Stacks and Moduli

working draft

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Abstract

These notes provide the foundations of moduli theory in algebraic geometry with the goal of providing self-contained proofs of the following theorems:

Theorem A. *The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is a smooth, proper, and irreducible Deligne–Mumford stack of dimension $3g - 3$ which admits a projective coarse moduli space.*

Theorem B. *The moduli space $\underline{\mathrm{Bun}}_{r,d}^{\mathrm{ss}}(C)$ of semistable vector bundles of rank r and degree d over a smooth, connected, and projective curve C of genus g is a smooth, universally closed, and irreducible algebraic stack of dimension $r^2(g - 1)$ which admits a projective good moduli space.*

Along the way we develop the foundations of algebraic spaces and stacks, which provide a convenient language to discuss and establish geometric properties of moduli spaces. Introducing these foundations requires developing several themes at the same time including:

- using the functorial and groupoid perspective in algebraic geometry: we will introduce the new algebro-geometric structures of algebraic spaces and stacks,

- replacing the Zariski topology on a scheme with the étale topology: we will introduce Grothendieck topologies proving a generalization of topological spaces, and we will systematically use descent theory for étale morphisms, and

- relying on several advanced topics not typically seen in a first algebraic geometry course: properties of flat, étale and smooth morphisms of schemes, algebraic groups and their actions, deformation theory, and the birational geometry of surfaces.

Choosing a linear order in presenting the foundations is no easy task. We attempt to mitigate this challenge by relegating much of the background to appendices. We keep the main body of the notes always focused on developing moduli theory with the above two theorems in mind.

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Chapter 0

Introduction and motivation

Moduli spaces arise as solutions to one of the most fundamental problems in mathematics:

> **Classification problem**: Can we classify the isomorphism classes of all algebro-geometric objects of a certain type?

There are many types of objects that we may want to classify:

- subspaces $V \subset \mathbb{C}^n$ of dimension k;
- plane curves $C \subset \mathbb{P}^2$ of degree d;
- curves C of genus g together with a degree d morphisms $C \to \mathbb{P}^1$;
- line bundles on a fixed projective variety X;
- representations of a group, e.g. an absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, the fundamental group $\pi_1(\Sigma)$ of a topological surface of genus g, or the path algebra of quiver.

Our primary interest is in the following two examples, which will be used throughout this book to illustrate the concepts of moduli.

(1) smooth (or more generally stable) curves of genus g, and

(2) vector bundles (or more specifically semistable vector bundles) on a *fixed* smooth curve C.

> A **moduli space** is a *space* whose points are in *natural* bijection with isomorphism classes of the given algebro-geometric objects.

The keyword above is 'natural', and it is probably not clear to you what this could mean. Indeed, one of the main challenges in developing moduli theory is formulating precisely what this means. After all, any two complex manifolds or varieties of positive dimension are bijective as they both have the cardinality of the continuum. Our approach to clarify a 'natural bijection' will be to introduce the notion of a *family of objects* and require a certain relationship between maps to the moduli space and families of objects.

Moreover, the structure of the 'space' depends on the context: if we are classifying topological objects, we might hope for the structure of a topological space, while if we are classifying analytic objects, we might hope for the structure of a manifold. In

this book, we are mainly focused on classifying algebraic structures, and we desire a moduli space with the structure of an algebraic variety, and ideally a projective variety.

Once you start viewing spaces through the lens of moduli, everything begins looking like a moduli space! This is true in a precise sense: every space M is the moduli space of its points. It is of course more interesting when there are equivalent descriptions with geometric meanings. Projective space $\mathbb{P}^1$ is the set of points in $\mathbb{P}^1$ (not so interesting), or lines in the plane passing through the origin (more interesting), or subschemes of $\mathbb{A}^2$ of length 2 supported at the origin (also interesting), or isomorphism classes of stable elliptic curves (very interesting). Fascinatingly, moduli spaces often afford equivalent viewpoints across disparate mathematical fields, as is the case in our primary two examples:

(1) $\mathcal{M}_g$ is the moduli of smooth projective algebraic curves of genus g, or the moduli of compact Riemann surfaces of genus g, or the moduli of ...

(2) $\mathrm{Bun}_{r,d}^{\mathrm{ss}}(C)$ is the moduli of semistable algebraic vector bundle on a fixed curve C, or the moduli of holomorphic vector bundles on C with flat unitary connection, or the moduli of irreducible unitary representation of $\pi_1(C)$.

This leads to a rich interplay between algebraic, analytic, and topological approaches. As Mumford writes in the preface of [Mum04]:

> Besides being a form of cartography, the theory of moduli spaces has the wonderful feature of having many doors, many techniques by which this theory can be developed. Of course, there is traditional algebraic geometry, but there is also invariant theory, complex-analytic techniques such as Teichmüller theory, global topological techniques, and purely characteristic p methods such as counting objects over finite fields. This is another part of its charm.

Discrete vs continuous moduli. Depending on the types of objects, the moduli space could be discrete or continuous, or a combination of the two. We illustrate this with the following examples:

- The moduli space of line bundles on $\mathbb{P}^1$ is the discrete set $\mathbb{Z}$: every line bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for a unique integer $n \in \mathbb{Z}$.

- The moduli space parameterizing quadric plane curves $C \subset \mathbb{P}^2$ is the connected space $\mathbb{P}^5$: a curve defined by $a_0 x^2 + a_1 xy + a_2 xz + a_3 y^2 + a_4 yz + a_5 z^2$ is uniquely determined by the point $[a_0, \ldots, a_5] \in \mathbb{P}^5$, and as a plane curve varies continuously (i.e. by varying the coefficients a_i), the corresponding point in $\mathbb{P}^5$ does too.

- For smooth curves, the genus g is a discrete invariant while for smooth curves of a fixed genus g, the moduli space M_g is a variety (it is in fact an *irreducible* quasi-projective variety but we are now getting far ahead of ourselves). The moduli space of all smooth curves can thus be viewed as the disjoint union $\coprod_g M_g$.

- For vector bundles on a smooth curve C, the rank r and degree d are discrete invariants while the moduli space $\mathrm{Bun}_{C,r,d}^{\mathrm{ss}}$ of semistable bundles of rank r and degree d is an irreducible projective variety.

Why study moduli spaces? Properties of moduli spaces can inform us about the properties of the objects themselves. Many properties of objects are best formulated

in terms of moduli spaces. For instance, to express the condition that a general genus 3 curve can be parameterized by an explicit coordinate system—namely a general genus 3 curve is canonically embedded into $\mathbb{P}^2$ as a plane quartic and thus parameterized by a point in the space $\mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}(4))) \cong \mathbb{P}^{14}$—we could say that the moduli space M_3 is *unirational*, i.e. there is a dominant rational map $\mathbb{P}^{14} \dashrightarrow M_3$.

quantizedhyperbolicity

Do moduli spaces exist? Before we can even begin to discuss the geometry of moduli spaces, we need to show that they exist. This is no easy task and is one of the major goals of this book. We develop the foundations of moduli theory in order to prove that there is a projective moduli space parameterizing stable curves of genus g (Theorem A) and a projective moduli space parameterizing semistable vector bundles of rank r and degree d on a fixed smooth curve (Theorem B). It seems like a miraculous coincidence that these moduli spaces exist as varieties! Their existence is the starting point of moduli theory.

Trichotomy of moduli. A recurring theme in moduli is the impact of automorphism groups on both the properties of a moduli space and the techniques used to study its geometry. There is a trichotomy in moduli theory depending on the size of the automorphism groups: (1) no automorphisms, (2) finite automorphisms, and (3) infinite automorphisms. In (3), the moduli spaces are particularly well-behaved when the *closed* points of the moduli stack have *reductive* automorphisms.

	No Auts	Finite Auts	Reductive Auts
Type of space	Scheme/algebraic space	Deligne–Mumford stack	algebraic stack
Defining property	Zariski/étale-locally an affine scheme	étale-locally an affine scheme	smooth-locally an affine scheme
Examples	$\mathbb{P}^n$, $\mathrm{Gr}(k,n)$, Hilb, Quot	$\mathcal{M}_g$	$\underline{\mathrm{Bun}}_{r,d}(C)$
Quotient stacks $[X/G]$	action is free	finite stabilizers	reductive stabilizers
Existence of moduli space	fine moduli space	coarse moduli space	good moduli space

Table 0.1: Trichotomy of moduli

Our approach. In this chapter, we motivate the approach of this text by gradually adding more enriched structures to sets and groupoids. We first introduce families of objects and the functorial worldview (§0.3) and then develop the groupoid perspective (§0.4). After motivating the étale topology (§0.5), we combine these perspectives by introducing moduli stacks (§0.6). We then sketch our main techniques to construct a projective moduli space (§0.7).
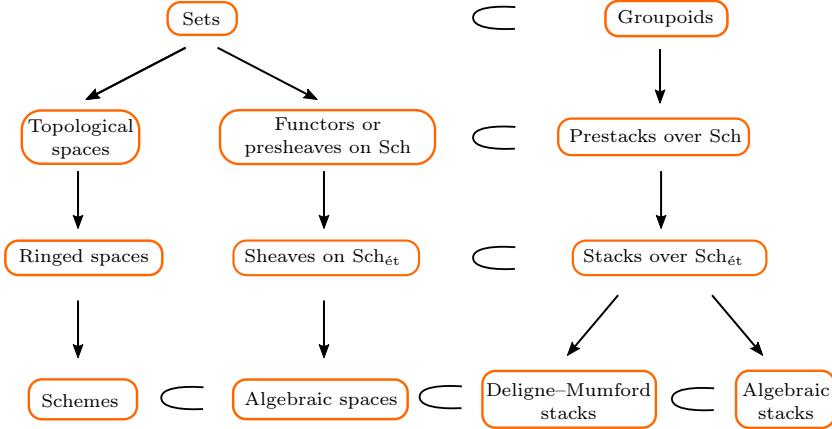
Figure 0.1: Schematic diagram algebro-geometric enrichments of sets and groupoids.

0.1 A brief history of moduli

We quickly discuss the historical development of moduli theory to provide a first glimpse of many themes in moduli.

0.1.1 Riemann and the origins of M_g

> *Die $3p-3$ übrigen Verzweigungswerthe in jenen Systemen gleichverzweigter μ werthiger Functionen können daher beliebige Werthe annehmen; und es hängt also eine Klasse von Systemen gleichverzweigter $\overline{2p+1}$ fach zusammenhängender Functionen und die zu ihr gehörende Klasse algebraischer Gleichungen von $3p-3$ stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen.*
>
> *Translation: The remaining $3p-3$ branch points in these systems of similarly branching μ-valued functions can therefore be assigned any given values; and thus a class of systems of similarly branching functions with connectivity $2p+1$, and the corresponding class of algebraic equations, depends on $3p-3$ continuous variables, which we shall call the moduli of the class.*

<div align="right">

RIEMANN [RIE57, PG.33]

</div>

This is a remarkable sentence in a remarkable paper—Riemann both introduces the concept of 'moduli' and computes that the 'number of moduli' of M_g is $3g-3$. Riemann's idea went something like this: instead of considering abstract smooth curves, let's view curves as branched covers over $\mathbb{P}^1$ and consider the moduli space

$$\mathrm{Hur}_{d,g} = \left\{ [C \to \mathbb{P}^1] \;\middle|\; \begin{array}{l} \bullet \; C \text{ is a smooth curve of genus } g \\ \bullet \; C \to \mathbb{P}^1 \text{ is a simply branched covering of degree } d \end{array} \right\}. \tag{0.1.1}$$

Formally studied later by Hurwitz [Hur91], these moduli spaces—which are now referred to as *Hurwitz spaces*—also play an essential role in irreducibility arguments for M_g (see §5.7).

A *simply branched covering* is a finite map of smooth curves where the ramification indices are at most two and every fiber has at most one ramification point; see also Definition 5.7.2. By Riemann–Hurwitz (5.7.3), every simply branched covering $C \to \mathbb{P}^1$ is branched over $2g + 2d - 2$ distinct points of $\mathbb{P}^1$. This gives a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hur}_d(C) & \hookrightarrow & \mathrm{Hur}_{d,g} \\
\end{array}
\tag{0.1.2}
$$

$$
\{[C]\} \hookrightarrow M_g \qquad \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1
$$

where

- the map $\mathrm{Hur}_{d,g} \to \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1$ sends a covering $[C \to \mathbb{P}^1]$ to the $2g + 2d - 2$ branched points; here $\mathrm{Sym}^N\,\mathbb{P}^1 = (\mathbb{P}^1)^N/S_N$ is the space classifying N *unordered* points,
- the map $\mathrm{Hur}_{d,g} \to M_g$ is defined by $[C \to \mathbb{P}^1] \mapsto [C]$, and
- $\mathrm{Hur}_d(C)$ is the preimage of $[C] \in M_g$ under $\mathrm{Hur}_{d,g} \to M_g$, i.e. $\mathrm{Hur}_d(C)$ classifies simply branched coverings $C \to \mathbb{P}^1$ where C is fixed.

If d is sufficiently large, then for every general collection of $2d + 2g - 2$ points of $\mathbb{P}^1$, there exists a genus g curve C and a simply branched covering $C \to \mathbb{P}^1$ branched over precisely these points, and moreover there are at most finitely many such maps. In other words, $\mathrm{Hur}_{d,g} \to \mathrm{Sym}^{2d+2g-2}\,\mathbb{P}^1$ has dense image and finite fibers; see §5.7.2 for precise details. Therefore,

$$
\begin{aligned}
\dim M_g &= \dim \mathrm{Hur}_{d,g} - \dim \mathrm{Hur}_d(C) \\
&= 2d + 2g - 2 - \dim \mathrm{Hur}_d(C)
\end{aligned}
\tag{0.1.3}
$$

To compute the dimension of $\mathrm{Hur}_d(C)$, we observe that a simply branched covering $C \to \mathbb{P}^1$ is the data of a degree d line bundle L and two base point free sections such that the induced map to $\mathbb{P}^1$ is simply branched. Since a general choice of two sections defines a simply branched covering (Lemma 5.7.13), we can compute

$$
\dim \mathrm{Hur}_d(C) = \dim \mathrm{Pic}_d(C) + 2h^0(C, L) - 1,
$$

where we subtract one since scaling any two sections will define the same map to $\mathbb{P}^1$. Riemann–Roch (5.1.4) tells us that $h^0(C, L) = d + 1 - g$. On the other hand $\dim \mathrm{Pic}_d = \dim \mathrm{Pic}_0 = g$; this can be seen using the exponential sequence: $0 \to \mathbb{Z} \to \mathcal{O}_C \xrightarrow{\exp} \mathcal{O}_C^* \to 0$ yields a long exact sequence

$$
\underbrace{H^1(C, \mathbb{Z})}_{\mathbb{Z}^{2g}} \to \underbrace{H^1(C, \mathcal{O}_C)}_{\mathbb{C}^g} \to \underbrace{H^1(C, \mathcal{O}_C^*)}_{\mathrm{Pic}(C)} \xrightarrow{\deg} \underbrace{H^2(C, \mathbb{Z})}_{\mathbb{Z}},
\tag{0.1.4}
$$

and provides an identification $\mathrm{Pic}_0(C) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$. We conclude that $\dim \mathrm{Pic}_0(C) = g$ and $\dim \mathrm{Hur}_d(C) = g + 2(d + 1 - g) - 1 = 2d - g + 1$. Plugging this into (0.1.3) yields

$$
\dim M_g = (2d + 2g - 2) - (2d - g + 1) = 3g - 3.
$$

Riemann in fact gave several other heuristic arguments computing the dimension of M_g. See [GH78, pg. 255-257] or [Mir95, pg. 211-215] for further discussion on the number of moduli of M_g, and see [AJP16] for historical background of Riemann's computations.

> **Riemann's moduli problem**: Does M_g exist as a complex analytic space?

While Riemann's argument can be made completely rigorous with today's methods (as we do ourselves later in this text), there are foundational issues with Riemann's method—today we would say that Riemann computed the dimension of a 'local deformation space.' Most notably, M_g was not known to exist and it wasn't clear what type of space M_g was supposed to be. Despite this, Riemann had an instinctive grasp of its geometry—in fact in the same paper [Rie57], Riemann introduced the word 'Mannigfaltigkeit' (or 'manifold') to describe its geometry. Manifolds were not formally defined until much later in the 1940s following Teichmüller, Chern, and Weil. Riemann was aware of the foundational issues in geometry:

> *It is well known that geometry presupposes not only the concept of space but also the first fundamental notions for constructions in space as given in advance. It only gives nominal definitions for them,... while the relationship of these presumptions is left in the dark... From Euclid to Legendre, to name the most renowned of modern writers on geometry, this darkness has been lifted neither by the mathematicians nor the philosophers who have laboured upon it.* —Riemann 1854

Riemann's work has inspired countless mathematicians to lift us out of this darkness.

> *The spirit of Riemann will move future generations as it has moved us.* —Ahlfors [Ahl53, p. 53]

> *It is difficult to recall another example in the history of 19th century mathematics when a struggle for a rigorous proof led to such productive results.* —Monastyrsky [Mon87, p. 41]

0.1.2 Moduli of curves of low genus

Genus 0. For $n \geq 3$, the moduli space $M_{0,n}$ of smooth genus 0 curves with n ordered distinct points can be described as

$$M_{0,n} = \left(\mathbb{P}^1 \setminus \{0, 1, \infty\}\right)^{n-3} \setminus (\text{all diagonals}).$$

Indeed, given n ordered distinct points $p_1, \ldots, p_n$ on $\mathbb{P}^1$, there is a unique automorphism $g \in \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2$ taking (p_1, p_2, p_3) to $(0, 1, \infty)$. When $n = 4$, we obtain that a bijection $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ given by the classical cross-ratio of four points in $\mathbb{P}^1$ first discovered by Pappus of Alexandria [Ale86] in 300 AD (see also Example 6.7.9).

Genus 1. Every elliptic curve (E, p), i.e. a smooth genus 1 curve E with a marked point $p \in E$, can be described as a plane cubic in Weierstrass form

$$E_\lambda = V(y^2 z - x(x - z)(x - \lambda z)) \subset \mathbb{P}^2$$

for some $\lambda \neq 0, 1$, where $p = [0 : 1 : 0] \in E_\lambda$. However, the choice of λ is not unique: the values λ, $1/\lambda$, $1 - \lambda$, $1/(1 - \lambda)$, $\lambda/(\lambda - 1)$, and $(\lambda - 1)/\lambda$ determine isomorphic

elliptic curves. In other words, the map $\mathbb{A}^1 \setminus \{0, 1\} \to M_{1,1}$ given by $\lambda \mapsto [E_\lambda]$ is a 6-to-1 surjective map. The j-invariant on the other hand

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

uniquely determines the isomorphism class of the curve and thus gives a bijection $M_{1,1} \cong \mathbb{A}^1$. For a modern treatment, see [Har77, §4].

Genus 2. Every smooth genus 2 curve C is hyperelliptic and can be written as a double cover $y^2 = (x - a_1) \cdots (x - a_6)$ over $\mathbb{P}^1$. This is a consequence of the sheaf of differentials Ω_C being a base point free line bundle of degree 2 with 2 global sections; the induced map $C \to \mathbb{P}^1$ is ramified over 6 points by Riemann–Hurwitz (5.7.3). We obtain the description that

$$M_2 = \left(\Gamma(\mathbb{P}^1, \mathcal{O}(6)) \setminus \Delta\right) / \mathrm{GL}_2,$$

where $\Delta \subset \Gamma(\mathbb{P}^1, \mathcal{O}(6))$ denotes the locus of binary sextics with a double root. After a projective change of coordinates on $\mathbb{P}^1$, we can arrange that the curve is ramified over 0, 1, ∞ and 3 other points $a_4, a_5, a_6 \in \mathbb{P}^1 \setminus (0, 1, \infty)$. In this way, we obtain a surjective map $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^3 \setminus \Delta \to M_2$.

Invariant theory of binary sextics (see [Cle70]) provides an even sharper description: the ring of invariant polynomials, i.e. polynomials in $a_0, \ldots, a_6$ that are invariant under automorphisms of $\mathbb{P}^1$, is generated by invariants J_2, J_4, J_6, J_8, J_{10}, and J_{15}, whose degree is indicated by the subscript, with a single relation $J_{15}^2 = G(J_2, J_4, J_6, J_{10})$ for some polynomial G. The invariant J_{10} is the discriminant while J_{15} does not affect the scheme structure. This yields that M_2 is an open subset of weighted projective space

$$M_2 = \mathrm{Proj}\, \mathbb{C}[J_2, J_4, J_6, J_{10}] \setminus \{J_{10} = 0\},$$

and thus M_2 is an affine variety embedded into $\mathbb{A}^8$ via

$$\frac{J_2^5}{J_{10}}, \frac{J_2^3 J_4}{J_{10}}, \frac{J_2^2 J_6}{J_{10}}, \frac{J_2 J_4^2}{J_{10}}, \frac{J_2 J_6^3}{J_{10}^2}, \frac{J_4^5}{J_{10}^2}, \frac{J_4 J_6}{J_{10}}, \frac{J_6^5}{J_{10}^3}.$$

We can identify this coordinate ring with the invariant ring of the action on $\mathbb{Z}/5$ on $\mathbb{A}^3$ where a generator $\zeta \in \mathbb{Z}/5$ acts via $\zeta \cdot (x, y, z) = (\zeta x, \zeta^2 y, \zeta^3 z)$; the above functions are identified with the invariants x^5, $x^3 y$, $x^2 z$, xy^2, xz^3, y^5, yz, z^5. This yields the rather elegant global description

$$M_2 = \mathbb{A}^3 / (\mathbb{Z}/5).$$

This was studied classically by Bolza [Bol87] and more recently by Igusa [Igu60].

Genus 3. A non-hyperelliptic smooth genus 3 curve embeds as a quartic in $\mathbb{P}^2$ under the canonical embedding. Letting $\Delta \subset \Gamma(\mathbb{P}^1, \mathcal{O}(4))$ be the locus of singular quartics, we obtain a description of the open locus of non-hyperelliptic curves as the quotient $\left(\Gamma(\mathbb{P}^1, \mathcal{O}(4)) \setminus \Delta\right) / \mathrm{GL}_3$—this is the first time that our description only describes a *general* curve. On the other hand, a hyperelliptic genus 3 curve is double cover of $\mathbb{P}^1$ ramified over 8 points, and we obtain a *set-theoretic* decomposition

$$M_3 = \underbrace{\left(\Gamma(\mathbb{P}^1, \mathcal{O}(4)) \setminus \Delta\right) / \mathrm{GL}_3}_{\dim=6} \coprod \underbrace{\left(\Gamma(\mathbb{P}^1, \mathcal{O}(8)) \setminus \Delta\right) / \mathrm{GL}_2}_{\dim=5},$$

suggesting that the locus of hyperelliptic curves is a divisor in M_3.

Genus 4. A non-hyperelliptic smooth curve C of genus 4 embeds into $\mathbb{P}^3$ under its canonical embedding, and can be realized as the intersection $C = Q \cap S$ of a quadric Q and cubic S. This gives a rational map $\Gamma(\mathbb{P}^3, \mathcal{O}(2)) \times \Gamma(\mathbb{P}^3, \mathcal{O}(3)) \dashrightarrow M_4$ whose image is the locus of non-hyperelliptic curves; as above the hyperelliptic locus can be parameterized by $\left(\Gamma(\mathbb{P}^1, \mathcal{O}(10)) \backslash \Delta\right) / \operatorname{GL}_2$. Alternatively, a general non-hyperelliptic smooth genus 4 curve can be realized as the normalization of a plane quintic with precisely two nodes, or as a degree 3 cover of $\mathbb{P}^1$ branched over 12 points.

Genera 5–10. Classically, general curves were described either as plane curves with prescribed singularities via the image of a map $C \to \mathbb{P}^2$, or as branched covers $C \to \mathbb{P}^1$. For a general genus g curve C, the smallest degree d such that C is realized as normalization of a singular plane curve is $d = \lfloor \frac{2g+8}{3} \rfloor$. If the plane curve had at worst nodal singularities, then the number of nodes is $\delta := (d-1)(d-2)/2 - g$. Meanwhile, the minimum degree of a map $C \to \mathbb{P}^1$ is $\lfloor \frac{g+3}{2} \rfloor$.

g	$d = $ min degree of $\operatorname{im}(C \to \mathbb{P}^2)$	$\delta = $ # of nodes	$\frac{(d+1)(d+2)}{2} - 3\delta$	min degree of $C \to \mathbb{P}^1$	# of branch pts
0	1	0	3	1	0
1	3	0	10	2	4
2	4	1	12	2	6
3	4	0	15	3	10
4	5	2	15	3	12
5	6	5	13	4	16
6	6	4	16	4	18
7	7	8	12	5	22
8	8	13	6	5	24
9	8	12	9	6	28
10	9	18	1	6	30
11	10	25	-9	7	34

Table 0.2: General curves of low genus

In [Sev15] and [Sev21], Severi used such descriptions to show that M_g is unirational for $g \leq 10$. Like other mathematicians of his era, Severi did not precisely formulate what it meant for M_g to be a moduli space.

What goes wrong for $g \geq 11$? As the genus grows, it becomes more difficult to describe a general genus g curve. To give an indication of the challenges for $g \geq 11$, let's try to describe a genus g curve as a degree $d = \lfloor \frac{2g+8}{3} \rfloor$ planar curve with $\delta = (d-1)(d-2)/2 - g$ nodes at prescribed points $p_1, \ldots, p_\delta \in \mathbb{P}^2$. If the plane curve defined by $f \in \Gamma(\mathbb{P}^2, \mathcal{O}(d))$ has a node at each p_i, then the equations $f_x(p_i) = f_y(p_i) = f_z(p_i) = 0$ imposes 3δ linear equations on $\Gamma(\mathbb{P}^2, \mathcal{O}(d))$. For such nodal plane curves to exist, we would need

$$\dim \Gamma(\mathbb{P}^2, \mathcal{O}(d)) - 3\delta = \frac{(d+1)(d+2)}{2} - 3\left(\frac{(d-1)(d-2)}{2} - g\right) > 0.$$

As illustrated by Table 0.2, $g = 11$ is the first case where this does not hold!

Severi's conjecture.   Severi conjectured that M_g is unirational for all g: "Ritengo probabile che la varietà sia razionale o quanto meno che sia riferibile ad un'involuzione di gruppi di punti in uno spazio lineare..." [Sev15, pg. 881]. While this conjecture turned out to be false, it motivated mathematicians for decades: "Whether more M_g's, $g \geq 11$, are unirational or not is a very interesting problem, but one which looks very hard too, especially if g is quite large" [Mum75a, pg. 37]. In the 1980s, Eisenbud, Harris, and Mumford disproved this conjecture and showed that in some sense quite the opposite is true in large genus: M_g is general type for $g \geq 24$ [HM82], [EH87].

Petri's description of canonical curves.   While most 19th and early 20th century mathematicians described curves as either plane singular curves or as covers of $\mathbb{P}^1$, Petri's explicit description [Pet23] of canonically embedded curves was an exception and is more reminiscent of modern approaches. As Mumford writes in [Mum75a, p.17], Petri's approach "is unavoidably a bit messy, but just to be able to brag, I think it is a good idea to be able to say 'I have seen *every* curve once.'" Building on M. Noether's result [Noe80] that the canonical embedding $C \hookrightarrow \mathbb{P}^{g-1}$ of a non-hyperelliptic smooth curve C is projectively normal—that is, $\varphi \colon \operatorname{Sym}^* \Gamma(C, \Omega_C) \to \bigoplus_{d \geq 0} \Gamma(C, \Omega_C^{\otimes d})$ is surjective—and also building on work of Enriques [Enr19] and Babbage [Bab39], Petri showed that the homogeneous ideal $I = \ker \varphi$ is generated by quadrics unless C is a plane quintic ($g = 6$) or trigonal (i.e. a triple covering of $\mathbb{P}^1$) in which case I is generated in degree 2 and 3. Petri's analysis was remarkably constructive leading to explicit equations in $\mathbb{P}^{g-1}$ cutting out C along with explicit syzygies among the equations. Petri's work continues to inspire research in the theory of moduli and syzygies. We won't cover this perspective further in this text but we recommend [SD73], [Mum75a, pgs. 17-21], [AS78], [Gre82], [Gre84], and [ACGH85, §III.3].

0.1.3   Analytic approaches and the Teichmüller space

In the late 19th and early 20th century, Riemann surfaces were described as quotients of the upper half plane by a discrete subgroup of $\operatorname{PSL}_2(\mathbb{R})$; such subgroups are named *Fuchsian groups* after Fuchs [Fuc66]. Fricke and Klein classified Fuchsian groups using the theory of automorphic functions in their 1300 page volumes [FK92], [FK12]. They constructed what's now known as the Teichmüller space, showed that it is a contractible space, and even exhibited complex structures. Torelli showed that a Riemann surface can be constructed from its Jacobian [Tor13], and Siegel constructed the moduli space A_g of abelian varieties of dimension g as an analytic space [Sie35].

Teichmüller[1] was the first to give a precise formulation of Riemann's moduli problem, to construct M_g as a complex analytic space, and to interpret $3g - 3$ as its complex dimension [Tei40], [Tei44]. Teichmüller constructed the *Teichmüller space* T_g parameterizing complex structures on a topological surface Σ_g of genus g up to homeomorphism. The space T_g is homeomorphic to a ball in $\mathbb{C}^{3g-3}$ and inherits an action of the *mapping class group* Γ_g of diffeomorphisms of Σ_g modulo the subgroup of diffeomorphisms isotopic to the identity. This action is properly discontinuous, and M_g is realized as the quotient T_g/Γ_g. Although largely forgotten for nearly 20 years, Teichmüller theory was later greatly expanded by Ahlfors, Bers,

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[1]Bers' use of the famous quote of Plutarch (Perikles 2.2) to describe Teichmüller in [Ber60a] seems fitting: "It does not of necessity follow that, if the work delights you with its grace, the one who wrought it is worthy of your esteem."

and Weil among others; see [Wei57], [Wei58], [AB60], [Ber60b], and [Ahl61]. For modern expository treatments, see [Ber72], [Hub06], and [FM12].

Teichmüller also introduced the notion of families of Riemann surfaces and showed that the Teichmüller space satisfies a universal property. Grothendieck, in a series of lectures at Cartan's seminar [Gro61], developed a general theory of analytic moduli spaces in the language of categories and functors, reformulated Teichmüller theory in this setting, and showed that T_g represents a functor parameterizing families of Riemann surfaces. This set the stage for Grothendieck's later work on algebraic moduli: "One can hope that we shall be able one day to eliminate analysis completely from the theory of Teichmüller space, which should be purely geometric" [Gro61, Lecture I].

0.1.4 The origins of algebraic moduli theory

> As for M_g there is virtually no doubt that it can be provided with the structure of an algebraic variety.
>
> WEIL [WEI58, PG. 383]

Cayley, Gordan, and Hilbert. The invariant theorists of the 19th century were interested in classifying homogeneous polynomials of degree d in n variables up to projective automorphisms, or in other words in the moduli space $\Gamma(\mathbb{P}^{n-1}, \mathcal{O}(d))/\operatorname{PGL}_n$. They attempted to describe this moduli space by exhibiting explicit invariant polynomials in the coefficients a_I of a polynomial $f = \sum_I a_I x^I$. The origins of invariant theory lie in work by Boole [Boo41] and Cayley [Cay45], and were further developed by Gordan and Hilbert along with many others. Gordan exhibited explicit generators of the ring of invariants of binary forms ($n = 2$) [Gor68] and Hilbert later proved that the ring of invariants is finitely generated for any ring [Hil90], [Hil93].

Cayley constructed the moduli space of curves in $\mathbb{P}^3$ [Cay60], [Cay62], which is now referred to as the *Chow variety*. His idea was to associate to a degree d curve $C \subset \mathbb{P}^3$ the set of lines $L \subset \mathbb{P}^3$ meeting C; this is a hypersurface of degree d in $\operatorname{Gr}(1, \mathbb{P}^3)$, and the Chow variety is the closure of all hypersurfaces in $\Gamma(\operatorname{Gr}(1, \mathbb{P}^3), \mathcal{O}(d))$ obtained this way. The general theory of Chow varieties parameterizing subschemes in $\mathbb{P}^n$ of any dimension was later developed by Chow and van der Waerdan [CW37].

Weil. In Weil's work on the Riemann hypothesis for curves over finite fields [Wei48], he needed to construct the Jacobian of a curve parameterizing degree 0 line bundles. At that point, varieties had only been considered as embedded in affine or projective space, and in his foundational work [Wei62] Weil enlarged the category to *abstract varieties*. This was enough to construct the Jacobian and give a proof—in fact his second proof—of the Riemann hypothesis for curves. Later Weil and Chow independently showed that the Jacobian was projective.

Baily. Baily constructed the moduli space A_g of principally polarized abelian varieties as a quasi-projective variety [Bai60a], [Bai60b], showed that Satake's topological compactification [Sat56] is algebraic [Bai58], and together with Borel introduced what's now known as the *Baily–Borel compactification* [BB66]. Using the period map $M_g \to A_g$ associating a curve to its Jacobian and Torelli's theorem that this map is injective, Baily concluded that M_g has the structure of a quasi-projective variety. However, he did not prove that this provided a 'natural' structure of a

variety nor that it had any uniqueness properties, i.e. that M_g is a coarse moduli space.

> *Thirdly, in order to call $\mathcal{E}$ the variety of moduli of Riemann surfaces of genus n, one should be able to state that it is unique and in some sense universal among normal parameter varieties of algebraic systems of curves of genus n. Namely, given any normal algebraic system of curves of genus n there should exist a natural map of the parameter variety of the nonsingular members of this system into $\mathcal{E}$.* — Baily [Bai60b, pgs. 59-60]

Mumford credits Baily for the quasi-projectivity of M_g in [Mum75a, p. 98] just as Gieseker does in his commentary in [Mum04].

Grothendieck. After Grothendieck's formalization of analytic moduli theory, in his 'FGA series' [FGA$_\text{I}$]–[FGA$_\text{VI}$], he applied his functorial approach to algebraic geometry. He defined the Hilbert, Quot, and Picard functors, and showed that they are representable by projective schemes. Grothendieck of course later reformulated the entire foundations of algebraic geometry by developing scheme theory. His profound influence on algebraic geometry and more broadly mathematics helped shape the future of moduli theory.

Although he didn't publish on M_g, Grothendieck was nevertheless very much interested in the existence of M_g as a quasi-projective variety and its connectedness in any characteristic—see his email correspondence with Mumford in the early 1960s [Mum10, §II]. Grothendieck was aware that the presence of automorphisms obstructed the representability of the functor parameterizing smooth families of curves. He rigidified the moduli problem by also parameterizing a level structure on a curve. While he could show that the functor of smooth curves with level structure $r \geq 3$ was representable by a scheme, he struggled to show that it was quasi-projective. The idea was to construct M_g as a quotient of the rigidified moduli space by modding out by the finite group acting on the choice of level n structure. The lack of quasi-projectivity impeded this approach as the quotient of a non-quasi-projective variety by a finite group need not exist as a variety (see Example 0.5.5).

Mumford. Motivated by Riemann's moduli space as well as by constructions of Chow varieties, Picard varieties, and the moduli of abelian varieties in the early 20th century, Mumford made immense contributions to the foundations of moduli theory. He was the first to systematically study their geometry.

> *The point is, I love maps, that is "maps" in the sense of "maps of the world," "charts of the ocean," "atlases of the sky"! I think one of the key things that attracted me to the group of problems was the hope of making a* map *of some parts of the world of algebraic varieties. An algebraic variety felt like a tangible thing in the lectures of Oscar Zariski, so why shouldn't you venture out, like Magellan, and uncover their geography?*
> —Mumford, preface of [Mum04]

By integrating Grothendieck's formalism of scheme theory with 19th century invariant theory, Mumford developed a theory of quotients in algebraic geometry now known as *Geometric Invariant Theory* (or *GIT*), and then applied this theory to construct both M_g and A_g. His theory was originally sketched in [Mum61] and fully worked out in the definitive text [GIT].

Later, Mumford constructed a quasi-projective variety parameterizing *stable* vector bundles on a fixed smooth curve [Mum63], and Seshadri then showed the moduli space of *semistable* vector bundles provides a projective compactification [Ses67]. In the seminal joint work [DM69], Deligne and Mumford not only introduce the compactification $\overline{\mathcal{M}}_{g,n}$ and apply it (as noted above) to prove the connectedness of M_g in all characteristics, they also first introduce the notion of an algebraic stack—now referred to as *Deligne–Mumford stacks*.

Artin.   The theory of algebraic spaces and stacks was developed by M. Artin. Similar to Weil's enlargement of affine and projective varieties to abstract varieties, enlarging the category of schemes to algebraic spaces allows us to construct the quotient of finite group actions or more generally any étale equivalence relation.[2] Knutson, a student of Artin, was the first to write down the theory of algebraic spaces [Knu71].

In 1969, Artin proved two crucial results in moduli theory—Artin Approximation (Theorem B.5.26) and Artin Algebraization (Theorem C.6.8). In his groundbreaking paper [Art74], Artin not only introduced the concept of algebraic stacks broadening the definition of Deligne and Mumford, but he also provided a local deformation-theoretic characterization of algebraic stacks. This is known as 'Artin's Criteria' and can be used to verify the algebraicity of a moduli stack (see §C.7).

As Faltings said: "The notion of stack came up in the sixties. But to swallow schemes was already enough for one generation of mathematicians" [Sto95, p.45]. The theory of stacks once had a formidable reputation and a somewhat questionable foundation. This could be in part due to the abstract categorical nonsense involved in its formulation and to 2-categorical subtleties, or perhaps due to shifting conditions on the diagonal of an algebraic stack in the literature. Or it could be partly due to that Deligne and Mumford did not give proofs for their results on algebraic stacks in [DM69, p. 76]—they write: "Full details on the basic properties and theorems for algebraic stacks will be given elsewhere." Sure enough, future mathematicians worked out the details, and there are now excellent textbooks on stacks such as [LMB00] and [Ols16]. Of course, the *Stacks Project* [SP] has now provided an unquestionably solid foundation.

For further historical background, we recommend [Mum75a], [Oor81], [Kle05], [JP13], [AJP16], and [Kol18].

0.2   Moduli sets of curves, vector bundles, and triangles

To define a moduli space as a set entails specifying two things:

  (1) a class of certain types of objects, and

  (2) an equivalence relation on objects.

Here's our first attempt at defining M_g:

Example 0.2.1 (Moduli set of smooth curves)**.** The objects of the *moduli set of smooth curves*, denoted as M_g, are smooth, connected, and projective curves of genus g over $\mathbb{C}$. Two curves are declared equivalent if they are isomorphic. There are

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[2]Matsusaka also built a theory of $\mathbb{Q}$-*varieties* by considering certain quotients of equivalence relations [Mat64] but it was not as robust as algebraic spaces.

many variants obtained by parameterizing additional structures or choosing different equivalence relations.

- We already saw the Hurwitz moduli set $\mathrm{Hur}_{d,g}$ in (B.1.6) parameterizing branched covers $C \to \mathbb{P}^1$ of degree d.

- The moduli set $M_{g,n}$ of n-pointed smooth genus g curves parameterizes the data of a smooth curve C together with n ordered distinct points $p_1, \ldots, p_n \in C$; two objects $(C, p_i) \sim (C', p_i')$ are equivalent if there is an isomorphism $\alpha \colon C \to C'$ with $\alpha(p_i) = p_i'$.

- The moduli set $M_g[n]$ of smooth genus g curves with level n structure parameterizes smooth, connected, and projective curves C of genus g over $\mathbb{C}$ together with a basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ of $H_1(C, \mathbb{Z}/n\mathbb{Z})$ such that the intersection pairing is symplectic, while two objects $(C, \alpha_i, \beta_i) \sim (C', \alpha_i', \beta_i')$ are declared equivalent if there is an isomorphism $C \to C'$ taking α_i and β_i to α_i' and β_i'.

- For the moduli set whose objects are plane curves $C \subset \mathbb{P}^2$, there are several choices for equivalence relations $C \sim C'$: (a) C and C' are equal as subschemes, (b) C and C' are projectively equivalent (i.e. there is an automorphism of $\mathbb{P}^2$ taking C to C'), or (c) C and C' are abstractly isomorphic.

Example 0.2.2 (Moduli set of vector bundles on a curve). The moduli set $\mathrm{Bun}_{r,d}(C)$ parameterizes vector bundles of rank r and degree d on a fixed smooth, connected, and projective curve C; the equivalence relation here is isomorphism. The special case of $r = 1$ yields the set $\mathrm{Pic}^d(C)$ parameterizing degree d line bundles on C. This is non-canonically identified with with the abelian variety $\mathrm{H}^1(C, \mathcal{O}_C)/\mathrm{H}^1(C, \mathbb{Z}) = \mathbb{C}^g/\mathbb{Z}^{2g}$ via the exponential exact sequence (0.1.4).

A recurring theme in moduli is the exhibition of moduli spaces as quotients of group actions.

Example 0.2.3 (Moduli set of orbits). Given a group action of a group G on a set X, we define the *moduli set of orbits* by taking the objects to be all elements $x \in X$ and by declaring x to be equivalent to x' if they have the same orbit $Gx = Gx'$. In other words, the moduli set of orbits is the quotient set X/G.

Some examples to keep in mind are the $\mathbb{Z}/2$-action on $\mathbb{A}^1$ via $(-1) \cdot x = -x$ and the usual scaling action of $\mathbb{G}_m$ on $\mathbb{A}^n$ via $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$. The quotient set $(\mathbb{A}^n \setminus 0)/\mathbb{G}_m$ is identified with $\mathbb{P}^{n-1}$. The quotient $\mathbb{A}^n/\mathbb{G}_m$ including the origin—and particularly the case of $\mathbb{A}^1/\mathbb{G}_m$—shows up repeatedly in this text. Another interesting example is the $\mathbb{G}_m$-action on $\mathbb{A}^2$ given by $t \cdot (x, y) = (tx, t^{-1}y)$.

0.2.1   Toy example: moduli of triangles

Before diving deeper into M_g and $\mathrm{Bun}_{r,d}(C)$, let's study the simple yet surprisingly fruitful example of the moduli of triangles. These moduli spaces are easy to visualize and, as M. Artin has remarked, are useful to illustrate various themes of stacks and moduli.

Example 0.2.4 (Labelled triangles). A *labeled triangle* is a triangle in $\mathbb{R}^2$ where the vertices are labeled with '1', '2' and '3', and the distances of the edges are denoted as a, b, and c. We require that triangles have nonzero area or equivalently that their vertices are not collinear.
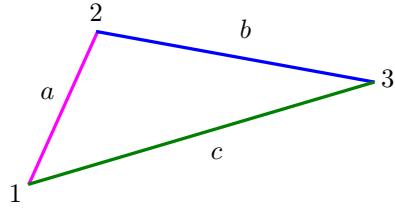
Figure 0.2: To keep track of the labeling, we color the edges.

We define the *moduli set of labeled triangles M* as the set of labeled triangles where two triangles are said to be equivalent if they are the same triangle in $\mathbb{R}^2$ with the same vertices and same labeling. By writing (x_1, y_1), (x_2, y_2) and (x_3, y_3) as the coordinates of the labeled vertices, we obtain a bijection

$$M \cong \left\{ (x_1, y_1, x_2, y_2, x_3, y_3) \;\middle|\; \det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \neq 0 \right\} \subset \mathbb{R}^6 \qquad (0.2.5)$$

with the open subset of $\mathbb{R}^6$ whose complement is the codimension 1 closed subset defined by the condition that the vectors $(x_2, y_2) - (x_1, y_1)$ and $(x_3, y_3) - (x_1, y_1)$ are linearly dependent.
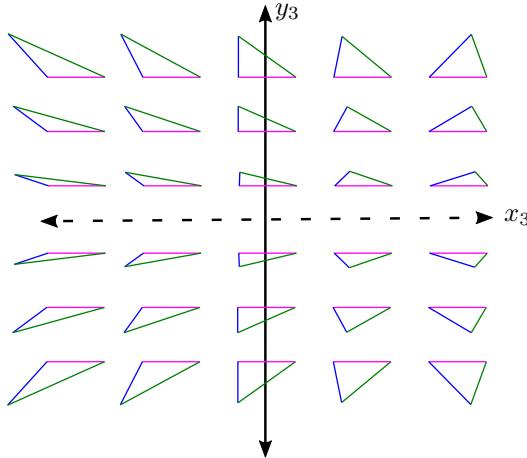
Figure 0.3: Picture of the slice of the moduli space M where $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (1, 0)$. Triangles are described by their third vertex (x_3, y_3) with $y_3 \neq 0$. We've drawn representative triangles for a handful of points in the $x_3 y_3-$ plane.

Example 0.2.6 (Labelled triangles up to similarity)**.** We define the *moduli set of labeled triangles up to similarity*, denoted by M^{lab}, by taking the same class of objects as in the previous example—labeled triangles—but changing the equivalence relation to label-preserving similarity.
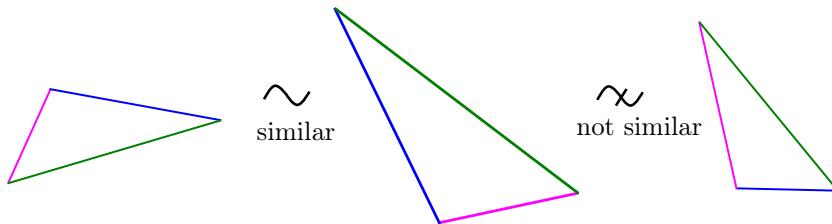
Figure 0.4: The two triangles on the left are similar, but the third is not.
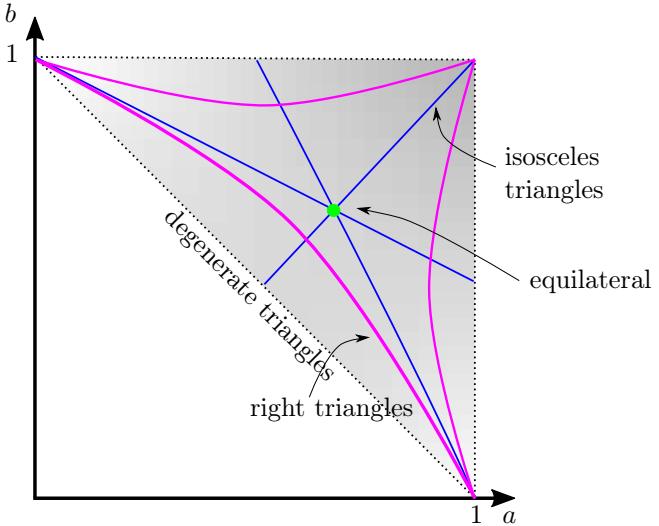
Every labeled triangle is similar to a unique labeled triangle with perimeter

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$a + b + c = 2$. We have the description

$$M^{\mathrm{lab}} = \left\{ (a,b,c) \, \middle| \, \begin{array}{c} a + b + c = 2 \\ 0 < a < b + c \\ 0 < b < a + c \\ 0 < c < a + b \end{array} \right\}. \tag{0.2.7}$$

By setting $c = 2 - a - b$, we may visualize M^{lab} as the analytic open subset of $\mathbb{R}^2$ defined by pairs (a, b) satisfying $0 < a, b < 1$ and $a + b > 1$.

Figure 0.5: M^{lab} is the shaded triangle. The pink lines represent the right triangles defined by $a^2 + b^2 = c^2$, $a^2 + c^2 = b^2$ and $b^2 + c^2 = a^2$, the blue lines represent isosceles triangles defined by $a = b$, $b = c$ and $a = c$, and the green point is the unique equilateral triangle defined by $a = b = c$.

Example 0.2.8 (Unlabeled triangles up to similarity)**.** We now turn to the moduli of unlabeled triangles up to similarity, which reveals a new feature not seen in to the two previous examples: symmetry!

We define the *moduli set of unlabeled triangles up to similarity*, denoted by M^{unl}, where the objects are unlabeled triangles in $\mathbb{R}^2$ and the equivalence relation is similarity. We can describe an unlabeled triangle uniquely by the ordered tuple (a, b, c) of increasing side lengths as follows:

$$M^{\mathrm{unl}} = \left\{ (a,b,c) \, \middle| \, \begin{array}{c} 0 < a \le b \le c < a + b \\ a + b + c = 2 \end{array} \right\}. \tag{0.2.9}$$
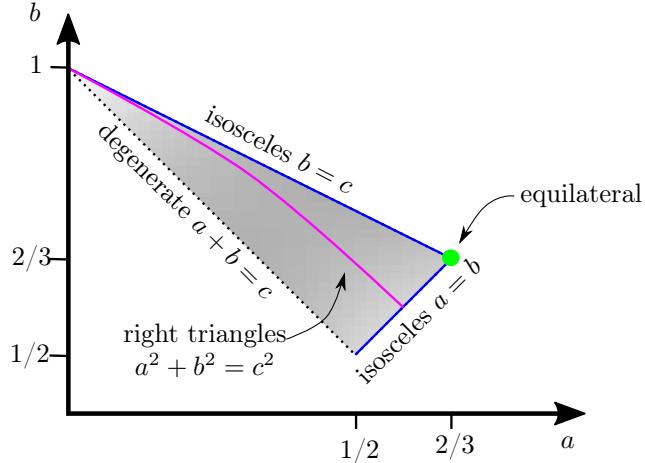
Figure 0.6: Picture of M^{unl}

The isosceles triangles with $a = b$ or $b = c$ and the equilateral triangle with $a = b = c$ have symmetry groups of $\mathbb{Z}/2$ and S_3, respectively. This is unfortunately not encoded into our description M^{unl} above. Note that we can identify M^{unl} as the quotient M^{lab}/S_3 under the natural action of S_3 on the labelings, and that the stabilizers of isosceles and equilateral triangles are precisely their symmetry groups $\mathbb{Z}/2$ and S_3. The action of S_3 on the locus of triangles that are not isosceles or equilateral is free.

