \le i \le s$.

(c) *The convergents converge to d/k, but in an oscillating fashion:*

$$\frac{p_1}{q_1} < \frac{p_3}{q_3} < \cdots \le \frac{d}{k} \le \cdots < \frac{p_4}{q_4} < \frac{p_2}{q_2}. \qquad \Box$$

Ordinary Continued Fractions and Convex Hulls. Felix Klein discovered a lovely geometric interpretation of ordinary continued fractions. Given a basis $u^{\mathrm{o}}_{-1}, u^{\mathrm{o}}_0$ of $N \simeq \mathbb{Z}^2$ and relatively prime integers $d > k > 0$, compute the continued fraction $d/k = [a_1,\ldots,a_s]$ and set

$$u^{\mathrm{o}}_i = q_i u^{\mathrm{o}}_{-1} + p_i u^{\mathrm{o}}_0, \quad 1 \le i \le s. \tag{10.2.13}$$

In this notation, the superscript "o" stands for "ordinary." Then $u^{\mathrm{o}}_s = k u^{\mathrm{o}}_{-1} + d u^{\mathrm{o}}_0$, and part (c) of Proposition 10.2.10 implies that the u^{o}_i lie on one side of the ray $\mathrm{Cone}(u^{\mathrm{o}}_s)$ for even indices and on the other side for odd indices. To give a careful description of what is happening, we introduce the cones

$$\sigma_{-1} = \mathrm{Cone}(u^{\mathrm{o}}_{-1}, u^{\mathrm{o}}_s),\ \sigma_0 = \mathrm{Cone}(u^{\mathrm{o}}_0, u^{\mathrm{o}}_s)$$

and associated convex hulls $\Theta_i = \Theta_{\sigma_i} = \mathrm{Conv}(\sigma_i \cap (N \setminus \{0\}))$, $i = -1, 0$.

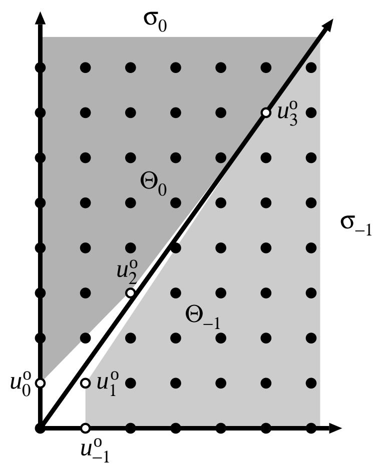


Figure 5. The cones σ_{-1}, σ_0, the convex hulls Θ_{-1}, Θ_0, and their vertices

Example 10.2.11. For $d = 7,\ k = 5$, the expansion $7/5 = [1,2,2]$ gives the vectors $u_{-1}^{\circ}, \ldots, u_3^{\circ}$ shown in Figure 5. In this figure, it is clear that the u_i° are the vertices of the convex hulls Θ_{-1} and Θ_0. $\Diamond$

This example is a special case of the following general result.

Theorem 10.2.12. *For $u_{-1}^{\circ}, \ldots, u_s^{\circ}$ and Θ_{-1}, Θ_0 as above, we have:*

(a) Θ_{-1} *has vertex set* $\{u_{2j-1}^{\circ} \mid 1 \leq j \leq \lfloor s/2 \rfloor\} \cup \{u_s^{\circ}\}$.

(b) Θ_0 *has vertex set* $\{u_{2j}^{\circ} \mid 0 \leq j \leq \lfloor s/2 \rfloor\} \cup \{u_s^{\circ}\}$.

(c) *For* $1 \leq i \leq s$, $\overline{u_{i-2}^{\circ} u_i^{\circ}}$ *is an edge of* Θ_{-1} *(resp.* Θ_0*) for i odd (resp. even) with* $a_i + 1$ *lattice points.*

Proof. First note that by Proposition 10.2.10, the vectors u_i° satisfy the recursion

$$u_i^{\circ} = a_i u_{i-1}^{\circ} + u_{i-2}^{\circ}, \quad 1 \leq i \leq s. \tag{10.2.14}$$

Since u_{i-1}° is primitive (Proposition 10.2.10), part (c) follows from parts (a) and (b).

We now prove the theorem using induction on the length s of the continued fraction expansion. Consider Figure 6 on the next page, which shows the first quadrant determined by $u_{-1}^{\circ}, u_0^{\circ}$, together with the vector u_s°. In the picture, σ_0 (darker) lies above the ray determined by u_s° and σ_{-1} (lighter) lies below. The figure also shows u_1° and the smaller cone $\sigma_1 = \mathrm{Cone}(u_1^{\circ}, u_s^{\circ}) \subseteq \sigma_{-1} = \mathrm{Cone}(u_{-1}^{\circ}, u_s^{\circ})$.

Since $u_s^{\circ} = k u_{-1}^{\circ} + d u_0^{\circ}$, the ray starting from u_{-1}° through u_1° passes through the upper edge of σ_{-1} at a point between $u_{-1}^{\circ} + \lfloor d/k \rfloor u_1^{\circ}$ and $u_{-1}^{\circ} + (\lfloor d/k \rfloor + 1) u_1^{\circ}$. But by the computation of the ordinary continued fraction, $\lfloor d/k \rfloor = a_1$. Therefore the segment from u_{-1}° to u_1° is the first bounded edge of Θ_{-1}. It follows that

$$\Theta_{-1} \cap \mathrm{Cone}(u_{-1}^{\circ}, u_1^{\circ})$$

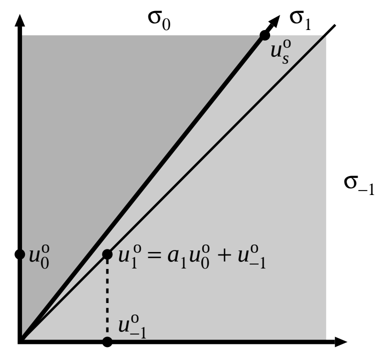


Figure 6. The cones $\sigma_0,\ \sigma_1 \subseteq \sigma_{-1}$ and vectors $u_{-1}^{\circ},\ u_0^{\circ},\ u_1^{\circ},\ u_s^{\circ}$

has vertices $u_{-1}^{\circ}, u_1^{\circ}$. It remains to understand Θ_0 and $\Theta_{-1} \cap \sigma_1$. This is where induction comes into play.

Apply the above construction to the new basis u_0°, u_1° and the continued fraction

$$\frac{k}{d-a_1k} = \frac{1}{\frac{d}{k}-a_1} = [a_2,\dots,a_s]$$

of length $s-1$. If we start numbering at 0 rather than at -1, then the vectors $u_0^{\circ}, u_1^{\circ}, \dots, u_s^{\circ}$ are the same as before by the recursion (10.2.14). This gives the cones σ_0, σ_1 shown in Figure 6. By induction, the vectors $u_0^{\circ}, u_1^{\circ}, \dots, u_s^{\circ}$ give the vertices of the corresponding convex hulls $\Theta_0 = \mathrm{Conv}(\sigma_0 \cap (N \setminus \{0\}))$ and $\Theta_1 = \mathrm{Conv}(\sigma_1 \cap (N \setminus \{0\}))$. It follows easily that the theorem holds for continued fraction expansions of length s. $\square$

A discussion of Klein's formulation of Theorem 10.2.12 can be found in [**231**]. In Exercise 10.2.6 you will apply the theorem to the resolution of pairs of singular points on certain toric surfaces. Geometric pictures similar to Figure 5 have also appeared in recent work of McDuff on symplectic embeddings of 4-dimensional ellipsoids (see [**202**]).

Ordinary Continued Fractions and the Supplementary Cone. To relate the above theorem to toric geometry, we follow the approach of [**231**]. Given a cone $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2) \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^2$ in normal form, its *supplement* is the cone $\sigma' = \mathrm{Cone}(-e_2, de_1 - ke_2)$. Thus $\sigma \cup \sigma'$ is the right half-plane, and the cones σ, σ' give the convex hulls

$$\begin{aligned}\Theta_\sigma &= \mathrm{Conv}(\sigma \cap (N \setminus \{0\}))\\ \Theta_{\sigma'} &= \mathrm{Conv}(\sigma' \cap (N \setminus \{0\})).\end{aligned}$$

Example 10.2.13. When $d = 7,\ k = 5$, Figure 7 on the next page shows the cones σ, σ' and the convex hulls Θ_σ, $\Theta_{\sigma'}$. The open circles in the figure are the vertices

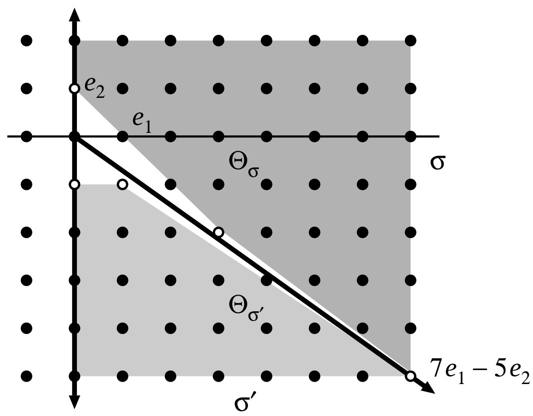


Figure 7. The cones σ,σ', the convex hulls $\Theta_\sigma,\Theta_{\sigma'}$, and their vertices

of Θ_σ and $\Theta_{\sigma'}$. Also observe that the fourth quadrant portion of Figure 7 becomes Figure 5 after a 90° counterclockwise rotation. $\Diamond$

For $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$, the ordinary continued fraction expansion $d/k = [a_1, a_2, \ldots, a_s]$ gives the sequences p_i, q_i, $0 \le i \le s$, defined in Proposition 10.2.10. Then define the vectors

$$u^\circ_{-1} = -e_2,\ u^\circ_i = p_i e_1 - q_i e_2,\ 0 \le i \le s. \tag{10.2.15}$$

These vectors enable us to describe the vertices of $\Theta_\sigma, \Theta_{\sigma'}$ as follows.

Theorem 10.2.14. *Let $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ be a cone in normal form with supplement σ'. Also let u°_i, $-1 \le i \le s$, be as defined in* (10.2.15). *Then:*

(a) *The set of vertices of $\Theta_{\sigma'}$ is*

$$\{u^\circ_{2j-1} \mid 1 \le j \le \lfloor s/2 \rfloor\} \cup \{u^\circ_s\}.$$

(b) *If $a_1 = 1$, then the set of vertices of Θ_σ is*

$$\{e_2\} \cup \{u^\circ_{2j} \mid 1 \le j \le \lfloor s/2 \rfloor\} \cup \{u^\circ_s\}.$$

(c) *If $a_1 > 1$, then the set of vertices of Θ_σ is*

$$\{e_2\} \cup \{u^\circ_{2j} \mid 0 \le j \le \lfloor s/2 \rfloor\} \cup \{u^\circ_s\}.$$

Proof. First note that since $u^\circ_{-1} = -e_2$ and $u^\circ_0 = e_1$, we can rewrite (10.2.15) as

$$u^\circ_i = p_i e_1 - q_i e_2 = q_i u_{-1} + p_i u_0, \quad 0 \le i \le s.$$

Thus we are in the situation of Theorem 10.2.12, where we have the cones $\sigma_{-1} = \mathrm{Cone}(u^\circ_{-1}, u^\circ_s)$ and $\sigma_0 = \mathrm{Cone}(u^\circ_0, u^\circ_s)$ and associated convex hulls $\Theta_{-1} = \Theta_{\sigma_{-1}}$ and $\Theta_0 = \Theta_{\sigma_0}$. Then Theorem 10.2.12 implies that the vectors $u^\circ_{-1}, u^\circ_0, \ldots, u^\circ_s$ give the vertices of Θ_{-1} and Θ_0.

However, $\sigma = \mathrm{Cone}(e_1, e_2) \cup \sigma_0$ and $\sigma' = \sigma_{-1}$. In particular, $\Theta_{\sigma'} = \Theta_{-1}$, so that part (a) of the theorem follows immediately. For parts (b) and (c), note that

$$\Theta_\sigma = (\Theta_\sigma \cap \mathrm{Cone}(e_1, e_2)) \cup \Theta_0.$$

The intersection $\Theta_\sigma \cap \mathrm{Cone}(e_1, e_2)$ has vertices $e_1 = u_0^{\mathrm{o}}, e_2$, while Θ_0 has vertices $u_0^{\mathrm{o}}, u_2^{\mathrm{o}}, \ldots, u_s^{\mathrm{o}}$. If $a_1 = 1$, then $e_2, u_0^{\mathrm{o}}, u_2^{\mathrm{o}}$ are collinear, so that u_0^{o} is not a vertex. This proves part (b) of the theorem. Finally, if $a_1 > 1$, then one can prove without difficulty that u_0^{o} is a vertex (Exercise 10.2.7), and part (c) follows. □

Example 10.2.15. Figure 7 above illustrates Theorem 10.2.12 for the cone $\sigma = \mathrm{Cone}(e_2, 7e_1 - 5e_2)$. Since $7/5 = [1,2,2]$, we use part (b) of the theorem. ◊

Ordinary Continued Fractions and the Dual Cone. For a cone σ in normal form, one surprise is that its supplement σ' is essentially the dual of σ.

Lemma 10.2.16. *Given a cone $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2) \subseteq N_{\mathbb{R}}$ in normal form, its supplementary cone $\sigma' \subseteq N_{\mathbb{R}}$ is isomorphic to $\sigma^\vee \subseteq M_{\mathbb{R}}$.*

Proof. Let e_1^*, e_2^* be the basis of M dual to the basis e_1, e_2 of N. Then the isomorphism defined by $e_1 \mapsto e_2^*$ and $e_2 \mapsto -e_1^*$ takes $\sigma' = \mathrm{Cone}(-e_2, de_1 - ke_2)$ to

$$\mathrm{Cone}(-(-e_1^*), d(e_2^*) - k(-e_1^*)) = \mathrm{Cone}(e_1^*, ke_1^* + de_2^*) = \sigma^\vee. \qquad \square$$

The isomorphism $\sigma' \simeq \sigma^\vee$ from this lemma leads to some nice results about $\sigma^\vee$ and the associated convex hull $\Theta_{\sigma^\vee} = \mathrm{Conv}(\sigma^\vee \cap (M \setminus \{0\}))$. Specifically, this isomorphism takes the vectors

$$u_{-1}^{\mathrm{o}} = -e_2,\ u_i^{\mathrm{o}} = p_i e_1 - q_i e_2, \quad 0 \le i \le s$$

from (10.2.15) to the dual vectors

$$m_{-1} = e_1^*,\ m_i = q_i e_1^* + p_i e_2^*, \quad 0 \le i \le s.$$

In particular, $m_s = ke_1^* + de_2^*$, so that $\sigma^\vee = \mathrm{Cone}(m_{-1}, m_s)$ in this notation. Then Theorem 10.2.14 implies that the vertex set of the convex hull $\Theta_{\sigma^\vee}$ is

$$\{m_{2j-1} \mid 1 \le j \le \lfloor s/2 \rfloor\} \cup \{m_s\}. \tag{10.2.16}$$

Hence, in the language of §7.1, $\Theta_{\sigma^\vee} \subseteq M_{\mathbb{R}}$ is a full dimensional lattice polyhedron. We can describe its recession cone and normal fan as follows.

Proposition 10.2.17. *Let $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2) \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^2$ be in normal form and set $P = \Theta_{\sigma^\vee} = \mathrm{Conv}(\sigma^\vee \cap (M \setminus \{0\}))$. Then:*

(a) *$P \subseteq M_{\mathbb{R}}$ is a full dimensional lattice polyhedron with recession cone $\sigma^\vee$.*

(b) *The normal fan Σ_P of P is the refinement of σ obtained by adding the minimal generators*

$$\begin{array}{ll} u_0^{\mathrm{o}},\ u_2^{\mathrm{o}},\ u_4^{\mathrm{o}},\ \ldots,\ u_{s-1}^{\mathrm{o}} & (s \text{ odd}) \\ u_0^{\mathrm{o}},\ u_2^{\mathrm{o}},\ u_4^{\mathrm{o}},\ \ldots,\ u_{s-2}^{\mathrm{o}}, u_s^{\mathrm{o}} - u_{s-1}^{\mathrm{o}} & (s \text{ even}). \end{array}$$

(c) *The toric morphism $X_P \to U_\sigma$ is projective and X_P is Gorenstein with at worst rational double points.*

Remark 10.2.18. When s is even, we can explain $u_s^\circ - u_{s-1}^\circ$ as follows. By Theorem 10.2.14, u_{s-2}° and u_s° give an edge Θ_σ. The vector determined by this edge is $u_s^\circ - u_{s-2}^\circ = a_s u_{s-1}^\circ$. Hence the edge has $a_s + 1$ lattice points since u_{s-1}° is primitive. Thus, if we start from u_s°, the next lattice point along the edge is

$$u_s^\circ - u_{s-1}^\circ.$$

Since $d > k > 0$ and $d/k = [a_1, \ldots, a_s]$ is computed using the Euclidean algorithm, we have $a_s \geq 2$ (Exercise 10.2.8). It follows that $u_s^\circ - u_{s-1}^\circ$ is not a vertex of Θ_σ.

Proof. Part (a) is straightforward (Exercise 10.2.9). For part (b), we will assume that s is even and leave the case when s is odd to the reader (Exercise 10.2.9).

If we write $s = 2\ell$, then the vertices (10.2.16) of P are $m_{-1}, m_1, \ldots, m_{2\ell-1}, m_{2\ell}$. Thus the bounded edges of $P = \Theta_{\sigma^\vee}$ are

$$\overline{m_{-1}m_1},\ \overline{m_1m_3},\ \ldots,\ \overline{m_{2\ell-3}m_{2\ell-1}},\ \overline{m_{2\ell-1}m_{2\ell}}.$$

The inward-pointing normals of these edges give the rays that refine σ in the normal fan of P.

The m_i's satisfy the same recursion $m_i = a_i m_{i-1} + m_{i-2}$, $1 \leq i \leq s$, as the u_i°'s in (10.2.14). Hence, for the edges $\overline{m_{2j-1}m_{2j+1}}$, $1 \leq j \leq \lfloor s/2 \rfloor$, we have

$$m_{2j+1} - m_{2j-1} = a_{2j+1} m_{2j} = a_{2j+1}(q_{2j}e_1^* + p_{2j}e_2^*).$$

Since $u_{2j}^\circ = p_{2j}e_1 - q_{2j}e_2$, one easily computes that $\langle m_{2j+1} - m_{2j-1}, u_{2j}^\circ \rangle = 0$. It follows that u_{2j}° is the inward-pointing normal of this edge since it lies in σ and is primitive. This takes care of all of the bounded edges except for $\overline{m_{2\ell-1}m_{2\ell}}$. Here, we compute

$$m_{2\ell} - m_{2\ell-1} = (q_{2\ell} - q_{2\ell-1})e_1^* + (p_{2\ell} - p_{2\ell-1})e_2^*.$$

This is clearly normal to $u_{2\ell}^\circ - u_{2\ell-1}^\circ = (p_{2\ell} - p_{2\ell-1})e_1 - (q_{2\ell} - q_{2\ell-1})e_2$. The latter vector is easily seen to be primitive by part (b) of Proposition 10.2.10. Furthermore, $u_{2\ell}^\circ - u_{2\ell-1}^\circ \in \sigma$ by Remark 10.2.18. Hence this is the inward-pointing normal of the final bounded edge $\overline{m_{2\ell-1}m_{2\ell}}$.

For part (c), note that $X_P \to U_\sigma$ is projective by Theorem 7.1.10. To complete the proof, we need to show that each maximal cone of Σ_P gives a Gorenstein affine toric variety. For simplicity, we assume that s is odd (see Exercise 10.2.9 for the even case). Since $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2) = \mathrm{Cone}(e_2, u_s^\circ)$, the maximal cones of Σ_P consist of two "boundary cones" $\mathrm{Cone}(e_1, u_0^\circ)$ and $\mathrm{Cone}(u_{s-1}^\circ, u_s^\circ)$, plus the "interior cones" $\mathrm{Cone}(u_{2j-2}, u_{2j}) \in \Sigma_P$, $1 \leq j \leq \lfloor s/2 \rfloor$. The boundary cones are easily seen to be smooth and hence Gorenstein (Exercise 10.2.9). For an interior cone $\mathrm{Cone}(u_{2j-2}, u_{2j})$, we use part (b) of Proposition 10.2.10 to compute

$$\begin{aligned}\langle m_{2j-1}, u^{\mathrm{o}}_{2j-2}\rangle &= \langle q_{2j-1}e_1^* + p_{2j-1}e_2^*, p_{2j-2}e_1 - q_{2j-2}e_2\rangle \\ &= -(p_{2j-1}q_{2j-2} - p_{2j-2}q_{2j-1}) = -(-1)^{2j-1} = 1,\end{aligned}$$

and a similar computation gives $\langle m_{2j-1}, u^{\mathrm{o}}_{2j}\rangle = p_{2j}q_{2j-1} - p_{2j-1}q_{2j} = (-1)^{2j} = 1$. By Proposition 8.2.12, we conclude that the corresponding affine toric variety is Gorenstein, as desired. Then the singular points of X_P are rational double points by Example 10.1.5 and Proposition 10.1.6. $\square$

In §11.3 we will revisit this result, where we will learn that the morphism $X_P \to U_\sigma$ from Proposition 10.2.17 is the blowup of the singular point of U_σ.

Just as the morphism $X_P \to U_\sigma$ is projective, one can show more generally that the resolution of singularities $X_\Sigma \to U_\sigma$ from Theorem 10.2.3 is a projective morphism. This requires finding a lattice polyhedron with the correct normal fan. You will explore one way of doing this in Exercise 10.2.10.

We next consider the Hilbert basis $\mathscr{H}$ of $\sigma^\vee \cap M$. Recall from Lemma 1.3.10 that $|\mathscr{H}|$ is the dimension of the Zariski tangent space at the singular point of U_σ and is the dimension of the most efficient embedding of U_σ into affine space.

Theorem 10.2.8 tells us that the Hilbert basis of $\sigma \cap N$ is computed using the Hirzebruch-Jung continued fraction expansion of d/k. Since $\sigma^\vee$ has parameters $d, d-k$ (part (b) of Exercise 10.1.2), it follows that we need the Hirzebruch-Jung continued fraction expansion of $d/(d-k)$ to get the Hilbert basis of $\sigma^\vee \cap M$. By Exercise 10.2.4, the ordinary continued fraction

$$d/k = [a_1, \dots, a_s]$$

gives the Hirzebruch-Jung continued fraction

$$d/(d-k) = \begin{cases} [[(2)^{a_1-1}, a_2+2, (2)^{a_3-1}, a_4+2, \dots, (2)^{a_{s-1}-1}, a_s+1]] & s \text{ even} \\ [[(2)^{a_1-1}, a_2+2, (2)^{a_3-1}, a_4+2, \dots, a_{s-1}+2, (2)^{a_s-1}]] & s \text{ odd}. \end{cases}$$

Theorem 10.2.8, applied to this expansion, gives the Hilbert basis of $\sigma^\vee \cap M$. To see the underlying geometry, we need the following observation (Exercise 10.2.11):

(10.2.17) In Theorem 10.2.5, three consecutive lattice points u_{i-1}, u_i, u_{i+1} are collinear if and only if $u_{i-1} + u_{i+1} = 2u_i$, i.e., $b_i = 2$.

This means that a string of consecutive 2's in a Hirzebruch-Jung continued fraction of $d/(d-k)$ gives a string of lattice points in the relative interior of a bounded edge of the convex hull $\Theta_{\sigma^\vee}$. For the vertices $m_{-1}, m_1, \dots$ of $\Theta_{\sigma^\vee}$, this gives two ways to think about lattice points on an edge connecting two adjacent vertices. For example, the edge $\overline{m_{-1}m_1}$ has $a_1 - 1$ lattice points in its relative interior because:

- The Hirzebruch-Jung expansion of $d/(d-k)$ starts with $a_1 - 1$ consecutive 2's.
- $m_1 - m_{-1} = a_1 u_0$, u_0 primitive, gives $a_1 + 1$ lattice points on the edge.

This pattern continues for the other bounded edges of $\Theta_{\sigma^\vee}$.

Oda gives a different argument for this pattern in [**218**, Sec. 1.6] and uses it to relate the Hirzebruch-Jung continued fractions for d/k and $d/(d-k)$. His relation also follows from our approach (see part (d) of Exercise 10.2.4). The connection between continued fractions and toric surfaces is surprisingly rich and varied and is one of the reasons why toric varieties are so much fun to study. See [**75**] and [**231**] for a further discussion of this wonderful topic.

Exercises for §10.2.

10.2.1. Prove part (a) of Proposition 10.2.2: Show that the sequences P_i and Q_i from (10.2.5) are increasing sequences of nonnegative numbers. Hint: Use $b_i \geq 2$ for all i.

10.2.2. In §10.1, we constructed several resolutions of singularities in a rather ad hoc way. In this exercise, we will see that the resolutions given by Theorem 10.2.3 are the same as what we saw before.

(a) When σ has parameters $d, 1$, show that the Hirzebruch-Jung continued fraction method gives the same resolution of U_σ as the one given in Example 10.1.8.

(b) Do the same for Example 10.1.9. Hint: First show that
$$\frac{d}{d-1} = [[2, 2, \ldots, 2]],$$
where there are $d-1$ 2's.

10.2.3. Verify the last claim in the proof of Proposition 10.2.6.

10.2.4. This exercise will consider some relations between ordinary continued fraction expansions and Hirzebruch-Jung continued fraction expansions.

(a) Given integers $a_1, a_2 > 0$ and a variable x, prove that
$$[a_1, a_2, x] = [[a_1 + 1, (2)^{a_2-1}, x+1]],$$
where for any $l \geq 0$, $(2)^l$ denotes a string of l 2's. Hint: Argue by induction on a_2.

(b) Use part (a) to prove the equality $[1,1,1,5] = [[2,3,2,2,2,2]]$ from Example 10.2.9.

(c) Given $d/k = [a_1, \ldots, a_s]$, prove that $d/(d-k)$ has the Hirzebruch-Jung expansion given in the discussion leading up to (10.2.17). Hint: You will want to consider the cases $a_1 = 1$ and $a_1 > 1$ separately, but the formula can be written as in (10.2.17) in either case. If you get stuck, see [**231**].

(d) Starting from the ordinary continued fraction for d/k, use parts (a) and (c) to show that if $d/k = [[b_1, \ldots, b_r]]$ and $d/(d-k) = [[c_1, \ldots, c_s]]$, then $(\sum_{i=1}^r b_i) - r = (\sum_{j=1}^s c_j) - s$.

10.2.5. In this exercise we will consider Proposition 10.2.10.

(a) Prove the proposition. Hint: For part (a), argue by induction on the length $s \geq 1$ of the expansion. The expression $[a_1, a_2, \ldots, a_i]$ is well-defined when the a_i are positive rational numbers, so we can write
$$[a_1, a_2, \ldots, a_{i-1}, a_i] = [a_1, a_2, \ldots, a_{i-1} + 1/a_i].$$
Then use (10.2.12) and follow the reasoning from the proof of Proposition 10.2.2.

(b) Suppose we modify the initialization and the recurrences (10.2.12) as follows. Let
$$(r_{-1}, s_{-1}) = (1, 0) \quad \text{and} \quad (r_0, s_0) = (0, 1).$$

Then, for all $1 \le i \le s$, compute

$$\begin{aligned} r_i &= r_{i-2} - a_i r_{i-1} \\ s_i &= s_{i-2} - a_i s_{i-1} \end{aligned}$$

(note the change in sign!). What is true about $r_i d + s_i k$ for all i? What do we get with $i = s$? Hint: This fact is the basis for the extended Euclidean algorithm.

10.2.6. In this exercise, you will show that the ordinary continued fraction expansion of a rational number d/k can be used to construct *simultaneous resolutions* of pairs of singularities of certain toric surfaces. Let $1 < k < d$ be relatively prime integers, and let P be the triangle $\mathrm{Conv}(0, de_1, ke_2)$ in $\mathbb{R}^2$.

(a) Draw the normal fan Σ_P and show that X_P has exactly two singular points. Note that X_P is isomorphic to the weighted projective plane $\mathbb{P}(1,k,d)$.

(b) Adapt Theorem 10.2.12 to produce a resolution of singularities of X_P from the ordinary continued fraction expansion of d/k. Hint: First refine Σ_P by introducing 1-dimensional cones $\mathrm{Cone}(-e_1)$ and $\mathrm{Cone}(-e_2)$. Then apply Theorem 10.2.12 to the third quadrant of your drawing.

10.2.7. Complete the proof of Theorem 10.2.14 by showing that u_0° is a vertex of Θ_σ if and only if $a_1 > 1$.

10.2.8. For relative prime integers $d > k > 0$, we used the Euclidean algorithm to construct $d/k = [a_1, \ldots, a_s]$. Prove that $a_s \ge 2$.

10.2.9. Prove part (a) of Proposition 10.2.17. Also prove part (b) for s odd and part (c) for s even.

10.2.10. In this exercise, given a cone σ in normal form with parameters d,k, you will see how to construct an unbounded polyhedron P in $M_\mathbb{R} \simeq \mathbb{R}^2$ whose recession cone is $\sigma^\vee = \mathrm{Cone}(e_1, ke_1 + de_2)$, and whose normal fan defines the resolution of U_σ from Theorem 10.2.3. Let P_i, Q_i be the sequences constructed in Proposition 10.2.2. Let m be the smallest positive integer such that $md > P_1 + \cdots + P_{r-1}$ and $mk > Q_1 + \cdots + Q_{r-1}$. Starting from the point $A_r = (mk, md)$ on the upper boundary ray of $\sigma^\vee$, construct the points

$$\begin{aligned} A_{r-1} &= (mk - Q_{r-1}, md - P_{r-1}) \\ A_{r-2} &= (mk - Q_{r-1} - Q_{r-2}, md - P_{r-1} - P_{r-2}) \\ &\ \ \vdots \\ A_1 &= (mk - \textstyle\sum_{i=1}^{r-1} Q_i, md - \sum_{i=1}^{r-1} P_i) \\ A_0 &= (mk - \textstyle\sum_{i=1}^{r-1} Q_i, 0). \end{aligned}$$

Then let P be the polyhedron with one edge along the positive x-axis starting at A_0, edges $\overline{A_i A_{i+1}}$ for $i = 0, \ldots, r-1$, and one edge along the upper edge of $\sigma^\vee$ starting from A_r.

(a) Draw the polyhedron P for the cone with parameters $(d,k) = (7,5)$. What is the integer m in this case?

(b) Show that $\sigma^\vee$ is the recession cone of P.

(c) Show that the normal fan of P is the fan giving the resolution of U_σ constructed in Theorem 10.2.3.

10.2.11. Prove (10.2.11).

10.2.12. Let d/k be a rational number in lowest terms with $0 < k < d$, and let

$$d/k = [[b_1, \dots, b_r]] = [a_1, \dots, a_s]$$

be its continued fraction expansions. You will show that the sequences P_i, Q_i and p_i, q_i considered in this section can be expressed in matrix form.

(a) Let $M^-(b) = \begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix}$. Show that for all $1 \le i \le r$,

$$\begin{pmatrix} P_i & -P_{i-1} \\ Q_i & -Q_{i-1} \end{pmatrix} = M^-(b_1)M^-(b_2)\cdots M^-(b_i).$$

(b) Let $M^+(a) = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$. Show that for all $1 \le i \le s$,

$$\begin{pmatrix} p_i & p_{i-1} \\ q_i & q_{i-1} \end{pmatrix} = M^+(a_1)M^+(a_2)\cdots M^+(a_i).$$

10.2.13. In this exercise, you will apply the results of this section to the weighted projective plane $\mathbb{P}(q_0, q_1, q_2)$ from §2.0 and Example 3.1.17.

(a) Construct a resolution of singularities for any $\mathbb{P}(1, 1, q_2)$, where $q_2 \ge 2$. What smooth toric surface is obtained in this way? A complete classification of the smooth complete toric surfaces will be developed in §10.4.

(b) Do the same for $\mathbb{P}(1, q_1, q_2)$ in general. Hint: Exercise 10.2.6.

§10.3. Gröbner Fans and McKay Correspondences

The fans obtained by resolving the singularities of the affine toric toric surfaces U_σ have unexpected descriptions that involve Gröbner bases and representation theory. In this section we will present these ideas, following [**156**] and [**157**].

Gröbner Bases and Gröbner Fans. We assume the reader knows about Gröbner bases (see [**69**]) and Gröbner fans (see [**70**, Ch. 8, §4] or [**264**]). A nonzero ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ has a unique reduced Gröbner basis with respect to each monomial order $>$ on the polynomial ring. However, the set of distinct *reduced marked Gröbner bases* for I (i.e., reduced Gröbner bases with marked leading terms in each polynomial) is finite. Hence the ideal I has a finite *universal Gröbner basis*, i.e., a finite subset $\mathscr{U} \subset I$ that is a Gröbner basis for all monomial orders simultaneously.

Let $\mathbf{w} \in \mathbb{R}^n_{\ge 0}$ be a weight vector in the positive orthant (so $\mathbf{w}$ could be taken as the first row of a weight matrix defining a monomial order). Let

$$\mathscr{G} = \{g_1, \dots, g_t\}$$

be one of the reduced marked Gröbner bases for I, where

$$g_i = \underline{x^{\alpha(i)}} + \textstyle\sum_\beta c_{i\beta} x^\beta,$$

and $x^{\alpha(i)}$ is marked as the leading term of g_i. If $\mathbf{w} \cdot \alpha(i) > \mathbf{w} \cdot \beta$ whenever $c_{i\beta} \ne 0$, then I will have Gröbner basis $\mathscr{G}$ with respect to any monomial order defined by a weight matrix with first row $\mathbf{w}$. The set

$$C_{\mathscr{G}} = \{\mathbf{w} \in \mathbb{R}^n_{\geq 0} \mid \mathbf{w} \cdot \alpha(i) \geq \mathbf{w} \cdot \beta \text{ whenever } c_{i\beta} \neq 0\} \tag{10.3.1}$$

is the intersection of a finite collection of half-spaces, hence has the structure of a closed convex polyhedral cone in $\mathbb{R}^n_{\geq 0}$. The cones $C_{\mathscr{G}}$ as $\mathscr{G}$ runs over all distinct marked Gröbner bases of I, together with all of their faces, have the structure of a fan in $\mathbb{R}^n_{\geq 0}$ called the *Gröbner fan* of I. In particular, for each pair $\mathscr{G}, \mathscr{G}'$ of marked Gröbner bases, the cones $C_{\mathscr{G}}$ and $C_{\mathscr{G}'}$ intersect along a common face where the $\mathbf{w}$-weights of terms in some polynomials in $\mathscr{G}$ (and in $\mathscr{G}'$) coincide.

A First Example. Let σ be a cone in normal form with parameters d, k, and recall $\gcd(d,k) = 1$ by hypothesis. By Proposition 10.1.2, the group

$$G_{d,k} = \{(\zeta, \zeta^k) \in (\mathbb{C}^*)^2 \mid \zeta^d = 1,\ 0 \leq k \leq d-1\} \simeq \mu_d$$

acts on $\mathbb{C}^2$ by componentwise multiplication

$$(\zeta, \zeta^k) \cdot (x, y) = (\zeta x, \zeta^k y), \tag{10.3.2}$$

with quotient $\mathbb{C}^2/G_{d,k} \simeq U_\sigma$.

Let $\mathbf{I}(G_{d,k})$ be the ideal defining $G_{d,k}$ as a variety in $\mathbb{C}^2$. In the next extended example, we will introduce the first main result of this section.

Example 10.3.1. Let $d = 7$, $k = 5$ and $I = \mathbf{I}(G_{7,5})$. It is easy to check that

$$I = \langle x^7 - 1,\ y - x^5 \rangle.$$

Moreover, for lexicographic order with $y > x$, the set

$$\mathscr{G}^{(1)} = \{\underline{x^7} - 1,\ \underline{y} - x^5\}$$

is the reduced marked Gröbner basis for I, where the underlines indicate the leading terms. The corresponding cone in the Gröbner fan of I is

$$C_{\mathscr{G}^{(1)}} = \{\mathbf{w} = (a,b) \in \mathbb{R}^2_{\geq 0} \mid b \geq 5a\} = \mathrm{Cone}(e_2, e_1 + 5e_2).$$

There are three other marked reduced Gröbner bases of I:

$$\begin{aligned}
\mathscr{G}^{(2)} &= \{\underline{x^5} - y,\ \underline{x^2y} - 1,\ \underline{y^2} - x^3\},\\
\mathscr{G}^{(3)} &= \{\underline{x^3} - y^2,\ \underline{x^2y} - 1,\ \underline{y^3} - x\},\\
\mathscr{G}^{(4)} &= \{\underline{y^7} - 1,\ \underline{x} - y^3\}.
\end{aligned}$$

It is easy to check that each of these sets is a Gröbner basis for I using Buchberger's criterion. The corresponding cones are

$$\begin{aligned}
C_{\mathscr{G}^{(2)}} &= \{(a,b) \in \mathbb{R}^2_{\geq 0} \mid b \leq 5a,\ 2b \geq 3a\} = \mathrm{Cone}(e_1 + 5e_2, 2e_1 + 3e_2),\\
C_{\mathscr{G}^{(3)}} &= \{(a,b) \in \mathbb{R}^2_{\geq 0} \mid 2b \leq 3a,\ 3b \geq a\} = \mathrm{Cone}(2e_1 + 3e_2, 3e_1 + e_2),\\
C_{\mathscr{G}^{(4)}} &= \{(a,b) \in \mathbb{R}^2_{\geq 0} \mid 3b \leq a\} = \mathrm{Cone}(3e_1 + e_2, e_1).
\end{aligned}$$

Since these three cones fill out rest of the first quadrant in $\mathbb{R}^2$, the Gröbner fan of I consists of the four cones $C_{\mathscr{G}(i)}$ and their faces, as shown in Figure 8. We will denote this fan by Γ in the following.

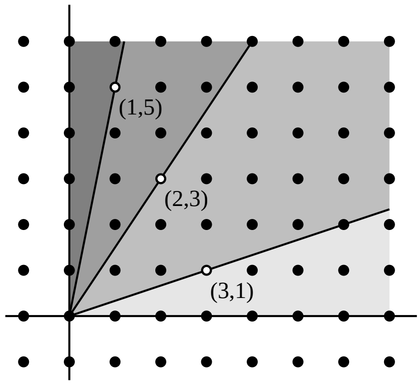


Figure 8. The Gröbner fan Γ

Next, let us consider the resolution of singularities

$$X_\Sigma \longrightarrow U_\sigma$$

for the cone σ with parameters $d = 7, k = 5$ computed in Example 10.2.4 in the last section. The reader can check that the linear transformation $T : N_\mathbb{R} \to N_\mathbb{R}$ with matrix relative to the basis e_1, e_2 given by

$$A = \begin{pmatrix} 7 & 0 \\ -5 & 1 \end{pmatrix} \tag{10.3.3}$$

maps the cones $C_{\mathscr{G}(i)}$ in the Gröbner fan Γ to the corresponding cones σ_i in the fan Σ. The matrix in (10.3.3) is invertible, but its inverse is not an integer matrix. The image of the lattice $N = \mathbb{Z}^2$ under T is the proper sublattice $7\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and T^{-1} maps N to the lattice

$$N' = \{(a/7, b/7) \mid a,\ b \in \mathbb{Z}, b \equiv 5a \bmod 7\} = N + \mathbb{Z}\left(\frac{1}{7}e_1 + \frac{5}{7}e_2\right).$$

There is an exact sequence

$$0 \longrightarrow N \longrightarrow N' \longrightarrow G \longrightarrow 0$$

induced by the map $N' \to (\mathbb{C}^*)^2$ defined by $(a/7, b/7) \mapsto (e^{2\pi i a/7}, e^{2\pi i b/7})$. Letting $\tau = \operatorname{Cone}(e_1, e_2)$, the corresponding toric morphism $U_{\tau,N} \longrightarrow U_{\tau,N'}$ is the quotient mapping $\mathbb{C}^2 \longrightarrow \mathbb{C}^2/G$.

It is easy to check that $\mathbf{w}_0 = (0,1)$ and $\mathbf{w}_1 = (1/7, 5/7)$ form a basis of the lattice N'. In Figure 9 on the next page, the fan defined by the cones with ray

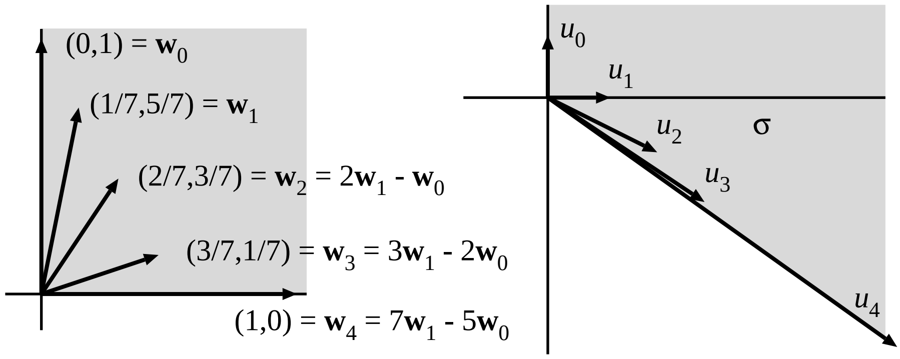


Figure 9. The toric variety $X_{\Gamma,N'}$ is the resolution of U_σ in Example 10.3.1

generators on the left is the same as in Figure 8 above, and the fan on the right is the same as in Figure 3 above. Note that $T(\mathbf{w}_i) = u_i$ for $0 \le i \le 4$ in Figure 9.

With respect to N', the cones in the Gröbner fan Γ are smooth cones, and it follows from the discussion of toric morphisms in §3.3 that the toric surfaces $X_{\Gamma,N'}$ and X_Σ are isomorphic. In other words, the Gröbner fan Γ of the ideal I encodes the structure of the resolution of singularities of U_σ.

Example B.2.4 shows how to compute this example using `GFan` [**161**]. $\Diamond$

A Tale of Two Fans. We next show that the last observation in Example 10.3.1 holds in general. As in the example, consider the ideal $I = \mathbf{I}(G_{d,k})$ and the action of $G_{d,k}$ given in (10.3.2). Each monomial in $\mathbb{C}[x,y]$ is equivalent modulo I to one of the monomials x^j, $j = 0,\dots,d-1$. This may be seen, for instance, from the remainders on division by the lexicographic Gröbner basis $\{\underline{x^d} - 1,\ \underline{y} - x^k\}$. As a result, we have a direct sum decomposition of the coordinate ring of the variety $G_{d,k}$ as a $\mathbb{C}$-vector space:

$$\mathbb{C}[x,y]/I \simeq \bigoplus_{j=0}^{d-1} V_j, \tag{10.3.4}$$

where V_j is the 1-dimensional subspace spanned by $x^j \bmod I$.

The following result establishes a first connection between the ideal I and the resolution of singularities of U_σ described in Theorem 10.2.3.

Proposition 10.3.2. *Let* $I = \mathbf{I}(G_{d,k})$ *where* $0 < k < d$ *and* $\gcd(d,k) = 1$. *Consider*

$$\begin{aligned} u_0 &= e_2 \\ u_i &= P_{i-1}e_1 - Q_{i-1}e_2, \quad i = 1,\dots,r+1, \end{aligned}$$

from Theorem 10.2.3. Let $T : N_\mathbb{R} \to N_\mathbb{R}$ *be the linear transformation with matrix*

$$A = \begin{pmatrix} d & 0 \\ -k & 1 \end{pmatrix}$$

and let $\mathbf{w}_i = T^{-1}(u_i)$ *for* $i = 0,\dots,r+1$. *Write* $\mathbf{w}_i = \frac{1}{d}(a_i e_1 + b_i e_2)$ *and define*

$$g_i = x^{b_i} - y^{a_i}, \quad 0 \le i \le r+1.$$

(a) *The polynomials* $g_0,\dots,g_{r+1}$ *are contained in the ideal* I.

(b) $\mathsf{S} = \{ae_1 + be_2 \in N \mid a,b \ge 0,\ x^b - y^a \in I\} \subseteq N$ *is an additive semigroup.*

(c) $\{d\mathbf{w}_i \mid i = 0,\dots,r+1\}$ *is the Hilbert basis of the semigroup* S *of part* (b).

Proof. It is an easy calculation to show T^{-1} maps $\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ to the first quadrant $\mathbb{R}^2_{\ge 0}$. Moreover, $\mathbf{w}_0 = e_2$, and for $i = 1,\dots,r+1$,

$$\mathbf{w}_i = \frac{1}{d}\big(P_{i-1}e_1 + (kP_{i-1} - dQ_{i-1})e_2\big).$$

Therefore, $g_0 = x^d - 1$ and

$$g_i = x^{kP_{i-1} - dQ_{i-1}} - y^{P_{i-1}}$$

for $i = 1,\dots,r+1$. Since $\zeta^d = 1$, these polynomials clearly vanish at $(\zeta,\zeta^k) \in G_{d,k}$. Therefore $g_i \in \mathbf{I}(G_{d,k}) = I$ for all i.

The proof of part (b) is left to the reader as Exercise 10.3.3. For part (c), it follows from parts (a) and (b) that $d\mathbf{w}_i$ is contained in S. On the other hand, let $ae_1 + be_2 \in \mathsf{S}$. Then $x^b - y^a \in I$, which implies that $b \equiv ak \bmod d$. Hence

$$T\Big(\frac{1}{d}(ae_1 + be_2)\Big) = ae_1 + \frac{b - ak}{d}e_2$$

must be an element of $\sigma \cap N$. Since the u_i are the Hilbert basis for the semigroup $\sigma \cap N$ by Theorem 10.2.8, this vector is a nonnegative integer combination of the u_i. Hence $ae_1 + be_2$ is a nonnegative integer combination of the $d\mathbf{w}_i$. It follows that S is generated by the $d\mathbf{w}_i$. The $d\mathbf{w}_i$ are irreducible in S because the corresponding $u_i = T(\mathbf{w}_i)$ are irreducible in the semigroup $\sigma \cap N$. $\square$

We also have a first result about reduced Gröbner bases of the ideal I.

Lemma 10.3.3. *Every element of a reduced Gröbner basis of* $I = \mathbf{I}(G_{d,k})$ *is either of the form* $x^b - y^a$ *or of the form* $x^s y^t - 1$ *for* $s,t > 0$.

Proof. Since I is generated by $x^d - 1, y - x^k$, the Buchberger algorithm implies that a reduced Gröbner basis $\mathscr{G}$ of I consists of binomials. By taking out common factors, every $g \in \mathscr{G}$ can be written $g = x^i y^j h$, where $h = x^b - y^a$ or $x^s y^t - 1$. Then $h \in I$ since it vanishes on $G_{d,k}$. Its leading term divisible by the leading term of an element of $\mathscr{G}$, which is impossible in a reduced Gröbner basis unless $g = h$. $\square$

The Gröbner bases in Example 10.3.1 give a nice illustration of Lemma 10.3.3. Our next lemma relates the polynomials $g_i = x^{b_i} - y^{a_i}$ from Proposition 10.3.2 to the reduced Gröbner bases of I.

Lemma 10.3.4. *Let* $g_i = x^{b_i} - y^{a_i}$ *be as in Proposition 10.3.2 and fix a monomial order* $>$ *on* $\mathbb{C}[x,y]$.

(a) *The a_i are increasing and the b_i are decreasing with i.*

(b) *There is some index $i = i_0$ (depending on $>$) such that*

$$\text{LT}_>(g_i) = x^{b_i} \text{ for all } i \le i_0, \text{ and } \text{LT}_>(g_i) = y^{a_i} \text{ for all } i > i_0.$$

(c) *If $i = i_0$ is the index from part* (b)*, then g_{i_0} and g_{i_0+1} are elements of the reduced Gröbner basis of $I = \mathbf{I}(G_{d,k})$ with respect to $>$.*

Proof. You will prove parts (a) and (b) in Exercise 10.3.4. For part (c), let $\mathscr{G}$ be the reduced Gröbner basis of I with respect to $>$. Since $g_{i_0} \in I$ by part (a) of Proposition 10.3.2, there is $g \in \mathscr{G}$ whose leading term divides $\text{LT}_>(g_{i_0}) = x^{b_{i_0}}$. By Lemma 10.3.3, it follows that $g = x^b - y^a$ with $\text{LT}_>(g) = x^b$. In particular, $b \le b_{i_0}$ and $ae_1 + be_2 \in \mathsf{S}$, where S is the semigroup from part (b) of Proposition 10.3.2. Then part (c) of the same proposition implies that $ae_1 + be_2$ must be a nonnegative integer combination

$$ae_1 + be_2 = \textstyle\sum_{i=0}^{r+1} \ell_i\, d\mathbf{w}_i = \sum_{i=0}^{r+1} \ell_i(a_i e_1 + b_i e_2), \quad \ell_i \in \mathbb{N}. \tag{10.3.5}$$

Since $b \le b_{i_0}$ and the b_i decrease with i, (10.3.5) can include only the $d\mathbf{w}_i$ with $i \ge i_0$. Suppose that $d\mathbf{w}_i$ appears in (10.3.5) with $i > i_0$. Then $a \ge a_i$ and $y^{a_i} > x^{b_i}$, so that $\text{LT}_>(g_i) = y^{a_i}$ divides y^a. Since $g_i \in I$, $\text{LT}_>(g_i)$ is divisible by the leading term of some $h \in \mathscr{G}$. Hence $\text{LT}_>(h)$ divides y^a, which is a term of $g = x^b - y^a \in \mathscr{G}$. This is impossible in a reduced Gröbner basis, hence $i > i_0$ cannot occur in (10.3.5). From here, it follows easily that $g = g_{i_0}$, giving $g_{i_0} \in \mathscr{G}$ as desired.

The statement for g_{i_0+1} follows by an argument parallel to the one above. The details are left to the reader (Exercise 10.3.4). □

We are now ready for the first major result of this section.

Theorem 10.3.5. *Let Γ be the fan in $\mathbb{R}^2$ with maximal cones*

$$\gamma_i = \text{Cone}(a_{i-1}e_1 + b_{i-1}e_2, a_i e_1 + b_i e_2), \quad 1 \le i \le r+1,$$

for a_i, b_i as in Proposition 10.3.2. Then Γ is the Gröbner fan of the ideal $\mathbf{I}(G_{d,k})$.

Proof. The cones γ_i fill out the first quadrant $\mathbb{R}^2_{\ge 0}$. Take any $\mathbf{w} = ae_1 + be_2$ lying in the interior of some γ_i and let $>$ be a monomial order defined by a weight matrix with $\mathbf{w}$ as first row. This gives a reduced Gröbner basis $\mathscr{G}$. The theorem will follow once we prove that γ_i is the Gröbner cone $C_{\mathscr{G}}$ of $\mathscr{G}$.

First observe that for $>$, we must have must have $i_0 = i - 1$ in part (b) of Lemma 10.3.4. It follows from part (c) of the lemma that g_{i-1} and g_i are elements of $\mathscr{G}$. Since a reduced Gröbner basis has only one element with leading term a power of x and only one with leading term a power of y, all other elements of $\mathscr{G}$ will have the form $x^s y^t - 1$, $s, t > 0$, by Lemma 10.3.3. Therefore, the Gröbner cone $C_{\mathscr{G}}$ is exactly γ_i and we are done. □

As in Example 10.3.1, the following statement is an immediate consequence.

Corollary 10.3.6. *Let N' be the lattice*

$$N' = \{(a/d, b/d) \mid a,b \in \mathbb{Z}, b \equiv k \bmod d\} = \mathbb{Z}\big(\tfrac{1}{d}e_1 + \tfrac{k}{d}e_2\big) \oplus \mathbb{Z}e_2.$$

The toric surfaces X_Σ and $X_{\Gamma,N'}$ are isomorphic.

In other words, the Gröbner fan of $\mathbf{I}(G_{d,k})$ can be used to construct a resolution of singularities of the affine toric surface U_σ when σ has parameters d,k.

Connections with Representation Theory. We now consider the above results from a different point of view. We assume the reader is familiar with the beginnings of representation theory for finite abelian groups.

The group $G = G_{d,k}$ from (10.3.2) acts on $V = \mathbb{C}^2$ by the 2-dimensional linear representation of the group μ_d of dth roots of unity defined by

$$\begin{aligned} \rho : \mu_d &\longrightarrow \mathrm{GL}(V) = \mathrm{GL}(2,\mathbb{C}) \\ \zeta &\longmapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^k \end{pmatrix}. \end{aligned} \tag{10.3.6}$$

Since μ_d is abelian, its irreducible representations are 1-dimensional over $\mathbb{C}$, and hence each is defined by a character

$$\begin{aligned} \chi_j : \mu_d &\longrightarrow \mathbb{C}^* \\ \zeta &\longmapsto \zeta^{-j} \end{aligned}$$

for $j = 0,\dots,d-1$. The reason for the minus sign will soon become clear.

Via (10.3.2), we get the induced action of μ_d on the polynomial ring $\mathbb{C}[x,y]$ by

$$\zeta \cdot x = \zeta^{-1}x, \quad \zeta \cdot y = \zeta^{-j}y,$$

as explained in §5.0. Each monomial $x^a y^b$ spans an invariant subspace where the action of μ_d is given by the irreducible representation with character χ_{-j} for $j \equiv a + kb \bmod d$. We call $a + kb \bmod d$ the *weight* of the monomial $x^a y^b$ with respect to this action of μ_d. Since the ideal of the group $G \subset (\mathbb{C}^*)^2$ is invariant, the action descends to the quotient $\mathbb{C}[x,y]/\mathbf{I}(G)$, and we have a representation of μ_d on $\mathbb{C}[x,y]/\mathbf{I}(G)$. The direct sum decomposition (10.3.4) shows that the irreducible representation with character χ_{-j} appears exactly once in this representation, as the subspace V_j in (10.3.4). This means that the representation on $\mathbb{C}[x,y]/\mathbf{I}(G)$ is isomorphic to the *regular representation* of μ_d (Exercise 10.3.5).

A 2-dimensional McKay Correspondence. In 1979, McKay pointed out that there is a one-to-one correspondence between the irreducible representations of μ_d and the components of the exceptional divisor in the resolution $\phi : X_\Sigma \longrightarrow U_\sigma$ when σ was a cone in normal form with parameters $k = d-1$, as in Example 10.1.5. In this case, the singular point of U_σ is a rational double point, and the image of the representation ρ from (10.3.6) lies in $\mathrm{SL}(2,\mathbb{C})$. A great deal of research was devoted to explaining the original McKay correspondence in representation-theoretic and geometric terms (work of Gonzalez-Sprinberg, Artin, and Verdier). However, for $1 < k < d-1$, $\rho(\mu_d)$ is a subgroup of $\mathrm{GL}(2,\mathbb{C})$, not $\mathrm{SL}(2,\mathbb{C})$, and there are more irreducible representations of μ_d than components of the exceptional divisor. The McKay correspondence can be extended to these cases by identifying certain *special representations* that correspond to the components of the exceptional divisor (work of Wunram, Esnault, Ito and Nakamura, Kidoh, and others).

We will describe a generalized McKay correspondence that applies for all d,k. Writing $G = G_{d,k}$ as before, consider the ring of invariants $\mathbb{C}[x,y]^G$. You will prove the following in Exercise 10.3.6.

Lemma 10.3.7. *Let V_j be an irreducible representation of μ_d with character χ_{-j}, and consider the action of $G = G_{d,k} \simeq \mu_d$ on $\mathbb{C}[x,y] \otimes_{\mathbb{C}} V_j$. Then the subspace of invariants $(\mathbb{C}[x,y] \otimes_{\mathbb{C}} V_j)^G$ has the structure of a module over the ring $\mathbb{C}[x,y]^G$.* □

Example 10.3.8. Let $G = G_{7,5}$ as in Example 10.3.1. The ring of invariants is $\mathbb{C}[x,y]^G = \mathbb{C}[x^7, x^2y, xy^4, y^7]$ in this case. If v_j is the basis of the representation V_j, then it is easy to check that $x^a y^b \otimes v_j$ is invariant under G if and only if $a + kb - j \equiv 0 \bmod 7$, or in other words if and only if $x^a y^b$ has weight j under this action of μ_7.

First consider the case $j = 1$ in Lemma 10.3.7. The monomials in the complement of the monomial ideal

$$\mathsf{M} = \langle x^7, x^2y, xy^4, y^7 \rangle$$

that have weight 1 are x and y^3. Then $x \otimes v_1$ and $y^3 \otimes v_1$ generate the module $(\mathbb{C}[x,y] \otimes V_1)^G$. Since x and y^3 have the same weight with respect to this action of μ_7, the difference $x - y^3$ is an element of the ideal $\mathbf{I}(G)$, and this is one of the polynomials g_i as in the proof of Proposition 10.3.2.

On the other hand, if $j = 2$, then there are three monomials with weight 2 in the complement of M, and these give three generators of $(\mathbb{C}[x,y] \otimes V_2)^G$, namely $x^2 \otimes v_2, xy^3 \otimes v_2$, and $y^6 \otimes v_2$. It is still true that $x^2 - xy^3$, $x^2 - y^6$, and $xy^3 - y^6$ are elements of $\mathbf{I}(G)$, but these polynomials cannot appear in a reduced Gröbner basis for $\mathbf{I}(G)$. Moreover, no proper subset of the three generators generates the whole module $(\mathbb{C}[x,y] \otimes V_2)^G$. ◇

Definition 10.3.9. Let $G = G_{d,k} \simeq \mu_d$ as above. We say that the representation V_j is ***special with respect to*** k if $(\mathbb{C}[x,y] \otimes V_j)^G$ is minimally generated as a module over the invariant ring $\mathbb{C}[x,y]^G$ by two elements.

Hence, in Example 10.3.8, V_1 is special while V_2 is not. According to our definition, the trivial representation V_0 is never special, since $(\mathbb{C}[x,y] \otimes V_0)^G$ is generated by the single monomial 1 over the invariant ring. Our next theorem gives a rudimentary form of a McKay correspondence for the group $G_{d,k}$.

Theorem 10.3.10 (McKay Correspondence)**.** *Let σ be a cone with parameters d,k, where $0 < k < d$ and $\gcd(d,k) = 1$. Then there is a one-to-one correspondence between the representations of μ_d that are special with respect to k and the components of the exceptional divisor for the minimal resolution $\phi : X_\Sigma \to U_\sigma$.*

Proof. Write $G = G_{d,k}$ as above and consider the set B of monomials in the complement of the ideal M generated by the G-invariant monomials. This set contains

$$L = \{1,x,x^2,\dots,x^{d-1},y,y^2,\dots,y^{d-1}\}.$$

Since $\gcd(d,k) = 1$, for each $1 \le j \le d-1$, there is an integer $1 \le a_j \le d-1$ such that x^j and y^{a_j} have equal weight (equal to j) for the action of G. The representation V_j is special with respect to k if and only if these are the *only* two monomials of weight j in the set B, and nonspecial if and only if there is some monomial $x^a y^b$ with $a,b > 0$ in B which also has weight j. Since $x^j - y^{a_j} \in \mathbf{I}(G)$, saying V_j is special with respect to k is in turn equivalent to saying that the corresponding vector $a_j e_1 + je_2$ is an irreducible element in the semigroup from part (b) of Proposition 10.3.2 (Exercise 10.3.8). By Theorem 10.3.5, $\mathrm{Cone}(a_j e_1 + je_2)$ is one of the 1-dimensional cones of the Gröbner fan of $\mathbf{I}(G)$. Then V_j corresponds to one of the 1-dimensional cones in the fan Σ and hence to one of the irreducible components of the exceptional divisor. □

The original McKay correspondence is the following special case.

Corollary 10.3.11. *When $k = d-1$, there is a one-to-one correspondence between the set of all irreducible representations of μ_d and the components of the exceptional divisor of the minimal resolution $\phi : X_\Sigma \longrightarrow U_\sigma$.*

Proof. In this case, the invariant ring is $\mathbb{C}[x^d,xy,y^d]$, so the sets L and B in the proof of the theorem coincide. □

There has also been much work devoted to extend the McKay correspondence to finite abelian subgroups $G \subset \mathrm{GL}(n,\mathbb{C})$ for $n \ge 3$, and several other ways to understand these constructions have also been developed, including the theory of G-Hilbert schemes. See Exercise 10.3.10 for the beginnings of this.

Exercises for §10.3.

10.3.1. In this exercise, you will verify the claims made in Example 10.3.1, and extend some of the observations there.

(a) Show that each of the $\mathscr{G}^{(i)}$ is a Gröbner basis of $\mathbf{I}(G_{7,5})$.

(b) Show that $\mathscr{G} = \{x^5-1, y-x^3, x^2y-1, x-y^2, y^5-1\}$ is a universal Gröbner basis for $\mathbf{I}(G_{7,5})$.

(c) Determine the cones $C_{\mathscr{G}^{(i)}}$ using (10.3.1).

(d) Verify the final claim that linear transformation defined by the matrix A from (10.3.3) maps the Gröbner cones $C_{\mathscr{G}^{(i)}}$ to the σ_i for $i = 1,2,3$.

10.3.2. Verify the conclusions of Proposition 10.3.2 and Theorem 10.3.5 for the case $d = 17, k = 11$.

10.3.3. Show that the set S defined in part (b) of Proposition 10.3.2 is an additive semigroup. Hint: A direct proof starts from two general elements $ae_1 + be_2$ and $a'e_1 + b'e_2$ in S. Consider $(x^b - y^a)(x^{b'} + y^{a'})$ and $(x^b + y^a)(x^{b'} - y^{a'})$.

10.3.4. In this exercise you will complete the proof of Lemma 10.3.4.

(a) Show that for each i, $kP_i - dQ_i = k_i$, where the k_i are produced by the modified Euclidean algorithm from (10.2.1).

(b) Prove part (a) of Lemma 10.3.4.

(c) Verify that there is an index i_0 as in part (b) of Lemma 10.3.4.

(d) Verify that if i_0 is as in part (c), then g_{i_0+1} is contained in the reduced Gröbner basis.

10.3.5. If G is any finite group, the (*left*) *regular representation* of G is defined as follows. Let W be a vector space over $\mathbb{C}$ of dimension $|G|$ with a basis $\{e_h \mid h \in G\}$ indexed by the elements of G. For each $g \in G$ let $\rho(g) : W \longrightarrow W$ be defined by $\rho(g)(e_h) = e_{gh}$.

(a) Show that $g \mapsto \rho(g)$ is a group homomorphism from G to $\mathrm{GL}(W)$.

(b) Now let G be the cyclic group μ_d of order d. Show that W is the direct sum of 1-dimensional invariant subspaces W_j, $j = 0, \ldots, d-1$ on which G acts by the character χ_j defined in the text, so that W decomposes as $W \simeq \oplus_{j=0}^{d-1} V_j$.

10.3.6. In this exercise you will consider the module structures from Lemma 10.3.7.

(a) Prove Lemma 10.3.7.

(b) Verify the claims made in Example 10.3.8.

10.3.7. In this exercise you will prove an alternate characterization of the special representations with respect to k from Definition 10.3.9. We write $G = G_{d,k} \simeq \mu_d$ as usual.

(a) Let $\Omega^2_{\mathbb{C}^2} = \{f\, dx \wedge dy \mid f \in \mathbb{C}[x,y]\}$. Show that $\zeta \cdot x^a y^b dx \wedge dy = \zeta^{a+b+1+k} x^a y^b dx \wedge dy$ defines an action of G on $\Omega^2_{\mathbb{C}^2}$.

(b) Show that the spaces of G-invariants $(\Omega^2_{\mathbb{C}^2})^G$ and $(\Omega^2_{\mathbb{C}^2} \otimes V_j)^G$ have the structure of modules over the invariant ring $\mathbb{C}[x,y]^G$.

(c) Show that V_j is special with respect to k if and only if the "multiplication map"
$$(\Omega^2_{\mathbb{C}^2})^G \otimes (\mathbb{C}[x,y] \otimes V_j)^G \longrightarrow (\Omega^2_{\mathbb{C}^2} \otimes V_j)^G$$
is surjective.

10.3.8. In the proof of the McKay correspondence, show that $a_j e_1 + j e_2$ is irreducible in the semigroup S from Proposition 10.3.2 if and only if the representation V_j is special with respect to k.

10.3.9. Let g_i, $0 \le i \le r+1$, be the binomials constructed in Proposition 10.3.2. Show that
$$\mathscr{U} = \{g_1, \ldots, g_r\} \cup \{x^a y^b - 1 \mid x^a y^b \text{ is } G_{d,k}\text{-invariant}\}$$
is a universal Gröbner basis for $\mathbf{I}(G_{d,k})$ (not always minimal, however).

10.3.10. Let $G = G_{d,k}$ act on $\mathbb{C}^2$ as in (10.3.2). As a point set, G can be viewed as the orbit of the point $(1,1)$ under this action. The ideal $I = \mathbf{I}(G)$ is invariant under the action of G on $\mathbb{C}[x,y]$ and as we have seen, the corresponding representation on $\mathbb{C}[x,y]/I$ is isomorphic to the regular representation of G.

(a) Show that if $p = (\xi, \eta)$ is any point in $\mathbb{C}^2$ other than the origin, the ideal I of the orbit of p is another G-invariant ideal and the corresponding representation of G on $\mathbb{C}[x,y]/I$ is also isomorphic to the regular representation of G.

(b) The *G-Hilbert scheme* can be defined as the set of all G-invariant ideals in $\mathbb{C}[x,y]$ such that the representation of G on $\mathbb{C}[x,y]/I$ is isomorphic to the regular representation of G. Show that every such ideal has a set of generators of the form

$$\{x^a - \alpha y^c, y^b - \beta x^d, x^{a-d}y^{b-c} - \alpha\beta\}$$

for some $\alpha, \beta \in \mathbb{C}$ and where x^a and y^c (resp. y^b and x^d) have equal weights for the action of G. It can be seen from this result that the G-Hilbert scheme is also isomorphic to the minimal resolution of singularities of U_σ.

§10.4. Smooth Toric Surfaces

This section will use §10.1 and §10.2 to classify smooth complete toric surfaces and study the relation between continued fractions and intersection products of divisors on the resulting resolutions of singularities.

Classification of Smooth Toric Surfaces. We will show that smooth complete toric surfaces are all obtained by toric blowups from either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or one of the Hirzebruch surfaces $\mathscr{H}_r$ with $r \geq 2$ from Example 3.1.16. The proof will be based on the following facts.

First, Proposition 3.3.15 implies that if $\sigma = \mathrm{Cone}(u_1, u_2)$ is a smooth cone and we refine σ by inserting the new 1-dimensional cone $\tau = \mathrm{Cone}(u_1 + u_2)$, then on the resulting toric surface, the smooth point p_σ is blown up to a copy of $\mathbb{P}^1$.

For the second ingredient of the proof, we introduce the following notation. If Σ is a smooth complete fan, then list the ray generators of the 2-dimensional cones in Σ as $u_0, u_1, \ldots, u_{r-1}$ in clockwise order around the origin in $N_\mathbb{R}$, and we will consider the indices as integers modulo r, so $u_r = u_0$. Then we have the following statement parallel to (10.2.8).

Lemma 10.4.1. *Let $u_0, \ldots, u_r$ be the ray generators for a smooth complete fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^2$. There exist integers b_i, $i = 0, \ldots, r-1$, such that*

$$u_{i-1} + u_{i+1} = b_i u_i. \tag{10.4.1}$$

Proof. This is a special case of the wall relation (6.4.4). □

We also have the following result.

Lemma 10.4.2. *Let Σ be a smooth fan that refines a smooth cone σ. Then Σ is obtained from σ by a sequence of star subdivisions as in Definition 3.3.13.*

Proof. Suppose Σ has r cones of dimension 2, with ray generators $u_0, \ldots, u_r$, listed clockwise starting from u_0. We argue by induction on $r = |\Sigma(2)|$. If $r = 1$, there is only one cone in Σ and there is nothing to prove. Assume the result has been proved for all Σ with $|\Sigma(2)| = r$, and consider a fan Σ with $|\Sigma(2)| = r+1$. There are r "interior" rays that by (6.4.4) give wall relations $u_{i-1} + u_{i+1} = b_i u_i$, $b_i \in \mathbb{Z}$, for $1 \le i \le r$. Note that σ is strongly convex so $b_i > 0$ for all i. We claim that there exists some i such that $b_i = 1$. If not, i.e., if $b_i \ge 2$ for all i, then as in §10.2, the Hirzebruch-Jung continued fraction

$$[[b_1, b_2, \dots, b_r]]$$

represents a rational number d/k and the cone σ has parameters d,k with $d > k > 0$. But then $d \ge 2$, which would contradict the assumption that σ is a smooth cone.

Hence there exists an i, $1 \le i \le r$, such that

$$u_{i-1} + u_{i+1} = u_i.$$

In this situation, $\mathrm{Cone}(u_{i-1}, u_{i+1})$ is also smooth (Exercise 10.4.1). Moreover, $\mathrm{Cone}(u_{i-1}, u_i)$ and $\mathrm{Cone}(u_i, u_{i+1})$ are precisely the cones in the star subdivision of $\mathrm{Cone}(u_{i-1}, u_{i+1})$. Then we are done by induction. $\square$

We are now ready to state our classification theorem.

Theorem 10.4.3. *Every smooth complete toric surface X_Σ is obtained from either*

$$\mathbb{P}^2,\ \mathbb{P}^1 \times \mathbb{P}^1,\ \textit{or}\ \mathscr{H}_r,\ r \ge 2$$

by a finite sequence of blowups at fixed points of the torus action.

Proof. We follow the notation of Lemma 10.4.1. As in the proof of Lemma 10.4.2, if $b_i = 1$ in (10.4.1) for some i, then our surface is a blowup of the smooth surface corresponding to the fan where u_i is removed. Hence we only need to consider the case in (10.4.1) where $b_i \neq 1$ for all i.

Suppose first that $u_j = -u_i$ for some $i < j$. We relabel the vertices to make $u_j = -u_1$ for some j. Note that $j > 2$ since the cones must be strongly convex. Then from (10.4.1),

$$u_0 = -u_2 + b_1 u_1.$$

Using the basis u_1, u_2 of N, we get the picture shown in Figure 10 on the next page. Comparing this with the fans for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathscr{H}_a$ (Figures 2, 3 and 4 from §3.1), we see that Σ is a refinement of the fan of $\mathscr{H}_r$ if $r = b_1 > 2$. The same follows if $b_1 < -2$ and $r = |b_1|$ (see Exercise 10.4.2). Since $b_1 \neq 1$, the remaining possibilities are $b_1 = 0$ or -1, where we get a refinement of the fan of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ respectively. Then the theorem follows from Lemma 10.4.2 in this case.

We will complete the proof using primitive collections (Definition 5.1.5). The first step is to note that X_Σ is projective by Proposition 6.3.25. This will allow us to use Proposition 7.3.6, which asserts that Σ has a primitive collection whose minimal generators sum to 0.

If $\Sigma(1)$ has only three elements, it is easy to see that $X_\Sigma = \mathbb{P}^2$ since Σ is smooth and complete (Exercise 10.4.3). If $|\Sigma(1)| > 3$, every primitive collection

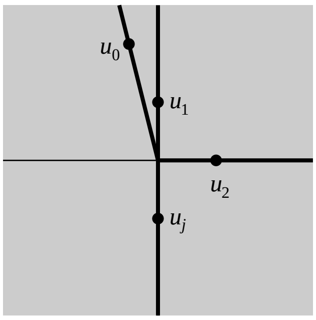


Figure 10. The ray generators u_0, u_1, u_2, u_j when $u_j = -u_1$

of Σ has exactly two elements (Exercise 10.4.3). By Proposition 7.3.6, one of these primitive collections must have minimal generators u_i, u_j that satisfy $u_i + u_j = 0$. Hence $u_j = -u_i$, and we are done by the earlier part of the proof. □

Since there is also a Hirzebruch surface $\mathscr{H}_1$, the statement of this theorem might seem puzzling. The reason that $\mathscr{H}_1$ is not included is that this surface is actually a blowup of $\mathbb{P}^2$ (Exercise 10.4.4).

The problem of classifying smooth compete toric varieties of higher dimension is much more difficult. We did this when $\operatorname{rank}\operatorname{Pic}(X_\Sigma) = 2$ in Theorem 7.3.7. See [**14**] for the case when $\operatorname{rank}\operatorname{Pic}(X_\Sigma) = 3$.

Intersection Products on Smooth Surfaces. A fundamental feature of the theory of smooth surfaces is the intersection product on divisors. In §6.3, we defined $D \cdot C$ when D is a Cartier divisor and C is a complete irreducible curve. On a smooth complete surface, this means that the intersection product $D \cdot C$ is defined for all divisors D and C. In particular, taking $D = C$ gives the *self-intersection* $D \cdot D = D^2$.

Here is a useful result about intersection numbers on a smooth toric surface.

Theorem 10.4.4. *Let D_ρ be the divisor on a smooth toric surface X_Σ corresponding to $\rho = \operatorname{Cone}(u)$ which is the intersection of 2-dimensional cones $\operatorname{Cone}(u, u_1)$ and $\operatorname{Cone}(u, u_2)$ in Σ. Then:*

(a) *$D_\rho \cdot D_\rho = -b$, where $u_1 + u_2 = bu$ as in* (10.4.1).

(b) *For a divisor $D_{\rho'} \neq D_\rho$, we have*

$$D_{\rho'} \cdot D_\rho = \begin{cases} 1 & \rho' = \operatorname{Cone}(u_i),\ i = 1, 2 \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Since X_Σ is smooth, part (a) follows from Lemma 6.4.4 once you compare (10.4.1) to (6.4.4). Part (b) follows from Corollary 6.4.3 and Lemma 6.4.4. □

Example 10.4.5. Let $\sigma = \mathrm{Cone}(u_1, u_2)$ be a cone in a smooth fan Σ, and consider the star subdivision, in which $\rho = \mathrm{Cone}(u_1 + u_2)$ is inserted to subdivide σ into two cones. Call the refined fan Σ'. Then the exceptional divisor $E = D_\rho$ of the blowup

$$\phi : X_{\Sigma'} \to X_\Sigma$$

satisfies $E \cdot E = -1$ on $X_{\Sigma'}$. $\Diamond$

Complete curves with self-intersection number -1 on a smooth surface are called *exceptional curves of the first kind*. They can always be *contracted* to a smooth point on a birationally equivalent surface, as in the above example.

One of the foundational results in the theory of general algebraic surfaces is that every smooth complete surface S has at least one *relatively minimal model*. This means that there is a birational morphism $S \to \overline{S}$, where $\overline{S}$ is a smooth surface with the property that if $\phi : \overline{S} \to S'$ is a birational morphism to another smooth surface S', then ϕ is necessarily an isomorphism. This is proved in [**131**, V.5.8]. Interestingly, the possible relatively minimal models for rational surfaces are precisely the surfaces $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathscr{H}_r$, $r \geq 2$, from Theorem 10.4.3.

On a smooth complete toric surface X_Σ, the intersection product can be regarded as a $\mathbb{Z}$-valued symmetric bilinear form on $\mathrm{Pic}(X_\Sigma)$. Here is an example.

Example 10.4.6. Consider the Hirzebruch surface $\mathscr{H}_r$. Using the fan shown in Figure 3 of Example 4.1.8, we get divisors $D_1, \ldots, D_4$ corresponding to minimal generators $u_1 = -e_1 + re_2,\ u_2 = e_2,\ u_3 = e_1,\ u_4 = -e_2$. By Theorem 10.4.4, we have the self-intersections

$$D_1 \cdot D_1 = D_3 \cdot D_3 = 0,\ D_2 \cdot D_2 = -r,\ D_4 \cdot D_4 = r.$$

The Picard group $\mathrm{Pic}(\mathscr{H}_r)$ is generated by the classes of D_3 and D_4. Note also that

$$D_3 \cdot D_4 = D_4 \cdot D_3 = 1$$

by Theorem 10.4.4. The intersection product is described by the matrix

$$\begin{pmatrix} D_3 \cdot D_3 & D_3 \cdot D_4 \\ D_4 \cdot D_3 & D_4 \cdot D_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & r \end{pmatrix}.$$

If $D \sim aD_3 + bD_4$ and $E \sim cD_3 + dD_4$ are any two divisors on the surface, then

$$D \cdot E = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & r \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = bc + ad + rbd. \tag{10.4.2}$$

For instance, with $D = E = D_2 \sim -rD_3 + D_4$, we obtain

$$D_2 \cdot D_2 = 1 \cdot (-r) + (-r) \cdot 1 + r \cdot 1 \cdot 1 = -r.$$

The self-intersection numbers $D_1 \cdot D_1 = D_3 \cdot D_3 = 0$ reflect the fibration structure on $\mathscr{H}_r$ studied in Example 3.3.20. The divisors D_1 and D_3 are fibers of the mapping $\mathscr{H}_r \to \mathbb{P}^1$. Such curves always have self-intersection equal to zero. You will compute several other intersection products on $\mathscr{H}_r$ in Exercise 10.4.5. $\Diamond$

Resolution of Singularities Reconsidered. Another interesting class of smooth toric surfaces consists of those that arise from a resolution of singularities of the affine toric surface U_σ of a 2-dimensional cone σ. Here is a simple example.

Example 10.4.7. Let $\sigma = \mathrm{Cone}(e_2, de_1 - e_2)$ have parameters $d,1$ where $d > 1$. The resolution of singularities $X_\Sigma \to U_\sigma$ constructed in Example 10.1.8 uses the smooth refinement of σ obtained by adding $\mathrm{Cone}(e_1)$. This gives the exceptional divisor E on X_Σ. Since

$$e_2 + (de_1 - e_2) = de_1,$$

we see that

$$E \cdot E = -d$$

is the self-intersection number of E. $\Diamond$

More generally, suppose that the smooth toric surface X_Σ is obtained via a resolution of singularities of U_σ, where the 2-dimensional cone σ has parameters d,k with $d > 1$. Let the Hirzebruch-Jung continued fraction expansion of d/k be

$$d/k = [[b_1, b_2, \dots, b_r]].$$

Recall from Theorem 10.2.3 that Σ is obtained from $\sigma = \mathrm{Cone}(u_0, u_{r+1})$ by adding rays generated by $u_1, \dots, u_r$, and by Theorem 10.2.5, we have

$$u_{i-1} + u_{i+1} = b_i u_i, \quad 1 \le i \le r.$$

It follows that $D_1, \dots, D_r$ are complete curves in X_Σ. They are the irreducible components of the exceptional fiber, with self-intersections

$$D_i \cdot D_i = -b_i, \quad 1 \le i \le r,$$

by Theorem 10.4.4. Then the intersection matrix $(D_i \cdot D_j)_{1 \le i,j \le r}$ is given by

$$D_i \cdot D_j = \begin{cases} -b_i & \text{if} \quad j = i \\ 1 & \text{if} \quad |i-j| = 1 \\ 0 & \text{otherwise}. \end{cases} \tag{10.4.3}$$

In Exercise 10.4.6 you will show that the associated quadratic form is negative definite. This condition is necessary for the *contractibility* of a complete curve C on a smooth surface S, i.e., the existence of a proper birational morphism $\pi : S \to \overline{S}$, where $\pi(C)$ is a (possibly singular) point on $\overline{S}$.

The resolutions described here have another important property.

Definition 10.4.8. A resolution of singularities $\phi : Y \to X$ is *minimal* if for every resolution of singularities $\psi : Z \to X$, there is a morphism $\rho : Z \to Y$ such that

$$\begin{tikzcd} & Y \arrow[d, "\phi"] \\ Z \arrow[ur, dashed, "\rho"] \arrow[r, "\psi"'] & X \end{tikzcd}$$

is a commutative diagram, i.e., $\psi = \phi \circ \rho$.

It is easy to see that if a minimal resolution of X exists, then it is unique up to isomorphism. If X has a unique singular point p, then using the theory of birational morphisms of surfaces it is not difficult to show that a resolution of singularities $\phi : Y \to X$ is minimal if the exceptional fiber contains no irreducible components E with $E \cdot E = -1$ (see Exercise 10.4.7). By Theorem 10.4.4 and the fact that $b_i \geq 2$ in Hirzebruch-Jung continued fractions, this holds for the resolutions constructed in Theorem 10.2.3. Hence we have the following.

Corollary 10.4.9. *The resolution of singularities of the affine toric surface U_σ constructed in Theorem 10.2.3 is minimal.* □

Rational Double Points Reconsidered. If σ has parameters $d, d-1$, then from Exercise 10.2.2, the Hirzebruch-Jung continued fraction expansion of $d/(d-1)$ is given by
$$d/(d-1) = [[2,2,\ldots,2]],$$
with $d-1$ terms. Hence $b_i = 2$ for all i, and (10.4.3) gives the $(d-1)\times(d-1)$ matrix
$$\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$
representing the intersection product on the subgroup of $\mathrm{Pic}(X_\Sigma)$ generated by the components of the exceptional divisor for the resolution of a rational double point of type A_{d-1}. We can now fully explain the terminology for these singularities.

The problem of classifying lattices
$$\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_s$$
with negative definite bilinear forms B satisfying $B(e_i, e_i) = -2$ for all i arises in many areas within mathematics, most notably in the classification of complex simple Lie algebras via root systems. The matrix above is the (negative of) the Cartan matrix for the root system of type A_{d-1}, which is often represented by the Dynkin diagram:

with $d-1$ vertices. The vertices represent the lattice basis vectors. The edges connect the pairs with $B(e_i, e_j) \neq 0$ and $B(e_i, e_i) = -2$ for all i as above. In our case, the vertices represent the components D_i of the exceptional divisor, and the bilinear form is the intersection product.

A precise definition of a surface *rational double point* follows.

Definition 10.4.10. A singular point p of a normal surface X is a ***rational double point*** or ***Du Val singularity*** if X has a minimal resolution of singularities $\phi : Y \to X$ such that if K_Y is a canonical divisor on Y, then every irreducible component E_i of the exceptional divisor E over p satisfies

$$K_Y \cdot E_i = 0.$$

We can relate these concepts to the toric case as follows.

Proposition 10.4.11. *Assume σ has parameters $d > k > 0$ and let $\phi : X_\Sigma \to U_\sigma$ be the resolution of singularities constructed in Theorem 10.2.3. Then the singular point of U_σ is a rational double point if and only if $k = d-1$.*

Proof. The canonical divisor of X_Σ is $K_{X_\Sigma} = -\sum_{i=0}^{r+1} D_i$, and one computes that

$$K_{X_\Sigma} \cdot D_i = b_i - 2, \quad 1 \leq i \leq r.$$

Thus the singular point is a rational double point if and only if $b_i = 2$ for all i. This easily implies $k = d-1$. You will verify these claims in Exercise 10.4.8. □

There is much more to say about rational double points. For example, one can show that $E = E_1 + \cdots + E_r$ satisfies $E \cdot E = -2$ (you will prove this in the toric case in Exercise 10.4.8). From a more sophisticated point of view, $E \cdot E = -2$ implies that the canonical sheaf on Y is the pullback of the canonical sheaf on X under ϕ. We will explore this in Proposition 11.2.8. See [**85**] for more on rational double points.

Exercises for §10.4.

10.4.1. Here you will verify several statements made in the proof of Lemma 10.4.2.

(a) Show that if the cone σ is strictly convex, then the integers b_i in (10.4.1) must be strictly positive.

(b) Show that if $u_{i-1} + u_{i+1} = u_i$, then $\mathrm{Cone}(u_{i-1}, u_{i+1})$ must also be smooth.

10.4.2. In the proof of Theorem 10.4.3, verify that if $u_j = -u_1$ and $u_0 = -u_2 + b_1 u_1$ with $b_1 < -2$, then Σ is a refinement of a fan Σ' with $X_{\Sigma'} \simeq \mathscr{H}_r$, where $r = |b_1|$.

10.4.3. In this exercise, you will prove some facts used in the proof of Theorem 10.4.3. Let Σ be a smooth complete fan in $N_{\mathbb{R}} \simeq \mathbb{R}^2$.

(a) If $|\Sigma(1)| = 3$, prove that $X_\Sigma \simeq \mathbb{P}^2$.

(b) If $|\Sigma(1)| > 3$, prove that every primitive collection of $\Sigma(1)$ has two elements.

10.4.4. In the statement of Theorem 10.4.3, you might have noticed the absence of the Hirzebruch surface $\mathscr{H}_1$. Show that this surface is isomorphic to the blowup of $\mathbb{P}^2$ at one of its torus-fixed points. See Exercise 3.3.8 for more details.

10.4.5. This exercise studies several further examples of the intersection product on $\mathscr{H}_r$.

(a) Compute $D_1 \cdot D_1$ using (10.4.2) and also directly from Theorem 10.4.4.

(b) Compute $K^2 = K \cdot K$ on $\mathscr{H}_r$, where $K = K_{\mathscr{H}_r}$ is the canonical divisor.

10.4.6. Show that the matrix defined by (10.4.3) has a negative-definite associated quadratic form. Hint: Recall that if $B(x,y)$ is a bilinear form, the associated quadratic form is $Q(x) = B(x,x)$.

10.4.7. Let X have a unique singular point p and let $\phi : Y \to X$ be a resolution of singularities such that no component E of the exceptional fiber $\phi^{-1}(p)$ has $E \cdot E = -1$. In this exercise, you will show that Y is a minimal resolution of X according to Definition 10.4.8. Let $\psi : Z \to X$ be another resolution of singularities and consider the possibly singular surface $S = Z \times_X Y$. Let R be a resolution of S. Then we have a commutative diagram of morphisms

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & Y \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ Z & \xrightarrow[\psi]{} & X. \end{array}$$

(a) Explain why it suffices to show that β must be an isomorphism.

(b) If not, apply [**131**, V.5.3] to show that β factors as a sequence of blowups of points. Hence R must contain curves L with $L \cdot L = -1$ in the exceptional fiber over p.

(c) Let L be an irreducible curve on R with $L \cdot L = -1$ and show that $E = \alpha(L)$ satisfies $E \cdot E = -1$.

(d) Deduce that β is an isomorphism, hence $\phi : Y \to X$ is a minimal resolution.

10.4.8. This exercise deals with the proof of Proposition 10.4.11.

(a) Show that $K_{X_\Sigma} \cdot D_i = b_i - 2$ for $1 \le i \le r$.

(b) Show that $d/k = [[2, \dots, 2]]$ if and only if $k = d - 1$. Hint: Exercise 10.2.2.

(c) Show that $E = D_1 + \cdots + D_r$ satisfies $E \cdot E = -2$.

10.4.9. Let σ have parmeters $d, d-1$, so that the singular point of U_σ is a rational double point. By Proposition 10.1.6, U_σ is Gorenstein, so that its canonical sheaf ω_{U_σ} is a line bundle. Let $\phi : X_\Sigma \to U_\sigma$ be the resolution constructed in Theorem 10.2.3. Prove that $\phi^*\omega_{U_\sigma}$ is the canonical sheaf of X_Σ.

10.4.10. Another interesting numerical fact about the integers b_i from (10.4.1) is the following. Suppose a smooth fan Σ has 1-dimensional cones labeled as in Lemma 10.4.1. Then

$$b_0 + b_1 + \cdots + b_{r-1} = 3r - 12. \tag{10.4.4}$$

This exercise will sketch a proof of (10.4.4).

(a) Show that (10.4.4) holds for the standard fans of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathscr{H}_r$, $r \ge 2$.

(b) Show that if (10.4.4) holds for a smooth fan Σ, then it holds for the fan obtained by performing a star subdivision on one of the 2-dimensional cones of Σ.

(c) Deduce that (10.4.4) holds for all smooth fans using Theorem 10.4.3.

§10.5. Riemann-Roch and Lattice Polygons

Riemann-Roch theorems are a class of results about the dimensions of sheaf cohomology groups. The original statement along these lines was the theorem of

Riemann and Roch concerning sections of line bundles on algebraic curves. This result and its generalizations to higher-dimensional varieties can be formulated most conveniently in terms of the Euler characteristic of a sheaf, defined in §9.4.

Riemann-Roch for Curves. A modern form of the Riemann-Roch theorem for curves states that if D is a divisor on a smooth projective curve C, then

$$\chi(\mathscr{O}_C(D)) = \deg(D) + \chi(\mathscr{O}_C), \tag{10.5.1}$$

where the degree $\deg(D)$ is defined in Definition 6.3.2. This equality can be rewritten using Serre duality as follows. Namely, if K_C is a canonical divisor on C, then we have

$$\begin{aligned} H^1(C, \mathscr{O}_C(D)) &\simeq H^0(C, \mathscr{O}_C(K_C - D))^\vee \\ H^1(C, \mathscr{O}_C) &\simeq H^0(C, \mathscr{O}_C(K_C))^\vee. \end{aligned}$$

The integer $g = \dim H^0(C, \mathscr{O}_C(K_C))$ is the *genus* of the curve C. Then (10.5.1) can be rewritten in the form commonly used in the theory of curves:

$$\dim H^0(C, \mathscr{O}_C(D)) - \dim H^0(C, \mathscr{O}_C(K_C - D)) = \deg(D) + 1 - g. \tag{10.5.2}$$

A proof of this theorem and a number of its applications are given in [**131**, Ch. IV]. Also see Exercise 10.5.1 below. As a first consequence, note that if $D = K_C$ is a canonical divisor, then

$$\deg(K_C) = 2g - 2. \tag{10.5.3}$$

We will need to use (10.5.2) most often in the simple case $X \simeq \mathbb{P}^1$. Then $g = 0$ and the Riemann-Roch theorem for $\mathbb{P}^1$ is the statement for all divisors D on $\mathbb{P}^1$,

$$\chi(\mathscr{O}_{\mathbb{P}^1}(D)) = \deg(D) + 1. \tag{10.5.4}$$

The Adjunction Formula. For a smooth curve C contained in a smooth surface X, the canonical sheaves ω_C of the curve and ω_X of the surface are related by

$$\omega_C \simeq \omega_X(C) \otimes_{\mathscr{O}_X} \mathscr{O}_C. \tag{10.5.5}$$

This follows without difficulty from Example 8.2.2 (Exercise 10.5.2) and has the following consequence for the intersection product on X.

Theorem 10.5.1 (Adjunction Formula)**.** *Let C be a smooth curve contained in a smooth complete surface X. Then*

$$K_X \cdot C + C \cdot C = 2g - 2,$$

where g is the genus of the curve C.

Proof. Let $i : C \hookrightarrow X$ be the inclusion map. Then

$$\omega_C \simeq \omega_X(C) \otimes_{\mathscr{O}_X} \mathscr{O}_C = i^*\omega_X(C) = i^*\mathscr{O}_X(K_X + C),$$

so that

$$2g - 2 = \deg(\omega_C) = \deg(i^*\mathscr{O}_X(K_X + C)) = (K_X + C) \cdot C,$$

where the first equality is (10.5.3) and the last is the definition of $(K_X+C)\cdot C$ given in §6.3. $\square$

Riemann-Roch for Surfaces. The statement for surfaces corresponding to (10.5.1) is given next.

Theorem 10.5.2 (Riemann-Roch for Surfaces)**.** *Let D be a divisor on a smooth projective surface X with canonical divisor K_X. Then*

$$\chi(\mathscr{O}_X(D)) = \frac{D\cdot D - D\cdot K_X}{2} + \chi(\mathscr{O}_X).$$

We will only prove this for X a smooth complete toric surface; there is a simple and concrete proof in this case.

Proof. The theorem certainly holds for $D=0$ since $\mathscr{O}_X(D)=\mathscr{O}_X$ in this case. Our proof will use the special properties of smooth complete toric surfaces. Recall that if $X = X_\Sigma$, then $\mathrm{Pic}(X)$ is generated by the classes of the divisors D_i, $i=1,\dots,r$, corresponding to the 1-dimensional cones in Σ. Hence, to prove the theorem, it suffices to show that if the theorem holds for a divisor D, then it also holds for $D+D_i$ and $D-D_i$ for all i.

Assume the theorem holds for D. By Proposition 4.0.28, the sequence

$$0 \longrightarrow \mathscr{O}_X(-D_i) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{D_i} \longrightarrow 0$$

is exact. Tensoring this with $\mathscr{O}_X(D+D_i)$ gives the exact sequence

$$0 \longrightarrow \mathscr{O}_X(D) \longrightarrow \mathscr{O}_X(D+D_i) \longrightarrow \mathscr{O}_{D_i}(D+D_i) \longrightarrow 0.$$

By (9.4.1), it follows that

$$\chi(\mathscr{O}_X(D+D_i)) = \chi(\mathscr{O}_X(D)) + \chi(\mathscr{O}_{D_i}(D+D_i)).$$

By the induction hypothesis,

$$\chi(\mathscr{O}_X(D)) = \frac{D\cdot D - D\cdot K_X}{2} + \chi(\mathscr{O}_X). \tag{10.5.6}$$

For $\chi(\mathscr{O}_{D_i}(D+D_i))$, recall that $D_i \simeq \mathbb{P}^1$. Hence, by the Riemann-Roch theorem for $\mathbb{P}^1$ given in (10.5.4), we have

$$\chi(\mathscr{O}_{D_i}(D+D_i)) = D\cdot D_i + D_i\cdot D_i + 1. \tag{10.5.7}$$

Combining (10.5.6) and (10.5.7), we obtain

$$\begin{aligned}\chi(\mathscr{O}_X(D+D_i)) &= \frac{D\cdot D - D\cdot K_X}{2} + D\cdot D_i + D_i\cdot D_i + 1 + \chi(\mathscr{O}_X)\\ &= \frac{(D+D_i)\cdot(D+D_i) + D_i\cdot D_i - D\cdot K_X + 2}{2} + \chi(\mathscr{O}_X)\end{aligned}$$

However, using $K_X = -(D_1+\cdots+D_r)$ and Theorem 10.4.4, one computes that

$$D_i\cdot K_X = -D_i\cdot D_i - 2. \tag{10.5.8}$$

Substituting this into the above expression for $\chi(\mathscr{O}_X(D+D_i))$ and simplifying, we obtain

$$\chi(\mathscr{O}_X(D+D_i)) = \frac{(D+D_i)\cdot(D+D_i)-(D+D_i)\cdot K_X}{2} + \chi(\mathscr{O}_X),$$

which shows that the theorem holds for $D+D_i$

The proof for $D-D_i$ is similar and is left to the reader (Exercise 10.5.3). □

The following statement is sometimes considered as the topological part of the Riemann-Roch theorem for surfaces.

Theorem 10.5.3 (Noether's Theorem). *Let X be a smooth projective surface with canonical divisor K_X. Then*

$$\chi(\mathscr{O}_X) = \frac{K_X\cdot K_X + e(X)}{12},$$

where $e(X)$ is the topological Euler characteristic of X defined by

$$e(X) = \sum_{k=0}^{4}(-1)^k \dim H^k(X,\mathbb{C}).$$

As before, we will give a proof only for a smooth complete toric surface. We will also use the Hodge decomposition

$$H^k(X,\mathbb{C}) \simeq \bigoplus_{p+q=k} H^q(X,\Omega_X^p)$$

from (9.4.11).

Proof. Demazure vanishing (Theorem 9.2.3) implies that for a smooth complete toric surface $X = X_\Sigma$,

$$\begin{aligned}\chi(\mathscr{O}_X) &= \dim H^0(X,\mathscr{O}_X) - \dim H^1(X,\mathscr{O}_X) + \dim H^2(X,\mathscr{O}_X)\\ &= 1-0+0 = 1.\end{aligned} \tag{10.5.9}$$

Thus Noether's theorem for a smooth complete toric surface is equivalent to

$$K_X\cdot K_X + e(X) = 12. \tag{10.5.10}$$

We prove this as follows. Set $r = |\Sigma(1)|$ and let the minimal generators of the rays be $u_0,\ldots,u_{r-1}$ as in Lemma 10.4.1. Since $K_X = -\sum_{i=0}^{r-1} D_i$, (10.5.8) implies

$$K_X\cdot K_X = -\sum_{i=0}^{r-1} D_i\cdot K_X = -\sum_{i=0}^{r-1}(-D_i\cdot D_i - 2) = 2r + \sum_{i=0}^{r-1} D_i\cdot D_i.$$

If $u_{i-1}+u_{u+i} = b_i u_i$ as in (10.4.1), then $D_i\cdot D_i = -b_i$ by Theorem 10.4.4. Hence

$$K_X\cdot K_X = 2r - \sum_{i=0}^{r-1} b_i = 2r-(3r-12) = 12-r,$$

where the equality $\sum_{i=0}^{r-1} b_i = 3r-12$ is from Exercise 10.4.10.

We next compute $e(X)$. Proposition 11.2.8 shows that Σ is the normal fan of a polygon with $r = |\Sigma(1)|$ sides. Then the formula for $\dim H^q(X,\Omega_X^p)$ given in Theorem 9.4.11 implies

$$\begin{aligned}
\dim H^0(X,\mathbb{C}) &= \dim H^0(X,\mathscr{O}_X) = 1\\
\dim H^1(X,\mathbb{C}) &= \dim H^0(X,\Omega_X^1) + \dim H^1(X,\mathscr{O}_X) = 0+0 = 0\\
\dim H^2(X,\mathbb{C}) &= \dim H^0(X,\Omega_X^2) + \dim H^1(X,\Omega_X^1) + \dim H^2(X,\mathscr{O}_X)\\
&= 0 + f_1 - 2f_2 = r-2\\
\dim H^3(X,\mathbb{C}) &= \dim H^1(X,\Omega_X^2) + \dim H^2(X,\Omega_X^1) = 0+0 = 0\\
\dim H^4(X,\mathbb{C}) &= \dim H^2(X,\Omega_X^2) = 1.
\end{aligned}$$

where $f_1 = r$ and $f_2 = 1$ are the face numbers of a polygon with r sides. It follows that $e(X) = 1 + (r-2) + 1 = r$. Then (10.5.10) follows easily from the above computation of $K_X \cdot K_X$. □

We will give a topological proof of $e(X) = r$ in Chapter 12, and in Chapter 13, we will interpret $e(X)$ in terms of the Chern classes of the tangent bundle.

The Riemann-Roch theorems for curves and surfaces have been vastly generalized by results of Hirzebruch and Grothendieck, and the precise relation of Noether's theorem to the Riemann-Roch theorem for surfaces is a special case of their approach. We will discuss Riemann-Roch theorems for higher-dimensional toric varieties in Chapter 13.

Lattice Polygons. For the remainder of this section, we will explore the relation between toric surfaces and the geometry and combinatorics of lattice polygons. We will see that the results from §9.4 for lattice polytopes have an especially nice form for lattice polygons.

Let $X = X_\Sigma$ be a smooth complete toric surface. Since $\chi(\mathscr{O}_X) = 1$ by (10.5.9), Riemann-Roch for a divisor D on X becomes

$$\chi(\mathscr{O}_X(D)) = \frac{D\cdot D - D\cdot K_X}{2} + 1, \tag{10.5.11}$$

Thus, for any $\ell \in \mathbb{Z}$,

$$\chi(\mathscr{O}_X(\ell D)) = \frac{\ell D\cdot \ell D - \ell D\cdot K_X}{2} + 1 = \tfrac{1}{2}(D\cdot D)\,\ell^2 - \tfrac{1}{2}(D\cdot K_X)\,\ell + 1. \tag{10.5.12}$$

The theory developed in §9.4 guarantees that $\chi(\mathscr{O}_X(\ell D))$ is a polynomial in ℓ; the above computation gives explicit formulas for the coefficients in terms of intersection products.

Here is an example of how this formula works.

Example 10.5.4. Let X be the Hirzebruch surface $\mathscr{H}_2$. We will use the notation of Example 10.4.6. The divisor $D = D_1 + D_2$ is clearly effective, but the inequalities (9.3.6) defining the nef cone show that D is not nef. Using $K_X = -D_1 - \cdots - D_4$ and Example 10.4.6, one computes that

$$D \cdot D = 0, \quad D \cdot K_X = -2.$$

Then (10.5.12) implies that

$$\chi(\mathscr{O}_X(\ell D)) = \ell + 1.$$

An easy application of Serre duality gives $H^2(X, \mathscr{O}_X(\ell D)) = 0$ (Exercise 10.5.4). Thus

$$\dim H^0(X, \mathscr{O}_X(\ell D)) - \dim H^1(X, \mathscr{O}_X(\ell D)) = \ell + 1.$$

Things get more surprising when we compute $\dim H^0(X, \mathscr{O}_X(\ell D))$. Using the ray generators $u_1, \ldots, u_4$ from Example 10.4.6, the polygon P_D corresponding to $D = D_1 + D_2$ is defined by the inequalities

$$\langle m, u_1 \rangle \geq -1, \ \langle m, u_2 \rangle \geq -1, \ \langle m, u_3 \rangle \geq 0, \ \langle m, u_4 \rangle \geq 0.$$

The polygon P_D is shown in Figure 11.

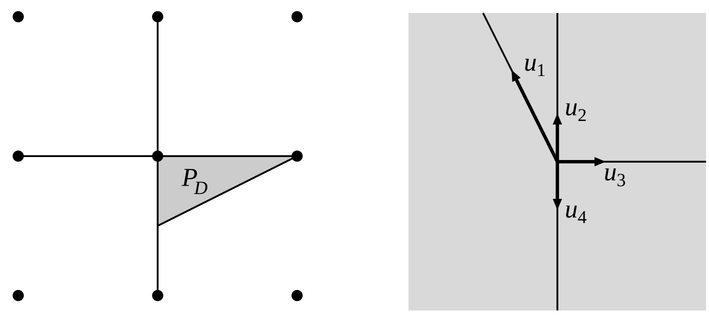


Figure 11. The polygon of the divisor D and the fan of $\mathscr{H}_2$

Even though P_D is *not* a lattice polytope, Proposition 4.3.3 still applies. Thus

$$\dim H^0(X, \mathscr{O}_X(\ell D)) = |P_{\ell D} \cap M| = |\ell P_D \cap M| = \begin{cases} \frac{1}{4}\ell^2 + \ell + 1 & \ell \text{ even} \\ \frac{1}{4}\ell^2 + \ell + \frac{3}{4} & \ell \text{ odd,} \end{cases}$$

where the final equality follows from Exercise 9.4.13. Combining this with the above computation of $\chi(\mathscr{O}_X(\ell D))$, we obtain

$$\dim H^1(X, \mathscr{O}_X(\ell D)) = \begin{cases} \frac{1}{4}\ell^2 & \ell \text{ even} \\ \frac{1}{4}\ell^2 - \frac{1}{4} & \ell \text{ odd.} \end{cases}$$

This is a vivid example how the Euler characteristic smooths out the complicated behavior of the individual cohomology groups. $\Diamond$

On the other hand, if D is nef, the higher cohomology is trivial by Demazure vanishing, so that the Euler characteristic reduces to $\dim H^0$. We exploit this as follows. Suppose that $P \subseteq M_{\mathbb{R}} \simeq \mathbb{R}^2$ is a lattice polygon. By Theorem 9.4.2 and Example 9.4.4, the Ehrhart polynomial $\mathrm{Ehr}_P(x) \in \mathbb{Q}[x]$ of P satisfies

$$\mathrm{Ehr}_P(\ell) = |(\ell P)\cap M| = \mathrm{Area}(P)\,\ell^2 + \tfrac{1}{2}|\partial P \cap M|\,\ell + 1 \tag{10.5.13}$$

for $\ell \in \mathbb{N}$. We next describe this polynomial in terms of intersection products.

By the results of §2.3, we get the projective toric surface X_P coming from the normal fan Σ_P of P. In general X_P will not be smooth, so we compute a minimal resolution of singularities

$$\phi : X_\Sigma \longrightarrow X_P$$

using the methods of this chapter. Recall that X_P has the ample divisor D_P whose associated polygon is P.

Proposition 10.5.5. *There is unique torus-invariant nef divisor D on X_Σ such that*

(a) *The support function of D equals the support function of D_P.*

(b) $\chi(\mathscr{O}_{X_\Sigma}(\ell D))$ *is the Ehrhart polynomial of P.*

Proof. Proposition 6.2.7 implies that X_Σ has a divisor D that satisfies part (a). As in §6.1, we call D the *pullback* of D_P. Since D_P has a convex support function, the same is true for D, so that D is nef. Furthermore, P is the polytope associated to D_P and hence is the polytope associated to D since the polytope of a nef divisor is determined by its support function (Theorem 6.1.7).

It follows that $\dim H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D)) = |P_{\ell D}\cap M| = |(\ell P)\cap M|$ when $\ell \geq 0$, so that $H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D))$ equals the Ehrhart polynomial of P when $\ell \geq 0$. However, ℓD is nef when $\ell \geq 0$ and hence has trivial higher cohomology by Demazure vanishing (Theorem 9.2.3). Thus $\ell \geq 0$ implies

$$\chi(\mathscr{O}_{X_\Sigma}(\ell D)) = \dim H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D)) = |(\ell P)\cap M|.$$

Since $\chi(\mathscr{O}_{X_\Sigma}(\ell D))$ is a polynomial in ℓ, it must be the Ehrhart polynomial of P. □

Theorem 10.5.5 and (10.5.12) imply that the Ehrhart polynomial of P is

$$\mathrm{Ehr}_P(\ell) = \tfrac{1}{2}(D\cdot D)\,\ell^2 - \tfrac{1}{2}(D\cdot K_{X_\Sigma})\,\ell + 1.$$

Comparing this to the formula (10.5.13) for $\mathrm{Ehr}_P(\ell)$, we get the following result.

Proposition 10.5.6. *Let P be a lattice polygon and let D be the pullback of D_P constructed in Proposition 10.5.5. Then*

$$\begin{aligned} D\cdot D &= 2\,\mathrm{Area}(P)\\ -D\cdot K_{X_\Sigma} &= |\partial P\cap M|. \end{aligned}$$

□

Example 10.5.7. Take the fan of $\mathscr{H}_2$ shown in Figure 11 from Example 10.5.4 and combine the two 2-dimensional cones containing u_2 into a single cone. The resulting fan has minimal generators u_1, u_3, u_4 that satisfy $u_1 + u_3 + 2u_4 = 0$, so the resulting toric variety is $\mathbb{P}(1,1,2)$.

Let $P = \mathrm{Conv}(0, 2e_1, -e_2) \subseteq M_{\mathbb{R}}$, which is the double of the polytope shown in Figure 11. The normal fan of P is the fan of $\mathbb{P}(1,1,2)$. The minimal generators u_1, u_3, u_4 of this fan give divisors D'_1, D'_3, D'_4 on $\mathbb{P}(1,1,2)$, and the divisor D_P is easily seen to be the ample divisor $2D'_1$.

Since the fan of $\mathscr{H}_2$ refines the fan of $\mathbb{P}(1,1,2)$, the resulting toric morphism $\mathscr{H}_2 \to \mathbb{P}(1,1,2)$ is a resolution of singularities. By considering the support function of D_P, we find that the pullback of $D_P = 2D'_1$ is $D = 2D_1 + 2D_2$. We leave it as Exercise 10.5.5 to compute $D \cdot D$ and $D \cdot K_{\mathscr{H}_2}$ and verify that they give the numbers predicted by Proposition 10.5.6. $\Diamond$

Sectional Genus. The divisor D_P on X_P is very ample since $\dim P = 2$. Hence it gives a projective embedding $X_P \hookrightarrow \mathbb{P}^s$ such that $\mathscr{O}_{\mathbb{P}^s}(1)$ restricts to $\mathscr{O}_{X_\Sigma}(D_P)$. In geometric terms, this means that hyperplanes $H \subseteq \mathbb{P}^s$ give curves $X_P \cap H \subseteq X_P$ that are linearly equivalent to D_P. For some hyperplanes, the intersection $X_P \cap H$ can be complicated. Since X_P has only finitely many singular points, the *Bertini theorem* (see [**131**, II 8.18 and III 7.9.1]) guarantees that when H is generic, $C = X_P \cap H$ is a smooth connected curve contained in the smooth locus of X_P. The genus g of C is called the *sectional genus* of the surface X_P.

We will compute g in terms of the geometry of P using the adjunction formula. Since we need a smooth surface for this, we use a resolution $\phi : X_\Sigma \to X_P$ and note that C can be regarded as a curve in X_Σ since ϕ is an isomorphism away from the singular points of X_P. Since $C \sim D_P$ on X_P, we have $C \sim D$ on X_Σ, where D is the pullback of D_P. Then the adjunction formula (Theorem 10.5.1) implies

$$2g - 2 = K_{X_\Sigma} \cdot C + C \cdot C = K_{X_\Sigma} \cdot D + D \cdot D,$$

so that

$$g = \tfrac{1}{2} D \cdot (K_{X_\Sigma} + D) + 1. \tag{10.5.14}$$

Then we have the following result.

Proposition 10.5.8. *The sectional genus of X_P is $g = |\mathrm{Int}(P) \cap M|$.*

Proof. Pick's formula from Example 9.4.4 can be written as

$$|\mathrm{Int}(P) \cap M| = \mathrm{Area}(P) - \tfrac{1}{2}|\partial P \cap M| + 1,$$

which by Proposition 10.5.6 becomes

$$|\mathrm{Int}(P) \cap M| = \tfrac{1}{2} D \cdot D + \tfrac{1}{2} D \cdot K_{X_\Sigma} + 1.$$

The right-hand side is g by (10.5.14), completing the proof. $\square$

Example 10.5.9. Let $P = d\Delta_2 = \mathrm{Conv}(0, de_1, de_2)$. Then X_P is the projective plane $\mathbb{P}^2$ in its dth Veronese embedding, and $D_P \sim dL$, where $L \subseteq \mathbb{P}^2$ is a line. The hyperplane sections are the curves of degree d in $\mathbb{P}^2$, and the smooth ones have genus

$$g = |\mathrm{Int}(d\Delta_2) \cap M| = \frac{(d-1)(d-2)}{2}.$$

You will check this assertion and another example in Exercise 10.5.6. $\Diamond$

The curves $C \subseteq X_\Sigma$ studied here can be generalized to the study of hypersurfaces in projective toric varieties coming from sections of a nef line bundle. The geometry and topology of these hypersurfaces have been studied in many papers, including [**15**], [**19**], [**77**] and [**197**].

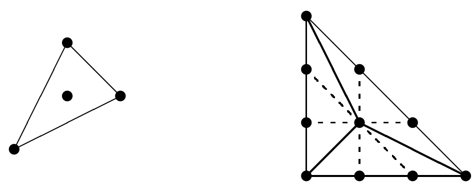
Reflexive Polygons and The Number 12. Our final topic gives a way to understand a somewhat mysterious formula we noted in the last section of Chapter 8. Recall from Theorem 8.3.7 that there are exactly 16 equivalence classes of reflexive lattice polytopes in $\mathbb{R}^2$, shown in Figure 3 of §8.3. The article [**230**] gives four different proofs of the following result.

Theorem 10.5.10. *Let P be a reflexive lattice polygon in $M_\mathbb{R} \simeq \mathbb{R}^2$. Then*

$$|\partial P \cap M| + |\partial P^\circ \cap N| = 12.$$

One proof consists of a case-by-case verification of the statement for each of 16 equivalence classes. You proved the theorem this way in Exercise 8.3.5. The argument was straightforward but not very enlightening! Here we will give another proof using Noether's theorem.

Proof. Since Noether's theorem requires a smooth surface, we need to refine the normal fan Σ_P of $P \subseteq M_\mathbb{R}$. Since P is reflexive, we can do this using the dual polygon $P^\circ \subseteq N_\mathbb{R}$. We know from (8.3.2) that the vertices of P° are the minimal generators of Σ_P. Let Σ be the refinement of Σ_P whose 1-dimensional cones are generated by the rays through the lattice points on the boundary of P°. This is illustrated in Figure 12.



P $\qquad$ P°

Figure 12. A reflexive polygon P and its dual P°

The fan Σ has the following properties:

- For each cone of Σ, its minimal generators and the origin form a triangle whose only lattice points are the vertices. Thus Σ is smooth by Exercise 8.3.4.
- The minimal generators of Σ are the lattice points of P° lying on the boundary. Thus $|\Sigma(1)| = |\partial P^\circ \cap N|$.

From the first bullet, we get a resolution $\phi : X_\Sigma \to X_P$. Recall that $D_P = -K_{X_P}$ since P is reflexive. The wonderful fact is that its pullback via ϕ is again anticanonical, i.e., $D = -K_{X_\Sigma}$. To prove this, recall that D and D_P have the same support function φ, which takes the value 1 at the vertices of P° since $D_P = -K_{X_P}$. It follows that $\varphi = 1$ on the boundary of P°. Then $D = -K_{X_\Sigma}$ because the minimal generators of Σ all lie on the boundary.

Now apply Noether's theorem to the toric surface X_Σ. By (10.5.10), we have

$$K_{X_\Sigma} \cdot K_{X_\Sigma} + e(X_\Sigma) = 12.$$

We analyze each term on the left as follows. First, $D = -K_{X_\Sigma}$ implies

$$K_{X_\Sigma} \cdot K_{X_\Sigma} = -D \cdot K_{X_\Sigma} = |\partial P \cap M|,$$

where the last equality follows from Proposition 10.5.6. Second, $e(X_\Sigma)$ is the number of minimal generators of Σ by the proof of Theorem 10.5.3. In other words,

$$e(X_\Sigma) = |\Sigma(1)| = |\partial P^\circ \cap N|,$$

where the second equality follows from the above analysis of Σ. Hence the theorem is an immediate consequence of Noether's theorem. □

A key step in the above proof was showing that the pullback of the canonical divisor on X_P was the canonical divisor on X_P. This may fail for a general resolution of singularities. We will say more about this when we study *crepant resolutions* in Chapter 11.

Exercises for §10.5.

10.5.1. The Riemann-Roch theorem for curves, in the form (10.5.1), can be proved by much the same method as used in the proof of Theorem 10.5.2. Namely, show that if (10.5.1) holds for a divisor D then it also holds for the divisors $D+P$ and $D-P$, where P is an arbitrary point on the curve.

10.5.2. Prove the adjunction formula (Theorem 10.5.1) using (10.5.3) and (10.5.5).

10.5.3. Complete the proof of Theorem 10.5.2 by showing that if the theorem holds for D, then it also holds for $D - D_i$ where D_i is any one of the divisors corresponding to the 1-dimensional cones in Σ.

10.5.4. Let $D = \sum_\rho a_\rho D_\rho$ be an effective $\mathbb{Q}$-Cartier Weil divisor on a complete toric variety X_Σ of dimension n. Use Serre duality (Theorem 9.2.10) to prove that $H^n(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = 0$.

10.5.5. In Example 10.5.7, compute $D \cdot D$ and $D \cdot K_{\mathscr{H}_2}$ and check that they agree with the numbers given by Proposition 10.5.6.

10.5.6. This exercise studies the sectional genus of toric surfaces.

(a) Verify the formula given in Example 10.5.9 for the sectional genus of $\mathbb{P}^2$ in its dth Veronese embedding.

(b) Let $P = \operatorname{Conv}(0, ae_1, be_2, ae_1 + be_2)$. What is the smooth toric surface X_P in this case? Show that its sectional genus is $(a-1)(b-1)$.

10.5.7. Let P be a reflexive polygon.

(a) Prove that the singularities (if any) of the toric surface X_P are rational double points. Hint: Proposition 10.1.6.

(b) Prove that X_P has sectional genus $g = 1$. This means that smooth anticanonical curves in X_P are all elliptic curves.

(c) Explain how part (b) relates to Exercise 10.5.6.

10.5.8. According to Theorem 10.4.3, every smooth toric surface is a blowup of either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathscr{H}_r$ for $r \geq 2$. For each of the 16 reflexive polygons in Figure 3 of §8.3, the process described in the proof of Theorem 10.5.10 produces a smooth toric surface X_Σ. Where does X_Σ fit in this classification in each case? (This gives a classification of smooth toric Del Pezzo surfaces.)

10.5.9. As in §9.3, the *p-Ehrhart polynomials* of a lattice polygon $P \subset M_\mathbb{R}$ are defined by

$$\operatorname{Ehr}_P^p(\ell) = \chi(\widehat{\Omega}^p_{X_P}(\ell D_P)), \quad p = 0, 1, 2.$$

We know that Ehr_P^0 is the usual Ehrhart polynomial Ehr_P, and then $\operatorname{Ehr}_P^2(x) = \operatorname{Ehr}_P(-x)$ by Theorem 9.4.7. The remaining case is Ehr_P^1. Prove that

$$\operatorname{Ehr}_P^1(x) = 2\operatorname{Area}(P)\,x^2 + f_1 - 2,$$

where f_1 is the number of edges of P. Hint: Use Theorem 9.4.11 for the constant term and part (c) of Theorem 9.4.7 for the coefficient of x. For the leading coefficient, tensor the exact sequence of Theorem 8.1.6 with $\mathscr{O}_{X_P}(\ell D_P)$, take the Euler characteristic, and then let $\ell \to \infty$.

Chapter 11

Toric Resolutions and Toric Singularities

This chapter will study the singularities of a normal toric variety. We begin in §11.1 with the existence of toric resolutions of singularities. In §11.2 we consider more special resolutions, including simple normal crossing resolutions, crepant resolutions, log resolutions, and embedded resolutions. There are also relations with ideal sheaves, Rees algebras, and multiplier ideals, to be studied in §11.3, and finally, in §11.4 we consider some important classes of toric singularities.

§11.1. Resolution of Singularities

The *singular locus* of an irreducible variety X, denoted X_{sing}, is the set of all singular points of X. One can prove that X_{sing} is a proper closed subvariety of X (see [**131**, Thm. I.5.3]). We call the complement $X \setminus X_{\mathrm{sing}}$ the *smooth locus* of X.

Definition 11.1.1. Given an irreducible variety X, a ***resolution of singularities*** of X is a morphism $f : X' \to X$ such that:

(a) X' is smooth and irreducible.

(b) f is proper.

(c) f induces an isomorphism of varieties $f^{-1}(X \setminus X_{\mathrm{sing}}) \simeq X \setminus X_{\mathrm{sing}}$.

Furthermore, $f : X' \to X$ is a ***projective resolution*** if f is a projective morphism.

A resolution of singularities modifies X to make it smooth without changing the part that is already smooth. In particular, a resolution of singularities is a birational morphism. Hironaka [**143**] proved the existence of a resolution of singularities over an algebraically closed field of characteristic 0. An introduction to Hironaka's proof and more recent developments can be found in [**138**].

Fortunately, the toric case is much simpler, partly because toric varieties cannot have arbitrarily bad singularities. Hence we will be able to give a complete proof of resolution of singularities for toric varieties. The key idea is that given a fan Σ in $N_{\mathbb{R}}$, the resolution will be the toric morphism

$$\phi : X_{\Sigma'} \longrightarrow X_\Sigma$$

coming from a suitable refinement Σ' of Σ.

The Singular Locus. The first step is to determine the singular locus of X_Σ. This has the following purely toric description.

Proposition 11.1.2. *Let X_Σ be the toric variety of the fan Σ. Then:*

$$(X_\Sigma)_{\mathrm{sing}} = \bigcup_{\sigma \text{ not smooth}} V(\sigma)$$
$$X_\Sigma \setminus (X_\Sigma)_{\mathrm{sing}} = \bigcup_{\sigma \text{ smooth}} U_\sigma,$$

where $V(\sigma) = \overline{O(\sigma)}$ is the closure of the T_N-orbit corresponding to σ.

Proof. Recall that σ is smooth if its minimal generators can be extended to a basis of N and that σ is smooth if and only if U_σ is smooth. Also observe that

- if σ is smooth, then so is every face of σ, hence
- if σ is not smooth, then so is every cone of Σ containing σ.

The first bullet tells us that the smooth cones of Σ form a fan whose toric variety is $\bigcup_{\sigma \text{ smooth}} U_\sigma$. This open set of X_Σ is clearly smooth, and its complement in X_Σ is $\bigcup_{\sigma \text{ not smooth}} O(\sigma)$ by the Orbit-Cone correspondence.

We also have $\bigcup_{\sigma \text{ not smooth}} V(\sigma) = \bigcup_{\sigma \text{ not smooth}} O(\sigma)$ by the second bullet and the Orbit-Cone correspondence. Hence the proof will be complete once we show that every point of $O(\sigma)$ is singular in X_Σ when σ is not smooth. It suffices to work in the affine open set U_σ. Let $N_\sigma = \mathrm{Span}(\sigma) \cap N$ and pick a complement $N_2 \subseteq N$ such that $N = N_\sigma \oplus N_2$. By (1.3.2), we have

$$U_{\sigma,N} \simeq U_{\sigma,N_\sigma} \times T_{N_2}.$$

Since $\dim \sigma = \operatorname{rank} N_\sigma$, the orbit $O_{N_\sigma}(\sigma) \subseteq U_{\sigma,N_\sigma}$ consists of the unique fixed point of the action of T_{N_σ}. Since σ is not smooth with respect to N_σ, the proof of Theorem 1.3.12 shows that this fixed point is singular in U_{σ,N_σ}. Then (1.3.3) shows that every point of $O_{N_\sigma}(\sigma) \times T_{N_2}$ is singular in $U_{\sigma,N_\sigma} \times T_{N_2}$. Since the above product decomposition induces

$$O_N(\sigma) \simeq O_{N_\sigma}(\sigma) \times T_{N_2}$$

(Exercise 11.1.1), we see that all points of $O(\sigma) = O_N(\sigma)$ are singular in X_Σ. $\square$

It follows that when constructing refinements of a fan Σ, we need to be sure to leave the smooth cones of Σ unchanged.

Star Subdivisions. We will construct refinements using a generalization of the star subdivision $\Sigma^*(\sigma)$ defined in §3.3. Given a fan Σ in $N_{\mathbb{R}}$ and a primitive element $v \in |\Sigma| \cap N$, let $\Sigma^*(v)$ be the set of the following cones:

(a) σ, where $v \notin \sigma \in \Sigma$.

(b) $\mathrm{Cone}(\tau, v)$, where $v \notin \tau \in \Sigma$ and $\{v\} \cup \tau \subseteq \sigma \in \Sigma$.

We call $\Sigma^*(v)$ the *star subdivision* of Σ at v. This terminology is justified by the following result.

Lemma 11.1.3. *$\Sigma^*(v)$ is a refinement of Σ.*

Proof. Let $\mathrm{Cone}(\tau, v)$ be a cone as in (b) above. Then observe that $v \notin \mathrm{Span}(\tau)$ since $v \notin \tau$ and $\tau = \sigma \cap \mathrm{Span}(\tau)$. It follows that $\dim \mathrm{Cone}(\tau, v) = \dim \tau + 1$.

To show that $\Sigma^*(v)$ is a fan, first consider a face of a cone in $\Sigma^*(v)$. If the cone arises from (a) above, then the face clearly lies in $\Sigma^*(v)$. If the cone arises from (b), then we have a face γ of $\mathrm{Cone}(\tau, v)$. Write $\gamma = H_m \cap \mathrm{Cone}(\tau, v)$ for $m \in \mathrm{Cone}(\tau, v)^\vee$ and observe that $m \in \tau^\vee$ and $\langle m, v\rangle \geq 0$. There are two cases:

- $\langle m, v\rangle = 0$, in which case $\gamma = \mathrm{Cone}(H_m \cap \tau, v) \in \Sigma^*(v)$.
- $\langle m, v\rangle > 0$, in which case $\gamma = H_m \cap \tau \in \Sigma^*(v)$.

You will provide the details of these cases in Exercise 11.1.2.

Next consider two cones in $\Sigma^*(v)$. If neither contains v, then the intersection is clearly in $\Sigma^*(v)$. Now suppose that one does not contain v and the other does. This gives cones $v \notin \sigma$ and $\mathrm{Cone}(\tau, v)$ as in (a) and (b) above. We claim that

$$\sigma \cap \mathrm{Cone}(\tau, v) = \sigma \cap \tau, \tag{11.1.1}$$

which will imply that $\sigma \cap \mathrm{Cone}(\tau, v) \in \Sigma^*(v)$. One inclusion of (11.1.1) is obvious; for the other inclusion, let $u \in \sigma \cap \mathrm{Cone}(\tau, v)$ and write $u = u_0 + \lambda v$ for $u_0 \in \tau$ and $\lambda \geq 0$. Also let $\sigma' \in \Sigma$ be a cone containing $\{v\} \cup \tau$. Then

$$u = u_0 + \lambda v \in \sigma \cap \mathrm{Cone}(\tau, v) \subseteq \sigma \cap \sigma'.$$

Since $u_0, \lambda v \in \sigma'$ and $\sigma \cap \sigma'$ is a face of σ', it follows that $u_0, \lambda v \in \sigma \cap \sigma'$. But $v \notin \sigma$, so that $\lambda = 0$ and hence $u = u_0 \in \tau$. Thus $u \in \sigma \cap \tau$, as desired.

The remaining case to consider is when both cones contain v. If the cones are $\mathrm{Cone}(\tau_1, v)$ and $\mathrm{Cone}(\tau_2, v)$, then we claim that

$$\mathrm{Cone}(\tau_1, v) \cap \mathrm{Cone}(\tau_2, v) = \mathrm{Cone}(\tau_1 \cap \tau_2, v), \tag{11.1.2}$$

which will imply that $\mathrm{Cone}(\tau_1, v) \cap \mathrm{Cone}(\tau_2, v) \in \Sigma^*(v)$. As above, one inclusion is obvious, and for the other, take $u \in \mathrm{Cone}(\tau_1, v) \cap \mathrm{Cone}(\tau_2, v)$. Then

$$u = u_1 + \lambda_1 v = u_2 + \lambda_2 v,$$

where $u_i \in \tau_i$ and $\lambda_i \geq 0$. We may assume that $\lambda_1 \geq \lambda_2$, in which case $u_2 = u_1 + (\lambda_2 - \lambda_1)\nu \in \mathrm{Cone}(\tau_1, v)$. Thus $u_2 \in \tau_2 \cap \mathrm{Cone}(\tau_1, v)$, which by (11.1.1) implies

$$u_2 \in \tau_2 \cap \mathrm{Cone}(\tau_1, v) = \tau_1 \cap \tau_2.$$

Thus $u = u_2 + \lambda_2 v \in \mathrm{Cone}(\tau_1 \cap \tau_2, v)$, as desired.

Finally, we need to show that $\Sigma^*(v)$ is a refinement of Σ. Since every cone of $\Sigma^*(v)$ is clearly contained in a cone of Σ, we need only show that each $\sigma \in \Sigma$ is a union of cones in $\Sigma^*(v)$. When $v \notin \sigma$, this is obvious since $\sigma \in \Sigma^*(v)$. On the other hand, when $v \in \sigma$, pick cone generators $m_i \in \sigma^\vee$, $i = 1, \ldots, r$, and note that $v \in \sigma$ implies $\langle m_i, v\rangle \geq 0$ for all i. Then set

$$\lambda = \min_{\langle m_i, v\rangle > 0} \frac{\langle m_i, u\rangle}{\langle m_i, v\rangle}. \tag{11.1.3}$$

We claim that $u_0 = u - \lambda v$ lies in σ. To prove this, take $m = \sum_{i=1}^r \mu_i m_i \in \sigma^\vee$, $\mu_i \geq 0$, and compute

$$\begin{aligned}\langle m, u_0\rangle &= \big\langle \textstyle\sum_{i=1}^r \mu_i m_i, u - \lambda v\big\rangle \\ &= \textstyle\sum_{\langle m_i, v\rangle = 0} \mu_i \langle m_i, u\rangle + \sum_{\langle m_i, v\rangle > 0} \mu_i \langle m_i, v\rangle \Big(\dfrac{\langle m_i, u\rangle}{\langle m_i, v\rangle} - \lambda\Big).\end{aligned}$$

The first sum is ≥ 0 since $u \in \sigma$, and the second is ≥ 0 by the definition of λ. Hence $u_0 \in (\sigma^\vee)^\vee = \sigma$, as claimed.

The definition (11.1.3) implies that we can pick i with $\langle m_i, v\rangle > 0$ and $\lambda = \frac{\langle m_i, u\rangle}{\langle m_i, v\rangle}$. Then $\langle m_i, u_0\rangle = 0$, so that u_0 lies in the face $\tau = H_{m_i} \cap \sigma$ of σ. This face does not contain v since $\langle m_i, v\rangle > 0$, hence $u \in \mathrm{Cone}(\tau, v) \in \Sigma^*(v)$. $\square$

Example 11.1.4. Let $\sigma = \mathrm{Cone}(u_1, \ldots, u_r) \in \Sigma$ be a smooth cone in $N_{\mathbb{R}} \simeq \mathbb{R}^n$. In §3.3, we defined the star subdivision $\Sigma^*(\sigma)$ using $u_0 = u_1 + \cdots + u_r$. It is easy to check that this is the star subdivision $\Sigma^*(u_0)$ since $u_0 \in \sigma \cap N$ is primitive. See Figure 9 in §3.3 for a picture when $n = 3$. $\Diamond$

Example 11.1.5. Figure 1 shows a fan Σ in $\mathbb{R}^3$ consisting of two 3-dimensional cones (one simplicial, the other not) that meet along a 2-dimensional face τ. The

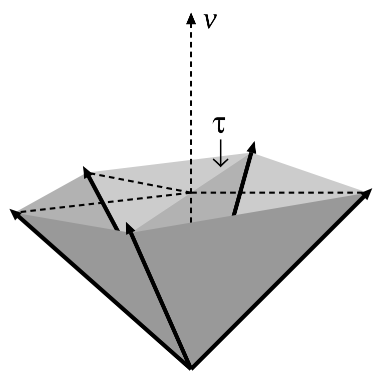


Figure 1. A fan Σ and its star subdivision $\Sigma^*(v)$

figure also shows the star subdivision induced by a primitive vector $v \in \mathrm{Relint}(\tau)$. The fan $\Sigma^*(v)$ has five 3-dimensional cones. $\Diamond$

The definition of $\Sigma^*(v)$ implies that v generates a 1-dimensional cone of $\Sigma^*(v)$ and hence gives a divisor on $X_{\Sigma^*(v)}$. Recall that a Weil divisor is $\mathbb{Q}$-Cartier if some positive integer multiple is Cartier.

Proposition 11.1.6. *The star subdivision $\Sigma^*(v)$ has the following properties:*

(a) *The 1-dimensional cones of $\Sigma^*(v)$ consist of the 1-dimesional cones of Σ plus the ray $\rho_v = \mathrm{Cone}(v)$ generated by v, i.e.,*

$$\Sigma^*(v)(1) = \Sigma(1) \cup \{\rho_v\}.$$

(b) *The torus-invariant divisor prime D_{ρ_v} is $\mathbb{Q}$-Cartier on $X_{\Sigma^*(v)}$.*

(c) *The induced toric morphism $\phi : X_{\Sigma^*(v)} \to X_\Sigma$ is projective.*

Proof. The first assertion is Exercise 11.1.3. To show that D_{ρ_v} is $\mathbb{Q}$-Cartier, we construct a support function $\varphi : |\Sigma^*(v)| \to \mathbb{R}$ as follows. For $\mathrm{Cone}(\tau, v) \in \Sigma^*(v)$, consider the linear function defined on $\mathrm{Span}(\mathrm{Cone}(\tau, v)) = \mathrm{Span}(\tau) + \mathbb{R}v$ that maps $\mathrm{Span}(\tau)$ to 0 and v to 1. This is well-defined since $v \notin \mathrm{Span}(\tau)$. Then define $\varphi|_{\mathrm{Cone}(\tau,v)}$ to be the restriction of this linear map to $\mathrm{Cone}(\tau, v)$. The remaining cones in $\Sigma^*(v)$ are the cones $\sigma \in \Sigma$ not containing v. We set $\varphi|_\sigma = 0$ for each of these. By (11.1.1) and (11.1.2), we get a well defined map $\varphi : |\Sigma^*(v)| \to \mathbb{R}$ that is linear on each cone of $\Sigma^*(v)$. Furthermore, φ is clearly defined over $N_\mathbb{Q}$ since τ is rational and $v \in N$. It follows that some positive integer multiple $k\varphi$ is integral with respect to N, i.e., $k\varphi \in \mathrm{SF}(\Sigma^*(v), N)$. Then by Theorem 4.2.12,

$$D_{k\varphi} = -\sum_{\rho \in \Sigma^*(v)(1)} k\varphi(u_\rho) D_\rho$$

is a Cartier divisor on $X_{\Sigma^*(v)}$. However, v is primitive and hence is the ray generator of ρ_v. Since φ vanishes on all other ray generators, we have $D_{k\varphi} = -k\varphi(v) D_{\rho_v} = -kD_{\rho_v}$. This proves that $-D_{\rho_v}$ and hence D_{ρ_v} are $\mathbb{Q}$-Cartier.

For the final assertion, we use the projectivity criterion from Theorem 7.2.12. Since $\Sigma^*(v)$ refines Σ, it suffices to find a torus-invariant Cartier divisor on $X_{\Sigma^*(v)}$ whose support function is strictly convex with respect to the fan

$$\Sigma^*(v)_\sigma = \{\sigma' \in \Sigma^*(v) \mid \sigma' \subseteq \sigma\}$$

for every $\sigma \in \Sigma$. The required Cartier divisor is easy to find: it is the divisor $-kD_{\rho_v}$, where $k > 0$ is from the previous paragraph, with support function $k\varphi$. We can ignore the factor of k when thinking about strict convexity. Hence the proof will be complete once we prove that φ has the desired strict convexity.

When $v \notin \sigma$, this is obvious since $\sigma \in \Sigma^*(v)$ in this case. So it remains to consider what happens when $v \in \sigma$. By restricting to $\mathrm{Span}(\sigma)$, we can assume that σ is full dimensional. Then, as noted in the discussion leading up to Proposition 7.2.3, the strict convexity criteria from Lemma 6.1.13 apply to this situation.

The analysis of $\Sigma^*(v)$ given in the proof of Lemma 11.1.3 shows that $\sigma = \bigcup_\tau \mathrm{Cone}(\tau, v)$, where the union is over all facets τ of σ not containing v. Now fix such a facet τ and consider the cone $\sigma' = \mathrm{Cone}(\tau, v) \in \Sigma^*(v)$. Pick $m \in \sigma^\vee \cap M$ such that $H_m \cap \sigma = \tau$. Since $v \in \sigma \setminus \tau$, we have

$$\langle m, u_\rho \rangle = 0,\ \rho \in \tau(1) \ \text{ and } \ \langle m, v \rangle > 0. \tag{11.1.4}$$

We claim that after rescaling m by a positive constant, we have $\varphi(u) = \langle m, u \rangle$ for all $u \in \sigma'$. To see why, note that φ is linear on σ', vanishes on τ, and takes the value 1 on v. Comparing this to (11.1.4), our claim follows immediately.

Given any $\rho \in \sigma(1) \setminus \tau(1)$, the way we picked m implies that

$$\langle m, u_\rho \rangle > 0.$$

However, m represents φ on σ', and

$$\sigma(1) \setminus \tau(1) = \Sigma^*(v)_\sigma(1) \setminus \sigma'(1)$$

by part (a) and the definitions of $\Sigma^*(v)$ and $\Sigma^*(v)_\sigma$. By (f) $\Rightarrow$ (a) of Lemma 6.1.13, it follows that $\varphi|_\sigma$ is strictly convex with respect to $\Sigma^*(v)_\sigma$. □

Simplicialization. As an application of star subdivisions, we prove that every fan has an efficient simplicial refinement. Here is the precise result.

Proposition 11.1.7. *Every fan Σ has a refinement Σ' with the following properties:*

(a) *Σ' is simplicial.*

(b) $\Sigma'(1) = \Sigma(1)$.

(c) *Σ' contains every simplicial cone of Σ.*

(d) *Σ' is obtained from Σ by a sequence of star subdivisions.*

(e) *The induced toric morphism $X_{\Sigma'} \to X_\Sigma$ is projective.*

Proof. First observe that part (c) follows from part (b). To see why, suppose that Σ' is a refinement of Σ satisfying $\Sigma'(1) = \Sigma(1)$ and let $\sigma \in \Sigma$ be simplicial. If $\sigma' \in \Sigma'$ lies in σ, then $\Sigma'(1) = \Sigma(1)$ implies $\sigma'(1) \subseteq \sigma(1)$, and hence σ' is a face of σ because the latter is simplicial. Since $\sigma = \bigcup_{\sigma' \in \Sigma', \sigma' \subseteq \sigma} \sigma'$, it follows that some σ' in this union must equal σ, i.e., $\sigma \in \Sigma'$.

Note also that part (e) is a consequence of part (d). This follows because, first, star subdivisions give projective morphisms by Proposition 11.1.6, and second, compositions of projective morphisms are projective by Proposition 7.0.5.

Hence it suffices to find a sequence of star subdivisions that lead to a simplicial fan with the same 1-dimensional cones as Σ. Our proof, based on an idea of Thompson [**270**], uses complete induction on

$$r = |\{\rho \in \Sigma(1) \mid D_\rho \text{ is not } \mathbb{Q}\text{-Cartier on } X_\Sigma\}|.$$

If $r = 0$, then every D_ρ is $\mathbb{Q}$-Cartier. Since the D_ρ generate $\mathrm{Cl}(X_\Sigma)$, every Weil divisor on X_Σ is $\mathbb{Q}$-Cartier. By Proposition 4.2.7, we conclude that Σ is simplicial.

If $r > 0$, then we can pick $\rho \in \Sigma(1)$ such that D_ρ is not $\mathbb{Q}$-Cartier. Consider the star subdivision $\Sigma^*(u_\rho)$, where u_ρ is the ray generator of ρ. Proposition 11.1.6 implies that

$$\Sigma^*(u_\rho)(1) = \Sigma(1) \cup \{\rho\} = \Sigma(1)$$

and that D_ρ is $\mathbb{Q}$-Cartier on $X_{\Sigma^*(u_\rho)}$. Note also that if $D_{\rho'}$ is $\mathbb{Q}$-Cartier on X_Σ, then it remains $\mathbb{Q}$-Cartier on $X_{\Sigma^*(u_\rho)}$ (think support functions). It follows easily that

$$\{\rho' \in \Sigma^*(u_\rho)(1) \mid D_{\rho'} \text{ is not } \mathbb{Q}\text{-Cartier on } X_{\Sigma^*(u_\rho)}\}$$

has strictly fewer than r elements. Our inductive hypothesis gives a refinement Σ' of $\Sigma^*(u_\rho)$ which is easily seen to be the desired refinement of Σ. □

In Exercise 11.1.4 you will show that the number of star subdivisions needed to create the simplicial refinement Σ' of Σ described in Proposition 11.1.7 is at most the rank of $\mathrm{Cl}(X_\Sigma)/\mathrm{Pic}(X_\Sigma)$. A different proof of Proposition 11.1.7 can be found in [**99**]. See also [**93**, Thm. V.4.2].

The Multiplicity of a Simplicial Cone. When we have a fan that is simplicial but not smooth, we need to subdivide the non-smooth cones to make them smooth. The elegant approach used in Chapter 10 does not generalize to higher dimensions. Instead, we will use the *multiplicity* of a simplicial cone from §6.4 and show how a carefully chosen star subdivision of a non-smooth cone can lower its multiplicity.

Let us recall the definition of multiplicity. Given a simplicial cone $\sigma \subset N_\mathbb{R}$ with generators $u_1, \ldots, u_d$, let $N_\sigma = \mathrm{Span}(\sigma) \cap N$ and note that $\mathbb{Z}u_1 + \cdots + \mathbb{Z}u_d \subseteq N_\sigma$ has finite index. Then the *multiplicity* or *index* of σ is the index

$$\mathrm{mult}(\sigma) = [N_\sigma : \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_d]. \tag{11.1.5}$$

The multiplicity of a simplicial cone has the following properties.

Proposition 11.1.8. *Let $\sigma \subseteq N_\mathbb{R}$ be a simplicial cone and let $u_1, \ldots, u_d$ and N_σ be as in* (11.1.5)*. Then:*

(a) *σ is smooth if and only if* $\mathrm{mult}(\sigma) = 1$.

(b) $\mathrm{mult}(\sigma)$ *is the number of points in $P_\sigma \cap N$, where*

$$P_\sigma = \left\{\textstyle\sum_{i=1}^d \lambda_i u_i : 0 \le \lambda_i < 1\right\}.$$

(c) *Let $e_1, \ldots, e_d$ be a basis of N_σ and write $u_i = \sum_{j=1}^d a_{ij} e_j$. Then*

$$\mathrm{mult}(\sigma) = |\det(a_{ij})|.$$

(d) $\mathrm{mult}(\tau) \le \mathrm{mult}(\sigma)$ *whenever $\tau \preceq \sigma$.*

Proof. The proof of part (a) is straightforward, and part (b) is equally easy since the composition

$$P_\sigma \cap N \hookrightarrow N_\sigma \longrightarrow N_\sigma/(\mathbb{Z}u_1 + \cdots + \mathbb{Z}u_d)$$

is a bijection (Exercise 11.1.5). Part (c) is the standard fact that $|\det(a_{ij})|$ is the index of $\mathbb{Z}u_1 + \cdots + \mathbb{Z}u_d$ in N_σ (see [**242**, Corollary 9.63]). Finally, if $\tau \preceq \sigma$, then $\tau(1) \subseteq \sigma(1)$. This implies $P_\tau \subseteq P_\sigma$, and then part (d) follows from part (b). $\square$

Some further properties of $\mathrm{mult}(\sigma)$ are discussed in Exercise 11.1.5.

The Resolution. We can now construct a fan that resolves the singularities of a normal toric variety X_Σ.

Theorem 11.1.9. *Every fan Σ has a refinement Σ' with the following properties:*

(a) *Σ' is smooth*

(b) *Σ' contains every smooth cone of Σ.*

(c) *Σ' is obtained from Σ by a sequence of star subdivisions.*

(d) *The toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ is a projective resolution of singularities.*

Proof. We first observe that part (d) follows from parts (a), (b) and (c). Since the refinement Σ' comes from a sequence of star subdivisions, $\phi : X_{\Sigma'} \to X_\Sigma$ is projective by the proof of Proposition 11.1.7. Furthermore, ϕ is the identity on the smooth locus of X_Σ by Proposition 11.1.2 and $X_{\Sigma'}$ is smooth Theorem 3.1.19. Hence ϕ is a resolution of singularities.

To produce a refinement that satisfies (a), (b) and (c), we begin with Proposition 11.1.7, which uses star subdivisions to construct a simplicial refinement with the same simplicial cones as Σ. This refinement does not change the smooth cones since smooth cones are simplicial.

Hence we may assume that Σ is simplicial. To measure how close Σ is to being smooth, we define

$$\mathrm{mult}(\Sigma) = \max_{\sigma \in \Sigma} \mathrm{mult}(\sigma).$$

Note that $\mathrm{mult}(\Sigma) = 1$ if and only if Σ is a smooth fan. We will show that whenever $\mathrm{mult}(\Sigma) > 1$, we can find a star subdivision $\Sigma^*(v)$ that does not change the smooth cones of Σ and satisfies either

$$\begin{aligned} &\mathrm{mult}(\Sigma^*(v)) < \mathrm{mult}(\Sigma), \text{ or} \\ &\mathrm{mult}(\Sigma^*(v)) = \mathrm{mult}(\Sigma) \text{ and } \Sigma^*(v) \text{ has fewer cones of this multiplicity}. \end{aligned} \tag{11.1.6}$$

Once this is done, the theorem will follow immediately.

Suppose $\mathrm{mult}(\Sigma) > 1$ and let $\sigma_0 \in \Sigma$ have maximal multiplicity. By part (b) of Proposition 11.1.8, there is $v \in P_{\sigma_0} \cap N \setminus \{0\}$. The star subdivision $\Sigma^*(v)$ consists of the cones of Σ not containing v together with the cones $\mathrm{Cone}(\tau, v)$ where $v \notin \tau$ and τ is a face of a cone $\sigma \in \Sigma$ containing v. We claim that

$$\mathrm{mult}(\mathrm{Cone}(\tau, v)) < \mathrm{mult}(\sigma). \tag{11.1.7}$$

Assume (11.1.7) for the moment. Part (b) of Proposition 11.1.8 shows that v lies in no smooth cone of Σ, so that $\Sigma^*(v)$ contains all smooth cones of Σ. Since all cones of $\Sigma^*(v)$ either lie in Σ or satisfy the inequality (11.1.7), it follows that when we replace Σ to $\Sigma^*(v)$, we lose at least one cone of maximum multiplicity (namely, σ_0) and create no new cones of this multiplicity. Thus $\Sigma^*(v)$ satisfies (11.1.6).

It remains to prove (11.1.7). Given $v \in P_{\sigma_0} \cap N \in \{0\}$ as above, we can write

$$v = \lambda_1 u_1 + \cdots + \lambda_d u_d, \quad 0 < \lambda_i < 1,$$

where $u_1, \dots, u_d$ are the ray generators of the minimal face τ_0 of σ_0 containing v. We also have $\mathrm{Cone}(\tau, v) \subseteq \sigma$. Since $v \in \sigma$ implies that τ_0 is a face of σ, we can write the ray generators of σ as

$$u_1, \dots, u_d, u_{d+1}, \dots, u_s, \quad s = \dim \sigma.$$

Also, $v \notin \tau$ implies that there is $i \in \{1, \dots, d\}$ such that $u_i \notin \tau$. Then

$$\begin{aligned}
\mathrm{mult}(\mathrm{Cone}(\tau, v)) &\le \mathrm{mult}(\mathrm{Cone}(u_1, \dots, \widehat{u}_i, \dots, u_d, u_{d+1}, \dots, u_s, v) \\
&= |\det(u_1, \dots, \widehat{u}_i, \dots, u_d, u_{d+1}, \dots, u_s, v)| \\
&= |\det(u_1, \dots, \widehat{u}_i, \dots, u_d, u_{d+1}, \dots, u_s, \lambda_i u_i)| \\
&= \lambda_i\, |\det(u_1, \dots, u_s)| = \lambda_i \,\mathrm{mult}(\sigma) < \mathrm{mult}(\sigma).
\end{aligned}$$

The first line follows from part (d) of Proposition 11.1.8 and the second follows from part (c). Here, $\det(w_1, \dots, w_d) = \det(b_{ij})$ where $w_i = \sum_{j=1}^d b_{ij} e_j$. The last two lines follow from $0 < \lambda_i < 1$ and standard properties of determinants. This completes the proof of (11.1.7), and the theorem follows. □

The Exceptional Locus. The resolution $\phi : X_{\Sigma'} \to X_\Sigma$ from Theorem 11.1.9 is an isomorphism above the smooth locus of X_Σ. It remains to consider what happens above the singular locus. The *exceptional locus* of ϕ is $\mathrm{Exc}(\phi) = \phi^{-1}((X_\Sigma)_{\mathrm{sing}})$.

Proposition 11.1.2 implies that the irreducible components of $(X_\Sigma)_{\mathrm{sing}}$ are given by the orbit closures

$$(X_\Sigma)_{\mathrm{sing}} = V(\sigma_1) \cup \cdots \cup V(\sigma_s)$$

where $\sigma_1, \dots, \sigma_s$ are the minimal non-smooth cones of Σ. Hence, to understand the exceptional locus $\mathrm{Exc}(\phi)$, it suffices to describe $\phi^{-1}(V(\sigma_i))$ for $1 \le i \le s$. We will use the following more general result.

Proposition 11.1.10. *Let Σ' be a refinement of Σ with induced toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$, and fix $\sigma \in \Sigma$. Then the irreducible components of $\phi^{-1}(V(\sigma))$ are*

$$\phi^{-1}(V(\sigma)) = V(\sigma'_1) \cup \cdots \cup V(\sigma'_r),$$

where $\sigma'_1, \dots, \sigma'_r$ are the minimal cones of Σ' that meet the relative interior of σ.

Proof. Since ϕ is equivariant, it maps $T_{N'}$-orbits into T_N-orbits. More precisely, we know from Lemma 3.3.21 that for $\sigma \in \Sigma$, a cone $\sigma' \in \Sigma'$ satisfies $\phi(O(\sigma')) \subseteq O(\sigma)$ if and only if σ is the minimal cone of Σ containing σ'. Furthermore, the latter happens if and only if σ' intersects the relative interior of σ. From here, the proof is an easy application of the Orbit-Cone correspondence (Exercise 11.1.6). $\square$

Example 11.1.11. Our first example of a toric resolution was in Example 10.1.9, where we considered the refinement Σ of the cone σ shown in Figure 2. Here, U_σ

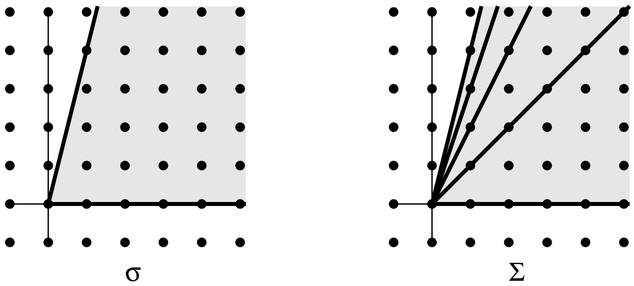


Figure 2. A cone σ with smooth refinement Σ

has a unique singular point, and the minimal cones of Σ that meet the interior of σ are the interior rays of Σ. Hence the exceptional locus of $X_\Sigma \to U_\sigma$ is a divisor. $\Diamond$

Example 11.1.12. Consider $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3) \subseteq \mathbb{R}^3$. Figure 3 on the next page shows σ and three smooth refinements $\Sigma_1, \Sigma_2, \Sigma_3$ that give

$$\begin{tikzcd} & X_{\Sigma_3} \arrow[dl] \arrow[dr] \arrow[dd] & \\ X_{\Sigma_1} \arrow[dr] & & X_{\Sigma_2} \arrow[dl] \\ & U_\sigma & \end{tikzcd}$$

For Σ_1, Σ_2, the smallest cone meeting the interior of σ is 2-dimensional, so the exceptional locus is $\mathbb{P}^1$ in each case. On the other hand, the ray τ in Figure 3 is the smallest cone of Σ_3 meeting the interior of σ. Hence the exceptional locus is $V(\tau)$, which by Example 3.2.8 is $\mathbb{P}^1 \times \mathbb{P}^1$.