0.3   The functorial worldview

Defining a moduli functor requires specifying:

(1) families of objects,

(2) when two families of objects are equivalent, and

(3) how families pull back under morphisms.

In the algebraic category, this is packaged with a contravariant functor

$$F \colon \mathrm{Sch} \to \mathrm{Sets}$$

where, for a scheme S, $F(S)$ denotes the set of families of objects over S, and, for a map $f \colon S \to T$, the $F(f) \colon F(T) \to F(S)$ gives the pull back map from a family over T to a family over S. (To be a functor, pull back maps must commute with composition: $F(g \circ f) = F(f) \circ F(g)$ for maps $f \colon S \to T$ and $g \colon T \to U$.)

0.3.1   Family matters

Families allow us to provide a precise formulation of a moduli space M. A family of objects $\mathcal{C}$ over a space S defines a set-theoretic map

$$S \to M, \qquad s \mapsto [\mathcal{C}_s], \tag{0.3.1}$$

where the fiber $[\mathcal{C}_s] \in M$ is the pullback of $\mathcal{C}$ under the inclusion $\{s\} \hookrightarrow S$. In the topological (resp. algebraic) category, we desire that the map $S \to M$ is continuous (resp. an algebraic map). Ideally, there is a bijective correspondence between families

over S and morphisms $S \to M$, or in other words that the space M *represents* the functor $S \mapsto \{$families over $S\}$. If this happens, we call M a *fine moduli space*, but we will see shortly that this is often too much to hope for.

Defining moduli spaces via families has advantages:

– We can endow the moduli set M with enriched structures. To provide M a topology, we declare a subset $U \subset M$ to be *open* if for every family of objects $\mathcal{C}$ over S, the locus $\{s \in S \mid [\mathcal{C}_s] \in U\}$ is an open subset of S. A global function f on M can be defined as the data of compatible global functions $f_{\mathcal{X}}$ on S for every family $\mathcal{X}$ over S.

– When M is a fine moduli space, the identity map $\mathrm{id} \colon M \to M$ corresponds to a family of objects $\mathcal{U}$ over the moduli space M. This is the *universal family*: for any other family $\mathcal{C}$ over S, there is a unique morphism $S \to M$ (given set-theoretically by (0.3.1)) such that the universal family $\mathcal{U}$ pulls back to $\mathcal{C}$.

This is certainly a giant leap in abstraction! And it may seem that we just made life more difficult: rather than introducing a space by specifying its points, its topology, and possibly other structures, we must specify an immense amount of categorical data. In practice, however, it is usually quite straightforward to define good notions of families.

Example 0.3.2 (Families of labeled triangles)**.** Revisiting the moduli of labeled triangles up to similarity introduced in Example 0.2.6, we define a *family of labeled triangles over a topological space S* as a tuple $(\mathcal{T}, \sigma_1, \sigma_2, \sigma_3)$ where $\mathcal{T} \to S$ is a fiber bundle with three sections $\sigma_i \colon S \to \mathcal{T}$ equipped with a continuous distance function $d \colon \mathcal{T} \times_S \mathcal{T} \to \mathbb{R}_{\geq 0}$ such that for every point $s \in S$, the restriction $d_s \colon \mathcal{T}_s \times \mathcal{T}_s \to \mathbb{R}_{\geq 0}$ is a metric on the fiber $\mathcal{T}_s$ with $\mathcal{T}_s$ isometric to a triangle with vertices $\sigma_i(s)$.

We say two families $(\mathcal{T}, (\sigma_i))$ and $(\mathcal{T}', (\sigma_i'))$ of labeled triangles over $S \in \mathrm{Top}$ are *similar* if there is a homeomorphism $f \colon \mathcal{T} \to \mathcal{T}'$ over S compatible with the sections (i.e. $f \circ \sigma_i = \sigma_i'$) such that for each $s \in S$, the induced map $\mathcal{T}_s \to \mathcal{T}_s'$ on fibers is a similarity of triangles, i.e. an isometry after rescaling. Given a family $\mathcal{T} \to S$ of labeled triangles and a continuous map $S' \to S$, the pullback family is defined as the fiber product $\mathcal{T} \times_S S'$ of sets together with the pullback sections $\sigma_i' \colon S' \to \mathcal{T}'$ and its inherited distance function.
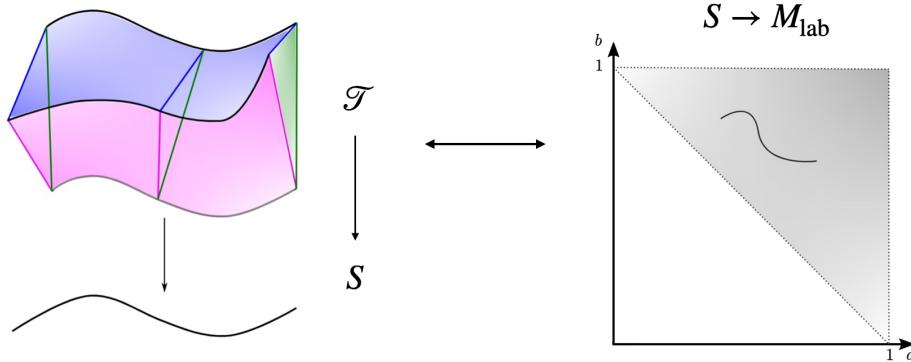
Figure 0.7: A family of labeled triangles over a curve corresponds to an arc in the moduli space.

We define the *moduli functor of labeled triangles* as

$$F_{M^{\mathrm{lab}}} \colon \mathrm{Top} \to \mathrm{Sets}, \quad S \mapsto \{\text{families } (\mathcal{T} \to S, \sigma_i) \text{ of labeled triangles}\}/(\text{similarity}).$$

Recall from (0.2.7) that the assignment of a triangle to its side lengths yields a bijection between $F_{M^{\mathrm{lab}}}$ and

$$M^{\mathrm{lab}} = \left\{ (a, b, c) \,\middle|\, \begin{array}{c} a + b + c = 2 \\ 0 < a < b + c \\ 0 < b < a + c \\ 0 < c < a + b \end{array} \right\}.$$

Since this extends to a compatible isomorphisms of $F_{M^{\mathrm{lab}}}(S) \to \mathrm{Mor}(S, M^{\mathrm{lab}})$ for every space S, the topological space M^{lab} represents the functor $F_{M^{\mathrm{lab}}}$. Consequently, there is a universal family $\mathcal{T}_{\mathrm{univ}} \subset M^{\mathrm{lab}} \times \mathbb{R}^2$ with $\sigma_i \colon M^{\mathrm{lab}} \to \mathcal{T}_{\mathrm{univ}}$.
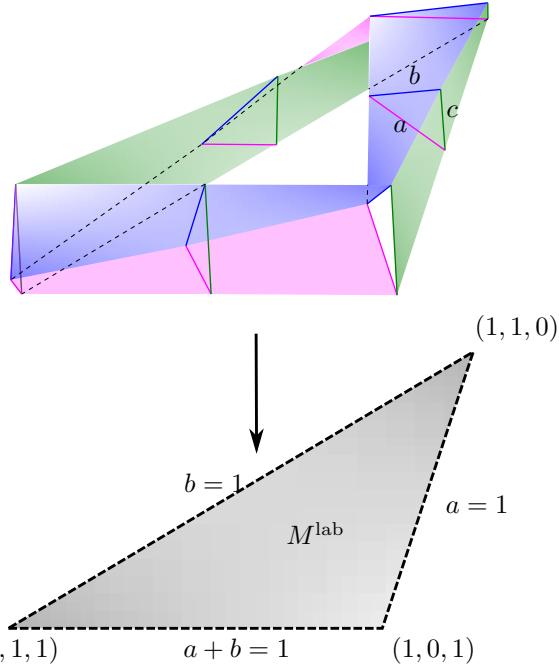
Figure 0.8: The universal family $U^{\mathrm{lab}} \to M^{\mathrm{lab}}$ of labeled triangles up to similarity.

Example 0.3.3 (Families of unlabeled triangles)**.** Revisiting Example 0.2.8, we define a *family of unlabeled triangles* as a fiber bundle $\mathcal{T} \to S$ equipped with a continuous distance function $d \colon \mathcal{T} \times_S \mathcal{T} \to \mathbb{R}_{\geq 0}$ that restricts to a metric on every fiber and such that every fiber is isometric to a triangle. Two families $\mathcal{T} \to S$ and $\mathcal{T}' \to S$ are *similar* if there is a homeomorphism $f \colon \mathcal{T} \to \mathcal{T}'$ over S compatible with the sections inducing similarities of triangles on fibers.

We define the functor

$$F \colon \mathrm{Top} \to \mathrm{Sets}, \quad S \mapsto \{\text{families } \mathcal{T} \subset S \times \mathbb{R}^2 \text{ of triangles}\}/\text{similarity}$$

but we can already see complications arising from the presence of symmetries of our objects—each equilateral triangle has symmetry group S_3 while the isosceles triangles have symmetry groups $\mathbb{Z}/2$. This functor is *not* representable as there are non-trivial families of triangles $\mathcal{T}$ such that all fibers are similar triangles (Proposition 0.3.21). For instance, we construct a non-trivial family of triangles over S^1 by gluing two trivial families via a symmetry of an equilateral triangle.

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Figure 0.9: A trivial (left) and non-trivial (right) family of equilateral triangles. Image taken from a video produced by Jonathan Wise: see http://math.colorado.edu/~jonathan.wise/visual/moduli/index.html.

0.3.2 Moduli functors of curves, vector bundles, and orbits

Defining a moduli functor $F \colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ in the category of $\mathbb{C}$-schemes entails specifying for every $\mathbb{C}$-scheme S a set $F(S)$ of families of objects over S, and pull back maps $F(S) \to F(S')$ for morphisms $S' \to S$ which are compatible under composition.

To gain intuition for a moduli functor, it is always useful to plug in special test schemes. For instance, by plugging in $S = \mathrm{Spec}\,\mathbb{C}$, we obtain the underlying moduli set $F(\mathrm{Spec}\,\mathbb{C})$ of objects. By plugging in $S = \mathbb{C}[\epsilon]/(\epsilon^2)$, we obtain a set of pairs consisting of a $\mathbb{C}$-point and a tangent vector, and plugging in a curve (or a DVR) gives families of objects over the curve.

Example 0.3.4 (Moduli functor of smooth curves). A *family of smooth curves of genus g* is a smooth, proper morphism $\mathcal{C} \to S$ of schemes such that for every $s \in S$, the fiber $\mathcal{C}_s$ is a connected curve of genus g.
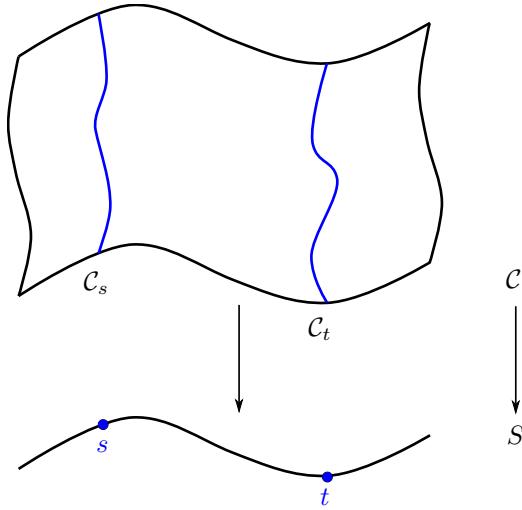
Figure 0.10: A family of curves over a curve S.

The *moduli functor of smooth curves of genus g* is

$$F_{M_g} \colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}, \quad S \mapsto \{\text{families of smooth curves } \mathcal{C} \to S \text{ of genus } g\}\,/\sim,$$

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where two families $\mathcal{C} \to S$ and $\mathcal{C}' \to S$ are equivalent if there is a isomorphism $\mathcal{C} \to \mathcal{C}'$ over S. If $S' \to S$ is a map of schemes and $\mathcal{C} \to S$ is a family of curves, the pullback is defined as the family $\mathcal{C} \times_S S' \to S'$.

Example 0.3.5 (Moduli functor of vector bundles on a curve)**.** Let C be a fixed smooth, connected, and projective curve over $\mathbb{C}$. A *family of vector bundles of rank r and degree d* over a scheme S is a vector bundle $\mathcal{E}$ on $C \times S$ such that for every $s \in S$, the restriction $\mathcal{E}_s := \mathcal{E}|_{C_{\kappa(s)}}$ of $\mathcal{E}$ to $C_{\kappa(s)} := C \times_{\mathbb{C}} \kappa(s)$ has rank r and degree d. The *moduli functor of vector bundles on C of rank r and degree d* is

$$\mathrm{Sch}/\mathbb{C} \to \mathrm{Sets} \qquad S \mapsto \left\{ \begin{array}{c} \text{vector bundles } \mathcal{E} \text{ on } C \times S \\ \text{of rank } r \text{ and degree } d \end{array} \right\} /(\text{isomorphism}),$$

If $f \colon S' \to S$ is a map of schemes and E is a vector bundle on $C \times S$, the pullback is defined as the vector bundle $(\mathrm{id} \times f)^* \mathcal{E}$ on $C \times S'$.

We will see in Section 0.3.5 that these two functors are not representable, and correspondingly that there is no fine moduli space.

Example 0.3.6 (Moduli functor of orbits)**.** Consider the action of an algebraic group G over $\mathbb{C}$ acting on a scheme X. For every scheme S, the abstract group $G(S)$ acts on the set $X(S)$—in fact, giving such actions functorial in S uniquely specifies the group action (Exercise B.2.11). We can consider the functor

$$\mathrm{Sch}/\mathbb{C} \to \mathrm{Sets} \qquad S \mapsto X(S)/G(S).$$

This is a very naive candidate for a moduli functor of a quotient, and very far from being representable even for free actions (see Exercise 0.3.27). We will modify this example in §0.6.5.

In some cases, you may know precisely which objects you want to parameterize, but it may not be straightforward to introduce a notion of families. Or there may be several candidate notions for a family of objects, which could translate to different scheme structures on the same topological space. This happens for instance for the moduli of higher dimensional varieties.

0.3.3   Yoneda's lemma and representable functors

Following Grothendieck, we study a scheme X by studying all maps to it! That schemes are determined by maps into them is a completely formal fact that is true in every category. This is the Yoneda Lemma: for an object X of a category $\mathcal{C}$, the contravariant functor

$$h_X \colon \mathcal{C} \to \mathrm{Sets}, \qquad S \mapsto \mathrm{Mor}(S, X)$$

recovers the object X itself.

Lemma 0.3.7 (Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category and X be an object. For every contravariant functor $G \colon \mathcal{C} \to \mathrm{Sets}$, the map*

$$\mathrm{Mor}(h_X, G) \to G(X), \qquad \alpha \mapsto \alpha_X(\mathrm{id}_X)$$

is bijective and functorial with respect to both X and G, where the left-hand side denotes the set of natural transformations $h_X \to G$ and α_X denotes the map $h_X(X) = \mathrm{Mor}(X, X) \to G(X)$.

Caution 0.3.8. Throughout this book, we will consistently abuse notation by conflating an element $g \in G(X)$ and the corresponding morphism $h_X \to G$, which we will often write simply as $X \to G$.

Exercise 0.3.9.

(a) Spell out precisely what 'functorial with respect to both X and G' means.

(b) Prove Yoneda's lemma.

Definition 0.3.10 (Representable functors and fine moduli spaces)**.** We say that a functor $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is *representable by a scheme* if there exists a scheme X and an isomorphism of functors $F \xrightarrow{\sim} h_X$.

When F is a moduli functor representable by a scheme M, we say that M is a *fine moduli space.*

By the Yoneda Lemma (0.3.7), if a functor is representable, then it is representable by a *unique* scheme. One of our aims is to understand when a given moduli functor F has a fine moduli space, i.e. is representable by a scheme.

Example 0.3.11 (Projective space as a functor)**.** By [Har77, Thm. II.7.1], there is a functorial bijection

$$\mathrm{Mor}(S, \mathbb{P}^n_{\mathbb{Z}}) \cong \left\{ (L, s_0, \ldots, s_n) \;\middle|\; \begin{array}{l} L \text{ is a line bundle on } S \text{ globally} \\ \text{generated by } s_0, \ldots, s_n \in \Gamma(S, L) \end{array} \right\} / \sim,$$

where $(L, (s_i)) \sim (L', (s'_i))$ if there exists an isomorphism $\alpha\colon L \to L'$ and $t \in \Gamma(S, \mathcal{O}_S)^*$ such that $s_i = t\alpha^* s'_i$ for all i. In other words, the functor on the right is representable by the scheme $\mathbb{P}^n_{\mathbb{Z}}$. The condition that the sections s_i are globally generated translates to the condition that for every $x \in S$, at least one section $s_i(x) \in L \otimes \kappa(t)$ is nonzero, or equivalently to the surjectivity of $(s_0, \ldots, s_n)\colon \mathcal{O}_S^{n+1} \to L$.

Example 0.3.12 (The Grassmannian functor)**.** As a set, the Grassmannian $\mathrm{Gr}(k, n)$ parameterizing k-dimensional *quotients* of n-dimensional space.[3] But what are families of k-dimensional quotients over a scheme S? A naive guess might be quotients $q\colon \mathcal{O}_S^n \twoheadrightarrow \mathcal{O}_S^k$ but this has no chance to be representable (see Exercise 0.3.27). The case of projective space suggests we define the *Grassmannian functor* as

$$\mathrm{Gr}(k, n)\colon \mathrm{Sch} \to \mathrm{Sets}$$

$$S \mapsto \left\{ [\mathcal{O}_S^n \twoheadrightarrow Q] \;\middle|\; Q \text{ is a vector bundle of rank } k \right\} / \sim$$

where $[\mathcal{O}_S^n \xrightarrow{q} Q] \sim [\mathcal{O}_S^n \xrightarrow{q'} Q']$ if there exists an isomorphism $\Psi\colon Q \xrightarrow{\sim} Q'$ such that

$$\begin{array}{ccc} \mathcal{O}_S^n & \xrightarrow{\;q\;} & Q \\ & {\scriptstyle q'} \searrow & \big\downarrow {\scriptstyle \Psi} \\ & & Q' \end{array}$$

commutes (i.e. $q' = \Psi \circ q$), or equivalently if $\ker(q) = \ker(q')$. Pullbacks are defined in the obvious manner.

We will later show that $\mathrm{Gr}(k, n)$ is representable by a scheme projective over $\mathbb{Z}$ (Theorem 1.1.1). The proof of this result is a good illustration of the utility of the functorial approach and a warmup for the representability of Hilb and Quot (Theorems 1.1.2 and 1.1.3).

---

[3]Alternatively, the points could be considered as k-dimensional *subspaces* but in these notes, we will follow Grothendieck's convention of quotients.

These exercises will give you some practice.

Exercise 0.3.13 (Affine and Projective Space)**.** Let S be a scheme and E be a vector bundle on S.

(a) Show that the affine space $\mathbb{A}(E) := \mathcal{S}pec_S(\mathrm{Sym}^* E)$ represents the functor assigning $f\colon T \to S$ to $\Gamma(T, f^*E^\vee)$, where $E^\vee := \mathscr{H}om_{\mathcal{O}_S}(E, \mathcal{O}_S)$. Note that a $\Bbbk$-point of $\mathbb{A}(E)$ is an element of $E^\vee \otimes_{\mathcal{O}_S} \Bbbk$, and the special case $\mathbb{A}(\mathcal{O}_S^{\oplus n}) \cong \mathbb{A}_S^n$.

(b) Show that the projectivization $\mathbb{P}(E) := \mathcal{P}roj_S(\mathrm{Sym}^* E)$ of E represents the functor

$$\mathrm{Sch}/S \to \mathrm{Sets}$$

$$(T \xrightarrow{f} S) \mapsto \{\text{quotients } q\colon f^*E \twoheadrightarrow L \text{ where } L \text{ is a line bundle on } T\}/\sim$$

where $[q\colon f^*E \twoheadrightarrow L] \sim [q'\colon f^*E \twoheadrightarrow L']$ if $\ker(q) = \ker(q')$ (or equivalently there is an isomorphism $\alpha\colon L \to L'$ with $q' = \alpha \circ q$). Note that E is naturally identified with the pushforward of $\mathcal{O}_{\mathbb{P}(E)}(1)$ along $\mathbb{P}(E) \to S$, and the special case $\mathbb{P}(\mathcal{O}_S^{n+1})$.

(Our definitions are consistent with [Har77, Exer. 5.18, Def. p.162], [EGA, II.4.1.1], and [SP, Tag 01OB] but beware that some authors use the dual $E^\vee$ instead of E in defining $\mathbb{A}(E)$ and $\mathbb{P}(E)$.)

Exercise 0.3.14. Provide functorial descriptions of:

(a) $\mathbb{A}^n \setminus 0$;

(b) the blowup $\mathrm{Bl}_p \mathbb{P}^n$ of $\mathbb{P}^n$ at a point;

(c) the normalization $\widetilde{X}$ of a reduced scheme X;

(d) $\mathcal{S}pec_S \mathcal{A}$ where $\mathcal{A}$ is a quasi-coherent sheaf of algebras on a scheme S; and

(e) $\mathrm{Proj}\, R$ where R is a positively graded ring.

Exercise 0.3.15.

(a) Let X be a scheme, and let E and F be vector bundles on X. Show that the functor

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(E, F)\colon \mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \mathrm{Hom}_{\mathcal{O}_{X \times T}}(E_T, F_T),$$

where E_T and F_T denote the pullbacks of E and F under $X \times T \to T$, is representable by $\mathbb{A}(\mathscr{H}om_{\mathcal{O}_X}(E, F)^\vee)$.

Note the distinction between $\mathrm{Hom}_{\mathcal{O}_X}(E, F)$ *(group of $\mathcal{O}_X$-module homomorphisms),* $\mathscr{H}om_{\mathcal{O}_X}(E, F)$ *(sheaf of $\mathcal{O}_X$-module homomorphisms), and* $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(E, F)$ *(functor or scheme parameterizing $\mathcal{O}_X$-module homomorphisms).*

(b) Let $E \to F$ be a homomorphism of coherent sheaves on a noetherian scheme X. Show that the subfunctor of X (or more precisely of $h_X = \mathrm{Mor}(-, X)$) defined by

$$\mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \{\text{morphisms } T \to X \text{ such that } E_T \to F_T \text{ is zero}\}$$

is representable by a closed subscheme of X.

(c) Recall that the group $\mathrm{Ext}^1_{\mathcal{O}_X}(Q, K)$ classifies extensions $0 \to K \to E \to Q \to 0$ of $\mathcal{O}_X$-modules, where two extensions are identified if there is an isomorphism of short exact sequences inducing the identity map on K and Q [Har77, Exer. III.6.1]. Assuming that Q and K are vector bundles, show that the affine scheme $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(Q, K) := \mathbb{A}(\mathscr{E}xt^1_{\mathcal{O}_X}(Q, K)^\vee)$ represents the functor

$$\mathrm{Sch} \to \mathrm{Sets}, \quad T \mapsto \mathrm{Ext}^1_{\mathcal{O}_{X \times T}}(Q_T, E_T).$$

0.3.4 Universal families

Definition 0.3.16. If $F\colon \mathrm{Sch} \to \mathrm{Sets}$ is a moduli functor representable by a scheme M via an isomorphism $\alpha\colon F \xrightarrow{\sim} h_M$ of functors, then the *universal family* of F is the object $u \in F(M)$ corresponding under α to the identity morphism $\mathrm{id}_M \in h_M(M) = \mathrm{Mor}(M, M)$.

Exercise 0.3.17 (easy but important)**.** Let $F\colon \mathrm{Sch} \to \mathrm{Sets}$ be a functor representable by a scheme M, and let $u \in F(X)$ be the universal family. Show that if $a \in F(T)$ is an object over a scheme T corresponding to a map $f_a\colon T \to M$, the object a is the pullback of u under f_a, i.e. $F(f_a)(a) = u$.

Suspend your skepticism for a moment and suppose that there actually exists a scheme M_g representing the moduli functor of smooth curves of genus g (Example 0.3.4). Then corresponding to the identity map $M_g \to M_g$ is a family of genus g curves $U_g \to M_g$ satisfying the following universal property: for every smooth family of curves $\mathcal{C} \to S$ over a scheme S, there is a unique map $S \to M_g$ and cartesian diagram

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & U_g \\
\downarrow & \square & \downarrow \\
S & \longrightarrow & M_g.
\end{array}
$$

The map $S \to M_g$ sends a point $s \in S$ to the curve $[\mathcal{C}_s] \in M_g$. The above ideas are made precise in §sec:universal-families.
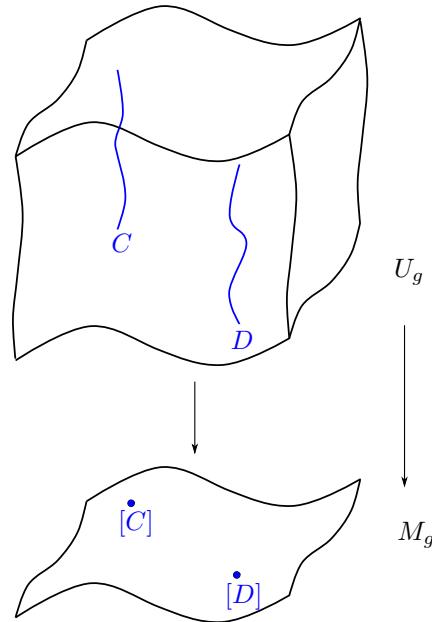
Figure 0.11: Visualization of a (non-existent) universal family over M_g.

Example 0.3.18. The universal family of the moduli functor of projective space (Example 0.3.11) is the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ together with the sections $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Example 0.3.19 (Universal extensions)**.** If X is a scheme with vector bundles E and G, the universal family for the moduli functor $\underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G, E)$ of extensions of Exercise 0.3.15(c) is an extension $0 \to p_1^* G \to \mathcal{F} \to p_1^* E \to 0$ of vector bundle on $X \times \underline{\mathrm{Ext}}^1_{\mathcal{O}_X}(G, E)$ whose restriction to $X \times \{t\}$ is the extension corresponding to $t \in \mathrm{Ext}^1_{\mathcal{O}_X}(G, E)$.

Example 0.3.20 (Classifying spaces in algebraic topology)**.** Let G be a topological group and $\mathrm{Top}^{\mathrm{para}}$ be the category of paracompact topological spaces where morphisms are defined up to homotopy. It is a theorem in algebraic topology that the functor

$$\mathrm{Top}^{\mathrm{para}} \to \mathrm{Sets}, \qquad S \mapsto \{\text{principal } G\text{-bundles } P \to S\}/\sim,$$

where $\sim$ denotes isomorphism, is represented by a topological space, which we denote by BG and call the *classifying space*. The universal family is usually denoted by $EG \to BG$. For example, the classifying space $B\mathbb{C}^*$ is the infinite dimensional manifold $\mathbb{CP}^\infty$.

0.3.5   Examples of non-representable moduli functors

If $F \colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ is a moduli functor, then an object $E \in F(\mathbb{C})$ with a non-trivial automorphism can prevent the functor F from being representable. This is because we may glue trivial families using the automorphism to construct a *non-trivial* family $\mathcal{E}$ over a scheme S such that every fiber $\mathcal{E}_s$ (i.e. the pullback of $\mathcal{E}$ along $\mathrm{Spec}\,\mathbb{C} \to S$) is isomorphic to E.

Proposition 0.3.21. *Let $F \colon \mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ be a moduli functor. If there is a family of objects $\mathcal{E} \in F(S)$ over a variety S such that*

  (a) the fibers $\mathcal{E}_s$ are isomorphic for $s \in S(\mathbb{C})$; and

  (b) the family $\mathcal{E}$ is non-trivial, i.e. is not equal to the pullback of an object $E \in F(\mathbb{C})$ along the structure map $S \to \mathrm{Spec}\,\mathbb{C}$,

then F is not representable.

Proof. Suppose by way of contradiction that F is represented by a scheme X. By condition (a), the restriction $E := \mathcal{E}_s$ is independent of $s \in S(\mathbb{C})$ and defines a unique point $x \in X(\mathbb{C})$. As S is reduced, the map $S \to X$ factors as $S \to \mathrm{Spec}\,\mathbb{C} \xrightarrow{x} X$. This implies that the family $\mathcal{E}$ is the pullback under the constant map $S \to \mathrm{Spec}\,\mathbb{C} \xrightarrow{x} X$, i.e. $\mathcal{E}$ is a trivial family, which contradicts condition (b). $\square$

Example 0.3.22 (Moduli of elliptic curves)**.** An *elliptic curve* is a pair (E, p) where E is a smooth, connected, and projective curve E of genus 1 and $p \in E(\mathbb{C})$. A *family of elliptic curves* over a scheme S is a pair $(\mathcal{E} \to S, \sigma)$ where $\mathcal{E} \to S$ is smooth proper morphism with a section $\sigma \colon S \to \mathcal{E}$ such that for every $s \in S$, the fiber $(\mathcal{E}_s, \sigma(s))$ is an elliptic curve over the residue field $\kappa(s)$. The *moduli functor of elliptic curves* is

$$\begin{aligned} F_{M_{1,1}} \colon \mathrm{Sch} &\to \mathrm{Sets} \\ S &\mapsto \{\text{families } (\mathcal{E} \to S, \sigma) \text{ of elliptic curves } \}/\sim, \end{aligned}$$

where $(\mathcal{E} \to S, \sigma) \sim (\mathcal{E}' \to S, \sigma')$ if there is a S-isomorphism $\alpha \colon \mathcal{E} \to \mathcal{E}'$ compatible with the sections (i.e. $\sigma' = \alpha \circ \sigma$).

Exercise 0.3.23 (good practice)**.** Consider the family of elliptic curves defined over $\mathbb{A}^1 \setminus 0$ (with coordinate t) by

$$\mathcal{E} := V(y^2 z - x^3 + t z^3) \lhook\joinrel\longrightarrow (\mathbb{A}^1 \setminus 0) \times \mathbb{P}^2$$
$$\downarrow$$
$$\mathbb{A}^1 \setminus 0$$

with section $\sigma \colon \mathbb{A}^1 \setminus 0 \to \mathcal{E}$ given by $t \mapsto [0, 1, 0]$. Show that $(\mathcal{E} \to \mathbb{A}^1 \setminus 0, \sigma)$ satisfies (a) and (b) in Proposition 0.3.21.

Example 0.3.24 (Moduli functor of smooth curves)**.** Let C be a curve with a non-trivial automorphism $\alpha \in \mathrm{Aut}(C)$ and let N be the nodal cubic curve, which we can think of as $\mathbb{P}^1$ with the points 0 and ∞ glued together. We can construct a family $\mathcal{C} \to N$ by taking the trivial family $\pi \colon C \times \mathbb{P}^1 \to \mathbb{P}^1$ and gluing the fiber $\pi^{-1}(0)$ with $\pi^{-1}(\infty)$ via the automorphism α. To show that the moduli functor $\mathrm{Sch}/\mathbb{C} \to \mathrm{Sets}$ of smooth curves is not representable, it suffices to show that $\mathcal{C} \to N$ is non-trivial.
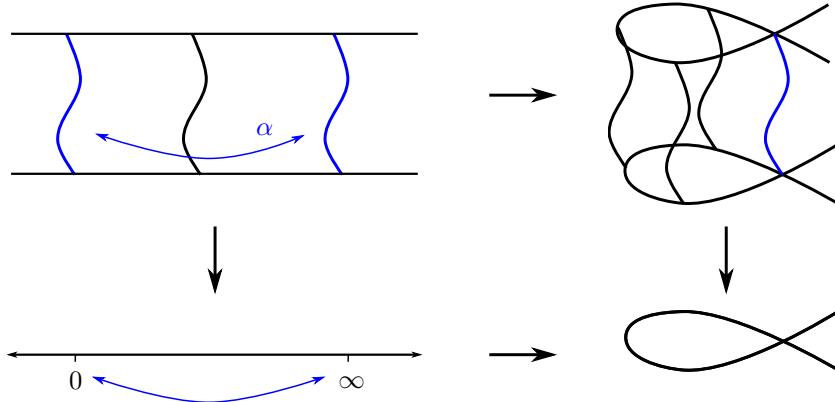
Figure 0.12: Family of curves over the nodal cubic obtaining by gluing the fibers over 0 and ∞ of the trivial family over $\mathbb{P}^1$ via α. (It would be more illustrative to draw a Mobius band as the family of curves over the nodal cubic.)

Exercise 0.3.25 (details)**.** Show that $\mathcal{C} \to N$ is a non-trivial family.

Exercise 0.3.26 (good practice)**.** Show that the moduli functor of vector bundles over a curve C is not representable.

0.3.6 Schemes are sheaves in the big Zariski topology

If $F \colon \mathrm{Sch} \to \mathrm{Sets}$ is representable by a scheme X, then F is necessarily a *sheaf in the big Zariski topology*, that is, for every scheme S, the presheaf on the Zariski topology of S, defined by assigning to an open subset $U \subset S$ the set $F(U)$, is a sheaf on the Zariski topology of S. This is a restatement that morphisms into the fixed scheme X glue uniquely. The failure to be a sheaf therefore provides another obstruction to the representability of a given moduli functor F.

Exercise 0.3.27.

(a) Show that the following naive Grassmannian functor

$$F\colon \text{Sch} \to \text{Sets}, \quad S \mapsto \{\text{quotients } q\colon \mathcal{O}_S^n \twoheadrightarrow \mathcal{O}_S^k\}/\sim$$

is not representable.

(b) Under the usual scaling action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1} \setminus 0$ with the usual scaling action, show that the functor $S \mapsto (\mathbb{A}^{n+1} \setminus 0)(S)/\mathbb{G}_m(S)$ is not a sheaf.

The presence of non-trivial automorphisms often implies that a given moduli functor is not a sheaf in the big Zariski topology.

Example 0.3.28. Consider the moduli functor F_{M_g} of smooth curves from Example 0.3.4. Let $\{S_i\}$ be a Zariski open covering of a scheme S, and suppose that $\mathcal{C}_i \to S_i$ are families of smooth curves $\mathcal{C}_i \to S_i$ with isomorphisms $\alpha_{ij}\colon \mathcal{C}_i|_{S_{ij}} \xrightarrow{\sim} \mathcal{C}_j|_{S_{ij}}$ on the intersection $S_{ij} := S_i \cap S_j$. The requirement that F_{M_g} be a sheaf (when restricted to the Zariski topology on S) implies that the families $\mathcal{C}_i \to S_i$ glue uniquely to a family of curves $\mathcal{C} \to S$. However, we have not required the isomorphisms α_i to be compatible on the triple intersection (i.e. $\alpha_{ij}|_{S_{ijk}} \circ \alpha_{jk}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$), which is necessary for gluing schemes [Har77, Exer II.2.12]. For this reason, F_{M_g} fails to be a sheaf.

Exercise 0.3.29. Show that the moduli functors of smooth curves and elliptic curves are not sheaves by explicitly exhibiting a scheme S, an open cover $\{S_i\}$, and families of curves over S_i that do not glue to a family over S.

0.3.7 The yoga of functors

Contravariant functors $F\colon \text{Sch} \to \text{Sets}$ form a category $\text{Fun}(\text{Sch}, \text{Sets})$ where morphisms are natural transformations. This category has fiber products: given morphisms $F \xrightarrow{\alpha} G$ and $G' \xrightarrow{\beta} G$, we define

$$F \times_G G'\colon \text{Sch} \to \text{Sets}$$
$$S \mapsto \{(a, b) \in F(S) \times G'(S) \,|\, \alpha_S(a) = \beta_S(b)\}$$

Exercise 0.3.30. Show that $F \times_G G'$ satisfies the universal property for fiber products in $\text{Fun}(\text{Sch}, \text{Sets})$.

Definition 0.3.31.

(1) We say that a morphism $F \to G$ of contravariant functors is *representable by schemes* if for every map $S \to G$ from a scheme S, the fiber product $F \times_G S$ is representable by a scheme.

(2) We say that a morphism $F \to G$ is an *open immersion* or that a subfunctor $F \subset G$ is *open* if for every morphism $S \to G$ from a scheme S, $F \times_G S$ is representable by an open subscheme of S.

(3) We say that a set of open subfunctors $\{F_i\}$ of F is a *Zariski open cover* if for every morphism $S \to F$ from a scheme S, $\{F_i \times_F S\}$ is a Zariski open cover of S (and in particular each F_i is an open subfunctor of F).

Each of these conditions can be checked on *affine* schemes.

These definitions give a recipe for checking that a given functor F is representable by a scheme: find a Zariski open cover $\{F_i\}$ where each F_i is representable.

Exercise 0.3.32.

(a) Let $F\colon \mathrm{Sch} \to \mathrm{Sets}$ be a functor which is a sheaf in the big Zariski topology and $\{F_i\}$ be a Zariski open cover of F. Show that if each F_i is representable by a scheme, then so is F.

(b) Show that a collection of open subfunctors $\{F_i\}$ of F is a Zariski open cover if and only if the map $\coprod_i F_i(\Bbbk) \to F(\Bbbk)$ is surjective for each algebraically closed field $\Bbbk$.

(c) Given morphisms of schemes $X \to Y$ and $Y' \to Y$, reprove the existence of the fiber product $X \times_Y Y'$ in the category of schemes by exhibiting a Zariski open cover $\{F_i\}$ of $X \times_Y Y'$ where each F_i is representable by an affine scheme.

Exercise 0.3.33. Let $X \to Y$ be a morphism of schemes each proper over a scheme S. Show that the subfunctor of $\mathrm{Mor}(-, S)$, parameterizing maps $T \to S$ of schemes such that $X_T \xrightarrow{\sim} Y_T$ is an isomorphism, is an open.

Exercise 0.3.34. Show that a scheme can be equivalently defined as a contravariant functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ on the category of affine schemes (or covariant functor on the category of rings) as follows. Let $\mathcal{C}$ be a full subcategory of the category $\mathrm{Fun}(\mathrm{AffSch}, \mathrm{Sets})$ of contravariant functors. Extending Definitions 0.3.10 and 0.3.31, we define a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ to be *representable in $\mathcal{C}$* if there exists an object $X \in \mathcal{C}$ and a functorial equivalence $F(S) = \mathrm{Mor}(S, X)$ for every $S \in \mathrm{AffSch}$. We say that a map $F \to G$ of functors from AffSch to Sets is *representable by open immersions in $\mathcal{C}$* if for every morphism $\mathrm{Spec}\, B \to G$, the fiber product $F \times_G \mathrm{Spec}\, B$ is representable by an object $X \in \mathcal{C}$ which is an open subscheme of $\mathrm{Spec}\, B$. Finally, we say that a collection $\{F_i\}$ of subfunctors of F is a *Zariski open $\mathcal{C}$-cover* if each $F_i \to F$ is representable by open immersions in $\mathcal{C}$ and for each algebraically closed field $\Bbbk$, the map $\coprod_i F_i(\Bbbk) \to F(\Bbbk)$ is surjective.

(a) Show that a scheme with affine diagonal can be equivalently defined as a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ such that there exists a Zariski open AffSch-cover $\{F_i\}$ of F with each F_i representable in AffSch.

(b) Give a similar characterization of separated schemes.

(c) Letting $\mathcal{C}$ be the category of schemes with affine diagonal, show that a scheme can be equivalently defined as a functor $F\colon \mathrm{AffSch} \to \mathrm{Sets}$ such that there exists a Zariski open $\mathcal{C}$-cover $\{F_i\}$ with each F_i representable in $\mathcal{C}$.

Replacing Zariski opens with étale morphisms leads to the definition of an algebraic space.

0.4   Moduli groupoids

We now change our perspective: rather than specifying *when* two objects are identified, we specify *how*!

One of the most desirable properties of a moduli space is the existence of a universal family (see §0.3.4) and the presence of automorphisms obstructs its existence (see §0.3.5). Encoding automorphisms into our descriptions will allow us to get around this problem.

To define a moduli groupoid, we need to specify

(1) objects; and

(2) a set of equivalences (possibly empty) between any two objects.

Shortly we will combine the functorial worldview of the last section with this groupoid perspective to define moduli stacks.

0.4.1 Groupoids

A convenient mathematical structure to encode objects and their identifications is a *groupoid*.

Definition 0.4.1. A *groupoid* is a category $\mathcal{C}$ where every morphism is an isomorphism.

    Two groupoids $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* if there is an equivalence of categories $F\colon \mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$, i.e. there is a functor $G\colon \mathcal{C}_2 \to \mathcal{C}_1$ such that $F \circ G$ and $G \circ F$ are isomorphic to the identity functors, or equivalently F is fully faithful and essentially surjective.

Example 0.4.2 (Sets are groupoids). If Σ is a set, the category $\mathcal{C}_\Sigma$, whose objects are elements of Σ and whose morphisms consist of only the identity morphism, is a groupoid.
    We say that a groupoid $\mathcal{C}$ *is equivalent to a set* Σ if there is an equivalence of categories $\mathcal{C} \to \mathcal{C}_\Sigma$.

Example 0.4.3 (Classifying groupoid). If G is a group, the *classifying groupoid* BG of G is defined as the category with one object $\star$ such that $\mathrm{Aut}(\star) = \mathrm{Mor}(\star, \star) = G$.

Example 0.4.4. The category FB of finite sets where morphisms are bijections is a groupoid. The isomorphism classes of FB are in bijection with $\mathbb{N}$ while $\mathrm{Aut}(\{1, \ldots, n\}) = S_n$ is the permutation group.

Example 0.4.5 (Projective space). Projective space is identified with the moduli groupoid of lines $L \subset \mathbb{A}^{n+1}$ through the origin where the only morphisms are the identity maps. Alternatively, the objects are nonzero linear maps $x = (x_0, \ldots, x_n)\colon \mathbb{C} \to \mathbb{C}^{n+1}$ and there is a unique morphism $x \to x'$ if and only if $\mathrm{im}(x) = \mathrm{im}(x') \subset \mathbb{C}^{n+1}$ (i.e. there exists a $\lambda \in \mathbb{C}^*$ such that $x' = \lambda x$).

0.4.2 Moduli groupoid of orbits

Example 0.4.6 (Moduli groupoid of orbits). Given an action of a group G on a set X, we define the *moduli groupoid of orbits* $[X/G]$[4] by taking the objects to be all elements $x \in X$ and by declaring $\mathrm{Mor}(x, x') = \{g \in G \mid x' = gx\}$.
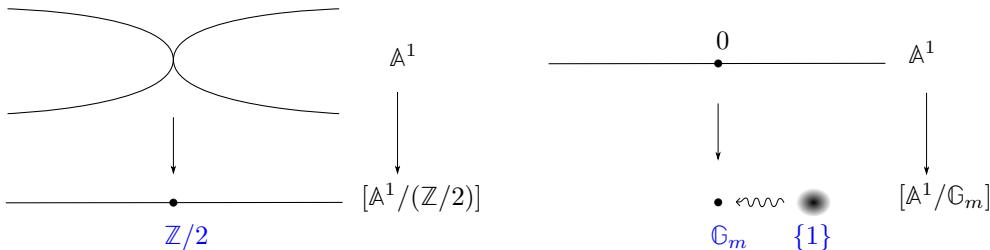
Figure 0.13: Pictures of the scaling actions of $\mathbb{Z}/2 = \{\pm 1\}$ and $\mathbb{G}_m$ on $\mathbb{A}^1$ over $\mathbb{C}$ with the automorphism groups listed in blue. Note that $[\mathbb{A}^1/\mathbb{G}_m]$ has two isomorphism classes of objects—0 and 1—corresponding to the two orbits—0 and $\mathbb{A}^1 \setminus 0$—such that $0 \in \overline{\{1\}}$ if the set $\mathbb{A}^1/\mathbb{G}_m$ is endowed with the quotient topology.

---

[4]We use brackets to distinguish the groupoid quotient $[X/G]$ from the set quotient X/G. Later when G is an algebraic group and X is a scheme, $[X/G]$ will denote the quotient stack.

Exercise 0.4.7. Show that the moduli groupoid of orbits $[X/G]$ in Example 0.4.6 is equivalent to a set if and only if the action of G on X is free.

Example 0.4.8. Consider the category $\mathcal{C}$ with two objects x_1 and x_2 such that $\mathrm{Mor}(x_i, x_j) = \{\pm 1\}$ for $i, j = 1, 2$ where composition of morphisms is given by multiplication. Then $\mathcal{C}$ is equivalent $B\mathbb{Z}/2$.
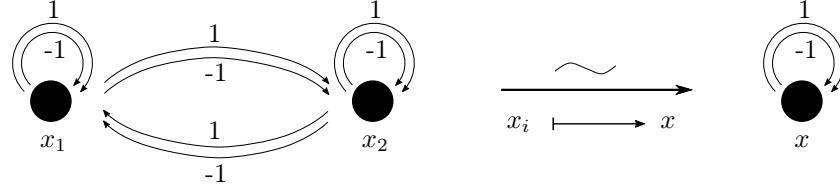
Figure 0.14: An equivalence of groupoids

Exercise 0.4.9. In Example 0.4.8, show that there is an equivalence of categories inducing a bijection on objects between $\mathcal{C}$ and either $[(\mathbb{Z}/2)/(\mathbb{Z}/4)]$ or $[(\mathbb{Z}/2)/(\mathbb{Z}/2 \times \mathbb{Z}/2)]$ where the action is given by surjections $\mathbb{Z}/4 \to \mathbb{Z}/2$ or $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$.

Example 0.4.10 (Projective space as a quotient). The moduli groupoid of projective space (Example 0.4.5) can also be described as the moduli groupoid of orbits $[(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m]$. We can also consider the quotient groupoid $[\mathbb{A}^{n+1}/\mathbb{G}_m]$, which is equivalent to the groupoid whose objects are (possibly zero) linear maps $x = (x_0, \ldots, x_n) \colon \mathbb{C} \to \mathbb{C}^{n+1}$ such that $\mathrm{Mor}(x, x') = \{t \in \mathbb{C}^* \mid x'_i = t x_i \text{ for all } i\}$. In this way, $\mathbb{P}^n$ is a subgroupoid of $[\mathbb{A}^{n+1}/\mathbb{G}_m]$.

Exercise 0.4.11. If a group G acts on a set X and $x \in X$ is a point, show that there is a fully faithful functor $BG_x \to [X/G]$. If the action is transitive, show that it is an equivalence.

A *morphisms of groupoids* $\mathcal{C}_1 \to \mathcal{C}_2$ is by definition a functor. The category $\mathrm{Mor}(\mathcal{C}_1, \mathcal{C}_2)$ has functors as objects and natural transformations as morphisms.

Exercise 0.4.12. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are groupoids, show that $\mathrm{Mor}(\mathcal{C}_1, \mathcal{C}_2)$ is a groupoid.

Exercise 0.4.13. If H and G are groups, show that there is an equivalence

$$\mathrm{Mor}(BH, BG) = \coprod_{\phi \in \mathrm{Hom}(H,G)/G} B\big(C^G(\mathrm{im}\,\phi)\big)$$

where $\mathrm{Hom}(H, G)/G$ denotes equivalence classes of homomorphisms $H \to G$ up to conjugation by G, and $C^G(\mathrm{im}\,\phi)$ denotes the centralizer of $\mathrm{im}\,\phi$ in G.

Exercise 0.4.14. Provide an example of group actions of H and G on sets X and Y and a map $[X/H] \to [Y/G]$ of groupoids that *does not* arise from a group homomorphism $\phi \colon H \to G$ and a ϕ-equivariant map $X \to Y$.

0.4.3 Examples of moduli groupoids

Example 0.4.15 (Moduli groupoid of smooth curves). In this case, the objects are smooth, connected, and projective curves of genus g over $\mathbb{C}$ and for two curves C, C', the set of morphisms is defined as the set of isomorphisms

$$\mathrm{Mor}(C, C') = \{\text{isomorphisms } \alpha \colon C \xrightarrow{\sim} C'\}.$$
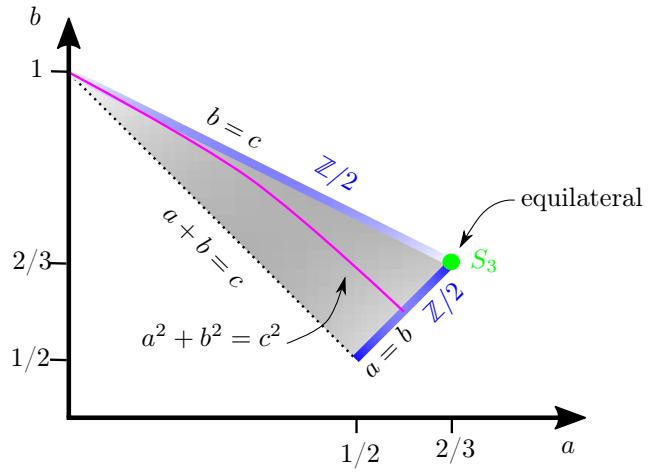
Example 0.4.16 (Moduli groupoid of vector bundles on a curve)**.** The objects are vector bundles E of rank r and degree d, and the morphisms are isomorphisms of vector bundles.

Example 0.4.17 (Moduli groupoid of unlabeled triangles)**.** Let's revisit the moduli M^{unl} of unlabeled triangles up to similarity from Example 0.2.8. Recall that we have already introduced families of unlabeled triangles and shown that this functor is not representable (Example 0.3.3).

We define the *moduli groupoid of unlabeled triangles up to similarity*, denoted by $\mathcal{M}^{\mathrm{unl}}$ (note the calligraphic font), where the objects are unlabeled triangles and the morphisms are similarities. For example, an isosceles triangle and an equilateral triangle have automorphism groups $\mathbb{Z}/2$ and S_3.

We can draw essentially the same picture as Figure 0.6 except we record the automorphisms.

Figure 0.15: Picture of the moduli groupoid $\mathcal{M}^{\mathrm{unl}}$ with non-trivial automorphism groups labeled.

There is a functor

$$\mathcal{M}^{\mathrm{unl}} \to M^{\mathrm{unl}},$$

from the moduli groupoid to the moduli set, which is an equivalence on isomorphism classes of objects and collapses all morphisms to the identity. This is a first example of a *coarse moduli space*.

Exercise 0.4.18. Recalling the description of the moduli set M^{lab} of labeled triangles up to similarity from (0.2.5), show that there is a natural action of S_3 on the moduli set M^{lab} of labeled triangles up to similarity and that there is an identification $\mathcal{M}^{\mathrm{unl}} \cong [M^{\mathrm{lab}}/S_3]$.

Exercise 0.4.19. Define a moduli groupoid of *oriented triangles* and investigate its relation to the moduli sets/groupoids of labeled/unlabeled triangles.

For a more detailed exposition of the moduli stack of triangles, see [Beh14].

0.5   Motivation: why the étale topology?

Moduli stacks will be introduced in the next section by combining moduli functors with groupoids—one needs to specify families of objects over every scheme S (along

with identifications and pullbacks). For such data to define a *stack*, we will require that objects and their morphisms glue in the *étale topology*!

Why is the Zariski topology not sufficient for our purposes? The short answer is that there are not enough Zariski-open subsets and that étale morphisms serve as a good replacement of analytic open subsets.

0.5.1 What is an étale morphism anyway?

I've sometimes been baffled when a student is intimidated by étale morphisms, especially when she has already mastered the conceptually more difficult notions of say properness and flatness. One factor could be the fact that the definition is buried in [Har77, Exercises III.10.3-6] and its importance is not highlighted there.
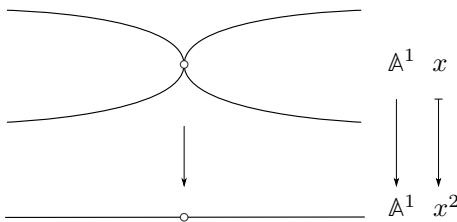
Figure 0.16: Picture of an étale double cover of $\mathbb{A}^1 \setminus 0$

The geometric picture to have in your mind is a covering space. There are several ways in which we can formulate an étale morphism $f \colon X \to Y$ of schemes of finite type over $\mathbb{C}$:

- f is smooth of relative dimension 0 (i.e. f is flat and all fibers are smooth of dimension 0);

- f is flat and unramified (i.e. for all $y \in Y(\mathbb{C})$, the scheme-theoretic fiber X_y is isomorphic to a disjoint union $\coprod_i \operatorname{Spec} \mathbb{C}$ of points);

- f is flat and $\Omega_{X/Y} = 0$;

- for all $x \in X(\mathbb{C})$, the induced map $\widehat{\mathcal{O}}_{Y,f(x)} \to \widehat{\mathcal{O}}_{X,x}$ on completions is an isomorphism; and

- assuming in addition that X and Y are smooth: for all $x \in X(\mathbb{C})$, the induced map $T_{X,x} \to T_{Y,f(x)}$ on tangent spaces is an isomorphism.

We say that f is *étale at* $x \in X$ if there is an open neighborhood U of x such that $f|_U$ is étale. See §A.3 for more background. These characterizations are all equivalent, but by no means this should be clear to you—some of the proofs are quite involved. Nevertheless, if you can take the equivalences on faith, it requires very little effort to not only internalize the concept but to master its use.

Exercise 0.5.1. Show that $f \colon \mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^2$ is étale over $\mathbb{A}^1 \setminus 0$ but is not étale at the origin. *Try to show this for as many of the above characterizations as you can.*

0.5.2 What can you see in the étale topology?

Working with the étale topology is like putting on a better pair of glasses allowing you to see what you couldn't before. Or perhaps more accurately, it is like getting

magnifying lenses for your algebraic geometry glasses allowing you to see what you already could with your differential geometry glasses.

Example 0.5.2 (Reducibility of a node)**.** Consider the plane nodal cubic C defined by $y^2 = x^2(x-1)$ in the plane. While there is an analytic open neighborhood of the node $p = (0,0)$ which is reducible, there is no such Zariski-open neighborhood. However, taking a 'square root' of $x - 1$ yields a reducible étale neighborhood. More specifically, define $C' = \operatorname{Spec} \mathbb{C}[x,y,t]_t / (y^2 - x^3 + x^2, t^2 - x + 1)$ and consider

$$C' \to C, \qquad (x, y, t) \mapsto (x, y)$$

Since $y^2 - x^3 + x^2 = (y - xt)(y + xt)$, we see that C' is reducible. This is illustrated by the following picture, which is also featured in [Har77, Exer. III.10.6].
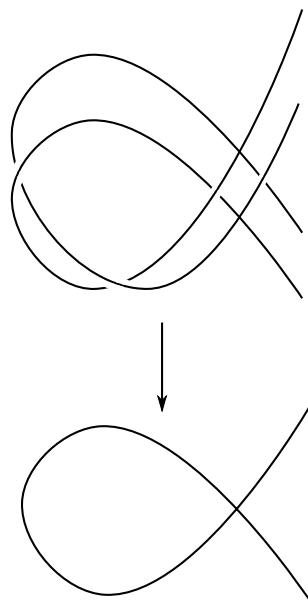
Figure 0.17: After an étale cover, the nodal cubic becomes reducible.

Example 0.5.3 (Étale cohomology)**.** Sheaf cohomology for the Zariski topology can be extended to the étale topology leading to the extremely robust theory of *étale cohomology*. For example, for a smooth projective curve C of genus g over $\mathbb{C}$, the étale cohomology $\mathrm{H}^1(C_{\text{ét}}, \mathbb{Z}/n)$ of the finite constant sheaf $\mathbb{Z}/2$ is isomorphic to $(\mathbb{Z}/n)^{2g}$ just like the ordinary cohomology groups, while the sheaf cohomology $\mathrm{H}^1(C, \mathbb{Z}/n)$ in the Zariski topology is 0. Finally, we would be remiss without mentioning the spectacular application of étale cohomology to prove the Weil conjectures.

Example 0.5.4 (Étale fundamental group)**.** Have you ever thought that there is a similarity between the bijection in Galois theory between intermediate field extensions and subgroups of the Galois group, and the bijection in algebraic topology between covering spaces and subgroups of the fundamental group? Well, you're in good company—Grothendieck also considered this and developed a beautiful theory of the *étale fundamental group* which packages Galois groups and fundamental groups in the same framework.

Example 0.5.5 (Quotients by free actions of finite groups)**.** If G is a finite group acting freely on a projective variety X, then there exists a quotient X/G as a

projective variety. The essential reason for this is that every G-orbit (or in fact every finite set of points) is contained in an affine variety U, which is the complement of some hypersurface. Then the intersection $V = \bigcap_g gU$ of the G-translates is a G-invariant affine open containing Gx and $V/G = \operatorname{Spec} \Gamma(V, \mathcal{O}_V)^G$ (see Corollary 4.2.8). These local quotients glue to form X/G.

However, if X is not projective, the quotient does not necessarily exist as a scheme. As with most phenomena for smooth proper varieties that are not projective, a counterexample can be constructed by using Hironaka's examples of smooth, proper 3-folds; [Har77, App. B, Ex. 3.4.1]. There is a smooth, proper 3-fold with a free action by $G = \mathbb{Z}/2$ such that there is an orbit Gx not contained in any G-invariant affine open. This shows that X/G cannot exist as a scheme; indeed, if it did, then the image of x under the finite morphism $X \to X/G$ would be contained in some affine and its inverse would be an affine open containing Gx. See [Knu71, Ex. 1.3] or [Ols16, Ex. 5.3.2] for details.

Nevertheless, for a free action of a finite group G on a scheme X, then every point $x \in X$ has a G-equivariant étale neighborhood $U_x \to X$ where U_x is an affine scheme, and the quotients U_x/G can be glued in the *étale topology* to construct X/G as an *algebraic space* (Corollary 3.1.14). The upshot is that we can always take quotients of free actions by finite groups. This is a very desirable feature given the ubiquity of group actions in algebraic geometry but it comes at the cost of enlarging our category from schemes to algebraic spaces.

Example 0.5.6 (Artin Approximation)**.** Artin Approximation (B.5.26) is a powerful and extremely deep result, due to Michael Artin, which implies that most properties which hold for the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring is also true in an étale neighborhood of x. See for a precise statement. For instance, since the completion of the local ring at a nodal singularity is reducible, Artin Approximation implies that there is a reducible étale neighborhood.

0.5.3   Working with the étale topology: descent theory

Another reason why the étale topology is so useful is that many properties of schemes and their morphisms can be checked on étale covers. In fact, almost every property that can be checked on a Zariski-open cover $\{U_i\}$ of scheme X can also be checked on an étale cover $\{U_i \to X\}$; here each map $U_i \to X$ is étale and $\coprod_i U_i \to X$ is surjective. Descent theory is developed in Section B.1 and is used to prove just about everything about algebraic spaces and stacks.

0.6   Moduli stacks

As promised, we now synthesize moduli functors with the groupoid perspective. To define a moduli stack, we need to specify

(1) families of objects;

(2) how two families of objects are isomorphic; and

(3) how families pull back under morphisms.

Notice the difference from specifying a moduli functor is that rather than specifying *when* two families are isomorphic, we specify *how*. In other words, we need to specify an assignment

$$F \colon \operatorname{Sch} \to \operatorname{Groupoids}, \quad S \mapsto \operatorname{Fam}_S.$$

taking a scheme S to a *groupoid* of families of objects over S. But what exactly do we mean by this? Groupoids form a '2-category' as they have objects (groupoids), morphisms (functors between groupoids), and 2-morphisms (natural transformations between functors). How can we precisely formulate such an assignment in down-to-earth terms? Well, we certainly need pullback functors $f^* \colon \mathrm{Fam}_T \to \mathrm{Fam}_S$ for each morphism $f \colon S \to T$. Given a composition $S \xrightarrow{f} T \xrightarrow{g} U$ of schemes, we should also have an isomorphism of functors (i.e. a 2-morphism) $\mu_{f,g} \colon (f^* \circ g^*) \xrightarrow{\sim} (g \circ f)^*$. Should the isomorphisms $\mu_{f,g}$ satisfy a compatibility condition under triples $S \xrightarrow{f} T \xrightarrow{g} U \xrightarrow{h} V$? Yes! This leads to the notion of a *pseudo-functor* but we won't spell it out here; we encourage the reader to work it out (or to look it up in [Vis05, Def. 3.10] or [SP, Tag 003N]). We take a slightly different approach which is technically more convenient, but it is nevertheless useful to think of a prestack as an assignment Sch $\to$ Groupoids.

0.6.1   Motivating the definition of a prestack

Instead of trying to define an assignment $S \mapsto \mathrm{Fam}_S$, we will build one massive category $\mathcal{X}$ encoding all of the groupoids Fam_S which will live over the category Sch of schemes. Loosely speaking, the objects of $\mathcal{X}$ will be a family a of objects over a scheme S, i.e. $a \in \mathrm{Fam}_S$, and a morphism $a \to b$ between a family a over S and a family b over T will be the data of a morphism $f \colon S \to T$ together with an isomorphism $a \xrightarrow{\sim} f^* b$ of a and the pullback family of b.

A *prestack* over Sch is a category $\mathcal{X}$ together with functor $p \colon \mathcal{X} \to$ Sch, which we visualize as

$$
\begin{array}{ccc}
\mathcal{X} & \quad & a \xrightarrow{\ \alpha\ } b \\
\downarrow{\scriptstyle p} & & \big\downarrow \qquad \big\downarrow \\
\mathrm{Sch} & & S \xrightarrow{\ f\ } T
\end{array}
$$

where the lower case letters a, b are objects in $\mathcal{X}$ and the upper case letters S, T are schemes. We say that *a is over S* and that $\alpha \colon a \to b$ *is over $f \colon S \to T$*. Moreover, we need to require that certain natural axioms hold for $p \colon \mathcal{X} \to$ Sch. Loosely speaking, we require the existence and uniqueness of pullbacks: given a map $S \to T$ and object $b \in \mathcal{X}$ over T, there should exist an arrow $a \xrightarrow{\alpha} b$ over f satisfying a suitable universal property; see Definition 2.3.1.

Given a scheme S, the *fiber category* $\mathcal{X}(S)$ is defined as the category of objects over S whose morphisms are over the identity. If $\mathcal{X}$ is built from the groupoids Fam_S as above, then $\mathcal{X}(S) = \mathrm{Fam}_S$.

Example 0.6.1 (Viewing a functor as a prestack)**.** A moduli functor $F \colon$ Sch $\to$ Sets can be encoded as a moduli prestack as follows: we define the category $\mathcal{X}_F$ of pairs (S, a) where S is a scheme and $a \in F(S)$. A map $(S', a) \to (S, a)$ is a map $f \colon S' \to S$ such that $a' = f^* a$, where f^* is convenient shorthand for $F(f) \colon F(S) \to F(S')$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$.

Example 0.6.2 (Moduli prestack of smooth curves)**.** The *moduli prestack of smooth curves* is the category $\mathcal{M}_g$ of families of smooth curves $\mathcal{C} \to S$ together with the functor $p \colon \mathcal{M}_g \to$ Sch defined by $(\mathcal{C} \to S) \mapsto S$. A morphism $(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$

is the data of maps $\alpha \colon \mathcal{C}' \to \mathcal{C}$ and $f \colon S' \to S$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\ \alpha\ } & \mathcal{C} \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow{\ f\ } & S \end{array}$$

is cartesian. Note that in the fiber category $\mathcal{M}_g(\mathbb{C})$, an object is a smooth curve C and the set of morphisms $C \to C$ is identified with the automorphism group $\mathrm{Aut}(C)$.

Example 0.6.3 (Moduli prestack of vector bundles)**.** The *moduli prestack of vector bundles* on a smooth curve C is the category $\underline{\mathrm{Bun}}_{r,d}(C)$ of pairs (E, S) where S is a scheme and E is a vector bundle on $C_S = C \times_{\mathbb{C}} S$ together with the functor $p \colon \underline{\mathrm{Bun}}_{r,d}(C) \to \mathrm{Sch}/\mathbb{C}, (E, S) \mapsto S$. A map $(E', S') \to (E, S)$ consists of a map of schemes $f \colon S' \to S$ together with an isomorphism $E' \xrightarrow{\sim} (\mathrm{id} \times f)^* E$

0.6.2 Motivating the definition of a stack

A stack is to a prestack as a sheaf is to a presheaf. The concept could not be more intuitive: we require that objects and morphisms glue uniquely.

Example 0.6.4 (Moduli stack of sheaves)**.** Define the category $\mathcal{X}$ over Sch of pairs (E, S) where E is a sheaf of abelian groups on a scheme S, and the functor $p \colon \mathcal{X} \to \mathrm{Sch}$ is given by $(E, S) \mapsto S$. A map $(E', S') \to (E, S)$ in $\mathcal{X}$ is a map of schemes $f \colon S' \to S$ together with a map $E \to f_* E'$ of $\mathcal{O}_{S'}$-modules whose adjoint is an isomorphism.

You already know that morphisms of sheaves glue: let E and F be sheaves on schemes S and T and let $f \colon S \to T$ be a map. If $\{S_i\}$ is a Zariski open cover of S, then giving a morphism $\alpha \colon (E, S) \to (F, T)$ is the same data as giving morphisms $\alpha_i \colon (E|_{S_i}, S_i) \to (F, T)$ such that $\alpha_i|_{S_{ij}} = \alpha_j|_{S_{ij}}$ [Har77, Exer. II.1.15]. You also know how sheaves glue—it is more complicated than gluing morphisms since sheaves have automorphisms and given two sheaves, we prefer to say that they are isomorphic rather than equal. If $\{S_i\}$ is a Zariski open cover of a scheme S, then giving a sheaf E on S is equivalent to giving a sheaf E_i on S_i and isomorphisms $\phi_{ij} \colon E_i|_{S_{ij}} \to E_j|_{S_{ij}}$ such that $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on the triple intersection S_{ijk} [Har77, Exer. II.1.22].

In a similar way, we could have considered the stack of $\mathcal{O}$-modules, quasi-coherent sheaves, or vector bundles, or we could have stacks of sheaves/$\mathcal{O}$-modules/quasi-coherent sheaves/vector bundles over a given scheme X where an object over a scheme S is sheaf on $X \times S$.

The definition of a stack (Definition 2.4.1) simply axiomitizes these two natural gluing concepts.

0.6.3 Motivating the definition of an algebraic stack

For a stack to be a geometric object, we need to specify that it is locally like a scheme in a suitable sense. Without such a condition would be like trying to study the geometry of an arbitrary ringed space $(X, \mathcal{O}_X)$ or a (possibly non-representable) functor $F \colon \mathrm{Sch} \to \mathrm{Sets}$ which is a sheaf in the big Zariski topology. If we wish to utilize our algebraic geometry toolkit (e.g. coherent sheaves, commutative algebra, cohomology, ...) to study stacks in a similar way that we study schemes, we must impose an algebraicity condition.

The conditions we impose are quite natural. In increasing generality, we define:

(1) A functor $X\colon \mathrm{Sch} \to \mathrm{Sets}$ is an *algebraic space* if objects of X glue uniquely in the étale topology and there is an étale cover $\{U_i \to X\}$ where each U_i is an affine scheme.

(2) A stack $\mathcal{X} \to \mathrm{Sch}$ is *Deligne–Mumford* if there is an étale cover $\{U_i \to \mathcal{X}\}$ where each U_i is an affine scheme.

(3) A stack $\mathcal{X} \to \mathrm{Sch}$ is *algebraic* if there is a smooth cover $\{U_i \to \mathcal{X}\}$ where each U_i is an affine scheme.

Of course, we need to make precise the notions of étale and smooth covers. For a first approximation, we say that $\{U_i \to X\}$ is an étale cover, we require that for every map $T \to X$ of functors where T is representable by a scheme, the fiber product of functors is representable by a scheme T_i, and moreover that $T_i \to T$ is étale and $\coprod T_i \to T$ is surjective. Note that in (1), if we replace 'étale' with 'Zariski', we would recover the notion of a scheme; see Exercise 0.3.32. It will take some time to develop the foundations to make this completely rigorous; precise definitions are postponed until §3.1.

Algebro-geometric space	Type of object	Obtained by gluing
Schemes	ringed space/ sheaf	affine schemes in the Zariski topology
Algebraic spaces	sheaf	affine schemes in the étale topology
Deligne–Mumford stacks	stack	affine schemes in the étale topology
Algebraic stacks	stack	affine schemes in the smooth topology

Table 0.3: Schemes, algebraic spaces, Deligne–Mumford stacks, and algebraic stacks are obtained by gluing affine schemes.

Why smooth covers? After all the fuss motivating étale morphisms above, you might be surprised to see that an algebraic stack is *smooth-locally* a scheme. For Deligne–Mumford stacks—which turn out to be precisely algebraic stacks with finite automorphism groups—étale covers are sufficient. But for algebraic stacks like $\underline{\mathrm{Bun}}_{r,d}(C)$ with infinite automorphism groups, we need smooth covers. For instance, we would like to be able to form the quotient $[\mathrm{Spec}\,\mathbb{C}/\mathbb{G}_m]$ (which we will call the *classifying stack* $B\mathbb{G}_m$) of the trivial action of $\mathbb{G}_m$ (or $\mathbb{C}^*$) on a point, and this will have no étale cover by a scheme.

0.6.4 Examples of moduli stacks

Constructing a smooth cover of a given moduli stack is a geometric problem inherent to the moduli problem. It can often be solved by rigidifying the moduli problem by parameterizing additional information. This concept is best absorbed in examples.

Example 0.6.5 (Moduli stack of elliptic curves). An elliptic curve (E, p) is embedded into $\mathbb{P}^2$ via $\mathcal{O}_E(3p)$ such that E is defined by a Weierstrass equation

$y^2z = x(x-z)(x-\lambda z)$ for some $\lambda \neq 0, 1$ [Har77, Prop. IV.4.6]. Setting $U = \mathbb{A}^1 \setminus \{0,1\}$ with coordinate λ, the family $\mathcal{E} \subset U \times \mathbb{P}^2$ of elliptic curves defined by the Weierstrass equation defines map $U \to \mathcal{M}_{1,1}$ which is an étale cover.

Example 0.6.6 (Moduli stack of smooth curves)**.** For a smooth curve C of genus $g \geq 2$, the line bundle $\Omega_C^{\otimes 3}$ is very ample and defines an embedding $C \hookrightarrow \mathbb{P}(\Gamma(C, \Omega_C^{\otimes 3})) \cong \mathbb{P}^{5g-6}$. There is a Hilbert scheme H (see Theorem 1.1.2) parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with the same Hilbert polynomial as $C \subset \mathbb{P}^{5g-6}$, and there is a locally closed subscheme $H' \subset H$ parameterizing *smooth* subschemes such that $\Omega_C^{\otimes 3} \cong \mathcal{O}_C(1)$. The universal subscheme over H' defines a map $H' \to \mathcal{M}_g$ which is a smooth cover (see Theorem 3.1.17 for details) and thus $\mathcal{M}_g$ is an algebraic stack. We will show that it is Deligne–Mumford in Corollary 3.6.10.

Example 0.6.7 (Moduli stack of vector bundles)**.** For every vector bundle E of rank r and degree d on a smooth curve C, the twist $E(m)$ is globally generated for sufficiently large m. Taking $N_m = \mathrm{h}^0(C, E(m))$, we can view E as a quotient $\mathcal{O}_C(-m)^{N_m} \twoheadrightarrow E$. There is a Quot scheme Q_m (see Theorem 1.1.3) parameterizing such quotients that have the same Hilbert polynomial as E and there is a locally closed subscheme $Q'_m \subset Q$ parameterizing vector bundle quotients $\pi \colon \mathcal{O}_C(-m)^{N_m} \twoheadrightarrow E$ such that the induced map $\Gamma(\pi \otimes \mathcal{O}_C(m)) \colon \mathbb{C}^{N_m} \to \Gamma(C, E(m))$ is an isomorphism. The universal quotient over Q'_m defines a map $Q'_m \to \underline{\mathrm{Bun}}_{r,d}(C)$ which is smooth and the collection $\{Q'_m \to \underline{\mathrm{Bun}}_{r,d}(C)\}$ for $m \gg 0$ defines a smooth cover. This shows that an $\underline{\mathrm{Bun}}_{r,d}(C)$ is an algebraic stack; see Theorem 3.1.21 for details. It is not a Deligne–Mumford stack.

0.6.5 Quotient stacks

One of the most important examples of a stack is a quotient stack $[X/G]$ arising from an action of an algebraic group G on a scheme X. The geometry of $[X/G]$ couldn't be simpler: it is the G-equivariant geometry of X (see Table 0.4).

Similar to how toric varieties provide concrete examples of schemes, quotient stacks provide both concrete examples useful to gain geometric intuition of general algebraic stacks and a fertile testing ground for conjectural results. On the other hand, it turns out that many algebraic stacks are quotient stacks or are at least locally quotient stacks, and most properties that hold for quotient stacks also hold for many algebraic stacks.

Quotient prestacks. Given an action of an algebraic group G on a scheme X, the *quotient prestack* $[X/G]^{\mathrm{pre}}$ is the prestack whose fiber category $[X/G]^{\mathrm{pre}}(S)$ over a scheme S is the quotient groupoid (or the moduli groupoid of orbits) $[X(S)/G(S)]$. This will not satisfy the gluing axioms of a stack; even when the action is free, the quotient functor $\mathrm{Sch} \to \mathrm{Sets}$ defined by $S \mapsto X(S)/G(S)$ is not a sheaf (see Exercise 0.3.27). How can we make it into a stack? Well, instead of thinking of an object of $[X/G]^{\mathrm{pre}}$ over a scheme S as a morphism $f \colon S \to X$, let's think of it as a trivial G-bundle together with a map to X:

$$G \times S \xrightarrow{\widetilde{f}} X, \qquad (g,s) \longmapsto g \cdot f(s)$$
$$\downarrow {\scriptstyle p_2}$$
$$S.$$

Given two maps $f_1, f_2 \colon S \to X$, an element of $\alpha \in G(S)$ with $f_2 = \alpha \cdot f_1$ is the same data as an isomorphism of trivial G-bundles $G \times S \to G \times S$, $(g,s) \mapsto (g\alpha(s), s)$;

this is because any such isomorphism must be G-equivariant and commute with the structure maps to S. From this perspective, it is even more clear that $[X/G]^{\mathrm{pre}}$ is not stack even when X is a point: given a Zariski cover $\{S_i\}$ of a scheme S, trivial G-bundles $G \times S_i \to S_i$ together with isomorphisms over $S_i \cap S_j$ satisfying a cocycle condition will glue to a principal G-bundle $P \to S$ but it will *not* necessarily be trivial. This suggests that we should define objects of a quotient stack to be principal G-bundles (Definition B.2.16).

Quotient stacks. We define the *quotient stack* $[X/G]$ as the category over $\mathrm{Sch}/\mathbb{C}$ whose objects over a $\mathbb{C}$-scheme S are diagrams

$$
\begin{array}{ccc}
P & \xrightarrow{\ f\ } & X \\
\downarrow & & \\
S & &
\end{array}
$$

where $P \to S$ is a principal G-bundle and $f\colon P \to X$ is a G-equivariant morphism. A morphism is the data of a commutative diagram

$$
\begin{array}{ccccc}
P' & \xrightarrow{\ \varphi\ } & P & \xrightarrow{\ f\ } & X \\
\downarrow & \square & \downarrow & & \\
S' & \xrightarrow{\ g\ } & S & &
\end{array}
$$

where the left square is cartesian. There is an object of $[X/G]$ over X given by the diagram

$$
\begin{array}{ccc}
G \times X & \xrightarrow{\ \sigma\ } & X \\
\downarrow{\scriptstyle p_2} & & \\
X, & &
\end{array}
$$

where σ denotes the action map. By the 2-Yoneda Lemma (2.3.20), this defines a map $X \to [X/G]$. Even if the action of G on X is not free, the map $X \to [X/G]$ is a principal G-bundle. Let's pause to appreciate that:

> The map $X \to [X/G]$ is a principal G-bundle even if the action of G on X is not free.

This is one of the great advantages of working with stacks. At the expense of enlarging our category from schemes to algebraic stacks, we are able to tautologically construct the quotient $[X/G]$ as a 'geometric space' with desirable properties.

Example 0.6.8 (Classifying stack). We define the *classifying stack* of an algebraic group G as the category $BG := [\operatorname{Spec} \mathbb{C}/G]$ of principal G-bundles $P \to S$. The projection $\operatorname{Spec}\mathbb{C} \to BG$ is not only a principal G-bundle; it is the *universal principal G-bundle*. Given any other principal G-bundle $P \to S$, there is a unique map $S \to BG$ and a cartesian diagram

$$
\begin{array}{ccc}
P & \longrightarrow & \operatorname{Spec}\mathbb{C} \\
\downarrow & \square & \downarrow \\
S & \longrightarrow & BG.
\end{array}
$$

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Example 0.6.9 (Quotients by finite groups). Quotients by free actions of finite groups exist as algebraic spaces! See Corollary 3.1.14.

Exercise 0.6.10. What is the universal family over the quotient stack $[X/G]$?

Moduli stacks can often be described as quotient stacks, and these descriptions can be leveraged to establish properties of the moduli stack.

Example 0.6.11 (Moduli stack of smooth curves as a quotient). Reexamining Example 0.6.6, we see that the embedding of a smooth curve C via $|\Omega_C^{\otimes 3}| \colon C \hookrightarrow \mathbb{P}^{5g-6}$ depends on a choice of basis $\Gamma(C, \Omega_C^{\otimes 3}) \cong \mathbb{C}^{5g-5}$ and therefore is only unique up to a projective automorphism, i.e. an element of $\mathrm{PGL}_{5g-5} = \mathrm{Aut}(\mathbb{P}^{5g-6})$. The algebraic group PGL_{5g-5} acts on the scheme H' parameterizing smooth tricanonically embedded curves such that $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-6}]$.

Example 0.6.12 (Moduli stack of smooth curves as a quotient). For the moduli stack of vector bundles (Example 0.6.7), the presentation of a vector bundle E as a quotient $\mathcal{O}_C(-m)^{N_m} \twoheadrightarrow E$ depends on a choice of basis $\Gamma(C, E(m)) \cong \mathbb{C}^{N_m}$. The algebraic group PGL_{N_m-1} acts on the scheme Q'_m and there is an identification

$$\underline{\mathrm{Bun}}_{r,d}(C) \cong \bigcup_{m \gg 0} [Q'_m / \mathrm{PGL}_{N_m-1}]$$

.

Geometry of a quotient stack. While the definition of the quotient stack $[X/G]$ may appear abstract, its geometry is very familiar. The table below provides a dictionary between the geometry of a quotient stack $[X/G]$ and the G-equivariant geometry of X. The stack-theoretic concepts on the left-hand side will be introduced later.

Geometry of $[X/G]$	G-equivariant geometry of X
$\mathbb{C}$-point $\overline{x} \in [X/G]$	orbit Gx of $\mathbb{C}$-point $x \in X$ (with $\overline{x}$ the image of x under $X \to [X/G]$)
automorphism group $\mathrm{Aut}(\overline{x})$	stabilizer G_x
function $f \in \Gamma([X/G], \mathcal{O}_{[X/G]})$	G-equivariant function $f \in \Gamma(X, \mathcal{O}_X)^G$
map $[X/G] \to Y$ to a scheme Y	G-equivariant map $X \to Y$
line bundle	G-equivariant line bundle (or G-linearization)
quasi-coherent sheaf	G-equivariant quasi-coherent sheaf
tangent space $T_{[X/G], \overline{x}}$	normal space $T_{X,x}/T_{Gx,x}$ to the orbit
coarse moduli space $[X/G] \to Y$	geometric quotient $X \to Y$
good moduli space $[X/G] \to Y$	good GIT quotient $X \to Y$

Table 0.4: Dictionary between the geometry of $[X/G]$ and the G-equivariant geometry of X

0.7 Constructing projective moduli spaces

Our motivation for algebraic stacks was to ensure that a given moduli problem $\mathcal{M}$ is representable with a universal family. While many geometric questions can be studied (and arguably should be studied) on the moduli stack $\mathcal{M}$ itself, it is often very convenient to make a trade-off: by sacrificing the existence of a universal family, we can sometimes construct a more familiar geometric space, ideally a projective variety. This allows us to utilize the much larger toolkit of projective geometry (e.g. birational geometry, intersection theory, Hodge theory, ...) to study the moduli problem.

We highlight two approaches to construct projective moduli spaces:

(1) Geometric Invariant Theory (GIT), and

(2) Intrinsic construction of coarse/good moduli spaces.

There is a beautiful interplay between the intrinsic and extrinsic approaches. Ideas from GIT have inspired techniques in each of the six steps of the intrinsic approach, and conversely the intrinsic approach sheds light back on GIT. GIT is also deeply intertwined with 19th century invariant theory, and determining the GIT semistable locus is an interesting and important problem on its own. It is valuable to keep both approaches in mind.

0.7.1 GIT approach

Outline of the GIT strategy

(A) Express the moduli stack $\mathcal{M}$ as a substack

$$\mathcal{M} \subset [X/G],$$

where G is reductive and $X \hookrightarrow \mathbb{P}(V)$ is G-equivariantly embedded into the projectivization of a G-representation V.

(B) Show that a point $x \in X$ is GIT semistable if and only if $x \in \mathcal{M}$, or in other words that $\mathcal{M} = [X^{\mathrm{ss}}/G]$.

For Step A, there are often natural ways to rigidify the moduli problem by parameterizing additional data. For smooth curves, we can parameterize a basis of $\Gamma(C, \Omega_C^{\otimes 3})$ along with a curve C to obtain a tricanonical embedding $C \hookrightarrow \mathbb{P}^{5g-6}$ (Example 0.6.11), or we can parameterize a basis of $\Gamma(C, \Omega_C^{\otimes k})$ for the kth canonical embedding $C \hookrightarrow \mathbb{P}^{(2k-1)(g-1)-1}$. For vector bundles on a smooth curve, we can parameterize a basis of $\Gamma(C, E(m))$ along with a vector bundle E, after making a *choice* of a sufficiently large integer m. The rigidified moduli problem should have a compactification which is represented by a projective variety X—which is Hilb and Quot in our two examples—and the choice of additional data should be governed by an action of a group G. For the GIT approach to succeed, we need that G is reductive and that $\mathcal{M}$ is a substack of $[X/G]$. Finally, we need to choose a G-equivariant embedding $X \hookrightarrow \mathbb{P}(V)$ where V is a finite dimensional G-representation, or equivalently choose a G-linearization of an ample line bundle on X.

Step B is the hardest: we must show that $\mathcal{M}$ is precisely the open substack of $[X/G]$ of GIT semistable points. Using the Hilbert–Mumford Criterion we can translate the problem to the following: a point $x \in X$ represents an object of the moduli problem $\mathcal{M}$ if and only if *Hilbert–Mumford index* $\mu(x, \lambda) \geq 0$ for every

one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$. This often reduces the goal to a tractable (but often still daunting) combinatorial problem.

The GIT quotient $M := X^{\mathrm{ss}}/\!/G$ is necessarily projective. One beautiful feature of GIT is that even if the moduli stack $\mathcal{M}$ is not compact, the GIT strategy provides a compactification! If $\mathcal{M}$ has only finite automorphisms or equivalently there are no strictly semistable points, then $X^{\mathrm{ss}} \to M$ is a geometric quotient and $\mathcal{M} = [X^{\mathrm{ss}}/G] \to M$ is a coarse moduli space. In the presence of infinite automorphisms, $X^{\mathrm{ss}} \to M$ is a good quotient and $\mathcal{M} \to M$ is a good moduli space.

The GIT approach is covered in detail in §6.7. We sketch the GIT construction of $\overline{M}_g$ in §5.8.4 and present a complete GIT construction of $\mathrm{Bun}_{r,d}(C)$ in Chapter 7.

0.7.2 Intrinsic approach

> *However I do not claim at all that [GIT] should be avoided, but only that sometimes it may be good to have an alternative.*

FALTINGS [FAL93]

Six steps toward projective moduli

① **Algebraicity:** Express the moduli stack $\mathcal{M}$ as a substack

$$\mathcal{M} \subset \mathcal{X}$$

of a larger moduli stack $\mathcal{X}$. Define an object $x \in \mathcal{X}$ to be *semistable* if it is in $\mathcal{M}$; this allows us to think of $\mathcal{M}$ as the semistable locus $\mathcal{X}^{\mathrm{ss}}$. Show that $\mathcal{X}$ is an algebraic stack locally of finite type over $\mathbb{C}$.

② **Openness of semistability:** Show that semistability is an open condition, i.e. $\mathcal{M} = \mathcal{X}^{\mathrm{ss}} \subset \mathcal{X}$ is an open substack.

③ **Boundedness of semistability:** Show that semistability is bounded, i.e. $\mathcal{M} = \mathcal{X}^{\mathrm{ss}}$ is of finite type over $\mathbb{C}$.

④ **Semistable reduction:** Show that $\mathcal{M}$ satisfies the existence part of the valuative criterion for properness.

⑤ **Existence of a moduli space:** Show that there is a fine/coarse/good moduli space $\mathcal{M} \to M$ where M is a proper algebraic space.[5]

⑥ **Projectivity:** Show that a tautological line bundle on $\mathcal{M}$ descends to an ample line bundle on M, i.e. M is projective.

GIT magically solves all these steps at once! In Step A of the GIT approach, expressing the moduli stack $\mathcal{M}$ as a substack $[X/G]$ already implies 'boundedness.' Since GIT semistability is always an open condition, the identification in Step B of $\mathcal{M}$ with the semistable locus $[X^{\mathrm{ss}}/G]$ gives 'openness of semistability' and thus 'algebraicity' of $\mathcal{M}$. Strikingly, GIT also implies each of the other steps: 'semistable reduction,' 'existence of a moduli space', and 'projectivity.'

Step 1 (Algebraicity). Many moduli stacks have natural enlargements. The stack $\mathcal{M}_g$ of smooth curves and the stack $\overline{\mathcal{M}}_g$ of stable curves are both contained in

---

[5]The calligraphic font $\mathcal{M}$ denotes the stack while the Roman font M denotes the space. This convention will be followed throughout the text.

the stack of all curves. The stack of semistable vector bundles on a smooth curve is contained in the stack of all vector bundles or even the stack of all coherent sheaves. It is usually easier to first show that the enlargement $\mathcal{X}$ is an algebraic stack, and then use Steps 2 and 3 to conclude that $\mathcal{M}$ itself is algebraic.

To get started, we need to define the stacks $\mathcal{M}$ and its enlargement $\mathcal{X}$—this entails specifying families of objects along with pullbacks and identifications. To check that $\mathcal{X}$ is algebraic requires finding a smooth cover $U \to \mathcal{X}$ by a scheme. In many cases, we can even show that $\mathcal{X}$ is identified with a quotient stack $[U/G]$ in which case $U \to [U/G]$ provides a presentation. Alternatively, it is often possible to use Artin's Criteria (Theorem C.7.4) to establish algebraicity; this essentially amounts to verifying local properties of the moduli problem and in particular requires an understanding of the deformation and obstruction theory.

Step 2 (Openness of semistability). This translates to the following condition: for every family $\mathcal{E}$ of objects of $\mathcal{X}$ over a scheme S, the subset

$$\{s \in S \mid \mathcal{E}_s \text{ is semistable}\}, \tag{0.7.1}$$

where $\mathcal{E}_s$ is the pullback of $\mathcal{E}$ along $\operatorname{Spec} \kappa(s) \to S$, is an open subset of S. This is precisely what it means for the inclusion $\mathcal{M} = \mathcal{X}^{\mathrm{ss}} \hookrightarrow \mathcal{X}$ to be representable by open immersions: for every map $S \to \mathcal{X}$ (corresponding to the family $\mathcal{E}$), the fiber product $\mathcal{M} \times_{\mathcal{X}} S$ (which is identified set-theoretically with (0.7.1)) is an open subscheme of S. This step ensures that $\mathcal{M}$ is also an algebraic stack locally of finite type over $\mathbb{C}$.

Step 3 (Boundedness of semistability). By boundedness, we mean that the moduli stack $\mathcal{M}$ is of finite type over $\mathbb{C}$. Since algebraicity implies that $\mathcal{M}$ is *locally* of finite type over $\mathbb{C}$, boundedness translates into quasi-compactness of $\mathcal{M}$. More concretely, boundedness is equivalent to the existence of a scheme Z of *finite type* over $\mathbb{C}$ and a family of objects $\mathcal{E}$ over Z such that every object E of $\mathcal{M}$ is isomorphic to $\mathcal{E}_z$ for some (not necessarily unique) $z \in Z$.

For example, $\mathcal{M}_g$ is bounded but the stack of all proper curves of genus g and the stack $\coprod_g \mathcal{M}_g$ of all smooth curves (of any genus) are not bounded. For vector bundles, the stack $\underline{\mathrm{Bun}}^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles of fixed rank and degree is bounded. The stack of all vector bundles $\underline{\mathrm{Bun}}_{r,d}(C)$ of fixed rank and degree is not bounded, nor is the stack of semistable vector bundles of arbitrary rank and degree.

Step 4 (Semistable reduction). The existence part of the valuative criterion for properness is the assertion that for every DVR R (which you can think of as a local model of a smooth curve) with fraction field K (or punctured curve) then every object $\mathcal{E}^{\times}$ over K extends to a family of objects $\mathcal{E}$ over R after possibly replacing R with an extension of DVRs. In other words, every diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\mathcal{E}^{\times}} & \mathcal{M} \\ \downarrow & \nearrow^{\mathcal{E}} & \\ \operatorname{Spec} R, & & \end{array} \tag{0.7.2}$$

has an extension after replacing R with an extension. If the extension $\mathcal{E}$ over R is also unique, then we say that $\mathcal{M}$ satisfies the valuative criterion for properness, and this implies properness (Theorem 3.8.2) and in particular separatedness. Arguably the usefulness of valuative criteria in algebraic geometry is best witnessed in moduli theory.

The moduli stack of smooth curves is not compact and does not satisfy the existence part of the valuative criterion.
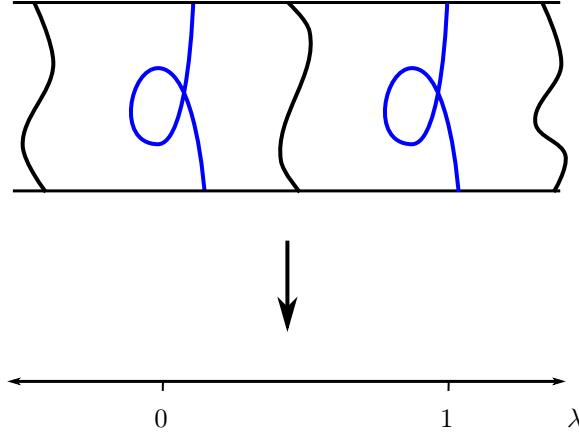
Figure 0.18: The family of elliptic curves $y^2 z = x(x-z)(x-\lambda z)$ degenerates to the nodal cubic over $\lambda = 0, 1$.

Projective varieties are of course compact and satisfy the valuative criterion. If there's any hope to construct a projective moduli space, then the moduli stack better satisfy the existence part of the valuative criterion. Properness of $\overline{\mathcal{M}}_g$ was first proven by Deligne and Mumford in their influential paper [DM69]. We prove semistable reduction in characteristic 0 in §5.5.

For the moduli of vector bundles, semistable reduction was first proved by Mumford and Seshadri as a consequence of the GIT construction [Ses67]. An intrinsic geometric argument was later proved by Langton [Lan75]. Note that unlike stable curves, the stack $\underline{\mathrm{Bun}}^{\mathrm{ss}}_{r,d}(C)$ is not separated as there may exist several non-isomorphic extensions of a vector bundle on C_K to C_R. Nevertheless, the moduli stack $\underline{\mathrm{Bun}}^{\mathrm{ss}}_{r,d}(C)$ satisfies a weaker notion of separatedness called *S-completeness*.

Step 5 (Existence of a moduli space). We would like to construct an algebraic space that is the best possible approximation of the moduli stack. This step depends on the automorphisms of the moduli problem:

- *No automorphisms*: in this case, the moduli stack $\mathcal{M}$ is already an algebraic space M. In other words, M is a *fine moduli space*: for a scheme S, there is a natural bijection between objects over a scheme S and maps $S \to M$.

- *Finite automorphisms*: we must show that $\mathcal{M}$ is separated or in other words that $\mathcal{M}$ satisfies the uniqueness part (in addition to the existence part) of the valuative criterion. The Keel–Mori theorem (Theorem 4.3.12) then establishes the existence of a *coarse moduli space* $\mathcal{M} \to M$ where M is a proper algebraic space. The map $\mathcal{M} \to M$ induces a bijection of $\mathbb{C}$-points and satisfies the universal property that any other map $\mathcal{M} \to Y$ to an algebraic space Y factors uniquely through M.

- *Reductive automorphisms*: we must show that $\mathcal{M}$ is Θ-*complete* and *S-complete*—these are valuative criteria about extending $\mathbb{G}_m$-equivariant families of objects over a punctured surface which are introduced in Section 6.8.2. Given these properties, Theorem 6.8.1 yields a *good moduli space* $\mathcal{M} \to M$ where M is a proper algebraic space. The map $\mathcal{M} \to M$ is no longer a bijection

of $\mathbb{C}$-points as it identifies points whose closures intersect in an analogous way to the orbit closure equivalence relation in GIT. But $\mathcal{M} \to M$ does induce a bijection between *closed* $\mathbb{C}$-points of $\mathcal{M}$ (sometimes called *polystable* objects) and the $\mathbb{C}$-points of M, and it also satisfies the universal property for maps to algebraic spaces.

Step 6 (Projectivity).   This is usually the hardest step. It requires a solid understanding of the geometry of the moduli problem and sometimes relies on sophisticated techniques in birational geometry. Kollár introduced a strategy in [Kol90] to verify projectivity for moduli stacks of varieties and applied it to verify the projectivity of $\overline{M}_g$. We cover Kollár's method in §5.8. Faltings constructed projective moduli spaces of vector bundles in [Fal93] without using the theory of GIT, and we borrow several of his ideas in our construction in Chapter 7.

Chapter 1

Hilbert and Quot schemes

We prove that the Grassmannian, Hilbert and Quot functors are representable by projective schemes. These results serve as the backbone of many results in moduli theory and more widely algebraic geometry. In particular, they are essential for establishing properties about the moduli stacks $\overline{\mathcal{M}}_g$ of stable curves and $\mathcal{V}_{r,d}^{\mathrm{ss}}$ of vector bundles over a curve. While the reader could safely treat these results as black boxes (and we encourage some readers to do this), it is also worthwhile to dive into the details. We follow Mumford's simplification [Mum66] of Grothendieck's original construction of Hilbert of Quot schemes [FGA$_{\mathrm{IV}}$]. Specifically, we exploit the theory of Castelnuovo–Mumford regularity (Section 1.3) and flattening stratifications (Theorem A.2.16), which are interesting results on their own with wide-ranging applications outside moduli theory.

1.1 The Grassmannian, Hilbert, and Quot functors

1.1.1 Statements of the main theorems

The representability theorems below are formulated for a *strongly projective* morphism $X \to S$ of noetherian schemes, i.e. there exists a closed immersion $X \hookrightarrow \mathbb{P}(E)$ over S where E is a vector bundle on S. This is a stronger condition than the *projectivity* of $X \to S$ which only requires that E is a coherent sheaf [EGA, §II.5], [SP, Tag 01W8]. On the other hand, the definition of projectivity in [Har77, II.4] requires that X embeds into projective space $\mathbb{P}_S^n$ over S.

Theorem 1.1.1. *Let S be a noetherian scheme and V be a vector bundle on S of rank n. For an integer $0 < k < n$, the functor*

$$\mathrm{Gr}_S(k, V) \colon \mathrm{Sch}/S \to \mathrm{Sets}$$

$$(T \xrightarrow{f} S) \mapsto \left\{ \text{ vector bundle quotients } V_T = f^*V \to Q \text{ of rank } k \ \right\}$$

is represented by a scheme strongly projective over S.

If $S = \mathrm{Spec}\,\mathbb{Z}$ and $V = \mathcal{O}_S^n$, then $\mathrm{Gr}_S(k, V)$ is equal to the functor $\mathrm{Gr}(k, n)$ defined in Example 0.3.12. In addition, when $k = 1$, the Grassmannian $\mathrm{Gr}_S(1, V)$ is identified with the projectivization $\mathbb{P}(V)$ of V as discussed in Exercise 0.3.13. For arbitrary S, we sometimes denote $\mathrm{Gr}_S(k, n) := \mathrm{Gr}_S(k, \mathcal{O}_S^n)$ and we sometimes drop the subscript S when we are working over a fixed base such as $S = \mathrm{Spec}\,\mathbb{k}$ or $S = \mathrm{Spec}\,\mathbb{Z}$.

In the formulation of the following two theorems, we will use the convention that if $X \to S$ and $T \to S$ are morphisms of schemes, then $X_T := X \times_S T$. Similarly, if F is a sheaf on X, then F_T denotes the pullback of F under $X_T \to X$. If $s \in S$ is a point, then $X_s := X \times_S \operatorname{Spec} \kappa(s)$ and $F_s := F|_{X_s} = F_{\operatorname{Spec} \kappa(s)}$. If $X \to S$ is a projective morphism, $\mathcal{O}_X(1)$ is relatively ample and $s \in S$ is a point, the *Hilbert polynomial of F_s* is

$$P_{F_s}(z) = \chi(X_s, F_s(z)),$$

where $F_s(z) = F_s \otimes \mathcal{O}_{X_s}(z)$. It is a fact that this defines a polynomial $P_{F_s} \in \mathbb{Q}[z]$ (c.f. [Har77, Exer III.5.2]]); for $z \gg 0$, we have $P_{F_s}(z) = \mathrm{h}^0(X_s, F_s(z))$.