The exceptional loci in the above resolutions are related by the diagram

$$\begin{tikzcd} & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl] \arrow[dr] \arrow[dd] & \\ \mathbb{P}^1 \arrow[rr, dashed] \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\ & \{\mathrm{pt}\} & \end{tikzcd} \tag{11.1.8}$$

where the nontrivial maps in (11.1.8) are the projections onto the factors of $\mathbb{P}^1 \times \mathbb{P}^1$ (Exercise 11.1.7).

Example B.6.1 shows how to find the refinement Σ_1 of σ using `Sage` [**262**]. $\Diamond$

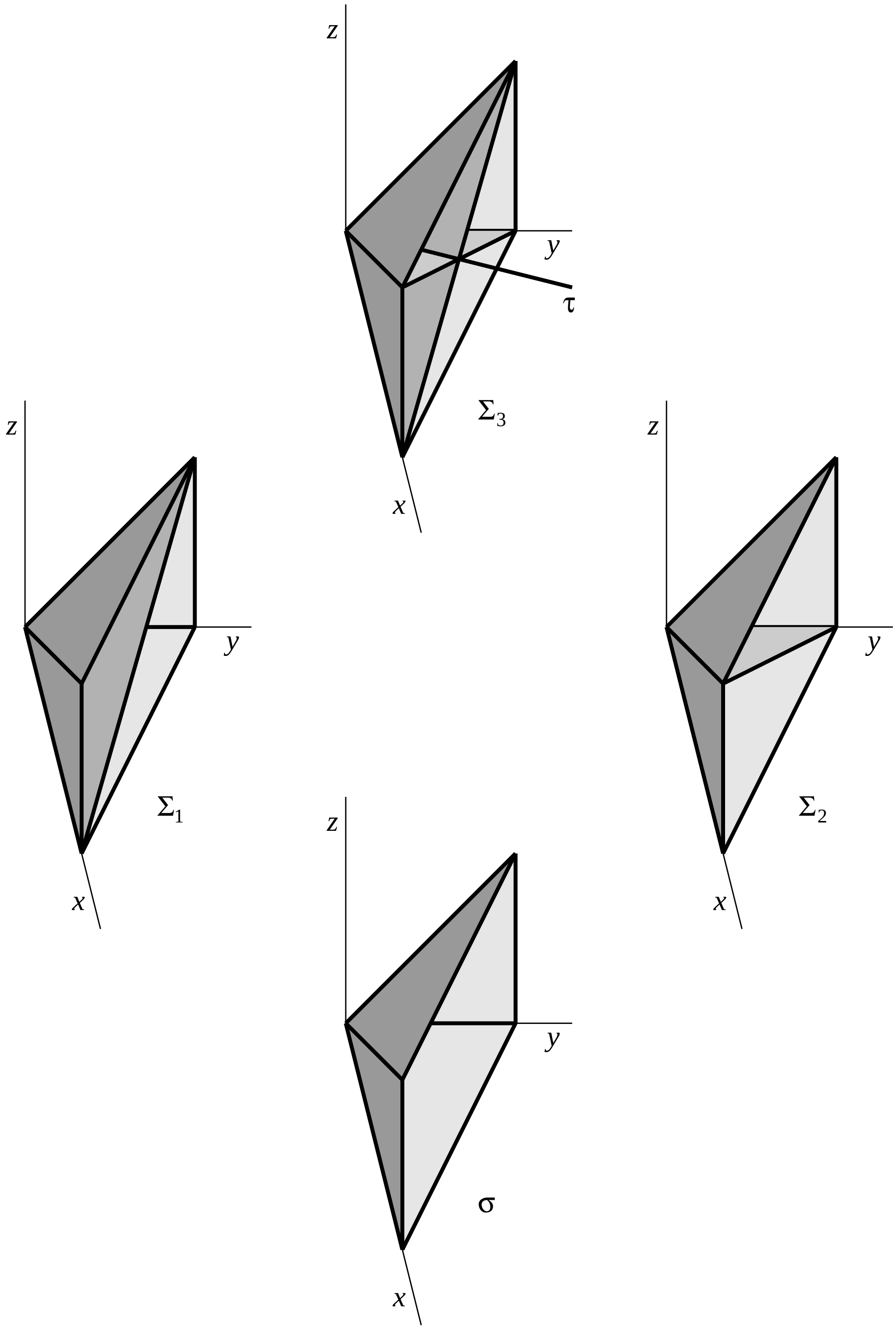


Figure 3. The cone σ with smooth refinements $\Sigma_1, \Sigma_2, \Sigma_3$

Example 11.1.12 was first described by Atiyah [**8**] in 1958. The rational map $X_1 \dashrightarrow X_2$ in (11.1.8) is an isomorphism outside of a set of codimension 2. In the minimal model program, such rational maps are called *flips* or *flops*, depending on the context. We will say more about this in Chapter 15.

In §10.4, we noted that when X_Σ is a toric surface, there is a unique minimal toric resolution of singularities $X_{\Sigma'} \to X_\Sigma$. Example 11.1.12 shows that unique minimal resolutions do not exist in dimensions ≥ 3.

Exercises for §11.1.

11.1.1. The proof of Proposition 11.1.2 constructed an isomorphism $U_{\sigma,N} \simeq U_{\sigma,N_\sigma} \times T_{N_2}$ and claimed that it induced an isomorphism of orbits

$$O_N(\sigma) \simeq O_{N_\sigma}(\sigma) \times T_{N_2}.$$

Prove this.

11.1.2. Let $\mathrm{Cone}(\tau, v)$ be cone of the star subdivision $\Sigma^*(v)$ containing v, and let γ be the face of $\mathrm{Cone}(\tau, v)$ determined by $m \in \mathrm{Cone}(\tau, v)^\vee$. Thus $\gamma = H_m \cap \mathrm{Cone}(\tau, v)$. As noted in the text, we have $m \in \sigma^\vee$ and $\langle m, v\rangle \geq 0$.

(a) If $\langle m, v\rangle = 0$, prove that $\gamma = \mathrm{Cone}(H_m \cap \tau, v) \in \Sigma^*(v)$.

(b) If $\langle m, v\rangle > 0$, in which case $\gamma = H_m \cap \tau \in \Sigma^*(v)$.

11.1.3. Let $\Sigma^*(v)$ be the star subdivision determined by v and let $\rho_v = \mathrm{Cone}(v)$. Prove that $\Sigma^*(v)(1) = \Sigma(1) \cup \{\rho_v\}$.

11.1.4. Recall that if G is a finitely generated abelian group and G_{tor} is the subgroup of elements of finite order, then $G/G_{\mathrm{tor}} \simeq \mathbb{Z}^r$, where r is called the *rank* of G. Prove that the number of star subdivisions need to create the simplicial refinement Σ' of Σ described in Proposition 11.1.7 is at most the rank of $\mathrm{Cl}(X_\Sigma)/\mathrm{Pic}(X_\Sigma)$.

11.1.5. This exercise will study the multiplicity $\mathrm{mult}(\sigma)$ of a simplicial cone σ.

(a) Prove part (a) of Proposition 11.1.8.

(b) Prove part (b) of Proposition 11.1.8.

(c) Prove that $\mathrm{mult}(\sigma)$ is the normalized volume of $P_\sigma \subseteq (N_\sigma)_\mathbb{R} = \mathrm{Span}(\sigma)$, where "normalized" means that the parallelotope determined by a basis of N_σ has volume 1. Hint: Use part (c) of Proposition 11.1.8 and remember how to compute the volume of a parallelotope (see [**263**, Theorem 15.2.1]).

(d) Let τ be a face of σ and let $u_1, \ldots, u_d$ be the minimal generators of σ. Prove that

$$\mathrm{mult}(\sigma) = \mathrm{mult}(\tau)[N_\sigma : N_\tau + \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_d]$$

and use this to give another proof of part (d) of Proposition 11.1.8.

11.1.6. Complete the proof of Proposition 11.1.10.

11.1.7. Verify the claims made in Example 11.1.12.

11.1.8. Let Σ be the fan in $\mathbb{R}^{n+1}$ consisting of $\mathrm{Cone}(e_0, \ldots, e_n) \subseteq \mathbb{R}^{n+1}$ and its faces. Let $v = \sum_{i=0}^n q_i e_i$, where the q_i are positive integers satisfying $\gcd(q_0, \ldots, q_n) = 1$. The star subdivision $\Sigma^*(v)$ gives a toric morphism $\phi : X_{\Sigma^*(v)} \to X_\Sigma = \mathbb{C}^{n+1}$. Prove that $\phi^{-1}(0) \simeq \mathbb{P}(q_0, \ldots, q_n)$.

11.1.9. Consider the complete fan Σ in $\mathbb{R}^3$ with six 3-dimensional cones and minimal generators $(\pm 1, \pm 1, \pm 1)$. Thus Σ consists of the cones over the faces of a cube centered at the origin in $\mathbb{R}^3$.

(a) Show that the simplicialization process described in Theorem 11.1.7 requires exactly two star subdivisions to produce a simplicial refinement Σ' of Σ.

(b) Show that Σ' is smooth, so that $X_{\Sigma'} \to X_\Sigma$ is a projective resolution of singularities.

(c) Find a smooth refinement Σ'' of Σ such that $\Sigma''(1) = \Sigma(1)$ but $X_{\Sigma''}$ is not projective. Conclude that $X_{\Sigma''} \to X_\Sigma$ is not a projective resolution of singularities.

11.1.10. The simplicial fan constructed in Proposition 11.1.7 is efficient (no new rays) but noncanonical (no canonical choice for the star subdivisions used in construction). Here you will explore a different method for simplicialization that is canonical but introduces many new rays. The *barycenter* v_σ of a nonzero cone $\sigma \subseteq N_\mathbb{R}$ is the minimal generator of $\mathrm{Cone}(\sum_{\rho\in\sigma(1)} u_\rho) \cap N$. Now list the nonzero cones of the fan Σ as $\sigma_1, \dots, \sigma_r$, where $\dim \sigma_1 \le \cdots \le \dim \sigma_r$. Then the *barycentric subdivision* of Σ is the fan

$$\beta(\Sigma) = \big(\cdots\big(\Sigma^*(v_{\sigma_r})\big)^*(v_{\sigma_{r-1}})\cdots\big)^*(v_{\sigma_1}).$$

Thus $\beta(\Sigma)$ is obtained from Σ by a sequence of star subdivisions where we use the barycenters of the cones, starting with the biggest cones and working down.

(a) Consider the fan Σ in $\mathbb{R}^3$ from Exercise 11.1.9. Draw two pictures to illustrate the construction of $\beta(\Sigma)$. The first picture, drawn on the surface of the cube, should illustrate the intermediate fan obtained by taking star subdivisions at the barycenters of the 3-dimensional cones. The second picture should show how this gets further subdivided to obtain $\beta(\Sigma)$.

(b) When the cones are listed by increasing dimension, cones of the same dimension can appear in any order. Hence the list is not unique. However, they all give the same barycentric subdivision $\beta(\Sigma)$. Prove this.

(c) Prove that $\beta(\Sigma)$ is a simplicial fan.

Part (a) of this exercise was inspired by [**93**, p. 74].

§11.2. Other Types of Resolutions

We next consider *simple normal crossing*, *crepant*, *log*, and *embedded* resolutions, which are important refinements of what it means to be a resolution of singularities.

SNC Resolutions. Example 11.1.12 illustrates that the exceptional locus of a resolution of singularities need not always be a divisor. Yet in many situations in algebraic geometry, one wants a divisor, and in fact, one often wants a simple normal crossing divisor, as mentioned in §8.1. Recall that a divisor $D = \sum_i D_i$ on a smooth variety X has *simple normal crossings* (*SNC* for short) if every D_i is smooth and irreducible, and for all $p \in X$, the divisors containing p meet as follows: if $I_p = \{i \mid p \in D_i\}$, then the tangent spaces $T_p(D_i) \subseteq T_p(X)$ meet transversely, i.e.,

$$\mathrm{codim}\Big(\bigcap_{i\in I_p} T_p(D_i)\Big) = |I_p|.$$

For example, given a smooth toric variety and a nonempty subset $A \subseteq \Sigma(1)$, the divisor $D = \sum_{\rho \in A} D_\rho$ is SNC.

Definition 11.2.1. A resolution of singularities $f : X' \to X$ is called ***SNC*** if the exceptional locus $\mathrm{Exc}(f)$ is a divisor with simple normal crossings.

Theorem 11.2.2. *Every fan Σ has a refinement Σ' as in Theorem 11.1.9 such that the toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ is a projective SNC resolution of singularities.*

Proof. By Theorem 11.1.9, Σ has a smooth refinement Σ_0 that gives a resolution of singularities. Let $V(\sigma_i)$ be an irreducible component of $(X_\Sigma)_{\mathrm{sing}}$ and let $\sigma \in \Sigma_0$ be a minimal cone that meets the relative interior of σ_i. If σ is not a ray, then consider the star subdivision $\Sigma_0^*(v_\sigma)$, where $v_\sigma = \sum_{\rho \in \sigma(1)} u_\rho$. Since Σ_0 is smooth, this is the fan denoted $\Sigma_0^*(\sigma)$ in Definition 3.3.17. In the discussion that followed the definition in §3.3, we saw that corresponding toric morphism is the blowup $\mathrm{Bl}_{V(\sigma)}(X_{\Sigma_0}) \to X_{\Sigma_0}$. Since v_σ lies in the relative interior of σ, the inverse image of $V(\sigma)$ is the divisor corresponding to the ray of $\Sigma_0^*(v_\sigma)$ generated by v_σ.

The composed map $\mathrm{Bl}_{V(\sigma)}(X_{\Sigma_0}) \to X_\Sigma$ is a resolution of singularities whose exceptional locus has one less non-divisorial component. Repeating this finitely many times gives a projective resolution (star subdivisions give projective morphisms) whose exceptional locus is a divisor, automatically SNC by the remark following Definition 11.2.1. $\square$

Example 11.2.3. The resolution $X_{\Sigma_1} \to U_\sigma$ constructed in Example 11.1.12 is not SNC. Applying the process described in the proof of Theorem 11.2.2 gives the resolution $X_{\Sigma_3} \to U_\sigma$, which is SNC. $\Diamond$

Log Resolutions. There are several types of log resolutions. Here we discuss the one for $\mathbb{Q}$-divisors.

Definition 11.2.4. Let $D = \sum_i d_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety X. Then a projective resolution of singularities $f : X' \to X$ is a ***log resolution of*** D if $\bigcup_i f^{-1}(D_i) \cup \mathrm{Exc}(f)$ is a SNC divisor, where as usual $\mathrm{Exc}(f)$ is the exceptional locus of f.

For a toric variety X_Σ, the natural choice for D is a torus-invariant $\mathbb{Q}$-divisor $D = \sum_\rho d_\rho D_\rho$, $a_\rho \in \mathbb{Q}$. In this case, a projective SNC resolution $X_{\Sigma'} \to X_\Sigma$ is automatically a log resolution of (X, D) since any collection of torus-invariant prime divisors on $X_{\Sigma'}$ is SNC.

In §11.3 we will discuss a different type of log resolution, where the given data consists of a normal variety X and a sheaf of ideals $\mathfrak{a} \subseteq \mathscr{O}_X$. See [**186**, 9.1.B] for more on log resolutions.

Crepant Resolutions. Given the importance of the canonical sheaf ω_X, it is natural to ask how ω_X changes in a resolution of singularities $f : X' \to X$. The nicest case is when X is normal and $\mathbb{Q}$-Gorenstein, i.e., $\omega_X = \mathscr{O}_X(K_X)$ for a $\mathbb{Q}$-Cartier canonical divisor K_X. (Note that if one canonical divisor is $\mathbb{Q}$-Cartier, then they all are.)

Suppose that ℓK_X is Cartier, where $\ell > 0$ is an integer. It follows that $f^*(\ell K_X)$ is a Cartier divisior, and then f^*K_X is defined to be the $\mathbb{Q}$-divisor

$$f^*K_X = \tfrac{1}{\ell}\, f^*(\ell K_X).$$

Example 11.2.7 below will show that f^*K_X need not be integral.

Definition 11.2.5. Let X be a normal $\mathbb{Q}$-Gorenstein variety. Then a resolution of singularities $f : X' \to X$ is ***crepant*** if $f^*(K_X)$ is integral and $K_{X'} \sim f^*(K_X)$.

Since $K_{X'}$ is only unique up to linear equivalence, the definition of crepant is often stated as $K_{X'} = f^*K_X$. The term "crepant" is due to Reid [**235**] and signals that there is no discrepancy between $K_{X'}$ and f^*K_X.

Example 11.2.6. The toric variety U_σ of $\sigma = \mathrm{Cone}(e_1,e_2,e_1+e_3,e_2+e_3) \subseteq \mathbb{R}^3$ is Gorenstein since the minimal generators of σ lie in the plane $\langle m,u\rangle = -1$, for $m = -e_1-e_2$. Thus the canonical divisor is $K_{U_\sigma} = \mathrm{div}(\chi^m)$.

For the fan Σ_1 from Figure 3 of Example 11.1.12, the resolution $X_{\Sigma_1} \to U_\sigma$ is crepant since

$$\phi^*(K_{U_\sigma}) = \phi^*(\mathrm{div}_{U_\sigma}(\chi^m)) = \mathrm{div}_{X_\Sigma}(\chi^m) = K_{X_\Sigma}.$$

The subscripts indicate where the divisor is being computed. You will verify this computation in Exercise 11.2.1, where you will also show that the SNC resolution $X_{\Sigma_3} \to U_\sigma$ from Example 11.1.12 is not crepant. ◊

Example 11.2.7. Consider $\sigma = \mathrm{Cone}(e_1,e_2,e_3) \subseteq N_\mathbb{R} = \mathbb{R}^3$ relative to the lattice

$$N = \{(a,b,c) \in \mathbb{Z}^3 \mid a+b+c \equiv 0 \bmod 2\}.$$

The minimal generators are $u_i = 2e_i$, $1 \le i \le 3$, and the class group of U_σ is $\mathbb{Z}/2\mathbb{Z}$, where the torus-invariant divisors D_i all map to the generator. It follows that the canonical divisor $K_{U_\sigma} = -D_1-D_2-D_3$ is not Cartier, but $2K_{U_\sigma} = \mathrm{div}(\chi^m)$ for $m = -e_1-e_2-e_3 \in M_\mathbb{R}$ (Exercise 11.2.2).

The star subdivision Σ with respect to $u_0 = e_1+e_2+e_3 \in N_\mathbb{R}$ gives a resolution of singularities $\phi : X_\Sigma \to U_\sigma$. The four minimal generators of Σ give divisors denoted D_0,D_1,D_2,D_3 by abuse of notation. Then

$$\begin{aligned}\phi^*(2K_{U_\sigma}) = \phi^*(\mathrm{div}_{U_{\sigma,N}}(\chi^m)) &= \mathrm{div}_{X_\Sigma}(\chi^m)\\ &= -3D_0-2D_1-2D_2-2D_3 = 2K_{X_\Sigma} - D_0,\end{aligned}$$

so that $K_{X_\Sigma} = \phi^*(K_{U_\sigma}) + \frac{1}{2}D_0$ (Exercise 11.2.2). Thus ϕ is not crepant. Note also that $\phi^*(K_{U_\sigma})$ is not an integral divisor even though K_{U_σ} is. ◊

For a toric surface, we have the minimal resolution $X_{\Sigma'} \to X_\Sigma$ constructed in §10.2. It is easy to characterize when this resolution is crepant.

Proposition 11.2.8. *Let X_Σ be a toric surface and consider the minimal resolution of singularities $\phi : X_{\Sigma'} \to X_\Sigma$ from §10.2. Then the following are equivalent:*

(a) *ϕ is crepant.*

(b) *X_Σ is Gorenstein.*

(c) *The singularties of X_Σ are at worst rational double points.*

Proof. The equivalence (b) $\Leftrightarrow$ (c) follows from Proposition 10.1.6. For (a) $\Leftrightarrow$ (c), one can reduce to the case of an affine surface U_σ with ray generators u_1, u_2. Here, the key observation is that (a) is equivalent to saying that the minimal smooth refinement of σ has ray generators lying on the line segment connecting u_1 and u_2,

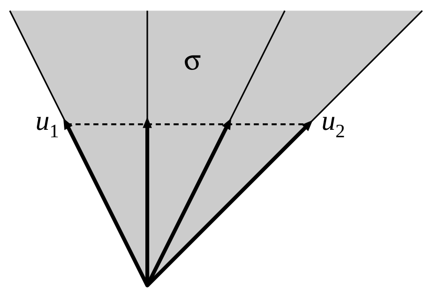


Figure 4. Cone σ with refinement ray generators on a line segement

as illustrated in Figure 4. Using the methods of §10.2, one sees that the rational double points are the only toric surface singularities where is happens. We leave the details as Exercise 11.2.3. $\square$

Here is an example that we have already seen.

Example 11.2.9. Let P be a reflexive polygon in $\mathbb{R}^2$. In Theorem 10.5.10, we proved that the minimal resolution $X_\Sigma \to X_P$ comes from the fan given by the lattice points on the boundary of P°. Figure 5 makes it clear that $X_\Sigma \to X_P$ is crepant, a fact we used in the proof of Theorem 10.5.10. $\Diamond$

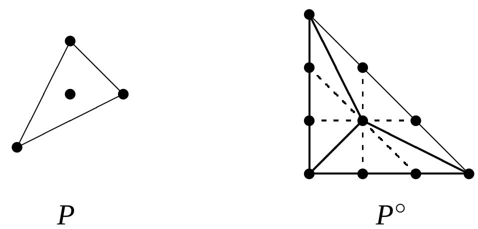


Figure 5. A reflexive polygon P and its dual P°

More generally, crepant resolutions of toric varieties coming from reflexive polytopes are important in mirror symmetry because they give rise to Calabi-Yau varieties. A smooth projective variety Y is *Calabi-Yau* if ω_Y is trivial and

$$H^1(Y,\mathscr{O}_Y) = \cdots = H^{d-1}(Y,\mathscr{O}_Y) = 0,\ d = \dim Y. \tag{11.2.1}$$

Now fix a reflexive polyope $P \subseteq M_{\mathbb{R}}$ of dimension ≥ 2. We know from §8.3 that X_P is a Gorenstein Fano toric variety, so that $-K_{X_P}$ is Cartier and ample. Assume also that $\phi : X_\Sigma \to X_P$ is a crepant resolution. Then $K_{X_\Sigma} = \phi^*(K_{X_P})$ is basepoint free. By the Bertini theorem [**131**, Cor. III.10.9 and Ex. III.11.3], a generic section of $\mathscr{O}_{X_\Sigma}(-K_{X_\Sigma})$ gives a smooth connected hypersurface $Y \subseteq X_\Sigma$ such that $Y \sim -K_{X_\Sigma}$. We call Y an *anticanonical hypersurface*.

Proposition 11.2.10. *With the above hypotheses, Y is Calabi-Yau.*

Proof. Since $\mathscr{I}_Y = \mathscr{O}_{X_\Sigma}(-Y)$ by Proposition 4.0.28, we have $\mathscr{I}_Y \simeq \mathscr{O}_{X_\Sigma}(K_{X_\Sigma}) \simeq \Omega^n_{X_\Sigma}$, $n = \dim X_\Sigma$. Then the desired vanishing (11.2.1) follows from the long exact cohomology sequence associated to

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathscr{O}_{X_\Sigma} \longrightarrow \mathscr{O}_Y \longrightarrow 0$$

plus the vanishing of $H^p(X_\Sigma,\Omega^n_{X_\Sigma})$ for $p < n$ from Theorem 9.3.2.

To compute the canonical bundle of Y, we use the adjunction formula from Example 8.2.2:

$$\omega_Y \simeq \omega_{X_\Sigma} \otimes_{\mathscr{O}_{X_\Sigma}} (\mathscr{I}_Y/\mathscr{I}_Y^2)^\vee \simeq \omega_{X_\Sigma}(Y) \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{O}_Y,$$

where we use $\mathscr{I}_Y/\mathscr{I}_Y^2 \simeq \mathscr{I}_Y \otimes_{\mathscr{O}_{X_\Sigma}} (\mathscr{O}_{X_\Sigma}/\mathscr{I}_Y) = \mathscr{O}_{X_\Sigma}(-Y) \otimes_{\mathscr{O}_{X_\Sigma}} \mathscr{O}_Y$. Since

$$\omega_{X_\Sigma} = \mathscr{O}_{X_\Sigma}(K_{X_\Sigma}) \simeq \mathscr{O}_{X_\Sigma}(-Y),$$

one sees immediately that $\omega_Y = \Omega^{n-1}_Y$ is trivial. □

Example 11.2.11. Continuing with Example 11.2.9, let P be a reflexive polygon with minimal resolution $X_\Sigma \to X_P$. In this case, a generic section of $\mathscr{O}_{X_\Sigma}(-K_{X_\Sigma})$ is a smooth connected curve $C \subseteq X_\Sigma$. We have two ways to compute its genus g:

(a) Recall from §10.5 that g is the sectional genus of X_P, which equals $|\mathrm{Int}(P)\cap M|$ by Proposition 10.5.8. Since P is reflexive, the origin is its unique interior point. Hence $g = 1$.

(b) The canonical bundle of C is trivial by Proposition 11.2.10. The canonical divisor has degree 0, so $0 = 2g-2$ by (10.5.3), hence $g = 1$.

Thus C is an elliptic curve, which is a Calabi-Yau variety of dimension 1. ◊

Unfortunately, for a reflexive polytope of dimension ≥ 3, X_P need not have a crepant resolution. Here, the best one can do is the *maximal projective crepant partial desingularization* described by Batyrev in [**16**]. The resulting anticanonical hypersurfaces are singular Calabi-Yau varieties (see [**68**, Def. 1.4.1]).

We will study toric singularities in more detail in §11.4.

Embedded Resolutions. Many treatments of the resolution of singularities use *embedded resolutions*. For an embedded resolution, we have an irreducible variety X contained in a smooth variety W, which we think of as the "ambient space." The idea is to repeatedly blow up the ambient space along smooth subvarieties (thereby changing the ambient space) until the proper transform of X becomes smooth.

Here is a more careful description. An irreducible smooth subvariety $Z \subseteq W$ gives the blowup $f : \mathrm{Bl}_Z(W) \to W$, where $\mathrm{Bl}_Z(W)$ is smooth and the *exceptional locus* $\mathrm{Exc}(f)$ (the subset of $\mathrm{Bl}_Z(W)$ where f is not an isomorphism) is the divisor given by the projective bundle of the normal bundle of $Z \subseteq W$. We call Z the *center* of the blowup. Then $X \subset W$ has a *proper transform* in $\mathrm{Bl}_Z(W)$ defined as the Zariski closure

$$\overline{f^{-1}(X \setminus Z)} \subseteq \mathrm{Bl}_Z(W).$$

Then one seeks a series of smooth centers $Z_i \subseteq W_i$, $0 \le i \le \ell$, such that $W_0 = W$, $W_{i+1} = \mathrm{Bl}_{Z_i}(W_i)$, and Z_{i+1} is transverse to the exceptional locus of $W_{i+1} \to W$. Furthermore, the final map $f_\ell : W_\ell \to W$ needs to have the following properties:

- The induced map $f|_{X_\ell} : X_\ell \to X$ is a resolution of singularities, where X_ℓ is the proper transform of X in W_ℓ.
- X_ℓ is transverse to the exceptional locus of f_ℓ.

The last bullet implies that $f|_{X_\ell} : X_\ell \to X$ is a SNC resolution of singularities, and one can show that $f|_{X_\ell}$ is a projective morphism. A careful explanation of what this means can be found in [**138**], along with a discussion of further properties one can impose on an embedded resolution.

In the toric version of this, we have an equivariant embedding $X \hookrightarrow W$ of toric varieties, with W smooth, and at each stage of the above process, W_i is toric, and the center $Z_i \subseteq W_i$ is an orbit closure. Then Z_i is smooth, $W_{i+1} = \mathrm{Bl}_{Z_i}(W_i)$ is toric, and $W_{i+1} \to W_i$ is a toric morphism. When such an embedded resolution exists, the result is a toric SNC resolution of singularities $X_\ell \to X$ achieved entirely through blowups of torus-invariant centers in smooth toric ambient spaces. The papers [**25**], [**32**] and [**117**] prove the existence of embedded toric resolution, and more importantly, they give *algorithms* for finding the centers $Z_0, Z_1, \ldots$ needed at each stage of the resolution process. See also [**268**]. In contrast, the resolution constructed in Theorem 11.1.9 depended on many choices and is far from unique.

The proof of toric embedded resolution, though much simpler than the general case, is still not easy. Hence we will confine ourselves to some examples, together with some remarks about nonnormal toric varieties.

Example 11.2.12. The toric variety $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ is the affine toric variety U_σ of the cone σ from Example 11.1.12. We can resolve the singularity of U_σ by blowing up the origin in $\mathbb{C}^4$. The proper transform of $\mathbf{V}(xy - zw)$ in $\mathrm{Bl}_0(\mathbb{C}^4)$ is the toric variety X_{Σ_3} from Example 11.1.12. There are two ways to see this.

First, the isomorphism $U_\sigma \simeq \mathbf{V}(xy - zw)$ induces the isomorphism $(\mathbb{C}^*)^3 \simeq \mathbf{V}(xy - zw) \cap (\mathbb{C}^*)^4$ given by $(t_1,t_2,t_3) \mapsto (t_1,t_2,t_3,t_1t_2t_3^{-1})$ (see Example 1.1.5). At the level of one-parameter subgroups, this gives the embedding

$$\overline{\phi} : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^4, \quad (a,b,c) \longmapsto (a,b,c,a+b-c).$$

Since $\sigma = \overline{\phi}_{\mathbb{R}}^{-1}(\mathbb{R}^4_{\geq 0})$ and $\mathbb{R}^4_{\geq 0}$ is the cone for $\mathbb{C}^4$, the corresponding toric morphism is the embedding $\phi : U_\sigma \to \mathbb{C}^4$ with image $\mathbf{V}(xy - zw)$.

Let Σ be the star subdivision of $\mathbb{R}^4_{\geq 0}$ at $(1,1,1,1)$. This gives the toric variety $X_\Sigma = \mathrm{Bl}_0(\mathbb{C}^4)$. Via $\overline{\phi}_{\mathbb{R}}$, Σ induces the refinement Σ_3 of σ described in Example 11.1.12 (Exercise 11.2.4). This gives a toric morphism $\phi : X_{\Sigma_3} \to \mathrm{Bl}_0(\mathbb{C}^4)$ whose image is the proper transform (we will say more about this below).

Second, the proper transform can studied by the method of Example 5.2.11. The blowup has quotient representation

$$\mathrm{Bl}_0(\mathbb{C}^4) = (\mathbb{C}^5 \setminus \mathbb{C} \times \{(0,0,0,0)\})/\mathbb{C}^*,$$

where $\lambda \cdot (t,x,y,z,w) = (\lambda^{-1}t, \lambda x, \lambda y, \lambda z, \lambda w)$, and the blowup map $\mathrm{Bl}_0(\mathbb{C}^4) \to \mathbb{C}^4$ is given by

$$(t,x,y,z,w) \longmapsto (tx,ty,tz,tw).$$

Via this map, $0 = xy - zw$ becomes $0 = (tx)(ty) - (tz)(tw) = t^2(xy - zw)$, so that the proper transform is defined by $xy - zw = 0$ in $\mathrm{Bl}_0(\mathbb{C}^4)$. There are two cases:

- If $t \neq 0$, then $t \cdot (t,x,y,z,w) = (1,tx,ty,tz,tw)$, so that this part of the proper transform is isomorphic to $\mathbf{V}(xy - zw) \setminus \{0\} \subseteq \mathbb{C}^4$.
- If $t = 0$, then this part of the proper transform is the fiber over $0 \in \mathbf{V}(xy - zw)$ and is the subvariety of $\mathbb{P}^3$ defined by $xy = zw$. This quadric surface is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$.

One can also show that the proper transform is smooth by checking locally on the affine pieces of $\mathrm{Bl}_0(\mathbb{C}^4)$, along the lines of Example 5.2.11 (Exercise 11.2.4). $\Diamond$

The general situation has an interesting wrinkle. Given a smooth toric variety X_{Σ_0} with torus T_{N_0}, any subtorus $T \subseteq T_{N_0}$ gives the toric variety

$$X = \overline{T} \subseteq X_{\Sigma_0} \tag{11.2.2}$$

with torus T. However, X *need not be normal*, as shown by the simple example where $T = \{(t^2,t^3) \mid t \in \mathbb{C}^*\} \subseteq (\mathbb{C}^*)^2$ and $X = \overline{T} = \mathbf{V}(x^3 - y^2) \subseteq \mathbb{C}^2$.

For this reason, the embedded resolution of toric varieties described in [**25**], [**32**] and [**117**] works with an equivariant embedding $X \subseteq W$ of toric varieties where W is smooth but X is not assumed to be normal. These papers use cones and fans for the ambient space W and its blowups, but use binomial equations for X and its proper transforms, just as we did in the second half of Example 11.2.12.

Example 11.2.13. In Example 5.2.11 we showed that the proper transform of the nonnormal toric variety $\mathbf{V}(x^3-y^2) \subseteq \mathbb{C}^2$ is smooth in $\mathrm{Bl}_0(\mathbb{C}^2)$. This not an embedded resolution since the proper transform is not transverse to the exceptional locus. Two more blowups are needed to get an embedded resolution (Exercise 11.2.5). $\Diamond$

While the toric variety X from (11.2.2) need not be normal, its normalization is easy to construct. Suppose in general that X_{Σ_0} is a normal toric variety, where Σ_0 is a fan in $(N_0)_{\mathbb{R}}$. Then subtori of T_{N_0} correspond bijectively to sublattices of N_0 with torsion-free quotient (Exercise 11.2.6). Let T_N be the subtorus coming from $N \subseteq N_0$ and let $X = \overline{T_N} \subseteq X_{\Sigma_0}$ be its Zariski closure. Using the subspace $N_{\mathbb{R}} \subseteq (N_0)_{\mathbb{R}}$, the fan of the normalization of X is described as follows.

Proposition 11.2.14. *Let X_{Σ_0} be the toric variety of a fan Σ_0 in $(N_0)_{\mathbb{R}}$. Fix a subtorus $T_N \subseteq T_{N_0}$ and let $X = \overline{T_N} \subseteq X_{\Sigma_0}$. Then X is a (possibly nonnormal) toric variety with torus T_N whose normalization is the toric variety X_Σ of the fan*

$$\Sigma = \{\sigma \cap N_{\mathbb{R}} \mid \sigma \in \Sigma_0\}$$

in $N_{\mathbb{R}}$. In particular, if X is normal, then $X \simeq X_\Sigma$.

Proof. The proof is similar to the proof of Theorem 3.A.5. $\square$

Example 11.2.15. As in Example 11.2.12, consider $U_\sigma \simeq \mathbf{V}(xy-zw) \subseteq \mathbb{C}^4$, and let X be the proper transform of $\mathbf{V}(xy-zw)$ in $\mathrm{Bl}_0(\mathbb{C}^4)$. Then X is the Zariski closure of its torus in $\mathrm{Bl}_0(\mathbb{C}^4)$, and we also know that X is smooth by the second half of Example 11.2.12. Using Proposition 11.2.14, it is now easy to show that X is the toric variety X_{Σ_3}, as claimed in Example 11.2.12 (Exercise 11.2.4). $\Diamond$

Exercises for §11.2.

11.2.1. In Exercise 11.2.6,

(a) Prove that $\phi^*(\mathrm{div}_{U_\sigma}(\chi^m)) = \mathrm{div}_{X_\Sigma}(\chi^m)$ for any $m \in M$. Hint: Use Proposition 6.2.7.

(b) Show that the resolution $X_{\Sigma_3} \to U_\sigma$ from Example 11.1.12 is not crepant.

11.2.2. Fill in the details omitted in Example 11.2.7.

11.2.3. Complete the proof of Proposition 11.2.8. Hint: If σ has parameters d,k, then the results of §10.2 give a smooth refinement Σ of σ with minimal generators $u_0,\ldots,u_{r+1}$ such that $u_0 = e_2$, $u_1 = e_1$, $u_{r+1} = de_1 - ke_2$. When are u_0,u_1,u_{r+1} collinear?

11.2.4. This exercise concerns Example 11.2.12.

(a) In the first half of the example, prove the claim that Σ_3 is the refinement of σ induced by the fan for $\mathrm{Bl}_0(\mathbb{C}^4)$ via the map $\overline{\phi}_{\mathbb{R}}$.

(b) Fill in the details omitted in the second half of the example.

(c) Use parts (a) and (b) together Proposition 11.2.14 to show that X_{Σ_3} is isomorphic to the proper transform of $\mathbf{V}(xy-zw)$ in $\mathrm{Bl}_0(\mathbb{C}^4)$.

11.2.5. Explain why the blowup $\mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2$ does not give an embedded resolution of singularities of $\mathbf{V}(x^3-y^2) \subseteq \mathbb{C}^2$. Then construct a blowup of $\mathrm{Bl}_0(\mathbb{C}^2)$ that does.

11.2.6. Fix a lattice N_0. Prove that there is a bijection between sublattices of N_0 with torision-free quotient and subtori of T_{N_0}.

11.2.7. Consider the complete fan Σ in $\mathbb{R}^3$ consisting of the cones over the faces of the cube with vertices $(\pm 1, \pm 1, \pm 1)$. In Exercise 11.1.9 you constructed a refinement Σ' giving a projective resolution of singularities.

(a) Explain why $X_{\Sigma'} \to X_\Sigma$ is not SNC.

(b) Find a nice refinement of Σ' that gives a projective SNC resolution of singularities.

11.2.8. The goal of this exercise and the next is to construct an embedded resolution of the rational double point singularity A_k from Example 10.1.5. We will change notation slightly and consider $\mathbf{V}(xz-y^{k+1}) \subseteq \mathbb{C}^3$, $k \geq 1$. The blowup $\mathrm{Bl}_0(\mathbb{C}^3)$ has homogeneous coordinates t,x,y,z in the sense of Chapter 5 such that $\mathrm{Bl}_0(\mathbb{C}^3) \to \mathbb{C}^3$ is given by $(t,x,y,z) \mapsto (tx,ty,tz)$.

(a) Show that the proper transform of $\mathbf{V}(xz-y^{k+1})$ is defined by $xz = t^{k-1}z^{k+1}$.

(b) $\mathrm{Bl}_0(\mathbb{C}^3)$ is covered by three affine open subsets isomorphic to $\mathbb{C}^3$. By the discussion of local coordinates in §5.2, these affine charts are given by setting one of x,y,z equal to 1. Show that the proper transform is smooth in the $x=1$ and $z=1$ charts and is defined by $xz = t^{k-1}$ in the $y=1$ chart. Conclude that the proper transform is normal.

(c) Show that the embedded resolution of $\mathbf{V}(xz-y^{k+1}) \subseteq \mathbb{C}^3$ can be done using $\lceil k/2 \rceil$ blowups at smooth points of the ambient space. Hint: Use part (b) and recursion.

11.2.9. We continue the study of $\mathbf{V}(xz-y^{k+1}) \subseteq \mathbb{C}^3$ begun in Exercise 11.2.8, this time focusing on the fan. First observe that $\mathbf{V}(xz-y^{k+1})$ is the Zariski closure of the subtorus

$$\{(t_1,t_2,t_1^{-1}t_2^{k+1}) \mid t_1,t_2 \in \mathbb{C}^*\} \subseteq (\mathbb{C}^*)^3$$

corresponding to the sublattice

$$N = \mathbb{Z}(1,0,-1) + \mathbb{Z}(0,1,k+1) \subseteq \mathbb{Z}^3.$$

Since $\mathbf{V}(xz-y^{k+1})$ is normal (Example 10.1.5), Proposition 11.2.14 implies that it is the affine toric variety of the cone $\sigma = N_\mathbb{R} \cap \mathbb{R}^3_{\geq 0}$ since $\mathbb{R}^3_{\geq 0}$ is the cone for $\mathbb{C}^3$.

(a) Show that $\sigma = \mathrm{Cone}((k+1,1,0),(0,1,k+1))$.

(b) $\mathrm{Bl}_0(\mathbb{C}^3)$ is the toric variety of the star subdivision of $\mathbb{R}^3_{\geq 0}$ at $\nu = (1,1,1)$. This induces a refinement Σ of σ, shown for $k=2$ in Figure 6 on the next page. Prove that for $k \geq 1$, Σ is the fan of the proper transform. Hint: Part (b) of the previous exercise.

(c) When $k=1$, show that Σ is smooth with exactly two 2-dimensional cones. Draw a picture similar to Figure 6 and explain why it is simpler.

(d) When $k \geq 2$, show that the fan Σ has three 2-dimensional cones, where the outer two cones are smooth and the middle cone gives A_{k-1}.

The methods of §10.2 resolve U_σ by subdividing σ into $k+1$ smooth cones (see Exercise 10.2.2). When $k \geq 2$, the fan Σ constructed in part (b) has the outermost cones of the smooth refinement plus a middle cone that combines the remaining cones of the smooth refinement. When $k \geq 3$, blowing up $\mathrm{Bl}_0(\mathbb{C}^3)$ refines Σ by adding two more cones from the smooth refinement, and each successive blowup adds in two more cones from the smooth refinement until the process is complete.

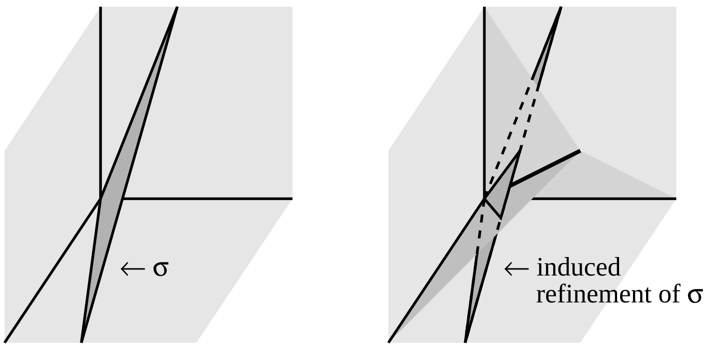


Figure 6. σ for $k = 2$ and refinement induced by blowing up $0 \in \mathbb{C}^3$ in Exercise 11.2.9

§11.3. Rees Algebras and Multiplier Ideals

So far, we have constructed resolution of singularities using repeated blowups. By using the blowup of a sheaf of ideals, it is often possible to construct the resolution in a single blowup. From an abstract point of view, this is a standard fact. Given any birational projective morphism $f : X \to Y$ of irreducible varieties with Y quasiprojective, there is an ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_Y$ such that $f : X \to Y$ is the blowup map $\pi : \mathrm{Bl}_{\mathscr{I}}(Y) \to Y$. This is proved in [**131**, Thm. II.7.17]. In the toric case, we will see that the blowup has an especially nice interpretation.

Blowups and Polyhedra. We begin with a full dimensional lattice polyhedron P in $M_{\mathbb{R}}$. Thus $P = Q + C$, where Q is a lattice polytope and C is strongly convex. In order to get a blowup, we assume in addition that C is full dimensional, so that $\sigma = C^\vee \subseteq N_{\mathbb{R}}$ is also full dimensional and strongly convex. Then $P = Q + \sigma^\vee$ and the normal fan Σ_P is a refinement of σ. Hence we have a birational toric morphism

$$\phi : X_P \longrightarrow U_\sigma,$$

which is projective by Theorem 7.1.10. Our goal is to represent this morphism as the blowup of an ideal $\mathfrak{a}$ in the coordinate ring $R = \mathbb{C}[\sigma^\vee \cap M]$ of U_σ.

Translating P by an element of M has no effect on X_P. For $m \in \mathrm{Int}(\sigma^\vee) \cap M$, one sees easily that $P + \lambda m \subseteq \sigma^\vee$ for integers $\lambda \gg 0$. Hence we may assume that $P \subseteq \sigma^\vee$. This gives the ideal

$$\mathfrak{a} = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m \subseteq R = \mathbb{C}[\sigma^\vee \cap M] = \bigoplus_{m \in \sigma^\vee \cap M} \mathbb{C} \cdot \chi^m, \tag{11.3.1}$$

and from $\mathfrak{a} \subseteq R$, we get the *Rees algebra*

$$R[\mathfrak{a}] = \bigoplus_{k=0}^{\infty} \mathfrak{a}^k t^k,$$

where t is a dummy variable that keeps track of the grading. This graded R-algebra is generated by its elements of degree 1 as an R-algebra and satisfies $R[\mathfrak{a}]_0 = R$.

Using the Proj construction described in the appendix to Chapter 7, we get a projective morphism

$$X = \mathrm{Proj}(R[\mathfrak{a}]) \longrightarrow \mathrm{Spec}(R) = U_\sigma. \tag{11.3.2}$$

This is the *blowup of* $\mathrm{Spec}(R)$ *with respect to* $\mathfrak{a}$, denoted $\mathrm{Bl}_{\mathfrak{a}}(\mathrm{Spec}(R))$. For us, a key property of this blowup involves the ideal sheaf $\mathfrak{a}\mathscr{O}_X$, defined as follows. On an affine open subset $U = \mathrm{Spec}(A) \subseteq X$, the map $U \to \mathrm{Spec}(R)$ comes from a ring homomorphism $R \to A$, and $\mathfrak{a}\mathscr{O}_U$ is the sheaf associated to the ideal $\mathfrak{a}A \subseteq A$. For the blowup morphism (11.3.2) the sheaf $\mathfrak{a}\mathscr{O}_X$ is a line bundle—this is [**131**, Prop. II.7.13]. See [**131**, II.7] for a discussion of blowing up ideals and ideal sheaves.

Besides the Rees algebra, the polyhedron P gives a second graded algebra. Recall from §7.1 that from $P \subseteq M_{\mathbb{R}}$ we get the cone $C(P) \subseteq M_{\mathbb{R}} \times \mathbb{R}$, where the slice at height $k > 0$ is kP and the slice at height 0 is $\sigma^\vee$ (see Figure 1 from §7.1). The associated semigroup algebra $S_P = \mathbb{C}[C(P) \cap (M \times \mathbb{Z})]$ is graded by height, and $(S_P)_0 = R$ by the discussion following the proof of Theorem 7.1.13.

Proposition 11.3.1. *With the above setup, there is an inclusion* $R[\mathfrak{a}] \hookrightarrow S_P$ *that makes* S_P *the integral closure of* $R[\mathfrak{a}]$*. Furthermore, this inclusion is an equality if and only* P *is normal.*

Proof. First observe that $R[\mathfrak{a}]$ is a semigroup algebra. Let

$$\mathscr{A} = \big((\sigma^\vee \cap M) \times \{0\}\big) \cup \big((P \cap M) \times \{1\}\big) \subseteq M \times \mathbb{Z}.$$

We leave it to reader to show that $R[\mathfrak{a}] = \mathbb{C}[\mathbb{N}\mathscr{A}]$ (Exercise 11.3.1). Also note:

- We have $\mathbb{Z}\mathscr{A} = M \times \mathbb{Z}$ since $\sigma^\vee$ is full dimensional and $P \subseteq \sigma^\vee$.
- $\mathrm{Cone}(\mathscr{A}) = C(P)$.

Then Proposition 1.3.8 implies that $S_P = \mathbb{C}[C(P) \cap (M \times \mathbb{Z})]$ is the integral closure of $R[\mathfrak{a}] = \mathbb{C}[\mathbb{N}\mathscr{A}]$.

Since $R[\mathfrak{a}]$ and S_P are equal in degrees ≤ 1, the final assertion of the proposition is an easy consequence of the definition of normal polyhedron. □

To translate Proposition 11.3.1 into geometric terms, we will use the Cartier divisor D_P of the polyhedron P. Recall that the Cartier data of D_P is given by the vertices $\mathsf{v} \in P$. This means that the support function of D_P satisfies $\varphi(u) = \langle \mathsf{v}, u \rangle$ when u lies in the cone $\sigma_{\mathsf{v}} \in \Sigma_P$ corresponding to v. Thus $\varphi(u) \geq 0$ since $v \in P \subseteq \sigma^\vee$ and $u \in \sigma_{\mathsf{v}} \subseteq \sigma$. It follows that $D = \sum_\rho \varphi(u_\rho) D_\rho$ is an effective divisor. Since $D_P = -\sum_\rho \varphi(u_\rho) D_\rho$, we conclude that $D_P = -D$, where $D \geq 0$.

In particular, $\mathscr{O}_{X_P}(D_P) = \mathscr{O}_{X_P}(-D)$ is an ideal sheaf of $\mathscr{O}_{X_P}$. This compares to $\mathfrak{a}\mathscr{O}_{X_P}$ as follows.

Corollary 11.3.2. *With the same hypotheses as Proposition 11.3.1, we have:*

(a) *The map $X_P \to U_\sigma$ factors*

$$X_P \longrightarrow \mathrm{Bl}_{\mathfrak{a}}(U_\sigma) \longrightarrow U_\sigma,$$

where the first morphism is normalization and the second is the blowup of $\mathfrak{a}$.

(b) $\mathfrak{a}\mathscr{O}_{X_P} = \mathscr{O}_{X_P}(D_P)$, *where D_P is the Cartier divisor associated to P.*

Proof. For part (a), recall from Theorem 7.1.13 that $X_P = \mathrm{Proj}(S_P)$, where S_P is the integral closure of $R[\mathfrak{a}]$ by Proposition 11.3.1. You will prove in Exercise 11.3.2 that the inclusion $R[\mathfrak{a}] \hookrightarrow S_P$ induces the normalization map

$$X_P = \mathrm{Proj}(S_P) \longrightarrow \mathrm{Proj}(R[\mathfrak{a}]).$$

For part (b), take a vertex $\mathsf{v} \in P$. This gives the cone $\sigma_{\mathsf{v}} \in \Sigma_P$, and since $P = \mathrm{Conv}(m \mid \chi^m \in \mathfrak{a})$, one sees easily that $\sigma_{\mathsf{v}}^\vee = \mathrm{Cone}(m - \mathsf{v} \mid \chi^m \in \mathfrak{a})$. Then any $\chi^m \in \mathfrak{a}$ can be written

$$\chi^m = \chi^{\mathsf{v}} \cdot \chi^{m-\mathsf{v}} \in \chi^{\mathsf{v}}\mathbb{C}[\sigma_{\mathsf{v}}^\vee \cap M],$$

from which we conclude that $\mathfrak{a}\mathbb{C}[\sigma_{\mathsf{v}}^\vee \cap M] = \chi^{\mathsf{v}}\,\mathbb{C}[\sigma_{\mathsf{v}}^\vee \cap M]$ since $\chi^{\mathsf{v}} \in \mathfrak{a}$. Since X_P is covered by the affine open subsets $U_{\sigma_{\mathsf{v}}}$, it follows that $\mathfrak{a}\mathbb{C}[\sigma_{\mathsf{v}}^\vee \cap M]$ is the line bundle of the Cartier divisor whose Cartier data is given by the vertices of P. As noted above, this divisor is D_P, which completes the proof. □

This corollary shows that the map $X_P \to U_\sigma$ is a single blowup followed by a normalization. Moreover, replacing P with a sufficiently high multiple, we may assume that P is normal (Theorem 7.1.9). This does not change the normal fan, so that in this case, the map $X_P \to U_\sigma$ is the blowup of the ideal $\mathfrak{a}$.

By Theorem 7.2.4, it follows that any projective toric resolution $X_\Sigma \to U_\sigma$ is the blowup of a suitable ideal $\mathfrak{a} \subseteq \mathbb{C}[\sigma^\vee \cap M]$. For the special case of blowing up the origin $0 \in \mathbb{C}^n$ or more generally a coordinate subspace $\{0\} \times \mathbb{C}^{n-r} \subseteq \mathbb{C}^n$, the ideal $\mathfrak{a}$ is easy to describe (Exercise 11.3.3).

More on the Ideal. If we fix the cone $\sigma \subseteq N_{\mathbb{R}}$, then we can give a purely algebraic description of the ideals that arise from (11.3.1). We first need a definition.

Definition 11.3.3. An ideal $\mathfrak{a}$ in a ring R is ***integrally closed*** if whenever an element $r \in R$ satisfies an equation

$$r^k + a_1 r^{k-1} + \cdots + a_{k-1} r + a_k = 0$$

with $a_i \in \mathfrak{a}^i$ for $1 \le i \le k$, we have $r \in \mathfrak{a}$.

Then we have the following result.

Proposition 11.3.4. *Fix a full dimensional strongly convex cone $\sigma \subseteq N_{\mathbb{R}}$. Then the maps*

$$P \longmapsto \mathfrak{a} = \bigoplus_{m \in P\cap M} \mathbb{C}\cdot\chi^m$$
$$\mathfrak{a} \longmapsto P = \operatorname{Conv}(m \mid \chi^m \in \mathfrak{a})$$

induce a bijection

$$\left\{\begin{array}{c}\text{lattice polyhedra } P \subseteq \sigma^\vee\\ \text{with recession cone } \sigma^\vee\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{nonzero integrally closed}\\ \text{ideals } \mathfrak{a} \subseteq \mathbb{C}[\sigma^\vee \cap M]\\ \text{generated by characters}\end{array}\right\}.$$

Proof. Given a lattice polyhedron $P \subseteq \sigma^\vee$ with recession cone $\sigma^\vee$, we need to show that $\mathfrak{a}$ is integrally closed. Standard arguments (see [**89**, p. 137] or [**266**, Cor. 1.3.1]) imply that

$$\overline{\mathfrak{a}} = \{r \in \mathbb{C}[\sigma^\vee \cap M] \mid r^k + a_1 r^{k-1} + \cdots + a_k = 0 \text{ for some } a_i \in \mathfrak{a}^i\}$$

is an integrally closed ideal containing $\mathfrak{a}$. Hence it suffices to prove $\mathfrak{a} = \overline{\mathfrak{a}}$.

Since $\mathfrak{a}$ is generated by characters, it is invariant under the T_N-action on R, which easily implies that $\overline{\mathfrak{a}}$ is T_N-invariant. Then Lemma 1.1.16 implies that $\overline{\mathfrak{a}}$ is also generated by characters (see also [**266**, Prop. 1.4.2]). Thus, to prove $\mathfrak{a} = \overline{\mathfrak{a}}$, we need to show that if a character $\chi^m \in \overline{\mathfrak{a}}$ satisfies

$$\chi^{km} + a_1\chi^{(k-1)m} + \cdots + a_{k-1}\chi^m + a_k = 0$$

with $a_i \in \mathfrak{a}^i$, then $\chi^m \in \mathfrak{a}$. Solving for the first term of the above equation gives

$$\chi^{km} = -a_1\chi^{(k-1)m} - \cdots - a_{k-1}\chi^m - a_k.$$

Writing a_i as a linear combination of i-fold products of characters in $\mathfrak{a}$, it follows that for some $1 \le i \le k$, we must have an equality

$$\chi^{km} = \chi^{m_1}\cdots\chi^{m_i}\chi^{(k-i)m}$$

with $\chi^{m_1},\ldots,\chi^{m_i} \in \mathfrak{a}$, i.e., $m_1,\ldots,m_i \in P$. This implies

$$m = \tfrac{1}{i}(m_1 + \cdots + m_i),$$

which lies in P since P is convex. Hence $\chi^m \in \mathfrak{a}$, proving that $\mathfrak{a}$ is integrally closed.

In the other direction, a nonzero ideal $\mathfrak{a} = \langle \chi^{m_1},\ldots,\chi^{m_s}\rangle \subseteq \mathbb{C}[\sigma^\vee \cap M]$ gives the convex set $P = \operatorname{Conv}(m \mid \chi^m \in I) \subseteq \sigma^\vee$. In Exercise 7.1.12 you showed that $P = \operatorname{Conv}(m_1,\ldots,m_s) + \sigma^\vee$. Hence P is a lattice polyhedron contained in I with recession cone $\sigma^\vee$.

It remains to show that these maps are inverses of each other. One direction is easy, for if $P \mapsto \mathfrak{a}$, then $\mathfrak{a} \mapsto \operatorname{Conv}(m \mid m \in P\cap M)$, which equals P since P is a lattice polyhedron. The other direction takes more thought. A nonzero integrally closed ideal $\mathfrak{a}$ gives $P = \operatorname{Conv}(m \mid \chi^m \in \mathfrak{a})$. We need to show that $m \in P\cap M$ implies $\chi^m \in \mathfrak{a}$. To prove this, first note that $m \in P \cap M$ implies $m \in \operatorname{Conv}(m_1,\ldots,m_s)$ where $\chi^{m_1},\ldots,\chi^{m_s} \in \mathfrak{a}$. By Carathéodory's theorem (see [**281**, Prop. 1.15]), we can assume that $m_1, \dots, m_s$ are affinely independent. Since $m, m_1, \dots, m_s \in M$, it follows that

$$m = \textstyle\sum_{i=1}^{s} \lambda_i m_i, \text{ where } \lambda_i \geq 0,\ \sum_{i=1}^{s} \lambda_i = 1,\ \lambda_i \in \mathbb{Q}.$$

Alfter clearing denominators, we get $km = \sum_{i=1}^{s} \mu_i m_i$ where $k, \mu_i, \dots, \mu_s \in \mathbb{N}$ and $\sum_{i=1}^{s} \mu_i = k > 0$. Then $r = \chi^m$ satisfies the equation

$$r^k - \textstyle\prod_{i=1}^{s} (\chi^{m_i})^{\mu_i} = 0.$$

Since $\prod_{i=1}^{s} (\chi^{m_i})^{\mu_i} \in \mathfrak{a}^k$ and $\mathfrak{a}$ is integrally closed, we must have $r = \chi^m \in \mathfrak{a}$. $\square$

Example 11.3.5. For the cone $\sigma \subseteq \mathbb{R}^3$ of Example 11.1.12, Figure 7 shows the polyhedron $P \subseteq \sigma^\vee$ obtained by pushing one of the facets of $\sigma^\vee$ one unit to the right.

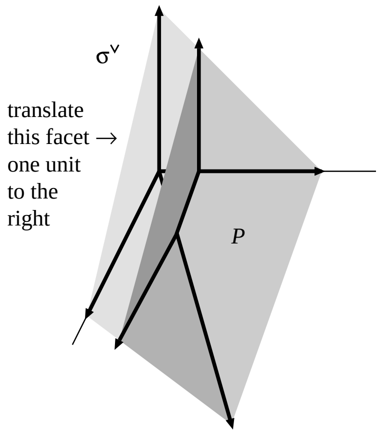


Figure 7. The polyhedron P and the dual cone $\sigma^\vee$

The corresponding ideal $\mathfrak{a}$ is generated by the vertices of P (a rare occurrence, but true here), so that

$$\mathfrak{a} = \langle \chi^{e_2}, \chi^{e_1+e_2-e_3} \rangle \subseteq \mathbb{C}[\sigma^\vee \cap M].$$

Looking at P, one sees that its normal fan is the fan Σ_2 from Figure 3 in Example 11.1.12. Since P is normal (Exercise 11.3.4), it follows that $X_{\Sigma_2} = \mathrm{Bl}_{\mathfrak{a}}(U_\sigma)$.

We can also think about this from the point of view of the simplicialization process described in the proof of Proposition 11.1.7. Since σ is not simplicial, we can take $\rho \in \sigma(1)$ such that D_ρ is not Cartier, say $\rho = \mathrm{Cone}(e_2)$. The star subdivision at e_2 gives the fan Σ_2 such that D_ρ is becomes Cartier on X_{Σ_2}. Then one can check in that P is the polyhedron of $-D_\rho$, so that

$$\mathfrak{a}\mathscr{O}_{X_{\Sigma_2}} = \mathscr{O}_{X_{\Sigma_2}}(-D_\rho)$$

(Exercise 11.3.4). $\Diamond$

Example 11.3.6. The maximal ideal $\mathfrak{m} = \bigoplus_{m \in \sigma^\vee \cap (M \setminus \{0\})} \mathbb{C} \cdot \chi^m \subseteq \mathbb{C}[\sigma^\vee \cap M]$ is clearly integrally closed. The polyhedron associated to $\mathfrak{m}$ is the convex hull $P_\mathfrak{m} = \mathrm{Conv}(\sigma^\vee \cap (M \setminus \{0\}))$.

When σ has dimension 2, this convex hull was denoted $\Theta_{\sigma^\vee}$ in §10.2, i.e., $P_\mathfrak{m} = \Theta_{\sigma^\vee}$. In Proposition 10.2.17, we described the properties of the associated projective toric morphism $X_{P_\mathfrak{m}} \to U_\sigma$. Since lattice polyhedra are normal in dimension 2, Proposition 11.3.1 and Corollary 11.3.2 imply that this morphism is the blowup of the ideal $\mathfrak{m}$, i.e., $X_{P_\mathfrak{m}} = \mathrm{Bl}_\mathfrak{m}(U_\sigma)$. $\Diamond$

Example 11.3.7. Consider the graph G in Figure 8, where the vertices are labeled with variables x, y, z. This is gives the *edge ideal* $\mathfrak{a} = \langle xy, yz, zx \rangle \subseteq \mathbb{C}[x, y, z]$ whose generators correspond to the edges of the graph. Note that $\mathfrak{a}$ is radical since it is

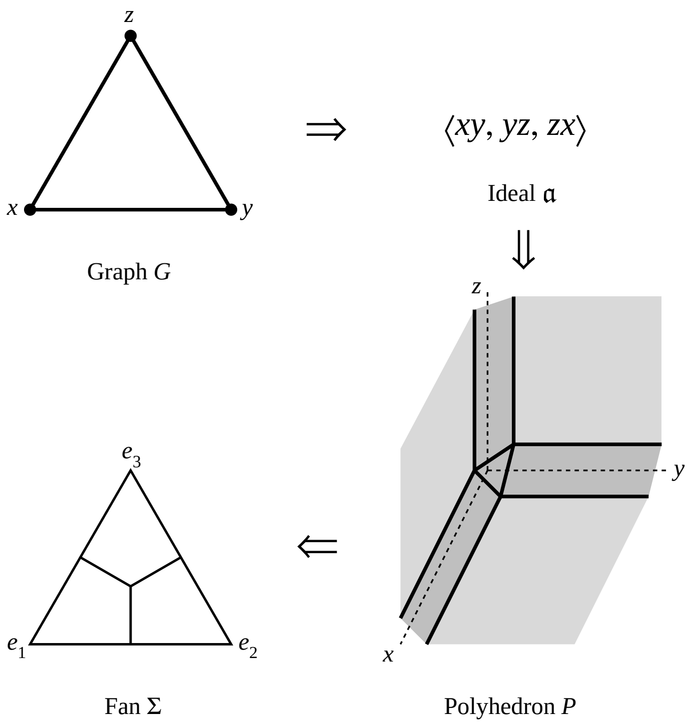


Figure 8. Graph, ideal, polyhedron, and fan

generated by square-free monomials and hence is integrally closed since radical ideals are automatically integrally closed (Exercise 11.3.5).

The ideal $\mathfrak{a}$ gives the polyhedron P shown in Figure 8. Since P sits inside the first octant in $\mathbb{R}^3$, the shaded facets form the "back" of P from our point of view. The vertices of P correspond to the generators of $\mathfrak{a}$, though in general, the ideal of a polyhedron usually has more generators than just the characters of the vertices.

One can show that P is normal (Exercise 11.3.5), so that

$$X_\Sigma = X_P = \mathrm{Bl}_{\mathfrak{a}}(\mathbb{C}^3) = \mathrm{Proj}(R[\mathfrak{a}]), \quad R = \mathbb{C}[x,y,z],$$

by Proposition 11.3.1 and Corollary 11.3.2. Another proof of normality comes from [**250**, Thm. 6.3], which implies that $R[\mathfrak{a}]$ is normal since G is an odd cycle. Hence P is normal Proposition 11.3.1. $\Diamond$

When $\mathfrak{a} \subseteq \mathbb{C}[\sigma^\vee \cap M]$ is generated by characters but not integrally closed, we have the following description of its integral closure.

Proposition 11.3.8. *Let $\mathfrak{a} \subseteq \mathbb{C}[\sigma^\vee \cap M]$ be an ideal generated by characters and set $P = \mathrm{Conv}(m \mid \chi^m \in \mathfrak{a})$. Then:*

(a) $\overline{\mathfrak{a}} = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m$*, so $P \leftrightarrow \overline{\mathfrak{a}}$ in the correspondence of Proposition 11.3.4.*

(b) $\mathfrak{a}\mathscr{O}_{X_P} = \overline{\mathfrak{a}}\mathscr{O}_{X_P} = \mathscr{O}_{X_P}(D_P)$.

Proof. We leave the proof of part (a) as Exercise 11.3.6. For part (b), applying Corollary 11.3.2 to P and $\overline{\mathfrak{a}}$ gives $\overline{\mathfrak{a}}\mathscr{O}_{X_P} = \mathscr{O}_{X_P}(D_P)$. Furthermore, the proof of the corollary shows that

$$\overline{\mathfrak{a}}\mathbb{C}[\sigma_{\mathsf{v}}^\vee \cap M] = \chi^{\mathsf{v}}\mathbb{C}[\sigma_{\mathsf{v}}^\vee \cap M]$$

when v is a vertex of P. However, $P = \mathrm{Conv}(m \mid \chi^m \in \mathfrak{a})$ implies that the vertices come from $\mathfrak{a}$, i.e., $\chi^{\mathsf{v}} \in \mathfrak{a}$. Hence $\overline{\mathfrak{a}}\mathbb{C}[\sigma_{\mathsf{v}}^\vee \cap M] = \mathfrak{a}\mathbb{C}[\sigma_{\mathsf{v}}^\vee \cap M]$, and part (b) follows immediately. $\square$

Part (a) of the proposition is well-known when $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a monomial ideal. See, for example, [**266**, Prop. 1.4.6]. Another nice result of "monomial commutative algebra" is the *monomial Briançon-Skoda theorem*, which asserts that if $\mathfrak{a} \subseteq \mathbb{C}[\sigma^\vee \cap M]$ is an ideal generated by characters, then

$$\overline{\mathfrak{a}^{n+\ell-1}} \subseteq \mathfrak{a}^\ell$$

where $n = \mathrm{rank}\, M$ and $\ell \geq 1$ is arbitrary. A proof can be found in [**268**, §4]. Monomial ideals are an active area of research in commutative algebra (see [**274**] and the references therein).

Global Aspect. Given a projective toric resolution of singularities $X_{\Sigma'} \to X_\Sigma$, it is natural to ask if $X_{\Sigma'}$ can be represented as the blowup $\mathrm{Bl}_{\mathscr{I}}(X_\Sigma)$ for some torus-invariant sheaf of ideals $\mathscr{I} \subseteq \mathscr{O}_{X_\Sigma}$. When X_Σ is quasiprojective, this can be done by adapting the technique used in the proof of [**131**, Thm. II.7.17]. In general, it might not be possible to find a sheaf of ideals, but one can always find a sheaf of *fractional ideals* that works. This is explained in [**172**, Ch. I, Thms. 10 and 11].

Log Resolutions. Suppose that we have a sheaf of ideals $\mathscr{I} \subseteq \mathscr{O}_X$, where X is smooth. Then $\mathscr{I}\mathscr{O}_Y$ is a line bundle on $Y = \mathrm{Bl}_{\mathscr{I}}(X)$, though Y may be rather singular. A log resolution for $\mathscr{I}$ asks for a smooth version of this. Here is the precise definition.

Definition 11.3.9. Let $\mathscr{I} \subseteq \mathscr{O}_X$ be an ideal sheaf on a smooth variety X. Then a projective birational morphism $f : X' \to X$ is a ***log resolution of*** $\mathscr{I}$ if X' is smooth, $\mathscr{I}\mathscr{O}_{X'} = \mathscr{O}_{X'}(-D)$ for a divisor D on X', and $\mathrm{Supp}(D) \cup \mathrm{Exc}(f)$ is a SNC divisor.

This type of log resolution is often called a *principalization*. In the toric case, we have the following principalization result of Howald [**147**] (see also [**268**]).

Theorem 11.3.10. *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1,\dots,x_n]$ be a monomial ideal. Then there is a toric morphism $\phi : X_\Sigma \to \mathbb{C}^n$ that is a log resolution of $\mathfrak{a}$.*

Proof. Let $P = \mathrm{Conv}(m \mid \chi^m \in \mathfrak{a})$ as in Proposition 11.3.8. This gives $X_P \to \mathbb{C}^n$ with $\mathfrak{a}\mathscr{O}_{X_P} = \mathscr{O}_{X_P}(D_P)$. Since X_P may be singular, we take a projective resolution of singularities $X_\Sigma \to X_P$. The composed map $\phi : X_P \to \mathbb{C}^n$ is projective, toric, and birational, though its exceptional locus $\mathrm{Exc}(\phi)$ need not be a divisor. This is easily fixed by adapting the strategy used in the proof of Theorem 11.2.2. Hence we may assume that $\mathrm{Exc}(\phi)$ is a divisor.

Then let $-D$ be the pullback of D_P via $X_\Sigma \to X_P$. One easily checks that $\mathfrak{a}\mathscr{O}_{X_\Sigma} = \mathscr{O}_{X_\Sigma}(-D)$, and $\mathrm{Supp}(-D) \cup \mathrm{Exc}(\phi)$ is SNC since it is a union of torus-invariant prime divisors on a smooth toric variety. $\square$

A stronger principalization result, proved by Goward in [**121**], asserts that the log resolution $\phi : X_\Sigma \to \mathbb{C}^n$ can be chosen to be a composition of blowups with smooth torus-invariant centers. This is analogous to embedded resolution of a toric variety, which also uses blowups with smooth torus-invariant centers. See Exercise 11.3.7 for an example.

Multiplier Ideals. Given a proper birational morphism $f : X' \to X$ where X' and X are smooth, we have the *relative canonical divisor*

$$K_{X'/X} = K_{X'} - f^*K_X,$$

As noted in [**186**, 9.1.B], this divisor is supported on the exceptional locus $\mathrm{Exc}(f)$ and hence satisfies

$$f_*\mathscr{O}_{X'}(K_{X'/X}) \simeq \mathscr{O}_X. \tag{11.3.3}$$

In the toric context, these assertions are easy to understand and are covered in Exercise 11.3.8.

One consequence of (11.3.3) is that if E is any effective divisor on X', then

$$\mathscr{O}_{X'}(K_{X'/X} - E) \subseteq \mathscr{O}_{X'}(K_{X'/X}),$$

so that $f_*\mathscr{O}_{X'}(K_{X'/X} - E) \subseteq f_*\mathscr{O}_{X'}(K_{X'/X}) \simeq \mathscr{O}_X$ is an ideal sheaf. When applied to the log resolution of an ideal sheaf, this gives the following definition.

Definition 11.3.11. Let $\mathscr{I} \subseteq \mathscr{O}_X$ be a nonzero ideal sheaf on a smooth variety X and let $f : X' \to X$ be a log resolution of $\mathscr{I}$ such that $\mathscr{I}\mathscr{O}_{X'} = \mathscr{O}_{X'}(-D)$ for an effective divisor D on X'. Given any rational number $c > 0$, the ***multiplier ideal*** $\mathcal{J}(c\cdot\mathscr{I})$ is the ideal sheaf defined by

$$\mathcal{J}(c\cdot\mathscr{I}) = f_*\mathscr{O}_{X'}(K_{X'/X} - \lfloor cD \rfloor).$$

One can show that this definition is independent of the log resolution [**186**, Thm. 9.2.18]. See [**34**] for an introduction to multiplier ideals and [**186**, Ch. 9–10] for a thorough discussion and many applications.

In the toric context, Howald [**147**] showed how to compute the multiplier ideal $\mathcal{J}(c\cdot\mathfrak{a})$ of a monomial ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1,\dots,x_n]$.

Theorem 11.3.12. *Let $P = \mathrm{Conv}(m \mid \chi^m \in \mathfrak{a})$ be the polyhedron associated to a monomial ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1,\dots,x_n]$. Then, for any $c > 0$ in $\mathbb{Q}$, we have*

$$\mathcal{J}(c\cdot\mathfrak{a}) = \bigoplus_{m+e_0\in \mathrm{Int}(cP)} \mathbb{C}\cdot\chi^m.$$

were $e_0 = (1,\dots,1) \in \mathbb{Z}^n$.

Proof. We use the log resolution $\phi : X_\Sigma \to \mathbb{C}^n$ constructed in Theorem 11.3.10. Thus $\mathfrak{a}\mathscr{O}_{X_\Sigma} = \mathscr{O}_{X_\Sigma}(-D)$, where $-D$ is the pullback of D_P on X_P, and it follows that the polyhedron of $-D$ is $P_{-D} = P$. If we write $D = \sum_\rho a_\rho D_\rho$, $a_\rho \geq 0$, then

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho\rangle \geq a_\rho \text{ for all } \rho \in \Sigma(1)\},$$

hence

$$cP = \{m \in M_{\mathbb{R}} \mid \langle m, u_\rho\rangle \geq ca_\rho \text{ for all } \rho \in \Sigma(1)\}. \tag{11.3.4}$$

Also note that $\mathrm{div}_{\mathbb{C}^n}(\chi^{-e_0}) = K_{\mathbb{C}^n}$, so that $\phi^*K_{\mathbb{C}^n} = \mathrm{div}_{X_\Sigma}(\chi^{-e_0})$. Hence

$$K_{X_\Sigma/\mathbb{C}^n} - \lfloor cD\rfloor = K_{X_\Sigma} - \phi^*K_{\mathbb{C}^n} - \lfloor cD\rfloor = \sum_\rho (-1 + \langle e_0, u_\rho\rangle - \lfloor ca_\rho\rfloor)D_\rho.$$

The multiplier ideal $\mathcal{J}(c\cdot\mathfrak{a})$ is given by the global sections of the sheaf of this divisor. By Proposition 4.3.3, we obtain

$$\begin{aligned} \chi^m \in \mathcal{J}(c\cdot\mathfrak{a}) &\iff \langle m, u_\rho\rangle \geq 1 - \langle e_0, u_\rho\rangle + \lfloor ca_\rho\rfloor && \text{for all } \rho \in \Sigma(1)\\ &\iff \langle m + e_0, u_\rho\rangle \geq 1 + \lfloor ca_\rho\rfloor && \text{for all } \rho \in \Sigma(1)\\ &\iff \langle m + e_0, u_\rho\rangle > ca_\rho && \text{for all } \rho \in \Sigma(1). \end{aligned}$$

By (11.3.4), we conclude that $\chi^m \in \mathcal{J}(c\cdot\mathfrak{a})$ if and only if $m + e_0$ lies in the interior of cP. This completes the proof. $\square$

Theorem 11.3.12 shows that the characters in the multiplier ideal $\mathcal{J}(c\cdot\mathfrak{a})$ come from translates by $-e_0 = (-1,\dots,-1)$ of interior points of c times the polyhedron. This result is also discussed in [**186**, Thm. 9.3.27].

Example 11.3.13. Consider the monomial ideal $\mathfrak{a} = \langle x^4, x^2y, y^3\rangle \subseteq \mathbb{C}[x,y]$. The multiplier ideal $\mathcal{J}(1\cdot\mathfrak{a}) = \mathcal{J}(\mathfrak{a})$ is given by

$$\mathcal{J}(\mathfrak{a}) = \langle x^2, xy, y^2\rangle.$$

This follows from Figure 9, which shows the polyhedron P (light shading), its

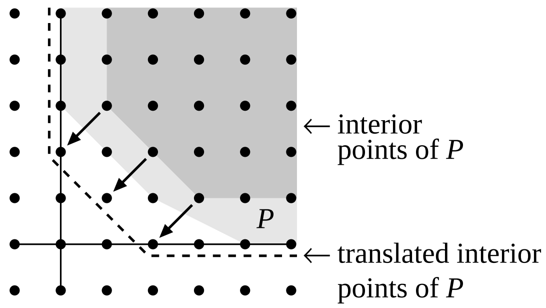


Figure 9. The polyhedron P, its interior lattice points, and their translates

interior lattice points (in the dark shading), and the result of translating these lattice points by $(-1,-1)$ (bounded by the dashed line). ◊

Example 11.3.14. For the ideal $\mathfrak{a} = \langle xy, yz, zx\rangle \subseteq \mathbb{C}[x,y,z]$ of Example 11.3.7, one sees easily that $(1,1,1)$ lies in the interior of P. It follows immediately that the multiplier ideal $\mathcal{J}(\mathfrak{a})$ is trivial. ◊

Exercises for §11.3.

11.3.1. Prove the claim made in the proof of Proposition 11.3.1 that $R[\mathfrak{a}] = \mathbb{C}[\mathbb{N}\mathscr{A}]$.

11.3.2. In Proposition 11.3.1, we showed that $S_P = \mathbb{C}[C(P)\cap(M\times\mathbb{Z})]$ is the integral closure of $R[\mathfrak{a}] = \mathbb{C}[\mathbb{N}\mathscr{A}]$. Prove that $\mathrm{Proj}(S_P) \to \mathrm{Proj}(R[\mathfrak{a}])$ is the normalization map. Hint: Look at the proof of Theorem 7.1.13. A character $\chi^m \in \mathfrak{a}$ gives an element $\chi^m t$ of degree 1 in $R[\mathfrak{a}]$ and S_P, which in turn gives open sets

$$U = \mathrm{Spec}(R[\mathfrak{a}]_{(\chi^m t)}) \text{ and } U' = \mathrm{Spec}((S_P)_{(\chi^m t)})$$

of $\mathrm{Proj}(R[\mathfrak{a}])$ and $\mathrm{Proj}(S_P)$ respectively. Show that these open sets cover $\mathrm{Proj}(R[\mathfrak{a}])$ and $\mathrm{Proj}(S_P)$ and that U' is the normalization of U.

11.3.3. For the blowup of the origin $0 \in \mathbb{C}^n$, find an explicit ideal $\mathfrak{a} \subseteq R = \mathbb{C}[x_1,\dots,x_n]$ such that $\mathrm{Bl}_0(\mathbb{C}^n) = \mathrm{Proj}(R[\mathfrak{a}])$. Then do the same for the blowup of the coordinate subspace $\{0\}\times\mathbb{C}^{n-r} \subseteq \mathbb{C}^n$.

11.3.4. Fill in the details omitted in the discussion of Example 11.3.5.

11.3.5. This exercise is concerned with Example 11.3.7.

(a) Prove that an ideal in $\mathbb{C}[x_1,\dots,x_n]$ generated by square-free monomials is radical.

(b) Prove that every radical ideal is integrally closed.

(c) Prove that the polytope P pictured in Figure 8 of Example 11.3.7 is normal.

11.3.6. Prove Proposition 11.3.8.

11.3.7. Let $\mathfrak{a} = \langle x^3, y^2\rangle \subseteq \mathbb{C}[x,y]$.

(a) Construct a log resolution $\phi : X_\Sigma \to \mathbb{C}^2$ of $\mathfrak{a}$ using the method described in the proof of Theorem 11.3.10.

(b) Interpret the fan X_Σ constructed in part (a) as a sequence of three star subdivisions, each of which blows up a fixed point of the torus action.

(c) Let E_i be the proper transform in X_Σ exceptional locus of the ith blowup. Show that $\mathfrak{a}\mathscr{O}_{X_\Sigma} = \mathscr{O}_{X_\Sigma}(-2E_1 - 3E_2 - 6E_3)$.

This exercise is taken from [**34**].

11.3.8. Let $\phi : X_{\Sigma'} \to X_\Sigma$ be the toric morphism coming from a smooth refinement Σ' of a smooth fan Σ.

(a) Prove that $K_{X_{\Sigma'}} - \phi^* K_{X_\Sigma}$ is supported on the exceptional locus $\mathrm{Exc}(\phi)$ and is ≥ 0.

(b) Let D be an effective divisor supported on $\mathrm{Exc}(\phi)$. Prove that $\phi_*\mathscr{O}_{X'_\Sigma}(D) \simeq \mathscr{O}_{X_\Sigma}$. Hint: Reduce to the affine case.

11.3.9. Compute the multiplier ideal $\mathscr{J}(\mathfrak{a})$ of the following monomial ideals:

(a) $\mathfrak{a} = \langle x^8, y^6\rangle \subseteq \mathbb{C}[x,y]$ (see [**147**, Ex. 2]).

(b) $\mathfrak{a} = \langle x^7, x^3y, xy^2, y^6\rangle \subseteq \mathbb{C}[x,y]$ (see [**186**, Ex. 9.3.28]).

11.3.10. Let $\mathfrak{a} = \langle x_1^{a_1},\dots,x_n^{a_n}\rangle \subseteq \mathbb{C}[x_1,\dots,x_n]$, where $a_1,\dots,a_n$ are positive integers. Prove that $\mathscr{J}(c\cdot\mathfrak{a})$ is trivial if and only if $c < \sum_{i=1}^n \frac{1}{a_i}$ (see [**147**, Ex. 3]).

11.3.11. Consider the monomial ideal $\mathfrak{a} = \langle xy, yz, zx\rangle \subseteq \mathbb{C}[x,y,z]$ and polyhedron P from Figure 8 in Example 11.3.7. The toric morphism $X_P \to \mathbb{C}^3$ satisfies $\mathfrak{a}\mathscr{O}_{X_P} = \mathscr{O}_{X_P}(D_P)$. To get a log resolution of $\mathfrak{a}$ as in Theorem 11.3.10, we need a smooth refinement of the normal fan Σ of P. Figure 10 shows Σ and two smooth refinements Σ_1 and Σ_2.

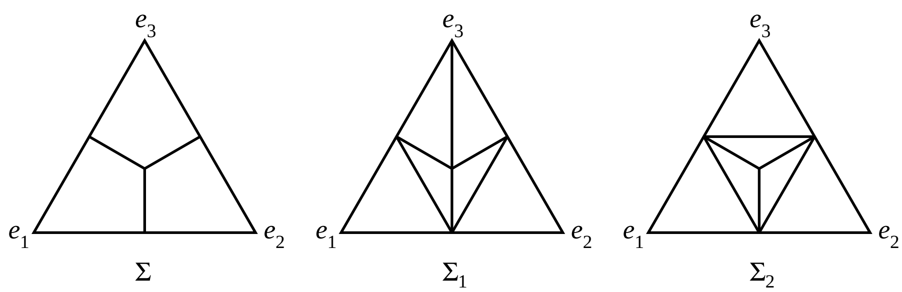


Figure 10. The fan Σ and smooth refinements Σ_1 and Σ_2

(a) Explain how Σ_1 and Σ_2 can be obtained from Σ by a series of star subdivisions. Hence $X_{\Sigma_1} \to \mathbb{C}^3$ and $X_{\Sigma_2} \to \mathbb{C}^3$ are birational projective toric morphisms.

(b) Show that $X_{\Sigma_1} \to \mathbb{C}^3$ is a composition of blowups at smooth torus-invariant centers.

(c) Show that $X_{\Sigma_2} \to \mathbb{C}^3$ is not a composition of blowups at smooth torus-invariant centers.

11.3.12. In this exercise and the next we follow [**33**] and explore the toric case of two variants of multiplier ideals. Let $U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ be an affine toric variety. Fix an effective torus-invariant $\mathbb{Q}$-divisor D and an ideal $\mathfrak{a} \subset \mathbb{C}[\sigma^\vee \cap M]$ generated by characters. Also assume that $K_{U_\sigma} + D$ is $\mathbb{Q}$-Cartier. Thus there are $m_0 \in M$ and $r > 0$ in $\mathbb{Z}$ with $\mathrm{div}(\chi^{-m_0}) = r(K_{U_\sigma} + D)$. Also let $P \subseteq \sigma^\vee$ be the polyhedron of $\mathfrak{a}$.

(a) Prove that there is a toric morphism $\phi : X_\Sigma \to U_\sigma$ which is a log resolution for both D and $\mathfrak{a}$. Write $\mathfrak{a}\mathscr{O}_{X_\Sigma} = \mathscr{O}_{X_\Sigma}(-A)$, where A is an effective divisor.

(b) Given $c > 0$ in $\mathbb{Q}$, prove that we have a natural inclusion

$$\phi_*\mathscr{O}_{X_\Sigma}(K_{X_\Sigma} - \lfloor \phi^*(K_{U_\sigma} + D) + cA \rfloor) \subseteq \mathscr{O}_{U_\sigma}.$$

This gives a multiplier ideal $\mathcal{J}(D, c\cdot\mathfrak{a}) \subseteq \mathbb{C}[\sigma^\vee \cap M]$.

(c) Adapt the proof of Theorem 11.3.12 to show that

$$\mathcal{J}(D, c\cdot\mathfrak{a}) = \bigoplus_{m + \frac{1}{r}m_0 \in \mathrm{Int}(cP)} \mathbb{C}\cdot\chi^m.$$

Part (c) is [**33**, Thm. 1]. Note that Theorem 11.3.12 is a special case of this result. Earlier, we defined the multiplier ideal $\mathcal{J}(c\cdot\mathfrak{a})$ for $\mathbb{C}^n$. This definition extends to $\mathbb{Q}$-Gorenstein affine toric varieties. However, when U_σ fails to be $\mathbb{Q}$-Gorenstein, then we use the log version, since $K_{U_\sigma} + D$ could be $\mathbb{Q}$-Cartier for some $\mathbb{Q}$-divisor D. See [**186**, 9.3.G].

11.3.13. To define a multiplier ideal that works for any affine toric variety U_σ, we proceed as follows. Let $\mathfrak{a} \subset \mathbb{C}[\sigma^\vee \cap M]$ be generated by characters and let $\phi : X_\Sigma \to U_\sigma$ be a log resolution such that $\mathfrak{a}\mathscr{O}_{X_\Sigma} = \mathscr{O}_{X_\Sigma}(-A)$, $A \geq 0$. Following [**33**], the *multiplier module* is

$$\phi_*\mathscr{O}_{X_\Sigma}(K_{X_\Sigma} - \lfloor cA \rfloor) \subseteq \omega_{U_\sigma},\ c > 0 \text{ in } \mathbb{Q}.$$

By Proposition 8.2.9, we can regard ω_{U_σ} as an ideal of $\mathbb{C}[\sigma^\vee \cap M]$ generated by characters coming from lattice points in the interior of $\sigma^\vee$. Then the multiplier module gives an ideal of $\mathbb{C}[\sigma^\vee \cap M]$ denoted $\mathcal{J}_\omega(c\cdot\mathfrak{a})$. Prove that this ideal is given by

$$\mathcal{J}_\omega(c\cdot\mathfrak{a}) = \bigoplus_{m \in \mathrm{Int}(cP)} \mathbb{C}\cdot\chi^m,$$

where as usual P is the polyhedron of $\mathfrak{a}$. This is [**33**, Thm. 2].

11.3.14. Let $\mathfrak{a} \subseteq R = \mathbb{C}[x_1, \ldots, x_n]$ be a monomial ideal. In the text, we used a log resolution to define the multiplier ideal $\mathcal{J}(c\cdot\mathfrak{a})$. This exercise will present a different construction of $\mathcal{J}(c\cdot\mathfrak{a})$. Let S be the integral closure of the Rees algebra $R[\mathfrak{a}]$. This gives the projective morphism $\phi : X = \mathrm{Proj}(S) \to \mathbb{C}^n$. Prove that

$$\mathcal{J}(c\cdot\mathfrak{a}) = \phi_*\mathscr{O}_X(K_X - \phi^*K_{\mathbb{C}^n} - \lfloor cD \rfloor),$$

where $\mathfrak{a}\mathscr{O}_X = \mathscr{O}_X(-D)$. Hint: Use Proposition 11.3.1. (For experts, this works because X has rational singularities.)

11.3.15. This exercise is for readers who know about symbolic powers of ideal sheaves (see, for example, [**186**, Def. 9.3.4]). Let $D = \sum_i D_i$ be a sum of distinct prime divisors on a normal variety X. Then $\mathscr{O}_X(-D) \subseteq \mathscr{O}_X$ is a sheaf of radical ideals. For an integer $n \geq 1$, show that $\mathscr{O}_X(-nD)$ equals the nth symbolic power $\mathscr{O}_X(-D)^{\langle n \rangle}$. Hint: Exercise 4.0.14.

§11.4. Toric Singularities

We next discuss the singularities of normal toric varieties. This is an active area of research, and our treatment will omit many important topics.

General Properties of Toric Singularities. Although a normal toric variety need not be smooth, it is at least Cohen-Macaulay. Furthermore, its singularities are of the following special type.

Definition 11.4.1. A normal variety X has ***rational singularities*** if for every resolution of singularities $f : X' \to X$, we have

$$R^p f_* \mathscr{O}_{X'} = 0 \text{ for all } p > 0.$$

One can show that if the above vanishing holds for one resolution of singularities, then it holds for all [**179**, Thm. 5.10]. Hence, in the toric case, we can use a toric resolution of singularities.

Theorem 11.4.2. *A normal toric variety X_Σ has rational singularities.*

Proof. Let $\phi : X_{\Sigma'} \to X_\Sigma$ be a toric resolution of singularities. In particular, ϕ is proper, so that $R^p\phi_*\mathscr{O}_{X_{\Sigma'}} = 0$ for $p > 0$ by Theorem 9.2.5. Then we are done by the above remark. □

The reader should consult [**179**, Sec. 5.1] for a careful discussion of rational singularities.

The Simplicial Case. We next give several characterizations of the singularities of a simplicial toric variety. The first is topological, based on the following idea. If X is a smooth irreducible variety of dimension n, then any $x \in X$ has a neighborhood $x \in U$ in the classical topology that is homeomorphic to an open ball in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. By excision and the long exact sequence for relative cohomology, one obtains

$$H^p(X, X \setminus \{x\}, \mathbb{Z}) \simeq H^p(U, U \setminus \{x\}, \mathbb{Z}) \simeq \begin{cases} 0 & p \neq 2n \\ \mathbb{Z} & p = 2n. \end{cases}$$

When we weaken the coefficients from $\mathbb{Z}$ to $\mathbb{Q}$, we get the following definition.

Definition 11.4.3. An irreducible variety X of dimension n is ***rationally smooth*** if for every $x \in X$, we have

$$H^p(X, X \setminus \{x\}, \mathbb{Q}) \simeq \begin{cases} 0 & p \neq 2n \\ \mathbb{Q} & p = 2n. \end{cases}$$

Smooth varieties are obviously rationally smooth. Here is a example that will be useful below.

Example 11.4.4. If $G \subseteq \mathrm{GL}(n, \mathbb{C})$ is a finite group, then $\mathbb{C}^n/G$ is rationally smooth by [**49**, Prop. A.1]. ◊

Before giving our second characterization, we need some terminology. The structure sheaf $\mathscr{O}_X$ of a variety X is a sheaf in the Zariski topology. When we switch to the classical topology, the corresponding sheaf is the sheaf of analytic functions on X, denoted $\mathscr{O}_X^{\mathrm{an}}$ (see [**131**, App. B] for a brief description). Then any classical open subset $U \subseteq X$ gives the *analytic variety* $(U, \mathscr{O}_U^{\mathrm{an}}) = (U, \mathscr{O}_X^{\mathrm{an}}|_U)$, and varieties X_1 and X_2 are *locally analytically equivalent* at $p_1 \in X_1$ and $p_2 \in X_2$ if there are classical neighborhoods $p_1 \in U_1 \subseteq X_1$ and $p_2 \in U_2 \subseteq X_2$ such that $U_1 \simeq U_2$ as analytic varieties, where the isomorphism take p_1 to p_2.

Definition 11.4.5. Let X be an irreducible variety of dimension n.

(a) A point $p \in X$ is a ***finite quotient singularity*** if there is a finite subgroup $G \subseteq \mathrm{GL}(n,\mathbb{C})$ such that $p \in X$ is locally analytically equivalent to $0 \in \mathbb{C}^n/G$.

(b) X is an ***orbifold*** or is ***quasismooth*** or has ***finite quotient singularities*** if every point of X is a finite quotient singularity.

(c) X has ***abelian finite quotient singularities*** if every point of X is a finite quotient singularity such that the finite group G in part (a) is abelian.

Here is a lemma whose proof is almost obvious.

Lemma 11.4.6. *Let $G \subseteq \mathrm{GL}(n,\mathbb{C})$ be a finite subgroup.*

(a) *$\mathbb{C}^n/G$ is an orbifold.*

(b) *If G is abelian, then $\mathbb{C}^n/G$ has abelian finite quotient singularities.*

Proof. Definition 11.4.5 guarantees that $0 \in \mathbb{C}^n/G$ is a finite quotient singularity. But what about the other points of $\mathbb{C}^n/G$? For $v \in \mathbb{C}^n$, let $G_v = \{g \in G \mid g \cdot v = v\}$ be its isotropy subgroup. We will show that $0 \in \mathbb{C}^n/G_v$ is locally analytically equivalent to $v \in \mathbb{C}^n/G$.

Note that $w \mapsto w + v$ is equivariant with respect to G_v and hence induces a local analytic equivalence from $0 \in \mathbb{C}^n/G_v$ to $v \in \mathbb{C}^n/G_v$. To complete the proof, we need to find a local analytic equivalence from $v \in \mathbb{C}^n/G_v$ to $v \in \mathbb{C}^n/G$.

Pick coset representatives so that $G = \bigcup_i g_i \cdot G_v$. The points $g_i \cdot v$ are distinct in $\mathbb{C}^n/G_v$, so there is a classical neighborhood $v \in U$ such that the $g_i \cdot U$ are disjoint, and replacing U with $\bigcap_{g \in G_v} g \cdot U$, we can assume that U is G_v-invariant. Then $V = \bigcup_i g_i \cdot U$ is a G-invariant neighborhood of v, and one sees easily that $U/G_v \simeq V/G$. This gives the desired local analytic equivalence. □

On $\mathbb{C}^n/G$, the coordinates $x_1, \dots, x_n$ on $\mathbb{C}^n$ become "local coordinates" on the quotient. The intuition is that anything G-invariant in the x_i descends to $\mathbb{C}^n/G$. For example, one can show that $\Gamma(\mathbb{C}^n/G, \widehat{\Omega}^p_{\mathbb{C}^n/G}) \simeq \Gamma(\mathbb{C}^n, \Omega^p_{\mathbb{C}^n})^G$, so that $\widehat{\Omega}^p_{\mathbb{C}^n/G}$ comes from G-invariant p-forms in the local coordinates.

Example 11.4.7. If $G \subseteq \mathrm{SL}(n,\mathbb{C})$, then $dx_1 \wedge \cdots \wedge dx_n$ is G-invariant. It follows that $\omega_{\mathbb{C}^n/G} \simeq \widehat{\Omega}^n_{\mathbb{C}^n/G}$ is trivial, i.e., $\mathbb{C}^n/G$ has trivial canonical divisor. This is why finite subgroups of $\mathrm{SL}(n,\mathbb{C})$ are so important. ◊

In the analytic category, the existence of local coordinates enables one to do analysis on orbifolds. See [**68**, App. A.3] for further discussion and references.

We now characterize simplicial toric varieties in terms of their singularities.

Theorem 11.4.8. *Given a normal toric variety X_Σ, the following are equivalent:*

(a) Σ *is simplicial.*

(b) X_Σ *has abelian finite quotient singularities.*

(c) X_Σ *is an orbifold.*

(d) X_Σ *is rationally smooth.*

Proof. First assume that Σ is simplicial and take $\sigma \in \Sigma$. If N' is the sublattice of $N \simeq \mathbb{Z}^n$ generated by the minimal generators of σ, then $U_{\sigma,N'} \simeq \mathbb{C}^k$, where $k = \dim\sigma$. Then Proposition 3.3.11 implies that there is a finite abelian group G with

$$U_\sigma = U_{\sigma,N} \simeq (U_{\sigma,N'} \times (\mathbb{C}^*)^{n-k})/G \simeq (\mathbb{C}^k \times (\mathbb{C}^*)^{n-k})/G \subseteq \mathbb{C}^n/G.$$

Since G is a subgroup of $(\mathbb{C}^*)^n \subseteq \mathrm{GL}(n,\mathbb{C})$, Lemma 11.4.6 implies that U_σ and hence X_Σ have abelian finite quotient singularities.

This proves (a) $\Rightarrow$ (b). Then (b) $\Rightarrow$ (c) is obvious, and (c) $\Rightarrow$ (d) follows from Example 11.4.4. It remains to prove (d) $\Rightarrow$ (a). For this, we can assume that $\sigma \subseteq N_\mathbb{R} \simeq \mathbb{R}^n$ is a cone such that U_σ is rationally smooth. A face $\tau \preceq \sigma$ gives the following objects:

- The sublattice $N_\tau = \mathrm{Span}(\tau) \cap N$ and quotient lattice $N(\tau) = N/N_\tau$.
- The exact sequence $1 \to T_{N_\tau} \to T_N \to T_{N(\tau)} \to 1$ of tori.
- The orbit closure $V(\tau) = \overline{O(\tau)} \subseteq U_\sigma$ of dimension $n - \dim\tau$.

Since $O(\tau) \simeq T_{N(\tau)}$ by Theorem 3.2.6, it follows easily that $V(\tau)$ is the fixed point set of the action of T_{N_τ} on U_σ (Exercise 11.4.1). In the standard notation for fixed-point sets, this means $U_\sigma^{T_{N_\tau}} = V(\tau)$. Hence

$$\dim U_\sigma^{T_{N_\tau}} = n - \dim\tau. \tag{11.4.1}$$

Now let $\tau_1,\ldots,\tau_r$ be the facets of σ. Then each $T_i = T_{N_{\tau_i}}$ is a subtorus of $T = T_{N_\sigma}$ of codimension 1 (Exercise 11.4.1). Furthermore, for any $p \in V(\sigma)$,

$$\begin{aligned} \dim_p U_\sigma - \dim_p U_\sigma^T &= n - (n - \dim\sigma) = \dim\sigma \\ \dim_p U_\sigma^{T_i} - \dim_p U_\sigma^T &= (n - \dim\tau_i) - (n - \dim\sigma) = 1 \end{aligned} \tag{11.4.2}$$

by (11.4.1). Since U_σ is rationally smooth at p, a result of Brion [**49**, Thm., p. 130] implies that

$$\dim_p U_\sigma - \dim_p U_\sigma^T = \sum_{T'} \left(\dim_p U_\sigma^{T'} - \dim_p U_\sigma^T\right),$$

where the sum is over all subtori $T' \subseteq T$ of codimension 1. Using (11.4.2), we obtain

$$\dim \sigma \geq \sum_{i=1}^{r} \left(\dim_p U_\sigma^{T_i} - \dim_p U_\sigma^T\right) = r.$$

For any cone, the number of facets is at least its dimension, with equality if and only if the cone is simplicial (Exercise 11.4.1). Hence the above inequality implies that σ is simplicial, and the proof is complete. □

We note one other important characterization of simplicial toric varieties: a normal toric variety is simplicial if and only if it is $\mathbb{Q}$-*factorial*, meaning that every Weil divisor is $\mathbb{Q}$-Cartier. We proved this in Proposition 4.2.7.

Terminal and Canonical Singularities. Given a normal variety X, recall from §11.2 that a resolution of singularities $f : X' \to X$ is crepant if X is $\mathbb{Q}$-Gorenstein and $K_{X'} = f^*K_X$. Having a crepant resolution is rare, so for most varieties, we have $K_{X'} \neq f^*K_X$ no matter which resolution we use. Measuring the difference between these divisors—their *discrepancy*—leads to some interesting classes of singularities.

In general, let K_X be a canonical divisor on a normal $\mathbb{Q}$-Gorenstein variety X. Given a proper birational morphism $f : X' \to X$ with X' smooth, one can find a canonical divisor $K_{X'}$ on X' such that

$$K_{X'} = f^*K_X + \sum_i a_i E_i, \tag{11.4.3}$$

where $a_i \in \mathbb{Q}$ and the E_i are the irreducible divisors lying in the exceptional locus $\mathrm{Exc}(f)$. This is proved in [**194**, Rem. 4-1-2].

Definition 11.4.9. Let X be a normal $\mathbb{Q}$-Gorenstein variety.

(a) X has ***terminal singularities*** if there is a proper birational $f : X' \to X$ with X' smooth, such that the coefficients a_i in (11.4.3) satisfy $a_i > 0$ for all i.

(b) X has ***canonical singularities*** if there is a proper birational $f : X' \to X$ with X' smooth, such that the coefficients a_i in (11.4.3) satisfy $a_i \geq 0$ for all i.

If X has terminal (resp. canonical) singularities, then the inequalities $a_i > 0$ (resp. $a_i \geq 0$) hold for *all* proper birational morphisms $f : X' \to X$ with X' smooth [**194**, Rem. 4-1-2 and 4-2-2]. Note that $f : X' \to X$ may fail to be a resolution of singularities since (for instance) some smooth points of X may get blown up by f. However, it is often useful to be able to work with these more general morphisms.

If X is not smooth and has terminal singularities, then X cannot have a SNC crepant resolution. Terminal singularities are very important in the minimal model program, since a minimal model is a $\mathbb{Q}$-factorial projective variety that has only terminal singularities. Canonical singularities are also relevant, since (roughly speaking) canonical singularities are the those that appear on canonical models of varieties of general type. See [**194**, Ch. 4] for a general discussion of terminal and canonical singularities. We will use terminal singularities in Chapter 15 when we discuss the toric minimal model program.

Let us apply these ideas in the toric context. Given a normal $\mathbb{Q}$-Gorenstein toric variety X_Σ, we can find a toric resolution of singularities $\phi : X_{\Sigma'} \to X_\Sigma$. Here, we have the toric canonical divisors $K_{X_{\Sigma'}}$ and K_{X_Σ}, and the relation (11.4.3) can be described explicitly as follows.

Lemma 11.4.10. *Let φ be the support function of the $\mathbb{Q}$-Cartier divisor K_{X_Σ} and let $\phi : X_{\Sigma'} \to X_\Sigma$ be the toric morphism coming from a refinement Σ' of Σ. Then:*

$$K_{X_{\Sigma'}} = \phi^* K_{X_\Sigma} + \sum_{\rho \in \Sigma'(1)\setminus\Sigma(1)} (\varphi(u_\rho) - 1) D_\rho.$$

Proof. The key observation is that K_{X_Σ} and its pullback $\phi^* K_{X_\Sigma}$ have the same support function. Thus $\phi^* K_{X_\Sigma} = -\sum_{\rho\in\Sigma'} \varphi(u_\rho) D_\rho$. Then the desired equation follows immediately since $\varphi(u_\rho) = 1$ for all $\rho \in \Sigma(1)$. $\square$

Here is an easy consequence of this lemma.

Proposition 11.4.11. *Let X_Σ be a normal Gorenstein toric variety. Then X_Σ has canonical singularities.*

Proof. The support function φ of K_{X_Σ} is integral with respect to N since X_Σ is Gorenstein. Now let Σ' be a smooth refinement of Σ. Given $u_\rho \in \Sigma'(1)$, we have:

- $\varphi(u_\rho) \in \mathbb{Z}$ since $u_\rho \in |\Sigma| \cap N$.
- $\varphi(u_\rho) > 0$ since u_ρ lies in a cone $\sigma \in \Sigma$ and $\varphi > 0$ on σ since φ takes the value 1 on every minimal generator of σ.

It follows that $\varphi(u_\rho) \geq 1$, and we are done by Lemma 11.4.10. $\square$

Our next result shows that we can determine when an affine toric variety U_σ has terminal or canonical singularities by studying the lattice points of the polytope

$$\Pi_\sigma = \mathrm{Conv}(0, u_\rho \mid \rho \in \sigma(1)).$$

Proposition 11.4.12. *Let U_σ and Π_σ be as above.*

(a) *The following are equivalent:*
  (i) *U_σ is $\mathbb{Q}$-Gorenstein*
  (ii) *There is $m \in M_\mathbb{Q}$ such that $\langle m, u_\rho\rangle = 1$ for all $\rho \in \sigma(1)$.*
  (iii) *Π_σ has a unique facet not containing the origin.*

(b) *If U_σ is $\mathbb{Q}$-Gorenstein, then:*

(i) *U_σ has terminal singularities $\Leftrightarrow$ the only lattice points of Π_σ are given by its vertices.*

(ii) *U_σ has canonical singularities $\Leftrightarrow$ the only nonzero lattice points of Π_σ lie in the facet not containing the origin.*

Proof. We begin with part (a). On an affine toric variety, the canonical divisor is $\mathbb{Q}$-Cartier if and only some multiple is the divisor of a character. This easily gives the equivalence of (i) and (ii). Furthermore, given $m \in M_\mathbb{Q}$ as in (ii), Π_σ lies on one side of the hyperplane defined by $\langle m,u\rangle = 1$. The face cut out by this hyperplane contains all of the u_ρ and hence is the unique facet of Π_σ not containing the origin, proving (iii). We leave (iii) $\Rightarrow$ (ii) as Exercise 11.4.2.

For part (b), first note that the support function of K_{U_σ} is $\varphi(u) = \langle m,u\rangle$ for $m \in M_\mathbb{Q}$ is as in the previous paragraph. Then

$$\Pi_\sigma = \{u \in \sigma \mid \langle m,u\rangle \le 1\}.$$

Now take any primitive vector $v \in \sigma \cap N$ different from the u_ρ. By taking a smooth refinement Σ of the star subdivision given by v, we get a proper birational morphism $\phi : X_\Sigma \to U_\sigma$ such that v is the minimal generator of $\rho = \mathrm{Cone}(v) \in \Sigma(1)$. By Lemma 11.4.10, the coefficient of D_ρ in $K_{X_\Sigma} - \phi^* K_{U_\sigma}$ is

$$\varphi(v) - 1 = \langle m,v\rangle - 1. \tag{11.4.4}$$

If U_σ has terminal singularities, then (11.4.4) is positive. Hence $\langle m,v\rangle > 1$, so that $v \notin \Pi_\sigma$. It follows that the only lattice points of Π_σ are its vertices. Conversely, suppose that Π_σ satisfies this condition and let Σ be a smooth refinement of σ. Given any $u_\rho \in \Sigma(1) \setminus \sigma(1)$, we have $u_\rho \notin \Pi_\sigma$, so that (11.4.4) is positive. Hence U_σ has terminal singularities.

The proof for canonical singularities is similar. The only difference is that the coefficient (11.4.4) is allowed to be zero, which happens only for lattice points lying on the facet of Π_σ not containing the origin. You will supply the details in Exercise 11.4.2. □

Example 11.4.13. Consider the lattice $N = \{(a,b,c) \in \mathbb{Z}^3 \mid a+b+c \equiv 0 \bmod 2\}$ and cone $\sigma = \mathrm{Cone}(e_1,e_2,e_3)$ from Example 11.2.7. There are two ways to see that U_σ has terminal singularities:

- The resolution $\phi : X_\Sigma \to U_\sigma$ constructed in Example 11.2.7 satisfies $K_{X_\Sigma} = \phi^* K_{U_\sigma} + \frac{1}{2} D_0$. Since $\frac{1}{2} > 0$ and D_0 is the exceptional locus, U_σ has terminal singularities by Definition 11.4.9.
- The minimal generators of σ with respect to N are $2e_1, 2e_2, 2e_3$, so that $\Pi_\sigma = \mathrm{Conv}(0, 2e_1, 2e_2, 2e_3)$. One easily checks that $\Pi_\sigma \cap N$ consists only of vertices, so that U_σ has terminal singularities by Proposition 11.4.12. ◊

In the surface case, terminal and canonical singularities are easy to understand.

Theorem 11.4.14. *Let X_Σ be a toric surface. Then:*

(a) *X_Σ has terminal singularities $\Leftrightarrow$ X_Σ is smooth.*

(b) *X_Σ has canonical singularities $\Leftrightarrow$ X_Σ is Gorenstein $\Leftrightarrow$ X_Σ has at worst rational double points.*

Proof. Part (a) follows because a 2-dimensional cone $\sigma = \mathrm{Cone}(u_1,u_2)$ such that $\Pi_\sigma \cap N = \mathrm{Conv}(0,u_1,u_2)\cap N = \{0,u_1,u_2\}$ is smooth (Exercise 8.3.4).

For part (b), Proposition 11.2.8 implies that being Gorenstein is equivalent to having rational double points, and if X_Σ is Gorenstein, then it has canonical singularities by Proposition 11.4.11. To complete the proof, we will show that an affine toric surface U_σ with canonical singularities has a rational double point. Let d,k be the parameters of σ, so that σ has minimal generators $e_2, de_1 - ke_2$. Then

$$\Pi_\sigma = \{u \in \sigma \mid \langle m,u\rangle \le 1\}, \quad m = \tfrac{k+1}{d}e_1 + e_2.$$

If $k < d-1$, then $\langle m, e_1\rangle = \frac{k+1}{d} < 1$, so that e_1 is an interior point of Π_σ. This is impossible since U_σ has canonical singularities. The only remaining case is $k = d-1$, which gives a rational double point by Example 10.1.5. $\square$

Reduction to Canonical and Terminal Singularities. We next discuss how any toric singularity can be made first canonical and then terminal. We begin with the canonical case.

Proposition 11.4.15. *A strongly convex cone $\sigma \subseteq N_\mathbb{R}$ has a canonically determined refinement Σ_{can} such that $X_{\Sigma_{\mathrm{can}}}$ has canonical singularities and the induced toric morphism $\phi : X_{\Sigma_{\mathrm{can}}} \to U_\sigma$ is projective.*

Proof. Let $\Theta_\sigma = \mathrm{Conv}(\sigma\cap(N\setminus\{0\}))$. In Exercise 11.4.3 you will show that Θ_σ is a lattice polyhedron with σ as recession cone. You will also show that taking cones over bounded faces of Θ_σ gives a fan Σ_{can} in $N_\mathbb{R}$ such that the morphism $\phi : X_{\Sigma_{\mathrm{can}}} \to U_\sigma$ is projective.

Take $\sigma' \in \Sigma_{\mathrm{can}}$ and suppose σ' is the cone over the bounded face F of Θ_σ. Then F is the unique facet of $\Pi_{\sigma'}$ not containing the origin, so $U_{\sigma'}$ is $\mathbb{Q}$-Gorenstein by Proposition 11.4.12. Since F is a facet of $\Theta_\sigma = \mathrm{Conv}(\sigma\cap(N\setminus\{0\}))$, the only nonzero lattice points of $\Pi_{\sigma'}$ lie in F, so that $U_{\sigma'}$ has canonical singularities by Proposition 11.4.12. $\square$

Example 11.4.16. When σ is 2-dimensional, the refinement Σ_{can} has an elegant description as follows. Recall from Proposition 8.2.9 that the canonical sheaf ω_{U_σ} comes from the ideal

$$\Gamma(U_\sigma, \omega_{U_\sigma}) = \bigoplus_{m\in \mathrm{Int}(\sigma^\vee)\cap M} \mathbb{C}\cdot\chi^m.$$

This ideal is integrally closed with associated polyhedron

$$P_\omega = \mathrm{Conv}(\mathrm{Int}(\sigma)\cap N)$$

as in Proposition 11.3.4. The reasoning in Example 11.3.6 shows that $X_{P_\omega} \to U_\sigma$ is the blowup of this ideal. We claim that

$$\Sigma_{\mathrm{can}} \text{ is the normal fan of } P_\omega,$$

i.e., $X_{\Sigma_{\mathrm{can}}} = X_{P_\omega}$. This will imply that X_{P_ω} has at worst rational double points by Proposition 11.4.14 and Theorem 11.4.15.

In dimension 2, a refinement of σ is determined uniquely by its ray generators. Thus it suffices to prove that

$$\text{the vertices of } \Theta_\sigma \text{ are the ray generators of the normal fan of } P_\omega. \tag{11.4.5}$$

We show this as follows. Recall that by Proposition 10.2.17, the ray generators of the normal fan of $P_{\mathfrak{m}} = \Theta_{\sigma^\vee} = \mathrm{Conv}(\sigma^\vee \cap (M \setminus \{0\}))$ are *almost* the vertices of Θ_σ. The difference arises from slight complications that can occur at the edges of σ, as can be seen by looking at Theorem 10.2.12 and Proposition 10.2.17.

Let us compare $P_{\mathfrak{m}}$ and P_ω. Since the latter uses only interior lattice points, these polyhedra differ along the edges of $\sigma^\vee$. To see how this affects their normal fans, consider Figure 11, which shows what can happen at an edge $\tau \preceq \sigma^\vee$. Let a_1 be the "lattice length" (number of lattice points -1) of the bounded edge of $P_{\mathfrak{m}}$ that

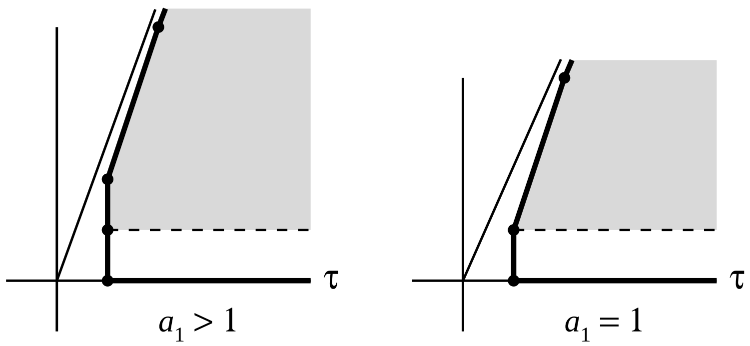


Figure 11. Two examples of $P_{\mathfrak{m}}$ (outlined in bold) and P_ω (shaded)

touches τ. As you can see, $P_{\mathfrak{m}}$ and P_ω have the same inner normal vectors near τ when $a_1 > 1$, but P_ω has one less inner normal when $a_1 = 1$. The same thing happens at the other edge of $\sigma^\vee$. Then (11.4.5) follows by comparing Theorem 10.2.12 and Proposition 10.2.17 to Figure 11 (Exercise 11.4.4). $\Diamond$

We should also mention that P_ω is the polyhedron denoted Θ° in [**218**, p. 28].

Before we can take a canonical singularity and make it terminal, we need to extend the definition of crepant, which was originally defined only for resolutions of singularities. Given normal varieties X, Y where Y is $\mathbb{Q}$-Gorenstein, a morphism $f : X \to Y$ is *crepant* if $K_X = f^*K_Y$. Then we have the following result.

Proposition 11.4.17. *If X_Σ has canonical singularities, then we can find a simplicial refinement Σ' of Σ such that $X_{\Sigma'}$ has terminal singularities and the induced toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ is projective and crepant. Furthermore, $X_{\Sigma'}$ is Gorenstein if and only if X_Σ is Gorenstein.*

Proof. Suppose that X_Σ has canonical singularities. To measure how far X_Σ is from being terminal, set

$$t(\Sigma) = \sum_{\sigma \in \Sigma_{\max}} |(\Pi_\sigma \cap N) \setminus \{\text{vertices}\}|.$$

By Proposition 11.4.12, X_Σ has terminal singularities if and only if $t(\Sigma) = 0$.

If $t(\Sigma) > 0$, pick a non-vertex $\nu \in \Pi_\sigma \cap N$ for some $\sigma \in \Sigma_{\max}$. Then the star subdivision $\Sigma' = \Sigma^*(\nu)$ satisfies the following:

- $X_{\Sigma'}$ has canonical singularities with $t(\Sigma') = t(\Sigma) - 1$.
- $X_{\Sigma'}$ is Gorenstein if and only if X_Σ is.
- $X_{\Sigma'} \to X_\Sigma$ is crepant.

You will verify these properties in Exercise 11.4.5. Figure 12 shows what happens for the affine toric variety of a cone $\sigma \subseteq \mathbb{R}^3$ with $t(\sigma) = 2$. Here, ν is one of the

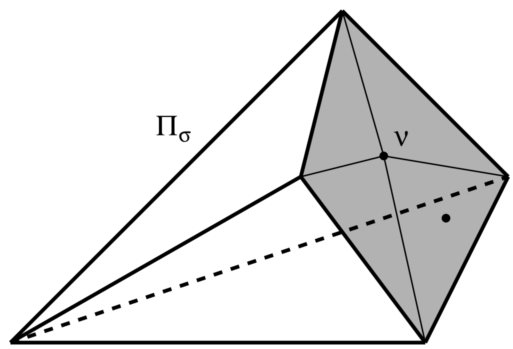


Figure 12. The polyedron Π_σ and the star subdivision of σ coming from ν

two non-vertex lattice points of Π_σ and the star subdivision Σ' has $t(\Sigma') = 1$.

By induction on $t(\Sigma)$, a sequence of these star subdivisions gives a refinement Σ' of Σ such that $X_{\Sigma'}$ has terminal singularities, and $X_{\Sigma'}$ is Gorenstein if and only if X_Σ is. If Σ' is simplicial, then we are done. If not, then the simplicial refinement constructed in Proposition 11.1.7 is easily seen to have the desired properties. $\square$

Terminal Singularities. Propositions 11.4.15 and 11.4.17 allow us to focus on toric varieties with terminal singularities. We begin with the affine case.

Gorenstein affine toric varieties with terminal singularities can be classified in terms of *empty lattice polytopes*, which are lattice polytopes whose only lattice points are their vertices. Here is the precise result.

Proposition 11.4.18. *Classifying Gorenstein affine toric varieties with terminal singularities that come from n-dimensional cones $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ is equivalent to classifying $(n-1)$-dimensional empty lattice polytopes.*

Proof. First assume that U_σ is Gorenstein with terminal singularities. Then there is $m \in M$ such that $\langle m, u_\rho \rangle = 1$ for all $\rho \in \sigma(1)$. Extend m to a basis of M and consider the dual basis of $N_{\mathbb{R}} \simeq \mathbb{R}^n$. Since $\dim \sigma = n$, it follows that $\sigma = \mathrm{Cone}(F)$, where $F \subseteq \mathbb{R}^{n-1} \times \{1\}$ is the facet of Π_σ not containing the origin. Then F is a lattice polytope of dimension $n-1$ and is empty since U_σ has terminal singularities (Proposition 11.4.12).

Conversely, given an empty lattice polytope $P \subseteq \mathbb{R}^{n-1}$ of dimension $n-1$, we get the n-dimensional cone $\sigma = \mathrm{Cone}(P \times \{1\}) \subseteq \mathbb{R}^n$. In Exercise 11.4.6 you will show that U_σ is Gorenstein and has terminal singularities since P is empty. $\square$

Proposition 11.4.18 has some nice consequences in the 3-dimensional case.

Proposition 11.4.19. *Let X_Σ be a* 3*-dimensional Gorenstein toric variety. Then:*

(a) *If X_Σ is simplicial, then X_Σ has terminal singularities $\Leftrightarrow$ X_Σ is smooth.*

(b) *X_Σ has a resolution of singularities $\phi : X_{\Sigma'} \to X_\Sigma$ such that ϕ is projective and crepant.*

Proof. For part (a), assume that X_Σ is simplicial with terminal singularities and take $\sigma \in \Sigma(3)$. Since 2-dimensional empty lattice simplices are lattice equivalent to the standard 2-simplex (Exercise 8.3.4), Proposition 11.4.18 implies that σ is smooth. When $\sigma \in \Sigma(2)$, Proposition 11.4.12 implies that Π_σ is a 2-dimensional empty lattice simplex, which as already noted is smooth. Hence X_Σ is smooth.

For part (b), recall that X_Σ has canonical singularities by Proposition 11.4.11. Then Proposition 11.4.17 gives a simplicial refinement Σ' such that $\phi : X_{\Sigma'} \to X_\Sigma$ is projective and crepant, $X_{\Sigma'}$ has terminal singularities, and $X_{\Sigma'}$ is Gorenstein since X_Σ is. Then $X_{\Sigma'}$ is smooth by part (a). One can also check that the refinement constructed in Proposition 11.4.17 does not affect the smooth cones of Σ (Exercise 11.4.7). Hence ϕ is the desired resolution of singularities. $\square$

Corollary 11.4.20. *The toric variety of a* 3*-dimensional reflexive polytope has a projective crepant resolution.*

Proof. This follows immediately from Proposition 11.4.19 since the toric variety of a reflexive polytope is a Gorenstein Fano variety by Theorem 8.3.4. $\square$

In the 3-dimensional case, there are also results about terminal singularities that do not assume that the variety is Gorenstein. To state our result, we define the *index* of a $\mathbb{Q}$-Gorenstein variety X to be the smallest positive integer r such that rK_X is Cartier. The X is Gorenstein if and only if it is $\mathbb{Q}$-Gorenstein of index 1.

Then we can classify all 3-dimensional cones that give terminal singularities as follows.

Theorem 11.4.21. *Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^3$ be a 3-dimensional cone whose affine toric variety U_σ has terminal singularities.*

(a) *If σ is simplicial, then N has a basis u_1, u_2, u_3 such that*

$$\sigma = \mathrm{Cone}(u_1, u_2, u_1 + pu_2 + qu_3),$$

where $0 \leq p < q$ are relatively prime. Furthermore, U_σ is $\mathbb{Q}$-Gorenstein of index q, and

$$U_\sigma \simeq \mathbb{C}^3/\mu_q,$$

where $\mu_q = \{\zeta \in \mathbb{C}^* \mid \zeta^q = 1\}$ acts on $\mathbb{C}^3$ via $\zeta \cdot (x,y,z) = (\zeta^{-1}x, \zeta^{-p}y, \zeta z)$.

(b) *If σ is not simplicial, then N has a basis u_1, u_2, u_3 such that*

$$\sigma = \mathrm{Cone}(u_1, u_2, u_1 + u_3, u_2 + u_3).$$

Furthermore, U_σ is Gorenstein and $U_\sigma \simeq \mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$.

Proof. For part (a), first note that Π_σ is a 3-dimensional empty lattice simplex. By the terminal lemma [**218**, §1.6], N has a basis such that the vertices of Π_σ are $0, u_1, u_2, u_1 + pu_2 + qu_3$, where $0 \leq p < q$ are relatively prime. This gives the desired formula for σ, and the quotient construction from Chapter 5 easily gives the representation $U_\sigma \simeq \mathbb{C}^3/\mu_q$ in the statement of the theorem (Exercise 11.4.8). To compute the index, let $m = (1, 1, -p/q)$ relative to the dual basis of M. Then

$$\langle m, u_1 \rangle = \langle m, u_2 \rangle = \langle m, u_3 \rangle = 1.$$

Since $\gcd(p,q) = 1$, qm is the smallest positive multiple of m that is a lattice point, hence the index is q. See [**103**, Thm. 2.2] for part (b). $\square$

In [**218**, §1.6], Oda discusses the terminal lemma, with references. In the literature, the action of μ_q on $\mathbb{C}^3$ is often written differently. Given the action $\zeta \cdot (x,y,z) = (\zeta^{-1}x, \zeta^{-p}y, \zeta z)$ as in the theorem, let $0 \leq a < q$ be the multiplicative inverse of $-p$ modulo q. Using the automorphism of μ_q given by $\zeta \mapsto \zeta^a$ and changing coordinates in $\mathbb{C}^3$, the action becomes $\zeta \cdot (x,y,z) = (\zeta^a x, \zeta^{-a} y, \zeta z)$. See [**218**, §1.6] for references to the classification of 3-dimensional cones whose toric varieties have canonical singularities. Also, [**13**] and the references therein discuss the classification of empty lattice simplices in dimensions 3 and 4.

Part (b) of Theorem 11.4.21 is especially nice, since it shows that one of our favorite examples, $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$, is even more interesting than we realized.

The singularities of a normal variety have codimension ≥ 2. In Exercise 11.4.9 you will prove that the following result, which tells us that the codimension is higher when the singularities are terminal.

Proposition 11.4.22. *Assume that X_Σ has only terminal singularities.*

(a) *The singular locus $(X_\Sigma)_{\mathrm{sing}}$ of X_Σ has codimension* $\mathrm{codim}(X_\Sigma)_{\mathrm{sing}} \geq 3$.

(b) *If in addition X_Σ is Gorenstein and simplicial, then* $\mathrm{codim}(X_\Sigma)_{\mathrm{sing}} \geq 4$. □

At the end of §8.3, we noted that in dimension 3, there are 4319 isomorphism classes of Fano toric varieties, corresponding to the 4319 classes of 3-dimensional reflexive polytopes. They all have canonical singularities by Proposition 11.4.11 since Fano varieties are Gorenstein. But only 100 of these have terminal singularities by [**165**], and as noted in §8.3, only 18 are smooth. This indicates the special nature of terminal singularities. See also [**166**].

Log Singularities. Besides terminal and canonical singularities, there are other classes of singularities that are important for the minimal model program. We will consider two, *log canonical* and *Kawamata log terminal*. The latter is usually abbreviated *klt*. These singularities are defined for a *pair* (X,D), where X is a normal variety and D is a $\mathbb{Q}$-divisor with coefficients in the interval $[0,1]$. See [**179**, p. 98] or [**194**, Ch. 11] for a nice discussion of why pairs are useful. A more recent reference is [**30**]. Pairs are sometimes called *log pairs*.

Let (X,D), $D = \sum_i d_i D_i$, be a pair. Also fix a proper birational morphism $f : X' \to X$ such that X' is smooth and $\mathrm{Exc}(f) \cup \bigcup_i f^{-1}(D_i)$ is a SNC divisor. Note that f may fail be a log resolution of D. We define the *birational transform D'* of D to be the divisor on X' defined by $D' = \sum_i d_i D_i'$, where $D_i' = \overline{f^{-1}(D_i \cap U)}$ and $U \subseteq X$ is the largest open subset where f^{-1} is a morphism (see also §15.4).

Let (X,D) and $f : X' \to X$ be as above, and assume that $K_X + D$ is $\mathbb{Q}$-Cartier. By [**179**, Sec. 2.3], we can pick a canonical divisor $K_{X'}$ such that

$$K_{X'} + D' = f^*(K_X + D) + \sum_i a_i E_i, \tag{11.4.6}$$

where $a_i \in \mathbb{Q}$ and the E_i are the irreducible divisors lying in the exceptional locus. This is the log version of (11.4.3).

Definition 11.4.23. Let (X,D) be a pair such that $K_X + D$ is $\mathbb{Q}$-Cartier, and write $D = \sum_i d_i D_i$ with $d_i \in [0,1] \cap \mathbb{Q}$.

(a) (X,D) has ***log canonical singularities*** if there is $f : X' \to X$ as above such that the coefficients a_i in (11.4.6) satisfy $a_i \geq -1$ for all i.

(b) (X,D) has ***klt singularities*** if $d_i \in [0,1)$ for all i and there is $f : X' \to X$ as above such that the coefficients a_i in (11.4.6) satisfy $a_i > -1$ for all i.

One can show that this definition is independent of the morphism $f : X' \to X$. In practice, one often says "(X,D) is log canonical" instead of "(X,D) has log canonical singularities," and "(X,D) is klt" has a similar meaning.

In the toric case, we consider pairs (X_Σ, D) where D is torus-invariant.

Proposition 11.4.24. *Let X_Σ be a normal toric variety, and let $D = \sum_\rho d_\rho D_\rho$, where $d_\rho \in [0,1] \cap \mathbb{Q}$. If $K_{X_\Sigma} + D$ is $\mathbb{Q}$-Cartier, then:*

(a) *(X_Σ, D) is log canonical.*

(b) *If in addition $d_\rho \in [0,1)$ for all $\rho \in \Sigma(1)$, then (X_Σ, D) is klt.*

Proof. Let φ be the support function of $K_{X_\Sigma} + D$, so that $\varphi(u_\rho) = 1 - d_\rho$ for all $\rho \in \Sigma(1)$. Let $\phi : X_{\Sigma'} \to X_\Sigma$ be a toric log resolution. For the rest of the proof, D_ρ will denote the divisor on $X_{\Sigma'}$ corresponding to $\rho \in \Sigma'(1)$. Thus the birational transform of D is $D' = \sum_{\rho \in \Sigma(1)} d_\rho D_\rho$.

To simplify notation, let $\Sigma'(1) = A \cup B$, where $A = \Sigma(1)$ and $B = \Sigma'(1) \setminus A$. Since φ is the support function of $\phi^*(K_{X_\Sigma} + D)$, we obtain

$$\begin{aligned}
\phi^*(K_{X_\Sigma} + D) &= -\sum_{\rho \in A \cup B} \varphi(u_\rho) D_\rho = \sum_{\rho \in A} (-1 + d_\rho) D_\rho - \sum_{\rho \in B} \varphi(u_\rho) D_\rho \\
&= -\sum_{\rho \in A \cup B} D_\rho + \sum_{\rho \in A} d_\rho D_\rho + \sum_{\rho \in B} (1 - \varphi(u_\rho)) D_\rho \\
&= \quad K_{X_{\Sigma'}} \quad + \quad D' \quad + \sum_{\rho \in B} (1 - \varphi(u_\rho)) D_\rho.
\end{aligned}$$

Hence

$$K_{X_{\Sigma'}} + D' = \phi^*(K_{X_\Sigma} + D) + \sum_{\rho \in B} (\varphi(u_\rho) - 1) D_\rho. \tag{11.4.7}$$

This is the log version of Lemma 11.4.10.

To analyze the coefficients, take $\rho \in B = \Sigma'(1) \setminus \Sigma(1)$. Since Σ' refines Σ, we have $u_\rho \in \sigma$ for some $\sigma \in \Sigma$. Thus $u_\rho = \sum_{\gamma \in \sigma(1)} \lambda_\gamma u_\gamma$, where $\lambda_\gamma \geq 0$. Hence

$$\varphi(u_\rho) = \sum_{\gamma \in \sigma(1)} \lambda_\gamma \varphi(u_\gamma) = \sum_{\gamma \in \sigma(1)} \lambda_\gamma (1 - d_\gamma). \tag{11.4.8}$$

We have $d_\gamma \leq 1$ by assumption, so that $1 - d_\gamma \geq 0$. It follows that $\varphi(u_\rho) \geq 0$, and thus the coefficients $\varphi(u_\rho) - 1$ in (11.4.7) are all ≥ -1. This proves that (X_Σ, D) is log canonical.

Now assume in addition that $d_\gamma < 1$ for all $\gamma \in \Sigma(1)$. Then $1 - d_\gamma > 0$ in (11.4.8), and since the λ_γ are not all 0 ($u_\rho \neq 0$), we conclude that $\varphi(u_\rho) > 0$ for all $\rho \in B$. This shows that the coefficients $\varphi(u_\rho) - 1$ in (11.4.7) are all > -1. Hence (X_Σ, D) is klt, as claimed. □

Here is an easy corollary that you will prove in Exercise 11.4.10.

Corollary 11.4.25. *Let X_Σ be a normal toric variety. Then:*

(a) *$(X_\Sigma, \sum_\rho D_\rho)$ is log canonical.*

(b) *If X_Σ is $\mathbb{Q}$-Gorenstein, then $(X_\Sigma, 0)$ is klt.* □

Here is an example based on an observation of Chen and Shokurov [**62**].

Example 11.4.26. Recall from §8.3 that a projective normal variety X is Gorenstein Fano if $-K_X$ is ample. The log version says that a normal projective variety X is of *Fano type* if there is $\mathbb{Q}$-divisor D such that $-(K_X+D)$ is $\mathbb{Q}$-ample (meaning that some positive multiple is Cartier and ample) and (X,D) is klt. Let us show that every projective toric variety X_Σ is of Fano type.

Since X_Σ is projective, there is a lattice polytope P that gives an ample divisor D_P on X_Σ. Multiplying P by a suitably large integer, we can assume that P has an interior lattice point m, and translating by m (which gives a linearly equivalent divisor), we can assume that 0 is an interior point. Since $D_P = \sum_\rho a_\rho D_\rho$ and

$$P = \{m \in M_\mathbb{R} \mid \langle m, u_\rho\rangle \geq -a_\rho\},$$

it follows that $a_\rho > 0$ for all ρ. Then set

$$D = \sum_\rho (1-\varepsilon a_\rho) D_\rho,$$

where $\varepsilon \in \mathbb{Q}$ is positive and satisfies $\varepsilon a_\rho < 1$ for all ρ. Then the coefficients of D lie in $[0,1)$ for all ρ.

By construction, D satisfies $K_{X_\Sigma} + D = -\varepsilon D_P$, so that $K_{X_\Sigma}+D$ is $\mathbb{Q}$-Cartier. Then (X_Σ, D) is klt by Proposition 11.4.24. It follows that X_Σ is of Fano type since $-(K_{X_\Sigma}+D) = \varepsilon D_P$ is $\mathbb{Q}$-ample. $\Diamond$

In the minimal model program, many results assume that (X,D) is a klt pair. See [**30**, Sec. 4.3] for some examples. In Chapter 15 we will see that klt pairs arise naturally in the toric minimal model program.

Exercises for §11.4.

11.4.1. In this exercise you will supply some details omitted from the proof of (d) $\Rightarrow$ (a) in Theorem 11.4.8.

(a) Prove that $V(\tau)$ is the unique fixed point of the action of T_{N_τ} on U_σ.

(b) Explain why T_{N_τ} is a subtorus of T_N of codimension 1 when τ is a facet of σ.

(c) Prove that σ is simplicial if and only if its dimension equals the number of its facets.

11.4.2. This exercise is concerned with the proof of Proposition 11.4.12.

(a) Prove (iii) $\Rightarrow$ (ii) in part (a).

(b) Prove the characterization of canonical singularities given in part (b).

11.4.3. As in Proposition 11.4.15, let $\Theta_\sigma = \mathrm{Conv}(\sigma \cap N \setminus \{0\})$.

(a) Prove that Θ_σ is a lattice polyhedron with σ as recession cone.

(b) Prove that taking cones over bounded faces of Θ_σ gives a fan Σ_{can} refining σ.

(c) Prove that $\Sigma_{\mathrm{can}} \to U_\sigma$ is projective. Hint: Suppose $\sigma \in \Sigma_{\mathrm{can}}$ comes from a facet of Θ_σ. If the facet is defined by $\langle m, -\rangle = a$, then consider the support function φ on $\Theta_\sigma = |\Sigma_{\mathrm{can}}|$ whose restriction to σ is given by $\frac{1}{a}m$.

11.4.4. Complete the proof of (11.4.5) sketched in Example 11.4.16.

11.4.5. Prove the properties of $\Sigma' = \Sigma^*(\nu)$ stated in the three bullets in the proof of Proposition 11.4.17.

11.4.6. In the proof of Proposition 11.4.18, we saw that an elementary lattice polytope P gives a cone σ of one dimension higher. Prove that U_σ is Gorenstein with terminal singularities. Hint: P is the facet of Π_σ lying at "height 1."

11.4.7. Let $\phi : X_{\Sigma'} \to X_\Sigma$ be the "terminalization" constructed in Proposition 11.4.17. Prove that all terminal simplices of Σ lie in Σ' and explain why this implies that ϕ is an isomorphism above the $\mathbb{Q}$-factorial terminal locus of X_Σ.

11.4.8. Complete the proof of Theorem 11.4.21 by using the methods of Chapter 5 to describe the quotient representation of U_σ. Hint: Lemma 5.1.1 will be useful.

11.4.9. Prove Proposition 11.4.22. Hint: Study the cones $\sigma \in \Sigma$ of dimension ≤ 3 using the methods of the proof of Proposition 11.4.19.

11.4.10. Prove Corollary 11.4.25.

11.4.11. Here is a partial converse to Theorem 11.4.17. Assume that X_Σ is $\mathbb{Q}$-Gorenstein. Prove that if Σ has a refinement Σ' such that $X_{\Sigma'}$ has terminal singularities and $X_{\Sigma'} \to X_\Sigma$ is crepant, then X_Σ has canonical singularities.

11.4.12. Explain how Propositions 11.4.15 and 11.4.17 give the minimal resolution of singularities of a toric surface X_Σ. Draw some pictures to illustrate what is happening.

11.4.13. Let X_Σ be a $\mathbb{Q}$-Gorenstein toric variety with terminal singularities.

(a) Give an example where X_Σ has a crepant resolution of singularities. Hint: Try one of our favorite examples.

(b) If X_Σ is simplicial but not smooth, then prove that all resolutions of singularities are not crepant.

(c) If X_Σ is not smooth, then prove that all SNC resolutions of singularities are not crepant.

Chapter 12

The Topology of Toric Varieties

In this chapter, we will study some topological invariants of a toric variety X, always for the classical topology. The fundamental group $\pi_1(X)$ is studied in §12.1, and §12.2 addresses the moment map and alternative topological models of X. Methods for computing the singular cohomology groups $H^k(X,\mathbb{Z})$ are discussed in §12.3. A complete description of the cup product ring structure on $H^\bullet(X,\mathbb{Q}) = \bigoplus_k H^k(X,\mathbb{Q})$ for simplicial complete X is developed in §12.4 using the equivariant cohomology of X for the action of T_N. The chapter concludes with §12.5, where we consider the Chow ring and intersection cohomology.

Our goal is to understand the information these invariants provide about the topology and geometry of toric varieties and their applications to polytopes. We will freely use the definitions and various properties of homotopy, homology, and cohomology groups, referring to [**135**], [**210**], and [**255**] as our primary references.

§12.1. The Fundamental Group

Topology of Tori. The most basic toric varieties are the tori T_N, and their topology is correspondingly simple to understand. A choice of basis in N determines a homeomorphism and isomorphism of groups $T_N \simeq (\mathbb{C}^*)^n$. Let S^1 denote the unit circle in $\mathbb{C}^*$. The usual polar coordinate system in each factor gives a homeomorphism and isomorphism of multiplicative groups

$$(\mathbb{C}^*)^n \simeq (\mathbb{R}_{>0})^n \times (S^1)^n.$$

In the following, we will use the notation S_N for the compact real torus in T_N.

Proposition 12.1.1. *Let N be a lattice of rank n. Then:*

(a) *S_N is a deformation retract of $T_N \simeq (\mathbb{C}^*)^n$.*

(b) *There is an isomorphism*

$$\pi_1(T_N) \simeq \mathbb{Z}^n.$$

Proof. Since $\mathbb{R}_{>0}$ is contractible to the point $1 \in \mathbb{R}_{>0}$, we easily construct the deformation retraction. Part (b) follows from the standard facts $\pi_1(S^1) \simeq \mathbb{Z}$ and $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$. $\square$

There is also a more intrinsic way to understand this isomorphism. Recall that each $u \in N$ defines a one-parameter subgroup $\lambda^u : \mathbb{C}^* \to T_N$. Restricting λ^u to $S^1 \subseteq \mathbb{C}^*$ gives a closed path in T_N, whose homotopy class represents an element of $\pi_1(T_N)$. In Exercise 12.1.1, you will show that the resulting map $N \to \pi_1(T_N)$ is an isomorphism. Hence we will usually use this intrinsic form of Proposition 12.1.1:

$$\pi_1(T_N) \simeq N. \tag{12.1.1}$$

To determine the fundamental group of other toric varieties, we will use the fact that the algebraic condition of normality has a very important, though quite nontrivial, topological consequence.

Topological Implications of Normality. To see the pattern, we consider two toric examples we have encountered before.

Example 12.1.2. In Example 3.A.2, we studied the nonnormal affine toric surface

$$Y_{\mathscr{A}} = \mathbf{V}(y^2 - x^2z) \subseteq \mathbb{C}^3$$

with $\mathscr{A} = \{e_1, e_1 + e_2, 2e_2\} \subseteq \mathbb{Z}^2$. Note that the z-axis is clearly contained in $Y_{\mathscr{A}}$. In Exercise 12.1.2 you will show that the z-axis is precisely the singular locus of $Y_{\mathscr{A}}$. Letting $p = (0,0,z)$ with $z \neq 0$, you will also show that if U is any sufficiently

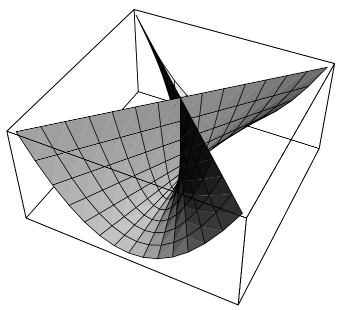


Figure 1. The surface $\mathbf{V}(y^2 - x^2z)$

small open neighborhood of p in $Y_{\mathscr{A}}$, then $U \setminus (U \cap \mathrm{Sing}(Y_{\mathscr{A}}))$ has two connected components, as suggested by the picture of the real points of the surface in Figure 1. These are called the *branches* of $Y_{\mathscr{A}}$ at p. $\Diamond$

Example 12.1.3. Consider the quadric $\widehat{C}_2 = \mathbf{V}(xz - y^2) \subseteq \mathbb{C}^3$, the normal affine toric surface from the cone $\sigma = \mathrm{Cone}(e_2, 2e_1 - e_2)$. The only singularity of this surface is the point $p = (0,0,0)$. From our standard parametrization, $\widehat{C}_2$ is the image of the map

$$\begin{aligned} \phi : \mathbb{C}^2 &\longrightarrow \mathbb{C}^3 \\ (t_1, t_2) &\longmapsto (t_1, t_1 t_2, t_1 t_2^2). \end{aligned}$$

The singular point p is the image of the line $\mathbf{V}(t_1)$ in $\mathbb{C}^2$. If we remove that line, then $\mathbb{C}^2 \setminus \mathbf{V}(t_1)$ is still connected, so

$$\widehat{C}_2 \setminus \{p\} = \phi(\mathbb{C}^2 \setminus \mathbf{V}(t_1))$$

is the continuous image of a connected set, hence connected. The same will be true if we intersect $\widehat{C}_2 \setminus \{p\}$ with any connected open neighborhood U of p in $\mathbb{C}^3$. $\Diamond$

These examples illustrate a general phenomenon. While a nonnormal variety can have more than one branch at a singular point, at each point x of a normal variety X, singular or nonsingular, X is *locally irreducible* or *unibranch* in the following sense.

Proposition 12.1.4. *Let X be a normal variety and take $x \in X$. Then there is a basis $\{V_\alpha \mid \alpha \in A\}$ of open neighborhoods of x in X such that $V_\alpha \setminus (V_\alpha \cap \mathrm{Sing}(X))$ is connected for all α.* $\square$

This is a topological version of the fundamental result known as *Zariski's main theorem*. We refer to [**208**, III.9] for algebraic versions of the result and their relation to Proposition 12.1.4.

Comparing $\pi_1(X_\Sigma)$ ***and*** $\pi_1(T_N)$**.** Let Σ be a fan in $N_\mathbb{R}$ and consider the normal toric variety X_Σ and the inclusion $i : T_N \hookrightarrow X_\Sigma$. There is a general fact that applies to the induced map $i_* : \pi_1(T_N) \to \pi_1(X_\Sigma)$ in this situation (see [**109**]).

Theorem 12.1.5. *Let X be a normal variety and let $i : U \hookrightarrow X$ be the inclusion of an open subvariety. Then the induced map $i_* : \pi_1(U) \longrightarrow \pi_1(X)$ is surjective.*

The proof of the theorem will consist of the following two lemmas. We recall that any suitably nice topological space X has a *universal covering space*, a simply-connected space $\widetilde{X}$ with an (unramified) covering map $p : \widetilde{X} \to X$. The universal cover of a variety over $\mathbb{C}$ is not always a variety, but it inherits the structure of a complex analytic space from X, which will be sufficient for our purposes.

Lemma 12.1.6. *If X is a normal variety and $i : U \hookrightarrow X$ is the inclusion of an open subvariety U, then $U \times_X \widetilde{X}$ is path-connected.*

Proof. Let $U = X \setminus Y$, where Y is Zariski closed in X. Then U is connected and path-connected in the classical topology. By Proposition 12.1.4 and the construction of the universal covering space, $\widetilde{X}$ is also locally irreducible. Since it is also connected, it is irreducible as an analytic space. Hence

$$\widetilde{X} \setminus p^{-1}(Y) = p^{-1}(X \setminus Y) = p^{-1}(U)$$

is also connected and path-connected in the classical topology. The analytic space

$$U \times_X \widetilde{X} = \{(u,\widetilde{x}) \mid p(\widetilde{x}) = u\}$$

is essentially the graph of p restricted to $p^{-1}(U)$. Since p is continuous, this is also connected and path-connected. □

Lemma 12.1.7. *Let X,Z be topological spaces and assume X has a universal covering space $p : \widetilde{X} \to X$. Let $f : Z \to X$ be continuous. If $Z \times_X \widetilde{X}$ is path-connected, then $f_* : \pi_1(Z) \to \pi_1(X)$ is a surjection.*

Proof. We must show that for each homotopy class $[\gamma] \in \pi_1(X)$, there exists a closed curve λ in Z such that $f_*([\lambda]) = [\gamma]$, or equivalently that $f \circ \lambda$ and γ are homotopic in X. Pick a base point x_0 on X with $x_0 = f(z_0)$. Let $\widetilde{x}_0 \in p^{-1}(x_0)$ in $\widetilde{X}$, and lift γ to $\widetilde{\gamma}$ in $\widetilde{X}$ starting from the point $\widetilde{x}_0$. The final point of $\widetilde{\gamma}$ will be some $\widetilde{x}_1 \in p^{-1}(x_0)$. Hence both $(z_0,\widetilde{x}_0)$ and $(z_0,\widetilde{x}_1)$ are points in

$$Z \times_X \widetilde{X} = \{(z,\tilde{x}) \in Z \times \widetilde{X} \mid f(z) = p(\widetilde{x})\}.$$

By hypothesis, we can find a path $\delta : [0,1] \to Z \times_X \widetilde{X}$ with $\delta(0) = (z_0,\widetilde{x}_0)$ and $\delta(1) = (z_0,\widetilde{x}_1)$. Let p_1, p_2 be the projections from $Z \times \widetilde{X}$ to the two factors. The projection $p_2 \circ \delta$ is homotopic to $\widetilde{\gamma}$ since $\widetilde{X}$ is simply-connected. Hence $p \circ p_2 \circ \delta$ is homotopic to γ. Moreover $p_1 \circ \delta = \lambda$ is a loop in Z such that $f \circ \lambda = p \circ p_2 \circ \delta$. This shows that f_* is surjective. □

Because of Theorem 12.1.5, to determine $\pi_1(X_\Sigma)$ we must determine the kernel of the homomorphism $i_* : \pi_1(T_N) \to \pi_1(X_\Sigma)$, and then

$$\pi_1(X_\Sigma) \simeq \pi_1(T_N)/\ker(i_*) \simeq N/\ker(i_*).$$

In particular, the fundamental group of a normal toric variety is always a finitely-generated abelian group. Here is one case where the kernel is easy to see.

Proposition 12.1.8. *Let $\rho = \mathrm{Cone}(u_\rho)$ be a 1-dimensional cone with u_ρ the primitive ray generator in N. Then $\ker(i_* : \pi_1(T_N) \to \pi_1(U_\rho)) = N_\rho = \mathbb{Z}u_\rho$.*

Proof. We use (12.1.1) and view i_* as a map from N to $\pi_1(U_\rho)$. Applying Proposition 3.2.2 to the cone ρ, we see that for all $u \in N$,

$$u \in \rho \Longleftrightarrow \lim_{z \to 0} \lambda^u(z) \text{ exists in } U_\rho. \tag{12.1.2}$$

Since $u_\rho \in \rho$, the limit point $\lim_{z\to 0} \lambda^{u_\rho}(z) = \gamma_\rho$ is in U_ρ. Restrict z to $S^1 \subseteq \mathbb{C}^*$ and take $t \in [0,1]$. Then the map

$$\eta(t,z) : [0,1] \times S^1 \longrightarrow X_\Sigma$$
$$(t,z) \longmapsto \begin{cases} \lambda^{u_\rho}((1-t)z) & 0 \le t < 1 \\ \gamma_\rho & t = 1 \end{cases}$$

is continuous. Moreover η gives a homotopy from a loop representing the homotopy class on T_N corresponding to $u_\rho \in N$ to a constant path at the distinguished point $\gamma_\rho \in U_\rho$, where γ_ρ is defined on page 116. This shows $N_\rho \subseteq \ker(i_*)$. The opposite inclusion follows since if $\ker(i_*)$ contained any $u \in N$ not in N_ρ, then

$$\operatorname{rank} \pi_1(U_\rho) = \operatorname{rank}(N/\ker(i_*))$$

would be $n-2$ or smaller. But $U_\rho \cong (\mathbb{C}^*)^{n-1} \times \mathbb{C}$, so $\pi_1(U_\rho) \simeq \mathbb{Z}^{n-1}$. □

The Affine Case. The idea behind the homotopy constructed in the proof of the last proposition can be generalized to prove the following fact.

Proposition 12.1.9. *Let σ be a strongly convex rational polyhedral cone in $N_\mathbb{R}$.*

(a) *The torus orbit $O(\sigma)$ is a T_N-equivariant deformation retract of U_σ.*

(b) *There are isomorphisms*

$$\pi_1(U_\sigma) \simeq \pi_1(T_{N(\sigma)}) \simeq N(\sigma),$$

where $N(\sigma) = N/N_\sigma$ as usual.

Proof. Part (b) follows immediately from part (a) because $O(\sigma)$ is equal to the torus $T_{N(\sigma)}$ by Lemma 3.2.5. To prove part (a), view the points $x \in U_\sigma$ as semigroup homomorphisms $x : \mathsf{S}_\sigma \to \mathbb{C}$, where $\mathsf{S}_\sigma = \sigma^\vee \cap M$. Choose any lattice point u in $\operatorname{Relint}(\sigma) \cap N$ and define the map

$$\eta : U_\sigma \times [0,1] \longrightarrow U_\sigma$$

by

$$\eta(x,t)(m) = \begin{cases} \chi^m(\lambda^u(1-t))x(m) & \text{if } 0 \le t < 1 \\ \gamma_\sigma(m)x(m) & \text{if } t = 1, \end{cases}$$

where γ_σ is the distinguished point corresponding to σ.