Theorem 1.1.2. *Let $X \to S$ be a strongly projective morphism of noetherian schemes and $\mathcal{O}_X(1)$ be a relatively ample line bundle on X. For every polynomial $P \in \mathbb{Q}[z]$, the functor*

$$\operatorname{Hilb}^P_{X/S} \colon \operatorname{Sch}/S \to \operatorname{Sets}$$

$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{subschemes } Z \subset X_T \text{ flat and finitely presented} \\ \text{over } T \text{ such that } Z_t \subset X_t \text{ has Hilbert} \\ \text{polynomial } P \text{ for all } t \in T \end{array} \right\}$$

is represented by a scheme strongly projective over S.

Theorem 1.1.3. *Let $\pi \colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on X, and F be a coherent sheaf on X which is the quotient of $\pi^*(W)(q)$ for a vector bundle W on S and an integer q. For every polynomial $P \in \mathbb{Q}[z]$, the functor*

$$\operatorname{Quot}^P_{X/S}(F) \colon \operatorname{Sch}/S \to \operatorname{Sets}$$

$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{quasi-coherent and finitely presented} \\ \text{quotients } F_T \to Q \text{ on } X_T \text{ such that } Q \text{ is} \\ \text{flat over } T \text{ and } Q|_{X_t} \text{ on } X_t \text{ has} \\ \text{Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

is represented by a scheme strongly projective over S.

The Grassmannian and the Hilbert scheme are special cases of the Quot scheme: $\operatorname{Gr}_S(k, V) \cong \operatorname{Quot}^P_{S/S}(V)$ where $P(z) = k$ is the constant polynomial and $\operatorname{Hilb}^P_{X/S} = \operatorname{Quot}^P_{X/S}(\mathcal{O}_X)$.

Remark 1.1.4.

(1) In the definition of the Grassmannian and Quot functor above, two quotients $V_T \xrightarrow{q} Q$ and $V_T \xrightarrow{q'} Q'$ are identified if $\ker(q) = \ker(q')$ as subsheaves of V_T, or equivalently there exists an isomorphism $Q \xrightarrow{\alpha} Q'$ such that the composition $V_T \xrightarrow{q} Q \xrightarrow{\alpha} Q'$ is equal to $V_T \xrightarrow{q'} Q'$. In the Hilbert functor, two subschemes of X_T are identified if they are equal as subschemes (or equivalently their ideal sheaves are equal as subsheaves of $\mathcal{O}_{X_T}$).

(2) The definitions $\operatorname{Hilb}^P_{X/S}$ and $\operatorname{Quot}^P_{X/S}(F)$ depend on the relatively ample line bundle $\mathcal{O}_X(1)$ but we have suppressed this from the notation.

(3) When T is noetherian, the conditions that Z be finitely presented and Q be of finite presentation in the definitions of $\operatorname{Hilb}^P_{X/S}$ and $\operatorname{Quot}^P_{X/S}(F)$ are superfluous.

(4) If we do not fix P, then $\mathrm{Hilb}(X/S)$ and $\mathrm{Quot}(F/X/S)$ are representable by schemes *locally* of finite type, and there are decompositions

$$\mathrm{Hilb}(X/S) = \coprod_P \mathrm{Hilb}^P_{X/S} \quad \text{and} \quad \mathrm{Quot}(F/X/S) = \coprod_P \mathrm{Quot}^P_{X/S}(F);$$

these functorial decompositions follows from the flatness of the quotient Q and the local constancy of the Hilbert polynomial (Proposition A.2.4).

(5) Suppose that S satisfies the *resolution property*, i.e. every coherent sheaf is the quotient of a vector bundle. This is satisfied if S has an ample line bundle or if S is regular. Then a projective morphism $X \to S$ is necessarily strongly projective. Moreover, if F is a coherent sheaf on X, then $\pi^*\pi_*(F(q)) \to F(q)$ is surjective for $q \gg 0$ and choosing a surjection $W \twoheadrightarrow \pi_*(F(q))$ from a vector bundle W on S, we have a surjection $\pi^*(W(-q)) \mapsto F$. Theorem 1.1.3 therefore implies that $\mathrm{Quot}^P_{X/S}(F)$ is strongly projective over S if $X \to S$ is projective and F is coherent.

Caution 1.1.5. We will abuse notation by using $\mathrm{Hilb}^P_{X/S}$, $\mathrm{Quot}^P_{X/S}(F)$ and $\mathrm{Gr}_S(k, V)$ to denote both the functor and the scheme that represents it.

1.1.2 Proof strategy

In §1.2, we show that $\mathrm{Gr}_S(k, V)$ is representable by a projective scheme by using the functorial Plücker embedding $\mathrm{Gr}_S(k, V) \to \mathbb{P}(\bigwedge^k V)$, which over an S-scheme T sends a quotient $V_T \to Q$ to the line bundle quotient $\bigwedge^k V_T \to \bigwedge^k Q$.

In §1.3, we introduce Castelnuovo–Mumford regularity and exploit Mumford's result on Boundedness of Regularity (Theorem 1.3.9) to show that under the hypotheses of Theorem 1.1.3, then for $d \gg 0$, the morphism of functors

$$\begin{aligned} \mathrm{Quot}^P_{X/S}(F) &\to \mathrm{Gr}_S(P(d), \pi_*F(d)) \\ [F_T \twoheadrightarrow Q] &\mapsto [\pi_{T,*}(F_T(d)) \to \pi_{T,*}(Q(d))], \end{aligned} \tag{1.1.6}$$

defined over an S-scheme T, is well-defined. Note that for a field-valued point $s\colon \mathrm{Spec}\,\Bbbk \to S$ a quotient $[F_s \mapsto Q]$ is mapped to $[\mathrm{H}^0(X_s, F_s(d)) \to \mathrm{H}^0(X_s, Q(d))]$.

We show that the above functor is representable by locally closed immersions (Proposition 1.4.1). This is established by reducing to the special case where $X = \mathbb{P}(V)$ and $F = \pi^*W$ where V and W are vector bundles on S; this is where Boundedness of Regularity (Theorem 1.3.9) is applied.

Since $\mathrm{Gr}_S(P(d), \pi_*F(d))$ is representable by a projective scheme over S (Theorem 1.1.1), this already establishes the representability and quasi-projectivity of $\mathrm{Quot}^P_{X/S}(F)$. Finally, we establish that $\mathrm{Quot}^P_{X/S}(F)$ is proper over S (Proposition 1.4.2) by checking the valuative criterion which implies that $\mathrm{Quot}^P_{X/S}(F)$ is projective over S.

Historical comments

Grothendieck established both the representability and projectivity of $\mathrm{Quot}^P_{\mathbb{P}^n_A/A}(F)$ where F is coherent sheaf on $\mathbb{P}^n_A$ and A is a noetherian ring [FGA$_{\mathrm{IV}}$, Thm. 3.2]. Our exposition largely follows Grothendieck's original strategy while incorporating Mumford's simplification to establish boundedness, i.e. finite typeness of $\mathrm{Quot}^P_{\mathbb{P}^n_A/A}(F)$. Boundedness is one of the hardest parts of the proof, and almost every boundedness argument for a moduli space in algebraic geometry ultimately relies on the

boundedness of Hilb or Quot. Grothendieck's approach for boundedness of Quot was a reduction argument to Hilb (i.e. the case where $F = \mathcal{O}_X$) and relied on Chow's boundedness result for the parameter space of reduced, pure-dimensional subschemes of fixed degree. In [Mum66], Mumford introduced the concept of regularity of a coherent sheaf—now called Castelnuovo–Mumford regularity—and proved that for sufficiently large integers m every subsheaf $F \subset \mathcal{O}_{\mathbb{P}^n}^r$ with fixed Hilbert polynomial is m-regular (Theorem 1.3.9). Mumford used this result to construct the Hilbert scheme of curves on a surface but his argument applies equally to construct $\mathrm{Quot}_{\mathbb{P}^n_A/A}^P(F)$.

Our formulation of Theorem 1.1.3 using the strong projectivity of $X \to S$ follows Altman and Kleiman [AK80, Thm. 2.6]. This chapter follows closely the excellent expositions of [Mum66, §14-15], [FGI$^+$05, §6], [Kol96, §1], [Laz04a, §1.8], and [AK80, §2].

1.2 Representability and projectivity of the Grassmannian

The Grassmannian provides a warmup to the functorial approach of constructing projective moduli spaces in these notes, and it is also used in the proof of the representability of Hilb and Quot. Given its importance, we present a slow-paced expository account of the representability and projectivity of the Grassmannian. We focus first on the Grassmannian $\mathrm{Gr}(k, n) = \mathrm{Gr}_{\mathbb{Z}}(k, \mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}^n)$ over $\mathbb{Z}$ parameterizing k-dimension quotients of a trivial vector bundle of rank n. The proof of the projectivity and representability of the relative Grassmannian $\mathrm{Gr}_S(k, V)$ is shown in §1.2.3.

1.2.1 Representability by a scheme

In this subsection, we show that $\mathrm{Gr}(k, n)$ is representable by a scheme (Proposition 1.2.4). Our strategy will be to find a Zariski open cover of $\mathrm{Gr}(k, n)$ by representable subfunctors; see Definition 0.3.31. Given a subset $I \subset \{1, \ldots, n\}$ of size k, let $\mathrm{Gr}_I \subset \mathrm{Gr}(k, n)$ be the subfunctor where for a scheme S, $\mathrm{Gr}(k, n)_I(S)$ is the subset of $\mathrm{Gr}(k, n)(S)$ consisting of surjections $\mathcal{O}_S^n \xrightarrow{q} Q$ such that the composition

$$\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \xrightarrow{q} Q$$

is an isomorphism, where e_I is the canonical inclusion.

Lemma 1.2.1. *For each $I \subset \{1, \ldots, n\}$ of size k, the functor Gr_I is representable by affine space $\mathbb{A}_{\mathbb{Z}}^{k \times (n-k)}$*

Proof. We may assume that $I = \{1, \ldots, k\}$. We define a map of functors $\phi \colon \mathbb{A}^{k \times (n-k)} \to \mathrm{Gr}_I$ where over a scheme S, a $k \times (n - k)$ matrix

$$f = \left(f_{i,j} \right)_{1 \leq i \leq k \,,\, 1 \leq j \leq n-k}$$

of global functions on S is mapped to the quotient

$$\begin{pmatrix} 1 & & & & f_{1,1} & \cdots & f_{1,n-k} \\ & 1 & & & f_{2,1} & \cdots & f_{2,n-k} \\ & & \ddots & & \vdots & & \\ & & & 1 & f_{k,1} & \cdots & f_{k,n-k} \end{pmatrix} : \mathcal{O}_S^n \to \mathcal{O}_S^k. \qquad (1.2.2)$$

The injectivity of $\phi(S)\colon \mathbb{A}^{k\times(n-k)}(S) \to \mathrm{Gr}_I(S)$ follows from the fact that any two quotients written in the form of (1.2.2) which are equivalent in Gr_I are necessarily defined by the same equations. To see surjectivity, let $[\mathcal{O}_S^n \xrightarrow{q} Q] \in \mathrm{Gr}_I(S)$ where by definition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \xrightarrow{q} Q$ is an isomorphism. The tautological commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_S^n & \xrightarrow{\;q\;} & Q \\
& \searrow{\scriptstyle (q\circ e_I)^{-1}\circ q} & \downarrow{\scriptstyle (q\circ e_I)^{-1}} \\
& & \mathcal{O}_S^I
\end{array}
$$

shows that $[\mathcal{O}_S^n \xrightarrow{q} Q] = [\mathcal{O}_S^n \xrightarrow{(q\circ e_I)^{-1}\circ q} \mathcal{O}_S^I] \in \mathrm{Gr}(k,n)(S)$. Since the composition $\mathcal{O}_S^I \xrightarrow{e_I} \mathcal{O}_S^n \xrightarrow{(q\circ e_I)^{-1}} \mathcal{O}_S^I$ is the identity, the $k\times n$ matrix corresponding to $(q\circ e_I)^{-1}\circ q$ is necessarily of the same form as (1.2.2) for functions $f_{i,j} \in \Gamma(S,\mathcal{O}_S)$. Therefore $\phi(S)(\{f_{i,j}\}) = [\mathcal{O}_S^n \xrightarrow{q} Q] \in \mathrm{Gr}(k,n)(S)$. $\qquad\square$

Lemma 1.2.3. $\{\mathrm{Gr}_I\}$ *is a Zariski open cover of* $\mathrm{Gr}(k,n)$ *where* I *ranges over all subsets of size* k.

Proof. For a fixed subset I, we first show that $\mathrm{Gr}_I \subset \mathrm{Gr}(k,n)$ is an open subfunctor. To this end, we consider a scheme S and a morphism $S \to \mathrm{Gr}(k,n)$ corresponding to a quotient $q\colon \mathcal{O}_S^n \to Q$. Let C denote the cokernel of the composition $q\circ e_I\colon \mathcal{O}_S^I \to Q$. Notice that if $C=0$, then $q\circ e_I$ is an isomorphism. The fiber product

$$
\begin{array}{ccc}
F_I & \longrightarrow & S \\
\downarrow & \square & \downarrow{\scriptstyle [\mathcal{O}_S^n \xrightarrow{q} Q]} \\
\mathrm{Gr}_I & \longrightarrow & \mathrm{Gr}(k,n)
\end{array}
$$

of functors is representable by the open subscheme $U = S \setminus \mathrm{Supp}(C)$ (the reader is encouraged to verify this claim). Note that if S is not noetherian, then $\mathrm{Supp}(C) \subset S$ is still closed as C is finitely presented as a quasi-coherent sheaf.

To check the surjectivity of $\coprod_I F_I \to S$, let $s \in S$ be a point. Since $\kappa(s)^n \xrightarrow{q\otimes\kappa(s)} Q \otimes \kappa(s)$ is a surjection of vector spaces, there is a nonzero $k\times k$ minor, given by a subset I, of the $k\times n$ matrix $q\otimes\kappa(s)$. This implies that $[\kappa(s)^n \xrightarrow{q\otimes\kappa(s)} Q \otimes \kappa(s)] \in F_I(\kappa(s))$. $\qquad\square$

Lemmas 1.2.1 and 1.2.3 together imply:

Proposition 1.2.4. *The functor* $\mathrm{Gr}(k,n)$ *is representable by a scheme.* $\qquad\square$

Exercise 1.2.5. Show that $\mathrm{Gr}(k,n)$ is an integral scheme of finite type over $\mathbb{Z}$.

Exercise 1.2.6. Use the valuative criterion of properness to show that $\mathrm{Gr}(k,n) \to \mathrm{Spec}\,\mathbb{Z}$ is proper.

1.2.2   Projectivity of the Grassmannian

We show that the Grassmannian scheme $\mathrm{Gr}(k,n)$ is projective (Proposition 1.2.7) by explicitly providing a projective embedding. The *Plücker embedding* is the map

of functors

$$P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge{}^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$$

$$[\mathcal{O}^n_S \xrightarrow{q} Q] \mapsto [\bigwedge{}^k \mathcal{O}^n_S \to \bigwedge{}^k Q]$$

defined above over a scheme S. As both sides are representable by schemes, the morphism P corresponds to a morphism of schemes via Yoneda's lemma.

Proposition 1.2.7. *The morphism $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$ of schemes is a closed immersion. In particular, $\mathrm{Gr}(k,n)$ is a strongly projective scheme over $\mathbb{Z}$.*

Proof. A subset $I \subset \{1,\dots,n\}$ corresponds to a coordinate x_I on $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$, and we set $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})_I$ to be the open locus where $x_I \neq 0$. Note that $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})_I \subset \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})$ is the subfunctor parameterizing line bundle quotients $\bigwedge^k \mathcal{O}^n_S \to L$ such that the composition $\mathcal{O}_S \xrightarrow{e_I} \bigwedge^k \mathcal{O}^n_S \to L$ (where the first map is the inclusion of the Ith term) is an isomorphism, or in other words $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})_I \cong \mathrm{Gr}(1, \binom{n}{k})_{\{I\}}$ viewing $\{I\}$ as the corresponding subset of $\{1,\dots,\binom{n}{k}\}$ of size 1. Using these functorial descriptions, one can check that there is a cartesian diagram of functors

$$
\begin{array}{ccc}
\mathrm{Gr}(k,n)_I & \xrightarrow{P_I} & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})_I \\
\downarrow & \square & \downarrow \\
\mathrm{Gr}(k,n) & \xrightarrow{P} & \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}}).
\end{array}
$$

Since $\{\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})_I\}$ is a Zariski open cover, it suffices to show that each $P_I\colon \mathrm{Gr}(k,n)_I \to \mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})_I$ is a closed immersion.

For simplicity, assume that $I = \{1,\dots,k\}$. Under the isomorphisms $\mathrm{Gr}(k,n)_I \cong \mathbb{A}^{k\times(n-k)}_{\mathbb{Z}}$ of Lemma 1.2.1 and $\mathbb{P}(\bigwedge^k \mathcal{O}^n_{\mathrm{Spec}\,\mathbb{Z}})_I \cong \mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$, the morphism P_I corresponds to the map

$$\mathbb{A}^{k\times(n-k)}_{\mathbb{Z}} \to \mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$$

assigning a $k \times (n-k)$ matrix $A = \{x_{i,j}\}$ to the element of $\mathbb{A}^{\binom{n}{k}-1}_{\mathbb{Z}}$ whose Jth coordinate, where $J \subset \{1,\dots,n\}$ is a subset of length k distinct from I, is the $\{1,\dots,k\} \times J$ minor of the $k \times n$ block matrix

$$
\left(
\begin{array}{cccc|ccc}
1 & & & & x_{1,1} & \cdots & x_{1,n-k} \\
& 1 & & & x_{2,1} & \cdots & x_{2,n-k} \\
& & \ddots & & \vdots & & \\
& & & 1 & x_{k,1} & \cdots & x_{k,n-k}
\end{array}
\right).
$$

The coordinate $x_{i,j}$ on $\mathbb{A}^{k\times(n-k)}_{\mathbb{Z}}$ is the pullback of the coordinate corresponding to the subset $\{1,\cdots,\widehat{i},\cdots,k,k+j\}$ (see Figure 1.1). This shows that the corresponding ring map is surjective thereby establishing that P_I is a closed immersion. $\square$

$$x_{i,j} = \det \begin{pmatrix} 1 & & & & x_{1,1} & \cdot \mid \cdot & x_{1,j} & \cdot \mid \cdot & x_{1,n-k} \\ & \ddots & & & \vdots & & \vdots & & \vdots \\ & & 1 & & x_{i,1} & \cdot \mid \cdot & x_{i,j} & \cdot \mid \cdot & x_{i,n-k} \\ & & & \ddots & \vdots & & \vdots & & \vdots \\ & & & 1 & x_{k,1} & \cdot \mid \cdot & x_{k,j} & \cdot \mid \cdot & x_{k,n-k} \end{pmatrix}$$

Figure 1.1: The minor obtained by removing the ith column and all columns $k+1, \ldots, n$ other than $k+j$ is precisely $x_{i,j}$.

Exercise 1.2.8. For a field $\Bbbk$, let $\mathrm{Gr}(k,n)_{\Bbbk}$ be the $\Bbbk$-scheme $\mathrm{Gr}(k,n) \times_{\mathbb{Z}} \Bbbk$, and $p \in \mathrm{Gr}(k,n)_{\Bbbk}$ be the point corresponding to a quotient $Q = \Bbbk^n/K$. Show that there is a natural bijection of the tangent space

$$T_p \mathrm{Gr}(k,n)_{\Bbbk} \xrightarrow{\sim} \mathrm{Hom}_{\Bbbk}(K, Q).$$

with the vector space of $\Bbbk$-linear maps $K \to Q$.

Exercise 1.2.9. Provide an alternative proof of the projectivity of $\mathrm{Gr}(k,n)$ as follows.

(a) Show that the functor $P \colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}^n)$ is injective on points and tangent spaces.

(b) Use a criterion for being a closed immersion (c.f. [Har77, Prop. II.7.3]) to show that $P \colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}^n)$ is a closed immersion.

(Alternatively, you could show that $P \colon \mathrm{Gr}(k,n) \to \mathbb{P}(\bigwedge^k \mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}^n)$ is a proper monomorphism and conclude that $\mathrm{Gr}(k,n)$ is projective over $\mathbb{Z}$.)

1.2.3 Relative version

We now prove the relative version of the representability and strong projectivity of the Grassmannian.

Proof of Theorem 1.1.1. If V is a vector bundle over S of rank n, there is the relative Plücker embedding

$$\begin{aligned} P \colon \mathrm{Gr}_S(k,V) &\to \mathbb{P}(\bigwedge^k V) \\ [V_T \xrightarrow{q} Q] &\mapsto [\bigwedge^k V_T \to \bigwedge^k Q] \end{aligned} \tag{1.2.10}$$

defined above over a S-scheme T. This is a morphism of functors over S. Since $\mathbb{P}(\bigwedge^k V)$ is projective over S, it suffices to show that this morphism is representable by closed immersions. This property can be checked Zariski-locally: if $U \subset S$ is an open subscheme where V is trivial, then the base change of $\mathrm{Gr}_S(k,V) \to \mathbb{P}(\bigwedge^k V)$ over U is the Plücker embedding $\mathrm{Gr}_U(k,\mathcal{O}_U^n) \to \mathbb{P}(\bigwedge^k \mathcal{O}_U^N)$ which is a closed immersion (Proposition 1.2.7). $\qquad\square$

Since the Grassmannian functor is representable, there is a *universal quotient* $\mathcal{O}_{\mathrm{Gr}_S(k,V)} \otimes_S V \to Q_{\mathrm{univ}}$; here $\mathcal{O}_{\mathrm{Gr}_S(k,V)} \otimes_S V$ denotes the pullback of V under the structure morphism $\mathrm{Gr}_S(k,V) \to S$. Under the Plücker embedding (1.2.10), the pullback of $\mathcal{O}(1)$ is identified with $\det(Q_{\mathrm{univ}})$, which we sometimes call the *Plücker line bundle*. Thus, we obtain:

Corollary 1.2.11. *The determinant* $\det(Q_{\mathrm{univ}})$ *of the universal quotient is a very ample line bundle on* $\mathrm{Gr}_S(k, V)$. $\qquad\square$

Remark 1.2.12. For projective space $\mathbb{P}^n = \mathrm{Gr}(1, n)$, the universal quotient yields an exact sequence $0 \to \Omega_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$, which is the dual of the Euler sequence [Har77, Ex. 8.20.1] twisted by $\mathcal{O}_{\mathbb{P}^n}(1)$.

1.3 Castelnuovo–Mumford regularity

The Cartan–Serre–Grothendieck theorem states that if F is a coherent sheaf on a projective variety $(X, \mathcal{O}_X(1))$, then for $d \gg 0$

(1) $F(d)$ is globally generated;

(2) $\mathrm{H}^i(X, F(d)) = 0$ for $i > 0$; and

(3) the multiplication map

$$\mathrm{H}^0(X, F(d)) \otimes \mathrm{H}^0(X, \mathcal{O}(p)) \to \mathrm{H}^0(X, F(d+p))$$

is surjective for all $p \geq 0$.

Castelnuovo–Mumford regularity provides a quantitative measure of the size of d necessary for the twist $F(d)$ to have the three above desired cohomological properties and in particular that the Hilbert polynomial $\chi(X, F(d))$ of F evaluated at d agrees with $\mathrm{h}^0(X, F(d))$.

1.3.1 Definition and basic properties

Definition 1.3.1. Let F be a coherent sheaf on projective space $\mathbb{P}^n$ over a field $\Bbbk$. For an integer m, we say that F is *m-regular* if

$$\mathrm{H}^i(\mathbb{P}^n, F(m-i)) = 0$$

for all $i \geq 1$.

The *regularity of F* is the smallest integer m such that $F(m)$ is m-regular.

While the requirement that the ith cohomology of the $(m-i)$th twist vanishes may appear mysterious at first, this definition is very convenient for induction arguments on the dimension n as indicated for instance by the following result.

Lemma 1.3.2. *Let F be an m-regular coherent sheaf on $\mathbb{P}^n$ over a field $\Bbbk$. If $H \subset \mathbb{P}^n$ is a hyperplane avoiding the associated points of F, then $F|_H$ is also m-regular.*

Proof. The hypotheses imply that over an affine open subscheme $U \subset \mathbb{P}^n$, the defining equation of H is a nonzerodivisor for the module $\Gamma(U, F)$. Thus $F(-1) \xrightarrow{H} F$ is injective, and for an integer $i > 0$ we have a short exact sequence

$$0 \to F(m-i-1) \to F(m-i) \to F|_H(m-i) \to 0$$

inducing a long exact sequence on cohomology

$$\cdots \to \mathrm{H}^i(\mathbb{P}^n, F(m-i)) \to \mathrm{H}^i(H, F|_H(m-i)) \to \mathrm{H}^{i+1}(\mathbb{P}^n, F(m-i-1)) \to \cdots$$

If F is m-regular, then $\mathrm{H}^i(\mathbb{P}^n, F(m-i)) = \mathrm{H}^{i+1}(\mathbb{P}^n, F(m-i-1)) = 0$. It follows that $\mathrm{H}^i(H, F|_H(m-i)) = 0$ for all $i > 0$, and thus $F|_H$ is also m-regular. $\qquad\square$

Remark 1.3.3. It follows from the definition of regularity that if F is m-regular, then $F(d)$ is $(m-d)$-regular. We will show in Lemma 1.3.6 that if F is m-regular, it also d-regular for all $d \geq m$.

Exercise 1.3.4.

  (a) Show that $\mathcal{O}(d)$ is $(-d)$-regular on $\mathbb{P}^n$.

  (b) Show that the structure sheaf of a hypersurface $H \subset \mathbb{P}^n$ of degree d is $(d-1)$-regular.

  (c) Show that the structure sheaf of a smooth curve $C \subset \mathbb{P}^n$ of genus g is $(2g-1)$-regular.

Exercise 1.3.5. Let F be a coherent sheaf on $\mathbb{P}^n$ resolved by a long exact sequence of coherent sheaves. Show that if each F_i is $(m+i)$-regular, then F is m-regular.

$$\cdots \to F_2 \to F_1 \to F_0 \to F \to 0$$

Another advantage of regularity is the following lemma due to Castelnuovo.

Lemma 1.3.6. *Let F be an m-regular coherent sheaf on $\mathbb{P}^n$.*

  (1) For $d \geq m$, F is d-regular.

  (2) The multiplication map

$$\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \to \mathrm{H}^0(\mathbb{P}^n, F(d+k))$$

  is surjective if $d \geq m$ and $k \geq 0$.

  (3) For $d \geq m$, $F(d)$ is globally generated and $\mathrm{H}^i(\mathbb{P}^n, F(d)) = 0$ for $i \geq 1$.

Proof. If $\Bbbk \to \Bbbk'$ is a field extension, then Flat Base Change (A.2.12) implies that $\mathrm{H}^i(\mathbb{P}^n_{\Bbbk}, F) \otimes_{\Bbbk} \Bbbk' = \mathrm{H}^i(\mathbb{P}^n_{\Bbbk'}, F \otimes_{\Bbbk} \Bbbk')$. As $\Bbbk \to \Bbbk'$ is faithfully flat, the assertions (1)–(3) can be checked after base change. We can thus assume that $\Bbbk$ is algebraically closed and in particular infinite.

For (1) and (2), we will argue by induction on n with the base case of $n = 0$ being clear. If $n > 0$, since $\Bbbk$ is infinite, we may choose a hyperplane $H \subset \mathbb{P}^n$ avoiding the associated points of F. Since the restriction $F|_H$ is m-regular (Lemma 1.3.2) on $H \cong \mathbb{P}^{n-1}$, the inductive hypothesis implies that (1) and (2) hold for $F|_H$.

We prove (1) by using induction also on d. The base case $d = m$ holds by hypothesis. For $d > m$, the short exact sequence $0 \to F(d-i-1) \to F(d-i) \to F|_H(d-i) \to 0$ induces a long exact sequence on cohomology

$$\cdots \to \mathrm{H}^i(\mathbb{P}^n, F(d-i-1)) \to \mathrm{H}^i(\mathbb{P}^n, F(d-i)) \to \mathrm{H}^i(H, F|_H(d-i)) \to \cdots$$

For $i > 0$, the first term vanishes by the induction hypothesis on d (F is $(d-1)$-regular so $\mathrm{H}^i(\mathbb{P}^n, F(d-1-i)) = 0$) and the third term vanishes by the inductive hypothesis on n ($F|_H$ is m-regular by Lemma 1.3.2 and thus d-regular by the inductive hypothesis on n so $\mathrm{H}^i(H, F|_H(d-i)) = 0$). Thus, the second term vanishes and we have established (1).

To show (2), we use induction on k in addition to n. We denote the multiplication map by

$$\mu_{d,k} \colon \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \to \mathrm{H}^0(\mathbb{P}^n, F(d+k)).$$

While the base case $k = 0$ is clear, the inductive argument will require us to directly establish the case $k = 1$. To this end, we consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) & \xrightarrow{\nu_d \otimes \mathrm{res}} & \mathrm{H}^0(H, F|_H(d)) \otimes \mathrm{H}^0(H, \mathcal{O}_H(1)) \\
\Big\downarrow{\scriptstyle \mu_{d,1}} & & \Big\downarrow \\
\mathrm{H}^0(\mathbb{P}^n, F(d)) \xrightarrow{\ \alpha\ } \mathrm{H}^0(\mathbb{P}^n, F(d+1)) & \xrightarrow{\ \nu_{d+1}\ } & \mathrm{H}^0(H, F|_H(d+1)).
\end{array}$$

(with $\mathrm{id} \otimes H$ arrow)

$$\tag{1.3.7}$$

As the map α is given by multiplication by $H \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1))$, α factors through the the map $\mathrm{id} \otimes H$ defined by $v \mapsto v \otimes H$. It follows that $\mathrm{im}(\alpha) \subset \mathrm{im}(\mu_{d,1})$. Since $\mathrm{H}^1(\mathbb{P}^n, F(d)) = 0$ by (2), the restriction map $\nu_d \colon \mathrm{H}^0(\mathbb{P}^n, F(d)) \to \mathrm{H}^0(H, F|_H(d))$ is surjective. Likewise, since $\mathrm{H}^1(\mathbb{P}^n, \mathcal{O}) = 0$, $\mathrm{res} \colon \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) \to \mathrm{H}^0(H, \mathcal{O}_H(1))$ is surjective. We conclude that the top horizontal arrow is surjective. The inductive hypothesis applied to $H = \mathbb{P}^{n-1}$ implies that the right vertical arrow is surjective. Therefore, the composition $\nu_{d+1} \circ \mu_{d,1}$ is surjective and it follows that $\mathrm{im}(\mu_{d,1})$ surjects onto $\mathrm{H}^0(H, F|_H(d+1))$. By exactness of the bottom row, we have that

$$\mathrm{H}^0(\mathbb{P}^n, F(d+1)) = \mathrm{im}(\mu_{d,1}) + \ker(\beta) = \mathrm{im}(\mu_{d,1}) + \mathrm{im}(\alpha) = \mathrm{im}(\mu_{d,1}),$$

which shows that $\mu_{d,1}$ is surjective.

If $k > 1$, we consider the commutative square

$$\begin{array}{ccc}
\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k-1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) & \longrightarrow & \mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \\
\Big\downarrow{\scriptstyle \mu_{d,k-1} \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \mu_{d,k}} \\
\mathrm{H}^0(\mathbb{P}^n, F(d+k-1)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(1)) & \xrightarrow{\ \mu_{d+k-1,1}\ } & \mathrm{H}^0(\mathbb{P}^n, F(d+k)).
\end{array}$$

The left vertical map and bottom horizontal arrow are surjective by the inductive hypothesis applied to $k - 1$ and $k = 1$, respectively. It follows that $\mu_{d,k}$ is surjective.

To show (3), we know that for $k \gg 0$, $F(d+k)$ is globally generated, i.e. $\gamma_{F(d+k)} \colon \mathrm{H}^0(\mathbb{P}^n, F(d+k)) \otimes \mathcal{O}_{\mathbb{P}^n} \to F(d+k)$ is surjective. Consider the commutative square

$$\begin{array}{ccc}
\mathrm{H}^0(\mathbb{P}^n, F(d)) \otimes \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O}_{\mathbb{P}^n} & \xrightarrow{\mu_{d,k} \otimes \mathrm{id}} & \mathrm{H}^0(\mathbb{P}^n, F(d+k)) \otimes \mathcal{O}_{\mathbb{P}^n} \\
\Big\downarrow{\scriptstyle \gamma_{F(d)} \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \gamma_{F(d+k)}} \\
F(d) \otimes \big(\mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(k)) \otimes \mathcal{O}_{\mathbb{P}^n}\big) & \xrightarrow{\ \mathrm{id} \otimes \gamma_{\mathcal{O}(k)}\ } & F(d) \otimes \mathcal{O}(k).
\end{array}$$

Since the top horizontal arrow is surjective by (1), the composition from the top left to the bottom right is surjective. Given the nature of the bottom horizontal map, we see that $\gamma_{F(d)}$ must be surjective (indeed, if $V = \mathrm{im}(\gamma_{F(d)}) \subset F(d)$, then $\mathrm{im}(\mathrm{id} \otimes \gamma_{\mathcal{O}(k)} \circ \gamma_{F(d)} \otimes \mathrm{id}) = V \otimes \mathcal{O}(k))$. Finally, to see the vanishing of the higher cohomology of $F(d)$ observe that for each $i > 0$, the sheaf F is $(d+i)$-regular by (2) and thus $\mathrm{H}^i(\mathbb{P}^n, F(d)) = 0$. $\qquad\square$

One easy consequence of (1) is that if F is m-regular, then the restriction map

$$\nu_d \colon\colon \mathrm{H}^0(\mathbb{P}^n, F(d)) \to \mathrm{H}^0(H, F|_H(d))$$

is surjective for all $d \geq m$. Indeed, (1) implies that F is also d-regular and the surjectivity follows from the vanishing of $\mathrm{H}^1(\mathbb{P}^n, F(d-1))$. The following lemma—which will be used in the proof of Theorem 1.3.9—shows that we can still arrange for the surjectivity of ν_d under weaker hypotheses.

Lemma 1.3.8. *Let F be a coherent sheaf on $\mathbb{P}^n$ and H be a hyperplane avoiding the associated points of F. If $F|_H$ is m-regular and ν_d is surjective for some $d \geq m$, then ν_p is surjective for all $p \geq d$.*

Proof. By staring at the square in diagram (1.3.7), we see that the top arrow $\nu_d \otimes \mathrm{res}$ is surjective (as both ν_d and res are surjective) and the vertical right multiplication morphism is surjective (by applying Lemma 1.3.6(2) to the m-regular sheaf $F|_H$). The statement follows. $\qquad\square$

1.3.2 Regularity bounds

We now turn to the following bound on the regularity of subsheaves of the trivial vector bundle established by Mumford in [Mum66, p.101].

Theorem 1.3.9 (Boundedness of Regularity). *For every pair of non-negative integers r and n, and for every polynomial $P \in \mathbb{Q}[z]$, there exists an integer m_0 with the following property: for every field $\Bbbk$, every subsheaf $F \subset \mathcal{O}_{\mathbb{P}^n_{\Bbbk}}^r$ with Hilbert polynomial P is m_0-regular.*

Proof. As in the proof of Lemma 1.3.6, we can assume that $\Bbbk$ is infinite. We will argue by induction on n. The base case of $n = 0$ holds as every sheaf F on $\mathbb{P}^0$ is m-regular for every integer m.

For $n \geq 1$ and a subsheaf $F \subset \mathcal{O}_{\mathbb{P}^n}^r$ with Hilbert polynomial P, we can choose a hyperplane $H \subset \mathbb{P}^n$ avoiding all associated points of $\mathcal{O}_{\mathbb{P}^n}^r/F$. This ensures that $\mathrm{Tor}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_H, \mathcal{O}_{\mathbb{P}^n}^r/F) = 0$ and that the short exact sequence $0 \to F \to \mathcal{O}_{\mathbb{P}^n}^r \to \mathcal{O}_{\mathbb{P}^n}^r/F \to 0$ restricts to a short exact sequence

$$0 \to F|_H \to \mathcal{O}_H^r \to \mathcal{O}_H^r/F \to 0. \qquad (1.3.10)$$

As $H \cong \mathbb{P}^{n-1}$, this will allow us to apply the inductive hypothesis to $F|_H \subset \mathcal{O}_H^r$.

On the other hand, since $F \subset \mathcal{O}_{\mathbb{P}^n}^r$ is torsion free, we have a short exact sequence

$$0 \to F(-1) \xrightarrow{H} F \to F|_H \to 0, \qquad (1.3.11)$$

and the Hilbert polynomial of $F|_H$ is $\chi(F|_H(d)) = \chi(F(d)) - \chi(F(d-1)) = P(d) - P(d-1)$. In particular, the Hilbert polynomial of $F|_H$ only depends on P and the inductive hypothesis applied to $F|_H \subset \mathcal{O}_H^r$ gives an integer m_1 such that $F|_H$ is m_1-regular.

For $m \geq m_1 - 1$, since $\mathrm{H}^i(H, F|_H(m)) = 0$ for all $i \geq 1$, we have a long exact sequence

$$0 \to \mathrm{H}^0(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^0(\mathbb{P}^n, F(m)) \to \mathrm{H}^0(H, F|_H(m)) \to$$
$$\mathrm{H}^1(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^1(\mathbb{P}^n, F(m)) \to 0. \quad (1.3.12)$$

For $i \geq 2$, we also have isomorphisms $\mathrm{H}^i(\mathbb{P}^n, F(m-1)) \to \mathrm{H}^i(\mathbb{P}^n, F(m))$, and since $\mathrm{H}^i(\mathbb{P}^n, F(d))$ vanishes for $d \gg 0$, we can conclude that $\mathrm{H}^i(\mathbb{P}^n, F(m-1)) = 0$.

To handle H^1, we use the inequalities $\mathrm{h}^1(\mathbb{P}^n, F(m_1)) \geq \mathrm{h}^1(\mathbb{P}^n, F(m_1 + 1)) \geq \cdots$, which eventually stabilize to 0. We claim that in fact that the inequalities $\mathrm{h}^1(\mathbb{P}^n, F(m_1)) > \mathrm{h}^1(\mathbb{P}^n, F(m_1 + 1)) > \cdots$ are strict until they become 0. To see this, we observe that there is an equality $\mathrm{h}^1(\mathbb{P}^n, F(m-1)) = \mathrm{h}^1(\mathbb{P}^n, F(m))$ for $m \geq m_1$ if and only if $\nu_m \colon \mathrm{H}^0(\mathbb{P}^n, F(m)) \to \mathrm{H}^0(H, F|_H(m))$ is surjective. If $\mathrm{h}^1(\mathbb{P}^n, F(m-1)) = \mathrm{h}^1(\mathbb{P}^n, F(m))$ for some $m \geq m_1$, then ν_m is surjective. Since $F|_H$ is m_1-regular, we may apply Lemma 1.3.8 to conclude that $\nu_{m'}$ is surjective for

all $m' \geq m$, which in turn implies that $\mathrm{h}^1(\mathbb{P}^n, F(m'))$ is constant for $m' \geq m$, and therefore zero. This establishes the claim. Setting $m_2 = m_1 + 1 + \mathrm{h}^1(\mathbb{P}^n, F(m_1))$, we see that $\mathrm{h}^1(\mathbb{P}^n, F(m_2 - 1)) = 0$ and that F is m_2-regular.

We now show that m_2 is bounded above by a constant m_0 independent of F. Since $F \subset \mathcal{O}^r_{\mathbb{P}^n}$, we have that $\mathrm{h}^0(\mathbb{P}^n, F(d)) \leq r\mathrm{h}^0(\mathbb{P}^n, \mathcal{O}(d)) = r\binom{n+d}{n}$ for any $d \geq 0$. Using the vanishing of $\mathrm{h}^i(\mathbb{P}^n, F(m_1))$ for $i \geq 2$, we have

$$\mathrm{h}^1(\mathbb{P}^n, F(m_1)) = \mathrm{h}^0(\mathbb{P}^n, F(m_1)) - \chi(F(m_1))$$
$$\leq r\binom{n+m_1}{n} + P(m_1).$$

Thus, defining $m_0 := m_1 + 1 + r\binom{n+m_1}{n} + P(m_1)$, we have that $m_2 \leq m_0$. $\qquad\square$

Remark 1.3.13. The above proof establishes in fact a stronger statement. To formulate the result, we recall that every numerical polynomial $P \in \mathbb{Q}[z]$ (i.e. $P(d) \in \mathbb{Z}$ for integers $d \gg 0$) of degree n can be uniquely written as

$$P(d) = \sum_{i=0}^{n} a_i \binom{d}{i}$$

for $a_i \in \mathbb{Z}$; this follows from a straightforward inductive argument (c.f. [Har77, Prop. I.7.3]). For non-negative integers r and n, there exists a polynomial $\Lambda_{r,n} \in \mathbb{Z}[x_0, \dots, x_n]$ with the following property: for every field $\Bbbk$, every subsheaf $F \subset \mathcal{O}^r_{\mathbb{P}^n_\Bbbk}$ with Hilbert polynomial $P(d) = \sum_{i=0}^n a_i \binom{d}{i}$ is m_0-regular for $m_0 = \Lambda_{r,n}(a_0, \dots, a_n)$.

Remark 1.3.14 (Optimal bounds). Although Mumford's result on Boundedness of Regularity (Theorem 1.3.9) provides an explicit bound and is sufficient for many applications including the construction of the Quot scheme as well as for other applications, there is a more optimal bound established by Gotzmann: for a projective scheme $X \subset \mathbb{P}^N$ over a field $\Bbbk$ with Hilbert polynomial P, there are unique integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ such that P can be expressed as

$$P(d) = \binom{d + \lambda_1 - 1}{\lambda_1 - 1} + \binom{d + \lambda_2 - 2}{\lambda_2 - 1} + \cdots + \binom{d + \lambda_r - r}{\lambda_r - 1},$$

and the ideal sheaf $\mathcal{I}_X$ of X is r-regular. See [Got78], [Gre89], [Gre98, §3] and [BH93, §4.3].

Exercise 1.3.15. Let $C \subset \mathbb{P}^n$ be a curve of degree d and genus g. Show that Gotzmann's bound implies that the ideal sheaf I_C of C is $(\binom{d}{2} + 1 - g)$-regular. Can you compare this to the bound given by the proof of Theorem 1.3.9, i.e. can you compute $\Lambda_{1,n}(1 - g, d)$ for an explicit polynomial satisfying Theorem 1.3.9?

Remark 1.3.16. It was shown in [GLP83] that the ideal sheaf I_C of an integral, non-degenerate curve $C \subset \mathbb{P}^N$ of degree d is $(d - N + 2)$-regular. It is conjectured more generally that the ideal sheaf of a smooth, non-degenerate projective variety $X \subset \mathbb{P}^N$ of dimension n and degree d is $(d - (N - n)) + 1)$-regular; see [GLP83] and [EG84].

Corollary 1.3.17. *Let $\pi\colon X \to S$ be a strongly projective morphism of noetherian schemes and $\mathcal{O}_X(1)$ be a relatively ample line bundle on X. Let F be quotient sheaf of $\pi^*(W)(q)$ for some vector bundle W on S and integer q. Let $P \in \mathbb{Q}[z]$ be a*

polynomial. There exists an integer m_0 satisfying the following property for every $d \geq m_0$: for every morphism $f \colon T \to S$ inducing a cartesian square

$$
\begin{array}{ccc}
X_T & \xrightarrow{\ f_T\ } & X \\
{\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \pi} \\
T & \xrightarrow{\ f\ } & S
\end{array}
$$

and every finitely presented quotient $Q = F_T/K$ flat over S such that every fiber Q_t on X_t has Hilbert polynomial P, then

(1) *$\pi_{T,*}Q(d)$ is a vector bundles of rank $P(d)$*

(2) *the comparison maps $f^*\pi_*Q(d) \to \pi_{T,*}f_T^*Q(d)$, $f^*\pi_*F(d) \to \pi_{T,*}f_T^*F(d)$ and $f^*\pi_*K(d) \to \pi_{T,*}f_T^*K(d)$ are isomorphisms;*

(3) *$R^1\pi_{T,*}K(d) = 0$ for $i > 0$; and*

(4) *the adjunction maps $\pi_T^*\pi_{T,*}Q(d) \to Q(d)$, $\pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$ and $\pi_T^*\pi_{T,*}K(d) \to K(d)$ are surjective.*

Proof. For (2), since $\pi \colon X \to S$ is strongly projective, there is a closed immersion $i \colon X \hookrightarrow \mathbb{P}(V)$ where V is a vector bundle on S. Since the statement is local on S (and S is quasi-compact), we may assume that S is affine and that V is the trivial vector bundle of rank $n+1$. We are given a surjection $\pi^*(W)(q) \twoheadrightarrow F$, and if Q is a quotient of i_*F with Hilbert polynomial P, then $Q(-q)$ is a quotient of π^*W with Hilbert polynomial P' where $P'(z) = P(z+d)$. We can therefore replace (F, X, P) with $(\pi^*(W), \mathbb{P}(V), P')$. In particular, for every field-valued point $s \colon \operatorname{Spec}\Bbbk \to S$, $\mathbb{P}(V)_s \cong \mathbb{P}^n_{\Bbbk}$ and $F_s \cong \mathcal{O}^k_{\mathbb{P}^n_{\Bbbk}}$ where $\operatorname{rk}(V) = n+1$ and $\operatorname{rk}(W) = k$.

By Boundedness of Regularity (Theorem 1.3.9), there exists an integer m_0 depending on n, r and P such that for every field-valued point $s \colon \operatorname{Spec}\Bbbk \to S$, the kernel K_s is m_0-regular. As K_s is also $(m_0 + 2)$-regular (Lemma 1.3.6) and $F_s \cong \mathcal{O}^k_{\mathbb{P}^n_{\Bbbk}}$ is (m_0+1)-regular (in fact, it is 0-regular), it follows that Q_s is m_0-regular (Exercise 1.3.4). By Lemma 1.3.6, for $d \geq m_0 + 2$, $K_s(d)$, $F_s(d)$ and $Q_s(d)$ are each globally generated with vanishing higher cohomology. Since K, F and Q are flat over S, statements (1)–(3) follow from applying Cohomology and Base Change in the form of Exercise A.6.9. For (4), to verify the surjectivity of the adjunction map $\pi_T^*\pi_{T,*}K(d) \to K(d)$ (and likewise for F_T and Q), it suffices to check that the restriction

$$(\pi_T^*\pi_{T,*}K(d))|_{X_t} \to K_t(d) \tag{1.3.18}$$

is surjective one each fiber X_t over $t \in T$. Using (2), we have identifications

$$(\pi_T^*\pi_{T,*}K(d))|_{X_t} \cong \pi_t^*(\pi_{T,*}K(d) \otimes \kappa(t)) \cong \pi_t^*\pi_{t,*}K_t(d),$$

where $\pi_t \colon X_t \to \operatorname{Spec}\kappa(t)$ and thus (1.3.18) corresponds to the adjunction map $\pi_t^*\pi_{t,*}K_t(d) \to K_t(d)$, which we know is surjective as $K_t(d)$ is globally generated. $\square$

1.4 Representability and projectivity of Hilb and Quot

In this section, we prove the representability and projectivity of Quot (Theorem 1.1.3) and as a consequence we obtain the same for the Hilbert scheme (Theorem 1.1.2).

As before, $\pi\colon X \to S$ is a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ is a relatively ample line bundle on X, F is a quotient sheaf of $\pi^*(W)(q)$ for some vector bundle W on S and integer q, and $P \in \mathbb{Q}[z]$ is a polynomial. Our strategy is to use the morphism of functors

$$\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Gr}_S(P(d), \pi_*F(d))$$
$$[F_T \twoheadrightarrow Q] \mapsto [\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)],$$

defined above over an S-scheme T. For $d \gg 0$, Corollary 1.3.17 implies that the above morphism is well-defined: indeed part (1) shows that $\pi_{T,*}Q(d)$ is a vector bundle of rank $P(d)$, part (2) shows the pullback of the coherent sheaf $\pi_*F(d)$ under $T \to S$ is identified with $\pi_{T,*}F_T(d)$, and part (3) shows that $R^1\pi_{T,*}K(d) = 0$ which implies the surjectivity of $\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)$.

1.4.1 Quot is locally closed in a Grassmannian

We prove that the map $\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Gr}_S(P(d), \pi_*F(d))$ is representable by locally closed immersions.

Proposition 1.4.1. *Let $\pi\colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on X, and F be a coherent sheaf on X which is the quotient of $\pi^*(W)(q)$ for a vector bundle W on S and an integer q. For $d \gg 0$, the morphism $\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Gr}_S(P(d), \pi_*F(d))$ is representable by locally closed immersions, i.e. for every morphism $T \to \mathrm{Gr}_S(P(d), \pi_*F(d))$ from a scheme, the fiber product*

$$T \times_{\mathrm{Gr}_S(P(d), \pi_*F(d))} \mathrm{Quot}^P_{X/S}(F)$$

is representable by a locally closed subscheme of T.

Proof. We first reduce to the special case that $X = \mathbb{P}(V)$ and $F = \pi^*W$ for trivial vector bundles V and W. Let $i\colon X \hookrightarrow \mathbb{P}(V)$ be a closed immersion where V is a vector bundle on S. The morphism of functors $\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Gr}_S(P(d), \pi_*F(d))$ is defined over S and its base change to an open subscheme $U \subset S$ is identified with the morphism $\mathrm{Quot}^P_{X_U/U}(F_U) \to \mathrm{Gr}_S(P(d), \pi_{U,*}F_U(d))$. Since the property of being a locally closed immersion is Zariski-local on the target, the statement is Zariski-local on S. We may therefore assume that S is affine and that V is the trivial vector bundle of rank $n+1$.

First, observe that since there is an isomorphism of functors

$$\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Quot}^P_{\mathbb{P}(V)/S}(i_*F),$$

we may replace (F, X) with $(i_*F, \mathbb{P}(V))$. Next using the surjection $\pi^*(W)(q) \twoheadrightarrow F$, we obtain a morphism of functors

$$\mathrm{Quot}^P_{\mathbb{P}(V)/S}(F) \to \mathrm{Quot}^{P'}_{\mathbb{P}(V)/S}(\pi^*W)$$
$$[F_T \to Q] \mapsto [(\pi^*W)_T \to F(-q)_T \to Q(-q)],$$

defined over an S-scheme T, where $P'(z) = P(z - q)$. We claim that this morphism is representable by closed immersions. This claims boils down to the statement that for an S-scheme T and quotient $\pi^*W(q)_T \twoheadrightarrow Q$, there is a closed subscheme $Z \subset T$ such that a morphism $U \to T$ factors through Z if and only if the restriction

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$\pi^*W(q)_U \twoheadrightarrow G_U$ factors through F_U. Defining $K = \ker(\pi^*W(q)_T \to F_T)$ and considering the diagram

$$0 \longrightarrow K \longrightarrow \pi^*W(q)_T \longrightarrow F_T \longrightarrow 0$$
$$\searrow \qquad \downarrow$$
$$G,$$

we see that the claim is satisfied by taking $Z \subset T$ to be vanishing scheme of the morphism $K \to G$ (see).

Finally, using that $\pi_*(\pi^*W(d)) = W \otimes \operatorname{Sym}^d V$, we have a commutative diagram

$$\begin{array}{ccc}
\operatorname{Quot}^P_{\mathbb{P}(V)/S}(F) & \hookrightarrow & \operatorname{Quot}^{P'}_{\mathbb{P}(V)/S}(\pi^*W) \\
\downarrow & & \uparrow \\
\operatorname{Gr}_S(P(d), \pi_*F(d)) & \longrightarrow & \operatorname{Gr}_S(P'(d), W \otimes \operatorname{Sym}^d V).
\end{array}$$

By the above claim, the top horizontal map is a closed immersion. As $\operatorname{Gr}_S(P(d), \pi_*F(d))$ and $\operatorname{Gr}_S(P'(d), W \otimes \operatorname{Sym}^d V)$ are projective (), the bottom horizontal map is projective and in particular separated. If the proposition holds for $\operatorname{Quot}^{P'}_{\mathbb{P}(V)/S}(\pi^*W)$ and the right vertical map is a locally closed immersion, then the left vertical map is also a closed immersion by the cancellation property.

We now handle the special case. We first claim that $\operatorname{Quot}^P_{X/S}(F) \to \operatorname{Gr}_S(P(d), W \otimes \operatorname{Sym}^d V)$ is a monomorphism, i.e.

$$\operatorname{Quot}^P_{X/S}(F)(T) \to \operatorname{Gr}_S(P(d), \pi_*F(d))(T)$$

is injective for each scheme T. To see this, observe that if $F_T = Q/K$ is a quotient with Hilbert polynomial P, then implies that there is a map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_T^*\pi_{T,*}K(d) & \longrightarrow & \pi_T^*\pi_{T,*}F_T(d) & \longrightarrow & \pi_T^*\pi_{T,*}Q(d) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K(d) & \longrightarrow & F_T(d) & \longrightarrow & Q(d) & \longrightarrow & 0
\end{array}$$

where the vertical maps are surjections. Thus $F_T(d) \to Q(d)$ can be recovered from $\pi_{T,*}F_T(d) \to \pi_{T,*}Q(d)$ by taking the cokernel of the composition $\pi_T^*\pi_{T,*}K(d) \to \pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$.

Let $T \to \operatorname{Gr}_S(P(d), W \otimes \operatorname{Sym}^d V)$ be a morphism determined by a vector bundle quotient $\gamma\colon \pi_{T,*}F_T(d) = W_T \otimes \operatorname{Sym}^d V_T \to G$ of rank $P(d)$. Define Q as the quotient sheaf of F_T with the property that $F_T(d) \twoheadrightarrow Q(d)$ is identified with the cokernel of $\ker(\pi_T^*\gamma) \to \pi_T^*\pi_{T,*}F_T(d) \to F_T(d)$. The fiber product

$$\begin{array}{ccc}
Z & \longrightarrow & T \\
\downarrow & & \downarrow \\
\operatorname{Quot}^P_{X/S}(F) & \longrightarrow & \operatorname{Gr}_S(P(d), W \otimes \operatorname{Sym}^d V)
\end{array}$$

is identified with the subfunctor of T (or more precisely the subfunctor of $\operatorname{Mor}_S(-, T)$) consisting of morphisms $T' \to T$ such that $Q_{T'}$ is flat over T' with Hilbert polynomial

P (in other words, a map $T' \to T$ factors through Z if and only if $Q_{T'}$ is flat over T' with Hilbert polynomial P). By Existence of Flattening Stratifications (Theorem A.2.16), Z is representable by a locally closed subscheme of T. $\qquad\square$

1.4.2 Valuative criteria for Hilb and Quot

To establish that Quot is projective, it will be sufficient to know that it is proper.

Proposition 1.4.2. *For every projective morphism $X \to S$ of noetherian schemes, relatively ample line bundle $\mathcal{O}_X(1)$, coherent sheaf F on X and polynomial $P \in \mathbb{Q}[x]$, the functor $\mathrm{Quot}^P_{X/S}(F)$ satisfies the valuative criterion for properness, i.e. for every DVR R over S with fraction field K, every flat coherent quotient $F_K \to Q^\times$ on X_K with Hilbert polynomial P extends uniquely to a flat coherent quotient $F_R \to Q$ on X_R with Hilbert polynomial P.*

Remark 1.4.3. In other words, the proposition implies that for every commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}\,K & \longrightarrow & \mathrm{Quot}^P_{X/S}(F) \\
\downarrow & \nearrow & \downarrow \\
\mathrm{Spec}\,R & \longrightarrow & S,
\end{array}
$$

of solid arrows, there is a unique dotted arrow filling in the diagram. See §3.8 for a further discussion of the valuative criterion for functors and stacks.

Proof. If we write $j\colon X_K \hookrightarrow X_R$ as the open immersion, we define Q as the image of the composition $F_R \to j_* F_K \to j_* Q^\times$ (where the first map is given by the adjunction $F_R \to j_* j^* F_R = j_* F_K$). Since Q is a subsheaf of $j_* Q^\times$, it is torsion free over R and thus flat (as R is a DVR). (Locally, if $S = \mathrm{Spec}\,B$ is affine and $U = \mathrm{Spec}\,A \subset X$ is an affine open, then we can write $F|_U = \widetilde{M}$ for a finitely generated A-module M and we have a quotient $M \otimes_B K \to N^\times$ of $A \otimes_B K$-modules where $Q^\times|_{U_K} = \widetilde{N^\times}$. Then $Q = \widetilde{N}$ where N is the $A \otimes_B R$-module defined by $N := \mathrm{im}(M \otimes_B R \to M \otimes_B K \to N^\times)$. Since the R-module N is a subsheaf of the K-module $N^\times$, we see that N is torsion free and thus flat.) Finally, since Q if flat over R and $\mathrm{Spec}\,R$ is connected, its Hilbert polynomial is constant. $\qquad\square$

Remark 1.4.4. For $\mathrm{Hilb}^P_{X/S}$, the argument translates into the following: the unique extension of a closed subscheme $Z^\times \subset X_K$ is the scheme-theoretic image $Z = \mathrm{im}(Z^\times \to X_K \hookrightarrow X_R)$. The scheme Z is flat over R as all associated points live over the generic point of $\mathrm{Spec}\,R$.

1.4.3 Projectivity

The proof of the main theorem of this section (Theorem 1.1.3) follows from the following proposition.

Proposition 1.4.5. *Let $\pi\colon X \to S$ be a strongly projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle on X, and F be a coherent sheaf on X which is the quotient of $\pi^*(W)(q)$ for a vector bundle W on S and an integer q. For $d \gg 0$, the morphism $\mathrm{Quot}^P_{X/S}(F) \to \mathrm{Gr}_S(P(d), \pi_* F(d))$ is a closed immersions.*

Proof. For $d \gg 0$, the morphism
$$\operatorname{Quot}^P_{X/S}(F) \to \operatorname{Gr}_S(P(d), \pi_* F(d))$$

is a locally closed immersion of schemes defines over S (Proposition 1.4.1). Since $\operatorname{Quot}^P_{X/S}(F)$ is proper over S (Proposition 1.4.2), this map is a closed immersion. Since $\operatorname{Gr}_S(P(d), \pi_* F(d))$ is strongly projective over S (Theorem 1.1.1), so is $\operatorname{Quot}^P_{X/S}(F)$. $\qquad\square$

Consider the diagram

As $\operatorname{Quot}^P_{X/S}(F)$ represents the Quot functor, there is a *universal quotient* $p_1^* F \to \mathcal{Q}_{\mathrm{univ}}$ on $X \times_S \operatorname{Quot}^P_{X/S}(F)$. For $d \gg 0$, we also have the universal quotient $g^* \pi_* F(d) \to Q_{\mathrm{univ}}$ on $\operatorname{Gr}_S(P(d), \pi_* F(d))$ and a composition of closed immersions

$$\operatorname{Quot}^P_{X/S}(F) \hookrightarrow \operatorname{Gr}_S(P(d), \pi_* F(d)) \hookrightarrow \mathbb{P}(\textstyle\bigwedge^{P(d)}(\pi_* F(d)))$$

$$[F_T \twoheadrightarrow Q] \longmapsto [\pi_{T,*} F_T(d) \to \pi_{T,*} Q(d)] \longmapsto \big[\textstyle\bigwedge^{P(d)}(\pi_{T,*} F_T(d)) \to \bigwedge^{P(d)}(\pi_{T,*} Q(d))\big]$$

The pullback of $\mathcal{O}(1)$ on $\mathbb{P}(\bigwedge^{P(d)}(\pi_* F(d)))$ pulls back to the Plücker line bundle $\det(Q_{\mathrm{univ}})$ (Corollary 1.2.11) which in turn pulls back to $\det\big(p_{2,*}(\mathcal{Q}_{\mathrm{univ}}(d))\big)$ on $\operatorname{Quot}^P_{X/S}(F)$. We obtain:

Corollary 1.4.6. *For $d \gg 0$, the line bundle* $\det\big(p_{2,*}(\mathcal{Q}_{\mathrm{univ}}(d))\big)$ *is very ample on* $\operatorname{Quot}^P_{X/S}(F)$. $\qquad\square$

Exercise 1.4.7.

(a) Show that if S is a noetherian scheme and V is a *coherent* sheaf on S, then functor $\operatorname{Gr}_S(k, V)$ defined analogously to Theorem 1.1.1 is represented by a scheme projective (but not necessarily strongly projective) over S.

(b) Show that if $X \to S$ is a projective morphism of noetherian scheme and F is a coherent sheaf on X *flat* over S, then $\operatorname{Quot}^P_{X/S}(F) \to \operatorname{Gr}_S(P(d), \pi_* F(d))$ is well-defined for $d \gg 0$ and $\operatorname{Quot}^P_{X/S}(F)$ is projective over S.

1.4.4 Generalizations

If $\pi \colon X \to S$ is a strongly quasi-projective morphism of noetherian schemes (i.e. there is a locally closed immersion $X \hookrightarrow \mathbb{P}(V)$ where V is a vector bundle on S), $\mathcal{O}_X(1)$ is a relatively ample line bundle, F is a coherent sheaf on X which is a quotient of $\pi^*(W)(q)$ for a vector bundle W on S and integer q, and $P \in \mathbb{Q}[z]$ is a polynomial, we can modify the functors of Hilb and Quot as follows:

$$\mathrm{Hilb}^P_{X/S}\colon \mathrm{Sch}/S \to \mathrm{Sets}$$

$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{subschemes } Z \subset X_T \text{ flat, proper, and finitely} \\ \text{presented over } T \text{ such that } Z_t \subset X_t \\ \text{has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

$$\mathrm{Quot}^P_{X/S}(F)\colon \mathrm{Sch}/S \to \mathrm{Sets}$$

$$(T \to S) \mapsto \left\{ \begin{array}{l} \text{quasi-coherent quotients } F_T \to Q \text{ on } X_T \\ \text{of finite presentation which are flat and have} \\ \text{proper support over } T \text{ such that } Q|_{X_t} \text{ on } X_t \\ \text{has Hilbert polynomial } P \text{ for all } t \in T \end{array} \right\}$$

Then $\mathrm{Hilb}^P_{X/S}$ and $\mathrm{Quot}^P_{X/S}(F)$ are represented by strongly quasi-projective schemes over S; see [FGA$_{\mathrm{IV}}$, §4], [AK80] or [FGI$^+$05, §5.6]

If $X \to S$ is merely a separated morphism of noetherian schemes, then one can define functors $\mathrm{Hilb}(X/S)$ and $\mathrm{Quot}(F/X/S)$ as above dropping the condition on the Hilbert polynomial P. These functors are representable by algebraic spaces separated and locally of finite type over S; see [Art69b, Thm. 6.1][1] and [SP, Tag 09TQ]. Examples of Hironaka produce smooth proper (but not projective) 3-folds X over $\mathbb{C}$ such that $\mathrm{Hilb}^P_{X/S}$ is not a scheme.

There are further variants and generalizations:

- Vistoli's Hilbert stack parameterizing finite and unramified morphisms to a separated scheme X (or stack) [Vis91].

- Alexeev and Knutson's moduli of branch varieties parameterizing finite morphisms from a geometrically reduced proper scheme to a separated scheme X [AK10].

- If $X \to S$ is not separated, then Hall and Rydh show that there is an algebraic stack locally of finite type over S parameterizing quasi-finite morphism $Z \to X$ from a proper scheme [HR14].

Exercise 1.4.8 (Schemes of morphisms)**.** For projective morphisms $X \to S$ and $Y \to S$ of noetherian schemes, consider the functor

$$\underline{\mathrm{Mor}}_S(X,Y)\colon \mathrm{Sch}/S \to \mathrm{Sets}$$
$$(T \to S) \mapsto \mathrm{Mor}_T(X_T, Y_T)$$

assigning an S-scheme T to the set of T-morphisms $X_T \to Y_T$. By using a suitable Hilbert scheme $\mathrm{Hilb}^P_{X \times_S Y/X}$ parameterizing graphs $X \subset X \times_S Y$ of morphisms $X \to Y$, show that $\underline{\mathrm{Mor}}_S(X,Y)$ is representable by a projective scheme over S. Can we weaken the hypothesis on X and Y?

1.5 An invitation to the geometry of Hilbert schemes

1.5.1 First examples

In this section, we work over an algebraically closed field $\Bbbk$.

---

[1]As pointed out in [Art74, Appendix], the representability is not true without the separated hypothesis on $X \to S$.

The Hilbert polynomial $P(z) = \sum_{i=0}^{d} a_i z^i$ of a projective scheme $X \subset \mathbb{P}^n$ encodes invariants of X. For instance, $\dim X$ is the degree d of P and $\deg X$ is the normalized leading coefficient $d! a_d$. Applying Riemann–Roch in the case of a smooth curve $C \subset \mathbb{P}^n$ gives $P(z) = \deg(C)z + (1-g)$ and for a surface $S \subset \mathbb{P}^n$ gives $P(z) = \frac{1}{2}(zH \cdot (zH - K)) + (1 - p_a)$ where H is a hyperplane divisor, K is the canonical divisor and $p_a = 1 - \chi(\mathcal{O}_S)$ is the arithmetic genus. In arbitrary dimension, Hirzebruch–Riemann–Roch implies that $P(z) = \int_X \text{ch}(\mathcal{O}_X(z))\text{td}(X)$, where $\text{ch}(\mathcal{O}_X(z))$ is the Chern character and $\text{td}(X)$ the Todd class.

Example 1.5.1 (Hypersurfaces and linear subspaces). A hypersurface $H \subset \mathbb{P}^n$ of degree d has Hilbert polynomial

$$P(z) = \chi(\mathcal{O}_{\mathbb{P}^n}(z)) - \chi(\mathcal{O}_{\mathbb{P}^n}(z-d)) = \binom{n+z}{n} - \binom{n+z-d}{n}$$

(coming from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0$). We claim that $\text{Hilb}_{\mathbb{P}^n}^P \cong \mathbb{P}(\Gamma(\mathbb{P}^n, \mathcal{O}(d)))$. We encourage the reader to show this and in particular establish that every subscheme $Z \subset \mathbb{P}^n$ with Hilbert polynomial P is a hypersurface.

Similarly, a linear subspace $L \subset \mathbb{P}^n$ of dimension k has Hilbert polynomial $P(z) = \binom{z+k}{k}$ and $\text{Hilb}_{\mathbb{P}^n}^P = \text{Gr}(k+1, n+1)$.

Example 1.5.2 (Hilbert scheme of points on a curve). If C is a smooth projective curve, then the Hilbert scheme of n points Hilb_C^n (viewing n as the constant polynomial) is a smooth irreducible projective variety isomorphic to the symmetric product
$$\text{Sym}^n C := \underbrace{C \times \cdots \times C}_{n} / S_n,$$

where S_n acts by permuting the factors. The quotient exists as a projective variety since $C \times \cdots \times C$ is projective; see Exercise 4.2.9.

Example 1.5.3 (Hilbert scheme of points on a surface). T.S. Gustavsen, D. Laksov, R.M. Skjelnes

If S is a smooth irreducible projective surface, then the Hilbert scheme of n points Hilb_S^n is a smooth irreducible projective variety [Fog68]. See also [Nak99a] and [Mac07, §4]. There is a birational morphism
$$\text{Hilb}_S^n \to \text{Sym}^n(S) := \underbrace{S \times \cdots \times S}_{n} / S_n,$$

of projective varieties. The symmetric product $\text{Sym}^n(S)$ is not smooth for $n > 1$ and this provides a resolution of singularities. For an unordered collection of (possibly non-distinct) points $(p_1, \ldots, p_n) \in \text{Sym}^n(S)$, the fiber consists of all possible scheme structures on $\{p_1, \ldots, p_n\}$ of length n.

When $n = 1$, $\text{Hilb}_S^1 = S$. For $n = 2$ and the points p_1 and p_2 are equal, there is a $\mathbb{P}^1$ of scheme structures given by $k[x,y]/(x^2, xy, y^2, ay - bx)$ (with coordinates such that $p_1 = p_2 = 0$) parameterized by their "tangent direction" $[a : b] \in \mathbb{P}^1$. In this case, $\text{Hilb}_S^2 \to \text{Sym}^2(S)$ is the blowup of the diagonal $S \hookrightarrow \text{Sym}^2(S)$ given by $p \mapsto (p, p)$. In fact, for $n > 2$, the map $\text{Hilb}_S^n \to \text{Sym}^n(S)$ is a blowup along some ideal sheaf [Hai98] but the description of the ideal sheaf is more complicated.

When X is of arbitrary dimension, Hilb_X^n is smooth at (reduced) closed subschemes $Z \subset X$ consisting of n distinct smooth points of X. If X is reduced, there is an open subscheme of Hilb_X^n dimension $n \dim(X)$ parameterizing n distinct smooth points. Another result of Fogarty is that Hilb_X^n is connected as long as X is

connected [Fog68]. Moreover, for every projective scheme X, there is an irreducible component Hilb_X^n, called the "good component," that can be identified with the blowup of $\mathrm{Sym}^n(X)$ along some ideal sheaf [ES14].

Example 1.5.4 (Twisted cubics). The Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^3}^{3z+1}$ consists of the union of two smooth rational irreducible components H and H' of dimensions 12 and 15 intersecting transversely along a smooth rational subvariety of dimension 11 [PS85].

The locus H is the closure of the locus H_0 consisting of *twisted cubics*, i.e. rational smooth curves in $\mathbb{P}^3$ of degree 3. Each twisted cubic can be represented by a map $\mathbb{P}^1 \to \mathbb{P}^3$ given by the line bundle $\mathcal{O}_{\mathbb{P}^1}(3)$ and a choice of basis of $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$, and this representation is unique up to automorphisms of $\mathbb{P}^1$. All such curves are projectively equivalent, i.e. differ by an automorphism of $\mathbb{P}^3$, so we see that H_0 is identified with the homogeneous space $\mathrm{Aut}(\mathbb{P}^3)/\mathrm{Aut}(\mathbb{P}^1) = \mathrm{SL}_4 / \mathrm{SL}_2$, which is smooth and irreducible of dimension 12. The locus H_0 is not proper as it includes families such as $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $[x, y] \mapsto [x^3, x^2 y, xy^2, ty^3]$ parameterized by $t \in \mathbb{A}^1$ whose limit is a singular curve C_0 supported on a nodal cubic in $V(w) = \mathbb{P}^2$ (where w is the 4th coordinate) but with an embedded point at the node; see [Har77, Ex. 9.8.4].

The locus H' is the closure of the locus H_0' consisting of subschemes $C \amalg \{p\}$ where C is a smooth cubic curve contained in a hyperplane H and $p \in \mathbb{P}^3 \setminus C$. To count the dimension, observe that the choice of hyperplane $H \in \mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(1)))$ is given by 3 parameters, the choice of plane cubic $C \in \mathbb{P}(\mathrm{H}^0(H, \mathcal{O}_H(3)))$ is given by 9 parameters and the point $p \in \mathbb{P}^3 \setminus C$ is given by 3 parameters. The locus H_0' is smooth and irreducible of dimension 15. Again, the locus H_0' is not proper and its closure contains the limits of for instance degenerating the point p to lie on the curve whose limit can be curves like C_0.

The intersection $H \cap H'$ consists of plane singular cubic curves with an embedded point at the singular point. This locus contains curves such as C_0 above but it also contains even more degenerate curves such as a triple line with an embedded point. Every curve $C \in H \cap H'$ is in fact projectively equivalent to the curve defined by $V(xz, yz, z^2, q(x, y, w))$ where $q(x, y, w)$ is a homogeneous cubic polynomial with a singular point at $(0, 0, 1)$. This depends on 11 parameters.

1.5.2 Geometric properties

Non-emptiness. The Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^n}^P$ is non-empty if and only if the Hilbert polynomial P can be written as as

$$P(z) = \binom{z + \lambda_1 - 1}{\lambda_1 - 1} + \binom{z + \lambda_2 - 2}{\lambda_2 - 1} + \cdots + \binom{z + \lambda_r - r}{\lambda_r - 1}, \tag{1.5.5}$$

integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$. This is a result of Hartshorne [Har66b, Cor. 5.7] The necessity of this condition was already mentioned in Remark 1.3.14 in the context of Gotzmann's bounds on regularity.

Connectedness. Hartshorne's Connectedness Theorem asserts that the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^n}^P$ is connected for every Hilbert polynomial P [Har66b]. More generally, for every connected noetherian scheme S, $\mathrm{Hilb}_{\mathbb{P}_S^n/S}^P$ is connected.

The strategy of the argument is to show that every closed subscheme $Z \subset \mathbb{P}^n$ degenerates to a subscheme $V(I)$ defined by a monomial ideal. This reduces the question to the combinatorial question of connecting any two monomial ideals by a family over $\mathbb{A}^1$. This turns out to be a purely deformation and combinatorial

question or as Hartshorne writes: "It also appears that the Hilbert scheme is never actually needed in the proof."

See also [Mac07, §3].

Murphy's Law.

> Murphy's Law for Hilbert Schemes: There is no geometric possibility so horrible that it cannot be found generically on some component of the Hilbert scheme. [HM98, p.18]

The first pathology was exhibited by Mumford: there is an irreducible component of $\mathrm{Hilb}_{\mathbb{P}^3}^{14z-23}$ which is generically non-reduced [Mum62]. Ellia—Hirschowitz—Mezzetti show that the number of irreducible components in $\mathrm{Hilb}_{\mathbb{P}^3}^{az+b}$ is not bounded by a polynomial in a, b [EHM92].

Murphy's Law was made precise by Vakil [Vak06]: for every scheme X finite type over $\mathbb{Z}$ and point $x \in X$, there exists a point $q = [Z \subset \mathbb{P}^n] \in \mathrm{Hilb}_{\mathbb{P}^n}^P$ of some Hilbert scheme and an isomorphism

$$\widehat{\mathcal{O}}_{X,p}[\![x_1, \ldots, x_s]\!] \cong \widehat{\mathcal{O}}_{\mathrm{Hilb}_{\mathbb{P}^n}^P, q}[\![y_1, \ldots, y_t]\!]$$

for integers s, t. In other words, if we introduce the equivalence relation on pointed schemes (Z, z) generated by $(Z, z) \sim (Z', z')$ if there exists a smooth pointed morphism $(Z', z') \to (Z, z)$, then (X, p) is equivalent to $(\mathrm{Hilb}_{\mathbb{P}^n}^P, q)$.

In fact, it can be arranged that the Hilbert scheme parameterizes smooth curves in $\mathbb{P}^n$ for some n, or that it parameterizes smooth surfaces in $\mathbb{P}^5$ (resp. surfaces in $\mathbb{P}^4$). It turns out that various other moduli spaces also satisfy Murphy's Law: Kontsevich's moduli space of maps, moduli of canonically polarized smooth surfaces, moduli of curves with linear systems, and the moduli space of stable sheaves.

Smoothness. Despite Murphy's Law, many Hilbert schemes are in fact smooth. We've seen before that the Hilbert scheme of points on a smooth surface is smooth. Moreover, it is not hard to see that the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^n/\Bbbk}^P$ of projective space over a field is smooth at every complete intersection $Z \subset \mathbb{P}^n$ (despite the obstruction space $\mathrm{H}^1(Z, N_{Z/X})$ being potentially nonzero).

A theorem of Skjelnes–Smith [SS20] states that the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^n}^P$ is smooth if and only if $P(z)$ can be written as (1.5.5) for a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$ such that one of the seven condition holds:

(1) $n = 2$;

(2) $\lambda_r \geq 2$;

(3) $\lambda = (1)$ or $\lambda = (n^{r-2}, \lambda_{r-1}, 1) = (\underbrace{n, \ldots, n}_{r-2}, \lambda_{r-1}, 1)$ where $r \geq 2$ and $n \geq \lambda_{r-1} \geq 1$;

(4) $\lambda = (n^{r-s-3}, \lambda_{r-s-2}^{s+2}, 1)$ where $r - s \geq s \geq 0$ and $n - 1 \geq \lambda_{r-s-2} \geq 3$;

(5) $\lambda = (n^{r-s-5}, 2^{s+4}, 1)$ where $r - 5 \geq s \geq 0$;

(6) $\lambda = (n^{r-3}, 1^s)$ where $r \geq 3$;

(7) $\lambda = (n + 1)$ or $r = 0$.

Local properties.

Exercise 1.5.6. Let X be a projective scheme over a field $\Bbbk$ and F be a coherent sheaf on X.

(a) Let $p \in \mathrm{Quot}^P_{X/\Bbbk}(F)$ be the point corresponding to a quotient $Q = F/K$. Show that $T_p \mathrm{Quot}^P_{X/\Bbbk}(F) \cong \mathrm{Hom}_{\mathcal{O}_X}(K, Q)$. This generalizes the exercise computing the tangent space of the Grassmannian (Exercise 1.2.8).

(b) Conclude that if $p \in \mathrm{Hilb}^P_{X/\Bbbk}$ is a point corresponding to a closed subscheme $Z \subset X$ defined by a sheaf of ideals I, then $T_p \mathrm{Hilb}^P_{X/\Bbbk} \cong \mathrm{H}^0(Z, N_{Z/X})$ where $N_{Z/X}$ is the normal sheaf $\mathrm{Hom}_{\mathcal{O}_Z}(I_Z/I_Z^2, \mathcal{O}_Z)$.

Chapter 2

Sites, sheaves, and stacks

2.1 Grothendieck topologies and sites

We would like to form a topology on a scheme where étale morphisms replace Zariski open subsets. This doesn't quite make sense using the conventional notion of a topological space so we instead adapt our definitions. Grothendieck topologies and stacks were introduced in [SGA4]. Our exposition closely follows [Art62], [FGI+05, Part 1], [Ols16, §2], and [SP, Tag 00UZ].

2.1.1 Definitions and examples

Definition 2.1.1 (Sites)**.** A *Grothendieck topology* on a category $\mathcal{S}$ consists of the following data: for each object $X \in \mathcal{S}$, there is a set $\mathrm{Cov}(X)$ consisting of *coverings* of X, i.e. collections of morphisms $\{X_i \to X\}_{i \in I}$ in $\mathcal{S}$. We require that:

(1) (identity) If $X' \to X$ is an isomorphism, then $(X' \to X) \in \mathrm{Cov}(X)$.

(2) (restriction) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is a morphism, then the fiber products $X_i \times_X Y$ exist in $\mathcal{S}$ and the collection $\{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(Y)$.

(3) (composition) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then $\{X_{ij} \to X_i \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(X)$.

A *site* is a category $\mathcal{S}$ with a Grothendieck topology.

Caution 2.1.2. The definition requires that $\mathrm{Cov}(X)$ is a *set*, but this is not true in many cases of interest such as the big étale site defined below. We however ignore set-theoretic issues in this book—they can be addressed by working with a suitable containing all morphisms of interest and which defines the same category of sheaves; see [SP, Tag 00VI].

Example 2.1.3 (Topological spaces)**.** If X is a topological space, let $\mathrm{Op}(X)$ denote the category of open sets $U \subset X$. There is a unique morphism $U \to V$ if and only if $U \subset V$. We say that a covering of U (i.e. an element of $\mathrm{Cov}(U)$) is a collection of open immersions $\{U_i \to U\}_{i \in I}$ such that $U = \bigcup_{i \in I} U_i$. This defines a Grothendieck topology on $\mathrm{Op}(X)$. In particular, if X is a scheme, the Zariski topology on X defines a site X_{Zar}, called the *small Zariski site on X*.

By replacing Zariski open immersions with étale morphisms, we obtain the small étale site.

Example 2.1.4 (Small étale site). If X is a scheme, the *small étale site on X* is the category $X_{\text{ét}}$ of étale morphisms $U \to X$ such that a morphism $(U \to X) \to (V \to X)$ is simply an X-morphism $U \to V$ (which is necessarily étale). In other words, $X_{\text{ét}}$ is the full subcategory of $\text{Sch}\,/X$ consisting of schemes étale over X. A covering of an object $(U \to X) \in X_{\text{ét}}$ is a collection of étale morphisms $\{U_i \to U\}$ such that $\coprod_i U_i \to U$ is surjective. Later we will introduce the small étale site $\mathcal{X}_{\text{ét}}$ of an algebraic space or Deligne–Mumford stack (Definition 4.1.1), which we use to define sheaves on $\mathcal{X}$.

Big sites. The big étale site $\text{Sch}_{\text{ét}}$ is the most frequently used site in these notes. It is used to define the most central notions in this book: an algebraic space is a sheaf on $\text{Sch}_{\text{ét}}$ that is étale locally a scheme (Definition 3.1.2) while an algebraic stack is a stack over $\text{Sch}_{\text{ét}}$ that is smooth-locally a scheme (Definition 3.1.6).

Example 2.1.5 (Big étale site). The *big étale site* is the category Sch where a covering of a scheme U is a collection of étale morphisms $\{U_i \to U\}$ in Sch such that $\coprod_i U_i \to U$ is surjective. We denote this site as $\text{Sch}_{\text{ét}}$.

There are various analogous constructions.

Example 2.1.6 (Big topological site). Let Top be the category of topological spaces. A covering of $U \in \text{Top}$ is a collection of open subspaces $\{U_i \hookrightarrow U\}_{i \in I}$ such that $U = \bigcup_{i \in I} U_i$.

Example 2.1.7 (Big Zariski site). Replacing étale morphisms in Example 2.1.5 with open immersions defines the *big Zariski site* Sch_{Zar}.

Unlike the big Zariski site, the big fppf site Sch_{fppf} is frequently used in moduli theory. In [SP], algebraic spaces and stacks are defined using Sch_{fppf} rather than $\text{Sch}_{\text{ét}}$, but these two sites nevertheless define equivalent notions [SP, Tag 076U]. In this text, the big fppf site is not essential and is used only in our discussion of gerbes in §6.2.

Example 2.1.8 (Big fppf site). A morphism of schemes is *fppf* if it is surjective, flat, and locally of finite presentation (see Definition A.2.20). The *big fppf site* Sch_{fppf} is the category Sch of schemes where a covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is fppf, i.e. each $U_i \to U$ is flat and locally of finite presentation, and $\coprod_i U_i \to U$ is surjective.

There are serious set-theoretic issues in defining an fpqc site, which arise from the presence of too many fpqc covers—given any nonzero ring R, there does not exist a set of fpqc coverings of $\text{Spec}\,R$ which can refine every fpqc covering. See [SP, Tag 0BBK] and the discussion in [SP, Tag 03NV]. If one defines the big fpqc site ignoring set-theoretic issues as in example below, there are presheaves that don't have a sheafification [Wat75, Thm. 5.5]. Fortunately, we have no need for fpqc sites in this text. We do occasionally use the notion of an fpqc sheaf or stack (e.g. Proposition 2.2.9); this notion is well-defined, but even this use of the fpqc topology is not essential for us as well we only need the fppf (or even étale) version.

Example 2.1.9 (Big fpqc site). A morphism of schemes is *fpqc* if it is surjective, flat, and every quasi-compact subset of the target is the image of a quasi-compact subset (see Definition A.2.20). The *big fpqc site* Sch_{fppf} is the category Sch where a covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is fppf.

Additional sites. We will later introduce the lisse-étale site of an algebraic stack (Definition 6.1.1) as a means to defining sheaves.

Example 2.1.10 (Lisse-étale site). On a scheme X, the *lisse-étale site* $X_{\text{lis-ét}}$ is the category of schemes smooth over X where morphisms in $X_{\text{lis-ét}}$ are (not necessarily smooth) morphisms of schemes over X. A covering $\{U_i \to U\}$ of an X-scheme U is a collection of X-morphisms such that $\coprod_i U_i \to U$ is surjective and étale.

Example 2.1.11 (Localized categories and sites). If $\mathcal{S}$ is a category and $S \in \mathcal{S}$, define the category $\mathcal{S}/S$ whose objects are maps $T \to S$ in $\mathcal{S}$. A morphism $(T' \to S) \to (T \to S)$ is a map $T' \to T$ over S. If $\mathcal{S}$ is a site, $\mathcal{S}/S$ is also a site where a covering of $T \to S$ in $\mathcal{S}/S$ is a covering $\{T_i \to T\}$ in $\mathcal{S}$. Applying this construction to a scheme S yields the big Zariski, étale, fppf, and fpqc sites $(\text{Sch}/S)_{\text{Zar}}$, $(\text{Sch}/S)_{\text{ét}}$, $(\text{Sch}/S)_{\text{fppf}}$, and $(\text{Sch}/S)_{\text{fpqc}}$.

Example 2.1.12 (Grothendieck topologies on the category of affine schemes). In the literature, authors sometimes use the big sites $\text{AffSch}_{\text{Zar}}$, $\text{AffSch}_{\text{ét}}$, $\text{AffSch}_{\text{fppf}}$ and $\text{AffSch}_{\text{lis-ét}}$ on the category of affine schemes. These define the same categories of sheaves as the corresponding big sites on Sch.

2.2 Presheaves and sheaves

Recall that if X is a topological space, a presheaf of sets on X is simply a contravariant functor $F\colon \text{Op}(X) \to \text{Sets}$ on the category $\text{Op}(X)$ of open sets. The sheaf axiom translates succinctly into the condition that for each covering $U = \bigcup_i U_i$, the sequence

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is exact (i.e. is an equalizer diagram), where the two maps $F(U_i) \rightrightarrows F(U_i \cap U_j)$ are induced by the two inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$. Also note that the intersections $U_i \cap U_j$ can also be viewed as fiber products $U_i \times_X U_j$.

2.2.1 Definitions

Definition 2.2.1 (Presheaves). A *presheaf* on a category $\mathcal{S}$ is a contravariant functor $\mathcal{S} \to \text{Sets}$.

Remark 2.2.2. If $F\colon \mathcal{S} \to \text{Sets}$ is a presheaf and $f\colon S \to T$ is a map in $\mathcal{S}$, then the pullback $F(f)(b)$ of an element $b \in F(T)$ is sometimes denoted as f^*b or $b|_S$.

Definition 2.2.3 (Sheaves). A *sheaf* on a site $\mathcal{S}$ is a presheaf $F\colon \mathcal{S} \to \text{Sets}$ such that for every object $S \in \mathcal{S}$ and covering $\{S_i \to S\} \in \text{Cov}(S)$, the sequence

$$F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j) \tag{2.2.4}$$

is exact, where the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are induced by the two maps $S_i \times_S S_j \to S_i$ and $S_i \times_S S_j \to S_i$.

Remark 2.2.5. The exactness of (2.2.4) means that it is an equalizer diagram: $F(S)$ is precisely the subset of $\prod_i F(S_i)$ consisting of elements whose images under the two maps $F(S_i) \rightrightarrows F(S_i \times_S S_j)$ are equal.

Exercise 2.2.6. Let F be a presheaf on Sch. Show that the following are equivalent:

   (1) F is a sheaf on $\text{Sch}_{\text{ét}}$ (resp. Sch_{fppf}, Sch_{fpqc}),

(2) F sends coproducts to products (i.e. $F(\coprod_i U_i) = \prod_i F(U_i)$ for schemes U_i) and for every surjective étale (resp. fppf, faithfully flat) morphism $S' \to S$ of schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

(3) F is a sheaf in the big Zariski topology $\mathrm{Sch}_{\mathrm{Zar}}$ and for every surjective étale (resp. fppf, faithfully flat) morphism $S' \to S$ of *affine* schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

Hint: Given a covering $\{S_i \to S\}$, consider the map $\coprod_i S_i \to S$. You will likely need to apply that étale and fppf morphisms are open (Proposition A.2.14), or the defining property of an fpqc cover (Definition A.2.20).

Exercise 2.2.7. If F and G are sheaves on a site $\mathcal{S}$, show that the presheaf $\mathrm{Mor}_{\mathcal{S}}(F, G)$, defined by $S \mapsto \mathrm{Mor}_{\mathrm{Sets}}(F(S), G(S))$, is a sheaf on $\mathcal{S}$.

Exercise 2.2.8 (Gluing sheaves). Let $\mathcal{S}$ be a site and $(X_i \to X)$ be a covering in $\mathcal{S}$. If F_i are sheaves on the restricted sites $\mathcal{S}/X_i$ and $\alpha_{ij} \colon F_i|_{X_{ij}} \to F_j|_{X_{ij}}$ are isomorphisms of sheaves on $\mathcal{S}/X_{ij}$ (where $X_{ij} := X_i \times_X X_j$) satisfying the cocycle condition $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on $\mathcal{S}/X_{ijk}$ (where $X_{ijk} = X_i \times_X X_j \times_X X_k$), then there exists a unique sheaf F on $\mathcal{S}$ and isomorphisms $\phi_i \colon F|_{X_i} \to F_i$ satisfying $\phi_i|_{X_{ij}} = \phi_j|_{X_{ij}} \circ \alpha_{ij}$.

Proposition 2.2.9 (Schemes are Sheaves in the Fpqc Topology). *If $X \to S$ is a morphism of schemes, then $\mathrm{Mor}_S(-, X) \colon \mathrm{Sch}/S \to \mathrm{Sets}$ is a sheaf on $(\mathrm{Sch}/S)_{\mathrm{fpqc}}$ and therefore also a sheaf on $(\mathrm{Sch}/S)_{\text{ét}}$ and $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.*

Proof. As $\mathrm{Mor}_S(-, X)$ is a sheaf in the big Zariski topology, it suffices by Exercise 2.2.6 to show that if $T' \to T$ is a faithfully flat morphism of affine schemes over S, then the sequence

$$\mathrm{Mor}_S(T, X) \to \mathrm{Mor}_S(T', X) \rightrightarrows \mathrm{Mor}_S(T' \times_T T', X)$$

is exact. This follows from fpqc descent for morphisms of schemes (Proposition B.1.7). $\square$

2.2.2 Morphisms and fiber products

A *morphism* of presheaves or sheaves is by definition a natural transformation. By Yoneda's lemma (Lemma 0.3.7), if X is a scheme and F is a presheaf on Sch, a morphism $\alpha \colon X \to F$ (which we interpret as a morphism of presheaves $\mathrm{Mor}(-, X) \to F$) corresponds to an element in $F(X)$, which by abuse of notation we also denote by α.

Exercise 2.2.10. Recall from Proposition 2.2.9 that a scheme can be viewed as a sheaf in the big fpqc topology.

(a) Show that a surjective étale (resp. fppf) morphism of schemes is an epimorphism of sheaves on $\mathrm{Sch}_{\text{ét}}$ (resp. $\mathrm{Sch}_{\mathrm{fppf}}$).

(b) Show that a surjective smooth morphism of schemes is an epimorphism sheaves on $\mathrm{Sch}_{\text{ét}}$.

Given morphisms $F \xrightarrow{\alpha} G$ and $G' \xrightarrow{\beta} G$ of presheaves on a category $\mathcal{S}$, define the presheaf $F \times_G G'$ by $(F \times_G G')(S) = F(S) \times_{G(S)} G'(S)$, i.e.

$$F \times_G G' \colon \mathcal{S} \to \mathrm{Sets}$$
$$S \mapsto \{(a, b) \in F(S) \times G'(S) \mid \alpha_S(a) = \beta_S(b)\}. \tag{2.2.11}$$

Exercise 2.2.12.

(a) Show that (2.2.11) is a fiber product $F \times_G G'$ in $\mathrm{Pre}(\mathcal{S})$. (This is a generalization of Exercise 0.3.30 but the same proof should work.)

(b) Show that if F, G and G' are sheaves on a site $\mathcal{S}$, then so is $F \times_G G'$. In particular, (2.2.11) is also a fiber product $F \times_G G'$ in $\mathrm{Sh}(\mathcal{S})$.

2.2.3 Sheafification

Theorem 2.2.13 (Sheafification)**.** *Let $\mathcal{S}$ be a site. The forgetful functor $\mathrm{Sh}(\mathcal{S}) \to \mathrm{Pre}(\mathcal{S})$ admits a left adjoint $F \mapsto F^{\mathrm{sh}}$, called the sheafification.*

Proof. A presheaf F on $\mathcal{S}$ is called *separated* if for every covering $\{S_i \to S\}$ of an object S, the map $F(S) \to \prod_i F(S_i)$ is injective (i.e. if sections glue, they glue uniquely). Let $\mathrm{Pre}(\mathcal{S})$ and $\mathrm{Sh}(\mathcal{S})$ be the categories of presheaves and sheaves, and let $\mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \subset \mathrm{Pre}(\mathcal{S})$ be the full subcategory of separated presheaves. We will construct left adjoints

$$\mathrm{Sh}(\mathcal{S}) \overset{\mathrm{sh}_2}{\underset{\hookrightarrow}{\longleftarrow}} \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S}) \overset{\mathrm{sh}_1}{\underset{\hookrightarrow}{\longleftarrow}} \mathrm{Pre}(\mathcal{S}).$$

For $F \in \mathrm{Pre}(\mathcal{S})$, we define $\mathrm{sh}_1(F)$ by $S \mapsto F(S)/\sim$ where $a \sim b$ if there exists a covering $\{S_i \to S\}$ such that $a|_{S_i} = b|_{S_i}$ for all i.

For $F \in \mathrm{Pre}^{\mathrm{sep}}(\mathcal{S})$, we define $\mathrm{sh}_2(F)$ by

$$S \mapsto \left\{ (\{S_i \to S\}, \{a_i\}) \,\middle|\, \begin{array}{l} \text{where } \{S_i \to S\} \in \mathrm{Cov}(S) \text{ and } a_i \in F(S_i) \\ \text{such that } a_i|_{S_{ij}} = a_j|_{S_{ij}} \text{ for all } i, j \end{array} \right\} / \sim$$

where $(\{S_i \to S\}, \{a_i\}) \sim (\{S_j' \to S\}, \{a_j'\})$ if $a_i|_{S_i \times_S S_j'} = a_j'|_{S_i \times_S S_j'}$ for all i, j. The details are left to the reader. $\square$

Remark 2.2.14 (Topos)**.** A *topos* is a category equivalent to the category of sheaves on a site. Two different sites may have equivalent categories of sheaves, and the topos is a more fundamental invariant. While topoi are undoubtedly important in moduli theory, they will not play a role in these notes.

2.2.4 Effective descent for sheaves

Proposition 2.2.15 (Effective Descent)**.** *Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: open immersion, closed immersion, locally closed immersion, affine, quasi-affine, or separated and locally quasi-finite. Let $X \to Y$ be a surjective smooth (resp. fppf, fpqc) morphism of schemes. Let F be a sheaf on $(\mathrm{Sch}/Y)_{\text{ét}}$ (resp. $(\mathrm{Sch}/Y)_{\mathrm{fppf}}$). Consider the fiber product*

$$
\begin{array}{ccc}
F_X & \longrightarrow & X \\
\downarrow & \square & \downarrow \\
F & \longrightarrow & Y
\end{array}
$$

of sheaves. If F_X is a scheme and $F_X \to X$ has $\mathcal{P}$, then F is a scheme and $F \to Y$ has $\mathcal{P}$.

Proof. As F_X is the pullback of F, there is a canonical isomorphism $\alpha\colon p_1^* F_X \to p_2^* F_X$ on $X \times_Y X$ satisfying the cocycle condition $p_{23}^* \alpha \circ p_{12}^* \alpha = p_{13}^* \alpha$. By Proposition B.1.8, there exists a morphism of schemes $W \to Y$ satisfying $\mathcal{P}$ that pulls back to $F_X \to X$. The sheaf F is identified with the sheafification of the presheaf on $\mathrm{Sch}\,/Y$ defined by

$$F^{\mathrm{pre}}\colon (T \to Y) \mapsto \mathrm{Eq}\left(F_X(T \to Y) \rightrightarrows F_X \times_X F_X(T \to Y)\right).$$

By Schemes are Sheaves in the Fpqc Topology (2.2.9), W is a sheaf and there is a morphism $F^{\mathrm{pre}} \to W$ of presheaves. By the universal property of sheafification, there is a morphism $\alpha\colon F \to W$ of sheaves which pulls back under $X \to Y$ to an isomorphism $\alpha_X\colon F_X \to W \times_Y X$. It is not hard to see that this implies that α is an isomorphism. For instance, since $\mathrm{Hom}(W, F)$ is a sheaf (Exercise 2.2.7) and the inverse β_X of α_X defines a section over $X \to Y$ whose two pullbacks to $X \times_Y X$ agree, the inverse β_X descends to a section β of $\mathrm{Hom}(W, F)$ over $\mathrm{id}\colon Y \to Y$. Since $\mathrm{Hom}(F, F)$ and $\mathrm{Hom}(W, W)$ are sheaves, it follows that $\beta \circ \alpha = \mathrm{id}_F$ and $\alpha \circ \beta = \mathrm{id}_W$. See also [SP, Tag 02W5]. $\square$

2.3 Prestacks

In §0.6.1, we motivated the concept of a prestack on a category $\mathcal{S}$ as a generalization of a presheaf $\mathcal{S} \to \mathrm{Sets}$. By trying to keep track of automorphisms, we were naively led to consider a 'functor'

$$F\colon \mathcal{S} \to \mathrm{Groupoids}\,.$$

While this is a good way to think about prestacks, we decided instead to package this data into one large category $\mathcal{X}$ over $\mathcal{S}$ parameterizing pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$.

2.3.1 Definition of a prestack

Let $\mathcal{S}$ be a category and $p\colon \mathcal{X} \to \mathcal{S}$ be a functor of categories. We visualize this data as

$$
\begin{array}{ccc}
\mathcal{X} & & a \xrightarrow{\;\alpha\;} b \\
\downarrow{\scriptstyle p} & & \downarrow \qquad \downarrow \\
\mathcal{S} & & S \xrightarrow{\;f\;} T
\end{array}
$$

where the lower case letters a, b are objects of $\mathcal{X}$ and the upper case letters S, T are objects of $\mathcal{S}$. We say that a is over S and $\alpha\colon a \to b$ is over $f\colon S \to T$.

Definition 2.3.1 (Prestacks)**.** A functor $p\colon \mathcal{X} \to \mathcal{S}$ is a *prestack over a category* $\mathcal{S}$ if

(1) (pullbacks exist) for every diagram

$$
\begin{array}{ccc}
a & \dashrightarrow & b \\
\uparrow & & \uparrow \\
\downarrow & & \downarrow \\
S & \longrightarrow & T
\end{array}
$$

of solid arrows, there exists a morphism $a \to b$ over $S \to T$; and

(2) (universal property for pullbacks) for every diagram

$$
\begin{array}{ccc}
a \dashrightarrow b \longrightarrow c \\
\downarrow \qquad \downarrow \qquad \downarrow \\
R \longrightarrow S \longrightarrow T
\end{array}
$$

of solid arrows, there exists a unique arrow $a \to b$ over $R \to S$ filling in the diagram.