For all (x,t), it is not hard to see that $\eta(x,t)$ is a semigroup homomorphism from S_σ to $\mathbb{C}$, hence a point of U_σ (Exercise 12.1.3). Moreover, $\eta(x,t)$ is continuous in x and t.

We have $\chi^m(\lambda^u(1-t)) = (1-t)^{\langle m,u\rangle}$, so $\eta(x,0)$ is the identity map on U_σ. Moreover if $m \in \sigma^\perp \cap M$, $\chi^m(\lambda^u(1-t)) = 1$ for all $0 \le t < 1$. From Lemma 3.2.5,

$$O(\sigma) = \{\gamma : \mathsf{S}_\sigma \to \mathbb{C} \mid \gamma(m) \neq 0 \Leftrightarrow m \in \sigma^\perp \cap M\}.$$

Chasing the definitions, one sees that $\eta(x,t) = x$ for all t when $x \in O(\sigma)$.

Finally,

$$\lim_{t\to 1}(1-t)^{\langle m,u\rangle} = \begin{cases} 1 & \text{if } m \in \mathsf{S}_\sigma \cap \sigma^\perp = \sigma^\perp \cap M \\ 0 & \text{otherwise,} \end{cases}$$

which implies that $\eta(x,1) \in O(\sigma)$ for all x. Combining all of this, we see that η is a deformation retraction. You will show T_N-equivariance in Exercise 12.1.3. $\square$

The General Case. For a general fan Σ we will use the last proposition and the Van Kampen theorem to complete the computation of $\pi_1(X_\Sigma)$.

Theorem 12.1.10. *Let Σ be a fan in $N_\mathbb{R}$ and let N_Σ be the sublattice of N generated by $|\Sigma| \cap N$. Then $\pi_1(X_\Sigma) \simeq N/N_\Sigma$.*

Proof. We will prove the theorem by induction on the number of cones in Σ. If Σ consists of a single cone $\{0\}$, then the claim follows from (12.1.1).

Now assume the result has been proved for all fans with $k-1$ cones or fewer and let Σ contain k cones. Pick any cone σ of maximal dimension in Σ, and let $\Sigma' = \Sigma \setminus \{\sigma\}$. Then

$$X_\Sigma = X_{\Sigma'} \cup U_\sigma,$$

with $X_{\Sigma'} \cap U_\sigma = X_{\Sigma''}$ for the fan Σ'' consisting of the proper faces of σ. Since $X_{\Sigma'}$, U_σ, and $X_{\Sigma''}$ are all path connected, we can apply the Van Kampen theorem [**135**, Thm. 1.20] to determine $\pi_1(X_\Sigma)$. From the diagram of inclusions

$$\begin{tikzcd} & X_{\Sigma'} \arrow[dr, "j_1"] & \\ X_{\Sigma''} \arrow[ur, "i_1"] \arrow[rr, "h"'] \arrow[dr, "i_2"'] & & X_\Sigma \\ & U_\sigma \arrow[ur, "j_2"'] & \end{tikzcd}$$

we obtain the corresponding diagram of fundamental groups

$$\begin{tikzcd} & \pi_1(X_{\Sigma'}) \arrow[dr, "j_{1*}"] & \\ \pi_1(X_{\Sigma''}) \arrow[ur, "i_{1*}"] \arrow[rr, "h_*"'] \arrow[dr, "i_{2*}"'] & & \pi_1(X_\Sigma). \\ & \pi_1(U_\sigma) \arrow[ur, "j_{2*}"'] & \end{tikzcd}$$

The Van Kampen theorem identifies $\pi_1(X_\Sigma)$ as the free product of $\pi_1(X_{\Sigma'})$ and $\pi_1(U_\sigma)$, modulo the normal subgroup generated by all elements of the form

$$(j_1 i_1)_*([\omega])((j_2 i_2)_*([\omega]))^{-1},$$

where $[\omega] \in \pi_1(X_{\Sigma''})$. Our induction hypothesis gives isomorphisms

$$\begin{aligned}\pi_1(X_{\Sigma'}) &\simeq N/N_{\Sigma'}\\ \pi_1(X_{\Sigma''}) &\simeq N/N_{\Sigma''}\\ \pi_1(U_\sigma) &\simeq N/N_\sigma.\end{aligned}$$

By using presentations of these groups in terms of generators and relations, it is easy to see that

$$\pi_1(X_\Sigma) \simeq N/(N_{\Sigma'} + N_\sigma) = N/N_\Sigma.$$

You will complete the details in Exercise 12.1.4. $\square$

Theorem 12.1.10 implies that X_Σ is simply connected if and only if the support of Σ contains a basis for N. This is the case, for instance, if Σ contains an n-dimensional cone. It follows that a simply connected toric variety has no torus factors. The converse of this assertion is not true, though, as the following example shows.

Example 12.1.11. Let N have rank 2, and let Σ be the fan consisting of the cones

$$\{0\}, \operatorname{Cone}(e_1), \operatorname{Cone}(e_1 + de_2).$$

The corresponding toric variety X_Σ is the complement of the origin in the rational normal cone $\widehat{C}_d$. Clearly, $N_\Sigma = \mathbb{Z}e_1 + d\mathbb{Z}e_2$ and $\pi_1(X_\Sigma) \simeq N/N_\Sigma \simeq \mathbb{Z}/d\mathbb{Z}$. $\Diamond$

Exercise 12.1.6 shows that for any finitely generated abelian group G, there exists a normal toric variety X_Σ with $\pi_1(X_\Sigma) \simeq G$. Some information on the higher homotopy group $\pi_2(X_\Sigma)$ will be obtained later in Exercise 12.3.10.

Exercises for §12.1.

12.1.1. Let $[\gamma]$ be the homotopy class of a closed path γ. Show that the map

$$\begin{aligned}N &\longrightarrow \pi_1(T_N)\\ u &\longmapsto [\lambda^u|_{S^1}]\end{aligned}$$

is an isomorphism of groups.

12.1.2. In this exercise you will verify the claims made in Example 12.1.2 about the surface $Y_{\mathscr{A}} = \mathbf{V}(y^2 - x^2z) \subseteq \mathbb{C}^3$, where $\mathscr{A} = \{e_1, e_1 + e_2, 2e_2\}$.

(a) Show that the singular locus of $Y_{\mathscr{A}}$ is the z-axis.

(b) Show that if p is any point of $Y_{\mathscr{A}}$ with $z \neq 0$ and V is a sufficiently small open neighborhood of p in $Y_{\mathscr{A}}$, then $V \setminus (V \cap \operatorname{Sing}(Y_{\mathscr{A}}))$ has two connected components.

12.1.3. In this exercise, you will complete the details of the proof of Proposition 12.1.9.

(a) Show that for all $x \in U_\sigma$ and $t \in [0,1]$, $\eta(x,t) : \mathsf{S}_\sigma \to \mathbb{C}$ is a semigroup homomorphism, hence a point of U_σ.

(b) Verify the claim in the proof of Proposition 12.1.9 that $\eta(x,t)$ is continuous in x and t.

(c) Show that the deformation retraction is equivariant for the T_N-action.

12.1.4. Complete the details of the argument using the Van Kampen theorem in the proof of Theorem 12.1.10.

12.1.5. Let Σ be the fan in $N_{\mathbb{R}} \simeq \mathbb{R}^2$ with cones

$$\{0\}, \mathrm{Cone}(e_1+e_2), \mathrm{Cone}(e_1-e_2), \mathrm{Cone}(-e_1+e_2), \mathrm{Cone}(-e_1-e_2).$$

Show that $\pi_1(X_\Sigma) \simeq \mathbb{Z}/2\mathbb{Z}$.

12.1.6. Show that given any finitely generated abelian group G, there exists a normal toric variety whose fundamental group is isomorphic to G.

12.1.7. Prove that $\pi_1(X_\Sigma)$ is finite if and only if X_Σ has no torus factors.

§12.2. The Moment Map

The compact real torus $S_N \simeq (S^1)^n$ is a subgroup of T_N and hence also acts on the toric variety X_Σ. We will show first that the quotient space X_Σ/S_N can be identified with a certain subset $(X_\Sigma)_{\geq 0}$ of X_Σ.

The Nonnegative Part of a Toric Variety. Recall the interpretation of points of an affine toric variety U_σ as semigroup homomorphisms $\gamma : \mathsf{S}_\sigma \to \mathbb{C}$. We can define subsets of U_σ by placing restrictions on the image of γ. For instance, the nonnegative part of U_σ is defined formally as

$$(U_\sigma)_{\geq 0} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{S}_\sigma, \mathbb{R}_{\geq 0}).$$

The absolute value map $z \mapsto |z|$ on $\mathbb{C}$ gives a retraction of $\mathbb{C}$ onto $\mathbb{R}_{\geq 0}$. So there is a corresponding retraction $U_\sigma \to (U_\sigma)_{\geq 0}$ defined by mapping $\gamma \mapsto |\gamma|$. If X_Σ is the toric variety of a fan Σ, it is easy to check that the $(U_\sigma)_{\geq 0}$ for $\sigma \in \Sigma$ glue together to form a closed subset $(X_\Sigma)_{\geq 0}$ of X_Σ in the classical topology. Moreover the retraction maps glue properly to give a retraction

$$X_\Sigma \longrightarrow (X_\Sigma)_{\geq 0}.$$

By considering semigroup homomorphisms with images in $\mathbb{R}$ or $\mathbb{R}_{>0}$, we can define real or positive real points of toric varieties as well. The real points of toric varieties are discussed for instance in [**254**].

If $\phi : X \to Y$ is a toric morphism, then the above discussion implies that ϕ restricts to a map

$$\phi_{\geq 0} : X_{\geq 0} \longrightarrow Y_{\geq 0}.$$

Moreover, $\phi_{\geq 0}$ fits in a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & Y \arrow[d] \\ X_{\geq 0} \arrow[r, "\phi_{\geq 0}"'] & Y_{\geq 0} \end{tikzcd} \tag{12.2.1}$$

where the vertical arrows are the above retractions (Exercise 12.2.1). This implies that a character χ^m of $T_N \simeq (\mathbb{C}^*)^n$ maps $(T_N)_{\geq 0} \simeq (\mathbb{R}_{>0})^n$ to $(\mathbb{C}^*)_{\geq 0} = \mathbb{R}_{>0}$.

We can also apply (12.2.1) to the quotient construction of toric varieties from Chapter 5. By Proposition 5.1.9, the quotient map

$$\pi : \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \longrightarrow X_\Sigma.$$

is a toric morphism, so that $\pi_{\geq 0}$ is defined.

Proposition 12.2.1. *Let X_Σ be a normal toric variety without torus factors. Then*

$$\pi_{\geq 0} : (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))_{\geq 0} \longrightarrow (X_\Sigma)_{\geq 0}$$

is surjective.

Proof. Theorem 5.1.11 implies that π is a good categorical quotient. Hence π is surjective by Theorem 5.0.6. Surjectivity of $\pi_{\geq 0}$ follows from the commutativity of the diagram (12.2.1). $\square$

Here is one way to use Proposition 12.2.1.

Example 12.2.2. Consider $X_\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^1$ for the fan Σ in $N_\mathbb{R} \simeq \mathbb{R}^2$ given in Example 3.1.12. By Example 5.1.8, we have

$$\mathbb{P}^1 \times \mathbb{P}^1 = U/(\mathbb{C}^*)^2,$$

where $U = \mathbb{C}^4 \setminus ((\{(0,0)\} \times \mathbb{C}^2) \cup (\mathbb{C}^2 \times \{(0,0)\}))$ and $(\mathbb{C}^*)^2$ acts via

$$(\lambda, \mu) \cdot (a,b,c,d) = (\lambda a, \lambda b, \mu c, \mu d).$$

By Proposition 12.2.1, a point in $(\mathbb{P}^1 \times \mathbb{P}^1)_{\geq 0}$ comes from $(a,b,c,d) \in U \cap \mathbb{R}^4_{\geq 0}$. Since $a, b \geq 0$ cannot both vanish, we can rescale by a positive number so that $a+b = 1$. Doing the same for c,d, we see that $(\mathbb{P}^1 \times \mathbb{P}^1)_{\geq 0}$ can be written as

$$(\mathbb{P}^1 \times \mathbb{P}^1)_{\geq 0} = \{(a,b,c,d) \in \mathbb{R}^4 : a,b,c,d \geq 0 \text{ and } a+b = c+d = 1\},$$

which is a square in a two-dimensional affine subspace in $\mathbb{R}^4$. Note that this is the same (combinatorially, at least) as a plane polygon P whose normal fan Σ_P coincides with the original fan Σ. We will see shortly that this is no coincidence.

In homogeneous coordinates, the retraction $\mathbb{P}^1 \times \mathbb{P}^1 \to (\mathbb{P}^1 \times \mathbb{P}^1)_{\geq 0}$ is given by

$$(a,b,c,d) \longmapsto \left(\frac{|a|}{|a|+|b|}, \frac{|b|}{|a|+|b|}, \frac{|c|}{|c|+|d|}, \frac{|d|}{|c|+|d|}\right).$$

Note the structure of the fibers of this map. Over the four corners of the square (that is $(a,b) = (1,0)$ or $(0,1)$ and similarly for (c,d)), the fiber consists of a single point. On an edge but not at a vertex, the fiber consists of a copy of S^1. Finally at an interior point of the square, the fiber consists of a copy of $S^1 \times S^1 = T^2$. Thus $\mathbb{P}^1 \times \mathbb{P}^1$ has a stratification by fiber bundles with compact real torus fibers. $\Diamond$

The Quotient of X_Σ by S_N. Now we consider the quotient of X_Σ by the compact real torus S_N. To prepare for our next result, note that the isomorphism $\mathbb{C}^* \simeq \mathbb{R}_{>0} \times S^1$ used in §12.1, composed with the real logarithm map $\mathbb{R}_{>0} \to \mathbb{R}$ on the first factor, yields an isomorphism $\mathbb{C}^* \simeq \mathbb{R} \times S^1$. In Exercise 12.2.2, you will show that S_N can be identified with $\mathrm{Hom}_\mathbb{Z}(M, S^1) \subseteq \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*)$, where S^1 is the unit circle in $\mathbb{C}^*$. Putting all of this together, we obtain isomorphisms

$$T_N = \mathrm{Hom}_\mathbb{Z}(M, \mathbb{C}^*) \simeq \mathrm{Hom}_\mathbb{Z}(M, \mathbb{R}) \times \mathrm{Hom}_\mathbb{Z}(M, S^1) \simeq N_\mathbb{R} \times S_N. \tag{12.2.2}$$

The structure of the quotient X_Σ/S_N is as follows.

Proposition 12.2.3. *Let X_Σ be a normal toric variety.*

(a) *The retraction $X_\Sigma \to (X_\Sigma)_{\geq 0}$ induces a homeomorphism $X_\Sigma/S_N \cong (X_\Sigma)_{\geq 0}$.*

(b) *For each cone σ in Σ, the fiber of $X_\Sigma \to (X_\Sigma)_{\geq 0}$ over a point in $O(\sigma)_{\geq 0}$ can be identified with $S_{N(\sigma)}$, a compact real torus of dimension $n - \dim\sigma$.*

Proof. For part (a), consider how S_N acts on each torus orbit $O(\sigma)$. Recall from Lemma 3.2.5 that $O(\sigma) \simeq \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq T_{N(\sigma)}$, where $N(\sigma) = N/N_\sigma$ is the dual lattice of $\sigma^\perp \cap M$. Then (12.2.2) implies that

$$O(\sigma) \simeq N(\sigma)_\mathbb{R} \times S_{N(\sigma)}.$$

Using the real logarithm again, $O(\sigma)$ retracts to

$$O(\sigma)_{\geq 0} = \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{R}_{>0}) \simeq \mathrm{Hom}_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{R}) = N(\sigma)_\mathbb{R}.$$

However, S_N acts on $O(\sigma)$ via the compact real torus $S_{N(\sigma)}$ in $T_{N(\sigma)}$. As a result, $O(\sigma)/S_N \cong N(\sigma)_\mathbb{R} \cong O(\sigma)_{\geq 0}$. The assertion for X_Σ follows. Part (b) follows by similar reasoning. □

Example 12.2.4. The toric variety $\mathbb{P}^1$ is homeomorphic to the real 2-sphere S^2. The torus S_N in this case is S^1, a circle acting on $\mathbb{P}^1$ fixing two points shown as the north and south poles in Figure 2. Every other point has a circle as orbit. The quotient space is homeomorphic to a closed interval.

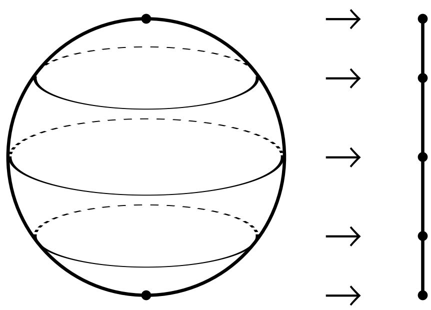


Figure 2. The quotient $\mathbb{P}^1/S^1 \cong (\mathbb{P}^1)_{\geq 0}$

Note also that the map $\mathbb{P}^1 \times \mathbb{P}^1 \to (\mathbb{P}^1 \times \mathbb{P}^1)_{\geq 0}$ from Example 12.2.2 can be visualized as the Cartesian product of two copies of this picture. $\Diamond$

The Moment Map. Recall that in Example 12.2.2 above we saw that $(\mathbb{P}^1 \times \mathbb{P}^1)_{\geq 0}$ was combinatorially the same as a polygon with normal fan equal to the standard fan for $\mathbb{P}^1 \times \mathbb{P}^1$. We now turn to a general connection between $(X_P)_{\geq 0}$ and the polytope P.

Let P be a full dimensional lattice polytope in $M_{\mathbb{R}}$. Corresponding to P, we have the normal fan Σ_P and the toric variety $X_P = X_{\Sigma_P}$. Replacing P by a multiple ℓP if necessary, we will assume that $P \cap M = \{m_0, \ldots, m_s\}$ defines an embedding

$$\begin{aligned} \varphi : X_P &\hookrightarrow \mathbb{P}^s \\ x &\longmapsto (\chi^{m_0}(x), \ldots, \chi^{m_s}(x)). \end{aligned}$$

Following the online version of [**254**], we define the *algebraic moment map* by

$$\begin{aligned} f : X_P &\longrightarrow M_{\mathbb{R}} \\ x &\longmapsto \frac{1}{\sum_{m \in P\cap M} |\chi^m(x)|} \sum_{m \in P \cap M} |\chi^m(x)| m. \end{aligned} \tag{12.2.3}$$

The *symplectic moment map* corresponding to the action of the compact real torus S_N (as defined in symplectic geometry) is the closely related map

$$\begin{aligned} \mu : X_P &\longrightarrow M_{\mathbb{R}} \\ x &\longmapsto \frac{1}{\sum_{m \in P\cap M} |\chi^m(x)|^2} \sum_{m \in P \cap M} |\chi^m(x)|^2 m. \end{aligned} \tag{12.2.4}$$

Note that f and μ are invariant under the action of $S_N \subset T_N$. The behavior we saw in Example 12.2.2 is a special case of the next result.

Theorem 12.2.5. *Let P be a full dimensional lattice polytope in $M_{\mathbb{R}}$. The restricted algebraic moment map*

$$\bar{f} = f|_{(X_P)_{\geq 0}} : (X_P)_{\geq 0} \longrightarrow M_{\mathbb{R}},$$

is a homeomorphism from $(X_P)_{\geq 0}$ to P.

Proof. The characters χ^m for $m \in P \cap M$ map $x \in (X_P)_{\geq 0}$ to nonnegative real numbers, so that $|\chi^m(x)| = \chi^m(x)$. Rescaling as in Example 12.2.2 gives $\varphi(x) = (a_0, \ldots, a_s) \in \mathbb{P}^s$, where $a_0 + \cdots + a_s = 1$ and $a_i \geq 0$ for all i. This implies that

$$\bar{f}(x) = a_0 m_0 + \cdots + a_s m_s.$$

It is clear from (12.2.3) that $\bar{f}$ is continuous. We will show that $\bar{f}$ is bijective and leave the verification that $\bar{f}^{-1}$ is continuous as an exercise. The proof of bijectivity will be accomplished by showing that $\bar{f}$ maps each torus orbit in X_P

bijectively to the relative interior of the corresponding face of P. We will give the details for $T_N \simeq (\mathbb{C}^*)^n$ in X_P and show that $(T_N)_{\geq 0}$ maps bijectively to the interior of P. The result for the other orbits will then follow by similar reasoning.

First we establish surjectivity. Fixing an isomorphism $M \simeq \mathbb{Z}^n$, write $m_i = (m_{i1}, \dots, m_{in})$ where $m_{ij} \in \mathbb{Z}$ for all i, j. Given v in the interior of P, there are in general many different ways to write v as a convex linear combination of the m_i. If $a = (a_0, \dots, a_s)$ is one vector of coefficients in such a combination and $a' = (a'_0, \dots, a'_s)$ is a second, then

$$v = a_0 m_0 + \cdots + a_s m_s = a'_0 m_0 + \cdots + a'_s m_s$$

with $a_i, a'_i \geq 0$ and $a_0 + \cdots + a_s = a'_0 + \cdots + a'_s = 1$. It follows that the $b_i = a'_i - a_i$ satisfy the linear equations $b_0 m_0 + \cdots + b_s m_s = 0$ and $b_0 + \cdots + b_s = 0$. With the notation

$$W_P = \{(b_0, \dots, b_s) \in \mathbb{R}^{s+1} \mid b_0 m_0 + \cdots + b_s m_s = 0 \text{ and } b_0 + \cdots + b_s = 0\},$$

the vectors $a' = (a'_0, \dots, a'_s)$ satisfying $v = \sum_{i=0}^s a'_i m_i$ are precisely the elements of $(a + W_P) \cap \mathbb{R}^{s+1}_{\geq 0}$.

Our proof will show, in fact, that v can be written as $v = \tilde{a}_0 m_0 + \cdots + \tilde{a}_s m_s$, where $\tilde{a}_i = \gamma \chi^{m_i}(x)$ where $\gamma > 0$ and $x = (x_1, \dots, x_n) \in T_N$ with $x_i > 0$ for all i. This will show the surjectivity of $\overline{f}$ as a map from the positive real points in T_N in X_P to the interior of P. The particular representation of v we want will come from minimizing a certain function on $(a + W_P) \cap \mathbb{R}^{s+1}_{\geq 0}$.

First note that $g(x) = x \log(x) - x$ can be defined for all $x \geq 0$ in $\mathbb{R}$ since L'Hôpital's rule implies that $\lim_{x \to 0^+} g(x) = 0$. Using g, we define

$$\begin{aligned} G : (\mathbb{R}_{\geq 0})^{s+1} &\longrightarrow \mathbb{R} \\ (x_0, \dots, x_s) &\longmapsto g(x_0) + \cdots + g(x_s). \end{aligned}$$

Since $g''(x) = \frac{1}{x}$, the Hessian (second derivative) of G is positive definite at any point $(x_0, \dots, x_s)$ with $x_i > 0$ for all i. In other words, G is *concave up* at all such points. You will show in Exercise 12.2.7 that G, restricted to $(a + W_P) \cap \mathbb{R}^{s+1}_{\geq 0}$, has a unique critical point $\tilde{a} = (\tilde{a}_0, \dots, \tilde{a}_s)$, necessarily a minimum.

We claim that $\tilde{a}_i > 0$ for all i. If not, then $I = \{i \mid \tilde{a}_i = 0\}$ is nonempty. In Exercise 12.2.7, you will show that there exists $b \in W_P$ satisfying $b_i > 0$ for all $i \in I$. Let $\lambda(t) = \tilde{a} + tb$ be the line through $\tilde{a}$ with direction vector b, and let $\lambda_i(t)$ be the components. Then $0 < \lambda_i(t) < 1$ for all i and all t in some sufficiently small open interval $(0, \varepsilon)$. It follows that

$$\lim_{t \to 0^+} \frac{d}{dt}(G \circ \lambda)(t) = \lim_{t \to 0^+} \sum_{i=0}^{s} b_i \log(\tilde{a}_i + t b_i) = -\infty.$$

But this contradicts the concavity properties of G. Hence $\tilde{a}$ must have all nonzero components.

For an arbitrary b in W_P, let $\lambda(t) = \tilde{a} + bt$ and consider the restriction of G to this line. Since $\tilde{a}_i > 0$ for all i, we have $0 < \lambda_i(t) < 1$ when t lies in some symmetric interval about 0. Then $(G \circ \lambda)(t)$ has a local minimum at $t = 0$, so by the chain rule,

$$0 = \frac{d}{dt}(G \circ \lambda)|_{t=0} = \sum_{i=0}^{s} b_i \log(\tilde{a}_i).$$

Since this is true for all $b \in W_P$, linear algebra implies that the overdetermined system of linear equations

$$\begin{aligned} \log(\tilde{a}_0) &= m_{01}y_1 + \cdots + m_{0n}y_n + c \\ &\vdots \\ \log(\tilde{a}_s) &= m_{s1}y_1 + \cdots + m_{sn}y_n + c \end{aligned} \tag{12.2.5}$$

in $n+1$ variables $(y_1, \ldots, y_n, c)$ has a real solution. Let $y_i = \log(x_i)$ and $c = \log(\gamma)$ where x_i, γ are real and positive. Exponentiating, $\tilde{a}_i = \gamma x_1^{m_{i1}} \cdots x_n^{m_{in}} = \gamma \chi^{m_i}(x)$ for all $i = 0, \ldots, s$. Since $\sum_{i=0}^{s} \tilde{a}_i = 1$, the constant γ must be

$$\gamma = \frac{1}{\sum_{i=0}^{s} \chi^{m_i}(x)}.$$

This completes the proof of surjectivity of the restricted algebraic moment map.

The injectivity of

$$\bar{f} : (X_P)_{\geq 0} \longrightarrow M_{\mathbb{R}}$$

is now a consequence of the constructions already made. Suppose v is in the interior of P and $v = \bar{f}(x)$ for x a positive real point in the T_N in X_P. Then the definition of $\bar{f}$ gives $v = \bar{f}(x) = a_0 m_0 + \cdots + a_s m_s$ with $a_i = \gamma x_1^{m_{i1}} \cdots x_n^{m_{in}}$ and $\gamma = (\sum_{i=0}^{s} \chi^{m_i}(x))^{-1}$. Writing $y_i = \log(x_i)$ and $c = \log(\gamma)$, we have a system

$$\begin{aligned} \log(a_0) &= m_{01}y_1 + \cdots + m_{0n}y_n + c \\ &\vdots \\ \log(a_s) &= m_{s1}y_1 + \cdots + m_{sn}y_n + c \end{aligned}$$

of the same form as (12.2.5). Since the system is consistent, we must have

$$0 = \sum_{i=0}^{s} b_i \log(a_i) \tag{12.2.6}$$

for all $b \in W_P$. However, by the same computations done before, the sum on the right of (12.2.6) is the derivative at $t = 0$ of $(G \circ \lambda)(t)$, where $\lambda(t) = a + bt$. By the concavity of G, this implies that $a = \tilde{a}$, the unique critical point of G on $a + W_P$. But then, since there are n linearly independent vectors among the m_i, it follows that the $a_i = \gamma x_1^{m_{i1}} \cdots x_n^{m_{in}}$ are uniquely determined. Since we assume that χ^m for $m \in P \cap M$ define an embedding of X_P, this shows that x is also uniquely determined. $\square$

This proof basically follows the presentation in [**93**, VII.7]; [**105**] gives a different argument. The result of Theorem 12.2.5 remains true if we replace the algebraic moment map by the symplectic moment map from (12.2.4) (Exercise 12.2.8).

Topological Models of Toric Varieties. Combining Theorems 12.2.3 and 12.2.5 gives a way to understand the underlying topological space of the projective toric variety X_P of a polytope P, similar to what we observed in Example 12.2.2. Indeed, it is not difficult to see that there is an S_N-equivariant homeomorphism

$$X_P \cong (S_N \times P)/\sim, \tag{12.2.7}$$

where two points (s_1, x_1) and (s_2, x_2) are identified if $x_1 = x_2$, and s_1 and s_2 are congruent modulo the subtorus S_{N_σ} of S_N for the cone $\sigma \in \Sigma_P$ corresponding to the minimal face of P containing x_1 (Exercise 12.2.9).

An analogous topological model can be given for any complete normal toric variety X_Σ using the unit ball B^n. Namely, a complete fan Σ in $N_\mathbb{R}$ determines a spherical complex C_Σ on the unit sphere S^n in $N_\mathbb{R}$ by intersecting each cone σ with the sphere. Then each σ determines a *spherical dual* $\widehat{\sigma}$ in B^n as follows. By a process analogous to that described in Exercise 11.1.10, one constructs a barycentric subdivision of C_Σ. If $\sigma = \{0\}$ then $\widehat{\sigma} = B^n$. Otherwise, $\widehat{\sigma}$ is the union of all spherical simplices in the barycentric subdivision whose vertices are barycenters of $\tau \cap S^n$ with $\sigma \prec \tau$. Then

$$X_\Sigma \cong (S_N \times B^n)/\sim, \tag{12.2.8}$$

where two points (s_1, x_1) and (s_2, x_2) are identified if and only if $x_1 = x_2$ and s_1 is congruent to s_2 modulo S_{N_σ} for the unique σ such that x_1 is in the relative interior of $\widehat{\sigma}$. See [**162**], where this construction of a space homeomorphic to X_Σ is attributed to MacPherson, and [**97**]. These references also discuss generalizations for toric varieties associated to noncomplete fans. Some recent work in toric topology essentially takes this topological construction as the *definition* of a toric variety. The article [**59**] gives a nice overview of toric topology.

Symplectic Geometry and Toric Varieties. We conclude with a brief discussion of the relations between moment maps, symplectic geometry, and toric varieties, without proofs. In symplectic geometry, one studies Hamiltonian actions of a compact connected Lie group $G_\mathbb{R}$ on a symplectic manifold X. Such an action has a symplectic moment map

$$\mu : X \longrightarrow \mathfrak{g}_\mathbb{R}^*,$$

where $\mathfrak{g}_\mathbb{R}$ is the Lie algebra of $G_\mathbb{R}$. For example, the action of S_N on the toric variety X_P gives the symplectic moment map (12.2.4). This follows because the the dual of the Lie algebra of S_N is $N_\mathbb{R}^* = M_\mathbb{R}$.

Another example comes from quotient construction of a projective simplicial toric variety

$$X_\Sigma \simeq (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))/G$$

from Chapter 5. The maximal compact subgroup of $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$ is $G_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), S^1)$ and the Lie algebra of $G_{\mathbb{R}}$ can be identified with $\mathrm{Cl}(X_\Sigma)_{\mathbb{R}}$. Here, the symplectic moment map

$$\mu_\Sigma : \mathbb{C}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma)_{\mathbb{R}}$$

factors as

$$\mathbb{C}^{\Sigma(1)} \xrightarrow{\mu} \mathbb{R}^{\Sigma(1)} \xrightarrow{\beta} \mathrm{Cl}(X_\Sigma)_{\mathbb{R}}.$$

The map μ is given by

$$\mu(z_1, \ldots, z_r) = \frac{1}{2}(|z_1|^2, \ldots, |z_r|^2),$$

and the map β comes from the exact sequence

$$0 \longrightarrow M_R \longrightarrow \mathbb{R}^{\Sigma(1)} \xrightarrow{\beta} \mathrm{Cl}(X_\Sigma)_{\mathbb{R}} \longrightarrow 0$$

obtained by tensoring (5.1.1) by $\mathbb{R}$.

Let $[D]$ be the class of an ample divisor on a complete simplicial toric variety X_Σ. Then $\mu_\Sigma^{-1}([D]) \subseteq \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$ and results of Guillemin imply that the natural map

$$\mu_\Sigma^{-1}([D])/G_{\mathbb{R}} \longrightarrow X_\Sigma \tag{12.2.9}$$

is a diffeomorphism. The divisor D determines a symplectic structure on X_Σ. The natural symplectic structure from $\mathbb{C}^{\Sigma(1)}$ descends to the quotient $\mu_\Sigma^{-1}([D])/G_{\mathbb{R}}$ and the diffeomorphism in (12.2.9) preserves the cohomology classes of the symplectic forms. Discussions of these results and references to detailed proofs can be found in [**66**, §4].

Example 12.2.6. Let $X_\Sigma = \mathbb{P}^n$, and recall that $\mathrm{Cl}(\mathbb{P}^n)_{\mathbb{R}} \simeq \mathbb{R}$. The symplectic moment map μ_Σ above is given by

$$\mu_\Sigma(z_1, \ldots, z_{n+1}) = \frac{1}{2}\sum_{i=1}^{n+1} |z_i|^2.$$

The class of an ample $[D]$ corresponds to a positive real value, so that $\mu_\Sigma^{-1}([D])$ is diffeomorphic to S^{2n+1}. The group $G_{\mathbb{R}} \simeq S^1$ in this case, and the diffeomorphism (12.2.9) is

$$S^{2n+1}/S^1 \xrightarrow{\sim} \mathbb{P}^n.$$

We obtain the identification of $\mathbb{P}^n$ as the base space of the *Hopf fibration* with total space S^{2n+1} and fiber S^1 [**135**, p. 337]. $\Diamond$

For symplectic geometers, (12.2.9) shows that toric varieties can be defined by a construction known as *symplectic reduction*.

Results along the lines of Theorem 12.2.5 were originally developed in a more general setting. A compact connected real $2n$-dimensional symplectic manifold (M, ω) is said to be *toric* if it has an effective Hamiltonian $(S^1)^n$ action. Results of Atiyah, Guillemin and Sternberg show that the image of the symplectic moment map of such a manifold is a polytope P in $\mathbb{R}^n$. Moreover, the polytopes that appear have been characterized by Delzant in [**81**]. They are polytopes having n edges incident at each vertex p, of the form $p + tu_i$ for some $u_i \in \mathbb{Z}^n$ forming a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. The vertices need not be lattice points, but if they are then P is a smooth polytope. In any case, M is diffeomorphic to a smooth projective toric variety.

Exercises for §12.2.

12.2.1. This exercise supplies some details for the construction of the nonnegative part of a toric variety and the behavior of a toric morphism on the nonnegative part.

(a) Let X_Σ be a normal toric variety. Show that the $(U_\sigma)_{\geq 0}$ for all σ in Σ glue together to form a closed subset $(X_\Sigma)_{\geq 0}$ of X_Σ (in the classical topology).

(b) Show that the retraction maps $U_\sigma \to (U_\sigma)_{\geq 0}$ for $\sigma \in \Sigma$ glue together properly to give a retraction $X_\Sigma \to (X_\Sigma)_{\geq 0}$.

(c) Let $\phi : X \to Y$ be a toric morphism. Show that ϕ restricts to a map $\phi_{\geq 0} : X_{\geq 0} \to Y_{\geq 0}$ commuting with the retractions on X, Y as in (12.2.1).

12.2.2. Let M, N be dual lattices Show that the compact torus $S_N \subseteq T_N$ is identified with $\mathrm{Hom}_{\mathbb{Z}}(M, S^1) \subseteq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$.

12.2.3. A generating set $m_1, \ldots, m_s$ of S_σ gives an embedding $U_\sigma \hookrightarrow \mathbb{C}^s$. Prove that $(U_\sigma)_{\geq 0} = U_\sigma \cap \mathbb{R}^s_{\geq 0}$. Hint: See the proof of Proposition 1.3.1.

12.2.4. Check the claims about the fibers of the retraction $\mathbb{P}^1 \times \mathbb{P}^1 \to (\mathbb{P}^1 \times \mathbb{P}^1)_{\geq 0}$ made in Example 12.2.2.

12.2.5. Determine the image of the algebraic moment maps for each of the following toric varieties directly from the associated polytope P without using Theorem 12.2.5.

(a) The projective plane $\mathbb{P}^2$, with $P = \mathrm{Conv}(0, e_1, e_2)$. Generalize to $\mathbb{P}^n$.

(b) The rational normal scroll X_P for $P = \mathrm{Conv}(0, 3e_1, -e_2, e_1 - e_2)$ (isomorphic to the Hirzebruch surface $\mathscr{H}_2$).

12.2.6. Complete the proof that $\bar{f}$ in Theorem 12.2.5 is a homeomorphism by showing that $\bar{f}^{-1}$ is continuous.

12.2.7. This exercise concerns some details in the proof of Theorem 12.2.5. Let $G = \sum_{i=0}^s x_i \log(x_i) - x_i$ be the function on $\mathbb{R}^{s+1}$ defined in that proof. Recall that G has positive definite Hessian (second derivative) at all points with all positive coordinates.

(a) Show that if W is a translate of a linear subspace of $\mathbb{R}^{s+1}$, and $W_{\geq 0}$ is the subset of W consisting of points with all coordinates nonnegative, then $G|_{W_{\geq 0}}$ has a unique minimum when $W_{\geq 0} \neq \emptyset$. Hint: Argue by contradiction. Let a and b be two minima and consider G restricted to the line containing a and b.

(b) Show that if G attains its minimum on $(a + W_P) \cap \mathbb{R}^{s+1}_{\geq 0}$ at $\tilde{a}$ such that $I = \{i \mid \tilde{a}_i = 0\}$ is nonempty, then there exists $b \in W_P$ with $b_i > 0$ for all $i \in I$. Hint: Recall that the point v is assumed to lie in the interior of P.

12.2.8. Show that the result of Theorem 12.2.5 is still true if we replace the algebraic moment map by the symplectic moment map from (12.2.4).

12.2.9. Use Theorems 12.2.5 and 12.2.3 to construct a homeomorphism (12.2.7).

§12.3. Singular Cohomology of Toric Varieties

In this section, we will study the singular cohomology groups of a toric variety X_Σ. We first describe these groups using the singular cohomology of the affine toric varieties U_σ for $\sigma \in \Sigma_{\max}$. We then give a different description that uses the singular cohomology of the torus orbits $O(\sigma)$ for $\sigma \in \Sigma$. In both cases, the machinery of spectral sequences (see Appendix C) will establish the connection.

The Picard Group of a Toric Variety and H^2**.** We begin with a lovely application of spectral sequences that connects the Picard group of a toric variety to $H^2(X_\Sigma, \mathbb{Z})$. Let Σ be a fan in $N_\mathbb{R} \simeq \mathbb{R}^n$. In Chapter 4, we constructed the map

$$\begin{aligned} \bigoplus_{\sigma_i \in \Sigma_{\max}} M/M(\sigma_i) &\longrightarrow \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j) \\ (m_i)_i &\longmapsto (m_i - m_j)_{i<j}, \end{aligned}$$

where $M(\sigma) = \sigma^\perp \cap M$. Proposition 4.2.9 provides a natural isomorphism

$$\mathrm{CDiv}_{T_N}(X_\Sigma) \simeq \ker\big(\textstyle\bigoplus_i M/M(\sigma_i) \to \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\big),$$

where $\mathrm{CDiv}_{T_N}(X_\Sigma)$ is the group of torus-invariant Cartier divisors on X_Σ. Then Theorem 4.2.1 relates this to $\mathrm{Pic}(X_\Sigma)$ via the exact sequence

$$M \longrightarrow \mathrm{CDiv}_{T_N}(X_\Sigma) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0.$$

The map $M \to \mathrm{CDiv}_{T_N}(X_\Sigma)$ is given by $m \mapsto \mathrm{div}(\chi^m)$. Our goal is to relate $\mathrm{Pic}(X_\Sigma)$ to the topological object $H^2(X_\Sigma, \mathbb{Z})$.

We begin by computing the singular cohomology of an affine toric variety.

Proposition 12.3.1. *Let σ be a cone in $N_\mathbb{R}$. Then*

$$H^\bullet(U_\sigma, \mathbb{Z}) \simeq H^\bullet(T_{N(\sigma)}, \mathbb{Z}) \simeq \textstyle\bigwedge^\bullet M(\sigma).$$

Proof. By Proposition 12.1.9, $T_{N(\sigma)}$ is a deformation retract of U_σ, and this implies the first isomorphism. The cohomology of a torus was given in Example 9.0.12 and the second isomorphism follows from the duality of $N \to N(\sigma)$ and $M(\sigma) \subseteq M$. $\square$

The Picard group relates to the singular cohomology of X_Σ as follows.

Theorem 12.3.2. *Let Σ be a fan in $N_\mathbb{R} \simeq \mathbb{R}^n$ with all maximal cones n-dimensional. Then*

$$\mathrm{Pic}(X_\Sigma) \simeq H^2(X_\Sigma, \mathbb{Z}).$$

Proof. Consider the open cover of X_Σ given by

$$\mathscr{U} = \{U_\sigma\}_{\sigma \in \Sigma_{\max}} = \{U_\sigma\}_{\sigma \in \Sigma(n)}.$$

As noted in §9.0, for the spaces we are considering, singular cohomology with coefficients in $\mathbb{Z}$ is the sheaf cohomology of the constant sheaf (in the classical topology) given by $\mathbb{Z}$. Hence the spectral sequence of the covering $\mathscr{U}$ (see (9.0.10) and Theorem C.2.2) becomes

$$E_1^{p,q} = \bigoplus_{\gamma=(i_0,\dots,i_p)\in I_p} H^q(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}, \mathbb{Z}) \Rightarrow H^{p+q}(X_\Sigma, \mathbb{Z}).$$

Our strategy will be to compute $E_2^{p,q}$ for small values of p,q.

The first observation is that

$$E_1^{p,0} = \bigoplus_{(i_0,\dots,i_p)\in I_p} \mathbb{Z}$$

since $U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}} = U_{\sigma_\gamma}$, $\sigma_\gamma = \sigma_{i_0} \cap \cdots \cap \sigma_{i_p}$, is connected for all γ. Hence we get the Koszul complex (9.1.15) with $M = \mathbb{Z}$, minus its first term. Thus

$$E_2^{p,0} = \begin{cases} 0 & p > 0 \\ \mathbb{Z} & p = 0. \end{cases}$$

Since the maximal cones in Σ are n-dimensional, Proposition 12.3.1 implies

$$E_1^{0,q} = \bigoplus_{\sigma_i \in \Sigma(n)} H^q(U_{\sigma_i}, \mathbb{Z}) = 0, \text{ for all } q > 0.$$

It follows that

$$E_2^{0,q} = 0 \quad \text{for all } q > 0.$$

Thus the E_2 sheet of the spectral sequence (with differentials shown only in the $p = 1$ column) is:

$$\begin{tikzcd}
0 & E_2^{1,2} \arrow[rrd, "d_2^{1,2}"] & E_2^{2,2} & E_2^{3,2} \\
0 & E_2^{1,1} \arrow[rrd, "d_2^{1,1}"] & E_2^{2,1} & E_2^{3,1} \\
\mathbb{Z} & 0 & 0 & 0.
\end{tikzcd}$$

Then $E_r^{2,0}$ and $E_r^{0,2}$ must be zero for all $r \geq 2$. Moreover, the differentials into and out of $E_r^{1,1}$ for all $r \geq 2$ must be zero and as a result,

$$E_2^{1,1} = E_\infty^{1,1} \simeq H^2(X_\Sigma, \mathbb{Z}).$$

However, we also know that $E_2^{1,1}$ is the kernel of the map

$$E_1^{1,1} = \bigoplus_{i<j} H^1(U_{\sigma_i} \cap U_{\sigma_j}, \mathbb{Z}) \longrightarrow E_1^{2,1} = \bigoplus_{i<j<k} H^1(U_{\sigma_i} \cap U_{\sigma_j} \cap U_{\sigma_k}, \mathbb{Z})$$

since $E_1^{0,1} = 0$. By Proposition 12.3.1,

$$H^1(U_{\sigma_i} \cap U_{\sigma_j}, \mathbb{Z}) \simeq M(\sigma_i \cap \sigma_j), \quad H^1(U_{\sigma_i} \cap U_{\sigma_j} \cap U_{\sigma_k}, \mathbb{Z}) \simeq M(\sigma_i \cap \sigma_j \cap \sigma_k).$$

Hence we get the commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H^2(X_\Sigma,\mathbb{Z}) \arrow[r] & \bigoplus_{i<j} M(\sigma_i\cap\sigma_j) \arrow[r] \arrow[d] & \bigoplus_{i<j<k} M(\sigma_i\cap\sigma_j\cap\sigma_k) \arrow[d] \\
M \arrow[r] & \bigoplus_i M \arrow[r] \arrow[dr, "\psi"] & \bigoplus_{i<j} M \arrow[r] \arrow[d] & \bigoplus_{i<j<k} M \arrow[d] \\
 & & \bigoplus_{i<j} M/M(\sigma_i\cap\sigma_j) \arrow[r] \arrow[d] & \bigoplus_{i<j<k} M/M(\sigma_i\cap\sigma_j\cap\sigma_k) \arrow[d] \\
 & & 0 & 0
\end{tikzcd}$$

The last two columns are obviously exact, and the first row is exact by the analysis of $H^2(X_\Sigma,\mathbb{Z})$ given above. The second row is also exact since it is the Koszul complex from (9.1.15). Then an easy diagram chase gives the exact sequence

$$M \longrightarrow \ker(\psi) \longrightarrow H^2(X_\Sigma,\mathbb{Z}) \longrightarrow 0.$$

However, σ_i has dimension n for all i, so that $M(\sigma_i) = 0$ for all i. Thus

$$\ker(\psi) = \ker\big(\bigoplus_i M/M(\sigma_i) \to \bigoplus_{i<j} M/M(\sigma_i\cap\sigma_j)\big) \simeq \mathrm{CDiv}_{T_N}(X_\Sigma),$$

and it follows immediately that $H^2(X_\Sigma,\mathbb{Z}) \simeq \mathrm{Pic}(X_\Sigma)$. □

This is a wonderful illustration of how to use spectral sequences.

Computing the Other Cohomology Groups. The spectral sequence of the cover $\mathscr{U} = \{U_\sigma\}_{\sigma\in\Sigma_{\max}}$ can be used to study the $H^k(X_\Sigma,\mathbb{Z})$ for all k. However, if there are many cones in Σ it becomes somewhat unwieldy to derive detailed information about H^k for $k > 2$ this way. In remainder of this section, we will see how to use the decomposition of X_Σ into torus orbits to compute the cohomology groups $H^k(X_\Sigma,\mathbb{Z})$, and hence the additive structure of $H^\bullet(X_\Sigma,\mathbb{Z})$, more efficiently. The ring structure of $H^\bullet(X_\Sigma,\mathbb{Z})$ will be discussed in §12.4.

A Family of Complexes. The method for computing the cohomology groups of X_Σ that we present comes from [**162**]. To begin, we discuss a notion of orientation for a pair of cones $\sigma \prec \tau$ with $\dim\tau = \dim\sigma + 1$. First, for each cone σ, we may arbitrarily pick an orientation of the linear subspace $(N_\sigma)_\mathbb{R}$ spanned by σ, determined by a choice of basis. For instance, in the special case that Σ is a simplicial fan, we can number the one-dimensional cones $\rho_i = \mathrm{Cone}(u_i)$ in a fixed way and if $\sigma = \rho_{i_1} + \cdots + \rho_{i_p}$ with $i_1 < \cdots < i_p$, then we can specify the orientation via the element

$$u_{i_1}\wedge\cdots\wedge u_{i_p} \in \textstyle\bigwedge^p N_\sigma.$$

Now if $\dim\tau = \dim\sigma + 1$, let v be any vector in τ not contained in σ. Then v together with a basis for $(N_\sigma)_{\mathbb{R}}$ forms a basis for $(N_\tau)_{\mathbb{R}}$, and defines an orientation. We define an *orientation coefficient*

$$c_{\sigma,\tau} = \begin{cases} +1 & \text{if the orientation of } \tau \text{ determined by } \sigma \text{ agrees with the chosen one} \\ -1 & \text{if not} \\ 0 & \text{if } \sigma \text{ is not a face of } \tau. \end{cases}$$

Given any fan Σ, fix an integer q, $0 \le q \le n$, and consider the abelian groups and maps

$$C^\bullet(\Sigma, \textstyle\bigwedge^q) = \{(C^p(\Sigma, \textstyle\bigwedge^q), \delta^p) \mid p \in \mathbb{Z}\}$$

defined as follows. First, we take

$$C^p(\Sigma, \textstyle\bigwedge^q) = \displaystyle\bigoplus_{\tau \in \Sigma(n-p)} \textstyle\bigwedge^q M(\tau),$$

where $M(\tau) = \tau^\perp \cap M$ as usual. This is free abelian with

$$\operatorname{rank} C^p(\Sigma, \textstyle\bigwedge^q) = \dbinom{p}{q} |\Sigma(n-p)|.$$

Then

$$\delta^p : C^p(\Sigma, \textstyle\bigwedge^q) \longrightarrow C^{p+1}(\Sigma, \textstyle\bigwedge^q)$$

is the map defined on the components corresponding to cones (τ, σ) in the two direct sums by the following rule. If σ is not a face of τ, that component of δ^p is defined to be zero. On the other hand, if $\sigma \prec \tau$, then δ^p is defined by

$$c_{\sigma,\tau}\, i^q_{\sigma,\tau}\,,$$

where $c_{\sigma,\tau}$ are the orientation coefficients, and

$$i^q_{\sigma,\tau} : \textstyle\bigwedge^q M(\tau) \longrightarrow \textstyle\bigwedge^q M(\sigma)$$

is induced by the inclusion $\tau^\perp \subseteq \sigma^\perp$. In other words, the component of δ^p in the summand for the cone σ in $C^{p+1}(\Sigma, \bigwedge^q)$ is given by

$$\sum_{\sigma \prec \tau} c_{\sigma,\tau}\, i^q_{\sigma,\tau}.$$

Note that $C^p(\Sigma, \bigwedge^q)$ is nonzero only for $0 \le q \le p \le n$.

Lemma 12.3.3. *$C^\bullet(\Sigma, \bigwedge^q)$ is a complex, i.e., $\delta^{p+1} \circ \delta^p = 0$ for all p.*

Proof. If γ is a codimension 2 face of a cone τ , then there are exactly two facets of τ containing γ, and you will show in Exercise 12.3.1 that

$$\sum_{\gamma \prec \sigma \prec \tau} c_{\gamma,\sigma}\, c_{\sigma,\tau} = 0. \tag{12.3.1}$$

Now, for any cone $\tau \in \Sigma(n-p)$,

$$\begin{aligned}\delta^{p+1} \circ \delta^p|_{M(\tau)} &= \delta^{p+1} \circ \sum_{\substack{\sigma\in\Sigma(n-p-1)\\ \tau\succ\sigma}} c_{\sigma,\tau}\, i^q_{\sigma,\tau} \\ &= \sum_{\substack{\gamma\in\Sigma(n-p-2)\\ \sigma\succ\gamma}} \Big(\sum_{\substack{\sigma\in\Sigma(n-p-1)\\ \tau\succ\sigma}} c_{\sigma,\tau}\, c_{\gamma,\sigma}\, i^q_{\gamma,\sigma} \circ i^q_{\sigma,\tau}\Big) \\ &= \sum_{\gamma\in\Sigma(n-p-2)} \Big(\sum_{\substack{\sigma\in\Sigma(n-p-1)\\ \gamma\prec\sigma\prec\tau}} c_{\gamma,\sigma}\, c_{\sigma,\tau}\Big) i^q_{\gamma,\tau} \\ &= 0,\end{aligned}$$

using (12.3.1). □

Example 12.3.4. Consider the fan defining $\mathbb{P}^2$, shown for instance in Figure 2 from Example 3.1.9. Denote $\rho_i = \mathrm{Cone}(e_i)$ for $i = 0,1,2$. We show the complexes $(C^\bullet(\Sigma, \bigwedge^q), \delta^\bullet)$ for $q = 0,1,2$ in the following diagram:

$$\begin{array}{lll} & q=2: & 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \\ (12.3.2) & q=1: & 0 \longrightarrow 0 \longrightarrow \mathbb{Z}^3 \xrightarrow{C} \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \cdots \\ & q=0: & 0 \longrightarrow \mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^3 \xrightarrow{B} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots. \end{array}$$

For instance, on the row for $q = 1$, the group $C^1(\Sigma, \bigwedge^1)$ is

$$C^1(\Sigma, \textstyle\bigwedge^1) = \displaystyle\bigoplus_{\rho\in\Sigma(1)} \textstyle\bigwedge^1 M(\rho) \simeq \mathbb{Z}^3,$$

since

$$\begin{aligned} \textstyle\bigwedge^1 M(\rho_0) &= \mathbb{Z}(e_1 - e_2) \\ \textstyle\bigwedge^1 M(\rho_1) &= \mathbb{Z}e_2 \\ \textstyle\bigwedge^1 M(\rho_2) &= \mathbb{Z}e_1. \end{aligned}$$

Similarly,

$$C^2(\Sigma, \textstyle\bigwedge^1) = \bigwedge^1 M(\{0\}) = M \simeq \mathbb{Z}^2.$$

Use orientation coefficients determined by the numbering of the one-dimensional cones in the order ρ_0, ρ_1, ρ_2 and the two-dimensional cones listed in the order $\sigma_1, \sigma_2, \sigma_0$. The map denoted by C in (12.3.2) is defined by the matrix

$$C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

The maps denoted by A and B in (12.3.2) are

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

Note that $BA = 0$ in agreement with Lemma 12.3.3. $\Diamond$

The cohomology groups

$$H^p(\Sigma, \textstyle\bigwedge^q) = \ker(\delta^p)/\mathrm{im}(\delta^{p-1})$$

of these complexes will appear shortly.

Spectral Sequence of a Filtered Topological Space. For each integer p, the toric variety X_Σ contains the union of the closures of torus orbits of dimension p,

$$X_p = \bigcup_{\sigma \in \Sigma(n-p)} V(\sigma) = \coprod_{\tau \in \Sigma(\ell),\, \ell \geq n-p} O(\tau).$$

By definition, these subsets give an increasing filtration of X_Σ,

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X_\Sigma. \tag{12.3.3}$$

For technical reasons, when working with general Σ, where X_Σ may not be compact, we will consider *cohomology with compact supports*, denoted $H^k_c(X_\Sigma, \mathbb{Z})$ (see [**135**, p. 242]). When Σ is complete the groups $H^k_c(X_\Sigma, \mathbb{Z})$ and the ordinary singular cohomology groups $H^k(X_\Sigma, \mathbb{Z})$ coincide. For orientable noncompact manifolds of real dimension d, it is the cohomology groups with compact supports that appear in the statement of *Poincaré duality* (see [**135**, Thm. 3.35]). Thus for instance, if X is homeomorphic to an open ball in $\mathbb{R}^d$,

$$H^k_c(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = d \\ 0 & \text{otherwise.} \end{cases} \tag{12.3.4}$$

Corresponding to the filtration (12.3.3), we have a first quadrant cohomology spectral sequence $E_r^{p,q}$ with

$$E_1^{p,q} = H_c^{p+q}(X_p, X_{p-1}, \mathbb{Z}) \Rightarrow H_c^{p+q}(X_\Sigma, \mathbb{Z}). \tag{12.3.5}$$

See Theorem C.2.5 in Appendix C, and [**136**, Ch. 1] or [**255**, §9.4] for more details on the construction.

Proposition 12.3.5. *For $p, q \geq 0$, the spectral sequence* (12.3.5) *has*

$$E_1^{p,q} \simeq \bigoplus_{\tau \in \Sigma(n-p)} \textstyle\bigwedge^q M(\tau) = C^p(\Sigma, \textstyle\bigwedge^q).$$

Moreover, the differentials $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ agree with the coboundary maps in the complex $C^\bullet(\Sigma, \bigwedge^q)$, so that

$$E_2^{p,q} = H^p(\Sigma, \textstyle\bigwedge^q).$$

Proof. By the excision property of cohomology with compact supports, we have

$$E_1^{p,q} \simeq \bigoplus_{\tau \in \Sigma(n-p)} H_c^{p+q}(O(\tau), \mathbb{Z}).$$

Furthermore, the homeomorphism $O(\tau) \cong \mathbb{R}^p_{>0} \times S_{N(\tau)}$ and the Künneth formula imply that

$$H_c^{p+q}(O(\tau),\mathbb{Z}) \simeq \bigoplus_{k+\ell=p+q} H_c^k(\mathbb{R}^p_{>0},\mathbb{Z}) \otimes_{\mathbb{Z}} H_c^\ell(S_{N(\tau)},\mathbb{Z}).$$

To analyze the terms in this direct sum, note that by (12.3.4)

$$H_c^k(\mathbb{R}^p_{>0},\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = p \\ 0 & \text{otherwise.} \end{cases}$$

Using Proposition 12.3.1, this shows that for each cone τ of dimension $n-p$,

$$H_c^{p+q}(O(\tau),\mathbb{Z}) \simeq H^q(S_{N(\tau)},\mathbb{Z}) \simeq \textstyle\bigwedge^q M(\tau).$$

Hence $E_1^{p,q} \simeq C^p(\Sigma, \bigwedge^q)$ as claimed.

To complete the proof, we need to show that the coboundary maps in the complex $C^\bullet(\Sigma, \bigwedge^q)$ agree with the differentials in the spectral sequence. By the construction of the spectral sequence as in [**255**], the differential

$$d_1^{p,q} : E_1^{p,q} \longrightarrow E_1^{p+1,q}$$

comes from the connecting homomorphism in the long exact cohomology sequence of the triple (X_{p+1}, X_p, X_{p-1}):

$$H_c^{p+q}(X_p, X_{p-1},\mathbb{Z}) \longrightarrow H_c^{p+q+1}(X_{p+1}, X_p,\mathbb{Z}).$$

By considering how this connecting homomorphism arises, it can be seen that $d_1^{p,q}$ coincides with δ^p from the complex $C^\bullet(\Sigma, \bigwedge^q)$ (Exercise 12.3.2). □

Since $E_1^{p,q}$ is zero for all p,q outside the triangular region $0 \le q \le p \le n$, it follows easily that for r sufficiently large, all of the higher differentials

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

will be zero and the spectral sequence will degenerate with

$$E_r^{p,q} = E_{r+1}^{p,q} = \cdots = E_\infty^{p,q}$$

for all p,q. When this happens, the $E_\infty^{p,q}$ with $p+q=k$ are successive quotients in a filtration of $H_c^k(X_\Sigma,\mathbb{Z})$. In relatively small dimensions, and in many other good cases as well, this information can be used to determine these cohomology groups completely. We will illustrate this with the following examples.

Example 12.3.6. We continue from (12.3.2) to compute the E_2 sheet of the spectral sequence (12.3.5) arising from the fan for $\mathbb{P}^2$. By a direct computation, the $q=0$ row is $E_2^{0,0} = \mathbb{Z}$ and $E_2^{1,0} = E_2^{2,0} = 0$ (see also Exercise 12.3.3).

On the second row, the kernel of C is 1-dimensional and the image of C is $\mathbb{Z}^2$. Hence $E_2^{1,1} \simeq \mathbb{Z}$, and $E_2^{2,1} = 0$. Finally $E_2^{2,2} = \mathbb{Z}$. Hence the E_2 sheet of the spectral sequence is reduced to:

$$
\begin{array}{ccccc}
0 & 0 & E_2^{2,2} = \mathbb{Z} & 0 & \cdots \\
0 & E_2^{1,1} = \mathbb{Z} & 0 & 0 & \cdots \\
E_2^{0,0} = \mathbb{Z} & 0 & 0 & 0 & \cdots .
\end{array}
\tag{12.3.6}
$$

The E_2 differentials are $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$. It can be seen directly in this case that the spectral sequence degenerates at E_2. Since there is at most one nonzero group for each possible value of $p+q$, it follows that

$$H^0(\mathbb{P}^2,\mathbb{Z}) \simeq \mathbb{Z}, \quad H^2(\mathbb{P}^2,\mathbb{Z}) \simeq \mathbb{Z}, \quad H^4(\mathbb{P}^2,\mathbb{Z}) \simeq \mathbb{Z},$$

and all of the other $H^k(\mathbb{P}^2,\mathbb{Z})$ (including all the odd-numbered ones) are zero (see Proposition C.1.5). The computations here can be generalized without difficulty to the case of $X_\Sigma = \mathbb{P}^n$ defined by the fan with one-dimensional cones $\rho_i = \mathrm{Cone}(e_i)$, $i = 0,\dots,n$ and $e_0 = -e_1 - \cdots - e_n$. The result is that

$$H^{2k}(\mathbb{P}^n,\mathbb{Z}) \simeq \mathbb{Z}, \quad k = 0,\dots,n,$$

and the odd-numbered cohomology groups are all zero. $\Diamond$

Our next examples show that the integral cohomology groups of a toric variety can have torsion.

Example 12.3.7. Let Σ in $N_\mathbb{R} \simeq \mathbb{R}^2$ be the complete fan with 1-dimensional cones ρ_i and ray generators u_i for $i = 1,\dots,r$, listed counterclockwise around the origin. The 2-dimensional cones are $\sigma_i = \rho_i + \rho_{i+1}$, $i = 1,\dots,r$ if we take indices modulo r. The rows of the E_1 sheet of the spectral sequence (12.3.5) for $q = 0,1,2$ are

$$
\begin{array}{l}
0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \\
0 \longrightarrow 0 \longrightarrow \mathbb{Z}^r \xrightarrow{C} \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \cdots \\
0 \longrightarrow \mathbb{Z}^r \xrightarrow{A} \mathbb{Z}^r \xrightarrow{B} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots .
\end{array}
\tag{12.3.7}
$$

By Exercise 12.3.3, $E_2^{p,0} = 0$ for $p = 1,2$ and $E_2^{0,0} \simeq \mathbb{Z}$.

The map C on the $q = 1$ row is the defined by the $2 \times r$ matrix with columns given by the vectors $u_i = (a_i,b_i)$. Since each pair u_i,u_{i+1} spans a two-dimensional cone σ_i, the kernel of C has rank $r-2$ and the image of C is a rank 2 sublattice of M. The E_2 sheet of the spectral sequence has the form

$$
\begin{array}{ccc}
0 & 0 & E_2^{2,2} = \mathbb{Z} \\
0 & E_2^{1,1} = \mathbb{Z}^{r-2} & E_2^{2,1} = \mathbb{Z}/m\mathbb{Z} \\
E_2^{0,0} = \mathbb{Z} & 0 & 0.
\end{array}
\tag{12.3.8}
$$

The integer m is given by

$$m = \gcd\left\{\det\begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix} \;\middle|\; 1 \le i < j \le r\right\}.$$

Once again, because of the placement of the nonzero terms in E_2, all of the higher differentials must be zero, so $E_2 = E_\infty$ and there is at most one nonzero $E_\infty^{p,q}$ for each value of $p+q$. Hence

$$\begin{aligned} H^0(X_\Sigma,\mathbb{Z}) \simeq \mathbb{Z},\quad H^1(X_\Sigma,\mathbb{Z}) = 0,\quad H^2(X_\Sigma,\mathbb{Z}) \simeq \mathbb{Z}^{r-2},\\ H^3(X_\Sigma,\mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z},\quad H^4(X_\Sigma,\mathbb{Z}) \simeq \mathbb{Z}. \end{aligned}$$

By the universal coefficient theorem for cohomology [**135**, Thm. 3.2], the torsion also appears in $H_2(X_\Sigma,\mathbb{Z})$. Also, note that $H_1(X_\Sigma,\mathbb{Z}) = 0$ as we expect from §12.1 since X_Σ is simply connected and $H_1(X_\Sigma,\mathbb{Z})$ is the abelianization of $\pi_1(X_\Sigma)$. You will study some special cases in Exercise 12.3.6. $\Diamond$

We conclude with a nonsimplicial example that illustrates most of the general behavior of these cohomology groups.

Example 12.3.8. Consider the fan Σ in $\mathbb{R}^3$ whose maximal cones are the cones over the faces of the cube with vertices $(\pm1,\pm1,\pm1)$. This cube is shown in Figure 8 of Example 2.3.11 and Σ is the normal fan of the octahedron shown in the same figure. The E_1 sheet in the spectral sequence (12.3.5) is:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots\\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}^8 & \xrightarrow{F} & \mathbb{Z}^3 & \longrightarrow & 0 & \longrightarrow & \cdots\\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}^{12} & \xrightarrow{D} & \mathbb{Z}^{16} & \xrightarrow{E} & \mathbb{Z}^3 & \longrightarrow & 0 & \longrightarrow & \cdots\\
0 & \longrightarrow & \mathbb{Z}^6 & \xrightarrow{A} & \mathbb{Z}^{12} & \xrightarrow{B} & \mathbb{Z}^8 & \xrightarrow{C} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots.
\end{array}$$

Suitable orientation coefficients can be obtained by placing any orientations on the edges of the cube in $\mathbb{R}^3$. We leave it to the reader to prove the following claims. First, the matrices A,B,C on the $q=0$ row of the E_1 sheet make that row exact except at the left and A has a rank 1 kernel, so $E_2^{0,0} \simeq \mathbb{Z}$, while

$$E_2^{1,0} = E_2^{2,0} = E_2^{3,0} = 0.$$

For the $q=1$ and $q=2$ rows, we must determine the $M(\sigma)$ for the cones in Σ. When this is done, it is routine to show that D has rank 11 and E has rank 3. The image of E is a sublattice of index 2 in $\bigwedge^1 M \simeq \mathbb{Z}^3$. Hence, $E_2^{1,1} \simeq \mathbb{Z}$, $E_2^{2,1} \simeq \mathbb{Z}^2$, and $E_2^{3,1} \simeq \mathbb{Z}/2\mathbb{Z}$. With respect to obvious choices of bases, the matrix F is

$$F = \begin{pmatrix} 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

It is easy to check that F has rank 3, but that the image is a sublattice of index 4 in $\bigwedge^2 M \simeq \mathbb{Z}^3$. The quotient is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. As a result the E_2 sheet of the spectral sequence is:

$$\begin{matrix} 0 & 0 & 0 & E_2^{3,3} = \mathbb{Z} \\ 0 & 0 & E_2^{2,2} = \mathbb{Z}^5 & E_2^{3,2} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\ 0 & E_2^{1,1} = \mathbb{Z} & E_2^{2,1} = \mathbb{Z}^2 & E_2^{3,1} = \mathbb{Z}/2\mathbb{Z} \\ E_2^{0,0} = \mathbb{Z} & 0 & 0 & 0. \end{matrix} \tag{12.3.9}$$

In this case too, all of the higher differentials are zero, so the spectral sequence degenerates at this point, and the E_∞ sheet coincides with (12.3.9). You will verify the details in Exercise 12.3.7 and show that

$$\begin{aligned} H^0(X_\Sigma, \mathbb{Z}) &\simeq \mathbb{Z}, & H^1(X_\Sigma, \mathbb{Z}) &= 0 \\ H^2(X_\Sigma, \mathbb{Z}) &\simeq \mathbb{Z}, & H^3(X_\Sigma, \mathbb{Z}) &\simeq \mathbb{Z}^2 \\ H^4(X_\Sigma, \mathbb{Z}) &\simeq \mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z}, & H^5(X_\Sigma, \mathbb{Z}) &\simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \\ H^6(X_\Sigma, \mathbb{Z}) &\simeq \mathbb{Z}. \end{aligned}$$

Note that unlike the previous examples, some of the odd cohomology groups would be nonzero in this example even if coefficients in $\mathbb{Q}$ were used. One can do these computations using the Maple package `torhom` developed by Franz [**98**]. $\Diamond$

The Topological Euler Characteristic. The spectral sequence (12.3.5) can be used to deduce several connections between the topology of the toric variety X_Σ and the combinatorics of the fan Σ. First, we consider the *topological Euler characteristic* of X_Σ, which by [**105**, p. 95] and [**162**, Prop. 3.1.2] equals

$$e(X_\Sigma) = \sum_{k=0}^{2n} (-1)^k \operatorname{rank} H_c^k(X_\Sigma, \mathbb{Z}) = \sum_{k=0}^{2n} (-1)^k \dim H_c^k(X_\Sigma, \mathbb{Q}). \tag{12.3.10}$$

Theorem 12.3.9. *For an n-dimensional toric variety X_Σ, the Euler characteristic is the number of n-dimensional cones in Σ, i.e.,*

$$e(X_\Sigma) = |\Sigma(n)|.$$

Proof. For each sheet of the spectral sequence (12.3.5), define

$$e(E_r) = \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \operatorname{rank} E_r^{p,q}.$$

Since E_{r+1} is the cohomology of E_r with respect to $d_r^{p,q}$, it follows that

$$e(E_{r+1}) = e(E_r)$$

for all r. The $E_\infty^{p,q}$ with $p+q=k$ are the quotients of a filtration of $H^k(X_\Sigma,\mathbb{Z})$, and by Proposition 12.3.5,

$$e(E_1) = \sum_{p,q\in\mathbb{Z}} (-1)^{p+q}\operatorname{rank}\big(\bigoplus_{\tau\in\Sigma(n-p)} {\textstyle\bigwedge}^q M(\tau)\big) = \sum_{p,q\in\mathbb{Z}} (-1)^{p+q}\binom{p}{q}|\Sigma(n-p)|$$

since $M(\tau)\simeq\mathbb{Z}^p$ when $\tau\in\Sigma(n-p)$. Hence

$$e(X_\Sigma) = e(E_\infty) = e(E_1) = \sum_{p=0}^{n}\Big(\sum_{q=0}^{p}(-1)^{p+q}\binom{p}{q}\Big)|\Sigma(n-p)| = |\Sigma(n)|,$$

where the last equality follows since the inner sums vanish for $p>0$ by properties of binomial coefficients. $\square$

When X_Σ is complete and simplicial, the individual Betti numbers (the ranks of the $H^k(X_\Sigma,\mathbb{Z})$) can also be determined from the structure of Σ.

Rational Coefficients. To suppress torsion in cohomology, we switch coefficients from $\mathbb{Z}$ to $\mathbb{Q}$. In this case, there is also a significant simplification in the behavior of the cohomology spectral sequence

$$E_1^{p,q} = H_c^{p+q}(X_p,X_{p-1},\mathbb{Q}) \Rightarrow H_c^{p+q}(X_\Sigma,\mathbb{Q}). \tag{12.3.11}$$

The same argument given in Proposition 12.3.5 shows that

$$E_1^{p,q} \simeq \bigoplus_{\tau\in\Sigma(n-p)} {\textstyle\bigwedge}^q M(\tau)_{\mathbb{Q}}. \tag{12.3.12}$$

Proposition 12.3.10. *The spectral sequence* (12.3.11) *degenerates at E_2.*

Proof. We show that $d_r^{p,q}=0$ for all $r\geq 2$ and all (p,q), so that $E_2^{p,q}=E_\infty^{p,q}$. Recall from Example 3.3.6 that for any positive integer ℓ, the multiplication map

$$\overline{\phi}_\ell : N \longrightarrow N, \qquad a \longmapsto \ell\cdot a$$

is compatible with Σ so there is a corresponding toric morphism $\phi_\ell : X_\Sigma\to X_\Sigma$ whose restriction to $T_N\subseteq X_\Sigma$ is the group homomorphism

$$\phi_\ell|_{T_N}(t_1,\ldots,t_n) = (t_1^\ell,\ldots,t_n^\ell),$$

and similarly on each torus orbit. Because ϕ_ℓ respects the orbit decomposition of X_Σ, it respects the filtration from (12.3.3) and induces homomorphisms

$$\phi_\ell^* : E_r^{p,q} \longrightarrow E_r^{p,q}$$

for each r. These commute with the differentials since (12.3.11) is functorial with respect to maps that preserve the filtration.

In Exercise 12.3.8 you will use $E_1^{p,q}\simeq\bigoplus_{\tau\in\Sigma(n-p)} H_c^q(O(\tau),\mathbb{Q})$ to show that ϕ_ℓ^* acts on $E_1^{p,q}$ by multiplication by ℓ^q. Then the same holds for all r since $E_{r+1}^{p,q}$ is a quotient of a subspace of $E_r^{p,q}$.

Let $\beta \in E_r^{p,q}$ for $r \geq 2$. Since $d_r^{p,q}(\beta) \in E_r^{p+r,q-r+1}$, we have

$$\begin{aligned} \ell^{q-r+1} d_r^{p,q}(\beta) &= \phi_\ell^*(d_r^{p,q}(\beta)) \\ &= d_r^{p,q}(\phi_\ell^*(\beta)) \\ &= d_r^{p,q}(\ell^q \beta) \\ &= \ell^q d_r^{p,q}(\beta). \end{aligned}$$

Since we use coefficients in $\mathbb{Q}$, this implies $d_r^{p,q}(\beta) = 0$ for all β. $\square$

Proposition 12.3.10 shows that computing $H_c^k(X_\Sigma, \mathbb{Q})$ using the cohomology spectral sequence is especially simple. Franz has shown in [**96**] that if X_Σ is smooth, the spectral sequence with integer coefficients also degenerates at E_2, and no additional extension data is needed to determine $H_c^*(X_\Sigma, \mathbb{Z})$.

A Vanishing Theorem for Singular Cohomology. We will now focus on complete simplicial toric varieties. In this case, the following result of Oda [**220**, Thm. 4.1] implies that the spectral sequence (12.3.11) simplifies even further.

Theorem 12.3.11. *If X_Σ is complete and simplicial, then $E_2^{p,q} = 0$ when $p \neq q$ in the spectral sequence* (12.3.11). *Thus:*

(a) $H^{2k+1}(X_\Sigma, \mathbb{Q}) = 0$ *for all k.*

(b) $H^{2k}(X_\Sigma, \mathbb{Q}) \simeq E_2^{k,k}$ *for all k.*

Proof. Our proof will use some results from Chapter 9. Theorem 9.3.2 tells us that

$$H^p(X_\Sigma, \widehat{\Omega}^q_{X_\Sigma}) = 0 \tag{12.3.13}$$

when $p \neq q$, and the Hodge decomposition (9.4.11) states that

$$H^k(X_\Sigma, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^p(X_\Sigma, \widehat{\Omega}^q_{X_\Sigma}). \tag{12.3.14}$$

Here, $\widehat{\Omega}^q_{X_\Sigma}$ is the sheaf of Zariski q-forms on X_Σ defined in §8.0.

It is now easy to show that $E_2^{p,q} = 0$ when $p+q$ is odd, since (12.3.13) and (12.3.14) imply that $H^k(X_\Sigma, \mathbb{C}) = 0$ for odd k. When we combine this with the degeneration proved in Proposition 12.3.10, we obtain

$$0 = \dim H^k(X_\Sigma, \mathbb{Q}) = \textstyle\sum_{p+q=k} \dim E_2^{p,q}.$$

When $p+q$ is even, we change notation slightly and consider $p+q = 2k$. The idea is to study how the maps $\phi_\ell : X_\Sigma \to X_\Sigma$ from the proof of Proposition 12.3.10 act on each side of (12.3.14). On the left-hand side, the degeneration implies that $H^{2k}(X_\Sigma, \mathbb{C})$ has a filtration $0 = F^{2k} \subseteq \cdots \subseteq F^0 = H^{2k}(X_\Sigma, \mathbb{C})$ such that

$$F^p/F^{p-1} \simeq (E_2^{p,q})_{\mathbb{C}} = E_2^{p,q} \otimes_{\mathbb{Q}} \mathbb{C}, \quad p+q = 2k.$$

By the proof of Proposition 12.3.10, $\phi_{\ell *} : (E_2^{p,q})_{\mathbb{C}} \to (E_2^{p,q})_{\mathbb{C}}$ is multiplication by ℓ^q. Hence $\phi_{\ell *} : H^{2k}(X_\Sigma, \mathbb{C}) \to H^{2k}(X_\Sigma, \mathbb{C})$ is multiplication by ℓ^q on F^p/F^{p-1}.

Now consider the right-hand side of (12.3.14) and assume for the moment that X_Σ is smooth. Recall from §8.1 that

$$\Omega^1_{X_\Sigma} \hookrightarrow M \otimes_{\mathbb{Z}} \mathscr{O}_{X_\Sigma} \simeq \Omega^1_{X_\Sigma}(\log D).$$

Since $\chi^m \circ \phi_\ell = \chi^{\ell m}$, we have $\phi_\ell^*(d\chi^m/\chi^m) = d\chi^{\ell m}/\chi^{\ell m} = \ell\, d\chi^m/\chi^m$. It follows that over the affine open subset U_σ, $\sigma \in \Sigma$, ϕ_ℓ^* acts on sections by multiplication by ℓ. Hence, for $\Omega^q_{X_\Sigma} = \bigwedge^q \Omega^1_{X_\Sigma}$, ϕ_ℓ^* acts on sections over U_σ by multiplication by ℓ^q. Computing cohomology via the Čech complex, we conclude that ϕ_ℓ^* acts on $H^p(X_\Sigma, \Omega^q_{X_\Sigma})$ by multiplication by ℓ^q. In Exercise 12.3.9, you will show that in the general case, ϕ_ℓ^* continues to be multiplication by ℓ^q on $H^p(X_\Sigma, \widehat{\Omega}^q_{X_\Sigma})$.

Since $H^{2k}(X_\Sigma, \mathbb{C}) \simeq H^k(X_\Sigma, \widehat{\Omega}^k_{X_\Sigma})$ by (12.3.13) and (12.3.14), it follows that $\phi_{\ell *}$ is multiplication by ℓ^k on $H^{2k}(X_\Sigma, \mathbb{C})$. Comparing this with our earlier analysis, we see that the filtration $\{F^p\}$ must collapse, giving $(E_2^{p,q})_{\mathbb{C}} = 0$ when $p+q = 2k$ and $p \neq q$. This proves the theorem. □

You should check that Examples 12.3.6 and 12.3.7 illustrate the vanishing of odd cohomology asserted by part (a) of Theorem 12.3.11. On the other hand, we see from Example 12.3.8 that this can fail for nonsimplicial X_Σ. There is also a more refined vanishing statement due to Brion [**48**, p. 5], which states that if X_Σ is complete of dimension n and all cones $\sigma \in \Sigma$ of dimension $\leq m$ are simplicial, then $E_2^{p,q} = 0$ unless $0 \leq p - q \leq n - m$.

Some Combinatorial Consequences. There are interesting relations between the numbers of cones of various dimensions in simplicial fans and the Betti numbers of the corresponding toric varieties, which are the numbers

$$b_{2k}(X_\Sigma) = \dim H^{2k}(X_\Sigma, \mathbb{Q}).$$

Theorem 12.3.12. *Let Σ be a complete simplicial fan in $N_{\mathbb{R}} \simeq \mathbb{R}^n$. Then the Betti numbers of X_Σ are given by*

$$b_{2k}(X_\Sigma) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} |\Sigma(n-i)|$$

and satisfy

$$b_{2k}(X_\Sigma) = b_{2n-2k}(X_\Sigma).$$

Proof. The $E_2^{p,q}$ terms of the spectral sequence (12.3.11) are the cohomology of the $E_1^{p,q}$ terms, and we also have $E_1^{p,q} = 0$ for $p < q$ by (12.3.12). Since $E_2^{p,q} = 0$ unless $p = q$ by Theorem 12.3.11, it follows that

$$0 \longrightarrow E_2^{k,k} \longrightarrow E_1^{k,k} \longrightarrow E_1^{k+1,k} \longrightarrow \cdots$$

is exact. Hence

$$b_{2k}(X_\Sigma) = \dim E_2^{k,k} = \sum_{i=k}^{n} (-1)^{i-k} \dim E_1^{i,k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} |\Sigma(n-i)|,$$

where the last equality uses (12.3.12). The second assertion follows from Poincaré duality, which is discussed in more detail in §12.4. □

If P is a full dimensional simple lattice polytope and X_P is the corresponding simplicial toric variety, then $|\Sigma(n-i)| = f_i$, the number of i-dimensional faces of P. An immediate corollary of the theorem is the formula

$$b_{2k}(X_P) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} f_i = h_k, \tag{12.3.15}$$

where the h_k are the combinations of the face numbers introduced in §9.4. The Dehn-Sommerville equations $h_k = h_{n-k}$ are a consequence of the symmetry of the Betti numbers in Theorem 12.3.12.

A final comment is that as in the proof of Theorem 12.3.11, (12.3.13) and (12.3.14) imply that

$$H^{2k}(X_P, \mathbb{C}) \simeq H^k(X_P, \widehat{\Omega}^k_{X_P})$$

for P as above. Since we also have $\dim H^k(X_P, \widehat{\Omega}^k_{X_P}) = h_k$ by Theorem 9.4.11, we get another proof of (12.3.15). This was noted earlier in (9.4.12).

Exercises for §12.3.

12.3.1. Show that if γ is a face of τ of codimension 2, then

$$\sum_{\gamma \prec \sigma \prec \tau} c_{\gamma,\sigma}\, c_{\sigma,\tau} = 0,$$

where the sum has two terms corresponding to the two facets of τ containing γ.

12.3.2. Complete the proof of Proposition 12.3.5 by showing that

$$d_1^{p,q} : H^{p+q}(X_p, X_{p-1}, \mathbb{Z}) \longrightarrow H^{p+q+1}(X_{p+1}, X_p, \mathbb{Z})$$

coincides with $\delta^p : C^p(\Sigma, \bigwedge^q) \to C^{p+1}(\Sigma, \bigwedge^q)$ under the isomorphisms given in the first part of the proof. Hint: If you get stuck, see [**162**].

12.3.3. This exercise will show that when X_Σ is complete, the $q = 0$ row of the E_2 sheet of the spectral sequence (12.3.5) is given by

$$E_2^{p,0} = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let S^{n-1} be the unit sphere in $N_\mathbb{R}$ and let

$$\mathcal{C} = \{\sigma \cap S^{n-1} \mid \sigma \in \Sigma, \sigma \neq \{0\}\}$$

be the spherical cell complex determined by Σ.

(a) Show that the complex $(C^\bullet(\Sigma, \bigwedge^0), \delta^\bullet)$ is isomorphic to the augmented cellular chain complex of the spherical complex $\mathcal{C}$ (see [**135**, p. 139]).

(b) Show that $E_2^{p,0}$ is isomorphic to the reduced homology group $\widetilde{H}_{n-p-1}(\mathcal{C},\mathbb{Z})$.

(c) If Σ is complete, then $\mathcal{C}$ is a subdivision of S^{n-1}. Deduce the above formula for $E_2^{p,0}$.

12.3.4. Let $\mathrm{Bl}_0(\mathbb{C}^2)$ denote the blowup of $\mathbb{C}^2$ at the origin. Compute the cohomology groups $H^k(\mathrm{Bl}_0(\mathbb{C}^2),\mathbb{Z})$ directly from the spectral sequence (12.3.5). Generalize this by computing $H^k(\mathrm{Bl}_0(\mathbb{C}^n),\mathbb{Z})$ for all $n \geq 2$.

12.3.5. Let Σ be the complete fan in $N_{\mathbb{R}} \simeq \mathbb{R}^2$ with four maximal cones $\mathrm{Cone}(\pm e_1, \pm e_2)$, so $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$. Compute $H^k(X_\Sigma,\mathbb{Z})$ directly via the spectral sequence (12.3.5), and compare with the result in Example 12.3.7. Generalize your computations to the complete fans Σ in $N_{\mathbb{R}} \simeq \mathbb{R}^n$ with maximal cones $\mathrm{Cone}(\pm e_1,\ldots,\pm e_n)$.

12.3.6. This exercise deals with Example 12.3.7.

(a) Show that if X_Σ is a smooth toric surface, then the cohomology groups $H^k(X_\Sigma,\mathbb{Z})$ are torsion free.

(b) Let X_Σ be the toric variety of the complete fan in $\mathbb{R}^2$ whose minimal ray generators are $u = (\pm 1, \pm 1)$. Show that $H^3(X_\Sigma,\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

12.3.7. In this exercise, you will verify details of the computation in Example 12.3.8.

(a) Show that the E_∞ sheet of the cohomology spectral sequence is given by (12.3.9).

(b) Show that the cohomology groups are as given in the example. Note that Proposition C.1.5 applies for all of the groups except $H^4(X_\Sigma,\mathbb{Z})$. Show that $H^4(X_\Sigma,\mathbb{Z}) \simeq \mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z}$ directly by considering the filtration with quotients given by $E_\infty^{p,q}$ with $p+q=4$.

12.3.8. Show that ϕ_ℓ is the ℓth power map on $O(\tau)$ and conclude that ϕ_ℓ^* is multiplication by ℓ^q on $H_c^q(O(\tau),\mathbb{Q}) \simeq H^q(S_{N(\tau)},\mathbb{Q})$.

12.3.9. Here are some details from the proof of Theorem 12.3.11

(a) Use the properties of reflexive sheaves from §8.0 to prove the assertion in the text that over U_σ, ϕ_ℓ^* acts on sections $\widehat{\Omega}^q_{X_\Sigma}$ by multiplication by ℓ^q.

(b) Explain carefully how computing cohomology via the Čech complex implies that ϕ_ℓ^* acts on $H^p(X_\Sigma,\widehat{\Omega}^q_{X_\Sigma})$ by multiplication by ℓ^q.

12.3.10. In this exercise, you will show that if Σ in $N_{\mathbb{R}} \simeq \mathbb{R}^n$ is smooth and complete, then the higher homotopy group $\pi_2(X_\Sigma)$ can be identified with $\mathrm{Pic}(X_\Sigma)^\vee$.

(a) Show that $\pi_2(X_\Sigma) \simeq H_2(X_\Sigma,\mathbb{Z})$. Hint: Apply the Hurewicz theorem as stated in [**135**, Thm. 4.32].

(b) Prove that $H_2(X_\Sigma,\mathbb{Z}) \simeq (H^2(X_\Sigma,\mathbb{Z}))^\vee$ under these hypotheses. Hint: You will need to show that $H_2(X_\Sigma,\mathbb{Z})$ has no torsion.

(c) Conclude that $\pi_2(X_\Sigma)$ is isomorphic to $\mathrm{Pic}(X_\Sigma)^\vee$.

12.3.11. To what extent is the cohomology of a toric variety X_Σ determined by the combinatorics of the fan Σ (the numbers of cones of each dimension, and their intersection relations)? In this exercise, drawing on [**200**], you will see that for $n \geq 3$, the Betti numbers of a toric variety are not necessarily determined by the combinatorial type of Σ.

(a) To begin, deduce from Example 12.3.7 that the Betti numbers of a complete toric surface *are* determined by the number of 2-dimensional cones in Σ, and that this is the only combinatorial invariant of Σ.

(b) Now as in Example 12.3.8, consider the fan Σ over the faces of the polytope P in $\mathbb{R}^3$ given by
$$P = \mathrm{Conv}\big(\pm e_1, \pm e_2, \pm e_3, \pm\tfrac{1}{2}e_1 \pm \tfrac{1}{2}e_2 \pm \tfrac{1}{2}e_3\big),$$
where we take all possible choices of signs, so there are 14 points in all in the set. Show that the Betti numbers of X_Σ are
$$1, 0, 2, 3, 11, 0, 1.$$
Note that X_Σ is not simplicial and Poincaré duality does not hold.

(c) Now let P' be the convex hull of the 14 points
$$e_1, e_2, -\tfrac{1}{2}e_1 + \tfrac{1}{2}e_3, -e_1, -e_2, -e_3, \tfrac{2}{5}e_1 + \tfrac{3}{5}e_2 + \tfrac{1}{5}e_3, \tfrac{1}{2}e_1 + \tfrac{1}{2}e_2 - \tfrac{1}{2}e_3,$$
$$\tfrac{2}{5}e_1 - \tfrac{3}{5}e_2 + \tfrac{1}{5}e_3, \tfrac{1}{2}e_1 - \tfrac{1}{2}e_2 - \tfrac{1}{2}e_3, -\tfrac{2}{3}e_1 + \tfrac{1}{3}e_2 + \tfrac{1}{3}e_3, -\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2 - \tfrac{1}{2}e_3,$$
$$-\tfrac{2}{3}e_1 - \tfrac{1}{3}e_2 + \tfrac{1}{3}e_3, -\tfrac{1}{2}e_1 - \tfrac{1}{2}e_2 - \tfrac{1}{2}e_3.$$
If Σ' is the fan over the faces of P', show that Σ and Σ' are combinatorially equivalent, but $X_{\Sigma'}$ has Betti numbers
$$1, 0, 1, 2, 11, 0, 1.$$

(d) Modify this example to produce similar examples in all dimensions $n \geq 3$.

12.3.12. The *Poincaré polynomial* $P_X(t) = \sum_{k=0}^{2n} b_k(X)t^k$ of a variety X of dimension n is the generating function for its Betti numbers. Show that Theorem 12.3.12 is equivalent to the assertion that $P_{X_\Sigma}(t) = \sum_{\sigma\in\Sigma}(t^2-1)^{\dim\sigma}$.

§12.4. The Cohomology Ring

In this section, we will make the standing assumption that X_Σ is complete and simplicial. Our goal is to prove a general theorem describing the ring structure given by cup product on
$$H^\bullet(X_\Sigma, \mathbb{Q}) = \bigoplus_{k=0}^{2n} H^k(X_\Sigma, \mathbb{Q}), \quad n = \dim X_\Sigma.$$
Along the way, we will also describe the equivariant cohomology ring $H^\bullet_{T_N}(X_\Sigma, \mathbb{Q})$.

Rationally Smooth Varieties. By Theorem 11.4.8, simplicial toric varieties are rationally smooth. The basic intuition is that rationally smooth varieties behave like smooth varieties, provided that one works over $\mathbb{Q}$.

We begin with some properties of the cohomology ring of an n-dimensional complete rationally smooth variety X:

- Poincaré duality holds between cohomology and homology, so that
$$H^k(X, \mathbb{Q}) \simeq H_{2n-k}(X, \mathbb{Q}).$$
- The isomorphism $H^{2n}(X, \mathbb{Q}) \simeq H_0(X, \mathbb{Q}) = \mathbb{Q}$ induces a map
$$\int_X : H^\bullet(X, \mathbb{Q}) \to \mathbb{Q},$$
where elements of $H^k(X, \mathbb{Q})$ map to zero when $k < 2n$.

- Poincaré duality implies that the cup product pairing
$$H^k(X,\mathbb{Q}) \times H^{2n-k}(X,\mathbb{Q}) \longrightarrow \mathbb{Q}$$
defined by $(\alpha,\beta) \mapsto \int_X \alpha \smile \beta$ is nondegenerate. Thus $H^{2n-k}(X,\mathbb{Q})$ is isomorphic to the dual vector space of $H^k(X,\mathbb{Q})$ and their dimensions are equal.
- An irreducible subvariety $W \subseteq X$ of codimension k has a *refined cohomology class* $[W]_r \in H^{2n-2k}(X, X \setminus W, \mathbb{Q})$. By mapping this to $H^{2n-2k}(X,\mathbb{Q})$, W has a *cohomology class* $[W] \in H^{2n-2k}(X,\mathbb{Q})$.
- The cohomology class of a divisor $D = \sum_i a_i D_i$ is $[D] = \sum_i a_i [D_i] \in H^2(X,\mathbb{Q})$, and linearly equivalent divisors give the same cohomology class.

See [**135**, Sec. 3.3] for the manifold case. For rationally smooth varieties these statements follow, for instance, from the properties of intersection cohomology developed by Goresky and MacPherson in [**119**] since over $\mathbb{Q}$, intersection cohomology coincides with ordinary cohomology for rationally smooth varieties. The assertions about refined cohomology classes and linearly equivalent Cartier divisors can be found in [**107**, Ch. 19]. We also note that if X is smooth, then the above properties hold over $\mathbb{Z}$. See also the appendix to Chapter 13.

These properties enable us to generalize the intersection products defined in §6.3. Let $V, W \subseteq X$ be irreducible subvarieties with $\dim V + \dim W = n$. The cup product $[V] \smile [W]$ lies in $H^{2n}(X,\mathbb{Q})$ and hence gives the *intersection product*

$$V \cdot W = \int_X [V] \smile [W] \in \mathbb{Q}. \tag{12.4.1}$$

When D is an irreducible Cartier divisor and C is an irreducible curve, this agrees with the intersection product $D \cdot C$ from Definition 6.3.6. See Exercise 12.5.9 and [**107**, §2.3].

Statement of the Main Theorem. Let X_Σ be a complete simplicial toric variety and fix a numbering $\rho_1, \ldots, \rho_r$ for the rays in $\Sigma(1)$. Also let u_i be the minimal generator of ρ_i and introduce a variable x_i for each ρ_i. In the ring $\mathbb{Q}[x_1,\ldots,x_r]$, let $\mathscr{I}$ be the monomial ideal with square-free generators as follows:

$$\mathscr{I} = \langle x_{i_1} \cdots x_{i_s} \mid i_j \text{ are distinct and } \rho_{i_1} + \cdots + \rho_{i_s} \text{ is not a cone of } \Sigma \rangle. \tag{12.4.2}$$

We call $\mathscr{I}$ the *Stanley-Reisner ideal*. Also let $\mathscr{J}$ be the ideal generated by the linear forms

$$\sum_{i=1}^r \langle m, u_i \rangle x_i, \tag{12.4.3}$$

where m ranges over M (or equivalently, over some basis for M). We define

$$R_{\mathbb{Q}}(\Sigma) = \mathbb{Q}[x_1,\ldots,x_r]/(\mathscr{I} + \mathscr{J}).$$

Since $\mathscr{I} + \mathscr{J}$ is homogeneous for the standard grading on $\mathbb{Q}[x_1,\ldots,x_r]$, $R_{\mathbb{Q}}(\Sigma)$ is a graded ring. Let $R_{\mathbb{Q}}(\Sigma)_d$ denote the graded piece in degree d.

Our next task is to show that $x_i \to [D_i] \in H^2(X_\Sigma,\mathbb{Q})$ induces a well-defined ring homomorphism

$$R_{\mathbb{Q}}(\Sigma) \longrightarrow H^\bullet(X_\Sigma,\mathbb{Q}). \tag{12.4.4}$$

First note that $\sum_{i=1}^r \langle m,u_i\rangle D_i = \mathrm{div}(\chi^m) \sim 0$, which by the above properties of cohomology classes implies that $\sum_{i=1}^r \langle m,u_i\rangle [D_i] = 0 \in H^2(X_\Sigma,\mathbb{Q})$. This shows that the ideal $\mathscr{J}$ maps to zero in $H^\bullet(X_\Sigma,\mathbb{Q})$.

Moreover, if $\rho_{i_1}+\cdots+\rho_{i_s}$ is not a cone of Σ, then $D_{i_1}\cap\cdots\cap D_{i_s} = \emptyset$ in X_Σ. Since cup product in relative cohomology satisfies

$$\alpha \in H^k(X,A,\mathbb{Q}) \text{ and } \beta \in H^\ell(X,B,\mathbb{Q}) \Longrightarrow \alpha\smile\beta \in H^{k+\ell}(X,A\cup B,\mathbb{Q})$$

(see [**210**, Ch. 5]), the cup product of the refined classes $[D_{i_j}]_r$ is

$$[D_{i_1}]_r \smile\cdots\smile [D_{i_s}]_r \in H^{2s}(X_\Sigma, \textstyle\bigcup_{j=1}^s (X_\Sigma\setminus D_{i_j}),\mathbb{Q}).$$

This relative cohomology group vanishes since $D_{i_1}\cap\cdots\cap D_{i_s} = \emptyset$. Then

$$[D_{i_1}]\smile\cdots\smile[D_{i_s}] = 0 \in H^{2s}(X_\Sigma,\mathbb{Q})$$

since $[D_{i_j}]_r$ maps to $[D_{i_j}]$. Hence the Stanley-Reisner ideal $\mathscr{I}$ also maps to zero in $H^\bullet(X_\Sigma,\mathbb{Q})$. This gives the ring homomorphism (12.4.4).

Our main theorem will show that $R_{\mathbb{Q}}(\Sigma)$ and $H^\bullet(X_\Sigma,\mathbb{Q})$ are isomorphic.

Theorem 12.4.1. *Let Σ be complete and simplicial. Then the map* (12.4.4) *is an isomorphism:*

$$R_{\mathbb{Q}}(\Sigma) \simeq H^\bullet(X_\Sigma,\mathbb{Q}).$$

Thus, in even degrees, $H^{2k}(X_\Sigma,\mathbb{Q})$ is isomorphic to $R_{\mathbb{Q}}(\Sigma)_k$, and in odd degrees, $H^{2k+1}(X_\Sigma,\mathbb{Q})$ is zero.

The proof of this theorem will be given later in the section.

Example 12.4.2. As a toric variety, $\mathbb{P}^n$ comes from the fan with ray generators $u_i = e_i$ for $i = 1,\dots,n$ and $u_0 = -e_1-\cdots-e_n$. Then it is easy to check that

$$\mathscr{I} = \langle x_0\cdots x_n\rangle,$$

and using the basis $e_1,\dots,e_n$ for M to obtain generators for $\mathscr{J}$, we have

$$\mathscr{J} = \langle x_1-x_0,\dots,x_n-x_0\rangle.$$

Then Theorem 12.4.1 gives an isomorphism

$$\begin{aligned} H^\bullet(\mathbb{P}^n,\mathbb{Q}) &\simeq \mathbb{Q}[x_0,x_1,\dots,x_n]/\langle x_0\cdots x_n, x_1-x_0,\dots,x_n-x_0\rangle \\ &\simeq \mathbb{Q}[x_0]/\langle x_0^{n+1}\rangle. \end{aligned}$$

This agrees with Example 9.0.13. See also Example 12.3.6. $\Diamond$

Example 12.4.3. Consider the Hirzebruch surface $\mathscr{H}_r$ and label the cones in the fan as in Example 10.4.6, so $\rho_1 = \mathrm{Cone}(-e_1+re_2)$, $\rho_2 = \mathrm{Cone}(e_2)$, $\rho_3 = \mathrm{Cone}(e_1)$, and $\rho_4 = \mathrm{Cone}(-e_2)$. Theorem 12.4.1 gives

$$H^\bullet(\mathscr{H}_r,\mathbb{Q}) \simeq \mathbb{Q}[x_1,x_2,x_3,x_4]/\langle x_1x_3, x_2x_4, -x_1+x_3, rx_1+x_2-x_4\rangle.$$

The two linear relations from $\mathscr{J}$ can be used to eliminate x_1,x_2, giving

$$H^\bullet(\mathscr{H}_r,\mathbb{Q}) \simeq \mathbb{Q}[x_3,x_4]/\langle x_3^2,\, x_4^2 - rx_3x_4\rangle.$$

Note that $1,x_3,x_4,x_3x_4$ form a $\mathbb{Q}$-basis of $H^\bullet(\mathscr{H}_r,\mathbb{Q})$. Hence

$$H^0(\mathscr{H}_r,\mathbb{Q}) \simeq \mathbb{Q},\quad H^2(\mathscr{H}_r,\mathbb{Q}) \simeq \mathbb{Q}^2,\quad \text{and}\quad H^4(\mathscr{H}_r,\mathbb{Q}) \simeq \mathbb{Q},$$

as we expect from §12.3. Also note that cup product in $H^\bullet(\mathscr{H}_r,\mathbb{Q})$ is defined by

$$x_3^2 = 0,\quad x_4^2 = rx_3x_4.$$

By (12.4.1), we recover the intersection form on the divisor classes, described by the matrix

$$(D_i\cdot D_j) = \begin{pmatrix} 0 & 1\\ 1 & r\end{pmatrix}$$

in Example 10.4.6. ◊

Theorem 12.4.1 also holds over $\mathbb{Z}$ when X_Σ is smooth. The examples just given show this for $\mathbb{P}^n$ and the Hirzebruch surface $\mathscr{H}_r$. Here is the precise result.

Theorem 12.4.4 (Jurkiewicz-Danilov). *Let X_Σ be a smooth complete toric variety. For the polynomial ring $\mathbb{Z}[x_1,\dots,x_r]$ with variables indexed by $\rho_1,\dots,\rho_r\in\Sigma(1)$, let $\mathscr{I}$ and $\mathscr{J}$ be the ideals in $\mathbb{Z}[x_1,\dots,x_r]$ generated by the polynomials in* (12.4.2) *and* (12.4.3)*, and define*

$$R(\Sigma) = \mathbb{Z}[x_1,\dots,x_r]/(\mathscr{I}+\mathscr{J}).$$

Then $x_i\mapsto [D_{\rho_i}]$ induces a ring isomorphism $R(\Sigma)\simeq H^\bullet(X_\Sigma,\mathbb{Z})$. □

See [**76**, Thm. 10.8], [**105**, Sec. 5.2] (which gives a proof under an additional hypothesis that is satisfied, for instance, if X_Σ is projective), and [**218**, Sec. 3.3]. Note that Theorem 12.4.1 is simultaneously less general (coefficients in $\mathbb{Q}$ instead of $\mathbb{Z}$) and more general (X_Σ is simplicial instead of smooth) than Theorem 12.4.4.

Let us make one final comment about the cohomology ring $H^\bullet(X_\Sigma,\mathbb{Q})$. Recall from Theorem 12.3.12 that the Betti numbers $b_k(X_\Sigma) = \dim H^k(X_\Sigma,\mathbb{Q})$ depend only on the combinatorial structure of the fan Σ in the simplicial case. Does this extend to the ring structure on $H^\bullet(X_\Sigma,\mathbb{Q})$? Sometimes the answer is yes.

Example 12.4.5. Given nonnegative integers $r\neq s$, the Hirzebruch surfaces $\mathscr{H}_r$ and $\mathscr{H}_s$ have cohomology rings

$$H^\bullet(\mathscr{H}_r,\mathbb{Q}) \simeq \mathbb{Q}[x,y]/\langle x^2, y^2-rxy\rangle,\ \ H^\bullet(\mathscr{H}_s,\mathbb{Q}) \simeq \mathbb{Q}[x,y]/\langle x^2, y^2-sxy\rangle$$

by Example 12.4.3. One easily checks that $(x,y)\mapsto (x, y+\frac{1}{2}(s-r)x)$ induces a ring isomorphism $H^\bullet(\mathscr{H}_r,\mathbb{Q})\simeq H^\bullet(\mathscr{H}_s,\mathbb{Q})$. ◊

In general, however, the ring structure on $H^\bullet(X_\Sigma,\mathbb{Q})$ is not a combinatorial invariant, even when Σ is simplicial (Exercise 12.4.1).

Equivariant Cohomology. We will prove Theorem 12.4.1 by first computing the *equivariant cohomology* of X_Σ for the action of the torus T_N and then passing from equivariant cohomology to ordinary singular cohomology. This method of proof comes from an extensive study of equivariant cohomology by many authors over the past 30 years. Our presentation draws mostly on [**47**] and [**108**].

Let G be a Lie group acting on a topological space X on the left. When the action of G on X is not free, the quotient X/G can be badly behaved. As a replacement for X/G when the action is not free, in [**38**], Borel introduced the following idea. Let EG be a contractible space on which G acts freely on the right, and let $BG = EG/G$. Then $EG\times X$ is homotopy equivalent to X and G acts freely on this space. Hence we can form the quotient

$$EG\times_G X = EG\times X/\sim, \text{ where } (e\cdot g,x)\sim(e,g\cdot x) \text{ for } g\in G,$$

as replacement for X/G. Moreover, $EG\times_G X$ has the structure of a fiber bundle over BG with fiber equal to X. It is a theorem that suitable, possibly infinite-dimensional, EG always exist. The homotopy type of $EG\times_G X$ is also independent of the EG used. Hence we have a well-defined new cohomology theory.

Definition 12.4.6. Let X be a topological space with a left action of the Lie group G. The ***equivariant cohomology*** of X with respect to G with coefficients in a ring R, denoted $H^\bullet_G(X,R)$, is defined as

$$H^\bullet_G(X,R) = H^\bullet(EG\times_G X,R),$$

where the right-hand side is ordinary singular cohomology.

We note an important special case of this definition that has some useful consequences. If $X=\{\mathrm{pt}\}$ is a single point with the trivial action of G, then

$$EG\times_G\{\mathrm{pt}\}\cong EG/G = BG.$$

Hence $H^\bullet_G(\{\mathrm{pt}\},\mathbb{Z}) = H^\bullet(BG,\mathbb{Z})$ is an algebraic invariant depending only on the group G. We will call this ring Λ_G. If X is any other space on which G acts, the constant map $X\to\{\mathrm{pt}\}$ induces a map

$$\Lambda_G = H^\bullet_G(\{\mathrm{pt}\},\mathbb{Z}) \longrightarrow H^\bullet_G(X,\mathbb{Z}) \tag{12.4.5}$$

that makes $H^\bullet_G(X,\mathbb{Z})$ into a module over the ring Λ_G. Since the product in Λ_G is the cup product, Λ_G may fail to be commutative if $H^k(BG,\mathbb{Z})\neq 0$ for some odd k.

The Torus. For us, the group G will always be a torus $T\simeq(\mathbb{C}^*)^n$. Here, the spaces EG and BG and the ring Λ_G can be described very concretely.

First consider the 1-dimensional torus $\mathbb{C}^*$. Let $\mathbb{C}^\infty$ be the space of all vectors $(a_0,a_1,\ldots)$ with $a_i\in\mathbb{C}$ for $i\in\mathbb{N}$, and $a_i=0$ for $i\gg 0$. Note that $\mathbb{C}^\infty$ is the union of $\mathbb{C}^{\ell+1}$ for $\ell \geq 1$ if we embed $\mathbb{C}^{\ell+1}$ as the set of vectors with zeros after the first $\ell+1$ entries. In Exercise 12.4.2 you will show the unexpected fact that

$$E\mathbb{C}^* = \mathbb{C}^\infty \setminus \{0\} \tag{12.4.6}$$

is a contractible space. The torus $\mathbb{C}^*$ acts freely on $E\mathbb{C}^*$ in the obvious way:

$$t \cdot (a_0, a_1, \ldots) = (ta_0, ta_1, \ldots).$$

By analogy with the finite-dimensional construction, $B\mathbb{C}^*$ is denoted by $\mathbb{P}^\infty$:

$$B\mathbb{C}^* = E\mathbb{C}^*/\mathbb{C}^* = \mathbb{P}^\infty.$$

Note that $\mathbb{P}^\infty = \bigcup_{\ell=0}^\infty \mathbb{P}^\ell$. Thus, when computing cohomology in a fixed degree, we can replace $B\mathbb{C}^*$ with the finite-dimensional approximation $\mathbb{P}^\ell$ for $\ell \gg 0$. To see how this works, note that

$$H^\bullet(\mathbb{P}^\ell, \mathbb{Z}) \simeq \mathbb{Z}[t]/\langle t^{\ell+1}\rangle$$

since the computations in Example 12.4.2 work over $\mathbb{Z}$. Letting $\ell \to \infty$, we obtain

$$\Lambda_{\mathbb{C}^*} = H^\bullet(B\mathbb{C}^*, \mathbb{Z}) = H^\bullet(\mathbb{P}^\infty, \mathbb{Z}) \simeq \mathbb{Z}[t], \tag{12.4.7}$$

where the generator t is in degree 2. Note that in this case $\Lambda_{\mathbb{C}^*}$ is a commutative ring since $H^k(\mathbb{P}^\infty, \mathbb{Z}) = 0$ when k is odd.

For the n-dimensional torus $T = (\mathbb{C}^*)^n$, we can take

$$ET = E\mathbb{C}^* \times \cdots \times E\mathbb{C}^*,$$

which implies

$$BT = B\mathbb{C}^* \times \cdots \times B\mathbb{C}^*.$$

Therefore by the Künneth formula,

$$H_T^\bullet(\{\mathrm{pt}\}, \mathbb{Z}) = H^\bullet(BT, \mathbb{Z}) = \Lambda_{T_N} \simeq \mathbb{Z}[t_1, \ldots, t_n]. \tag{12.4.8}$$

Note that Λ_{T_N} is commutative and the t_i all have degree 2.

For a torus $T_N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$, we can describe (12.4.8) more intrinsically as follows. Recall that M has the *symmetric algebra*

$$\mathrm{Sym}_{\mathbb{Z}}(M) = \mathbb{Z} \oplus M \oplus \mathrm{Sym}^2(M) \oplus \cdots.$$

To map this to $H^\bullet(BT_N, \mathbb{Z})$, take $m \in M$. By the functorial properties of $G \mapsto BG$ (see [**271**]), $\chi^m : T_N \to \mathbb{C}^*$ induces $B\chi^m : BT_N \to B\mathbb{C}^*$. Taking H^2, we obtain

$$\mathbb{Z} \cdot t \simeq H^2(B\mathbb{C}^*, \mathbb{Z}) \xrightarrow{(B\chi^m)^*} H^2(BT_N, \mathbb{Z}).$$

Then define $\mathsf{s} : M \to H^2(BT_N, \mathbb{Z})$ by $m \mapsto (B\chi^m)^*(t)$. This gives a canonical isomorphism

$$\mathsf{s} : \mathrm{Sym}_{\mathbb{Z}}(M) \xrightarrow{\sim} H^\bullet(BT_N, \mathbb{Z}).$$

To simplify notation, we will often write T instead of T_N in what follows. Then, when we switch from $\mathbb{Z}$ to $\mathbb{Q}$, these isomorphisms become

$$\mathsf{s} : \mathrm{Sym}_{\mathbb{Q}}(M) \xrightarrow{\sim} H_T^\bullet(\{\mathrm{pt}\}, \mathbb{Q}) = H^\bullet(BT, \mathbb{Q}) = (\Lambda_T)_{\mathbb{Q}}, \tag{12.4.9}$$

where $\mathrm{Sym}_{\mathbb{Q}}(M)$ is our simplified notation for $\mathrm{Sym}_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Sym}_{\mathbb{Q}}(M_{\mathbb{Q}})$.

The Fixed Points. Now let $T = T_N$ be the torus of X_Σ and let X_Σ^T denote the set of fixed points for the torus action. The Orbit-Cone Correspondence implies that there are finitely many fixed points x_σ, one for each cone $\sigma \in \Sigma(n)$.

The inclusion $X_\Sigma^T \hookrightarrow X_\Sigma$ induces a ring homomorphism

$$H_T^\bullet(X_\Sigma, \mathbb{Z}) \longrightarrow H_T^\bullet(X_\Sigma^T, \mathbb{Z}). \tag{12.4.10}$$

Since X_Σ^T is finite, we have

$$H_T^\bullet(X_\Sigma^T, \mathbb{Q}) \;=\; \bigoplus_{\sigma\in\Sigma(n)} H_T^\bullet(\{x_\sigma\}, \mathbb{Q}) \;=\; \bigoplus_{\sigma\in\Sigma(n)} (\Lambda_T)_{\mathbb{Q}},$$

so that (12.4.10) can be written

$$\delta : H_T^\bullet(X_\Sigma, \mathbb{Q}) \;\longrightarrow\; \bigoplus_{\sigma\in\Sigma(n)} (\Lambda_T)_{\mathbb{Q}}. \tag{12.4.11}$$

The map δ is used in the *localization theorem*, which is a key tool in equivariant cohomology. We will say more about this in Corollary 12.4.9.

Comparing Equivariant and Ordinary Cohomology. We next study the relation between equivariant and singular cohomology for a toric variety. Let $T = T_N$ be the torus of X_Σ and note that $H_T^\bullet(X_\Sigma, \mathbb{Q})$ is a $(\Lambda_T)_{\mathbb{Q}}$-module by (12.4.5). This is compatible with the ring structure, so that $H_T^\bullet(X_\Sigma, \mathbb{Q})$ is a $(\Lambda_T)_{\mathbb{Q}}$-algebra.

The next proposition shows that the equivariant cohomology of a toric variety can be obtained from its singular cohomology by a change of scalars.

Proposition 12.4.7. *Let X_Σ be a complete simplicial toric variety and let $T = T_N$. Then there is a isomorphism of $(\Lambda_T)_{\mathbb{Q}}$-modules*

$$H_T^\bullet(X_\Sigma, \mathbb{Q}) \simeq (\Lambda_T)_{\mathbb{Q}} \otimes_{\mathbb{Q}} H^\bullet(X_\Sigma, \mathbb{Q}).$$

In particular, $H_T^\bullet(X_\Sigma, \mathbb{Q})$ is a free $(\Lambda_T)_{\mathbb{Q}}$-module of finite rank.

Proof. By the Borel construction, $ET \times_T X_\Sigma$ is a fiber bundle over $BT = (\mathbb{P}^\infty)^n$ with fiber equal to X_Σ. The Serre spectral sequence for singular cohomology (see [**136**, Ch. 1], [**199**, Thm. 5.2], [**255**, Sec. 9.4], and Theorem C.2.6) computes the cohomology of the total space $ET \times_T X_\Sigma$ in terms of the cohomologies of the base and the fiber. Since we use rational coefficients, the E_2 sheet has the form

$$E_2^{p,q} = H^p(BT, H^q(X_\Sigma, \mathbb{Q})) \simeq H^p(BT, \mathbb{Q}) \otimes_{\mathbb{Q}} H^q(X_\Sigma, \mathbb{Q})$$

by the universal coefficient theorem, and the spectral sequence converges to

$$H^{p+q}(ET \times_T X_\Sigma, \mathbb{Q}) = H_T^{p+q}(X_\Sigma, \mathbb{Q}).$$

However, $H^q(X_\Sigma, \mathbb{Q}) = 0$ for odd q by Theorem 12.3.11, and similarly $H^p(BT, \mathbb{Q})$ vanishes for odd p by the comments following (12.4.8). Since the rth differential is $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, it follows that all of the d_r are zero for $r \geq 2$. Hence the Serre spectral sequence degenerates at E_2. Then the Leray-Hirsch theorem (see [**135**, Thm. 4D.1]) implies that

$$H^\bullet(ET \times_T X_\Sigma, \mathbb{Q}) = H^\bullet_T(X_\Sigma, \mathbb{Q}) \simeq H^\bullet(BT, \mathbb{Q}) \otimes_{\mathbb{Q}} H^\bullet(X_\Sigma, \mathbb{Q}).$$

This is a $(\Lambda_T)_{\mathbb{Q}}$-module isomorphism since $ET \times_T X_\Sigma \to BT$ induces the edge homomorphism $E^{p,0} = H^p(BT, \mathbb{Q}) \to H^p_T(X_\Sigma, \mathbb{Q})$ by Theorem C.2.6. □

Example 12.4.15 below will show that the isomorphism in Proposition 12.4.7 may fail to preserve the ring structure.

In general, the action of a torus T on a space X is called *equivariantly formal* if the spectral sequence in Proposition 12.4.7 degenerates at E_2. Starting in [**38**], equivariantly formal spaces have been studied extensively in topology and symplectic geometry. A good reference is the paper [**118**] by Goresky, Kottwitz and MacPherson. For toric varieties, the paper [**97**] by Franz gives conditions under which Proposition 12.4.7 remains true over $\mathbb{Z}$.

We next describe two useful consequences of Proposition 12.4.7. The first uses the ideal $I_T \subseteq (\Lambda_T)_{\mathbb{Q}}$ generated by elements of positive degree. The quotient $(\Lambda_T)_{\mathbb{Q}}/I_T \simeq \mathbb{Q}$ gives $\mathbb{Q}$ the structure of a $(\Lambda_T)_{\mathbb{Q}}$-module. Here is the result.

Corollary 12.4.8. *The ordinary cohomology ring $H^\bullet(X_\Sigma, \mathbb{Q})$ is determined by the $(\Lambda_T)_{\mathbb{Q}}$-algebra structure of $H^\bullet_T(X_\Sigma, \mathbb{Q})$ via the isomorphism*

$$H^\bullet(X_\Sigma, \mathbb{Q}) \simeq H^\bullet_T(X_\Sigma, \mathbb{Q})/I_T H^\bullet_T(X_\Sigma, \mathbb{Q}) \simeq H^\bullet_T(X_\Sigma, \mathbb{Q}) \otimes_{(\Lambda_T)_{\mathbb{Q}}} \mathbb{Q}.$$

Proof. We have to be careful since $H^\bullet_T(X_\Sigma, \mathbb{Q}) \simeq (\Lambda_T)_{\mathbb{Q}} \otimes_{\mathbb{Q}} H^\bullet(X_\Sigma, \mathbb{Q})$ need not be a ring isomorphism. When we combine this isomorphism with $(\Lambda_T)_{\mathbb{Q}} \to \mathbb{Q}$, we get a surjective homomorphism of graded $\mathbb{Q}$-vector spaces

$$H^\bullet_T(X_\Sigma, \mathbb{Q}) \to H^\bullet(X_\Sigma, \mathbb{Q}) \tag{12.4.12}$$

with kernel $I_T H^\bullet_T(X_\Sigma, \mathbb{Q})$. If we can show that (12.4.12) is a ring homomorphism, then the corollary will follow.

The isomorphism of Proposition 12.4.7 comes from the Serre spectral sequence used in the proof. Degeneration at E_2 implies that

$$H^q_T(X_\Sigma, \mathbb{Q}) = \bigoplus_{a+b=q} E_2^{a,b} = \bigoplus_{a+b=q} H^a(BT, \mathbb{Q}) \otimes_{\mathbb{Q}} H^b(X_\Sigma, \mathbb{Q}).$$

Since $I_T \subseteq (\Lambda_T)_{\mathbb{Q}} = H^\bullet(BT, \mathbb{Q})$ is generated by the elements of positive degree, the map (12.4.12) in degree q is the edge homomorphism $H^q_T(X_\Sigma, \mathbb{Q}) \to E_2^{0,q}$, which by Theorem C.2.6 is the map induced by the inclusion of the fiber

$$i_{X_\Sigma} : X_\Sigma \hookrightarrow ET \times_T X_\Sigma$$

in the fiber bundle given by the Borel construction. Hence (12.4.12) is the ring homomorphism

$$i^*_{X_\Sigma} : H^\bullet_T(X_\Sigma, \mathbb{Q}) \to H^\bullet(X_\Sigma, \mathbb{Q}). \tag{12.4.13}$$

The map $i^*_{X_\Sigma}$ is often called "taking the nonequivariant limit." □

The second corollary concerns the map $\delta : H_T^\bullet(X_\Sigma,\mathbb{Q}) \to \bigoplus_{\sigma\in\Sigma(n)}(\Lambda_T)_\mathbb{Q}$ that we constructed in (12.4.10) using the fixed points of the torus action.

Corollary 12.4.9. *$\delta : H_T^\bullet(X_\Sigma,\mathbb{Q}) \to \bigoplus_{\sigma\in\Sigma(n)}(\Lambda_T)_\mathbb{Q}$ is injective.*

Proof. By (12.4.10) and (12.4.11), δ is the map $H_T^\bullet(X_\Sigma,\mathbb{Q}) \to H_T^\bullet(X_\Sigma^T,\mathbb{Q})$ induced by the inclusion of the fixed point set $X_\Sigma^T \subseteq X_\Sigma$. Let K be the field of fractions of $(\Lambda_T)_\mathbb{Q} \simeq \mathbb{Q}[t_1,\dots,t_n]$. The localization theorem in equivariant cohomology states that δ becomes an isomorphism after tensoring with K (see [**9**]). This implies that the kernel of δ is a $(\Lambda_T)_\mathbb{Q}$-torsion module. However, $H_T^\bullet(X_\Sigma,\mathbb{Q})$ is free over $(\Lambda_T)_\mathbb{Q}$ by Proposition 12.4.7, so $\ker(\delta)$ is torsion-free. Hence the kernel vanishes. $\square$

Other proofs of this corollary can be found in [**50**, 3.4] and [**118**, Thm. 7.2]. Both papers explain how one can choose a single element $f \in (\Lambda_T)_\mathbb{Q}$ such that $H_T^\bullet(X,\mathbb{Q})_f \to H_T^\bullet(X^T,\mathbb{Q})_f$ is an isomorphism. See also Exercise 12.4.3.

The Stanley-Reisner Ring. Our strategy for proving Theorem 12.4.1 will be to compute $H_T^\bullet(X_\Sigma,\mathbb{Q})$ and then take its nonequivariant limit via Corollary 12.4.8. For this purpose, we introduce a modification of the ring $R_\mathbb{Q}(\Sigma)$ appearing in the statement of Theorem 12.4.1.

Definition 12.4.10. The ***Stanley-Reisner ring*** of a fan Σ in $N_\mathbb{R}$ is

$$\mathrm{SR}_\mathbb{Q}(\Sigma) = \mathbb{Q}[x_1,\dots,x_r]/\mathscr{I},$$

where $\Sigma(1) = \{\rho_1,\dots,\rho_r\}$ and as in (12.4.2), $\mathscr{I}$ is the Stanley-Reisner ideal

$$\mathscr{I} = \langle x_{i_1}\cdots x_{i_s} \mid i_j \text{ distinct and } \rho_{i_1}+\cdots+\rho_{i_s} \text{ is not a cone of } \Sigma\rangle.$$

Example 12.4.11. Let Σ_r be the usual fan defining the Hirzebruch surface $\mathscr{H}_r$ as in Example 12.4.3. Then $\mathrm{SR}_\mathbb{Q}(\Sigma_r) = \mathbb{Q}[x_1,x_2,x_3,x_4]/\langle x_1x_3,x_2x_4\rangle$. $\Diamond$

The Stanley-Reisner ring $\mathrm{SR}_\mathbb{Q}(\Sigma)$ also has a natural structure as an algebra over $\mathrm{Sym}_\mathbb{Q}(M)$. This comes from the ring homomorphism

$$\mathrm{Sym}_\mathbb{Q}(M) \longrightarrow \mathrm{SR}_\mathbb{Q}(\Sigma)$$

induced by $m \mapsto [-\sum_{i=1}^r \langle m,u_i\rangle x_i]$ for $m\in M$, where $[f]\in \mathrm{Sym}_\mathbb{Q}(M)$ is the image of $f\in\mathbb{Q}[x_1,\dots,x_r]$. As a $\mathrm{Sym}_\mathbb{Q}(M)$-module, an element $m\in M$ acts on $\mathrm{SR}_\mathbb{Q}(\Sigma)$ by multiplication by the class of the linear form $-\sum_{i=1}^r\langle m,u_i\rangle x_i$. The unexpected minus sign will become clear once we discuss equivariant cohomology classes at the end of the section.

Here is a simple but illuminating example.

Example 12.4.12. Let us compute the $\mathrm{Sym}_\mathbb{Q}(M)$-module structure on $\mathrm{SR}_\mathbb{Q}(\Sigma_r) = \mathbb{Q}[x_1,x_2,x_3,x_4]/\langle x_1x_3,x_2x_4\rangle$ from Example 12.4.11. Let e_1,e_2 be the standard basis of $M = \mathbb{Z}^2$. Then

$$e_1 \text{ acts by multiplication by } -\textstyle\sum_{i=1}^4 \langle e_1, u_i\rangle x_i = x_1 - x_3$$
$$e_2 \text{ acts by multiplication by } -\textstyle\sum_{i=1}^4 \langle e_2, u_i\rangle x_i = -r x_1 - x_2 + x_4.$$

For $r \geq 0$, the surfaces $\mathscr{H}_r$ all have isomorphic Stanley-Reisner rings. But once we introduce the $\mathrm{Sym}_{\mathbb{Q}}(M)$-module structure, then the action of e_2 tells us which Hirzebruch surface we have. ◊

The Equivariant Cohomology Class of a Divisor. A divisor D on a toric variety X_Σ has a cohomology class $[D] \in H^2(X_\Sigma, \mathbb{Q})$. If D is torus-invariant, then it also has an *equivariant cohomology class* $[D]_T \in H^2_T(X_\Sigma, \mathbb{Q})$ as follows.

Proposition 12.4.13. *Let X_Σ be a simplicial toric variety. Then a torus-invariant divisor has a class $[D]_T \in H^2_T(X_\Sigma, \mathbb{Q})$ with the following properties:*

(a) $[D_1 + D_2]_T = [D_1]_T + [D_2]_T$.

(b) $[\mathrm{div}(\chi^m)]_T = -\mathsf{s}(m) \cdot 1$, *where* $1 \in H^0_T(X_\Sigma, \mathbb{Q})$ *and* $\mathsf{s}(m) \in (\Lambda_T)_{\mathbb{Q}}$ *by* (12.4.9).

(c) $i^*[D]_T = [D|_U]_T$, *where $i : U \subseteq X_\Sigma$ is the inclusion of a torus-invariant open subset of X_Σ.*

(d) $i^*_{X_\Sigma}[D]_T = [D]$, *where* $i^*_{X_\Sigma} : H^\bullet_T(X_\Sigma, \mathbb{Q}) \to H^\bullet(X_\Sigma, \mathbb{Q})$ *is the nonequivariant limit map from* (12.4.13).

We defer the proof until the the end of the section.

The Equivariant Cohomology Ring of a Toric Variety. Recall that $H^\bullet_T(X_\Sigma, \mathbb{Q})$ is a $(\Lambda_T)_{\mathbb{Q}}$-algebra. Then the isomorphism $\mathsf{s} : \mathrm{Sym}_{\mathbb{Q}}(M) \simeq (\Lambda_T)_{\mathbb{Q}}$ from (12.4.9) makes $H^2_T(X_\Sigma, \mathbb{Q})$ into a $\mathrm{Sym}_{\mathbb{Q}}(M)$-algebra. Then we have the following theorem.

Theorem 12.4.14. *The map $x_i \mapsto [D_i]_T \in H^2_T(X_\Sigma, \mathbb{Q})$ induces an isomorphism of* $\mathrm{Sym}_{\mathbb{Q}}(M)$*-algebras* $\mathrm{SR}_{\mathbb{Q}}(\Sigma) \simeq H^\bullet_T(X_\Sigma, \mathbb{Q})$.

Before starting the proof, we give two applications. The first is a proof of the main result of this section.

Proof of Theorem 12.4.1. The ideal $\langle M\rangle \subseteq \mathrm{Sym}_{\mathbb{Q}}(M)$ corresponds to $I_T \subseteq (\Lambda_T)_{\mathbb{Q}}$. Thus Theorem 12.4.14 gives an isomorphism

$$\mathrm{SR}_{\mathbb{Q}}(\Sigma)/\langle M\rangle \simeq H^\bullet_T(X_\Sigma, \mathbb{Q})/I_T H^\bullet_T(X_\Sigma, \mathbb{Q}).$$

The definition of $\mathrm{Sym}_{\mathbb{Q}}(M)$-module structure on $\mathrm{SR}_{\mathbb{Q}}(\Sigma)$ easily implies that

$$\mathrm{SR}_{\mathbb{Q}}(\Sigma)/\langle M\rangle \simeq \mathbb{Q}[x_1, \ldots, x_r]/(\mathscr{I} + \mathscr{J}) = R_{\mathbb{Q}}(\Sigma),$$

and $H^\bullet_T(X_\Sigma, \mathbb{Q})/I_T H^\bullet_T(X_\Sigma, \mathbb{Q})$ is isomorphic to $H^\bullet(X_\Sigma, \mathbb{Q})$ by Corollary 12.4.8. This gives a ring isomorphism $R_{\mathbb{Q}}(\Sigma) \simeq H^\bullet(X_\Sigma, \mathbb{Q})$ that takes the class of x_i to the nonequivariant limit of $[D_i]_T$, which is $[D_i]$ by Proposition 12.4.12. □

We next give a concrete example of Theorem 12.4.14.

Example 12.4.15. Combining Example 12.4.12 and Theorem 12.4.14, we see that the Hirzebruch surface $\mathscr{H}_r$ has equivariant cohomology ring

$$H_T^\bullet(\mathscr{H}_r,\mathbb{Q}) \simeq \mathrm{SR}_\mathbb{Q}(\Sigma_r) = \mathbb{Q}[x_1,x_2,x_3,x_4]/\langle x_1x_3,\, x_2x_4\rangle,$$

where Σ_r is the usual fan for $\mathscr{H}_r$. This has two interesting consequences:

- $H_T^\bullet(\mathscr{H}_r,\mathbb{Q}) \simeq (\Lambda_T)_\mathbb{Q} \otimes_\mathbb{Q} H^\bullet(\mathscr{H}_r,\mathbb{Q})$ by Proposition 12.4.7. This is not a ring isomorphism, for it were, we would have an injective ring homomorphism
$$H^\bullet(\mathscr{H}_r,\mathbb{Q}) \hookrightarrow H_T^\bullet(\mathscr{H}_r,\mathbb{Q}).$$
This is impossible since elements of $H^2(\mathscr{H}_r,\mathbb{Q})$ are nilpotent in $H^\bullet(\mathscr{H}_r,\mathbb{Q})$, yet nonzero elements of $H_T^2(\mathscr{H}_r,\mathbb{Q})$ are never nilpotent in $H_T^\bullet(\mathscr{H}_r,\mathbb{Q})$.
- As a ring, $H_T^\bullet(\mathscr{H}_r,\mathbb{Q}) \simeq \mathbb{Q}[x_1,x_2,x_3,x_4]/\langle x_1x_3,\, x_2x_4\rangle$ does not depend on r. However, r appears in $\mathrm{Sym}_\mathbb{Q}(M)$-module structure from Example 12.4.12. In Exercise 12.4.4 you will show that r is an isomorphism invariant of the $\mathrm{Sym}_\mathbb{Q}(M)$-algebra structure of $H_T^\bullet(\mathscr{H}_r,\mathbb{Q})$.

In contrast, we saw in Example 12.4.5 that r is *not* an isomorphism invariant of the ring structure of $H^\bullet(\mathscr{H}_r,\mathbb{Q})$. $\Diamond$

***Preparation for the Proof*.** We need two lemmas and a diagram before we can prove Theorem 12.4.14. The first lemma embeds $\mathrm{SR}_\mathbb{Q}(\Sigma)$ into an exact sequence of $\mathrm{SR}_\mathbb{Q}(\Sigma)$-modules. For a cone $\gamma = \rho_{i_1} + \cdots + \rho_{i_t}$ in Σ, let

$$\mathbb{Q}[\gamma] = \mathbb{Q}[x_{i_1},\ldots,x_{i_t}] = \mathbb{Q}[x_i \mid \rho_i \in \gamma(1)] \simeq \mathbb{Q}[x_1,\ldots,x_r]/\langle x_j \mid \rho_j \notin \gamma(1)\rangle.$$

Thus we can view $\mathbb{Q}[\gamma]$ as a module over $\mathbb{Q}[x_1,\ldots,x_r]$ where $x_j\mathbb{Q}[\tau] = 0$ whenever $\rho_j \notin \gamma(1)$. Hence $\mathbb{Q}[\gamma]$ may also be viewed as a module over $\mathrm{SR}_\mathbb{Q}(\Sigma)$, since each generator of the ideal $\mathscr{I}$ acts by zero on $\mathbb{Q}[\tau]$. Then consider the sequence

$$0 \longrightarrow \mathrm{SR}_\mathbb{Q}(\Sigma) \xrightarrow{\alpha} \textstyle\bigoplus_{\sigma\in\Sigma(n)} \mathbb{Q}[\sigma] \xrightarrow{\beta} \bigoplus_{\tau\in\Sigma(n-1)} \mathbb{Q}[\tau], \tag{12.4.14}$$

where α and β are defined as follows:

- Since $f \mapsto \big(f|_{x_i=0 \text{ for } \rho_i\notin\sigma(1)}\big)_{\sigma\in\Sigma(n)}$ sends the Stanley-Reisner ideal to zero, we get a well-defined map
$$\alpha([f]) = \big(f|_{x_i=0 \text{ for } \rho_i\notin\sigma(1)}\big)_{\sigma\in\Sigma(n)} \ \text{ for } [f] \in \mathrm{SR}_\mathbb{Q}(\Sigma).$$
- Given $\mathbf{f} = (f_\sigma)_{\sigma\in\Sigma(n)}$ and $\tau \in \Sigma(n-1)$, we define the τ-component $\beta(\mathbf{f})_\tau$ as follows. Let the two cones of $\Sigma(n)$ containing τ be $\sigma = \tau + \rho_i$ and $\sigma' = \tau + \rho_j$ where $i < j$. Then
$$\beta(\mathbf{f})_\tau = (f_\sigma)|_{x_i=0} - (f_{\sigma'})|_{x_j=0}.$$

It is easy to see that α and β are $\mathrm{SR}_\mathbb{Q}(\Sigma)$-module homomorphisms.

Lemma 12.4.16. *The sequence* (12.4.14) *is exact.*

Proof. We leave it to the reader to show that α is injective and that $\beta \circ \alpha = 0$ (Exercise 12.4.5). If we write the sequence as $0 \to \mathrm{SR}_{\mathbb{Q}}(\Sigma) \to C_n \to C_{n-1}$, then it remains to prove exactness at C_n.

We will use the $\mathbb{N}^r$-grading on $\mathbb{Q}[x_1,\ldots,x_r]$ where the graded piece in degree $\mathbf{a} \in \mathbb{N}^r$ is $\mathbb{Q}\cdot x^{\mathbf{a}}$. When we regard (12.4.14) as a sequence of $\mathbb{Q}[x_1,\ldots,x_r]$-modules, the $\mathbb{N}^r$-grading is preserved since the Stanley-Reisner ideal is generated by monomials. Hence it suffices to check exactness in degree $\mathbf{a} \in \mathbb{N}^r$. The case $\mathbf{a} = 0$ is easy, since an element of $(C_n)_0$ is $\mathbf{f} = (\lambda_\sigma)_{\sigma\in\Sigma(n)}$ where $\lambda_\sigma \in \mathbb{Q}$. Then $\beta(\mathbf{f}) = 0$ forces the λ_σ to be all equal since Σ is a complete fan.

If $\mathbf{a} = (a_1,\ldots,a_r) \neq 0$, set $\gamma = \sum_{a_i>0}\rho_i$. If $\gamma \notin \Sigma$, exactness holds trivially since all graded pieces in degree $\mathbf{a}$ vanish. So we may assume $\gamma \in \Sigma$. As in (3.2.8), let $\overline{\Sigma} = \mathrm{Star}(\gamma)$ be the fan in $N(\tau)_{\mathbb{R}} = (N/N_\tau)_{\mathbb{R}}$ consisting of the projections of cones of Σ containing γ. Note that $\overline{\Sigma}$ is complete and simplicial since Σ is.

By Exercise 12.4.5, $\mathrm{SR}_{\mathbb{Q}}(\Sigma)_{\mathbf{a}} \to (C_n)_{\mathbf{a}} \to (C_{n-1})_{\mathbf{a}}$ is isomorphic to

$$\mathrm{SR}_{\mathbb{Q}}(\overline{\Sigma})_0 \longrightarrow \textstyle\bigoplus_{\overline{\sigma}\in\overline{\Sigma}(n-k)} \mathbb{Q}[\overline{\sigma}]_0 \longrightarrow \textstyle\bigoplus_{\overline{\tau}\in\overline{\Sigma}(n-k-1)} \mathbb{Q}[\overline{\tau}]_0,$$

where $k = \dim\gamma$. This is exact by our previous observation about degree 0. □

We next consider the following large diagram that will be used in proof of Theorem 12.4.14:

(12.4.15)
$$\begin{tikzcd}
0 \arrow[r] & \mathrm{SR}_{\mathbb{Q}}(\Sigma) \arrow[r, "\alpha"] \arrow[d, dotted] & \bigoplus_{\sigma\in\Sigma(n)} \mathbb{Q}[\sigma] \arrow[r, "\beta"] \arrow[d, "A"] & \bigoplus_{\tau\in\Sigma(n-1)} \mathbb{Q}[\tau] \arrow[d, "B"] \\
0 \arrow[r] & H_T^\bullet(X_\Sigma) \arrow[r, "\alpha'"] & \bigoplus_{\sigma\in\Sigma(n)} H_T^\bullet(U_\sigma) \arrow[r, "\beta'"] & \bigoplus_{\tau\in\Sigma(n-1)} H_T^\bullet(U_\tau).
\end{tikzcd}$$

All cohomology is computed over $\mathbb{Q}$ in this diagram. The dotted arrow is the isomorphism we are trying to construct. The top row is exact by Lemma 12.4.14, and we will prove below that the square on the right commutes and that A and B are isomorphisms. As for the bottom row, exactness can be proved using [**118**, (6.3)]. Our approach is different and will in particular give an elementary proof of the exactness of the bottom row of (12.4.15).

Let us describe the maps in the diagram. We know α and β, and α' is induced by the inclusions $U_\sigma \subseteq X_\Sigma$ for $\sigma \in \Sigma(n)$. Similarly, if $\tau \in \Sigma(n-1)$ is a facet of $\sigma \in \Sigma(n)$, then on the σ and τ components in the bottom row of (12.4.15), β' is induced by the inclusion $U_\tau \subseteq U_\sigma$, with the same $\pm$ sign as in the definition of β.

For the vertical map A, let $\sigma \in \Sigma(n)$. For each $\rho_i \in \sigma(1)$, the torus-invariant divisor $D_i \cap U_\sigma \subseteq U_\sigma$ has a class $[D_i \cap U_\sigma]_T \in H_T^2(U_\sigma,\mathbb{Q})$. Then $x_i \mapsto [D_i \cap U_\sigma]_T$ gives a map $\mathbb{Q}[\sigma] \to H_T^\bullet(U_\sigma)$. This defines A, and B is defined similarly.

Lemma 12.4.17. *In the diagram* (12.4.15)*, we have:*

(a) *The square on the right commutes.*

(b) $\beta' \circ \alpha' = 0$.

(c) α' *is injective.*

(d) *The vertical maps A and B are isomorphisms.*

Proof. Proposition 12.4.13 implies that $[D_i \cap U_\sigma]_T$ maps to $[D_i \cap U_\tau]_T$, and part (a) follows immediately. For part (b), note that $\tau \in \Sigma(n-1)$ is a face of exactly two different maximal cones σ and σ' since Σ is complete. The commutative diagram of inclusions

$$\begin{tikzcd} & X_\Sigma & \\ U_\sigma \arrow[ur] & & U_{\sigma'} \arrow[ul] \\ & U_\tau \arrow[ul] \arrow[ur] & \end{tikzcd}$$

induces the commutative diagram of maps on equivariant cohomology

$$\begin{tikzcd} & H_T^\bullet(X_\Sigma,\mathbb{Q}) \arrow[dl] \arrow[dr] & \\ H_T^\bullet(U_\sigma,\mathbb{Q}) \arrow[dr] & & H_T^\bullet(U_{\sigma'},\mathbb{Q}) \arrow[dl] \\ & H_T^\bullet(U_\tau,\mathbb{Q}). & \end{tikzcd}$$

In Exercise 12.4.6, you will check that the component of β' mapping to $\mathbb{Q}[\tau]$ is the difference of these two restrictions. It follows that β' vanishes on the image of α', which is what we wanted to show.

For part (c), the inclusions $\{x_\sigma\} \subseteq U_\sigma$ of the fixed points for $\sigma \in \Sigma(n)$ give a commutative diagram

$$\begin{tikzcd} H_T^\bullet(X_\Sigma,\mathbb{Q}) \arrow[r, "\alpha'"] \arrow[dr, "\delta"'] & \bigoplus_{\sigma\in\Sigma(n)} H_T^\bullet(U_\sigma,\mathbb{Q}) \arrow[d] \\ & \bigoplus_{\sigma\in\Sigma(n)} H_T^\bullet(\{x_\sigma\},\mathbb{Q}) \end{tikzcd}$$

where δ is the injective map from Lemma 12.4.9. Hence α' is injective.

For part (d), take $\sigma \in \Sigma(n)$ and for simplicity assume $\sigma(1) = \{\rho_1,\dots,\rho_n\}$. Since σ is simplicial, $D_i \cap U_\sigma$ is $\mathbb{Q}$-Cartier, so that $\ell_i D_i \cap U_\sigma$ is Cartier for some integer $\ell_i > 0$. The Picard group of an affine toric variety is trivial, so $\ell_i D_i \cap U_\sigma = \mathrm{div}(\chi^{m_i})$ for some $m_i \in M$. By Proposition 12.4.13, we obtain

$$\ell_i [D_i \cap U_\sigma]_T = [\ell_i D_i \cap U_\sigma]_T = [\mathrm{div}(\chi^{m_i})]_T = -\mathsf{s}(m_i)\cdot 1 \in H_T^2(U_\sigma,\mathbb{Q}). \tag{12.4.16}$$

Note also that $\ell D_i \cap U_\sigma = \mathrm{div}(\chi^{m_i})$ implies that $\langle m_i, u_j\rangle = \ell_i \delta_{ij}$. The u_i are a basis of $N_\mathbb{Q}$ since $\sigma \in \Sigma(n)$, and hence the $\ell_i^{-1} m_i$ are a basis of $M_\mathbb{Q}$. Thus $x_i \mapsto -\ell_i^{-1} m_i$ defines an isomorphism $\bar{A}_\sigma : \mathbb{Q}[\sigma] = \mathbb{Q}[x_1,\dots,x_n] \simeq \mathrm{Sym}_\mathbb{Q}(M)$.

Let A_σ be the σ-component of A and let $i^* : H_T^2(U_\sigma,\mathbb{Q}) \to H_T^2(\{x_\sigma\},\mathbb{Q}) = (\Lambda_T)_\mathbb{Q}$ be induced by the inclusion of the fixed point $i : \{x_\sigma\} \to U_\sigma$. Using the isomorphism $\mathsf{s} : \mathrm{Sym}_\mathbb{Q}(M) \simeq (\Lambda_T)_\mathbb{Q}$ from (12.4.9), we get a diagram

$$\begin{tikzcd} & \mathbb{Q}[\sigma] \arrow[dl, "A_\sigma"'] \arrow[dr, "\bar{A}_\sigma"] & \\ H_T^\bullet(U_\sigma,\mathbb{Q}) \arrow[dr, "i^*"'] & & \mathrm{Sym}_\mathbb{Q}(M) \arrow[dl, "\mathsf{s}"] \\ & (\Lambda_T)_\mathbb{Q} & \end{tikzcd}$$

which is commutative by (12.4.16). By Proposition 12.1.9, there is an equivariant deformation retraction from U_σ to x_σ. Thus i^* is an isomorphism. Since $\bar{A}_\sigma$ and s are also isomorphisms, the same is true for A_σ.

The proof that B is an isomorphism is similar (with some small differences) and will be covered in Exercise 12.4.7. □

We now have everything needed to compute the equivariant cohomology of X_Σ.

Proof of Theorem 12.4.14. Consider the map $\mathbb{Q}[x_1,\dots,x_r] \to H_T^\bullet(X_\Sigma,\mathbb{Q})$ defined by $x_i \mapsto [D_i]_T$. To show that this map kills the Stanley-Reisner ideal, take distinct indices $i_1,\dots,i_s$ such that $\rho_{i_1}+\cdots+\rho_{i_s} \notin \Sigma$. We need to prove that

$$[D_{i_1}]_T \smile \cdots \smile [D_{i_s}]_T = 0 \in H_T^\bullet(X_\Sigma,\mathbb{Q}).$$

For each $\sigma \in \Sigma(n)$, this cup product maps to $[D_{i_1}\cap U_\sigma]_T \smile \cdots \smile [D_{i_s}\cap U_\sigma]_T$ in $H_T^\bullet(U_\sigma,\mathbb{Q})$ by Proposition 12.4.13. This vanishes in $H_T^\bullet(U_\sigma,\mathbb{Q})$ since $\rho_{i_j} \notin \sigma(1)$ for some $j \in \{1,\dots,s\}$. Thus the map α' from (12.4.15) maps $[D_{i_1}]_T \smile \cdots \smile [D_{i_s}]_T$ to zero. Since α' is injective by Lemma 12.4.17, the cup product is zero in $H_T^\bullet(X_\Sigma,\mathbb{Q})$.

Thus we have a well-defined ring homomorphism $\gamma : \mathrm{SR}_\mathbb{Q}(\Sigma) \to H_T^\bullet(X_\Sigma,\mathbb{Q})$. Also, $m \in M$ acts on $\mathrm{SR}_\mathbb{Q}(\Sigma)$ by multiplication by the image of $-\sum_{i=1}^r \langle m,u_i\rangle x_i$ in $\mathrm{SR}_\mathbb{Q}(\Sigma)$. This maps to

$$\big[-\textstyle\sum_{i=1}^r \langle m,u_i\rangle D_i\big]_T = -[\mathrm{div}(\chi^m)]_T = \mathsf{s}(m)\cdot 1,$$

where the last equality follows from Proposition 12.4.13. It follows that γ is a $\mathrm{Sym}_\mathbb{Q}(M)$-algebra homomorphism.

Now add γ to the diagram (12.4.15):

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{SR}_\mathbb{Q}(\Sigma) \arrow[r, "\alpha"] \arrow[d, "\gamma"] & \bigoplus_{\sigma\in\Sigma(n)} \mathbb{Q}[\sigma] \arrow[r, "\beta"] \arrow[d, "A"] & \bigoplus_{\tau\in\Sigma(n-1)} \mathbb{Q}[\tau] \arrow[d, "B"] \\ 0 \arrow[r] & H_T^\bullet(X_\Sigma) \arrow[r, "\alpha'"] & \bigoplus_{\sigma\in\Sigma(n)} H_T^\bullet(U_\sigma) \arrow[r, "\beta'"] & \bigoplus_{\tau\in\Sigma(n-1)} H_T^\bullet(U_\tau). \end{tikzcd}$$

The argument used in in Proposition 12.4.13 shows that the square on the left commutes. Hence we have a commutative diagram where the top row is exact

(Lemma 12.4.16), the maps A and B are isomorphisms (Lemma 12.4.17), and α' is injective (Lemma 12.4.17). From here, an easy diagram chase shows that γ is bijective, and the theorem is proved. $\square$

The above proof implies that the bottom row of our big diagram

$$0 \longrightarrow H_T^\bullet(X_\Sigma,\mathbb{Q}) \longrightarrow \bigoplus_{\sigma\in\Sigma(n)} H_T^\bullet(U_\sigma,\mathbb{Q}) \longrightarrow \bigoplus_{\tau\in\Sigma(n-1)} H_T^\bullet(U_\tau,\mathbb{Q})$$

is exact. There are equivariant deformation retractions $U_\sigma \to O(\sigma) = \{x_\sigma\}$ and $U_\tau \to O(\tau)$ by Proposition 12.1.9. Thus we can rewrite the exact sequence as

$$0 \longrightarrow H_T^\bullet(X_\Sigma,\mathbb{Q}) \longrightarrow \bigoplus_{\sigma\in\Sigma(n)} H_T^\bullet(O(\sigma),\mathbb{Q}) \longrightarrow \bigoplus_{\tau\in\Sigma(n-1)} H_T^\bullet(O(\tau),\mathbb{Q}). \tag{12.4.17}$$

This sequence, called a Chang-Skjelbred sequence, tells us that the structure of $H_T^\bullet(X_\Sigma,\mathbb{Q})$ (and hence that of $H^\bullet(X_\Sigma,\mathbb{Q})$) is completely determined by three things:

- The equivariant cohomology of the fixed point set
$$X_0 = X_\Sigma^T = \textstyle\bigcup_{\sigma\in\Sigma(n)} O(\sigma) = \bigcup_{\sigma\in\Sigma(n)} \{x_\sigma\}.$$
- The equivariant cohomology of the 1-dimensional orbits
$$X_1 = \textstyle\bigcup_{\tau\in\Sigma(n-1)} O(\tau).$$
- The inclusions $X_0 \subseteq \overline{X_1} \subseteq X_\Sigma$.

In [**118**, §7.2], Goresky, Kottwitz and MacPherson show that the same is true when a variety X is equivariantly formal for the action of a torus and there are finitely many fixed points and 1-dimensional orbits. If X_Σ is smooth, then the same arguments can be carried out with coefficients in $\mathbb{Z}$ rather than in $\mathbb{Q}$, and this gives a proof of the Jurkiewicz-Danilov theorem (Theorem 12.4.4).

Piecewise-Polynomial Functions. The exact sequence (12.4.17) has an interesting interpretation. For $\sigma \in \Sigma(n)$, $H_T^\bullet(O(\sigma),\mathbb{Q}) \simeq H_T^\bullet(U_\sigma,\mathbb{Q}) \simeq \mathrm{Sym}_{\mathbb{Q}}(M)$, and for $\tau \in \Sigma(n-1)$, you will construct an isomorphism

$$H_T^\bullet(O(\tau),\mathbb{Q}) \simeq \mathrm{Sym}_{\mathbb{Q}}(M_\tau)$$

in Exercise 12.4.7, where $M_\tau = M/(\tau^\perp \cap M)$. It follows that the exact sequence for equivariant cohomology can be written

$$0 \longrightarrow H_T^\bullet(X_\Sigma,\mathbb{Q}) \longrightarrow \bigoplus_{\sigma\in\Sigma(n)} \mathrm{Sym}_{\mathbb{Q}}(M) \longrightarrow \bigoplus_{\tau\in\Sigma(n-1)} \mathrm{Sym}_{\mathbb{Q}}(M_\tau). \tag{12.4.18}$$

Now observe the following:

- For each $\sigma \in \Sigma(n)$, $\mathrm{Sym}_{\mathbb{Q}}(M)$ is naturally isomorphic to the ring of polynomial functions on σ with coefficients in $\mathbb{Q}$.
- Similarly, for each $\tau \in \Sigma(n-1)$, $\mathrm{Sym}_{\mathbb{Q}}(M_\tau)$ is naturally isomorphic to the ring of polynomial functions on τ with coefficients in $\mathbb{Q}$.

- The exactness of (12.4.18) means that an element of $H_T^\bullet(X_\Sigma,\mathbb{Q})$ can be thought of as a collection $\{f_\sigma\}_{\sigma\in\Sigma(n)}$, where f_σ is a polynomial function on σ, such that $f_\sigma|_\tau = f_{\sigma'}|_\tau$ whenever $\sigma\cap\sigma' = \tau \in \Sigma(n-1)$.

Thus we can identify the equivariant cohomology of X_Σ with the ring of *piecewise polynomial functions with rational coefficients* on the simplicial decomposition of $N_\mathbb{R}$ defined by Σ with the usual pointwise sum and product. Such functions are also known as *polynomial splines* and are an important tool in numerical analysis and applications. This gives perhaps the most concrete way to compute and understand the equivariant cohomology of X_Σ.

Example 12.4.18. The torus $T \simeq (\mathbb{C}^*)^2$ acts as usual on $\mathbb{P}^2$. In Exercise 12.4.8, you will verify the following claims. We write $\mathrm{Sym}_\mathbb{Q}(\mathbb{Z}^2) = \mathbb{Q}[x,y]$. First, by (12.4.18), $H_T^\bullet(\mathbb{P}^2,\mathbb{Q})$ is equal to the set of triples $(f_0,f_1,f_2)\in\mathbb{Q}[x,y]^3$ such that

$$\begin{aligned} f_1 - f_0 &\in \langle x\rangle\\ f_2 - f_0 &\in \langle y\rangle\\ f_2 - f_1 &\in \langle y-x\rangle. \end{aligned}$$

Therefore,

$$H_T^\bullet(\mathbb{P}^2,\mathbb{Q}) = \{(f_0, f_0+xg, f_0+yg+y(y-x)h) \mid f_0,g,h\in\mathbb{Q}[x,y]\}.$$

This shows that as a module over $\mathbb{Q}[x,y]$, $H_T^\bullet(\mathbb{P}^2,\mathbb{Q})$ is generated by $(1,1,1)$, $(0,x,y)$, and $(0,0,y(y-x))$. Since we have one generator in each degree $0,1,2$, remembering the doubling of degrees in Theorem 12.4.1, we obtain an explicit description of the $(\Lambda_T)_\mathbb{Q}$-module isomorphism

$$H_T^\bullet(\mathbb{P}^2,\mathbb{Q}) \simeq (\Lambda_T)_\mathbb{Q}\otimes_\mathbb{Q} H^\bullet(\mathbb{P}^2,\mathbb{Q}) \simeq \mathrm{Sym}_\mathbb{Q}(\mathbb{Z}^2)\otimes_\mathbb{Q} H^\bullet(\mathbb{P}^2,\mathbb{Q})$$

from Proposition 12.4.7 in this case. $\Diamond$

Properties of Equivariant Cohomology Classes. The final task of this section is to define the equivariant cohomology class of a torus-invariant divisor on a simplicial toric variety (not necessarily complete) and to prove that these classes have the properties listed in Proposition 12.4.13. Our construction will use *Chern classes*.

In §13.1 we give an overview of the Chern classes of locally free sheaves on varieties. Here, we combine the algebraic approach with a more topological version of Chern classes that applies to complex vector bundles over topological spaces.

Let D be a torus-invariant Cartier divisor on a simplicial toric variety X_Σ. This gives the invertible sheaf $\mathscr{O}_{X_\Sigma}(D)$, which by Theorem 6.0.18 is the sheaf of sections of a rank 1 vector bundle $\pi: V_\mathscr{L}\to X_\Sigma$ for $\mathscr{L} = \mathscr{O}_{X_\Sigma}(D)$. Here, we write V_D instead of $V_\mathscr{L}$. In Proposition 7.3.1 we constructed a fan $\Sigma\times D$ in $N_\mathbb{R}\times R$ such that V_D is the toric variety of $\Sigma\times D$. Hence the torus of V_D is $T_{N\oplus\mathbb{Z}} = T_N\times\mathbb{C}^*$. In particular, the torus $T = T_N$ of X_Σ acts on V_D such that the projection map π is equivariant. This makes V_D into an *equivariant vector bundle* over X_Σ.

In the classical topology, $\pi : V_{\mathscr{L}} \to X_\Sigma$ has a Chern class $c_1(V_D) \in H^2(X_\Sigma, \mathbb{Z})$, which equals the algebraic Chern class $c_1(\mathscr{O}_{X_\Sigma}(D))$ discussed in §13.1. Thus

$$c_1(V_D) = c_1(\mathscr{O}_{X_\Sigma}(D)) = [D] \in H^2(X_\Sigma, \mathbb{Z}), \tag{12.4.19}$$

where the last equality uses the properties of Chern classes stated in §13.1.

To construct the cohomology class $[D]_T \in H^2_T(X_\Sigma, \mathbb{Z})$, recall that $H^\bullet_T(X_\Sigma, \mathbb{Z}) = H^\bullet(ET \times_T X_\Sigma, \mathbb{Z})$. Since $\pi : V_D \to X_\Sigma$ is T-equivariant, the rank 1 vector bundle

$$(V_D)_T = ET \times_T V_D \longrightarrow ET \times_T X_\Sigma$$

has Chern class $[D]_T = c_1\big((V_D)_T\big) \in H^2(ET \times_T X_\Sigma, \mathbb{Z}) = H^\bullet_T(X_\Sigma, \mathbb{Z})$.

Proof of Proposition 12.4.13. We need to show that the classes $[D]_T$ satisfy:

(a) $[D_1 + D_2]_T = [D_1]_T + [D_2]_T$.

(b) $[\mathrm{div}(\chi^m)]_T = -\mathsf{s}(m) \cdot 1$, where $1 \in H^0_T(X_\Sigma, \mathbb{Q})$ and $\mathsf{s}(m) \in (\Lambda_T)_\mathbb{Q}$ by (12.4.9).

(c) $i^*[D]_T = [D|_U]_T$, where $i : U \subseteq X_\Sigma$ is the inclusion of a torus-invariant open subset of X_Σ.

(d) $i^*_{X_\Sigma}[D]_T = [D]$, where $i^*_{X_\Sigma} : H^\bullet_T(X_\Sigma, \mathbb{Q}) \to H^\bullet(X_\Sigma, \mathbb{Q})$ is the nonequivariant limit map from (12.4.13).

For part (a), note that $V_{D_1+D_2} = V_{D_1} \otimes V_{D_2}$, so $(V_{D_1+D_2})_T = (V_{D_1})_T \otimes (V_{D_2})_T$ as vector bundles over $ET \times_T X_\Sigma$. Then we are done by the properties of Chern classes of rank 1 vector bundles.

Part (c) is also easy. Given $i : U \subseteq X_\Sigma$, one checks that $V_{D|_U}$ is the pullback of V_D via i. Thus $(V_{D|_U})_T$ is the pullback of $(V_D)_T$ via $i_T : ET \times_T U \subseteq ET \times_T X_\Sigma$. Then we are done since c_1 is compatible with pullbacks.

For part (d), recall that the map $ET \times_T X_\Sigma \to BT$ is a fibration with fiber X_Σ. If $i_{X_\Sigma} : X_\Sigma \hookrightarrow ET \times_T X_\Sigma$ is the inclusion of the fiber, then V_D is the pullback of $(V_D)_T$ via i_{X_Σ}. Since c_1 is compatible with pullbacks, we obtain

$$i^*_{X_\Sigma}[D]_T = i^*_{X_\Sigma} c_1\big((V_D)_T\big) = c_1(i^*_{X_\Sigma}(V_D)_T) = c_1(V_D) = [D],$$

where the last equality uses (12.4.19).

The above proofs work over $\mathbb{Z}$ since we are assuming that D is Cartier. If D is an arbitrary divisor on X_Σ, then ℓD is Cartier for some integer $\ell > 0$ since X_Σ is simplicial. Then

$$[D]_T = \ell^{-1}[\ell D]_T \in H^\bullet(X_\Sigma, \mathbb{Q})$$

is well-defined. It follows that properties (a), (c) and (d) hold over $\mathbb{Q}$.

The proof of part (b) takes a bit more work. Take $m \in M$ and set $D = \mathrm{div}(\chi^m)$. Also let $p : X_\Sigma \to \{\mathrm{pt}\}$ be the map that takes every element of X_Σ to the same point, and let $V_{\chi^m} = \mathbb{C} \to \{\mathrm{pt}\}$ be the vector bundle over $\{\mathrm{pt}\}$ where T acts on $\mathbb{C}$ via χ^m.

In Exercise 12.4.9 you will show that there is a pullback diagram

$$\begin{tikzcd} V_D \arrow[r] \arrow[d] & V_{\chi^m} \arrow[d] \\ X_\Sigma \arrow[r, "p"] & \{\mathrm{pt}\} \end{tikzcd}$$

of T-equivariant vector bundles. In concrete terms, this says that V_D is the trivial bundle $X_\Sigma \times \mathbb{C} \to X_\Sigma$ where T acts on $\mathbb{C}$ via χ^m.

It follows that $(V_D)_T$ is the pullback of $(V_{\chi^m})_T$ via $p_T : ET \times_T X_\Sigma \to BT$. For the moment, set $\lambda = c_1\big((V_{\chi^m})_T\big) \in H_T^2(\{\mathrm{pt}\},\mathbb{Z}) = \Lambda_T$. Then

$$[\mathrm{div}(\chi^m)]_T = c_1\big((V_D)_T\big) = p_T^* c_1\big((V_{\chi^m})_T\big) = p_T^*\lambda = \lambda \cdot 1$$

The last equality follows from the definition of $H_T^\bullet(X_\Sigma,\mathbb{Z})$ as a Λ_T-module.

Hence it suffices to prove that $\lambda = -\mathsf{s}(m)$, where s is defined in the discussion leading up to (12.4.9). First observe that $\chi^m : T \to \mathbb{C}^*$ induces $B\chi^m : BT \to B\mathbb{C}^*$. By Exercise 12.4.9, there is a pullback diagram

$$\begin{tikzcd} (V_{\chi^m})_T \arrow[r] \arrow[d] & (\mathbb{C})_{\mathbb{C}^*} \arrow[d] \\ BT \arrow[r, "B\chi^m"] & B\mathbb{C}^*, \end{tikzcd}$$

where $\mathbb{C} \to \{\mathrm{pt}\}$ where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication. This implies that

$$\lambda = c_1\big((V_{\chi^m})_T\big) = (B\chi^m)^*\big(c_1\big((\mathbb{C})_{\mathbb{C}^*}\big)\big). \tag{12.4.20}$$

So we need to compute $c_1((\mathbb{C})_{\mathbb{C}^*}) \in H^2(B\mathbb{C}^*,\mathbb{Z})$. Recall that $(\mathbb{C})_{\mathbb{C}^*} = E\mathbb{C}^* \times_{\mathbb{C}^*} \mathbb{C}$, where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication. Also, $E\mathbb{C}^* = \bigcup_{\ell\geq 0}(\mathbb{C}^{\ell+1}\setminus\{0\})$ and $B\mathbb{C}^* = \bigcup_{\ell\geq 0}\mathbb{P}^\ell = \mathbb{P}^\infty$. When computing equivariant cohomology in a specific degree, one can always restrict to a finite-dimensional approximation. Hence, for ℓ sufficiently large, $c_1((\mathbb{C})_{\mathbb{C}^*})$ is the Chern class of the rank 1 vector bundle

$$V_\ell = (\mathbb{C}^{\ell+1}\setminus\{0\}) \times_{\mathbb{C}^*} \mathbb{C} \longrightarrow \mathbb{P}^\ell.$$

We have seen this bundle before. Since $\mathbb{C}^*$ acts on $(\mathbb{C}^{\ell+1}\setminus\{0\}) \times \mathbb{C}$ via $t\cdot(p,\lambda) = (t^{-1}p, t\lambda)$, the map $(p,\lambda) \mapsto ([p],\lambda p) \in \mathbb{P}^\ell \times \mathbb{C}^{\ell+1}$ is constant on $\mathbb{C}^*$ orbits. Here $[p]$ denotes the point in $\mathbb{P}^\ell$ whose homogeneous coordinates are $p \in \mathbb{C}^{\ell+1}\setminus\{0\}$. Hence we get a map

$$V_\ell \longrightarrow \mathbb{P}^\ell \times \mathbb{C}^{\ell+1}.$$

In Exercise 12.4.9, you will prove that the image of this map is the vector bundle $V \to \mathbb{P}^\ell$ from Example 6.0.19. Thus $V_\ell \simeq V$ as vector bundles over $\mathbb{P}^\ell$. Since the sheaf of sections of V is $\mathscr{O}_{\mathbb{P}^\ell}(-1)$ by Example 6.0.21, we have

$$c_1\big((\mathbb{C})_{\mathbb{C}^*}\big) = c_1(V_\ell) = c_1(V) = c_1(\mathscr{O}_{\mathbb{P}^\ell}(-1)) \tag{12.4.21}$$

in the cohomology group

$$\mathbb{Z}\cdot t = H^2(\mathbb{P}^\ell,\mathbb{Z}) = H^2(\mathbb{P}^\infty,\mathbb{Z}) = H^2(B\mathbb{C}^*,\mathbb{Z}).$$

Here we are using $H^\bullet(\mathbb{P}^\ell,\mathbb{Z}) = \mathbb{Z}[t]/\langle t^{\ell+1}\rangle$ and $H^\bullet(\mathbb{P}^\infty,\mathbb{Z}) = \mathbb{Z}[t]$ from (12.4.7). A standard fact about Chern classes is that

$$c_1(\mathscr{O}_{\mathbb{P}^\ell}(1)) = t \in \mathbb{Z}\cdot t.$$

This is the normalization axiom for Chern classes—see [**144**, §I.4]. It follows that $c_1(\mathscr{O}_{\mathbb{P}^\ell}(-1)) = -t$. Combining this with (12.4.20) and (12.4.21), we have

$$\begin{aligned}\lambda = c_1\big((V_{\chi^m})_T\big) = (B\chi^m)^*\big(c_1\big((\mathbb{C})_{\mathbb{C}^*}\big)\big) &= (B\chi^m)^*\big(c_1(\mathscr{O}_{\mathbb{P}^\ell}(-1))\big)\\ &= (B\chi^m)^*(-t) = -(B\chi^m)^*(t)\\ &= -\mathsf{s}(m),\end{aligned}$$

where the final equality is the definition of $\mathsf{s}(m)$ in (12.4.9). This completes the proof of Proposition 12.4.13. $\square$

The proof of the proposition explains the minus sign in Proposition 12.4.13—it comes from the Chern class of the vector bundle over $B\mathbb{C}^* = \mathbb{P}^\infty$ determined by the equvariant bundle $\mathbb{C} \to \{\mathrm{pt}\}$ where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication.

Exercises for §12.4.

12.4.1. The following example is taken from [**95**].

(a) Let Σ denote the complete fan in $N_{\mathbb{R}} \simeq \mathbb{R}^2$ with minimal generators

$$u_1 = e_1,\ u_2 = e_2,\ u_3 = -e_1,\ u_4 = -e_2$$

and Σ' the complete fan with minimal generators

$$u_1' = e_1,\ u_2' = e_2,\ u_3' = -e_1+e_2,\ u_4' = -e_1-e_2.$$

Use Theorem 12.4.1 to show that

$$\begin{aligned}H^\bullet(X_\Sigma,\mathbb{Q}) &\simeq \mathbb{Q}[x,y]/\langle xy, x^2+y^2\rangle\\ H^\bullet(X_{\Sigma'},\mathbb{Q}) &\simeq \mathbb{Q}[x,y]/\langle xy, x^2+2y^2\rangle.\end{aligned}$$

(b) Deduce that the cohomology rings $H^\bullet(X_\Sigma,\mathbb{Q})$ and $H^\bullet(X_{\Sigma'},\mathbb{Q})$ are not isomorphic. Hint: Show that $H^\bullet(X_{\Sigma'},\mathbb{Q})$ has no elements α,β of degree 2 satisfing $\alpha^2 = -\beta^2 \neq 0$ and $\alpha\beta = 0$.

(c) Show that if we extend scalars to $\mathbb{R}$, however, then the cohomology rings become isomorphic:

$$H^\bullet(X_{\Sigma'},\mathbb{R}) = H^\bullet(X_{\Sigma'},\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{R} \simeq H^\bullet(X_\Sigma,\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{R} = H^\bullet(X_\Sigma,\mathbb{R}).$$

The paper [**95**] shows that the same is true for all complete toric surfaces—the real cohomology algebra depends only on the number of rays in the fan. This result does not extend to higher-dimensional toric varieties.

12.4.2. In this exercise, you will show that $\mathbb{C}^\infty\setminus\{0\}$ is contractible.

(a) Show that $\mathbb{C}^\infty\setminus\{0\}$ is homotopy equivalent to the unit sphere $S^\infty \subseteq \mathbb{C}^\infty$, where S^∞ is the set of all $(a_0,a_1,\dots)\in\mathbb{C}^\infty$ such that $\sum_{i=0}^\infty |a_i|^2 = 1$.

(b) Show that the shift map $T : S^\infty \to S^\infty$ defined by $T(a_0,a_1,\dots) = (0,a_0,a_1,\dots)$ is homotopic to the identity map $\mathrm{id} : S^\infty \to S^\infty$. Hint: Consider the 1-parameter family of maps $(1-t)\,\mathrm{id} + t\,T$, $t\in[0,1]$, and note that T has no eigenvalues or eigenvectors.

(c) Finally, show that T is homotopic to the constant map $U(a_0,a_1,\dots) = (1,0,\dots)$. Hint: Consider the 1-parameter family

$$\cos(\pi t/2)\,T(a_0,a_1,\dots) + \sin(\pi t/2)\,U(a_0,a_1,\dots).$$

12.4.3. For each $\tau \in \Sigma(n-1)$, let m_τ be a nonzero element of $\sigma^\perp \cap M$ and let $f \in (\Lambda_T)_\mathbb{Q}$ correspond to $\prod_{\tau\in\Sigma(n-1)} m_\tau \in \mathrm{Sym}_\mathbb{Q}(M)$. Prove that the map δ from (12.4.11) becomes an isomorphism after localizing at f. Hint: Use (12.4.18).

12.4.4. Consider $\mathrm{SR}_\mathbb{Q}(\Sigma_r) = \mathbb{Q}[x_1,x_2,x_3,x_4]/\langle x_1x_3,\, x_2x_4\rangle$, which for $M = \mathbb{Z}^2$ becomes a $\mathrm{Sym}_\mathbb{Q}(M)$-algebra where e_1 acts as $x_1 - x_3$ and e_1 acts as $-rx_1 - x_2 + x_4$.

(a) $\mathrm{SR}_\mathbb{Q}(\Sigma_r)$ has graded ring automorphisms given by rescaling the variables by elements of $\mathbb{Q}^*$, plus the maps induced by $x_1 \leftrightarrow x_3$, by $x_2 \leftrightarrow x_4$, and by $(x_1,x_3) \leftrightarrow (x_2,x_4)$. Show that these generate the group of all graded ring automorphisms of $\mathrm{SR}_\mathbb{Q}(\Sigma_r)$

(b) Prove that if $\mathrm{SR}_\mathbb{Q}(\Sigma_r) \simeq \mathrm{SR}_\mathbb{Q}(\Sigma_s)$ as graded $\mathrm{Sym}_\mathbb{Q}(M)$-algebras, then $r = s$.

12.4.5. Complete the proof of Lemma 12.4.16.

12.4.6. In Lemma 12.4.17, show that if $\tau \in \Sigma(n-1)$ is a face of $\sigma, \sigma' \in \Sigma(n)$, then the component of the map β' corresponding to τ can be identified with the difference of the restriction maps $H_T^\bullet(U_\sigma,\mathbb{Q}) \to H_T^\bullet(U_\tau,\mathbb{Q})$ and $H_T^\bullet(U_{\sigma'},\mathbb{Q}) \to H_T^\bullet(U_\tau,\mathbb{Q})$.

12.4.7. We will use the notation of Lemma 12.4.17. Let $\tau \in \Sigma(n-1)$. Define the map $B_\tau : \mathbb{Q}[\tau] \to H_T^\bullet(U_\tau,\mathbb{Q})$ by $x_i \mapsto [D_i \cap U_\tau]_T$ for $\rho_i \in \tau(1)$. Also let $N_\tau = \mathrm{Span}(\tau) \cap N \subseteq N$, with dual $M \to M_\tau = M/(\tau^\perp \cap N)$.

(a) As in the proof of Lemma 12.4.17, there are $\ell_i > 0$ and $m_i \in M$ such that $\ell_i D_i \cap U_\tau = \mathrm{div}(\chi^{m_i})$. Explain why m_i is not unique and show that $x_i \mapsto \ell_i^{-1}[m_i] \in (M_\tau)_\mathbb{Q}$ defines an isomorphism $\bar{B}_\tau : \mathbb{Q}[\tau] \simeq \mathrm{Sym}_\mathbb{Q}(M_\tau)$.

(b) Use Proposition 12.1.9 to show that $H_T^\bullet(U_\tau,\mathbb{Q}) \simeq H_T^\bullet(O(\tau),\mathbb{Q})$.

(c) Let x_τ be the identity element of $O(\tau)$. Show that T_{N_t} is the isotropy subgroup of $x_\tau \in O(\tau)$ under the action of $T = T_N$.

(d) Show that $N = N_\tau \oplus N'$, where $N' \simeq \mathbb{Z}$. Conclude that $T = T_{N_\tau} \times \mathbb{C}^*$, where T_{N_t} acts trivially on $O(\tau)$ and $\mathbb{C}^*$ acts freely and transitively on $O(\tau)$.

(e) Suppose a product group $G \times H$ acts on a space X such that G acts trivially and H acts freely and transitively. Prove that $H_{G\times H}^\bullet(X,\mathbb{Q}) \simeq H_G^\bullet(\{\mathrm{pt}\},\mathbb{Q})$. Hint: Use $E(G\times H) = EG \times EH$ and regard $G \times H$ as acting on $\{\mathrm{pt}\} \times X$.

(f) Explain why $H_T^\bullet(O(\tau),\mathbb{Q}) \simeq H_{T_{N_\tau}}^\bullet(\{\mathrm{pt}\},\mathbb{Q})$ and conclude that B_τ is an isomorphism. Hint: (12.4.16) holds for τ.

12.4.8. Verify the claims in Example 12.4.18.

12.4.9. In proof of part (b) of Proposition 12.4.13

(a) Prove that V_D is the pullback of V_{χ^m} via $p : X \to \{\mathrm{pt}\}$.

(b) Prove that $(V_{\chi^m})_T$ is the pullback of $(\mathbb{C})_{\mathbb{C}^*}$ via $B\chi^m : BT \to B\mathbb{C}^*$.

(c) Let $\mathbb{C}^*$ act on $\mathbb{C}$ by multiplication. Prove that $(\mathbb{C}^{\ell+1} \setminus \{0\}) \times_{\mathbb{C}^*} \mathbb{C} \to \mathbb{P}^\ell$ is the vector bundle $V \to \mathbb{P}^\ell$ described in Example 6.0.19.

12.4.10. In the situation of Theorem 12.4.1, take $\sigma \in \Sigma$ and write $\sigma(1) = \{i_1,\dots,i_d\}$, where $d = \dim\sigma$. Then let

$$\mathsf{x}^\sigma = [x_{i_1}\cdots x_{i_d}] \in R_\mathbb{Q}(\Sigma)_d.$$

In the text, σ was often an element of $\Sigma(n)$; here, it represents an arbitrary element of Σ. In this exercise you will prove that $R_{\mathbb{Q}}(\Sigma)$ is spanned over $\mathbb{Q}$ by the x^σ for $\sigma \in \Sigma$. (The x^σ may be linearly dependent, however.)

(a) Show that
$$\mathrm{x}^\sigma \mathrm{x}^{\sigma'} = \begin{cases} \mathrm{x}^{\sigma+\sigma'} & \text{if } \sigma \cap \sigma' = \{0\} \text{ and } \sigma+\sigma' \in \Sigma \\ 0 & \text{if } \sigma+\sigma' \notin \Sigma. \end{cases}$$

(b) Now consider a monomial $x^a = x_{i_1}^{a_1} \cdots x_{i_s}^{a_s}$ with $i_1 < i_2 < \cdots < i_s$ and $a_j \geq 1$. We may assume $\rho_{i_1} + \cdots + \rho_{i_s} \in \Sigma$ since otherwise $[x^a] = 0$ in $R_{\mathbb{Q}}(\Sigma)$. If $a_1 > 1$, then use the relations from $\mathcal{J}$ to show that $[x_{i_1}]$ is a linear combination with coefficients in $\mathbb{Q}$ of the $[x_\ell]$ for $\ell \notin \{i_1, \ldots, i_s\}$ Hint: $u_{i_1}, \ldots, u_{i_s}$ are part of a basis of $N_{\mathbb{Q}}$.

(c) Write $x^a = x_{i_1} \cdot x_{i_1}^{a_1-1} \cdots x_{i_s}^{a_s}$ and replace the first x_{i_1} with the linear combination found in part (b). Then repeat part (b) for each term the resulting linear combination. Show that this process eventually expresses $[x^a]$ as a linear combination of cosets of square-free monomials. This will prove our claim. Hint: Induction on $\sum_{i=1}^s (a_i - 1)$.

§12.5. The Chow Ring and Intersection Cohomology

Here we explore two alternative descriptions of the cohomology ring of a complete simplicial toric variety. The first involves the Chow ring, while the second uses intersection cohomology.

The Chow Ring. We will show that the description of the cohomology ring given in Theorem 12.4.1 applies equally well to the *rational Chow ring* $A^\bullet(X_\Sigma)_{\mathbb{Q}}$ of a complete simplicial toric variety X_Σ. We begin by sketching the construction of the Chow ring. We refer the reader to [**107**] for the details.

Let $Z_k(X)$ denote the group of k-dimensional algebraic cycles on a variety X. In other words, $Z_k(X)$ is the additive group of finite formal linear combinations
$$\sum_i n_i [V_i]$$
of irreducible subvarieties V_i of dimension k with coefficients $n_i \in \mathbb{Z}$. A k-cycle α on X is defined to be *rationally equivalent to zero* if there are finitely many $(k+1)$-dimensional irreducible subvarieties $W_i \subseteq X$ and nonzero rational functions $f_i \in \mathbb{C}(W_i)$ such that
$$\alpha = \sum_i [\mathrm{div}_{W_i}(f_i)],$$
where $\mathrm{div}_{W_i}(f_i)$ is the divisor of the rational function f_i on W_i. Note that W_i may fail to be normal, so this requires a more general definition of $\mathrm{div}(f_i)$ than the one given in §4.0. The cycles rationally equivalent to zero form a subgroup $\mathrm{Rat}_k(X)$ of $Z_k(X)$ (Exercise 12.5.1). Then the group of k-cycles modulo rational equivalence is defined to be
$$A_k(X) = Z_k(X)/\mathrm{Rat}_k(X).$$

When X is normal of dimension n, $A_{n-1}(X)$ is the group $\mathrm{Cl}(X)$ of Weil divisors modulo linear equivalence defined in §4.0 (Exercise 12.5.1).

For a general n-dimensional variety X, the group of cycles of *codimension* k modulo rational equivalence is defined to be $A^k(X) = A_{n-k}(X)$. If in addition X is smooth and projective, then with a substantial amount of work (see [**107**, Ch. 8]), it can be shown that there is a product

$$A^k(X) \times A^\ell(X) \longrightarrow A^{k+\ell}(X),$$

which coincides with the geometric intersection of cycles in the case of transverse intersections. We will write the product as $\alpha \cdot \beta$ for $\alpha \in A^k(X)$ and $\beta \in A^\ell(X)$. It can be shown that this intersection product is commutative and associative, and this makes $A^\bullet(X) = \bigoplus_{k=0}^n A^k(X)$ into a graded ring, called the *Chow ring* of X. $A^\bullet(X)$ can be viewed as a sort of algebraic version of the integral cohomology ring $H^\bullet(X,\mathbb{Z})$. The intersection of cycles corresponds to cup product in cohomology, so there is a ring homomorphism

$$A^\bullet(X) \longrightarrow H^\bullet(X,\mathbb{Z}) \tag{12.5.1}$$

defined as follows. Let $n = \dim X$. Then a cycle $\alpha \in A^k(X) = A_{n-k}(X)$ gives a homology class in $H_{2n-2k}(X,\mathbb{Z})$ whose Poincaré dual is in $H^{2k}(X,\mathbb{Z})$. In particular, the map (12.5.1) doubles degrees.

The Toric Case. If X_Σ is a complete simplicial toric variety of dimension n, then the intersection product can be defined on rational cycles, making

$$A^\bullet(X_\Sigma)_\mathbb{Q} = A^\bullet(X_\Sigma) \otimes_\mathbb{Z} \mathbb{Q} = \bigoplus_{k=0}^n A^k(X_\Sigma) \otimes_\mathbb{Z} \mathbb{Q}$$

into a graded ring. The key point here is that X_Σ has a covering by open affine varieties of the form $\mathbb{C}^n/G$ for some finite group G. In this case,

$$A^\bullet(\mathbb{C}^n/G)_\mathbb{Q} \simeq A^\bullet(\mathbb{C}^n)^G_\mathbb{Q},$$

the G-invariant subring. As in the smooth case, each subvariety of codimension k in X_Σ defines a class in $H_{2n-2k}(X_\Sigma,\mathbb{Q})$, hence by Poincaré duality, a class in $H^{2k}(X_\Sigma,\mathbb{Q})$. Hence we also have a ring homomorphism

$$A^\bullet(X_\Sigma)_\mathbb{Q} \longrightarrow H^\bullet(X_\Sigma,\mathbb{Q}). \tag{12.5.2}$$

For each $\sigma \in \Sigma$, the orbit closure $V(\sigma)$ is a subvariety of codimension $\dim\sigma$. Write $[V(\sigma)]$ for its rational equivalence class in $A^{\dim\sigma}(X_\Sigma)$.

Lemma 12.5.1. *The* $[V(\sigma)]$ *for* $\sigma \in \Sigma$ *generate* $A^\bullet(X_\Sigma)$ *as an abelian group.*

Proof. We will show that the $[V(\sigma)]$ for σ of dimension $n-k$ generate $A_k(X_\Sigma)$. In general, when $Y \subseteq X$ is a closed subvariety of a variety X, there is a short exact sequence of Chow groups

$$A_k(Y) \longrightarrow A_k(X) \longrightarrow A_k(X \setminus Y) \longrightarrow 0, \tag{12.5.3}$$

where the first map comes from the inclusion $Y \hookrightarrow X$, and the second comes by restriction. See [**107**, Prop. 1.8]) for a proof. If X is normal of dimension n, then (12.5.3) for $k = n-1$ is precisely Theorem 4.0.20 (Exercise 12.5.2).

Consider the filtration of X_Σ as in (12.3.3),

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X_\Sigma,$$

where X_p is the union of the closures $V(\sigma)$ of the torus orbits of dimension p. By the Orbit-Cone Correspondence, we have

$$X_p \setminus X_{p-1} = \coprod_{\sigma \in \Sigma(n-p)} O(\sigma).$$

Then (12.5.3) gives an exact sequence

$$A_k(X_{p-1}) \longrightarrow A_k(X_p) \longrightarrow \bigoplus_{\sigma \in \Sigma(n-p)} A_k(O(\sigma)) \longrightarrow 0.$$

But $O(\sigma)$ is a p-dimensional torus when $\sigma \in \Sigma(n-p)$ and hence can be viewed as an affine open subset of $\mathbb{C}^p$. In Exercise 12.5.3 you will show that

$$A_k(\mathbb{C}^p) = \begin{cases} \mathbb{Z}[\mathbb{C}^p] & k = p \\ 0 & \text{otherwise.} \end{cases}$$

Using (12.5.3) for the inclusion $O(\sigma) \hookrightarrow \mathbb{C}^p$, we see that

$$A_k(O(\sigma)) = \begin{cases} \mathbb{Z}[O(\sigma)] & k = p \\ 0 & \text{otherwise.} \end{cases}$$

When $p > k$, it follows that the above exact sequence simplifies to

$$A_k(X_{p-1}) \longrightarrow A_k(X_p) \longrightarrow 0.$$

Now fix k and let p increase from k to n. It follows that $A_k(X_k)$ surjects onto $A_k(X_\Sigma)$. By Exercise 12.5.2, we also have

$$A_k(X_k) = \bigoplus_{\sigma \in \Sigma(n-k)} \mathbb{Z}[V(\sigma)]$$

since $X_k = \bigcup_{\sigma \in \Sigma(n-k)} V(\sigma)$ and each $V(\sigma)$ is irreducible of dimension k. The lemma now follows easily. □

When $\rho \in \Sigma(1)$, the orbit closure $V(\rho)$ is the divisor denoted D_ρ. For $m \in M$, recall from Proposition 4.1.2 that

$$\mathrm{div}(\chi^m) = \textstyle\sum_\rho \langle m, u_\rho \rangle D_\rho.$$

Since $A^1(X_\Sigma) = \mathrm{Cl}(X_\Sigma)$ by Exercise 12.5.2, we obtain

$$\textstyle\sum_\rho \langle m, u_\rho \rangle [D_\rho] = 0 \text{ in } A^1(X_\Sigma). \tag{12.5.4}$$

Here is an example of how to compute intersection products in the Chow ring.

Lemma 12.5.2. *Assume that X_Σ is complete and simplicial. If $\rho_1,\dots,\rho_d \in \Sigma(1)$ are distinct, then in $A^\bullet(X_\Sigma)_{\mathbb{Q}}$, we have*

$$[D_{\rho_1}]\cdot[D_{\rho_2}]\cdots[D_{\rho_d}] = \begin{cases} \frac{1}{\mathrm{mult}(\sigma)}[V(\sigma)] & \textit{if } \sigma = \rho_1+\cdots+\rho_d \in \Sigma \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Let $\rho \in \Sigma(1)$ and σ be a cone of Σ not containing ρ. We will show that

$$[D_\rho]\cdot[V(\sigma)] = \begin{cases} \frac{\mathrm{mult}(\sigma)}{\mathrm{mult}(\tau)}[V(\tau)] & \text{if } \tau = \rho+\sigma \in \Sigma \\ 0 & \text{otherwise}. \end{cases} \tag{12.5.5}$$

In Exercise 12.5.4 you will prove that the lemma follows from (12.5.5). You should also check that (12.5.5) generalizes the intersection formulas from parts (a) and (b) of Proposition 6.4.4.

We noted earlier that intersection products can be subtle. Life is easier when intersecting with a divisor, but even here, some care is needed. The relevant theory is developed in [**107**, §2.3]. We will use the following special case. Let D be a Cartier divisor on X_Σ and let $i : V \hookrightarrow X_\Sigma$ be a normal subvariety of dimension k. The line bundle $\mathscr{O}_{X_\Sigma}(D)$ pulls back to a line bundle $i^*\mathscr{O}_{X_\Sigma}(D)$ on V, so that by Theorem 6.0.20, $i^*\mathscr{O}_{X_\Sigma}(D) \simeq \mathscr{O}_V(D')$ for some Cartier divisor D' on V. Then D' is a $(k-1)$-cycle on X_Σ and by [**107**, §2.3] gives the intersection product

$$[D]\cdot[V] = [D']. \tag{12.5.6}$$

In Exercise 12.5.5 you will use this definition to show that $[D]\cdot[V] = 0$ when $\mathrm{Supp}(D)\cap V = \emptyset$.

Since X_Σ is simplicial, D_ρ is $\mathbb{Q}$-Cartier, so there is some positive multiple $D = \ell D_\rho$ that is Cartier on X_Σ. If $\rho+\sigma \notin \Sigma$, then $\mathrm{Supp}(D)\cap V(\sigma) = D_\rho \cap V(\sigma) = \emptyset$ by the Orbit-Cone Correspondence. Then $[D]\cdot[V(\sigma)] = 0$ by the previous paragraph, which implies $[D_\rho]\cdot[V(\sigma)] = 0$. Hence we may assume that $\tau = \rho+\sigma \in \Sigma$. We need to prove that

$$[D]\cdot[V(\sigma)] = \ell\,\frac{\mathrm{mult}(\sigma)}{\mathrm{mult}(\tau)}[V(\tau)]. \tag{12.5.7}$$

The first step is to find a Cartier divisor D' on $V(\sigma)$ such that $i^*\mathscr{O}_{X_\Sigma}(D) \simeq \mathscr{O}_{V(\sigma)}(D')$, where $i : V(\sigma) \hookrightarrow X_\Sigma$ is the inclusion. We will use the support function $\varphi_D : N_{\mathbb{R}} \to \mathbb{R}$ of D, which satisfies

$$\varphi_D(u_{\rho'}) = \begin{cases} -\ell & \text{if } \rho' = \rho \\ 0 & \text{otherwise}. \end{cases}$$

Note in particular that φ_D vanishes on $\mathrm{Span}(\sigma)$ since $\rho \notin \sigma(1)$.

Recall from Proposition 3.2.7 that $V(\sigma)$ is the toric variety of the fan $\mathrm{Star}(\sigma)$ in $N(\sigma)_{\mathbb{R}}$, where $N(\sigma) = N/(\mathrm{Span}(\sigma)\cap N)$. The previous paragraph shows that φ_D

factors through the quotient map $N_{\mathbb{R}} \to N(\sigma)_{\mathbb{R}}$. By Exercise 12.5.6, the induced map $N(\sigma)_{\mathbb{R}} \to \mathbb{R}$ is the support function $\varphi_{D'}$ of the divisor D' we want.

The rays of $\mathrm{Star}(\sigma)$ are images of rays of Σ, so that $\varphi_{D'}$ vanishes on all rays of $\mathrm{Star}(\sigma)$ except for $\overline{\rho} = \overline{\rho+\sigma} = \overline{\tau}$. Thus D' is the divisor on $V(\sigma)$ given by

$$D' = -\varphi_{D'}(u_{\overline{\rho}})D_{\overline{\rho}},$$

where $u_{\overline{\rho}}$ is the minimal generator of $\overline{\rho} \cap N(\sigma)$. In particular, the image of u_ρ in $N(\sigma)$ is $\overline{u}_\rho = \beta u_{\overline{\rho}}$ for some integer β. Thus

$$-\varphi_{D'}(u_{\overline{\rho}}) = \frac{-1}{\beta}\varphi_{D'}(\beta u_{\overline{\rho}}) = \frac{-1}{\beta}\varphi_{D'}(\overline{u}_\rho) = \frac{-1}{\beta}\varphi_D(u_\rho) = \frac{\ell}{\beta}.$$

Since $\beta = \mathrm{mult}(\tau)/\mathrm{mult}(\sigma)$ by the proof of Lemma 6.4.2, we obtain

$$D' = \ell\,\frac{\mathrm{mult}(\sigma)}{\mathrm{mult}(\tau)}D_{\overline{\rho}}.$$

However, since $\tau = \rho+\sigma$, the divisor $D_{\overline{\rho}}$ in $V(\sigma)$ is the subvariety $V(\tau)$ in $V(\sigma)$. From here, (12.5.7) follows easily, and the proof is complete. □

Similar to what happened in §6.4, Lemma 12.5.2 shows that rational numbers appear naturally when doing intersection theory on simplicial toric varieties.

The Chow Ring of a Toric Variety. As in §12.4, write $\Sigma(1) = \{\rho_1,\dots,\rho_r\}$. This gives the ring

$$R_{\mathbb{Q}}(\Sigma) = \mathbb{Q}[x_1,\dots,x_r]/(\mathscr{I}+\mathscr{J})$$

for $\mathscr{I}$ and $\mathscr{J}$ as in (12.4.2) and (12.4.3). Then Lemma 12.5.2 and (12.5.4) imply that $[x_i] \mapsto [D_{\rho_i}] \in A^1(X_\Sigma)_{\mathbb{Q}}$ defines a ring homomorphism

$$R_{\mathbb{Q}}(\Sigma)_{\mathbb{Q}} \longrightarrow A^\bullet(X_\Sigma)_{\mathbb{Q}}. \tag{12.5.8}$$

We also have the ring homomorphism $A^\bullet(X_\Sigma)_{\mathbb{Q}} \to H^\bullet(X_\Sigma,\mathbb{Q})$ from (12.5.2).

Theorem 12.5.3. *If X_Σ is complete and simplicial, then*

$$R_{\mathbb{Q}}(\Sigma)_{\mathbb{Q}} \simeq A^\bullet(X_\Sigma)_{\mathbb{Q}} \simeq H^\bullet(X_\Sigma,\mathbb{Q}),$$

where the maps are given by (12.5.8) *and* (12.5.2).

Proof. The composition of the maps $R_{\mathbb{Q}}(\Sigma)_{\mathbb{Q}} \to A^\bullet(X_\Sigma)_{\mathbb{Q}} \to H^\bullet(X_\Sigma,\mathbb{Q})$ from (12.5.8) and (12.5.2) is given by $[x_i] \mapsto [D_{\rho_i}] \in H^2(X_\Sigma,\mathbb{Q})$. Since this defines the isomorphism $R_{\mathbb{Q}}(\Sigma)_{\mathbb{Q}} \simeq H^\bullet(X_\Sigma,\mathbb{Q})$ from Theorem 12.4.1, it follows that the map $R_{\mathbb{Q}}(\Sigma)_{\mathbb{Q}} \to A^\bullet(X_\Sigma)_{\mathbb{Q}}$ is injective.

For surjectivity, take $\sigma = \rho_{i_1} + \cdots + \rho_{i_d} \in \Sigma$. By Lemma 12.5.2, $[x_{i_1}\cdots x_{i_d}] = [x_{i_1}]\cdots[x_{i_d}]$ maps to the product

$$[D_{\rho_{i_1}}]\cdots[D_{\rho_{i_d}}] = \frac{1}{\mathrm{mult}(\sigma)}[V(\sigma)].$$

Then surjectivity follows since the $[V(\sigma)]$ span $A^\bullet(X_\Sigma)_\mathbb{Q}$ by Lemma 12.5.1. Hence (12.5.8) is an isomorphism, and then the same must be true for (12.5.2). $\square$

The isomorphism here explains the connection between the cup product in cohomology and the intersection formula from (12.4.1). We should also note that there are isomorphisms

$$R(\Sigma) \simeq A^\bullet(X_\Sigma) \simeq H^\bullet(X_\Sigma, \mathbb{Z}) \tag{12.5.9}$$

when X_Σ is smooth, where $R(\Sigma)$ is the ring from Theorem 12.4.4.

There is also an *equivariant Chow ring* $A_T^\bullet(X_\Sigma)$ associated to the action of $T = T_N$ on X_Σ. If X_Σ is not simplicial, then Payne has shown in [**224**] that $A_T^\bullet(X_\Sigma)$ still coincides with the ring of continuous piecewise polynomial functions on the polyhedral decomposition of $N_\mathbb{R}$ defined by Σ, even over $\mathbb{Z}$. However, this ring need not coincide with equivariant cohomology ring in this case.

An Exact Sequence for the Chow Group. The proof of Lemma 12.5.1 gives a surjective map $\beta : \bigoplus_{\sigma\in\Sigma(n-k)} \mathbb{Q}[V(\sigma)] \to A_k(X_\Sigma)_\mathbb{Q}$. In [**111**], Fulton and Sturmfels describe generators of the kernel using the sequence

$$\bigoplus_{\tau\in\Sigma(n-k-1)} M(\tau)_\mathbb{Q} \xrightarrow{\ \alpha\ } \bigoplus_{\sigma\in\Sigma(n-k)} \mathbb{Q}[V(\sigma)] \xrightarrow{\ \beta\ } A_k(X_\Sigma)_\mathbb{Q} \longrightarrow 0. \tag{12.5.10}$$

To define α, take $\tau \in \Sigma(n-k-1)$, so $\dim V(\tau) = k+1$, and note that $M(\tau)$ is the character group of the torus of the toric variety $V(\tau)$. If $m \in M(\tau)$, then χ^m is a rational function on $V(\tau)$ and hence $\mathrm{div}(\chi^m)$ is a divisor on $V(\tau)$. This gives the k-cycle $\alpha(m) = [\mathrm{div}(\chi^m)]$ on X_Σ. In Exercise 12.5.7 you will show that

$$\alpha(m) \ = \sum_{\sigma\in\Sigma(n-k),\, \tau\prec\sigma} \langle m, u_{\rho,\tau}\rangle [V(\sigma)], \tag{12.5.11}$$

where $u_{\rho,\tau} \in \sigma$ generates $N_\sigma/N_\tau \simeq \mathbb{Z}$. Then the definition of rational equivalence guarantees that (12.5.10) is a complex, i.e., $\beta\circ\alpha = 0$.

Theorem 12.5.4. *The complex* (12.5.10) *is exact if X_Σ is complete and simplicial.*

Proof. Since β is surjective, (12.5.10) gives a surjective map $\mathrm{coker}(\alpha) \to A_k(X_\Sigma)_\mathbb{Q}$. Now consider the dual of α, which can be written

$$\bigoplus_{\sigma\in\Sigma(n-k)} \big(\textstyle\bigwedge^0 M(\sigma)_\mathbb{Q}\big)^* \xrightarrow{\ \alpha^*\ } \bigoplus_{\tau\in\Sigma(n-k-1)} \big(\textstyle\bigwedge^1 M(\tau)_\mathbb{Q}\big)^*$$

Since $\dim M(\sigma)_\mathbb{Q} = k$ and $\dim M(\tau)_\mathbb{Q} = k+1$, we have

$$\begin{aligned}\big(\textstyle\bigwedge^0 M(\sigma)_\mathbb{Q}\big)^* &\simeq \textstyle\bigwedge^k M(\sigma)_\mathbb{Q} \otimes_\mathbb{Q} \bigwedge^k N(\sigma)_\mathbb{Q}\\ \big(\textstyle\bigwedge^1 M(\tau)_\mathbb{Q}\big)^* &\simeq \textstyle\bigwedge^k M(\tau)_\mathbb{Q} \otimes_\mathbb{Q} \bigwedge^{k+1} N(\tau)_\mathbb{Q}.\end{aligned}$$

Picking bases of $N(\sigma)$ and $N(\tau)$ and thinking in terms of orientation coefficients, one sees that α^* can be identified with the map

$$C^k(\Sigma,\textstyle\bigwedge^k)_{\mathbb{Q}} = \displaystyle\bigoplus_{\sigma\in\Sigma(n-k)} \textstyle\bigwedge^k M(\sigma)_{\mathbb{Q}} \xrightarrow{\delta^k} C^{k+1}(\Sigma,\textstyle\bigwedge^k)_{\mathbb{Q}} = \displaystyle\bigoplus_{\tau\in\Sigma(n-k-1)} \textstyle\bigwedge^k M(\tau)_{\mathbb{Q}},$$

where $C^\bullet(\Sigma,\bigwedge^k)$ is the complex defined in the discussion preceding Lemma 12.3.3. The spectral sequence $E_1^{p,q} \Rightarrow H^{p+q}(X_\Sigma,\mathbb{Q})$ from (12.3.11) has $E_1^{p,q} = C^p(\Sigma,\bigwedge^q)_{\mathbb{Q}}$ by (12.3.12). Since $E_1^{k-1,k} = C^{k-1}(\Sigma,\bigwedge^k) = 0$ for dimension reasons, we obtain

$$E_2^{k,k} = \ker(E_1^{k,k} \to E_1^{k+1,k}) = \ker(\delta^k) = \ker(\alpha^*).$$

Theorem 12.3.11 implies that $E_2^{k,k} \simeq H^{2k}(X_\Sigma,\mathbb{Q})$, which in turn is isomorphic to $H^{2n-2k}(X_\Sigma,\mathbb{Q}) \simeq A^{n-k}(X_\Sigma)_{\mathbb{Q}}$ by Poincaré duality and Theorem 12.5.3. Thus

$$\dim \operatorname{coker}(\alpha) = \dim \ker(\alpha^*) = \dim A^{n-k}(X_\Sigma)_{\mathbb{Q}} = \dim A_k(X_\Sigma)_{\mathbb{Q}}.$$

It follows that the surjective map $\operatorname{coker}(\alpha) \to A_k(X_\Sigma)_{\mathbb{Q}}$ is an isomorphism. This proves that (12.5.10) is exact. □

The result proved in [**111**, Prop. 2.1] is stronger than Theorem 12.5.4 since it holds over $\mathbb{Z}$ and applies to arbitrary toric varieties. Note also that when $k = n-1$, the exact sequence (12.5.10) becomes the familiar exact sequence

$$M_{\mathbb{Q}} \xrightarrow{\alpha} \textstyle\bigoplus_\rho \mathbb{Q}[D_\rho] \xrightarrow{\beta} A_{n-1}(X_\Sigma)_{\mathbb{Q}} = \operatorname{Cl}(X_\Sigma)_{\mathbb{Q}} \longrightarrow 0$$

from Chapter 4. For general k, you should check that (12.5.10) can be regarded as the purely "toric" version of rational equivalence.

Intersection Cohomology. As we saw in §12.3, most of the nice properties of the cohomology ring $H^\bullet(X_\Sigma,\mathbb{Q})$ for simplicial X_Σ (the vanishing of the odd-degree groups, Poincaré duality, and so forth) can fail for nonsimplicial toric varieties. The theory of intersection homology and cohomology was first developed by Goresky and MacPherson in the 1970's to study spaces with singularities more complicated than finite quotient singularities, and many of the nice properties are recovered in this cohomology theory. See [**175**] for a general introduction.

Intersection homology was originally defined for spaces with a special sort of stratification (see [**119**] and [**120**]). In the following, $\operatorname{cone}(S)$ is the open topological cone over S, namely the quotient space of $S\times[0,1)$ modulo the relation that identifies all $(s,0)$ for $s\in S$.

Definition 12.5.5. A ***topological pseudomanifold*** X is a paracompact Hausdorff space with a filtration

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$$

such that:

(a) For each point p in $X_i \setminus X_{i-1}$, there is a neighborhood N of p in X, a compact Hausdorff space L with an $n-i-1$ dimensional topological stratification

$$\emptyset = L_{-1} \subseteq L_0 \subseteq \cdots \subseteq L_{n-i-1} = L,$$

and a homeomorphism $\phi : \mathbb{R}^i \times \operatorname{cone}(L) \to N$ that maps each $\mathbb{R}^i \times \operatorname{cone}(L_j)$ homeomorphically to $N \cap X_{i+j+1}$.

(b) $X_{n-1} = X_{n-2}$.

When the first condition holds, the second condition implies that $X \setminus X_{n-2}$ is an n-dimensional real manifold. It is known that all projective or quasiprojective varieties over $\mathbb{C}$ are topological pseudomanifolds.

A *perversity* is a function $p : \{2,3,\ldots\} \to \mathbb{N}$ such that $p(2) = 0$ and both $p(k)$ and $k-2-p(k)$ are nonnegative and nondecreasing functions of k. The *complementary perversity* of p is $q(k) = k-2-p(k)$. The constant function $\min(k) = 0$ is the minimal perversity; its complement is the maximal perversity $\max(k) = k-2$. The *middle perversity* is $p(k) = \lfloor \frac{k-2}{2} \rfloor$. Perversities are used to specify allowable intersections of cycles with the strata X_i in the definition of intersection homology.

Definition 12.5.6. The ***intersection homology groups*** $IH_i^p(X)$ of a topological pseudomanifold X with respect to a given perversity p are the homology groups of a subcomplex $IC_\bullet^p(X)$ of the singular chains on X as follows. A singular i-simplex $\sigma : \Delta_i \to X$ is said to be *p-allowable* if for every k, $\sigma^{-1}(X_{n-k} \setminus X_{n-k-1})$ is contained in the union of the faces of Δ_i of dimension $i-k+p(k)$. Then a singular chain is said to be p-allowable if it is a linear combination of p-allowable singular simplices. Finally, $IC_i^p(X)$ consists of all p-allowable i-chains c such that ∂c is also p-allowable.

Following an idea of Deligne, this definition was later reformulated using the derived category of the category of sheaves of $\mathbb{Z}$-modules on X (see [**120**]). This way of framing the theory has numerous technical advantages, but we will not pursue this approach here. The definition of the $IH_i^p(X)$ appears to depend on the stratification, but Goresky and MacPherson proved that the $IH_i^p(X)$ are actually topological invariants of X.

Example 12.5.7. Let X be an irreducible variety of dimension n with a single isolated singular point x. For the middle perversity, it is not difficult to see that

$$IH_i(X) = \begin{cases} H_i(X) & i > n \\ \operatorname{im}(H_i(X \setminus \{x\}) \to H_i(X)) & i = n \\ H_i(X \setminus \{x\}) & i < n. \end{cases}$$

You will prove this in Exercise 12.5.8. ◊

The following theorem summarizes the properties of intersection homology that we will discuss.

Theorem 12.5.8. *Let X be a normal projective variety of dimension n.*

(a) *$IH_i^p(X)$ is finitely generated for any i and any perversity p.*

(b) *For the minimal and maximal perversities, we have*

$$IH_i^{\min}(X) \simeq H^{2n-i}(X,\mathbb{Z}), \quad IH_i^{\max}(X) \simeq H_i(X,\mathbb{Z}).$$

Now let p be the middle perversity and define the ***intersection cohomology groups*** *as $IH^i(X)_{\mathbb{Q}} = IH^p_{2n-i}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then:*

(c) *Poincaré duality holds for the intersection cohomology. In particular,*

$$\dim IH^i(X)_{\mathbb{Q}} = \dim IH^{2n-i}(X)_{\mathbb{Q}}.$$

(d) *$IH^\bullet(X)_{\mathbb{Q}}$ is a module over the cohomology ring $H^\bullet(X,\mathbb{Q})$.*

(e) *Let $\omega \in H^2(X,\mathbb{Q})$ be the class of an ample divisor on X. Then for all $0 \le i \le n$, multiplication by ω^{n-i} defines an isomorphism*

$$\omega^{n-i} : IH^i(X)_{\mathbb{Q}} \longrightarrow IH^{2n-i}(X)_{\mathbb{Q}}.$$

This is the ***hard Lefschetz theorem****.*

(f) *Let ω be as in part* (e)*. Then for $0 \le i < n$, multiplication by ω*

$$\omega : IH^i(X)_{\mathbb{Q}} \longrightarrow IH^{i+2}(X)_{\mathbb{Q}}$$

is injective. □

See [**120**] for parts (a)–(d) of this theorem and [**23**] for the hard Lefschetz theorem stated in part (e). Note that part (f) follows from part (e).

The intersection cohomology $IH^\bullet(X)_{\mathbb{Q}}$ does not have a natural ring structure. However, when X_Σ is a complete simplicial toric variety, Comment 6.4 in [**119**] implies that the odd-degree intersection cohomology groups are zero and that

$$IH^{2i}(X_\Sigma)_{\mathbb{Q}} \simeq H^{2i}(X_\Sigma,\mathbb{Q}) \simeq A^i(X_\Sigma)_{\mathbb{Q}}, \quad i = 0,\ldots,n. \tag{12.5.12}$$

Applications to Polytopes. In 1980, Stanley [**257**] used toric varieties and their cohomology to prove the McMullen conjecture for a simplicial polytope P. His proof applied hard Lefschetz to the toric variety X_Σ, where Σ is the fan consisting of the cones over the proper faces of P. This is the normal fan of the dual polytope P°, which is simple. Hence X_Σ comes from a simplicial fan and we know hard Lefschetz is true in this case by (12.5.12) and part (e) of Theorem 12.5.8. As explained in Appendix A, Stanley's paper was an important event in the history of toric geometry.

Let $f_0^P,\ldots,f_n^P$ denote the face numbers of an n-dimensional polytope P. If P is simplicial, then (9.4.6), Exercise 9.4.10 and Theorem 12.3.12 imply that the numbers

$$h_k(P) = \sum_{i=k}^{n} (-1)^{i-k}\binom{i}{k} f^P_{n-i-1} = \sum_{i=k}^{n} (-1)^{i-k}\binom{i}{k} f_i^{P^\circ} \tag{12.5.13}$$

are the even Betti numbers $b_{2k} = \dim H^{2k}(X_\Sigma, \mathbb{Q})$ of X_Σ. Hence they satisfy the Dehn-Sommerville equations $h_k(P) = h_{n-k}(P)$ by Poincaré duality for the simplicial toric variety X_Σ. Also observe that

$$1 = h_0(P) \le h_1(P) \le \cdots \le h_m(P), \quad m = \lfloor \tfrac{n}{2} \rfloor$$

by the injectivity of part (f) of Theorem 12.5.8. This monotone property of the $h_k(P)$ is used by Stanley in [**257**].

For a more general (i.e., nonsimplicial) polytope P, some of the $h_k(P)$ in (12.5.13) can be negative, so a different approach is necessary if we want to relate these numbers to cohomology groups. Following [**258**], we define polynomials $h(P,t)$ and $g(P,t)$ recursively as follows. Let $g(\emptyset,t) = 1$ and $\dim \emptyset = -1$. Now assume that $g(Q,t)$ has been defined for all polytopes Q of dimension $< n$. Then, for a polytope P of dimension n, define

$$h(P,t) = \sum_{Q \prec P} g(Q,t)\,(t-1)^{n-1-\dim Q},$$

where is sum is over all proper faces Q of P, including $Q = \emptyset$. Then write $h(P,t) = \sum_{k=0}^{n} h_k(P)\, t^k$ and define

$$g(P,t) = \sum_{k=0}^{\lfloor n/2 \rfloor} (h_k(P) - h_{k-1}(P))\, t^k.$$

In particular, the numbers $h_k(P)$ are defined for any polytope P, and one can show that they agree with (12.5.13) when P is simplicial. We call

$$h(P,t) = \sum_{k=0}^{n} h_k(P)\, t^k$$

the *h-polynomial* of P. Here is a result from [**258**].

Theorem 12.5.9. *Let P be a full dimensional lattice polytope in $\mathbb{R}^n$ containing the origin as an interior point and let Σ be the fan whose cones are the cones over the proper faces of P. Then:*

(a) *The $h_k(P)$ are the intersection cohomology Betti numbers of X_Σ, i.e.,*

$$h_k(P) = \dim IH^{2k}(X_\Sigma)_{\mathbb{Q}}.$$

(b) $h_k(P) = h_{n-k}(P)$ *for* $k = 0, \ldots, n$.

(c) *Let $m = \lfloor \frac{n}{2} \rfloor$. Then*

$$1 = h_0(P) \le h_1(P) \le \cdots \le h_m(P). \qquad \square$$

The equalities in part (b) of the theorem generalize the Dehn-Sommerville equations. They follow from part (a) and Poincaré duality for intersection cohomology. Similarly, part (c) follows from part (f) of Theorem 12.5.8, which in turn is a consequence of hard Lefschetz.

Motivated by this application to polytopes, a purely combinatorial version of intersection cohomology for toric varieties has been studied by many researchers, including Barthel, Brasselet, Bressler, Fieseler, Karu, Kaup and Lunts. See [**42**] for a discussion of this topic and references to the original papers.

Exercises for §12.5.

12.5.1. This exercise will study rational equivalence on a variety X of dimension n.

(a) Show that $\mathrm{Rat}_k(X)$ is a subgroup of $Z_k(X)$ for all k.

(b) When X is normal, prove that two $(n-1)$ cycles on X are rationally equivalent if and only if they are linearly equivalent as defined in §4.0. Conclude that $A_{n-1}(X) = \mathrm{Cl}(X)$.

(c) Explain why $A_{n-1}(X) = \mathrm{Pic}(X)$ when X is smooth.

12.5.2. Let X be a variety of dimension n.

(a) In general, X may be reducible and may have irreducible components of dimension strictly smaller than n. Prove that $A_n(X) = Z_n(X) = \bigoplus_Y \mathbb{Z}[Y]$, where the sum is over all irreducible components $Y \subseteq X$ of dimension n.

(b) Assume in addition that X is normal. Then show that the exact sequence (12.5.3) is Theorem 4.0.20 when $k = n-1$.

12.5.3. Show directly from the definitions that the Chow groups of affine space are

$$A_k(\mathbb{C}^p) = \begin{cases} \mathbb{Z}[\mathbb{C}^p] & \text{if } k = p \\ 0 & \text{otherwise.} \end{cases}$$

12.5.4. Prove that Lemma 12.5.2 follows from (12.5.5).

12.5.5. Use (12.5.6) to prove that $[D]\cdot[V] = 0$ when $\mathrm{Supp}(D)\cap V = \emptyset$. Hint: Explain why $\mathrm{Supp}(D)\cap V = \emptyset$ implies that $i^*\mathscr{O}_{X_\Sigma}(D)$ is trivial on V.

12.5.6. In the proof of Lemma 12.5.2, we claimed that $\varphi_D : N_\mathbb{R} \to \mathbb{R}$ is the composition of $N_\mathbb{R} \to N(\sigma)_\mathbb{R}$ and $\varphi_{D'} : N(\sigma)_\mathbb{R} \to \mathbb{R}$. Prove this. Hint: Adapt the proof of Proposition 6.2.7.

12.5.7. Prove (12.5.11) when $\tau \in \Sigma(n-k-1)$ and $m \in M(\tau)$.

12.5.8. Use the definition of intersection homology to prove the formula for $IH_i(X)$ given in Example 12.5.7.

12.5.9. Let X be a complete smooth variety of dimension n. Given a Cartier divisor D and a smooth curve C on X, we defined the intersection number $D\cdot C$ in §6.3. In this exercise you will use (12.5.6) to prove that

$$D\cdot C = \int_X [D]\smile[C],$$

where $[D]$ and $[C]$ are the corresponding cohomology classes in $H^\bullet(X,\mathbb{Z})$.

(a) Explain why a point $p \in X$ gives a class $[p] \in H^{2n}(X,\mathbb{Z})$ such that $\int_X [x] = 1$.

(b) Let $i : C \hookrightarrow X$ be the inclusion map and suppose that $i^*\mathscr{O}_X(D) = \mathscr{O}_C\big(\sum_{i=1}^s \ell_i p_i\big)$ for $\ell_i \in \mathbb{Z}$ and $p_i \in C$. Explain why $D\cdot C = \sum_{i=1}^s \ell_i$ and $[D]\cdot[C] = \sum_{i=1}^s \ell_i[p_i]$ in $A^\bullet(X)$.

(c) Use the ring homomorhpism $A^\bullet(X) \to H^\bullet(X,\mathbb{Z})$ to prove $D\cdot C = \int_X [D]\smile[C]$.

12.5.10. Explain how Exercise 12.4.10 relates to the results of this section.

Chapter 13

Toric Hirzebruch-Riemann-Roch

The Hirzebruch-Riemann-Roch theorem (called HRR) shows how to compute the Euler characteristic of a coherent sheaf $\mathscr{F}$ on a smooth complete n-dimensional variety X using intersection theory. The formula is

$$\chi(\mathscr{F}) = \int_X \operatorname{ch}(\mathscr{F})\operatorname{Td}(X). \tag{13.0.1}$$

The term on the left is the Euler characteristic of $\mathscr{F}$ from Chapter 9, defined by

$$\chi(\mathscr{F}) = \chi(X,\mathscr{F}) = \sum_{i=0}^{n}(-1)^i \dim H^i(X,\mathscr{F}).$$

The right-hand side of (13.0.1) introduces two objects we have not seen before, the *Chern character* $\operatorname{ch}(\mathscr{F}) \in H^\bullet(X,\mathbb{Q})$ and the *Todd class* $\operatorname{Td}(X) \in H^\bullet(X,\mathbb{Q})$. The integrand on the right-hand side of (13.0.1) is the cup product $\operatorname{ch}(\mathscr{F}) \smile \operatorname{Td}(X)$, which is written $\operatorname{ch}(\mathscr{F})\operatorname{Td}(X)$ or $\operatorname{ch}(\mathscr{F})\cdot\operatorname{Td}(X)$ in practice.

This chapter will prove HRR for a line bundle $\mathscr{L} = \mathscr{O}_X(D)$ on a smooth complete toric variety $X = X_\Sigma$. Our strategy will be to first prove an equivariant Riemann-Roch theorem for X and then derive HRR via the nonequivariant limit.

In §13.1, we begin by defining the terms involved in (13.0.1) and working out some specific examples of HRR. To prepare for the equivariant version of Riemann-Roch, we discuss some remarkable equalities due to Brion in §13.2. Then §13.3 proves equivariant Riemann-Roch in the smooth case, following the paper [**50**] of Brion and Vergne. The final two sections §13.4 and §13.5 apply these ideas to lattice points in polytopes, including the volume polynomial and the Khovanskii-Pukhlikov theorem. We will also sketch a second proof of HRR.

§13.1. Chern Characters, Todd Classes, and HRR

In order to state our version of the toric HRR, we need to define Chern characters and Todd classes. Both of these require knowledge of Chern classes, which we now discuss.

Chern Classes. A locally free sheaf $\mathscr{E}$ of rank r on a variety X has *Chern classes* $c_i(\mathscr{E}) \in H^{2i}(X,\mathbb{Z})$ for $0 \le i \le r$. Its *total Chern class* is $c(\mathscr{E}) = c_0(\mathscr{E}) + \cdots + c_r(\mathscr{E})$. Here are the key properties we will need:

- $c_0(\mathscr{E}) = 1$ for any $\mathscr{E}$ and $c(\mathscr{O}_X) = 1$ for any X.
- $c_i(f^*\mathscr{E}) = f^*c_i(\mathscr{E})$ when $f : X \to Y$ is a morphism and $\mathscr{E}$ is locally free on Y.
- $c(\mathscr{F}) = c(\mathscr{E}) \cdot c(\mathscr{G})$ when $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ is an exact sequence.
- $c_i(\mathscr{E}^\vee) = (-1)^i c_i(\mathscr{E})$, where $\mathscr{E}^\vee$ is the dual of $\mathscr{E}$.
- $c_1(\bigwedge^r \mathscr{E}) = c_1(\mathscr{E})$ when $\mathscr{E}$ has rank r.
- If D is a Cartier divisor on a smooth complete variety X, then
$$c_1(\mathscr{O}_X(D)) = [D] \in H^2(X,\mathbb{Z}).$$

We refer the reader to [**107**] or [**144**] for a further discussion of Chern classes. Later in the section we will say a few words about Chern classes of coherent sheaves.

Here is a nice example that shows how to use the properties of Chern classes.

Example 13.1.1. Let $X = X_\Sigma$ be a smooth complete toric variety. Its cotangent bundle Ω^1_X fits into the generalized Euler sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \textstyle\bigoplus_\rho \mathscr{O}_X(-D_\rho) \longrightarrow \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathscr{O}_X \longrightarrow 0 \tag{13.1.1}$$

from Theorem 8.1.6. Since $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathscr{O}_X \simeq \mathscr{O}_X^{r-n}$ for $r = |\Sigma(1)|$, the total Chern class of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathscr{O}_X$ is 1. Hence

$$c(\Omega^1_X) = c\big(\textstyle\bigoplus_\rho \mathscr{O}_X(-D_\rho)\big) = \prod_\rho c(\mathscr{O}_X(-D_\rho)) = \prod_\rho (1 - [D_\rho]),$$

where $\prod$ means cup product in $H^\bullet(X,\mathbb{Z})$. Thus $c_1(\Omega^1_X) = [-\sum_\rho D_\rho]$.

Let $\dim X = n$. Then the canonical bundle $\omega_X = \Omega^n_X$ has first Chern class

$$c_1(\omega_X) = c_1(\textstyle\bigwedge^n \Omega^1_X) = c_1(\Omega^1_X) = [-\sum_\rho D_\rho].$$

This gives a Chern class proof that X has canonical divisor $K_X = -\sum_\rho D_\rho$. $\Diamond$

The Chern classes of the tangent bundle $\mathscr{T}_X$ of a smooth complete variety X are called the *Chern classes* of X. We often write $c_i = c_i(\mathscr{T}_X)$. As we will soon see, the Todd class of X is a special polynomial in the c_i. The top Chern class is $c_n = c_n(\mathscr{T}_X)$, $n = \dim X$, which is usually called the *Euler class* of X. The name comes from the well-known formula

$$\int_X c_n(\mathscr{T}_X) = e(X), \tag{13.1.2}$$

where $e(X) = \sum_{i=0}^{2n}(-1)^i \operatorname{rank} H^i(X,\mathbb{Z})$ is the topological Euler characteristic of X. See, for example, [**144**, Thm. 4.10.1].

In the toric case, the Euler class is easy to compute. Recall from (12.5.9) that $H^\bullet(X,\mathbb{Z})$ is isomorphic to the Chow ring $A^\bullet(X)$ when X is a smooth complete toric variety. In particular, a point of X gives a class $[\mathrm{pt}] \in A^n(X) = H^{2n}(X,\mathbb{Z})$, where $n = \dim X$. Then we have the following result.

Proposition 13.1.2. *Let $X = X_\Sigma$ be a smooth complete n-dimensional toric variety with tangent bundle $\mathscr{T}_X$. Then:*

(a) $c(\mathscr{T}_X) = \prod_\rho (1+[D_\rho]) = \sum_{\sigma\in\Sigma}[V(\sigma)]$.

(b) $c_1 = c_1(\mathscr{T}_X) = [\sum_\rho D_\rho] = [-K_X]$.

(c) $c_n = c_n(\mathscr{T}_X) = \kappa\,[\mathrm{pt}]$, *where* $\kappa = |\Sigma(n)|$.

Proof. The first equality of part (a) follows easily by taking the dual of the exact sequence (13.1.1) and applying the method of Example 13.1.1. Next observe that Lemma 12.5.2 implies that if $\rho_1,\dots,\rho_n \in \Sigma(1)$ are distinct, then

$$[D_{\rho_1}]\cdots[D_{\rho_n}] = \begin{cases} [V(\sigma)] & \sigma = \rho_1+\cdots+\rho_n \in \Sigma(n) \\ 0 & \text{otherwise}, \end{cases}$$

since X is smooth. Then multiplying out $\prod_\rho(1+[D_\rho])$ gives the second equality.

Part (b) follows, and for part (c), note that if $\sigma \in \Sigma(n)$, then $V(\sigma)$ is a point, so that $[V(\sigma)] = [\mathrm{pt}]$. From here, the formula for c_n follows immediately. $\square$

Since $\int_X [\mathrm{pt}] = 1$, (13.1.2) and Proposition 13.1.2 imply that the topological Euler characteristic of X is

$$e(X) = \int_X c_n(\mathscr{T}_X) = \int_X \kappa\,[\mathrm{pt}] = \kappa = |\Sigma(n)|.$$

This agrees with the result proved in Theorem 12.3.9.

Using Chern classes, it is easy to define the Chern character of a line bundle.

Definition 13.1.3. Let X be a variety of dimension n. The *Chern character* of a line bundle $\mathscr{L}$ is

$$\mathrm{ch}(\mathscr{L}) = 1 + c_1(\mathscr{L}) + \frac{c_1(\mathscr{L})^2}{2!} + \frac{c_1(\mathscr{L})^3}{3!} + \cdots \in H^\bullet(X,\mathbb{Q}). \tag{13.1.3}$$

Truncating the Taylor series for e^x in degrees greater than n and then substituting $c_1(\mathscr{L}) \in H^2(X,\mathbb{Q})$ for x shows that we may write this more compactly as $e^{c_1(\mathscr{L})}$.

We will see later that Chern characters can also be defined for coherent sheaves.

The Todd Class. To define the Todd class $\mathrm{Td}(X)$ of a smooth complete variety X of dimension n, we begin by introducing symbolic objects $\xi_1,\dots,\xi_n$ such that

$$c(\mathscr{T}_X) = \prod_{i=1}^{n}(1+\xi_i).$$

We regard the ξ_i as living in $H^2(X,\mathbb{Q})$. The ξ_i are the *Chern roots* of the tangent bundle $\mathscr{T}_X$. In practical terms, this means that $c_i = c_i(\mathscr{T}_X) = \sigma_i(\xi_1,\dots,\xi_n)$, where σ_i is the ith elementary symmetric polynomial.

Definition 13.1.4. Let X be a smooth complete n-dimensional variety and let $\xi_1,\dots,\xi_n$ be the Chern roots of its tangent bundle. Then the ***Todd class*** of X is

$$\mathrm{Td}(X) = \prod_{i=1}^{n}\frac{\xi_i}{1-e^{-\xi_i}}.$$

When we expand the right-hand side of this formula in $H^\bullet(X,\mathbb{Q})$, we get a symmetric polynomial in the ξ_i with rational coefficients. The theory of symmetric polynomials implies that this is a polynomial expression in $c_i = \sigma_i(\xi_1,\dots,\xi_n)$ for $1 \le i \le n$. Hence the Todd class of X is a polynomial in its Chern classes $c_1,\dots,c_n$. The key fact is that the *same* polynomial is used for *all* smooth complete varieties of dimension n.

To compute these polynomials, we use the well-known power series expansion

$$\text{(13.1.4)}\quad \frac{x}{1-e^{-x}} = 1+\frac{1}{2}x+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}x^{2k} = 1+\frac{1}{2}x+\frac{1}{12}x^2-\frac{1}{720}x^4+\cdots,$$

where $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42},\dots$ are the Bernoulli numbers (see [**144**, p. 13]). Here is a classic example.

Example 13.1.5. A smooth complete surface X has Chern classes $c_1 = \xi_1+\xi_2$ and $c_2 = \xi_1\xi_2$, where ξ_1, ξ_2 are the Chern roots. Since the Chern roots have degree 2 and all monomials in ξ_1,ξ_2 of total degree ≥ 3 vanish (X is a surface), we obtain

$$\begin{aligned}\mathrm{Td}(X) &= (1+\tfrac{1}{2}\xi_1+\tfrac{1}{12}\xi_1^2)(1+\tfrac{1}{2}\xi_2+\tfrac{1}{12}\xi_2^2)\\ &= 1+\frac{1}{2}(\xi_1+\xi_2)+\frac{1}{12}(\xi_1^2+3\xi_1\xi_2+\xi_2^2)\\ &= 1+\frac{1}{2}c_1+\frac{1}{12}(c_1^2+c_2).\end{aligned}\qquad\Diamond$$

When X has dimension n, one can show that $\mathrm{Td}(X) = T_0+\cdots+T_n$ such that T_ℓ is a weighted homogeneous polynomial of weighted degree ℓ in $c_1,\dots,c_\ell$, provided

c_i has weight i. We call T_i the ith *Todd polynomial*. The first few are

$$T_0 = 1,\ \ T_1 = \frac{1}{2}c_1,\ \ T_2 = \frac{1}{12}(c_1^2 + c_2),\ \ T_3 = \frac{1}{24}c_1c_2$$
$$T_4 = \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4).$$

You will compute T_3 in Exercise 13.1.1. See [**144**, pp. 13–14] for more on Todd polynomials. At the end of the section we will say more about the intuition behind Definition 13.1.4.

The Todd Class of a Smooth Toric Variety. In the toric setting, the cohomology ring is generated by the classes $[D_\rho]$ for $\rho \in \Sigma(1)$ by Theorems 12.4.1 and 12.4.4. The Todd class can be expressed in terms of these classes as follows.

Theorem 13.1.6. *The Todd class of a smooth complete toric variety $X = X_\Sigma$ is*

$$\mathrm{Td}(X) = \prod_{\rho\in\Sigma(1)} \frac{[D_\rho]}{1 - e^{-[D_\rho]}} \in H^\bullet(X,\mathbb{Q}).$$

Proof. Recall that $c(\mathscr{T}_X) = \prod_\rho (1 + [D_\rho])$ by Proposition 13.1.2. If we set $\Sigma(1) = \{\rho_1,\ldots,\rho_r\}$ and $D_i = D_{\rho_i}$, then

$$c_i = \sigma_i([D_1],\ldots,[D_r]).$$

On the other hand, the Chern roots ξ_i satisfy $c_i = \sigma_i(\xi_1,\ldots,\xi_n)$, where $n = \dim X$. So $[D_1],\ldots,[D_r]$ behave like the Chern roots, except that $r > n$ since $r = |\Sigma(1)|$ and Σ is a complete fan in $N_\mathbb{R} \simeq \mathbb{R}^n$.

We can avoid this problem by computing $\prod_{i=1}^n \frac{\xi_i}{1-e^{-\xi_i}}$ and $\prod_{i=1}^r \frac{[D_i]}{1-e^{-[D_i]}}$ purely symbolically. We first consider the formal power series ring $\mathbb{Q}[[x_1,\ldots,x_r]]$ with maximal ideal $\mathfrak{m} = \langle x_1,\ldots,x_r\rangle$. Then there is a polynomial P of n variables such that

$$\prod_{i=1}^r \frac{x_i}{1-e^{-x_i}} \equiv P\big(\sigma_1(x_1,\ldots,x_r),\ldots,\sigma_n(x_1,\ldots,x_r)\big) \bmod \mathfrak{m}^{n+1}. \tag{13.1.5}$$

We can stop at n since $\sigma_i(x_1,\ldots,x_r) \in \mathfrak{m}^{n+1}$ for $i > n$. Setting $x_i = [D_i]$, we obtain

$$\prod_{i=1}^r \frac{[D_i]}{1-e^{-[D_i]}} = P(c_1,\ldots,c_n).$$

Now let $\mathfrak{n} = \langle x_1,\ldots,x_n\rangle \subseteq \mathbb{Q}[[x_1,\ldots,x_n]]$. Since $\frac{x_i}{1-e^{-x_i}} = 1 + \frac{1}{2}x_i + \cdots$ equals 1 when $x_i = 0$, we see that setting $x_i = 0$ for $n+1 \le i \le r$ in (13.1.5) gives the identity

$$\begin{aligned}\prod_{i=1}^n \frac{x_i}{1-e^{-x_i}} &\equiv P\big(\sigma_1(x_1,\ldots,x_n,0,\ldots,0),\ldots,\sigma_n(x_1,\ldots,x_n,0,\ldots,0)\big) \bmod \mathfrak{n}^{n+1}\\ &\equiv P\big(\sigma_1(x_1,\ldots,x_n),\ldots,\sigma_n(x_1,\ldots,x_n)\big) \bmod \mathfrak{n}^{n+1}\end{aligned}$$

in $\mathbb{Q}[x_1,\dots,x_n]$. The substitution $x_i = \xi_i$ gives

$$\prod_{i=1}^{n} \frac{\xi_i}{1-e^{-\xi_i}} = P(c_1,\dots,c_n),$$

and the theorem is proved. □

The HRR Theorem. Now that we have defined Chern characters and the Todd class, we can begin to play with HRR. If D is a divisor on a smooth complete variety X, then HRR gives the equality

$$\chi(\mathscr{O}_X(D)) = \int_X \mathrm{ch}(\mathscr{O}_X(D))\,\mathrm{Td}(X)$$

Here is what this looks like for a surface.

Example 13.1.7. Let X be a smooth complete surface, so that

$$\mathrm{Td}(X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2).$$

Also recall that $c_1 = [-K_X]$, where K_X is the canonical divisor.

We first apply HRR to $D = 0$. Since $\mathrm{ch}(\mathscr{O}_X) = 1$, we obtain

$$\chi(\mathscr{O}_X) = \int_X \mathrm{Td}(X) = \int_X \big(1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2+c_2)\big) = \frac{1}{12}\int_X c_1^2 + c_2$$

since $\int_X$ is trivial away from the top degree. It follows that the formula for $\mathrm{Td}(X)$ can be written

$$\mathrm{Td}(X) = 1 - \frac{1}{2}[K_X] + \chi(\mathscr{O}_X)[\mathrm{pt}]. \tag{13.1.6}$$

Furthermore, since $\int_X c_1^2 = \int_X [-K_X]^2 = (-K_X)\cdot(-K_X) = K_X \cdot K_X$ by (12.4.1) and $\int_X c_2 = e(X)$ by (13.1.2), the above formula for $\chi(\mathscr{O}_X)$ simplifies to

$$\chi(\mathscr{O}_X) = \frac{1}{12}\big(K_X\cdot K_X + e(X)\big),$$

which is Noether's theorem (Theorem 10.5.3). This result expresses the topological invariant $e(X)$ in terms of the algebro-geometric invariants $\chi(\mathscr{O}_X)$ and $K_X \cdot K_X$.

In Exercise 13.1.2 you will show more generally that HRR applied to $\mathscr{O}_X(D)$ gives the formula

$$\chi(\mathscr{O}_X(D)) = \frac{1}{2}(D\cdot D - D\cdot K_X) + \chi(\mathscr{O}_X). \tag{13.1.7}$$

This is the version of Riemann-Roch stated in Theorem 10.5.2. ◊

It should be no surprise that the Riemann-Roch theorem for smooth complete curves is also a special case of HRR. See Exercise 13.1.3.

Example 13.1.8. The Hirzebruch surface $X = \mathscr{H}_2$ was studied in Example 4.1.8. In particular, the Picard group $\operatorname{Pic}(X)$ is generated by the classes of D_3 and D_4. We will use the Riemann-Roch theorem to compute $\chi(\mathscr{O}_X(D))$ for the divisor $D = 3D_3 - 5D_4$ on X. Example 4.1.8 shows that $D_1 \simeq D_3$ and $D_2 \simeq -2D_3 + D_4$. Thus

$$K_X = -(D_1 + D_2 + D_3 + D_4) = -2D_4.$$

The intersection pairing on $X = \mathscr{H}_2$ was determined in Example 10.4.6, which showed that

$$D_3^2 = 0,\ \ D_3 \cdot D_4 = 1,\ \ D_4^2 = 2.$$

Also note that $\chi(\mathscr{O}_X) = 1$ by Demazure vanishing. Then applying Riemann-Roch from (13.1.7) to the divisor $D = 3D_3 - 5D_4$, we obtain

$$\begin{aligned}\chi(\mathscr{O}_X(D)) &= \tfrac{1}{2}D \cdot (D - K_X) + 1\\ &= \tfrac{1}{2}(3D_3 - 5D_4) \cdot (3D_3 - 3D_4) + 1\\ &= \tfrac{1}{2}(9D_3^2 - 24D_3 \cdot D_4 + 15D_4^2) + 1 = 4.\end{aligned}$$

In Example B.7.1 we show how to do this computation using `Sage` [**262**]. $\Diamond$

Grothendieck-Riemann-Roch. Hirzebruch proved his version of Riemann-Roch for vector bundles on smooth projective varieties in 1954. This was generalized by Grothendieck in 1957 to proper morphisms $f : X \to Y$ between smooth quasiprojective varieties and coherent sheaves on X. His proof was published by Borel and Serre in 1958. Bott's wonderful review [**41**] of their paper points out that Grothendieck's theorem explains why the Todd class from Definition 13.1.4 makes sense. For this reason, we will briefly discuss Grothendieck-Riemann-Roch (called GRR) in the special case that uses smooth complete varieties and replaces the Chow ring used by Grothendieck with the cohomology ring.

For a smooth complete variety X, we first define the group $K(X)$ generated by coherent sheaves on X, modulo the relations

$$[\mathscr{G}] = [\mathscr{F}] + [\mathscr{H}]$$

for all short exact sequences $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$. One can also prove that a coherent sheaf $\mathscr{F}$ on X has a *Chern character* $\operatorname{ch}(\mathscr{F}) \in H^\bullet(X,\mathbb{Q})$ such that

- $\operatorname{ch}(\mathscr{L}) = e^{c_1(\mathscr{L})}$ when $\mathscr{L}$ is a line bundle.
- $\operatorname{ch}(\mathscr{G}) = \operatorname{ch}(\mathscr{F}) + \operatorname{ch}(\mathscr{H})$ when $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is exact.

It follows that the Chern character induces a group homomorphism

$$\operatorname{ch} : K(X) \longrightarrow H^\bullet(X,\mathbb{Q}).$$

Now suppose that $f : X \to Y$ is a proper morphism between smooth complete varieties. If $\mathscr{F}$ is coherent on X, then one can prove that the higher direct images

$R^i f_*\mathscr{F}$ are coherent on Y. By the long exact sequence in cohomology, the map

$$f_![\mathscr{F}] = \sum_{i\geq 0}(-1)^i [R^i f_*\mathscr{F}] \in K(Y)$$

defines a homomorphism $f_! : K(X) \to K(Y)$. On the cohomology side, we also have a map $f_! : H^\bullet(X,\mathbb{Q}) \to H^\bullet(Y,\mathbb{Q})$, which for $f : X \to Y$ as above is the Poincaré dual of $f_* : H_\bullet(X,\mathbb{Q}) \to H_\bullet(Y,\mathbb{Q})$. We call $f_!$ a *generalized Gysin map*. Here are two important examples of these maps:

- When $f : X \to \{\mathrm{pt}\}$, $f_!$ is the map $\int_X : H^\bullet(X,\mathbb{Q}) \to \mathbb{Q}$ from §12.4.
- When $i : X \hookrightarrow Y$ is an inclusion map, then $i_! : H^k(X,\mathbb{Q}) \to H^{k+2\,\mathrm{codim}\,X}(Y,\mathbb{Q})$ is the Gysin map, which satisfies $i_! i^*\alpha = \alpha\cdot[X]$ for $\alpha \in H^k(Y,\mathbb{Q})$.

We will say more about the maps $f_!$ in the appendix at the end of the chapter.

These definitions show that a proper morphism $f : X \to Y$ gives a diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathrm{ch}} & H^\bullet(X,\mathbb{Q}) \\ {\scriptstyle f_!}\big\downarrow & & \big\downarrow{\scriptstyle f_!} \\ K(Y) & \xrightarrow{\mathrm{ch}} & H^\bullet(Y,\mathbb{Q}) \end{array}$$

which unfortunately does not commute. This is why Todd classes are needed: GRR states that the modified diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(X)} & H^\bullet(X,\mathbb{Q}) \\ {\scriptstyle f_!}\big\downarrow & & \big\downarrow{\scriptstyle f_!} \\ K(Y) & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(Y)} & H^\bullet(Y,\mathbb{Q}) \end{array} \tag{13.1.8}$$

does commute. In Exercise 13.1.4 you will prove that (13.1.8) reduces to HRR when $Y = \{\mathrm{pt}\}$.

Following [**41**], we now specialize to the case where $i : X \hookrightarrow Y$ is the inclusion of a smooth hypersurface in a smooth complete variety, and we apply (13.1.8) to $[\mathscr{O}_X] \in K(X)$. First observe that $i_![\mathscr{O}_X] = [i_*\mathscr{O}_X]$ since i is an affine morphism. Then the exact sequence $0 \to \mathscr{O}_Y(-X) \to \mathscr{O}_Y \to i_*\mathscr{O}_X \to 0$ and the properties of the Chern character imply that

$$\begin{aligned}\mathrm{ch}(i_![\mathscr{O}_X])\cdot\mathrm{Td}(Y) = \mathrm{ch}([i_*\mathscr{O}_X])\cdot\mathrm{Td}(Y) &= \mathrm{ch}\big([\mathscr{O}_Y] - [\mathscr{O}_Y(-X)]\big)\cdot\mathrm{Td}(Y)\\ &= (1-e^{-[X]})\cdot\mathrm{Td}(Y).\end{aligned}$$

This is encouraging since expressions like $1-e^{-[X]}$ appear in the denominator in the formula for the Todd class given in Definition 13.1.4. We can do more since $S = [X]/(1-e^{-[X]})$ is invertible in $H^\bullet(Y,\mathbb{Q})$, so that

$$(1-e^{-[X]})\cdot\mathrm{Td}(Y) = (S^{-1}\mathrm{Td}(Y))\cdot[X] = i_! i^*(S^{-1}\mathrm{Td}(Y)), \tag{13.1.9}$$

where the last equality follows since $i_!$ is a Gysin map. In order for (13.1.8) to commute, (13.1.9) must equal $i_!\big(\mathrm{ch}([\mathscr{O}_X])\cdot \mathrm{Td}(X)\big) = i_!(\mathrm{Td}(X))$. In other words, we want $\mathrm{Td}(X) = i^*(S^{-1}\mathrm{Td}(Y))$, which is equivalent to the equation

$$i^*\mathrm{Td}(Y) = \mathrm{Td}(X)\cdot i^*\Big(\frac{[X]}{1-e^{-[X]}}\Big). \tag{13.1.10}$$

In Exercise 13.1.5, you will show that the formula of Definition 13.1.4 guarantees that Todd classes satisfy this identity.

Exercises for §13.1.

13.1.1. Use the method of Example 13.1.5 to compute the Todd polynomial T_3.

13.1.2. Derive the Riemann-Roch theorem for surfaces (13.1.7) from (13.0.1).

13.1.3. Derive the Riemann-Roch theorem for curves (10.5.1) from (13.0.1).

13.1.4. Prove that HRR is the special case of the commutative diagram (13.1.8) when $Y = \{\mathrm{pt}\}$. Thus GRR implies HRR.

13.1.5. Let Y be a smooth complete variety and let $i : X \hookrightarrow Y$ be a smooth hypersurface. Then we have the exact sequence of locally free sheaves

$$0 \longrightarrow \mathscr{T}_X \longrightarrow i^*\mathscr{T}_Y \longrightarrow \mathscr{N}_{X/Y} \longrightarrow 0$$

from Theorem 8.0.18, where $\mathscr{N}_{X/Y} = (\mathscr{I}_X/\mathscr{I}_X^2)^\vee$.

(a) Prove that $\mathscr{N}_{X/Y} \simeq i^*\mathscr{O}_Y(X)$. Hint: The ideal sheaf of $i : X \hookrightarrow Y$ is $\mathscr{I}_X = \mathscr{O}_Y(-X)$ since X has codimension 1.

(b) Prove the identity $c(i^*\mathscr{T}_Y) = c(\mathscr{T}_X)\cdot c(i^*\mathscr{O}_Y(X))$ of total Chern classes.

(c) Show that $c(i^*\mathscr{O}_Y(X)) = 1 + i^*[X]$ and use part (b) to explain why $i^*[X]$ can be regarded as a Chern root of $i^*\mathscr{T}_Y$.

(d) Finally, show that the Todd classes $\mathrm{Td}(X)$ and $\mathrm{Td}(Y)$ satisfy (13.1.10).

13.1.6. Let $\mathscr{E}$ be a locally free sheaf of rank r on a smooth variety X. Similar to what we did with the tangent bundle, the *Chern roots* of $\mathscr{E}$ are symbolic objects ξ_i such that $c(\mathscr{E}) = \prod_{i=1}^r (1+\xi_i)$. Then the *Chern character* $\mathrm{ch}(\mathscr{E})$ is defined as

$$\mathrm{ch}(\mathscr{E}) = \sum_{k=0}^{\infty} \frac{1}{k!}(\xi_1^k + \cdots + \xi_r^k).$$

(a) Explain why $\mathrm{ch}(\mathscr{E})$ is a polynomial in the Chern class of $\mathscr{E}$.

(b) Let $\mathscr{E} = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r$ be a sum of line bundles. Use the definition of Chern character to show that $\mathrm{ch}(\mathscr{E}) = \mathrm{ch}(\mathscr{L}_1) + \cdots + \mathrm{ch}(\mathscr{L}_r)$.

(c) If $\mathscr{E}$ has rank 2, show that

$$\mathrm{ch}(\mathscr{E}) = 2 + c_1(\mathscr{E}) + \frac{1}{2}\big(c_1(\mathscr{E})^2 - 2c_2(\mathscr{E})\big) + \frac{1}{6}\big(c_1(\mathscr{E})^3 - 3c_1(\mathscr{E})c_2(\mathscr{E})\big) + \cdots.$$

§13.2. Brion's Equalities

When we prove the toric version of equivariant Riemann-Roch in §13.3, we will create equivariant versions of Euler characteristics, Chern characters and Todd classes, and then decompose them into sums of local terms. In this section, we will see how this works for the Euler characteristic of a line bundle.

When D is a torus-invariant divisor on a complete toric variety $X = X_\Sigma$ of dimension n, the sheaf $\mathscr{L} = \mathscr{O}_X(D)$ has Euler characteristic

$$\chi(\mathscr{L}) = \chi(X,\mathscr{L}) = \sum_{i=0}^{n} (-1)^i \dim H^i(X_\Sigma,\mathscr{L}).$$

Also recall from §9.1 that we used the decomposition

$$H^i(X,\mathscr{L}) = \bigoplus_{m\in M} H^i(X,\mathscr{L})_m$$

to compute sheaf cohomology of $\mathscr{L}$. Combining these two formulas, we define

$$\widetilde{\chi}(\mathscr{L}) = \widetilde{\chi}(X,\mathscr{L}) = \sum_{i=0}^{n} (-1)^i \dim H^i(X_\Sigma,\mathscr{L})_m \chi^m.$$

This lives in the semigroup algebra $\mathbb{Z}[M]$. The goal of this section is to write $\widetilde{\chi}(X,\mathscr{L})$ as a sum of local terms that live in a certain localization of $\mathbb{Z}[M]$.

The Formal Semigroup Module. The local terms will involve cohomology over affine open subsets, which can lead to infinite sums. Following [**46**], we introduce the abelian group $\mathbb{Z}[[M]]$ consisting of all formal sums $\sum_{m\in M} a_m \chi^m$ for $a_m \in \mathbb{Z}$. This is not a ring, but there is a multiplication map

$$\mathbb{Z}[M] \times \mathbb{Z}[[M]] \longrightarrow \mathbb{Z}[[M]]$$

that makes $\mathbb{Z}[[M]]$ into a $\mathbb{Z}[M]$-module. We say that $\mathbb{Z}[[M]]$ is the *formal semigroup module* of M.

Now let $X = X_\Sigma$ be an arbitrary n-dimensional toric variety, which need not be complete. Given a torus-invariant Weil divisor D on X, let $\mathscr{L} = \mathscr{O}_X(D)$ and define

$$\widetilde{\chi}(X,\mathscr{L}) = \sum_{m\in M} \sum_{i=0}^{n} (-1)^i \dim H^i(X,\mathscr{L})_m \chi^m \in \mathbb{Z}[[M]]. \tag{13.2.1}$$

In Chapter 9, we computed $H^\bullet(X,\mathscr{L})$ using the Čech complex $\check{C}^\bullet(\mathscr{U},\mathscr{L})$. For the open cover $\mathscr{U} = \{U_\sigma\}_{\sigma\in\Sigma_{\max}}$, let $\Sigma_{\max} = \{\sigma_1,\ldots,\sigma_\ell\}$ and define

$$\check{C}^p(\mathscr{U},\mathscr{L}) \;=\; \bigoplus_{\gamma\in[\ell]_p} H^0(U_{\sigma_\gamma},\mathscr{L}),$$

where we refer to (9.1.1) for the precise meaning of $[\ell]_p$ and σ_γ. This gives a "Čech" formula for $\widetilde{\chi}(X,\mathscr{L})$ as follows.

Lemma 13.2.1. $\widetilde{\chi}(X,\mathscr{L}) = \sum_{p\geq 0}(-1)^p\big(\sum_{\gamma\in[\ell]_p}\widetilde{\chi}(U_{\sigma_\gamma},\mathscr{L})\big)$ *in* $\mathbb{Z}[[M]]$.

Proof. Just as in §9.1, the M-grading on $H^\bullet(X_\Sigma,\mathscr{L})$ and $\check{C}^\bullet(\mathscr{U},\mathscr{L})$ implies that $H^\bullet(X,\mathscr{L})_m$ is the cohomology of the complex $\check{C}^\bullet(\mathscr{U},\mathscr{L})_m$. Since a complex and its cohomology have the same Euler characteristic, it follows that

$$\sum_{i=0}^n(-1)^i\dim H^i(X,\mathscr{L})_m = \sum_{p=0}^n(-1)^p\sum_{\gamma\in[\ell]_p}\dim H^0(U_{\sigma_\gamma},\mathscr{L})_m.$$

However $\dim H^0(U_{\sigma_\gamma},\mathscr{L})_m = \sum_{i=0}^n(-1)^i\dim H^i(U_{\sigma_\gamma},\mathscr{L})_m$ since U_{σ_γ} is affine. From here, the lemma follows easily. $\square$

Summable Series. For any cone $\sigma\in\Sigma$, the vanishing of $H^i(U_\sigma,\mathscr{L})$ for $i>0$ implies that

$$\widetilde{\chi}(U_\sigma,\mathscr{L}) = \sum_{m\in M}\dim H^0(U_\sigma,\mathscr{L})_m\,\chi^m\in\mathbb{Z}[[M]]. \tag{13.2.2}$$

We will soon see that these sums are reasonably behaved in the following sense.

Definition 13.2.2. An element $f\in\mathbb{Z}[[M]]$ is ***summable*** if there exists $g\in\mathbb{Z}[M]$ and a finite set $I\subseteq M\setminus\{0\}$ such that in $\mathbb{Z}[[M]]$, we have

$$f\cdot\prod_{m\in I}(1-\chi^m) = g.$$

Also let $\mathbb{Z}[[M]]_{\mathrm{Sum}}\subseteq\mathbb{Z}[[M]]$ be the subset of summable elements of $\mathbb{Z}[[M]]$.

By Exercise 13.2.1, $\mathbb{Z}[[M]]_{\mathrm{Sum}}$ is a $\mathbb{Z}[M]$-submodule of $\mathbb{Z}[[M]]$. Here are some examples of summable elements.

Example 13.2.3. Let $M=\mathbb{Z}$ and let $\chi=\chi^1$. Then $\sum_{\ell=0}^\infty\chi^\ell$ is summable since

$$(1-\chi)(1+\chi+\chi^2+\cdots)=1.$$

A more surprising example is $\sum_{\ell=-\infty}^\infty\chi^\ell$, which is summable since

$$(1-\chi)(\cdots+\chi^{-2}+\chi^{-1}+1+\chi+\chi^2+\cdots)=0.$$

In Example 13.2.7 we will use this to show that $\sum_{\ell=-\infty}^\infty\chi^\ell$ "sums" to zero. $\lozenge$

Here is a basic summability result.

Lemma 13.2.4. *Let D be a torus-invariant Cartier divisor on $X=X_\Sigma$ and set $\mathscr{L}=\mathscr{O}_X(D)$. Then $\widetilde{\chi}(U_\sigma,\mathscr{L})$ is summable for every $\sigma\in\Sigma$. Furthermore:*

(a) *If* $\dim\sigma<n$, *then*

$$\widetilde{\chi}(U_\sigma,\mathscr{L})\cdot(1-\chi^{m_0})=0$$

for some $m_0\in M\setminus\{0\}$.

(b) *If* $\dim\sigma = n$ *and* σ *is smooth, then*

$$\widetilde{\chi}(U_\sigma,\mathscr{L})\cdot\prod_{i=1}^{n}(1-\chi^{m_{\sigma,i}}) = \chi^{m_\sigma},$$

where $m_\sigma \in M$ *satisfies* $D|_{U_\sigma} = \operatorname{div}(\chi^{-m_\sigma})|_{U_\sigma}$ *and* $m_{\sigma,1},\dots,m_{\sigma,n} \in M$ *are the minimal generators of* $\sigma^\vee$.

Proof. If $\dim\sigma < n$, then $\sigma^\vee$ is not strongly convex, so that we can find a nonzero $m_0 \in \sigma^\vee\cap(-\sigma^\vee)\cap M$. Then $\chi^{m_0}\in\mathbb{C}[\sigma^\vee\cap M]$ is invertible, which means that multiplication by χ^{m_0} gives an isomorphism $H^0(U_\sigma,\mathscr{L})\simeq H^0(U_\sigma,\mathscr{L})$. Hence

$$H^0(U_\sigma,\mathscr{L})_m \simeq H^0(U_\sigma,\mathscr{L})_{m+m_0}$$

for all $m\in M$. This implies $\chi^{m_0}\cdot\widetilde{\chi}(U_\sigma,\mathscr{L}) = \widetilde{\chi}(U_\sigma,\mathscr{L})$, and part (a) follows.

For part (b), assume σ is smooth and let $u_1,\dots,u_n$ be the minimal generators of $\rho_1,\dots,\rho_n\in\sigma(1)$. They form a basis of N since σ is smooth. Then the dual basis $\{m_1,\dots,m_n\}\subseteq M$ consists of the minimal generators of $\sigma^\vee$. If we write $D|_{U_\sigma} = \sum_{i=1}^n a_i D_{\rho_i}$, then $m_\sigma = -\sum_{i=1}^n a_i m_i$ satisfies $D|_{U_\sigma} = \operatorname{div}(\chi^{-m_\sigma})|_{U_\sigma}$.

The polyhedron of $\mathscr{L}|_{U_\sigma}$ is $P_{D,\sigma} = \{m\in M \mid \langle m,u_i\rangle \ge -a_i,\ 1\le i\le n\}$. Using the above formula for m_σ, we obtain $P_{D,\sigma}\cap M = m_\sigma+\sigma^\vee\cap M$. We also have $H^0(U_\sigma,\mathscr{L}) = \bigoplus_{m\in P_{D,\sigma}\cap M}\mathbb{C}\cdot\chi^m$ by (4.3.3). Then (13.2.2) implies that

$$\widetilde{\chi}(U_\sigma,\mathscr{L}) = \sum_{m\in m_\sigma+\sigma^\vee\cap M}\chi^m = \chi^{m_\sigma}\cdot\sum_{m\in\sigma^\vee\cap M}\chi^m. \tag{13.2.3}$$

Since $\sigma^\vee\cap M = \mathbb{N}m_1+\cdots+\mathbb{N}m_n$, the sum on the right lives in the formal series ring $\mathbb{Q}[[\chi^{m_1},\dots,\chi^{m_n}]]$. In this ring, one computes that

$$\sum_{m\in\sigma^\vee\cap M}\chi^m = \prod_{i=1}^n\Big(\sum_{\ell=0}^\infty\chi^{\ell m_i}\Big) = \prod_{i=1}^n(1-\chi^{m_i})^{-1}.$$

Part (b) of the proposition now follows easily.

Finally, suppose that $\sigma\in\Sigma(n)$ is not smooth. Since D is Cartier, we have $D|_{U_\sigma} = \operatorname{div}(\chi^{-m_\sigma})|_{U_\sigma}$ for some $m_\sigma\in M$. Then (13.2.3) still holds, so it suffices to show that $\sum_{m\in\sigma^\vee\cap M}\chi^m$ is summable. The methods of §11.1 imply that $\sigma^\vee\subseteq M_\mathbb{R}$ has a smooth refinement. Let the maximal cones of this refinement be $C_1,\dots,C_s$. By inclusion-exclusion, we obtain

$$\sum_{m\in\sigma^\vee\cap M}\chi^m = \sum_i\sum_{m\in C_i\cap M}\chi^m - \sum_{i<j}\sum_{m\in C_i\cap C_j\cap M}\chi^m + \cdots. \tag{13.2.4}$$

Since an intersection $C = C_{i_1}\cap\cdots\cap C_{i_\ell}$ is smooth, the minimal generators of $C\cap M$ are a subset of a basis of M. Then the proof of part (b) implies that $\sum_{m\in C\cap M}\chi^m$ is summable. We conclude that $\widetilde{\chi}(U_\sigma,\mathscr{L})$ is summable. □

The Simplicial Case. When $\sigma \in \Sigma(n)$ is smooth, part (b) of Lemma 13.2.4 shows that $\widetilde{\chi}(U_\sigma, \mathscr{L})$ satisfies a nice identity. With a little work, this can be generalized to the simplicial case.

Let $\sigma \subseteq \Sigma(n)$ be simplicial and let $m_{\sigma,1}, \ldots, m_{\sigma,n}$ be the minimal generators of $\sigma^\vee \subseteq M_\mathbb{R}$. Recall from Proposition 11.1.8 that

$$\text{(13.2.5)} \qquad \operatorname{mult}(\sigma^\vee) = |P_{\sigma^\vee}|, \quad P_{\sigma^\vee} = \{\textstyle\sum_{i=1}^n \lambda_i m_{\sigma,i} \mid 0 \le \lambda_i < 1\},$$

where $\operatorname{mult}(\sigma^\vee) = [M : \mathbb{Z}m_{\sigma,1} + \cdots + \mathbb{Z}m_{\sigma,n}]$. You will explore the relation between $\operatorname{mult}(\sigma)$ and $\operatorname{mult}(\sigma^\vee)$ in Exercise 13.2.3.

Lemma 13.2.5. *If D is a torus-invariant Cartier divisor on $X = X_\Sigma$ and $\sigma \in \Sigma(n)$ is simplicial, then for $\mathscr{L} = \mathscr{O}_X(D)$, we have*

$$\widetilde{\chi}(U_\sigma, \mathscr{L}) \cdot \prod_{i=1}^n (1 - \chi^{m_{\sigma,i}}) = \chi^{m_\sigma} \cdot \sum_{m \in P_{\sigma^\vee} \cap M} \chi^m,$$

where $D|_{U_\sigma} = \operatorname{div}(\chi^{-m_\sigma})|_{U_\sigma}$, and $m_{\sigma,1}, \ldots, m_{\sigma,n}$ and $P_{\sigma^\vee}$ are as above.

Proof. For simplicity, write $m_i = m_{\sigma,i}$. Since $\widetilde{\chi}(U_\sigma, \mathscr{L}) = \chi^{m_\sigma} \cdot \sum_{m \in \sigma^\vee \cap M} \chi^m$ by (13.2.3), it suffices to prove that

$$\text{(13.2.6)} \qquad \Big(\sum_{m \in \sigma^\vee \cap M} \chi^m\Big) \cdot \prod_{i=1}^n (1 - \chi^{m_i}) = \sum_{m \in P_{\sigma^\vee} \cap M} \chi^m$$

in $\mathbb{Z}[[M]]$. Let $A = \mathbb{N}m_1 + \cdots + \mathbb{N}m_n \subseteq M$. We claim that

$$\text{(13.2.7)} \qquad m \in \sigma^\vee \cap M \iff m = m' + m'' \text{ for } m' \in A \text{ and } m'' \in P_{\sigma^\vee} \cap M.$$

One direction is obvious. For the other direction, take $m \in \sigma^\vee \cap M$. Since the m_i generate $\sigma^\vee$, we can write $m = \sum_{i=1}^n c_i m_i$ for $c_i \ge 0$. Then writing $c_i = a_i + b_i$ for $a_i \in \mathbb{N}$ and $0 \le b_i < 1$ easily implies that m has the desired form, proving (13.2.7).

By Exercise 13.2.4, the decomposition $m = m' + m''$ in (13.2.7) is unique when it exists. It follows that

$$\text{(13.2.8)} \qquad \sum_{m \in \sigma^\vee \cap M} \chi^m = \Big(\sum_{m \in A} \chi^m\Big) \cdot \Big(\sum_{m \in P_{\sigma^\vee} \cap M} \chi^m\Big).$$

Arguing as in the proof of Theorem 13.2.8, we also have

$$\Big(\sum_{m \in A} \chi^m\Big) \cdot \prod_{i=1}^n (1 - \chi^{m_i}) = 1.$$

Hence multiplying each side of (13.2.8) by $\prod_{i=1}^n (1 - \chi^{m_i})$ gives (13.2.6). $\square$

Example 13.2.6. Consider the cone $\sigma = \operatorname{Cone}(e_1, -e_1 - 2e_2) \subseteq N_\mathbb{R}$, with dual $\sigma^\vee = \operatorname{Cone}(-e_2, 2e_1 - e_2) \subseteq M_\mathbb{R}$. One computes easily that $P_{\sigma^\vee} = \{0, e_1 - e_2\}$. Setting $t_i = \chi^{e_i}$, Lemma 13.2.5 implies that

$$\widetilde{\chi}(U_\sigma, \mathscr{O}_{U_\sigma}) \cdot (1 - t_2^{-1})(1 - t_1^2 t_2^{-1}) = 1 + t_1 t_2^{-1}. \qquad \Diamond$$

In [**50**, 1.3], Brion and Vergne prove that $\widetilde{\chi}(U_\sigma,\mathscr{L})$ is summable under the weaker hypothesis that D is a Weil divisor, and in [**50**, 2.1], they give a version of Lemma 13.2.5 that applies to Weil divisors and simplicial cones.

Brion's Equalities. Consider the multiplicative set $\overline{S} \subseteq \mathbb{Z}[M]$ consisting of all finite products of the form $\prod_{i=1}^{s}(1-\chi^{m_i})$, where $m_1,\dots,m_s \in M\setminus\{0\}$. Note that $1\in\overline{S}$ because of the empty product. This gives the localization $\mathbb{Z}[M]_{\overline{S}}$.

Given a summable element $f \in \mathbb{Q}[[M]]_{\mathrm{Sum}}$, there are $h\in\overline{S}$ and $g\in\mathbb{Z}[M]$ such that $h\cdot f = g$. Then set

$$\mathcal{S}(f) = \frac{g}{h} \in \mathbb{Z}[M]_{\overline{S}}.$$

In Exercise 13.2.1 you will prove that the map

$$\mathcal{S} : \mathbb{Z}[[M]]_{\mathrm{Sum}} \longrightarrow \mathbb{Z}[M]_{\overline{S}} \tag{13.2.9}$$

is a well-defined $\mathbb{Z}[M]$-module homomorphism. We call $\mathcal{S}$ the *sum function*.

Example 13.2.7. Let $M=\mathbb{Z}$ and χ be as in Example 13.2.3. Then

$$\begin{aligned}\mathcal{S}\big(\textstyle\sum_{\ell=0}^{\infty}\chi^\ell\big) &= \frac{1}{1-\chi} &&\text{since } (1-\chi)(1+\chi+\chi^2+\cdots)=1\\ \mathcal{S}\big(\textstyle\sum_{\ell=-\infty}^{\infty}\chi^\ell\big) &= \ 0 &&\text{since } (1-\chi)(\cdots+\chi^{-1}+1+\chi+\cdots)=0.\end{aligned} \qquad \Diamond$$

This example shows that $\mathcal{S}$ is not injective. In fact, it has a rather large kernel, which is key to proving the following remarkable equalities due to Brion.

Theorem 13.2.8. *Let $X = X_\Sigma$ be a complete toric variety of dimension n. If $\mathscr{L} = \mathscr{O}_X(D)$ for a torus-invariant Cartier divisor D with Cartier data $\{m_\sigma\}_{\sigma\in\Sigma(n)}$, then:*

(a) $\widetilde{\chi}(X,\mathscr{L}) \ = \displaystyle\sum_{\sigma\in\Sigma(n)} \mathcal{S}(\widetilde{\chi}(U_\sigma,\mathscr{L}))$ *in* $\mathbb{Q}[M]_{\overline{S}}$.

(b) *If $\sigma\in\Sigma(n)$ is simplicial and $m_{\sigma,1},\dots,m_{\sigma,n}\in M$ are the minimal generators of $\sigma^\vee \subseteq M_{\mathbb{R}}$, then*

$$\mathcal{S}(\widetilde{\chi}(U_\sigma,\mathscr{L})) = \frac{\chi^{m_\sigma}\cdot\sum_{m\in P_{\sigma^\vee}\cap M}\chi^m}{\prod_{i=1}^{n}(1-\chi^{m_{\sigma,i}})}$$

for $P_{\sigma^\vee}$ as in (13.2.5). *In particular, if σ is smooth, then*

$$\mathcal{S}(\widetilde{\chi}(U_\sigma,\mathscr{L})) = \frac{\chi^{m_\sigma}}{\prod_{i=1}^{n}(1-\chi^{m_{\sigma,i}})}.$$

Proof. First, $\mathcal{S}(\widetilde{\chi}(X,\mathscr{L})) = \widetilde{\chi}(X,\mathscr{L})$ since X is complete. Then Lemma 13.2.1 implies that

$$\widetilde{\chi}(X_\Sigma,\mathscr{L}) = \sum_{p\geq 0}(-1)^p\Big(\sum_{\gamma\in[\ell]_p}\mathcal{S}(\widetilde{\chi}(U_{\sigma_\gamma},\mathscr{L}))\Big).$$

But we also know that $\mathcal{S}(\widetilde{\chi}(U_{\sigma_\gamma},\mathscr{L}))=0$ when $\dim\sigma_\gamma < n$ by Lemma 13.2.4. Since X is complete, the open cover used in the Čech complex is $\mathscr{U} = \{U_\sigma\}_{\sigma\in\Sigma(n)}$, and the only place n-dimensional cones appear is in the first term $\check{C}^0(\mathscr{U},\mathscr{L}) = \bigoplus_{\sigma\in\Sigma(n)} H^0(U_\sigma,\mathscr{L})$. Hence all terms in the above sum drop out except for those for $p=0$, which correspond to $\sigma\in\Sigma(n)$. This proves part (a). Part (b) now follows immediately from Lemmas 13.2.4 and 13.2.5. $\square$

Example 13.2.9. For the cone $\sigma = \mathrm{Cone}(e_1,-e_1-2e_2)\subseteq N_{\mathbb{R}}$ of Example 13.2.6, part (b) of Theorem 13.2.8 implies that

$$\mathcal{S}(\widetilde{\chi}(U_\sigma,\mathscr{O}_{U_\sigma})) = \frac{1+t_1t_2^{-1}}{(1-t_2^{-1})(1-t_1^2t_2^{-1})}.$$

This also follows from Example 13.2.6 by the definition of $\mathcal{S}$. $\Diamond$

Here is a wonderful application of Theorem 13.2.8.

Corollary 13.2.10 (Brion's Equalities)**.** *Let $P\subseteq M_{\mathbb{R}}\simeq\mathbb{R}^n$ be a full dimensional lattice polytope, and for each vertex $\mathsf{v}\in P$, let $C_{\mathsf{v}} = \mathrm{Cone}(P\cap M-\mathsf{v})\subseteq M_{\mathbb{R}}$. Then:*

(a) $\displaystyle\sum_{m\in P\cap M}\chi^m = \sum_{\mathsf{v}\text{ vertex}}\chi^{\mathsf{v}}\cdot\mathcal{S}\Big(\sum_{m\in C_{\mathsf{v}}\cap M}\chi^m\Big).$

(b) *If P is simple and $\mathsf{v}\in P$ is a vertex, let $m_{\mathsf{v},1},\dots,m_{\mathsf{v},n}\in M$ be the minimal generators of C_{v}. Then*

$$\sum_{m\in P\cap M}\chi^m = \sum_{\mathsf{v}\text{ vertex}}\frac{\chi^{\mathsf{v}}\cdot\sum_{m\in P_{\mathsf{v}}\cap M}\chi^m}{\prod_{i=1}^n(1-\chi^{m_{\mathsf{v},i}})}$$

for $P_{\mathsf{v}} = P_{C_{\mathsf{v}}}$ as in (13.2.5)*. In particular, if P is smooth, then*

$$\sum_{m\in P\cap M}\chi^m = \sum_{\mathsf{v}\text{ vertex}}\frac{\chi^{\mathsf{v}}}{\prod_{i=1}^n(1-\chi^{m_{\mathsf{v},i}})}.$$

Proof. We will apply Proposition 13.2.8 to the toric variety X_P and the line bundle $\mathscr{L} = \mathscr{O}_{X_P}(D_P)$. By definition, the maximal cones of the normal fan Σ_P are $\sigma_{\mathsf{v}} = C_{\mathsf{v}}^\vee$ for $\mathsf{v}\in P$ a vertex. Also recall from (4.2.8) that the v's are the Cartier data of D_P.

The higher cohomology of $\mathscr{L}$ vanishes since D_P is ample, so that

$$\widetilde{\chi}(X_P,\mathscr{L}) = \sum_{m\in P\cap M}\chi^m.$$

Furthermore, $\sigma_{\mathsf{v}}^\vee = C_{\mathsf{v}}$ and (13.2.3) imply that

$$\widetilde{\chi}(U_{\sigma_{\mathsf{v}}},\mathscr{L}) = \chi^{\mathsf{v}}\cdot\sum_{m\in\sigma_{\mathsf{v}}^\vee\cap M}\chi^m = \chi^{\mathsf{v}}\cdot\sum_{m\in C_{\mathsf{v}}\cap M}\chi^m.$$

Then part (a) follows from Theorem 13.2.8, and the same is true for part (b). $\square$

Brion's original proof [**46**] of Corollary 13.2.10 used equivariant Riemann-Roch. In [**155**], Ishida gave an elementary proof of Brion's equalities and used it prove a special case of HRR. Brion and Vergne prove Theorem 13.2.8 in [**50**, 1.3] as part of their toric proof of equivariant Riemann-Roch. A completely elementary approach to Corollary 13.2.10 can be found in [**22**, Thm. 9.7].

Let us give a simple example of Brion's equalities.

Example 13.2.11. Consider the interval $[0,d] \subseteq \mathbb{R}$ with vertices $\mathsf{v}_1 = 0$ and $\mathsf{v}_2 = d$. Note that $C_{\mathsf{v}_1} = [0,\infty)$ and $C_{\mathsf{v}_2} = (-\infty,0]$. Since

$$\mathcal{S}\Big(\sum_{\ell\in[0,\infty)\cap\mathbb{Z}} \chi^\ell\Big) = \mathcal{S}\Big(\sum_{\ell=0}^{\infty}\chi^\ell\Big) = \frac{1}{1-\chi}$$

$$\mathcal{S}\Big(\sum_{\ell\in(-\infty,0]\cap\mathbb{Z}} \chi^\ell\Big) = \mathcal{S}\Big(\sum_{\ell=0}^{-\infty}\chi^\ell\Big) = \frac{1}{1-\chi^{-1}},$$

the right-hand side of Brion's equality is

$$\begin{aligned}\frac{1}{1-\chi} + \chi^d\frac{1}{1-\chi^{-1}} &= \frac{1-\chi^{-1}+\chi^d(1-\chi)}{(1-\chi)(1-\chi^{-1})}\\ &= \frac{(1-\chi^{d+1})(1-\chi^{-1})}{(1-\chi)(1-\chi^{-1})}\\ &= 1+\chi+\chi^2+\cdots+\chi^d.\end{aligned}$$

The final result is $\sum_{\ell\in[0,d]\cap\mathbb{Z}} \chi^\ell$, as predicted by Brion's equality. $\Diamond$

Here is a more substantial example that uses Theorem 13.2.8.

Example 13.2.12. For the Hirzebruch surface $X = \mathscr{H}_2$ and divisor $D = 3D_3 - 5D_4$, we found that $\chi(\mathscr{L}) = 4$ for $\mathscr{L} = \mathscr{O}_X(D)$ in Example 13.1.8. Here, $M = \mathbb{Z}^2$, and

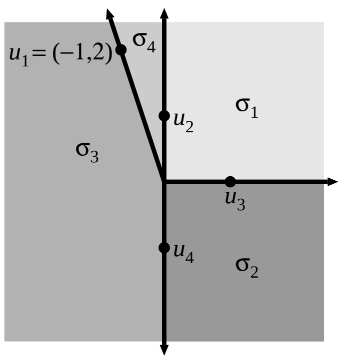


Figure 1. The fan for $X = \mathscr{H}_2$

we set $t_1 = \chi^{e_1}$ and $t_2 = \chi^{e_2}$. In Exercise 13.2.2 you will check that with labellings on $\Sigma(2)$ shown in Figure 1, we have

$$\begin{aligned}
\mathcal{S}(\widetilde{\chi}(U_{\sigma_1},\mathscr{L})) &= \frac{t_1^{-3}}{(1-t_1)(1-t_2)}\\
\mathcal{S}(\widetilde{\chi}(U_{\sigma_2},\mathscr{L})) &= \frac{t_1^{-3}t_2^{-5}}{(1-t_1)(1-t_2^{-1})}\\
\mathcal{S}(\widetilde{\chi}(U_{\sigma_3},\mathscr{L})) &= \frac{t_1^{-10}t_2^{-5}}{(1-t_1^{-1})(1-t_1^{-2}t_2^{-1})}\\
\mathcal{S}(\widetilde{\chi}(U_{\sigma_4},\mathscr{L})) &= \frac{1}{(1-t_1^{-1})(1-t_1^{2}t_2)}.
\end{aligned}$$

Adding these up and doing a lot of algebra yields

$$t_1^{-4}t_2^{-3}+t_1^{-5}t_2^{-3}+t_1^{-4}t_2^{-4}+t_1^{-5}t_2^{-4}+t_1^{-6}t_2^{-4}+t_1^{-7}t_2^{-4}-t_1^{-3}t_2^{-1}-t_1^{-2}t_2^{-1} \tag{13.2.10}$$

In Figure 2, the six exponent vectors $m=(a,b)$ where $t_1^at_2^b$ appears with a positive sign in (13.2.10) are plotted as solid dots, and the two exponent vectors where $t_1^at_2^b$ appears with a negative sign are plotted as hollow dots.

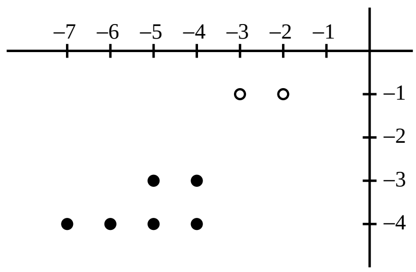


Figure 2. Exponent vectors of the monomials in (13.2.10)

In Exercise 13.2.2 you will show that $P_D=\emptyset$, so that $H^0(X,\mathscr{L})=0$. Hence

$$\widetilde{\chi}(X,\mathscr{L}) = -\sum_{m\in\mathbb{Z}^2}\dim H^1(X,\mathscr{L})_m\,t^m+\sum_{m\in\mathbb{Z}^2}\dim H^2(X,\mathscr{L})_m\,t^m.$$

where $t^m=t_1^at_2^b$ for $m=(a,b)\in\mathbb{Z}^2$. This equals (13.2.10) by Theorem 13.2.8. The sign patterns from Proposition 9.1.6 show that the same m can't appear in both $H^1(X,\mathscr{L})$ and $H^2(X,\mathscr{L})$. Thus solid dots correspond to m with $H^2(X,\mathscr{L})_m\neq 0$ and hollow dots correspond to m with $H^1(X,\mathscr{L})_m\neq 0$. We will confirm these cohomology computations using `Macaulay2` [**123**] in Example B.5.1 and using `Sage` [**262**] in Example B.7.1.

$\Diamond$

When X is complete, $\chi^m \mapsto 1$ takes $\widetilde{\chi}(X,\mathscr{L})$ to $\chi(X,\mathscr{L})$. We will call this "taking the nonequivariant limit" in §13.3. For instance, Example 13.2.12 gives

$$\widetilde{\chi}(\mathscr{L}) = t_1^{-4}t_2^{-3} + t_1^{-5}t_2^{-3} + t_1^{-4}t_2^{-4} + t_1^{-5}t_2^{-4} + t_1^{-6}t_2^{-4} + t_1^{-7}t_2^{-4} - t_1^{-3}t_2^{-1} - t_1^{-2}t_2^{-1},$$

so that mapping $t_i = \chi^{e_i}$ to 1 gives

$$\chi(\mathscr{L}) = 1+1+1+1+1+1-1-1 = 4.$$

This agrees with the computation done in Example 13.1.8. The difference is that previously, we used Riemann-Roch, while here, all we needed was the explicit local decompostion of $\widetilde{\chi}(\mathscr{L})$. In the next section, this local decompostion will be an important part of our proof of equivariant Riemann-Roch for toric varieties.

Exercises for §13.2.

13.2.1. Consider $\mathbb{Q}[[M]]_{\mathrm{Sum}} \subseteq \mathbb{Q}[[M]]$ from Definition 13.2.2.

(a) Prove that $\mathbb{Q}[[M]]_{\mathrm{Sum}}$ is a $\mathbb{Q}[M]$-submodule of $\mathbb{Q}[[M]]$.

(b) Prove that (13.2.9) is well-defined and is a homomorphism of $\mathbb{Q}[M]$-modules.

13.2.2. Consider $X = \mathscr{H}_2$ and $\mathscr{L} = \mathscr{O}_X(D)$, $D = 3D_3 - 5D_4$, as in Example 13.2.12.

(a) Compute the polyhedron P_D of D and conclude that $H^0(X,\mathscr{L}) = 0$.

(b) This example computed the m's that occur in $H^1(X,\mathscr{L})$ and $H^2(X,\mathscr{L})$. Compute the chamber decomposition for D on X (see Example 9.1.8) and then use Proposition 9.1.6 to show that we get the same m's as in Figure 2.

13.2.3. Let $\sigma \subseteq N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a simplicial cone of dimension n with minimal generators $u_1,\dots,u_n \in N$, and let $m_1',\dots,m_n' \in M_{\mathbb{R}}$ be the dual basis in the sense of linear algebra. Also let ℓ_i be the smallest positive integer such that $m_i = \ell_i m_i' \in M$.

(a) Prove that $m_1,\dots,m_n$ are the minimal generators of $\sigma^\vee$.

(b) Prove that $\mathrm{mult}(\sigma)\,\mathrm{mult}(\sigma^\vee) = \prod_{i=1}^n \ell_i$. Hint: Show that $M \subseteq \mathbb{Z}m_1' + \cdots + \mathbb{Z}m_n'$ is dual to $\mathbb{Z}u_1 + \cdots + \mathbb{Z}u_n \subseteq N$.

13.2.4. Prove that the decomposition $m = m' + m''$ in (13.2.7) is unique when it exists.

13.2.5. Let $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(-d)$ for $d > 0$.

(a) Compute $\widetilde{\chi}(\mathbb{P}^1,\mathscr{L})$ using Batyrev-Borisov vanishing (Theorem 9.2.7).

(b) Compute $\widetilde{\chi}(\mathbb{P}^1,\mathscr{L})$ using Theorem 13.2.8.

(c) Compute $\chi(\mathbb{P}^1,\mathscr{L})$ using the Riemann-Roch theorem for curves (10.5.1) and explain how your answer relates to $\widetilde{\chi}(\mathbb{P}^1,\mathscr{L})$.

13.2.6. For $X = X_\Sigma$ and $\mathscr{L}$ as in Lemma 13.2.2, we have $\mathcal{S}(\widetilde{\chi}(U_\sigma,\mathscr{L})) = 0$ when $\sigma \in \Sigma$ satisfies $\dim\sigma < n = \dim X$. Here you will prove that $\mathcal{S}(\widetilde{\chi}(U_\sigma,\mathscr{L})) \neq 0$ when $\dim\sigma = n$. To begin, assume that there are $m_1,\dots,m_s \in M \setminus \{0\}$ with

$$\widetilde{\chi}(U_\sigma,\mathscr{O}_{U_\sigma}) \cdot \prod_{i=1}^{s}(1-\chi^{m_i}) = 0. \tag{13.2.11}$$

(a) Show there is $u \in \mathrm{Int}(\sigma \cap N)$ such that $\langle m_i, u\rangle \neq 0$ for all i.

(b) Show that we can assume $\langle m_i, u\rangle > 0$ for all i. Hint: If necessary, multiply (13.2.11) by χ^{-m_i}.

(c) Show that the constant term 1 on the left-hand side of (13.2.11) cannot cancel. Then conclude that $\mathcal{S}(\widetilde{\chi}(U_\sigma,\mathscr{L})) \neq 0$.

§13.3. Toric Equivariant Riemann-Roch

In §12.4 we described the cohomology ring of a complete simplicial toric variety by first computing its equivariant cohomology and then taking the "nonequivariant limit." We use the same strategy here: we will deduce HRR for a smooth complete toric variety from the following equivariant Riemann-Roch theorem.

Theorem 13.3.1. *For a smooth complete toric variety $X = X_\Sigma$ and a line bundle $\mathscr{L} = \mathscr{O}_X(D)$ of a torus-invariant divisor D on X, we have*

$$\chi^T(\mathscr{L}) = \int_{X^{eq}} \mathrm{ch}^T(\mathscr{L})\,\mathrm{Td}^T(X).$$

In this theorem, $\chi^T(\mathscr{L})$, $\int_{X^{eq}}$, $\mathrm{ch}^T(\mathscr{L})$ and $\mathrm{Td}^T(X)$ are equivariant versions of the corresponding objects appearing in (13.0.1). The equality of Theorem 13.3.1 takes place in the completion of the equivariant cohomology ring of a point, which we denote by $\widehat{\Lambda}$. All of this will be defined carefully in the course of the section.

We will follow [**50**], where Brion and Vergne prove equivariant Riemann-Roch for complete simplicial varieties. We will discuss their simplicial version of the theorem after completing the proof of the smooth case.

The Equivariant Euler Characteristic. As in §12.4, we set

$$(\Lambda_T)_{\mathbb{Q}} = H_T^\bullet(\{\mathrm{pt}\},\mathbb{Q}),$$

where $T = T_N$ is the torus associated to M. Since exponentials are infinite sums, we will use the completion of $H_T^\bullet(\{\mathrm{pt}\},\mathbb{Q}) = \bigoplus_{k=0}^\infty H_T^k(\{\mathrm{pt}\},\mathbb{Q})$, written

$$\widehat{\Lambda} = \widehat{H}_T^\bullet(\{\mathrm{pt}\},\mathbb{Q}) = \prod_{k=0}^\infty H_T^k(\{\mathrm{pt}\},\mathbb{Q}).$$

Recall from (12.4.9) that we have an isomorphism $\mathsf{s} : \mathrm{Sym}_{\mathbb{Q}}(M) \simeq (\Lambda_T)_{\mathbb{Q}}$, where $\mathrm{Sym}_{\mathbb{Q}}(M)$ is our notation for $\bigoplus_{k=0}^\infty \mathrm{Sym}_{\mathbb{Q}}^k(M_{\mathbb{Q}})$. Hence, if $m \in M$, then $\mathsf{s}(m)$ is a degree 2 element of $\widehat{\Lambda}$. It follows that $m \in M$ gives the exponential

$$e^{\mathsf{s}(m)} = 1 + \mathsf{s}(m) + \tfrac{1}{2}\mathsf{s}(m)^2 + \tfrac{1}{6}\mathsf{s}(m)^3 + \cdots \in \widehat{\Lambda}.$$

We can now define the equivariant Euler characteristic that appears on the left-hand side of the equivariant Riemann-Roch theorem.

Definition 13.3.2. If $X = X_\Sigma$ is complete and $\mathscr{L} = \mathscr{O}_X(D)$ is the line bundle of a torus-invariant divisor D on X, then the ***equivariant Euler characteristic*** of $\mathscr{L}$ is

$$\chi^T(\mathscr{L}) = \sum_{m\in M}\sum_{i=0}^n (-1)^i \dim H^i(X,\mathscr{L})_m\, e^{\mathsf{s}(m)} \in \widehat{\Lambda}, \tag{13.3.1}$$

where $H^i(X,\mathscr{L}) = \bigoplus_{m\in M} H^i(X,\mathscr{L})_m$ is the decomposition from §9.1.

We will see below that $\chi^T(\mathscr{L})$ is closely related to $\widetilde{\chi}(\mathscr{L})$ from §13.2.

The Nonequivariant Limit. Let $i_{\mathrm{pt}}^* : \widehat{\Lambda} \to \mathbb{Q}$ be the map that sends elements of positive degree to zero. Then

$$i_{\mathrm{pt}}^*(\chi^T(\mathscr{F})) = \chi(\mathscr{F}) = \sum_{i=0}^{n}(-1)^i \dim H^i(X,\mathscr{F}),$$

i.e., i_{pt}^* takes the equivariant Euler characteristic to the ordinary Euler characteristic.

Later in the section we will show that applying i_{pt}^* to the equivariant Riemann-Roch theorem gives HRR from (13.0.1). As mentioned in §12.4, the maps i_{pt}^* and $i_X^* : H_T^\bullet(X,\mathbb{Q}) \to H^\bullet(X,\mathbb{Q})$ are the "nonequivariant limit" that turns equivariant cohomology into ordinary cohomology.

Equivariant Chern Characters and Todd Classes. To define equivariant Chern characters, we replace the equivariant cohomology $H_T^\bullet(X,\mathbb{Q})$ with its completion

$$\widehat{H}_T^\bullet(X,\mathbb{Q}) = \prod_{k=0}^{\infty} H_T^k(X,\mathbb{Q}).$$

The nonequivariant limit map $i_X^* : H_T^\bullet(X,\mathbb{Q}) \to H^\bullet(X,\mathbb{Q})$ extends in the obvious way to a ring homomorphism

$$i_X^* : \widehat{H}_T^\bullet(X,\mathbb{Q}) \to H^\bullet(X,\mathbb{Q})$$

since elements of degree $> 2\dim X$ map to zero.

By Proposition (12.4.13), a torus-invariant divisor $D = \sum_\rho a_\rho D_\rho$ gives the equivariant cohomology class $[D]_T = \sum_\rho a_\rho [D_\rho]_T \in H_T^2(X,\mathbb{Q})$. Then we define the *equivariant Chern character* $\mathrm{ch}^T(\mathscr{L})$ of $\mathscr{L} = \mathscr{O}_X(D)$ to be

$$\mathrm{ch}^T(\mathscr{L}) = e^{[D]_T} = 1 + [D]_T + \tfrac{1}{2!}[D]_T^2 + \cdots \in \widehat{H}_T^\bullet(X,\mathbb{Q}).$$

Furthermore, since X is smooth, Theorem 13.1.6 implies that the ordinary Todd class of X is

$$\mathrm{Td}(X) = \prod_{i=1}^{r} \frac{[D_\rho]}{1 - e^{-[D_\rho]}}.$$

Hence it is natural to define the *equivariant Todd class* of X to be

$$\mathrm{Td}^T(X) = \prod_\rho \frac{[D_\rho]_T}{1 - e^{-[D_\rho]_T}}.$$

The power series (13.1.4) shows that $\mathrm{Td}^T(X) \in \widehat{H}_T^\bullet(X,\mathbb{Q})$. Since $i_X^*[D_\rho]_T = [D_\rho]$ by Proposition 12.4.13, we have

$$i_X^*\,\mathrm{ch}^T(\mathscr{L}) = \mathrm{ch}(\mathscr{L}) \quad \text{and} \quad i_X^*\,\mathrm{Td}^T(\mathscr{L}) = \mathrm{Td}(\mathscr{L}). \tag{13.3.2}$$

The Equivariant Integral. We saw in §13.1 that the map $\int_X : H^*(X,\mathbb{Q}) \to \mathbb{Q}$ is the generalized Gysin map of the constant function $p : X \to \{\mathrm{pt}\}$. In the appendix to this chapter, we will see that p also gives the *equivariant Gysin map*

$$p_! : H_T^\bullet(X,\mathbb{Q}) \to H_T^\bullet(\{\mathrm{pt}\},\mathbb{Q})$$

which maps $H_T^k(X,\mathbb{Q})$ to $H_T^{k-2\dim X}(\{\mathrm{pt}\},\mathbb{Q})$. We write $\int_{X^{eq}}$ instead of $p_!$, so that

$$\int_{X^{eq}} : \widehat{H}_T^\bullet(X,\mathbb{Q}) \to \widehat{H}_T^\bullet(\{\mathrm{pt}\},\mathbb{Q}) = \widehat{\Lambda}.$$

This is called the *equivariant integral*. We will prove later that $\int_{X^{eq}}$ and $\int_X$ are compatible with taking the nonequivariant limit. This will make it easy to derive HRR from equivariant Riemann-Roch.

Strategy of the Proof. We have now defined everything needed to make sense of the equivariant Riemann-Roch theorem for $X = X_\Sigma$ in Theorem 13.3.1:

- $\chi^T(\mathscr{L})$ lives in $\widehat{\Lambda}$.
- $\mathrm{ch}^T(\mathscr{L})$ and $\mathrm{Td}^T(X)$ live in the ring $\widehat{H}_T^\bullet(X,\mathbb{Q})$.
- $\int_{X^{eq}}$ maps $H_T^\bullet(X,\mathbb{Q})$ to $\widehat{\Lambda}$.

Thus equivariant RR is the equation $\chi^T(\mathscr{L}) = \int_{X^{eq}} \mathrm{ch}^T(\mathscr{L})\,\mathrm{Td}^T(X)$ in $\widehat{\Lambda}$.

The next step is to say a few words about how we will prove the theorem. The main idea is to use the map $\delta : H_T^\bullet(X,\mathbb{Q}) \to H_T^\bullet(X^T,\mathbb{Q})$ from (12.4.11) induced by the inclusion of the fixed point set $X^T \subseteq X$. Recall that $X^T = \{x_\sigma \mid \sigma \in \Sigma(n)\}$, where $n = \dim X$. The completed version of δ is

$$\delta : \widehat{H}_T^\bullet(X,\mathbb{Q}) \longrightarrow \bigoplus_{\sigma\in\Sigma(n)} \widehat{H}_T^\bullet(\{x_\sigma\},\mathbb{Q}) = \bigoplus_{\sigma\in\Sigma(n)} \widehat{\Lambda}.$$

This map is injective by Corollary 12.4.9, and the localization theorem (see [**9**]) implies that δ becomes an isomorphism after tensoring with the field of fractions of $(\Lambda_T)_\mathbb{Q} \subseteq \widehat{\Lambda}$. In our case, we can do better. Let S be the multiplicative set consisting of all finite products of nonzero degree 2 elements of $\widehat{\Lambda}$. The empty product shows that $1 \in S$. Then Exercise 12.4.3 implies that the localized map

$$\delta_S : \widehat{H}_T^\bullet(X,\mathbb{Q})_S \longrightarrow \bigoplus_{\sigma\in\Sigma(n)} \widehat{\Lambda}_S. \tag{13.3.3}$$

is an isomorphism. Here, $\widehat{H}_T^\bullet(X,\mathbb{Q})_S$ and $\widehat{\Lambda}_S$ are the localizations in the sense of commutative algebra. Thus we consider fractions whose denominators lie in S.

The strategy of the proof is to express each side of the desired equation as a sum of local terms in $\widehat{\Lambda}_S$ indexed by $\sigma \in \Sigma(n)$. The proof will then reduce to checking that the local terms on each side of the equation are equal in $\widehat{\Lambda}_S$. Many proofs in equivariant cohomology use this approach.

Local Euler Characteristics. Following the strategy outlined above, we first show that the equivariant Euler characteristic $\chi^T(\mathscr{L}) \in \widehat{\Lambda}$ decomposes as a sum of local terms

$$\chi^T(\mathscr{L}) = \sum_{\sigma\in\Sigma(n)} \chi^T_\sigma(\mathscr{L}), \quad \chi^T_\sigma(\mathscr{L}) \in \widehat{\Lambda}_S.$$

When $X = X_\Sigma$ is complete, Brion's equalities (Theorem 13.2.8) imply that

$$\widetilde{\chi}(X,\mathscr{L}) = \sum_{m\in M}\sum_{i=0}^{n}(-1)^i \dim H^i(X,\mathscr{L})_m\,\chi^m \in \mathbb{Z}[M] \tag{13.3.4}$$

from (13.2.1) as a sum of local terms

$$\widetilde{\chi}(X,\mathscr{L}) = \sum_{\sigma\in\Sigma(n)} \mathcal{S}(\widetilde{\chi}(U_\sigma,\mathscr{L})), \tag{13.3.5}$$

where each $\mathcal{S}(\widetilde{\chi}(U_\sigma,\mathscr{L}))$ lies in the localization $\mathbb{Z}[M]_{\overline{S}}$ and the sum function $\mathcal{S}$ is defined in (13.2.9).

We translate (13.3.5) into a decomposition in $\widehat{\Lambda}_S$ using the ring homomorphism

$$\Phi : \mathbb{Z}[M] \longrightarrow \widehat{\Lambda}, \quad \chi^m \longmapsto e^{\mathsf{s}(m)}.$$

If $m \in M$ is nonzero, then

$$\frac{1-e^{\mathsf{s}(m)}}{\mathsf{s}(m)} = \frac{-\mathsf{s}(m)-\frac{1}{2}\mathsf{s}(m)^2-\cdots}{\mathsf{s}(m)} = -1-\tfrac{1}{2}\mathsf{s}(m)-\cdots$$

is invertible in $\widehat{\Lambda}$. Thus

$$\begin{aligned}\Phi(1-\chi^m) = 1-e^{\mathsf{s}(m)} &= \mathsf{s}(m)\,\frac{1-e^{\mathsf{s}(m)}}{\mathsf{s}(m)}\\ &= \mathsf{s}(m)\times \text{invertible element of } \widehat{\Lambda}.\end{aligned}$$

Since $\mathbb{Q}[M]_{\overline{S}}$ inverts all $1-\chi^m$ and $\widehat{\Lambda}_S$ inverts all $\mathsf{s}(m)$, it follows that Φ extends to a ring homomorphism

$$\Phi : \mathbb{Q}[M]_{\overline{S}} \longrightarrow \widehat{\Lambda}_S.$$

The definitions of $\chi^T(\mathscr{L})$ from (13.3.1) and $\widetilde{\chi}(X_\Sigma,\mathscr{L})$ from (13.3.4) make it easy to see that

$$\Phi(\widetilde{\chi}(X,\mathscr{L})) = \chi^T(\mathscr{L})$$

when $X = X_\Sigma$ is complete. Using Φ, we now define the local version of $\chi^T(\mathscr{L})$.

Definition 13.3.3. Let $X = X_\Sigma$ be complete of dimension n and set $\mathscr{L} = \mathscr{O}_X(D)$ for a torus-invariant Cartier divisor D. Given $\sigma \in \Sigma(n)$, define

$$\chi^T_\sigma(\mathscr{L}) = \Phi(\mathcal{S}(\widetilde{\chi}(U_\sigma,\mathscr{L}))).$$

Here is the desired decomposition.

Theorem 13.3.4. *Let $X = X_\Sigma$ be complete of dimension n and set $\mathscr{L} = \mathscr{O}_X(D)$ for a torus-invariant Cartier divisor D with Cartier data $\{m_\sigma\}_{\sigma\in\Sigma(n)}$. Then:*

(a) $\chi^T(\mathscr{L}) = \sum_{\sigma\in\Sigma(n)} \chi^T_\sigma(\mathscr{L})$ *in* $\widehat{\Lambda}_S$.

(b) *If* $\sigma \in \Sigma(n)$ *is simplicial and* $m_{\sigma,1},\dots,m_{\sigma,n} \in M$ *are the minimal generators of* $\sigma^\vee \subseteq M_{\mathbb{R}}$*, then*

$$\chi^T_\sigma(\mathscr{L}) = \frac{e^{\mathsf{s}(m_\sigma)} \cdot \sum_{m\in P_{\sigma^\vee}\cap M} e^{\mathsf{s}(m)}}{\prod_{i=1}^n (1-e^{\mathsf{s}(m_{\sigma,i})})}$$

for $P_{\sigma^\vee}$ *as in* (13.2.5). *In particular, if* σ *is smooth, then*

$$\chi^T_\sigma(\mathscr{L}) = \frac{e^{\mathsf{s}(m_\sigma)}}{\prod_{i=1}^n (1-e^{\mathsf{s}(m_{\sigma,i})})}.$$

Proof. Apply Φ to Theorem 13.2.8. □

Fixed Points. After decomposing the equivariant Euler characteristic into local factors, the next step is to decompose the equivariant integral. This requires that we study the fixed points of the torus action on $X = X_\Sigma$. Here we will assume that X is complete and simplicial.

Given $\sigma \in \Sigma(n)$, note that $V(\sigma) = \{x_\sigma\}$ is a fixed point of the torus action. Let

$$i_\sigma : V(\sigma) = \{x_\sigma\} \hookrightarrow X$$

be the inclusion map. We also have the constant map $p : X \to \{\mathrm{pt}\}$. Since $p\circ i_\sigma$ is the canonical map that takes x_σ to pt, we get a commutative diagram

$$\begin{tikzcd} \widehat{H}^\bullet_T(X,\mathbb{Q}) \arrow[dr, "i^*_\sigma"] & \\ \widehat{H}^\bullet_T(\{\mathrm{pt}\},\mathbb{Q}) = \widehat{\Lambda} \arrow[u, "p^*"] \arrow[r, "\sim"] & \widehat{H}^\bullet(\{x_\sigma\},\mathbb{Q}), \end{tikzcd} \tag{13.3.6}$$

where the isomorphism on the bottom is $(p\circ i_\sigma)^* = i^*_\sigma \circ p^*$. Using this isomorphism to identify $\widehat{H}^\bullet(\{x_\sigma\},\mathbb{Q})$ with $\widehat{H}^\bullet_T(\{\mathrm{pt}\},\mathbb{Q}) = \widehat{\Lambda}$, we can write

$$i^*_\sigma(\alpha) \in \widehat{\Lambda}$$

when $\alpha \in \widehat{H}^\bullet_T(X,\mathbb{Q})$.

Lemma 13.3.5. *Let* $X = X_\Sigma$ *be complete and simplicial of dimension n. Also let* $\sigma \in \Sigma(n)$ *and* $\rho \in \Sigma(1)$*. Then:*

(a) *If* $\rho \notin \sigma(1)$*, then* $i^*_\sigma([D_\rho]_T) = 0$.

(b) *If* $\rho \in \sigma(1)$*, then* $i^*_\sigma([D_\rho]_T) = \frac{-1}{\ell}\,\mathsf{s}(m)$*, where* ℓ *be the smallest positive integer such that* ℓD_ρ *is Cartier and* $m \in M$ *satisfies* $\ell D_\rho \cap U_\sigma = \mathrm{div}(\chi^m)$.

Proof. Factor $i_\sigma : \{x_\sigma\} \hookrightarrow X$ as $\{x_\sigma\} \hookrightarrow U_\sigma \xhookrightarrow{j_\sigma} X$. Proposition 12.4.13 implies that for any $\rho \in \Sigma(1)$,

$$j^*_\sigma([D_\rho]_T) = [D_\rho \cap U_\sigma]_T.$$

If $\rho \notin \sigma(1)$, this is zero since $D_\rho \cap U_\sigma = \emptyset$. On the other hand, if $\rho \in \sigma(1)$, then on U_σ, we have $\ell D_\rho \cap U_\sigma = \mathrm{div}(\chi^m)$. Using Proposition 12.4.13 again, we obtain

$$j_\sigma^*([\ell D_\rho]_T) = [\ell D_\rho \cap U_\sigma]_T = -[\mathrm{div}(\chi^m)]_T = \mathsf{s}(m) \cdot 1 \in \widehat{H}_T^\bullet(U_\sigma, \mathbb{Q}).$$

Since $\mathsf{s}(m) \in \widehat{\Lambda}$ and everything is a module over $\widehat{\Lambda}$, mapping the above equation into $\widehat{H}_T^\bullet(\{x_\sigma\}, \mathbb{Q})$ implies $i_\sigma^*([\ell D_\rho]_T) = -\mathsf{s}(m) \cdot 1$. The desired formula follows. $\square$

We also have the equivariant Gysin map $i_{\sigma!} : \widehat{H}_T^\bullet(\{\mathrm{pt}\}, \mathbb{Q}) \to \widehat{H}_T^\bullet(X, \mathbb{Q})$ from the appendix to this chapter. Here is an important property of this map.

Proposition 13.3.6. *Let $X = X_\Sigma$ be complete and simplicial. If $\sigma \in \Sigma(n)$, then*

$$i_{\sigma!}(1) = \mathrm{mult}(\sigma) \prod_{\rho \in \sigma(1)} [D_\rho]_T.$$

Proof. Since we have not developed the full theory of equivariant cohomology classes, our argument will be somewhat ad-hoc. Note that $\prod_{\rho \in \sigma(1)} [D_\rho]_T$ lies in $H_T^{2n}(X, \mathbb{Q})$, and the same is true for $i_{\sigma!}(1)$ by Proposition 13.A.9.

The first step is to show that

$$\int_{X^{eq}} i_{\sigma!}(1) = 1 \ \text{ and } \int_{X^{eq}} \mathrm{mult}(\sigma) \prod_{\rho \in \sigma(1)} [D_\rho]_T = 1. \tag{13.3.7}$$

Since $\int_{X^{eq}} = p_!$, the integral on the left is $p_!(i_{\sigma!}(1))$, which by Proposition 13.A.9 is equal to $(p \circ i_\sigma)_!(1) = 1$ since $p \circ i_\sigma : \{x_\sigma\} \to \{\mathrm{pt}\}$. For the other integral in (13.3.7), observe that $\int_{X^{eq}} \prod_{\rho \in \sigma(1)} [D_\rho]_T \in \widehat{\Lambda} = \widehat{H}_T^\bullet(\{\mathrm{pt}\}, \mathbb{Q})$ has degree zero by Proposition 13.A.9. Since $\widehat{\Lambda}_0 = \mathbb{Q}$, it suffices to consider

$$i_{\mathrm{pt}}^* \int_{X^{eq}} \prod_{\rho \in \sigma(1)} [D_\rho]_T = \int_X i_X^* \Big(\prod_{\rho \in \sigma(1)} [D_\rho]_T \Big) = \int_X \prod_{\rho \in \sigma(1)} i_X^* [D_\rho]_T = \int_X \prod_{\rho \in \sigma(1)} [D_\rho],$$

where the first equality uses the commutative diagram from Proposition 13.A.11 and third uses Proposition 12.4.13. Then we are done by Lemma 12.5.2, which implies that $\prod_{\rho \in \sigma(1)} [D_\rho] = \mathrm{mult}(\sigma)^{-1} [V(\sigma)] = \mathrm{mult}(\sigma)^{-1} [\{x_\sigma\}]$.

The second step is to show that

$$i_{\sigma'}^*(i_{\sigma!}(1)) = 0 \ \text{ and } \ i_{\sigma'}^* \Big(\prod_{\rho \in \sigma(1)} [D_\rho]_T \Big) = 0 \ \text{ when } \ \sigma' \in \Sigma(n), \ \sigma' \neq \sigma. \tag{13.3.8}$$

In this case, the second equality is easy, since some $\rho \in \sigma(1)$ is not contained in $\sigma'(1)$, so that $i_{\sigma'}^*[D_\rho]_T = 0$ by Lemma 13.3.5. For the first, $\{x_\sigma\} \cap \{x_{\sigma'}\} = \emptyset$ and Proposition 13.A.10 give a commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \widehat{H}_T^\bullet(\{x_{\sigma'}\}, \mathbb{Q}) \simeq \widehat{\Lambda} \\ \big\uparrow & & \big\uparrow {\scriptstyle i_{\sigma'}^*} \\ \widehat{H}_T^\bullet(\{x_\sigma\}, \mathbb{Q}) & \xrightarrow{i_{\sigma!}} & \widehat{H}^\bullet(X, \mathbb{Q}). \end{array}$$

It follows that $i_{\sigma'}^* \circ i_{\sigma !} = 0$, and (13.3.8) is proved.

In Exercise 13.3.1 you will give the easy argument that (13.3.7) and (13.3.8) imply that $i_{\sigma !}(1)$ and $\mathrm{mult}(\sigma)\prod_{\rho\in\sigma(1)}[D_\rho]_T$ are equal in $H_T^{2n}(X,\mathbb{Q})$. □

Decomposing the Equivariant Integral. When X is smooth and complete, we get the following important formula for the equivariant integral.