Caution 2.3.2. When defining and discussing prestacks, we often write $\mathcal{X}$ instead of $\mathcal{X} \to \mathcal{S}$, but when necessary, we denote the projection by $p_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{S}$. We do not usually spell out the definition of the functor $\mathcal{X} \to \mathcal{S}$ as it should be clear to the reader. Moreover, when defining a prestack $\mathcal{X}$, we often only define the objects and morphisms in $\mathcal{X}$ and leave the composition law to the reader.

Remark 2.3.3. Axiom (2) above implies that the pullback in Axiom (1) is unique up to unique isomorphism. We often write $f^* b$ or simply $b|_S$ to indicate a *choice* of a pullback.

Definition 2.3.4 (Fiber categories). If $\mathcal{X}$ is a prestack over $\mathcal{S}$, the *fiber category* $\mathcal{X}(S)$ over $S \in \mathcal{S}$ is the category of objects in $\mathcal{X}$ over S with morphisms over id_S.

Exercise 2.3.5. Show that the fiber category $\mathcal{X}(S)$ is a groupoid.

Caution 2.3.6. Our terminology is not standard. Prestacks are usually referred to as *categories fibered in groupoids*. In the literature (c.f. [FGI+05, Part 1], [Ols16]) a prestack is sometimes defined as a category fibered in groupoids together with Axiom 2.4.1(1) of a stack.

It is also standard to call a morphism $b \to c$ in $\mathcal{X}$ *cartesian* if it satisfies the universal property in Axiom 2.4.1(2) and $p \colon \mathcal{X} \to S$ a *fibered category* if for every diagram as in Axiom 2.4.1(1), there exists a cartesian morphism $a \to b$ over $S \to T$. With this terminology, a prestack (as we've defined it) is a fibered category where every arrow is cartesian, or equivalently where every fiber category $\mathcal{X}(S)$ is a groupoid.

2.3.2 Examples

Example 2.3.7 (Presheaves are prestacks). If $F \colon \mathcal{S} \to \mathrm{Sets}$ is a presheaf, we can construct a prestack $\mathcal{X}_F$ as the category of pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$. A morphism $(a', S') \to (a, S)$ in $\mathcal{X}_F$ is a map $f \colon S' \to S$ such that $a' = f^* a$, where f^* is convenient shorthand for $F(f) \colon F(S) \to F(S')$. The projection $\mathcal{X}_F \to \mathcal{S}$ is defined by $(a, S) \mapsto S$. Observe that the fiber categories $\mathcal{X}_F(S)$ are equivalent (even equal) to the set $F(S)$. We will often abuse notation by conflating F and $\mathcal{X}_F$.

Example 2.3.8 (Schemes are prestacks). For a scheme X, applying the previous example to the functor $\mathrm{Mor}(-, X) \colon \mathrm{Sch} \to \mathrm{Sets}$ yields a prestack $\mathcal{X}_X$. This allows us to view a scheme X as the prestack $\mathcal{X}_X$ where an object over a scheme T is a morphism $T \to X$ of schemes. We often abuse notation by referring to $\mathcal{X}_X$ as X.

Example 2.3.9 (Prestack of smooth curves). We define the prestack $\mathcal{M}$ over Sch as the category of families of smooth curves $\mathcal{C} \to S$, i.e. smooth and proper morphisms $\mathcal{C} \to S$ of schemes such that every geometric fiber is a connected curve. A map

$(\mathcal{C}' \to S') \to (\mathcal{C} \to S)$ is the data of maps $\alpha \colon \mathcal{C}' \to \mathcal{C}$ and $f \colon S' \to S$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\ \alpha\ } & \mathcal{C} \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow{\ f\ } & S \end{array}$$

is cartesian. The projection $\mathcal{M} \to \mathrm{Sch}$ is given by $(\mathcal{C} \to S) \mapsto S$. The prestack $\mathcal{M}_g$ is defined as the full subcategory of $\mathcal{M}$ consisting of families of smooth curves $\mathcal{C} \to S$ where every geometric fiber has genus g. Note that the fiber category $\mathcal{M}_g(\Bbbk)$ over a field $\Bbbk$ is the groupoid of smooth, connected, and projective curves C over $\Bbbk$ of genus g such that $\mathrm{Mor}_{\mathcal{M}_g(\Bbbk)}(C, C') = \mathrm{Isom}_{\mathrm{Sch}/\Bbbk}(C, C')$.

Exercise 2.3.10. Verify that $\mathcal{M}$ and $\mathcal{M}_g$ are prestacks.

Example 2.3.11 (Prestack of coherent sheaves and vector bundles)**.** Let X be a scheme over a field $\Bbbk$. We define the prestack $\underline{\mathrm{QCoh}}(X)$ over $\mathrm{Sch}/\Bbbk$ as the category of pairs (E, S) where S is a scheme over $\Bbbk$ and E is a quasi-coherent sheaf on $X_S = X \times_\Bbbk S$ flat over S. A morphism $(E', S') \to (E, S)$ consists of a map of schemes $f \colon S' \to S$ together with an isomorphism $f^* E \to E'$ of $\mathcal{O}_{X_{S'}}$-modules.[1] The projection $\underline{\mathrm{QCoh}}(X) \to \mathrm{Sch}/\Bbbk$ is defined by $(E, S) \mapsto S$.

The substacks $\underline{\mathrm{Coh}}(X) \subset \underline{\mathrm{QCoh}}(X)$ is the full subcategory consisting of pairs (E, S) where E is a finitely presented, quasi-coherent sheaf on X_S (or equivalently a coherent sheaf when X_S is noetherian). Similarly, $\underline{\mathrm{Bun}}(X) \subset \underline{\mathrm{Coh}}(X)$ is the full subcategory where E is a vector bundle on X_S (i.e. locally free quasi-coherent sheaf of finite rank).

Exercise 2.3.12. Verify that $\underline{\mathrm{QCoh}}(X)$, $\underline{\mathrm{Coh}}(X)$, and $\underline{\mathrm{Bun}}(X)$ are prestacks.

Classifying stacks and quotient stacks. Classifying and quotient stacks were motivated in §0.6.5. Their definitions involve the notion of a principal G-bundle. For a smooth affine group scheme $G \to S$,[2] a *principal G-bundle over an S-scheme* T is a morphism $P \to T$ with an action of G on P via $\sigma \colon G \times_S P \to P$ such that $P \to T$ is a G-invariant fppf morphism and

$$(\sigma, p_2) \colon G \times_S P \to P \times_T P, \qquad (g, p) \mapsto (gp, p)$$

is an isomorphism (see Definition B.2.16); in other words G acts freely and transitively on P with quotient T. Equivalently, $P \to T$ is a principal G-bundle if there is étale cover $T' \to T$ such that $P \times_T T'$ is G-equivariantly isomorphic to the trivial principal G-bundle $G \times_S T'$ (Proposition B.2.19). See §B.2.3 for further background on principal bundles including many examples.

---

[1] This definition of a morphism is not completely precise as the pullback $f^* E$ is not canonical. Recall that $f^* E$ is defined as $f^{-1} E \otimes_{f^{-1} \mathcal{O}_S} \mathcal{O}_{S'}$ and while it exists and is unique up to unique isomorphism, a choice of pullback $f^* E$ involves a choice of a limit in the definition of $f^{-1} E$, choices of tensor products, and a choice of sheafification. Instead, one can define a morphism $(E', S') \to (E, S)$ as an equivalence class of triples (f, F', α) where $f \colon S' \to S$ is a map of schemes, F' is a *choice* of a pullback of E, and an isomorphism $\alpha \colon F' \to E'$ where $(f, F', \alpha) \sim (g, G', \beta)$ if $f = g$ and the canonical isomorphism $\gamma \colon F' \xrightarrow{\sim} G'$ satisfies $\alpha = \beta \circ \gamma$. Alternatively, since the pushforward $f_* E'$ is canonical, a morphism $(E', S') \to (E, S)$ can be defined as a map $f \colon S' \to S$ and a morphism $E \to f_* E'$ of $\mathcal{O}_{X_S}$-modules whose adjoint is an isomorphism.

[2] In §6.2.2, we will define principal G-bundles, classifying stacks, and quotient stacks more generally for fppf group schemes.

Definition 2.3.13 (Classifying stacks)**.** The *classifying stack* BG of a smooth affine group scheme $G \to S$ is the category over Sch$/S$ whose objects are principal G-bundles $P \to T$ and a morphism $(P' \to T') \to (P \to T)$ is the data of a G-equivariant morphism $P' \to P$ such that

$$
\begin{array}{ccc}
P' & \longrightarrow & P \\
\downarrow & \square & \downarrow \\
T' & \longrightarrow & T
\end{array}
$$

is cartesian.

Definition 2.3.14 (Quotient prestacks and stacks)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. The *quotient prestack* $[U/G]^{\mathrm{pre}}$ of an action of a smooth affine group scheme $G \to S$ on an S-scheme U is the category over Sch$/S$ consisting of pairs (T, u) where T is an S-scheme and $u \in U(T)$. A morphism $(T', u') \to (T, u)$ is the data of a map $f \colon T' \to T$ of S-schemes and an element $g \in G(T')$ such that $f^* u = g \cdot u'$. Note that the fiber category $[U/G]^{\mathrm{pre}}(T)$ is identified with the quotient groupoid $[U(T)/G(T)]$ from Example 0.4.6.

The *quotient stack* $[U/G]$ is the category over Sch$/S$ consisting of diagrams

$$
\begin{array}{ccc}
P & \longrightarrow & U \\
\downarrow & & \\
T & &
\end{array}
$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of S-schemes. A morphism $(P' \to T', P' \to U) \to (P \to T, P \to U)$ consists of a morphism $T' \to T$ and a G-equivariant morphism $P' \to P$ of schemes such that the diagram

$$
\begin{array}{ccc}
P' \longrightarrow P \longrightarrow U \\
\downarrow \quad \square \quad \downarrow \\
T' \longrightarrow T
\end{array}
$$

is commutative and the left square is cartesian.

Exercise 2.3.15. Verify that $[U/G]^{\mathrm{pre}}$ and $[U/G]$ are prestacks over Sch$/S$.

We show shortly that $[U/G]$ and $BG = [S/G]$ are stacks over $(\mathrm{Sch}/S)_{\text{ét}}$ (Proposition 2.4.12), which justifies our terminology of a 'quotient stack' and 'classifying stack'. We also show that $[U/G]$ is identified as the stackification of $[U/G]^{\mathrm{pre}}$ (Exercise 2.4.20), and later show that $[U/G]$ is algebraic (Theorem 3.1.10).

2.3.3 Morphisms of prestacks

Definition 2.3.16.

(1) A *morphism of prestacks* $f \colon \mathcal{X} \to \mathcal{Y}$ is a functor $f \colon \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\quad f \quad} & \mathcal{Y} \\
& {}_{p_{\mathcal{X}}}\searrow \quad \swarrow{}_{p_{\mathcal{Y}}} & \\
& \mathcal{S} &
\end{array}
$$

strictly commutes, i.e. for every object $a \in \mathrm{Ob}(\mathcal{X})$, there is an *equality* $p_{\mathcal{X}}(a) = p_{\mathcal{Y}}(f(a))$ of objects in $\mathcal{S}$.

(2) If $f, g\colon \mathcal{X} \to \mathcal{Y}$ are morphisms of prestacks, a *2-isomorphism* (or *2-morphism*) $\alpha\colon f \to g$ is a natural transformation $\alpha\colon f \to g$ such that for every object $a \in \mathcal{X}$, the morphism $\alpha_a\colon f(a) \to g(a)$ in $\mathcal{Y}$ (which is an isomorphism) is over the identity in $\mathcal{S}$. We often describe the 2-isomorphism α schematically as

$$\mathcal{X} \underset{g}{\overset{f}{\Rightarrow\alpha}} \mathcal{Y}.$$

(3) We define the category $\mathrm{Mor}(\mathcal{X}, \mathcal{Y})$ whose objects are morphisms of prestacks and whose morphisms are 2-isomorphisms.

(4) A *2-commutative diagram* (which we often call simply a *commutative diagram*) is a diagram

$$\begin{array}{ccc} \mathcal{X}' & \overset{f'}{\longrightarrow} & \mathcal{Y}' \\ {\scriptstyle g'}\downarrow & \overset{\Swarrow_\alpha}{} & \downarrow{\scriptstyle g} \\ \mathcal{X} & \underset{f}{\longrightarrow} & \mathcal{Y} \end{array}$$

together with a 2-isomorphism $\alpha\colon g \circ f' \overset{\sim}{\to} f \circ g'$.

(5) A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of prestacks is a *monomorphism* (resp. *epimorphism*) if f is fully faithful (resp. essentially surjective), and f is an *isomorphism* if there exists a morphism $g\colon \mathcal{Y} \to \mathcal{X}$ of prestacks and 2-isomorphisms $g \circ f \overset{\sim}{\to} \mathrm{id}_\mathcal{X}$ and $f \circ g \overset{\sim}{\to} \mathrm{id}_\mathcal{Y}$.

Exercise 2.3.17. Show that every 2-isomorphism is indeed an isomorphism of functors, or in other words that $\mathrm{Mor}(\mathcal{X}, \mathcal{Y})$ is a groupoid.

Exercise 2.3.18. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks over a category $\mathcal{S}$.

(a) Show that f is a monomorphism if and only if $f_S\colon \mathcal{X}(S) \to \mathcal{Y}(S)$ is fully faithful for every $S \in \mathcal{S}$.

(b) Show that f is an isomorphism if and only if f is fully faithful and essentially surjective.

A prestack $\mathcal{X}$ is *equivalent to a presheaf* if there is a presheaf F and an isomorphism between $\mathcal{X}$ and the prestack $\mathcal{X}_F$ corresponding to F (see Example 2.3.7).

Exercise 2.3.19. Show that G acts freely on U (i.e. the action map $(\sigma, p_2)\colon G \times_S U \to U \times_S U$ is a monomorphism) if and only if $[U/G]^{\mathrm{pre}}$ (resp. $[U/G]$) is equivalent to a presheaf. We often denote these presheaves by $(U/G)^{\mathrm{pre}}$ and U/G.

2.3.4 The 2-Yoneda Lemma

The Yoneda Lemma (0.3.7) states that for a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ on a category $\mathcal{S}$ and an object $S \in \mathcal{S}$, there is a bijection $\mathrm{Mor}(S, F) \overset{\sim}{\to} F(S)$. In particular, there is a fully faithful embedding $\mathcal{S} \to \mathrm{Pre}(\mathcal{S})$, from $\mathcal{S}$ into the category of presheaves on $\mathcal{S}$, given by $S \mapsto \mathrm{Mor}(-, S)$. We will need an analogue of Yoneda's lemma for prestacks. First we recall that an object $S \in \mathcal{S}$ can be viewed as a prestack over $\mathcal{S}$, which we also denote by S, whose objects over $T \in \mathcal{S}$ are morphisms $T \to S$ and a morphism $(T \to S) \to (T' \to S)$ is an S-morphism $T \to T'$.

Lemma 2.3.20 (The 2-Yoneda Lemma). *Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$ and $S \in \mathcal{S}$. The functor*

$$\mathrm{Mor}(S, \mathcal{X}) \to \mathcal{X}(S), \qquad f \mapsto f_S(\mathrm{id}_S)$$

is an equivalence of categories.

Proof. We will construct a quasi-inverse $\Psi \colon \mathcal{X}(S) \to \mathrm{Mor}(S, \mathcal{X})$ as follows.

On objects: For $a \in \mathcal{X}(S)$, we define $\Psi(a) \colon S \to \mathcal{X}$ as the morphism of prestacks sending an object $(T \xrightarrow{f} S)$ (of the prestack corresponding to S) over T to a *choice* of pullback $f^*a \in \mathcal{X}(T)$ and a morphism $(T' \xrightarrow{f'} S) \to (T \xrightarrow{f} S)$ given by an S-morphism $g \colon T' \to T$ to the morphism $f'^*a \to f^*a$ uniquely filling in the diagram

$$
\begin{array}{ccc}
f'^*a \dashrightarrow f^*a \longrightarrow a \\
\downarrow \qquad \downarrow \qquad \downarrow \\
T' \xrightarrow{\ g\ } T \xrightarrow{\ f\ } S,
\end{array}
$$

using Axiom (2) of a prestack.

On morphisms: If $\alpha \colon a' \to a$ is a morphism in $\mathcal{X}(S)$, then $\Psi(\alpha) \colon \Psi(a') \to \Psi(a)$ is defined as the morphism of functors which maps a morphism $T \xrightarrow{f} S$ (i.e. an object in S over T) to the unique morphism $f^*a' \to f^*a$ filling in the diagram

$$
\begin{array}{ccccc}
f^*a' \dashrightarrow f^*a & & & T \\
\downarrow \qquad \downarrow & & \text{over} & \downarrow f \\
a' \xrightarrow{\ \alpha\ } a & & & S
\end{array}
$$

using again Axiom (2) of a prestack.

We leave the verification that Ψ is a quasi-inverse to the reader. $\qquad\square$

Caution 2.3.21. We will use the 2-Yoneda Lemma, often without mention, throughout these notes in passing between morphisms $S \to \mathcal{X}$ and objects of $\mathcal{X}$ over S.

Remark 2.3.22. If $f, g \colon S \to \mathcal{X}$ are morphisms from a scheme S corresponding via the 2-Yoneda Lemma to objects $a, b \in \mathcal{X}(S)$, then a 2-isomorphism $\alpha \colon f \xrightarrow{\sim} g$ corresponds to an isomorphism $a \xrightarrow{\sim} b$ in $\mathcal{X}(S)$.

Remark 2.3.23 (Universal families). When $F \colon \mathrm{Sch} \to \mathrm{Sets}$ is the functor $h_M = \mathrm{Mor}(-, M)$ representable by a scheme M, the (usual) Yoneda Lemma (0.3.7) gives a bijection $\mathrm{Mor}(h_M, F) \cong F(M)$ and the object $U \in F(M)$ corresponding to the identity map is a universal family (see §subsec:universal-families). The Yoneda 2-Lemma does not immediately apply to give a universal family as the category $\mathcal{X}(\mathcal{X})$ of objects of $\mathcal{X}$ over $\mathcal{X}$ is not well-defined. Later in §3.1.7, we prove a Generalized 2-Yoneda Lemma (3.1.24) for algebraic stacks and use it to define universal families.

Example 2.3.24 (Quotient stack presentations). Consider the prestack $[U/G]$ in Definition 2.3.14 arising from a group action $\sigma \colon G \times_S U \to U$. The object of $[U/G]$ over U given by the diagram

$$
\begin{array}{ccc}
G \times_S U & \xrightarrow{\ \sigma\ } & U \\
\downarrow{\scriptstyle p_2} & & \\
U & &
\end{array}
$$

corresponds via the 2-Yoneda Lemma (2.3.20) to a morphism $U \to [U/G]$.

Exercise 2.3.25.

(a) Show that there is a morphism $p\colon U \to [U/G]^{\mathrm{pre}}$ and a 2-commutative diagram

$$
\begin{array}{ccc}
G \times_S U & \xrightarrow{\;\sigma\;} & U \\
{\scriptstyle p_2}\downarrow & \underset{\alpha}{\Swarrow} & \downarrow{\scriptstyle p} \\
U & \xrightarrow[\;p\;]{} & [U/G]^{\mathrm{pre}}
\end{array}
$$

(b) Show that $U \to [U/G]^{\mathrm{pre}}$ is a categorical quotient among prestacks, i.e. for every 2-commutative diagram

$$
\begin{array}{ccc}
G \times_S U & \xrightarrow{\;\sigma\;} & U \\
{\scriptstyle p_2}\downarrow & \underset{\alpha}{\Swarrow} & \downarrow{\scriptstyle p} \\
U & \xrightarrow[\;p\;]{} & [U/G]^{\mathrm{pre}}
\end{array}
\quad\raisebox{1.5em}{φ}
$$

$$
\xrightarrow[\;\varphi\;]{}\; \underset{\tau}{\Swarrow}\; \mathcal{Z}
$$

of prestacks, there exists a morphism $\chi\colon [U/G]^{\mathrm{pre}} \to \mathcal{Z}$ and a 2-isomorphism $\beta\colon \varphi \xrightarrow{\sim} \chi \circ p$ which is compatible with α and τ (i.e. the two natural transformations $\varphi \circ \sigma \xrightarrow{\beta \circ \sigma} \chi \circ p \circ \sigma \xrightarrow{\chi \circ \alpha} \chi \circ p \circ p_2$ and $\varphi \circ \sigma \xrightarrow{\tau} \varphi \circ p_2 \xrightarrow{\beta \circ p_2} \chi \circ p \circ p_2$ agree.

2.3.5 Warmup: fiber products of groupoids

Before we get to fiber products of prestacks, it is instructive to first compute fiber products of groupoids.

Construction 2.3.26. Let $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D}' \to \mathcal{D}$ be functors of groupoids. Define the groupoid $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ as the category of triples (c, d', γ) where $c \in \mathcal{C}$ and $d' \in \mathcal{D}'$ are objects, and $\gamma\colon f(c) \xrightarrow{\sim} g(d')$ is an isomorphism in $\mathcal{D}$. A morphism $(c_1, d'_1, \gamma_1) \to (c_2, d'_2, \gamma_2)$ is the data of morphisms $\chi\colon c_1 \xrightarrow{\sim} c_2$ and $\lambda\colon d'_1 \xrightarrow{\sim} d'_2$ such that

$$
\begin{array}{ccc}
f(c_1) & \xrightarrow{\;f(\chi)\;} & f(c_2) \\
{\scriptstyle \gamma_1}\downarrow & & \downarrow{\scriptstyle \gamma_2} \\
g(d'_1) & \xrightarrow{\;g(\lambda)\;} & g(d'_2)
\end{array}
$$

commutes.

Exercise 2.3.27 (details)**.** Formulate a university property for fiber products of groupoids and show that $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ satisfies it.

The following foreshadows important cartesian diagrams involving quotient stacks.

Exercise 2.3.28 (good practice)**.** Let G be a group acting on a set U via $\sigma\colon G \times U \to U$. Let $[U/G]$ denote the quotient groupoid as defined in Exercise 0.4.7: objects are elements $u \in U$ and a morphism $u \to u'$ is an element $g \in G$ with $u' = gu$. Let $p\colon U \to [U/G]$ denote the projection.

(a) Let P be a set with a free action of G and quotient $T = P/G$. If $f\colon P \to U$ is a G-equivariant map, show that there is a cartesian diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\;f\;} & U \\
\downarrow & \square & \downarrow{\scriptstyle p} \\
T & \longrightarrow & [U/G].
\end{array}
$$

(b) Show that there are cartesian diagrams

$$
\begin{array}{ccc}
G \times U & \xrightarrow{\;\sigma\;} & U \\
{\scriptstyle p_2}\downarrow & \square & \downarrow{\scriptstyle p} \\
U & \xrightarrow{\;p\;} & [U/G]
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
G \times U & \xrightarrow{(\sigma, p_2)} & U \times U \\
\downarrow & \square & \downarrow{\scriptstyle p \times p} \\
[U/G] & \xrightarrow{\;\Delta\;} & [U/G] \times [U/G].
\end{array}
$$

Exercise 2.3.29 (Good practice). Recall from Example 0.4.3 that the classifying groupoid BG of a group G is the category with one object $*$ with $\operatorname{Mor}(*, *) = G$.

(a) Let $\phi\colon H \to G$ be a homomorphism of groups. Show that there is an induced morphism $BH \to BG$ of groupoids and that $BH \times_{BG} \mathrm{pt} \cong [G/H]$, where pt denotes the groupoid with one object and one morphism.

(b) If $K \triangleleft G$ is a normal subgroup with quotient $Q = G/K$, show that there is a cartesian diagram

$$
\begin{array}{ccccc}
Q & \longrightarrow & BK & \longrightarrow & \mathrm{pt} \\
\downarrow & \square & \downarrow & \square & \downarrow \\
\mathrm{pt} & \longrightarrow & BG & \longrightarrow & BQ.
\end{array}
$$

(c) Let G be a group acting on a set U and let $[U/G]$ be the groupoid quotient. If $u \in U$, show that there is a morphism $BG_u \to [U/G]$ of groupoids and a cartesian diagram

$$
\begin{array}{ccc}
Gu & \longrightarrow & U \\
\downarrow & \square & \downarrow{\scriptstyle p} \\
BG_u & \longrightarrow & [U/G].
\end{array}
$$

Exercise 2.3.30 (Good practice). Show that a groupoid $\mathcal{C}$ is equivalent to a set if and only if $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is fully faithful.

2.3.6 Fiber products of prestacks

The fiber product of morphisms $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ of prestacks over a category $\mathcal{S}$ is the prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ whose fiber category over $S \in \mathcal{S}$ is the fiber product $\mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}'(S)$ of groupoids.

Construction 2.3.31. Let $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y}' \to \mathcal{Y}$ be morphisms of prestacks over a category $\mathcal{S}$. Define the prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ over $\mathcal{S}$ as the category of triples (x, y', γ) where $x \in \mathcal{X}$ and $y' \in \mathcal{Y}'$ are objects over the *same* object $S := p_{\mathcal{X}}(x) = p_{\mathcal{Y}'}(y') \in \mathcal{S}$, and $\gamma\colon f(x) \xrightarrow{\sim} g(y')$ is an isomorphism in $\mathcal{Y}(S)$. A morphism $(x_1, y_1', \gamma_1) \to (x_2, y_2', \gamma_2)$ consists of a triple (h, χ, λ) where $h\colon p_{\mathcal{X}}(x_1) = p_{\mathcal{Y}'}(y_1') \to$

$p_{\mathcal{Y}'}(y_2') = p_{\mathcal{X}}(x_2)$ is a morphism in $\mathcal{S}$, and $\chi\colon x_1 \to x_2$ and $\lambda\colon y_1' \to y_2'$ are morphisms in $\mathcal{X}$ and $\mathcal{Y}'$ over h such that

$$
\begin{array}{ccc}
f(x_1) & \xrightarrow{\ f(\chi)\ } & f(x_2) \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} \\
g(y_1') & \xrightarrow{\ g(\lambda)\ } & g(y_2')
\end{array}
$$

commutes.

Letting $p_1\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{X}$ and $p_2\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ denote the projections $(x, y', \gamma) \mapsto x$ and $(x, y', \gamma) \mapsto y'$, define the 2-isomorphism $\alpha\colon f \circ p_1 \xrightarrow{\sim} g \circ p_2$, which is defined on an object $(x, y', \gamma) \in \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ by setting $\alpha_{(x,y',\gamma)} := \gamma\colon f(x) \to g(y')$. This yields a 2-commutative diagram

$$
\begin{array}{ccc}
\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' & \xrightarrow{\ p_2\ } & \mathcal{Y}' \\
\downarrow{\scriptstyle p_1}\quad {\scriptstyle \alpha}\!\!\nearrow & & \downarrow{\scriptstyle g} \\
\mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y}.
\end{array} \qquad (2.3.32)
$$

Theorem 2.3.33. *The prestack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ together with the morphisms p_1 and p_2 and the 2-isomorphism α as in* (2.3.32) *satisfy the following universal property: for every 2-commutative diagram*

with 2-isomorphism $\tau\colon f \circ q_1 \xrightarrow{\sim} g \circ q_2$, there exists a morphism $h\colon \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ and 2-isomorphisms $\beta\colon q_1 \to p_1 \circ h$ and $\rho\colon q_2 \to p_2 \circ h$ yielding a 2-commutative diagram

such that

$$
\begin{array}{ccc}
f \circ q_1 & \xrightarrow{\ f(\beta)\ } & f \circ p_1 \circ h \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \alpha \circ h} \\
g \circ q_2 & \xrightarrow{\ g(\rho)\ } & g \circ p_2 \circ h
\end{array}
$$

commutes. The data (h, β, ρ) is unique up to unique isomorphism.

Proof. We define $h\colon \mathcal{T} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ on objects by $t \mapsto \big(q_1(t), q_2(t), \tau_t\colon f(q_1(t)) \overset{\sim}{\to} g(q_2(t))\big)$ and on morphisms as $(\Psi\colon t \to t') \mapsto (p_{\mathcal{T}}(\Psi), q_1(\Psi), q_2(\Psi))$. There are *equalities* of functors $q_1 = p_1 \circ h$ and $q_2 = p_2 \circ h$ so we can define β and ρ as the identity natural transformations. The remaining details are left to the reader. $\square$

Definition 2.3.34. We say that a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ \downarrow & \overset{\alpha}{\nearrow} & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y}. \end{array}$$

is *cartesian* if it satisfies the universal property of Theorem 2.3.33. We often write a cartesian diagram of stacks as

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\ \downarrow & \square & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Y}, \end{array}$$

where the existence of the 2-isomorphism α is implicit.

As usual, $\mathcal{X} \times \mathcal{Y}$ denotes the fiber product $\mathcal{X} \times_{\operatorname{Spec} \mathbb{Z}} \mathcal{Y}$ or, in the case that we are working over a field $\Bbbk$, the fiber product $\mathcal{X} \times_{\operatorname{Spec} \Bbbk} \mathcal{Y}$.

2.3.7 Examples

The following exercise is essential for establishing properties of quotient stacks such as their algebraicity (Theorem 3.1.10).

Exercise 2.3.35 (important, good practice)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S via $\sigma\colon G \times_S U \to U$, and let $[U/G]$ be the quotient stack (Definition 2.3.14).

(a) Let $T \to [U/G]$ be a morphism corresponding via the 2-Yoneda Lemma (2.3.20) to a principal G-bundle $P \to T$ and a G-equivariant map $f\colon P \to U$. Show that there is a cartesian diagram

$$\begin{array}{ccc} P & \overset{f}{\longrightarrow} & U \\ \downarrow & \square & \downarrow \\ T & \longrightarrow & [U/G]. \end{array}$$

(We will later see that $[U/G]$ is an algebraic stack and that $U \to [U/G]$ is principal G-bundle (Theorem 3.1.10). In particular, the principal G-bundle $U \to [U/G]$ with the identity map $U \to U$ is the universal family over $[U/G]$, corresponding via the 2-Yoneda Lemma to the identity map $[U/G] \to [U/G]$.)

(b) Show that there are cartesian diagrams

$$\begin{array}{ccc} G \times_S U & \overset{\sigma}{\longrightarrow} & U \\ {\scriptstyle p_2}\downarrow & \square & \downarrow{\scriptstyle p} \\ U & \overset{p}{\longrightarrow} & [U/G] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G \times_S U & \overset{(\sigma, p_2)}{\longrightarrow} & U \times_S U \\ \downarrow & \square & \downarrow{\scriptstyle p \times p} \\ [U/G] & \overset{\Delta}{\longrightarrow} & [U/G] \times_S [U/G]. \end{array}$$

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The diagram in the next exercise is utilized extensively, just as it is in the case of schemes. It will be used to define stabilizers and the inertia stack in §3.2.2.

Exercise 2.3.36 (Magic Square, important)**.** Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$. Let $S, T \in \mathcal{S}$ be objects which we can view as prestacks over $\mathcal{S}$ via Example 2.3.8. Show that for all morphisms $a \colon S \to \mathcal{X}$ and $b \colon T \to \mathcal{X}$, there is a cartesian diagram

$$
\begin{array}{ccc}
S \times_{\mathcal{X}} T & \longrightarrow & S \times T \\
\downarrow & \square & \downarrow{\scriptstyle a \times b} \\
\mathcal{X} & \xrightarrow{\ \Delta\ } & \mathcal{X} \times \mathcal{X},
\end{array}
$$

where the fiber products $S \times T$ and $\mathcal{X} \times \mathcal{X}$ are taken over $\operatorname{Spec}\mathbb{Z}$.

Exercise 2.3.37 (Isom presheaves, important)**.** Let $\mathcal{X}$ be a prestack over a category $\mathcal{S}$. For $S \in \mathcal{S}$, recall from Example 2.1.11 that the localized category $\mathcal{S}/S$ denotes the category whose objects are morphisms $T \to S$ in $\mathcal{S}$ and whose morphisms are S-morphisms.

(a) Show that for objects a and b of $\mathcal{X}$ over S that the functor

$$
\underline{\operatorname{Isom}}_{\mathcal{X}(S)}(a, b) \colon \mathcal{S}/S \to \operatorname{Sets}
$$

$$
(T \xrightarrow{f} S) \mapsto \operatorname{Mor}_{\mathcal{X}(T)}(f^*a, f^*b),
$$

where f^*a and f^*b are choices of a pullback, defines a presheaf on $\mathcal{S}/S$.

(b) Show that there is a cartesian diagram

$$
\begin{array}{ccc}
\underline{\operatorname{Isom}}_{\mathcal{X}(S)}(a, b) & \longrightarrow & S \\
\downarrow & \square & \downarrow{\scriptstyle (a,b)} \\
\mathcal{X} & \xrightarrow{\ \Delta\ } & \mathcal{X} \times \mathcal{X}.
\end{array}
$$

(c) Show that the presheaf $\underline{\operatorname{Aut}}_{\mathcal{X}(T)}(a) = \underline{\operatorname{Isom}}_{\mathcal{X}(T)}(a, a)$ is naturally a presheaf in groups.

(d) Show that $\mathcal{X}$ is equivalent to a presheaf if and only if the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is fully faithful.

Exercise 2.3.38 (good practice)**.**

(a) If $H \to G$ is a morphism of smooth affine group schemes over a scheme S, define a morphism of prestacks $BH \to BG$ over Sch/S by using Definition B.2.34 to construct a principal G-bundle from a principal H-bundle.

(b) Show that $BH \times_{BG} S \cong [G/H]$.

(c) If $1 \to K \to G \to Q \to 1$ is an exact sequence of smooth affine algebraic groups over a field $\mathbb{k}$, show that there is a cartesian diagram

$$
\begin{array}{ccccc}
Q & \longrightarrow & BK & \longrightarrow & \operatorname{Spec}\mathbb{k} \\
\downarrow & \square & \downarrow & \square & \downarrow \\
\operatorname{Spec}\mathbb{k} & \longrightarrow & BG & \longrightarrow & BQ.
\end{array}
$$

Exercise 2.3.39 (good practice)**.** Let G and H be smooth affine group schemes over a scheme S

(a) Show that $B(G \times H) \cong BG \times BH$.

(b) If X and Y are S-schemes with a actions by G and H, show that $[(X \times Y)/(G \times H)] \cong [X/G] \times [Y/H]$.

(c) Conclude that $[\mathbb{A}^n/\mathbb{G}_m^n] \cong \underbrace{[\mathbb{A}^1/\mathbb{G}_m] \times \cdots \times [\mathbb{A}^1/\mathbb{G}_m]}_{n \text{ times}}$.

Exercise 2.3.40 (important, good practice)**.** Analogous to the prestack $\mathcal{M}_g$ of smooth curves (Example 2.3.9), let $\mathcal{M}_{g,1}$ be the prestack where an object over a scheme S is a family of smooth curves $\mathcal{C} \to S$ and a section $\sigma \colon S \to \mathcal{C}$. Let $\mathcal{M}_{g,1} \to \mathcal{M}_g$ be the morphism of prestacks forgetting the section. Show that if $S \to \mathcal{M}_g$ is a morphism corresponding to a family of curves $\mathcal{C} \to S$, there is a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{M}_{g,1} \\
\downarrow & \square & \downarrow \\
S & \longrightarrow & \mathcal{M}_g.
\end{array}
$$

In other words, $\mathcal{M}_{g,1} \to \mathcal{M}_g$ is the universal family, as defined later in Definition 3.1.26.

Exercise 2.3.41 (good practice)**.** Let H and G be smooth affine group schemes over a scheme S. Let $\underline{\mathrm{Hom}}(H,G)$ be the sheaf on $(\mathrm{Sch}/S)_{\text{ét}}$ whose sections over an S-scheme T are homomorphisms $H_T \to G_T$, and let $\underline{\mathrm{Mor}}(BH, BG)$ be the stack over $\mathcal{S}$ whose objects over $T \in \mathcal{S}$ are morphisms $B(H_T) \to B(G_T)$. Show that

$$
\underline{\mathrm{Mor}}(BH, BG) \cong [\underline{\mathrm{Hom}}(H,G)/G],
$$

where G acts via conjugation.

2.4 Stacks

A stack over a site $\mathcal{S}$ is a prestack $\mathcal{X}$ such that objects and morphisms glue uniquely in the Grothendieck topology of $\mathcal{S}$. Results from descent theory (Section B.1) can be invoked to show that a given prestack over $\mathrm{Sch}_{\text{ét}}$ is a stack.

2.4.1 Definition of a stack

Given a covering $\{S_i \to S\}$ of a site, we will use the convention that S_{ij} denotes $S_i \times_S S_j$ and S_{ijk} denotes $S_i \times_S S_j \times_S S_k$.

Definition 2.4.1 (Stacks)**.** A prestack $\mathcal{X}$ over a site $\mathcal{S}$ is a *stack* if the following conditions hold for all coverings $\{S_i \to S\}$ of an object $S \in \mathcal{S}$:

(1) (morphisms glue) For objects a and b in $\mathcal{X}$ over S and morphisms $\phi_i \colon a|_{S_i} \to b$ such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$ as displayed in the diagram

there exists a unique morphism $\phi\colon a \to b$ over id_S with $\phi|_{S_i} = \phi_i$.

(2) (objects glue) For objects a_i over S_i and isomorphisms $\alpha_{ij}\colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$, as displayed in the diagram

satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}, then there exists an object a over S and isomorphisms $\phi_i\colon a|_{S_i} \to a_i$ over id_{S_i} such that $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}} \circ \alpha_{ij}$ on S_{ij}.

Remark 2.4.2. If the covering consists of a single map $S' \to S$ and $a' \in \mathcal{X}$ is an object over S', the cocycle condition for an isomorphism $\alpha\colon p_1^* a' \xrightarrow{\sim} p_2^* a'$ over $S' \times_S S'$ translates to the commutativity of

over $S' \times_S S' \times_S S'$. Axiom (2) requires the existence of $a \in \mathcal{X}(S)$ and an isomorphism $\phi\colon a' \xrightarrow{\sim} a|_{S'}$ satisfying $p_1^* \phi = p_2^* \phi \circ \alpha$.

Remark 2.4.3. Analogous to the sheaf axiom of a presheaf $F\colon \mathcal{S} \to \mathrm{Sets}$ requiring that $F(S) \to \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$ is exact for coverings $\{S_i \to S\}$, the stack axioms translate to the 'exactness' of

$$\mathcal{X}(S) \longrightarrow \prod_i \mathcal{X}(S_i) \rightrightarrows \prod_{i,j} \mathcal{X}(S_i \times_S S_j) \Rrightarrow \prod_{i,j,k} \mathcal{X}(S_i \times_S S_j \times_S S_k).$$

Exercise 2.4.4. Show that Axiom (1) is equivalent to the condition that for all objects a and b of $\mathcal{X}$ over $S \in \mathcal{S}$, the Isom presheaf $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ (see Exercise 2.3.37) is a sheaf on $\mathcal{S}/S$.

Exercise 2.4.5. Generalizing Exercise 2.2.6 from sheaves to shacks, show that a prestack $\mathcal{X}$ over Sch is a stack over $\mathrm{Sch}_{\text{ét}}$ (resp. $\mathrm{Sch}_{\mathrm{fppf}}$, $\mathrm{Sch}_{\mathrm{fpqc}}$) if and only if $\mathcal{X}$ is a prestack over $\mathrm{Sch}_{\mathrm{Zar}}$ and Axioms (1) and (2) hold for a surjective étale (resp. fppf, fpqc) map $S' \to S$.

A *morphism* of stacks is a morphism of prestacks.

Exercise 2.4.6 (Fiber product of stacks)**.** Show that if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ are morphisms of stacks over a site $\mathcal{S}$, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$ is also a stack over $\mathcal{S}$.

2.4.2 First examples of stacks

Example 2.4.7 (Sheaves and schemes are stacks)**.** A presheaf F on a site $\mathcal{S}$ defines a prestack $\mathcal{X}_F$ over $\mathcal{S}$ whose objects are pairs (a, S) where $S \in \mathcal{S}$ and $a \in F(S)$ (see Example 2.3.7), and F is a sheaf if and only if $\mathcal{X}_F$ is a stack. We often abuse notation by writing F also as the stack $\mathcal{X}_F$.

Since schemes are sheaves on $\mathrm{Sch}_{\text{ét}}$ (Proposition 2.2.9), a scheme X defines a stack—also denoted as X—over $\mathrm{Sch}_{\text{ét}}$, whose objects over a scheme S are morphisms $S \to X$.

Example 2.4.8 (Stacks of sheaves)**.** Let $\underline{\text{Sheaves}}$ be the prestack over Sch whose objects are pairs (T, F) where T is a scheme and F is a sheaf on the Zariski topology of T. A morphism $(T', F') \to (T, F)$ is pair (f, α) where $f \colon T' \to T$ is a map of schemes and $\alpha \colon f^{-1}F \to F'$ is an isomorphism of sheaves. The projection $\underline{\text{Sheaves}} \to$ Sch is defined by $(T, F) \mapsto T$. Because sheaves and their morphisms glue in the Zariski topology [Har77, Exer. II.1.15 and 22], $\mathcal{X}$ is a stack over the big Zariski site $\mathrm{Sch}_{\mathrm{Zar}}$.

Similarly, we define the $\underline{\text{QCoh}}$, $\underline{\text{Coh}}$, and $\underline{\text{Bun}}$ as the category over Sch consisting of pairs (T, F) where F is a quasi-coherent sheaf (resp. finitely presented, quasi-coherent sheaf, vector bundle) on a scheme T and a morphism $(T, F) \to (T', F')$ is a map $f \colon T \to T'$ and an isomorphism $\alpha \colon f^*F' \to F$. To see that $\underline{\text{QCoh}}$ is a stack over $\mathrm{Sch}_{\mathrm{fpqc}}$, by Exercise 2.4.5, it suffices to verify Axioms (1) and (2) with respect to an fpqc map $T' \to T$, and these translate literally to the two parts of Fpqc Descent of Quasi-coherent Sheaves (B.1.4). In particular, $\underline{\text{QCoh}}$ is a stack over $\mathrm{Sch}_{\text{ét}}$. By Fpqc Descent of Properties of Quasi-coherent Sheaves (B.1.11), $\underline{\text{Coh}}$ and $\underline{\text{Bun}}$ are also stacks over $\mathrm{Sch}_{\mathrm{fpqc}}$ and $\mathrm{Sch}_{\text{ét}}$.

Exercise 2.4.9. Show that the prestack $\underline{\text{Mod}}$ parameterizing pairs (T, F) where T is a scheme and F is a sheaf of $\mathcal{O}_T$-modules is not a stack over $\mathrm{Sch}_{\text{ét}}$.

Exercise 2.4.10. Define the prestack of sheaves over any site and apply Exercises 2.2.7 and 2.2.8 to conclude that it is a stack.

Example 2.4.11 (Stack of schemes)**.** Define $\underline{\text{Schemes}}$ as the prestack over Sch consists of morphisms $T \to S$ of schemes where a morphism $(T' \to S') \to (T \to S)$ consists of morphisms $T' \to T$ and $S' \to S$ that forms a cartesian diagram. The projection $\underline{\text{Schemes}} \to$ Sch takes $T \to S$ to S. Since schemes glue in the Zariski topology [Har77, Exer. II.2.12], $\underline{\text{Schemes}}$ is a stack over $\mathrm{Sch}_{\mathrm{Zar}}$. However, $\underline{\text{Schemes}}$ is *not* a stack over $\mathrm{Sch}_{\text{ét}}$. Schemes can be glued to algebraic spaces in the étale topology and there is a stack of algebraic spaces over $\mathrm{Sch}_{\text{ét}}$; see Exercise 4.4.15.

2.4.3 Classifying stacks and quotient stacks

Let $G \to S$ be a smooth affine group scheme acting on an S-scheme U, and let $[U/G]$ be the category over Sch/S defined prestack in Definition 2.3.14: an object

over an S-scheme T is a diagram

$$
\begin{array}{ccc}
P & \longrightarrow & U \\
\downarrow & & \\
T & &
\end{array}
$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of schemes.

Proposition 2.4.12. *If $G \to S$ be a smooth affine group scheme acting on an S-scheme U, then $[U/G]$ is a stack over $\mathrm{Sch}_{\text{ét}}$. In particular, the classifying stack $BG = [S/G]$ is a stack over $\mathrm{Sch}_{\text{ét}}$.*

Proof. We will show in fact that $[U/G]$ is a stack over $\mathrm{Sch}_{\text{fpqc}}$. Since schemes and their morphisms glue in the Zariski topology, it suffices by Exercise 2.4.5 to verify Axioms (1) and (2) with respect to an fpqc map $T' \to T$. For Axiom (1), let $(P' \to T, f' \colon P \to U)$ and $(P \to T, f \colon P \to U)$ be objects over an S-scheme T, $T' \to T$ be an fpqc map, and $\phi' \colon P'_{T'} \to P$ be a G-equivariant morphism identifying $P'_{T'}$ with $P \times_T T'$ and compatible with f and f'. By Fpqc Descent for Morphisms (B.1.7), there exists a unique morphism $P' \to P$ compatible with f' and f. By Fpqc Local Properties on the Target (B.1.9), $P' \to P$ is an isomorphism. For Axiom (2), let $(P' \to T', f' \colon P' \to U)$ be an object over T' and $\alpha \colon p_1^* P' \to p_2^* P'$ be an isomorphism commuting with f' and satisfying the cocycle condition $p_{23}^* \alpha \circ p_{12}^* \alpha = p_{13}^* \alpha$ on $T' \times_T T' \times T'$. The existence of a principal G-bundle $P \to T$ pulling back to $P' \to T$ follows from Fpqc Descent of Principal G-bundles (B.2.20). The existence of a G-equivariant morphism $P \to U$ follows from Fpqc Descent for Morphisms (B.1.7). $\square$

2.4.4 Moduli stack of curves

Let $\mathcal{M}_g$ denote the prestack of families of smooth curves $\mathcal{C} \to S$ of genus g as defined in Example 2.3.9.

Proposition 2.4.13 (Moduli stack of smooth curves). *If $g \geq 2$, then $\mathcal{M}_g$ is a stack over $\mathrm{Sch}_{\text{ét}}$.*

Proof. We check that $\mathcal{M}_g$ is a stack over $\mathrm{Sch}_{\text{fpqc}}$ as smooth curves and their morphisms glue in the Zariski topology, it suffices by Exercise 2.4.5 to verify Axioms (1) and (2) with respect to an fpqc map $S' \to S$. Axiom (1) translates to: for families of smooth curves $\mathcal{C} \to S$ and $\mathcal{D} \to S$ of genus g, every commutative diagram

$$
\begin{array}{ccccccc}
& & & & & \overset{f'}{\frown} & \\
\mathcal{C}_{S' \times_S S'} & \longrightarrow & \mathcal{C}_{S'} & \longrightarrow & \mathcal{C} & \overset{f}{\dashrightarrow} & \mathcal{D} \\
\downarrow & \square & \downarrow & \square & & \searrow & \downarrow \\
S' \times_S S' & \rightrightarrows & S' & \longrightarrow & & S &
\end{array}
$$

of solid arrows can be uniquely filled in. The existence and uniqueness of f follow from Fpqc Descent for Morphisms (B.1.7). The fact that f is an isomorphism also follows from étale descent (Proposition B.1.9).

Axiom (2) will require some geometry: we must show that given a diagram

$$
\begin{array}{ccccccc}
p_1^*\mathcal{C}' & \xrightarrow{\ \ \alpha\ \ } & p_2^*\mathcal{C}' & \longrightarrow & \mathcal{C}' & \dashrightarrow & \mathcal{C} \\
\downarrow & \swarrow & & \square & \downarrow & \square & \downarrow \\
& S' \times_S S' & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & & S' & \longrightarrow & S
\end{array}
$$

where $\mathcal{C}' \to S'$ is a family of smooth curves and $\alpha\colon p_1^*\mathcal{C}' \to p_2^*\mathcal{C}'$ is an isomorphism satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ on $T' \times_T T' \times T'$, there is family of smooth curves $\mathcal{C} \to S$ and an isomorphism $\phi\colon \mathcal{C}|_{S'} \to \mathcal{C}'$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$. We will apply Properties of Families of Smooth Curves (5.1.15): $\Omega_{\mathcal{C}'/S'}^{\otimes 3}$ is relatively very ample on S (as $g \geq 2$) and $E' := \pi_*(\Omega_{\mathcal{C}'/S'}^{\otimes 3})$ is a vector bundle on S' of rank $5(g-1)$. This implies that $\Omega_{\mathcal{C}'/S'}^{\otimes 3}$ defines a closed immersion $\mathcal{C}' \hookrightarrow \mathbb{P}(E')$ over S'. The isomorphism α induces an isomorphism $\beta\colon p_1^*E' \to p_2^*E'$ satisfying the cocycle condition $p_{23}^*\beta \circ p_{12}^*\beta = p_{13}^*\beta$ on $S' \times_S S' \times_S S'$. Fpqc Descent for Quasi-coherent Sheaves (B.1.4) yields a quasi-coherent sheaf E on S and isomorphisms $\psi\colon E' \to E|_{S'}$ such that $p_1^*\Psi = p_2^*\psi \circ \beta$. It follows from Fpqc Descent of Properties of Quasi-coherent Sheaves (B.1.11) that E is a vector bundle. Pictorially, we have

$$
\begin{array}{ccccccc}
\mathbb{P}(p_1^*E') & \xrightarrow{\ \ \beta\ \ } & \mathbb{P}(p_2^*E') & \longrightarrow & \mathbb{P}(E') & \longrightarrow & \mathbb{P}(E) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
p_1^*\mathcal{C}' & \xrightarrow{\ \ \alpha\ \ } & p_2^*\mathcal{C}' & \longrightarrow & \mathcal{C}' & \dashrightarrow & \mathcal{C} \\
\downarrow & \swarrow & & & \downarrow & & \downarrow \\
& S' \times_S S' & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & & S' & \longrightarrow & S.
\end{array}
$$

Under the identifications of $\mathbb{P}(p_1^*E')$ and $\mathbb{P}(p_2^*E')$ with $\mathbb{P}(E) \times_S (S' \times_S S')$, the preimages $p_1^*\mathcal{C}'$ and $p_2^*\mathcal{C}'$ are equal. By Effective Descent (B.1.8), there is a closed subscheme $\mathcal{C} \subset \mathbb{P}(E)$ pulling back to $\mathcal{C}'$. By Fpqc Local Properties on the Target (B.1.9), smoothness and properness are local properties on the target, and thus $\mathcal{C} \to S$ is smooth and proper. Every geometric fiber of $\mathcal{C} \to S$ is identified with a geometric fiber of $\mathcal{C}' \to S'$ and is thus a connected genus g curve. $\qquad\square$

Exercise 2.4.14.

(a) Use the correspondence between families of genus 0 curves and principal PGL_2-torsors (Exercise B.2.30) to show that the prestack $\mathcal{M}_0$ is a stack on $\mathrm{Sch}_{\text{ét}}$ isomorphic to $B\,\mathrm{PGL}_2$ over $\mathrm{Spec}\,\mathbb{Z}$.

(b) A *family of elliptic curves* over a scheme S is a pair $(\mathcal{E} \to S, \sigma)$ where $\mathcal{E} \to S$ is smooth proper morphism with a section $\sigma\colon S \to \mathcal{E}$ such that for every $s \in S$, the fiber $(\mathcal{E}_s, \sigma(s))$ is an elliptic curve over the residue field $\kappa(s)$. Show that the moduli stack $\mathcal{M}_{1,1}$, whose objects are families of elliptic curves, is a stack on $\mathrm{Sch}_{\text{ét}}$.

Remark 2.4.15 ($\mathcal{M}_1$). The prestack $\mathcal{M}_1$, whose objects are smooth and proper morphisms $\mathcal{C} \to S$ of *schemes* whose geometric fibers are connected genus 1 curves, is not a stack over $\mathrm{Sch}_{\text{ét}}$. Unlike $\mathcal{M}_g$ with $g \geq 2$ and $\mathcal{M}_{1,1}$, there is no natural line bundle defining an embedding into projective space that we can use as above to verify Axiom (2). Raynaud constructed a family $\mathcal{C}' \to S'$ of smooth genus 1 curves

which does not descend to a family $\mathcal{C} \to S$ [Ray70, XIII 3.2]. However, similar to Example 2.4.11, if we redefine $\mathcal{M}_1$ as the category of smooth and proper morphisms $\mathcal{C} \to S$ from an *algebraic space* such that every geometric fiber is a connected curve of genus 1, then $\mathcal{M}_1$ is a stack.

2.4.5 Moduli stack of coherent sheaves and vector bundles

If X is a scheme over a field $\Bbbk$, the prestacks $\underline{\mathrm{QCoh}}(X)$, $\underline{\mathrm{Coh}}(X)$, and $\underline{\mathrm{Bun}}(X)$ from Example 2.3.11 parameterize quasi-coherent sheaves, coherent sheaves, and vector bundles.

Proposition 2.4.16. *The prestacks $\underline{\mathrm{QCoh}}(X)$, $\underline{\mathrm{Coh}}(X)$, and $\underline{\mathrm{Bun}}(X)$ are stacks over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$.*

Proof. The same argument of Example 2.4.8 shows that each prestack is a stack over $(\mathrm{Sch}/\Bbbk)_{\text{fpqc}}$ using Fpqc Descent for Quasi-coherent Sheaves (B.1.4) and Fpqc Descent of Properties of Quasi-coherent Sheaves (B.1.11). $\qquad\square$

2.4.6 Stackification

Recall from Sheafification (2.2.13) that for any a presheaf F on a site $\mathcal{S}$, there is a map $F \to F^{\mathrm{sh}}$ which is a left adjoint to the inclusion, i.e. $\mathrm{Mor}(F^{\mathrm{sh}}, G) \to \mathrm{Mor}(F, G)$ is bijective for every sheaf G on $\mathcal{S}$. Similarly, there is a stackification $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of a prestack $\mathcal{X}$ over $\mathcal{S}$.

Theorem 2.4.17 (Stackification). *If $\mathcal{X}$ is a prestack over a site $\mathcal{S}$, there exists a stack $\mathcal{X}^{\mathrm{st}}$, which we call the stackification, and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that for every stack $\mathcal{Y}$ over $\mathcal{S}$, the induced functor*

$$\mathrm{Mor}(\mathcal{X}^{\mathrm{st}}, \mathcal{Y}) \to \mathrm{Mor}(\mathcal{X}, \mathcal{Y}) \tag{2.4.18}$$

is an equivalence of categories.

Proof. As in the construction of the sheafification in Theorem 2.2.13, we construct the stackification in stages. Most details are left to the reader.

First, given a prestack $\mathcal{X}$, we can construct a prestack $\mathcal{X}^{\mathrm{st}_1}$ satisfying Axiom (1) and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}_1}$ of prestacks such that

$$\mathrm{Mor}(\mathcal{X}^{\mathrm{st}_1}, \mathcal{Y}) \to \mathrm{Mor}(\mathcal{X}, \mathcal{Y})$$

is an equivalence for all prestacks $\mathcal{Y}$ satisfying Axiom (1). Specifically, the objects of $\mathcal{X}^{\mathrm{st}_1}$ are the same as $\mathcal{X}$, and for objects $a, b \in \mathcal{X}$ over $S, T \in \mathcal{S}$, the set of morphisms $a \to b$ in $\mathcal{X}^{\mathrm{st}_1}$ over a given morphism $f \colon S \to T$ is the global sections $\Gamma(S, \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, f^*b)^{\mathrm{sh}})$ of the sheafification of the Isom presheaf (Exercise 2.3.37).

Second, given a prestack $\mathcal{X}$ satisfying Axiom (1), we construct a stack $\mathcal{X}$ and a morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ of prestacks such that (2.4.18) is an equivalence for all stacks $\mathcal{Y}$. An object of $\mathcal{X}^{\mathrm{st}}$ over $S \in \mathcal{S}$ is given by a triple consisting of a covering $\{S_i \to S\}$, objects a_i of $\mathcal{X}$ over S_i, and isomorphisms $\alpha_{ij} \colon a_i|_{S_{ij}} \to a_j|_{S_{ij}}$ satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}. Morphisms

$$\big(\{S_i \to S\}, \{a_i\}, \{\alpha_{ij}\}\big) \to \big(\{T_\mu \to T\}, \{b_\mu\}, \{\beta_{\mu\nu}\}\big)$$

in $\mathcal{X}^{\mathrm{st}}$ over $S \to T$ are defined as follows: first consider the induced cover $\{S_i \times_S T_\mu \to S\}_{i,\mu}$ and choose pullbacks $a_i|_{S_i \times_S T_\mu}$ and $b_\mu|_{S_i \times_S T_\mu}$. A morphism is then the data of maps $\Psi_{i\mu} \colon a_i|_{S_i \times_S T_\mu} \to b_\mu|_{S_i \times_S T_\mu}$ for all i, μ which are compatible with α_{ij} and $\beta_{\mu\nu}$ (i.e. $\Psi_{j\nu} \circ \alpha_{ij} = \beta_{\mu\nu} \circ \Psi_{i\mu}$ on $S_{ij} \times_T T_{\mu\nu}$). $\qquad\square$

Exercise 2.4.19. Show that stackification commutes with fiber products: if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are morphisms of prestacks, then $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})^{\mathrm{st}} \cong \mathcal{X}^{\mathrm{st}} \times_{\mathcal{Y}^{\mathrm{st}}} \mathcal{Z}^{\mathrm{st}}$.

Exercise 2.4.20. Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. Recall from Definition 2.3.14 that the quotient prestack $[U/G]^{\mathrm{pre}}$ and quotient stack $[U/G]$ denote the prestacks over Sch/S classifying trivial principal G-bundles (resp. principal G-bundles) $P \to T$ and G-equivariant maps $P \to U$.

  (a) Show that $[U/G]^{\mathrm{pre}}$ satisfies Axiom (1) of a stack over $(\mathrm{Sch}/S)_{\text{ét}}$.

  (b) Show that the $[U/G]$ is isomorphic to the stackification of $[U/G]^{\mathrm{pre}}$ over $(\mathrm{Sch}/S)_{\text{ét}}$, and that $[U/G]^{\mathrm{pre}} \to [U/G]$ is fully faithful.

Exercise 2.4.21. Extending Exercise 2.3.25, show that $U \to [U/G]$ is a categorical quotient among stacks.

Chapter 3

Algebraic spaces and stacks

> *The notion of stacks came up in the sixties. But to swallow schemes was already enough for one generation of mathematicians.*

<div align="right">

FALTINGS [STO95, p.45]

</div>

3.1 Definitions of algebraic spaces and stacks

What are algebraic spaces, Deligne–Mumford stacks, and algebraic stacks? After giving their definitions, we will verify the algebraicity of quotient stacks $[U/G]$, the moduli stack of curves $\mathcal{M}_g$, and the moduli stack of vector bundles $\underline{\mathrm{Bun}}_{r,d}(C)$.

3.1.1 Algebraic spaces

Definition 3.1.1 (Morphisms representable by schemes). A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable by schemes* (or *schematic*) if for every morphism $T \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme.

If $\mathcal{P}$ is a property of morphisms of schemes stable under base change (e.g. surjective or étale), a morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks representable by schemes *has property* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme, the morphism $\mathcal{X} \times_{\mathcal{Y}} T \to T$ of schemes has property $\mathcal{P}$.

Definition 3.1.2 (Algebraic spaces). An *algebraic space* is a sheaf X on $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective étale morphism $U \to X$ representable by schemes.

The map $U \to X$ is called an *étale presentation*. Morphisms of algebraic spaces are by definition morphisms of sheaves. Every scheme is an algebraic space.

3.1.2 Deligne–Mumford stacks

Definition 3.1.3 (Representable morphisms and their properties). A morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks (or presheaves) over Sch is *representable* if for every morphism $T \to \mathcal{Y}$ from a scheme T, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space.

Let $\mathcal{P}$ be a property of morphisms of schemes stable under base change and étale-local on the source (i.e. if $X' \to X$ is a surjective étale morphism, then a morphism

$X \to Y$ of schemes has $\mathcal{P}$ if and only if $X' \to X \to Y$ has $\mathcal{P}$); examples include the properties of being surjective, étale, or smooth. We say that a representable morphism $\mathcal{X} \to \mathcal{Y}$ of prestacks *has property* $\mathcal{P}$ if for every morphism $T \to \mathcal{Y}$ from a scheme and étale presentation $U \to \mathcal{X} \times_\mathcal{Y} T$ by a scheme, the composition $U \to \mathcal{X} \times_\mathcal{Y} T \to T$ has property $\mathcal{P}$.

Definition 3.1.4 (Deligne–Mumford stacks)**.** A *Deligne–Mumford stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective, étale, and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called an *étale presentation*. Morphisms of Deligne–Mumford stacks are by definition morphisms of stacks. Every algebraic space is a Deligne–Mumford stack via Example 2.3.7. We show later that a Deligne–Mumford stack (or even an algebraic stack) that is a sheaf is an algebraic space (Theorems 3.6.6 and 4.4.10).

Remark 3.1.5. While the essential difference between an algebraic space and a Deligne–Mumford stack is that one is a sheaf while the other is a stack, there is also the technical difference that an étale presentation of an algebraic space is *representable by schemes* while an étale presentation of a Deligne–Mumford stack is only required to be *representable*. If the diagonal of a Deligne–Mumford stack is separated and quasi-compact, then it is representable by schemes and every presentation $U \to \mathcal{X}$ is representable by schemes (Corollary 4.4.8). Therefore, if you are like most other humans and are happy restricting to stacks with separated and quasi-compact diagonal, then you may treat algebraic spaces and Deligne–Mumford stacks on the same footing. On the other hand, you should be aware that there are Deligne–Mumford stacks whose diagonal is not quasi-compact, not separated, or not representable by schemes, e.g. BG for an étale group algebraic space that is not quasi-compact, not separated, or not a scheme; see Examples 3.9.20 to 3.9.22.

3.1.3 Algebraic stacks

Definition 3.1.6 (Algebraic stacks)**.** An *algebraic stack* is a stack $\mathcal{X}$ over $\mathrm{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective, smooth, and representable morphism $U \to \mathcal{X}$.

The morphism $U \to \mathcal{X}$ is called a *smooth presentation*. Morphisms of algebraic stacks are by definition morphisms of prestacks. Every scheme, algebraic space, or Deligne–Mumford stack is also an algebraic stack.

Caution 3.1.7. The definitions above are not standard as most authors add a representability condition on the diagonal. They are nevertheless equivalent to the standard definitions: we show in Theorem 3.2.1 that the diagonal of an algebraic space is representable by schemes and that the diagonal of an algebraic stack is representable.

Definition 3.1.8 (Open and closed substacks)**.** A substack $\mathcal{T} \subset \mathcal{X}$ of a stack over $\mathrm{Sch}_{\text{ét}}$ is called an *open substack* (resp. closed substack) if the inclusion $\mathcal{T} \to \mathcal{X}$ is representable by schemes and an open immersion (resp. closed immersion).

Exercise 3.1.9 (Fiber products)**.** Show that fiber products exist for algebraic spaces, Deligne–Mumford stacks, and algebraic stacks.

3.1.4 Algebraicity of quotient stacks

Theorem 3.1.10 (Algebraicity of Quotient Stacks)**.** *If $G \to S$ is a smooth affine group scheme acting on an algebraic space $U \to S$, the quotient stack $[U/G]$ is an algebraic stack over S such that $U \to [U/G]$ is a principal G-bundle and in particular surjective, smooth, and affine. In particular, the classifying stack $BG = [S/G]$ is algebraic.*

Remark 3.1.11. An eagle-eyed reader may have noticed that we only defined $[U/G]$ when U is a scheme. It is not hard to extend the definition. An *action* of a smooth affine group scheme $G \to S$ on an algebraic space $U \to S$ is a morphism $\sigma \colon G \times_S U \to U$ satisfying the same axioms as in Definition B.2.10, and we define the quotient stack $[U/G]$ as the stackification of the prestack $[U/G]^{\mathrm{pre}}$, whose fiber category over an S-scheme T is the quotient groupoid $[U(T)/G(T)]$, as in Definition 2.3.14. Objects of $[U/G]$ over an S-scheme T are principal G-bundles $P \to T$ and G-equivariant morphisms $P \to U$. Since morphisms to algebraic spaces glue uniquely in the étale topology, the argument of Proposition 2.4.12 extends to show that $[U/G]$ is a stack. Finally, we note that saying that $U \to [U/G]$ is a principal G-bundle means that it is representable by schemes and every base change by a map $T \to [U/G]$ from scheme is a principal G-bundle.

Proof. We will use the natural projection $U \to [U/G]$ corresponding via the 2-Yoneda Lemma (2.3.20) to the trivial principal G-bundle $p_2 \colon G \times U \to U$ and the G-equivariant map $\sigma \colon G \times U \to U$ given by multiplication. To show that $U \to [U/G]$ is a principal G-bundle, let $T \to [U/G]$ is a morphism from an S-scheme classified by a principal G-bundle $P \to T$ and a G-equivariant map $P \to U$. By Exercise 2.3.35, there is a cartesian diagram

$$
\begin{array}{ccc}
P & \longrightarrow & U \\
\downarrow & \square & \downarrow \\
T & \longrightarrow & [U/G].
\end{array}
$$

Since every base change is a principal G-bundle, so is $U \to [U/G]$. The map $U \to [U/G]$ is almost a smooth presentation except that U may not be a scheme: letting $U' \to U$ be an étale presentation with U' a scheme, the composition $U' \to U \to [U/G]$ provides a smooth presentation. $\square$

Example 3.1.12 ($B\mathbb{G}_m$, $B\operatorname{GL}_n$, and $B\operatorname{PGL}_n$)**.** Since principal $\mathbb{G}_m$-bundles correspond to line bundles (Exercise B.2.22), the classifying stack $B\mathbb{G}_m$ is equivalent to the category of pairs (S, V) consisting of a scheme S and a line bundle V on S. Similarly, $B\operatorname{GL}_n$ is the stack of pairs (S, V) where V is a vector bundle of rank n on a scheme S (Exercise B.2.27). The classifying stack $B\operatorname{PGL}_n$ can be described equivalently using either principal PGL_n-bundles, Brauer–Severi schemes, or Azumaya algebras over S (see Exercises B.2.29 and B.2.31).

Exercise 3.1.13 ($BO(q)$)**.** Let $\Bbbk$ be a field of $\operatorname{char}(\Bbbk) \neq 2$. For a non-degenerate quadratic form q on an n-dimensional vector space V, the orthogonal group $O(q)$ is the subgroup of $\operatorname{GL}(V)$ containing matrices preserving q. If q and q' are non-degenerate quardratic forms, show that $BO(q) \cong BO(q')$ even though $O(q)$ and $O(q')$ may be non-isomorphic.

Corollary 3.1.14. *Let G be an finite abstract group viewed as a group scheme over a scheme S. If G acts freely on an algebraic space U over S, then the quotient sheaf U/G is an algebraic space.*

Proof. Since the action is free, the quotient stack $[U/G]$ is equivalent to a sheaf, which we denote by U/G (see Exercise 2.3.19). Theorem 3.1.10 implies that U/G is an algebraic stack and that $U \to U/G$ is a principal G-bundle so in particular finite, étale, surjective and representable by schemes. Taking $U' \to U$ to be an étale presentation by a scheme, the composition $U' \to U \to U/G$ yields an étale presentation of U/G. $\square$

Remark 3.1.15. This resolves the troubling issue from Example 0.5.5 that the quotient of a finite group acting freely on a scheme need not exist as a scheme. In addition, it shows that the category of algebraic spaces itself is closed under taking quotients by free actions of finite groups so that we don't need to enlarge our category even more.

Exercise 3.1.16 (easy)**.** Let $G \to S$ be a smooth affine group scheme acting on S-schemes X and Y. Show that a G-equivariant morphism $X \to Y$ induces a morphism $[X/G] \to [Y/G]$ of algebraic stacks, and conversely that $[X/G] \to [Y/G]$ is induced by a G-equivariant morphism if and only if $[X/G] \to [Y/G]$ is a morphism over BG.

3.1.5 Algebraicity of $\mathcal{M}_g$

Why is $\mathcal{M}_g$ algebraic? Here's one reason: every smooth, connected, and projective curve C is tri-canonically embedded $C \hookrightarrow \mathbb{P}^{5g-6}$ by the very ample line bundle $\omega_C^{\otimes 3}$ and the locally closed subscheme $H' \subset \mathrm{Hilb}_P(\mathbb{P}^{5g-6})$ parameterizing smooth families of tri-canonically embedded curves provides a smooth presentation $H' \to \mathcal{M}_g$. The technical details however are a bit involved. The Algebraicity of $\mathrm{Bun}_{r,d}(C)$ (3.1.21) will be easier.

Theorem 3.1.17 (Algebraicity of $\mathcal{M}_g$)**.** *If $g \geq 2$, then $\mathcal{M}_g$ is an algebraic stack over* $\mathrm{Spec}\,\mathbb{Z}$.

Proof. As in the proof that $\mathcal{M}_g$ is a stack (Proposition 2.4.13), we will use Properties of Families of Smooth Curves (5.1.15): for a family of smooth curves $p \colon \mathcal{D} \to S$, $\omega_{\mathcal{D}/S}^{\otimes 3}$ is relatively very ample on S and $p_*(\omega_{\mathcal{D}/S}^{\otimes 3})$ is a vector bundle of rank $5(g-1)$. It follows that $\omega_{\mathcal{D}/S}^{\otimes 3}$ defines a closed immersion $\mathcal{D} \hookrightarrow \mathbb{P}(p_*(\omega_{\mathcal{D}/S}^{\otimes 3}))$ over S. By Riemann–Roch, the Hilbert polynomial of a fiber $\mathcal{D}_s \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(s)}$ is given by

$$P(n) := \chi(\mathcal{O}_{\mathcal{D}_s}(n)) = \deg(\omega_{\mathcal{D}_s}^{\otimes 3n}) + 1 - g = (6n-1)(g-1).$$

and we define

$$H := \mathrm{Hilb}^P(\mathbb{P}_{\mathbb{Z}}^{5g-6}/\mathbb{Z})$$

as the (projective) Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^{5g-6}$ with Hilbert polynomial P (Theorem 1.1.2). Let $\mathcal{C} \hookrightarrow \mathbb{P}^{5g-6} \times H$ be the universal closed subscheme and let $\pi \colon \mathcal{C} \to H$ be the projection. We claim that there is a locally closed subscheme $H' \subset H$ such that the family $\mathcal{C}_{H'} \to H'$ satisfies

(a) for every $h \in H'$, $\mathcal{C}_h \to \mathrm{Spec}\,\kappa(h)$ is smooth and geometrically connected (and thus geometrically integral),

(b) the natural map $p_{2,*}\mathcal{O}_{\mathbb{P}^{5g-6} \times H'}(1) \to p_{2,*}\mathcal{O}_{\mathcal{C}_{H'}}(1)$ induced by the closed immersion $\mathcal{C}' \hookrightarrow \mathbb{P}^{5g-6} \times H'$ is an isomorphism (or equivalently $\mathrm{H}^0(\mathbb{P}^{5g-6}_{\kappa(h)}, \mathcal{O}(1)) \to \mathrm{H}^0(\mathcal{C}_h, \mathcal{O}_{\mathcal{C}_h}(1))$ is an isomorphism for all $h \in H'$), and

(c) the line bundles $\omega_{\mathcal{C}_{H'}/H'}^{\otimes 3}$ and $\mathcal{O}_{\mathcal{C}_{H'}}(1)$ differ by a pullback of a line bundle from H'.

Moreover, if $T \to H$ is a morphism of schemes such that (a)–(c) hold for the family $\mathcal{C}_T \to T$, then $T \to H$ factors through H'. In particular, $H' \subset H$ is unique. Note that (b)–(c) imply that $\mathcal{C}_h \hookrightarrow \mathbb{P}_{\kappa(h)}^{5g-6}$ is embedded by the complete linear series $\omega_{\mathcal{C}_h/\kappa(h)}^{\otimes 3}$ for $h \in H$.

Since the condition that a fiber of a proper morphism (of finite presentation) is smooth is an open condition on the target (Corollary A.3.9), the condition on H that $\mathcal{C}_h$ is smooth is open. Consider the Stein factorization [Har77, Cor. 11.5] $\mathcal{C} \to \widetilde{H} = \mathrm{Spec}_H \pi_* \mathcal{O}_{\mathcal{C}} \to H$ where $\mathcal{C} \to \widetilde{H}$ has geometrically connected fibers and $\widetilde{H} \to H$ is finite. Since the kernel and cokernel of $\mathcal{O}_H \to \pi_* \mathcal{O}_{\mathcal{C}}$ have closed support (as they are coherent), $\widetilde{H} \to H$ is an isomorphism over an open subscheme of H, which is precisely where the fibers of $\mathcal{C} \to H$ are geometrically connected. In summary, the set of $h \in H$ satisfying (a) is an open subscheme of H, which we will denote by H_1. To arrange (b), observe that Cohomology and Base Change (A.6.8) implies that both $p_{2,*}\mathcal{O}_{\mathbb{P}^{5g-6} \times H_1}(1)$ and $p_{2,*}\mathcal{O}_{\mathcal{C}_{H_1}}(1)$ are vector bundles whose constructions commute with base change, and Riemann–Roch (5.1.1) further implies they have the same rank. Therefore, $\alpha \colon p_{2,*}\mathcal{O}_{\mathbb{P}^{5g-6} \times H_1}(1) \to p_{2,*}\mathcal{O}_{\mathcal{C}_{H_1}}(1)$ is an isomorphism if and only if it is surjective. Setting $H_2 := H_1 \setminus \mathrm{Supp}(\mathrm{coker}(\alpha))$, we conclude that (a)–(b) hold over H_2. It is also clear that any map $T \to H$ over which (a)–(b) hold factors through H_2. For (c), the relative canonical sheaf $\omega := \omega_{\mathcal{C}_{H_2}/H_2}$ is a line bundle. By Proposition A.6.17, there exists a closed subscheme $H_3 \hookrightarrow H_2$ such that a morphism $T \to H_2$ factors through H_3 if and only if $\omega^{\otimes 3}|_{\mathcal{C}_T}$ and $\mathcal{O}_{\mathcal{C}}(1)|_{\mathcal{C}_T}$ differ by the pullback of a line bundle on T. The subscheme $H' := H_3$ satisfies (a)–(c) along with the universal property.

The group scheme $\mathrm{PGL}_{5g-5} = \underline{\mathrm{Aut}}(\mathbb{P}_{\mathbb{Z}}^{5g-6})$ over $\mathbb{Z}$ acts naturally on H: if $g \in \mathrm{Aut}(\mathbb{P}_S^{5g-6})$ and $[\mathcal{D} \subset \mathbb{P}_S^{5g-6}] \in H(S)$, then $g \cdot [\mathcal{D} \subset \mathbb{P}_S^{5g-6}] = [g(\mathcal{D}) \subset \mathbb{P}_S^{5g-6}]$. The closed subscheme $H' \subset H$ is PGL_{5g-5}-invariant and we claim that $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$. This finishes the theorem by the Algebraicity of Quotient Stacks (3.1.10). Consider the morphism $H' \to \mathcal{M}_g$ defined by the restriction $\mathcal{C}' \to H'$ of the universal family of the Hilbert scheme. This morphism forgets the embedding, i.e. assigns a closed subscheme $\mathcal{D} \subset \mathbb{P}_S^{5g-6}$ to the family $\mathcal{D} \to S$. This morphism is PGL_{5g-5}-invariant and descends to a morphism $[H'/\mathrm{PGL}_{5g-5}]^{\mathrm{pre}} \to \mathcal{M}_g$ of prestacks. We claim that this map is fully faithful. To see this, let $S \to H'$ be a map corresponding to a closed subscheme

$$
\begin{array}{ccc}
\mathcal{D} & \hookrightarrow & \mathbb{P}^{5g-6} \times S \\
& {}^{q}\searrow & \downarrow{}^{p_2} \\
& & S
\end{array}
$$

We will exploit the equivalences

$$
\begin{aligned}
\mathrm{H}^0(\mathbb{P}_{\mathbb{Z}}^{5g-6}, \mathcal{O}(1)) \otimes \mathcal{O}_S &\cong p_{2,*}\mathcal{O}_{\mathbb{P}^{5g-6} \times S}(1) \\
&\cong q_* \mathcal{O}_{\mathcal{D}}(1) && \text{(property (c))} \\
&\cong q_*(\omega_{\mathcal{D}/S}^{\otimes 3} \otimes q^* M) && \text{(property (b) for } M \in \mathrm{Pic}(S)) \\
&\cong q_*(\omega_{\mathcal{D}/S}^{\otimes 3}) \otimes M && \text{(projection formula)}.
\end{aligned}
$$

Therefore, we see that an automorphism of $\mathcal{D} \to S$ induces an automorphism of $\omega_{\mathcal{D}/S}^{\otimes 3}$ and thus an automorphism of $q_*(\omega_{\mathcal{D}/S}^{\otimes 3}) \otimes M$, which in turn induces an automorphism

of $\mathbb{P}^{5g-6} \times S$ preserving $\mathcal{D}$. Since $\mathcal{M}_g$ is a stack (Theorem 3.1.10), the universal property of stackification yields a morphism $[H'/\operatorname{PGL}_{5g-5}] \to \mathcal{M}_g$; this map is fully faithful since $[H'/\operatorname{PGL}_{5g-5}]^{\mathrm{pre}} \to [H'/\operatorname{PGL}_{5g-5}]$ is fully faithful (Exercise 2.4.20). It remains to check that $[H'/\operatorname{PGL}_{5g-5}] \to \mathcal{M}_g$ is essentially surjective. For this, it suffices to check that if $q\colon \mathcal{D} \to S$ is a family of smooth curves, there exists an étale cover $\{S_i \to S\}$ such that each $\mathcal{D}_{S_i}$ is in the image of $H'(S_i) \to \mathcal{M}_g(S_i)$. Since $\omega_{\mathcal{D}/S}^{\otimes 3}$ defines a closed immersion $\mathcal{D} \hookrightarrow \mathbb{P}(q_*(\omega_{\mathcal{D}/S}^{\otimes 3}))$ over S and $q_*(\omega_{\mathcal{D}/S}^{\otimes 3})$ is locally free of rank $5g-5$, we may simply take $\{S_i\}$ to be a Zariski open cover (and thus étale cover) such that each $(q_*(\omega_{\mathcal{D}/S})^{\otimes 3})|_{S_i}$ is free. $\qquad\square$

Remark 3.1.18. The entire stack $\mathcal{M}$ of smooth curves (Example 2.3.9) is also algebraic since $\mathcal{M} = \coprod_g \mathcal{M}_g$.

Exercise 3.1.19. Recall from Exercise 2.4.14 that $\mathcal{M}_{1,1}$ denotes the stack over $\mathrm{Sch}_{\text{ét}}$ parameterizing families of elliptic curves.

(a) Show that $\mathcal{M}_{1,1}$ is an algebraic stack over $\mathbb{Z}$.

(b) Use the Weierstrass form $y^2 = x^3 + ax + b$ (see [Sil09, §3.1]) to show that if we invert the primes 2 and 3, there is an isomorphism

$$\mathcal{M}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \smallsetminus V(\Delta))/\mathbb{G}_m],$$

where the action is given by $t \cdot (a,b) = (t^4 a, t^6 b)$ and Δ is the discriminant $4a^3 + 27b^2$.

(c) Define a *stable elliptic curve* over a field $\Bbbk$ as a pair (E, p) where E is an irreducible projective curve over $\Bbbk$ of arithmetic genus 1 with at worst nodal singularities and $p \in E(\Bbbk)$ is a smooth point. Over a scheme S, a *family of stable elliptic curves over S* is a proper flat $\mathcal{E} \to S$ and a section $\sigma\colon S \to \mathcal{E}$ such that every fiber is a stable elliptic curve. Denoting $\overline{\mathcal{M}}_{1,1}$ as the stack over Sch classifying stable elliptic curves, show that

$$\overline{\mathcal{M}}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$$

with the same action as above.

Exercise 3.1.20. An *n-pointed family of genus 0 curves* is smooth, proper morphism $X \to S$ of schemes with n sections $\sigma_1, \ldots, \sigma_n\colon S \to X$ such that for every $s\colon \operatorname{Spec}\Bbbk \to S$, X_s is a genus 0 curve with and $\sigma_1(s), \ldots, \sigma_n(s) \in X_s$ are distinct. In Exercise 2.4.14, we've identified $\mathcal{M}_{0,0}$ with the classifying stack $B\operatorname{PGL}_2$.

(a) Show that the prestack $\mathcal{M}_{0,n}$ parameterizing n-pointed families of genus 0 curves is a stack over $\mathrm{Sch}_{\text{ét}}$.

(b) Show that $\mathcal{M}_{0,1} \cong BU_2$ where $U_2 \subset \operatorname{PGL}_2$ is the two-dimensional subgroup of upper triangular matrices.

(c) Show that $\mathcal{M}_{0,2} \cong B\mathbb{G}_m$.

(d) Show that $\mathcal{M}_{0,3} \cong \operatorname{Spec}\mathbb{Z}$.

(e) Show that for $n > 3$, $\mathcal{M}_{0,n} \cong (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \Delta$ where Δ is the closed subscheme where at least two of the $n-3$ points are equal.

3.1.6 Algebraicity of $\operatorname{Bun}(C)$

In Proposition 2.4.16, we show that $\underline{\operatorname{QCoh}}(X)$, $\underline{\operatorname{Coh}}(X)$, and $\underline{\operatorname{Bun}}(X)$ are stacks over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ for a scheme X over a field $\Bbbk$. We now specialize to the case of a smooth,

connected, and projective curve C over an algebraically closed field $\Bbbk$, even though the following result holds in far greater generality. We define

$$\underline{\mathrm{Coh}}_{r,d}(C) \subset \underline{\mathrm{Coh}}(C) \quad \text{and} \quad \underline{\mathrm{Bun}}_{r,d}(C) \subset \underline{\mathrm{Bun}}(C)$$

as the full subcategories parameterizing pairs (E, S) such that for every geometric point $\mathrm{Spec}\, K \to S$, E_K is a coherent sheaf on C_K of rank r and degree d.

Theorem 3.1.21 (Algebraicity of $\underline{\mathrm{Bun}}(C)$). *Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$. The stacks $\underline{\mathrm{Bun}}(C)$ and $\underline{\mathrm{Coh}}(C)$ are algebraic, and $\underline{\mathrm{Bun}}(C) \subset \underline{\mathrm{Coh}}(C)$ is an open substack. For integers $r \geq 0$ and d, $\underline{\mathrm{Bun}}_{r,d}(C)$ and $\underline{\mathrm{Coh}}_{r,d}(C)$ are algebraic stacks, and $\underline{\mathrm{Bun}}_{r,d}(C) \subset \underline{\mathrm{Bun}}(C)$ and $\underline{\mathrm{Coh}}_{r,d}(C) \subset \underline{\mathrm{Coh}}(C)$ are open and closed substacks.*

Remark 3.1.22. The theorem yields decompositions

$$\underline{\mathrm{Coh}}(C) = \coprod_{r,d} \underline{\mathrm{Coh}}_{r,d}(C) \quad \text{and} \quad \underline{\mathrm{Bun}}(C) = \coprod_{r,d} \underline{\mathrm{Bun}}_{r,d}(C).$$

While $\underline{\mathrm{Bun}}_{r,d}(C)$ and $\underline{\mathrm{Coh}}_{r,d}(C)$ are not quasi-compact (Definition 3.3.20), the proof below shows that every quasi-compact open substack of $\underline{\mathrm{Bun}}_{r,d}(C)$ or $\underline{\mathrm{Coh}}_{r,d}(C)$ is a quotient stack.

Proof. We first note that $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\underline{\mathrm{Bun}}_{r,d}(C)$ are stacks over $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$ since they are defined as the full subcategories of the stacks $\underline{\mathrm{Coh}}(C)$ and $\underline{\mathrm{Bun}}(C)$ by a condition on the geometric fiber. To see that $\underline{\mathrm{Bun}}(C) \subset \underline{\mathrm{Coh}}(C)$ is an open substack, let $S \to \underline{\mathrm{Coh}}(C)$ be a morphism classified by a finitely presented, quasi-coherent sheaf E on $C \times S$ flat over S. If $V \subset C \times S$ is the open locus where E is a vector bundle, then $S \setminus p_2(C \times S \setminus V)$ is open and identified with the fiber product $S \times_{\underline{\mathrm{Coh}}(C)} \mathrm{Bun}(C)$. By Riemann–Roch (5.1.1), the Hilbert polynomial of a coherent sheaf E on C of rank r and degree d is

$$P(n) := \chi(E(n)) = \deg(E(n)) + \mathrm{rk}(E(n))(1-g) = d + rn + r(1-g).$$

Since the Hilbert polynomial is locally constant in flat families (Proposition A.2.4), the inclusions $\underline{\mathrm{Bun}}_{r,d}(C) \hookrightarrow \underline{\mathrm{Bun}}(C)$ and $\underline{\mathrm{Coh}}_{r,d}(C) \hookrightarrow \underline{\mathrm{Coh}}(C)$ are open and closed substacks. It therefore suffices to show that $\underline{\mathrm{Coh}}_{r,d}(C)$ is algebraic.

If E is a coherent sheaf C of rank r and degree d, then Serre vanishing implies that $E(N)$ is globally generated and $\mathrm{H}^1(C, E(N)) = 0$ for $N \gg 0$. This yields a surjection $\Gamma(C, E(N)) \otimes_{\Bbbk} \mathcal{O}_C \twoheadrightarrow E(N)$ inducing an isomorphism on global sections. For each integer N, let

$$\mathcal{U}_N \subset \underline{\mathrm{Coh}}_{r,d}(C)$$

be the substack parameterizing pairs $(S, \mathcal{E}) \in \underline{\mathrm{Coh}}_{r,d}(C)$ such that for every $s \in S$, $\mathcal{E}_s(N) := \mathcal{E}(N)|_{C \times \mathrm{Spec}\, \kappa(s)}$ is globally generated on $C_s := C \times_{\Bbbk} \kappa(s)$ and $\mathrm{H}^1(C_s, \mathcal{E}_s(N)) = 0$. Note that since $P(N) = \mathrm{h}^0(C_s, \mathcal{E}_s(N)) - \mathrm{h}^1(C_s, \mathcal{E}_s(N))$, these conditions imply that $\mathrm{h}^0(C, \mathcal{E}_s(N)) = P(N)$ and that the surjection $\Gamma(C, \mathcal{E}(N)) \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{E}(N)$ induces an isomorphism on global sections.

We claim that $\mathcal{U}_N \subset \underline{\mathrm{Coh}}_{r,d}(C)$ is an open substack. To verify the claim, let S be a scheme and $\mathcal{E}$ a finitely presented, quasi-coherent sheaf on $C \times S$ flat over S with Hilbert polynomial P, and consider the diagram

$$
\begin{array}{ccc}
 & C \times S & \\
{\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\
C & & S.
\end{array}
$$

A simple application of Cohomology and Base Change (see Proposition A.6.11) implies that the locus $S' \subset S$ of points $s \in S$ such that $\mathrm{H}^1(C, \mathcal{E}_s(N)) = 0$ is open, and moreover that $\left(\mathrm{R}^1 p_{2,*}\mathcal{E}(N)\right)|_{S'} = 0$ and $\left(p_{2,*}\mathcal{E}(N)\right)|_{S'}$ is a vector bundle of rank $P(N)$ whose construction commutes base change. The sheaf $\mathcal{F} := \mathrm{coker}\left(p_2^* p_{2,*}\mathcal{E}(N) \to \mathcal{E}(N)\right)$ has closed support and the open subset $S'' := S' \setminus p_2(\mathrm{Supp}(\mathcal{F}))$ is the locus of points $s \in S$ such that $\mathcal{E}_s(N)$ is globally generated and $\mathrm{H}^1(C, \mathcal{E}_s(N)) = 0$. If $S \to \underline{\mathrm{Coh}}_{r,d}(C)$ is the map classifying $\mathcal{E}$, the base change $\mathcal{U}_N \times_{\underline{\mathrm{Coh}}_{r,d}(C)} S$ is identified with S'', and the claim is established.

For each N, consider the Quot scheme

$$Q_N := \mathrm{Quot}_C^P(\mathcal{O}_C(-N)^{P(N)})$$

parameterizing quotients $\mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F$ with Hilbert polynomial P (Theorem 1.1.3). A similar argument as above shows that there is an open subscheme $Q_N' \subset Q_N$ parameterizing quotients $q \colon \mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F$ such that $\mathrm{H}^0(q(N)) \colon \mathrm{H}^0(C, \mathcal{O}_C)^{P(N)} \to \mathrm{H}^0(C, F(N))$ is surjective and $\mathrm{H}^1(C, F(N)) = 0$. The Quot scheme Q_N inherits a natural action from $\mathrm{GL}_{P(N)}$: given $g \in \mathrm{GL}_{P(N)}$ and $[q \colon \mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F] \in Q_N$,

$$g \cdot q := [\mathcal{O}_C(-N)^{P(N)} \xrightarrow{g^{-1}} \mathcal{O}_C(-N)^{P(N)} \xrightarrow{q} F] \in Q_N.$$

The locus $Q_N' \subset Q_N$ is clearly $\mathrm{GL}_{P(N)}$-invariant. The morphism

$$Q_N' \to \mathcal{U}_N, \qquad [\mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F] \mapsto F,$$

is $\mathrm{GL}_{P(N)}$-invariant (i.e. $\mathrm{GL}_{P(N)}$-equivariant with respect to the trivial action on $\mathcal{U}_N$) and thus descends to a morphism $\Psi^{\mathrm{pre}} \colon [Q_N' / \mathrm{GL}_{P(N)}]^{\mathrm{pre}} \to \mathcal{U}_N$ of prestacks. To see that Ψ^{pre} is fully faithful, observe that Ψ^{pre} induces a map $\mathrm{Stab}(q) = (\mathrm{GL}_{P(N)})_q \to \mathrm{Aut}(F)$ from the stabilizer of a quotient $[q \colon \mathcal{O}_C(-N)^{P(N)} \twoheadrightarrow F] \in Q_N$ to the automorphism group of F. Conversely, an automorphism $\alpha \in \mathrm{Aut}(F)$ induces an automorphism $\mathrm{H}^0(\alpha(N))$ of $\mathrm{H}^0(C, F(N)) \cong \Bbbk^{P(N)}$, corresponding to an element of $\mathrm{GL}_{P(N)}$ fixing q. This defines an inverse to $\mathrm{Stab}(q) \to \mathrm{Aut}(F)$, and it is not hard to extend this construction to a family of quotients over a base S.

Since $\underline{\mathrm{Coh}}_{r,d}(C)$ is a stack (Proposition 2.4.16), there is an induced morphism $\Psi \colon [Q_N' / \mathrm{GL}_{P(N)}] \to \mathcal{U}_N$ of stacks which is fully faithful (by Exercise 2.4.20) and essentially surjective (by construction). We conclude that $\mathcal{U}_N = [Q_N' / \mathrm{GL}_{P(m)}]$ and that

$$\underline{\mathrm{Coh}}_{r,d}(C) = \bigcup_N \left[Q_N' / \mathrm{GL}_{P(N)} \right].$$

The algebraicity of quotient stacks (Theorem 3.1.10) implies the algebraicity of $\underline{\mathrm{Coh}}_{r,d}(C)$. $\qquad\square$

Exercise 3.1.23. Modify the above argument to show that $\underline{\mathrm{Bun}}(X)$ and $\underline{\mathrm{Coh}}(X)$ are algebraic stacks if X is a projective scheme over $\Bbbk$. More generally, show that if $X \to S$ is a strongly projective and flat morphism of noetherian schemes, then the stack $\underline{\mathrm{Coh}}(X/S)$, whose objects over an S-scheme T are finitely presented, quasi-coherent sheaves on X_T flat over T, is an algebraic stack.

3.1.7 Universal families

Lemma 3.1.24 (Generalized 2-Yoneda Lemma)**.** *Let $\mathcal{X}$ be a stack over $\mathrm{Sch}_{\text{ét}}$. If $\mathcal{T}$ is an algebraic stack and $T \to \mathcal{T}$ is a smooth presentation, define the category $\mathcal{X}(\mathcal{T})$*

as the equalizer

$$\mathcal{X}(\mathcal{T}) := \operatorname{Eq}\big(\, \mathcal{X}(T) \rightrightarrows \mathcal{X}(T \times_{\mathcal{T}} T) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \mathcal{X}(T \times_{\mathcal{T}} T \times_{\mathcal{T}} T) \,\big),$$

i.e. an object is the data of a pair (a, α) where $a \in \mathcal{X}(T)$ and $\alpha\colon p_1^* a \xrightarrow{\sim} p_2^* a$ is an isomorphism satisfying the cocycle condition $p_{23}^* \alpha \circ p_{12}^* \alpha = p_{13}^* \alpha$, while a morphism $(a, \alpha) \to (a', \alpha')$ is the data of a morphism $\beta\colon a \to a'$ satisfying $p_2^* \beta \circ \alpha = \alpha' \circ p_1^* \beta$. There is a natural equivalence of categories

$$\operatorname{Mor}(\mathcal{T}, \mathcal{X}) \to \mathcal{X}(\mathcal{T}). \tag{3.1.25}$$

In particular, $\mathcal{X}(\mathcal{T})$ is independent of the presentation.

Proof. Let $q\colon T \to \mathcal{T}$ denote the smooth presentation, and consider the 2-commutative diagram

$$\begin{array}{ccc} T \times_{\mathcal{T}} T & \xrightarrow{p_2} & T \\ {\scriptstyle p_1}\downarrow & \overset{\gamma}{\Longleftarrow} & \downarrow{\scriptstyle q} \\ T & \xrightarrow{q} & \mathcal{T}. \end{array}$$

Given a morphism $f\colon \mathcal{T} \to \mathcal{X}$, let $a \in \mathcal{X}(T)$ be the object corresponding via the 2-Yoneda Lemma (2.3.20) to the composition $f \circ q\colon T \to \mathcal{T} \to \mathcal{X}$. Composing the 2-isomorphism $\gamma\colon q \circ p_2 \to q \circ p_1$ with f induces a 2-isomorphism $\alpha\colon p_1^* a \xrightarrow{\sim} p_2^* a$. The map (3.1.25) is defined by taking the morphism f to the pair $(a, \alpha) \in \mathcal{X}(\mathcal{T})$.

To show that (3.1.25) is an equivalence, we first assume that it holds for algebraic spaces $\mathcal{T}$. Thus, (3.1.25) holds for T, $T \times_{\mathcal{T}} T$, and $T \times_{\mathcal{T}} T \times_{\mathcal{T}} T$. We will construct an inverse $\Psi\colon \mathcal{X}(\mathcal{T}) \to \operatorname{Mor}(\mathcal{T}, \mathcal{X})$ of (3.1.25). Let $(a, \alpha) \in \mathcal{X}(\mathcal{T})$ and $f_a\colon T \to \mathcal{X}$ be the map corresponding to a. To define a morphism $\Psi(a, \alpha)\colon \mathcal{T} \to \mathcal{X}$ of stacks, let $b \in \mathcal{T}$ be an object over a scheme S classified by a map $f_b\colon S \to \mathcal{T}$, and consider the commutative diagram

$$\begin{array}{ccccccccc} S_2 & \hookrightarrow & S_1 & \longrightarrow & S_0 & \longrightarrow & T & \xrightarrow{f_a} & \mathcal{X} \\ & & & & \downarrow & \square & \downarrow{\scriptstyle q} & & \\ & & & & S & \xrightarrow{f_b} & \mathcal{T}, & & \end{array}$$

where $S_0 = S \times_{\mathcal{T}} T$ is an algebraic space, $S_1 \to S_0$ is an étale presentation, and $S_2 \to S$ is an etale surjection of schemes factoring through S_1, which exists because smooth morphisms have sections étale locally (Corollary A.3.5). The composition in the top row defines an object c_2 of $\mathcal{X}$ over S_2. The isomorphism $\alpha\colon p_1^* a \xrightarrow{\sim} p_2^* a$ over $T \times_{\mathcal{T}} T$ induces an isomorphism $\beta\colon p_1^* c_2 \xrightarrow{\sim} p_2^* c_2$ over $S_2 \times_S S_2$. Since α satisfies the cocycle condition and (3.1.25) is an equivalence for $T \times_{\mathcal{T}} T \times_{\mathcal{T}} T$, β also satisfies the cocycle condition, and thus there is an object c of $\mathcal{X}$ over S pulling back to c_2. We set $\Psi(a, \alpha)(b) = c \in \mathcal{X}(S)$. We leave the details that this is an inverse to (3.1.25) to the reader. Finally, the same argument applies to show that (3.1.25) is an equivalence when $\mathcal{T}$ is an algebraic space by using an étale presentation $T \to \mathcal{T}$. $\quad\square$

Definition 3.1.26 (Universal family)**.** If $\mathcal{X}$ is an algebraic stack, the *universal family over $\mathcal{X}$* is the object $u \in \mathcal{X}(\mathcal{X})$ (unique up to unique isomorphism) corresponding to the identity morphism $\operatorname{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ under the equivalence (3.1.25).

Exercise 3.1.27 (details). Let $\mathcal{X}$ be an algebraic stack and $u \in \mathcal{X}(\mathcal{X})$ be the universal family. If $g\colon \mathcal{S} \to \mathcal{T}$ is a morphism of algebraic stacks, show that there is a natural pullback functor $g^*\colon \mathcal{X}(\mathcal{T}) \to \mathcal{X}(\mathcal{S})$, and that if $f_a\colon T \to \mathcal{X}$ is a map from scheme classified by an object $a \in \mathcal{X}(T)$, then a is isomorphic to the pullback $f_a^* u$.

In practice, for an algebraic stack $\mathcal{X}$ arising from a moduli problem, the geometric significance of objects of $\mathcal{X}(S)$ is usually clear.