Theorem 13.3.7. *If $X = X_\Sigma$ is an n-dimensional smooth complete toric variety and $\alpha \in \widehat{H}_T^\bullet(X,\mathbb{Q})$, then*

$$\int_{X^{eq}} \alpha = (-1)^n \sum_{\sigma\in\Sigma(n)} \frac{i_\sigma^*(\alpha)}{\prod_{i=1}^n \mathsf{s}(m_{\sigma,i})}$$

in $\widehat{\Lambda}_S$, where $m_{\sigma,1},\dots,m_{\sigma,n}$ are the minimal generators of $\sigma^\vee \subseteq M_\mathbb{R}$ for $\sigma \in \Sigma(n)$.

Proof. We will work in the localization $\widehat{H}_T^\bullet(X,\mathbb{Q})_S$. For $\sigma \in \Sigma(n)$, let

$$\Phi_\sigma = \lambda_\sigma\, i_{\sigma !}(1) \in \widehat{H}_T^\bullet(X,\mathbb{Q})_S, \ \text{ where } \ \lambda_\sigma = \frac{(-1)^n}{\prod_{i=1}^n \mathsf{s}(m_{\sigma,i})} \in \widehat{\Lambda}_S.$$

The two key properties of the Φ_σ are

$$\int_{X^{eq}} \alpha\,\Phi_\sigma = \lambda_\sigma\, i_\sigma^*(\alpha) \tag{13.3.9}$$

$$\sum_{\sigma\in\Sigma(n)} \Phi_\sigma = 1 \in \widehat{H}_T^\bullet(X,\mathbb{Q})_S. \tag{13.3.10}$$

Note that the theorem follows immediately from these properties.

For (13.3.9), recall that $\int_{X^{eq}}$ equals $p_!$ for the constant map $p : X \to \{\mathrm{pt}\}$. Then $\widehat{H}_T^\bullet(X,\mathbb{Q})$ becomes a $\widehat{\Lambda}$-module via p^* and $p_!$ is a $\widehat{\Lambda}$-module homomorphism since $p_!(p^*(\beta)\cdot\alpha) = \beta\cdot p_!(\alpha)$ by Proposition 13.A.9. It follows that $\int_{X^{eq}}$ extends to a $\widehat{\Lambda}_S$-module homomorphism between the localizations at S.

By Proposition 13.A.9, the equivariant Gysin map $i_{\sigma !}$ satisfies $\alpha\cdot i_{\sigma !}(1) = i_{\sigma !}(i_\sigma^*(\alpha))$. Hence

$$\alpha\,\Phi_\sigma = \lambda_\sigma\,\alpha\cdot i_{\sigma !}(1) = \lambda_\sigma\, i_{\sigma !}(i_\sigma^*(\alpha)).$$

Then (13.3.9) follows from the equalities

$$\int_{X^{eq}} \alpha\,\Phi_\sigma = p_!(\lambda_\sigma\, i_{\sigma !}(i_\sigma^*(\alpha)) = \lambda_\sigma\, p_!(i_{\sigma !}(i_\sigma^*(\alpha)) = \lambda_\sigma\, i_\sigma^*(\alpha),$$

where the last equality follows because $p_!\circ i_{\sigma !} = (p\circ i_\sigma)_!$ is the inverse of the map $(p\circ i_\sigma)^*$ used in (13.3.6) to identify $\widehat{H}_T^\bullet(\{x_\sigma\},\mathbb{Q})$ with $\widehat{H}_T^\bullet(\{\mathrm{pt}\},\mathbb{Q}) = \widehat{\Lambda}$.

For (13.3.10), we will use the map $\delta : \widehat{H}_T^\bullet(X,\mathbb{Q})_S \simeq \bigoplus_{\sigma\in\Sigma(n)} \widehat{\Lambda}_S$ from (13.3.3). The strategy is to show that for $\sigma \in \Sigma(n)$,

$$\delta(i_{\sigma !}(1)) = (-1)^n \textstyle\prod_{i=1}^n \mathsf{s}(m_{\sigma,i})\cdot e_\sigma. \tag{13.3.11}$$

Once this is proved, then $\delta(\Phi_\sigma) = e_\sigma$ follows by the definition of λ_σ. Thus
$$\delta\big(\textstyle\sum_{\sigma\in\Sigma(n)} \Phi_\sigma\big) = \sum_{\sigma\in\Sigma(n)} e_\sigma = \delta(1),$$
which implies (13.3.10) since δ is injective.

To prove (13.3.11), first note that $i^*_{\sigma'}(i_{\sigma !}(1)) = 0$ for $\sigma' \neq \sigma$ by (13.3.8). To calculate $i^*_\sigma(i_{\sigma !}(1))$, let u_i be the minimal generator of $\rho_i \in \sigma(1)$ for $i \le i \le n$. Since σ is smooth, the dual basis $m_i = m_{\sigma,i}$ gives the minimal generators of $\sigma^\vee$. Furthermore, $D_i = D_{\rho_i}$ is Cartier since X is smooth. Then
$$i^*_\sigma(i_{\sigma !}(1)) = \prod_{i=1}^n i^*_\sigma([D_i]_T) = \prod_{i=1}^n(-\mathsf{s}(m_i)) = (-1)^n \prod_{i=1}^n \mathsf{s}(m_i),$$
where the first equality uses Proposition 13.3.6 and the second uses Lemma 13.3.5 and $D_i \cap U_\sigma = \mathrm{div}(\chi^{m_i})$. This proves (13.3.11) and completes the proof. □

Theorem 13.3.7 is a special case of a general formula described in [**9**]. From this more sophisticated view, the denominator $\prod_{i=1}^n \mathsf{s}(m_{\sigma,i})$ is the equivariant Euler class of the normal bundle of $\{x_\sigma\} \subseteq X$. See Exercise 13.3.2 for further details.

We should also mention that there is a simplicial version of Theorem 13.3.7, which states that

$$\int_{X^{eq}} \alpha = (-1)^n \sum_{\sigma\in\Sigma(n)} \frac{\mathrm{mult}(\sigma^\vee)\, i^*_\sigma(\alpha)}{\prod_{i=1}^n \mathsf{s}(m_{\sigma,i})} \tag{13.3.12}$$

when X is complete and simplicial. You will prove this in Exercise 13.3.3.

Proof of Equivariant RR. We finally have the tools needed to prove our version of the equivariant Riemann-Roch theorem for smooth complete toric varieties, stated earlier as Theorem 13.3.1.

Proof of Theorem 13.3.1. We need to show that
$$\chi^T(\mathscr{L}) = \int_{X^{eq}} \alpha, \quad \alpha = \mathrm{ch}^T(\mathscr{L})\,\mathrm{Td}^T(X),$$
when $\mathscr{L} = \mathscr{O}_X(D)$ and D is a torus-invariant divisor on $X = X_\Sigma$. Let the Cartier data of D be $\{m_\sigma\}_{\sigma\in\Sigma(n)}$. By Theorem 13.3.4, we have
$$\chi^T(\mathscr{L}) = \sum_{\sigma\in\Sigma(n)} \frac{e^{\mathsf{s}(m_\sigma)}}{\prod_{i=1}^n(1-e^{\mathsf{s}(m_{\sigma,i})})}.$$
Comparing this to the decomposition of $\int_{X^{eq}} \alpha$ given in Theorem 13.3.7, we see that it suffices to prove that for $\sigma \in \Sigma(n)$, we have

$$\frac{e^{\mathsf{s}(m_\sigma)}}{\prod_{i=1}^n(1-e^{\mathsf{s}(m_i)})} = (-1)^n \frac{i^*_\sigma(\alpha)}{\prod_{i=1}^n \mathsf{s}(m_i)}, \quad \alpha = \mathrm{ch}^T(\mathscr{L})\,\mathrm{Td}^T(X). \tag{13.3.13}$$

where for simplicity we write $m_i = m_{\sigma,i}$. The m_i are dual to the minimal generators n_i of σ. Let $D_i = D_{\rho_i}$ be the divisor corresponding to $\rho_i = \mathrm{Cone}(u_i) \in \sigma(1)$.

Since $i_\sigma^*(\alpha) = i_\sigma^*(\mathrm{ch}^T(\mathscr{L}))\, i_\sigma^*(\mathrm{Td}^T(X))$, we need to compute $i_\sigma^*(\mathrm{ch}^T(\mathscr{L}))$ and $i_\sigma^*(\mathrm{Td}^T(X))$. We begin with the former. The Chern character is $\mathrm{ch}^T(\mathscr{L}) = e^{[D]_T}$. Adapting the proof of Lemma 13.3.5, one easily sees that

$$i_\sigma^*([D]_T) = \mathsf{s}(m_\sigma)$$

since the Cartier data of D satisfies $D|_{U_\sigma} = \mathrm{div}(\chi^{-m_\sigma})|_{U_\sigma}$. Hence

$$i_\sigma^*(\mathrm{ch}^T(\mathscr{L})) = i_\sigma^*(e^{[D]_T}) = e^{i_\sigma^*([D]_T)} = e^{\mathsf{s}(m_\sigma)}.$$

We next compute $i_\sigma^*(\mathrm{Td}^T(X))$. If $\rho \notin \sigma(1)$, then

$$i_\sigma^*\Big(\frac{[D_\rho]_T}{1-e^{-[D_\rho]_T}}\Big) = i_\sigma^*\big(1 + \tfrac{1}{2}[D_\rho]_T + \cdots\big) = 1$$

by Lemma 13.3.5. On the other hand, if $\rho = \rho_i$ for $1 \le i \le n$, then $D_i \cap U_\sigma = \mathrm{div}(\chi^{m_i})$ on U_σ, so the same lemma implies that

$$i_\sigma^*\Big(\frac{[D_\rho]_T}{1-e^{-[D_\rho]_T}}\Big) = \frac{i_\sigma^*[D_\rho]_T}{1-e^{-i_\sigma^*[D_\rho]_T}} = \frac{-\mathsf{s}(m_i)}{1-e^{\mathsf{s}(m_i)}}.$$

Since $\mathrm{Td}^T(X) = \prod_{\rho\in\Sigma(1)} [D_\rho]_T/(1-e^{-[D_\rho]_T})$, it follows that

$$i_\sigma^*(\mathrm{Td}^T(X)) = \prod_{i=1}^n \frac{-\mathsf{s}(m_i)}{1-e^{\mathsf{s}(m_i)}} = (-1)^n \prod_{i=1}^n \frac{\mathsf{s}(m_i)}{1-e^{\mathsf{s}(m_i)}}.$$

It is now easy to prove (13.3.13), since

$$\begin{aligned}\frac{e^{\mathsf{s}(m_\sigma)}}{\prod_{i=1}^n(1-e^{\mathsf{s}(m_i)})} &= \frac{(-1)^n}{\prod_{i=1}^n \mathsf{s}(m_i)} \cdot e^{\mathsf{s}(m_\sigma)} \cdot (-1)^n \prod_{i=1}^n \frac{\mathsf{s}(m_i)}{1-e^{\mathsf{s}(m_i)}}\\ &= \frac{(-1)^n}{\prod_{i=1}^n \mathsf{s}(m_i)} \cdot i_\sigma^*(\mathrm{ch}^T(\mathscr{L})) \cdot i_\sigma^*(\mathrm{Td}^T(X)),\end{aligned}$$

where we have used the above computations of $i_\sigma^*(\mathrm{ch}^T(\mathscr{L}))$ and $i_\sigma^*(\mathrm{Td}^T(X))$. This gives (13.3.13) since i_σ^* is a ring homomorphism. □

Passing from Equivariant RR to HRR. After a lot of hard work, our reward is an easy proof of the main result of this chapter.

Theorem 13.3.8. *For an invertible sheaf $\mathscr{L}$ on a smooth complete toric variety $X = X_\Sigma$, we have*

$$\chi(\mathscr{L}) = \int_X \mathrm{ch}(\mathscr{L})\,\mathrm{Td}(X).$$

Proof. We may assume that $\mathscr{L} = \mathscr{O}_X(D)$ for a torus-invariant divisor D on X. This ensures that $\chi^T(\mathscr{L})$ and $\mathrm{ch}^T(\mathscr{L})$ are defined. Then

$$\chi(\mathscr{L}) = i_{\mathrm{pt}}^*(\chi^T(\mathscr{L})) = i_{\mathrm{pt}}^* \int_{X^{eq}} \mathrm{ch}^T(\mathscr{L})\,\mathrm{Td}^T(X) = \int_X i_X^*\big(\mathrm{ch}^T(\mathscr{L})\,\mathrm{Td}^T(X)\big),$$

where the first equality is (13.3.1), the second is equivariant Riemann-Roch, and the third is the commutative diagram from Proposition 13.A.11. Since i_X^* is a ring homomorphism, (13.3.2) implies that the integral on the right is $\int_X \mathrm{ch}(\mathscr{L})\,\mathrm{Td}(X)$. We have proved the Hirzebruch-Riemann-Roch theorem! □

To get a better sense of how HRR relates to equivariant RR, let us work out a concrete example.

Example 13.3.9. Let $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(dD_0)$, where the minimal generators of the fan of $\mathbb{P}^1$ are $u_0 = -e_1$ and $u_1 = e_1$ in $N = \mathbb{Z}e_1$. The completed equivariant cohomology ring of $X = \mathbb{P}^1$ is

$$\widehat{H}_T^\bullet(\mathbb{P}^1,\mathbb{Q}) = \mathbb{Q}[[x_0,x_1]]/\langle x_0 x_1\rangle$$

by Theorem 12.4.14. We write $\widehat{\Lambda} = \mathbb{Q}[[t]]$, where $t = \mathsf{s}(e_1)$ for $e_1 \in M = \mathbb{Z}e_1$. As explained in the discussion following Example 12.4.11, t acts on $\widehat{H}_T^\bullet(\mathbb{P}^1,\mathbb{Q})$ by multiplication by $-(\langle e_1,u_0\rangle x_0 + \langle e_1,u_1\rangle x_1) = x_0 - x_1$.

Recall that $\widetilde{\chi}(\mathscr{L}) = 1+\chi+\cdots+\chi^d \in \mathbb{Z}[M]$ by Example 13.2.11. Since χ maps to $e^{\mathsf{s}(e_1)} = e^t$, we obtain

$$\chi^T(\mathscr{L}) = 1+e^t+\cdots+e^{dt} = d+1+\tbinom{d+1}{2}t+\cdots. \tag{13.3.14}$$

Using $x_i = [D_i]_T$, we see that the equivariant Todd class of $X = \mathbb{P}^1$ is

$$\begin{aligned}\mathrm{Td}^T(\mathbb{P}^1) = \frac{x_0}{1-e^{-x_0}}\frac{x_1}{1-e^{-x_1}} &= (1+\tfrac{1}{2}x_0+\tfrac{1}{12}x_0^2-\cdots)(1+\tfrac{1}{2}x_1+\tfrac{1}{12}x_1^2-\cdots)\\ &= 1+\tfrac{1}{2}(x_0+x_1)+\tfrac{1}{12}(x_0^2+x_1^2)+\cdots\end{aligned}$$

since $x_0x_1 = 0$ in $\mathbb{Q}[[x_0,x_1]]/\langle x_0x_1\rangle$. The equivariant Chern character of $\mathscr{L}$ is

$$\mathrm{ch}^T(\mathscr{L}) = e^{[dD_0]_T} = e^{dx_0} = 1+dx_0+\tfrac{1}{2}(dx_0)^2+\cdots,$$

and you will compute in Exercise 13.3.4 that

$$\mathrm{ch}^T(\mathscr{L})\,\mathrm{Td}^T(\mathbb{P}^1) = 1+(d+\tfrac{1}{2})x_0+\tfrac{1}{2}x_1+(\tfrac{1}{2}d^2+\tfrac{1}{2}d+\tfrac{1}{12})x_0^2+\tfrac{1}{12}x_1^2+\cdots.$$

The next step is to describe the equivariant integral $\int_{(\mathbb{P}^1)^{eq}} : \widehat{H}_T^\bullet(\mathbb{P}^1,\mathbb{Q}) \to \mathbb{Q}[[t]]$. In Exercise 13.3.4 you will explain why $\int_{(\mathbb{P}^1)^{eq}} x_0 = \int_{(\mathbb{P}^1)^{eq}} x_1 = 1$. Since $\int_{(\mathbb{P}^1)^{eq}}$ is a $\mathbb{Q}[[t]]$-module homomorphism, we have

$$\int_{(\mathbb{P}^1)^{eq}} x_0^2 = \int_{(\mathbb{P}^1)^{eq}} (x_0-x_1)x_0 = \int_{(\mathbb{P}^1)^{eq}} t\cdot x_0 = t\int_{(\mathbb{P}^1)^{eq}} x_0 = t.$$

A similar computation shows that $\int_{(\mathbb{P}^1)^{eq}} x_1^2 = -t$ (be sure you see where the minus sign comes form). Applying $\int_{(\mathbb{P}^1)^{eq}}$ to the above computation of $\mathrm{ch}^T(\mathscr{L})\,\mathrm{Td}^T(\mathbb{P}^1)$, we obtain

$$\begin{aligned}\int_{(\mathbb{P}^1)^{eq}} \mathrm{ch}^T(\mathscr{L})\,\mathrm{Td}^T(\mathbb{P}^1) &= 0+d+\tfrac{1}{2}+\tfrac{1}{2}+(\tfrac{1}{2}d^2+\tfrac{1}{2}d+\tfrac{1}{12})t+\tfrac{1}{12}(-t)+\cdots\\ &= d+1+\tbinom{d+1}{2}t+\cdots.\end{aligned}$$

The first two terms agree with what we computed in (13.3.14), and all terms agree by the equivariant Riemann-Roch theorem. Note that HRR is the equality of the constant terms. $\Diamond$

The Simplicial Case. The equivariant Riemann-Roch theorem proved by Brion and Vergne [**50**] applies when $X = X_\Sigma$ is complete and simplicial. The problem is that the formula

$$\mathrm{Td}^T(X) = \prod_\rho \frac{[D_\rho]_T}{1-e^{-[D_\rho]_T}}$$

for the equivariant Todd class no longer holds in the simplicial case. Similarly, for HRR, the formula

$$\mathrm{Td}(X) = \prod_\rho \frac{[D_\rho]}{1-e^{-[D_\rho]}} \tag{13.3.15}$$

needs to be modified when X is simplicial. Here is an example.

Example 13.3.10. Suppose that HRR holds for $X = \mathbb{P}(1,1,2)$. We will use the fan Σ shown in Figure 5 of Example 3.1.17, where the minimal generators are u_1, u_2 and $u_0 = -u_1 - 2u_2$. The relations in the class group are

$$D_1 \sim D_0 \text{ and } D_2 \sim 2D_0,$$

and the intersection pairing is determined by $D_1 \cdot D_2 = 1$ since u_1, u_2 generate a smooth cone. Note also that $X = X_P$ for $P = \mathrm{Conv}(0, 2e_1, e_2)$ and that $D_P = 2D_0$. In Exercise 13.3.5 you will check the details of the following computations.

The cohomology ring of X is $H^\bullet(X,\mathbb{Q}) \simeq \mathbb{Q} \oplus \mathbb{Q}[D_0] \oplus \mathbb{Q}[\mathrm{pt}]$, where $[D_0]^2 = \frac{1}{2}[\mathrm{pt}]$. If we apply the formula (13.3.15) to $X = \mathbb{P}(1,1,2)$, then one computes that

$$\text{“}\mathrm{Td}(X)\text{”} = 1 + 2[D_0] + \tfrac{7}{8}[\mathrm{pt}]. \tag{13.3.16}$$

The actual Todd class will have the form $a + b[D_0] + c[\mathrm{pt}]$. To determine the constants a, b, c, we apply HRR to $\mathscr{L} = \mathscr{O}_X(\ell D_P)$, which gives

$$\begin{aligned}
\chi(\mathscr{O}_X(\ell D_P)) &= \int_X (1 + [\ell D_P] + \tfrac{1}{2}[\ell D_P]^2)(a + b[D_0] + c[\mathrm{pt}])\\
&= \int_X (1 + [2\ell D_0] + \tfrac{1}{2}[2\ell D_0]^2)(a + b[D_0] + c[\mathrm{pt}])\\
&= \int_X a + (b + 2a\ell)[D_0] + (a\ell^2 + b\ell + c)[\mathrm{pt}]\\
&= a\ell^2 + b\ell + c.
\end{aligned}$$

On the other hand, $\chi(\mathscr{O}_X(kD_P)) = |(\ell P) \cap M|$ is the Ehrhart polynomial of P, which by (10.5.13) is

$$\mathrm{Ehr}_P(\ell) = \mathrm{Area}(P)\,\ell^2 + \tfrac{1}{2}|\partial P \cap M|\,\ell + 1 = \ell^2 + 2\ell + 1.$$

It follows that the actual Todd class is

$$\mathrm{Td}(X) = 1 + 2[D_0] + [\mathrm{pt}].$$

This differs from "$\mathrm{Td}(X)$" by $\frac{1}{8}[\mathrm{pt}]$. We will soon see the theoretical reason for this discrepancy. ◊

Returning to the equivariant case, let $X = X_\Sigma$ be a complete simplicial toric variety. To define its equivariant Todd class $\mathrm{Td}^T(X)$, consider the group G in the quotient construction of X from §5.1, namely

$$G = \big\{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid \textstyle\prod_\rho t_\rho^{\langle m,u_\rho\rangle} = 1 \text{ for all } m \in M\big\}.$$

Each $\rho \in \Sigma(1)$ gives the character

$$\chi_\rho : (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow \mathbb{C}^*$$

defined by projection on the ρth factor. Then for any cone $\sigma \in \Sigma$, let

$$\begin{aligned} G_\sigma &= \{g \in G \mid \chi_\rho(g) = 1 \text{ for all } \rho \notin \sigma(1)\} \\ &= \big\{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} \mid t_\rho = 1 \text{ for } \rho \notin \sigma(1),\ \textstyle\prod_{\rho\in\sigma(1)} t_\rho^{\langle m,u_\rho\rangle} = 1 \text{ for } m \in M\big\}. \end{aligned}$$

One can show that

$$G_\sigma \simeq N_\sigma / \big(\textstyle\sum_{\rho\in\sigma(1)} \mathbb{Z}u_\rho\big) \tag{13.3.17}$$

(Exercise 13.3.6), so that $|G_\sigma| = \mathrm{mult}(\sigma)$.

Then we set

$$G_\Sigma = \bigcup_{\sigma\in\Sigma} G_\sigma \subseteq G$$

and define the *equivariant Todd class* of X to be

$$\mathrm{Td}^T(X) = \sum_{g\in G_\Sigma} \prod_{\rho\in\Sigma(1)} \frac{[D_\rho]_T}{1 - \chi_\rho(g)\, e^{-[D_\rho]_T}} \tag{13.3.18}$$

when $X = X_\Sigma$ is complete and simplicial. Here is the result proved in [**50**].

Theorem 13.3.11. *Let $X = X_\Sigma$ be a complete simplicial toric variety. If $\mathscr{L} = \mathscr{O}_X(D)$ is the line bundle of a torus-invariant Cartier divisor D on X, then*

$$\chi^T(\mathscr{L}) = \int_{X^{eq}} \mathrm{ch}^T(\mathscr{L})\, \mathrm{Td}^T(X),$$

where $\mathrm{Td}^T(X)$ *is defined in* (13.3.18). □

This result has also been proved by Edidin and Graham [**87**] using different methods. Once we have a simplicial version of equivariant Riemann-Roch, the proof of Theorem 13.3.8 gives HRR in the simplicial case as follows.

Corollary 13.3.12. *Let $X = X_\Sigma$ be a complete simplicial toric variety. If $\mathscr{L}$ is a line bundle on X, then*

$$\chi(\mathscr{L}) = \int_X \mathrm{ch}(\mathscr{L})\,\mathrm{Td}(X),$$

where

$$\mathrm{Td}(X) = \sum_{g\in G_\Sigma} \prod_{\rho\in\Sigma(1)} \frac{[D_\rho]}{1-\chi_\rho(g)\,e^{-[D_\rho]}} \qquad \square$$

Let us apply this to the previous example.

Example 13.3.13. Consider $\mathbb{P}(1,1,2)$ as in Example 13.3.10. In Exercise 13.3.5 you will show that $G_\Sigma = \{(1,1,1),(-1,-1,1)\} \subseteq G = \{(t,t,t^2) \mid t \in \mathbb{C}^*\}$. Thus $\mathrm{Td}(X)$ is a sum of two terms. For $g = (1,1,1)$, we gave $\chi_\rho(g) = 1$ for all ρ, so that this term is what we get when we use the formula for the smooth case. In (13.3.16), we computed this to be

$$\text{“}\mathrm{Td}(X)\text{”} = 1 + 2[D_0] + \tfrac{7}{8}[\mathrm{pt}].$$

For $g = (-1,-1,1)$, we get the product

$$\frac{[D_0]}{1+e^{-[D_0]}} \cdot \frac{[D_1]}{1+e^{-[D_1]}} \cdot \frac{[D_2]}{1-e^{-[D_2]}},$$

which becomes

$$(\tfrac{1}{2}[D_0] + \tfrac{1}{4}[D_0]^2)\cdot(\tfrac{1}{2}[D_1] + \tfrac{1}{4}[D_1]^2)\cdot(1 + \tfrac{1}{2}[D_2] + \tfrac{1}{12}[D_1]^2) = \tfrac{1}{4}[D_0][D_1] = \tfrac{1}{8}[\mathrm{pt}].$$

The terms for $g = (1,1,1)$ and $g = (-1,-1,1)$ sum to $\mathrm{Td}(X) = 1 + 2[D_0] + [\mathrm{pt}]$, in agreement with Example 13.3.10. $\Diamond$

In [**229**], Pommersheim defines the *mock Todd class* of a complete simplicial toric variety to be the class computed using the formula for the smooth case, i.e.,

$$\text{“}\mathrm{Td}(X)\text{”} = \prod_\rho \frac{[D_\rho]}{1-e^{-[D_\rho]}}.$$

He shows that the difference between the actual and mock Todd classes depends on the codimension of the singular locus of X. He also expresses the difference in codimension 2 in terms of Dedekind sums.

We should also mention that a nice discussion of Todd classes of general toric varieties and their enumerative applications can be found in [**105**, Sec. 5.3].

Exercises for §13.3.

13.3.1. Let $X = X_\Sigma$ be complete and simplicial of dimension n, and for $\sigma \in \Sigma(n)$, let i_σ be as in (13.3.6). Let $\alpha,\beta \in H_T^{2n}(X,\mathbb{Q})$ satisfy $\int_{X^{eq}} \alpha = 1$ and $\int_{X^{eq}} \beta = 1$ and assume there is $\sigma \in \Sigma(n)$ such that $i_{\sigma'}^*(\alpha) = 0$ and $i_{\sigma'}^*(\beta) = 0$ for all $\sigma' \neq \sigma$. Prove that $\alpha = \beta$. Hint: Let $u = i_\sigma^*(\alpha)$ and $v = i_\sigma^*(\beta)$ in $\widehat{\Lambda}$. Then consider $v\alpha - u\beta \in H_T^{2n}(X,\mathbb{Q})$.

13.3.2. The decomposition of the equivariant integral given in Theorem 13.3.7 involves terms containing the product $(-1)^n \prod_{i=1}^n \mathsf{s}(m_{\sigma,i})$ for $\sigma \in \Sigma(n)$. Here you will show that this is the nth equivariant Chern class of the normal bundle of $\{x_\sigma\} \subseteq X$.

(a) Use $\{x_\sigma\} \subseteq U_\sigma \subseteq X$ to show that the normal bundle is $\mathbb{C}^n$ where T acts on the ith factor via $\chi^{m_{\sigma,i}}$. Thus the normal bundle is the direct sum of the rank 1 equivariant vector bundles given by the characters $\chi^{m_{\sigma,1}}, \dots, \chi^{m_{\sigma,n}}$.

(b) If a rank n vector bundle is a direct sum of rank 1 bundles, show that its nth Chern class is the product of the first Chern classes of the factors.

(c) Explain why the proof of Proposition 12.4.13 implies that the first equivariant Chern of the ith factor of the normal bundle is $-\mathsf{s}(m_{\sigma,i})$.

13.3.3. Prove (13.3.12). Hint: Adapt the proof of Theorem 13.3.7 using Lemma 13.3.5 and Exercise 13.2.3.

13.3.4. Supply the details omitted in Example 13.3.9.

13.3.5. Supply the details omitted in Examples 13.3.10 and 13.3.13.

13.3.6. Prove (13.3.17). Also show that G_σ acts on $\mathbb{C}^{\sigma(1)}$ with quotient $\mathbb{C}^{\sigma(1)}/G_\sigma \simeq U_\sigma$.

13.3.7. Our discussion of the equivariant Euler characteristic used the map $\Phi : \mathbb{Z}[M] \to \widehat{\Lambda}$ defined by $\chi^m \mapsto e^{\mathsf{s}(m)}$. This exercise will explore the canonical meaning of Φ. We first replace $\mathbb{Z}[M]$ with $\mathbb{Q}[M]$ and note that $\mathbb{Q}[M]$ is the rational representation ring of T since the χ^m are the irreducible representations of T. The *augmentation ideal* of $\mathbb{Q}[M]$ is

$$I = \big\{\textstyle\sum_{m\in M} a_m \chi^m \in \mathbb{Q}[M] \mid \sum_{m\in M} a_m = 0\big\}.$$

We also replace $\widehat{\Lambda}$ with $\widehat{\mathrm{Sym}} = \prod_{k=0}^\infty \mathrm{Sym}^k_{\mathbb{Q}}(M_{\mathbb{Q}})$. Then we can think of Φ as the map

$$\Phi : \mathbb{Q}[M] \longrightarrow \widehat{\mathrm{Sym}}, \quad \chi^m \longmapsto e^m.$$

Prove that Φ induces an isomorphism between $\widehat{\mathrm{Sym}}$ and the I-adic completion of $\mathbb{Q}[M]$ at the augmentation ideal I. This is the canonical meaning of Φ.

13.3.8. Let $X = X_P$ be a projective toric variety, where $P \subseteq M_{\mathbb{R}}$ is a full dimensional lattice polytope, and let $\mathscr{L} = \mathscr{O}_X(D_P)$. Then $\chi^T(\mathscr{L}) = \sum_{m\in P\cap M} e^{\mathsf{s}(m)} \in \widehat{\Lambda} = \widehat{\Lambda}_0 \times \widehat{\Lambda}_2 \times \cdots$.

(a) Show that the constant term of $\chi^T(\mathscr{L})$ is $|P \cap M|$.

(b) Show that the degree 2 term is $\sum_{m\in P\cap M} \mathsf{s}(m) = \mathsf{s}\big(\sum_{m\in P\cap M} m\big)$. It is interesting to note that $\frac{1}{|P\cap M|}\sum_{m\in P\cap M} m$ is the "average" lattice point of P.

In a similar way, the higher terms of $\chi^T(\mathscr{L})$ can be interpreted as various moments of the lattice points of P. The power of equivariant Riemann-Roch is that when P is smooth, all of these terms can be computed using equivariant intersection theory on $X = X_P$.

§13.4. The Volume Polynomial

In this section, we discuss the relation between cup product and the volume of a polytope, leading up to the *volume polynomial*. This will give a new description of the cohomology ring of a complete simplicial toric variety.

Cup Product and Normalized Volume. By Proposition 10.5.6, the Euclidean area of a lattice polygon P in the plane is given by the intersection formula

$$2\,\mathrm{Area}(P) = D\cdot D,$$

where D is a torus-invariant nef divisor on a smooth complete toric surface X_Σ such that $P = P_D$.

Recall from §9.5 that a polytope $P \subseteq M_\mathbb{R} \simeq \mathbb{R}^n$ has Euclidean volume $\mathrm{vol}(P)$, where a fundamental parallelotope determined by a basis of M has volume 1. Then we define the *normalized volume* of P by

$$\mathrm{Vol}(P) = n!\,\mathrm{vol}(P).$$

Thus the standard n-simplex $\Delta_n \subseteq \mathbb{R}^n$ has normalized volume $\mathrm{Vol}(\Delta_n) = 1$.

Theorem 13.4.1. *Let D be a torus-invariant divisor on a smooth complete toric variety X_Σ of dimension n and set $P = P_D \subseteq M_\mathbb{R}$.*

(a) *If D is very ample and $X_\Sigma \hookrightarrow \mathbb{P}^s$, $s+1 = \dim H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) = |P\cap M|$, is the projective embedding given by global sections of $\mathscr{O}_{X_\Sigma}(D)$, then*

$$\deg(X_\Sigma \subseteq \mathbb{P}^s) = \int_{X_\Sigma} [D]^n = \mathrm{Vol}(P).$$

(b) *If D is nef, then*

$$\int_{X_\Sigma} [D]^n = \mathrm{Vol}(P).$$

Proof. The degree of $X_\Sigma \subseteq \mathbb{P}^s$ is the number of points in the intersection $X_\Sigma \cap L$, where $L \subseteq \mathbb{P}^s$ is a generic linear subspace of codimension n. Bertini's theorem (see [**131**, II.8.18]) and (12.5.6) imply that if $V \subseteq \mathbb{P}^s$ is smooth and $H \subseteq \mathbb{P}^s$ is a sufficiently generic hyperplane, then in the Chow ring $A^\bullet(\mathbb{P}^s)$, we have $[H]\cdot[V] = [H\cap V]$, where $H\cap V$ is smooth. Hence in $H^\bullet(\mathbb{P}^s,\mathbb{Z})$ we have

$$\deg(X_\Sigma \subseteq \mathbb{P}^s) = \int_{\mathbb{P}^s} [H_1]\cdots[H_n]\cdot[X_\Sigma]$$

when $H_1,\ldots,H_n$ are sufficiently generic. However, if $i : X_\Sigma \hookrightarrow \mathbb{P}^s$ is the inclusion, the Gysin map $i_! : H^k(X_\Sigma,\mathbb{Z}) \to H^{k+2n}(\mathbb{P}^s,\mathbb{Z})$ satisfies $i_! i^*\alpha = \alpha \smile [X_\Sigma]$ for all $\alpha \in H^\bullet(\mathbb{P}^s,\mathbb{Z})$. Setting $\alpha = [H_1]\cdots[H_n]$, we get

$$\int_{\mathbb{P}^s} [H_1]\cdots[H_n]\cdot[X_\Sigma] = \int_{\mathbb{P}^s} \alpha \smile [X_\Sigma] = \int_{\mathbb{P}^s} i_! i^*\alpha = \int_{X_\Sigma} i^*\alpha = \int_{X_\Sigma} [D]^n$$

by properties of Gysin maps discussed in Theorem 13.A.6. Thus the degree is the intersection number $\int_{X_\Sigma} [D]^n$.

To bring volume into the picture, we note that the degree is also $n!$ times the leading coefficient of the Hilbert polynomial of the homogeneous coordinate ring $\mathbb{C}[X_\Sigma] = \mathbb{C}[x_0,\ldots,x_s]/\mathbf{I}(X_\Sigma)$ of $X_\Sigma \subseteq \mathbb{P}^s$. We proved in Proposition 9.4.3 that the Hilbert polynomial is the Ehrhart polynomial of P, and by the discussion following

Proposition 9.4.3, we see that the leading coefficient of the Ehrhart polynomial is $\mathrm{Vol}(P)/n!$. Hence the degree is $\mathrm{Vol}(P)$.

For part (b), the nef divisor D need not give a projective embedding. In fact, X_Σ need not be projective. So we instead use HRR. Let $\ell \geq 1$ be an integer. Then

$$|(\ell P)\cap M| = \dim H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(\ell D)) = \chi(\mathscr{O}_{X_\Sigma}(\ell D)),$$

where the first equality comes from Example 4.3.7, and the second follows from Demazure vanishing. By HRR,

$$|(\ell P)\cap M| = \chi(\mathscr{O}_{X_\Sigma}(\ell D)) = \int_{X_\Sigma} \mathrm{ch}(\mathscr{O}_{X_\Sigma}(\ell D))\,\mathrm{Td}(X_\Sigma).$$

In $\mathrm{ch}(\mathscr{O}_{X_\Sigma}(\ell D))$, the term containing the highest power of ℓ is $[\ell D]^n/n! = \ell^n[D]^n/n!$. Since the degree zero term of $\mathrm{Td}(X_\Sigma)$ is 1, we obtain

$$\int_{X_\Sigma} \mathrm{ch}(\mathscr{O}_{X_\Sigma}(\ell D))\,\mathrm{Td}(X_\Sigma) = \frac{\ell^n}{n!}\int_{X_\Sigma}[D]^n + \text{lower degree terms in } \ell.$$

Therefore, by (9.4.4), in the limit as $\ell \to \infty$, the normalized volume is given by

$$\mathrm{Vol}(P) = n! \lim_{\ell\to\infty} \frac{|(\ell P)\cap M|}{\ell^n} = \lim_{\ell\to\infty}\frac{n!}{\ell^n}\int_{X_\Sigma}\mathrm{ch}(\mathscr{O}_{X_\Sigma}(\ell D))\,\mathrm{Td}(X_\Sigma) = \int_{X_\Sigma}[D]^n,$$

which is what we wanted to show. □

If P is not full dimensional, then $\mathrm{Vol}(P) = 0$. As in the discussion following Lemma 9.3.9, D is a *big* divisor if $\mathrm{Vol}(P) > 0$. A different proof that the degree equals in the volume in the very ample case can be found in [**113**, Thm. 6.2.3].

We next extend part (b) of Theorem 13.4.1 to arbitrary complete toric varieties. We will need the following fact.

Lemma 13.4.2. *Assume that $f : X \to Y$ is a birational morphism between complete irreducible varieties of dimension n. If D is a Cartier divisor on Y and $D' = f^*D$, then*

$$\int_X [D']^n = \int_Y [D]^n.$$

Proof. Set $\alpha = [D]^n$ and let $p : Y \to \{\mathrm{pt}\}$ be the constant map. Example 13.A.3 explains that $\int_Y \alpha = p_*(\alpha \frown [Y])$, where $[Y] \in H_{2n}(Y,\mathbb{Q})$ is the fundamental class and $p_* : H_\bullet(Y,\mathbb{Q}) \to H_\bullet(\{\mathrm{pt}\},\mathbb{Q}) = \mathbb{Q}$. Then $p' = p\circ f$ is the constant map for X, so that

$$\begin{aligned}\int_X [D']^n = p'_*(f^*(\alpha)\frown[X]) &= p_* f_*(f^*(\alpha)\frown[X])\\ &= p_*(\alpha\frown f_*[X]) = p_*(\alpha\frown[Y]) = \int_Y [D]^n.\end{aligned}$$

Here, the second line uses Proposition 13.A.2, and in the third line, $f_*[X] = [Y]$ follows from Proposition 13.A.5. □

We can now prove a more general version of Theorem 13.4.1.

Theorem 13.4.3. *Let D be a torus-invariant nef Cartier divisor on a complete toric variety X_Σ of dimension n. Then*

$$\int_{X_\Sigma}[D]^n = \mathrm{Vol}(P_D).$$

Proof. Let $\phi : X_{\Sigma'} \to X_\Sigma$ be a toric resolution of singularities and set $D' = \phi^*D$. The pullback $D' = \phi^*D$ has the same support function as D by Proposition 6.2.7. Then D' and D have the same polytope by Lemma 6.1.6, i.e., $P_{D'} = P_D$. Thus

$$\mathrm{Vol}(P_D) = \mathrm{Vol}(P_{D'}) = \int_{X_{\Sigma'}}[D']^n = \int_{X_\Sigma}[D]^n,$$

where the second equality uses Theorem 13.4.1 and the third equality follows from Lemma 13.4.2. □

The Cartier hypothesis can also be relaxed slightly; Theorem 13.4.3 is also true for any nef $\mathbb{Q}$-Cartier divisor D (Exercise 13.4.1). Here is an example illustrating the theorem.

Example 13.4.4. See Figure 11 in Example 2.4.6 and consider the triangle $P = \mathrm{Conv}(-2e_1+e_2, -2e_1-e_2, 2e_1-e_2)$ in $M_\mathbb{R} = \mathbb{R}^2$. The toric variety X_P is the weighted projective plane $\mathbb{P}(1,1,2)$. Label the ray generators of the normal fan of P as $u_1 = e_1, u_2 = e_2, u_3 = -e_1-2e_2$, and let $\rho_i = \mathrm{Cone}(u_i)$, $D_i = D_{\rho_i}$. Then it is easy to see that $P = P_D$ for the divisor $D = 2D_1 + D_2$. On $X_P \simeq \mathbb{P}(1,1,2)$, we have

$$\int_{X_P}[D]^2 = \mathrm{Vol}(P) = 8$$

by Theorem 13.4.3. On the other hand, $\int_{X_P}[D]^2 = D^2$. Since $D_1^2 = \frac{1}{2}$ (D_1 is not Cartier), $D_1 \cdot D_2 = 1$, and $D_2^2 = 2$, one computes directly that $D^2 = 8$.

The proof of Theorem 13.4.3 shows that we can also do this computation using a resolution of singularities. If we refine the normal fan of X_P by introducing a new ray $\rho_0 = \mathrm{Cone}(-e_2)$, we get a smooth fan Σ'. Indeed, the resulting surface isomorphic to the Hirzebruch surface $\mathscr{H}_2$. Let $\phi : \mathscr{H}_2 \to X_P$ be the corresponding morphism. If we let $D' = \phi^*D = D_0 + 2D_1 + D_2$, then one can verify directly that $P_{D'} = P$, so that

$$\int_{\mathscr{H}_2}[D']^2 = \mathrm{Vol}(P) = 8$$

by Theorem 13.4.1. On $\mathscr{H}_2$, $D_1 \sim D_3$, $D_2 \sim D_0 + 2D_3$, so that $D' \sim 2D_0 + 4D_3$. Since $D_0^2 = -2$, $D_3^2 = 0$, and $D_0 \cdot D_3 = 1$, one computes that $(\phi^*D)^2 = 8$. You will check these assertions in Exercise 13.4.2. ◊

The Volume Polynomial. Let X_Σ be a complete simplicial toric variety of dimension n. The divisor $\sum_\rho t_\rho D_\rho$ is $\mathbb{Q}$-Cartier when the t_ρ are all rational and hence has a cohomology class in $H^2(X_\Sigma,\mathbb{Q})$. The resulting integral

$$\int_{X_\Sigma} \big[\textstyle\sum_\rho t_\rho D_\rho\big]^n \tag{13.4.1}$$

is a homogeneous polynomial of degree n in t_ρ, $\rho \in \Sigma(1)$. When $D = \sum_\rho t_\rho D_\rho$ is nef and Cartier, the integral equals $\mathrm{Vol}(P_D)$ by Theorem 13.4.3. For this reason, we call (13.4.1) the *volume polynomial* of X_Σ.

Example 13.4.5. Let $D_0,\dots,D_n$ be the torus-invariant prime divisors on $\mathbb{P}^n$. Then the volume polynomial is

$$\int_{\mathbb{P}^n} [t_0 D_0 + \cdots + t_n D_n]^n = (t_0 + \cdots + t_n)^n$$

since $D_i \sim D_0$ for $1 \le i \le n$ and $\int_{\mathbb{P}^n}[D_0]^n = 1$. Note that $D = t_0 D_0 + \cdots + t_n D_n$ is nef if and only if $t_0 + \cdots + t_n \ge 0$. In this case, the above computation and Theorem 13.4.1 imply that P_D has normalized volume $(t_0 + \cdots + t_n)^n$. $\Diamond$

Example 13.4.6. Let Σ be the usual fan for $(\mathbb{P}^1)^n$. Let the divisor D_{2i-1} correspond to the ray $\mathrm{Cone}(e_i)$ and D_{2i} correspond to $\mathrm{Cone}(-e_i)$ for $1 \le i \le n$. Then the polytope of a general divisor $D = t_1 D_1 + \cdots + t_{2n} D_{2n}$ with $t_1,\dots,t_{2n} \ge 0$ is the rectangular solid

$$P_D = [-t_1, t_2] \times \cdots \times [-t_{2n-1}, t_{2n}]$$

in $M_\mathbb{R} = \mathbb{R}^n$. The Euclidean volume is $(t_1 + t_2)\cdots(t_{2n-1} + t_{2n})$, which gives the normalized volume

$$\mathrm{Vol}(P_D) = n!\,(t_1 + t_2)\cdots(t_{2n-1} + t_{2n}). \tag{13.4.2}$$

The reason for the $n!$ becomes clear when we compute the volume polynomial in $H^\bullet((\mathbb{P}^1)^n,\mathbb{Q})$. By the calculations in §12.3 and §12.4,

$$H^\bullet((\mathbb{P}^1)^n,\mathbb{Q}) \simeq \mathbb{Q}[x_1,\dots,x_n]/\langle x_1^2,\dots,x_n^2\rangle,$$

with $x_i = [D_{2i-1}]$, $1 \le i \le n$. The top-dimensional component of the cohomology ring is $H^{2n}((\mathbb{P}^1)^n,\mathbb{Q})$, generated by the product $x_1\cdots x_n$. The cohomology class of D is $[D] = (t_1+t_2)x_1 + \cdots + (t_{2n-1}+t_{2n})x_n \in H^2((\mathbb{P}^1)^n,\mathbb{Q})$. Then

$$((t_1+t_2)x_1 + \cdots + (t_{2n-1}+t_{2n})x_n)^n = n!\,(t_1+t_2)\cdots(t_{2n-1}+t_{2n})\,x_1\cdots x_n,$$

since there are $n!$ ways to order the factors in $x_1\cdots x_n$, and all other terms in the nth power are zero. Since $\int_{(\mathbb{P}^1)^n} x_1\cdots x_n = 1$, we get (13.4.2). $\Diamond$

In general, as in these examples, if we start from a complete simplicial fan Σ with $|\Sigma(1)| = r$ and the divisors $D_1,\dots,D_r$ on X_Σ corresponding to $\rho_1,\dots,\rho_r$ in $\Sigma(1)$, then the volume polynomial (13.4.1) becomes

$$V(t_1,\dots,t_r) = \int_{X_\Sigma} [t_1 D_1 + \cdots + t_r D_r]^n.$$

The classes $[D_i]$ are not linearly independent in $H^2(X_\Sigma,\mathbb{Q})$. There is also a more efficient *reduced volume polynomial*, which is constructed using divisors D_i whose classes give a basis of $H^2(X_\Sigma,\mathbb{Q}) = \mathrm{Pic}(X_\Sigma)_\mathbb{Q}$. In Example 13.4.5, for instance, $\mathrm{Pic}(\mathbb{P}^n)_\mathbb{Q}$ has dimension 1, and the reduced volume polynomial is just $\overline{V}(t) = t^n$. In Example 13.4.6, on the other hand, $\mathrm{Pic}((\mathbb{P}^1)^n)_\mathbb{Q}$ has dimension n and the reduced volume polynomial is $\overline{V}(t_1,\ldots,t_n) = n!\,t_1\cdots t_n$.

The Volume Polynomial and the Cohomology Ring. An alternative description of the cohomology ring $H^\bullet(X_\Sigma,\mathbb{Q})$ for complete simplicial toric varieties was proved by Khovanskii and Pukhlikov in [**174**]. We will sketch the ideas following the presentation in [**170**] with some modifications.

The relation between the volume polynomial and the cohomology ring will come from the following algebraic fact.

Lemma 13.4.7. *Let $A = \bigoplus_{i=0}^m A_i$ be a finite-dimensional commutative graded algebra over $\mathbb{Q}$ satisfying:*

(a) $A_0 \simeq A_m \simeq \mathbb{Q}$.

(b) *A is generated by A_1 as a $\mathbb{Q}$-algebra.*

(c) *For all $i = 0,\ldots,m$, the bilinear map*

$$\begin{aligned} A_i \times A_{m-i} &\longrightarrow A_m \simeq \mathbb{Q}\\ (u,v) &\longmapsto uv \end{aligned}$$

is nondegenerate.

Let $s_1,\ldots,s_r$ span A_1 and define

$$P(t_1,\ldots,t_r) = (t_1s_1+\cdots+t_rs_r)^m \in A_m \simeq \mathbb{Q},$$

which we regard as a polynomial in $\mathbb{Q}[t_1,\ldots,t_r]$. Finally, define

$$I = \Big\{ f(x_1,\ldots,x_r) \in \mathbb{Q}[x_1,\ldots,x_r] \;\Big|\; f\Big(\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_r}\Big)P = 0\Big\}.$$

Then the map $x_i \mapsto s_i$ induces an isomorphism of graded $\mathbb{Q}$-algebras

$$\mathbb{Q}[x_1,\ldots,x_r]/I \simeq A.$$

Here $f\big(\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_r}\big)$ is the differential operator obtained from f by replacing each x_i with $\frac{\partial}{\partial t_i}$. We will postpone the proof of the lemma until we see how it applies to $H^\bullet(X_\Sigma,\mathbb{Q})$.

The rings satisfying the hypotheses in Lemma 13.4.7 are a special class of finite-dimensional graded *Gorenstein rings*. If X_Σ is complete and simplicial, then hypothesis (b) is satisfied by the cohomology ring $H^\bullet(X_\Sigma,\mathbb{Q})$ by Theorem 12.4.1. It is easy to see that hypothesis (a) holds from the cohomology spectral sequence in §12.3. Then hypothesis (c) follows from Poincaré duality on X_Σ. As a result, the cohomology rings of complete simplicial toric varieties are Gorenstein rings of this type. Here is the Khovanskii-Pukhlikov presentation of the cohomology ring.

Theorem 13.4.8. *Let X_Σ be a complete simplicial toric variety. Let $r = |\Sigma(1)|$ and $s = \dim H^2(X_\Sigma,\mathbb{Q}) = \dim \mathrm{Pic}(X_\Sigma)_\mathbb{Q}$. Then:*

(a) *Let $V(t_1,\dots,t_r)$ be the volume polynomial of X_Σ. Then there is an isomorphism of $\mathbb{Q}$-algebras*

$$H^\bullet(X_\Sigma,\mathbb{Q}) \simeq \mathbb{Q}[x_1,\dots,x_r]/I,$$

where $I = \big\{ f(x_1,\dots,x_r) \in \mathbb{Q}[x_1,\dots,x_r] \mid f\big(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_r}\big)V(t_1,\dots,t_r) = 0\big\}$. In addition, I equals the ideal $\mathscr{I}+\mathcal{J}$ from Theorem 12.4.1.

(b) *Let $\overline{V}(t_1,\dots,t_s)$ be a reduced volume polynomial for X_Σ. Then there is an isomorphism of $\mathbb{Q}$-algebras,*

$$H^\bullet(X_\Sigma,\mathbb{Q}) \simeq \mathbb{Q}[x_1,\dots,x_s]/J,$$

where $J = \big\{ f(x_1,\dots,x_s) \in Q[x_1,\dots,x_s] \mid f\big(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_s}\big)\overline{V}(t_1,\dots,t_s) = 0\big\}$.

Proof. The isomorphisms in parts (a) and (b) follow from Lemma 13.4.7, and the equality $I = \mathscr{I}+\mathcal{J}$ then follows from Theorem 12.4.1 (Exercise 13.4.3). □

Here are some examples.

Example 13.4.9. For instance, if $X_\Sigma = \mathbb{P}^n$, then from Example 13.4.5

$$V(t_0,\dots,t_n) = (t_0+\cdots+t_n)^n.$$

It is easy to check directly that the ideal I from Theorem 13.4.8 is

$$I = \langle x_1-x_0,\dots,x_n-x_0, x_0\cdots x_n\rangle = \mathscr{I}+\mathcal{J}$$

(Exercise 13.4.4). Using the reduced volume polynomial $\overline{V}(t) = t^n$, one computes that $J = \langle x^{n+1}\rangle$. Hence we get isomorphisms

$$\mathbb{Q}[x_0,\dots,x_n]/I \simeq \mathbb{Q}[x]/\langle x^{n+1}\rangle \simeq H^\bullet(\mathbb{P}^n,\mathbb{Q}).$$

Both isomorphisms were known previously; the surprise is seeing how they arise from volume polynomials. ◊

Example 13.4.10. The volume polynomial computed in Example 13.4.6 is

$$V(t_1,\dots,t_{2n}) = n!\,(t_1+t_2)\cdots(t_{2n-1}+t_{2n}).$$

Here, one computes that the ideal I from Theorem 13.4.8 is

$$I = \langle x_1-x_2,\dots,x_{2n-1}-x_{2n}, x_1x_2,\dots,x_{2n-1}x_{2n}\rangle = \mathscr{I}+\mathcal{J}$$

(Exercise 13.4.4). Furthermore, the reduced volume polynomial $\overline{V}(t_1,\dots,t_n) = n!\,t_1\cdots t_n$ gives the ideal

$$J = \langle x_1^2,\dots,x_n^2\rangle.$$

This gives the presentation of the cohomology ring used in Example 13.4.6. ◊

We now turn to the proof of Lemma 13.4.7.

Proof of Lemma 13.4.7. It is easy to prove that

$$I = \Big\{ f(x_1,\dots,x_r) \;\Big|\; f\Big(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_r}\Big)P = 0 \Big\}$$

is an ideal in $\mathbb{Q}[x_1,\dots,x_r]$.

Using hypothesis (a) and spanning set $\{s_1,\dots,s_r\}$ for A_1, we have a surjection

$$\begin{aligned} \phi : \mathbb{Q}[x_1,\dots,x_r] &\longrightarrow A, \\ f &\longmapsto f(s_1,\dots,s_r), \end{aligned}$$

and we need to show $\ker(\phi) = I$. Since $P(t_1,\dots,t_r)$ is a homogeneous polynomial of degree m, I will be generated by homogeneous polynomials, and I will contain all homogeneous polynomials of degree strictly larger than m. Note also that we have the expansion

$$P(t_1,\dots,t_n) = (t_1s_1+\cdots+t_rs_r)^m = \sum_{\alpha_1+\cdots+\alpha_r=m} \frac{m!}{\alpha_1!\cdots\alpha_r!}\, t_1^{\alpha_1}\cdots t_r^{\alpha_r} s_1^{\alpha_1}\cdots s_r^{\alpha_r}.$$

If $f(x_1,\dots,x_r)$ is homogeneous of degree m, then a direct calculation shows that

$$f\Big(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_r}\Big)P(t_1,\dots,t_r) = m!\, f(s_1,\dots,s_r)$$

(Exercise 13.4.5). Hence

$$f \in I \iff f(s_1,\dots,s_r) = 0 \iff f \in \ker(\phi)$$

in this case.

Now assume f is homogeneous of degree $p < m$. Suppose that $f \notin \ker(\phi)$, so $f(s_1,\dots,s_r) \neq 0$. By hypothesis (c) in the Proposition, there is a homogeneous polynomial g of degree $m-p$ such that $g(s_1,\dots,s_r)f(s_1,\dots,s_r) \neq 0 \in A_m$. Thus, by the first part of the proof, $gf \notin I$ so $f \notin I$. Thus $f \in I$ implies $f \in \ker(\phi)$. Finally, suppose $f \in \ker(\phi)$, so $f(s_1,\dots,s_r) = 0$. You will show in Exercise 13.4.5 that

$$f\Big(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_r}\Big)P(t_1,\dots,t_r) = f(s_1,\dots,s_r)\,h(t_1,\dots,t_r,s_1,\dots,s_r),$$

where $h(t_1,\dots,t_r,s_1,\dots,s_r)$ is given by

$$\sum_{\beta_1+\cdots+\beta_r=m-p} \frac{m!}{\beta_1!\cdots\beta_r!}\,(t_1s_1)^{\beta_1}\cdots(t_rs_r)^{\beta_r}.$$

Since we assume $f(s_1,\dots,s_r) = 0$, this shows that $f \in I$. $\square$

In [**170**], Kaveh shows that the results of this section can be generalized to *spherical varieties*, which are varieties with an action of a reductive algebraic group G such that some Borel subgroup of G has a dense orbit.

Exercises for §13.4.

13.4.1. Show that Theorem 13.4.3 is also true if the divisor D is nef and $\mathbb{Q}$-Cartier.

13.4.2. Check the assertions made in Example 13.4.4.

13.4.3. Complete the proof of part (a) of Theorem 13.4.8 by showing that if X_Σ is a complete simplicial toric variety, then the ideal I from the theorem is the ideal $\mathscr{I} + \mathcal{J}$ from Theorem 12.4.1. Hint: Think about the kernel of the homomorphism $\mathbb{Q}[x_1,\dots,x_r] \to H^\bullet(X_\Sigma,\mathbb{Q})$ that takes x_i to $[D_i]$.

13.4.4. In this exercise, you will check some claims made in Examples 13.4.9 and 13.4.10.

(a) In Example 13.4.9, verify that
$$I = \langle x_1 - x_0, \dots, x_n - x_0, x_0 \cdots x_n \rangle.$$
Hint: One inclusion is clear. For the other, show that the ideal on the right-hand side is $\langle x_1 - x_0, \dots, x_n - x_0, x_0^{n+1} \rangle$. Also note that $f \in \mathbb{Q}[x_0,\dots,x_n]$ can be written as $f = \sum_{i=1}^n (x_i - x_0) A_i(x_0,\dots,x_n) + g(x_0)$. Remember that I is homogeneous.

(b) In Example 13.4.10, verify that I is as claimed.

(c) In both examples, compute the ideal J coming from the reduced volume polynomial.

13.4.5. In this exercise, you will verify some details of the proof of Lemma 13.4.7.

(a) Show that if $f(x_1,\dots,x_r)$ is homogeneous of degree m, then
$$f\left(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_r}\right) P(t_1,\dots,t_r) = m!\, f(s_1,\dots,s_r).$$

(b) Show that if $f(x_1,\dots,x_r)$ is homogeneous of degree $p < m$, then
$$f\left(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_r}\right) P(t_1,\dots,t_r) = f(s_1,\dots,s_r) h(t_1,\dots,t_r,s_1,\dots,s_r),$$
where $h(t_1,\dots,t_r,s_1,\dots,s_r)$ is given by
$$\sum_{\beta_1+\cdots+\beta_r = m-p} \frac{m!}{\beta_1! \cdots \beta_r!} (t_1 s_1)^{\beta_1} \cdots (t_r s_r)^{\beta_r}.$$

13.4.6. Consider the Hirzebruch surface $\mathscr{H}_r$.

(a) Determine the volume polynomial and verify the isomorphism from Theorem 13.4.8 in this case.

(b) In the notation of Example 12.4.3, compute the reduced volume polynomial $\overline{V}(x_3,x_4)$ using the basis of $H^2(\mathscr{H}_r,\mathbb{Q}) = \mathrm{Pic}(\mathscr{H}_r)_\mathbb{Q}$ given by D_3, D_4. Then verify that this gives the presentation of the cohomology ring constructed in Example 12.4.3.

13.4.7. Let Σ be a complete simplicial fan in $N_\mathbb{R} \simeq \mathbb{R}^n$. Following [**170**], we present another way to think about the reduced volume polynomial when X_Σ is projective.

(a) Let S_Σ be the set of all lattice polytopes with normal fan Σ, modulo the equivalence relation $P \sim P'$ if P' is a translate of P. Show that S_Σ forms a semigroup under the operation of Minkowski sum of polytopes. Hint: Use Proposition 6.2.13.

(b) Let V_Σ be the vector space generated by S_Σ over $\mathbb{Q}$. Show that there is a natural isomorphism
$$V_\Sigma \simeq H^2(X_\Sigma,\mathbb{Q}).$$

(c) The normalized volume function $\mathrm{Vol} : S_\Sigma \to \mathbb{Q}$ is homogeneous of degree n. Show that there is $\overline{V} \in \mathrm{Sym}^n(V_\Sigma^*)$ such that $\overline{V} : V_\Sigma \to \mathbb{Q}$ extends Vol, i.e., $\overline{V}|_{S_\Sigma} = \mathrm{Vol}$. Hint: You may find it useful to first do part (d) of the exercise.

(d) Let $s = \dim H^2(X_\Sigma, \mathbb{Q})$ and consider a reduced volume polynomial $\overline{V}(t_1, \ldots, t_s)$ for X_Σ, which we construct using prime toric divisors D_i that give a basis of $H^2(X_\Sigma, \mathbb{Q})$. Show that this basis gives an isomorphism $V_\Sigma \simeq \mathbb{Q}^s$ that takes $\overline{V}$ to $\overline{V}(t_1, \ldots, t_s)$.

§13.5. The Khovanskii-Pukhlikov Theorem

Given a positive integer n and a function f on the interval $[0,n]$, the Euler-Maclaurin summation formula relates the sum of the values of f at the integers $0, \ldots, n$ to the integral $\int_0^n f(x)\,dx$. In [**174**], Khovanskii and Pukhlikov use Brion's equalities and Todd differential operators to give an analogous formula relating the sum of the values of a suitable function f at the lattice points in a polytope to integrals over polytopes. Specializing to the constant function $f = 1$ gives a formula for the number of lattice points in a polytope. We will see that the Khovanskii-Pukhlikov theorem gives another proof of Hirzebruch-Riemann-Roch for a smooth projective toric variety.

The Euler-Maclaurin Formula. In §13.1, we saw that the Bernoulli numbers B_k give the power series

$$\frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k}. \tag{13.5.1}$$

This series converges for all $x \in \mathbb{C}$ with $|x| < 2\pi$. Bernoulli numbers also play a key role in the Euler-Maclaurin formula, which can be stated as follows.

Theorem 13.5.1 (Euler-Maclaurin Summation)**.** *If f is a C^∞ function on $[0,n]$, then*

$$\begin{aligned}\int_0^n f(x)\,dx &= \frac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2}f(n)\\ &\quad + \sum_{k=1}^{\ell} (-1)^k \frac{B_k}{(2k)!}\big(f^{(2k-1)}(n) - f^{(2k-1)}(0)\big) + R_{2\ell},\end{aligned}$$

where $R_{2\ell}$ is a remainder term. Furthermore, if there are positive constants C and $\lambda < 2\pi$ such that $|f^{(\ell)}(x)| \le C\lambda^\ell$ for all $x \in [0,n]$ and all $\ell \ge 1$, then

$$\begin{aligned}\int_0^n f(x)\,dx &= \frac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2}f(n)\\ &\quad + \sum_{k=1}^{\infty} (-1)^k \frac{B_k}{(2k)!}\big(f^{(2k-1)}(n) - f^{(2k-1)}(0)\big).\end{aligned}$$

□

The explicit form of the remainder is not important for us here; it and a proof of the summation formula are given in [**5**]. The Euler-Maclaurin formula is often used as a generalization of the trapezoidal rule for approximating the integral of f. We should mention that there are a number of different competing conventions for writing Bernoulli numbers, so Euler-Maclaurin formulas in other sources may look different from Theorem 13.5.1.

Todd Operators. We will also use formal Todd differential operators obtained from the series expansion in (13.5.1). The definition is given by

$$\mathrm{Todd}(x) = 1 + \frac{1}{2}\frac{\partial}{\partial x} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} \frac{\partial^{2k}}{\partial x^{2k}}. \tag{13.5.2}$$

For notational convenience, we will write Todd operators as

$$\mathrm{Todd}(x) = \frac{\partial/\partial x}{1 - e^{-\partial/\partial x}},$$

but an expression of this form should always be interpreted as the series (13.5.2).

A key property of the Todd operator for us will be the equation

$$\mathrm{Todd}(x)(e^{xz}) = \frac{ze^{xz}}{1-e^{-z}}, \tag{13.5.3}$$

which follows easily by a direct calculation:

$$\begin{aligned} \mathrm{Todd}(x)(e^{xz}) &= e^{xz} + \frac{1}{2} ze^{xz} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k} e^{xz} \\ &= e^{xz}\Big(1 + \frac{1}{2} z + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k}\Big) \\ &= e^{xz} \cdot \frac{z}{1-e^{-z}}. \end{aligned}$$

This computation is valid for all $z \in \mathbb{C}$ satisfying $|z| < 2\pi$.

A Consequence of Brion's Equalities. Let P be a simple lattice polytope in $M_{\mathbb{R}}$. Recall from Brion's equalities (Corollary 13.2.10) that

$$\sum_{m \in P \cap M} \chi^m = \sum_{\mathsf{v}\ \text{vertex}} \chi^{\mathsf{v}} \frac{\sum_{m \in P_{\mathsf{v}} \cap M} \chi^m}{\prod_{i=1}^n (1 - \chi^{m_{\mathsf{v},i}})}. \tag{13.5.4}$$

Here, the $m_{\mathsf{v},i}$ are the minimal generators of the cone $C_{\mathsf{v}} = \mathrm{Cone}(P \cap M - \mathsf{v})$, and

$$P_{\mathsf{v}} = \{\textstyle\sum_{i=1}^n \lambda_i m_{\mathsf{v},i} \mid 0 \le \lambda_i < 1\}$$

is the fundamental parallelotope of the simplicial cone C_{v}. Equation (13.5.4) is an equality of two elements of the localization $\mathbb{Z}[M]_{\overline{S}}$. In §13.3, we mapped equations of this form to the localized ring $\widehat{\Lambda}_S$ that arises in equivariant cohomology via the map $\chi^m \mapsto e^{\mathsf{s}(m)}$, and we used the resulting equations in our proof of the equivariant Riemann-Roch theorem.

In this section, on the other hand, we will consider a different consequence of Brion's equalities. Take a point $z = \sum_i z_i u_i \in N_{\mathbb{C}} = N \otimes_{\mathbb{Z}} \mathbb{C}$, where $z_i \in \mathbb{C}$ and $u_i \in N$. Each u_i gives the one-parameter subgroup $\lambda^{u_i} : \mathbb{C}^* \to T_N$. The point $p_z = \prod_i \lambda^{u_i}(e^{z_i}) \in T_N$ depends only on z. Here, e is the base of the natural logarithm. Then observe that for any $m \in M$, we have

$$\chi^m(p_z) = e^{\langle m,z\rangle} \in \mathbb{C}^*.$$

Choose $z \in N_{\mathbb{C}}$ so that $\langle m_{\mathsf{v},i}, z\rangle \notin 2\pi i\mathbb{Z}$ for all $m_{\mathsf{v},i}$ in (13.5.4). Then evaluating (13.5.4) at the point p_z maps χ^m to $e^{\langle m,z\rangle}$ and hence gives the relation

$$\sum_{m\in P\cap M} e^{\langle m,z\rangle} = \sum_{\mathsf{v}\text{ vertex}} e^{\langle \mathsf{v},z\rangle} \frac{\sum_{m\in P_{\mathsf{v}}\cap M} e^{\langle m,z\rangle}}{\prod_{i=1}^n (1-e^{\langle m_{\mathsf{v},i},z\rangle})} \tag{13.5.5}$$

since none of the denominators vanish.

This shows the power of the what we did in §13.2. The version of Brion's equalities proved there yields different equalities depending on how we evaluate the χ^m, giving results that can be used in very different contexts.

From Discrete to Continuous. Brion's equalities in the form (13.5.5) deal with discrete sums over sets of lattice points in polytopes. Our next results deal with continuous analogs of these sums, namely integrals over polytopes, and their relation with discrete sums. For the application to the Khovanskii-Pukhlikov theorem, we need to consider a class of polytopes more general than lattice polytopes.

Theorem 13.5.2. *Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional simple polytope, and assume the rays of the vertex cones C_{v} are spanned by primitive vectors $m_{\mathsf{v},i} \in M$ for all vertices v. If $z \in N_{\mathbb{C}}$ satisfies $\langle m_{\mathsf{v},i}, z\rangle \notin 2\pi i\mathbb{Z}$ for all $m_{\mathsf{v},i}$, then*

$$\int_P e^{\langle x,z\rangle}\, dx = (-1)^n \sum_{\mathsf{v}\text{ vertex}} \frac{e^{\langle \mathsf{v},z\rangle}\cdot \mathrm{mult}(C_{\mathsf{v}})}{\prod_{i=1}^n \langle m_{\mathsf{v},i}, z\rangle}.$$

Proof. The integral on the left is equal to the limit of a sum as follows. Scale the lattice M by a factor of $\frac{1}{k}$, and consider the points $m \in P \cap \frac{1}{k}M$. Approximate P by a collection of small cubes of side $\frac{1}{k}$ centered at all $m \in P \cap \frac{1}{k}M$. Then evaluate the exponential function $e^{\langle m,u\rangle}$ at these points and multiply by the volumes of the cubes to form a Riemann sum. In the limit as $k \to \infty$ we have

$$\int_P e^{\langle x,z\rangle}\, dx = \lim_{k\to\infty} \frac{1}{k^n} \sum_{m\in P\cap \frac{1}{k}M} e^{\langle m,z\rangle}\ . \tag{13.5.6}$$

The next step is to apply (13.5.5) to P and the scaled lattice $\frac{1}{k}M$. The original fundamental parallelotope $P_{\mathsf{v}} = P_{\mathsf{v},M}$ depends on M. For $\frac{1}{k}M$, the primitive ray generators are $\frac{1}{k}m_{\mathsf{v},i}$, and it follows that the $\frac{1}{k}M$-lattice points of the new fundamental parallelotope are given by

$$P_{\mathsf{v},\frac{1}{k}M} \cap \tfrac{1}{k}M = \tfrac{1}{k}(P_{\mathsf{v}} \cap M) \tag{13.5.7}$$

(Exercise 13.5.1). When we combine this with (13.5.5), we see that the right-hand side of (13.5.6) can be written as

$$\lim_{k\to\infty} \frac{1}{k^n} \sum_{\mathsf{v} \text{ vertex}} e^{\langle \mathsf{v},z\rangle} \frac{\sum_{m\in P_{\mathsf{v}}\cap M} e^{\langle m/k,z\rangle}}{\prod_{i=1}^n (1-e^{\langle m_{\mathsf{v},i}/k,z\rangle})} = \lim_{k\to\infty} \sum_{\mathsf{v} \text{ vertex}} \frac{e^{\langle \mathsf{v},z\rangle} \sum_{m\in P_{\mathsf{v}}\cap M} e^{\langle m/k,z\rangle}}{\prod_{i=1}^n k(1-e^{\langle m_{\mathsf{v},i}/k,z\rangle})}.$$

For the term in the sum for the vertex v, the limit of the numerator is

$$\lim_{k\to\infty} e^{\langle \mathsf{v},z\rangle} \sum_{m\in P_{\mathsf{v}}\cap M} e^{\langle m/k,z\rangle} = e^{\langle \mathsf{v},z\rangle} |P_{\mathsf{v}} \cap M| = e^{\langle \mathsf{v},z\rangle} \operatorname{mult}(C_{\mathsf{v}})$$

by Proposition 11.1.8. For the denominator $\prod_{i=1}^n k(1-e^{\langle m_{\mathsf{v},i}/k,z\rangle})$, note that

$$\lim_{k\to\infty} k(1-e^{\langle m_{\mathsf{v},i}/k,z\rangle}) = \lim_{k\to\infty} \frac{1-e^{\langle m_{\mathsf{v},i},z\rangle/k}}{1/k} = -\langle m_{\mathsf{v},i}, z\rangle$$

by L'Hôpital's Rule. Reassembling the different parts of the computation, we get the desired expression for the integral. □

The Khovanskii-Pukhlikov Theorem. Let P be a simple lattice polytope defined by inequalities of the form

$$\langle m, u_F\rangle + a_F \geq 0,$$

where u_F are the facet normals. Let $h = (h_F)_{F \text{ facet}}$ be a vector with real entries indexed by the facets F of P. Consider the polytope $P(h)$ with shifted facets defined by the inequalities

$$\langle m, u_F\rangle + a_F + h_F \geq 0. \tag{13.5.8}$$

Note that the shifting factors h_F for the different facets are independent. It is not difficult to see that if all entries of h are sufficiently small in absolute value, then $P(h)$ is still simple (Exercise 13.5.2). If some of the h_F are not rational, then the vertices might not be rational points. We want to consider what happens when we apply differential operators with respect to the h_F to integrals over the corresponding polytopes, and then set $h = 0$.

The differential operators alluded to above are the formal multivariate Todd differential operators defined using (13.5.2):

$$\operatorname{Todd}(h) = \prod_{F \text{ facet}} \frac{\partial/\partial h_F}{1-e^{-\partial/\partial h_F}}.$$

We are now ready to state the Khovanskii-Pukhlikov theorem for smooth lattice polytopes, the main result of this section.

Theorem 13.5.3. *Let P be a smooth lattice polytope. Then*

$$\mathrm{Todd}(h)\Big(\int_{P(h)} e^{\langle x,z\rangle}\,dx\Big)\Big|_{h=0} = \sum_{m\in P\cap M} e^{\langle m,z\rangle}$$

provided $z \in N_{\mathbb{C}}$ satisfies $0 < |\langle m_{v,i}, z\rangle| < 2\pi$ for all primitive ray generators $m_{v,i}$ of the vertex cones C_v of P.

Before we give the proof, we want to indicate exactly how this result relates to the Euler-Maclaurin formula from the start of the section.

Example 13.5.4. Let $P = [0,n] \subseteq \mathbb{R}$. The endpoints of P are its facets, so the facet normals are ± 1, and the facet equations of P are

$$\langle m,1\rangle = m \geq 0 \text{ and } \langle m,-1\rangle = -m \geq -n.$$

Hence, the shifted polytope $P(h)$ is the interval $P(h) = [-h_0, n+h_1]$ where h_0, h_1 are real-valued independent variables. Fix $z \in \mathbb{C}$ such that $0 < |z| < 2\pi$ and let $f(x) = e^{xz}$. In Exercise 13.5.3, you will show that for $h = (h_0, h_1)$,

$$\begin{aligned}\mathrm{Todd}(h)\Big(\int_{-h_0}^{n+h_1} f(x)\,dx\Big)\Big|_{h_0=h_1=0} &= \int_0^n f(x)\,dx + \frac{1}{2}(f(n)+f(0))\\ &\quad + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}\big(f^{(2k-1)}(n) - f^{(2k-1)}(0)\big).\end{aligned} \tag{13.5.9}$$

However, Theorem 13.5.3 implies that

$$\mathrm{Todd}(h)\Big(\int_{-h_0}^{n+h_1} e^{xz}\,dx\Big)\Big|_{h_0=h_1=0} = \sum_{m\in[0,n]\cap\mathbb{Z}} e^{mz}.$$

Since $f(x) = e^{xz}$, we can write this as

$$\mathrm{Todd}(h)\Big(\int_{-h_0}^{n+h_1} f(x)\,dx\Big)\Big|_{h_0=h_1=0} = \sum_{m\in[0,n]\cap\mathbb{Z}} f(m) = f(0)+\cdots+f(n).$$

Combining this with (13.5.9) gives the Euler-Maclaurin formula

$$\begin{aligned}\int_0^n f(x)\,dx &= \frac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2}f(n)\\ &\quad + \sum_{k=1}^{\infty}(-1)^{k}\frac{B_k}{(2k)!}\big(f^{(2k-1)}(n) - f^{(2k-1)}(0)\big)\end{aligned}$$

from Theorem 13.5.1 for $f(x) = e^{xz}$ when $0 < |z| < 2\pi$. $\Diamond$

From this point of view, we see that Theorem 13.5.3 is the "polytope" version of the Euler-Maclaurin formula for exponential functions of the form $f(x) = e^{\langle x,z\rangle}$. In [**174**], Khovanskii and Pukhlikov study more general functions (polynomials times exponentials) and prove a related result in this case. We should also mention that there is a large and growing literature on various forms of generalized Euler-Maclaurin formulas on polytopes. See the notes at the end of [**22**, Ch. 10] for references.

Now we turn to the proof of Theorem 13.5.3.

Proof of Theorem 13.5.3. Since P is smooth, each vertex cone C_{v} has multiplicity $\operatorname{mult}(C_{\mathsf{v}}) = 1$. Then Theorem 13.5.2 implies that

$$\int_P e^{\langle x,z\rangle}\,dx = (-1)^n \sum_{\mathsf{v}\text{ vertex}} \frac{e^{\langle \mathsf{v},z\rangle}}{\prod_{i=1}^n \langle m_{\mathsf{v},i}, z\rangle}.$$

Our first goal is to determine the corresponding equation with P replaced by $P(h)$. For this, we need to determine how the vertices of $P(h)$ are related to the vertices of P. First we note that since each facet of P shifts to a parallel facet in $P(h)$, the $m_{\mathsf{v},i}$ do not change, and hence Theorem 13.5.2 will apply to all $P(h)$ as well.

If v is the intersection of facets $F_1(\mathsf{v}),\ldots,F_n(\mathsf{v})$, then the ray generators $m_{\mathsf{v},i}$ of the vertex cone at v are the basis of M dual to the basis $u_{F_1(\mathsf{v})},\ldots,u_{F_n(\mathsf{v})}$ of N. It follows that

$$\langle \mathsf{v} - \textstyle\sum_{i=1}^n h_{F_i(\mathsf{v})} m_{\mathsf{v},i}, u_{F_j}(\mathsf{v})\rangle = -a_{F_j(\mathsf{v})} - h_{F_j(\mathsf{v})}.$$

So the vertex v of P shifts to $\mathsf{v} - \sum_{i=1}^n h_{F_i(\mathsf{v})} m_{\mathsf{v},i}$ in $P(h)$. For simplicity, we will write $h_{F_i(\mathsf{v})}$ as $h_i(\mathsf{v})$ in the following. Hence, applying Theorem 13.5.2 to $P(h)$,

$$\int_{P(h)} e^{\langle x,z\rangle}\,dx = (-1)^n \sum_{\mathsf{v}\text{ vertex}} e^{\langle \mathsf{v} - \sum_i h_i(\mathsf{v}) m_{\mathsf{v},i}, u\rangle} \cdot \frac{1}{\prod_{i=1}^n \langle m_{\mathsf{v},i}, z\rangle}. \tag{13.5.10}$$

Now, we apply the Todd operator to both sides of (13.5.10). Since each summand of the right hand side factors as

$$\frac{e^{\langle \mathsf{v},z\rangle}}{\prod_{i=1}^n \langle m_{\mathsf{v},i}, z\rangle} \cdot e^{\langle -\sum_i h_i(\mathsf{v}) m_{\mathsf{v},i}, z\rangle} = c_{\mathsf{v}} \cdot e^{\langle -\sum_i h_i(\mathsf{v}) m_{\mathsf{v},i}, z\rangle},$$

where c_{v} does not depend on h, we have $\operatorname{Todd}(h)\Big(\displaystyle\int_{P(h)} e^{\langle x,z\rangle}\,dx\Big) =$

$$\begin{aligned} &= (-1)^n \sum_{\mathsf{v}\text{ vertex}} c_{\mathsf{v}} \cdot \operatorname{Todd}(h)\big(e^{\langle -\sum_i h_i(\mathsf{v}) m_{\mathsf{v},i}, z\rangle}\big) \\ &= (-1)^n \sum_{\mathsf{v}\text{ vertex}} c_{\mathsf{v}} \cdot \Big(e^{\langle -\sum_i h_i(\mathsf{v}) m_{\mathsf{v},i}, z\rangle} \prod_{i=1}^n \frac{-\langle m_{\mathsf{v},i}, z\rangle}{(1 - e^{-\langle m_{\mathsf{v},i}, z\rangle})}\Big), \end{aligned} \tag{13.5.11}$$

where the second line follows by applying (13.5.3) for each h_F (Exercise 13.5.4). Setting $h_F = 0$ for all F and some algebraic simplification between c_{v} and the other factor in each term yields

$$\sum_{\mathsf{v}\text{ vertex}} \frac{e^{\langle \mathsf{v},z\rangle}}{\prod_{i=1}^n (1 - e^{-\langle m_{\mathsf{v},i}, z\rangle})},$$

which is exactly the sum on the right in (13.5.5). (Recall that we are assuming the vertex cones of P are smooth, so $P_\mathsf{v} \cap M = \{0\}$ for all v). This completes the proof of the theorem because the other side of the equality in (13.5.5) is the sum $\sum_{m \in P\cap M} e^{\langle m,z\rangle}$ and this is what we wanted. $\square$

Theorem 13.5.3 has the following nice consequence.

Corollary 13.5.5. *For a smooth lattice polytope P*

$$\mathrm{Todd}(h)\Big(\mathrm{vol}(P(h))\Big)\Big|_{h=0} = |P \cap M|,$$

where vol *is the usual Euclidean volume.*

Proof. Theorem 13.5.3 applies to the exponential function $f(x) = e^{\langle x,z\rangle}$ for $z \in N_\mathbb{C}$ satisfying $0 < |\langle m_{\mathsf{v},i}, z\rangle| < 2\pi$ for all $m_{\mathsf{v},i}$. Then, taking the limit as $z \to 0$ in the theorem implies that

$$\mathrm{Todd}(h)\Big(\int_{P(h)} e^{\langle x,0\rangle}\, dx\Big)\Big|_{h=0} = \sum_{m\in P\cap M} e^{\langle m,0\rangle},$$

which gives the desired result. Taking the limit inside the Todd operator and then inside the integral takes some care. We omit the details. $\square$

Relation to HRR. It turns out that Corollary 13.5.5 can be used to prove HRR for smooth projective toric varieties.

Theorem 13.5.6. *Let $\mathscr{L}$ be a line bundle on a smooth projective toric variety $X = X_\Sigma$. Then*

$$\chi(\mathscr{L}) = \int_X \mathrm{ch}(\mathscr{L})\,\mathrm{Td}(X).$$

Proof. We will sketch the proof, though several points in the discussion require justification that we will omit. We begin with a very ample divisor $D = \sum_\rho a_\rho D_\rho$ and set $P = P_D$. Then part (a) of Theorem 13.4.1 implies that

$$\int_X [D]^n = \mathrm{Vol}(P). \tag{13.5.12}$$

Unlike part (b) of Theorem 13.4.1, the proof of part (a) does not use HRR. Note also that this equation holds when $D = \sum_\rho a_\rho D_\rho$ is ample since replacing D with ℓD multiplies each side by ℓ^n.

The next step is to let $h = (h_\rho)_{\rho\in\Sigma(1)}$, where each h_ρ is real, and set $D(h) = \sum_\rho (a_\rho + h_\rho) D_\rho$. The cohomology class $[D(h)]$ lives in $H^\bullet(X,\mathbb{R})$, and the polytope associated to $D(h)$ is defined by

$$\langle m, u_\rho\rangle + a_\rho + h_\rho \geq 0, \quad \rho \in \Sigma(1),$$

which as above is denoted by $P(h)$. While $P(h)$ is no longer a lattice polytope, its normal fan is still Σ when h is small. (Earlier, we indexed h by facets of P, which correspond to the rays ρ since $\Sigma = \Sigma_P$.) For h sufficiently small, (13.5.12) implies

$$\int_X [D(h)]^n = \mathrm{Vol}(P(h)) \tag{13.5.13}$$

for h rational (by rescaling) and then for h arbitrary (by continuity).

Note that the Todd operators $\mathrm{Todd}(h_\rho)$ act naturally on cohomology classes in $H^\bullet(X,\mathbb{R})$ that depend polynomially on h. An example is $e^{h_\rho[D_\rho]}$, which is a polynomial in h_ρ since $[D_\rho]$ has degree 2. The basic relation (13.5.3) implies that

$$\mathrm{Todd}(h_\rho)\big(e^{h_\rho[D_\rho]}\big) = \frac{[D_\rho]\,e^{h_\rho[D_\rho]}}{1-e^{-[D_\rho]}}.$$

Then we compute

$$\begin{aligned}
\mathrm{Todd}(h)\big(e^{[D(h)]}\big)\big|_{h=0} &= \Big(\prod_\rho \mathrm{Todd}(h_\rho)\big(e^{(a_\rho+h_\rho)[D_\rho]}\big)\Big)\Big|_{h=0}\\
&= \prod_\rho \frac{[D_\rho]\,e^{a_\rho[D_\rho]}}{1-e^{-[D_\rho]}}\\
&= \mathrm{ch}(\mathscr{L})\,\mathrm{Td}(X)
\end{aligned}$$

since $\mathrm{ch}(\mathscr{L}) = e^{[\sum_\rho a_\rho D_\rho]}$ and $\mathrm{Td}(X) = \prod_\rho [D_\rho]/(1-e^{-[D_\rho]})$ by Theorem 13.1.6.

The integral $\int_X : H^\bullet(X,\mathbb{R}) \to \mathbb{R}$ is a linear map, which implies that $\int_X$ and $\mathrm{Todd}(h)$ commute when applied to $e^{[D(h)]}$. Since $\int_X$ kills everything in degree different from $2n$, we obtain

$$\begin{aligned}
\int_X \mathrm{ch}(\mathscr{L})\,\mathrm{Td}(X) &= \int_X \mathrm{Todd}(h)\big(e^{[D(h)]}\big)\Big|_{h=0}\\
&= \mathrm{Todd}(h)\Big(\int_X e^{[D(h)]}\Big)\Big|_{h=0}\\
&= \mathrm{Todd}(h)\Big(\int_X \tfrac{1}{n!}\big[D(h)\big]^n\Big)\Big|_{h=0}\\
&= \mathrm{Todd}(h)\big(\tfrac{1}{n!}\mathrm{Vol}(P(h))\big)\big|_{h=0} = \mathrm{Todd}(h)\big(\mathrm{vol}(P(h))\big)\big|_{h=0},
\end{aligned}$$

where the fourth equality follows from (13.5.13).

Note that $P = P_D$ is a smooth lattice polytope since D is ample and X is smooth. Thus we can bring Corollary 13.5.5 into the picture, which implies

$$\mathrm{Todd}(h)\big(\mathrm{vol}(P(h))\big)\big|_{h=0} = |P\cap M| = \chi(\mathscr{O}_X(D)) = \chi(\mathscr{L}).$$

Combining this with the previous display gives the Hirzebruch-Riemann-Roch equality

$$\int_X \mathrm{ch}(\mathscr{L})\,\mathrm{Td}(X) = \chi(\mathscr{L}) \tag{13.5.14}$$

in this special case when $\mathscr{L} = \mathscr{O}_X(D)$ and D is ample.

Now pick divisors $D_1, \ldots, D_s$ whose classes give a basis of $\operatorname{Pic}(X)$ and let $D(a) = \sum_{i=1}^s a_i D_i$ and $\mathscr{L}(a) = \mathscr{O}_X(D(a))$. It suffices to prove that

$$\int_X \operatorname{ch}(\mathscr{L}(a))\operatorname{Td}(X) = \chi(\mathscr{L}(a)) \tag{13.5.15}$$

for all $a = (a_1, \cdots, a_s) \in \mathbb{Z}^s$. The left-hand side of (13.5.15) is clearly a polynomial in a, and the same is true for the Euler characterstic $\chi(\mathscr{L}(a))$ on the right-hand side. When D_1 is very ample, we proved in Exercise 9.4.2 that $\chi(\mathscr{O}_X(a_1 D_1))$ is a polynomial in a_1, and a similar result for $a_1 D_1 + \cdots + a_s D_s$ (for arbitrary $D_1, \ldots, D_s$) is proved in [**176**].

If $a \in \mathbb{Z}^s$ is chosen so that $D(a)$ is ample, then equality holds for (13.5.15) by (13.5.14). However, since X is projective, the ample divisor classes are the lattice points in the interior of the nef cone in $\operatorname{Pic}(X)_{\mathbb{R}}$. Since two polynomials on $\operatorname{Pic}(X)_{\mathbb{R}}$ that agree on the lattice points in an open cone must be equal (Exercise 13.5.5), we conclude that (13.5.15) is true for all $a \in \mathbb{Z}^s$. This completes our second proof of the Hirzebruch-Riemann-Roch theorem for toric varieties. □

The version of HRR proved here is not as general as Theorem 13.3.8, which only assumes that X is complete. Nevertheless, it is amazing how the relatively elementary techniques leading to the Khovanskii-Pukhlikov theorem are so closely related to a deep theorem about the geometry of toric varieties.

Exercises for §13.5.

13.5.1. Prove (13.5.7).

13.5.2. Show that if $|h_F|$ is sufficiently small for all F in (13.5.8), then P is still simple.

13.5.3. In this exercise, you will give two proofs of (13.5.9). Assume $0 < |z| < 2\pi$.

(a) Prove (13.5.9) using the power series definition of $\operatorname{Todd}(h) = \operatorname{Todd}(h_0)\operatorname{Todd}(h_1)$.

(b) Compute $\int_{-h_0}^{n+h_1} e^{xz}\,dx$ explicitly and then use (13.5.3) to show that

$$\operatorname{Todd}(h)\Big(\int_{-h_0}^{n+h_1} e^{xz}\,dx\Big)\Big|_{h_0=h_1=0} = \frac{e^{nz}}{1-e^{-z}} + \frac{1}{1-e^{z}}.$$

Then apply (13.5.1) to the right-hand side to derive (13.5.9).

(c) Use Example 13.2.11 to show that right-hand side of the equation from part (b) is equal to $1 + e^z + \cdots + e^{nz}$. Thus we get a direct proof of Theorem 13.5.3 in this case.

13.5.4. This exercise deals with some details in the proof of Theorem 13.5.3.

(a) Show that the second line of (13.5.11) equals the first by using (13.5.3) for each h_F.

(b) Show that the second line in (13.5.11) simplifies to yield the right-hand side of (13.5.5).

13.5.5. Assume that $f \in \mathbb{R}[x_1, \ldots, x_s]$ vanishes on all lattice points in the interior of a cone in $\mathbb{R}^s$ of dimension s. Prove that $f = 0$.

Appendix: Generalized Gysin Maps

Here we collect some facts we need about Borel-Moore homology and generalized Gysin maps. This material is discussed briefly in [**105**, Ch. 19] and in [**110**] in more detail. There is also a nice treatment of Borel-Moore homology in [**106**, App. B].

***Borel-Moore Homology*.** Besides ordinary homology, a suitably nice topological space X has *Borel-Moore* homology groups defined by

$$H_i^{\mathrm{BM}}(X,\mathbb{Q}) = H^{n-i}(\mathbb{R}^n,\mathbb{R}^n\setminus X,\mathbb{Q})$$

for a closed embedding $X \hookrightarrow \mathbb{R}^n$. These groups are independent of the embedding and are defined for any variety X in its classical topology.

Proposition 13.A.1 (Basic Properties)**.**

(a) *If X is compact, then $H_i^{\mathrm{BM}}(X,\mathbb{Q}) = H_i(X,\mathbb{Q})$.*

(b) *There is a cap product operation*

$$H^i(X,\mathbb{Q})\otimes_{\mathbb{Q}} H_j^{\mathrm{BM}}(X,\mathbb{Q}) \longrightarrow H_{j-i}^{\mathrm{BM}}(X,\mathbb{Q}), \quad \alpha\otimes\beta \longmapsto \alpha\frown\beta$$

such that

$$(\alpha\smile\beta)\frown\gamma = \alpha\frown(\beta\frown\gamma)$$

for all $\alpha,\beta \in H^\bullet(X,\mathbb{Q})$ and $\gamma \in H_\bullet^{\mathrm{BM}}(X,\mathbb{Q})$. □

The functorial properties of Borel-Moore homology are more complicated since a continuous map $f : X \to Y$ does not always induce a map $f_* : H_i^{\mathrm{BM}}(X,\mathbb{Q}) \to H_i^{\mathrm{BM}}(Y,\mathbb{Q})$.

Proposition 13.A.2 (Functorial Properties)**.**

(a) *A proper map $f : X \to Y$ induces $f_* : H_i^{\mathrm{BM}}(X,\mathbb{Q}) \to H_i^{\mathrm{BM}}(Y,\mathbb{Q})$ such that*

$$f_*(f^*(\alpha)\frown\beta) = \alpha\frown f_*(\beta)$$

for all $\alpha \in H^\bullet(Y,\mathbb{Q})$ and $\beta \in H_\bullet^{\mathrm{BM}}(X,\mathbb{Q})$.

(b) *If $f : X \to Y$ and $g : Y \to Z$ are proper, then $(g\circ f)_* = g_*\circ f_*$.*

(c) *An inclusion $j : U \hookrightarrow Y$ of an open set induces $j^! : H_i^{\mathrm{BM}}(X,\mathbb{Q}) \to H_i^{\mathrm{BM}}(U,\mathbb{Q})$ such that*

$$j^!(\alpha\frown\beta) = j^*(\alpha)\frown j^!(\beta)$$

for all $\alpha \in H^\bullet(X,\mathbb{Q})$ and $\beta \in H_\bullet^{\mathrm{BM}}(X,\mathbb{Q})$.

(d) *If $j' : U' \hookrightarrow U$ and $j : U \hookrightarrow X$ are open inclusions, then $(j\circ j')^! = j'^!\circ j^!$.*

(e) *If $f : X \to Y$ is proper and $j : U \hookrightarrow X$ is an open inclusion, then the diagram*

$$\begin{tikzcd}
H_\bullet^{\mathrm{BM}}(X,\mathbb{Q}) \arrow[r, "j'^!"] \arrow[d, "f_*"'] & H_\bullet^{\mathrm{BM}}(f^{-1}(U),\mathbb{Q}) \arrow[d, "f'_*"] \\
H_\bullet^{\mathrm{BM}}(Y,\mathbb{Q}) \arrow[r, "j^!"] & H_\bullet^{\mathrm{BM}}(U,\mathbb{Q})
\end{tikzcd}$$

commutes, where $f' = f|_{f^{-1}(U)}$ and $j' : f^{-1}(U) \hookrightarrow X$ is inclusion. □

Fundamental Classes and Refined Cohomology Classes. A variety X of dimension n has a canonically defined homology class $[X] \in H_{2n}^{\mathrm{BM}}(X,\mathbb{Q})$ called the *fundamental class* of X. Furthermore, if X is irreducible, then

$$H_{2n}^{\mathrm{BM}}(X,\mathbb{Q}) = \mathbb{Q}[X].$$

Example 13.A.3. Let X be a complete variety of dimension n. Then the constant map $p : X \to \{\mathrm{pt}\}$ is proper. Hence we can define $\int_X : H^\bullet(X,\mathbb{Q}) \to \mathbb{Q}$ such that the diagram

$$\begin{tikzcd} H^\bullet(X,\mathbb{Q}) \arrow[d, "\frown [X]"'] \arrow[dr, "\int_X"] & \\ H_\bullet^{\mathrm{BM}}(X,\mathbb{Q}) \arrow[r, "p_*"] & H_\bullet^{\mathrm{BM}}(\{\mathrm{pt}\},\mathbb{Q}) = \mathbb{Q} \end{tikzcd}$$

commutes, i.e., $\int_X \alpha = p_*(\alpha \frown [X])$. We use this map frequently in Chapters 12 and 13. $\Diamond$

When $Y \subseteq X$ is a closed subset, the cap product in Proposition 13.A.1 generalizes to

$$H^i(X, X\setminus Y,\mathbb{Q}) \otimes_{\mathbb{Q}} H_j^{\mathrm{BM}}(X,\mathbb{Q}) \longrightarrow H_{j-i}^{\mathrm{BM}}(Y,\mathbb{Q}), \quad \alpha\otimes\beta \longmapsto \alpha \frown \beta.$$

Applied to the fundamental class $[X]$, this gives a map

$$H^i(X, X\setminus Y,\mathbb{Q}) \longrightarrow H_{j-i}^{\mathrm{BM}}(Y,\mathbb{Q}), \quad \alpha \longmapsto \alpha \frown [X].$$

When X is irreducible and rationally smooth, we get some classical duality theorems.

Proposition 13.A.4 (Duality)**.** *If X is irreducible and rationally smooth of dimension n and $Y \subseteq X$ is closed, then we have isomorphisms*

(a) Poincaré Duality: $\frown [X] : H^i(X,\mathbb{Q}) \simeq H_{2n-i}^{\mathrm{BM}}(X,\mathbb{Q})$.

(b) Alexander Duality: $\frown [X] : H^i(X, X\setminus Y,\mathbb{Q}) \simeq H_{2n-i}^{\mathrm{BM}}(Y,\mathbb{Q})$. $\square$

This proposition explains why we are using coefficients in $\mathbb{Q}$. If we want Poincaré and Alexander duality to hold over $\mathbb{Z}$, then we need to assume that X is smooth.

For X as in Proposition 13.A.4, let $i : Y \hookrightarrow X$ be a d-dimensional irreducible subvariety. Then the fundamental class $[Y] \in H_{2d}^{\mathrm{BM}}(Y,\mathbb{Q})$ gives the following classes:

- The *refined cohomology class* $[Y]_r \in H^{2n-2d}(X, X\setminus Y,\mathbb{Q})$ maps to $[Y] \in H_{2d}^{\mathrm{BM}}(Y,\mathbb{Q})$ under Alexander duality.
- The *cohomology class* $[Y] \in H^{2n-2d}(X,\mathbb{Q})$ maps to $i_*[Y] \in H_{2d}^{\mathrm{BM}}(X,\mathbb{Q})$ under Poincaré duality.

The natural map $H^{2n-2d}(X, X\setminus Y,\mathbb{Q}) \to H^{2n-2d}(X,\mathbb{Q})$ takes $[Y]_r$ to $[Y]$. We use refined cohomology classes in §12.4.

We will also need the following property of fundamental classes.

Proposition 13.A.5. *If $f : X \to Y$ is a proper birational morphism between irreducible varieties, then $f_*[X] = [Y]$.* $\square$

Generalized Gysin Maps. Let $f : X \to Y$ be proper map such that Y is irreducible and rationally smooth. Besides the usual contravariant map $f^* : H^\bullet(Y,\mathbb{Q}) \to H^\bullet(X,\mathbb{Q})$, there is also a unique covariant map $f_! : H^\bullet(X,\mathbb{Q}) \to H^\bullet(Y,\mathbb{Q})$ such that the diagram

$$\begin{tikzcd} H^\bullet(X,\mathbb{Q}) \arrow[r, "f_!"] \arrow[d, "\frown [X]"'] & H^\bullet(Y,\mathbb{Q}) \arrow[d, "\frown [Y]"] \\ H^{\mathrm{BM}}_\bullet(X,\mathbb{Q}) \arrow[r, "f_*"] & H^{\mathrm{BM}}_\bullet(Y,\mathbb{Q}) \end{tikzcd}$$

commutes. This follows since the vertical map on the right is an isomorphism. We call $f_!$ a *generalized Gysin map*. These maps behave nicely as follows.

Proposition 13.A.6. *Assume that $f : X \to Y$ is proper and Y is irreducible and rationally smooth. Then:*

(a) $f_! : H^k(X,\mathbb{Q}) \to H^{k+2\dim Y - 2\dim X}(Y,\mathbb{Q})$.

(b) *If $g : Y \to Z$ is proper and Z is irreducible and rationally smooth, then $(g \circ f)_! = g_! \circ f_!$.*

(c) $f_! : H^\bullet(X,\mathbb{Q}) \to H^\bullet(Y,\mathbb{Q})$ *satisfies*

$$f_!(f^*(\alpha) \smile \beta) = \alpha \smile f_!(\beta)$$

for all $\alpha \in H^\bullet(Y,\mathbb{Q})$ and $\beta \in H^\bullet(X,\mathbb{Q})$. □

In particular, taking $\beta = 1$ in part (c) of the proposition gives $f_!(f^*(\alpha)) = \alpha \smile f_!(1)$. Here are two cases where $f_!(1)$ is explicitly known for f as in Proposition 13.A.6:

- If $f : X \to Y$ is proper and birational, then $f_!(1) = 1$. Thus $f_!(f^*(\alpha)) = \alpha$ for all $\alpha \in H^\bullet(Y,\mathbb{Q})$.
- If $i : Y \hookrightarrow X$ is the inclusion of an irreducible subvariety, then $i_!(1) = [Y]$. Thus $i_!(i^*(\alpha)) = \alpha \smile [Y]$ for all $\alpha \in H^\bullet(X,\mathbb{Q})$.

In the second bullet, the map $i_! : H^\bullet(Y,\mathbb{Q}) \to H^\bullet(X,\mathbb{Q})$ is called the *Gysin map*.

Here is an example of a generalized Gysin map.

Example 13.A.7. Let X be complete, irreducible and rationally smooth of dimension n. If $p : X \to \{\mathrm{pt}\}$ is the constant map, then $p_! : H^\bullet(X,\mathbb{Q}) \to H^\bullet(\{\mathrm{pt}\},\mathbb{Q}) \simeq \mathbb{Q}$ is the map $\int_X$ from Example 13.A.3. This follows easily from the definitions of $p_!$ and $\int_X$. Note also that this agrees with the ad-hoc definition of $\int_X$ given in §12.4. ◊

We will need the following compatibility result between pullbacks and generalized Gysin maps.

Proposition 13.A.8. *Suppose we have a commutative diagram of maps*

(13.A.1)
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where f is proper, Y and Y' are irreducible and rationally smooth, and $X' = X \times_Y Y'$. Then we have a commutative diagram in cohomology

$$\begin{tikzcd} H^\bullet(X',\mathbb{Q}) \arrow[r, "f'_!"] & H^\bullet(Y',\mathbb{Q}) \\ H^\bullet(X,\mathbb{Q}) \arrow[r, "f_!"] \arrow[u, "g'^*"] & H^\bullet(Y,\mathbb{Q}). \arrow[u, "g^*"'] \end{tikzcd}$$

□

The diagram (13.A.1) is called *Cartesian* when $X' = X \times_Y Y'$. It is a standard fact that when (13.A.1) is Cartesian, f' is proper whenever f is.

The complicated behavior of Borel-Moore homology and Gysin maps were one of the factors that inspired Fulton and MacPherson to develop the *bivariant theories* introduced in [**110**]. This is related to the *bivariant intersection theory* discussed in [**107**, Ch. 17].

Equivariant Gysin Maps. Let T be a torus acting on varieties X and Y and let $f : X \to Y$ be T-equivariant and proper. Our goal is to define an *equivariant Gysin map*

$$f_! : H_T^\bullet(X,\mathbb{Q}) \to H_T^\bullet(Y,\mathbb{Q}). \tag{13.A.2}$$

For simplicity, we will assume that Y is a simplicial toric variety where T acts on Y via a homomorphism $T \to T_N =$ the torus of Y.

To define $f_!$, we follow [**203**, App.]. Recall that $H_T^\bullet(Y,\mathbb{Q}) = H^\bullet(ET \times_T Y,\mathbb{Q})$, where $T \simeq (\mathbb{C}^*)^m$ implies $EG \simeq (\mathbb{C}^\infty \setminus \{0\})^m$. This follows from (12.4.6), where we showed that $E\mathbb{C}^* = \mathbb{C}^\infty \setminus \{0\}$. Then define the finite-dimensional approximation

$$ET_\ell \simeq (\mathbb{C}^{\ell+1} \setminus \{0\})^m$$

of ET. Taking the quotient by T, we get the approximation $BT_\ell \simeq (\mathbb{P}^\ell)^m$ of BT.

When Y is a simplicial toric variety, $Y_\ell = ET_\ell \times_T Y$ is irreducible and rationally smooth for all ℓ, so that the induced proper map $f_\ell : X_\ell \to Y_\ell$ has a generalized Gysin map

$$f_{\ell!} : H^\bullet(X_\ell,\mathbb{Q}) \to H^\bullet(Y_\ell,\mathbb{Q}).$$

Furthermore, the inclusions $X_\ell \subseteq X_{\ell+1}$ and $Y_\ell \subseteq Y_{\ell+1}$ give a Cartesian diagram

$$\begin{tikzcd} X_\ell \arrow[r, "f_\ell"] \arrow[d, hook] & Y_\ell \arrow[d, hook] \\ X_{\ell+1} \arrow[r, "f_{\ell+1}"] & Y_{\ell+1} \end{tikzcd}$$

It follows from Proposition 13.A.8 that the maps $f_{\ell!}$ are compatible as $\ell \to \infty$. Hence we get the desired map (13.A.2).

Here are the basic properties of equivariant Gysin maps.

Proposition 13.A.9. *Assume that $f : X \to Y$ is proper and T-equivariant, and assume that Y is a simplicial toric variety. Then:*

(a) $f_! : H_T^k(X,\mathbb{Q}) \to H_T^{k+2\dim Y - 2\dim X}(Y,\mathbb{Q})$.

(b) *If $g : Y \to Z$ is proper and T-equivariant and Z is a simplicial toric variety, then* $(g \circ f)_! = g_! \circ f_!$.

(c) $f_! : H_T^\bullet(X,\mathbb{Q}) \to H_T^\bullet(Y,\mathbb{Q})$ *satisfies*

$$f_!(f^*(\alpha) \smile \beta) = \alpha \smile f_!(\beta)$$

for all $\alpha \in H_T^\bullet(Y,\mathbb{Q})$ and $\beta \in H_T^\bullet(X,\mathbb{Q})$. □

Equivariant Gysin maps also work nicely in Cartesian squares.

Proposition 13.A.10. *Suppose we have a commutative diagram of maps*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

such that f is proper, f and g are equivariant, Y and Y' are simplicial toric varieties, and $X' = X \times_Y Y'$. Then we have a commutative diagram in equivariant cohomology

$$\begin{tikzcd} H_T^\bullet(X',\mathbb{Q}) \arrow[r, "f'_!"] & H_T^\bullet(Y',\mathbb{Q}) \\ H_T^\bullet(X,\mathbb{Q}) \arrow[r, "f_!"] \arrow[u, "g'^*"] & H_T^\bullet(Y,\mathbb{Q}). \arrow[u, "g^*"'] \end{tikzcd}$$ □

For the final property we need, let $X = X_\Sigma$ be a complete simplicial toric variety. Then the constant map $p : X \to \{\mathrm{pt}\}$ is proper and equivariant under the action of $T = T_N$. Hence we get

$$\int_{X^{eq}} = p_! : H_T^\bullet(X,\mathbb{Q}) \longrightarrow H_T^\bullet(\{\mathrm{pt}\},\mathbb{Q}).$$

This is the *equivariant integral*. We need to show that $\int_{X^{eq}}$ is compatible with the ordinary integral

$$\int_X = p_! : H^\bullet(X,\mathbb{Q}) \longrightarrow H^\bullet(\{\mathrm{pt}\},\mathbb{Q}) = \mathbb{Q}.$$

Here is the precise result we will use in §13.4.

Proposition 13.A.11. *In the above situation, we have a commutative diagram*

$$\begin{tikzcd} H^\bullet(X,\mathbb{Q}) \arrow[r, "\int_X"] & \mathbb{Q} \\ H_T^\bullet(X,\mathbb{Q}) \arrow[r, "\int_{X^{eq}}"] \arrow[u, "i_X^*"] & H_T^\bullet(\{\mathrm{pt}\},\mathbb{Q}). \arrow[u, "i_{\mathrm{pt}}^*"'] \end{tikzcd}$$

Proof. We will prove this using the commutative diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, hook] & \{\mathrm{pt}\} \arrow[d, hook] \\ X_\ell \arrow[r, "p_\ell"] & BT_\ell, \end{tikzcd}$$

where X is a complete simplicial toric variety. Here, $p_\ell : X_\ell = ET_\ell \times_T X \to BT_\ell$ is induced by projection on the first factor and pt is any point of BT_ℓ. This is a Cartesian diagram, so that by Proposition 13.A.8, we get a commutative diagram

$$\begin{tikzcd} H^\bullet(X,\mathbb{Q}) \arrow[r, "p_!"] & \mathbb{Q} \\ H^\bullet(X_\ell,\mathbb{Q}) \arrow[r, "p_{\ell !}"] \arrow[u, "i_X^*"] & H^\bullet(BT_\ell,\mathbb{Q}). \arrow[u, "i_{\mathrm{pt}}^*"'] \end{tikzcd}$$

where $i_X : X \hookrightarrow X_\ell$ and $i_{\mathrm{pt}} : \{\mathrm{pt}\} \hookrightarrow BT_\ell$ are the inclusions. Letting $\ell \to \infty$, we get the desired commutative diagram. □

Chapter 14

Toric GIT and the Secondary Fan

This chapter will explore a rich collection of ideas that give different ways to think about toric varieties. We begin in §14.1 with the *geometric invariant theory* of a closed subgroup $G \subseteq (\mathbb{C}^*)^r$ acting on $\mathbb{C}^r$, which uses a character χ of G to lift the G-action to a trivial line bundle over $\mathbb{C}^r$. In §14.2 we show that the GIT quotient is a semiprojective toric variety in this situation, and in §14.3 we use Gale duality to help us understand how the quotient depends on the character χ. The full story of what happens as χ varies is controlled by the *secondary fan*, which is the main topic of §14.4. The geometry of the secondary fan will be explored in Chapter 15.

§14.1. Introduction to Toric GIT

Geometric invariant theory was invented by Mumford [**209**] in 1965 to prove the existence of suitable projective compactifications of the moduli spaces he was studying. GIT, as it is now called, is a powerful tool in modern algebraic geometry. See [**83**] for a nice introduction to the subject.

We will consider the special case of a closed subgroup $G \subseteq (\mathbb{C}^*)^r$ acting on $\mathbb{C}^r = \operatorname{Spec}(\mathbb{C}[x_1,\dots,x_r])$. Thus $G \simeq (\mathbb{C}^*)^\ell \times H$, where H is finite. Since G is reductive, Proposition 5.0.9 gives the good categorical quotient

$$\mathbb{C}^r /\!/ G = \operatorname{Spec}(\mathbb{C}[x_1,\dots,x_r]^G).$$

However, such quotients can behave badly, as shown by the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^r$. The only invariants are the constant polynomials, so that

$$\mathbb{C}^r /\!/ \mathbb{C}^* = \operatorname{Spec}(\mathbb{C}[x_1,\dots,x_r]^{\mathbb{C}^*}) = \operatorname{Spec}(\mathbb{C}) = \{\mathrm{pt}\}.$$

A better model for the kind of quotient we want comes from the quotient construction of a toric variety X_Σ without torus factors. Here, Theorem 5.1.11 gives the almost geometric quotient

$$X_\Sigma \simeq (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))/\!\!/G.$$

Thus, if we take $\mathbb{C}^{\Sigma(1)}$ and throw away some points, we get an almost geometric quotient, which by definition means that removing further points gives a geometric quotient. As we will see, this is similar to what happens in GIT, where (roughly speaking) we first remove points that leave us with the set of *semistable points*, and then we remove even more points to get the set of *stable points*.

Linearized Line Bundles. In GIT, deciding which points to remove is governed by a lifting of the G-action on $\mathbb{C}^r$ to the rank 1 trivial vector bundle $\mathbb{C}^r \times \mathbb{C} \to \mathbb{C}^r$. Liftings are described by characters of G. Let the character group of G be

$$\widehat{G} = \{\chi : G \to \mathbb{C}^* \mid \chi \text{ is a homomorphism of algebraic groups}\}.$$

Then a character $\chi \in \widehat{G}$ gives the action of G on $\mathbb{C}^r \times \mathbb{C}$ defined by

$$g \cdot (p,t) = (g \cdot p, \chi(g)t), \quad g \in G,\ (p,t) \in \mathbb{C}^r \times \mathbb{C}.$$

This lifts the G-action on $\mathbb{C}^r$, and all possible liftings arise this way.

Let $\mathscr{L}_\chi$ denote the sheaf of sections of $\mathbb{C}^r \times \mathbb{C}$ with this G-action. We call $\mathscr{L}_\chi$ the *linearized line bundle* with character χ. Note that for $d \in \mathbb{Z}$, the tensor product $\mathscr{L}_\chi^{\otimes d}$ is the linearized line bundle with character χ^d, i.e., $\mathscr{L}_\chi^{\otimes d} = \mathscr{L}_{\chi^d}$.

If we forget the G-action, then $\mathscr{L}_\chi \simeq \mathscr{O}_{\mathbb{C}^r}$ as a line bundle on $\mathbb{C}^r$. It follows that a global section $s \in \Gamma(\mathbb{C}^r, \mathscr{L}_\chi)$ can be written

$$s(p) = (p, F(p)) \in \mathbb{C}^r \times \mathbb{C}, \quad p \in \mathbb{C}^r,$$

for a unique $F \in \mathbb{C}[x_1,\ldots,x_r]$. The group G acts on global sections as follows.

Lemma 14.1.1. *Let $\mathscr{L}_\chi$ be the linearized line bundle on $\mathbb{C}^r$ with character $\chi \in \widehat{G}$.*

(a) *If s is the global section of $\mathscr{L}_\chi$ given by F, then for any $g \in G$, $g \cdot s$ is the global section defined by*

$$(g\cdot s)(p) = (p, \chi(g)F(g^{-1}\cdot p)), \quad p \in \mathbb{C}^r.$$

(b) *The G-invariant global sections are described by the isomorphism*

$$\Gamma(\mathbb{C}^r, \mathscr{L}_\chi)^G \simeq \{F \in \mathbb{C}[x_1,\ldots,x_r] \mid F(g\cdot p) = \chi(g)F(p) \text{ for } g \in G,\ p \in \mathbb{C}^r\}.$$

Proof. By Exercise 14.1.1, $g \in G$ acts on a global section s by

$$(g\cdot s)(p) = g\cdot(s(g^{-1}\cdot p)), \quad p \in \mathbb{C}^r.$$

Since $s(p) = (p, F(p))$, part (a) follows immediately, and then we are done since part (b) is an easy consequence of part (a). □

Definition 14.1.2. The polynomials in part (b) of the lemma are (G,χ)-***invariant***.

Example 14.1.3. Consider the usual action of $\mathbb{C}^*$ on $\mathbb{C}^r$. Then $\widehat{\mathbb{C}^*} \simeq \mathbb{Z}$, where $d \in \mathbb{Z}$ gives the character $\chi(t) = t^d$ for $t \in \mathbb{C}^*$. One checks that $\Gamma(\mathbb{C}^r, \mathscr{L}_\chi)^G$ is isomorphic to the vector space of homogeneous polynomials of degree d. $\Diamond$

Example 14.1.3 has a nice toric generalization.

Example 14.1.4. Let X_Σ be a toric variety with no torus factors. The class group $\mathrm{Cl}(X_\Sigma)$ gives the algebraic group $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \mathbb{C}^*)$, and using the map $\mathbb{Z}^{\Sigma(1)} \to \mathrm{Cl}(X_\Sigma)$ from §4.1, we can regard G as a subgroup of $\mathbb{C}^{\Sigma(1)}$.

The total coordinate ring of $\mathbb{C}^{\Sigma(1)}$ is $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$, which is graded by $\mathrm{Cl}(X_\Sigma)$ as in §5.2. Note also that $\widehat{G} \simeq \mathrm{Cl}(X_\Sigma)$, where a divisor class $\beta \in \mathrm{Cl}(X_\Sigma)$ gives the character $\chi \in \widehat{G}$ defined by evaluation at β. In Exercise 14.1.2 you will construct an isomorphism

$$\Gamma(\mathbb{C}^{\Sigma(1)}, \mathscr{L}_\chi)^G \simeq S_\beta = \{F \in S \mid \deg(F) = \beta\}.$$

Thus S_β is the set of all (G, χ)-invariant polynomials. $\Diamond$

Here is an example that will appear several times in this section and the next.

Example 14.1.5. Let $G = \{(t, t^{-1}, u) \in (\mathbb{C}^*)^3 \mid t \in \mathbb{C}^*, u = \pm 1\} \simeq \mathbb{C}^* \times \mu_2$. One easily sees that

$$\mathbb{C}[x,y,z]^G = \mathbb{C}[xy, z^2],$$

so that $\mathbb{C}^3/\!/G \simeq \mathbb{C}^2$.

Now consider the character $\chi \in \widehat{G} \simeq \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ defined by $\chi(t, t^{-1}, u) = tu$. Then a monomial $x^a y^b z^c$ is (G, χ)-invariant if and only if

$$(tx)^a (t^{-1}y)^b (uz)^c = tu\, x^a y^b z^c \iff t^{a-b} u^c = tu \iff a = b+1,\ c \equiv 1 \bmod 2.$$

It follows easily that $\Gamma(\mathbb{C}^3, \mathscr{L}_\chi)^G \simeq xz\, \mathbb{C}[xy, z^2]$. $\Diamond$

Semistable and Stable Points. Given a global section s of $\mathscr{L}_\chi$, note that

$$(\mathbb{C}^r)_s = \{p \in \mathbb{C}^r \mid s(p) \neq 0\}$$

is an affine open subset of $\mathbb{C}^r$ since $s(p) = (p, F(p))$ and $s(p) \neq 0$ means $F(p) \neq 0$. Also observe that G acts on $(\mathbb{C}^r)_s$ when s is G-invariant.

Definition 14.1.6. Fix $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, with linearized line bundle $\mathscr{L}_\chi$.

(a) $p \in \mathbb{C}^r$ is ***semistable*** if there are $d > 0$ and $s \in \Gamma(\mathbb{C}^r, \mathscr{L}_{\chi^d})^G$ such that $p \in (\mathbb{C}^r)_s$.

(b) $p \in \mathbb{C}^r$ is ***stable*** if there are $d > 0$ and $s \in \Gamma(\mathbb{C}^r, \mathscr{L}_{\chi^d})^G$ such that $p \in (\mathbb{C}^r)_s$, the isotropy subgroup G_p is finite, and all G-orbits in $(\mathbb{C}^r)_s$ are closed in $(\mathbb{C}^r)_s$.

(c) The set of all semistable (resp. stable) points is denoted $(\mathbb{C}^r)^{\mathsf{ss}}_\chi$ (resp. $(\mathbb{C}^r)^{\mathsf{s}}_\chi$).

Since a global section $s \in \Gamma(\mathbb{C}^r, \mathscr{L}_{\chi^d})^G$ corresponds to a (G, χ^d)-invariant polynomial F, one can determine semistability and stability using (G, χ^d)-invariant polynomials. We will do this frequently in the remainder of the chapter.

In general treatments of GIT, where G acts on a variety X, one must also require that the nonvanishing subset of the section s be affine in the definition of semistable and stable point. This is automatic in our case. See [**83**, Ch. 8] and [**209**].

Example 14.1.7. Consider the usual action of $\mathbb{C}^*$ on $\mathbb{C}^r$ and let χ^d, $d \in \mathbb{Z}$, be as in Example 14.1.3. Then one can check without difficulty that

$$\begin{aligned} d > 0 &: (\mathbb{C}^r)^{\mathsf{ss}}_{\chi^d} = (\mathbb{C}^r)^{\mathsf{s}}_{\chi^d} = \mathbb{C}^r \setminus \{0\} \\ d = 0 &: (\mathbb{C}^r)^{\mathsf{ss}}_{\chi^d} = \mathbb{C}^r,\ (\mathbb{C}^r)^{\mathsf{s}}_{\chi^d} = \emptyset \\ d < 0 &: (\mathbb{C}^r)^{\mathsf{ss}}_{\chi^d} = (\mathbb{C}^r)^{\mathsf{s}}_{\chi^d} = \emptyset. \end{aligned}$$

This shows that the notions of semistable and stable depend strongly on which character we use. $\Diamond$

Example 14.1.8. For $G = \{(t,t^{-1},u) \mid t \in \mathbb{C}^*, u = \pm 1\}$ and $\chi(t,t^{-1},u) = tu$ from Example 14.1.5, the (G,χ^d)-invariant polynomials are

$$\Gamma(\mathbb{C}^3, \mathscr{L}_{\chi^d})^G \simeq \begin{cases} x^d z\, \mathbb{C}[xy,z^2] & d \text{ odd} \\ x^d\, \mathbb{C}[xy,z^2] & d \text{ even.} \end{cases} \tag{14.1.1}$$

In particular, x^2 is (G,χ^2)-invariant, which implies that all points in $\mathbb{C}^* \times \mathbb{C}^2$ are semistable. With more work (Exercise 14.1.3), one can show that

$$(\mathbb{C}^3)^{\mathsf{s}}_\chi = (\mathbb{C}^3)^{\mathsf{ss}}_\chi = \mathbb{C}^* \times \mathbb{C}^2. \qquad \Diamond$$

For a toric variety X_Σ, the group $G \subseteq (\mathbb{C}^*)^{\Sigma(1)}$ acts on $\mathbb{C}^{\Sigma(1)}$ as described in Example 14.1.4, and characters correspond to divisor classes since $\widehat{G} \simeq \mathrm{Cl}(X_\Sigma)$. For the character of an ample class, semistable and stable have a nice meaning.

Proposition 14.1.9. *Let X_Σ be a projective toric variety, and let $\Sigma' \subseteq \Sigma$ be the subfan consisting of all simplicial cones of Σ. If $\chi \in \widehat{G}$ comes from an ample divisor class $\beta \in \mathrm{Cl}(X_\Sigma)$, then*

$$\begin{aligned} \big(\mathbb{C}^{\Sigma(1)}\big)^{\mathsf{ss}}_\chi &= \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \\ \big(\mathbb{C}^{\Sigma(1)}\big)^{\mathsf{s}}_\chi &= \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma'). \end{aligned}$$

Proof. Let $\beta = [D]$, where $D = \sum_\rho a_\rho D_\rho$ is ample. Then the polytope $P_D \subseteq M_\mathbb{R}$ has facet presentation

$$P_D = \{m \in M_\mathbb{R} \mid \langle m, u_\rho\rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\},$$

and the vertices of P_D give the Cartier data $\{m_\sigma\}_{\sigma\in\Sigma(n)}$ of D. Also recall from §5.4 that if $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$, then the graded piece S_β is spanned by the monomials

$$x^{\langle m,D\rangle} = \textstyle\prod_\rho x_\rho^{\langle m,u_\rho\rangle + a_\rho}, \quad m \in P_D \cap M.$$

Now take $p \in \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$. Since $Z(\Sigma)$ is defined by the vanishing of $x^{\hat\sigma} = \prod_{\rho\notin\sigma(1)} x_\rho$, $\sigma \in \Sigma(n)$, there must be $\sigma \in \Sigma(n)$ such that $x^{\hat\sigma}$ does not vanish at p.

We claim that $x^{\langle m_\sigma, D\rangle} \in S_\beta$ does not vanish at p. Since $x^{\langle m_\sigma, D\rangle}$ is (G,χ)-invariant by Example 14.1.4, this will imply that p is semistable. To prove our claim, recall from the discussion before Example 5.4.5 that the exponent of x_ρ in $x^{\langle m, D\rangle}$ is the lattice distance from m to the facet of P_D whose normal is u_ρ. For m_σ, the lattice distance is zero when $\rho \in \sigma(1)$ since these ρ's give the facets containing the vertex m_σ, and all other lattice distances are positive. Thus $x^{\langle m_\sigma, D\rangle}$ and $x^{\hat{\sigma}}$ involve the same variables, so that $x^{\langle m_\sigma, D\rangle}$ does not vanish at p since the same is true for $x^{\hat{\sigma}}$.

Next assume $p \in \mathbb{C}^{\Sigma(1)}$ is semistable. Then there is $d > 0$ such that some element of $S_{d\beta}$ does not vanish at p. It follows that $x^{\langle m, dD\rangle}$ is nonzero at p for some $m \in (dP_D) \cap M$. Let $Q \preceq dP_D$ be the smallest face of dP_D containing m and let dm_σ be a vertex of this face. We claim that $x^{\hat{\sigma}}$ divides $x^{\langle m, dD\rangle}$. Since $x^{\langle m, dD\rangle}$ does not vanish at p, this will imply that the same is true for $x^{\hat{\sigma}}$, and $p \in \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$ will follow. To prove our claim, take $\rho \notin \sigma(1)$ and consider the facet $F \preceq dP_D$ with facet normal u_ρ. If $m \in F$, then $Q \preceq F$, which would give $dm_\sigma \in F$. This contradicts $\rho \notin \sigma(1)$ by the definition of normal fan, so that m has positive lattice distance to the facets corresponding to $\rho \notin \sigma(1)$. It follows that $x^{\hat{\sigma}}$ divides $x^{\langle m, dD\rangle}$.

It remains to consider stable points. Since $\Sigma'(1) = \Sigma(1)$ (rays are simplicial), X_Σ and $X_{\Sigma'}$ have the same class group and the same group G. The difference is that $\Sigma' \subseteq \Sigma$ implies an inclusion of irrelevant ideals $B(\Sigma') \subseteq B(\Sigma)$, which in turn gives the inclusion

$$\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma') \subseteq \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma).$$

Since Σ' is simplicial, Theorem 5.1.11 implies that $X_{\Sigma'} \simeq (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma'))/\!\!/G$ is a geometric quotient. It follows that $G \cdot p$ is closed for all $p \in \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma')$. Furthermore, Exercise 5.1.11 implies that the isotropy subgroup G_p is finite for these p's since Σ' is simplicial. Thus p is stable, hence $\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma') \subseteq \big(\mathbb{C}^{\Sigma(1)}\big)^{\mathsf{s}}_\chi$.

For the opposite inclusion, let $U = \big(\mathbb{C}^{\Sigma(1)}\big)^{\mathsf{s}}_\chi$ be the set of stable points. Using the trivial character, the action of $(\mathbb{C}^*)^{\Sigma(1)}$ lifts to an action on $\mathscr{L}_\chi$ that is easily seen to commute with the action of G on $\mathscr{L}_\chi$. This induces an action of $(\mathbb{C}^*)^{\Sigma(1)}$ on $\Gamma(\mathbb{C}^{\Sigma(1)}, \mathscr{L}_{\chi^d})^G$ for every $d \in \mathbb{N}$. It follows that if $p \in \mathbb{C}^{\Sigma(1)}$ is stable, so is its $(\mathbb{C}^*)^{\Sigma(1)}$-orbit. In other words, $(\mathbb{C}^*)^{\Sigma(1)}$ acts on U. Note that we also have $U \subseteq \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma)$ since stable points are semistable.

Now consider the quotient map

$$\pi : \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma) \longrightarrow X_\Sigma = (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))/\!\!/G.$$

Because G-orbits in U are closed, Theorem 5.0.6 and Proposition 5.0.7 imply that $\pi(U) \subseteq X_\Sigma$ is open. By the previous paragraph, $\pi(U)$ is stable under the action of $T_N = (\mathbb{C}^*)^{\Sigma(1)}/G$ and hence is a toric variety. This means it comes from a subfan of Σ. Note also that $\pi|_U : U \to \pi(U)$ is a geometric quotient since G-orbits are closed in U. By Theorem 5.1.11, this subfan is simplicial and hence is contained in Σ'. It follows easily that $U \subseteq \mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma')$, and we are done. $\square$

GIT Quotients. Given $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, our next task is to define the GIT quotient $\mathbb{C}^r /\!/_\chi G$. The basic idea is to take the quotient of $\big(\mathbb{C}^{\Sigma(1)}\big)^{\mathsf{ss}}$ under the action of G. As shown by the non-separated quotient from Example 5.0.15, care must be taken to ensure that the quotient is well-behaved.

The strategy will be to use the graded ring

$$R_\chi = \bigoplus_{d=0}^{\infty} \Gamma(\mathbb{C}^r, \mathscr{L}_{\chi^d})^G. \tag{14.1.2}$$

We first study the structure of this ring.

Lemma 14.1.10. *The graded ring R_χ is a finitely generated $\mathbb{C}$-algebra.*

Proof. Consider the action of G on $\mathbb{C}^r \times \mathbb{C}$ given by $g\cdot(p,t) = (g\cdot p, \chi^{-1}(g)t)$. Then G acts on $f \in \mathbb{C}[x_1,\ldots,x_r,w]$ by

$$(g\cdot f)(p,t) = f(g^{-1}\cdot(p,t)) = f(g^{-1}\cdot p, \chi^{-1}(g^{-1})t) = f(g^{-1}\cdot p, \chi(g)t).$$

In particular, if $F \in \mathbb{C}[x_1,\ldots,x_r]$, then $f = F\,w^d$ is G-invariant if and only if for all $(p,t) \in \mathbb{C}^r \times \mathbb{C}$ and all $g \in G$, we have

$$\begin{aligned} f(g^{-1}\cdot p, \chi(g)t) = f(p,t) &\iff F(g^{-1}\cdot p)(\chi(g)t)^d = F(p)t^d \\ &\iff F(g^{-1}\cdot p) = (\chi(g))^{-d}F(p) = \chi^d(g^{-1})F(p). \end{aligned}$$

Replacing g with g^{-1}, we see that $f = F\,w^d$ is G-invariant if and only if F is (G,χ^d)-invariant. Using Lemma 14.1.1, we obtain an isomorphism

$$R_\chi \simeq \mathbb{C}[x_1,\ldots,x_r,w]^G. \tag{14.1.3}$$

Since G is reductive, R_χ is a finitely generated $\mathbb{C}$-algebra by Proposition 5.0.9 □

We now define the GIT quotient using the Proj construction described in §7.0.

Definition 14.1.11. For $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, the ***GIT quotient*** $\mathbb{C}^r /\!/_\chi G$ is

$$\mathbb{C}^r /\!/_\chi G = \mathrm{Proj}(R_\chi).$$

Here are some easy properties of GIT quotients.

Proposition 14.1.12. *For $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, we have:*

(a) *There is a projective morphism $\mathbb{C}^r /\!/_\chi G \to \mathbb{C}^r /\!/ G = \mathrm{Spec}(\mathbb{C}[x_1,\ldots,x_r]^G)$.*

(b) $\mathbb{C}^r /\!/_\chi G \neq \emptyset$ *if and only if* $(\mathbb{C}^r)^{\mathsf{ss}}_\chi \neq \emptyset$.

(c) *The GIT quotient $\mathbb{C}^r /\!/_\chi G$ is a good categorical quotient of $(\mathbb{C}^r)^{\mathsf{ss}}_\chi$ under the action of G, i.e., $\mathbb{C}^r /\!/_\chi G \simeq (\mathbb{C}^r)^{\mathsf{ss}}_\chi /\!/ G$.*

(d) *If $(\mathbb{C}^r)^{\mathsf{s}}_\chi \neq \emptyset$, then the action of G on $(\mathbb{C}^r)^{\mathsf{s}}_\chi$ has a geometric quotient $(\mathbb{C}^r)^{\mathsf{s}}_\chi/G$ isomorphic to a nonempty open subset of $\mathbb{C}^r /\!/_\chi G$. Thus $\mathbb{C}^r /\!/_\chi G \simeq (\mathbb{C}^r)^{\mathsf{ss}}_\chi /\!/ G$ is an almost geometric quotient and* $\dim \mathbb{C}^r /\!/_\chi G = r - \dim G$.

Proof. The map $\mathrm{Proj}(R_\chi) \to \mathrm{Spec}((R_\chi)_0)$ is projective by Proposition 7.0.9, and then part (a) follows since $(R_\chi)_0 = \mathbb{C}[x_1,\dots,x_r]^G$. For part (b), first note that

$$(\mathbb{C}^r)^{\mathsf{ss}}_\chi = \emptyset \iff (R_\chi)_d = 0 \text{ for } d > 0.$$

The proof of Lemma 14.1.10 shows that R_χ is an integral domain, and then one sees easily that $\mathrm{Proj}(R_\chi) = \emptyset$ if and only if $(R_\chi)_d = 0$ for $d > 0$ (Exercise 14.1.5).

For part (c), let $R = \mathbb{C}[x_1,\dots,x_r,w]^G$ be the ring of invariants introduced in Lemma 14.1.10, which we grade by degree in w. The isomorphism of graded rings $R_\chi \simeq R$ from (14.1.3) implies that $\mathbb{C}^r /\!/_\chi G \simeq \mathrm{Proj}(R)$. Recall from §7.0 that $\mathrm{Proj}(R)$ is covered by open subsets $D_+(f) = \mathrm{Spec}(R_{(f)})$, where $f \in R_d$ is nonzero with $d > 0$ and

$$R_{(f)} = \Big\{ \frac{h}{f^\ell} \;\Big|\; h \in S_{\ell d},\ \ell \geq 0 \Big\}. \tag{14.1.4}$$

Now suppose that $f = F\,w^d$, where $F \in \mathbb{C}[x_1,\dots,x_r]$ is (G,χ^d)-invariant. Then F corresponds to global section $s \in \Gamma(\mathbb{C}^r, \mathscr{L}_{\chi^d})^G$ by Lemma 14.1.1. Furthermore, the nonvanishing set $(\mathbb{C}^r)_s \subseteq \mathbb{C}^r$ of s is

$$(\mathbb{C}^r)_s = (\mathbb{C}^r)_F = \mathrm{Spec}(\mathbb{C}[x_1,\dots,x_r]_F),$$

where $\mathbb{C}[x_1,\dots,x_r]_F$ is localization at F. In Exercise 14.1.6 you will show that

$$\frac{H}{F^\ell} \longmapsto \frac{H\,w^{\ell d}}{(F\,w^d)^\ell} = \frac{H\,w^{\ell d}}{f^\ell} \tag{14.1.5}$$

induces an isomorphism $(\mathbb{C}[x_1,\dots,x_r]_F)^G \simeq R_{(f)}$. This implies that

$$D_+(f) = \mathrm{Spec}(R_{(f)}) \simeq \mathrm{Spec}((\mathbb{C}[x_1,\dots,x_r]_F)^G) = (\mathbb{C}^r)_s /\!/ G.$$

Then part (c) follows since $\mathrm{Proj}(R)$ is covered by the open subsets $D_+(f)$ and $(\mathbb{C}^r)^{\mathsf{ss}}_\chi$ is covered by the affine open subsets $(\mathbb{C}^r)_s$. See [**83**, Sec. 8.2] for the details.

The first assertion of part (d) is a standard result in GIT (see [**83**, Thm. 8.1]). Thus the quotient map $\pi : (\mathbb{C}^r)^{\mathsf{ss}}_\chi \to \mathbb{C}^r /\!/_\chi G$ of part (c) is an almost geometric quotient as defined in §5.0. Finally, if p is a stable point, then $\pi^{-1}(\pi(p)) = G \cdot p \simeq G/G_p$, which has dimension $\dim G$ since p is stable. Then the dimension formula

$$\dim\, (\mathbb{C}^r)^{\mathsf{ss}}_\chi = \dim \mathbb{C}^r /\!/_\chi G \,+\, \dim \text{generic fiber of } \pi$$

(see [**245**, Thm. I.6.7]) shows that $\dim \mathbb{C}^r /\!/_\chi G = r - \dim G$. □

Remark 14.1.13. Part (d) of Proposition 14.1.12 has two useful consequences:

(a) If stable points exist, then $\mathbb{C}^r /\!/_\chi G$ has the *expected dimension* $r - \dim G$. In general, we always have $\dim \mathbb{C}^r /\!/_\chi G \leq r - \dim G$ (Exercise 14.1.4).

(b) G-orbits of stable points are closed in $(\mathbb{C}^r)^{\mathsf{ss}}_\chi$ (Exercise 14.1.4).

The properties of $\mathbb{C}^r /\!/_\chi G$ stated in Proposition 14.1.12 apply to more general GIT quotients, as explained in [**83**] and [**209**].

Here are two illustrations of Proposition 14.1.12.

Example 14.1.14. For $G = \{(t,t^{-1},u) \mid t \in \mathbb{C}^*, u = \pm 1\}$ and $\chi(t,t^{-1}u) = tu$, we have $(\mathbb{C}^3)^{\mathsf{ss}} = \mathbb{C}^* \times \mathbb{C}^2$ by Example 14.1.8. Since this is affine, Proposition 14.1.12 implies that

$$\mathbb{C}^3/\!/_\chi G \simeq (\mathbb{C}^* \times \mathbb{C}^2)/\!/G = \operatorname{Spec}(\mathbb{C}[x^{\pm 1},y,z]^G).$$

The ring of invariants is easily seen to be $\mathbb{C}[xy,z^2]$, so that

$$\mathbb{C}^3/\!/_\chi G \simeq \operatorname{Spec}(\mathbb{C}[xy,z^2]) \simeq \mathbb{C}^2.$$

Working directly from $\mathbb{C}^3/\!/_\chi G = \operatorname{Proj}(R_\chi)$ is harder since

$$R_\chi = \mathbb{C}[xy,z^2] \oplus xz\,\mathbb{C}[xy,z^2] \oplus x^2\,\mathbb{C}[xy,z^2] \oplus x^3z\,\mathbb{C}[xy,z^2] \oplus x^4\,\mathbb{C}[xy,z^2] \oplus \cdots$$

by (14.1.1). It is not obvious that Proj of this ring gives $\mathbb{C}^2$. We will see in §14.2 that the complications of this example come from a polyhedron whose vertices are not lattice points. $\Diamond$

Example 14.1.15. Consider $G = \{(t,t,u,u) \mid t,u \in \mathbb{C}^*\}$ and $\chi(t,t,u,u) = t$. Then a polynomial $F(x,y,z,w)$ is (G,χ^d)-invariant if and only if

$$F(tx,ty,uz,uw) = t^d F(x,y,z,w),$$

i.e., if and only if F has degree $(d,0)$ in the grading where x,y have degree $(1,0)$ and z,w have degree $(0,1)$. It follows that as graded rings,

$$R_\chi \simeq \mathbb{C}[x,y].$$

Thus $\mathbb{C}^4/\!/_\chi G = \operatorname{Proj}(\mathbb{C}[x,y]) = \mathbb{P}^1$. This is disconcerting since $\mathbb{C}^4$ has dimension 4 and G has dimension 2, yet the quotient only has dimension 1. The reason is that there are no stable points. In fact,

$$(\mathbb{C}^4)^{\mathsf{ss}}_\chi = \mathbb{C}^2 \times (\mathbb{C}^2 \setminus \{0\}),\ (\mathbb{C}^4)^{\mathsf{s}}_\chi = \emptyset.$$

Note that $(\mathbb{C}^4)^{\mathsf{s}}_\chi = \emptyset$ follows from part (d) Proposition 14.1.12 since the quotient does not have the expected dimension.

In contrast, if one uses the character defined by $\chi(t,t,u,u) = t^a u^b$ for $a,b > 0$, then $\mathbb{C}^4/\!/_\chi G = \mathbb{P}^1 \times \mathbb{P}^1$ and $(\mathbb{C}^4)^{\mathsf{s}} \neq \emptyset$. Once we introduce the secondary fan in §14.4, this example will be easy to understand. $\Diamond$

The GIT quotients in Examples 14.1.14 and 14.1.15 are toric varieties. This is no accident, as we will prove in the next section.

More General Toric GIT Quotients. So far, we have studied the quotient of $\mathbb{C}^r$ by a subgroup $G \subseteq (\mathbb{C}^*)^r$. An obvious generalization would be to consider the quotient of a toric variety X_Σ by a subgroup $G \subseteq T_N$. This question has been studied extensively in the literature. For more details about these quotients, interested readers should consult the papers [**3**] by A'Campo-Neuen and Hausen, [**149**] by Hu, and [**164**] by Kapranov, Sturmfels and Zelevinsky.

Exercises for §14.1.

14.1.1. Let $\pi : V \to X$ a vector bundle over a variety X. Suppose that an affine algebraic group G acts algebraically on X and V such that π is equivariant.

(a) Given a global section $s : X \to V$ of π, show that

$$(g \cdot s)(p) = g \cdot (s(g^{-1} \cdot p))$$

defines a global section of π.

(b) Show carefully that $g \cdot (h \cdot s) = (gh) \cdot s$ for $g, h \in G$.

14.1.2. Prove the isomorphism $\Gamma(\mathbb{C}^r, \mathscr{L}_{\chi^\beta})^G \simeq S_\beta$ from Example 14.1.4.

14.1.3. Prove that $(\mathbb{C}^3)^{\mathsf{s}}_\chi = (\mathbb{C}^3)^{\mathsf{ss}}_\chi = \mathbb{C}^* \times \mathbb{C}^2$ in Example 14.1.8.

14.1.4. This exercise concerns Remark 14.1.13.

(a) Prove that $\dim \mathbb{C}^r /\!\!/_\chi G \le r - \dim G$. Hint: Use [**245**, Thm. I.6.7].

(b) Prove that G-orbits of stable points are closed in $(\mathbb{C}^r)^{\mathsf{ss}}_\chi$. Hint: Use part (d) of Proposition 14.1.12 and remember that fibers are closed.

14.1.5. Let $R = \bigoplus_{d=0}^\infty R_d$ be a graded integral domain. Prove that $\mathrm{Proj}(R) = \emptyset$ if and only if $R_d = 0$ for all $d > 0$.

14.1.6. Prove that the map (14.1.5) in the proof of Proposition 14.1.12 induces a ring isomorphism $(\mathbb{C}[x_1, \dots, x_r]_F)^G \simeq R_{(f)}$.

14.1.7. Prove that if $\mathbb{C}[x_1, \dots, x_r]^G = \mathbb{C}$, then $\mathbb{C}^r /\!\!/_\chi G$ is projective. The converse is more subtle, as you will learn in the next two exercises.

14.1.8. Let $G = \{(t,t,1) \mid t \in \mathbb{C}^*\} \subseteq (\mathbb{C}^*)^3$ and $\chi(t,t,1) = t^{-1}$.

(a) Show that $(R_\chi)_0 = \mathbb{C}[x,y,z]^G = \mathbb{C}[z]$ and that $(R_\chi)_d = 0$ for $d > 0$.

(b) Conclude that the converse of Exercise 14.1.7 is false. Hint: $\emptyset$ is a projective variety.

14.1.9. The converse of Exercise 14.1.7 is false by Exercise 14.1.8. Fortunately, once we assume that $\mathbb{C}^r /\!\!/_\chi G \neq \emptyset$, everything is fine.

(a) Let $U \to V$ be a morphism of irreducible varieties such that the map $\mathbb{C}[V] \to \mathbb{C}[U]$ is injective. Prove that the image of U is Zariski dense in V.

(b) Suppose $(R_\chi)_d \neq 0$ for some $d > 0$. Use part (a) to prove that $\mathrm{Proj}(R_\chi) \to \mathrm{Spec}((R_\chi)_0)$ has Zariski dense image. Hint: Construct a map $(R_\chi)_0 \hookrightarrow (R_\chi)_{(f)}$ for $f \neq 0$ in $(R_\chi)_d$.

(c) Prove that if $\mathbb{C}^r /\!\!/_\chi G$ is projective and nonempty, then $\mathbb{C}[x_1, \dots, x_r]^G = \mathbb{C}$.

14.1.10. Prove that $\mathbb{C}^r /\!\!/_\chi G \simeq \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_r]^G)$ when χ is a torsion element of $\widehat{G}$.

§14.2. Toric GIT and Polyhedra

We continue to assume that $G \subseteq (\mathbb{C}^*)^r$ is an algebraic subgroup and $\chi \in \widehat{G}$ is a character. A full understanding of the GIT quotients introduced in §14.1 involves the interesting polyhedra associated to characters.

The Group G. Before discussing polyhedra, we need to study G. Since $\mathbb{Z}^r$ is the character group of $(\mathbb{C}^*)^r$, the inclusion $G \subseteq (\mathbb{C}^*)^r$ induces a homomorphism $\gamma : \mathbb{Z}^r \to \widehat{G}$ whose kernel we denote by M. Hence we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\delta} \mathbb{Z}^r \xrightarrow{\gamma} \widehat{G}. \tag{14.2.1}$$

The image of $\mathbf{a} \in \mathbb{Z}^r$ in $\widehat{G}$ will be written $\gamma(\mathbf{a}) = \chi^{\mathbf{a}}$. The dual of δ is a map $\mathbb{Z}^r \to N$, where N is the dual of M. The images of the standard basis $e_1,\dots,e_r \in \mathbb{Z}^r$ give elements $\nu_1,\dots,\nu_r \in N$, and the map δ in (14.2.1) can be written

$$\delta(m) = (\langle m,\nu_1\rangle,\dots,\langle m,\nu_r\rangle), \quad m \in M.$$

Maps like this appear in the exact sequence (4.1.3) for the class group of a toric variety, where the u_ρ, $\rho \in \Sigma(1)$, play the role of the ν_i. However, we will see in Example 14.2.4 below that the ν_i are not always as nice as the u_ρ.

Here is a basic result about G and its character group.

Lemma 14.2.1. *Let $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{C}^*)$.*

(a) *The map γ in* (14.2.1) *is surjective, so that we have an exact sequence*

$$0 \longrightarrow M \xrightarrow{\delta} \mathbb{Z}^r \xrightarrow{\gamma} \widehat{G} \longrightarrow 0. \tag{14.2.2}$$

(b) $(\mathbb{C}^*)^r/G \simeq T_N$, *so that we have an exact sequence of algebraic groups*

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^r \longrightarrow T_N \longrightarrow 1.$$

(c) $G = \{(t_1,\dots,t_r) \in (\mathbb{C}^*)^r \mid \prod_{i=1}^r t_i^{\langle m,\nu_i\rangle} = 1 \text{ for all } m \in M\}$.

Proof. We begin with part (b). Since G is reductive and G-orbits are closed in $(\mathbb{C}^*)^r$, we have a geometric quotient $(\mathbb{C}^*)^r/G = \mathrm{Spec}(\mathbb{C}[x_1^{\pm 1},\dots,x_r^{\pm 1}]^G)$. One checks that Laurent monomial $x^{\mathbf{b}}$ is G-invariant if and only $\gamma(\mathbf{b}) \in \widehat{G}$ is the trivial character. By (14.2.1), it follows that $\mathbb{C}[x_1^{\pm 1},\dots,x_r^{\pm 1}]^G = \mathbb{C}[M]$. Part (b) follows immediately since $T_N = \mathrm{Spec}(\mathbb{C}[M])$.

For part (c), consider $(\mathbb{C}^*)^r \to T_N = \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{C}^*)$. In Exercise 14.2.1 you will show that this map takes $(t_1,\dots,t_r) \in (\mathbb{C}^*)^r$ to $\phi \in \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{C}^*)$ defined by $\phi(m) = \prod_{i=1}^r t_i^{\langle m,\nu_i\rangle}$. Then we are done since G is the kernel of this map by part (b).

Finally, for part (a), let n be the rank of M, which is also the rank of δ. Then pick bases of M and $\mathbb{Z}^r$ such that δ has Smith normal form

$$\delta = \begin{pmatrix} d_1 & & \cdots & & 0 \\ & \ddots & & & \\ \vdots & & d_n & & \vdots \\ & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix},$$

where $d_1 | d_2 | \cdots | d_n$ are positive and all other entries are zero. Using the dual basis $e_1,\dots,e_n$ of N, this means that δ is given by $\nu_i' = d_i e_i$ for $1 \le i \le n$ and $\nu_i' = 0$ for

$i > n$. Then applying part (c) gives a description of G which makes it obvious that characters of G extend to characters of $(\mathbb{C}^*)^r$ (Exercise 14.2.1). $\square$

The Polyhedron of a Character. We will give two models of the polyhedron of $\chi \in \widehat{G}$. The first model is less intrinsic but has a nice relation to the treatment of polytopes and polyhedra given in Chapters 2 and 7.

For a character $\chi = \chi^{\mathbf{a}} \in \widehat{G}$, where $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$, define the polyhedron

$$P_{\mathbf{a}} = \{m \in M_{\mathbb{R}} \mid \langle m, \nu_i \rangle \geq -a_i,\ 1 \leq i \leq r\} \subseteq M_{\mathbb{R}}. \tag{14.2.3}$$

For fixed χ, the various choices of $\mathbf{a}$ differ by elements of $\delta(M)$, so the polyhedra $P_{\mathbf{a}}$ are translates of each other by elements of M.

Here is an important property of these polyhedra.

Lemma 14.2.2. *If $P_{\mathbf{a}} \neq \emptyset$, then the recession cone of $P_{\mathbf{a}}$ is strongly convex.*

Proof. Assume that $P_{\mathbf{a}}$ from (14.2.3) is nonempty. Then its recession cone is $C = \{m \in M_{\mathbb{R}} \mid \langle m, \nu_i \rangle \geq 0,\ 1 \leq i \leq r\}$, so that $C \cap (-C)$ is defined by $\langle m, \nu_i \rangle = 0$ for $1 \leq i \leq r$. Thus $C \cap (-C) = \{0\}$ since the map δ in (14.2.2) is injective. $\square$

Recall from §7.1 that $P \subseteq M_{\mathbb{R}}$ is a lattice polyhedron when its recession cone is strongly convex and its vertices are lattice points. Furthermore, when $\dim P = \dim M_{\mathbb{R}}$, we constructed the toric variety X_P with torus T_N. Lemma 14.2.2 shows that $P_{\mathbf{a}}$ has the right kind of recession cone. Unfortunately, it may have the wrong dimension and its vertices may fail to be lattice points. Here are some examples.

Example 14.2.3. Consider $G = \{(t,t,u,u) \mid t,u \in \mathbb{C}^*\}$ and $\chi(t,t,u,u) = t$. Then $\chi = \chi^{\mathbf{a}}$ for $\mathbf{a} = (1,0,0,0)$, and the polyhedron $P_{\mathbf{a}}$ is the line segment

$$P_{\mathbf{a}} = \mathrm{Conv}(0, e_1) \subseteq M_{\mathbb{R}} = \mathbb{R}^2$$

(Exercise 14.2.2). Hence $P_{\mathbf{a}}$ is not full dimensional. $\Diamond$

Example 14.2.4. For the group $G = \{(t, t^{-1}, u) \mid t \in \mathbb{C}^*, u = \pm 1\} \subseteq (\mathbb{C}^*)^3$, the exact sequence (14.2.2) is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\ \delta\ } \mathbb{Z}^3 \xrightarrow{\ \gamma\ } \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0,$$

where $\delta(m) = (\langle m, e_1 \rangle, \langle m, e_1 \rangle, \langle m, 2e_2 \rangle)$ and $\gamma(a,b,c) = (a-b, c \bmod 2)$. Thus $\nu_1 = \nu_2 = e_1$ and $\nu_3 = 2e_2$, so these ν_i are neither distinct nor primitive. This contrasts with toric case, where the ν_i are the ray generators of a fan and hence are distinct and primitive.

The character $\chi(t, t^{-1}u) = tu$ from Example 14.1.14 is $\chi = \chi^{\mathbf{a}}$ for $\mathbf{a} = (1,0,1)$. Then $P_{\mathbf{a}} \subseteq M_{\mathbb{R}} = \mathbb{R}^2$ is defined by the inequalities

$$\langle m, e_1 \rangle \geq -1,\ \langle m, e_1 \rangle \geq 0,\ \langle m, 2e_2 \rangle \geq -1.$$

Figure 1 on the next page shows $P_{\mathbf{a}}$. Note that the vertex of $P_{\mathbf{a}}$ is *not* a lattice point. We also see that the first inequality defining $P_{\mathbf{a}}$ is redundant. $\Diamond$

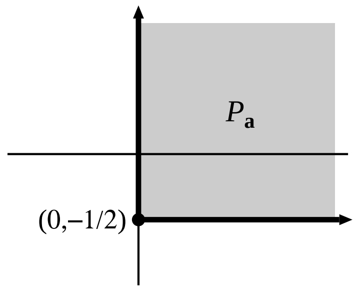


Figure 1. The polyhedron $P_{\mathbf{a}} \subseteq \mathbb{R}^2$ for $\mathbf{a} = (1,0,1)$ in Example 14.2.4

Example 14.2.5. For the usual action of $\mathbb{C}^*$ on $\mathbb{C}^r$, the exact sequence (14.2.2) is

$$0 \longrightarrow \mathbb{Z}^{r-1} \xrightarrow{\delta} \mathbb{Z}^r \xrightarrow{\gamma} \mathbb{Z} \longrightarrow 0,$$

where δ is defined using $\nu_i = e_i$, $1 \le i \le r-1$ and $\nu_r = -e_1 - \cdots - e_{r-1}$.

Let $\chi = \chi^{\mathbf{a}}$ for $\mathbf{a} = (0,\ldots,0,d) \in \mathbb{Z}^r$. Then:

$$P_{\mathbf{a}} = \begin{cases} d\Delta_{r-1} & d > 0 \\ \{0\} & d = 0 \\ \emptyset & d < 0. \end{cases}$$

Theorem 14.2.13 below will explain how this relates to Example 14.1.7. $\Diamond$

The polyhedron of $\chi \in \widehat{G}$ has a second, more intrinsic model P_χ described as follows. Tensoring (14.2.2) with $\mathbb{R}$ gives an exact sequence of vector spaces

$$0 \longrightarrow M_{\mathbb{R}} \xrightarrow{\delta_{\mathbb{R}}} \mathbb{R}^r \xrightarrow{\gamma_{\mathbb{R}}} \widehat{G}_{\mathbb{R}} \longrightarrow 0, \tag{14.2.4}$$

where $\widehat{G}_{\mathbb{R}} = \widehat{G} \otimes_{\mathbb{Z}} \mathbb{R}$. Also let $\chi \otimes 1 \in \widehat{G}_{\mathbb{R}}$ be the image of χ under the map $\widehat{G} \to \widehat{G}_{\mathbb{R}}$. Note that this map is not injective when $\widehat{G}$ has torsion, which happens precisely when G is not a torus.

Then we get the polyhedron $P_\chi \subseteq \mathbb{R}^r$ defined by

$$P_\chi = \{\mathbf{b} \in \mathbb{R}^r_{\ge 0} \mid \gamma_{\mathbb{R}}(\mathbf{b}) = \chi \otimes 1\} = \gamma_{\mathbb{R}}^{-1}(\chi \otimes 1) \cap \mathbb{R}^r_{\ge 0}. \tag{14.2.5}$$

It is straightforward to show that if $\chi = \chi^{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{Z}^r$, then

$$P_\chi = \delta_{\mathbb{R}}(P_{\mathbf{a}}) + \mathbf{a}.$$

In other words, if $\mathbf{a} = (a_1,\ldots,a_r)$, then

$$P_\chi = \{(\langle m,\nu_1\rangle + a_1, \ldots \langle m,\nu_r\rangle + a_r) \mid m \in P_{\mathbf{a}}\} \tag{14.2.6}$$

(Exercise 14.2.3). The key point is that the inequalities (14.2.3) defining $P_{\mathbf{a}}$ are equivalent to saying that $(\langle m,\nu_1\rangle + a_1, \ldots \langle m,\nu_r\rangle + a_r)$ lies in $\mathbb{R}^r_{\ge 0}$.

Via (14.2.6), lattice points of $P_{\mathbf{a}}$ correspond to the points

$$\delta_{\mathbb{R}}(P_{\mathbf{a}} \cap M) + \mathbf{a} = \gamma^{-1}(\chi) \cap \mathbb{N}^r \subseteq \gamma_{\mathbb{R}}^{-1}(\chi \otimes 1) \cap \mathbb{N}^r = P_\chi \cap \mathbb{Z}^r.$$

One subtle feature of P_χ is that the inclusion $\delta_\mathbb{R}(P_\mathbf{a} \cap M) + \mathbf{a} \subseteq P_\chi \cap \mathbb{Z}^r$ is strict when $\widehat{G}$ has torsion. Here is an example.

Example 14.2.6. For the polyhedron $P_\mathbf{a}$ shown in Figure 1 of Example 14.2.4, we have $\mathbf{a} = (1,0,1)$ and $\nu_1 = \nu_2 = e_1,\ \nu_3 = 2e_2$. Thus

$$P_\chi = \{(\langle m, e_1\rangle + 1, \langle m, e_1\rangle, \langle m, 2e_2\rangle + 1) \mid m \in P_\mathbf{a}\}.$$

Figure 2 shows $P_\mathbf{a}$ and P_χ. The vertex $(1,0,0) \in P_\chi \cap \mathbb{Z}^3$ does *not* come from a lattice point in $P_\mathbf{a} \cap M$. $\Diamond$

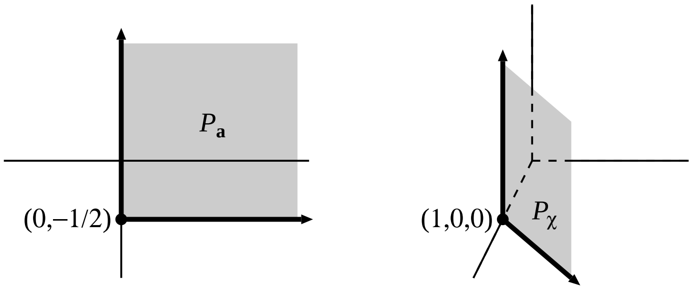


Figure 2. The polyhedra $P_\mathbf{a}$ and P_χ

In the literature, this difficulty is often avoided by assuming that G is a torus, so that $\widehat{G}$ is torsion-free (see, for example, [**137**]).

Our goal is to prove that $\mathbb{C}^r /\!/_\chi G$ is a toric variety. The rough intuition is that $\mathbb{C}^r /\!/_\chi G$ is the toric variety of $P_\mathbf{a} \subseteq M_\mathbb{R}$. However, we can't apply the theory of Chapter 7 directly since $P_\mathbf{a}$ may have the wrong kind of vertices and the wrong dimension. Our next task is to develop some tools to address these problems.

Veronese Subrings. The polyhedron $P_\mathbf{a} \subseteq M_\mathbb{R}$ need not be a lattice polyhedron since its vertices may not be lattice points. However, the vertices are rational, so some integer multiple will be a lattice polyhedron. We will see that when translated into algebra, this gives a *Veronese subring* of a graded semigroup algebra.

Before defining Veronese subrings in general, we recall a classic example that explains where the name comes from.

Example 14.2.7. For a positive integer ℓ, the line bundle $\mathscr{O}_{\mathbb{P}^n}(\ell)$ on $\mathbb{P}^n$ gives the ℓth Veronese embedding $\mathbb{P}^n \simeq X \subseteq \mathbb{P}^{\binom{n+\ell}{n}-1}$. If we let $\mathbb{C}[X]$ be the homogeneous coordinate ring of X as a subvariety of $\mathbb{P}^{\binom{n+\ell}{n}-1}$, then

$$\mathbb{P}^n \simeq X \simeq \mathrm{Proj}(\mathbb{C}[X]).$$

However, the total coordinate ring of $\mathbb{P}^n$ as a toric variety is $S = \mathbb{C}[x_0, \ldots, x_n]$, and the ℓth Veronese embedding is determined by the polytope $P = \ell\Delta_n$, where Δ_n is the standard n-simplex. Since P is normal, Theorem 5.4.8 implies that

$$\mathbb{C}[X] = \mathbb{C}[X_P] \simeq \bigoplus_{d=0}^{\infty} S_{d\ell} \subseteq \bigoplus_{d=0}^{\infty} S_d = S.$$

This is the ℓth *Veronese subring* of S. When we combine the above two displays, we see that $\mathbb{P}^n$ is the Proj of not only S but also of its Veronese subrings. ◊

More generally, let $R = \bigoplus_{d=0}^{\infty} R_d$ be a graded ring that is finitely generated as a $\mathbb{C}$-algebra. Given a positive integer ℓ, the ℓth *Veronese subring* of R is

$$R^{[\ell]} = \bigoplus_{d=0}^{\infty} R_{d\ell} \subseteq \bigoplus_{d=0}^{\infty} R_d = R,$$

so $(R^{[\ell]})_d = R_{d\ell}$. Then Example 14.2.7 generalizes as follows (Exercise 14.2.4).

Lemma 14.2.8. *Given R and a Veronese subring $R^{[\ell]}$ as above, there is a natural isomorphism* $\mathrm{Proj}(R^{[\ell]}) \simeq \mathrm{Proj}(R)$. □

Here is an example of how to use Lemma 14.2.8.

Example 14.2.9. The coordinate ring A of an affine variety gives the graded ring $A[x_0,\ldots,x_n]$, where all variables have degree 1. In Example 7.0.10, we noted that

$$\mathrm{Proj}(A[x_0,\ldots,x_n]) = \mathrm{Spec}(A) \times \mathbb{P}^n.$$

In particular, when $n = 0$, we have

$$\mathrm{Proj}(A[x_0]) = \mathrm{Spec}(A) \times \mathbb{P}^0 = \mathrm{Spec}(A).$$

Now consider the complicated graded ring R_χ from Example 14.1.14. The Veronese subring

$$R_\chi^{[2]} = \mathbb{C}[xy,z^2] \oplus x^2\,\mathbb{C}[xy,z^2] \oplus x^4\,\mathbb{C}[xy,z^2] \oplus x^6\,\mathbb{C}[xy,z^2] \oplus \cdots$$

is a polynomial ring in one variable over $\mathbb{C}[xy,z^2]$, i.e., $R_\chi^{[2]} = \mathbb{C}[xy,z^2][x^2]$. Thus

$$\mathbb{C}^3/\!/_\chi\, G = \mathrm{Proj}(R_\chi) \simeq \mathrm{Proj}(R_\chi^{[2]}) = \mathrm{Spec}(\mathbb{C}[xy,z^2]) \simeq \mathbb{C}^2.$$

We now understand this example from the Proj point of view. ◊

The Normal Fan of a Polyhedron. Previously, we have given three constructions of the normal fan:

- From §2.3: The normal fan of a full dimensional lattice polytope $P \subseteq M_\mathbb{R}$ is a complete fan in $N_\mathbb{R}$.
- From §6.2: The normal fan of a lattice polytope $P \subseteq M_\mathbb{R}$ is a complete generalized fan in $N_\mathbb{R}$. This fan is degenerate when P is not full dimensional.
- From §7.1: The normal fan of a full dimensional lattice polyhedron $P \subseteq M_\mathbb{R}$ is a fan in $N_\mathbb{R}$ whose support is the dual of the recession cone of P.

It can happen that none of these constructions apply to the polyhedra $P_{\mathbf{a}} \subseteq M_{\mathbb{R}}$ considered here. Hence we need a fourth construction of a normal fan. You may want to review the discussion of generalized fans given in §6.2.

Let $P \subseteq M_{\mathbb{R}}$ be a polyhedron with rational vertices such that the recession cone of P is strongly convex and rational. A vertex $\mathsf{v} \in P$ gives the cone

$$C_{\mathsf{v}} = \mathrm{Cone}(P \cap M_{\mathbb{Q}} - \mathsf{v}) \subseteq M_{\mathbb{R}}.$$

This is similar to our earlier constructions, except that we use $P \cap M_{\mathbb{Q}}$ instead of $P \cap M$ since P may have few lattice points (in fact, it could have none). The dual cone $C_{\mathsf{v}}^{\vee} \subseteq N_{\mathbb{R}}$ is a rational polyhedral cone, and these cones give a generalized fan as follows (Exercise 14.2.5).

Proposition 14.2.10. *Let $C \subseteq M_{\mathbb{R}}$ be the recession cone of P. Then*

$$\Sigma_P = \{\sigma \mid \sigma \preceq C_{\mathsf{v}}^{\vee},\ \mathsf{v} \text{ is a vertex of } P\}$$

*is a generalized fan in $N_{\mathbb{R}}$ called the **normal fan** of P. Furthermore, $|\Sigma_P| = C^{\vee}$, and Σ_P is a fan if and only if $P \subseteq M_{\mathbb{R}}$ is full dimensional.* □

The toric variety X_P is then defined to be the toric variety of the generalized fan X_{Σ_P}, i.e., $X_P = X_{\Sigma_P}$. Here is an example we studied in §6.2.

Example 14.2.11. Let D be a basepoint free torus-invariant Cartier divisor on a complete toric variety X_Σ. From D we get $P_D \subseteq M_{\mathbb{R}}$, which is a lattice polytope by Theorem 6.1.7, though it may fail to be full dimensional. The normal fan Σ_{P_D} is a generalized fan that gives the toric variety X_{P_D} featured in Theorem 6.2.8.

We note that generalized fans appear naturally in this situation. The vertices of P_D give the Cartier data $\{m_\sigma\}_{\sigma \in \Sigma(n)}$ of D, where $n = \dim X_\Sigma$. The m_σ are distinct when D is ample, but some may coincide in the basepoint free case. Combining those σ's for which m_σ has a common value, say $\mathsf{v} \in P_D$, gives the union

$$\sigma_{\mathsf{v}} = \bigcup_{\substack{\sigma \in \Sigma(n) \\ m_\sigma = \mathsf{v}}} \sigma$$

By Proposition 6.2.5, this union is a maximal cone in the normal fan of P_D in $N_{\mathbb{R}}$. You should reread the discussion following the proof of Proposition 6.2.5.

We will soon see that X_{P_D} has a natural interpretation as a GIT quotient. ◊

Similar to §7.1, a polyhedron $P \subseteq M_{\mathbb{R}}$ gives a cone $C(P) \subseteq M_{\mathbb{R}} \times \mathbb{R}$ whose slice at height $\lambda > 0$ (resp. $\lambda = 0$) is λP (resp. the recession cone of P). The associated semigroup algebra

$$S_P = \mathbb{C}[C(P) \cap (M \times \mathbb{Z})]$$

is graded as usual using the last coordinate. We now generalize Theorem 7.1.13.

Proposition 14.2.12. *For a polyhedron $P \subseteq M_{\mathbb{R}}$ as above, we have $X_P \simeq \mathrm{Proj}(S_P)$. Furthermore, X_P is semiprojective.*

Proof. The polyhedron P gives the (possibly degenerate) normal fan Σ_P. First note that multiplying P by a positive integer ℓ has no effect on the normal fan Σ_P. Also, Lemma 14.2.8 and the obvious isomorphism $S_P^{[\ell]} \simeq S_{\ell P}$ imply that

$$\mathrm{Proj}(S_P) \simeq \mathrm{Proj}(S_P^{[\ell]}) \simeq \mathrm{Proj}(S_{\ell P}).$$

Hence we may assume that P is a lattice polyhedron. Note also that translating P by $m \in M$ has no effect on Σ_P. Since $S_{P+m} \simeq S_P$, we may assume that $0 \in P$.

The minimal cone $\sigma_0 \in \Sigma_P$ is the largest subspace of $N_{\mathbb{R}}$ contained in every cone of the normal fan. As we saw in §6.2, Σ_P projects to a genuine fan $\overline{\Sigma}_P$ in $\overline{N}_{\mathbb{R}} = (N/\sigma_0)_{\mathbb{R}}$, and X_P is the toric variety of this fan. Since $0 \in P$, it follows that:

- $\sigma_0^\perp$ is the smallest subspace of $M_{\mathbb{R}}$ containing P.
- $\overline{M} = \sigma_0^\perp \cap M \hookrightarrow M$ is dual to the map $N \to \overline{N}$.
- P is a full dimensional lattice polytope in $\overline{M}_{\mathbb{R}}$ whose normal fan in $\overline{N}_{\mathbb{R}}$ is given by $\overline{\Sigma}_P$.

The straightforward proofs are left to the reader as Exercise 14.2.6. The last bullet and Theorem 7.1.13 imply that

$$X_P \simeq \mathrm{Proj}(\mathbb{C}[C(P) \cap (\overline{M} \times \mathbb{Z})]).$$

Then the first bullet implies

$$C(P) \cap (M \times \mathbb{Z}) = C(P) \cap (\overline{M} \times \mathbb{Z}),$$

which gives $X_P \simeq \mathrm{Proj}(S_P)$.

Finally, we show that X_P is semiprojective. As above, we may assume that $P \subseteq M_{\mathbb{R}}$ is a lattice polyhedron containing the origin. Then P is a full dimensional lattice polyhedron in $\overline{M}_{\mathbb{R}}$, so X_P is semiprojective by Proposition 7.2.9. □

GIT Quotients are Toric. We can now prove that $\mathbb{C}^r /\!/_\chi G$ is a toric variety.

Theorem 14.2.13. *For $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, pick $\mathbf{a} \in \mathbb{Z}^r$ such that $\chi = \chi^{\mathbf{a}}$. Then:*

(a) *The graded ring R_χ from* (14.1.2) *satisfies $R_\chi \simeq \mathbb{C}[C(P_{\mathbf{a}}) \cap (M \times \mathbb{Z})]$.*

(b) *The GIT quotient $\mathbb{C}^r /\!/_\chi G = \mathrm{Proj}(R_\chi)$ is the toric variety of $P_{\mathbf{a}} \subseteq M_{\mathbb{R}}$.*

Proof. We first note that R_χ is the semigroup algebra of the semigroup given by the disjoint union

$$\mathsf{S}_\chi = \textstyle\coprod_{d=0}^\infty \gamma^{-1}(\chi^d) \cap \mathbb{N}^r,$$

where γ is from (14.2.2). To see why, note that $x^{\mathbf{b}} \in \mathbb{C}[x_1,\dots,x_r]$ is (G,χ^d)-invariant if and only if $\gamma(\mathbf{b}) = \chi^d$. Then (14.1.3) gives an isomorphism

$$R_\chi \simeq \textstyle\bigoplus_{d=0}^\infty \{F \in \mathbb{C}[x_1,\dots,x_r] \mid F \text{ is } (G,\chi^d)\text{-invariant}\} = \mathbb{C}[\mathsf{S}_\chi].$$

Lemma 14.2.2 implies that $P_{\mathbf{a}} \subseteq M_{\mathbb{R}}$ has a strongly convex rational recession cone, and the description of $P_{\mathbf{a}}$ given in (14.2.3) shows that the vertices are rational.

Hence Propositions 14.2.10 and 14.2.12 apply to $P_{\mathbf{a}}$. In particular, we get the cone $C(P_{\mathbf{a}}) \subseteq M_{\mathbb{R}} \times \mathbb{R}$, and $(m,k) \mapsto \delta(m) + k\,\mathbf{a}$ induces a semigroup isomorphism

$$C(P_{\mathbf{a}}) \cap (M \times \mathbb{Z}) \simeq \mathsf{S}_{\chi} \tag{14.2.7}$$

(Exercise 14.2.7). It follows that $R_\chi \simeq \mathbb{C}[C(P_{\mathbf{a}}) \cap (M \times \mathbb{Z})]$. Thus

$$\mathbb{C}^r /\!/_\chi\, G = \mathrm{Proj}(R_\chi) \simeq \mathrm{Proj}(\mathbb{C}[C(P_{\mathbf{a}}) \cap (M \times \mathbb{Z})]) \simeq X_{P_{\mathbf{a}}},$$

where the last isomorphism follows from Proposition 14.2.12. □

In [**204**, Def. 10.8], Miller and Sturmfels *define* a toric variety to be a GIT quotient of $\mathbb{C}^r$ by G. They use the notation $\mathbb{C}^r /\!/_{\mathbf{a}}\, G$, which for us means $\mathbb{C}^r /\!/_{\chi^{\mathbf{a}}}\, G$. Versions of Theorem 14.2.13 appear in [**83**, Ch. 12], [**204**, Ch. 10] and [**232**].

In Theorem 14.2.13, we used the polyhedron $P_{\mathbf{a}}$ rather than the more intrinsic model P_χ. The reason goes back to Example 14.2.4, where we learned that lattice points in $P_\chi \subseteq \mathbb{R}^r$ relative to $\mathbb{Z}^r$ need not come from lattice points in $P_{\mathbf{a}}$. When $\widehat{G}$ has no torsion (i.e., when G is a torus), this problem goes away and $\mathbb{C}^r /\!/_\chi\, G$ is the toric variety of the polyhedron $P_\chi \subseteq \mathbb{R}^r$.

Here are some examples that illustrate Theorem 14.2.13.

Example 14.2.14. Let X_Σ be a projective toric variety with quotient construction $X_\Sigma = (\mathbb{C}^{\Sigma}(1) \setminus Z(\Sigma))/G$. Then a divisor class $[D] = [\sum_\rho a_\rho D_\rho] \in \mathrm{Cl}(X_\Sigma)$ gives a character $\chi \in \widehat{G}$. Note also that $\chi = \chi^{\mathbf{a}}$ for $\mathbf{a} = (a_\rho) \in \mathbb{Z}^{\Sigma(1)}$. Furthermore,

$$P_{\mathbf{a}} = P_D \subseteq M_{\mathbb{R}},$$

where P_D is from (4.3.2). There are two cases where we know the GIT quotient:

- When D is ample, Σ is the normal fan of $P_D = P_{\mathbf{a}}$. Hence

  $$\mathbb{C}^r /\!/_\chi\, G \simeq X_\Sigma$$

  by Theorem 14.2.13. This also follows from Propositions 14.1.9 and 14.1.12.
- When D is nef, Theorem 14.2.13 implies that

  $$\mathbb{C}^r /\!/_\chi\, G \simeq X_{P_D},$$

  where X_{P_D} is the toric variety from Example 14.2.11 and Theorem 6.2.8.

We will pursue this example in Chapter 15 when we study the secondary fan. ◊

Example 14.2.15. The affine space $\mathrm{Sym}_3(\mathbb{C})$ of 3×3 symmetric matrices contains the torus $G \simeq (\mathbb{C}^*)^3$ consisting of the rank 1 matrices

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} = \begin{pmatrix} t_1^2 & t_1t_2 & t_1t_3 \\ t_1t_2 & t_2^2 & t_2t_3 \\ t_1t_3 & t_2t_3 & t_3^2 \end{pmatrix}, \quad t_1, t_2, t_3 \in \mathbb{C}^*.$$

We will compute $\mathrm{Sym}_3(\mathbb{C})/\!/_\chi G$ for the character χ that sends the above matrix to $t_1^2t_2^2t_3^2$. Write $\mathrm{Sym}_3(\mathbb{C}) = \mathrm{Spec}(\mathbb{C}[x_{11},x_{12},x_{13},x_{22},x_{23},x_{33}]) \simeq \mathbb{C}^6$ and note that

$$\det\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33}\end{pmatrix} = x_{11}x_{22}x_{33} + 2x_{12}x_{23}x_{13} - x_{11}x_{23}^2 - x_{22}x_{13}^2 - x_{33}x_{12}^2.$$

The monomials appearing in the determinant form a basis of the (G,χ)-invariant polynomials. This explains our choice χ. Label these monomials as

$$X_0 = x_{12}x_{23}x_{13},\ X_1 = x_{11}x_{23}^2,\ X_2 = x_{22}x_{13}^2,\ X_3 = x_{33}x_{12}^2,\ X_4 = x_{11}x_{22}x_{33}$$

and note that $X_0^2X_4 = X_1X_2X_3$. Using coordinates $Y_0,\dots,Y_4$ on $\mathbb{P}^4$, we claim that

$$\mathrm{Sym}_3(\mathbb{C})/\!/_\chi G \simeq \mathbf{V}(Y_0^2Y_4 - Y_1Y_2Y_3) \subseteq \mathbb{P}^4. \tag{14.2.8}$$

We will sketch the proof, leaving the details as Exercise 14.2.8. The map $\mathbb{Z}^6 \to \widehat{G} \simeq \mathbb{Z}^3$ is given by $e_{ij} \mapsto e_i + e_j$, where $e_{11},e_{12},e_{13},e_{22},e_{23},e_{33}$ is the basis of $\mathbb{Z}^6$. Also observe that χ corresponds to $(2,2,2) \in \mathbb{Z}^3$, so that $\chi = \chi^{\mathbf{a}}$ for $\mathbf{a} = (0,1,1,0,1,0) \in \mathbb{Z}^6$. In Example 14.2.18 below we will see that the ν_i's are

$$\nu_1 = (1,0,0),\ \nu_2 = (-1,-1,1),\ \nu_3 = (-1,1,-1),$$
$$\nu_4 = (0,1,0),\ \nu_5 = (1,-1,-1),\ \nu_6 = (0,0,1).$$

This gives the polytope $P_{\mathbf{a}}$ in Figure 3. The only lattice points of $P_{\mathbf{a}}$ are its vertices $m_0,\dots,m_4$, which map to $\delta(m_0)+\mathbf{a},\dots,\delta(m_4)+\mathbf{a}$, the exponent vectors of the (G,χ)-invariant monomials $X_0,\dots,X_4$. Furthermore, since $P_{\mathbf{a}}$ is a normal polytope, the monomials X_i generate R_χ and give a projective embedding into $\mathbb{P}^4$. From here, (14.2.8) follows easily. We thank Igor Dolgachev for showing us this example. $\Diamond$

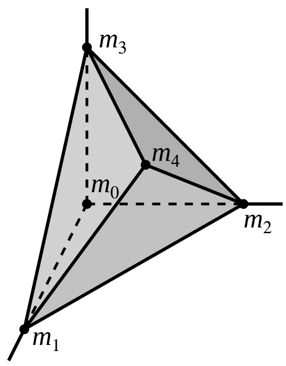


Figure 3. $P_{\mathbf{a}}$ and its lattice points $m_0,\dots,m_4$

Since $P_{\mathbf{a}} \simeq P_\chi$, Proposition 14.2.12 and Theorem 14.2.13 have the following immediate corollary.

Corollary 14.2.16. *If $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$, then the GIT quotient $\mathbb{C}^r /\!/_\chi G$ is a semiprojective toric variety of dimension*

$$\dim \mathbb{C}^r /\!/_\chi G = \dim P_\chi. \qquad \square$$

Multigraded Polynomial Rings. This section began with an algebraic subgroup G of $(\mathbb{C}^*)^r$. Taking character groups, we obtained the exact sequence

$$0 \longrightarrow M \xrightarrow{\ \delta\ } \mathbb{Z}^r \xrightarrow{\ \gamma\ } \widehat{G} \longrightarrow 0$$

from (14.2.2). This gives a multigrading on the polynomial ring $S = \mathbb{C}[x_1,\dots,x_r]$ by defining $\deg(x^{\mathbf{a}}) = \gamma(\mathbf{a}) \in \widehat{G}$. An example is given by the grading on the total coordinate ring of a toric variety without torus factors.

In particular, a character $\chi \in \widehat{G}$ gives the graded piece S_χ of S, and one checks that $\Gamma(\mathbb{C}^r, \mathscr{L}_\chi) \simeq S_\chi$. It follows that the graded ring R_χ used in the definition of $\mathbb{C}^r /\!/_\chi G$ is isomorphic to the ring

$$R_\chi \simeq \textstyle\bigoplus_{d=0}^{\infty} S_{\chi^d}. \tag{14.2.9}$$

We will see below that χ determines an irrelevant ideal $B(\chi) \subseteq S$ and leads to a quotient construction remarkably similar to what we did in Chapter 5.

Conversely, suppose that we start with a multigrading on $\mathbb{C}[x_1,\dots,x_r]$ given by a surjective homomorphism $\delta : \mathbb{Z}^r \to A$ for a finitely generated abelian group A. Applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$, we get an inclusion

$$G = \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{C}^*) \subseteq (\mathbb{C}^*)^r = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^r,\mathbb{C}^*)$$

such that $\widehat{G} \simeq A$. Hence the multigraded approach, featured in [**190**] and [**204**], is fully equivalent to what we are doing here.

Matrices. When G is a torus, we have $\widehat{G} \simeq \mathbb{Z}^s$. This implies $M \simeq \mathbb{Z}^{r-s}$, so that the exact sequence (14.2.2) can be written

$$0 \longrightarrow \mathbb{Z}^{r-s} \xrightarrow{\ B\ } \mathbb{Z}^r \xrightarrow{\ A\ } \mathbb{Z}^s \longrightarrow 0, \tag{14.2.10}$$

where B and A are integer matrices of respective sizes $r \times (r-s)$ and $s \times r$ such that $AB = 0$. Note that the r rows of B give $\nu_1,\dots,\nu_r \in N = \mathbb{Z}^{r-s}$.

Example 14.2.17. Let $r \in \mathbb{N}$ and consider the exact sequence

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\ B\ } \mathbb{Z}^4 \xrightarrow{\ A\ } \mathbb{Z}^2 \longrightarrow 0,$$

where

$$B = \begin{pmatrix} -1 & r \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & -r & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

This is the sequence $0 \to M \to \mathbb{Z}^4 \to \mathrm{Cl}(\mathscr{H}_r) \to 0$ that computes the class group of the Hirzebruch surface $\mathscr{H}_r$. The rows $\nu_1, \dots, \nu_4$ of B are the minimal generators u_i of the fan of $\mathscr{H}_r$ shown in Figure 3 of Example 4.1.8.

Here the group is $G = \{(t, t^{-r}u, t, u) \mid t, u \in \mathbb{C}^*\}$. Note that the first row of A gives the t-exponent of elements of G and the second row gives the u-exponent. Also, the last two columns of A show that the divisor classes $[D_3], [D_4]$ corresponding to u_3, u_4 give a basis of the class group, and the first two columns show how to express $[D_1], [D_2]$ in terms of this basis. $\Diamond$

Example 14.2.18. In Example 14.2.15, the exact sequence (14.2.2) can be written

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{B} \mathbb{Z}^6 \xrightarrow{A} \mathbb{Z}^3 \longrightarrow 0,$$

where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

The matrix A gives the map $e_{ij} \mapsto e_i + e_j$ from Example 14.2.15, and given A, it is easy to find a matrix B that makes the sequence exact. We call B the *Gale dual* of A. Note that the ν_i's are the rows of B. See §14.2 for more on Gale duality. $\Diamond$

As shown by Examples 14.2.17 and 14.2.18, the matrices B and A make it easy to determine the vectors ν_i and the group G.

Virtual Facets. The facets of a polyhedron contain a lot of information about its geometry and combinatorics. For a GIT quotient $\mathbb{C}^r /\!/_\chi G$, we have the polyhedra $P_\chi \simeq P_{\mathbf{a}}$ for $\chi = \chi^{\mathbf{a}}$. However, to capture the full story of what is going on, we need not only their facets but also their *virtual facets*.

This is easiest to see in $P_{\mathbf{a}}$, which for $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$ is defined by

$$P_{\mathbf{a}} = \{m \in M_{\mathbb{R}} \mid \langle m, \nu_i \rangle \geq -a_i,\ 1 \leq i \leq r\}$$

as in (14.2.3). For $1 \leq i \leq r$, we call the set

$$F_{i,\mathbf{a}} = \{m \in P_{\mathbf{a}} \mid \langle m, \nu_i \rangle = -a_i\}$$

a *virtual facet* of $P_{\mathbf{a}}$. The facets of $P_{\mathbf{a}}$ all occur among the $F_{i,\mathbf{a}}$, while other virtual facets may be quite different—some may be faces of smaller dimension, some may be empty, and some may be all of $P_{\mathbf{a}}$. We can also have $F_{i,\mathbf{a}} = F_{j,\mathbf{a}}$ for $i \neq j$.

Example 14.2.19. In Example 14.2.4, we have $M = \mathbb{Z}^3$, $\mathbf{a} = (1,0,1)$, and $\nu_1 = \nu_2 = e_1$, $\nu_3 = 2e_2$. Thus

$$P_{\mathbf{a}} = \{m \in M_{\mathbb{R}} \mid \langle m, e_1 \rangle \geq -1, \langle m, e_1 \rangle \geq 0, \langle m, 2e_2 \rangle \geq -1\}.$$

It follows that $F_{1,\mathbf{a}} = \emptyset$ while $F_{2,\mathbf{a}}$ and $F_{3,\mathbf{a}}$ are the facets of $P_{\mathbf{a}}$ shown in Figure 1 from Example 14.2.4. $\Diamond$

Example 14.2.20. Let $G = \{(t, t^{-r}u, t, u) \mid t, u \in \mathbb{C}^*\}$ as in Example 14.2.17. For $\mathbf{a} = (0,0,0,1)$, the polytope $P_{\mathbf{a}} \subseteq \mathbb{R}^2$ is shown in Figure 4. Here we have three gen-

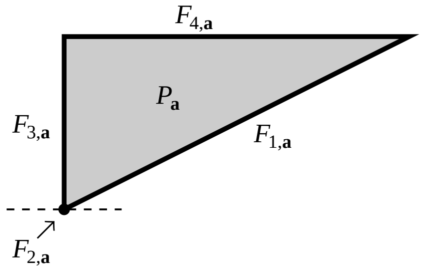


Figure 4. The polytope $P_{\mathbf{a}} \subseteq \mathbb{R}^2$ for $\mathbf{a} = (0,0,0,1)$ and its virtual facets

uine facets $F_{1,\mathbf{a}}, F_{3,\mathbf{a}}, F_{4,\mathbf{a}}$ and one virtual facet $F_{2,\mathbf{a}}$ which is a vertex. We will see in Example 14.3.7 below that $P_{\mathbf{a}}$ is the polytope of the divisor D_4 on the Hirzebruch surface $\mathscr{H}_r$. $\Diamond$

Virtual facets can also be defined for P_χ. Using the description of P_χ given in (14.2.6), one sees that $F_{i,\mathbf{a}} \subseteq P_{\mathbf{a}}$ corresponds to the subset of P_χ where the ith coordinate vanishes. We will denote this set by $F_{i,\chi}$, so that

$$F_{i,\chi} = P_\chi \cap \mathbf{V}(x_i).$$

Semistable Points. Virtual facets determine the semistable points of $\mathbb{C}^r$ as follows.

Proposition 14.2.21. *Let $p = (p_1, \ldots, p_r) \in \mathbb{C}^r$ and set $I(p) = \{i \mid 1 \le i \le r, p_i = 0\}$. Then $p \in (\mathbb{C}^r)^{\mathrm{ss}}_\chi$ if and only if $\bigcap_{i \in I(p)} F_{i,\chi} \neq \emptyset$.*

Proof. Let p be semistable, so that some (G, χ^d)-invariant polynomial F does not vanish at p. Writing F as a linear combination of (G, χ^d)-invariant monomials shows that some (G, χ^d)-invariant monomial $x^{\mathbf{b}}$ that does not vanish at p. Then $\mathbf{b} = (b_1, \ldots, b_r)$ lies in dP_χ. For $I(p)$ as above, we have $b_i = 0$ for all $i \in I(p)$ since $x^{\mathbf{b}}$ does not vanish at p. This implies $\mathbf{b} \in \bigcap_{i \in I(p)} dF_{i,\chi}$, hence $\bigcap_{i \in I(p)} F_{i,\chi} \neq \emptyset$.

The converse is also easy, except that we have to be careful about torsion. Assume that $\bigcap_{i \in I(p)} F_{i,\chi} \neq \emptyset$. Then the intersection contains a rational point, so that there is $\mathbf{b} \in (\bigcap_{i \in I(p)} dF_{i,\chi}) \cap \mathbb{Z}^r$ for some integer $d > 0$. This implies two things:

- $x^{\mathbf{b}}$ does not vanish at p.
- $\gamma_{\mathbb{R}}(\mathbf{b}) = \chi^d \otimes 1$, so that $\gamma(\mathbf{b})$ and χ^d differ by a torsion element of $\widehat{G}$.

It follows that for some integer $\ell > 0$, $x^{\ell\mathbf{b}}$ is $(G, \chi^{\ell d})$-invariant. Then p is semistable since $x^{\ell\mathbf{b}}$ does not vanish at p. $\square$

When we think about semistable points algebraically, we get the ideal

$$B(\chi) = \big\langle \textstyle\prod_{i\notin I} x_i \mid I \subseteq \{1,\dots,r\},\ \bigcap_{i\in I} F_{i,\chi} \neq \emptyset \big\rangle \subseteq S = \mathbb{C}[x_1,\dots,x_r], \tag{14.2.11}$$

called the *irrelevant ideal* of χ. The vanishing locus of $B(\chi)$ is the *exceptional set* $Z(\chi) = \mathbf{V}(B(\chi)) \subseteq \mathbb{C}^r$. This gives a nice description of the set of semistable points, which you will prove in Exercise 14.2.9.

Corollary 14.2.22. *$(\mathbb{C}^r)^{\mathsf{ss}}_\chi = \mathbb{C}^r \setminus Z(\chi)$, so that $\mathbb{C}^r /\!/_\chi G \simeq (\mathbb{C}^r \setminus Z(\chi))/\!/G$.* □

Exercises for §14.2.

14.2.1. Supply the details omitted in the proof of Lemma 14.2.1.

14.2.2. Supply the details omitted in Example 14.2.3.

14.2.3. Prove the description of P_χ given in (14.2.6).

14.2.4. The exercise will prove Lemma 14.2.8. As in the proof of Proposition 14.1.12, $\mathrm{Proj}(R)$ is covered by the affine open subsets $D_+(f) = \mathrm{Spec}(R_{(f)})$, where $f \in R_d$ is non-nilpotent and $R_{(f)}$ is the homogeneous localization (14.1.4).

(a) Note that $f \in R_d$ implies $f^\ell \in R_{d\ell} = (R^{[\ell]})_d$. Prove that

$$(R^{[\ell]})_{(f^\ell)} = R_{(f)}.$$

(b) Pick $f_1,\dots,f_s \in R$ homogeneous such that $\sqrt{\langle f_1,\dots,f_s\rangle} = R_+ = \bigoplus_{d>0} R_d$. Prove that

$$\sqrt{\langle f_1^\ell,\dots,f_s^\ell\rangle} = \sqrt{\langle f_1,\dots,f_s\rangle} = R_+.$$

as ideals of R, and conclude that $\sqrt{\langle f_1^\ell,\dots,f_s^\ell\rangle} = (R^{[\ell]})_+$ as ideals of $R^{[\ell]}$.

(c) Prove that $\mathrm{Proj}(R^{[\ell]}) \simeq \mathrm{Proj}(R)$. Hint: For $f_1,\dots,f_s \in R$ as in part (b), $\{D_+(f_i)\}_{i=1}^s$ is an open cover of $\mathrm{Proj}(R)$ by §7.0.

14.2.5. Prove Proposition 14.2.10.

14.2.6. Prove the assertions made in the three bullets in the proof of Proposition 14.2.12.

14.2.7. In (14.2.7) we claimed that the mapping $(m,k) \mapsto \delta(m) + k\mathbf{a}$ induces a semigroup isomorphism $C(P_\mathbf{a}) \cap (M\times\mathbb{Z}) \simeq \mathsf{S}_\chi$. Prove this.

14.2.8. This exercise will fill in some of the details omitted in Example 14.2.15.

(a) Show that $P_\mathbf{a} \subseteq \mathbb{R}^3 = M_\mathbb{R}$ is defined by $x \geq 0$, $y \geq 0$, $z \geq 0$, $x+y \leq 1+z$, $x+z \leq 1+y$ and $y+z \leq 1+x$. Also show that the vertices are the only lattice points of $P_\mathbf{a}$.

(b) Prove that $P_\mathbf{a}$ is normal. Hint: If $(a,b,c) \in \ell P_\mathbf{a}$, study $abc = 0$ and $abc > 0$ separately.

(c) Prove (14.2.8)

14.2.9. Prove Corollary 14.2.22. Hint: Use Proposition 14.1.12.

14.2.10. Consider the group $G = \{(t,t^{-1},u) \mid t \in \mathbb{C}^*, u = \pm 1\}$ and character $\chi(t,t^{-1},u) = tu$ from Example 14.1.5. Note that $\chi = \chi^\mathbf{a}$ for $\mathbf{a} = (1,0,1)$. By Example 14.1.8, the graded pieces of R_χ have a complicated behavior, and by Example 14.2.4, multiples of $P_\mathbf{a}$ have a complicated behavior since $P_\mathbf{a}$ is not a lattice polyhedron.

(a) Use the proof of Theorem 14.2.13 to explain how these complications are linked.

(b) In Example 14.2.9, we used the Veronese subring $R_\chi^{[2]}$ to show that $\mathrm{Proj}(R_\chi) \simeq \mathbb{C}^2$. Explain how this relates to the lattice polyhedron $2P_{\mathbf{a}}$.

14.2.11. Consider the Grassmannian $\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$ defined in Exercise 6.0.5. We use the Plücker embedding $\mathbb{G}(1,3) \hookrightarrow \mathbb{P}^5 = \mathbb{P}^{\binom{4}{2}-1}$ defined by

$$A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \longmapsto (p_{ij}) = (p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}),$$

where p_{ij} is the 2×2 minor of A formed by columns i and j. An elementary treatment of this Plücker embedding appears in [**69**, Ch. 8, §6]. The torus $G = (\mathbb{C}^*)^4$ acts on A by right multiplication. This gives an action on $\mathbb{G}(1,3)$ where $g = (t_0, t_1, t_2, t_3) \in G$ acts on Plücker coordinates via $g \cdot (p_{ij}) = (t_i t_j p_{ij})$. This lifts to an action on the affine cone $\widehat{\mathbb{G}}(1,3) \subseteq \mathbb{C}^6 = \mathbb{C}^{\binom{4}{2}}$. The action of G extends to an action on $\mathbb{C}^6$. For the character $\chi = \chi^{\mathbf{a}}$, $\mathbf{a} = (1, \dots, 1) \in \mathbb{Z}^6$, prove that $\mathbb{C}^6 /\!/_\chi G \simeq \mathbb{P}^2$. The more general GIT quotient $\widehat{\mathbb{G}}(1,3) /\!/_\chi G$ is described in [**83**, Ex. 12.4].

§14.3. Toric GIT and Gale Duality

This section begins our study of what happens to the GIT quotient $\mathbb{C}^r /\!/_\chi G$ as we vary χ. A key player is Gale duality.

Gale Duality. A toric variety X_Σ comes with two finite sets indexed by $\rho \in \Sigma(1)$:

- The minimal generators $u_\rho \in N$.
- The divisor classes $[D_\rho] \in \mathrm{Cl}(X_\Sigma)$.

When X_Σ has no torus factors, these are related by the exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0,$$

where the first map is defined by $m \mapsto (\langle m, u_\rho \rangle)_{\rho \in \Sigma(1)} \in \mathbb{Z}^{\Sigma(1)}$ and the second is $e_\rho \mapsto [D_\rho] \in \mathrm{Cl}(X_\Sigma)$.

Gale duality generalizes this situation in the context of real vector spaces. Let W be a vector space over $\mathbb{R}$ spanned by vectors $\beta_1, \dots, \beta_r$. Some of the β_i may be zero, and repetitions are allowed—we do not assume that the β_i are distinct. We will use $\boldsymbol{\beta}$ to denote this list of vectors. Then we have an exact sequence

$$0 \longrightarrow V \xrightarrow{\delta} \mathbb{R}^r \xrightarrow{\gamma} W \longrightarrow 0, \tag{14.3.1}$$

where $\gamma(e_i) = \beta_i$ and $\delta : V \to \mathbb{R}^r$ is the inclusion of the kernel of γ. Dualizing, we obtain the exact sequence

$$0 \longrightarrow W^* \xrightarrow{\gamma^*} \mathbb{R}^r \xrightarrow{\delta^*} V^* \longrightarrow 0, \tag{14.3.2}$$

where we use dot product to identify $\mathbb{R}^r$ with its dual. This gives vectors $\nu_i = \delta^*(e_i) \in V^*$. We will use $\boldsymbol{\nu}$ to denote this list of vectors. As in §14.2, the map δ in (14.3.1) is given by $\delta(\mathrm{v}) = (\langle \mathrm{v}, \nu_1 \rangle, \dots, \langle \mathrm{v}, \nu_r \rangle) \in \mathbb{R}^r$ for $\mathrm{v} \in V$.

Since (14.3.1) is the dual of (14.3.2), we could equally well start with the vectors $\boldsymbol{\nu}$ in V^* and then recover the vectors $\boldsymbol{\beta}$ by duality. This implies in particular that γ^* in (14.3.2) is given by $\gamma^*(w) = (\langle\beta_1, w\rangle, \dots, \langle\beta_r, w\rangle) \in \mathbb{R}^r$ for $w \in W^*$. The symmetry between $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$ will be used often in our discussion of Gale duality.

Our first result describes what happens when a subset of $\boldsymbol{\beta}$ or $\boldsymbol{\nu}$ is a basis.

Lemma 14.3.1. *Let $I \subseteq \{1,\dots,r\}$ and set $J = \{1,\dots,r\} \setminus I$. Then β_i, $i \in I$, form a basis of W if and only if ν_j, $j \in J$, form a basis of V^*.*

Proof. Note that $|I| = \dim W$ implies $|J| = \dim V = \dim V^*$. Assume ν_j, $j \in J$, give a basis of V^* and let v_j, $j \in J$, be the dual basis of V. Then we have $\delta(\mathrm{v}_j) = e_j + \sum_{i\in I}\langle \mathrm{v}_j, \nu_i\rangle e_i$ for $j \in J$. Applying γ, we see that $j \in J$ implies that

$$\beta_j = -\textstyle\sum_{i\in I}\langle \mathrm{v}_j, \nu_i\rangle \beta_i.$$

This proves that β_i, $i \in I$, span W, and since $|I| = \dim W$, they form a basis of W.

The other direction of the proof follows immediately by duality. $\square$

Using $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$ we get cones $C_{\boldsymbol{\beta}} = \mathrm{Cone}(\boldsymbol{\beta}) \subseteq W$ and $C_{\boldsymbol{\nu}} = \mathrm{Cone}(\boldsymbol{\nu}) \subseteq V^*$. Here is an example of how these cones are related.

Lemma 14.3.2. *Consider the cones $C_{\boldsymbol{\beta}} \subseteq W$ and $C_{\boldsymbol{\nu}} \subseteq V^*$ defined above.*

(a) *If $C_{\boldsymbol{\beta}} = W$, then $C_{\boldsymbol{\nu}}$ is strongly convex in V^*. Conversely, if $\boldsymbol{\nu}$ consists of nonzero vectors and $C_{\boldsymbol{\nu}}$ is strongly convex in V^*, then $C_{\boldsymbol{\beta}} = W$.*

(b) *If $C_{\boldsymbol{\nu}} = V^*$, then $C_{\boldsymbol{\beta}}$ is strongly convex in W. Conversely, if $\boldsymbol{\beta}$ consists of nonzero vectors and $C_{\boldsymbol{\beta}}$ is strongly convex in W, then $C_{\boldsymbol{\nu}} = V^*$.*

Proof. Assume $C_{\boldsymbol{\beta}} = W$. Pick $\mathrm{v} \in C_{\boldsymbol{\nu}} \cap (-C_{\boldsymbol{\nu}})$ and write

$$\mathrm{v} = \sum_{i=1}^{r} a_i \nu_i = -\sum_{i=1}^{r} b_i \nu_i$$

with $a_i, b_i \geq 0$. Then $\sum_{i=1}^r (a_i + b_i)\nu_i = 0$. Since (14.3.2) is exact, there is $w \in W^*$ such that $\gamma^*(w) = \sum_{i=1}^r (a_i + b_i) e_i$, so that $\langle \beta_i, w\rangle = a_i + b_i$ by the description of γ^* given in the discussion following (14.3.2). Since $a_i + b_i \geq 0$ for all i, we conclude that $w \in \mathrm{Cone}(\boldsymbol{\beta})^\vee = C_{\boldsymbol{\beta}}^\vee = \{0\}$, where the last equality follows from $C_{\boldsymbol{\beta}} = W$. Then $a_i + b_i = 0$ for all i, so that $a_i = b_i = 0$ for all i, hence $\mathrm{v} = 0$.

For the converse, take $w \in C_{\boldsymbol{\beta}}^\vee \subseteq W^*$. Then $\langle \beta_i, w\rangle \geq 0$ for all i. Since $\gamma^*(w) = \sum_{i=1}^r \langle \beta_i, w\rangle e_i$, we have $\sum_{i=1}^r \langle \beta_i, w\rangle \nu_i = 0$. Strong convexity implies that $\{0\} \subseteq C_{\boldsymbol{\nu}}$ is a face, and then $\langle \beta_i, w\rangle \nu_i = 0$ for all i by Lemma 1.2.7. Since the ν_i are nonzero, we must have $\langle \beta_i, w\rangle = 0$ for all i, which implies $w = 0$ since the β_i span W. This proves $C_{\boldsymbol{\beta}}^\vee = \{0\}$. Taking duals gives $C_{\boldsymbol{\beta}} = W$, completing the proof of part (a).

Part (b) of the lemma follow from part (a) by duality. $\square$

Gale duality also describes the faces of these cones. Here is the result for $C_{\boldsymbol{\beta}}$; the corresponding statment for $C_{\boldsymbol{\nu}}$ follows immediately by duality.

Lemma 14.3.3. *Let $I \subseteq \{1,\dots,r\}$ and set $J = \{1,\dots,r\} \setminus I$. Then the following are equivalent:*

(a) *There is a face $F \preceq C_{\boldsymbol{\beta}}$ such that $\beta_i \in F$ if and only if $i \in I$.*

(b) *There are positive numbers b_i, $i \in J$, such that $\sum_{i\in J} b_i \nu_i = 0$.*

Proof. Given (a), there is $w \in C_{\boldsymbol{\beta}}^\vee$ such that $\langle \beta_i, w\rangle = 0$ for $i \in I$ and $\langle \beta_i, w\rangle > 0$ for $i \in J$. Then (b) holds since $0 = \delta^* \circ \gamma^*(w) = \sum_{i\in J} \langle \beta_i, w\rangle \nu_i$. The proof of the converse is equally easy (Exercise 14.3.1). $\square$

The introduction to Gale duality given here follows [**222**]. There is *much* more to say about this subject—the full story involves ideas such as *circuits* and *oriented matroids*. See [**281**, Lec. 6] for a more complete treatment of Gale duality. Circuits will play an important role in Chapter 15.

Characters as Vectors. We return to the GIT quotient $\mathbb{C}^r /\!/_\chi G$, where $G \subseteq (\mathbb{C}^*)^r$ and $\chi \in \widehat{G}$. We will continue to regard $\widehat{G}$ as a multiplicative group, but since the tensor product $\widehat{G}_{\mathbb{R}} = \widehat{G} \otimes_{\mathbb{Z}} \mathbb{R}$ is a vector space over $\mathbb{R}$ of dimension $\dim G$, we will switch to additive notation when working in $\widehat{G}_{\mathbb{R}}$. Thus characters $\chi, \psi \in \widehat{G}$ give vectors $\chi \otimes 1, \psi \otimes 1 \in \widehat{G}_{\mathbb{R}}$ such that $(\chi\psi) \otimes 1 = \chi \otimes 1 + \psi \otimes 1$.

The inclusion $G \subseteq (\mathbb{C}^*)^r$ gives characters $\chi_i \in \widehat{G}$ defined by $\chi_i(g) = t_i$ for $g = (t_1,\dots,t_r) \in G$. In terms of (14.2.2), this means $\chi_i = \delta(e_i)$. These generate $\widehat{G}$, so that $\widehat{G}_{\mathbb{R}}$ is spanned by the vectors

$$\beta_i = \chi_i \otimes 1 = \delta_{\mathbb{R}}(e_i) \in \widehat{G}_{\mathbb{R}}, \quad 1 \le i \le r.$$

Note that $\mathbf{a} = (a_1,\dots,a_r) \in \mathbb{Z}^r$ gives the character $\prod_{i=1}^r \chi_i^{a_i}$, which was written $\gamma(\mathbf{a}) = \chi^{\mathbf{a}}$ in the discussion following (14.2.2). It follows that

$$\gamma_{\mathbb{R}}(\mathbf{a}) = \chi^{\mathbf{a}} \otimes 1 = \textstyle\sum_{i=1}^r a_i \beta_i \in \widehat{G}_{\mathbb{R}}.$$

The Two Cones. The vectors $\beta_i \in \widehat{G}_{\mathbb{R}}$ give a list we denote by $\boldsymbol{\beta}$, and the vectors $\nu_i = \delta^*(e_i) \in N \subseteq N_{\mathbb{R}} = M_{\mathbb{R}}^*$ give a list we denote by $\boldsymbol{\nu}$. Then the exact sequence

$$0 \longrightarrow M_{\mathbb{R}} \xrightarrow{\delta_{\mathbb{R}}} \mathbb{R}^r \xrightarrow{\gamma_{\mathbb{R}}} \widehat{G}_{\mathbb{R}} \longrightarrow 0 \tag{14.3.3}$$

from (14.2.4) satisfies

- $\gamma_{\mathbb{R}}(e_i) = \beta_i$ for $1 \le i \le r$.
- $\delta_{\mathbb{R}}(m) = (\langle m, \nu_1\rangle, \dots, \langle m, \nu_r\rangle)$ for $m \in M_{\mathbb{R}}$.

Hence Gale duality applies to this situation. We have the additional structure given by the lattices $M \subseteq M_{\mathbb{R}}$ and $\widehat{G} \otimes 1 \subseteq \widehat{G}_{\mathbb{R}}$. Furthermore, the cones

$$C_{\boldsymbol{\beta}} = \mathrm{Cone}(\boldsymbol{\beta}) = \mathrm{Cone}(\beta_1,\dots,\beta_r) \subseteq \widehat{G}_{\mathbb{R}}$$
$$C_{\boldsymbol{\nu}} = \mathrm{Cone}(\boldsymbol{\nu}) = \mathrm{Cone}(\nu_1,\dots,\nu_r) \subseteq N_{\mathbb{R}}$$

are full dimensional in $\widehat{G}_{\mathbb{R}}$ and $N_{\mathbb{R}}$ and are rational polyhedral with respect to the lattices $\widehat{G}\otimes 1$ and N. These cones will play an important role in this section.

Example 14.3.4. Let X_P be the toric variety of a full dimensional lattice polyhedron $P \subseteq M_{\mathbb{R}}$. Then (14.3.3) is obtained from the standard exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma_P(1)} \longrightarrow \mathrm{Cl}(X_P) \longrightarrow 0$$

by tensoring with $\mathbb{R}$. Thus:

- The β_i's come from the divisor classes $[D_\rho]$, so that the cone $C_{\boldsymbol{\beta}}$ is the cone of torus-invariant effective $\mathbb{R}$-divisor classes.
- The ν_i's are the minimal generators u_ρ, so that the cone $C_{\boldsymbol{\nu}}$ is the support of the normal fan Σ_P. $\Diamond$

Existence of Semistable and Stable Points. We first show the cone $C_{\boldsymbol{\beta}}$ controls when the GIT quotient is nonempty, i.e., when semistable points exist.

Proposition 14.3.5. *If $\chi \in \widehat{G}$ is a character, then the following are equivalent:*

(a) $\mathbb{C}^r /\!/_\chi G \neq \emptyset$.

(b) $(\mathbb{C}^r)^{\mathsf{ss}}_\chi \neq \emptyset$.

(c) $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$.

Proof. We have (a) $\Leftrightarrow$ (b) by Proposition 14.1.12. Using $\dim \mathbb{C}^r /\!/_\chi G = \dim P_\chi$ (Corollary 14.2.16) and $P_\chi = \gamma_{\mathbb{R}}^{-1}(\chi\otimes 1)\cap \mathbb{R}^r_{\geq 0}$, (a) $\Leftrightarrow$ (c) follows from

$$\begin{aligned} \mathbb{C}^r /\!/_\chi G \neq \emptyset \iff P_\chi \neq \emptyset \iff \text{there is } \mathbf{b} = (b_1,\dots,b_r) \in \gamma_{\mathbb{R}}^{-1}(\chi\otimes 1)\cap \mathbb{R}^r_{\geq 0} \\ \iff \chi\otimes 1 = \sum_{i=1}^r b_i\beta_i,\ b_i \geq 0 \iff \chi \otimes 1 \in C_{\boldsymbol{\beta}}. \quad \square \end{aligned}$$

Our next task is to determine when stable points exist. Here, the interior of $C_{\boldsymbol{\beta}}$ is the key player.

Proposition 14.3.6. *If $\chi\otimes 1 \in C_{\boldsymbol{\beta}}$, then the following are equivalent:*

(a) $(\mathbb{C}^*)^r \subseteq (\mathbb{C}^r)^{\mathsf{s}}_\chi$.

(b) $(\mathbb{C}^r)^{\mathsf{s}}_\chi \neq \emptyset$.

(c) $\chi \otimes 1$ *is contained in the interior of* $C_{\boldsymbol{\beta}}$.

(d) $F_{i,\chi}$ *is a proper face of* P_χ *for all i.*

Furthermore, the above conditions imply that

(e) $\dim \mathbb{C}^r /\!/_\chi G = r - \dim G$,

and when $\boldsymbol{\nu}$ *consists of nonzero vectors,* (e) *is equivalent to* (a)–(d).

Proof. Throughout the proof we pick $\mathbf{a} = (a_1,\dots,a_r) \in \mathbb{Z}^r$ such that $\chi = \chi^{\mathbf{a}}$.

The implication (a) $\Rightarrow$ (b) is obvious. Assume (b) and suppose that $\chi \otimes 1 \in F$, where F is a proper face of C_{β}. Let $I = \{i \mid \beta_i \in F\}$ and $J = \{1,\dots,r\} \setminus I$. Note that J is nonempty since F is a proper face. If $\mathbf{b} = (b_1,\dots,b_r) \in P_\chi$, then

$$\chi \otimes 1 = \textstyle\sum_{i=1}^r b_i \beta_i.$$

Then Lemma 1.2.7 and $\chi \otimes 1 \in F$ imply that $b_i\beta_i \in F$ for all i. Since $\beta_i \notin F$ for $i \in J$, we must have $b_i = 0$ for $i \in J$. Thus $F_{i,\chi} = P_\chi$ for $i \in J$. Relabeling if necessary, we may assume $1 \in J$. Then $F_{1,\chi} = P_\chi$ and (14.2.11) imply that the minimal generators of the ideal $B(\chi)$ do not involve x_1. By Corollary 14.2.22, we have

$$(\mathbb{C}^r)^{\mathsf{ss}}_\chi = \mathbb{C} \times U, \quad U \subseteq \mathbb{C}^{r-1} \text{ open.} \tag{14.3.4}$$

Translated to $P_{\mathbf{a}}$, $F_{1,\chi} = P_\chi$ means that

$$P_{\mathbf{a}} \subseteq \{m \in M_{\mathbb{R}} \mid \langle m, \nu_1\rangle = a_1\}.$$

Since $P_{\mathbf{a}}$ has full dimension in $M_{\mathbb{R}}$ by Proposition 14.1.12 and Theorem 14.2.13, this inclusion forces $\nu_1 = 0$. Using the description of G from Lemma 14.2.1, we get a product decomposition

$$G = \mathbb{C}^* \times G' \subseteq \mathbb{C}^* \times (\mathbb{C}^*)^{r-1} = (\mathbb{C}^*)^r.$$

Thus $\lambda(t) = (t,1) \in \mathbb{C}^* \times G' = G$ is a one-parameter subgroup of G. Via (14.3.4), pick $p = (p_1, p') \in \mathbb{C} \times U = (\mathbb{C}^r)^{\mathsf{ss}}_\chi$. Then $\lambda(t) \cdot p \in G \cdot p$, and the limit

$$\lim_{t\to 0} \lambda(t)\cdot p = \lim_{t \to 0}(tp_1, p') = (0, p') \in \mathbb{C} \times U$$

exists as point of $(\mathbb{C}^r)^{\mathsf{ss}}_\chi$. But this limit is not in $G \cdot p$ when $p_1 \neq 0$, so that $G \cdot p$ is not closed in $(\mathbb{C}^r)^{\mathsf{ss}}_\chi$ when $p_1 \neq 0$. Thus p cannot be stable by Remark 14.1.13. This is easily seen to contradict $(\mathbb{C}^r)^{\mathsf{s}}_\chi \neq \emptyset$, and (b) $\Rightarrow$ (c) follows.

Assume (c) and let $J = \{i \mid F_{i,\chi} = P_\chi\}$. Suppose $J \neq \emptyset$ and set $I = \{1,\dots,r\}\setminus J$. Since $F_{i,\chi} \subsetneq P_\chi$ for each $i \in I$, we can pick $\mathbf{c} = (c_1,\dots,c_r) \in P_\chi \setminus (\bigcup_{i\in I} F_{i,\chi})$. Then the ith coordinate of $\mathbf{c}$ is positive for $i \in I$ and is zero for $i \in J$. Hence

$$\chi \otimes 1 = \textstyle\sum_{i \in I} c_i \beta_i \tag{14.3.5}$$

Suppose that $\mathrm{Cone}(\nu_i \mid i \in J) \subseteq N_{\mathbb{R}}$ is strongly convex. Then $\{0\}$ is a face, which implies that there is $m \in M_{\mathbb{R}}$ such that $\langle m, \nu_i\rangle > 0$ for all $i \in J$. This gives $\mathbf{c}' = (\langle m,\nu_1\rangle,\dots,\langle m,\nu_r\rangle) \in \gamma_{\mathbb{R}}^{-1}(0)$ whose ith coordinate is positive for $i \in J$. Then

$$\mathbf{b} = \mathbf{c} + \varepsilon \mathbf{c}' \in \gamma_{\mathbb{R}}^{-1}(0)$$

for all ε, and when $\varepsilon > 0$ is sufficiently small, the ith coordinate of $\mathbf{b}$ is positive for all i. This follows easily by treating the cases $i \in I$ and $i \in J$ separately. Thus $\mathbf{b} \in P_\chi$ has positive coordinates, which contradicts $J = \{i \mid F_{i,\chi} = P_\chi\} \neq \emptyset$. We conclude that $\mathrm{Cone}(\nu_i \mid i \in J)$ is not strongly convex. This easily gives a nonempty subset $J' \subseteq J$ such that

$$\textstyle\sum_{i\in J'} \lambda_i \nu_i = 0, \ \lambda_i > 0$$

(Exercise 14.3.2). By Gale duality (Lemma 14.3.3), it follows that there is a face $F \preceq C_{\boldsymbol{\beta}}$ such that $\{i \mid \beta_i \in F\} = \{1,\dots,r\} \setminus J'$. This is a proper face since $J' \neq \emptyset$, and it contains $\chi \otimes 1$ by (14.3.5) and $I = \{1,\dots,r\} \setminus J$. This contradicts $\chi \otimes 1 \in \mathrm{Int}(C_{\boldsymbol{\beta}})$ and completes the proof of (c) $\Rightarrow$ (d).

Assume (d). Since $F_{i,\chi} \neq P_\chi$ for all i, P_χ has a rational point with positive coordinates. Arguing as in the proof of Proposition 14.2.21, there is $d > 0$ and $\mathbf{b} \in dP_\chi \cap \mathbb{Z}^r_{>0}$ such that $x^{\mathbf{b}}$ is (G,χ^d)-invariant. Since all exponents are positive, the nonvanishing set of $x^{\mathbf{b}}$ is $(\mathbb{C}^*)^r$. It follows easily that points of $(\mathbb{C}^*)^r$ are stable. This proves (d) $\Rightarrow$ (a).

The implication (b) $\Rightarrow$ (e) follows from Proposition 14.1.12. Now assume (e), so that $P_{\mathbf{a}}$ is full dimensional in $M_{\mathbb{R}}$. This implies that if $\nu_i \neq 0$, then $m \mapsto \langle m, \nu_i \rangle$ cannot be constant on $P_{\mathbf{a}}$, and hence $F_{i,\mathbf{a}} \neq P_{\mathbf{a}}$. Thus (e) $\Rightarrow$ (d) when $\boldsymbol{\nu}$ consists of nonzero vectors, and the proof is complete. □

Proposition 14.3.6 has many nice consequences concerning $\mathbb{C}^r /\!/_\chi G$ when $\boldsymbol{\nu}$ consists of nonzero vectors, including:

- Stable points exist if and only if the GIT quotient has the expected dimension.
- The GIT quotient is nonempty of smaller than expected dimension if and only if the character is on the boundary of $C_{\boldsymbol{\beta}}$.

Here is an example to illustrate the last bullet.

Example 14.3.7. Let $G = \{(t, t^{-r}u, t, u) \mid t, u \in \mathbb{C}^*\} \subseteq (\mathbb{C}^*)^4$. This group appeared in Example 14.2.17. For $r = 1$, the cone $C_{\boldsymbol{\beta}} \subseteq \widehat{G}_{\mathbb{R}} \simeq \mathbb{R}^2$ is the shaded region in Figure 5. The figure also shows $\beta_1, \beta_2, \beta_3, \beta_4$. Three of the P_χ's in the figure include dashed lines that represent virtual facets, two of which are empty and the other of which is a vertex. Note that $\dim P_\chi = \dim \mathbb{C}^4 /\!/_\chi G$ drops dimension precisely on the boundary of $C_{\boldsymbol{\beta}}$, as predicted by Proposition 14.3.6.

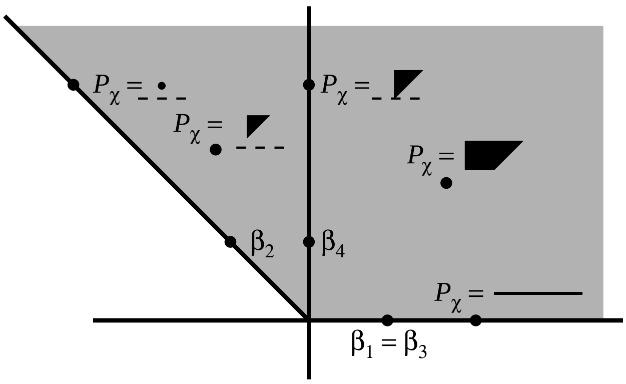


Figure 5. $C_{\boldsymbol{\beta}}$ and the polytopes P_χ for all $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$

Figure 5 is closely related to Figure 12 in Example 6.3.23. In that example, our focus was on the Hirzebruch surface $\mathscr{H}_r$, so there we saw only the first quadrant portion of Figure 5. Looking at all of Figure 5, we see that away from the boundary, $\mathbb{C}^r /\!/_\chi G$ is either $\mathbb{P}^2$ or $\mathscr{H}_1$, depending on which quadrant we are in. $\Diamond$

In many of the most interesting cases, $\boldsymbol{\nu}$ consists of nonzero vectors. We should nevertheless give an example where one of the ν_i's vanishes.

Example 14.3.8. Consider the group $G = \{(t,u,u) \mid t,u \in \mathbb{C}^*\}$ and the character $\chi(t,u,u) = u$. We leave it as Exercise 14.3.3 to show that:

- $\nu_1 = 0,\ \nu_2 = e_1,\ \nu_3 = -e_1$, where e_1 is a basis of $N_\mathbb{R} \simeq \mathbb{Z}$.
- $C_{\boldsymbol{\beta}} = \mathbb{R}^2_{\geq 0}$ and $\chi \otimes 1 = (0,1) \in \partial C_{\boldsymbol{\beta}}$.
- $P_\chi = \mathrm{Conv}((0,1,0),(0,0,1))$ and $F_{1,\chi} = P_\chi$.
- $(\mathbb{C}^3)^{\mathsf{ss}} = \mathbb{C} \times (\mathbb{C}^2 \setminus \{0\})$ and $(\mathbb{C}^3)^{\mathsf{s}} = \emptyset$.

Note that parts (a), (b), (c), (d) of Proposition 14.3.6 are false while part (e) is true since $\mathbb{C}^3 /\!/_\chi G \simeq \mathbb{P}^1$ has the expected dimension $3-2=1$. $\Diamond$

Projective and Birational GIT Quotients. By Proposition 14.1.12, the quotients we are studying come with a projective morphism

$$\mathbb{C}^r /\!/_\chi G \longrightarrow \mathbb{C}^r /\!/ G = \mathrm{Spec}(\mathbb{C}[x_1,\dots,x_r]^G). \tag{14.3.6}$$

Here we explore the extreme cases where $\mathbb{C}^r /\!/ G$ is very small or very large. We will need the following result about $C_{\boldsymbol{\nu}} = \mathrm{Cone}(\boldsymbol{\nu}) \subseteq N_\mathbb{R}$.

Lemma 14.3.9.

(a) $\mathbb{C}[x_1,\dots,x_r]^G = \mathbb{C}[C_{\boldsymbol{\nu}}^\vee \cap M]$, *so that* $\mathbb{C}^r /\!/ G = \mathrm{Spec}(\mathbb{C}[C_{\boldsymbol{\nu}}^\vee \cap M])$.

(b) *If* $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$, *then* $C_{\boldsymbol{\nu}}^\vee \subseteq M_\mathbb{R}$ *is the recession cone of* $P_{\mathbf{a}}$ *when* $\chi = \chi^{\mathbf{a}}$.

Proof. For part (a), note that $x^{\mathbf{b}}$ is G-invariant if and only if $\gamma(\mathbf{b})$ is trivial, which is equivalent to $\mathbf{b} = \delta(m)$ for some $m \in M$. Since $\mathbf{b} \in \mathbb{N}^r$, the formula for $\delta(m)$ shows that $\mathbb{C}[x_1,\dots,x_r]^G$ is the semigroup algebra of lattice points in the cone

$$\{m \in M_\mathbb{R} \mid \langle m, \nu_i \rangle \geq 0,\ 1 \leq i \leq r\} = \mathrm{Cone}(\nu_1,\dots,\nu_r)^\vee = C_{\boldsymbol{\nu}}^\vee. \tag{14.3.7}$$

For part (b), let $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$. Then $P_{\mathbf{a}}$ is nonempty by the proof of Proposition 14.3.5 and hence has recession cone (14.3.7) by the proof of Lemma 14.2.2. $\square$

Our first extreme case is when $\mathbb{C}^r /\!/ G = \mathrm{Spec}(\mathbb{C}[x_1,\dots,x_r]^G)$ is very small.

Proposition 14.3.10. *The following are equivalent:*

(a) $\mathbb{C}^r /\!/_\chi G$ *is projective for all* $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$.

(b) $\mathbb{C}^r /\!/_\chi G$ *is projective for some* $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$.

(c) $\mathbb{C}[x_1,\dots,x_r]^G = \mathbb{C}$.

(d) $\boldsymbol{\beta}$ *consists of nonzero vectors and* $C_{\boldsymbol{\beta}}$ *is strongly convex.*

Proof. First, (a) $\Rightarrow$ (b) is clear, and (b) $\Rightarrow$ (c) follows from Exercise 14.1.9 since $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$ implies $\mathbb{C}^r /\!/_{\chi} G \neq \emptyset$. We also have (c) $\Rightarrow$ (a) by Exercise 14.1.7.

If (d) is true, then $C_{\boldsymbol{\nu}} = N_{\mathbb{R}}$ by Gale duality (Lemma 14.3.2), and then we have $C_{\boldsymbol{\nu}}^{\vee} = \{0\}$. Assume $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$. Then $C_{\boldsymbol{\nu}}^{\vee} = \{0\}$ is the recession cone of $P_{\mathbf{a}}$ by Lemma 14.3.9, so that $P_{\mathbf{a}}$ is a polytope. It follows that $\mathbb{C}^r /\!/_{\chi} G \simeq X_{P_{\mathbf{a}}}$ is projective.

Finally, suppose that (c) is true. If $\beta_i = 0$, then $\beta_i = \chi_i \otimes 1$ implies that $\chi_i \in \widehat{G}$ has finite order, say ℓ. Since $\chi_i : G \to \mathbb{C}^*$ is projection onto the ith coordinate, it follows that $\mathbb{C}[x_i^{\ell}] \subseteq \mathbb{C}[x_1, \ldots, x_r]^G$, a contradiction. Hence the β_i are nonzero. Also, (c) and Lemma 14.3.9 imply that $C_{\boldsymbol{\nu}}^{\vee} = \{0\}$, so that $C_{\boldsymbol{\nu}} = N_{\mathbb{R}}$. Hence $C_{\boldsymbol{\beta}}$ is strongly convex by Gale duality (Lemma 14.3.2). $\square$

The other extreme is when $\mathbb{C}^r /\!/ G$ is large. Since $\dim \mathbb{C}^r /\!/ G \leq r - \dim G$ by Remark 14.1.13, $\mathbb{C}^r /\!/ G$ is as large as possible when $\dim \mathbb{C}^r /\!/ G = r - \dim G$. Here is our result.

Proposition 14.3.11. *The following conditions are equivalent:*

(a) $\dim \mathbb{C}^r /\!/ G = r - \dim G$.

(b) $C_{\boldsymbol{\nu}} \subseteq N_{\mathbb{R}}$ *is strongly convex.*

(c) *The map* $\mathbb{C}^r /\!/_{\chi} G \to \mathbb{C}^r /\!/ G$ *from* (14.3.6) *is birational for all* $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$.

Furthermore, when $\boldsymbol{\nu}$ *consists of nonzero vectors, we can replace* (b) *and* (c) *with:*

(b′) $C_{\boldsymbol{\beta}} = \widehat{G}_{\mathbb{R}}$.

(c′) $\mathbb{C}^r /\!/_{\chi} G \to \mathbb{C}^r /\!/ G$ *is birational for all* χ.

Proof. Lemma 14.3.9 implies $\dim \mathbb{C}^r /\!/ G = \dim C_{\boldsymbol{\nu}}^{\vee}$. Since $r - \dim G = \dim M_{\mathbb{R}}$, we obtain

$$\begin{aligned} \dim \mathbb{C}^r /\!/ G = r - \dim G &\iff \dim C_{\boldsymbol{\nu}}^{\vee} = \dim M_{\mathbb{R}} \\ &\iff C_{\boldsymbol{\nu}} \subseteq N_{\mathbb{R}} \text{ is strongly convex.} \end{aligned}$$

This proves (a) $\Leftrightarrow$ (b). Now assume (b). Then $C_{\boldsymbol{\nu}}$ is strongly convex, so that $\mathbb{C}^r /\!/ G$ is the affine toric variety of $C_{\boldsymbol{\nu}}$ by Lemma 14.3.9. Furthermore, $P_{\mathbf{a}}$ is full dimensional since its recession cone is $C_{\boldsymbol{\nu}}^{\vee}$ by Lemma 14.3.9. Hence the normal fan of $P_{\mathbf{a}}$ refines $C_{\boldsymbol{\nu}}$ by Theorem 7.1.6. Then (c) follows since $\mathbb{C}^r /\!/_{\chi} G \to \mathbb{C}^r /\!/ G$ is the birational map induced by this refinement. On the other hand, if (c) holds, then $\mathbb{C}^r /\!/_{\chi} G \to \mathbb{C}^r /\!/ G$ is birational for $\chi \otimes 1 \in \mathrm{Int}(C_{\boldsymbol{\beta}})$, so that

$$\dim \mathbb{C}^r /\!/ G = \dim \mathbb{C}^r /\!/_{\chi} G = r - \dim G,$$

where the last equality uses Proposition 14.3.6. Hence (c) $\Rightarrow$ (a).

Assume that $\boldsymbol{\nu}$ consists of nonzero vectors. By Gale duality (Lemma 14.3.2), $C_{\boldsymbol{\nu}} \subseteq N_{\mathbb{R}}$ is strongly convex if and only if $C_{\boldsymbol{\beta}} = \widehat{G}_{\mathbb{R}}$, which gives (b) $\Leftrightarrow$ (b′). Then (c) $\Leftrightarrow$ (c′) follows immediately. $\square$

Example 14.3.12. For $G = \{(t,t,t^{-1}) \mid t \in \mathbb{C}^*\}$, one gets $\beta_1 = \beta_2 = e_1$, $\beta_3 = -e_1$ in $\widehat{G}_{\mathbb{R}} \simeq \mathbb{R}$ and $\nu_1 = e_1$, $\nu_2 = e_2$, $\nu_3 = e_1 + e_2$ in $N_{\mathbb{R}} \simeq \mathbb{R}^2$. Thus $C_{\boldsymbol{\nu}} = \mathrm{Cone}(e_1, e_2)$ is strongly convex and $\nu_i \neq 0$ for all i. Hence all conditions of Proposition 14.3.11 apply, i.e., for all $\chi \in \widehat{G}$, we have a birational morphism

$$\mathbb{C}^3/\!/_\chi G \to \mathbb{C}^3/\!/G = \mathrm{Spec}(\mathbb{C}[xz, yz]) = \mathbb{C}^2.$$

To understand $\mathbb{C}^r/\!/_\chi G$, consider Figure 6, which shows the polyhedron P_χ for all χ. For "positive" χ, $\mathbb{C}^r/\!/_\chi G$ is the blowup of $\mathbb{C}^2$ at the origin, while for

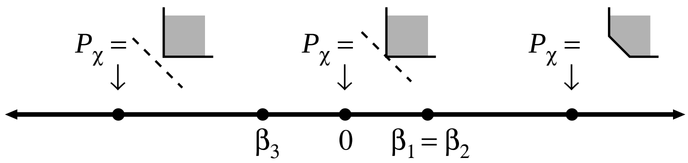


Figure 6. The polytopes P_χ for all χ

"negative" χ, the GIT quotient is $\mathbb{C}^2$. The dashed lines in the figure indicate the presence of a virtual facet, which is empty for negative χ and consists of the vertex when χ is the origin. In Exercise 14.3.4 you will compare this to the quotient construction of $\mathrm{Bl}_0(\mathbb{C}^2)$. $\Diamond$

Generic Characters. We next study what it means for a character to be generic.

Definition 14.3.13. A character $\chi \in \widehat{G}$ is ***generic*** if $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$ and for every subset $\boldsymbol{\beta}' \subseteq \boldsymbol{\beta}$ with $\dim \mathrm{Cone}(\boldsymbol{\beta}') < \dim C_{\boldsymbol{\beta}} = \dim G$, we have $\chi \otimes 1 \notin \mathrm{Cone}(\boldsymbol{\beta}')$.

We can determine when a character is generic as follows.

Theorem 14.3.14. *For $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$, the following are equivalent:*

(a) *χ is generic.*

(b) *Every vertex of P_χ has precisely $\dim G$ nonzero coordinates.*

(c) *P_χ is simple of dimension $r - \dim G$, every virtual facet $F_{i,\chi} \subseteq P_\chi$ is either empty or a genuine facet, and $F_{i,\chi} \neq F_{j,\chi}$ if $i \neq j$ and $F_{i,\chi}, F_{j,\chi}$ are nonempty.*

(d) $(\mathbb{C}^r)^{\mathrm{s}}_\chi = (\mathbb{C}^r)^{\mathrm{ss}}_\chi$.

Proof. We will prove that (a), (c) and (d) are each equivalent to (b).

For (a) $\Rightarrow$ (b), assume $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$ is generic. Then it does not lie on the boundary of $C_{\boldsymbol{\beta}}$, which implies $\dim \mathbb{C}^r/\!/_\chi G = r - \dim G$ by Proposition 14.3.6. In other words, $\dim P_\chi = r - \dim G$.

Now take any vertex $\mathbf{b} \in P_\chi$. Then $\dim P_\chi = r - \dim G$ implies that at least this many facets meet at $\mathbf{b}$. Since all facets of P_χ occur among the virtual facets $F_{i,\chi} = P_\chi \cap \mathbf{V}(x_i)$, it follows that at least $r - \dim G$ coordinates of $\mathbf{b}$ vanish. Thus $\mathbf{b}$ has at most $\dim G$ nonzero coordinates. Suppose that $\mathbf{b}$ has $s < \dim G$ nonzero coordinates. Writing $\mathbf{b} = \sum_{j=1}^{s} \mu_j e_{i_j}$, $u_j > 0$, we obtain $\chi \otimes 1 = \sum_{j=1}^{s} \mu_j \beta_{i_j}$, which is impossible since χ is generic and $s < \dim G$. This proves (a) $\Rightarrow$ (b).

To prove (b) $\Rightarrow$ (a), assume that χ is not generic. By Carathédory's theorem, $\chi \otimes 1 = \sum_{j=1}^{s} \mu_j \beta_{i_j}$, where $u_j > 0$ and $s < \dim G$ (Exercise 14.3.5). Then $\mathbf{b} = \sum_{j=1}^{s} \mu_j e_{i_j} \in P_\chi$ has fewer than $\dim G$ nonzero coordinates. In Exercise 14.3.5 you will prove that this gives a vertex of P_χ with the same property, contradicting (b).

To prove (b) $\Rightarrow$ (c), assume (b) and take a vertex $\mathbf{b} \in P_\chi$. Arguing as above, we see that $\dim P_\chi = r - \dim G$ and that the actual facets of P_χ make at least $r - \dim G$ coordinates of $\mathbf{b}$ vanish. But exactly $r - \dim G$ vanish by (b), so that the only virtual facets containing $\mathbf{b}$ correspond to actual facets. Then (c) follows easily.

The proof of (c) $\Rightarrow$ (b) is similar and is left to the reader (Exercise 14.3.5).

Before proving (b) $\Leftrightarrow$ (d), we first describe the semistable points. Given a subset $I \subseteq \{1,\dots,r\}$, note that $\bigcap_{i\in I} F_{i,\chi} \neq \emptyset$ if and only if there is a vertex $\mathbf{b} \in P_\chi$ with $\mathbf{b} \in F_{i,\chi}$ for all $i \in I$. Hence the maximal such subsets are those given by $\{i \mid \mathbf{b} \in F_{i,\chi}\}$, $\mathbf{b} \in P_\chi$ a vertex. It follows that the ideal $B(\chi)$ defined in (14.2.11) is given by

$$B(\chi) = \big\langle \textstyle\prod_{\mathbf{b}\notin F_{i,\chi}} x_i \mid \mathbf{b} \text{ is a vertex of } P_\chi \big\rangle. \tag{14.3.8}$$

By Corollary 14.2.22, it follows that

$$(\mathbb{C}^r)^{\mathsf{ss}}_\chi = \bigcup_{\mathbf{b} \text{ a vertex}} \mathbb{C}^r_{\mathbf{b}}. \tag{14.3.9}$$

where $\mathbb{C}^r_{\mathbf{b}} = \{(p_1,\dots,p_r) \in \mathbb{C}^r \mid p_i \neq 0 \text{ when } \mathbf{b} \notin F_{i,\chi}\}$.

We now turn to (b) $\Rightarrow$ (d). The first step is to show that G-orbits are closed in $(\mathbb{C}^r)^{\mathsf{ss}}_\chi$. By (14.3.9), we need only show that G-orbits of points in $\mathbb{C}^r_{\mathbf{b}}$ are closed in $\mathbb{C}^r_{\mathbf{b}}$ for every vertex of $\mathbf{b}$. By hypothesis, $\mathbf{b}$ has precisely $r - \dim G$ coordinates that vanish, i.e., precisely this many virtual facets $F_{i,\chi}$ contain $\mathbf{b}$. Let $I = \{i \mid \mathbf{b} \in F_{i,\chi}\}$. By the earlier part of the proof, we know that P_χ is simple of dimension $r - \dim G$. Hence the $F_{i,\chi}$, $i \in I$, are the genuine facets containing $\mathbf{b}$, so the facet normals ν_i, $i \in I$, form a basis of $M_{\mathbb{R}}$.

Since the connected component of the identity $G^\circ \subseteq G$ has finite index, it suffices to show that $G^\circ \cdot p$ is closed in $\mathbb{C}^r_{\mathbf{b}}$ for all $p \in \mathbb{C}^r_{\mathbf{b}}$. Take $\overline{p} \in \overline{G^\circ \cdot p} \subseteq \mathbb{C}^r_{\mathbf{b}}$. Using Lemma 5.1.10 and arguing as in the proof of Theorem 5.1.11, we can find a one-parameter subgroup $\lambda : \mathbb{C}^* \to G^\circ$ and a point $g \in G^\circ$ such that

$$\overline{p} = \lim_{t\to 0} \lambda(t) g \cdot p. \tag{14.3.10}$$

Since $G^\circ \subseteq (\mathbb{C}^*)^r$, we can write $\lambda(t) = (t^{a_1},\dots,t^{a_r})$ for exponents $a_i \in \mathbb{Z}$, and

$$\textstyle\sum_{i=1}^{r} a_i \nu_i = 0, \tag{14.3.11}$$

follows from (14.2.2) since λ is a one-parameter subgroup of G (Exercise 14.3.5). The coordinates of $p, \overline{p}, g$ are nonzero for $i \notin I$, hence the limit (14.3.10) implies that $a_i = 0$ for $i \notin I$. Thus (14.3.11) becomes $\sum_{i\in I} a_i \nu_i = 0$, so that $a_i = 0$ for all

i since the ν_i, $i \in I$, are linearly independent. Hence λ is the trivial one-parameter subgroup. which implies $\overline{p} = g \cdot p \in G^\circ \cdot p$ by (14.3.10). We conclude that $G^\circ \cdot p$ is closed in $\mathbb{C}^r_{\mathbf{b}}$.

In Exercise 14.3.5 you will show that the isotropy subgroup G_p is finite for $p \in (\mathbb{C}^r)^{\mathsf{ss}}_\chi$. This and the previous paragraphs imply that every semistable point is stable, which completes the proof of (b) $\Rightarrow$ (d).

Finally, for (d) $\Rightarrow$ (b), first note that $(\mathbb{C}^r)^{\mathsf{s}}_\chi \neq \emptyset$, so that $\dim P_\chi = \dim \mathbb{C}^r /\!/_\chi G = r - \dim G = \dim M_\mathbb{R}$ by Proposition 14.1.12. As noted earlier, this implies that every vertex of P_χ has at least $r - \dim G$ coordinates that vanish. Now assume (b) fails, so that some vertex $\mathbf{b} \in P_\chi$ has $I = \{i \mid \mathbf{b} \in F_{i,\chi}\}$ with $|I| > r - \dim G$. Thus the $\nu_i \in M_\mathbb{R}$, $i \in I$, are linearly dependent since $\dim M_\mathbb{R} = r - \dim G$. Then there is a relation $\sum_{i\in I} a_i \nu_i = 0$ where $a_i \in \mathbb{Z}$ and $a_i > 0$ for at least one i. If we set $a_i = 0$ for $i \notin I$, then

$$\lambda(t) = (t^{a_1}, \ldots, t^{a_r}) \in (\mathbb{C}^*)^r$$

is a one-parameter subgroup of G by Exercise 14.3.5. Now define the point $p = (p_1, \ldots, p_r) \in \mathbb{C}^r_{\mathbf{b}}$ by

$$p_i = \begin{cases} 1 & a_i \geq 0 \\ 0 & a_i < 0. \end{cases}$$

Consider $\lim_{t\to 0} \lambda(t) \cdot p$. The limit exists in $\mathbb{C}^r$ since $p_i = 0$ for $a_i < 0$, and if $i \notin I$, then the ith coordinate of $\lambda(t) \cdot p$ is 1 for all t, so that the limit $\overline{p} = \lim_{t\to 0} \lambda(t) \cdot p$ lies in $\mathbb{C}^r_{\mathbf{b}}$. Since there is $i_0 \in I$ with $a_{i_0} > 0$, the i_0th coordinate of $\overline{p} = \lim_{t\to 0} \lambda(t) \cdot p$ is zero. However, the i_0th coordinate of p is nonzero, so the same is true for $G \cdot p$. Hence $G \cdot p$ is not closed in $\mathbb{C}^r_{\mathbf{b}}$, and it follows that p is a point that is semistable but not stable. This contradicts (d) and completes the proof of the theorem. □

The equivalence (a) $\Leftrightarrow$ (b) in Theorem 14.3.14 appears in [**137**], while (c) is taken from [**275**]. Our treatment of (d) was inspired by [**83**, Prop. 12.1].

Example 14.3.15. In Example 14.3.12, every nontrivial character is generic. When χ is trivial, Figure 6 shows that part (c) of Theorem 14.3.14 fails because there is a nonempty virtual facet that is not a facet. ◊

Example 14.3.16. Figure 5 of Example 14.3.7 makes it clear that the non-generic elements of $C_{\boldsymbol{\beta}}$ lie on the three rays generated by the β_i's. Hence the generic elements form two "chambers," a term that will be defined in §14.4. In the chamber on the right in Figure 5, P_χ is a quadrilateral, while in the chamber on the left, P_χ is a triangle with an empty virtual facet indicated by the dashed line. When χ lies on the vertical axis, P_χ is also a triangle but is non-generic because of the nonempty virtual facet where the dashed line touches the triangle. This illustrates the importance of the virtual facets, as described in part (c) of Theorem 14.3.14.

We will see in §14.4 that Figure 5 is an example of a secondary fan. ◊

Example 14.3.17. The polytope in Figure 3 of Example 14.2.15 is not simple, so that the character χ in that example is not generic. Since χ was chosen because it gave the monomials appearing in the determinant, this shows that geometrically interesting characters may fail to be generic. $\Diamond$

Let us give one more detailed example of the theorem.

Example 14.3.18. Consider $G = \{(t,t,t^{-1}) \mid t \in \mathbb{C}^*\}$ as in Example 14.3.12. If we choose the trivial character $\chi = 1$, then the GIT quotient is

$$\mathbb{C}^3/\!/_\chi G = \mathbb{C}^3/\!/G = \mathrm{Spec}(\mathbb{C}[x,y,z]^G) = \mathrm{Spec}(\mathbb{C}[xz,yz]) = \mathbb{C}^2.$$

Every part of Theorem 14.3.14 fails in this example. It is instructive to see how this happens:

(a) The trivial character is never generic, as follows from the definition.

(b) P_χ is the middle polyhedron in Figure 6 of Example 14.3.12. All three virtual facets meet at the vertex of P_χ, which means that too many coordinates vanish.

(c) As shown by Figure 6, P_χ has a nonempty virtual facet that is not a genuine facet. Yet P_χ is simple and has the correct dimension.

(d) $\chi = 1$ easily implies that $(\mathbb{C}^3)^{\mathrm{ss}}_\chi = \mathbb{C}^3$. Since $G \cdot (1,0,0) = \mathbb{C}^* \times \{(0,0)\}$ is not closed in $(\mathbb{C}^3)^{\mathrm{ss}}_\chi$, it follows that $(1,0,0) \notin (\mathbb{C}^3)^{\mathrm{s}}_\chi$.

For the quotient $X_\Sigma \simeq (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))/\!/G$ considered in Proposition 14.1.9, all semistable points were stable when the toric variety is simplicial. For general GIT quotients such as the one considered here, things can be more complicated. $\Diamond$

See also Exercise 14.3.6 for an example where the same genuine facet occurs for two distinct indices.

The following immediate corollary of Theorem 14.3.14 summarizes the nice properties of GIT quotients when the character is generic.

Corollary 14.3.19. *If $\chi \in \widehat{G}$ is generic, then $\mathbb{C}^r/\!/_\chi G$ is a simplicial semiprojective toric variety of the expected dimension $r - \dim G$ with $(\mathbb{C}^r)^{\mathrm{s}}_\chi = (\mathbb{C}^r)^{\mathrm{ss}}_\chi$.* $\square$

Generic characters are clearly very nice. In the next section we will see that generic characters determine a refinement of C_β, the so-called *secondary fan*, that tells the full story of $\mathbb{C}^r/\!/_\chi G$ as we vary χ.

Exercises for §14.3.

14.3.1. Complete the proof of Lemma 14.3.3.

14.3.2. Show that a cone $\mathrm{Cone}(\mathscr{A})$ is not strongly convex if and only there is a nonempty subset $\mathscr{A}' \subseteq \mathscr{A}$ and $\lambda_v > 0$ for $v \in \mathscr{A}'$ such that $\sum_{v \in \mathscr{A}'} \lambda_v v = 0$.

14.3.3. Supply the details omitted in Example 14.3.8.

14.3.4. Explain how Example 14.3.12 relates to the quotient construction of $\mathrm{Bl}_0(\mathbb{C}^2)$ from Example 5.1.16. See also Example 7.1.4.

14.3.5. This exercise is devoted to the proof of Theorem 14.3.14.

(a) In the proof of (b) $\Rightarrow$ (a), we claimed that Carathédory's theorem (see the proof of Theorem 2.2.12) implies that if χ is not generic, then $\chi \otimes 1 = \sum_{j=1}^{s} \mu_j \beta_{i_j}$, where $u_j > 0$ and $s < \dim G$. Prove this.

(b) Suppose that a point $\mathbf{b} \in P_\chi$ has fewer than $\dim G$ nonzero coordinates. Prove that there is a vertex of P_χ with the same property. Hint: Write $\mathbf{b}$ as a convex combination of vertices plus an element of the recession cone.

(c) Given $(a_1,\ldots,a_r) \in \mathbb{Z}^r$, prove that $\lambda(t) = (t^{a_1},\ldots,t^{a_r})$ defines a one-parameter subgroup of G if and only if $\sum_{i=1}^{r} a_i \nu_i = 0$.

(d) Complete the proof of (b) $\Rightarrow$ (d) by showing that the isotropy subgroup G_p is finite for every $p \in \mathbb{C}^r_{\mathbf{b}}$. Hint: Let $I = \{i \mid \mathbf{b} \in F_{i,\chi}\}$. Then show two things: first, that $g = (t_1,\ldots,t_r) \in G_p$ has $g_i = 1$ for $i \notin I$ and second, that the ν_i, $i \in I$, form a basis of $N_\mathbb{R}$. Now use Exercise 14.2.1 to show that G has finite order. You may also want to look at Exercise 5.1.11.

14.3.6. Consider $G = \{(t,t^{-1}) \mid t \in \mathbb{C}^*\}$ with the trivial character $\chi = 1$.

(a) Show that $\nu_1 = \nu_2 = e_1$ and $\beta_1 = e_1$, $\beta_2 = -e_1$.

(b) Show that $P_\chi = \mathrm{Cone}(e_1) \subseteq M_\mathbb{R} \simeq \mathbb{R}$ with $F_{1,\chi} = F_{2,\chi} = \{0\}$.

(c) Show that $\mathbb{C}^2 /\!/_\chi G = \mathbb{C}$.

(d) Show that $(\mathbb{C}^2)^{\mathsf{ss}}_\chi = \mathbb{C}^2$ and $(\mathbb{C}^2)^{\mathsf{s}}_\chi = (\mathbb{C}^*)^2$.

This is our first example where a character fails to be generic because the same genuine facet occurs for two separate indices.

14.3.7. In the situation of Gale duality, let $I \subseteq \{1,\ldots,r\}$ and $J = \{1,\ldots,r\} \setminus I$. Prove that the β_i, $i \in I$, are linearly independent in W if and only if the ν_j, $j \in J$, span V^*. Hint: Enlarge I to get a basis of W.

14.3.8. Suppose that $W \simeq \mathbb{R}$ with basis e_1. Then pick vectors $\beta_1 = e_1$ and $\beta_2 = 0$ and define V as in (14.3.1).

(a) Show that V has a basis e_1 such that $\nu_1 = 0$ and $\nu_2 = e_1$.

(b) Use this example to explain why the nonzero vector hypothesis is needed in parts (b) and (d) of Lemma 14.3.2.

(c) Adapt this situation to $\mathbb{Z}$ and give an example where part (e) of Proposition 14.3.6 is true but the other parts of the proposition are false.

14.3.9. An *augmented polyhedron* consists of a polyhedron P and a list of faces F_i of P for $1 \le i \le r$. We allow $F_i = \emptyset$ or $F_i = P$, and we can also have $F_i = F_j$ for $i \neq j$. However, we require that all actual facets of P occur among the F_i. An example of an augmented polyhedron is given by P_χ with its virtual facets $F_{i,\chi}$, $1 \le i \le r$.

(a) Define what it means for an augmented polyhedron to be *simple* in terms of the number of virtual facets containing a vertex.

(b) Restate part (c) of Theorem 14.3.14 using augmented polyhedra. It will be elegant.

(c) Use one of the examples given in the text to show that there is a non-simple augmented polyhedron whose underlying polyhedron is simple.

§14.4. The Secondary Fan

For $G \subseteq (\mathbb{C}^*)^r$, we study the GIT quotient $\mathbb{C}^r /\!/_\chi G$ as $\chi \in \widehat{G}$ varies. The cones

$$C_{\boldsymbol{\beta}} = \mathrm{Cone}(\boldsymbol{\beta}) = \mathrm{Cone}(\beta_1, \ldots, \beta_r) \subseteq \widehat{G}_{\mathbb{R}}$$
$$C_{\boldsymbol{\nu}} = \mathrm{Cone}(\boldsymbol{\nu}) = \mathrm{Cone}(\nu_1, \ldots, \nu_r) \subseteq N_{\mathbb{R}}$$

introduced in §14.3 will play a key role in our discussion. Recall that $\widehat{G}_{\mathbb{R}}$ and $N_{\mathbb{R}}$ have the lattices $\widehat{G} \otimes 1$ and N.

A first observation is that there are only finitely many distinct GIT quotients up to isomorphism. To see why, recall from Corollary 14.2.22 that

$$\mathbb{C}^r /\!/_\chi G \simeq \big(\mathbb{C}^r \setminus Z(\chi)\big) /\!/ G,$$

where $Z(\chi)$ is the vanishing locus of the irrelevant ideal (14.2.11) defined by

$$B(\chi) = \big\langle \textstyle\prod_{i \notin I} x_i \mid I \subseteq \{1, \ldots, r\},\ \bigcap_{i \in I} F_{i,\chi} \neq \emptyset \big\rangle \subseteq \mathbb{C}[x_1, \ldots, x_r]. \tag{14.4.1}$$

There are only finitely many such ideals, hence only finitely many GIT quotients. This decomposes the lattice points of $C_{\boldsymbol{\beta}}$ into finitely many disjoint subsets where $\mathbb{C}^r /\!/_\chi G$ is constant on each subset. The basic idea is that each subset lies in the relative interior of a cone of the *secondary fan*, whose support is $C_{\boldsymbol{\beta}}$.

A Shift in Focus. Giving a rigorous construction of the secondary fan will require some new ideas, and we will also need to work in $\mathbb{R}^r$ rather than in $\widehat{G}_{\mathbb{R}}$. Given $\mathbf{a} \in \mathbb{Z}^r$, the polyhedron $P_{\mathbf{a}} \subseteq M_{\mathbb{R}}$ gives two objects of interest:

- The generalized fan Σ, which is the normal fan of $P_{\mathbf{a}}$ in $N_{\mathbb{R}}$.
- The set $I_\emptyset = \{i \mid F_{i,\mathbf{a}} = \emptyset\}$, which determines the empty virtual facets of $P_{\mathbf{a}}$.

We can characterize the pairs $(\Sigma, I_\emptyset)$ that occur as follows.

Proposition 14.4.1. *Given a generalized fan Σ in $N_{\mathbb{R}}$ and subset $I_\emptyset \subseteq \{1, \ldots, r\}$, then Σ and $I_\emptyset$ come from some $\mathbf{a} \in \mathbb{Z}^r$ if and only if*

(a) $|\Sigma| = C_{\boldsymbol{\nu}}$.

(b) *X_Σ is semiprojective.*

(c) $\sigma = \mathrm{Cone}(\nu_i \mid \nu_i \in \sigma,\ i \notin I_\emptyset)$ *for every* $\sigma \in \Sigma$.

Proof. First suppose that Σ and $I_\emptyset$ come from $\mathbf{a} \in \mathbb{Z}^r$. Since Σ is the normal fan of $P_{\mathbf{a}}$, Lemma 14.2.2 and Proposition 14.2.10 imply $|\Sigma| = C_{\boldsymbol{\nu}}$. Furthermore, X_Σ is the toric variety of $P_{\mathbf{a}}$ and hence is semiprojective by Proposition 14.2.12. Thus Σ satisfies conditions (a) and (b).

For (c), it suffices to consider maximal cones, which correspond to vertices $\mathsf{v} \in P_{\mathbf{a}}$. In Exercise 14.4.1 you will show that v gives the maximal cone

$$\mathrm{Cone}(\nu_i \mid \langle \mathsf{v}, \nu_i \rangle = -a_i) = \mathrm{Cone}(\nu_i \mid \mathsf{v} \in F_{i,\mathbf{a}}).$$

Then (c) follows since $\mathsf{v} \in F_{i,\mathbf{a}}$ implies $i \notin I_\emptyset$, Conversely, suppose that Σ and $I_\emptyset$ satisfy (a), (b) and (c). Let σ_0 be the minimal cone of Σ and set $\overline{N} = N/\sigma_0$. Then Σ gives a genuine fan $\overline{\Sigma}$ in $\overline{N}_{\mathbb{R}}$. Since $X_\Sigma = X_{\overline{\Sigma}}$ is semiprojective, Theorem 7.2.4 and Proposition 7.2.9 imply that $X_{\overline{\Sigma}}$ has a torus-invariant Cartier divisor whose support function is strictly convex on $|\overline{\Sigma}|$. Composing this with the map $N_{\mathbb{R}} \to \overline{N}_{\mathbb{R}}$ gives a strictly convex support function φ on $|\Sigma|$ that takes integer values on N. Then define $\mathbf{a} = (a_1,\dots,a_r) \in \mathbb{Z}^r$ by

$$a_i = \begin{cases} \varphi(\nu_i) & i \notin I_\emptyset \\ \varphi(\nu_i) - 1 & i \in I_\emptyset. \end{cases}$$

In Exercise 14.4.1 you will show that Σ and $I_\emptyset$ come from $\mathbf{a}$. □

GKZ Cones. Following Gel′fand, Kapranov and Zelevinsky [**113**, Ch. 7] (see also [**112**]), we construct the secondary fan using cones defined in terms of support functions. Our treatment is also influenced by [**141**] and [**222**].

Support functions with respect to a fan were defined in Definition 4.2.11. This definition extends to generalized fans without change. Let $\mathrm{SF}(\Sigma)$ denote the set of all support functions with respect the generalized fan Σ. Since Σ has full dimensional convex support by Proposition 14.4.1, we can define

$$\mathrm{CSF}(\Sigma) = \{\varphi \in \mathrm{SF}(\Sigma) \mid \varphi \text{ is convex}\}. \tag{14.4.2}$$

The convexity criteria from §6.1 and §7.2 apply to our situation.

Definition 14.4.2. Let Σ and $I_\emptyset \subseteq \{1,\dots,r\}$ be as in Proposition 14.4.1. Then the ***GKZ cone*** of Σ and $I_\emptyset$ is the set

$$\begin{aligned}\widetilde{\Gamma}_{\Sigma,I_\emptyset} = \big\{(a_1,\dots,a_r) \in \mathbb{R}^r \mid &\text{ there is } \varphi \in \mathrm{CSF}(\Sigma) \text{ such that}\\ &\varphi(\nu_i) = -a_i \text{ for } i \notin I_\emptyset \text{ and } \varphi(\nu_i) \geq -a_i \text{ for } i \in I_\emptyset\big\}.\end{aligned}$$

We will see below that $\widetilde{\Gamma}_{\Sigma,I_\emptyset}$ is a cone. Note that when $\mathbf{a} \in \widetilde{\Gamma}_{\Sigma,I_\emptyset}$, the support function φ is unique. This follows from Proposition 14.4.1 since φ is linear on the cones of Σ and satisfies $\varphi(\nu_i) = -a_i$ for $i \notin I_\emptyset$. Hence we write φ as $\varphi_{\mathbf{a}}$.

Here are some easy properties of GKZ cones.

Proposition 14.4.3. *Let Σ and $I_\emptyset$ be as above. Then:*

(a) *$\widetilde{\Gamma}_{\Sigma,I_\emptyset}$ is a rational polyhedral cone in $\mathbb{R}^r$.*

(b) *The minimal face of $\widetilde{\Gamma}_{\Sigma,I_\emptyset}$ is $\ker(\gamma_{\mathbb{R}}) = \mathrm{im}(\delta_{\mathbb{R}}) \simeq M_{\mathbb{R}}$.*

Proof. We begin with a different model of $\widetilde{\Gamma}_{\Sigma,I_\emptyset}$ that will be useful later in the section. Consider the direct sum $W = \mathbb{R}^r \oplus \bigoplus_{\sigma\in\Sigma_{\max}} M_{\mathbb{R}}$. We write points of W as $(\mathbf{a},\{m_\sigma\})$, where $\mathbf{a} = (a_1,\dots,a_r)$ and $\{m_\sigma\} = \{m_\sigma\}_{\sigma\in\Sigma_{\max}}$. Then let $\mathcal{C}_{\Sigma,I_\emptyset} \subseteq W$ be the subset consisting of all points $(\mathbf{a},\{m_\sigma\})$ satisfying

$$\begin{aligned}\langle m_\sigma,\nu_i\rangle &= -a_i, \text{ for } \nu_i \in \sigma \text{ and } i \notin I_\emptyset\\ \langle m_\sigma,\nu_i\rangle &\geq -a_i, \text{ for } \nu_i \notin \sigma \text{ or } i \in I_\emptyset.\end{aligned} \tag{14.4.3}$$

Let $\pi : W \to \mathbb{R}^r$ be the projection map. We claim that π induces a bijection

$$\mathcal{C}_{\Sigma,I_\emptyset} \simeq \widetilde{\Gamma}_{\Sigma,I_\emptyset}. \tag{14.4.4}$$

To prove this, take $(\mathbf{a},\{m_\sigma\}) \in \mathcal{C}_{\Sigma,I_\emptyset}$. Then $\varphi(u) = \min_{\sigma\in\Sigma_{\max}}\langle m_\sigma, u\rangle$ is convex by Lemma 6.1.5, and by (14.4.3), we have $\varphi(\nu_i) = -a_i$, $i \notin I_\emptyset$, and $\varphi(\nu_i) \geq -a_i$, $i \in I_\emptyset$. If $\sigma \in \Sigma_{\max}$, then (14.4.3) also implies that $\langle m_\sigma, \nu_i\rangle = -a_i$ for $\nu_i \in \sigma$, $i \notin I_\emptyset$. Since these ν_i's generate σ by Proposition 14.4.1, we have $\varphi \in \mathrm{CSF}(\Sigma)$ and thus $\mathbf{a} \in \widetilde{\Gamma}_{\Sigma,I_\emptyset}$. Conversely, let $\mathbf{a} \in \widetilde{\Gamma}_{\Sigma,I_\emptyset}$. For each $\sigma \in \Sigma_{\max}$ there is $m_\sigma \in M_{\mathbb{R}}$ such that $\varphi_{\mathbf{a}}(u) = \langle m_\sigma, u\rangle$ for $u \in \sigma$. By convexity and Lemma 6.1.5, $\langle m_\sigma, u\rangle \geq \varphi_{\mathbf{a}}(u)$ for all $u \in |\Sigma|$. From here, one sees easily that $(\mathbf{a},\{m_\sigma\}) \in \mathcal{C}_{\Sigma,I_\emptyset}$.

For part (a), observe that $\mathcal{C}_{\Sigma,I_\emptyset}$ is a rational polyhedral cone in W relative to the lattice $\mathbb{Z}^r \oplus \bigoplus_{\sigma\in\Sigma_{\max}} M$ since $\nu_i \in N$ for all i. Then (14.4.4) implies that $\widetilde{\Gamma}_{\Sigma,I_\emptyset}$ is a rational polyhedral cone in $\mathbb{R}^r$.

For part (b), note that for $m \in M_{\mathbb{R}}$, the point $\delta_{\mathbb{R}}(m) = (\langle m,\nu_1\rangle,\ldots,\langle m,\nu_r\rangle)$ lies in $\widetilde{\Gamma}_{\Sigma,I_\emptyset}$ because of the convex support function defined by $\varphi(u) = -\langle m,u\rangle$. Now take $\mathbf{a} \in \widetilde{\Gamma}_{\Sigma,I_\emptyset} \cap (-\widetilde{\Gamma}_{\Sigma,I_\emptyset})$. Then $\varphi_{\mathbf{a}}$ and $-\varphi_{\mathbf{a}} = \varphi_{-\mathbf{a}}$ are convex, so that $\varphi_{\mathbf{a}}$ is linear, say $\varphi_{\mathbf{a}}(u) = -\langle m,u\rangle$ for $u \in C_{\boldsymbol{\nu}}$. Since $\varphi_{\mathbf{a}}(\nu_i) \geq -a_i$ and $-\varphi_{\mathbf{a}}(\nu_i) \geq -a_i$ for all i, we obtain $\langle m,\nu_i\rangle = -a_i$ for all i, i.e., $\delta_{\mathbb{R}}(m) = \mathbf{a}$. □

We will soon see that the GKZ cones form a generalized fan in $\mathbb{R}^r$. The first step is Proposition 14.4.3, which shows that GKZ cones are rational polyhedral cones. The other properties we need will be covered in three lemmas that describe the relative interiors, union, and faces of GKZ cones.

Three Lemmas. Our first lemma describes the relative interior of a GKZ cone.

Lemma 14.4.4. *If $\widetilde{\Gamma}_{\Sigma,I_\emptyset}$ is a GKZ cone, then*

$$\mathrm{Relint}(\widetilde{\Gamma}_{\Sigma,I_\emptyset}) = \{\mathbf{a} \in \widetilde{\Gamma}_{\Sigma,I_\emptyset} \mid \varphi_{\mathbf{a}} \text{ is strictly convex and } \varphi_{\mathbf{a}}(\nu_i) > -a_i,\ i \in I_\emptyset\}.$$

Furthermore, if $\mathbf{a} \in \mathrm{Relint}(\widetilde{\Gamma}_{\Sigma,I_\emptyset})$, then:

(a) *Σ is the normal fan of $P_{\mathbf{a}}$.*

(b) *$I_\emptyset = \{i \mid \varphi_{\mathbf{a}}(\nu_i) > -a_i\} = \{i \mid F_{i,\mathbf{a}} = \emptyset\}$.*

Proof. Let $\mathcal{C}_{\Sigma,I_\emptyset} \subseteq W$ be the cone from the proof of Proposition 14.4.3. The first line of (14.4.3) defines a subspace $W_0 \subseteq W$ containing $\mathcal{C}_{\Sigma,I_\emptyset}$, and then the second line shows that $\mathcal{C}_{\Sigma,I_\emptyset} \subseteq W_0$ is defined by the inequalities

$$\begin{aligned} &\langle m_\sigma,\nu_i\rangle \geq -a_i, \text{ for } \nu_i \notin \sigma \text{ and } i \notin I_\emptyset \\ &\langle m_\sigma,\nu_i\rangle \geq -a_i, \text{ for } i \in I_\emptyset. \end{aligned} \tag{14.4.5}$$

By assumption, Σ and $I_\emptyset$ come from a polytope $P_{\mathbf{a}}$, $\mathbf{a} \in \mathbb{Z}^r$. Let the vertices of $P_{\mathbf{a}}$ be m_σ for $\sigma \in \Sigma_{\max}$. This gives the point $(\mathbf{a},\{m_\sigma\}) \in \mathcal{C}_{\Sigma,I_\emptyset}$ that makes all of the inequalities in (14.4.5) strict. It follows easily that the relative interior of $\mathcal{C}_{\Sigma,I_\emptyset}$ is the subset of $\mathcal{C}_{\Sigma,I_\emptyset}$ defined by

$$\text{(14.4.6)} \qquad \begin{aligned} \langle m_\sigma, \nu_i\rangle &> -a_i, \text{ for } \nu_i \notin \sigma \text{ and } i \notin I_\emptyset\\ \langle m_\sigma, \nu_i\rangle &> -a_i, \text{ for } i \in I_\emptyset.\end{aligned}$$

We claim that these strict inequalities correspond via (14.4.4) to the subset

$$\{\mathbf{a} \in \widetilde{\Gamma}_{\Sigma,I_\emptyset} \mid \varphi \text{ is strictly convex and } \varphi(\nu_i) > -a_i,\ i \in I_\emptyset\}.$$

To prove this, note that an arbitrary point $(\mathbf{a}, \{m_\sigma\}) \in \mathcal{C}_{\Sigma,I_\emptyset}$ gives a support function $\varphi_{\mathbf{a}}$ such that $\varphi_{\mathbf{a}}(u) = \langle m_\sigma, u\rangle$ for $u \in \sigma$. Since $\sigma \in \Sigma_{\max}$ is built from the ν_i for $i \notin I_\emptyset$, the first line of (14.4.6) is equivalent to the strict convexity of $\varphi_{\mathbf{a}}$ by (a) $\Leftrightarrow$ (f) of Lemma 6.1.13, and the second line is equivalent to $\varphi_{\mathbf{a}}(\nu_i) > -a_i$ for $i \in I_\emptyset$. This proves our claim, and our description of $\mathrm{Relint}(\widetilde{\Gamma}_{\Sigma,I_\emptyset})$ follows.

Now take $\mathbf{a} \in \mathrm{Relint}(\widetilde{\Gamma}_{\Sigma,I_\emptyset})$ and suppose that $(\mathbf{a}, \{m_\sigma\}) \in \mathcal{C}_{\Sigma,I_\emptyset}$ maps to $\mathbf{a}$. Then m_σ represents $\varphi_{\mathbf{a}}$ on $\sigma \in \Sigma_{\max}$, and the convexity of $\varphi_{\mathbf{a}}$ implies that

$$\varphi_{\mathbf{a}}(u) = \min_{\sigma\in\Sigma_{\max}} \langle m_\sigma, u\rangle.$$

by Lemma 6.1.5. Then Theorem 7.2.2 shows that the m_σ are the vertices of $P_{\mathbf{a}}$. The cone of the normal fan corresponding to this vertex is

$$\mathrm{Cone}(P_{\mathbf{a}} - m_\sigma)^\vee,$$

which is easily seen to be σ (Exercise 14.4.1). This proves part (a).

For part (b), take $i \in I_\emptyset$. Then Proposition 14.4.3 implies that $\varphi_{\mathbf{a}}(\nu_i) > -a_i$ since $\mathbf{a}$ is in the relative interior. On the other hand, if $i \notin I_\emptyset$, then pick $\sigma \in \Sigma_{\max}$ with $\nu_i \in \sigma$. The first line of (14.4.3) implies that $\varphi_{\mathbf{a}}(\nu_i) = -a_i$, and then part (b) follows easily. □

A nice consequence of Lemma 14.4.4 is that the relative interiors of two GKZ cones are either equal or disjoint.

Our second lemma describes the union of the GKZ cones.

Lemma 14.4.5. $\bigcup_{\Sigma,I_\emptyset} \widetilde{\Gamma}_{\Sigma,I_\emptyset} = \gamma_{\mathbb{R}}^{-1}(C_{\boldsymbol{\beta}})$, *where the union is over all* Σ *and* $I_\emptyset$ *satisfying Proposition 14.4.1.*

Proof. If $\mathbf{a}$ is in the union, then $P_{\mathbf{a}}$ is nonempty, so that $\gamma_{\mathbb{R}}(\mathbf{a}) \in C_{\boldsymbol{\beta}}$ by the proof of Proposition 14.3.5. Conversely, suppose that $\mathbf{a} \in \gamma_{\mathbb{R}}^{-1}(C_{\boldsymbol{\beta}}) \cap \mathbb{Q}^r$. Then some positive multiple $\ell\mathbf{a} \in \gamma_{\mathbb{R}}^{-1}(C_{\boldsymbol{\beta}}) \cap \mathbb{Z}^r$. By Proposition 14.4.1, $\ell\mathbf{a}$ gives $\Sigma, I_\emptyset$ with $\mathbf{a} = (1/\ell)(\ell\mathbf{a}) \in \widetilde{\Gamma}_{\Sigma,I_\emptyset}$. Thus

$$\gamma_{\mathbb{R}}^{-1}(C_{\boldsymbol{\beta}}) \cap \mathbb{Q}^r \subseteq \bigcup_{\Sigma,I_\emptyset} \widetilde{\Gamma}_{\Sigma,I_\emptyset},$$

and then $\gamma_{\mathbb{R}}^{-1}(C_{\boldsymbol{\beta}}) \subseteq \bigcup_{\Sigma,I_\emptyset} \widetilde{\Gamma}_{\Sigma,I_\emptyset}$ follows by taking the closure. □

The third lemma concerns the faces of a GKZ cone.

Lemma 14.4.6. *Every face of a GKZ cone is again a GKZ cone, and* $\widetilde{\Gamma}_{\Sigma',I'_\emptyset}$ *is a face of* $\widetilde{\Gamma}_{\Sigma,I_\emptyset}$ *if and only if* Σ *refines* Σ' *and* $I'_\emptyset \subseteq I_\emptyset$.

Proof. We will use the cone $\mathcal{C}_{\Sigma,I_\emptyset} \subseteq W$ from the proof of Proposition 14.4.3. Recall that $\mathcal{C}_{\Sigma,I_\emptyset} \subseteq W_0$, where $W_0 \subseteq W = \mathbb{R}^r \oplus \bigoplus_{\sigma\in\Sigma_{\max}} M_\mathbb{R}$ is defined by the first line of (14.4.3) and $\mathcal{C}_{\Sigma,I_\emptyset} \subseteq W_0$ is defined by the inequalities from the second line, namely

$$(14.4.7) \qquad \begin{aligned} &\langle m_\sigma, \nu_i\rangle \geq -a_i, \text{ for } \nu_i \notin \sigma \text{ and } i \notin I_\emptyset \\ &\langle m_\sigma, \nu_i\rangle \geq -a_i, \text{ for } i \in I_\emptyset. \end{aligned}$$

The idea is to study the faces of $\mathcal{C}_{\Sigma,I_\emptyset}$, which map to faces of $\widetilde{\Gamma}_{\Sigma,I_\emptyset}$ by (14.4.4).

Recall that $(\mathbf{a},\{m_\sigma\}) \in \mathcal{C}_{\Sigma,I_\emptyset}$ gives a polyhedron $P_\mathbf{a}$ with virtual facets $F_{i,\mathbf{a}}$, normal fan $\Sigma_\mathbf{a}$, and index set $I_{\emptyset,\mathbf{a}} = \{i \mid F_{i,\mathbf{a}} = \emptyset\}$. Also note the following:

- The m_σ are the vertices of $P_\mathbf{a}$ by the proof of Lemma 14.4.4.
- Σ refines $\Sigma_\mathbf{a}$ since $\varphi_\mathbf{a} \in \mathrm{SF}(\Sigma)$.
- $I_{\emptyset,\mathbf{a}} \subseteq I_\emptyset$. To see why, take $i \notin I_\emptyset$ and pick $\sigma \in \Sigma_{\max}$ with $\nu_i \in \sigma$. This implies $\langle m_\sigma, \nu_i\rangle = -a_i$, so that $m_\sigma \in F_{i,\mathbf{a}}$. Hence $i \notin I_{\emptyset,\mathbf{a}}$.

Now suppose that $F \preceq \mathcal{C}_{\Sigma,I_\emptyset}$ is a proper face. Then F is defined by turning some of the inequalities (14.4.7) into equalities. To keep track of which ones, define

$$\mathcal{I}_F = \{(\sigma,i) \mid \nu_i \notin \sigma \text{ or } i \in I_\emptyset, \text{ and } \langle m_\sigma,\nu_i\rangle = -a_i \text{ for all } (\mathbf{a},\{m_\sigma\}) \in F\}.$$

We claim that $\Sigma_\mathbf{a}$ and $I_{\emptyset,\mathbf{a}}$ are equal for all $(\mathbf{a},\{m_\sigma\}) \in \mathrm{Relint}(F)$. For $I_{\emptyset,\mathbf{a}}$, note that since the m_σ are the vertices of $P_\mathbf{a}$, we have

$$\begin{aligned} F_{i,\mathbf{a}} = \emptyset &\iff \langle m_\sigma,\nu_i\rangle > -a_i \text{ for all } \sigma \in \Sigma_{\max} \\ &\iff i \in I_\emptyset \text{ and } (\sigma,i) \notin \mathcal{I}_F \text{ for all } \sigma \in \Sigma_{\max}. \end{aligned}$$

You will prove the second equivalence in Exercise 14.4.2. This shows that $I_{\emptyset,\mathbf{a}}$ depends only on F when $(\mathbf{a},\{m_\sigma\}) \in \mathrm{Relint}(F)$.

The argument for $\Sigma_\mathbf{a}$ will take more work. Recall from §6.1 that convexity can be detected by looking at walls. We say that $\langle m_\sigma,\nu_i\rangle \geq -a_i$ is a (σ,σ',ν_i)-*wall inequality* if

$$\sigma,\sigma' \in \Sigma_{\max},\ \sigma\cap\sigma' \text{ is a wall},\ \nu_i \in \sigma'\setminus\sigma,\ i \notin I_\emptyset.$$

Every wall inequality appears among the inequalities in the first line of (14.4.7). Furthermore, given $(\mathbf{a},\{m_\sigma\}) \in \mathcal{C}$ and a (σ,σ',ν_i)-wall inequality, we have

$$(14.4.8) \qquad \langle m_\sigma,\nu_i\rangle = -a_i \iff m_\sigma = m_{\sigma'}.$$

This follows since $\sigma\cap\sigma'$ is a wall and $\langle m_\sigma - m_{\sigma'}, u\rangle = 0$ for all $u \in \sigma\cap\sigma'$.

The face F gives an equivalence relation on $\Sigma_{\max}$ as follows. Given cones $\sigma,\sigma' \in \Sigma_{\max}$, we say that $\sigma \sim_F \sigma'$ if either $\sigma = \sigma'$ or there exists a chain of cones $\sigma = \sigma_0,\sigma_1,\dots,\sigma_{s-1},\sigma_s = \sigma'$ in $\Sigma_{\max}$ and $\nu_{i_1},\dots,\nu_{i_s}$ such that for $1 \leq j \leq s$, we have a $(\sigma_{j-1},\sigma_j,\nu_{i_j})$-wall inequality with $(\sigma_{j-1},\nu_{i_j})\in\mathcal{I}_F$. Thus $\sigma\sim_F\sigma'$ means that σ and σ' are connected by a chain of compatible wall inequalities that become equalities in F. For $(\mathbf{a},\{m_\sigma\})\in\operatorname{Relint}(F)$, we will show that

$$m_\sigma = m_{\sigma'} \iff \sigma\sim_F\sigma'.$$

One direction is easy: if $\sigma\sim_F\sigma'$, then (14.4.8) and $(\sigma_{j-1},\nu_{i_j})\in\mathcal{I}_F$ imply that $m_{\sigma_{j-1}} = m_{\sigma_j}$ for $1\le j\le s$, and $m_\sigma = m_{\sigma'}$ follows immediately. For the other direction, suppose that $m_\sigma = m_{\sigma'}$. Then m_σ and $m_{\sigma'}$ give the same vertex of $P_{\mathbf{a}}$, which implies that σ and σ' are contained in the same maximal cone $\widetilde{\sigma}\in\Sigma_{\mathbf{a}}$. Then a suitably chosen line segment connecting interior points of σ,σ' lies in $\tilde{\sigma}$ and meets only walls of Σ. This gives a compatible chain of wall inequalities that are equalities. Since $(\mathbf{a},\{m_\sigma\})\in\operatorname{Relint}(F)$, inequalities indexed by $(\sigma,\nu_i)\notin\mathcal{I}_F$ are strict. Thus the wall inequalities in our chain come from $\mathcal{I}_F$, hence $\sigma\sim_F\sigma'$.

It follows that each maximal cone of $\Sigma_{\mathbf{a}}$ is the union of σ's contained in an equivalence class of $\sim_F$. This proves that $\Sigma_{\mathbf{a}}$ depends only on the face F when $(\mathbf{a},\{m_\sigma\})\in\operatorname{Relint}(F)$. The same is true for $I_{\emptyset,\mathbf{a}}$, so that via (14.4.4), we have

$$F\simeq\widetilde{\Gamma}_{\Sigma_{\mathbf{a}},I_{\emptyset,\mathbf{a}}}$$

when $(\mathbf{a},\{m_\sigma\})\in\operatorname{Relint}(F)$. Thus every face of a GKZ cone is a GKZ cone. From here, Lemma 14.4.4 makes it straightforward to prove the final assertion of the proposition. We leave the details to the reader (Exercise 14.4.2). □

The Secondary Fan. The above results show that the set of GKZ cones

$$\widetilde{\Sigma}_{\mathrm{GKZ}} = \{\widetilde{\Gamma}_{\Sigma,I_\emptyset}\mid \Sigma, I_\emptyset \text{ as in Proposition 14.4.1}\}$$

has many nice properties. To see that $\widetilde{\Sigma}_{\mathrm{GKZ}}$ is a generalized fan, we need one further observation: a set $\widetilde{\Sigma}$ of rational polyhedral cones is a generalized fan if and only if

- Every face of an element of $\widetilde{\Sigma}$ lies in $\widetilde{\Sigma}$.
- The relative interiors of the elements of $\widetilde{\Sigma}$ are pairwise disjoint.

You will prove this Exercise 14.4.3.

It follows that $\widetilde{\Sigma}_{\mathrm{GKZ}}$ is a generalized fan: GKZ cones are rational polyhedral by Proposition 14.4.3, their faces are again GKZ cones by Lemma 14.4.6, and their relative interiors are pairwise disjoint by Lemma 14.4.4. Note also that $|\widetilde{\Sigma}_{\mathrm{GKZ}}| = \gamma_{\mathbb{R}}^{-1}(C_{\boldsymbol{\beta}})$ by Lemma 14.4.5.

We know from §6.2 that we can turn a generalized fan into an actual fan by taking the quotient by its minimal face, which is the minimal face of every cone in the fan. For $\widetilde{\Sigma}_{\mathrm{GKZ}}$, this minimal face is $\ker(\gamma_{\mathbb{R}})$ by Proposition 14.4.3. It follows that the *GKZ cones*

$$\Gamma_{\Sigma,I_\emptyset} = \widetilde{\Gamma}_{\Sigma,I_\emptyset}/\ker(\gamma_{\mathbb{R}})\subseteq\mathbb{R}^r/\ker(\gamma_{\mathbb{R}}) = \widehat{G}_{\mathbb{R}}$$

form the *secondary fan*

$$\Sigma_{\mathrm{GKZ}} = \{\Gamma_{\Sigma,I_\emptyset} \mid \Sigma, I_\emptyset \text{ as in Proposition 14.4.1}\}.$$

Note that $|\Sigma_{\mathrm{GKZ}}| = C_{\boldsymbol{\beta}}$, so the maximal cones of Σ_{GKZ} have dimension $\dim G$. The maximal cones of Σ_{GKZ} are the *chambers* of the secondary fan. Another name for the secondary fan is the *GKZ decomposition*, hence the notation Σ_{GKZ}.

Here are some properties of the secondary fan.

Theorem 14.4.7.
(a) Σ_{GKZ} *is a fan in* $\widehat{G}_{\mathbb{R}}$ *with* $|\Sigma_{\mathrm{GKZ}}| = C_{\boldsymbol{\beta}}$.
(b) $\Gamma_{\Sigma',I'_\emptyset} \preceq \Gamma_{\Sigma,I_\emptyset}$ *if and only if* Σ *refines* Σ' *and* $I'_\emptyset \subseteq I_\emptyset$.
(c) $\mathbb{C}^r /\!\!/_\chi G \simeq X_\Sigma$ *when* $\chi \otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma,I_\emptyset})$.

Remark 14.4.8. Part (c) of the theorem tells us that the GIT quotient is constant on the relative interiors of the GKZ cones.

Proof. We proved part (a) above, and part (b) follows from Lemma 14.4.6. For part (c), assume that $\chi \otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma,I_\emptyset})$ and write $\chi = \chi^{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{Z}^r$. The GIT quotient $\mathbb{C}^r /\!\!/_\chi G$ is the toric variety of $P_{\mathbf{a}}$ by Theorem 14.2.13. However, Σ is the normal fan of $P_{\mathbf{a}}$ by Lemma 14.4.4, and $\mathbb{C}^r /\!\!/_\chi G \simeq X_\Sigma$ follows. $\square$

Our approach to the secondary fan is more general than what one finds in the literature. The papers [**141**] and [**222**] assume that the ν_i are nonzero and generate distinct rays in $N_{\mathbb{R}}$ ([**141**] also assumes that the ν_i are primitive). We will study this special case in §15.1, where we will see that the secondary fan has an especially nice structure.

Generic Characters. Before giving examples of the secondary fan, we need to relate GKZ cones to the generic characters introduced in §14.3.

Proposition 14.4.9. *If* $\chi \otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma,I_\emptyset})$, *then the following are equivalent:*
(a) χ *is generic.*
(b) Σ *is simplicial and* $i \mapsto \mathrm{Cone}(\nu_i)$ *induces a bijection* $\{1,\ldots,r\} \setminus I_\emptyset \simeq \Sigma(1)$.
(c) $\Gamma_{\Sigma,I_\emptyset}$ *is a chamber of the secondary fan.*

Remark 14.4.10. Using this proposition, one can show that the generic characters determine the chambers of the secondary fan uniquely (Exercise 14.4.4).

Proof. Write $\chi = \chi^{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{Z}^r$. By Theorem 14.3.14, χ is generic if and only if $P_{\mathbf{a}}$ is simple of dimension $r - \dim G$ and the nonempty virtual facets $F_{i,\mathbf{a}}$, $i \notin I_\emptyset$, are the actual facets of $P_{\mathbf{a}}$, with no duplications. Then (a) $\Leftrightarrow$ (b) follows easily since Σ is the normal fan of $P_{\mathbf{a}}$ by Lemma 14.4.4.

To prove (b) $\Rightarrow$ (c), note that $\Gamma_{\Sigma,I_\emptyset}$ is a chamber if and only if $\dim \widetilde{\Gamma}_{\Sigma,I_\emptyset} = r$. Furthermore, the proof of Lemma 14.4.4 shows that

$$\dim \widetilde{\Gamma}_{\Sigma,I_\emptyset} = \dim \mathcal{C}_{\Sigma,I_\emptyset} = \dim W_0,$$

where $W_0 \subseteq \mathbb{R}^r \oplus \bigoplus_{\sigma\in\Sigma_{\max}} M_\mathbb{R}$ is defined by

$$\langle m_\sigma, \nu_i\rangle = a_i, \text{ for } \nu_i \in \sigma \text{ and } i \notin I_\emptyset.$$

Hence it suffices to prove that (b) implies that $\dim W_0 = r$.

Assume (b) and take $\sigma \in \Sigma_{\max}$. Then σ is simplicial of maximal dimension, so that generators of its rays form a basis of $N_\mathbb{R}$. By assumption, the ray generators can be chosen to be the $\nu_i \in \sigma$ with $i \notin I_\emptyset$. It follows that for any $\mathbf{b} \in \mathbb{R}^r$ and $\sigma \in \Sigma_{\max}$, the equations

$$\langle m_\sigma, \nu_i\rangle = -b_i, \text{ for } \nu_i \in \sigma \text{ and } i \notin I_\emptyset$$

have a unique solution $m_\sigma \in M_\mathbb{R}$. When we do this for all $\sigma \in \Sigma_{\max}$, we see that for any $\mathbf{b} \in \mathbb{R}^r$, the equations defining W_0 have a unique solution. Thus $\dim W_0 = r$.

Finally, assume (c). First observe that since (b) involves only Σ and $I_\emptyset$, the equivalence (a) $\Leftrightarrow$ (b) means that if (a) holds for one $\chi' \otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma,I_\emptyset}) = \mathrm{Int}(\Gamma_{\Sigma,I_\emptyset})$, then it holds for all such $\chi' \otimes 1$. Hence it suffices to find one generic character that gives an interior point of the chamber.

Let $U = C_{\boldsymbol{\beta}} \setminus \big(\bigcup_{\boldsymbol{\beta}'} \mathrm{Cone}(\boldsymbol{\beta}')\big)$, where the union is over all $\boldsymbol{\beta}' \subseteq \boldsymbol{\beta}$ such that $\dim \mathrm{Cone}(\boldsymbol{\beta}') < \dim G$. Then U is open and dense in $C_{\boldsymbol{\beta}}$, and $\chi' \in \widehat{G}$ is generic if and only if $\chi' \otimes 1 \in U$. Since $\Gamma_{\Sigma,I_\emptyset}$ is a chamber, its interior must have nonempty intersection with U. This easily implies the existence of the required χ', and the proof is complete. $\square$

We can finally give an example of a secondary fan.

Example 14.4.11. Consider Figure 5 from Example 14.3.7, which we reproduce as Figure 7 on the next page. As we noted in Example 14.3.16, the non-generic characters of $C_{\boldsymbol{\beta}}$ lie on the three rays generated by the β_i's. This shows that the secondary fan has two chambers. You should compute Σ and $I_\emptyset$ for each of the six GKZ cones in the secondary fan, and you should check that part (b) of Proposition 14.4.9 holds only in the interior of the chambers. $\Diamond$

Another secondary fan is Figure 6 in §14.3, and Exercises 14.4.5–14.4.8 give more secondary fans. Further examples will be given in Chapter 15.

Degenerate Fans. Recall that a generalized fan Σ in $N_\mathbb{R}$ is called *degenerate* when it is not an actual fan. Equivalently,

$$\text{(14.4.9)} \qquad \Sigma \text{ is degenerate } \iff \dim X_\Sigma < \dim N_\mathbb{R}.$$

It is easy to determine where degenerate fans occur in the GKZ decomposition.

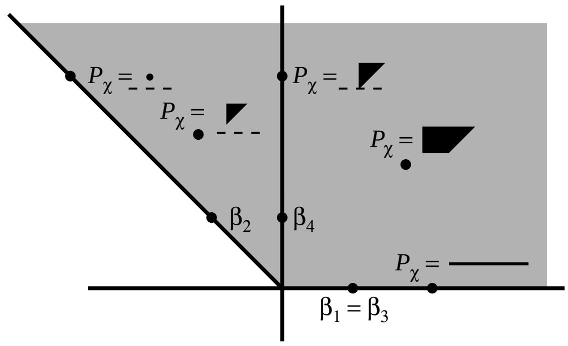


Figure 7. C_β and the polytopes P_χ for $\chi \otimes 1 \in C_\beta$ in Example 14.4.11

Proposition 14.4.12. *If $\chi \otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma,I_\emptyset})$, then*

$$\Sigma \text{ is degenerate } \iff \dim \mathbb{C}^r /\!/_\chi G < r - \dim G. \tag{14.4.10}$$

Furthermore:

(a) *If Σ is degenerate, then $\Gamma_{\Sigma,I_\emptyset}$ is contained in the boundary of $C_{\boldsymbol\beta}$.*

(b) *If $\boldsymbol\nu = (\nu_1, \ldots, \nu_r)$ consists of nonzero vectors, then*

$$\partial C_{\boldsymbol\beta} = \bigcup_{\Sigma \text{ degenerate}} \Gamma_{\Sigma,I_\emptyset}.$$

Proof. Take $\chi \otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma,I_\emptyset})$. Since $\dim N_\mathbb{R} = r - \dim G$ and $\mathbb{C}^r /\!/_\chi G \simeq X_\Sigma$ (Theorem 14.4.7), the equivalence (14.4.10) follows from (14.4.9).

Recall from (c) $\Rightarrow$ (e) of Proposition 14.3.6 that $\dim \mathbb{C}^r /\!/_\chi G$ has the expected dimension $r - \dim G$ when $\chi \otimes 1$ lies in the interior of the cone $C_{\boldsymbol\beta}$. Hence $\chi \otimes 1$ must lie in the boundary when the dimension of $\dim \mathbb{C}^r /\!/_\chi G$ drops. Then when part (a) follows from (14.4.10). Also recall that (c) $\Leftrightarrow$ (e) in Proposition 14.3.6 when $\boldsymbol\nu$ consists of nonzero vectors. In other words, when $\boldsymbol\nu$ satisfies this condition, the boundary of $C_{\boldsymbol\beta}$ is precisely where the dimension of the GIT quotient drops. Then part (b) follows from (14.4.10). $\square$

Irrelevant Ideals. At the beginning of the section, we motivated the existence of the secondary fan by noting that there are only finitely many possible irrelevant ideals (14.4.1). Here we will show that the secondary fan can be described completely in terms of irrelevant ideals.

Recall that $\chi \otimes 1 \in C_{\boldsymbol\beta}$ gives the polyhedron $P_\chi \subseteq \mathbb{R}^r$ with virtual facets $F_{i,\chi} = P_\chi \cap \mathbf{V}(x_i)$, and by (14.4.1), we have the irrelevant ideal

$$B(\chi) = \big\langle \textstyle\prod_{i \notin I} x_i \mid I \subseteq \{1, \ldots, r\},\ \bigcap_{i \in I} F_{i,\chi} \neq \emptyset \big\rangle.$$

By (14.3.8), the vertices of P_χ determine the minimal generators of $B(\chi)$. Thus

$$B(\chi) = \big\langle \textstyle\prod_{\mathbf{b} \notin F_{i,\chi}} x_i \mid \mathbf{b} \text{ is a vertex of } P_\chi \big\rangle. \tag{14.4.11}$$

Example 14.4.13. Figure 8 shows an example of P_χ and its virtual facets $F_{i,\chi}$. This picture and (14.4.11) make it easy to compute that $B(\chi) = \langle x_1x_2, x_2x_3, x_4\rangle$. Note that this is the polytope from Figure 7 when χ comes from the vertical axis. ◊

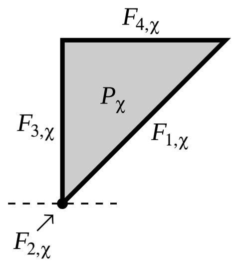


Figure 8. A polytope P_χ and its virtual facets

Given a GKZ cone $\Gamma_{\Sigma,I_\emptyset}$, define the ideal

$$B(\Sigma, I_\emptyset) = \big\langle \textstyle\prod_{\nu_i\notin\sigma \text{ or } i\in I_\emptyset} x_i \mid \sigma \in \Sigma_{\max}\big\rangle. \tag{14.4.12}$$

The ideals $B(\chi)$ and $B(\Sigma, I_\emptyset)$ relate to GKZ cones as follows.

Proposition 14.4.14. *If $\chi\otimes 1 \in C_\beta$, then:*

(a) $\chi\otimes 1 \in \Gamma_{\Sigma,I_\emptyset}$ *if and only if* $B(\Sigma, I_\emptyset) \subseteq B(\chi)$.

(b) $\chi\otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma,I_\emptyset})$ *if and only if* $B(\Sigma, I_\emptyset) = B(\chi)$.

(c) χ *is generic if and only if every minimal generator of* $B(\chi)$ *has degree* $\dim G$.

Proof. We begin with two easy observations about the ideals $B(\Sigma, I_\emptyset)$:

- Σ and $I_\emptyset$ are easy to recover from $B(\Sigma, I_\emptyset)$. To see why, observe first that $I_\emptyset = \{i \mid B(\Sigma, I_\emptyset) \subseteq \langle x_i\rangle\}$ and second that the minimal generators of $B(\Sigma, I_\emptyset)$ determine the maximal cones of Σ by (14.4.12) and Proposition 14.4.1.
- $B(\Sigma, I_\emptyset) \subseteq B(\Sigma', I'_\emptyset)$ if and only if Σ refines Σ' and $I'_\emptyset \subseteq I_\emptyset$. We leave the easy proof to the reader.

Now suppose that $\chi\otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma,I_\emptyset})$ and write $\chi = \chi^{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{Z}^r$. Then Σ is the normal fan of $P_{\mathbf{a}}$ and $I_\emptyset = \{i \mid F_{i,\mathbf{a}} = \emptyset\}$ by Lemma 14.4.4, and it follows easily from Proposition 14.4.1 that $B(\Sigma, I_\emptyset) = B(\chi)$. The converse follows easily from the first bullet above. This proves part (a) of the proposition, and part (b) follows easily from the second bullet and the description of the face relation given in Lemma 14.4.6.

For part (c), suppose that χ is generic. Then Proposition 14.3.14 implies that every vertex of P_χ is contained in precisely $r - \dim G$ virtual facets. By (14.4.11), the corresponding minimal generator of $B(\chi)$ has degree $\dim G$. The converse is equally straightforward. □

Propositions 14.3.14, 14.4.9 and 14.4.14 give a robust understanding of what it means for a character to be generic. Proposition 14.4.14 also gives an alternate description of the face relation in the GKZ fan.

Corollary 14.4.15. *Given GKZ cones $\Gamma_{\Sigma,I_\emptyset}$ and $\Gamma_{\Sigma',I'_\emptyset}$, we have*

$$\Gamma_{\Sigma',I'_\emptyset} \preceq \Gamma_{\Sigma,I_\emptyset} \iff B(\Sigma,I_\emptyset) \subseteq B(\Sigma',I'_\emptyset). \qquad \square$$

It follows that the ideals $B(\Sigma,I_\emptyset)$ completely determine the secondary fan. Here is what this looks like for a familiar example.

Example 14.4.16. Figure 7 from Example 14.4.11 shows P_χ and its virtual facets for five different χ's. Computing the ideals $B(\chi)$ (going from left to right) gives

$$\langle x_2\rangle \supseteq \langle x_1x_2, x_2x_3, x_2x_4\rangle \subseteq \langle x_1x_2, x_2x_3, x_4\rangle \supseteq \langle x_1x_2, x_2x_3, x_3x_4, x_1x_4\rangle \subseteq \langle x_1, x_3\rangle.$$

These five ideals are contained in the ideal of the trivial character, $B(1) = \langle 1\rangle$. The inclusion relation of these ideals determines the face relation of the GKZ fan, so that the chambers correspond to the minimal ideals $\langle x_1x_2, x_2x_3, x_2x_4\rangle$ and $\langle x_1x_2, x_2x_3, x_3x_4, x_1x_4\rangle$. The minimal generators of these two ideals all have degree equal to 2. It follows that generic characters give elements in the interiors of the corresponding cones, exactly as predicted by Proposition 14.4.14. $\Diamond$

Exercises for §14.4.

14.4.1. This exercise is concerned with the proof of Proposition 14.4.1.

(a) Let $\mathbf{a} \in \mathbb{R}^r$ and assume that $P_\mathbf{a} \neq \emptyset$. For a (possibly non-rational) vertex $\mathsf{v} \in P_\mathbf{a}$, define the cone $C_\mathsf{v} = \mathrm{Cone}(P_\mathbf{a} - \mathsf{v})$. Prove that

$$C_\mathsf{v}^\vee = \mathrm{Cone}(\nu_i \mid \langle \mathsf{v}, \nu_i\rangle = -a_i) = \mathrm{Cone}(\nu_i \mid \mathsf{v} \in F_{i,\mathbf{a}}).$$

(b) Prove the final assertion made in the proof of Proposition 14.4.1.

14.4.2. This exercise is concerned with the proof of Lemma 14.4.6.

(a) As in the proof of the lemma, let $(\mathbf{a}, \{m_\sigma\}) \in \mathrm{Relint}(F)$. Prove that $\langle m_\sigma, \nu_i\rangle > -a_i$ for all $\sigma \in \Sigma_{\max}$ if and only if $i \in I_\emptyset$ and $(\sigma, i) \notin \mathcal{I}_F$ for all $\sigma \in \Sigma_{\max}$.

(b) Complete the proof of the lemma.

14.4.3. Let $\widetilde{\Sigma}$ be a collection of polyhedral cones in a finite dimensional vector space W. Assume that every face of a cone in $\widetilde{\Sigma}$ lies in $\widetilde{\Sigma}$ and that the relative interiors of the cones in $\widetilde{\Sigma}$ are pairwise disjoint. The goal of this exercise is to prove that $\widetilde{\Sigma}$ is closed under intersection. For this purpose, take $\sigma, \sigma' \in \widetilde{\Sigma}$.

(a) Given $u \in \sigma \cap \sigma'$, let τ (resp. τ') be the minimal face of σ (resp. σ') containing u. Prove that $\tau = \tau'$.

(b) Conclude that $\sigma \cap \sigma'$ is a union $\tau_1 \cup \cdots \cup \tau_\ell$ with $\tau_i \in \widetilde{\Sigma}$ for $1 \leq i \leq \ell$.

(c) Show that there is some i such that $\sigma \cap \sigma' = \tau_i$. Hint: $\sigma \cap \sigma'$ is closed under addition. Take $u_i \in \mathrm{Relint}(\tau_i)$ and apply Lemma 1.2.7 to $u_1 + \cdots + u_\ell$.

14.4.4. Suppose that $\widetilde{\Sigma}$ is a fan in $\widetilde{G}_{\mathbb{R}}$ consisting of strongly convex rational polyhedral cones. Assume that $|\widetilde{\Sigma}| = C_{\beta}$ and that every generic $\chi \otimes 1$ lies in the interior of a maximal cone of $\widetilde{\Sigma}$. Prove that $\widetilde{\Sigma} = \Sigma_{\mathrm{GKZ}}$. This justifies the uniqueness claimed in Remark 14.4.8.

14.4.5. Use the secondary fan to explain what is happening in Example 14.1.15.

14.4.6. Assume that $G \subseteq (\mathbb{C}^*)^6$ with $\widehat{G} \simeq \mathbb{Z}^3$ and $\beta_1, \dots, \beta_6$ correspond to $(\pm 2, \pm 1, 1)$, $(\pm 1, 0, 1) \in \mathbb{Z}^3$. Determine the number of chambers in the secondary fan. Hint: Think about generic characters and draw a picture.

14.4.7. Let $G = \{(t, t^{-r}u, t, u) \mid t, u \in \mathbb{C}^*\} \subseteq (\mathbb{C}^*)^4$. In Examples 14.3.7 and 14.4.11, we drew the secondary fan when $r = 1$ and showed that for generic characters, the GIT quotient was either $\mathscr{H}_1$ or $\mathbb{P}^2$. Draw the secondary fan for general $r \geq 2$ and determine the generic GIT quotients. (One will be $\mathscr{H}_r$, while the other will be a weighted projective plane.) You should also draw the generalized fan for each cone in the secondary fan.

14.4.8. The projective bundle $X_\Sigma = \mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1))$ is a 3-dimensional smooth projective toric variety. The description given in Example 7.3.5 shows that Σ has five minimal generators. Since $\mathrm{Pic}(X_\Sigma) \simeq \mathbb{Z}^2$, the quotient construction of X_Σ uses a subgroup $G \subseteq (\mathbb{C}^*)^5$ with $G \simeq (\mathbb{C}^*)^2$. In this situation, the ν_i are the minimal generators u_ρ.

(a) Determine the β_i. Hint: In terms of the matrices in the exact sequence (14.2.10), you know the matrix B. Use this to find A. Three of the β_i are equal.

(b) Use Lemma 14.2.1 to give an explicit description of the group G.

(c) Determine the secondary fan and compute all possible GIT quotients $\mathbb{C}^5/\!/_\chi G$ as we vary χ. Hint: There are five distinct quotients.

(d) What happens when $X_\Sigma = \mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(a))$ for $a \geq 2$? Do you see the similarity to Exercise 14.4.7?

(e) Generalize part (d) by computing the secondary fan coming from the quotient construction of $X_\Sigma = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a))$. Hint: Treat the cases $a = 0$ and $a \geq 1$ separately.

Chapter 15

Geometry of the Secondary Fan

This chapter will continue our study of the GIT quotients $\mathbb{C}^r /\!/_\chi G$ as $\chi \in \widehat{G}$ varies. Recall the key players introduced in Chapter 14:

- $G \subseteq (\mathbb{C}^*)^r$ gives $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_r)$, $\beta_i \in \widehat{G}_{\mathbb{R}}$ and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_r)$, $\nu_i \in N_{\mathbb{R}}$. We also have the cones $C_{\boldsymbol{\beta}} = \mathrm{Cone}(\boldsymbol{\beta}) \subseteq \widehat{G}_{\mathbb{R}}$ and $C_{\boldsymbol{\nu}} = \mathrm{Cone}(\boldsymbol{\nu}) \subseteq N_{\mathbb{R}}$.
- The secondary fan Σ_{GKZ} consists of GKZ cones $\Gamma_{\Sigma, I_\emptyset}$ and has support $|\Sigma_{\mathrm{GKZ}}| = C_{\boldsymbol{\beta}}$. A chamber is a full dimensional GKZ cone.
- If $\chi \otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma, I_\emptyset})$, then $\mathbb{C}^r /\!/_\chi G \simeq X_\Sigma$, where X_Σ is semiprojective and Σ is a generalized fan with $|\Sigma| = C_{\boldsymbol{\nu}}$. Furthermore, the cones of Σ are generated by the ν_i for $i \notin I_\emptyset$, as described in Proposition 14.4.1.

The goal of this chapter is to understand the structure of the GKZ cones and what happens to the associated toric varieties as we move around the secondary fan. We will discuss a variety of topics, including nef cones, moving cones, Gale duality, triangulations, wall crossings, and the toric minimal model program.

§15.1. The Nef and Moving Cones

Before we can study GKZ cones, we first need to generalize the results of §6.3 and §6.4 concerning the nef and Mori cones of toric varieties.

The Nef Cone of a Toric Variety. Given a generalized fan Σ in $N_{\mathbb{R}}$, the vector space of support functions $\mathrm{SF}(\Sigma)$ contains the lattice

$$\mathrm{SF}(\Sigma, N) = \{\varphi \in \mathrm{SF}(\Sigma) \mid \varphi(|\Sigma| \cap N) \subseteq \mathbb{Z}\}.$$

When $|\Sigma|$ is convex, we also have $\mathrm{CSF}(\Sigma) = \{\varphi \in \mathrm{SF}(\Sigma) \mid \varphi \text{ is convex}\}$.

In §6.3 we studied the nef cone $\mathrm{Nef}(X_\Sigma)$ for a fan Σ with full dimensional convex support. Our results apply to generalized fans as follows.

Theorem 15.1.1. *Let Σ be a generalized fan with full dimensional convex support in $N_\mathbb{R}$. Then:*

(a) *There is an exact sequence*

$$0 \longrightarrow M \longrightarrow \mathrm{SF}(\Sigma, N) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0.$$

(b) *A Cartier divisor on X_Σ is basepoint free if and only if it is nef.*

(c) *The nef cone* $\mathrm{Nef}(X_\Sigma)$ *is contained in* $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$*, and its inverse image in* $\mathrm{SF}(\Sigma)$ *is* $\mathrm{CSF}(\Sigma)$.

(d) *If X_Σ is semiprojective, then* $\mathrm{Nef}(X_\Sigma) \subseteq \mathrm{Pic}(X_\Sigma)_\mathbb{R}$ *is full dimensional.*

Proof. As explained in §6.2, the generalized fan Σ in $N_\mathbb{R}$ gives a genuine fan $\overline{\Sigma}$ in $\overline{N}_\mathbb{R}$, where $\overline{N} = N/(\sigma_0 \cap N)$ and σ_0 is the minimal cone of Σ. Let $\overline{M} \subseteq M$ be the dual of $N \to \overline{N}$. Since $X_\Sigma = X_{\overline{\Sigma}}$, the correspondence between Cartier divisors and support functions (Theorem 4.2.12) gives an exact sequence

$$0 \longrightarrow \overline{M} \longrightarrow \mathrm{SF}(\overline{\Sigma}, \overline{N}) \longrightarrow \mathrm{Pic}(X_\Sigma) \longrightarrow 0.$$

To relate this to the generalized fan Σ, note that composition with $N_\mathbb{R} \to \overline{N}_\mathbb{R}$ induces an injection $\mathrm{SF}(\overline{\Sigma}, \overline{N}) \hookrightarrow \mathrm{SF}(\Sigma, N)$. This gives the commutative diagram (ignore the dotted arrow for now):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathrm{SF}(\Sigma, N) & \dashrightarrow & \mathrm{Pic}(X_\Sigma) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \overline{M} & \longrightarrow & \mathrm{SF}(\overline{\Sigma}, \overline{N}) & \longrightarrow & \mathrm{Pic}(X_\Sigma) & \longrightarrow & 0. \end{array} \tag{15.1.1}$$

However, the vertical injections have isomorphic cokernels, because:

- A element of $\mathrm{SF}(\Sigma, N)$ comes from $\mathrm{SF}(\overline{\Sigma}, \overline{N})$ if and only if it vanishes on the kernel of $N_\mathbb{R} \to \overline{N}_\mathbb{R}$.
- Every element of $\mathrm{SF}(\Sigma, N)$ is linear on the kernel of $N_\mathbb{R} \to \overline{N}_\mathbb{R}$ since the kernel is the minimal cone of Σ.

It follows that the dotted arrow exists such that the above diagram commutes and has exact rows. This proves part (a).

Part (b) follows from Theorem 6.3.12. For part (c), (6.3.2) implies

$$\mathrm{Nef}(X_\Sigma) \subseteq N^1(X_\Sigma) = \mathrm{Pic}(X_\Sigma)_\mathbb{R}.$$

In Exercise 15.1.1 you will show that the convexity criteria of Theorem 7.2.2 apply to generalized fans. This plus part (b) complete the proof of part (c).

If X_Σ is semiprojective and Σ is a fan, then Theorem 7.2.4 and Proposition 7.2.9 imply that $\mathrm{CSF}(\Sigma)$ contains a strictly convex support function. Strict convexity is an open condition, so that $\mathrm{CSF}(\Sigma)$ is full dimensional in $\mathrm{SF}(\Sigma)$. Then part (c) implies that $\mathrm{Nef}(X_\Sigma)$ is full dimensional $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$, as claimed in part (d).

To complete the proof, we need to consider the case of a generalized fan. Here, the proof follows easily from (15.1.1) and the previous paragraph. □

The Toric Cone Theorem. We can also describe the dual of $\mathrm{Nef}(X_\Sigma)$ in terms of walls when Σ is a generalized fan with full dimensional convex support. A wall τ of such a fan is a facet of full dimensional cones $\sigma \neq \sigma' \in \Sigma$. In Exercise 15.1.2 you will show that the orbit closure $V(\tau)$ is a complete torus-invariant curve in X_Σ and that all complete torus-invariant curves arise this way.

The toric cone theorem stated in Theorem 6.3.20 generalizes as follows.

Theorem 15.1.2. *If Σ is a generalized fan with full dimensional convex support, then we have*

$$\overline{\mathrm{NE}}(X_\Sigma) = \mathrm{NE}(X_\Sigma) = \sum_{\tau \text{ a wall}} \mathbb{R}_{\geq 0}[V(\tau)].$$

Furthermore, $\overline{\mathrm{NE}}(X_\Sigma)$ is strongly convex when X_Σ is semiprojective.

Proof. Let $\overline{\Sigma}$ be the genuine fan determined by Σ. Since walls of Σ correspond bijectively to walls of $\overline{\Sigma}$, Theorem 6.3.20 gives the desired formula for $\overline{\mathrm{NE}}(X_\Sigma)$. If X_Σ is semiprojective, then $\overline{\mathrm{NE}}(X_\Sigma)$ is strongly convex since its dual $\mathrm{Nef}(X_\Sigma)$ is full dimensional by Theorem 15.1.1. □

When X_Σ is semiprojective, $\overline{\mathrm{NE}}(X_\Sigma)$ is minimally generated by its edges, called *extremal rays*. If $\mathcal{R}$ is the extremal ray generated by the class of a curve C, then the corresponding facet of $\mathrm{Nef}(X_\Sigma)$ is defined by $D \cdot C = 0$ for $[D] \in \mathrm{Nef}(X_\Sigma)$.

Also, in the situation of Theorem 15.1.2, we have a proper map $\phi : X_\Sigma \to U_\Sigma$, where U_Σ is the affine toric variety associated to $|\Sigma|$ as in (7.2.1). Then

$$N_1(X_\Sigma) = N_1(X_\Sigma/U_\Sigma).$$

Here, $N_1(X_\Sigma/U_\Sigma)$ is generated by irreducible curves lying in fibers of ϕ, modulo numerical equivalence, which equals $N_1(X_\Sigma)$ since ϕ is proper and U_Σ is affine. Hence we are in what is often called the *relative case*.

The Simplicial Case. When Σ is simplicial with full dimensional convex support, we have an exact sequence

$$0 \longrightarrow M_\mathbb{R} \longrightarrow \mathbb{R}^{\Sigma(1)} \longrightarrow \mathrm{Pic}(X_\Sigma)_\mathbb{R} \longrightarrow 0, \tag{15.1.2}$$

and since $N_1(X_\Sigma)$ is dual to $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$ under intersection product, taking duals gives the exact sequence

$$0 \longrightarrow N_1(X_\Sigma) \longrightarrow \mathbb{R}^{\Sigma(1)} \longrightarrow N_\mathbb{R} \longrightarrow 0. \tag{15.1.3}$$

Thus we can interpret classes in $N_1(X_\Sigma)$ as relations among the u_ρ's. In particular, the class of an irreducible complete curve $C \subseteq X_\Sigma$ is represented by the relation

$$\sum_\rho (D_\rho \cdot C) u_\rho = 0. \tag{15.1.4}$$

This follows from Proposition 6.4.1 and (6.4.2).

Nef Cones and the Secondary Fan. Now consider the secondary fan determined by $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$ as at the beginning of the chapter. A GKZ cone $\Gamma_{\Sigma,I_\emptyset}$ gives the toric variety

$$\mathbb{C}^r /\!/_\chi G \simeq X_\Sigma$$

for all $\chi \otimes 1 \in \operatorname{Relint}(\Gamma_{\Sigma,I_\emptyset})$. The nef cone $\operatorname{Nef}(X_\Sigma)$ relates to $\Gamma_{\Sigma,I_\emptyset}$ as follows.

Proposition 15.1.3. *For any GKZ cone $\Gamma_{\Sigma,I_\emptyset}$, we have:*

(a) $\Gamma_{\Sigma,I_\emptyset} \simeq \operatorname{Nef}(X_\Sigma) \times \mathbb{R}^{I_\emptyset}_{\geq 0}$.

(b) $\dim \Gamma_{\Sigma,I_\emptyset} = \dim \operatorname{Pic}(X_\Sigma)_\mathbb{R} + |I_\emptyset|$.

Proof. Since GIT quotients are semiprojective, part (b) follows from part (a) by Theorem 15.1.1. For part (a), recall from Definition 14.4.2 that $\mathbf{a} \in \Gamma_{\Sigma,I_\emptyset}$ has convex support function $\varphi_\mathbf{a} \in \operatorname{CSF}(\Sigma)$. Then we have an isomorphism of cones

$$\widetilde{\Gamma}_{\Sigma,I_\emptyset} \simeq \operatorname{CSF}(\Sigma) \oplus \mathbb{R}^{I_\emptyset}_{\geq 0},$$

where $\mathbf{a} \in \Gamma_{\Sigma,I_\emptyset}$ maps to $(\varphi_\mathbf{a}, (\varphi_\mathbf{a}(\nu_i) + a_i)_{i \in I_\emptyset})$. Since $\widetilde{\Gamma}_{\Sigma,I_\emptyset}/M_\mathbb{R} \simeq \Gamma_{\Sigma,I_\emptyset}$ and $\operatorname{CSF}(\Sigma)/M_\mathbb{R} \simeq \operatorname{Nef}(X_\Sigma)$ by Theorem 15.1.1, part (a) follows immediately. □

The Moving Cone of the Secondary Fan. The nicest case of Proposition 15.1.3 is when $I_\emptyset = \emptyset$, for here the GKZ cone is $\Gamma_{\Sigma,\emptyset} \simeq \operatorname{Nef}(X_\Sigma)$. There may be many GKZ cones with $I_\emptyset = \emptyset$. Let

$$\operatorname{Mov}_{\text{GKZ}} = \bigcup \Gamma_{\Sigma,\emptyset} \tag{15.1.5}$$

be the union of all GKZ cones with $I_\emptyset = \emptyset$. This union has a nice structure.

Proposition 15.1.4. $\operatorname{Mov}_{\text{GKZ}}$ *is a convex polyhedral cone in $\widehat{G}_\mathbb{R}$.*

Proof. We need only prove convexity. Take $\beta_1 \in \Gamma_{\Sigma_1,\emptyset}$ and $\beta_2 \in \Gamma_{\Sigma_2,\emptyset}$ and consider

$$\beta = t\beta_1 + (1-t)\beta_2, \quad t \in [0,1].$$

Write $\beta_j = \gamma_\mathbb{R}(\mathbf{a}_j)$ for $\mathbf{a}_j \in \mathbb{R}^r$ and set $\mathbf{a} = t\mathbf{a}_1 + (1-t)\mathbf{a}_2$. By hypothesis, the virtual facets $F_{i,\mathbf{a}_1} \subseteq P_{\mathbf{a}_1}$ and $F_{i,\mathbf{a}_2} \subseteq P_{\mathbf{a}_2}$ are nonempty. An easy calculation shows that

$$tF_{i,\mathbf{a}_1} + (1-t)F_{i,\mathbf{a}_2} \subseteq F_{i,\mathbf{a}},$$

which implies that the virtual facets of $P_\mathbf{a}$ are also nonempty. If $\beta \in \operatorname{Relint}(\Gamma_{\Sigma,I_\emptyset})$, then $I_\emptyset$ is the set of indices of empty virtual facets of $P_\mathbf{a}$ by Lemma 14.4.4. Hence $I_\emptyset = \emptyset$, so that $\beta \in \operatorname{Mov}_{\text{GKZ}}$. □

We call $\mathrm{Mov}_{\mathrm{GKZ}}$ the *moving cone* of the secondary fan. The name "moving cone" will be explained later in the section.

Here is an example where the moving cone is small.

Example 15.1.5. Let $G = \{(t,u,tu) \mid t,u \in \mathbb{C}^*\}$. One easily computes that $\widehat{G}_{\mathbb{R}} \simeq \mathbb{R}^2$ with basis β_1, β_2, and $\beta_3 = \beta_1 + \beta_2$. Furthermore $\nu_1 = \nu_2 = e_1$ and $\nu_3 = -e_1$ in N. The secondary fan is illustrated in Figure 1. Virtual facets are indicated by dotted lines. Hence the moving cone is the diagonal ray generated by β_3. $\Diamond$

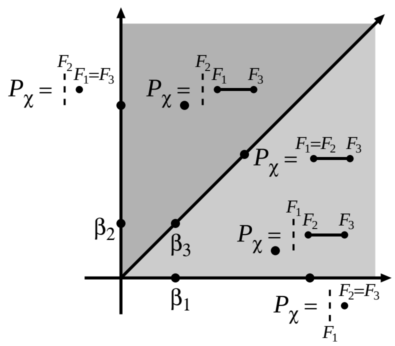


Figure 1. A secondary fan with small moving cone

The Geometric Case. Our next result characterizes when the secondary fan has a chamber satisfying $I_\emptyset = \emptyset$. We say that $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_r)$ is *geometric* if every ν_i is nonzero and the ν_i generate distinct rays in $N_{\mathbb{R}}$.

Proposition 15.1.6. *The secondary fan has a chamber with $I_\emptyset = \emptyset$ if and only if $\boldsymbol{\nu}$ is geometric.*

Proof. Suppose we have a chamber of the form $\Gamma_{\Sigma,\emptyset}$ and pick $\chi \otimes 1 \in \mathrm{Int}(\Gamma_{\Sigma,\emptyset})$. Then χ is generic, so that by Proposition 14.4.9, $i \mapsto \mathrm{Cone}(\nu_i)$ gives a bijection $\{1,\ldots,r\} \simeq \Sigma(1)$ since $I_\emptyset = \emptyset$. It follows immediately that $\boldsymbol{\nu}$ is geometric.

Conversely, assume that $\boldsymbol{\nu}$ is geometric and for each $1 \le i \le r$, let u_i be the minimal generator of $\mathrm{Cone}(\nu_i) \cap N$. By hypothesis, this gives r primitive elements of N. Now take any chamber $\Gamma_{\Sigma', I'_\emptyset}$. By Proposition 14.4.9, we have a bijection $\{1,\ldots,r\} \setminus I'_\emptyset \simeq \Sigma'(1)$ defined by $i \mapsto \mathrm{Cone}(\nu_i)$. If $I'_\emptyset = \emptyset$, then we are done. If not, let Σ be the refinement of Σ' obtained by successive star subdivisions (as defined in §11.1) at u_i for $i \in I'_\emptyset$. The fan Σ will depend on the order in which we perform the star subdivisions, but all such Σ have the following properties:

- Σ is simplicial since Σ' is simplicial and the star subdivision of a simplicial fan is simplicial.
- $i \mapsto \mathrm{Cone}(\nu_i)$ is a bijection $\{1,\ldots,r\} \simeq \Sigma(1)$ since the star subdivisions add the rays $\mathrm{Cone}(\nu_i)$, $i \in I'_\emptyset$, to the original fan Σ'. This follows since $\boldsymbol{\nu}$ is geometric.

- X_Σ is semiprojective since $X_{\Sigma'}$ is semiprojective and $X_\Sigma \to X_{\Sigma'}$ is projective, being a composition of projective morphisms.

It follows that Σ and $I_\emptyset = \emptyset$ satisfy the conditions of Proposition 14.4.1 and hence give the GKZ cone $\Gamma_{\Sigma,\emptyset}$. Furthermore, the first two bullets and Proposition 14.4.9 imply that this cone is a chamber. $\square$

Besides being geometric, a stronger condition is when $\boldsymbol{\nu}$ is *primitive geometric*, which means that $\boldsymbol{\nu}$ is geometric and each $\nu_i \in N$ is primitive. Here is an example.

Example 15.1.7. Let X_Σ be a projective toric variety. The quotient construction $X_\Sigma \simeq (\mathbb{C}^{\Sigma(1)} \setminus Z(\Sigma))/\!\!/G$ from §5.1 is a GIT quotient by Example 14.2.14. As noted in Example 14.3.4, β_i's come from the divisor classes $[D_\rho]$ and the ν_i's are the minimal generators u_ρ. Thus $\boldsymbol{\nu}$ is primitive geometric. $\Diamond$

The Pseudoeffective and Moving Cones of a Normal Variety. Let X be a normal variety. In addition to the nef cone $\mathrm{Nef}(X)$, other interesting cones live in $N^1(X)$, in particular, the *pseudoeffective cone* $\overline{\mathrm{Eff}}(X)$ and the *moving cone* $\overline{\mathrm{Mov}}(X)$. These cones are related to the nef cone by the inclusions

$$\mathrm{Nef}(X) \subseteq \overline{\mathrm{Mov}}(X) \subseteq \overline{\mathrm{Eff}}(X) \subseteq N^1(X).$$

The pseudoeffective cone $\overline{\mathrm{Eff}}(X)$ is easy to define: it is the closure of the cone generated by effective $\mathbb{R}$-Cartier divisor classes. Here is one case where the pseudoeffective cone is easy to describe.

Lemma 15.1.8. *If X_Σ is simplicial and semiprojective, then*

$$\overline{\mathrm{Eff}}(X_\Sigma) = \mathrm{Eff}(X_\Sigma) = \mathrm{Cone}([D_\rho] \mid \rho \in \Sigma(1)).$$

Proof. We know that $N^1(X_\Sigma) = \mathrm{Pic}(X_\Sigma)_\mathbb{R}$ since Σ has full dimensional convex support, and every D_ρ is $\mathbb{Q}$-Cartier since X_Σ is simplicial. The classes $[D_\rho]$ are clearly effective. Conversely, if D is effective, then $H^0(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)) \neq 0$. We may assume that D is torus-invariant, and then Proposition 4.3.2 implies that there is $m \in M$ with $D + \mathrm{div}(\chi^m) \geq 0$. Then $[D] = [D + \mathrm{div}(\chi^m)] \in \mathrm{Cone}([D_\rho])$. $\square$

The moving cone $\overline{\mathrm{Mov}}(X)$ was introduced in [**171**]. An effective Weil divisor $D \geq 0$ on X has a *fixed component* if there is an effective divisor $D_0 \neq 0$ such that every effective divisor $D' \sim D$ satisfies $D' \geq D_0$. This implies that $D = E + D_0$, $E \geq 0$, where every effective divisor $D' \sim D$ of the form $D' = E' + D_0$, $E' \geq 0$. In other words, as we vary $D = E + D_0$ in its linear equivalence class, D_0 is fixed.

We can formulate this in terms of sheaves as follows. If $D \geq D_0 \geq 0$ and $D_0 \neq 0$, then D_0 is a fixed component of D if and only if the natural inclusion of sheaves $\mathscr{O}_X(D - D_0) \to \mathscr{O}_X(D)$ induces an isomorphism

$$H^0(X, \mathscr{O}_X(D - D_0)) \simeq H^0(X, \mathscr{O}_X(D)) \tag{15.1.6}$$

(Exercise 15.1.3). Here is an example of a divisor with a fixed component.

Example 15.1.9. For the Hirzebruch surface $\mathscr{H}_2$, we have the usual divisors D_1, D_2, D_3, D_4 such that the classes of D_3, D_4 generate the nef cone. Let $D = D_4$ and $D' = D_2 + D_4$. Figure 4 from Example 6.1.2 shows that the divisors D and D' have the same polytopes and hence their sheaves have the same global sections by Proposition 4.3.3. Since $D' - D_2 = D$, the discussion leading up to (15.1.6) implies that D_2 is a fixed component of D'. $\Diamond$

A Weil divisor D on X is *movable* if it doesn't have a fixed component, and then the *moving cone* of X is defined by

$$\overline{\mathrm{Mov}}(X) = \overline{\mathrm{Cone}([D] \mid D \text{ is Cartier and movable})} \subseteq N^1(X). \tag{15.1.7}$$

The Moving Cone in the Primitive Geometric Case. We now explain how the moving cone $\mathrm{Mov}_{\mathrm{GKZ}}$ defined in (15.1.5) relates to the moving cone (15.1.7).

Theorem 15.1.10. *If $\boldsymbol{\nu}$ is primitive geometric, then the moving cone $\mathrm{Mov}_{\mathrm{GKZ}}$ is the union of the GKZ chambers with $I_\emptyset = \emptyset$. Furthermore, if $\Gamma_{\Sigma,\emptyset}$ is a chamber, then:*

(a) *X_Σ is simplicial and semiprojective, with $\boldsymbol{\nu}$ as minimal generators of $\Sigma(1)$.*

(b) *If $\chi \otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma,\emptyset})$, then the GIT quotient*

$$\mathbb{C}^r /\!/_\chi G = \big(\mathbb{C}^r \setminus Z(\chi)\big) /\!/ G \simeq X_\Sigma$$

is precisely the quotient construction described in Theorem 5.1.11.

(c) *There is a natural isomorphism $\mathrm{Pic}(X_\Sigma)_\mathbb{R} \simeq \widehat{G}_\mathbb{R}$ that takes the cones*

$$\mathrm{Nef}(X_\Sigma) \subseteq \overline{\mathrm{Mov}}(X_\Sigma) \subseteq \overline{\mathrm{Eff}}(X_\Sigma)$$

to the cones

$$\Gamma_{\Sigma,\emptyset} \;\subseteq \mathrm{Mov}_{\mathrm{GKZ}} \;\subseteq\; C_{\boldsymbol{\beta}}.$$

Proof. $\mathrm{Mov}_{\mathrm{GKZ}}$ is the union of chambers with $I_\emptyset = \emptyset$ by Propositions 15.1.4 and 15.1.6. Now let $\Gamma_{\Sigma,\emptyset}$ be a chamber. By Proposition 14.4.9, Σ is simplicial and $i \mapsto \rho_i = \mathrm{Cone}(\nu_i)$ gives a bijection $\{1,\ldots,r\} \simeq \Sigma(1)$. Thus $\Sigma(1) = \{\rho_1,\ldots,\rho_r\}$, and the corresponding divisors will be denoted $D_1,\ldots,D_r$. This proves part (a).

Since $\boldsymbol{\nu}$ is primitive, the map $\delta : M \to \mathbb{Z}^r$ from (14.2.2) is the same as the corresponding map in the exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} = \mathbb{Z}^r \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0$$

from (5.1.1) in Chapter 5. This gives a natural isomorphism $\mathrm{Cl}(X_\Sigma) \simeq \widehat{G}$, which shows that G is the group from the quotient construction in §5.1. Furthermore, if $B(\Sigma)$ is the irrelevant ideal from §5.1, then $B(\chi) = B(\Sigma,\emptyset) = B(\Sigma)$, where the first equality follows from Proposition 14.4.14 and the second from (14.4.12). From here, is it straightforward to complete the proof of part (b).

For part (c), the isomorphism $\mathrm{Cl}(X_\Sigma) \simeq \widehat{G}$ induces $\mathrm{Pic}(X_\Sigma)_\mathbb{R} \simeq \widehat{G}_\mathbb{R}$ since X_Σ is simplicial. This takes $\mathrm{Nef}(X_\Sigma)$ to $\Gamma_{\Sigma,\emptyset}$ by Proposition 15.1.3. Since $[D_i] \mapsto \beta_i$, this isomorphism takes $\overline{\mathrm{Eff}}(X_\Sigma)$ to $C_{\boldsymbol{\beta}}$ by Lemma 15.1.8.

Let $\mathrm{Mov}(X_\Sigma) \subseteq \mathrm{Pic}(X_\Sigma)_{\mathbb{R}}$ be the cone generated by classes of movable divisors (remember that every Weil divisor is $\mathbb{Q}$-Cartier since X_Σ is simplicial). We prove $\mathrm{Nef}(X_\Sigma) \subseteq \mathrm{Mov}(X_\Sigma)$ as follows. A nef Cartier divisor $D = \sum_{i=1}^r a_i D_i$ has no basepoints by Theorem 15.1.1, so that its Cartier data $\{m_\sigma\}_{\sigma\in\Sigma_{\max}}$ lies in the polyhedron P_D by Theorem 7.2.2. This implies that the coefficient of D_i in $D + \mathrm{div}(\chi^{m_\sigma}) \geq 0$ vanishes when $\rho_i \in \sigma(1)$, hence D is movable.

We also note that X_Σ contains the toric variety $X_{\Sigma(1)}$ whose fan consists of the rays in $\Sigma(1)$, together with the trivial cone $\{0\}$. Since $X_{\Sigma(1)} \subseteq X_\Sigma$ is an open subset whose complement has codimension ≥ 2, X_Σ and $X_{\Sigma(1)}$ have the same Weil divisors via the restriction $D \mapsto D \cap X_{\Sigma(1)}$. They also have the same principal divisors and same class group. It follows easily that

(15.1.8) $\qquad X_\Sigma$ and $X_{\Sigma(1)}$ have the same movable divisors.

For the remainder of the proof, fix a chamber $\Gamma_{\Sigma,\emptyset}$ and identify $\mathrm{Pic}(X_\Sigma)_{\mathbb{R}}$ with $\widehat{G}_{\mathbb{R}}$. If $\Gamma_{\Sigma',\emptyset}$ is another chamber, then $\Sigma'(1) = \{\rho_1,\ldots,\rho_r\} = \Sigma(1)$, so that

$$X_\Sigma \supseteq X_{\Sigma(1)} = X_{\Sigma'(1)} \subseteq X_{\Sigma'}. \tag{15.1.9}$$

When we identify $\mathrm{Pic}(X_\Sigma)_{\mathbb{R}}$ and $\mathrm{Pic}(X_{\Sigma'})_{\mathbb{R}}$ with $\widehat{G}_{\mathbb{R}}$, these inclusions and (15.1.8) imply that X_Σ and $X_{\Sigma'}$ have the same movable divisors, i.e., $\mathrm{Mov}(X_\Sigma) = \mathrm{Mov}(X_{\Sigma'})$. Since $\Gamma_{\Sigma',\emptyset} = \mathrm{Nef}(X_{\Sigma'}) \subseteq \mathrm{Mov}(X_{\Sigma'})$, it follows $\Gamma_{\Sigma',\emptyset} \subseteq \mathrm{Mov}(X_\Sigma)$. This proves that $\mathrm{Mov}_{\mathrm{GKZ}} \subseteq \mathrm{Mov}(X_\Sigma)$ since $\mathrm{Mov}_{\mathrm{GKZ}}$ is the union of the chambers $\Gamma_{\Sigma',\emptyset}$.

For the opposite inclusion, take an effective Cartier divisor $D = \sum_{i=1}^r a_i D_i$ and assume that $[D] \notin \mathrm{Mov}_{\mathrm{GKZ}}$. Then $[D] \in \mathrm{Relint}(\Gamma_{\Sigma',I'_\emptyset})$ with $I'_\emptyset \neq \emptyset$. We claim that $D_0 = \sum_{i\in I'_\emptyset} D_i$ is a fixed component of D, which will imply $[D] \notin \mathrm{Mov}(X_\Sigma)$.

By (15.1.6) and Proposition 4.3.3, the claim will follow once we prove that $P_{D-D_0} \cap M = P_D \cap M$. The inclusion $P_{D-D_0} \cap M \subseteq P_D \cap M$ is trivial since $D_0 \geq 0$. For the other direction, take any $m \in P_D \cap M$. In the notation of §14.2, $P_D = P_{\mathbf{a}}$ for $\mathbf{a} = (a_1,\ldots,a_r)$, so that the inequality $\langle m, \nu_i\rangle \geq -a_i$ must be strict for $i \in I'_\emptyset$ since $I'_\emptyset$ consists of the indices of empty virtual facets. Since $\langle m,\nu_i\rangle$ and a_i are integers, it follows that $\langle m,\nu_i\rangle \geq -(a_i - 1)$, hence $m \in P_{D-D_0} \cap M$. This proves our claim.

We have now shown that $\mathrm{Mov}_{\mathrm{GKZ}} = \mathrm{Mov}(X_\Sigma)$. Since $\mathrm{Mov}_{\mathrm{GKZ}}$ is closed, the same is true for $\mathrm{Mov}(X_\Sigma)$. Thus $\mathrm{Mov}_{\mathrm{GKZ}} = \overline{\mathrm{Mov}}(X_\Sigma)$ and part (c) is proved. □

The proof of Theorem 15.1.10 shows that $\overline{\mathrm{Mov}}(X_\Sigma) = \mathrm{Mov}(X_\Sigma)$, and we saw that $\overline{\mathrm{Eff}}(X_\Sigma) = \mathrm{Eff}(X_\Sigma)$ in Lemma 15.1.8. Furthemore, these cones are polyhedral, as is $\mathrm{Nef}(X_\Sigma)$. This is typical of how life is simpler in the toric case. In general, these cones can be much more complicated.

For a simplicial semiprojective toric variety X_Σ, Theorem 15.1.10 gives a vivid description of the moving cone of X_Σ. Namely,

$$\mathrm{Mov}(X_\Sigma) = \bigcup \mathrm{Nef}(X_{\Sigma'}),$$

where the union is over all simplicial fans Σ' in $N_\mathbb{R}$ such that $X_{\Sigma'}$ is semiprojective and $\Sigma'(1) = \Sigma(1)$. Notice that $X_{\Sigma'}$ is isomorphic to X_Σ in codimension 1 by (15.1.9). In §15.3 we will give a careful description of the birational isomorphism $X_{\Sigma'} \dashrightarrow X_\Sigma$.

Here is an easy example of the moving cone.

Example 15.1.11. For the Hirzebruch surface $\mathscr{H}_2$ considered in Example 15.1.9, we have the secondary fan pictured in Figure 2, where $\beta_i = [D_i]$. We showed in Example 15.1.9 that $D_2 + D_4$ has a fixed component, so that the class $[D_2 + D_4]$ is not in the moving cone. This is also clear since $I_\emptyset \neq \emptyset$ in the left chamber. Thus the moving cone is the right chamber, which is $\mathrm{Nef}(\mathscr{H}_2)$. $\Diamond$

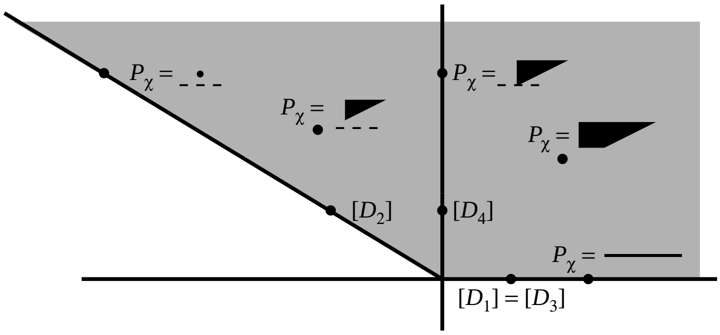


Figure 2. The secondary fan of $\mathscr{H}_2$

In §15.2 we will give a more substantial example of a moving cone when we study an alternate description of the moving cone and explore the relation between the secondary fan and triangulations.

In the literature, the nef and moving cones are often defined in the relative case of a projective morphism $f : X \to S$ (see [**171**]). Here, numerical equivalence is defined using complete curves in X that map to a point under f, and $N^1(X/S)$ is defined using this notion of numerical equivalence. When the base S is affine, every complete curve in X maps to a point, so that the definitions of numerical equivalence and $N^1(X)$ given in §6.2 agree with the relative versions. This applies in particular to a semiprojective toric variety, since such a variety is projective over an affine toric variety. See also [**71**, Sec. 1.4].

Exercises for §15.1.

15.1.1. Prove that the convexity criteria of Theorem 7.2.2 apply to generalized fans with full dimensional convex support.

15.1.2. Let X_Σ be the toric variety of a generalized fan Σ in $N_\mathbb{R}$.

(a) Prove that there is a bijection $\sigma \mapsto O(\sigma)$ between cones of Σ and T_N-orbits in X_Σ such that $\dim O(\sigma) = \operatorname{codim} \sigma$.

(b) Prove that if τ is a wall, then the orbit closure $V(\tau)$ is isomorphic to $\mathbb{P}^1$.

15.1.3. Let D and D_0 be effective Weil divisors on a normal variety X.

(a) Construct a natural inclusion of sheaves $\mathscr{O}_X(D-D_0) \hookrightarrow \mathscr{O}_X(D)$.

(b) Prove that D_0 is a fixed component of D if and only if the map of part (a) induces an isomorphism of global sections as in (15.1.6).

15.1.4. Consider the exact sequence $0 \to \mathbb{Z}^2 \xrightarrow{B} \mathbb{Z}^4 \xrightarrow{A} \mathbb{Z}^2 \to 0$, where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -1 \end{pmatrix}.$$

As in §14.2, the ν_i's are the rows of B and the β_i's are the columns of A.

(a) Determine the group $G \subseteq (\mathbb{C}^*)^4$.

(b) Compute the secondary fan and, for each chamber, draw the corresponding fan that gives the GIT quotient in that chamber.

(c) What is the moving cone in this case?

15.1.5. Suppose that $\boldsymbol{\nu} = (\nu_1,\dots,\nu_r)$ consists of vectors in $\mathbb{Z}^2$. Also assume that $\boldsymbol{\nu}$ is geometric and $C_{\boldsymbol{\nu}} \subseteq \mathbb{R}^2$ is strongly convex with $C_{\boldsymbol{\nu}} = \mathrm{Cone}(\nu_1,\nu_r)$.

(a) Use Proposition 14.4.9 to prove that Σ_{GKZ} has 2^{r-2} chambers. Hint: First show that if $\Gamma_{\Sigma,I_\emptyset}$ is a chamber, then $I_\emptyset \subseteq \{2,\dots,r-1\}$ and that Σ is uniquely determined by $I_\emptyset$.

(b) Explain why $\mathrm{Mov}_{\mathrm{GKZ}}$ consists of a single chamber.

15.1.6. As in Example 15.1.7, a projective toric variety X_Σ gives a GIT quotient where $\boldsymbol{\nu}$ consists of the u_ρ, $\rho \in \Sigma(1)$. The GKZ cone $\Gamma_{\Sigma,\emptyset}$ equals $\mathrm{Nef}(X_\Sigma)$ by Proposition 15.1.3. Let D be a torus-invariant nef divisor on X_Σ and set $\chi = [D] \in \mathrm{Pic}(X_\Sigma)$. By Theorem 14.4.7, $\chi \otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma',\emptyset})$, where Σ refines Σ'. Prove that $\mathbb{C}^r /\!/_\chi G \simeq X_{\Sigma'}$ is the toric variety X_{P_D} in Example 14.2.11 and Theorem 6.2.8. See also Example 14.2.14.

15.1.7. Let X_Σ be the toric variety of a generalized fan Σ in $N_\mathbb{R}$. Parallel to the definition of semiprojective, we say that X_Σ is *semicomplete* if the map $X_\Sigma \to \mathrm{Spec}(H^0(X_\Sigma,\mathscr{O}_{X_\Sigma}))$ is proper and X_Σ has a torus fixed point. Prove that X_Σ is semicomplete if and only if Σ has full dimensional convex support. Hint: See the proof of Proposition 7.2.9.

15.1.8. Assume that $\boldsymbol{\nu}$ is geometric. Prove that there is a bijective correspondence between chambers of secondary fan and simplicial fans Σ such that $\Sigma(1) \subseteq \{\mathrm{Cone}(\nu_i) \mid 1 \le i \le r\}$, $|\Sigma| = C_{\boldsymbol{\nu}}$, and X_Σ is semiprojective.

§15.2. Gale Duality and Triangulations

In §15.1, we described various cones in $C_{\boldsymbol{\beta}}$ in terms of divisors on toric varieties. In this section we will instead focus on the combinatorial structure of these cones. We begin with the chambers of the secondary fan. Our goal is to show that they can be constructed in a purely combinational way that makes nice use of Gale duality.

A subset $J \subseteq \{1,\dots,r\}$ is a ***β**-basis* if $|J| = \dim G$ and the vectors β_i, $i \in J$, form a basis of $\widehat{G}_\mathbb{R}$. Then $\mathrm{Cone}(\boldsymbol{\beta}_J) = \mathrm{Cone}(\beta_i \mid i \in J)$ is a full dimensional simplicial cone in $\widehat{G}_\mathbb{R}$. These cones relate nicely to the chambers of Σ_{GKZ}.

To see how this works, fix a chamber $\Gamma_{\Sigma,I_\emptyset}$. The lattice points in its interior are all generic by Proposition 14.4.9. On the other hand, for a $\boldsymbol{\beta}$-basis J, the lattice points in the boundary of $\mathrm{Cone}(\boldsymbol{\beta}_J)$ are nongeneric by definition. It follows easily that for any $\boldsymbol{\beta}$-basis J, we have

$$\text{either } \Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\boldsymbol{\beta}_J) \ \text{ or } \ \mathrm{Int}(\Gamma_{\Sigma,I_\emptyset}) \cap \mathrm{Cone}(\boldsymbol{\beta}_J) = \emptyset. \tag{15.2.1}$$

The surprise is that the $\boldsymbol{\beta}$-bases J that satisfy $\Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\boldsymbol{\beta}_J)$ have a very nice structure and that the chamber $\Gamma_{\Sigma,I_\emptyset}$ is uniquely determined by these bases. Here is the precise statement.

Proposition 15.2.1. *Let $\Gamma_{\Sigma,I_\emptyset}$ be a chamber of the secondary fan. Then:*

(a) *If $\sigma \in \Sigma_{\max}$, then $J_\sigma = \{i \mid \nu_i \notin \sigma \text{ or } i \in I_\emptyset\}$ is a $\boldsymbol{\beta}$-basis.*

(b) *if J is a $\boldsymbol{\beta}$-basis, then $\Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\boldsymbol{\beta}_J)$ if and only if $J = J_\sigma$ for some $\sigma \in \Sigma_{\max}$.*

(c) $\Gamma_{\Sigma,I_\emptyset} = \bigcap_{\sigma\in\Sigma_{\max}} \mathrm{Cone}(\boldsymbol{\beta}_{J_\sigma})$.

Proof. Our proof is based on [**28**, Lem. 4.2]. Since $\Gamma_{\Sigma,I_\emptyset}$ is a chamber, we know that Σ is simplicial and that $i \mapsto \mathrm{Cone}(\nu_i)$ induces a bijection $\{1,\dots,r\}\setminus I_\emptyset \simeq \Sigma(1)$. Thus, given a maximal cone $\sigma \in \Sigma_{\max}$, the $\nu_i \in \sigma$ with $i \notin I_\emptyset$ form a basis of $N_\mathbb{R}$. By Gale duality (Lemma 14.3.1), the β_i with $\nu_i \notin \sigma$ or $i \in I_\emptyset$ form a basis of $\widehat{G}_\mathbb{R}$, i.e., $J_\sigma = \{i \mid \nu_i \notin \sigma \text{ or } i \in I_\emptyset\}$ is a $\boldsymbol{\beta}$-basis. This proves part (a).

For part (b), we first prove that if $\sigma \in \Sigma_{\max}$, then

$$\Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\boldsymbol{\beta}_{J_\sigma}).$$

Take $\chi\otimes 1 \in \mathrm{Int}(\Gamma_{\Sigma,I_\emptyset})$ and let $\mathbf{b} \in P_\chi$ be the vertex corresponding to σ. This easily implies that $\mathbf{b} = \sum_{i\in J_\sigma} \lambda_i e_i$ with $\lambda_i \geq 0$, so that $\chi\otimes 1 = \gamma_\mathbb{R}(\mathbf{b}) \in \mathrm{Cone}(\boldsymbol{\beta}_{J_\sigma})$. Then the desired inclusion follows from (15.2.1).

Next assume $\Gamma_{\Sigma,I_\emptyset} \subseteq \mathrm{Cone}(\boldsymbol{\beta}_J)$ for a $\boldsymbol{\beta}$-basis J and take $\chi\otimes 1 \in \mathrm{Int}(\Gamma_{\Sigma,I_\emptyset})$. Write $\chi\otimes 1 = \sum_{i\in J}\lambda_i e_i$ with $\lambda_i > 0$. Then $\mathbf{b} = \sum_{i\in J}\lambda_i e_i \in \mathbb{R}^r$ is a point of P_χ with precisely $\dim G$ nonzero coordinates. Since χ is generic, the vertices of P_χ have the same property by Proposition 14.3.14. It follows that $\mathbf{b}$ is a vertex and hence corresponds to a maximal cone $\sigma \in \Sigma_{\max}$. Since J is the set of indices of nonzero coordinates of $\mathbf{b}$, it follows easily that $J = J_\sigma$.

Part (b) implies $\Gamma_{\Sigma,I_\emptyset} \subseteq \bigcap_{\sigma\in\Sigma_{\max}} \mathrm{Cone}(\boldsymbol{\beta}_{J_\sigma})$, and then part (c) will follow once we prove the opposite inclusion. Take $\beta \in \bigcap_{\sigma\in\Sigma_{\max}} \mathrm{Cone}(\boldsymbol{\beta}_{J_\sigma})$ and pick $\mathbf{a} \in \mathbb{R}^r$ with $\beta = \gamma_\mathbb{R}(\mathbf{a})$. It suffices to prove that $\mathbf{a} \in \widetilde{\Gamma}_{\Sigma,I_\emptyset}$. This means finding a support function $\varphi \in \mathrm{CSF}(\Sigma)$ such that $\varphi(\nu_i) \geq -a_i$ for all i and $\varphi(\nu_i) = -a_i$ for $i \notin I_\emptyset$.

For $\sigma \in \Sigma_{\max}$, our hypothesis on β implies that $\beta \in \mathrm{Cone}(\boldsymbol{\beta}_{J_\sigma})$, so that we can write $\beta = \sum_{i\in J_\sigma}\lambda_i\beta_i$ with $\lambda_i \geq 0$. Hence there is $m_\sigma \in M_\mathbb{R}$ such that $\mathbf{a} + \delta_\mathbb{R}(m_\sigma) = \sum_{i\in J_\sigma}\lambda_i e_i$. Thus

$$\begin{aligned} &\langle m_\sigma, \nu_i\rangle \geq -a_i \ \text{ for all } i \\ &\langle m_\sigma, \nu_i\rangle = -a_i \ \text{ for all } i \notin J_\sigma, \text{ i.e., for all } \nu_i \in \sigma,\ i \notin I_\emptyset. \end{aligned} \tag{15.2.2}$$

Now define

$$\varphi(u) = \min_{\sigma\in\Sigma_{\max}} \langle m_\sigma, u\rangle, \quad u \in C_{\boldsymbol{\nu}}.$$

This function is clearly convex, and by (15.2.2) it satisfies

$$\begin{aligned}&\varphi(\nu_i) \geq -a_i \ \text{ for all } i\\ &\varphi(\nu_i) = \langle m_\sigma, \nu_i\rangle = -a_i \ \text{ for all } \nu_i \in \sigma,\ i \notin I_\emptyset.\end{aligned}$$

The second line implies that $\varphi \in \mathrm{SF}(\Sigma)$, and it follows that φ has the required properties. This completes the proof. $\square$

Here is an example that will appear several times in this section.

Example 15.2.2. Consider the exact sequence $0 \to \mathbb{Z}^3 \xrightarrow{B} \mathbb{Z}^5 \xrightarrow{A} \mathbb{Z}^2 \to 0$, where

$$B = \begin{pmatrix} 2 & 1 & 1\\ 2 & -1 & 1\\ 1 & 0 & 1\\ -2 & 1 & 1\\ -2 & -1 & 1\end{pmatrix}, \ A = \begin{pmatrix} 1 & 2 & -4 & 1 & 0\\ 2 & 1 & -4 & 0 & 1\end{pmatrix}.$$

As in §14.2, the ν_i's are the rows of B and the β_i's are the columns of A. The secondary fan shown in Figure 3 has five chambers. The shaded chamber $\Gamma_{\Sigma,\emptyset}$ is

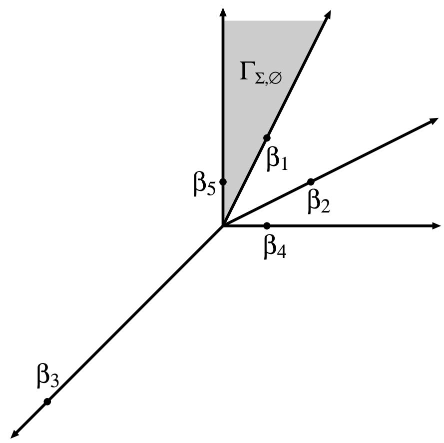


Figure 3. The secondary fan and a selected chamber $\Gamma_{\Sigma,\emptyset}$

contained in the cones

$$\mathrm{Cone}(\beta_1,\beta_5),\ \mathrm{Cone}(\beta_1,\beta_3),\ \mathrm{Cone}(\beta_4,\beta_5),\ \mathrm{Cone}(\beta_2,\beta_5).$$

By Proposition 15.2.1, the maximal cones of Σ are the “complementary” cones

$$\mathrm{Cone}(\nu_2,\nu_3,\nu_4),\ \mathrm{Cone}(\nu_2,\nu_4,\nu_5),\ \mathrm{Cone}(\nu_1,\nu_2,\nu_3),\ \mathrm{Cone}(\nu_1,\nu_3,\nu_4).$$

These lie in $\mathbb{R}^3$, but since the ν_i's lie on the plane $z = 1$, we can intersect Σ with this plane to get the triangulation of the rectangle $\mathrm{Conv}(\boldsymbol{\nu})$ shown in Figure 4. $\Diamond$

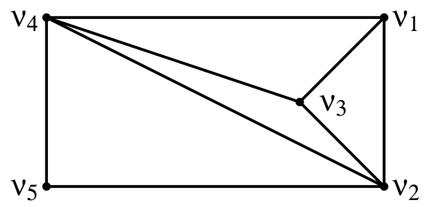


Figure 4. The fan Σ intersected with the plane $z = 1$

Our second example is related to the classification theorem proved in §7.3.

Example 15.2.3. Let X_Σ be a smooth projective toric variety of dimension n whose fan in $N_\mathbb{R} \simeq \mathbb{R}^n$ has $n+2$ rays, with ray generators $u_1,\ldots,u_{n+2}$. Then we have a geometric quotient $X_\Sigma \simeq (\mathbb{C}^{n+2} \setminus Z(\Sigma))/G$, where $G \simeq (\mathbb{C}^*)^2$. Furthermore, $\widehat{G}_\mathbb{R} = \mathrm{Pic}(X_\Sigma)_\mathbb{R} \simeq \mathbb{R}^2$. The β_i, $1 \le i \le n+2$, are the classes of the torus-invariant prime divisors, and $C_{\boldsymbol{\beta}} = \mathrm{Cone}(\boldsymbol{\beta})$ is the effective cone of X_Σ. The secondary fan refines $C_{\boldsymbol{\beta}}$, and the GKZ cone $\Gamma_{\Sigma,\emptyset}$ is the nef cone $\mathrm{Nef}(X_\Sigma)$ (Proposition 15.1.3). Note also that $C_{\boldsymbol{\beta}}$ is strongly convex since Σ is complete (Proposition 14.3.10).

Fix an ample class $\alpha \in \mathrm{Nef}(X_\Sigma)$. Since $\mathrm{Pic}(X_\Sigma)_\mathbb{R} \simeq \mathbb{R}^2$, the line through α divides $\mathrm{Pic}(X_\Sigma)_\mathbb{R}$ into two half-planes H^+ and H^-. Let

$$P = \{i \mid \beta_i \in H^+\},\ |P| = r+1$$
$$Q = \{j \mid \beta_j \in H^-\}, |Q| = s+1,$$

as illustrated by Figure 5. Take any $i \in P$ and $j \in Q$. Then β_i, β_j form a basis of

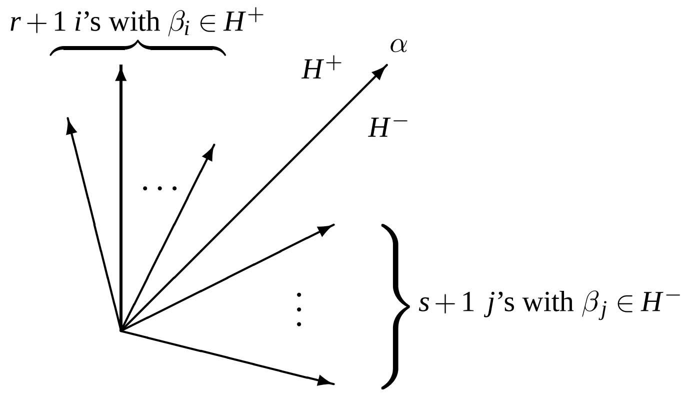


Figure 5. The partition of $\boldsymbol{\beta}$ induced by α

$\mathrm{Pic}(X_\Sigma)_\mathbb{R}$, so that $\{i, j\}$ is a $\boldsymbol{\beta}$-basis. Note also that $\alpha \in \mathrm{Cone}(\beta_i, \beta_j)$ since β_i and

β_j lie on opposite sides of the line determined by α. By (15.2.1), it follows that $\Gamma_{\Sigma,\emptyset} \subseteq \mathrm{Cone}(\beta_i, \beta_j)$. Using Proposition 15.2.1, we conclude that

$$\sigma_{i,j} = \mathrm{Cone}(u_k \mid k \neq i, j)$$

is a maximal cone of Σ and that all maximal cones of Σ are of this form. This proves (7.3.8), which is a key step in the proof of Kleinschmidt's classification theorem of smooth projective toric varieties of Picard number 2 (Theorem 7.3.7). The theory developed here makes (7.3.8) easy to see. $\Diamond$

The cones $\boldsymbol{\beta}_{J_\sigma}$, $\sigma \in \Sigma_{\max}$, from Proposition 15.2.1 are an example of a *bunch of cones* in the terminology of Berchtold and Hausen. Their paper [**24**] shows that many aspects of toric geometry can be explained in the language of bunches. For example, they give a "bunch" proof of Kleinschmidt's classification theorem from §7.3 in [**24**, Prop. 10.1].

The Moving Cone. Proposition 15.2.1 shows how to represent the chambers of the secondary fan as intersections of cones generated by certain subsets of $\boldsymbol{\beta}$. Our next result, taken from [**134**], shows that the moving cone $\mathrm{Mov}_{\mathrm{GKZ}}$ has a similar representation.

Proposition 15.2.4. *The moving cone of the secondary fan is the intersection*

$$\mathrm{Mov}_{\mathrm{GKZ}} = \bigcap_{i=1}^{r} \mathrm{Cone}(\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_r).$$

Proof. Take $\chi \otimes 1 \in C_{\boldsymbol{\beta}}$. The key observation is that

$$F_{i,\chi} \neq \emptyset \iff \chi \otimes 1 \in \mathrm{Cone}(\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_r).$$

This follows from $F_{i,\chi} = P_\chi \cap \mathbf{V}(x_i)$ and $P_\chi = \gamma_{\mathbb{R}}^{-1}(\chi \otimes 1) \cap \mathbb{R}^r_{\geq 0}$. Now suppose that $\chi \otimes 1 \in \mathrm{Relint}(\Gamma_{\Sigma,I_\emptyset})$. Since $I_\emptyset = \{i \mid F_{i,\chi} = \emptyset\}$ by Lemma 14.4.4, we obtain

$$I_\emptyset = \emptyset \iff \chi \otimes 1 \in \textstyle\bigcap_{i=1}^r \mathrm{Cone}(\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_r).$$

Then we are done since $\mathrm{Mov}_{\mathrm{GKZ}}$ is the union of all GKZ cones with $I_\emptyset = \emptyset$. $\square$

In the primitive geometric case, Theorem 15.1.10 implies that the moving cone is $\overline{\mathrm{Mov}}(X_\Sigma)$ for any chamber $\Gamma_{\Sigma,\emptyset}$. In this case, $\beta_i = [D_i]$, and a divisor $D = \sum_{i=1}^r a_i D_i$ is movable if for each i, we can move it away from D_i, i.e., we can find a linear equivalence $D \sim \sum_{j \neq i} b_j D_j \geq 0$. You should check that this translates exactly into the condition of Proposition 15.2.4.

Here are two examples to illustrate this proposition.

Example 15.2.5. In Figure 2 from Example 15.1.11, we have $\beta_i = [D_i]$. Looking at the figure, we see that $\mathrm{Cone}(\beta_1, \beta_3, \beta_4)$ is the right chamber and

$$\mathrm{Cone}(\beta_1, \beta_2, \beta_3) = \mathrm{Cone}(\beta_1, \beta_3, \beta_4) = \mathrm{Cone}(\beta_2, \beta_3, \beta_4) = C_{\boldsymbol{\beta}}.$$

By Proposition 15.2.4, the moving cone is the intersection of these cones, which is clearly the right chamber. This confirms the result of Example 15.1.11. $\Diamond$

Example 15.2.6. In Figure 3 from Example 15.2.2, we see that $\mathrm{Cone}(\beta_1, \beta_2, \beta_4, \beta_5)$ is the first quadrant and that $\mathrm{Cone}(\beta_1, \dots, \widehat{\beta_i}, \dots, \beta_5) = \mathbb{R}^2$ for $i = 1, 2, 4, 5$. Then Proposition 15.2.4 implies that the moving cone is the first quadrant, which by Figure 3 is the union of three GKZ chambers. $\Diamond$

Triangulations. The triangulation in Example 15.2.2 generalizes nicely. Before giving the general theory, let us explore this example in more detail.

Example 15.2.7. Let $\boldsymbol{\beta}$ and $\boldsymbol{\nu}$ be as in Example 15.2.2, and recall that $\nu_1, \dots, \nu_5$ lie in the plane $z = 1$. The convex hull $Q_{\boldsymbol{\nu}} = \mathrm{Conv}(\boldsymbol{\nu})$ is a rectangle. Figures 3 and 4 illustrate the triangulation of $Q_{\boldsymbol{\nu}}$ obtained from one of the GKZ chambers. Doing this for every chamber gives the picture shown in Figure 6.

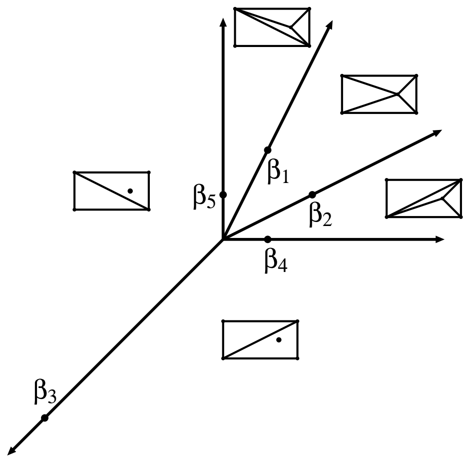


Figure 6. The secondary fan and its associated triangulations in Example 15.2.7

Note that each GKZ chamber gives a triangulation of $Q_{\boldsymbol{\nu}}$ such that the vertices of the triangles lie in $\boldsymbol{\nu}$. Some elements of $\boldsymbol{\nu}$ can be omitted provided that we triangulate all of $Q_{\boldsymbol{\nu}}$. The triangulations that use every element of $\boldsymbol{\nu}$ correspond to the chambers that make up the moving cone. $\Diamond$

To generalize this construction, take distinct elements $\nu_1, \dots, \nu_r \in N$ lying on an integral affine hyperplane $H \subseteq N_{\mathbb{R}}$ with $0 \notin H$. Thus there are $m \in M$ and $a \in \mathbb{Z} \setminus \{0\}$ such that $\langle m, \nu_i \rangle = a$ for all i. This gives the lattice polytope

$$Q_{\boldsymbol{\nu}} = \mathrm{Conv}(\boldsymbol{\nu}) = \mathrm{Conv}(\nu_1, \dots, \nu_r)$$

lying in the hyperplane H. We assume that $Q_{\boldsymbol{\nu}}$ has full dimension in H. Thus $Q_{\boldsymbol{\nu}}$ has codimension 1 in $N_{\mathbb{R}}$ and the cone $C_{\boldsymbol{\nu}} = \mathrm{Cone}(\boldsymbol{\nu})$ has full dimension in $N_{\mathbb{R}}$. Note also that $C_{\boldsymbol{\nu}}$ is strongly convex.

A *triangulation* $\mathcal{T}$ of $\boldsymbol{\nu}$ is a collection of simplices satisfying:

- Each simplex in $\mathcal{T}$ has codimension 1 in $N_{\mathbb{R}}$ with vertices in $\boldsymbol{\nu}$.
- The intersection of any two simplices in $\mathcal{T}$ is a face of each.
- The union of the simplices in $\mathcal{T}$ is $Q_{\boldsymbol{\nu}}$.

Figure 6 shows all triangulations of the vectors $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_5)$ given by the rows of the matrix B in Example 15.2.2. See [**80**] for an introduction to triangulations.

There is bijective correspondence between triangulations $\mathcal{T}$ of $\boldsymbol{\nu}$ and simplicial fans Σ such that $|\Sigma| = C_{\boldsymbol{\nu}}$ and $\Sigma(1) \subseteq \{\mathrm{Cone}(\nu_i) \mid 1 \leq i \leq r\}$. The correspondence is easy to describe:

- Given $\mathcal{T}$, the cones of Σ are the cones over the simplices of $\mathcal{T}$ and their faces.
- Given Σ, the simplices of $\mathcal{T}$ are $\sigma \cap Q_{\boldsymbol{\nu}}$ for $\sigma \in \Sigma_{\max}$.

This correspondence, written $\Sigma \mapsto \mathcal{T} = \Sigma \cap Q_{\boldsymbol{\nu}}$, will be used frequently in the our discussion of triangulations.

We next define an especially nice class of triangulations, following [**264**]. Given nonnegative weights $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r) \in \mathbb{R}^r_{\geq 0}$, we get the cone

$$C_{\boldsymbol{\nu},\boldsymbol{\omega}} = \mathrm{Cone}((\nu_1, \omega_1), \ldots, (\nu_r, \omega_r)) \subseteq N_{\mathbb{R}} \times \mathbb{R}.$$

When $\boldsymbol{\nu}$ consists of five equally spaced points on a line in $\mathbb{R}^2$, Figure 7 shows $C_{\boldsymbol{\nu},\boldsymbol{\omega}}$ for one choice of the weights $\boldsymbol{\omega}$. In the figure, the lengths of the dashed lines are

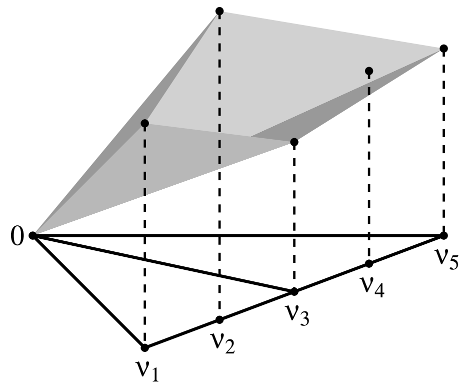


Figure 7. The cone $C_{\nu,\omega}$ determined by ν and weights ω

determined by $\boldsymbol{\omega}$, and (ν_4, ω_4) is in the interior of the cone. Projection onto $N_{\mathbb{R}}$ maps $C_{\boldsymbol{\nu},\boldsymbol{\omega}}$ onto $C_{\boldsymbol{\nu}}$.

The *lower hull* of $C_{\boldsymbol{\nu},\boldsymbol{\omega}}$ consists of all facets of $C_{\boldsymbol{\nu},\boldsymbol{\omega}}$ whose inner normal has a positive last coordinate. Projecting the facets in the lower hull and their faces gives a fan $\Sigma_{\boldsymbol{\omega}}$ in $N_{\mathbb{R}}$ such that $|\Sigma_{\boldsymbol{\omega}}| = C_{\boldsymbol{\nu}}$ and $\Sigma_{\boldsymbol{\omega}}(1) \subseteq \{\mathrm{Cone}(\nu_i) \mid 1 \leq i \leq r\}$.

Definition 15.2.8. A triangulation $\mathcal{T}$ of $\boldsymbol{\nu}$ is ***regular*** if there are weights $\boldsymbol{\omega}$ such that $\Sigma_{\boldsymbol{\omega}}$ is simplicial and $\mathcal{T} = \Sigma_{\boldsymbol{\omega}} \cap Q_{\boldsymbol{\nu}}$.

To see how this relates to the secondary fan, note that $\boldsymbol{\nu}$ gives an injection $M \to \mathbb{Z}^r$ defined by $m \mapsto (\langle m, \nu_1\rangle, \dots, \langle m, \nu_r\rangle)$ as in §14.2. The cokernel of this map is the character group $\widehat{G}$ of a subgroup $G \subseteq (\mathbb{C}^*)^r$. Hence the theory developed in Chapter 14 applies. In particular, Lemma 14.3.2 implies that $C_{\boldsymbol{\beta}} = \widehat{G}_{\mathbb{R}}$, so that the secondary fan is complete in this situation.

Proposition 15.2.9. *For $\boldsymbol{\nu}$ as above, the map $\Gamma_{\Sigma,I_\emptyset} \mapsto \mathcal{T} = \Sigma \cap Q_{\boldsymbol{\nu}}$ is a bijection between chambers of the secondary fan and regular triangulations of $\boldsymbol{\nu}$.*

Proof. If $\Gamma_{\Sigma,I_\emptyset}$ is a chamber, then Σ is a simplicial fan such that $|\Sigma| = C_{\boldsymbol{\nu}}$ and $\Sigma(1) \subseteq \{\mathrm{Cone}(\nu_i) \mid 1 \le i \le r\}$. Thus $\mathcal{T} = \Sigma \cap Q_{\boldsymbol{\nu}}$ is a triangulation of $\boldsymbol{\nu}$. If another chamber $\Gamma_{\Sigma',I'_\emptyset}$ maps to the same triangulation $\mathcal{T}$, then $\Sigma' = \Sigma$ since the maximal cones of the fan are the cones over the simplices of $\mathcal{T}$. Then Proposition 14.4.9 implies that $I'_\emptyset = I_\emptyset$. Hence the map from chambers to triangulations is injective.

We prove that $\mathcal{T} = \Sigma \cap Q_{\boldsymbol{\nu}}$ is regular as follows. Take $\beta \in \mathrm{Relint}(\Gamma_{\Sigma,I_\emptyset})$ and pick $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{R}^r_{\ge 0}$ that maps to β. By Lemma 14.4.4, the support function $\varphi_{\mathbf{a}}$ is strictly convex with respect to Σ. Furthermore,

$$\varphi_{\mathbf{a}}(\nu_i) = -a_i \text{ for } i \notin I_\emptyset,\ \varphi_{\mathbf{a}}(\nu_i) > -a_i \text{ for } i \in I_\emptyset.$$

For weights given by $\mathbf{a}$, you will prove in Exercise 15.2.1 that

$$\begin{aligned} \varphi_{\mathbf{a}} \text{ convex} &\iff \text{the lower hull of } C_{\boldsymbol{\nu},\mathbf{a}} \text{ is the graph of } -\varphi_{\mathbf{a}} \\ \varphi_{\mathbf{a}} \text{ strictly convex} &\iff \text{facets of the lower hull map to maximal cones of } \Sigma. \end{aligned}$$

Hence $\Sigma = \Sigma_{\mathbf{a}}$, so that $\mathcal{T} = \Sigma \cap Q_{\boldsymbol{\nu}}$ is a regular triangulation of $\boldsymbol{\nu}$ when $\Gamma_{\Sigma,I_\emptyset}$ is a chamber.

It remains to prove that every regular triangulation of $\boldsymbol{\nu}$ arises in this way. Suppose that $\mathcal{T} = \Sigma_{\boldsymbol{\omega}} \cap Q_{\boldsymbol{\nu}}$. Then $\boldsymbol{\omega} \in \mathbb{R}^r_{\ge 0}$ implies that $\boldsymbol{\omega} \in \mathrm{Relint}(\widetilde{\Gamma}_{\Sigma,I_\emptyset})$ for some GKZ cone $\widetilde{\Gamma}_{\Sigma,I_\emptyset}$. The associated support function $\varphi_{\boldsymbol{\omega}}$ is strictly convex with respect to Σ and satisfies

$$\varphi_{\boldsymbol{\omega}}(\nu_i) = -\omega_i \text{ for } i \notin I_\emptyset,\ \varphi_{\boldsymbol{\omega}}(\nu_i) > -\omega_i \text{ for } i \in I_\emptyset.$$

This implies $\Sigma = \Sigma_{\boldsymbol{\omega}}$. However, $\Gamma_{\Sigma,I_\emptyset}$ need not be a chamber, even though Σ is simplicial. Figure 8 on the next page gives an example where $I_\emptyset = \{2\}$ and Σ has maximal cones $\mathrm{Cone}(\nu_1,\nu_3)$ and $\mathrm{Cone}(\nu_3,\nu_4,\nu_5)$. This is easy to fix—for any $i \notin I_\emptyset$ with $\mathrm{Cone}(\nu_i) \notin \Sigma(1)$ (such as $i = 4$ in Figure 8), increase ω_i so that $\varphi_{\boldsymbol{\omega}}(\nu_i) > -\omega_i$. When we do this, Σ stays the same but we now have $I_\emptyset = \{i \mid \mathrm{Cone}(\nu_i) \notin \Sigma(1)\}$. Since Σ is also simplicial, Proposition 14.4.9 implies that $\Gamma_{\Sigma,I_\emptyset}$ is a chamber. $\square$

The Secondary Polytope. In the situation of Proposition 15.2.9, the secondary fan is complete. It is natural to ask if the secondary fan is the normal fan a polytope. Gel′fand, Kapranov and Zelevinsky [**112**] proved the existence of such a polytope, called the *secondary polytope*. Other proofs have been given in [**28**] and [**222**]. We will present a proof from [**28**] that uses Gale duality.

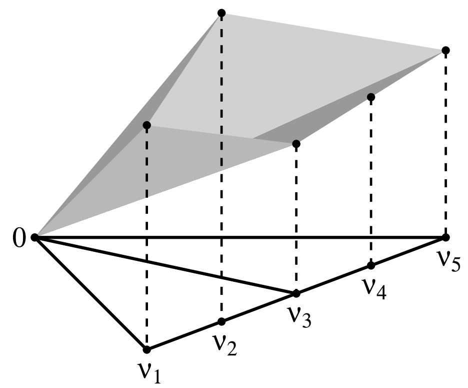


Figure 8. Weights where Σ is simplicial but $\Gamma_{\Sigma,I_\emptyset}$ is not a chamber in Proposition 15.2.9

A $\boldsymbol{\beta}$-basis $J \subseteq \{1,\dots,r\}$ gives the cone $\mathrm{Cone}(\boldsymbol{\beta}_J) = \mathrm{Cone}(\beta_i \mid i \in J)$ featured in Proposition 15.2.1. Given a real number $\varepsilon > 0$, consider the polytope

$$P_J = \mathrm{Conv}(\beta_i,\ \varepsilon\beta_j \mid i \in J,\ j \notin J) \subseteq \widehat{G}_{\mathbb{R}}.$$

We assume that ε is chosen sufficiently small so that $\mathrm{Conv}(\beta_J) = \mathrm{Conv}(\beta_i \mid i \in J)$ is a facet of P_J. Since $C_{\boldsymbol{\beta}} = \mathrm{Cone}(\boldsymbol{\beta}) = \widehat{G}_{\mathbb{R}}$, the origin is an interior point of P_J. Hence taking cones over proper faces of P_J gives a complete fan Σ_J in $\widehat{G}_{\mathbb{R}}$. Our choice of ε guarantees that $\mathrm{Cone}(\boldsymbol{\beta}_J)$ is a cone of Σ_J.

We will use the following notation. Given two complete fans Σ_1 and Σ_2, the set of all intersections $\sigma_1 \cap \sigma_2$ for $\sigma_i \in \Sigma_i$, $i = 1,2$, is a fan denoted $\Sigma_1 \wedge \Sigma_2$. Thus

$$\Sigma_1 \wedge \Sigma_2 = \{\sigma_1 \cap \sigma_2 \mid \sigma_i \in \Sigma_i,\ i = 1,2\}$$

Any common refinement of Σ_1 and Σ_2 also refines $\Sigma_1 \wedge \Sigma_2$.

Lemma 15.2.10. *As above, let Σ_J be the fan constructed from the $\boldsymbol{\beta}$-basis J. Then:*

(a) *The secondary fan Σ_{GKZ} refines Σ_J.*

(b) $\Sigma_{\mathrm{GKZ}} = \bigwedge_{J \text{ is a } \boldsymbol{\beta}\text{-basis}} \Sigma_J$.

Proof. Extending Definition 14.3.13, we say that $\beta \in C_{\boldsymbol{\beta}}$ is *generic* if $\beta \notin \mathrm{Cone}(\boldsymbol{\beta}')$ for all subsets $\boldsymbol{\beta}' \subseteq \boldsymbol{\beta}$ with $\dim \mathrm{Cone}(\boldsymbol{\beta}') < \dim G$.

Faces of P_J are of the form $\mathrm{Conv}(\beta_i,\ \varepsilon\beta_j \mid i \in A,\ j \notin B)$ for suitable sets A and B. It follows that the cones of Σ_J are generated by subsets of $\boldsymbol{\beta}$. In particular, any cone of Σ_J of dimension $< \dim G$ consists entirely of nongeneric elements.

Take any chamber $\Gamma_{\Sigma,I_\emptyset}$. The interior points of $\Gamma_{\Sigma,I_\emptyset}$ are generic (this follows easily from the proof of Proposition 14.4.9), so that $\mathrm{Int}(\Gamma_{\Sigma,I_\emptyset})$ is disjoint from any cone of Σ_J of dimension $< \dim G$. This implies that $\Gamma_{\Sigma,I_\emptyset}$ is contained in a maximal cone of Σ_J, and part (a) follows without difficulty.

For part (b), first note that by part (a), Σ_{GKZ} refines $\bigwedge_{J \text{ is a } \beta\text{-basis}} \Sigma_J$. Then observe that every maximal cone of Σ_{GKZ} appears in $\bigwedge_{J \text{ is a } \beta\text{-basis}} \Sigma_J$ by Proposition 15.2.1. It follows that the two fans must be equal, which completes the proof of part (b). □

We now construct the secondary polytope.

Proposition 15.2.11. *There is a full dimensional lattice polytope $P_{\mathrm{GKZ}} \subseteq \widehat{G}^*_{\mathbb{R}}$ whose normal fan in $\widehat{G}_{\mathbb{R}}$ is the secondary fan Σ_{GKZ}.*

Proof. Let J be a β-basis. Since $P_J \subseteq \widehat{G}_{\mathbb{R}}$ contains the origin as an interior point, we have the dual polytope $P_J^\circ \subseteq \widehat{G}^*_{\mathbb{R}}$ defined in §2.2, and Σ_J is the normal fan of P_J° by Exercise 2.3.4. Recall from Proposition 6.2.13 that if polytopes P_1, P_2 have normal fans $\Sigma_{P_1}, \Sigma_{P_2}$, then the Minkowski sum $P_1 + P_2$ has normal fan $\Sigma_{P_1} \wedge \Sigma_{P_2}$. Now consider the Minkowski sum

$$P_{\mathrm{GKZ}} = \textstyle\sum_{J \text{ is a } \beta\text{-basis}} P_J^\circ \subseteq \widehat{G}^*_{\mathbb{R}}.$$

It follows that P_{GKZ} has normal fan $\bigwedge_{J \text{ is a } \beta\text{-basis}} \Sigma_J$, which by Lemma 15.2.10 is the secondary fan. □

It is customary to call P_{GKZ} "the" *secondary polytope* even though it is far from unique. A specific choice for P_{GKZ}, based on a different but quite elegant construction, is given in [**113**] and [**112**] (see also [**28**]).

Since the vertices of a polytope correspond to maximal cones of the normal fan, Propositions 15.2.9 and 15.2.11 have the following nice corollary.

Corollary 15.2.12. *There is a bijective correspondence between vertices of the secondary polytope $P_{\mathrm{GKZ}} \subseteq \widehat{G}^*_{\mathbb{R}}$ and regular triangulations of $\boldsymbol{\nu}$.* □

The secondary polytope P_{GKZ} has dimension $\dim G$. For $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_r)$, we can compute this dimension as follows. First, the convex hull $Q_{\boldsymbol{\nu}} = \mathrm{Conv}(\boldsymbol{\nu})$ has dimension $\dim Q_{\boldsymbol{\nu}} = \dim N_{\mathbb{R}} - 1$ by assumption. Since $\dim G = r - \dim N_{\mathbb{R}}$, we get the dimension formula

$$\dim P_{\mathrm{GKZ}} = |\boldsymbol{\nu}| - \dim Q_{\boldsymbol{\nu}} - 1.$$

Here is a famous example of a secondary polytope.

Example 15.2.13. The *associahedron* is the secondary polytope for a configuration where $N_{\mathbb{R}} = \mathbb{R}^3$ and $\boldsymbol{\nu}$ consists of the vertices of a convex r-gon sitting in the plane $z = 1$. This polytope has dimension is $r-3$ and is sometimes denoted $\mathcal{K}^{r-3}$. Another name for the associahedron is the *Stasheff polytope* because of its origins in algebraic topology.

The number of vertices of $\mathcal{K}^{r-3}$ is the Catalan number C_{r-2}, where

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Besides counting vertices of the associahedron, the Catalan number C_{r-2} counts many things, including:

- The number of triangulations of a convex r-gon that introduce no new vertices.
- The number of rooted binary trees with $r-1$ leaves.
- The number of ways pairs of parentheses can be introduced in a nonassociative product $x_1 \cdots x_{r-1}$. For $r-1=4$, one gets

$$(((x_1x_2)x_3)x_4,\ (x_1(x_2x_3))x_4,\ (x_1x_2)(x_3x_4),\ x_1((x_2x_3)x_4),\ x_1(x_2(x_3x_4))).$$

See Exercise 15.2.2 and [**80**, Thms. 1.1.2 and 1.1.3].

The associahedron $\mathcal{K}^{r-3}$ is easy to work out when $r=4$ or 5 (Exercise 15.2.3). When $r=6$, $\mathcal{K}^3$ has $C_4=14$ vertices. In Figure 9, we show $\mathcal{K}^3$ together with the triangulation at each vertex. The figure uses Loday's construction [**188**] of the associahedron, and the labeling of the vertices in Figure 9 was inspired by [**80**, Fig. 1.15]. See Example B.2.3 for more on how to construct this figure. $\Diamond$

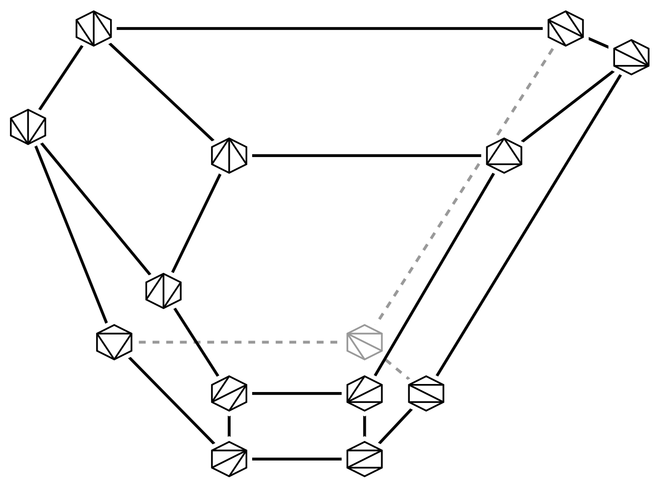


Figure 9. The associahedron $\mathcal{K}^3$ with triangulations at the vertices

Final comments. There is a *lot* more to say about the secondary fan. Here are some highlights:

- By Proposition 15.2.11, the secondary fan is the normal fan of the secondary polytope when ν lies on an affine hyperplane not passing through the origin in $N_{\mathbb{R}}$. One can ask more generally if the secondary fan is the normal fan of some polyhedron for arbitrary ν. By[**29**], the answer is yes when ν is geometric.
- The secondary polytope is related to Chow polytopes and the Newton polytope of the principal A-determinant. See [**113**] and [**112**].

- The secondary fan is also related to the Gröbner fan defined in §10.3. See [**80**, Sec. 9.4] and [**264**].
- The secondary fan is used in mirror symmetry. See [**66**] and [**68**].
- Chapter 5 of the book [**80**] is devoted entirely to regular triangulations and the secondary fan.

Another important topic is what happens to the toric variety X_Σ of a chamber $\Gamma_{\Sigma,I_\emptyset}$ when we cross a wall in the secondary fan. This will be studied in the next section.

Exercises for §15.2.

15.2.1. Prove the claims about $\varphi_{\mathbf{a}}$ made in the proof of Proposition 15.2.9.

15.2.2. The goal of this exercise is to relate triangulations, binary trees, and parentheses. For $r \geq 3$, fix an r-gon with vertices labeled $1,\dots,r$. Also consider symbols $x_1,\dots,x_{r-1}$, which we arrange in a diagram with an empty box before and after each symbol. The boxes are numbered $1,\dots,r$ corresponding to the vertices of the r-gon:

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 \cdots & & r-1 & & r \\ \square & x_1 & \square & x_2 & \square \cdots & & \square & x_{r-1} & \square \end{array} \tag{15.2.3}$$

A triangulation $\mathcal{T}$ of the r-gon is determined by adding certain diagonals. Each diagonal is oriented so that we go from a smaller vertex to a bigger one. Also, at vertex i, we order the diagonals containing i according to the label of the other vertex of the diagonal.

(a) Given $\mathcal{T}$, fill in the ith box in the diagram (15.2.3) with open or close parentheses, one for each diagonal containing i. The parentheses are placed in the same order as the diagonals containing i, and we use an open parenthesis if the diagonal starts at i and a close parenthesis if the diagonal ends at i. Prove that this defines a bijection between triangulations and parenthesis placements in the symbols.

(b) Given a rooted binary tree with leaves $x_1,\dots,x_{r-1}$, each internal node different from the root is root of a subtree, and we put parentheses around the rightmost and leftmost leaves this subtree. Prove that this gives a bijection between binary trees and parenthesis placements in the symbols.

Figure 10 shows an example of parts (a) and (b) of the exercise when $r = 6$. See also [**80**, Thm. 1.1.3].

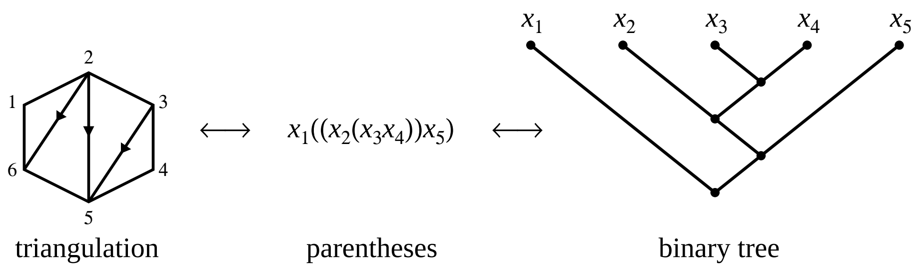


Figure 10. The correspondences described in Exercise 15.2.2

15.2.3. Consider the associahedron $\mathcal{K}^{r-3}$ from Example 15.2.13.

(a) Draw $\mathcal{K}^1$ and $\mathcal{K}^2$. At each vertex, give the corresponding triangulation of the square and pentagon, similar to Figure 9.

(b) Consider the top pentagonal facet of $\mathcal{K}^3$ in Figure 9. All triangulations lying in this facet contain a common diagonal that forms a triangle with two edges of the hexagon. Use this and part (a) to explain why this facet is a pentagon. Then explain why $\mathcal{K}^3$ has six such facets.

(c) Pick one of the four-sided facets of Figure 9. Show that all triangulations lying in this facet contain a common diagonal that bisects the hexagon. Use this to explain why the facet is a quadrilateral and why there are three such facets.

15.2.4. The secondary fan computed in Exercise 15.1.4 has four chambers. Apply the method of Example 15.2.2 to each of these chambers and determine the maximal cones of the corresponding fan.

15.2.5. Consider the cone $\sigma = \mathrm{Cone}(e_1, e_1 + de_2) \subseteq \mathbb{R}^2$ and let Σ be the smooth refinement whose ray generators are $\nu_i = e_1 + ie_2$ for $0 \le i \le d$. We assume $d \ge 2$. By Exercise 15.1.5, the secondary fan has 2^{d-1} chambers corresponding to subsets $I_\emptyset \subseteq \{1, \dots, d-1\}$. This exercise will study the chambers where $I_\emptyset = \emptyset$ and $I_\emptyset = \{1, \dots, d-1\}$.

(a) The ν_i's are the rows of the $(d+1) \times 2$ matrix

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & d \end{pmatrix}$$

that fits into an exact sequence $0 \to \mathbb{Z}^2 \xrightarrow{B} \mathbb{Z}^{d+1} \xrightarrow{A} \mathbb{Z}^{d-1} \to 0$. Show that A can be chosen to be the $(d-1) \times (d+1)$ matrix

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \end{pmatrix}.$$

Then the β_i's are the columns of A.

(b) Suppose that $I_\emptyset = \{1, \dots, d-1\}$. Show that U_σ is the toric variety of this chamber and use Proposition 15.2.1 to show that the chamber is the simplicial cone generated by the columns $2, 3, \dots, d-1$ of A.

(c) Suppose that $I_\emptyset = \emptyset$. Show that X_Σ is the toric variety of this chamber. Proposition 15.2.1 represents this chamber as the intersection of d simplicial cones. The next parts of the exercise will show that this chamber is $\mathbb{R}^{d-1}_{\ge 0}$.

(d) Show that the chamber is $\mathrm{Nef}(X_\Sigma)$ and show that $D = \sum_{i=0}^d a_i D_i$ is nef if and only if $a_{i-1} - 2a_i + a_{i+1} \ge 0$ for $i = 1, \dots, d-1$. Hint: Use Example 6.4.7.

(e) Show $\mathrm{Nef}(X_\Sigma) = \mathbb{R}^{d-1}_{\ge 0}$. Hint: $[D] = \sum_{i=0}^d a_i \beta_i$, where β_i is the ith column of A.

This exercise was inspired by [**64**, App. A], where the toric variety of the secondary fan has a nice moduli interpretation.

15.2.6. The definition of regular triangulation involved choosing a weight vector in $\mathbb{R}^r_{\ge 0}$.

(a) Define what it means for a weight vector to be generic. Hint: The weights used in Figure 7 are generic while those in Figure 8 are not.

(b) Explain how the definition given in part (a) relates to definition of generic given in the proof of Lemma 15.2.10.

15.2.7. Let $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_6)$ be the rows of the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

(a) Show that $Q_{\boldsymbol{\nu}} = \mathrm{Conv}(\boldsymbol{\nu})$ is a 3-dimensional prism in the hyperplane $H \subseteq \mathbb{R}^4$ where the last coordinate is 1. The secondary polytope has dimension $r - \dim Q_{\boldsymbol{\nu}} - 1 = 2$.

(b) Show that the secondary polytope is a hexagon and each triangulation has 3 simplices.

See [**228**, Fig. 11] for a splendid picture of the secondary polytope with the corresponding triangulation of $Q_{\boldsymbol{\nu}}$ at each vertex.

§15.3. Crossing a Wall

The goal of this section is to understand what happens when we cross a wall in the secondary fan. We begin with some examples.

Examples. In the general case when $\boldsymbol{\nu}$ has elements that vanish or are repeated, it is possible that very little happens when crossing a wall. Here is a simple example.

Example 15.3.1. In Example 15.1.5 we have $\beta_3 = \beta_1 + \beta_2$ in $\widehat{G}_{\mathbb{R}} \simeq \mathbb{R}^2$, and $\nu_1 = \nu_2 = e_1$, $\nu_3 = -e_1$ in $N_{\mathbb{R}} \simeq \mathbb{R}$. Figure 11 shows the secondary fan together with

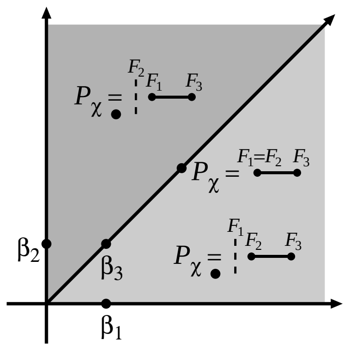


Figure 11. A boring wall crossing

P_χ and its virtual facets F_1, F_2, F_3 for three choices of χ. These χ's were chosen to illustrate that very little happens when crossing the wall between the two chambers:

the GIT quotient is $\mathbb{P}^1$ on either side of the wall and is also $\mathbb{P}^1$ on the wall itself. The only thing that changes is which virtual facet is empty. $\Diamond$

What allows Example 15.3.1 to be so boring is the equality $\nu_1 = \nu_2$. For the rest of this section, we will assume that $\boldsymbol{\nu}$ is geometric, so that the ν_i are nonzero and generate distinct rays in $N_{\mathbb{R}}$. We will see that this assumption rules out the behavior illustrated in Example 15.3.1.

Here are some more interesting wall crossings.

Example 15.3.2. Figure 6 from Example 15.2.7 shows a secondary fan with five chambers. Here, $\boldsymbol{\nu}$ consists of five vectors in $\mathbb{R}^3$ lying in the plane $z = 1$. Each triangulation in Figure 6 is the intersection of a fan in $\mathbb{R}^3$ with $z = 1$.

We will study how these fans change as we go clockwise around the origin, starting with the fan Σ_0 in the large chamber to the left in in Figure 6. Figure 12

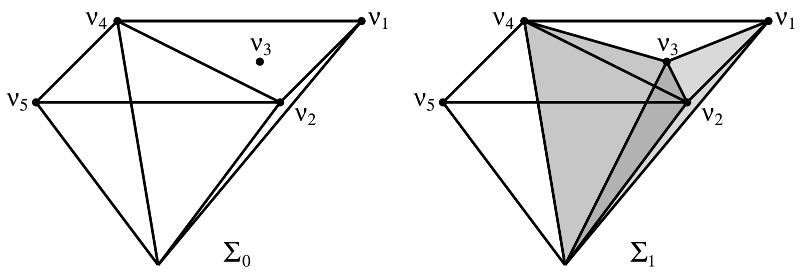


Figure 12. The fans to the left and right of the wall generated by β_5

shows that as we cross the vertical wall generated by β_5, Σ_0 changes to Σ_1, which is the star division of Σ_0 at the point ν_3, as indicated by the shading.

Continuing our loop, we cross the wall generated by β_1. This takes us from Σ_1 to the fan Σ_2 shown in Figure 13. Here, the only change is that the shaded wall in Σ_1 "flips" to become the shaded wall in Σ_2 in Figure 13. We've seen this type of flip before—the large illustration given in Figure 3 of Example 11.1.12 has fans (also called Σ_1 and Σ_2) that are related by the same type of flip.

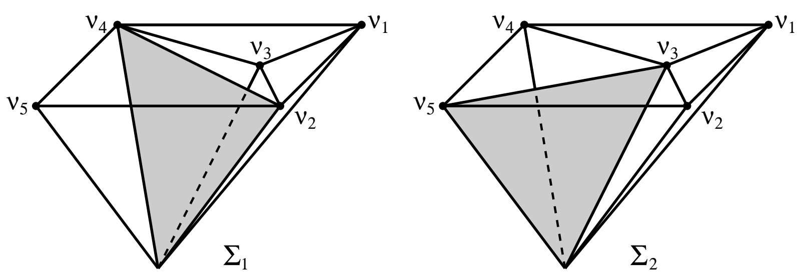


Figure 13. The fans to the left and right of the wall generated by β_1

To finish the clockwise loop, we have three more walls to cross in Figure 6:

- Crossing the wall generated by β_2 is another flip.
- Crossing the wall generated by β_4 "undoes" a star subdivision.
- Crossing the wall generated by β_3 is a flip. $\Diamond$

Example 15.3.3. In the situation of Propositions 15.2.9 and 15.2.11, the secondary fan is the normal fan of the secondary polytope. Thus a wall between two chambers of Σ_{GKZ} corresponds to an edge of the secondary polytope P_{GKZ}.

Now look at the associahedron $\mathcal{K}^3$ shown in Figure 9 of Example 15.2.13. Every edge comes from a "flip" of the triangulations at the vertices of the edge. $\Diamond$

Example 15.3.4. Figure 2 of Example 15.1.11 shows a secondary fan with exactly two chambers. You should check that the corresponding fans are related by a star subdivision. $\Diamond$

With the exception of Example 15.3.1, the wall crossings in these examples all come from flips or star subdivisions. We will soon see that this is no accident.

For the rest of this section we assume that $\boldsymbol{\nu}$ is geometric.

Facets of GKZ Cones. Before we can understand wall crossings, we need to learn more about the facets of a GKZ cone. Recall from Proposition 15.1.3 that

$$\Gamma_{\Sigma,I_\emptyset} \simeq \mathrm{Nef}(X_\Sigma) \times \mathbb{R}^{I_\emptyset}_{\geq 0}.$$

Using this and the face relation described in Theorem 14.4.7, one easily obtains the following result (Exercise 15.3.1).

Lemma 15.3.5. *The facets of* $\Gamma_{\Sigma,I_\emptyset}$ *have one of two forms:*

(a) $\Gamma_{\Sigma',I_\emptyset} \simeq$ (*a facet of* $\mathrm{Nef}(X_\Sigma)) \times \mathbb{R}^{I_\emptyset}_{\geq 0}$, *where* Σ *refines* Σ', *or*

(b) $\Gamma_{\Sigma,I_\emptyset\setminus\{i\}} \simeq \mathrm{Nef}(X_\Sigma) \times \mathbb{R}^{I_\emptyset\setminus\{i\}}_{\geq 0}$, *where* $i \in I_\emptyset$. $\square$

Part (b) of the lemma describes exactly what the facets look like, while part (a) does not. Later in the section we will use the toric cone theorem to say more the facets from part (a).

Note also that if $\Gamma_{\Sigma,I_\emptyset}$ is a chamber of the secondary fan, then its facets are either walls of the secondary fan or lie on the boundary of $C_{\boldsymbol{\beta}}$. Since $\boldsymbol{\nu}$ is geometric, Proposition 14.4.12 implies that the latter happens only for facets of the form $\Gamma_{\Sigma',I_\emptyset}$ where Σ' is degenerate. This can only occur in part (a) of Lemma 15.3.5.

Star Subdivisions. There is one type of wall crossing in the secondary fan that is very easy to describe.

Theorem 15.3.6. *Assume that* $\boldsymbol{\nu}$ *is geometric and let* $\Gamma_{\Sigma,I_\emptyset}$ *be a chamber of the secondary fan with* $I_\emptyset \neq \emptyset$. *If* $i \in I_\emptyset$, *then:*

(a) $\Gamma_{\Sigma,I_\emptyset\setminus\{i\}}$ *is a facet of* $\Gamma_{\Sigma,I_\emptyset}$ *and is a wall of the secondary fan.*

(b) *The chamber on the other side of the wall is* $\Gamma_{\Sigma',I_\emptyset\setminus\{i\}}$, *where* Σ' *is the star subdivision of* Σ *at the minimal generator of* $\mathrm{Cone}(\nu_i)\cap N$.

(c) *The exceptional locus of the toric morphism* $X_{\Sigma'} \to X_\Sigma$ *has codimension* 1.

Proof. $\Gamma_{\Sigma,I_\emptyset\setminus\{i\}}$ is a facet of $\Gamma_{\Sigma,I_\emptyset}$ by Lemma 15.3.5 and is not contained in the boundary of C_β since Σ is nondegenerate. Part (a) follows. Let Σ' be the star subdivision of Σ at the generator of $\mathrm{Cone}(\nu_i)\cap N$, as defined in §11.1. The proof of Proposition 15.1.6 shows that Σ' is simplicial with $X_{\Sigma'}$ semiprojective.

Since $\Gamma_{\Sigma,I_\emptyset}$ is a chamber, $j \mapsto \mathrm{Cone}(\nu_j)$ is a bijection $\{1,\dots,r\}\setminus I_\emptyset \simeq \Sigma(1)$ by Proposition 14.4.9. This extends to a bijection $\{1,\dots,r\}\setminus(I_\emptyset\setminus\{i\}) \simeq \Sigma'(1)$ since $\boldsymbol{\nu}$ is geometric, and it follows that $\Gamma_{\Sigma',I_\emptyset\setminus\{i\}}$ is a chamber by Proposition 14.4.9. Furthermore, $\Gamma_{\Sigma,I_\emptyset\setminus\{i\}}$ is a face of this chamber by Theorem 14.4.7. Thus

$$\Gamma_{\Sigma,I_\emptyset} \succeq \Gamma_{\Sigma,I_\emptyset\setminus\{i\}} \preceq \Gamma_{\Sigma',I_\emptyset\setminus\{i\}}.$$

The cone in the middle has codimension 1 (it is a facet of the chamber $\Gamma_{\Sigma,I_\emptyset}$) and hence must be the wall between the two chambers.

We get a birational toric morphism $\phi : X_{\Sigma'} \to X_\Sigma$ since Σ' is a refinement of Σ. To describe the exceptional locus, let $D_i \subseteq X_{\Sigma'}$ be the divisor corresponding to $\mathrm{Cone}(\nu_i) \in \Sigma'(1)$ and let $\gamma \in \Sigma$ be the smallest cone containing ν_i. Then $\phi(D_i) = V(\gamma)$ and ϕ induces an isomorphism $X_{\Sigma'}\setminus D_i \simeq X_\Sigma \setminus V(\gamma)$ by the definition of star subdivision. Since $\operatorname{codim} V(\gamma) \geq 2$, D_i is the exceptional locus. □

Types of Walls. Theorem 15.3.6 describes the walls of the secondary fan that arise from star subdivisions, which we call *divisorial walls* because of part (c) of the proposition. The remaining walls are the intersection of two GKZ chambers that satisfy part (a) of Lemma 15.3.5. We will denote such a wall by $\Gamma_{\Sigma_0,I_\emptyset}$, and the chambers on either side will be written

$$\Gamma_{\Sigma,I_\emptyset} \succeq \Gamma_{\Sigma_0,I_\emptyset} \preceq \Gamma_{\Sigma',I_\emptyset}. \tag{15.3.1}$$

These are the only GKZ cones containing $\Gamma_{\Sigma_0,I_\emptyset}$ since it is a wall. We call $\Gamma_{\Sigma_0,I_\emptyset}$ a *flipping wall*. This terminology will be justified below. Here is a preliminary result about flipping walls.

Lemma 15.3.7. *If* $\Gamma_{\Sigma_0,I_\emptyset}$ *is a flipping wall between* $\Gamma_{\Sigma_0,I_\emptyset}$ *and* $\Gamma_{\Sigma_0,I_\emptyset}$, *then:*

(a) Σ_0 *is not simplicial.*

(b) $\Sigma_0(1) = \Sigma(1) = \Sigma'(1)$.

(c) Σ_0 *is the coarsest common refinement of* Σ *and* Σ'.

(d) *The exceptional loci of the toric morphisms* $X_\Sigma \to X_{\Sigma_0}$ *and* $X_{\Sigma'} \to X_{\Sigma_0}$ *have codimension* ≥ 2.

Proof. We first prove part (b). Note $\Sigma_0(1) \subseteq \Sigma(1)$ since Σ refines Σ_0. Suppose that $\mathrm{Cone}(\nu_i) \in \Sigma(1) \setminus \Sigma_0(1)$. Then $\Gamma_{\Sigma_0,I_\emptyset} \preceq \Gamma_{\Sigma_0,I_\emptyset \cup \{i\}}$ by Theorem 14.4.7. This is impossible by (15.3.1), and part (b) follows easily.

For part (a), we have $\{1,\dots,r\} \setminus I_\emptyset \simeq \Sigma(1)$ by Proposition 14.4.9 since $\Gamma_{\Sigma,I_\emptyset}$ is a chamber. If Σ_0 were simplicial, then $\Sigma_0(1) = \Sigma(1)$ and Proposition 14.4.9 would imply that $\Gamma_{\Sigma_0,I_\emptyset}$ is a chamber, which is clearly impossible. Hence Σ_0 is not simplicial.

Part (c) is straightforward and is left to the reader (Exercise 15.3.2).

Since Σ is a refinement of Σ_0 satsifying $\Sigma_0(1) = \Sigma(1)$, the birational toric morphism $X_\Sigma \to X_{\Sigma_0}$ is an isomorphism in codimension 1 (see (15.1.9)). Hence the exceptional locus must have codimension ≥ 2. The same holds for $X_{\Sigma'} \to X_{\Sigma_0}$, and part (d) follows. □

Later we will give a careful description of the exceptional loci that occur in part (d) of Lemma 15.3.7. The terms "divisorial wall" and "flipping wall" are taken from the minimal model program. We will say more about this in §15.4.

The Geometry of Circuits. Our next task is to study a flipping wall given by (15.3.1). The starting point is that $\Gamma_{\Sigma_0,I_\emptyset} \preceq \Gamma_{\Sigma,I_\emptyset}$ comes from a facet of the nef cone $\mathrm{Nef}(X_\Sigma)$ by Lemma 15.3.5. This corresponds to an edge of the Mori cone $\overline{\mathrm{NE}}(X_\Sigma) \subseteq N_1(X_\Sigma)$, which by the toric cone theorem is generated by the class of an orbit closure $V(\tau)$ of a wall $\tau = \sigma \cap \sigma'$ of Σ.

Since Σ is simplicial, we can describe τ using the notation of Figure 17 from §6.4. Let $n = \dim N_\mathbb{R}$. After renumbering the ν_i's, we can assume that

$$\begin{aligned} \sigma &= \mathrm{Cone}(\nu_1,\dots,\nu_n)\\ \sigma' &= \mathrm{Cone}(\nu_2,\dots,\nu_{n+1}) \qquad (15.3.2)\\ \tau &= \mathrm{Cone}(\nu_2,\dots,\nu_n). \end{aligned}$$

The vectors $\nu_1,\dots,\nu_{n+1}$ are linearly dependent with a nontrivial linear relation

$$\sum_{i=1}^{n+1} b_i \nu_i = 0 \qquad (15.3.3)$$

that is unique up to multiplication by a nonzero constant. This relation played an important role in §6.4, where it was called a *wall relation* (see (6.4.4)).

Here we will consider the more general situation where $\nu_1,\dots,\nu_{n+1} \in N \simeq \mathbb{Z}^n$ are nonzero and span $N_\mathbb{R}$. A result of Reid [**236**] describes some simplicial fans associated to $\nu_1,\dots,\nu_{n+1}$. Our treatment is based on [**80**, Lem. 2.4.2]. Using the relation (15.3.3), we define the sets

$$J_- = \{i \mid b_i < 0\},\ J_0 = \{i \mid b_i = 0\},\ J_+ = \{i \mid b_i > 0\}.$$

The vectors ν_i for $i \in J_- \cup J_+$ form a *circuit* since they are linearly dependent but every proper subset is linearly independent. Its *orientation* is determined by J_-

and J_+. Multiplying (15.3.3) by a negative number switches J_- and J_+ and hence changes the orientation. Then define the following two sets of simplicial cones

$$\begin{aligned}\Sigma_- &= \{\sigma \mid \sigma \preceq \mathrm{Cone}(\nu_1,\ldots,\widehat{\nu_i},\ldots,\nu_{n+1}),\ i \in J_+\}\\ \Sigma_+ &= \{\sigma \mid \sigma \preceq \mathrm{Cone}(\nu_1,\ldots,\widehat{\nu_i},\ldots,\nu_{n+1}),\ i \in J_-\}.\end{aligned} \tag{15.3.4}$$

Let us give some examples of Σ_- and Σ_+.

Example 15.3.8. The rows of the matrix B from Example 15.2.2 are the points $\nu_1 = (2,1,1),\ \nu_2 = (2,-1,1),\ \nu_3 = (1,0,1),\ \nu_4 = (-2,1,1),\ \nu_5 = (-2,-1,1)$ in $\mathbb{R}^3$. Then:

- ν_1,ν_2,ν_3,ν_4 satisfy the relation $-\nu_1 - 2\nu_2 + 4\nu_3 - \nu_4 = 0$, so that

$$J_- = \{1,2,4\},\ J_0 = \emptyset,\ J_+ = \{3\}.$$

  Computing Σ_- and Σ_+, we obtain:

$$\Sigma_- \text{ has maximal cone } \mathrm{Cone}(\nu_1,\nu_2,\nu_4)$$

$$\Sigma_+ \text{ has maximal cones } \begin{cases}\mathrm{Cone}(\nu_2,\nu_3,\nu_4),\ \mathrm{Cone}(\nu_1,\nu_3,\nu_4),\\ \mathrm{Cone}(\nu_1,\nu_2,\nu_3).\end{cases}$$

  In Figure 12, Σ_- is a subfan of Σ_0 while Σ_+ is a subfan of Σ_1.

- ν_2,ν_3,ν_4,ν_5 satisfy the relation $-3\nu_2 + 4\nu_3 - 2\nu_4 + \nu_5 = 0$. Thus

$$J_- = \{2,4\},\ J_0 = \emptyset,\ J_+ = \{3,5\}.$$

  Computing Σ_+ and Σ_-, we obtain:

$$\Sigma_- \text{ has maximal cones } \mathrm{Cone}(\nu_2,\nu_4,\nu_5),\ \mathrm{Cone}(\nu_2,\nu_3,\nu_4)$$

$$\Sigma_+ \text{ has maximal cones } \mathrm{Cone}(\nu_3,\nu_4,\nu_5),\ \mathrm{Cone}(\nu_2,\nu_3,\nu_5).$$

  In Figure 13, Σ_- is a subfan of Σ_1 while Σ_+ is a subfan of Σ_2.

Thus the fan changes observed in Figures 12 and 13 can be explained in terms of replacing Σ_- with Σ_+. $\diamondsuit$

Here is a 4-dimensional example where things are more complicated.

Example 15.3.9. Consider the five points in $\mathbb{R}^4$ given by $\nu_1 = (0,-1,0,1),\ \nu_2 = (0,1,0,1),\ \nu_3 = (1,0,1,1),\ \nu_4 = (-1,0,1,1),\ \nu_5 = (0,0,-2,1)$. The relation is $-3\nu_1 - 3\nu_2 + 2\nu_3 + 2\nu_4 + 2\nu_5 = 0$, so

$$J_- = \{1,2\},\ J_0 = \emptyset,\ J_+ = \{3,4,5\}.$$

These points lie in the hyperplane $H \simeq \mathbb{R}^3$ where the last coordinate is 1. If we slice the fans Σ_- and Σ_+ with H, we get Figure 14 on the next page. For the fan Σ_-, the three maximal cones "spin about" the common face corresponding to J_- (the thick line segment on the left), and for Σ_+, the two maximal cones meet along the common face corresponding to J_+ (the shaded triangle on the right).

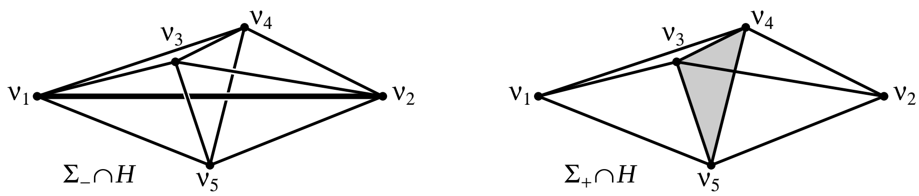


Figure 14. The intersections $\Sigma_- \cap H$ and $\Sigma_+ \cap H$

In Figure 14, going from Σ_- to Σ_+ is a "flip" where the three maximal cones of Σ_- that meet along $\mathrm{Cone}(\nu_1,\nu_2)$ are replaced with the two maximal cones of Σ_+ that meet along $\mathrm{Cone}(\nu_3,\nu_4,\nu_5)$. $\Diamond$

The fans Σ_- and Σ_+ defined in (15.3.4) have the following properties.

Lemma 15.3.10. *Let $\nu_1,\dots,\nu_{n+1} \in N \simeq \mathbb{Z}^n$ be nonzero, and assume that $\sigma_0 =$ $\mathrm{Cone}(\nu_1,\dots,\nu_{n+1})$ is strongly convex of full dimension. Then:*

(a) *J_- and J_+ are nonempty.*

(b) *Σ_- and Σ_+ are simplicial fans with support σ_0 such that $\Sigma_-(1) = \Sigma_+(1) =$ $\{\mathrm{Cone}(\nu_i) \mid 1 \le i \le n+1\}$. Furthermore, X_{Σ_-} and X_{Σ_+} are semiprojective.*

(c) *Σ_- and Σ_+ are the only fans with the properties listed in part* (b).

Proof. Note that $\nu_1,\dots,\nu_{n+1}$ span $N_{\mathbb{R}}$ since σ_0 has full dimension. We will first compute the Gale dual of $\nu_1,\dots,\nu_{n+1}$. The relation $\sum_{i=1}^{n+1} b_i\nu_i = 0$ gives the exact sequence

$$0 \longrightarrow M_{\mathbb{R}} \xrightarrow{\delta_{\mathbb{R}}} \mathbb{R}^{n+1} \xrightarrow{\gamma_{\mathbb{R}}} \mathbb{R} \longrightarrow 0,$$

where $\delta_{\mathbb{R}}(m) = (\langle m,\nu_1\rangle,\dots,\langle m,\nu_{n+1}\rangle)$ and $\gamma_{\mathbb{R}}(e_i) = b_i$. Thus the Gale dual is the vector of coefficients $\boldsymbol{\beta} = (b_1,\dots,b_{n+1})$.

Since $\mathrm{Cone}(\nu_1,\dots,\nu_{n+1})$ is strongly convex and the ν_i are nonzero, $C_{\boldsymbol{\beta}} = \mathbb{R}$ by Lemma 14.3.2. Part (a) follows immediately. Also, $C_{\boldsymbol{\beta}} = \mathbb{R}$ implies that the secondary fan has two chambers, hence there are precisely two fans satisfying the conditions of part (b). Thus part (c) will follow as soon as we prove part (b).

First take the GKZ cone $\mathbb{R}_{\ge 0} \subseteq \mathbb{R}$. We will use Proposition 15.2.1 to determine the corresponding fan. The $\boldsymbol{\beta}$-bases are the nonzero b_i's, and $\mathbb{R}_{\ge 0} \subseteq \mathrm{Cone}(b_i)$ if and only if $i \in J_+$. By Proposition 15.2.1, the maximal cones of the fan are $\mathrm{Cone}(\nu_1,\dots,\widehat{\nu_i},\dots,\nu_{n+1})$ for $i \in J_+$ (you should check this carefully). This gives Σ_-, which proves that Σ_- is a fan with the required properties. Similarly, applying Proposition 15.2.1 to $\mathbb{R}_{\le 0}$ shows that Σ_+ also satisfies part (b). $\square$

The assumption in Lemma 15.3.10 that $\mathrm{Cone}(\nu_1,\dots,\nu_{n+1})$ is strongly convex is needed to ensure that we have two fans. You will explore what happens when strong convexity fails in Exercise 15.3.4.

The proof of Lemma 15.3.10 showed that the secondary fan of $\nu_1,\dots,\nu_{n+1}$ is the complete fan in $\mathbb{R}$. Thus the origin is a wall between the two chambers. To describe the resulting wall crossing, we will use the following notation:

$$\sigma_J = \mathrm{Cone}(\nu_i \mid i \in J) \ \text{ for } \ J \subseteq \{1,\dots,n\}.$$

Then the fans Σ_- and Σ_+ from (15.3.4) can be written as

$$\begin{aligned} \Sigma_- &= \{\sigma_J \mid J_+ \not\subseteq J\} \\ \Sigma_+ &= \{\sigma_J \mid J_- \not\subseteq J\} \end{aligned} \tag{15.3.5}$$

(Exercise 15.3.5). Here is our result.

Lemma 15.3.11. *Let $\nu_1,\dots,\nu_{n+1} \in N \simeq \mathbb{Z}^n$ be nonzero, and assume that $\sigma_0 = \mathrm{Cone}(\nu_1,\dots,\nu_{n+1})$ is strongly convex of full dimension. Then:*

(a) *If $J_- = \{i\}$, then the origin in the secondary fan of $\nu_1,\dots,\nu_{n+1}$ is a divisorial wall. Furthermore, σ_0 is simplicial, Σ_+ consists of σ_0 and its faces, and Σ_- is the star subdivision of σ_0 at the minimal generator of $\mathrm{Cone}(\nu_i) \cap N$.*

(b) *If $J_+ = \{i\}$, then part (a) holds with the roles of $+$ and $-$ reversed.*

(c) *If J_- and J_+ have at least two elements, then the origin is a flipping wall. Furthermore, σ_0 is nonsimplicial, and the refinements Σ_- and Σ_+ of σ_0 give a commutative diagram of surjective toric morphisms*

$$\begin{tikzcd}
V(\sigma_{J_-}) \arrow[r, hook] \arrow[ddrr] & X_{\Sigma_-} \arrow[dr, "\phi_-"] & & X_{\Sigma_+} \arrow[dl, "\phi_+"'] & V(\sigma_{J_+}) \arrow[l, hook'] \arrow[ddll] \\
& & U_{\sigma_0} & & \\
& & V(\sigma_{J_- \cup J_+}) \arrow[u, hook] & &
\end{tikzcd}$$

such that $\phi_\pm$ is birational with exceptional locus $V(\sigma_{J_\pm})$. In addition,

$$\begin{aligned} \dim V(\sigma_{J_-}) &= |J_0| + |J_+| - 1 = n - |J_-| \\ \dim V(\sigma_{J_+}) &= |J_0| + |J_-| - 1 = n - |J_+| \\ \dim V(\sigma_{J_- \cup J_+}) &= |J_0|. \end{aligned}$$

Proof. If $J_- = \{i\}$, then $\nu_i \in \mathrm{Cone}(\nu_1,\dots,\widehat{\nu_i},\dots,\nu_{n+1})$, which implies that $\sigma_0 = \mathrm{Cone}(\nu_1,\dots,\widehat{\nu_i},\dots,\nu_{n+1})$, so σ_0 is simplicial, and (15.3.4) shows that σ_0 is the unique maximal cone of Σ_+. The relation (15.3.3) implies $-b_i\nu_i = \sum_{j\in J_+} b_j\nu_j$, where $b_i < 0$ and $b_j > 0$ for $j \in J_+$. Thus σ_{J_+} is the minimal cone of Σ_+ containing ν_i. Using the definition of star subdivison from §11.1, one can check that Σ_- is the star subdivison of Σ_+ at the generator of $\mathrm{Cone}(\nu_i) \cap N$ (Exercise 15.3.6). Parts (a) and (b) follow easily.

For part (c), suppose the origin is a divisorial wall. Then some ν_i is a nonnegative combination of ν_j for $j \neq i$. This implies $J_+ = \{i\}$ or $J_- = \{i\}$, a contradiction. Hence the origin is a flipping wall and σ_0 is nonsimplicial by Lemma 15.3.7.

To analyze $\phi_- : \Sigma_- \to U_{\sigma_0}$, first note that $\sigma_{J_-} \in X_{\Sigma_-}$ by (15.3.5). By the Orbit-Cone Correspondence, $X_{\Sigma_-} \setminus V(\sigma_{J_-})$ is the toric variety of the fan

$$\begin{aligned}\Sigma_- \setminus \{\sigma \in \Sigma_- \mid \sigma_{J_-} \preceq \sigma\} &= \Sigma_- \setminus \{\sigma_J \in \Sigma_- \mid J_- \subseteq J\}\\ &= \{\sigma_J \mid J_- \not\subseteq J,\ J_+ \not\subseteq J\},\end{aligned}\tag{15.3.6}$$

where the second equality uses (15.3.5). The next step is to describe the faces of σ_0. If we set $\boldsymbol{\nu} = (\nu_1, \dots, \nu_{n+1})$, then $\sigma_0 = C_{\boldsymbol{\nu}}$, so that we can use Lemma 14.3.3 to describe its faces. Given $J \subseteq \{1, \dots, n+1\}$, the lemma implies that there is a face $\gamma \preceq \sigma_0$ with $J = \{i \mid \nu_i \in \gamma\}$ if and only if there are $a_j > 0$ for $j \notin J$ such that $\sum_{j \notin J} a_j \nu_j = 0$. Using the partition $\{1, \dots, n+1\} = J_- \cup J_0 \cup J_+$, you will prove in Exercise 15.3.6 that σ_0 has two types of faces:

$$\begin{aligned}&\sigma_J \text{ for } J_- \cup J_+ \subseteq J \ \ (\text{all nonsimplicial})\\ &\sigma_J \text{ for } J_- \not\subseteq J,\ J_+ \not\subseteq J \ \ (\text{all simplicial}).\end{aligned}\tag{15.3.7}$$

The cones in the first line of (15.3.7) are the ones we need to remove to get the fan for $U_{\sigma_0} \setminus V(\sigma_{J_- \cup J_+})$, while the cones in the second line are the cones of (15.3.6). This proves that ϕ_- induces an isomorphism

$$X_{\Sigma_-} \setminus V(\sigma_{J_-}) \simeq U_{\sigma_0} \setminus V(\sigma_{J_- \cup J_+}).\tag{15.3.8}$$

Now we compute some dimensions. Since σ_{J_-} is simplicial, we have

$$\dim V(\sigma_{J_-}) = n - \dim \sigma_{J_-} = n - |J_-| = |J_0| + |J_+| - 1$$

since $n+1 = |J_+| = |J_0| + |J_-|$. The ν_i, $i \in J_- \cup J_+$, form a circuit, so that

$$\dim V(\sigma_{J_- \cup J_+}) = n - \dim \sigma_{J_- \cup J_+} = n - (|J_-| + |J_+| - 1) = |J_0|.$$

Then $\dim V(\sigma_{J_-}) > \dim V(\sigma_{J_+ \cup J_-})$ since $|J_+| \geq 2$. This and (15.3.8) show that $V(\sigma_{J_-})$ is the exceptional locus of ϕ_-. Note also that the exceptional locus has codimension $|J_-| \geq 2$.

We get the same picture for ϕ_+ with the roles of $-$ and $+$ reversed. This gives the diagram in the statement of the lemma and completes the proof of part (c). $\square$

The Flipping Theorem. Before stating our main result, we need an example of a more complicated wall crossing.

Example 15.3.12. Consider the six points in $\mathbb{R}^4$ given by $\nu_1 = (0,-1,0,1)$, $\nu_2 = (0,1,0,1)$, $\nu_3 = (1,0,1,1)$, $\nu_4 = (0,0,1,1)$, $\nu_5 = (0,0,0,1)$, $\nu_6 = (1,0,0,1)$, which lie in the hyperplane $H \simeq \mathbb{R}^3$ where the last coordinate is 1. Figure 15 on the next page shows the secondary fan in $\mathbb{R}^2$. There are four chambers, two divisorial walls, and two flipping walls. In Exercise 15.3.7 you will compute this secondary fan and the fan of each chamber.

We are most interested in the chambers $\Gamma_{\Sigma_1,\emptyset}$ and $\Gamma_{\Sigma_2,\emptyset}$, which meet along the flipping wall $\Gamma_{\Sigma_0,\emptyset}$ in Figure 15. Similar to what we did in Example 15.3.9, we visualize the fans Σ_1, Σ_2, and Σ_0 in $\mathbb{R}^4$ by slicing them with the hyperplane H to get the picture shown in Figure 16 on the next page.

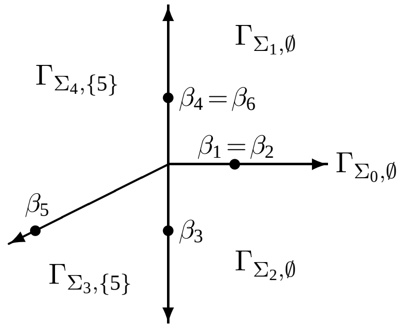


Figure 15. A secondary fan in $\mathbb{R}^2$ with two flipping walls

To understand Figure 16, first focus on the left-hand side of the intersections, which involve $\nu_1,\nu_3,\nu_4,\nu_5,\nu_6$. Using the relation $-\nu_3+\nu_4-\nu_5+\nu_6=0$, we get $J_- = \{3,5\}$ and $J_+ = \{4,6\}$. Then Lemma 15.3.11 explains the "left half" of the fans we see in Figure 16.

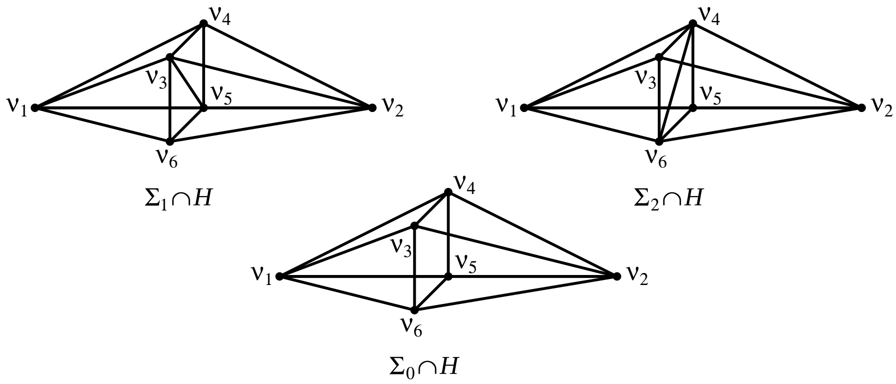


Figure 16. The intersections $\Sigma_1 \cap H$, $\Sigma_2 \cap H$, and $\Sigma_0 \cap H$

Now shift your focus to the right-hand side of the intersections in the figure, which involve $\nu_2,\nu_3,\nu_4,\nu_5,\nu_6$. Lemma 15.3.11 still applies, which explains the "right half" of each fan. The key observation is that both sides use the *same* relation $-\nu_3+\nu_4-\nu_5+\nu_6=0$. In other words, the wall crossing is completely determined by the single oriented circuit $-\nu_3+\nu_4-\nu_5+\nu_6=0$. $\Diamond$

We now return to the general situation of two chambers in the secondary fan separated by a flipping wall, which by (15.3.1) is written

$$\Gamma_{\Sigma,I_\emptyset} \succeq \Gamma_{\Sigma_0,I_\emptyset} \preceq \Gamma_{\Sigma',I_\emptyset}.$$

Recall that Σ_0 is not simplicial by Lemma 15.3.7.

Similar to what we did in Lemma 15.3.11, given $\boldsymbol{\nu} = (\nu_1, \dots, \nu_r)$, we set

$$\sigma_J = \mathrm{Cone}(\nu_i \mid i \in J) \ \text{ for } \ J \subseteq \{1, \dots, r\} \setminus I_\emptyset.$$

Then we have the following result.

Theorem 15.3.13. *Assume $\boldsymbol{\nu}$ is geometric and let $\Gamma_{\Sigma_0, I_\emptyset}$ be a flipping wall between chambers $\Gamma_{\Sigma, I_\emptyset}$ and $\Gamma_{\Sigma', I_\emptyset}$ of the secondary fan. Then there is an oriented circuit*

$$\sum_{i \in J_-} b_i \nu_i + \sum_{i \in J_+} b_i \nu_i = 0 \tag{15.3.9}$$

with $J_- \cup J_+ \subseteq \{1, \dots, r\} \setminus I_\emptyset$, $J_- \cap J_+ = \emptyset$, and $b_i < 0$ for $i \in J_-$, $b_i > 0$ for $i \in J_+$. Furthermore:

(a) *$\sigma_{J_-} \in \Sigma$, $\sigma_{J_+} \in \Sigma'$, and $\sigma_{J_- \cup J_+} \in \Sigma_0$.*

(b) *$X_{\Sigma_0} \setminus V(\sigma_{J_- \cup J_+})$ is the simplicial locus of X_{Σ_0}.*

(c) *The refinements Σ and Σ' of Σ_0 give a commutative diagram of surjective toric morphisms*

$$\begin{tikzcd} V(\sigma_{J_-}) \arrow[r, hook] \arrow[ddrr] & X_\Sigma \arrow[dr, "\phi"] & & X_{\Sigma'} \arrow[dl, "\phi'"'] & V(\sigma_{J_+}) \arrow[l, hook'] \arrow[ddll] \\ & & X_{\Sigma_0} & & \\ & & V(\sigma_{J_- \cup J_+}) \arrow[u, hook] & & \end{tikzcd}$$

such that ϕ and ϕ' are birational with exceptional loci $V(\sigma_{J_-})$ and $V(\sigma_{J_+})$. Also, $\operatorname{codim} V(\sigma_{J_-}) = |J_-| \geq 2$ *and* $\operatorname{codim} V(\sigma_{J_+}) = |J_+| \geq 2$.

(d) *If $\sigma_0 \in (\Sigma_0)_{\max}$ is nonsimplicial, then there is $J_0 \subseteq \{1, \dots, r\} \setminus I_\emptyset$ such that the disjoint union $J_- \cup J_0 \cup J_+$ has cardinality* $\dim \sigma_0 + 1$ *and satisfies $\sigma_0 = \sigma_{J_- \cup J_0 \cup J_+}$. Furthermore, $\Sigma|_{\sigma_0} = \Sigma_-$ and $\Sigma'|_{\sigma_0} = \Sigma_+$, so that restricting the diagram of part* (c) *to $U_{\sigma_0} \subseteq X_{\Sigma_0}$ gives the diagram of Lemma 15.3.11*

Proof. As noted in the discussion leading up to (15.3.3), $\Gamma_{\Sigma_0, I_\emptyset} \preceq \Gamma_{\Sigma, I_\emptyset}$ comes from a facet $\mathrm{Nef}(X_{\Sigma_0}) \preceq \mathrm{Nef}(X_\Sigma)$ by Lemma 15.3.5. This facet is defined by an extremal ray $\mathcal{R}$ of $\overline{\mathrm{NE}}(X_\Sigma) \subseteq N_1(X_\Sigma)$, which by the toric cone theorem is generated by the class of $V(\tau)$ for a wall τ of Σ.

Let $n = \dim N_\mathbb{R}$ and let $\sigma_0 \in \Sigma_0(n)$ be nonsimplicial.

Claim 1. If $\tau \in \Sigma(n-1)$ is any wall that meets the interior of σ_0, then the class $[V(\tau)]$ lies in the ray $\mathcal{R}$.

To prove this, let $D = \sum_\rho a_\rho D_\rho$ be a Cartier divisor on X_Σ whose class lies in the relative interior of $\mathrm{Nef}(X_{\Sigma_0}) \preceq \mathrm{Nef}(X_\Sigma)$. Note that the support function of D is linear on σ_0. If $\tau = \sigma \cap \sigma'$ is a wall of Σ that meets $\mathrm{Int}(\sigma_0)$, then the Cartier data of D satisfies $m_\sigma = m_{\sigma'}$ since σ and σ' lie in σ_0. Hence $D \cdot V(\tau) = 0$ by Proposition 6.3.8. Our hypothesis on D then implies that $[V(\tau)] \in \mathcal{R}$.

To construct the oriented circuit (15.3.9), let $\tau = \sigma \cap \sigma'$ be a wall of Σ that meets the interior of σ_0. We use the notation of (15.3.2), where the ν_i are renumbered so that τ is generated by $\nu_2, \ldots, \nu_n$, and σ (resp. σ') is obtained from τ by adding ν_1 (resp. ν_{n+1}). The linear relation satisfied by $\nu_1, \ldots, \nu_{n+1}$ can be written

$$\sum_{i=1}^{n+1} b_i \nu_i = 0, \quad b_1, b_{n+1} > 0. \tag{15.3.10}$$

Then (15.3.9) is defined to be the relation $\sum_{i \in J_- \cup J_+} b_i \nu_i = 0$, where

$$J_- = \big\{ i \in \{1, \ldots, n+1\} \mid b_i < 0 \big\},\ J_+ = \big\{ i \in \{1, \ldots, n+1\} \mid b_i > 0 \big\}.$$

Let D_i be the divisor corresponding to $\mathrm{Cone}(\nu_i) \in \Sigma(1)$ for $i \in \{1, \ldots, r\} \setminus I_\emptyset$. Then we can describe J_- and J_+ in terms of the intersections $D_i \cdot V(\tau)$ as follows:

$$\begin{aligned} J_- &= \{ i \mid D_i \cdot V(\tau) < 0\} \\ J_+ &= \{ i \mid D_i \cdot V(\tau) > 0\}. \end{aligned} \tag{15.3.11}$$

This follows from the intersection formulas of Proposition 6.4.4 since X_Σ is simplicial and $b_1, b_{n+1} > 0$ in (15.3.10). Here is an observation of Reid [**236**].

Claim 2. If $i \in J_+$, then $\sigma_i = \mathrm{Cone}(\nu_1, \ldots, \widehat{\nu_i}, \ldots, \nu_{n+1}) \in \Sigma$.

First note that $1, n+1 \in J_+$ since $b_1, b_{n+1} > 0$ in (15.3.10). For $i = 1, n+1$, the claim holds since $\sigma_1 = \sigma'$ and $\sigma_{n+1} = \sigma$, both of which lie in Σ. Now take $i \neq 1, n+1$ in J_+. Then $\gamma = \mathrm{Cone}(\nu_2, \ldots, \widehat{\nu_i}, \ldots, \nu_n)$ is a facet of τ and is in the subfan $\Sigma|_{\sigma_0}$ of Σ consisting of cones of Σ contained in σ_0. Let $N(\gamma)_\mathbb{R} \simeq \mathbb{R}^2$ be the quotient of $N_\mathbb{R}$ by $\mathrm{Span}(\gamma)$. By Proposition 3.2.7, $V(\gamma)$ is the toric variety of

$$\mathrm{Star}(\gamma) = \{ \overline{\sigma}_* \subseteq N(\gamma)_\mathbb{R} \mid \gamma \preceq \sigma_* \in \Sigma|_{\sigma_0} \},$$

where $u \in N_\mathbb{R}$ gives $\overline{u} \in N(\gamma)_\mathbb{R}$. Since $\Sigma|_{\sigma_0}$ has full dimensional convex support, the same is true for $\mathrm{Star}(\gamma)$. Also, $b_1 \overline{\nu}_1 + b_i \overline{\nu}_i + b_{n+1} \overline{\nu}_{n+1} = 0$ in $N(\gamma)_\mathbb{R}$. Since $b_1, b_i, b_{n+1} > 0$, we conclude that $\mathrm{Star}(\gamma)$ is complete.

Consider the ray $\overline{\tau}'$ of $\overline{\sigma}$ generated by $\overline{\nu}_1$. On one side of $\overline{\nu}_1$ we have $\overline{\sigma} = \mathrm{Cone}(\overline{\nu}_1, \overline{\nu}_i) \in \mathrm{Star}(\gamma)$ as shown in Figure 17 on the next page. By completeness, there must be $\gamma \preceq \sigma_* \in \Sigma|_{\sigma_0}$ such that $\overline{\sigma}_*$ is on the other side of the wall generated by $\overline{\nu}_1$. Hence there is $\ell \neq i$ such that $1 \le \ell \le r$ and $\overline{\sigma}_* = \mathrm{Cone}(\overline{\nu}_1, \overline{\nu}_\ell)$. Thus

$$\sigma_* = \gamma + \mathrm{Cone}(\nu_1) + \mathrm{Cone}(\nu_\ell) = \mathrm{Cone}(\nu_1, \nu_2, \ldots, \widehat{\nu_i}, \ldots, \nu_n, \nu_\ell) \in \Sigma.$$

Note that $\tau' = \sigma \cap \sigma_* = \mathrm{Cone}(\nu_1, \nu_2, \ldots, \widehat{\nu_i}, \ldots, \nu_n)$ is a wall of $\Sigma|_{\sigma_0}$.

If $\ell = n+1$, then $\sigma_* = \sigma_i$ and the claim follows. So suppose $\ell > n+1$. Then

$$\begin{aligned} D_\ell \cdot V(\tau') > 0 &\text{ by Proposition 6.4.4 since } \sigma_* = \tau' + \mathrm{Cone}(\nu_\ell) \\ D_\ell \cdot V(\tau) = 0 &\text{ by Proposition 6.4.4 since } \nu_\ell \notin \{\nu_1, \ldots, \nu_{n+1}\}. \end{aligned} \tag{15.3.12}$$

This is impossible since $[V(\tau)],\ [V(\tau')] \in \mathcal{R}$ by Claim 1, and Claim 2 is proved.

Claim 3. $\sigma_0 = \mathrm{Cone}(\nu_1, \ldots, \nu_{n+1})$ and $\Sigma|_{\sigma_0} = \Sigma_-$.

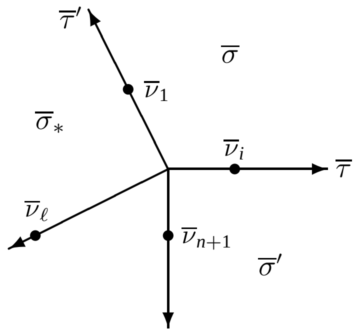


Figure 17. The fan $\mathrm{Star}(\gamma)$ in $N(\gamma)_\mathbb{R} \simeq \mathbb{R}^2$

By Lemma 15.3.10, $\mathrm{Cone}(\nu_1,\ldots,\nu_{n+1}) = \bigcup_{i\in J_+} \sigma_i \subseteq \sigma_0$, hence $\sigma_i \in \Sigma|_{\sigma_0}$ by Claim 2. Suppose $\Sigma|_{\sigma_0}$ contains a maximal cone different from the σ_i. Since σ_0 is convex, there must be a wall $\tau' = \sigma_i \cap \sigma_*$ where $\sigma_* \in \Sigma|_{\sigma_0}(n)$ is different from the σ_i. Writing $\sigma_* = \tau' + \mathrm{Cone}(\nu_\ell)$, we have $D_\ell \cdot V(\tau') > 0$ by the first line of (15.3.12). Then the second line of (15.3.12) and Claim 1 imply that ν_ℓ is one of $\nu_1,\ldots,\nu_{n+1}$. It follows that $\sigma_* \subseteq \mathrm{Cone}(\nu_1,\ldots,\nu_{n+1})$. This is impossible since $\mathrm{Cone}(\nu_1,\ldots,\nu_{n+1}) = \bigcup_{i\in J_-} \sigma_i$ and σ_* is different from the σ_i. Claim 3 follows.

We are now ready to prove the theorem. First note that Σ_0 is nonsimplicial and hence has a nonsimplicial maximal cone σ_0. By Claim 3 and Lemma 15.3.11, we have $\sigma_{J_-} \in \Sigma_- = \Sigma|_{\sigma_0} \subseteq \Sigma$ and $\sigma_{J_-\cup J_+} \preceq \sigma_0 \in \Sigma_0$.

Also observe that $-\mathcal{R}$ is the extremal ray defining $\mathrm{Nef}(X_{\Sigma_0}) \preceq \mathrm{Nef}(X_{\Sigma'})$ since the nef cones are on opposite sides of the wall defined by $\mathcal{R}$. Hence, if $\sigma_0 \in \Sigma_0(n)$ is nonsimplicial, then walls τ' of Σ' that meet the interior of σ_0 satisfy $[V(\tau')] \in -\mathcal{R}$, which is the Σ' version of Claim 1. The other claims modify in similar ways; in particular, the Σ' version of Claim 3 states that $\Sigma'|_{\sigma_0} = \Sigma_+$. Arguing as above, we get $\sigma_{J_+} \in \Sigma'$. This proves part (a).

For part (b), let σ_* be any nonsimplicial cone of Σ_0. It is contained in a nonsimplicial maximal cone σ_0. The above claims imply that Lemma 15.3.11 applies to σ_0. Then (15.3.7) from the proof of the lemma show that $\sigma_{J_-\cup J_+} \preceq \sigma_*$. Thus $\Sigma_0 \setminus \{\sigma_* \in \Sigma_0 \mid \sigma_{J_-\cup J_+} \preceq \sigma_*\}$ is the subfan of simplicial cones of Σ_0. It follows from the Orbit-Cone Correspondence that $X_{\Sigma_0} \setminus V(\sigma_{J_-\cup J_+})$ is the simplicial locus of X_{Σ_0}, proving part (b).

For part (d), let $\sigma_0 \in \Sigma_0(n)$ be nonsimplicial. By Claim 3, we have $\Sigma|_{\sigma_0} = \Sigma_-$ and $\Sigma'|_{\sigma_0} = \Sigma_+$. It follows that over U_{σ_0}, the diagram of part (c) becomes the diagram in Lemma 15.3.11. This proves part (d).

Finally, for (c), first observe that ϕ and ϕ' induce isomorphisms

$$X_\Sigma \setminus V(\sigma_{J_-}) \simeq X_{\Sigma_0} \setminus V(\sigma_{J_-\cup J_+}) \simeq X_{\Sigma'} \setminus V(\sigma_{J_+}). \tag{15.3.13}$$

To see why, recall that $\Sigma(1) = \Sigma_0(1) = \Sigma'(1)$ by Lemma 15.3.7. It follows that any simplicial cone of Σ_0 is a cone of Σ since Σ refines Σ_0 (Exercise 15.3.8). The same is true for Σ', and the isomorphisms (15.3.13) follow.

Hence, to study ϕ and ϕ', it suffices to work over U_{σ_0} for $\sigma_0 \in \Sigma_0(n)$ nonsimplicial. The key point is that *all* such σ_0 use the *same* relation (15.3.9). Then we are done by part (c) and Lemma 15.3.11. □

The maps ϕ and ϕ' from Theorem 15.3.13 give a birational map

$$\phi'^{-1} \circ \phi : X_\Sigma \dashrightarrow X_{\Sigma'} \tag{15.3.14}$$

called an *elementary flip*. This birational map is equivariant with respect to the torus actions on X_Σ and $X_{\Sigma'}$ and is an isomorphism in codimension 1.

In §15.4 we will say more about the maps ϕ and ϕ' when we discuss the toric minimal model program. We will show that they are the *extremal contractions* determined by the extremal rays $\mathcal{R} \subseteq \overline{\mathrm{NE}}(X_\Sigma)$ and $-\mathcal{R} \subseteq \overline{\mathrm{NE}}(X_{\Sigma'})$.

An Application. Any two toric varieties X_Σ and $X_{\Sigma'}$ of dimension n are birational since they both contain a copy of $(\mathbb{C}^*)^n$. However, if we require the birational map to be equivariant and an isomorphism in codimension 1, then a much nicer picture emerges.

Theorem 15.3.14. *Let X_Σ and $X_{\Sigma'}$ be simplicial semiprojective toric varieties and let $\psi : X_\Sigma \dashrightarrow X_{\Sigma'}$ be an equivariant birational map that is an isomorphism in codimension* 1*. Then ψ is a composition of elementary flips and a toric isomorphism.*

Proof. First note that ψ induces an isomorphism $\overline{\psi} : N \to N'$ since ψ is equivariant and birational, and $\overline{\psi}(\Sigma(1)) = \Sigma'(1)$ since it is an isomorphism is codimension 1. Changing $X_{\Sigma'}$ by a toric isomorphism, we may assume that Σ' is a fan in $N_\mathbb{R}$ with $\Sigma'(1) = \Sigma(1)$ and that ψ is the identity on the torus T_N.

Now consider the secondary fan where $\boldsymbol{\nu}$ consists of the minimal generators u_ρ for $\rho \in \Sigma(1) = \Sigma'(1)$. The GKZ cones $\Gamma_{\Sigma,\emptyset}$ and $\Gamma_{\Sigma',\emptyset}$ are chambers of Σ_{GKZ} by Proposition 14.4.9 and lie in the moving cone $\mathrm{Mov}_{\mathrm{GKZ}}$. This cone is convex by Proposition 15.1.4, so that we can find a line segment connecting interior points of $\Gamma_{\Sigma,\emptyset}$ and $\Gamma_{\Sigma',\emptyset}$ which meets all intermediate walls at points of their relative interior. This gives chambers

$$\Gamma_{\Sigma,\emptyset},\ \Gamma_{\Sigma_1,\emptyset},\ \dots\ ,\Gamma_{\Sigma_\ell,\emptyset},\ \Gamma_{\Sigma',\emptyset}$$

such that consecutive chambers share a common wall. These walls are flipping walls since $I_\emptyset = \emptyset$ on each side of the wall. It follows from Theorem 15.3.13 that we get a composition of elementary flips

$$\psi' : X_\Sigma \dashrightarrow X_{\Sigma_1} \dashrightarrow \ \cdots\ \dashrightarrow X_{\Sigma_1} \dashrightarrow X_{\Sigma'}.$$

Thus ψ' is an equivariant birational map that is an isomorphism in codimension 1 and is the identity on T_N.

The composed birational map $\varepsilon = \psi'^{-1} \circ \psi : X_\Sigma \dashrightarrow X_\Sigma$ is the identity on T_N and hence is the identity as a birational map. It follows that the birational maps ψ and ψ' are equal. $\square$

When $\boldsymbol{\nu}$ is primitive and lies in an affine hyperplane not containing the origin, Theorem 15.3.6, Lemmas 15.3.10 and 15.3.11 and Theorem 15.3.13 can all be interpreted in terms of triangulations. This is explained nicely in the book [**80**], though one caution is that in [**80**], the term "flip" applies to both divisorial walls and flipping walls.

The material in this section is based on [**194**, Ch. 14], [**222**], [**236**] and [**269**].

Exercises for §15.3.

15.3.1. Prove Lemma 15.3.5. Hint: The facets of a product cone are easy to describe, as are the facets of $\mathbb{R}^{I_\emptyset}_{\geq 0}$.

15.3.2. Prove part (c) of Lemma 15.3.7. Hint: Let $\widetilde{\Sigma}$ be a common refinement of Σ and Σ'. Prove that $\Gamma_{\widetilde{\Sigma},I_\emptyset}$ is a face of $\Gamma_{\Sigma_0,I_\emptyset}$.

15.3.3. Prove part (d) of Proposition 15.3.10.

15.3.4. Let $\nu_1,\ldots,\nu_{n+1} \in N$ span $N_\mathbb{R} \simeq \mathbb{R}^n$ and assume that $\sigma_0 = \mathrm{Cone}(\nu_1,\ldots,\nu_{n+1})$ is not strongly convex.

(a) Prove that J_+ or J_- is empty. For the rest of the exercise, assume that $J_- = \emptyset$.

(b) Prove that the secondary fan of $\nu_1,\ldots,\nu_{n+1}$ consists has only one chamber.

(c) Conclude that $\sigma_0 = \mathrm{Cone}(\nu_1,\ldots,\nu_{n+1})$ has a unique simplicial refinement Σ_- with $|\Sigma_-| = \sigma_0$ and X_{Σ_-} semiprojective. Also explain why the maximal cones of Σ_- are given by $\mathrm{Cone}(\nu_1,\ldots,\widehat{\nu_i},\ldots,\nu_{n+1})$ for $i \in J_+$.

(d) Prove that σ_{J_+} is the minimal face of σ_0 and that the quotient σ_0/σ_{J_+} is simplicial.

(e) Draw a picture of what happens when $n = 2$, $J_+ = \{1,3\}$, and $J_- = \emptyset$.

15.3.5. Prove that (15.3.5) follows from (15.3.4).

15.3.6. This exercise will cover some details needed for the proof of Lemma 15.3.11.

(a) In part (a) of the lemma, show carefully that Σ_- is the star division of Σ_+ at the generator of $\mathrm{Cone}(\nu_i) \cap N$ when $J_- = \{i\}$.

(b) Prove the description of the faces of σ_0 given in (15.3.7).

15.3.7. Compute the secondary fan shown in Figure 15.3.12. Also compute the fans $\Sigma_1,\ldots,\Sigma_4$ mentioned in the figure. Hint: Use Proposition 15.2.1.

15.3.8. Suppose that we have fans Σ_1 and Σ_2 in $N_\mathbb{R}$ such that $\Sigma_1(1) = \Sigma_2(1)$ and Σ_1 refines Σ_2. Prove that every simplicial cone of Σ_2 is contained in Σ_1.

15.3.9. In the situation of Lemma 15.3.11, prove that the following are equivalent:

(a) $\mathrm{Cone}(\nu_1,\ldots,\nu_{n+1})$ is simplicial.

(b) $\nu_i \in \mathrm{Cone}(\nu_1,\ldots,\widehat{\nu_i},\ldots,\nu_{n+1})$ for some i.

(c) J_- or J_+ consists of one element.

(d) Σ_- and Σ_+ are related by a star subdivision.

(e) $\Sigma_-(1) \neq \Sigma_+(1)$.

15.3.10. In the situation of Lemma 15.3.11, prove that the following are equivalent:

(a) $\mathrm{Cone}(\nu_1, \dots, \nu_{n+1})$ is not simplicial.

(b) J_- and J_+ have at least two elements.

(c) $\Sigma_-(1) = \Sigma_+(1)$.

15.3.11. In the situation of Lemma 15.3.11, assume that J_- and J_+ have at least two elements.

(a) Prove that J_+ is the unique primitive collection for Σ_-.

(b) Prove that J_- is the unique primitive collection for Σ_+.

Hence the elementary flip $X_{\Sigma_-} \dashrightarrow X_{\Sigma_+}$ interchanges the primitive collections.

§15.4. Extremal Contractions and Flips

The *minimal model progam*, called the MMP, is a first step toward the birational classification of projective varieties. In dimension 1, birationally equivalent smooth projective curves are isomorphic. This fails in dimension 2, where instead smooth surfaces are successive blowups of smooth minimal surfaces. Minimal surfaces play a key role in the Enriques-Kodaira classification of smooth projective surfaces.

In dimension ≥ 3, two complications arise. First, smooth minimal models need not exist. Instead, a *minimal model* is a normal $\mathbb{Q}$-Gorenstein projective variety with only terminal singularities (Definition 11.4.9) and whose canonical divisor is nef. The second complication is that constructing the minimal model of a smooth projective variety may require successive blowdowns of divisors together with *flips*, which are a type of birational map not present in dimension 2.

Brief Sketch of the MMP. The goal of the MMP is to show that if X is a $\mathbb{Q}$-factorial normal projective variety with only terminal singularities, then there is a sequence of blowdowns of divisors and flips $X \dashrightarrow X' \dashrightarrow \cdots \dashrightarrow Y$ such that either

- Y is a minimal model, or
- There is $f : Y \to Z$ such that $\dim Z < \dim Y$ and all curves $C \subseteq Y$ mapping to a point in Z satisfy $K_Y \cdot C < 0$. We say that f is a *Mori fibration*.

To apply the MMP to a variety X as above, we can assume that K_X is not nef. Then one can show the following:

- There is a curve $C \subseteq X$ such that $K_X \cdot C < 0$ and C generates an edge of the Mori cone of X, a so-called *extremal ray*.
- An extremal ray gives an *extremal contraction* $f : X \to X'$.
- The contraction f is of one of three types: a Mori fibration, a contraction of a divisor, or a *small contraction*, where the exceptional locus of f has codimension ≥ 2.

The type of f dictates the next step in the MMP. When f is a small contraction, this leads to one of the flips mentioned above. Proving termination of this process is a very difficult problem in general.

The MMP has been proved in dimension 3 by the work of Kawamata, Kollár, Mori, Reid, Shokurov, and others. Recently, minimal models have been proved to exist for varieties of general type of arbitrary dimension [**31**]. The MMP is an active area of research in algebraic geometry. See [**179**] or [**194**] for an introduction to the minimal model program.

The Toric MMP. When we apply the MMP to toric varieties and toric morphisms, we get the *toric minimal model progam*. In one sense, this is uninteresting, since a projective toric variety X_Σ is birationally equivalent to $\mathbb{P}^n$ and its canonical class $K_{X_\Sigma} = -\sum_\rho D_\rho$ can never be made nef, so the toric MMP will always end with a Mori fibration. Nevertheless, there are good reasons to study the toric MMP:

- Many MMP constructions are easy to describe in the toric setting. For instance, extremal contractions and flips are part of the rich geometry of wall crossings.
- There are wonderful toric examples of the individual steps in the MMP.
- When we switch from the MMP to the relative or log MMP in §15.5, we will see that there are nontrivial toric examples of these versions of the MMP.

The goal of this section is to introduce the toric versions of extremal contractions and flips. Then, in §15.5, we will discuss the toric MMP.

Extremal Contractions. We begin with the following result that shows how an extremal ray in the Mori cone gives an interesting toric morphism.

Proposition 15.4.1. *Let Σ be a simplicial fan in $N_\mathbb{R}$ such that X_Σ is semiprojective and let $\mathcal{R} \subseteq \overline{\mathrm{NE}}(X_\Sigma)$ be an extremal ray. Then there is a generalized fan Σ_0 in $N_\mathbb{R}$ with the following properties:*

(a) *Σ refines Σ_0 and X_{Σ_0} is semiprojective.*

(b) *The toric morphism $\phi : X_\Sigma \to X_{\Sigma_0}$ has the property that for every wall τ of Σ,*

$$\phi(V(\tau)) = \{\mathrm{pt}\} \iff [V(\tau)] \in \mathcal{R}.$$

Proof. Consider the secondary fan when $\boldsymbol{\nu}$ consists of the minimal generators u_ρ for $\rho \in \Sigma(1)$. Then $\Gamma_{\Sigma,\emptyset}$ is chamber isomorphic to $\mathrm{Nef}(X_\Sigma)$ by Proposition 15.1.3, so that $\mathcal{R}$ defines a facet of $\Gamma_{\Sigma,\emptyset}$. By Proposition 14.4.6 this facet equals $\Gamma_{\Sigma_0,\emptyset}$ where the generalized fan Σ_0 satisfies part (a) of the proposition.

Let φ be the support function of a Cartier divisor D on X_Σ whose class lies in the relative interior of $\Gamma_{\Sigma_0,\emptyset}$. Then φ is strictly convex with respect to Σ_0. If $\{m_\sigma\}_{\sigma\in\Sigma(n)}$, $n = \dim N_\mathbb{R}$, is the Cartier data of D, then $m_\sigma = m_{\sigma'}$ if and only if σ, σ' lie in the same cone of Σ_0. This follows from the proof of Lemma 14.4.6.

Now suppose that $\tau = \sigma \cap \sigma'$ is a wall of Σ. Then

$$\begin{aligned}\phi(V(\tau)) = \{\mathrm{pt}\} &\iff \tau \text{ meets the interior of some } \sigma_0 \in \Sigma_0(n)\\ &\iff m_\sigma = m_{\sigma'}\\ &\iff D \cdot V(\tau) = 0\\ &\iff [V(\tau)] \in \mathcal{R}.\end{aligned}$$

The first equivalence is easy and the second follows from the previous paragraph. The third equivalence follows from Proposition 6.3.8 plus standard arguments (see the proof of Proposition 6.3.15), and the fourth follows since D lies in the relative interior of the facet of $\mathrm{Nef}(X_\Sigma)$ defined by $\mathcal{R}$. Thus Σ_0 satisfies part (b). $\square$

When Σ is complete, one can give an elementary proof of Proposition 15.4.1 using Proposition 6.2.5 and Theorem 6.2.8 (Exercise 15.4.1).

The morphism $\phi : X_\Sigma \to X_{\Sigma_0}$ constructed in Proposition 15.4.1 is called an *extremal contraction*. It is of one of three types:

- Fibering: $\dim X_{\Sigma_0} < \dim X_\Sigma$.
- Divisorial: ϕ is birational and its exceptional locus is a divisor.
- Flipping: ϕ is birational and its exceptional locus codimension ≥ 2.

This follows from Theorem 15.3.6 and Lemma 15.3.7.

Structure of Extremal Contractions. Our next task is to describe the contraction $\phi : X_\Sigma \to X_{\Sigma_0}$ from Proposition 15.4.1. Since Σ is simplicial, the discussion of (15.1.4) shows that elements of $N_1(X_\Sigma)$ come from relations among the u_ρ.

Let $\mathcal{R} \subseteq \overline{\mathrm{NE}}(X_\Sigma)$ be an extremal ray as in Proposition 15.4.1 and suppose that $\mathcal{R}$ is generated by a relation

$$\textstyle\sum_\rho b_\rho u_\rho = 0.$$

We will construct a very simple toric variety $X_\mathcal{R}$ that contains a lot of information about the extremal contraction ϕ. The construction of $X_\mathcal{R}$ will use only $\Sigma(1)$ and $\sum_\rho b_\rho u_\rho = 0$. The other cones of Σ will play no role.

We begin by defining the sets

$$\begin{aligned}J_- &= J_{\mathcal{R},-} = \{\rho \in \Sigma(1) \mid b_\rho < 0\}\\ J_+ &= J_{\mathcal{R},+} = \{\rho \in \Sigma(1) \mid b_\rho > 0\},\end{aligned} \tag{15.4.1}$$

and for a subset $J \subseteq \Sigma(1)$, let

$$\sigma_J = \mathrm{Cone}(u_\rho \mid \rho \in J).$$

Then we have the sublattices

$$N_{\sigma_{J_-}} = \mathrm{Span}(\sigma_{J_-}) \cap N \subseteq N_{\sigma_{J_- \cup J_+}} = \mathrm{Span}(\sigma_{J_- \cup J_+}) \cap N \subseteq N,$$

and we set $N_\mathcal{R} = N_{\sigma_{J_- \cup J_+}}/N_{\sigma_{J_-}}$. The map $N_{\sigma_{J_- \cup J_+}} \to N_\mathcal{R}$ will be denoted $u \mapsto \overline{u}$.

Lemma 15.4.2. $\Sigma_{\mathcal{R}} = \{\overline{\sigma}_J \mid J \subsetneq J_+\}$ *is a complete simplicial fan in* $(N_{\mathcal{R}})_{\mathbb{R}}$ *whose toric variety* $X_{\mathcal{R}}$ *has the following properties:*

(a) $X_{\mathcal{R}}$ *is* $\mathbb{Q}$*-factorial and* $\mathbb{Q}$*-Fano.*

(b) $X_{\mathcal{R}}$ *has dimension* $|J_+| - 1$ *and Picard number* 1.

(c) $X_{\mathcal{R}}$ *is a finite abelian quotient of a weighted projective space.*

Furthermore, we always have $|J_+| \geq 2$.

Remark 15.4.3.

(a) A complete normal variety X is $\mathbb{Q}$-*Fano* if K_X is $\mathbb{Q}$-Cartier and some positive integer multiple of $-K_X$ is Cartier and ample. In the literature, $\mathbb{Q}$-Fano varieties with Picard number 1 are sometimes called *unipolar* [**63**].

(b) In the literature, the toric varieties of part (b) are often called *fake weighted projective spaces* [**60**], [**167**].

Proof. The toric cone theorem tells us that Σ has a wall $\tau = \sigma \cap \sigma'$ such that $[V(\tau)]$ generates $\mathcal{R}$. This means that $\mathcal{R}$ is generated by the wall relation (15.1.4), which we write as

$$\sum_\rho b_\rho u_\rho = 0, \quad b_\rho = D_\rho \cdot V(\tau).$$

Then $|J_+| \geq 2$ since the rays $\rho \in \sigma(1)$ and $\rho' \in \sigma'(1)$ opposite the common facet τ satisfy $D_\rho \cdot V(\tau) > 0$ and $D_{\rho'} \cdot V(\tau) > 0$ by Proposition 6.4.4. Also note that the $\overline{u}_\rho$ for $\rho \in J_+$ span $N_{\mathcal{R}}$ and satisfy $\sum_{\rho \in J_+} b_\rho \overline{u}_\rho = 0$. Furthermore, $\dim N_{\mathcal{R}} = |J_+| - 1$ since the u_ρ, $\rho \in J_- \cup J_+$, form a circuit.

As in the proof of Lemma 15.3.10, the Gale transform of $\overline{u}_\rho$, $\rho \in J_+$, consists of the positive numbers $b_\rho > 0$ for $\rho \in J_+$. It follows that $\mathbb{R}_{\geq 0}$ is the unique chamber of the secondary fan. Each b_ρ is a β-basis in the sense of Proposition 15.2.1, so that $\sigma_{J_+ \setminus \{\rho\}}$ is a maximal cone of the fan corresponding to the chamber. Thus $\Sigma_{\mathcal{R}}$ is a fan and $X_{\mathcal{R}}$ is projective. Furthermore, since $\mathrm{Pic}(X_{\mathcal{R}})_{\mathbb{R}} \simeq \mathbb{R}$, any effective divisor (e.g., $-K_{\mathcal{R}}$) has a positive integer multiple that is ample. Hence $X_{\mathcal{R}}$ is $\mathbb{Q}$-Fano. This proves parts (a) and (b), and then (c) follows from Exercise 5.1.13. $\square$

Example 15.4.4. Suppose that Σ is a fan in $\mathbb{R}^4$ with minimal generators $u_1 = e_1$, $u_2 = e_2$, $u_3 = e_3$, $u_4 = e_1 + e_2 + e_3 + 2e_4$, $u_5 = e_2 + e_3 - 2e_4$ and maximal cones $\mathrm{Cone}(u_2, u_3, u_4, u_5)$, $\mathrm{Cone}(u_1, u_3, u_4, u_5)$, and $\mathrm{Cone}(u_1, u_2, u_4, u_5)$. There are three walls, all with the same relation

$$u_1 + 2u_2 + 2u_3 - u_4 - u_5 = 0.$$

This relation generates an extremal wall $\mathcal{R}$ in $\overline{\mathrm{NE}}(X_\Sigma)$ where J_+ consists of the rays generated by u_1, u_2, u_3. Then $N_{\mathcal{R}} \simeq \mathbb{Z}^2$ and $\overline{u}_1, \overline{u}_2, \overline{u}_3 \in N_{\mathcal{R}}$ generate a sublattice of index 2 and satisfy $\overline{u}_1 + 2\overline{u}_2 + 2\overline{u}_3 = 0$ (Exercise 15.4.2). It follows that

$$X_{\mathcal{R}} \simeq \mathbb{P}(1,2,2)/(\mathbb{Z}/2\mathbb{Z}). \qquad \lozenge$$

Given an extremal ray $\mathcal{R}$, we have the extremal contraction

$$\phi : X_\Sigma \longrightarrow X_{\Sigma_0}$$

from Proposition 15.4.1. The behavior of ϕ is determined by J_- and J_+ from (15.4.1) and $X_{\mathcal{R}}$ from Proposition 15.4.2 as follows.

Proposition 15.4.5. *Let $\phi : X_\Sigma \to X_{\Sigma_0}$ come from the extremal ray $\mathcal{R}$ with J_-, J_+ and $X_{\mathcal{R}}$ as above, and recall that $|J_+| \geq 2$ by Lemma 15.4.2. Then:*

(a) *ϕ is fibering $\Leftrightarrow$ Σ_0 is a degenerate fan $\Leftrightarrow$ $J_- = \emptyset$. Here, X_{Σ_0} is simplicial, and the fibers of ϕ are isomorphic to $X_{\mathcal{R}}$.*

(b) *ϕ is divisorial $\Leftrightarrow$ Σ_0 is a simplicial fan $\Leftrightarrow$ $|J_-| = 1$. Here, the exceptional locus $V(\sigma_{J_-})$ is a divisor, and the fibers of $\phi|_{V(\sigma_{J_-})}$ are isomorphic to $X_{\mathcal{R}}$.*

(c) *ϕ is flipping $\Leftrightarrow$ Σ_0 is a nonsimplicial fan $\Leftrightarrow$ $|J_-| > 1$. Here, the exceptional locus $V(\sigma_{J_-})$ has codimension $|J_-|$, and the fibers of $\phi|_{V(\sigma_{J_-})}$ are isomorphic to $X_{\mathcal{R}}$.*

Proof. We begin with part (c). Pick a wall τ of Σ so that $[V(\tau)]$ generates $\mathcal{R}$, which defines a flipping wall of $\mathrm{Nef}(X_\Sigma)$. Then $\mathcal{R}$ comes from the relation (15.1.4) for $V(\tau)$, and J_- and J_+ are defined in (15.4.1). On the other hand, Theorem 15.3.13 also uses sets denoted J_- and J_+, which are the same as here by (15.3.11).

This gives the equivalences of part (c). Also, Theorem 15.3.13 implies that ϕ induces an isomorphism

$$X_\Sigma \setminus V(\sigma_{J_-}) \simeq X_{\Sigma_0} \setminus V(\sigma_{J_- \cup J_+}) \tag{15.4.2}$$

and that the exceptional locus $V(\sigma_{J_-})$ has codimension $|J_+|$. It follows that $\phi|_{V(\sigma_{J_-})}$ is the map

$$\psi = \phi|_{V(\sigma_{J_-})} : V(\sigma_{J_-}) \longrightarrow V(\sigma_{J_- \cup J_+}). \tag{15.4.3}$$

In the notation of Proposition 3.2.7, the fan $\mathrm{Star}(\sigma_{J_-})$ of $V(\sigma_{J_-})$ lives in $N(\sigma_{J_-})_{\mathbb{R}} = (N/N_{\sigma_{J_-}})_{\mathbb{R}}$, and the fan $\mathrm{Star}(\sigma_{J_- \cup J_+})$ of $V(\sigma_{J_- \cup J_+})$ lives in $N(\sigma_{J_- \cup J_+})_{\mathbb{R}}$, defined similarly. Using the snake lemma, one easily obtains the exact sequence

$$0 \longrightarrow N_{\mathcal{R}} \longrightarrow N(\sigma_{J_-}) \xrightarrow{\overline{\psi}} N(\sigma_{J_- \cup J_+}) \longrightarrow 0.$$

We will show that the $\mathrm{Star}(\sigma_{J_-})$ is weakly split by $\Sigma_{\mathcal{R}}$ and $\mathrm{Star}(\sigma_{J_- \cup J_+})$ in the sense of Exercise 3.3.7. This will imply that the fibers of ψ are isomorphic to $X_{\mathcal{R}}$.

The proof of Theorem 15.3.13 shows that for $J \subseteq \Sigma(1)$,

$$\sigma_{J_-} \preceq \sigma_J \in \Sigma \iff J_+ \not\subseteq J \text{ and } J_- \subseteq J \subseteq \sigma_0(1) \text{ for some } \sigma_0 \in \Sigma_0.$$

When we project these into $N(\sigma_{J_-})_{\mathbb{R}}$, the elements in J_- map to zero. Thus

$$\mathrm{Star}(\sigma_{J_-}) = \{\overline{\sigma}_K \mid J_+ \not\subseteq K \text{ and } K \subseteq \sigma_0(1) \setminus J_- \text{ for some } \sigma_0 \in \Sigma_0\}.$$

It follows that $\mathrm{Star}(\sigma_{J_-})$ has two subfans:

- The cones with $K \subseteq J_+$ give $\Sigma_{\mathcal{R}} = \{\overline{\sigma}_K \mid K \subsetneq J_+\}$.
- The cones with $K \cap J_+ = \emptyset$ give $\widehat{\Sigma} = \{\overline{\sigma}_K \mid K \subseteq \sigma_0(1) \setminus (J_- \cup J_+),\ \sigma_0 \in \Sigma_0\}$.

It is easy to see that every cone of $\mathrm{Star}(\sigma_{J_-})$ can be written uniquely as a sum $\overline{\sigma}_{K_1} + \overline{\sigma}_{K_2}$ where $\overline{\sigma}_{K_1} \in \Sigma_{\mathcal{R}}$ and $\overline{\sigma}_{K_2} \in \widehat{\Sigma}$. In Exercise 15.4.3 you will show that $\overline{\psi}_{\mathbb{R}}$ maps $\overline{\sigma}_K \in \widehat{\Sigma}$ bijectively to $\overline{\psi}_{\mathbb{R}}(\overline{\sigma}_K) \in \mathrm{Star}(\sigma_{J_- \cup J_+})$ such that $\overline{\sigma}_K \mapsto \overline{\psi}_{\mathbb{R}}(\overline{\sigma}_K)$ defines a bijection $\widehat{\Sigma} \xrightarrow{\sim} \mathrm{Star}(\sigma_{J_- \cup J_+})$. This gives the desired weak splitting, and part (c) follows.

For part (b), the equivalences follow easily. If $J_- = \{\rho\}$, then the results of §15.3 imply that $\rho \notin \Sigma_0(1)$ and that Σ is the star subdivision of Σ_0 at u_ρ. A relation $\sum_\rho b_\rho u_\rho = 0$ generating $\mathcal{R}$ can be chosen so that $b_\rho = -1$, hence

$$u_\rho = \sum_{\rho' \in J_+} b_{\rho'} u_{\rho'}, \quad b_{\rho'} > 0.$$

Thus σ_{J_+} is the minimal cone of Σ_0 containing u_ρ, and $\sigma_{J_- \cup J_+} = \sigma_{J_+}$. It follows that as in part (c), ϕ induces an isomorphism (15.4.2) and $\phi|_{\sigma_{J_-}}$ is (15.4.3). Then the analysis of part (c) shows that the fibers of $\phi|_{\sigma_{J_-}}$ are isomorphic to $X_{\mathcal{R}}$.

Finally, for part (a), suppose that $J_- = \emptyset$. Then $\mathcal{R}$ is generated by a relation

$$\sum_{\rho \in J_+} b_\rho u_\rho = 0, \quad b_\rho > 0. \tag{15.4.4}$$

Let σ_0 be a maximal cone of the degenerate fan Σ_0. Then there is at least one wall τ of Σ whose interior meets σ_0. The wall relation of τ must be (15.4.4). Then Claims 1 and 2 from the proof of Theorem 15.3.13 remain true in this situation, as does Claim 3, provided we interpret Σ_- as the fan constructed in Exercise 15.3.4.

By Exercise 15.3.4, σ_{J_+} is the minimal face of σ_0. Since σ_0 was an arbitrary maximal cone of Σ_0, it follows that σ_{J_+} is the minimal cone of the degenerate fan Σ_0. Exercise 15.3.4 also implies that σ_0/σ_{J_+} is simplicial, so that the genuine fan $\overline{\Sigma}_0$ constructed from Σ_0 is simplicial. Hence $X_{\Sigma_0} = X_{\overline{\Sigma}_0}$ is simplicial.

Since $J_- = \emptyset$ and σ_{J_+} is the minimal cone of Σ_0, we have $V(\sigma_{J_- \cup J_+}) = V(\sigma_{J_+}) = X_{\Sigma_0}$, and $V(\sigma_{J_-}) = X_\Sigma$ follows by a similar argument. Hence (15.4.3) is the map $\phi : X_\Sigma \to X_{\Sigma_0}$, and then the analysis of part (c) shows that the fibers of ϕ are isomorphic to $X_{\mathcal{R}}$. □

Example 15.4.6. Consider the complete fan Σ in $\mathbb{R}^3$ pictured in Figure 18 on the next page. The minimal generators of the rays $\rho_0, \ldots, \rho_4$ are

$$u_0 = (0,0,-1), u_1 = (0,-1,1), u_2 = (1,0,1), u_3 = (0,1,1), u_4 = (-1,0,1).$$

The wall $\tau = \mathrm{Cone}(u_2, u_4)$ gives an extremal ray $\mathcal{R} = \mathbb{R}_{\geq 0}[V(\tau)]$, whose extremal contraction is the toric morphism $X_\Sigma \to \mathbb{P}^1$ induced by projection onto the y-axis in Figure 18. The fibers of ϕ are isomorphic to $X_{\mathcal{R}} = \mathbb{P}(1,1,2)$. In Exercise 15.4.4

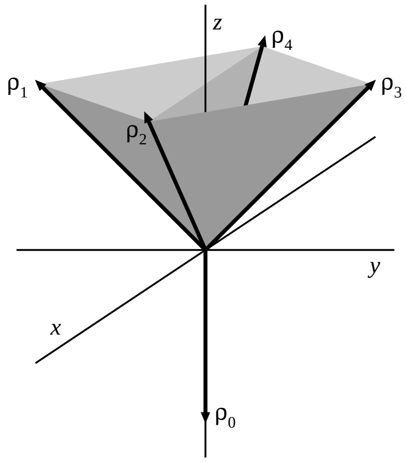


Figure 18. The fan Σ from Example 15.4.6

you will prove these claims and explain why Figure 19 from Example 6.4.12 is the secondary fan of this example. See also Example 15.4.12 below. $\Diamond$

Birational Transforms. Before discussing flips, we need to study how divisors move between birationally equivalent varieties. Suppose that $f : X \dashrightarrow X'$ is a birational map between normal varieties and let $U \subseteq X$ is the largest open subset of X where f is defined. A prime divisor $D \subseteq X$ gives a divisor on X' defined by

$$f_*D = \begin{cases} \overline{f(D\cap U)} & \text{if codim } \overline{f(D\cap U)} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the *birational transform by f* of a Weil divisor $D = \sum_i a_i D_i$ on X is the Weil divisor on X' defined by

$$f_*D = \textstyle\sum_i a_i f_*D_i.$$

The birational transform of a $\mathbb{Q}$-Weil divisor on X is defined similarly.

Example 15.4.7. Let $D = \sum_\rho a_\rho D_\rho$ be an effective torus-invariant divisor on a simplicial semiprojective toric variety X_Σ. By Theorem 15.1.10, the u_ρ for $\rho \in \Sigma(1)$ give a secondary fan where $\mathrm{Nef}(X_\Sigma)$ is the chamber $\Gamma_{\Sigma,\emptyset}$ and the class of D lies in the support C_β of the secondary fan. Hence there is a GKZ chamber with $[D] \in \Gamma_{\Sigma',I_\emptyset}$. Under the isomorphism

$$\Gamma_{\Sigma',I_\emptyset} \simeq \mathrm{Nef}(X_{\Sigma'}) \times \mathbb{R}^{I_\emptyset}$$

from Proposition 15.1.3, $[D]$ projects to the class $[D']$ of a nef divisor D' on $X_{\Sigma'}$. Since X_Σ and $X_{\Sigma'}$ have the same torus T_N, we get a birational map $\psi : X_\Sigma \dashrightarrow X_{\Sigma'}$. In Exercise 15.4.5 you will show that D' is the birational transform of D by ψ. $\Diamond$

This example shows that an effective divisor D on X_Σ becomes nef after a suitable birational transform. The toric MMP discussed in §15.5 will explain how to do this using only elementary flips and divisorial extremal contractions.

One has to be careful with birational transforms since they are sometimes not well-behaved. For example, Cartier divisors need not be preserved.

Example 15.4.8. Consider the toric varieties X_{Σ_-} and X_{Σ_+} from Example 15.3.9. The fans Σ_- and Σ_+ are pictured in Figure 14, and Lemma 15.3.11 shows that we have an elementary flip $X_{\Sigma_-} \dashrightarrow X_{\Sigma_+}$. In Exercise 15.4.6 you will show that

$$\mathrm{Cl}(X_{\Sigma_-}) = \mathrm{Cl}(X_{\Sigma_+}) \simeq \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}),$$

where

$$[\textstyle\sum_{i=1}^5 a_i D_i] = [\sum_{i=1}^5 a_i D_i'] \mapsto (3a_1 + 3a_2 - 2a_3 - 2a_4 - 2a_5, a_3 + a_4 \bmod 2).$$

Furthermore, you will show that the Picard groups do *not* have the same image in $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$:

$$\begin{aligned}\mathrm{Pic}(X_{\Sigma_-}) &= \mathbb{Z}[2D_5], \ [2D_5] \mapsto (-4, 0 \bmod 2)\\ \mathrm{Pic}(X_{\Sigma_+}) &= \mathbb{Z}[D_1], \quad [D_1] \ \mapsto \ (3, 0 \bmod 2).\end{aligned}$$

So $2D_5$ is Cartier on X_{Σ_-} but its birational transform $2D_5'$ is not Cartier on X_{Σ_+}. ◊

Another problem with birational transforms is that they are not functorial.

Example 15.4.9. The blowup of a smooth point p of a variety Y of dimension $n > 1$ gives a birational morphism $f : X \to Y$ whose exceptional locus E is a divisor. Then $f^{-1} : Y \dashrightarrow X$ is also birational. However,

$$f_*^{-1} f_* E = f_*^{-1} 0 = 0, \text{ yet } (f^{-1} \circ f)_* E = E,$$

since the largest open set where $f^{-1} \circ f$ is defined is all of X. ◊

Things are much nicer if we work with special types of birational maps.

Lemma 15.4.10. *Let $f : X \dashrightarrow X'$ be a birational map between normal varieties that is either an isomorphism in codimension* 1 *or a proper morphism defined on all of X. Then:*

(a) *If $D_1 \sim D_2$ on X, then $f_* D_1 \sim f_* D_2$ on X'.*

(b) *If $g : X' \dashrightarrow X''$ is birational of one of the same two types, then $g_* f_* = (g \circ f)_*$.*

Proof. Part (a) is easy if f is an isomorphism in codimension 1 (Exercise 15.4.7). So assume that $f : X \to X'$ is proper and birational. We claim that the birational transform is the same as the push-forward defined in [**107**, Sec. 1.4]. Once we prove this, then part (a) for f will follow from [**107**, Thm. 1.4].

If $D \subseteq X$ is a prime divisor, then $f(D)$ is closed in X'. The push-forward of D is

$$f_*D = \begin{cases} [\mathbb{C}(D) \colon \mathbb{C}(f(D))]\, f(D) & \text{if } \operatorname{codim} f(D) = 1 \\ 0 & \text{otherwise}, \end{cases} \tag{15.4.5}$$

where $[\mathbb{C}(D) \colon \mathbb{C}(f(D))]$ is the degree of the field extension $\mathbb{C}(f(D)) \subseteq \mathbb{C}(D)$. However, $f^{-1} : X' \dashrightarrow X$ is defined outside of a set of codimension ≥ 2. This is proved in [**131**, Lem. V.5.1] when X, X' are projective, but the same proof works when $f : X \to X'$ is proper and birational. In particular, f^{-1} is defined on an open subset of $f(D)$, so that $f|_D : D \to f(D)$ is birational. Hence the degree is 1, which shows that (15.4.5) agrees with the birational transform defined above.

Part (b) is straightforward and is left to the reader (Exercise 15.4.7). □

Note that Lemma 15.4.10 applies to elementary flips and divisorial extremal contractions. This will be important in our discussion of the toric MMP.

If $\psi : X \dashrightarrow X'$ is an isomorphism in codimension 1, the birational transform gives an isomorphism $\operatorname{Div}(X) \simeq \operatorname{Div}(X')$ that preserves linear equivalence and hence induces an isomorphism $\operatorname{Cl}(X) \simeq \operatorname{Cl}(X')$. It follows that if X_Σ and $X_{\Sigma'}$ are simplicial semiprojective toric varieties and $\psi : X_\Sigma \dashrightarrow X_{\Sigma'}$ is equivariant and an isomorphism in codimension 1, then ψ induces isomorphisms

$$\operatorname{Pic}(X_\Sigma)_\mathbb{R} \simeq \operatorname{Pic}(X_{\Sigma'})_\mathbb{R} \ \text{ and } \ N_1(X_\Sigma) \simeq N_1(X_{\Sigma'}).$$

The left isomorphism was used implicitly in §15.1 and §15.3. Also note that a ray $\mathcal{R}$ in $N_1(X_\Sigma)$ gives a ray in $N_1(X_{\Sigma'})$, which will be denoted by $\mathcal{R}$.

Existence of Toric Flips. A *flipping contraction* is a birational extremal contractions whose exceptional locus has codimension ≥ 2. An important part of the MMP is proving the existence of flips for flipping contractions. Our previous results make this easy to do in the toric setting.

Theorem 15.4.11. *Let X_Σ be a simplicial semiprojective toric variety and let $\mathcal{R}$ be an extremal ray that gives a flipping contraction $\phi : X_\Sigma \to X_{\Sigma_0}$. Then there is a commutative diagram of toric maps*

$$\begin{tikzcd} X_\Sigma \arrow[rr, dashed, "\psi"] \arrow[dr, "\phi"'] & & X_{\Sigma'} \arrow[dl, "\phi'"] \\ & X_{\Sigma_0} & \end{tikzcd}$$

with the following properties:

(a) *The birational map ψ is an elementary flip in the sense of* (15.3.14) *and in particular is equivariant and an isomorphism in codimension* 1.

(b) *$X_{\Sigma'}$ is simplicial and semiprojective.*

(c) *$\mathcal{R}' = -\mathcal{R}$ is an extremal ray for $X_{\Sigma'}$.*

(d) *ϕ' is a flipping extremal contraction for $\mathcal{R}'$.*

Proof. As in the proof of Proposition 15.4.1, Σ gives a chamber $\Gamma_{\Sigma,\emptyset} \simeq \mathrm{Nef}(X_\Sigma)$ in the secondary fan. The extremal ray $\mathcal{R}$ defines a flipping wall $\Gamma_{\Sigma_0,\emptyset}$, which has a chamber $\Gamma_{\Sigma',\emptyset} \simeq \mathrm{Nef}(X_{\Sigma'})$ on the other side. Then the diagram of the theorem follows from Theorem 15.3.13, and the proof of the theorem shows that ϕ' is the extremal contraction of $X_{\Sigma'}$ for the extremal ray $\mathcal{R}' = -\mathcal{R}$. □

In the situation of Theorem 15.4.11, we say that $\phi' : X_{\Sigma'} \to X_{\Sigma_0}$ is the *flip* of $\phi : X_\Sigma \to X_{\Sigma_0}$. Here is an example.

Example 15.4.12. Consider the toric variety X_Σ from Example 15.4.6. The left side of Figure 19 shows the secondary fan, whose chambers are the nef cones of the toric varieties X_Σ and $X_{\Sigma'}$. Here, $\beta_i = [D_i]$, and Σ' is obtained from Σ by replacing $\mathrm{Cone}(u_2,u_4)$ with $\mathrm{Cone}(u_1,u_3)$ in Figure 18 of Example 15.4.6. The right side of Figure 19 shows the corresponding Mori cones.

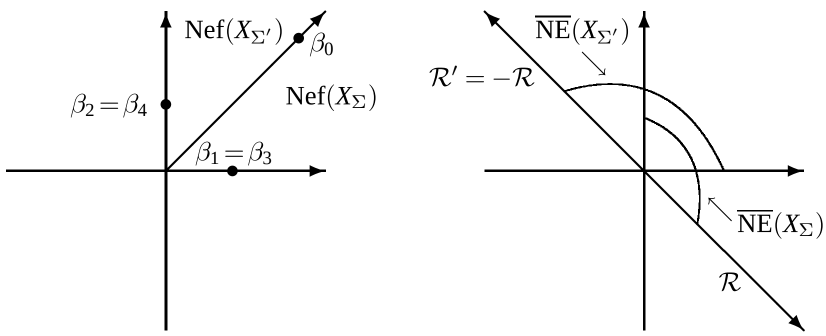


Figure 19. Extremal rays $\mathcal{R}$ for X_Σ and $\mathcal{R}' = -\mathcal{R}$ for $X_{\Sigma'}$

The ray generated by β_0 on the left side of Figure 19 is a flipping wall between X_Σ and $X_{\Sigma'}$. This flipping wall is defined by the extremal rays $\mathcal{R}$ for X_Σ and $\mathcal{R}' = -\mathcal{R}$ for $X_{\Sigma'}$, as pictured on the right side of the figure. ◊

Exercises for §15.4.

15.4.1. Here you will give an alternative proof of Proposition 15.4.1.

(a) Assume Σ is complete and let $\mathcal{R}$ be an extremal ray. Pick a divisor class $[D]$ in the relative interior of the facet of $\mathrm{Nef}(X_\Sigma)$ corresponding to $\mathcal{R}$. Use Proposition 6.2.5 and Theorem 6.2.8 to give an elementary proof of Proposition 15.4.1 in this case.

(b) Show that Proposition 6.2.5 and Theorem 6.2.8 apply when Σ has full dimensional convex support. This gives an elementary proof of Proposition 15.4.1 in general.

(c) Explain how Proposition 15.4.1 relates to to Examples 14.2.11 and 14.2.14.

Theorem 4.1 of [**104**] gives a yet different approach to Proposition 15.4.1 that avoids fans.

15.4.2. Verify the details of Example 15.4.4.

15.4.3. Consider part (c) of Proposition 15.4.5. Prove that $\overline{\psi}_{\mathbb{R}}$ maps $\overline{\sigma}_K \in \widehat{\Sigma}$ bijectively to $\overline{\psi}_{\mathbb{R}}(\overline{\sigma}_K) \in \mathrm{Star}(\sigma_{J_- \cup J_+})$ such that $\overline{\sigma}_K \mapsto \overline{\psi}_{\mathbb{R}}(\overline{\sigma}_K)$ gives a bijection $\widehat{\Sigma} \xrightarrow{\sim} \mathrm{Star}(\sigma_{J_- \cup J_+})$.

15.4.4. In Example 15.4.6, show that $X_{\mathcal{R}} \simeq \mathbb{P}(1,1,2)$ and explain how the secondary fan relates to Figure 19 of Example 6.4.12.

15.4.5. Prove the claim made in Example 15.4.7 that $D' = \psi_* D$.

15.4.6. Verify the details of Example 15.4.8.

15.4.7. In part (b) of Exercise 15.4.10, there are four cases to consider. When f and g are both proper birational morphisms, $g_* f_* = (g \circ f)_*$ follows from [**107**, Sec. 1.4]. Prove the remaining three cases.

15.4.8. Let X_Σ be a simplicial semiprojective toric variety. Recall that $P \subseteq \Sigma(1)$ is a primitive collection if P is not contained in $\sigma(1)$ for all $\sigma \in \Sigma$ but any proper subset is.

(a) Show that the definition of primitive relation given in Definition 6.4.10 applies to X_Σ. This gives an element $r(P) \in N_1(X_\Sigma)$ as explained in (6.4.8)

(b) Prove that $r(P) \in \overline{\mathrm{NE}}(X_\Sigma)$. Hint: Use (6.4.9).

(c) Let $\mathcal{R}$ be a flipping extremal ray. Use Lemma 15.3.11 and Theorem 15.3.13 to show that the rays generated by the u_i for $i \in J_+$ form a primitive collection whose primitive relation generates $\mathcal{R}$.

(d) Prove a similar result for fibering and divisorial extremal rays. Hint: See the proof of Proposition 15.4.5.

(e) Conclude that $\overline{\mathrm{NE}}(X_\Sigma) = \sum_P \mathbb{R}_{\geq 0} r(P)$, where the sum is over all primitive collections P of Σ.

This shows that Theorem 6.4.11 holds for simplicial semiprojective toric varieties.

§15.5. The Toric Minimal Model Program

Our discussion of the toric MMP is based on the papers [**104**] and [**236**] and the books [**179**] and [**194**]. As mentioned at the beginning of §15.4, the MMP starts with a $\mathbb{Q}$-factorial variety X with only terminal singularities and tries to make K_X nef using divisorial extremal contractions and flips. There are also relative, log, and equivariant versions of the MMP described in [**179**, Sec. 2.2]. We will see that all of these are relevant to the toric case.

Before giving the toric MMP, we first consider the extremal contractions and flips constructed in §15.4 from the point of view of the MMP.

Toric Flips via Proj. We will use the following terminology:

- Given a ray $\mathcal{R}$ in $N_1(X_\Sigma)$ and a $\mathbb{Q}$-Cartier divisor D, we write $D \cdot \mathcal{R}$ to mean $D \cdot C$ for any generator $[C] \in \mathcal{R}$. This is well-defined up to a positive constant.
- A ray $\mathcal{R}$ is *D-negative* if $D \cdot \mathcal{R} < 0$
- A ϕ-ample Cartier divisor was defined in Definition 7.2.5. Then a $\mathbb{Q}$-Cartier divisor D is *ϕ-ample* if a positive integer multiple is a ϕ-ample Cartier divisor.

Here is a different way to think about a toric flip.

Proposition 15.5.1. *Let $\phi' : X_{\Sigma'} \to X_{\Sigma_0}$ be the flip of $\phi : X_\Sigma \to X_{\Sigma_0}$ for the extremal ray $\mathcal{R}$ of flipping type as in Theorem 15.4.11. If D is a divisor on X_Σ with birational transform D' on $X_{\Sigma'}$, then*

$$D \text{ is } \phi\text{-ample} \iff D\cdot\mathcal{R} > 0 \iff -D' \text{ is } \phi'\text{-ample.}$$

Furthermore, when D is ϕ-ample, we have isomorphisms

$$\begin{aligned} X_\Sigma &\simeq \operatorname{Proj}\big(\textstyle\bigoplus_{\ell=0}^\infty \phi_*\mathscr{O}_{X_\Sigma}(\ell D)\big)\\ X_{\Sigma'} &\simeq \operatorname{Proj}\big(\textstyle\bigoplus_{\ell=0}^\infty \phi_*\mathscr{O}_{X_\Sigma}(-\ell D)\big).\end{aligned}$$

Proof. By Theorem 7.2.11, a Cartier divisor on X_Σ is ϕ-ample if and only if for every maximal cone $\sigma_0 \in \Sigma_0$, its support function is strictly convex on the fan $\Sigma|_{\sigma_0} = \{\sigma \in \Sigma \mid \sigma \subseteq \sigma_0\}$. The proof generalizes to $\mathbb{Q}$-Cartier divisors and hence applies to D since Σ is simplicial.

Assume $D\cdot\mathcal{R} > 0$ and take $\sigma_0 \in (\Sigma_0)_{\max}$. If $\sigma_0 \in \Sigma$, then the support function φ_D is clearly strictly convex on $\Sigma|_{\sigma_0}$. When $\sigma_0 \notin \Sigma$, let $\tau = \sigma\cap\sigma'$ be a wall of Σ that meets the interior of σ_0. Then ϕ maps $V(\tau)$ to a point, so $[V(\tau)] \in \mathcal{R}$. Hence

$$D\cdot V(\tau) > 0.$$

However, picking $u \in \sigma'\cap N$ as in Proposition 6.3.8 implies

$$D\cdot V(\tau) = \langle m_\sigma - m_{\sigma'}, u\rangle$$

where $m_\sigma, m_{\sigma'}$ are the Cartier data of D for σ, σ'. Since $u \in \sigma'$, we obtain

$$\varphi_D(u) = \langle m_{\sigma'}, u\rangle < \langle m_\sigma, u\rangle.$$

Hence φ_D is strictly convex on $\Sigma|_{\sigma_0}$ by Lemma 6.1.13, proving that D is ϕ-ample. The other direction is straightforward, so D is ϕ-ample if and only if $D\cdot\mathcal{R} > 0$.

Since ϕ' is the contraction associated to the extremal ray $\mathcal{R}'$, the equivalence just proved implies that $-D'\cdot\mathcal{R}' > 0$ if and only if $-D'$ is ϕ'-ample. Since $\mathcal{R}' = -\mathcal{R}$, the latter is equivalent to $D\cdot\mathcal{R} > 0$. This gives the desired equivalences.

Now assume that D is ϕ-ample on X_Σ. In Exercise 15.5.1 you will give a toric proof of $X_\Sigma \simeq \operatorname{Proj}(\bigoplus_{\ell=0}^\infty \phi_*\mathscr{O}_{X_\Sigma}(\ell D))$. Furthermore, $-D'$ is ϕ'-ample on $X_{\Sigma'}$, so

$$X_{\Sigma'} \simeq \operatorname{Proj}\big(\textstyle\bigoplus_{\ell=0}^\infty \phi'_*\mathscr{O}_{X_\Sigma}(-\ell D')\big).$$

However, $\phi : X_\Sigma \to X_{\Sigma_0}$ and $\phi' : X_{\Sigma'} \to X_{\Sigma_0}$ are isomorphisms in codimension 1. This means that the divisors D on X_Σ and D' on $X_{\Sigma'}$ have the same birational transform D_0 on X_{Σ_0}. Then it is straightforward to show that

$$\phi_*\mathscr{O}_{X_\Sigma}(D) \simeq \mathscr{O}_{X_{\Sigma_0}}(D_0) \simeq \phi'_*\mathscr{O}_{X_\Sigma}(D') \tag{15.5.1}$$

(Exercise 15.5.1), and $X_{\Sigma'} \simeq \operatorname{Proj}(\bigoplus_{\ell=0}^\infty \phi_*\mathscr{O}_{X_\Sigma}(-\ell D))$ follows easily. $\square$

The Canonical Divisor and Terminal Singularities. Suppose we apply the MMP to a toric variety X_Σ. Following strategy outlined at the beginning of §15.4, we start with a K_{X_Σ}-negative extremal ray $\mathcal{R}$ (i.e., $K_{X_\Sigma} \cdot \mathcal{R} < 0$). If $\mathcal{R}$ is of flipping type, then applying Proposition 15.5.1 with $D = -K_{X_\Sigma}$ gives a nice description of the resulting flip.

Proposition 15.5.2. *Suppose that $\phi : X_\Sigma \to X_{\Sigma_0}$ is a flipping contraction for the extremal ray $\mathcal{R}$ satisfying $K_{X_\Sigma} \cdot \mathcal{R} < 0$. Then the flip of X_Σ is given by*

$$X_{\Sigma'} \simeq \mathrm{Proj}\big(\textstyle\bigoplus_{\ell=0}^\infty \phi_* \mathscr{O}_{X_\Sigma}(\ell K_{X_\Sigma})\big). \qquad \square$$

In the MMP, the flip of a flipping extremal contraction $f : X \to Y$ satisfying $K_X \cdot \mathcal{R} < 0$ is constructed via Proj as in Proposition 15.5.2. However, in order to apply Proj, one has to know that $\bigoplus_{\ell=0}^\infty \phi_* \mathscr{O}_X(\ell K_X)$ is finitely generated. This is hard in general [**31**] but easy in the toric context (Exercise 15.5.1). Fujino and Sato [**104**, Thm. 4.5] use the latter to give a quick proof of the existence of toric flips.

Another feature of the MMP is that it applies to $\mathbb{Q}$-factorial varieties with only terminal singularities. A key step is showing that divisorial extremal contractions and flips preserves these singularities. In the toric case, we prove this as follows.

Proposition 15.5.3. *Let $\phi : X_\Sigma \to X_{\Sigma_0}$ be the extremal contraction for the extremal ray $\mathcal{R}$. If $\mathcal{R}$ is K_{X_Σ}-negative and X_Σ has only terminal singularities, then:*

(a) *If ϕ is divisorial, then X_{Σ_0} has only terminal singularities and $\phi_* K_{X_\Sigma} = K_{X_{\Sigma_0}}$.*

(b) *If ϕ is flipping with flip $\psi : X_\Sigma \dashrightarrow X_{\Sigma'}$, then $X_{\Sigma'}$ has only terminal singularities and $\psi_* K_{X_\Sigma} = K_{X_{\Sigma'}}$.*

Proof. We studied terminal singularities in §11.4. Given a simplicial cone $\sigma \subseteq N_\mathbb{R}$, recall from Proposition 11.4.12 that U_σ has only terminal singularities if and only if the only σ is terminal, meaning that lattice points of $\Pi_\sigma = \mathrm{Conv}(0, u_\rho \mid \rho \in \sigma(1))$ are its vertices.

Now take a wall $\tau = \sigma \cap \sigma'$ of Σ such that $[V(\tau)]$ generates $\mathcal{R}$ and set $n = \dim N_\mathbb{R}$. Similar to (15.3.2), we label the minimal generators of σ and σ' so that

$$\begin{aligned}\sigma &= \mathrm{Cone}(u_1, \dots, u_n)\\ \sigma' &= \mathrm{Cone}(u_2, \dots, u_{n+1})\\ \tau &= \mathrm{Cone}(u_2, \dots, u_n).\end{aligned}$$

Then $[V(\tau)]$ is represented by the wall relation of the form $\sum_{i=1}^{n+1} b_i u_i = 0$ with $b_1, b_{n+1} > 0$. Define J_- and J_+ as usual. Also let $\sigma_0 = \mathrm{Cone}(u_1, \dots, u_{n+1})$ and $\sigma_i = \mathrm{Cone}(u_1, \dots, \widehat{u_i}, \dots, u_{n+1})$ for $1 \le i \le n+1$.

Since $K_{X_\Sigma} = -\sum_\rho D_\rho$, it follows from (15.1.2) and (15.1.3) that

$$K_{X_\Sigma} \cdot V(\tau) = -\textstyle\sum_{i=1}^{n+1} b_i \tag{15.5.2}$$

(Exercise 15.5.2). Thus $K_{X_\Sigma} \cdot \mathcal{R} < 0$ is equivalent to $\sum_{i=1}^{n+1} b_i > 0$.

Now suppose that ϕ is a flipping contraction. By Theorem 15.3.13, $\Sigma|_{\sigma_0}$ has maximal cones σ_i for $i \in J_+$, and $\Sigma'|_{\sigma_0}$ has maximal cones σ_i for $i \in J_-$. Since X_Σ has only terminal singularities, σ_i is terminal when $i \in J_+$. We need to prove that the same is true for σ_i when $i \in J_-$.

We begin with two consequences of $\sum_{i=1}^{n+1} b_i > 0$. First,

$$\mathrm{Conv}(0,u_1,\dots,u_{n+1}) = \textstyle\bigcup_{i\in J_+} P_{\sigma_i}. \tag{15.5.3}$$

For the nontrivial inclusion, suppose $u = \sum_{i=1}^{n+1} \lambda_i u_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n+1} \lambda_i \leq 1$. Set $\lambda = \min_{i\in J_+} \lambda_i/b_i$. Since $\sum_{i=1}^{n+1} b_i > 0$ and $u = \sum_{i=1}^{n+1} (\lambda_i - \lambda b_i)u_i$, one sees easily that $u \in P_{\sigma_i}$ for any $i \in J_+$ with $\lambda = \lambda_i/b_i$. This proves (15.5.3).

The second consequence of $\sum_{i=1}^{n+1} b_i > 0$ is

$$u_i \notin P_{\sigma_i} \text{ for } i \in J_-. \tag{15.5.4}$$

To prove (15.5.4), assume otherwise, so that $u_i = \sum_{j\neq i} \lambda_j u_j$ with $\lambda_j \geq 0$ and $\sum_{j\neq i} \lambda_j \leq 1$. This implies $-u_i + \sum_{j\neq i} \lambda_j u_j = 0$, where the coefficient of u_i is negative and the sum of the coefficients is ≤ 0. Since $i \in J_-$, this relation is a positive multiple of $\sum_{i=1}^{n+1} b_i u_i = 0$. Thus $\sum_{i=1}^{n+1} b_i \leq 0$, a contradiction.

Using (15.5.3), we obtain

$$\begin{aligned}\mathrm{Conv}(0,u_1,\dots,u_{n+1}) \cap N &= \textstyle\bigcup_{i\in J_+} P_{\sigma_i} \cap N\\ &= \textstyle\bigcup_{i\in J_+} \{0,u_1,\dots,\widehat{u_i},\dots,u_{n+1}\}\\ &= \{0,u_1,\dots,u_{n+1}\}\end{aligned}$$

since each σ_i is terminal. This and (15.5.4) easily imply that σ_i is terminal for $i \in J_-$. This shows that $X_{\Sigma'}$ has only terminal singularities. We leave the proof of $\psi_* K_{X_\Sigma} = K_{X_{\Sigma'}}$ to the reader.

Finally, suppose that ϕ is a divisorial contraction, so that Σ is a star subdivision of Σ_0 at a primitive element we will call u_i. Suppose that $u_i \in \sigma_0 \in \Sigma_0(n)$ and let the minimal generators of σ_0 be $u_1,\dots,\widehat{u_i},\dots,u_{n+1}$, where things are labeled so that $i \neq 1, n+1$. Then $u_i = \sum_{j\neq i} \beta_j u_j$, $b_j \geq 0$, so that we get a relation $\sum_{j=1}^{n+1} b_j u_j = 0$ with $J_- = \{i\}$. Then one checks easily that (15.5.3) and (15.5.4) still hold and that $\Sigma|_{\sigma_0} = \Sigma_-$ in the notation of Theorem 15.3.13. The argument of the previous paragraph shows that σ_0 is terminal. It is also easy to see that $\phi_* K_{X_\Sigma} = K_{X_{\Sigma_0}}$. □

Example 15.5.4. Consider the smooth complete fan Σ in $\mathbb{R}^2$ shown in Figure 20 on the next page. The walls $\tau_1,\dots,\tau_5$ give curves $C_i = V(\tau_i)$ with Mori cone

$$\overline{\mathrm{NE}}(X_\Sigma) = \mathrm{Cone}([C_2],[C_3],[C_4]) \subseteq N_1(X_\Sigma) \simeq \mathbb{R}^3$$

(Exercise 15.5.3). The resulting extremal rays $\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ give three divisorial extremal contractions, as you can see by removing τ_2 or τ_3 or τ_4. Also,

$$K_{X_\Sigma} \cdot C_2 = 0,\ K_{X_\Sigma} \cdot C_3 = K_{X_\Sigma} \cdot C_4 = -1$$

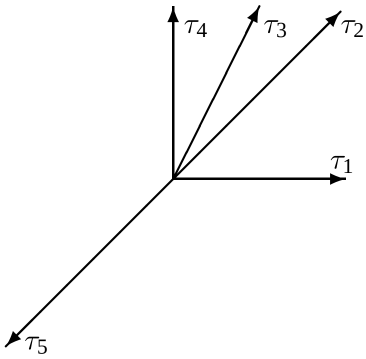


Figure 20. A fan Σ with three extremal contractions

(Exercise 15.5.3). By Proposition 15.5.3, the contractions for $\mathcal{R}_3, \mathcal{R}_4$ map to smooth varieties since terminal means smooth in dimension 2 (Theorem 11.4.14). However, the extremal contraction for $\mathcal{R}_2$ maps to a singular variety. It still has canonical singularities since $K_{X_\Sigma} \cdot C_2 = 0$, as you will prove in Exercise 15.5.4. $\Diamond$

The Toric MMP. For a toric variety, it may be impossible to make the canonical divisor nef. Hence we will modify the MMP for the toric case by replacing K_{X_Σ} with an arbitrary divisor D. The goal is to make D nef by using elementary flips and divisorial contractions. We do this as follows.

Procedure 15.5.5. Let X_Σ be simplicial and semiprojective, and let D be a Weil divisor on X_Σ. Then do the following steps:

(a) If D is nef, then stop.

(b) If D is not nef, then by the toric cone theorem, there is an extremal ray $\mathcal{R}$ with $D \cdot \mathcal{R} < 0$. Let $\phi : X_\Sigma \to X_{\Sigma_0}$ be the corresponding extremal contraction.

(c) If ϕ is a fibering contraction, then stop.

(d) If ϕ is a divisorial contraction, replace X_Σ and D with X_{Σ_0} and the birational transform $\phi_* D$. Note that X_{Σ_0} is simplicial and semiprojective with support $|\Sigma_0| = |\Sigma|$, and as we saw in the proof of Lemma 15.4.10, $\phi_* D$ in the push-forward of D in the sense of [**107**, Sec. 1.4]. Return to step (a) and continue.

(e) If ϕ is a flipping contraction, then we have the flip

$$\begin{tikzcd} X_\Sigma \arrow[rr, dashed, "\psi"] \arrow[dr, "\phi"'] & & X_{\Sigma'} \arrow[dl, "\phi'"] \\ & X_{\Sigma_0}. & \end{tikzcd}$$

Note that $X_{\Sigma'}$ is simplicial and semiprojective with $|\Sigma'| = |\Sigma|$. Replace X_Σ and D with $X_{\Sigma'}$ and the birational transform $\psi_* D$. Return to step (a) and continue.

As we run this procedure on X_Σ, the toric varieties that appear all have the same convex support, namely $|\Sigma|$. If U_Σ is the affine toric variety of $|\Sigma|$, then there is a projective morphism $X_\Sigma \to U_\Sigma$. The other toric varieties involved also map projectively to U_Σ, and the elementary flips and extremal contractions that occur commute with the projective morphisms to U_Σ. Hence:

- Procedure 15.5.5 is an example of the *relative MMP*, since everything is projective over the base U_Σ.
- Procedure 15.5.5 is also an example of the *equivariant MMP*, since the torus T_N acts on all toric varieties that occur and all maps are T_N-equivariant.

The procedure terminates with a composition $\psi : X_\Sigma \dashrightarrow X_{\Sigma'} \dashrightarrow \cdots \dashrightarrow X_{\Sigma_*}$ of D-negative divisorial extremal contractions and elementary flips such that either

- There is a D-negative fibering contraction from X_{Σ_*} to a toric variety of smaller dimension, or
- $\psi_* D$ is nef on X_{Σ_*}.

To explain what we mean by "D-negative," let $X_{\Sigma_i} \dashrightarrow X_{\Sigma_{i+1}}$ be an elementary flip that arises from running Procedure 15.5.5 on X and D. Then the previous steps of the MMP give a birational map $\psi_i : X_\Sigma \dashrightarrow X_{\Sigma_i}$, and the flip $X_{\Sigma_i} \dashrightarrow X_{\Sigma_{i+1}}$ is associated to a flipping extremal ray of X_{Σ_i} that is negative with respect to the birational transform $\psi_{i*}D$. We say that $X_{\Sigma_i} \dashrightarrow X_{\Sigma_{i+1}}$ is a *D-negative elementary flip* in this situation, and the terms *D-negative divisorial contraction* and *D-negative fibering contraction* have similar meanings. Note that we are making frequent use of the functoriality proved in Lemma 15.4.10.

Termination of Flips. The big question is whether Procedure 15.5.5 terminates. The secondary fan makes it easy to choose the extremal rays so that this happens.

Proposition 15.5.6. *Let D be a Weil divisor on a simplicial semiprojective toric variety X_Σ. Then the D-negative extremal rays in Procedure 15.5.5 for X_Σ and D can be chosen so that the procedure stops after finitely many iterations.*

Proof. We use induction on the rank of $\mathrm{Pic}(X_\Sigma)$. The base case is Exercise 15.5.5.

Now assume that $\mathrm{Pic}(X_\Sigma)$ has rank > 1. Consider the secondary fan where $\boldsymbol{\nu}$ consists of the u_ρ for $\rho \in \Sigma(1)$. Then $\Gamma_{\Sigma,\emptyset} = \mathrm{Nef}(X_\Sigma)$ is a chamber in the secondary fan in $\widehat{G}_\mathbb{R} \simeq \mathrm{Pic}(X_\Sigma)_\mathbb{R}$. Assume $[D] \notin \mathrm{Nef}(X_\Sigma)$ and draw a line segment between $[D]$ and a generic point in the interior of $\mathrm{Nef}(X_\Sigma)$ such that the segment always crosses from one chamber to another at a relative interior point of a wall.

Consider the facet $\Gamma_{\Sigma_0,\emptyset}$ where the line segment leaves $\Gamma_{\Sigma,\emptyset}$. This gives an extremal ray $\mathcal{R}$ such that $D \cdot \mathcal{R} < 0$, and $\phi : X_\Sigma \to X_{\Sigma_0}$ is the extremal contraction of $\mathcal{R}$. There are four possibilities to consider:

- If $\Gamma_{\Sigma_0,\emptyset}$ lies on the boundary the secondary fan, then Σ_0 is degenerate by Proposition 14.4.12, in which case we are done.

- If $\Gamma_{\Sigma_0,\emptyset}$ is divisorial wall, then we replace X_Σ and D with X_{Σ_0} and $\phi_* D$. Since the rank of the Picard drops by one, we are done by induction.
- If $\Gamma_{\Sigma_0,\emptyset}$ is flipping wall, then we have the flip $\psi : X_\Sigma \to X_{\Sigma'}$. If $\psi_* D$ is nef on $X_{\Sigma'}$, then we are done by Procedure 15.5.5.
- The remaining case is when $\Gamma_{\Sigma_0,\emptyset}$ is flipping wall and $\psi_* D$ is not nef on $X_{\Sigma'}$. Then our chosen line segment leaves $\Gamma_{\Sigma',\emptyset} = \mathrm{Nef}(X_{\Sigma'})$ at a facet defined by an extremal ray $\mathcal{R}'$ of $X_{\Sigma'}$ with $(\psi_* D) \cdot \mathcal{R}' < 0$. Then replace X_Σ, D, $\mathcal{R}$ with $X_{\Sigma'}$, $\psi_* D$, $\mathcal{R}'$ and continue, using the same line segment as before.

The fourth bullet can occur only finitely many times since the line segment meets only finitely many chambers. Hence the process must terminate. □

We will next improve Proposition 15.5.6 by showing that Procedure 15.5.5 terminates no matter which D-negative extremal rays are used in step (b) of the procedure. Since a divisorial extremal contraction lowers the rank of the Picard group by 1, only finitely many of these steps can occur. Hence the problem reduces to the *termination of flips*. We will need the following lemma.

Lemma 15.5.7. *Let $\psi : X_\Sigma \dashrightarrow X_{\Sigma'}$ be an elementary flip for an extremal ray $\mathcal{R}$. Pick a relation $\sum_\rho b_\rho u_\rho = 0$ generating $\mathcal{R}$ and scale it so that*

(15.5.5) $$u_0 = \textstyle\sum_{\rho \in J_+} b_\rho u_\rho = -\sum_{\rho \in J_-} b_\rho u_\rho$$

is a primitive element of N. Then:

(a) *We have an equality of star subdivisions $\Sigma^*(u_0) = \Sigma_0^*(u_0) = \Sigma'^*(u_0)$.*

(b) *Let Σ^* be the fan of part* (a). *Then we have a commutative diagram*

$$\begin{tikzcd} & X_{\Sigma^*} \arrow[dl, "\Phi"'] \arrow[dr, "\Phi'"] & \\ X_\Sigma \arrow[rr, dashed, "\psi"] \arrow[dr, "\phi"'] & & X_{\Sigma'} \arrow[dl, "\phi'"] \\ & X_{\Sigma_0}. & \end{tikzcd}$$

(c) *Let $D' = \psi_* D$ be the birational transform of a divisor D on X_Σ. Then*

$$\Phi^* D = \Phi'^* D' - (D \cdot C) D_0^*,$$

where D_0^* is the toric divisor on X_{Σ^*} corresponding to u_0 and C is a 1*-cycle on X_Σ whose class is represented by the relation $\sum_\rho b_\rho u_\rho = 0$.*

Proof. Part (a) is straightforward, and part (b) follows immediately from part (a) (Exercise 15.5.6). For part (c), suppose that $D = \sum_\rho a_\rho D_\rho$. The fan Σ^* refines Σ, which gives the proper birational map $\Phi : X_{\Sigma^*} \to X_\Sigma$. Since X_Σ is simplicial, D is $\mathbb{Q}$-Cartier and hence has a support function φ_D. By Proposition 6.2.7, $\Phi^* D$ and D have the same support function φ_D.

This makes Φ^*D easy to compute. First note that $\Sigma^*(1) = \Sigma(1) \cup \{\rho_0\}$, where $\rho_0 = \mathrm{Cone}(u_0)$. Let D^*_ρ (resp. D^*_0) be the divisor on X_{Σ^*} associated to $\rho \in \Sigma(1)$ (resp. ρ_0). Then

$$\Phi^*D = -\textstyle\sum_{\rho\in\Sigma(1)} \varphi_D(u_\rho)D^*_\rho - \varphi_D(u_0)D^*_0 = \textstyle\sum_{\rho\in\Sigma(1)} a_\rho D^*_\rho - \varphi_D(u_0)D^*_0,$$

where the second equality uses $\varphi_D(u_\rho) = -a_\rho$ for $\rho \in \Sigma(1)$. Theorem 15.3.13 implies that $\sigma_{J_-} = \mathrm{Cone}(u_\rho \mid \rho \in J_-) \in \Sigma$. Since φ_D is linear on σ_{J_-}, (15.5.5) yields $\varphi_D(u_0) = \sum_{\rho\in J_-} a_\rho b_\rho$. Hence

$$\Phi^*D = \textstyle\sum_{\rho\in\Sigma(1)} a_\rho D^*_\rho - \big(\textstyle\sum_{\rho\in J_-} a_\rho b_\rho\big)D^*_0.$$

Furthermore, $\sigma_{J_+} \in \Sigma'$ by Theorem 15.3.13, so repeating the above computation with D' on $X_{\Sigma'}$ gives

$$\Phi'^*D' = \textstyle\sum_{\rho\in\Sigma(1)} a_\rho D^*_\rho + \big(\textstyle\sum_{\rho\in J_+} a_\rho b_\rho\big)D^*_0.$$

We conclude that

$$\Phi^*D = \Phi'^*D' - \big(\textstyle\sum_{\rho\in\Sigma(1)} a_\rho b_\rho\big)D^*_0.$$

If C is a 1-cycle whose class is represented by the relation $\sum_\rho b_\rho u_\rho = 0$, then the quantity in parentheses is $D \cdot C$ by Exercise 15.5.2. $\square$

Example 15.5.8. Figure 3 from Example 11.1.12 is a classic example of part (b) of Lemma 15.5.7. In §11.1 we mentioned that Figure 3 was an example of a flip. But the flip only concerns the bottom half of the figure—the common subdivision Σ^* is what explains the top half. $\Diamond$

Here is an immediate consequence of Lemma 15.5.7.

Corollary 15.5.9. *Assume that $D \cdot \mathcal{R} < 0$ in the situation of Lemma 15.5.7. Then there is a nonzero effective divisor D^* on X_{Σ^*} such that*

$$\Phi^*D = \Phi'^*D' + D^*. \qquad \square$$

Corollary 15.5.9 is a special case of a result that applies more generally. See, for example, [**104**, Lem. 4.10], [**179**, Lem. 3.38], or [**194**, Lem. 9-1-3].

We can now prove termination.

Theorem 15.5.10 (Termination of Toric Flips)**.** *Given a divisor D_1 on a simplicial semiprojective toric variety X_{Σ_1}, there is no infinite sequence of elementary flips*

$$X_{\Sigma_1} \overset{\psi_1}{\dashrightarrow} X_{\Sigma_2} \overset{\psi_2}{\dashrightarrow} X_{\Sigma_3} \overset{\psi_3}{\dashrightarrow} \cdots$$

for flipping extremal rays $\mathcal{R}_i \in \overline{\mathrm{NE}}(X_{\Sigma_i})$ satisfying $D_i \cdot \mathcal{R}_i < 0$, where for $i \geq 2$, $D_i = (\psi_{i-1} \circ \cdots \circ \psi_1)_* D_1$.

Proof. Assume that such an infinite sequence exists. Since $\Sigma_1(1) = \Sigma_2(1) = \cdots$ and there are only finitely many fans with a given set of rays, the same fan must occur twice in the sequence. If we start numbering at this fan, then we obtain

$$X_{\Sigma_1} \overset{\psi_1}{\dashrightarrow} X_{\Sigma_2} \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_{\ell-1}}{\dashrightarrow} X_{\Sigma_\ell} = X_{\Sigma_1}.$$

Furthermore, $\psi_{\ell-1} \circ \cdots \circ \psi_1$ is the identity since it is the identity on $T_N \subseteq X_{\Sigma_1}$.

We will show that this is impossible by proving the existence of a commutative diagram

$$\begin{tikzcd} & X_{\widehat{\Sigma}_i} \arrow[dl, "\widehat{\Phi}_i"'] \arrow[dr, "\widehat{\Phi}'_i"] & \\ X_{\Sigma_1} \arrow[rr, dashed, "\psi_{i-1}\circ\cdots\circ\psi_1"'] & & X_{\Sigma_i} \end{tikzcd} \tag{15.5.6}$$

such that

$$\widehat{\Phi}_i^* D_1 = \widehat{\Phi}_i'^* D_i + \widehat{D}_i, \quad \widehat{D}_i \text{ nonzero and effective}. \tag{15.5.7}$$

Once this is proved, we get an easy contradiction. This is because when $i = \ell$, the diagram (15.5.6) implies that $\widehat{\Phi}_\ell = \widehat{\Phi}'_\ell$ since $\psi_{\ell-1} \circ \cdots \circ \psi_1$ is the identity. Then

$$\widehat{\Phi}_\ell^* D_1 = \widehat{\Phi}_\ell'^* D_\ell + \widehat{D}_\ell = \widehat{\Phi}_\ell^* D_1 + \widehat{D}_\ell$$

since $D_\ell = D_1$. This forces $\widehat{D}_\ell = 0$, contradicting (15.5.7).

We prove (15.5.6) and (15.5.7) by induction on i. The base case $i = 2$ follows from Corollary 15.5.9. For the inductive step, we will do $i = 3$ for simplicity. Consider the commutative diagram

$$\begin{tikzcd} & & X_{\widehat{\Sigma}_2} \arrow[dl, "\Psi"'] \arrow[dr, "\Psi'"] \arrow[ddll, bend right, "\widehat{\Phi}_2"'] \arrow[ddrr, bend left, "\widehat{\Phi}'_2"] & & \\ & X_{\Sigma_1^*} \arrow[dl, "\Phi_1"'] \arrow[dr, "\Phi'_1"] & & X_{\Sigma_2^*} \arrow[dl, "\Phi_2"'] \arrow[dr, "\Phi'_2"] & \\ X_{\Sigma_1} \arrow[rr, dashed, "\psi_1"'] & & X_{\Sigma_2} \arrow[rr, dashed, "\psi_2"'] & & X_{\Sigma_3}, \end{tikzcd}$$

where the two lower triangles come from Lemma 15.5.7 and $\widehat{\Sigma}_2$ is any common refinement of Σ_1^* and Σ_2^*. Then Corollary 15.5.7 implies that

$$\Phi_1^* D_1 = \Phi_1'^* D_2 + D_1^* \ \text{ and } \ \Phi_2^* D_2 = \Phi_1'^* D_3 + D_2^*,$$

where D_1^* and D_2^* are nonzero effective divisors. An easy diagram chase yields

$$\widehat{\Phi}_2^* D_1 = \widehat{\Phi}_2'^* D_3 + \Psi^* D_1^* + \Psi'^* D_2^*.$$

Then we are done since the pullback by surjective map of a nonzero effective divisor is nonzero and effective. $\square$

In the general version of the MMP, termination of flips has been proved only in dimension 3. However, a special case of termination is proved in [**31**], which is sufficient to prove the existence of minimal models of n-dimensional projective varieties of general type.

The Log Toric MMP. An important technical tool in the MMP is provided by pairs (X,D), where X is normal and $D = \sum_i a_i D_i$ is a $\mathbb{Q}$-divisor such that $a_i \in [0,1] \cap \mathbb{Q}$. These "log varieties" were introduced in §11.4, where we gave the references [**179**, p. 98] and [**194**, Ch. 11] that explain their usefulness.

We also defined log canonical and klt (kawamata log terminal) singularities of a pair (X,D) in Definition 11.4.23, which are important in the MMP. For a toric variety X_Σ, we use $D = \sum_\rho a_\rho D_\rho$. When $K_{X_\Sigma} + D$ is $\mathbb{Q}$-Cartier, Proposition 11.4.24 tells us that:

- If $a_\rho \in [0,1]$ for all $\rho \in \Sigma(1)$, then (X_Σ, D) is log canonical.
- If $a_\rho \in [0,1)$ for all $\rho \in \Sigma(1)$, then (X_Σ, D) is klt.

The standard MMP for X uses the divisor K_X and requires that X be $\mathbb{Q}$-factorial with only terminal singularities. Hence it should not be surprising that the log MMP for the pair (X,D) uses the divisor $K_X + D$ and requires that X be $\mathbb{Q}$-factorial with only klt singularities. In order to run the log MMP, one needs to show that these properties are preserved by the divisorial contractions and flips associated to $(K_X + D)$-negative extremal rays.

In the toric case, let X_Σ be simplicial and semiprojective, and take a divisor $D = \sum_\rho a_\rho D_\rho$ with $a_\rho \in [0,1) \cap \mathbb{Q}$ for all ρ. Then $K_{X_\Sigma} + D$ is $\mathbb{Q}$-Cartier and (X_Σ, D) is klt by Proposition 11.4.24. Then the log toric MMP for the pair (X_Σ, D) is described as follows.

Procedure 15.5.11. Let (X_Σ, D) be as above. Then do the following steps:

(a) If $K_{X_\Sigma} + D$ is nef, then stop.

(b) If $K_{X_\Sigma} + D$ is not nef, then there is an extremal ray $\mathcal{R}$ with $(K_{X_\Sigma} + D) \cdot \mathcal{R} < 0$. Let $\phi : X_\Sigma \to X_{\Sigma_0}$ be the corresponding extremal contraction.

(c) If ϕ is fibering, then stop.

(d) If ϕ is divisorial, then replace (X_Σ, D) with $(X_{\Sigma_0}, \phi_* D)$. Return to step (a) and continue.

(e) If ϕ is flipping with flip $\psi : X_\Sigma \dashrightarrow X_{\Sigma'}$, then replace (X_Σ, D) with $(X_{\Sigma'}, \psi_* D)$. Return to step (a) and continue.

The log toric MMP for (X_Σ, D) is the toric MMP for X and $K_{X_\Sigma} + D$ since

$$\begin{aligned} \phi_*(K_{X_\Sigma} + D) &= K_{X_{\Sigma_0}} + \phi_* D \ \text{ in step (d)} \\ \psi_*(K_{X_\Sigma} + D) &= K_{X_{\Sigma'}} + \psi_* D \ \text{ in step (e)} \end{aligned}$$

by Proposition 15.5.3. Also note that klt singularities are preserved when we run the log toric MMP since

$$\begin{aligned} &(X_{\Sigma_0}, \phi_* D) \ \text{ in step (d)}\\ &(X_{\Sigma'}, \psi_* D) \ \text{ in step (e)} \end{aligned}$$

are klt by Proposition 11.4.24. Finally, Proposition 15.5.6 (careful choice of $\mathcal{R}$ in step (b)) and Theorem 15.5.10 (any $(K_{X_\Sigma} + D)$-negative $\mathcal{R}$ in step (b)) guarantee that the log toric MMP terminates.

Example 15.5.12. Suppose that $\mathcal{R}$ is *any* extremal ray for X_Σ. Let us show that the associated extremal contraction $\phi : X_\Sigma \to X_{\Sigma_0}$ occurs in Procedure 15.5.11 for the pair (X_Σ, D), provided that D is chosen correctly.

Pick a Cartier divisor D_0 whose class lies in the interior of $\mathrm{Nef}(X_\Sigma)$. Replacing D_0 with a positive integer multiple, we may assume that the polyhedron P_{D_0} has an interior point m, and then replacing D_0 with $D_0 + \mathrm{div}(\chi^m)$, we may assume that $D_0 = \sum_\rho a_\rho D_\rho$, where $a_\rho > 0$ for all ρ. Now consider the $\mathbb{Q}$-divisor

$$D = -K_{X_\Sigma} - \varepsilon D_0 = \sum_\rho (1 - \varepsilon a_\rho) D_\rho,$$

where $\varepsilon \in \mathbb{Q}$ is positive and satisfies $0 < \varepsilon a_\rho < 1$ for all ρ. Then $K_{X_\Sigma} + D = -\varepsilon D_0$, which makes it easy to see that every extremal ray $\mathcal{R}$ is $(K_{X_\Sigma} + D)$-negative. $\Diamond$

Log Minimal Models and Flops. When we run the log toric MMP on (X_Σ, D), the result is a sequence $\psi : X_\Sigma \dashrightarrow X_{\Sigma'} \dashrightarrow \cdots \dashrightarrow X_{\Sigma_*}$ of $(K_{X_\Sigma} + D)$-negative divisorial extremal contractions and elementary flips such that either

- There is a $(K_{X_\Sigma} + D)$-negative fibering extremal contraction from X_{Σ_*} to a toric variety of smaller dimension, or
- $K_{X_{\Sigma_*}} + D_*$ is nef on X_{Σ_*}, where $D_* = \psi_* D$.

In the latter case, observe that (X_{Σ_*}, D_*) is klt and $K_{X_{\Sigma_*}} + D_*$ is nef. We say that (X_{Σ_*}, D_*) is a *log minimal model* of (X_Σ, D).

A basic result of the MMP states that log minimal models are isomorphic in codimension 1 [**179**, Thm. 3.52]. In our situation, two log minimal models (X_{Σ_*}, D_*) and $(X_{\Sigma'_*}, D'_*)$ of (X_Σ, D) satisfy

$$|\Sigma_*| = |\Sigma'_*| = |\Sigma|,\ \Sigma_*(1) \subseteq \Sigma(1),\ \Sigma'_*(1) \subseteq \Sigma(1).$$

Since X_{Σ_*} and $X_{\Sigma'_*}$ are isomorphic in codimension 1, it follows that

$$\Sigma_*(1) = \Sigma'_*(1). \tag{15.5.8}$$

This will enable us to show that the map connecting X_{Σ_*} and $X_{\Sigma'_*}$ is built from a special type of flip called a *flop*. Here is the definition.

Definition 15.5.13. Let $\mathcal{R}$ be a flipping extremal ray with flip $\psi : X_\Sigma \dashrightarrow X_{\Sigma'}$, and let D be a $\mathbb{Q}$-divisior on X_Σ. Then ψ is a ***D*-*flop*** if $D \cdot \mathcal{R} = 0$.

Here is a classic example.

Example 15.5.14. Figure 21 on the next page is Figure 3 of Example 11.1.12. For now, focus on the bottom half of the figure, which involves the cone σ and two smooth refinements Σ_1, Σ_2 that give toric morphisms

$$\begin{tikzcd} X_{\Sigma_1} \arrow[rr, dashed, "\psi"] \arrow[dr] & & X_{\Sigma_2} \arrow[dl] \\ & U_\sigma. & \end{tikzcd}$$

The birational map ψ is a flip (see, for instance, Figure 13 from Example 15.3.2). One easily computes that $K_{X_{\Sigma_1}} \sim 0$. Hence ψ is a $K_{X_{\Sigma_1}}$-flop. $\Diamond$

We can now describe the log minimal models of a toric pair (X_Σ, D).

Theorem 15.5.15. *Assume that X_Σ is simplicial and semiprojective, and let $D = \sum_\rho a_\rho D_\rho$, where $a_\rho \in [0,1) \cap \mathbb{Q}$ for all ρ. Then:*

(a) *Up to toric isomorphism, (X_Σ, D) has only finitely many log minimal models.*

(b) *Any two log minimal models of (X_Σ, D) are related by a finite sequence of $(K_{X_\Sigma} + D)$-flops.*

Proof. Any two log minimal models share the same rays by (15.5.8), so finiteness is immediate.

To relate the various log minimal models of (X_Σ, D), we start by fixing one, say (X_{Σ_*}, D_*). Consider the secondary fan where $\boldsymbol{\nu}$ consists of u_ρ for $\rho \in \Sigma_*(1)$. Then the GKZ cone $\Gamma_{\Sigma_*,\emptyset} = \mathrm{Nef}(X_{\Sigma_*})$ is a chamber of the secondary fan. Now let $(X_{\Sigma'_*}, D'_*)$ be a second log minimal model. Then (15.5.8) implies that $\Sigma_*(1) = \Sigma'_*(1)$, so that $\Gamma_{\Sigma'_*,\emptyset}$ is also a chamber, which (via the birational transform) is isomorphic to $\mathrm{Nef}(X_{\Sigma'_*})$. Under this identification, we see that $\Gamma_{\Sigma_*,\emptyset}$ and $\Gamma_{\Sigma'_*,\emptyset}$ both contain the class $[K_{X_{\Sigma_*}} + D_*]$.

Hence the log minimal models of (X_Σ, D) correspond to certain chambers of the secondary fan that contain $[K_{X_{\Sigma_*}} + D_*]$. Suppose for the moment that $\Gamma_{\Sigma_*,\emptyset}$ and $\Gamma_{\Sigma'_*,\emptyset}$ are chambers that contain $[K_{X_{\Sigma_*}} + D_*]$ and share a common wall, regardless of whether or not they are actual log minimal models. If $\mathcal{R}$ is an extremal ray for X_{Σ_*} that defines this wall, then $(K_{X_{\Sigma_*}} + D_*) \cdot \mathcal{R} = 0$ since $K_{X_{\Sigma_*}} + D_*$ is in both chambers and hence is contained in the wall where they intersect. The resulting flip $X_{\Sigma_*} \dashrightarrow X_{\Sigma'_*}$ is a $(K_{X_{\Sigma_*}} + D_*)$-flop. This is what we mean by "$(K_{X_\Sigma} + D)$-flop" in part (b) of the theorem.

Since any two chambers containing $[K_{X_{\Sigma_*}} + D_*]$ can be connected by a chain of such wall crossings, we see that any two log minimal models can be connected by a finite composition of $(K_{X_\Sigma} + D)$-flops. $\square$

Here is an example of the log toric MMP.

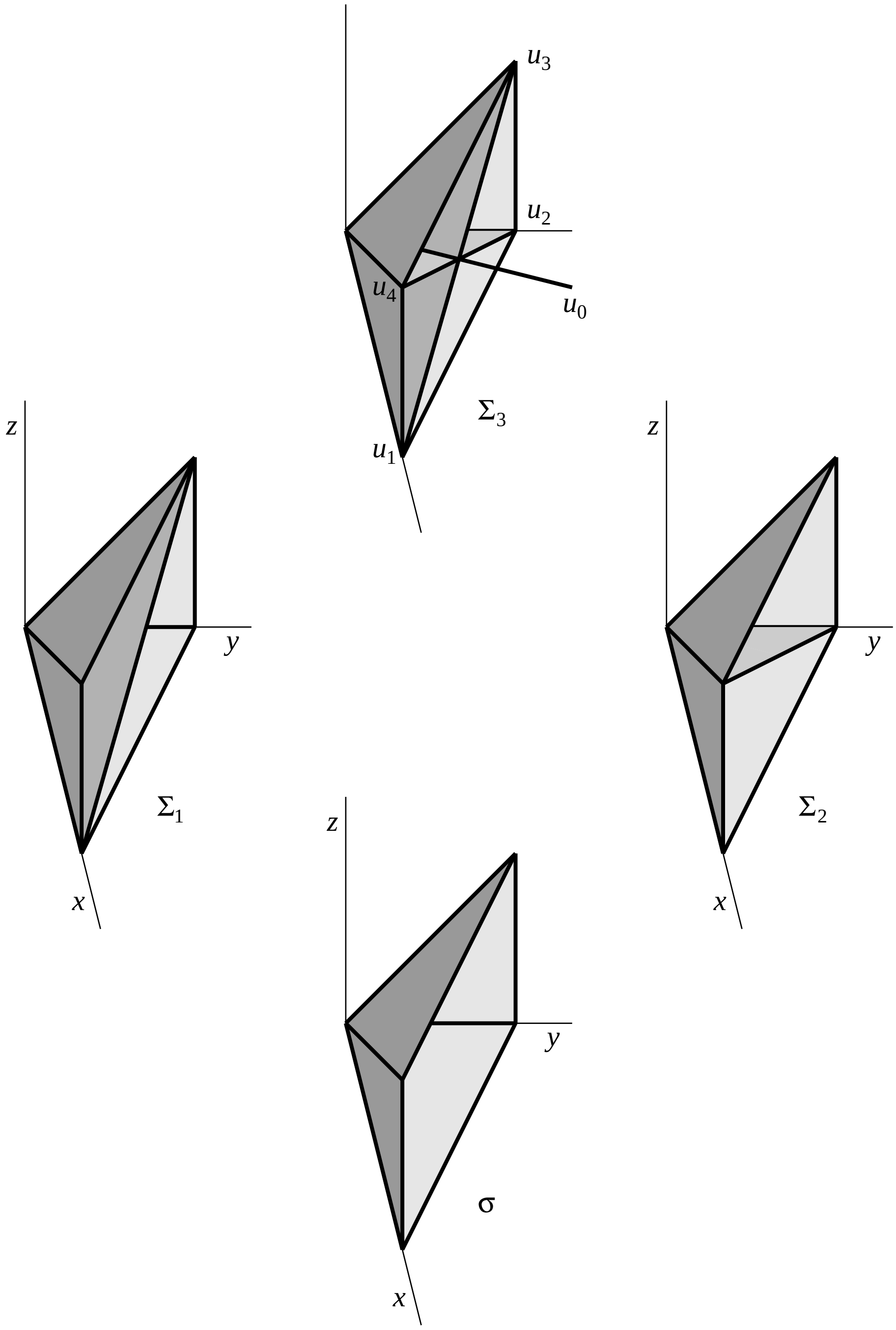


Figure 21. The cone σ with smooth refinements $\Sigma_1, \Sigma_2, \Sigma_3$ in Examples 15.5.14 and 15.5.16

Example 15.5.16. In Figure 21 from Example 15.5.14, the toric variety X_{Σ_3} on top has five minimal generators

$$u_0 = e_1 + e_2 + e_3,\ u_1 = e_1,\ u_2 = e_2,\ u_3 = e_2 + e_3,\ u_4 = e_1 + e_3.$$

We also have the four walls $\tau_i = \mathrm{Cone}(u_0, u_i)$ for $1 \le i \le 4$. The walls τ_1 and τ_3 share the wall relation $u_2 - u_0 + u_4 = 0$, so that $[V(\tau_1)] = [V(\tau_3)]$ in $N_1(X_{\Sigma_3})$. One similarly sees that $[V(\tau_2)] = [V(\tau_4)]$. This gives extremal rays

$$\mathcal{R}_1 = \mathbb{R}_{\ge 0}[V(\tau_1)] = \mathbb{R}_{\ge 0}[V(\tau_3)] \ \text{ and } \ \mathcal{R}_2 = \mathbb{R}_{\ge 0}[V(\tau_2)] = \mathbb{R}_{\ge 0}[V(\tau_4)].$$

The intersection formulas from §6.4 make it easy to compute that

$$K_{X_{\Sigma_3}} \cdot \mathcal{R}_1 = K_{X_{\Sigma_3}} \cdot \mathcal{R}_2 = -1.$$

Notice also that X_{Σ_3} is smooth.

The log toric MMP for $(X_{\Sigma_3}, 0)$ is the same as the toric MMP for X_{Σ_3} and $K_{X_{\Sigma_3}}$. We have two ways to begin, since both extremal rays are $K_{X_{\Sigma_3}}$-negative. They are also divisorial, and the resulting contractions give the commutative diagram

$$\begin{tikzcd} & X_{\Sigma_3} \arrow[dl] \arrow[dr] & \\ X_{\Sigma_1} \arrow[rr, dashed, "\psi"] & & X_{\Sigma_2}, \end{tikzcd} \tag{15.5.9}$$

where the fans Σ_1, Σ_2 are from Figure 21. You should check that $X_{\Sigma_3} \to X_{\Sigma_1}$ is contraction for $\mathcal{R}_2$ and $X_{\Sigma_3} \to X_{\Sigma_2}$ is contraction for $\mathcal{R}_1$.

We saw in Example 15.5.14 that $K_{X_{\Sigma_1}} \sim 0$, and similarly $K_{X_{\Sigma_2}} \sim 0$. It follows that X_{Σ_1} and X_{Σ_2} are both minimal models of X_{Σ_3} (we drop the "log" in this case since we are doing the MMP for the canonical divisor). In particular, X_{Σ_3} does not have a unique minimal model. Also, Example 15.5.14 shows that the birational map ψ in (15.5.9) is the flop connecting the two minimal models.

Figure 21 tells the full story of this example. You can also see the two minimal models of X_{Σ_3} by computing the secondary fan of X_{Σ_1}. $\Diamond$

Note that the cone σ at the bottom of Figure 21 gives the affine toric variety $\mathbf{V}(xy - zw) \subseteq \mathbb{C}^4$ from Example 1.1.5. It is amazing that this simple toric variety is still relevant 750 pages after we first encountered it. This captures perfectly the power of toric geometry to illustrate deep phenomena in algebraic geometry.

Exercises for §15.5.

15.5.1. This exercise will supply some details omitted in the proof of Proposition 15.5.1

(a) Prove $X_\Sigma \simeq \mathrm{Proj}(\bigoplus_{\ell=0}^{\infty} \phi_* \mathscr{O}_{X_\Sigma}(\ell D))$ when $\phi : X_\Sigma \to X_{\Sigma_0}$ is ϕ-ample. Hint: You can assume that X_{Σ_0} is affine. Use Proposition 7.2.3 and Theorems 7.2.4 and 7.1.13.

(b) Prove (15.5.1). Hint: ϕ and ϕ' are isomorphisms in codimension 1.

(c) Let D be any divisor on a toric variety X_Σ. Show that $\bigoplus_{\ell=0}^{\infty} H^0(X_\Sigma, \phi_* \mathscr{O}_{X_\Sigma}(\ell D))$ is a finitely generated $\mathbb{C}$-algebra. Hint: Interpret the ring in terms of the graded ring $\mathbb{C}[C(P_D) \cap (M \times \mathbb{Z})]$.

15.5.2. Use (15.1.2) and (15.1.3) to prove (15.5.2).

15.5.3. This exercise concerns the fan Σ from Example 15.5.4.

(a) Show that the Mori cone is generated by the classes of the cones C_2, C_3 and C_4.

(b) Verify the intersection products computed in Example 15.5.4.

(c) Show that running the toric MMP with K_{X_Σ} always leads to a fibering contraction. Most of the fibering contractions map to $\mathbb{P}^1$, though there is one that maps to a point.

15.5.4. Here we explore the relation between divisorial and flipping extremal contractions and the canonical class.

(a) Assume that X_Σ has only terminal singularities and let $\phi : X_\Sigma \to X_{\Sigma_0}$ be a divisorial extremal contraction for the extremal ray $\mathcal{R}$. Show that X_{Σ_0} has only canonical singularities if and only if $K_{X_\Sigma} \cdot \mathcal{R} \leq 0$.

(b) Let $\phi : X_\Sigma \to X_{\Sigma_0}$ be a flipping extremal contraction for the extremal ray $\mathcal{R}$. Prove that $K_{X_{\Sigma_0}}$ is $\mathbb{Q}$-Cartier if and only if $K_{X_\Sigma} \cdot \mathcal{R} = 0$.

(c) As in part (b), let $\phi : X_\Sigma \to X_{\Sigma_0}$ be a flipping extremal contraction for the extremal ray $\mathcal{R}$, but now assume that $K_{X_\Sigma} \cdot \mathcal{R} < 0$. Prove that X_{Σ_0} is neither $\mathbb{Q}$-factorial nor $\mathbb{Q}$-Gorenstein, i.e., Σ_0 is not simplicial and $K_{X_{\Sigma_0}}$ is not $\mathbb{Q}$-Cartier.

Part (c) shows that if $\mathcal{R}$ is a K_{X_Σ}-negative extremal ray, then the associated contraction ϕ maps to the badly behaved variety X_{Σ_0}. This explains why flips are essential in the MMP.

15.5.5. Prove that the toric MMP always terminates when $\mathrm{Pic}(X_\Sigma)$ has rank 1.

15.5.6. Prove parts (a) and (b) of Lemma 15.5.7.

15.5.7. Explain how Example 15.5.12 relates to Example 11.4.26.

15.5.8. Suppose that D is an effective divisor on a simplicial semiprojective toric variety X_Σ. Show that applying Procedure 15.5.5 to X_Σ and D will never result in a D-negative fibering contraction.

Appendix A

The History of Toric Varieties

This appendix discusses the origins of toric geometry, with brief remarks about recent developments. We will describe how the subject evolved and what its early successes were. We make no claim as to completeness and omit many topics.

§A.1. The First Ten Years

Specific examples of toric varieties, such as $\mathbb{C}^n$, $\mathbb{P}^n$, $\mathbb{P}^n \times \mathbb{P}^m$, and the Hirzebruch surfaces $\mathscr{H}_r$, have been known for a long time. The idea of a general toric variety is more recent.

Demazure. The first formal definition of toric variety came in 1970 in Demazure's paper *Sous-groupes algébriques de rang maximum du groupe de Cremona* [**82**]. The Cremona group is the (very large) group of birational automorphisms of $\mathbb{P}^n$. Demazure's main result is that automorphism groups of smooth toric varieties give interesting algebraic subgroups of the Cremona group. For him, toric varieties are

> certain $\mathbb{Z}$-schemes with a cellular decomposition obtained by
> adding certain "points at infinity" to a split torus.

Demazure worked with schemes over $\mathrm{Spec}(\mathbb{Z})$, and if M is a lattice, then the split torus $\mathrm{Spec}(\mathbb{Z}[M])$ is a group scheme over $\mathrm{Spec}(\mathbb{Z})$. This makes M the character group of the torus. So the notation for M was there from the beginning.

Here is Demazure's definition of fan [**82**, Def. 4.2.1].

Definition A.1.1. Let M^* be a free abelian group of finite type. One calls a ***fan*** (in French, ***éventail***) a finite set Σ of subsets of M^* such that

(a) Every element of Σ is a subset of a basis of M^*.

(b) Every subset of an element of Σ belongs to Σ.

(c) If $K, L \in \Sigma$, one has $\mathbb{N}K \cap \mathbb{N}L = \mathbb{N}(K \cap L)$.

This looks odd until you realize that Demazure is only considering the smooth case. His K is the set of minimal generators of a smooth cone σ. Also, M^* is the dual of M, which is N in our notation. Here are some further definitions from [**82**]:

- The *support* of Σ is $|\Sigma| = \bigcup_{K \in \Sigma} K$.
- $\Omega = \bigcup_{K \in \Sigma} \mathbb{N}K$.
- Σ is *complete* if $\Omega = M^*$.

If $|\Sigma|$ is the support of a fan Σ in the modern sense, then $|\Sigma| \cap N$ is what Demazure calls Ω. Thus his notion of complete is equivalent to ours.

Given a fan Σ as in Definition A.1.1, Demazure constructs its toric variety X as a scheme over $\mathrm{Spec}(\mathbb{Z})$ by gluing together affine varieties, similar to what we did in Chapter 3. Many basic results about smooth toric varieties appear in [**82**]. Here is a sample, with pointers to where the results appear in this book:

- For a field k, $X_k = X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k)$ is complete if and only if Σ is complete (Theorem 3.4.6).
- The exact sequence $M \to \mathbb{Z}^{\Sigma(1)} \to \mathrm{Pic}(X) \to 0$ (Theorem 4.2.1).
- If X is smooth and complete, then every ample line bundle on X is very ample (Theorem 6.1.15).
- $H^i(X, \mathscr{O}_X(D))_m \simeq \widetilde{H}^{i-1}(V_{D,m}, \mathbb{C})$ when $m \in M$ (Theorem 9.1.3).

When X is complete, Demazure applies the last bullet to prove that $H^p(X, \mathscr{O}_X) = 0$ for $p > 0$, the first toric vanishing theorem. He also gives criteria for a divisor $D = \sum_\rho a_\rho D_\rho$ with Cartier data $\{m_\sigma\}_{\sigma \in \Sigma_{\max}}$ to be basepoint free or ample. Support functions are not mentioned, so that, for example, his criterion for ampleness is

$$D \text{ is ample} \iff \langle m_\sigma, u_\rho \rangle > -a_\rho \text{ when } \rho \in \Sigma(1),\ \rho \notin \sigma(1).$$

For us, this follows from Lemma 6.1.13 and Theorem 6.1.14.

Given Demazure's focus on the Cremona group, it is not surprising that one of the major results of [**82**] is an explicit description of the automorphism group of a smooth complete toric variety (generalized to the simplicial case in [**65**]).

After Demazure. Besides [**82**], other papers touched on some of the same ideas, though often in very different contexts. An example is Satake's 1973 Bulletin article *On the arithmetic of tube domains* [**243**]. It did not take long for people to flesh out the wonderful ideas implicit in these papers. Two particularly important early works are the Springer lecture notes *Toroidal Embeddings I* by Kempf, Knudsen, Mumford and Saint-Donat, published in 1973 [**172**], and the paper *Almost homogeneous algebraic varieties under algebraic torus action* by Miyake and Oda, presented at a conference in 1973 and published in 1975 [**205**].

The spirit of the times is captured nicely in Mumford's introduction to [**172**], which begins

> The goal of these notes is to formalize and illustrate the power of a technique which has cropped up independently in the work of at least a dozen people, . . .

The aptness of the phrase "cropped up independently" is confirmed by a footnote that Mumford added at the end of the introduction:

> *) After this was written, I received a paper by K. Miyake and T. Oda entitled *Almost homogeneous algebraic varieties under algebraic torus action* also on this topic.

The introduction also shows that the elementary aspects of toric geometry were already recognized:

> When teaching algebraic geometry and illustrating simple singularities, varieties, and morphisms, one almost invariably tends to choose examples of a "monomial type" type: i.e., varieties defined by equations
>
> $$x_1^{a_1} \cdots x_r^{a_r} = x_{r+1}^{a_{r+1}} \cdots x_n^{a_n}$$
>
> and morphisms f for which
>
> $$f^*(y_i) = x_1^{a_{i1}} \cdots x_n^{a_{in}}.$$

In [**172**] we find the first appearance of M and N for the dual lattices used in toric geometry. Here is their definition of fan.

Definition A.1.2. A finite rational partial polyhedral decomposition (we abbreviate this to f.r.p.p decomposition) of $N_{\mathbb{R}}$ is a finite set $\{\sigma_\alpha\}$ of convex rational polyhedral cones in $N_{\mathbb{R}}$ such that:

(a) if σ is a face of σ_α, then $\sigma = \sigma_\beta$ for some β

(b) $\forall \alpha, \beta$, $\sigma_\alpha \cap \sigma_\beta$ is a face of σ_α and σ_β.

This accidentally omits the hypothesis of strong convexity, which is expressed elsewhere in [**172**] by saying that σ "does not contain any linear subspace." Given the unwieldy phrase "f.r.p.p decomposition," we should be grateful to Demazure for the elegant word "fan" now in use.

A key result of [**172**] and [**205**] is that if a normal variety is toric in the sense of Definition 3.1.1, then it is the toric variety of a fan. This is our Corollary 3.1.8, which follows from Sumihiro's theorem (Theorem 3.1.7), just as in [**172**] and [**205**]. We also find the Orbit-Cone Correspondence (Theorem 3.2.6) and the usual criteria for completeness (Theorem 3.4.6) and smoothness (Theorem 3.1.19) in [**172**] and [**205**].

The properness criterion from Theorem 3.4.11 is proved in [**172**]. Kempf et al. also treat convexity in the general setting of sheaves of torus-invariant complete fractional ideals. The support functions defined in Chapters 4 and 6 appear in part III.c of Theorem 9 on [**172**, pp. 28–29]. A major result is Theorem 13 on [**172**, pp. 48], which gives the criterion of Theorem 7.2.12 for a toric morphism to be projective. Our proof of this theorem uses results from EGA [**127**] described in the appendix to Chapter 7. These results are not mentioned in [**172**] since the authors, like many algebraic geometers of the time, knew much of EGA by heart.

Another important result in [**172**] is that any toric variety has a resolution of singularities given by a projective toric morphism, our Theorem 11.1.9.

The material on toric varieties appears in the first chapter of [**172**]. The second chapter introduces the more general *toroidal varieties*, which "locally" look like toric varieties in a suitable sense. Toroidal varieties are used in the main result of the book, the characteristic 0 semi-stable reduction theorem for surjective morphisms $f : X \to C$, where C is a smooth curve. More recently, toroidal varieties have been used to study the structure of birational morphisms—see [**1**] and [**279**].

Turning to [**205**], we find the first statement of the classification of smooth complete toric surfaces (Theorem 10.4.3), along with preliminary classification results for toric threefolds. Miyake and Oda also give an example of a 3-dimensional smooth complete nonprojective toric variety X that is simpler than the complicated one given by Demazure. The fan for their example is Figure 9 in Example 6.1.17.

The nonprojective toric threefold X reappears at the end of [**205**], where the authors use a series of smooth blowups followed by a series of smooth blowdowns to convert X into $\mathbb{P}^3$. They also conjecture that this can be done for any smooth complete toric threefold. This conjecture is still open, as is its generalization called the *strong Oda conjecture*, which applies to birational toric maps between smooth complete toric varieties of arbitrary dimension. However, if we allow the blowups and blowdowns to be intermixed, then one gets the *weak Oda conjecture*, which has been proved in [**2**] and [**278**].

We should mention two other notable papers from this period:

- Hochster's 1971 paper *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes* [**146**] proved that the semigroup algebra of a saturated affine semigroup is Cohen-Macaulay. This implies that normal toric varieties are Cohen-Macaulay (Theorem 9.2.9). Hochster's paper initiated the important interaction between toric geometry and commutative algebra.
- Ehlers' 1975 paper *Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einiger isolierter Singularitäten* [**88**] considers toric varieties as complex manifolds. Our proof of the compactness criterion in Theorem 3.4.1 uses his argument. Ehlers used infinite fans to resolve some interesting singularities.

Similar infinite fans were also used to construct smooth toroidal compactifications of bounded symmetric domains. See the books [**6**] from 1975 and [**213**] from 1980.

The Russian School. In the mid 1970s, Bernstein, Khovanskii and Kusnirenko were studying subvarieties of $(\mathbb{C}^*)^n$ defined by the vanishing of Laurent polynomials f_i. The Newton polytopes of the f_i play an important role their work. For example, the number of solutions of a generic system $f_1 = \cdots = f_n = 0$ is given by the mixed volume of the Newton polytopes of the f_i. This is described in [**70**, §7.5] and [**105**, Sec. 5.5].

Drawing on [**88**], [**146**] and especially [**172**], Khovanskii studies toric varieties in his 1977 paper *Newton polyhedra and toroidal varieties*. This paper is notable for several reasons:

- It introduced the term "support function" in the toric context.
- It proved the Demazure vanishing theorem $H^p(X, \mathscr{O}_X(D)) = 0$ for $p > 0$ when D is basepoint free (Theorem 9.2.3).
- It gives the first toric proof of the properties of the Ehrhart polynomial, similar to what we did in Theorem 9.4.2.

Khovanskii's paper also makes it clear that there is a deep connection between toric varieties and polytopes. We will soon see another important aspect of this paper.

Danilov's wonderful 1978 survey paper *Geometry of toric varieties* [**76**] is a high point of the era. He drew on all of the references mentioned so far, with the exception of [**205**], which was not known in Russia at the time. Danilov covers an amazing amount of material in [**76**]. In addition to the basic facts about toric varieties already mentioned, we also find:

- The inverse limit formula (4.2.5) for Cartier divisors.
- The isomorphism $\omega_{X_\Sigma} = \widehat{\Omega}^n_X \simeq \mathscr{O}_{X_\Sigma}(-\sum_\rho D_\rho)$ from Theorem 8.2.3.
- The formula for $\Gamma(U_\sigma, \widehat{\Omega}^p_{U_\sigma})$ given in Proposition 8.2.18.
- The Demazure vanishing theorem (Theorem 9.2.3).
- The toric proof of Serre duality (Exercise 9.2.12).
- The Bott-Steenbrink-Danilov vanishing theorem (Theorem 9.3.1).
- The fundamental group of a toric variety (Theorem 12.1.10).
- The Chow and cohomology rings of a smooth toric variety (Theorem 12.5.3).
- Riemann-Roch and lattice points in polytopes (Chapter 13).

Danilov's paper remains to this day one of the best introductions to toric geometry.

Toric Varieties. The name "toric variety" was not used until 1977. In earlier works, we find a variety of names, such as:

- In [**82**], Demazure says "the scheme defined by the fan Σ."
- In [**172**], Kempf et al. say "torus embedding."
- In [**205**], Miyake and Oda say "almost homogeneous algebraic variety under torus action."

The 1977 article of Khovanskii mentioned earlier originally appeared in Russian as Многогранники Ньютона и торические многообразия. This was translated as *Newton polyhedra and toroidal varieties*, but "toroidal" is not the right word for торические (toricheskie), because the toroidal varieties defined in [**172**] are slightly different from toric varieties. Something different was needed.

The same problem occurred with Danilov's 1978 survey, whose Russian title is Геометрия торических многообразий. One translation was *Geometry of toral varieties*, but fortunately for us, "toral" did not stick. When Miles Reid translated the paper into English for the Russian Math Surveys, he chose the title *Geometry of toric varieties*. This is the origin of the name "toric variety."

It took a while before "toric variety" became standard. For many years, the term "torus embedding" introduced in [**172**] was more common, especially since it was used in Oda's wonderful books [**217**] from 1978 and [**218**] from 1988. The switch to "toric variety" was confirmed by Oda's use of the term in title of his survey papers from 1991 [**219**] and 1994 [**221**].

Polytopes and Normal Fans. Polytopes have been objects of mathematical interest for over 2000 years. The study of lattice points is more recent, dating from the 19th century. For example, in 1844 Eisenstein gave a nice proof of quadratic reciprocity that involved the interior lattice points in the triangle $\mathrm{Conv}(0, \frac{p}{2}e_1, \frac{p}{2}e_1 + \frac{q}{2}e_2)$ for distinct odd primes p and q.

In toric geometry, the most interesting object associated to a lattice polytope is its normal fan. For many years, normal fans were only implicit in the polytope literature. Here are some examples:

- Proposition 6.2.13 relates refinements of normal fans and Minkowski sums. This result is implicit in the 1963 paper *Decomposable convex polyhedra* by Shephard [**249**].
- Given a polytope P, the cones $C_{\mathsf{v}} = \mathrm{Cone}(P - \mathsf{v})$ for v a vertex of P are studied in Exercise 3.4.9 of Grünbaum's classic 1967 book *Convex Polytopes* [**128**]. The duals of these cones are the maximal cones of the normal fan of P.
- Proposition 6.2.18 relates normal fans of zonotopes to central hyperplane arrangements. This is implicit in McMullen's 1971 paper *On zonotopes* [**201**].

In the toric literature, normal fans are implicit in Khovanskii's 1977 paper [**173**] and are described clearly for the first time in Oda's 1988 book [**218**], though the term "normal fan" did not appear in print until the 1990s (see, for example, [**28**]).

Toric varieties came to the attention of the polytope community when people started using them to prove theorems about polytopes, such as Khovanskii's 1977 toric proof of Ehrhart reciprocity [**173**]. Shortly thereafter, in 1979, Teissier [**268**] and Khovanskii (unpublished) used toric varieties and the Hodge index theorem to prove the Alexandrov-Fenchel inequality for mixed volumes, which states that

$$MV(P_1,P_2,P_3,\dots,P_n)^2 \geq MV(P_1,P_1,P_3\dots,P_n)\,MV(P_2,P_2,P_3,\dots,P_n)$$

for rational polytopes $P_1,\dots,P_n$ in $\mathbb{R}^n$. See [**105**, Sec. 5.4] for a toric proof.

The McMullen Conjecture. The deeper connection between toric varieties and polytopes appears in Stanley's 1980 paper [**257**], which completed the proof of the McMullen conjecture on the face numbers of an n-dimensional simplicial polytope P. If f_i is the number of i-dimensional faces of P, then Stanley defined

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{n-j}{n-i} f_{j-1} = \sum_{\ell=n-i}^{n} (-1)^{\ell-(n-i)} \binom{\ell}{n-i} f_{n-\ell-1}, \tag{A.1.1}$$

where $f_{-1} = 0$. Note that h_i equals the number $h_{n-i}(P)$ defined in (12.5.13). The h_i satisfy the Dehn-Sommerville equations

$$h_i = h_{n-i}, \tag{A.1.2}$$

and in 1971 McMullen conjectured that

$$h_i - h_{i-1} \geq 0, \quad 1 \leq i \leq \lfloor \tfrac{n}{2} \rfloor, \tag{A.1.3}$$

and that if

$$h_i - h_{i-1} = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_r}{r}$$

with $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ and $n_i > n_{i-1} > \cdots > n_r \geq r \geq 1$, then

$$h_{i+1} - h_i \leq \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_r+1}{r+1}. \tag{A.1.4}$$

Furthermore, McMullen conjectured that if positive integers $f_0,\dots,f_{n-1}$ satisfy (A.1.2)–(A.1.4), then they are the face numbers of an n-dimensional simplicial polytope. In other words, he conjectured that (A.1.2)–(A.1.4) are necessary and sufficient conditions for the f_i to come from a simplicial polytope.

The sufficiency of (A.1.2)–(A.1.4) was proved by Billera and Lee [**27**]. It thus remained to show necessity, specifically that (A.1.3) and (A.1.4) hold for any simplicial polytope. Stanley's paper [**257**] is three pages long, where the first page recalls the conjecture and the third is mostly references. It is a one-page proof!

Stanley first explains why one can assume that $P \subseteq N_{\mathbb{R}}$ is rational with the origin as an interior point. Then cones over the faces of P give a complete fan Σ in $N_{\mathbb{R}}$. The toric variety X_Σ is projective, which for us is easy since Σ is the normal fan of the dual polytope $P^\circ \subseteq M_{\mathbb{R}}$. Stanley had to work a little harder to prove this since the theory of normal fans was not fully developed at the time. Note also that X_Σ is an orbifold since P is simplicial. Hence:

- (A.1.2) follows from Poincaré duality.
- (A.1.3) follows from the hard Lefschetz theorem.

Stanley then proves (A.1.4) using hard Lefschetz and known results about graded algebras. A complication is that his arguments require a version of hard Lefschetz that applies to projective orbifolds, yet Steenbrink's proof in [**261**] has a gap. To get a complete proof of hard Lefschetz in this situation, one needs to use hard Lefschetz for intersection cohomology, which was proved by Saito in 1990. We discuss the intersection cohomology of toric varieties in §12.5 and give a careful statement of hard Lefschetz in Theorem 12.5.8. See also [**105**, Sec. 5.2].

When we studied polytopes in §9.4, we focused on simple polytopes $P \subseteq M_{\mathbb{R}}$ and used their face numbers to define the numbers h_i in Theorem 9.4.7. These differ from the h_i defined in (A.1.1), but are closely related since the dual of a simple polytope is simplicial. See Exercise 9.4.10.

A nice commentary on the first decade of toric geometry was written by Reid in 1983 [**237**], where he notes that the construction of the toric variety of a fan

> has been of considerable use within algebraic geometry in the last 10 years ... and has also been amazingly successful as a tool of algebro-geometric imperialism, infiltrating areas of combinatorics. For example, the hard Lefschetz theorem on the cohomology of projective varieties has been translated into combinatorics to complete the proof of a long-standing conjecture of P. McMullen giving an if and only if condition for the existence of a simplicial polytope with a given number f_i of i-dimensional faces.

Final Comment. We end with an observation from the "social history" of modern mathematics. The foundational works on toric varieties were written in a variety of languages, including English, French, German and Russian. This is quite different from the current era, where most mathematics is published in English.

§A.2. The Story Since 1980

After 1980, the study of toric varieties expanded rapidly, especially during the 1990s. We will say some brief words about some of these developments, though we caution the reader that many important topics will be omitted.

The 1980s. There was a steady stream of papers about toric varieties in the 1980s, some of which are featured in this book. Here are a few examples:

- In 1983, Reid published *Decomposition of toric morphisms* [**236**], cited in Chapter 15.
- In 1986, Danilov and Khovanskii published *Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers* [**77**], whose results give a vast generalization of the sectional genus formula proved in Proposition 10.5.8.
- In 1988, Kleinschmidt published *A classification of toric varieties with few generators* [**177**], discussed in §7.3.

- In 1989, Gel′fand, Kapranov and Zelevinsky published *Newton polyhedra of principal A-determinants* [**112**], which introduced the GKZ decomposition that we studied in Chapters 14 and 15.

The titles of these papers give a good sense of the wide range of topics relevant to toric geometry. A nice overview of the evolution of toric varieties in the 1980s can be found in Oda's 1989 survey paper [**219**].

One notable event of the decade was the 1988 publication of Oda's *Convex Bodies and Algebraic Geometry* [**218**]. This mature and polished book has had a major influence on the field.

The 1990s. In 1989, the first author had the good fortune to attend a conference at Washington University where Bill Fulton gave a series of lectures on toric varieties, based on notes that eventually became his wonderful 1993 book *Introduction to Toric Varieties* [**105**]. As a result, toric geometry began the 1990s with a rich body of work and the superb expositions of Oda and Fulton. This, coupled with two other events, led to an explosion of papers about toric varieties.

The first event was the quotient representation of toric varieties given in §5.1, which appeared in Audin's book *The Topology of Torus Actions on Symplectic Manifolds* [**11**] in 1991. As before, this construction "cropped up independently" in several places in the 1990s. The 1995 paper [**65**] notes five independent discoveries of Theorem 5.1.11. The total coordinate ring from §5.2 was introduced in [**65**] and has been used in non-toric situations, often under the name "Cox ring" (see [**150**]).

The second event was mirror symmetry in mathematical physics, which had a huge impact on algebraic geometry and on the study of toric varieties in particular. Here are three brief hints of the role played by toric geometry:

- The 1993 paper *Phases of $N = 2$ theories in two dimensions* [**277**] by Witten used the symplectic version of the quotient representation of a toric variety (see the end of §12.2) to construct a quantum field theory called a gauged linear sigma model.
- The 1994 paper *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties* by Batyrev [**16**] showed that the duality of reflexive polytopes (see §8.3 and Proposition 11.2.1) is related to mirror symmetry.
- The 1994 paper *Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory* by Aspinwall, Greene and Morrison [**7**] used ideas from the secondary fan (see Chapters 14 and 15) to study moduli spaces associated to certain quantum field theories.

The full story is summarized in the paper [**66**, Sec. 10] and developed in more detail in the book [**68**].

One way to understand the growth of the field during this decade is to consider the four survey papers that have been written in recent years about toric varieties:

- Oda's 1989 *Geometry of toric varieties* [**219**], with 114 references.
- Oda's 1994 *Recent topics on toric varieties* [**221**], with 64 references.
- Cox's 1997 *Recent developments in toric geometry* [**66**], with 157 references.
- Cox's 2001 *Update on toric geometry* [**67**], with 240 references.

We should also mention three notable books published in the 1990s:

- The 1993 book *Discriminants, Resultants and Multidimensional Determinants* [**113**] by Gel′fand, Kapranov and Zelevinsky included the first careful treatment of nonnormal toric varieties.
- The 1996 book *Combinatorial Convexity and Algebraic Geometry* by Ewald [**93**] was the first attempt to explain the classic theory of normal toric varieties to a wider audience than just algebraic geometers.
- The 1996 book *Gröbner Bases and Convex Polytopes* by Sturmfels [**264**] covers the elementary theory of affine and projective toric varieties and explains their relation to toric ideals, Gröbner Bases, and graded algebras.

Applications of Toric Varieties. Besides the applications to physics coming from mirror symmetry, toric varieties began to be applied to other contexts starting in the mid 1990s. Here are five examples:

- Relations between toric ideals and integer programming are discussed in the 1996 book *Gröbner Bases and Convex Polytopes* [**264**] mentioned above.
- Applications to solving systems of polynomial equations are described in the 1998 book *Using Algebraic Geometry* [**70**].
- Applications to geometric modeling are explored in the 2003 book *Topics in Algebraic Geometry and Geometric Modeling* [**116**].
- Applications to coding theory appear in the 2008 book *Advances in Algebraic Geometry Codes* [**192**].
- Applications to algebraic statistics are discussed in the 2009 book *Lectures on Algebraic Statistics* [**84**].

Toric Varieties in the 21st Century. Since 2001, research on toric varieties has continued at a rapid pace. One can measure the growth of the field by looking at the MathSciNet database. Starting in 1991, toric varieties have had their own classification number, 14M25, and during the period between January 2001 and August 2010, there were

- 229 papers that listed 14M25 as their primary classification, and
- another 434 papers that listed 14M25 as their secondary classification.

It is our hope that this book will help you understand the reasons for this amazing activity and encourage some of you to make your own contributions to this wonderful field of mathematics.

Appendix B

Computational Methods

There are a wide range of packages available for computations in toric geometry. We focus on packages supported by two open source systems:

(a) `Macaulay2` [**123**], by Dan Grayson, Mike Stillman and collaborators.

(b) `Sage` [**262**], by William Stein and collaborators.

Both `Magma` and `GAP` also have toric packages, and the open source algebra systems `Singular` [**78**] and `CoCoA` [**72**] are functionally similar to `Macaulay2`. The `Macaulay2` toric package `NormalToricVarieties` [**252**] is by Greg Smith, and the `Sage` toric package `ToricVarieties` [**44**] is by Volker Braun and Andrey Novoseltsev. There are many other programs which are useful for computations in toric and polyhedral geometry; an incomplete list might include:

(a) `Normaliz` by Bruns, Ichim and Söger [**57**].

(b) `LattE` by De Loera [**79**].

(c) `Polymake` by Gawrilow and Joswig [**114**].

(d) `Polyhedra(Sage)` by Braun, Hampton, Novoseltsev and collaborators [**43**].

(e) `4ti2` by Hemmecke, Köppe, Malkin and Walter [**140**].

(f) `Gfan` by Jensen [**161**].

(g) `TOPCOM` by Rambau [**233, 234**].

Many of the programs listed above have interfaces to `Macaulay2` and `Sage`, as well as interfaces to each other. There are also packages for applications, such as coding theory (`toriccodes` by Ilten [**153**], or `toric` by Joyner [**163**]) and physics (`PALP`, by Kreuzer and Skarke [**182, 183**]). In what follows, we will illustrate how to compute with toric varieties by working through a series of examples. To save space, we will often suppress superfluous output.

§B.1. The Rational Quartic

Example B.1.1. We begin with a "barehanded" example. In Exercise 1.1.7, we considered the map $\phi : \mathbb{C}^2 \to \mathbb{C}^4$ defined by

$$\phi(s,t) = (s^4, s^3t, st^3, t^4).$$

Since the monomials are homogeneous, ϕ also defines a map $\mathbb{P}^1 \to \mathbb{P}^3$. Note that these monomials correspond to the subset $\mathscr{A} \subseteq \mathbb{Z}^2$ given by the columns of

$$\begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

To find the image of ϕ via `Macaulay2`, we compute the kernel I of $g = \phi^* : S \to R$:

```
i1: R = QQ[s,t]
i2: S = QQ[x,y,z,w]
i3 : g = map(R,S,{s^4,s^3*t,s*t^3,t^4})
o3 : RingMap R <--- S
bi4 : I = kernel g       3      2     2    3    2
o4 = ideal (y*z - x*w, z  - y*w , x*z  - y w, y  - x z)
```

These are the equations appearing in Exercise 1.1.7. $\Diamond$

Example B.1.2. The map ϕ of the previous example gives the projective curve $X_{\mathscr{A}} \subseteq \mathbb{P}^3$. Example 2.1.10 showed that $X_{\mathscr{A}}$ is normal but not projectively normal. To see this computationally, we use Exercise 9.4.6, which implies that $X_{\mathscr{A}} \subseteq \mathbb{P}^3$ is projectively normal if and only if $X_{\mathscr{A}}$ is normal and $H^1(\mathbb{P}^3, \mathscr{I}_{X_{\mathscr{A}}}(\ell)) = 0$ for all $\ell \geq 0$. By local duality [**89**, Thm. A4.2] for the curve $X_{\mathscr{A}} \subseteq \mathbb{P}^3$,

$$H^1(\mathbb{P}^3, \mathscr{I}_{X_{\mathscr{A}}}(\ell)) \simeq \mathrm{Ext}^3_S(S/I_{X_{\mathscr{A}}}, S)_{-\ell-4}.$$

Since $X_{\mathscr{A}}$ is defined by the ideal I computed in Example B.1.1, we obtain:

```
i5 : E3 = Ext^3(coker gens I, S)
o5 = cokernel {-5} | w z y x |
i6 : hilbertFunction(-5,E3)
o6 = 1
```

Since $\dim H^1(\mathbb{P}^3, \mathscr{I}_{X_{\mathscr{A}}}(1)) = 1$, projective normality fails. This reflects our failure to use all lattice points of the polytope $P = \mathrm{Conv}(\mathscr{A})$ to map to projective space. Our computational check works in general: a normal variety $X \subseteq \mathbb{P}^n$ is projectively normal exactly when $\mathrm{Ext}^n_S(S/I_X, S) = 0$, where $S = \mathbb{C}[x_0, \ldots, x_n]$. $\Diamond$

Exercises for §B.1.

B.1.1. Use Chapter 9 to show that $H^i(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(\ell)) = 0$ for all $1 \leq i \leq n-1$ and all ℓ.

§B.2. Polyhedral Computations

Example B.2.1. The cone $\sigma = \mathrm{Cone}(e_1, e_2, e_1+e_3, e_2+e_3)$ is depicted in Figure 2 of Chapter 1, and Example 1.2.9 showed that the dual cone is given by $\sigma^\vee = \mathrm{Cone}(e_1, e_2, e_3, e_1+e_2-e_3) \subseteq \mathbb{R}^3$. The `Macaulay2` package `Polyhedra` is one option for computing with cones: for the example above, we have

```
i1 : loadPackage "Polyhedra"
i2 : M= matrix{{1,0,0},{0,1,0},{1,0,1},{0,1,1}}
i3 : C=posHull transpose M
o3 = {ambient dimension => 3           }
      dimension of lineality space => 0
      dimension of the cone => 3
      number of facets => 4
      number of rays => 4
i4 : rays C
o4 = | 1 0 1 0 |
     | 0 1 0 1 |
     | 0 0 1 1 |
i5 : fVector C
o5 = {1, 4, 4, 1}
i6 : Cv = dualCone C
i7 : rays Cv
o7 = | 1 0 1  0 |
     | 0 1 1  0 |
     | 0 0 -1 1 |
i8 : hilbertBasis C
o8 = {| 1 |, | 0 |, | 1 |, | 0 |}
      | 0 |  | 1 |  | 0 |  | 1 |
      | 0 |  | 0 |  | 1 |  | 1 |
```

◇

Example B.2.2. We now illustrate a fan computation. In Example 6.1.17 we studied the smooth fan `F` consisting of a subdivision of the positive orthant into cones `B1,..,B9`, and the remaining 7 orthants `C1,..,C7`.

```
o9 :  C1=posHull transpose matrix{{0,-1,0},{0,0,1},{-1,0,0}};
      (input cones C2..B9, supressed)
i25 : F=fan{C1,C2,C3,C4,C5,C6,C7,B1,B2,B3,B4,B5,B6,B7,B8,B9}
o25 = {ambient dimension => 3          }
       number of generating cones => 16
       number of rays => 10
       top dimension of the cones => 3
i26 : isPolytopal F
o26 = false
```

◇

Example B.2.3. In this example, we compute the secondary polytope for the convex hull of six points in the plane. The secondary polytope is an *associahedron*, and appears in Example 15.2.13. One package which computes the secondary polytope is `TOPCOM` by Rambau. Braun has created a `Sage` version of `TOPCOM`, which we use for the computation (you may need to install this package separately).

```
sage: pointmatrix = matrix([
              [ 1, 1, 0, -1, -1, 0 ],
              [ 0, 1, 1,  0, -1, -1 ]])
sage: pc = PointConfiguration(pointmatrix.columns())
sage: Tris = pc.triangulations_list()
sage: Tri = Tris[7]
sage: list(Tri)
      [[0, 1, 4], [0, 4, 5], [1, 2, 3], [1, 3, 4]]
sage: show(Tris[7].plot(axes=False))
```

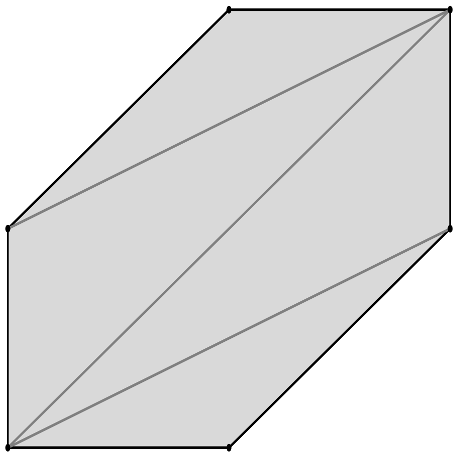


Figure 1. The seventh triangulation

Next, we generate the secondary polytope and face lattice

```
sage: K3 = pc.secondary_polytope(); K3
      A lattice polytope: 3-dimensional, 14 vertices.
sage: K3.faces()
     [[0],[1], (vertices: output suppressed) [13]],
     [[1,6],[5,6],[0,1],[0,5],[6,10],[1,4],[4,13],[10,13],
      [5,7],[7,9],[9,10],[3,4],[2,3],[0,2],[3,12],[12,13],
      [2,8],[8,11],[11,12],[7,8],[9,11]],
     [[0,1,5,6],[7,8,9,11],[0,2,5,7,8],[0,1,2,3,4],[3,4,12,13],
      [2,3,8,11,12],[5,6,7,9,10],[1,4,6,10,13],[9,10,11,12,13]]
```

```
sage: plotK3 = K3.plot3d() #view the secondary polytope
sage: plotK3.rotateZ(60*pi/180).show(viewer='tachyon')
```

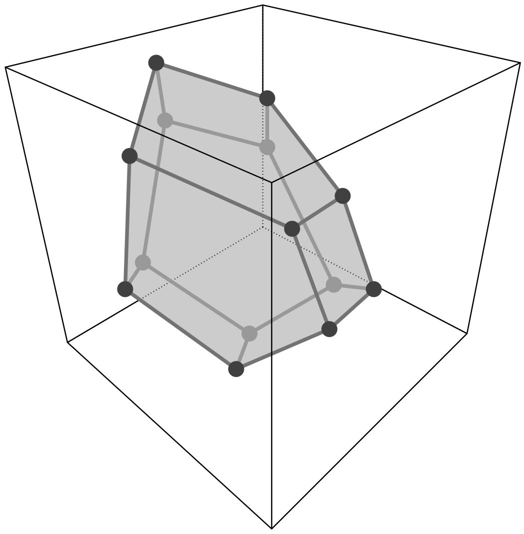


Figure 2. The secondary polytope $\mathcal{K}^3$

Figure 2 is a simplified version of the Sage output optimized for viewing in black and white. To see the 1-skeleton appearing in Figure 9 of Example 15.2.13, enter:

```
sage: Kp = Polyhedron(vertices=K3.vertices().columns())
sage: Egraph = Kp.graph()
sage: Egraph.show()
```

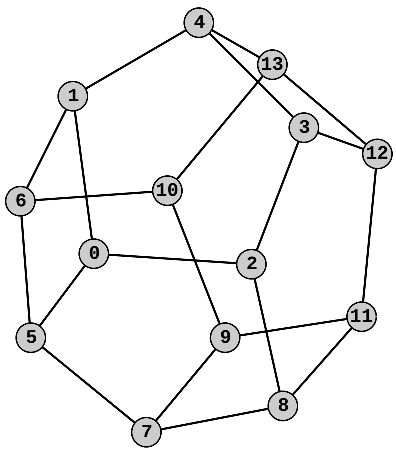


Figure 3. The edge graph of $\mathcal{K}^3$ ◊

Example B.2.4. In our final polyhedral example, we compute the Gröbner fan of the ideal $\langle x^7-1, y-x^5\rangle$ which appears in Example 10.3.1, using the package `Gfan`, which has both `Macaulay2` and `Sage` interfaces. In `Macaulay2`:

```
i1 : R=QQ[x,y]; I=ideal(x^7-1, y-x^5); gfan(I)
       +-------------+----------------------------+
       | 7           | 7          3               |
o1   = |{y , x}      |{y  - 1, - y  + x}          |
       +-------------+----------------------------+
       | 3   2    3 | 3        2         3    2   |
       |{y , x y, x }|{y  - x, x y - 1, x  - y }  |
       +-------------+----------------------------+
       | 2   2    5 |    3    2   2        5      |
       |{y , x y, x }|{- x  + y , x y - 1, x  - y}|
       +-------------+----------------------------+
       |     7       |    5       7               |
       |{y, x }      |{- x  + y, x  - 1}          |
       +-------------+----------------------------+
```

And in `Sage`:

```
sage: R.<x,y,z> = PolynomialRing(QQ)
sage: I=R.ideal([x^7-1,y-x^5]).groebner_fan()
sage: I.reduced_groebner_bases()
[[y^7 - 1, -y^3 + x], [y^3 - x, x^2*y - 1, x^3 - y^2],
[-x^3 + y^2, x^2*y - 1, x^5 - y], [-x^5 + y, x^7 - 1]]
sage: I.render()
```

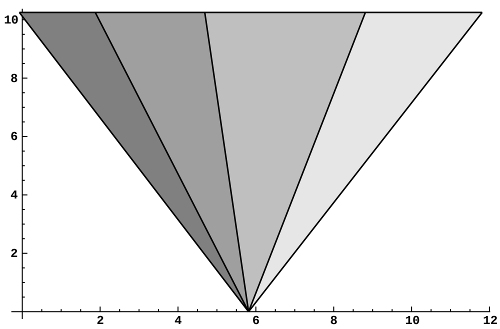


Figure 4. Output of the render command

The render command expects input in three variables (hence the three variable ring). The figure above is the intersection of the Gröbner fan with the standard two simplex, which after an affine transformation gives the fan of Example 10.3.1. $\diamondsuit$

§B.3. Normalization and `Normaliz`

The `Normaliz` package is designed for computations with affine monoids, vector configurations, lattice polytopes, and rational cones.

As input, `Normaliz` expects a file containing matrices; each matrix is preceded by two lines, the first indicating the number of rows and the second the number of columns.

Example B.3.1. We illustrate `Normaliz` for the very ample but not normal polytope of Example 2.2.20.

```
10
6
1 1 1 0 0 0
(8 additional lattice points, suppressed)
0 0 1 1 0 1
1
```

Save the file above as `VeryampleNonNormal.in`. `Normaliz` supports ten types of input, specified by the number on the line following the matrix. In the example above the `1` on the last line indicates we want to normalize the monoid generated by the rows of the matrix. To execute the code, use the graphical interface `jNormaliz`, or type directly from the command line:

```
% norm64 VeryampleNonNormal.in
```

The output is saved in the file `VeryampleNonNormal.out`, which reads:

```
11 Hilbert basis elements
10 height 1 Hilbert basis elements
10 extreme rays
22 support hyperplanes
rank = 6 (maximal)
index = 1
original monoid is not integrally closed
extreme rays are homogeneous via the linear form:
1 1 1 1 1 1
Hilbert basis elements are not homogeneous
multiplicity = 21
h-vector:
1 4 11 4 1 0
Hilbert polynomial:
1/1 157/60 25/8 53/24 7/8 7/40
****************************************************
11 Hilbert basis elements:
(output suppressed: original rays and 1 1 1 1 1 1)
```

```
10 extreme rays:
(output suppressed: original rays)
22 support hyperplanes:
 -1  2 -1  2  2 -1
 -1 -1  2  2  2 -1
(remaining 20 hyperplanes suppressed)
1 congruences:
 1 1 1 1 1 1 3
10 height 1 Hilbert basis elements:
(output suppressed: original rays)
```

`Normaliz` has interfaces for both `Macaulay2` and `Singular`. For example, start `Macaulay2` in the directory containing `Normaliz`, and enter the commands

```
i1 : installPackage "Normaliz"
i2 : A = matrix{{1,1,1,0,0,0},(suppressed),{0,0,1,1,0,1}}
i3 : normaliz(A,1)
```

◊

Example B.3.2. Consider the binomial ideal I computed in Example B.1.1. To find the associated monoid and normalization, we use the option `10` in `Normaliz`. We encode the first generator $yz-xw$ of I as `-1 1 1 -1`, and similarly for the other three generators. Hence our input file is

```
4
4
-1 1 1 -1
0 -1 3 -2
1 -2 2 -1
-2 3 -1 0
10
```

This yields output (with some items suppressed)

```
4 original generators:
 0 4
 3 1
 1 3
 4 0
5 Hilbert basis elements:
 4 0
 3 1
 2 2
 1 3
 0 4
```

You should compare this to what we computed in Example 1.3.9. ◊

Example B.3.3. In Example 9.4.5 we found the Ehrhart polynomial for the lattice simplex $\mathrm{Conv}\{(0,0,0),(1,0,0),(0,1,0),(1,1,3)\}$. Create an input file `B3-3.in` as below:

```
4
3
0 0 0
1 0 0
0 1 0
1 1 3
2
```

The command `norm64 -h B3-3.in` produces an output file `B3-3.out` as below. The `-h` option stipulates that the Ehrhart polynomial is to be included in the output data.

```
4 lattice points in polytope
4 extreme points of polytope
4 support hyperplanes
polytope is not integrally closed
dimension of the polytope = 3
normalized volume = 3
h-vector:
1 0 2 0
Ehrhart polynomial:
1/1 3/2 1/1 1/2
```

The translates to the polynomial $1+\frac{3}{2}x+x^2+\frac{1}{2}x^3$ from Example 9.4.5. $\Diamond$

§B.4. Sheaf Cohomology and Resolutions

Example B.4.1. Example 9.5.6 computed $H^1(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(-5))$ via Ext. Let $I = \langle x,y\rangle \subseteq S = \mathbb{C}[x,y]$. Then

$$H^1(\mathbb{P}^1,\mathscr{O}_{\mathbb{P}^1}(-5)) = H^2_I(S)_{-5} = \mathrm{Ext}^2_S(A,S)_{-5},$$

where $A = S/I^{[k]} = S/\langle x^k,y^k\rangle$ for k sufficiently large. Since $\dim H^2_I(S)_{-2} = 4$ by Example 9.5.3, we can use `Macaulay2` to find a k that works. Suppressing some outputs, one computes:

```
i1 : S = QQ[x,y]
i2 : A = coker matrix{{x^3,y^3}}
i3 : hilbertFunction(-5,Ext^2(A,S))
o3 = 2
```

So $k=3$ is not big enough. Repeating this with $k=4$ shows that $k=4$ works. $\Diamond$

Example B.4.2. In Example 9.5.13, we needed a resolution of the cotangent bundle of $\mathbb{P}^1$. This is done using the commands:

```
i1 : S = QQ[x,y,z]
i2 : OM = ker matrix{{x,y,z}}
i3 : F = res OM
i4 : F.dd
           3                  1
o4 = 0 : S  <-------------- S  : 1
              {2} | z  |
              {2} | x  |
              {2} | -y |
```

The last two commands compute the resolution and display the differentials. This gives the resolution shown in (9.5.9). ♢

§B.5. Sheaf Cohomology on the Hirzebruch Surface $\mathscr{H}_2$

This example illustrates the `NormalToricVarieties` package for `Macaulay2`, written by Greg Smith. Use `viewHelp` for documentation on the commands.

Example B.5.1. In Example 13.2.12 we studied the line bundle $\mathscr{L} = \mathscr{O}_X(D)$ for $D = 3D_3 - 5D_4$ on $X = \mathscr{H}_2$, computing that $H^0(X,\mathscr{L}) = 0$, $\dim H^1(X,\mathscr{L}) = 2$, and $\dim H^2(X,\mathscr{L}) = 6$. The command `hirzebruchSurface(2)` creates the toric variety $X = \mathscr{H}_2$:

```
i1 : installPackage "NormalToricVarieties"
i2 : H2 = hirzebruchSurface(2)
i3 : rays H2
o3 = {{1, 0}, {0, 1}, {-1, 2}, {0, -1}}
i4 : max H2
o4 = {{0, 1}, {0, 3}, {1, 2}, {2, 3}}
i5 : isFano H2
o5 = false
```

Now we bring D into the picture. In `NormalToricVarieties` a divisor is specified by a tuple corresponding to the rays of the fan, so with the rays ordered as above, D is $3D_0 - 5D_3$.

```
i6 : D = 3*H2_0 - 5*H2_3
o6 = 3*D  - 5*D
        0      3
i7 : isCartier D
o7 = true
i8 : isAmple D
o8 = false
```

To compute cohomology, we turn D into the sheaf $\mathscr{L} = \mathscr{O}_X(D)$, and use a loop to display the ranks of $H^i(X,\mathscr{L})$:

```
i9 : L = OO D
       1
o9 = OO (3,-5)
       H2
i10 :  for i to 2 list rank HH^i(H2, L)
o10 = {0, 2, 6}
```

◊

Example B.5.2. We continue to study cohomology on $X = \mathscr{H}_2$, but now turn our attention to Ω^1_X. In Table 4 of Example 9.5.14 we computed $\dim H^1(X,\Omega^1_X(a,b))$ for $(a,b) \in \{(-2,-2),\ldots,(2,2)\}$. We will build a script called `cohomologyTable` which takes as input an integer k reflecting the H^k to compute, a sheaf, and high and low values for the range of degrees in the Picard group. Since `cohomologyTable` builds on packages `BoijSoederberg` and `BGG`, we need to load these. Below, we check the computations in Example 9.5.14

```
i11 : loadPackage "BoijSoederberg"
i12 : loadPackage "BGG"
i13 : OM = cotangentSheaf H2

o13 = cokernel {2, 0}  | 2x_1x_3 |
               {-1, 2} | x_0     |
               {-1, 2} | -x_2    |
                                                  1           2
o13 : coherent sheaf on H2, quotient of OO  (-2,0)++OO  (1,-2)
                                          H2          H2
i14 : cohomologyTable := (k,F,lo,hi) ->
      (DegRange = toList(lo#0..hi#0);
      new CohomologyTally from select(flatten apply(DegRange,
               j -> apply(toList(lo#1..hi#1),
               i -> {(j,i-j), rank HH^k(variety F, F(i,j))})),
               p -> p#1 != 0));
i15 : cohomologyTable(1,OM,{-2,-2},{2,2})
          -2 -1 0 1 2
o15 =  2:  3  2 1 . .
       1:  4  2 1 . .
       0:  3  2 2 2 3
      -1:  .  . 1 2 4
      -2:  .  . 1 2 3
```

◊

Exercises for §B.5.

B.5.1. Compute the cohomology table for $H^i(\mathbb{P}^2,\Omega^1_{\mathbb{P}^2}(j))$ appearing in Example 9.5.13.

§B.6. Resolving Singularities

In this section we use the `ToricVarieties` package in `Sage`, written by Braun and Novoseltsev, to resolve singularities. The package finds a simplicial resolution automatically.

Example B.6.1. Let us revisit Example B.2.1. Consider the affine toric variety defined by the cone $\sigma = \mathrm{Cone}(e_1, e_2, e_1+e_3, e_2+e_3)$, which is the nonsimplicial affine threefold appearing in Example 11.1.12. We first find a simplicial resolution:

```
sage: C = Cone([(1,0,0), (0,1,0), (1,0,1), (0,1,1)]); C
3-d cone in 3-d lattice N
sage: QuadCone = AffineToricVariety(C)
sage: Orbif = QuadCone.resolve_to_orbifold()
sage: Orbif.fan().ngenerating_cones()
2
sage: [ cone.ambient_ray_indices() for cone in Orbif.fan() ]
[(2, 3, 0), (1, 3, 0)]
```

This is the refinement Σ_1 of σ appearing in Figure 3 of Example 11.1.12. Note that the rays are indexed by `0..3` rather than `1..4`. $\Diamond$

Example B.6.2. In this example, we resolve weighted projective space $\mathbb{P}(1,1,2)$, whose fan is pictured in Figure 11 of Example 2.4.6.

```
sage: s0 = Cone([(0,1), ( 1, 0)]);
sage: s1 = Cone([(0,1), (-1,-2)]);
sage: s2 = Cone([(1,0), (-1,-2)]);
sage: F = Fan([s0, s1, s2]);
sage: P112 = ToricVariety(F);  P112
2-d toric variety covered by 3 affine patches
sage: P112.is_orbifold()
True
sage: P112.is_smooth()
False
sage: lp = LatticePolytope( matrix(F.rays()).transpose() )
sage: lp.points()  # columns are all points in the convex hull
[ 0  1 -1  0  0]
[ 1  0 -2 -1  0]
sage: BlP112 = P112.resolve(new_rays=[(0,-1)]);  BlP112
2-d toric variety covered by 4 affine patches
sage: BlP112.is_smooth()
True
```

We used this resolution of $\mathbb{P}(1,1,2)$ in Example 13.4.4. $\Diamond$

§B.7. Intersection Theory and Hirzebruch-Riemann-Roch

Example B.7.1. In Example 13.2.12, we checked the Hirzebruch-Riemann-Roch theorem for the divisor $D = D_3 - 5D_4$ on $\mathscr{H}_2$. Using the `ToricVarieties` package in `Sage` and observing that the variety `BlP112` in the previous example is $X = \mathscr{H}_2$, we calculate

$$\int_X \mathrm{ch}(\mathscr{L}) \cdot \mathrm{Td}(X), \quad \mathscr{L} = \mathscr{O}_X(D).$$

```
sage: BlP112.fan().ray_matrix()
      [ 0  1 -1  0]
      [ 1  0 -2 -1]
sage: BlP112.cohomology_ring().gens()
     ([2*z2 + z3], [z2], [z2], [z3])
sage: d4 = BlP112.fan().cone_containing(0);d4
sage: D4 = d4.cohomology_class()
      [2*z2 + z3]
sage: d3 = BlP112.fan().cone_containing(1);d3
sage: D3 = d3.cohomology_class(); D3
      [z2]
sage: chL = (3*D3-5*D4).exp(); chL
      [-5*z3^2 - 7*z2 - 5*z3 + 1]
sage: tdX = BlP112.Todd_class(); tdX
      [-1/2*z3^2 + 2*z2 + z3 + 1]
sage: BlP112.integrate(tdX*chL)
      4
```

This agrees with our computation of $\chi(\mathscr{L})$ in §B.5. We can also check by extracting the degree two piece of $\mathrm{ch}(\mathscr{L}) \cdot \mathrm{Td}(X)$:

```
sage: chD*tdX
      [-2*z3^2 - 5*z2 - 4*z3 + 1]
sage: Deg2part=(chD*tdX).part_of_degree(2); Deg2part
      [-2*z3^2]
```

With labelling as in Example 4.1.8, `z3` corresponds to the ray u_2, and one checks that the corresponding divisor D_2 has self-intersection -2. Thus $-2D_2^2 = 4$. $\Diamond$

Example B.7.2. To compute sheaf cohomology using `ToricVarieties`, first we create a divisor from an list indexed by the rays, where the order corresponds to the ray matrix of the fan:

```
sage: Ddiv = BlP112.divisor([-5,3,0,0]);Ddiv
sage: Ddiv.cohomology()
     (0, 2, 6)
sage: M = BlP112.fan().dual_lattice()
```

```
sage: Ddiv.cohomology(M(-2,1))
     (0, 1, 0)
```

The last line gives $\dim H^i(\mathscr{H}_2, \mathscr{O}_{\mathscr{H}_2}(D))_{(-2,1)}$; the weight $(-2,1)$ corresponds to a weight of $(-2,-1)$ in Figure 2 in Example 13.2.12 since the fans are flipped. $\Diamond$

§B.8. Anticanonical Embedding of a Fano Toric Variety

Example B.8.1. The toric variety $X = X_\Sigma$ in Examples 9.1.2 and 9.1.8 is the blowup of $\mathbb{P}^2$ at the three torus-fixed points. We first show how to create this toric variety "from scratch" using the `Macaulay2` package `NormalToricVarieties`. The input is a list of rays and a list of maximal cones. The rays below are ordered as in Figure 2 of Example 9.1.2.

```
i2 : rayList = {{1,0},{1,1},{0,1},{-1,0},{-1,-1},{0,-1}}
i3 : coneList = {{0,1},{1,2},{2,3},{3,4},{4,5},{5,0}}
i4 : BlP2 = normalToricVariety(rayList,coneList)
```

Now let $D = -K_X$ be the anticanonical divisor, which is very ample since X is a smooth Fano surface. We analyze the equations that arise when we embed X into $\mathbb{P}(H^0(X, \mathscr{O}_X(D)))$. Below, we create a script which takes as input a toric divisor D, finds the sections, and returns the ideal defining $X \subseteq \mathbb{P}(H^0(X, \mathscr{L}))$ for $\mathscr{L} = \mathscr{O}_X(D)$. Make a file called `Embed` as below:

```
projEmb = (D)->(X = variety D;
                L = OO D;
                m = rank HH^0(X, L);
                S = ring X;
                R = QQ[y_0..y_(m-1)];
                phi = map(S, R, basis(- first degrees L, S));
                kernel phi)
```

Now we reap the benefits of our work:

```
i5 : D = BlP2_0+BlP2_1+BlP2_2+BlP2_3+BlP2_4+BlP2_5
i6 : load "Embed"
i7 : I = projEmb(D);
i8 : transpose mingens I
o8 = {-2} | u_4u_5-u_3u_6 | (more quadrics, output suppressed)
i9 : hilbertPolynomial(coker gens I, Projective=>false)
       2
o9 = 3i  + 3i + 1
```

$\Diamond$

Exercises for §B.8.

B.8.1. Compute the Ehrhart polynomial of P_{-K_X} using `Normaliz` as in Example B.3.3.

B.8.2. Compute $H^i(X, \mathscr{O}_X(D))$ for the divisor $D = -D_3 + 2D_5 - D_6$ from Example 9.1.8.

Appendix C

Spectral Sequences

Spectral sequences are important tools in modern algebraic geometry and algebraic topology that are used several times in Chapters 9 and 12. Here we give some background necessary for understanding these applications for readers who have not encountered these objects before. Our discussion is very far from complete; we refer the reader to [**115**], [**136**], [**199**], or [**276**] for more comprehensive treatments of spectral sequences.

§C.1. Definitions and Basic Properties

At first glance, spectral sequences may seem rather complicated. Fortunately, they are often straightforward to use in practice.

The Definition. The spectral sequences encountered in this book will all be of the following type and will be relatively simple to understand.

Definition C.1.1. A **(*cohomology*)** ***spectral sequence*** is a collection of abelian groups $E_r^{p,q}$ and homomorphisms $d_r^{p,q}$ with the following structure and properties:

(a) The groups are $E_r^{p,q}$, indexed by integers p,q,r. Fixing r, we obtain one *sheet* of the spectral sequence, which is visualized as a diagram of groups indexed by integer lattice points in the plane.

(b) In the rth sheet, there are homomorphisms

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all p,q,r. In other words, the rth sheet splits up into a collection of cochain complexes in which the differentials are all mappings of bidegree $(r, 1-r)$ for the indexing by p,q.

(c) The $(r+1)$st sheet $E_{r+1}^{p,q}$ is the cohomology of $(E_r^{p,q}, d_r^{p,q})$, i.e.,

$$E_{r+1}^{p,q} = \ker(d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}) \big/ \operatorname{im}(d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q}).$$

In many of the examples in the text, the $E_r^{p,q}$ are vector spaces over $\mathbb{Q}$ or $\mathbb{C}$ and the $d_r^{p,q}$ are linear mappings. There are also homology spectral sequences that appear in many applications; the only differences are that the groups are usually written $E_{p,q}^r$ and the differentials $d_{p,q}^r$ in a homology spectral sequence have bidegree $(-r, r-1)$.

We will always work with *first quadrant* spectral sequences, for which $E_r^{p,q} = 0$ when $p < 0$ or $q < 0$. Thus in each sheet, the nonvanishing terms lie in the quadrant where $p, q \geq 0$. The minimum value of r is usually 1 or 2, in which case we say that $(E_r^{p,q}, d_r^{p,q})$ is an E_1 or E_2 spectral sequence respectively.

Intuition and Meaning of Convergence. When working with an E_1 spectral sequence, for instance, we often know the differentials $d_1^{p,q}$ explicitly. In general, however, the differentials $d_r^{p,q}$ for $r \geq 2$ are much more difficult to describe. Fortunately, there are many cases where the structure of the nonzero terms in a spectral sequence forces many differentials to be zero. For instance, in the first quadrant spectral sequences we will encounter, the differentials mapping to and from $E_r^{p,q}$ for fixed p, q vanish when r is sufficiently large. It follows that for each p, q, there exists some r (depending on p, q) such that

$$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \cdots .$$

This common value is defined to be $E_\infty^{p,q}$.

Definition C.1.2. A first quadrant spectral sequence $(E_r^{p,q}, d_r^{p,q})$ ***converges*** to a sequence of abelian groups H^k, $k \geq 0$, if there is a filtration

$$0 = F^{k+1}H^k \subseteq F^k H^k \subseteq F^{k-1}H^k \subseteq \cdots \subseteq F^1 H^k \subseteq F^0 H^k = H^k$$

of H^k by subgroups such that

$$E_\infty^{p,q} \simeq F^p H^{p+q} / F^{p+1} H^{p+q}.$$

For an E_1 or E_2 spectral sequence, we write this as

$$E_1^{p,q} \Rightarrow H^{p+q} \quad \text{or} \quad E_2^{p,q} \Rightarrow H^{p+q}$$

respectively.

According to [**199**, Ch. 1], the "generic theorem" about spectral sequences says something like: there is a spectral sequence with $E_1^{p,q}$ or $E_2^{p,q}$ "something computable" and converging to H^{p+q}, "something desirable." Several specific examples of this are discussed later. The intuition behind a convergent E_1 spectral sequence is that we can usually compute the E_2 terms from the d_1 differentials explicitly. But we are primarily interested in what the spectral sequence converges to. We can think of $E_2^{p,q}$ as a first approximation to the convergent, and then $E_3^{p,q}, E_4^{p,q}, \ldots$

as better and better approximations. If the situation is favorable, enough of the higher differentials $d_r^{p,q}$ for $r \geq 2$ and enough of the $E_\infty^{p,q}$ may vanish so that the H^{p+q} can be determined.

Here is a simple first example where we can carry out this sort of analysis.

Proposition C.1.3. *Suppose $E_2^{p,q} \Rightarrow H^{p+q}$ is a first quadrant E_2 spectral sequence with the property that $E_2^{p,q} = 0$ for all $q > 0$. Then*

$$E_2^{p,0} \simeq H^p \quad \textit{for all } p \geq 0.$$

Proof. First observe that all differentials $d_r^{p,q}$ vanish for $r \geq 2$ since $E_2^{p,q} = 0$ for $q > 0$. It follows that $E_2^{p,q} = E_\infty^{p,q}$ for all p, q. Then we have

$$E_2^{p,0} \simeq F^p H^p / F^{p+1} H^p = F^p H^p$$

and $0 = E_2^{p-1,1} = E_2^{p-2,2} = \cdots = E_2^{0,p}$ implies

$$0 = F^{p-1} H^p / F^p H^p = F^{p-2} H^p / F^{p-1} H^p = \cdots = F^0 H^p / F^1 H^p.$$

Hence $F^p H^p = F^{p-1} H^p = \cdots = F^1 H^p = F^0 H^p = H^p$. Thus $E_2^{p,0} \simeq H^p$, as claimed. □

The hypothesis of Proposition C.1.3 implies that the differentials $d_2^{p,q}$ vanish for all p, q. This is an instance of the following general idea.

Definition C.1.4. We say that a spectral sequence ***degenerates*** at the E_r sheet if the differentials $d_s^{p,q} = 0$ for all p, q and all $s \geq r$.

Note that degeneration at E_r implies that $E_r^{p,q} \simeq E_\infty^{p,q}$ for all p, q, so we have a strong form of convergence in this case.

If we have a convergent spectral sequence, say $E_2^{p,q} \Rightarrow H^{p+q}$, it is important to note that there are situations where knowing $E_\infty^{p,q}$ for all p, q is not quite enough to determine H^{p+q}. This can happen for instance if there are several $E_\infty^{p,q}$ with $p + q = k$ for some k. Definition C.1.2 shows that we know the quotients

$$E_\infty^{p,q} \simeq F^p H^{p+q} / F^{p+1} H^{p+q}$$

in the filtration of $H^k = H^{p+q}$. However, additional extension data may be needed to determine the isomorphism type of the abelian group H^k in general. This situation does not arise when the $E_r^{p,q}$ are vector spaces over a field. In all cases, though, the proof of Proposition C.1.3 shows the following.

Proposition C.1.5. *Let $E_{r_0}^{p,q} \Rightarrow H^{p+q}$ be a spectral sequence. If $E_\infty^{p,q} = 0$ for all p, q with $p + q = k$ except possibly $p, q = p_0, q_0$, then $H^k \simeq E_\infty^{p_0,q_0}$.* □

Edge Homomorphisms. Another feature of a convergent first quadrant E_2 spectral sequence is that all differentials starting from $E_r^{p,0}$ vanish, so that the spectral sequence gives maps

$$E_2^{p,0} \longrightarrow E_3^{p,0} \longrightarrow \cdots \longrightarrow E_\infty^{p,0} \simeq F^pH^p/F^{p+1}H^p = F^pH^p \subseteq H^p.$$

Definition C.1.6. The map $E_2^{p,0} \to H^p$ is called an ***edge homomorphism***.

This leads to a more precise version of Proposition C.1.3.

Proposition C.1.7. *Suppose $E_2^{p,q} \Rightarrow H^{p+q}$ is a first quadrant E_2 spectral sequence with the property that $E_2^{p,q} = 0$ for all $q > 0$. Then the edge homomorphism*

$$E_2^{p,0} \longrightarrow H^p$$

is an isomorphism for all $p \geq 0$. □

A convergent first quadrant E_2 spectral sequence has second edge homomorphism since all differentials ending at $E_r^{0,q}$ vanish, so that we have the map

$$H^q \longrightarrow H^q/F^1H^q = F^0H^q/F^1H^q = E_\infty^{0,q} \subseteq \cdots \subseteq E_3^{0,q} \subseteq E_2^{0,q}.$$

Definition C.1.8. The map $H^q \to E_2^{0,q}$ is called an ***edge homomorphism***.

For many E_2 spectral sequences, we know some of the edge homomorphisms. For example, we know $E_2^{p,0} \to H^p$ in the two applications of Proposition C.1.7 given in §9.0, and $H^q \to E_2^{0,q}$ is known when we apply the Serre spectral sequence (Theorem C.2.6) in §12.4.

§C.2. Spectral Sequences Appearing in the Text

In the remainder of this appendix we discuss the first quadrant spectral sequences used in Chapters 9 and 12 to derive facts about singular and sheaf cohomology of toric varieties.

The Leray Spectral Sequence. The *Leray spectral sequence* of a continuous map is used in §9.0.

Theorem C.2.1. *Let $f : X \to Y$ be a continuous map of topological spaces, and let $\mathscr{F}$ be a sheaf of abelian groups on X. There is a spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_*\mathscr{F}) \Rightarrow H^{p+q}(X, \mathscr{F}).$$

Furthermore, when $q = 0$, the map $H^p(Y, f_*\mathscr{F}) \to H^p(X, \mathscr{F})$ is the edge homomorphism $E_2^{p,0} \to H^p(X, \mathscr{F})$ (see Definition C.1.6 above). □

This spectral sequence is covered in [**115**, II.4.17] and [**125**, p. 463], and its intuitive meaning is explained in the discussion leading up to Proposition 9.0.8.

The Spectral Sequence of a Covering. Next, let $\mathscr{U} = \{U_i\}$ be an open cover of X and $\mathscr{F}$ be a sheaf on X. If $\mathscr{U}$ consists of two sets and $\mathscr{F}$ is a constant sheaf, then the Mayer-Vietoris long exact sequence can be used to obtain information about the cohomology of X from the cohomology of U_1, U_2 and $U_1 \cap U_2$. More generally $H^\bullet(X,\mathscr{F})$ is determined by $H^\bullet(U_{i_0} \cap \cdots \cap U_{i_{p+1}}, \mathscr{F})$ as we vary over all p by way of the following *spectral sequence of an open cover*.

Theorem C.2.2. *Let $\mathscr{U} = \{U_i\}$ be an open cover of X and $\mathscr{F}$ be a sheaf on X. Then there is a spectral sequence*

$$E_1^{p,q} = \bigoplus_{(i_0,\dots,i_p)\in[\ell]_p} H^q(U_{i_0} \cap \cdots \cap U_{i_p}, \mathscr{F}) \Rightarrow H^{p+q}(X,\mathscr{F}),$$

where the differential $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ is induced by inclusion, with signs similar to the differential in the Čech complex. $\square$

This spectral sequence is discussed in [**115**, II.5.4]. We use it in §9.0 and §12.3. A variant of Theorem C.2.2 applies to a locally finite closed cover $\mathscr{C} = \{C_i\}$. This means $X = \bigcup_i C_i$, each C_i is closed in X, and each point $x \in X$ has a neighborhood meeting only finitely many C_i. A finite cover is clearly locally finite.

Theorem C.2.3. *Let $\mathscr{C} = \{C_i\}$ be a locally finite closed cover of X and $\mathscr{F}$ be a sheaf on X. Then there is a spectral sequence*

$$E_1^{p,q} = \bigoplus_{(i_0,\dots,i_p)\in[\ell]_p} H^q(C_{i_0} \cap \cdots \cap C_{i_p}, \mathscr{F}) \Rightarrow H^{p+q}(X,\mathscr{F}),$$

where the differential $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ is induced by inclusion, with signs similar to the differential in the Čech complex. $\square$

This spectral sequence is constructed in [**115**, II.5.2]. We will use it in §9.1 when we compute the sheaf cohomology of a toric variety.

The Spectral Sequence for Ext. Let X be a variety. As in §9.0, the Ext groups $\mathrm{Ext}^p_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ are the derived functors of the Hom functor $\mathscr{G} \mapsto \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ for fixed $\mathscr{F}$. The Ext sheaves $\mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))$ are defined similarly. These are related by the following spectral sequence.

Theorem C.2.4. *Let $\mathscr{F}$ and $\mathscr{G}$ be quasicoherent sheaves on a variety X. Then there is a spectral sequence*

$$E_2^{p,q} = H^p(X, \mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}). \qquad \square$$

We will use this in Theorem 9.2.12, which is part of our discussion of Serre duality. The spectral sequences from Theorems C.2.1 and C.2.4 are special cases of the Grothendieck spectral sequence of a composition of functors (see [**276**, §5.8]).

The Spectral Sequences of a Filtered Toplogical Space. First, given an increasing filtration of a topological space X,

(C.2.1) $$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X,$$

suppose we have information about the singular cohomology of the pairs (X_p, X_{p-1}) for all p. This information can be "stitched together" to produce the E_1 sheet of the *spectral sequence of a filtration*. See [**136**, Ch. 1] and [**255**, Sec. 9.4] for proofs.

Theorem C.2.5. *Let X be a topological space with a filtration of the form* (C.2.1) *and R be a ring. Then there is a spectral sequence*

$$E_1^{p,q} = H^{p+q}(X_p, X_{p-1}, R) \Rightarrow H^{p+q}(X,R),$$

where the filtration on $H^{p+q}(X,R)$ is given by

$$F^p H^{p+q}(X,R) = \ker(H^{p+q}(X,R) \longrightarrow H^{p+q}(X_p,R))$$

for all p,q. □

This is not always a first quadrant spectral sequence, though it will be when we apply Theorem C.2.5 in §12.3 to the filtration of a toric variety, where X_p is the union of the closures of the T_N-orbits of dimension p.

The Serre Spectral Sequence. The spectral sequence for a filtration can be used to derive the following *Serre spectral sequence* of a fibration. We recall that a fibration is a continuous mapping $p : E \to B$ satisfying the homotopy lifting property with respect to all spaces Y (see [**199**, §4.3]). It follows that if B is path-connected, then all of the fibers $p^{-1}(b)$ for $b \in B$ are homotopy equivalent. Hence we can speak of "the fiber" and denote it by F. Let $i : F \hookrightarrow E$ be the inclusion of the fiber.

Theorem C.2.6. *Let $p : E \to B$ be an orientable fibration over a path-connected and simply-connected base B with fiber F, and let R be a ring. Then there is a spectral sequence*

$$E_2^{p,q} = H^p(B, H^q(F,R)) \Rightarrow H^{p+q}(E,R)$$

for a suitable filtration on $H^{p+q}(E,R)$. Furthermore:

(a) *The edge homomorphism $E_2^{p,0} \to H^p(E,R)$ is $p^* : H^p(B,R) \to H^p(E,R)$.*

(b) *The edge homomorphism $H^q(E,R) \to E_2^{0,q}$ is $i^* : H^q(E,R) \to H^q(F,R)$.* □

If B is not simply connected, then a similar result is true, but the fundamental group $\pi_1(B)$ acts on the the cohomology of the fiber and the E_2 sheet involves the more complicated cohomology with local coefficients. This is discussed in [**136**, Ch. 1], [**199**, Thm. 5.2], and [**255**, Sec. 9.4]. We use the Serre spectral sequence in §12.4 to study the equivariant cohomology of a toric variety. In our case, $B = (\mathbb{P}^\infty)^n = (\bigcup_{\ell=1}^\infty \mathbb{P}^\ell)^n$, so the hypotheses of Theorem C.2.6 are satisfied.

The Serre spectral sequence can be seen as a special case of the Leray spectral sequence from Theorem C.2.6.

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