Example 3.1.28 (Universal family of $\mathcal{M}_g$). A *family of smooth curves* $\mathcal{C} \to \mathcal{T}$ over an algebraic stack $\mathcal{T}$ is a morphism of algebraic stacks which is representable by schemes, proper, and flat, and such that for every geometric point $\operatorname{Spec} \Bbbk \to \mathcal{T}$, the fiber $\mathcal{C}_s := \mathcal{C} \times_S \Bbbk$ is a connected, smooth, and projective curve of genus g. Descent theory provides an identification of $\mathcal{M}_g(\mathcal{T})$ with the category of family of smooth curves over $\mathcal{T}$. Let $[\mathcal{U}_g \to \mathcal{M}_g] \in \mathcal{M}_g(\mathcal{M}_g)$ denote the universal family. By Exercise 2.3.40, the universal family is identified with the map $\mathcal{M}_{g,1} \to \mathcal{M}_g$ forgetting a section, where $\mathcal{M}_{g,1}$ is the stack of smooth 1-pointed curves. For every morphism $S \to \mathcal{M}_g$ corresponding to a family $\mathcal{C} \to S$ of smooth curves, there is a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{U}_g = \mathcal{M}_{g,1} \\
\downarrow & \square & \downarrow \\
S & \longrightarrow & \mathcal{M}_g.
\end{array}
$$

See Figure 0.11 for a visualization of $\mathcal{U}_g \to \mathcal{M}_g$.

Example 3.1.29 (Universal family of $\underline{\operatorname{Bun}}_{r,d}(C)$). Later we will define vector bundles on an algebraic stack using the theory of quasi-coherent sheaves (see §6.1). By decent theory, $\underline{\operatorname{Bun}}_{r,d}(C)(\mathcal{T})$ is identified with the groupoid of vector bundles on $C \times \mathcal{T}$ of rank r and degree d. The universal family is a vector bundle $\mathcal{E}_{\mathrm{univ}}$ on $C \times \underline{\operatorname{Bun}}_{r,d}(C)$ with the property that for every scheme S and vector bundle E on $C \times S$ corresponding to a map $f\colon S \to \operatorname{Bun}_{r,d}(C)$, $E \cong (\mathrm{id} \times f)^* \mathcal{E}_{\mathrm{univ}}$.

3.1.8 Desideratum

We will develop the foundations of algebraic spaces and stacks in the forthcoming chapters but we first share some of the highlights.

The importance of the diagonal. When overhearing others discussing algebraic stacks, you may have wondered what's all the fuss about the diagonal. Well, I'll tell you—the diagonal encodes the stackiness!

First and foremost, the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ of an algebraic stack is representable and the diagonal $X \to X \times X$ of an algebraic space is representable by schemes (Theorem 3.2.1). The *automorphism group* or *stabilizer* G_x of a field-valued point $x\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is defined as the sheaf $\underline{\operatorname{Aut}}_{\mathcal{X}(\Bbbk)}(x) = \underline{\operatorname{Isom}}_{\mathcal{X}(\Bbbk)}(x, x)$ and is identified with the fiber product $\mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \operatorname{Spec} \Bbbk$ by Exercise 2.3.37. By Representability of the Diagonal (3.2.1) the stabilizer G_x is representable by a group algebraic space over $\Bbbk$. See §3.2.2 for a further discussion of stabilizers.

For schemes (resp. separated schemes), the diagonal is an immersion (resp. closed immersion). For algebraic spaces, the diagonal need not be an immersion, and for algebraic stacks, the diagonal need not even be a monomorphism as the fiber over $(x, x)\colon \operatorname{Spec} \Bbbk \to \mathcal{X} \times \mathcal{X}$, or in other words the stabilizer G_x, may be non-trivial. Theorems 3.6.4 and 3.6.6 characterizes algebraic spaces and Deligne–Mumford stacks in terms of the diagonal, as expressed by the following table:

Type of space	Property of the diagonal	Property of stabilizers
algebraic space	monomorphism	trivial
Deligne–Mumford stack	unramified	reduced finite groups[1]
algebraic stack	arbitrary	arbitrary

Table 3.1: Characterization of algebraic spaces and Deligne–Mumford stacks. See Table 0.1, Figure 0.1, and Table 0.3 for further diagramatic explanations of the trichotomy of moduli.

This characterization will allow us to conclude that $\mathcal{M}_g$ is Deligne–Mumford (Corollary 3.6.10). Extending this characterization, we show that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is representable if and only if the induced map on stabilizer groups $G_x \to G_{f(x)}$ is injective for every field-valued point $x \in \mathcal{X}(\Bbbk)$ (Corollaries 3.6.9 and 4.4.11).

Properties of algebraic spaces.

- (Algebraicity of Quotients by Equivalence Relations) If $R \rightrightarrows U$ is an étale equivalence relation of schemes, the quotient sheaf U/R is an algebraic space (Theorem 3.4.11). This is extended to smooth equivalence relations of algebraic spaces in Corollary 4.4.12 and fppf equivalence relations in Corollary 6.2.4, and in particular the quotient of a free action by an algebraic group on an algebraic space exists as an algebraic space.

- (Zariski's Main Theorem) If $X \to Y$ is a separated and quasi-finite morphism of noetherian algebraic spaces, then there exists a factorization $X \hookrightarrow \widetilde{X} \to Y$ where $X \hookrightarrow \widetilde{X}$ is an open immersion and $\widetilde{X} \to Y$ is finite (Theorem 4.4.9). In particular, $X \to Y$ is quasi-affine. Zariski's Main Theorem also holds for representable morphisms of Deligne–Mumford stacks, and is extended to representable morphisms of algebraic stacks in Theorem 6.1.10.

- (Affine Criteria) By Serre's Criterion for Affineness (4.4.16), an algebraic space X is an affine scheme if and only if $\Gamma(X, -)$ is exact on the category of quasi-coherent sheaves, and by Chevelley's Criterion for Affineness (4.4.20), if $X \to Y$ is a finite surjection of noetherian algebraic spaces and X is affine, then Y is also affine.

- (Algebraic spaces vs schemes) If X is a quasi-separated algebraic space, there exists a dense open subspace $U \subset X$ which is a scheme (Theorem 4.4.1). A quasi-separated group algebraic space locally of finite type over a field is a scheme (Theorem 4.4.28); in particular, the stabilizer of a point of an algebraic stack with quasi-separated diagonal is a group scheme. A separated one-dimensional algebraic spaces are schemes (Theorem 4.4.32)

Properties of Deligne–Mumford stacks.

- (Algebraicity of Quotients by Groupoids) If $R \rightrightarrows U$ is an étale groupoid of schemes, the quotient stack $[U/R]$ is a Deligne–Mumford stack (Theorem 3.4.11).

- (Local Structure of Deligne–Mumford Stacks) If $\mathcal{X}$ is a separated Deligne–Mumford stack and $x \in \mathcal{X}$ is a finite type point with stabilizer G_x, there exists an affine étale neighborhood $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ of x (Theorem 4.2.12).

---

[1]If the diagonal is not quasi-compact, the stabilizers will only be discrete and reduced.

- (Keel–Mori Theorem) If $\mathcal{X}$ is an Deligne–Mumford stack separated and of finite type over a noetherian scheme S, there exists a coarse moduli space $\mathcal{X} \to X$ where X is an algebraic space separated over S (Theorem 4.3.12).

- (Le Lemme de Gabber) If $\mathcal{X}$ is a Deligne–Mumford stack (e.g. algebraic space) separated and of finite type over a noetherian scheme S, there exists a scheme Z and a finite surjection $U \to \mathcal{X}$ (Theorem 4.5.1).

Properties of algebraic stacks.

- (Algebraicity of Quotients by Groupoids) If $R \rightrightarrows U$ is a smooth groupoid of schemes, the quotient stack $[U/R]$ is an algebraic stack (Theorem 3.4.11). This is extended to fppf groupoids of algebraic spaces in Corollary 6.2.4.

- (Residual Gerbes and Minimal Presentations) If $\mathcal{X}$ is a noetherian algebraic stack and $x \in |\mathcal{X}|$ is a finite type point, then the residual gerbe $\mathcal{G}_x$ exists and $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a locally closed immersion (Proposition 6.2.36), and if in addition the stabilizer G_x is smooth, there is a smooth presentation $(U, u) \to (\mathcal{X}, x)$ of relative dimesion $\dim G_x$ such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$ (Theorem 3.6.1). Later we establish the existence of residual gerbes for any point $x \in |\mathcal{X}|$ (Proposition 6.2.36).

- (Infinitesimal Lifting Criteria and Valuative Criteria) We establish these criteria for algebraic stacks in Theorem 3.7.1 and Theorem 3.8.2.

- (Local Structure of Algebraic Stacks) If $\mathcal{X}$ is an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal, every point $x \in \mathcal{X}(\Bbbk)$ with linearly reductive stabilizer G_x has an affine étale neighborhood $[\operatorname{Spec}(A)/G_x] \to \mathcal{X}$ of x (Theorem 6.5.1).

- (Existence of Good Moduli Spaces) Let $\mathcal{X}$ be an algebraic stack, of finite type over an algebraically closed field $\Bbbk$ of characteristic 0, with affine diagonal. If $\mathcal{X}$ is S-complete and Θ-complete, then there exists a good moduli space $\mathcal{X} \to X$ where X is a separated algebraic space of finite type over $\Bbbk$ (Theorem 6.8.1).

Historical comments

Deligne–Mumford and algebraic stacks were first introduced in [DM69] and [Art74]—and in both cases referred to as *algebraic stacks*—with conventions slightly different than ours. Namely, [DM69, Def. 4.6] assumed that algebraic stacks had an étale presentation $U \to \mathcal{X}$ which is representable by schemes (which as they point out in a footnote gives the "right" definition when the diagonal is separated and quasi-compact). On the other hand, [Art74, Def. 5.1] assumed that algebraic stacks are locally of finite type over an excellent Dedekind domain. The term *Artin stack*—which we refrain from using—is sometimes reserved for stacks that satisfy Artin's Axioms (C.7.1 or C.7.4) as first estsablished in [Art74, Thm. 5.3].

We follow the conventions of [LMB00], [Ols16], and [SP] with the following exceptions:

- [LMB00] imposes a separated and quasi-compact hypothesis on the diagonal, but we do not as in [Ols16] and [SP].

- We define algebraic spaces and stacks over $\mathrm{Sch}_{\text{ét}}$, while [LMB00] works over the étale site of affine schemes and [SP] works over $\mathrm{Sch}_{\text{fppf}}$. These gives equivalent notions of algebraic stacks (c.f. [SP, Tag 076U]).

- Perhaps confusingly, instead of our convention of calling a morphism $\mathcal{X} \to \mathcal{Y}$ of stacks *representable* (resp. *representable by schemes*), [SP, Tags 02ZW and 02Y7] uses *representable by algebraic spaces* (resp. *representable*).

3.2 Representability of the diagonal

3.2.1 Representability

Theorem 3.2.1 (Representability of the Diagonal)**.**

   (1) The diagonal of an algebraic space is representable by schemes.

   (2) The diagonal of an algebraic stack is representable.

Proof. Let X be an algebraic space and $U \to X$ be an étale presentation. Define the scheme $R := U \times_X U$. If $T \to X \times X$ is a morphism from a scheme, we need to show that the sheaf $Q_T = X \times_{X \times X} T$ is in fact a scheme. Since $U \to X$ is étale, surjective, and representable by schemes, so is $U \times U \to X \times X$. The base change of $T \to X \times X$ by $U \times U \to X \times X$ is a scheme T' which is surjective étale over T. In the cartesian cube

$$(3.2.2)$$

Q_T is a sheaf on $\mathrm{Sch}_{\text{ét}}$ while $Q_{T'}$ is a scheme. Since $R \to U \times U$ is a separated and locally quasi-finite morphism of schemes, so is $Q_{T'} \to T'$. (If X had a quasi-compact diagonal, then by Zariski's main theorem $R \to U \times U$ is quasi-affine and thus so is $Q_{T'} \to T'$.) Since Q_T is a sheaf in the étale topology that pulls back to a scheme $Q_{T'}$ separated and locally quasi-finite over T', we may apply Effective Descent (Proposition 2.2.15) to conclude that Q_T is a scheme.

If $\mathcal{X}$ is an algebraic stack and $U \to \mathcal{X}$ is a smooth presentation, we may imitate the above argument. The fiber product $R := U \times_{\mathcal{X}} U$ is an algebraic space. If $T \to \mathcal{X} \times \mathcal{X}$ is a morphism from a scheme, its base change along $U \times U \to \mathcal{X} \times \mathcal{X}$ yields an algebraic space T_1 which is surjective smooth over T. Choose an étale presentation $T_2 \to T_1$. Then $T_2 \to T$ is a surjective smooth morphism of schemes which has a section after an étale cover $T' \to T$ (Proposition A.3.4). The composition $T' \to T_2 \to T_1 \to U \times U$ provides a lift of $T \to \mathcal{X} \times \mathcal{X}$. We obtain a diagram similar to (3.2.2) but where the left and right squares are not necessarily cartesian. Note that Q_T is a sheaf as it is identified with $\underline{\mathrm{Isom}}_{\mathcal{X}(T)}(a, a)$ by Exercise 2.3.37, where $a \colon T \to \mathcal{X}$. The morphism $Q_{T'} \to Q_T$ is étale, surjective, and representable by schemes (as $T' \to T$ is). Choosing an étale presentation $V \to Q_{T'}$ of the algebraic space $Q_{T'}$, the composition $V \to Q_{T'} \to Q_T$ yields an étale presentation showing that Q_T is an algebraic space. $\qquad\square$

Corollary 3.2.3.

   (1) If the diagonal of a stack $\mathcal{X}$ is representable (resp. representable by a scheme), then every morphism $U \to \mathcal{X}$ from a scheme is representable (resp. representable by a scheme).

   (2) Every morphism from a scheme to an algebraic stack (resp. algebraic spaces) is representable (resp. representable by schemes).

Proof. The first part follows from the cartesian diagram

$$
\begin{array}{ccc}
T_1 \times_{\mathcal{X}} T_2 & \longrightarrow & T_1 \times T_2 \\
\downarrow & \square & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X}.
\end{array}
$$

associated to any two maps $T_1 \to \mathcal{X}$ and $T_2 \to \mathcal{X}$ from schemes to an algebraic stack. The second part follows directly from the first part and the Representability of the Diagonal (Theorem 3.2.1). $\qquad\square$

Exercise 3.2.4.

(a) If $\mathcal{X} \to \mathcal{Y}$ is a representable morphism of algebraic stacks (e.g. a morphism of algebraic spaces), then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by schemes.

(b) If $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable.

3.2.2 Stabilizer groups and the inertia stack

Now that we know the diagonal is representable, we can discuss its properties. One of the most important features of the diagonal is that it encodes the stabilizer groups.

Definition 3.2.5 (Stabilizers). If $\mathcal{X}$ is an algebraic stack and $x \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is a field-valued point, the *stabilizer of x* is defined as the group algebraic space $G_x := \underline{\operatorname{Aut}}_{\mathcal{X}(\Bbbk)}(x)$.

By Exercise 2.3.37, we can identify G_x with the fiber product

$$
\begin{array}{ccc}
G_x = \underline{\operatorname{Aut}}_{\mathcal{X}(\Bbbk)}(x) & \longrightarrow & \operatorname{Spec} \Bbbk \\
\downarrow & \square & \downarrow {\scriptstyle (x,x)} \\
\mathcal{X} & \overset{\Delta}{\longrightarrow} & \mathcal{X} \times \mathcal{X}.
\end{array}
$$

The sheaf G_x is representable by an algebraic space over K by Representability of the Diagonal (Theorem 3.2.1). The stabilizer G_x is a group algebraic space, i.e. an algebraic space G_x with multiplication, inverse, and identity morphisms satisfying the commutativity conditions of Definition B.2.1 (or equivalently a group object in the category of algebraic spaces). In fact, G_x is actually a group scheme locally of finite type as long as the diagonal of $\mathcal{X}$ is quasi-separated (Corollary 4.4.31).

Remark 3.2.6. Let G be a group scheme over a field $\Bbbk$ acting on a $\Bbbk$-scheme U via $\sigma \colon G \times U \to U$, and let $u \in U(\Bbbk)$. The stabilizer of the image of u in $[U/G]$ is the usual stabilizer group scheme, i.e. the fiber product of $(\sigma, p_2) \colon G \times U \to U \times U$ along $(u, u) \colon \operatorname{Spec} \Bbbk \to U \times U$.

Exercise 3.2.7.

(a) Show that the stabilizer of a field-valued point of a fiber product of algebraic stacks is the fiber product of stabilizers, i.e. for $x' \in (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}')(\Bbbk)$, then $G_{x'} = G_x \times_{G_y} G_{y'}$ where x, y and y' are the images of x'.

(b) Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and $x \in \mathcal{X}(\Bbbk)$ be a field valued point. Show that the fiber of the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ over the point $(x, x, \mathrm{id}) \in (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})(\Bbbk)$ is identified with $\ker(G_x \to G_y)$. What is the fiber of the diagonal over an arbitrary field-valued point of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$?

Exercise 3.2.8. Let $\mathcal{X}$ be a Deligne–Mumford stack (quasi-separated Deligne–Mumford stack). An algebraic stack is *quasi-separated* if for every morphism $(a, b)\colon S \to \mathcal{X} \times \mathcal{X}$ from a scheme, the fiber product $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b) \cong \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (a,b)} S$ is quasi-compact over S; see also Definition 3.3.10.

  (a) For a field-valued point $x \in \mathcal{X}(\Bbbk)$, show that G_x is a separated (resp. finite) étale group scheme over $\Bbbk$.

     Hint: First show that G_x is an etale group algebraic space over $\Bbbk$. If $\Bbbk = \overline{\Bbbk}$, use that a section of the structure morphism $G_x \to \mathrm{Spec}\,\Bbbk$ is an open immersion to give an open covering of G_x by schemes. Apply Proposition B.2.9 to conclude that G_x is separated. For the general case, apply Effective Descent (Proposition 2.2.15).

  (b) If $\Bbbk$ is algebraically closed, show that G_x is the discrete and reduced (resp. finite and reduced) group scheme corresponding to the abstract group $G_x(\Bbbk)$.

  (c) Show that the diagonal of $\mathcal{X}$ is unramified.

We will see later that these properties characterize Deligne–Mumford stacks; see Theorem 3.6.4.

Varying the point x of $\mathcal{X}$, the stabilizer group varies and naturally forms a family. We've already seen this: if $a\colon T \to \mathcal{X}$ is an object, then $\mathrm{Isom}_{\mathcal{X}(T)}(a) \to S$ is a group algebraic space such that the fiber over a point $s \in S$ is the stabilizer of the restriction $a|_{\mathrm{Spec}\,\kappa(s)}$ of a to $\mathrm{Spec}\,\kappa(s)$. Applying this to the identity map $\mathrm{id}_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$ yields the construction of the inertia stack.

Definition 3.2.9 (Inertia stack). The *inertia stack* of an algebraic stack $\mathcal{X}$ is the fiber product

$$
\begin{array}{ccc}
I_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\
\downarrow & \square & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X}.
\end{array}
$$

In the relative setting of a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks, the *relative inertia stack* is $I_{\mathcal{X}/\mathcal{Y}} := \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$.

The stack $I_{\mathcal{X}}$ is equivalent to the category of pairs (x, α) where $x \in \mathcal{X}$ is an object over $S \in \mathrm{Sch}$ and $\alpha \in \mathrm{Aut}_{\mathcal{X}(S)}(x)$ (see Exercise 3.2.14(a)). The fiber of $I_{\mathcal{X}} \to \mathcal{X}$ over a field-valued point $x\colon \mathcal{X}(\Bbbk)$ is precisely the stabilizer G_x. We can therefore think of $I_{\mathcal{X}}$ as a group scheme (or really group algebraic space) over $\mathcal{X}$ incorporating all of the stabilizers of $\mathcal{X}$. If we let $(\mathrm{Sch}\,/\mathcal{X})_{\text{ét}}$ be the big étale site of schemes over $\mathcal{X}$, then $I_{\mathcal{X}}$ can be viewed as a sheaf of groups on $(\mathrm{Sch}\,/\mathcal{X})_{\text{ét}}$ where $I_{\mathcal{X}}(a) = \mathrm{Aut}_S(a)$ for $a \in \mathcal{X}(S)$. If $a' \to a$ is a morphism over $S' \to S$, there is a natural pullback functor $\alpha^*\colon \mathrm{Aut}_S(a) \to \mathrm{Aut}_{S'}(a')$ defined as follows: for $\beta \in \mathrm{Aut}_S(a)$, the image $\alpha^*(\beta)$ is the unique dotted arrow (provided by Axiom (2) of a prestack (Definition 2.3.1)) making the diagram

$$
a' \overset{\beta \circ \alpha}{\underset{\alpha^*(\beta)}{\dashrightarrow}} a' \overset{}{\underset{\alpha}{\longrightarrow}} a \tag{3.2.10}
$$

commute. Note that if $\alpha\colon \alpha \to \alpha$ is an isomorphism over the identity, then $\alpha^*(\beta) = \alpha^{-1} \circ \beta \circ \alpha$ is conjugation by α.

Exercise 3.2.11. Let $G \to S$ be a group scheme acting on a scheme $U \to S$, and

let $\mathcal{X} = [U/G]$ be the quotient stack. Show that there is a cartesian diagram

$$
\begin{array}{ccc}
S_U & \longrightarrow & U \\
\downarrow & \square & \downarrow \\
I_{\mathcal{X}} & \longrightarrow & \mathcal{X}
\end{array}
$$

where $S_U \to U$ is the stabilizer group scheme, i.e. the fiber product of the action map $G \times U \to U \times U$ and the diagonal $U \to U \times U$.

Example 3.2.12. The inertia stack of the classifying stack $B\mathbb{G}_m$ is $I_{B\mathbb{G}_m} \cong \mathbb{G}_m \times B\mathbb{G}_m$. Similarly, if we let $\mathbb{G}_m$ act on $\mathbb{G}_m \times \mathbb{A}^1$ via the product of the trivial and the scaling action and we let $V(x(t-1)) \subset \mathbb{G}_m \times \mathbb{A}^1$ be the $\mathbb{G}_m$-invariant closed subscheme, then $I_{[\mathbb{A}^1/\mathbb{G}_m]} \cong [V(x(t-1))/\mathbb{G}_m]$.

Exercise 3.2.13.

(a) If G is a smooth affine algebraic group over a field $\Bbbk$, show that the inertia stack of BG is the quotient $[G/G]$ where G acts on itself via conjugation. In particular, if G is abelian then $I_{BG} \cong G \times BG$.

(b) More generally, show that if G acts on a $\Bbbk$-scheme U, show that $I_{[U/G]} \cong [(G \times U)/G]$ where the action is given by $g \cdot (h, u) = (ghg^{-1}, gu)$.

(c) Let G be a group scheme over $\Bbbk$ corresponding to a finite abstract group. If G acts on a $\Bbbk$-scheme U, then

$$
I_{[U/G]} = \coprod_{g \in \mathrm{Conj}(G)} [U^g / C_g],
$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of elements of G, C_g is the centralizer of g, and $U^g := \{x \in U \mid gx = x\}$ is the closed locus fixed by g.

(d) Explicitly compute the inertia stack for the quotient $[\mathbb{A}^3/S_3]$ of the permutation action.

Exercise 3.2.14. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.

(a) Show that $I_{\mathcal{X}/\mathcal{Y}}$ is equivalent to the category of pairs (x, α) where $x \in \mathcal{X}$ and $\alpha \colon x \xrightarrow{\sim} x$ is an isomorphism such that $f(\alpha) = \mathrm{id}_{f(x)}$, and show that the identity section $e \colon \mathcal{X} \to I_{\mathcal{X}/\mathcal{Y}}$ takes an object x to (x, id_x).

(b) Show that there are morphisms $I_{\mathcal{X}/\mathcal{Y}} \to I_{\mathcal{X}} \to I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ of algebraic stacks over $\mathcal{X}$ such that the induced morphisms on the fibers over a field-valued point $x \in \mathcal{X}(\Bbbk)$ correspond to a left exact sequence $1 \to K_x \to G_x \to G_{f(x)}$ of algebraic groups.

(c) Show that there is a cartesian diagram

$$
\begin{array}{ccc}
I_{\mathcal{X}} & \longrightarrow & I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X} \\
\downarrow & \square & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}.
\end{array}
$$

Hint for (c): An object of $I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X}$ over a scheme S is a quadruple (y, α, x, β) where $y \in \mathcal{X}(S)$, $\alpha \colon y \xrightarrow{\sim} y$, $x \in \mathcal{X}(S)$, and $\beta \colon y \xrightarrow{\sim} f(x)$. On the other hand, an object of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ over S is a triple (x_1, x_2, γ) where $x_1, x_2 \in \mathcal{X}(S)$ and $\gamma \colon f(x_1) \xrightarrow{\sim} f(x_2)$. Define $I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ on fiber categories by $(y, \alpha, x, \beta) \mapsto (x, x, \beta \circ \alpha \circ \beta^{-1})$. Construct a map $I_{\mathcal{X}}(S)$ to the fiber product of $\mathcal{X}(S)$ and $(I_{\mathcal{Y}} \times_{\mathcal{Y}} \mathcal{X})(S)$ over $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X})(S)$, and show that it is an equivalence.

3.3 First properties

3.3.1 Properties of morphisms

Recall that a morphism of prestacks $\mathcal{X} \to \mathcal{Y}$ over $\mathrm{Sch}_{\text{ét}}$ is representable by schemes (resp. representable) if for every morphism $T \to \mathcal{Y}$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme (resp. algebraic space); see Definitions 3.1.1 and 3.1.3. Both notions are clearly stable under base change. Morphisms representable by schemes are also clearly stable under composition, and the following lemma shows the same for representable morphisms.

Lemma 3.3.1.

(1) *If $\mathcal{X} \to Y$ is a representable morphism of prestacks over $\mathrm{Sch}_{\text{ét}}$ and Y is an algebraic space, then $\mathcal{X}$ is an algebraic space.*

(2) *The composition of representable morphisms is representable.*

Proof. For the first statement, if $V \to Y$ is an étale presentation by a scheme V, the base change $\mathcal{X}_V$ is an algebraic space. Since the diagonal $\mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ base changes under $V \to Y$ to a monomorphism $\mathcal{X}_V \to \mathcal{X}_V \times_V \mathcal{X}_V$, étale descent implies that $\mathcal{X} \to \mathcal{X} \times_Y \mathcal{X}$ is also a monomorphism. Hence, $\mathcal{X}$ is equivalent to a sheaf (Exercise 2.3.37). Moreover, the base change $\mathcal{X}_V \to \mathcal{X}$ is a morphism of algebraic spaces which is étale, surjective and representable by schemes. Letting $U \to X_V$ be an étale presentation, then the composition $U \to \mathcal{X}_V \to \mathcal{X}$ is étale, surjective, and representable by schemes, and thus $\mathcal{X}$ is an algebraic space. The second statement follows from the first. $\qquad\square$

Definition 3.3.2. Let $\mathcal{P}$ be a property of morphisms of schemes.

(1) We say that $\mathcal{P}$ is *étale local on the source* if for every étale surjection $X' \to X$ of schemes, a morphism $X \to Y$ satisfies $\mathcal{P}$ if and only if $X' \to X \to Y$ does, and that $\mathcal{P}$ is *étale local on the target* if for every étale surjection $Y' \to Y$ of schemes, a morphism $X \to Y$ satisfies $\mathcal{P}$ if and only if $X \times_Y Y' \to Y$ does. The notions of being *smooth (resp. fppf, fpqc) local on the source/target* are defined similarly.

(2) If $\mathcal{P}$ is stable under composition and base change and is étale local (resp. smooth local) on the source and target, a morphism $\mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks (resp. algebraic stacks) *has property $\mathcal{P}$* if for all étale (resp. smooth) presentations (equivalently there exists presentations) $V \to \mathcal{Y}$ and $U \to \mathcal{X} \times_{\mathcal{Y}} V$ yielding a diagram

$$\begin{array}{ccccc} U & \longrightarrow & \mathcal{X} \times_{\mathcal{Y}} V & \longrightarrow & V \\ & & \downarrow & \square & \downarrow \\ & & \mathcal{X} & \longrightarrow & \mathcal{Y}, \end{array}$$

the composition $U \to V$ has $\mathcal{P}$.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks representable by schemes *has property $\mathcal{P}$* if for every morphism $T \to \mathcal{Y}$ from a scheme, the base change $\mathcal{X} \times_{\mathcal{Y}} T \to T$ has $\mathcal{P}$. In particular, a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is an *isomorphism*, *open immersion*, *closed immersion*, *locally closed immersion*, *affine*, or *quasi-affine* if it is representable by schemes and has the corresponding property.

The properties of flatness, smoothness (resp. smoothness of relative dimension n), surjectivity, locally of finite presentation, and locally of finite type are smooth local on the source and target. By (2), these properties extend to morphisms of algebraic stacks. Likewise, étaleness and unramifiedness are étale local on the source and target, and thus extend to morphisms of Deligne–Mumford stacks. These properties are stable under composition and base change.

Exercise 3.3.3. Show that the diagonal of a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is locally of finite type.

We now show that certain representable morphisms are smooth local on the target. They are even fppf local, but this is postponed until Proposition 6.2.3.

Proposition 3.3.4. *Let $\mathcal{P}$ be one of the following properties of morphisms of algebraic stacks: representable, isomorphism, open immersion, closed immersion, locally closed immersion, affine, or quasi-affine. Consider a cartesian diagram*

$$
\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{Y}' \\
\downarrow & \square & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{Y}
\end{array}
$$

of algebraic stacks where $\mathcal{Y}' \to \mathcal{Y}$ is smooth and surjective. Then $\mathcal{X} \to \mathcal{Y}$ has $\mathcal{P}$ if and only if $\mathcal{X}' \to \mathcal{Y}'$ has $\mathcal{P}$.

Proof. We will show the ($\Leftarrow$) implications as the other directions are clear. For representability, we may assume that $\mathcal{Y}$ and $\mathcal{Y}'$ are schemes. It suffices to show that the every automorphism $\alpha \colon a \to a$ of an object $a \in \mathcal{X}$ over a scheme T is trivial. The base change T' of $a \colon T \to \mathcal{X}$ by $\mathcal{X}' \to \mathcal{X}$ is a scheme since it's also identified with $T \times_{\mathcal{Y}} \mathcal{Y}'$. Since smooth morphisms étale locally have sections (Corollary A.3.5), there is an étale cover $g \colon \widetilde{T} \to T$ that factors through T'. The automorphism α defines a section of $\underline{\mathrm{Aut}}_T(a)$ over T. Since $\underline{\mathrm{Aut}}_T(a)$ is a sheaf on $(\mathrm{Sch}/T)_{\text{ét}}$ and $g^*\alpha = \mathrm{id}$, we have that $\alpha = \mathrm{id}$.

For the other properties, we already know that $\mathcal{X} \to \mathcal{Y}$ is representable and it thus suffices to assume that $\mathcal{Y}$, $\mathcal{Y}'$ and $\mathcal{X}'$ are schemes and that $\mathcal{X}$ is an algebraic space. Fortunately we can apply Effective Descent (2.2.15) to conclude that $\mathcal{X}$ is a scheme and that $\mathcal{X} \to \mathcal{Y}$ has property $\mathcal{P}$. $\square$

Example 3.3.5. If $G \to S$ is a smooth affine group scheme acting on an algebraic space $U \to S$, then $[U/G] \to S$ is flat (resp. smooth, surjective, locally of finite presentation, locally of finite type) if and only if $U \to S$ is. In particular, using the quotient stack presentations in the proofs of Theorems 3.1.17 and 3.1.21, we conclude that $\mathcal{M}_g$ is locally of finite type over $\mathbb{Z}$ and that both $\underline{\mathrm{Coh}}_{r,d}(C)$ and $\mathrm{Bun}_{r,d}(C)$ are locally of finite type over $\Bbbk$.

3.3.2 Properties of algebraic spaces and stacks

Definition 3.3.6 (Properties of algebraic spaces and stacks)**.** Let $\mathcal{P}$ be a property of schemes which is étale (resp. smooth) local, i.e. if $X \to Y$ is an étale (resp. smooth) surjection of schemes, then X has $\mathcal{P}$ if and only if Y has $\mathcal{P}$. We say that a Deligne–Mumford stack (resp. algebraic stack) $\mathcal{X}$ *has property $\mathcal{P}$* if for an étale (resp. smooth) presentation (equivalently for all presentations) $U \to \mathcal{X}$, the scheme U has $\mathcal{P}$.

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The properties of being locally noetherian, reduced, or regular are smooth local (Proposition B.1.12).

Example 3.3.7. Let $G \to S$ be a smooth affine group scheme acting on a scheme U over S. Then $[U/G]$ is locally noetherian, reduced, or regular if and only if U is.

Definition 3.3.8 (Substacks). If $\mathcal{X}$ is an algebraic stack, a substack $\mathcal{Z} \subset \mathcal{X}$ is *closed* (resp. *open*, *locally closed*) if the induced morphism $\mathcal{Z} \to \mathcal{X}$ is a closed immersion (resp. open immersion, locally closed immersion).

Exercise 3.3.9. For an action of a smooth affine group scheme $G \to S$ on a scheme U over S, show that there is an equivalence between closed (resp. open) substacks of $[U/G]$ and G-invariant closed (resp. open) subschemes of U.

3.3.3   Separation properties

Separation properties for algebraic stacks are defined in terms of the diagonal.

Definition 3.3.10.

(1) A morphism of algebraic stack $\mathcal{X} \to \mathcal{Y}$ has *affine diagonal* (resp. *quasi-affine diagonal, separated diagonal*) if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is affine (resp. quasi-affine, separated). An algebraic stack $\mathcal{X}$ has *affine diagonal* (resp. *quasi-affine diagonal, separated diagonal*) if $\mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ does.

(2) A morphism of algebraic stack $\mathcal{X} \to \mathcal{Y}$ is *quasi-separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and second diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ are quasi-compact. An algebraic stack $\mathcal{X}$ is *quasi-separated* if it is quasi-separated over $\operatorname{Spec}\mathbb{Z}$.

(3) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, which is representable by schemes (Exercise 3.2.4), is proper. Separated morphisms are defined in general in Definition 3.8.1.

Conditions on the diagonal translate to conditions on the Isom sheaves since the base change of $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ by a morphism $(a, b) \colon S \to \mathcal{X} \times \mathcal{X}$ from a scheme S is identified with $\underline{\operatorname{Isom}}_{\mathcal{X}(S)}(a, b)$ (see Exercise 2.3.37), which is an algebraic space by Representability of the Diagonal (Theorem 3.2.1(2)). In particular, $\mathcal{X}$ has affine diagonal if and only if every algebraic space $\underline{\operatorname{Isom}}_{\mathcal{X}(S)}(a, b)$ is a scheme affine over S. Every algebraic stack with affine or quasi-affine diagonal is necessarily quasi-separated.

Lemma 3.3.11. *Let S be an affine scheme and $G \to S$ be a smooth affine group scheme acting on an algebraic space U over S. If U has affine diagonal (resp. quasi-affine diagonal), then so does $[U/G]$.*

Proof. Recall that we established that $[U/G]$ is an algebraic stack in Theorem 3.1.10. Representability of the Diagonal (Theorem 3.2.1(2)) implies that $[U/G] \to [U/G] \times_S [U/G]$ is a representable morphism. Applying smooth descent to the cartesian diagram

$$
\begin{array}{ccc}
G \times_S U & \longrightarrow & U \times_S U \\
\downarrow & \square & \downarrow \\
[U/G] & \longrightarrow & [U/G] \times_S [U/G]
\end{array}
$$

of Exercise 2.3.35, it suffices to show that $G \times_S U \to U \times_S U$ is affine (resp. quasi-affine). Since G is affine, so is the composition $G \times_S U \to U \times_S U \xrightarrow{p_2} U$. On

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the other hand, $p_2 \colon U \times_S U \to U$ has affine diagonal (resp. quasi-affine diagonal) since U does. The cancellation law implies that $G \times_S U \to U \times_S U$ is affine (resp. quasi-affine). $\qquad\square$

The condition of having affine diagonal is satisfied by most moduli problems (except for example $\mathcal{M}_1$).

Example 3.3.12. The moduli stacks $\mathcal{M}_g$ and $\underline{\mathrm{Bun}}_{r,d}(C)$ have affine diagonal and are thus quasi-separated. The statement for $\mathcal{M}_g$ follows from the above lemma and the quotient presentation $\mathcal{M}_g = [H' / \mathrm{PGL}_{5g-5}]$ in the proof of Theorem 3.1.17 where H' is locally closed subscheme of a projective Hilbert scheme. We will show later that $\mathcal{M}_g$ is separated or in other words that the diagonal of $\mathcal{M}_g$ is a finite morphism.

Similarly in Theorem 3.1.21, we expressed every quasi-compact open substack of $\underline{\mathrm{Bun}}_{r,d}(C)$ as a quotient stack $[Q' / \mathrm{GL}_N]$ where Q' is an open subscheme of a projective Quot scheme. To see that $\underline{\mathrm{Bun}}_{r,d}(C)$ has affine diagonal, it suffices to show that the base change of the along a morphism $\mathrm{Spec}\, A \to \underline{\mathrm{Bun}}_{r,d}(C) \times \underline{\mathrm{Bun}}_{r,d}(C)$ is affine. But such a morphism factors through $\mathcal{U} \times \mathcal{U}$ for some quasi-compact open substack $\mathcal{U} \subset \underline{\mathrm{Bun}}_{r,d}(C)$ and we know that $\mathcal{U}$ has affine diagonal.

Remark 3.3.13. A quasi-separated Deligne–Mumford stack has *finite* and reduced stabilizer groups (see Exercise 3.2.8).

For morphisms of schemes, the definition of separatedness above agrees with the usual notation as the diagonal of a morphism of schemes is a closed immersion if and only if it is proper. We postpone the definition of separatedness for non-representable morphisms until Definition 3.8.1.

Example 3.3.14. The non-separated union $\mathbb{A}^\infty \bigcup_{\mathbb{A}^\infty \setminus 0} \mathbb{A}^\infty$ is a typical example of a non-quasi-separated scheme. For algebraic spaces and stacks, there are additional pathologies coming from actions of non-quasi-compact group schemes. For instance, $[\mathbb{A}^1 / \mathbb{Z}]$ is a non-quasi-separated algebraic space (see Example 3.9.22) while $B\mathbb{Z}$ is a non-quasi-separated algebraic stack (see Example 3.9.20).

Exercise 3.3.15. An action of an algebraic group G over a field $\Bbbk$ on an algebraic space U is called *proper* if the action map

$$\Psi \colon G \times U \to U \times U, \quad (g, u) \mapsto (gu, u)$$

is proper.

(a) Show that the action of G on U is proper if and only if $[U/G]$ is separated.

(b) For $u \in U(\Bbbk)$, let $\Psi_u \colon G \to U$ be the map defined by $g \mapsto gu$ (viewing Ψ as a morphism over U via the projections on the second component, then Ψ_u is the fiber of Ψ over u). Show that the following are equivalent:

   (i) $\Psi_u \colon G \to U$ is proper,
   (ii) $u \colon \mathrm{Spec}\,\Bbbk \to [U/G]$ is proper,
   (iii) $Gu \subset U$ is closed and G_u is proper.

   Hint: To show that (i) or (ii) implies (iii), replace U with the reduced orbit Gu, use Generic Flatness (3.3.30) to show that $\mathrm{Spec}\,\Bbbk \to [U/G]$ is faithfully flat, and then use fppf descent.

3.3.4 The topological space of a stack

We can associate a topological space $|\mathcal{X}|$ to every algebraic stack $\mathcal{X}$.

Definition 3.3.16 (Topological space of an algebraic stack). If $\mathcal{X}$ is an algebraic stack, we define *the topological space of* $\mathcal{X}$ as the set $|\mathcal{X}|$ consisting of field-valued morphisms $x\colon \operatorname{Spec} K \to \mathcal{X}$. Two morphisms $x_1\colon \operatorname{Spec} K_1 \to \mathcal{X}$ and $x_2\colon \operatorname{Spec} K_2 \to \mathcal{X}$ are identified in $|\mathcal{X}|$ if there exists field extensions $K_1 \to K_3$ and $K_2 \to K_3$ such that $x_1|_{\operatorname{Spec} K_3}$ and $x_2|_{\operatorname{Spec} K_3}$ are isomorphic in $\mathcal{X}(K_3)$. A subset $U \subset |\mathcal{X}|$ is open if there exists an open substack $\mathcal{U} \subset \mathcal{X}$ such that $U = |\mathcal{U}|$.

A morphism of algebraic stacks $\mathcal{X} \to \mathcal{Y}$ induces a continuous map $|\mathcal{X}| \to |\mathcal{Y}|$.

Exercise 3.3.17. Show that if $\mathcal{X}$ is an algebraic stack and $U \subset |\mathcal{X}|$ is an open subset, then there exists a reduced closed substack $\mathcal{Z} \hookrightarrow \mathcal{X}$ such that $|\mathcal{Z}| = |\mathcal{X}| \setminus U$.

Example 3.3.18. The topological space of the quotient stack $|[\mathbb{A}^1_\Bbbk/\mathbb{G}_m]|$ with the standard scaling action consists of two points with representatives $x_0\colon \operatorname{Spec} k \xrightarrow{0} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ and $x_1\colon \operatorname{Spec} k \xrightarrow{1} \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$. In particular, the inclusion of the generic point $\operatorname{Spec} \Bbbk(x) \to \mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to x_1.

While the stabilizer group $G_{\overline{x}}$ depends on the choice of representative $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ of $x \in |\mathcal{X}|$, its dimension—which we denote by $\dim G_x$—is independent of this choice. Similarly, the properties of being smooth, unramified, affine, finite, and reduced are also independent of this choice.

Exercise 3.3.19. Let $x \in |\mathcal{X}|$ be a point of an algebraic stack with two representatives $x_1\colon \operatorname{Spec} \Bbbk_1 \to \mathcal{X}$ and $x_2\colon \operatorname{Spec} \Bbbk_2 \to \mathcal{X}$.

(1) Show that the stabilizer group G_{x_1} is smooth (resp. étale, unramified, affine, finite) if and only if G_{x_2} is.

(2) Show that $\dim G_{x_1} = \dim G_{x_2}$.

(3) If $\mathcal{X}$ is Deligne–Mumford and both $\Bbbk_1$ and $\Bbbk_2$ are algebraically closed, show that the abstract discrete groups corresponding to G_{x_1} and G_{x_2} (see Exercise 3.2.8) are isomorphic.

As a consequence of the above exercise, it makes sense to say that $x \in |\mathcal{X}|$ has *smooth* (resp. *étale, unramified, affine, finite*) *stabilizer*. For a Deligne–Mumford stack $\mathcal{X}$, we define the *geometric stabilizer of* x as the discrete group $G = G_{\overline{x}}(\Bbbk)$ where $\overline{x}\colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is a geometric point representing x.

We can now define topological properties of algebraic stacks and their morphisms.

Definition 3.3.20. We say that an algebraic stack $\mathcal{X}$ is *quasi-compact, connected,* or *irreducible* if $|\mathcal{X}|$ is, and we say that $\mathcal{X}$ is *noetherian* if it is locally noetherian, quasi-compact and quasi-separated.

Exercise 3.3.21. Show that an algebraic stack $\mathcal{X}$ is quasi-compact if and only if there exists a smooth presentation $\operatorname{Spec} A \to \mathcal{X}$ and that a quasi-separated algebraic stack $\mathcal{X}$ is noetherian if and only if there exists a smooth presentation $\operatorname{Spec} A \to \mathcal{X}$ where A is a noetherian ring.

Example 3.3.22. The moduli stack $\mathcal{M}_g$ is noetherian and in particular quasi-compact. This follows from the above exercise using the quotient presentation $\mathcal{M}_g = [H'/\operatorname{PGL}_{5g-5}]$ from Theorem 3.1.17. However, $\underline{\operatorname{Bun}}_{r,d}(C)$ is not quasi-compact.

Exercise 3.3.23.

(a) Show that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is surjective if and only if $|\mathcal{X}| \to |\mathcal{Y}|$ is surjective.

(b) Show that if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Y}' \to \mathcal{Y}$ are morphisms of algebraic stacks, then $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{X}| \times_{|\mathcal{Y}|} |\mathcal{Y}'|$ is surjective.

Exercise 3.3.24. If $\mathcal{X}$ is a quasi-compact and locally noetherian algebraic stack, show that $|\mathcal{X}|$ is a noetherian topological space.

Exercise 3.3.25. Since the property of being universally open for a morphism of schemes is smooth local on the source and target, we can define *universally open* morphisms of algebraic stacks using Definition 3.3.2(2). This property includes faithfully flat morphisms locally of finite presentation.

(a) If $f \colon \mathcal{X} \to \mathcal{Y}$ is a universally open morphism of algebraic stacks, show that $f(|\mathcal{X}|) \subset |\mathcal{Y}|$ is open and conclude that for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}'|$ is open.

Hint: Show that the image is identified with the open substack $\mathcal{V} \subset \mathcal{Y}$, whose objects over a scheme T consist of morphisms $T \to \mathcal{Y}$ such that $\mathcal{X}_T \to T$ is surjective.

(b) Show that if $U \to \mathcal{X}$ is a smooth presentation of an algebraic stack, then a set $\Sigma \subset |\mathcal{X}|$ is open (resp. closed) if and only if its preimage in U is.

Definition 3.3.26. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-compact* if for every morphism $\operatorname{Spec} B \to \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} B$ is quasi-compact. We say that $\mathcal{X} \to \mathcal{Y}$ is *of finite type* if $\mathcal{X} \to \mathcal{Y}$ is locally of finite type and quasi-compact.

Example 3.3.27. The moduli stack $\mathcal{M}_g$ is finite type over $\mathbb{Z}$. On the other hand, $\underline{\operatorname{Bun}}_{r,d}(C)$ is locally of finite type over $\Bbbk$ but not of finite type.

Remark 3.3.28. A quasi-compact morphism $\mathcal{X} \to \mathcal{Y}$ induces a quasi-compact morphism $|\mathcal{X}| \to |\mathcal{Y}|$ on topological spaces. The converse is true if $\mathcal{Y}$ is quasi-separated but not in general, e.g. $\operatorname{Spec} \Bbbk \to B_{\Bbbk}\mathbb{Z}$ (see Example 3.9.20).

Exercise 3.3.29.

(a) Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. For a point $x \in |\mathcal{X}|$, show that $f(\overline{\{x\}}) = \overline{\{f(x)\}}$.

(b) Generalize Chevalley's criterion to algebraic stacks: if $f \colon \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks locally of finite presentation, then the image $f(|\mathcal{X}|) \subset |\mathcal{Y}|$ is constructible.

(c) Show an open morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks (i.e. $|\mathcal{X}| \to |\mathcal{Y}|$ is open) satisfies the following lifting property: if $x \in |\mathcal{X}|$ is a point, then every specialization $y' \rightsquigarrow f(x)$ lifts to a specialization $x' \rightsquigarrow x$. Show that the converse is true for morphisms locally of finite presentation.

(d) If $\mathcal{X}$ is a quasi-separated algebraic stack, show that $|\mathcal{X}|$ is a sober topological space, i.e. every irreducible closed subset has a unique generic point.

Exercise 3.3.30 (Generic Flatness)**.** Generalize Theorem A.2.13 to algebraic stacks: if $\mathcal{X} \to \mathcal{Y}$ is a finite type morphism of algebraic stacks with $\mathcal{Y}$ reduced, then there exists a dense open substack $\mathcal{U} \subset \mathcal{Y}$ such that the base change $X_{\mathcal{U}} \to U$ is flat and of presentation.

Exercise 3.3.31. Extend the characterization of locally of finite presentation morphisms given in Proposition A.1.1 to algebraic stacks: a morphism $f: \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is locally of finite presentation if and only if for every directed system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of affine schemes over $\mathcal{Y}$, the natural map

$$\operatorname{colim}_\lambda \operatorname{Mor}_{\mathcal{Y}}(\operatorname{Spec} A_\lambda, \mathcal{X}) \to \operatorname{Mor}_{\mathcal{Y}}(\operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda), \mathcal{X})$$

is bijective.

3.3.5 Quasi-finite morphisms

A morphism of schemes is *locally quasi-finite* if it is locally of finite type and every fiber is discrete. Since this property is étale local on the source and target, we can extend this property to morphisms of *algebraic spaces* using Definition 3.3.2.

Definition 3.3.32.

(1) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *locally quasi-finite* if for every morphism $T \to \mathcal{Y}$ from a scheme, the algebraic space $\mathcal{X} \times_{\mathcal{Y}} T$ is locally quasi-finite over T.

(2) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *locally quasi-finite* if is locally of finite type, the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally quasi-finite, and for every morphism $\operatorname{Spec} \Bbbk \to \mathcal{Y}$ from a field, the topological space $|\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} \Bbbk|$ is discrete.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-finite* if it is locally quasi-finite and quasi-compact.

To understand condition (2), recall that the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is always a representable morphism (Exercise 3.2.4). The diagonal is quasi-finite (resp. locally quasi-finite) if and only if for every field-valued point $x \in \mathcal{X}(\Bbbk)$ with image $y \in \mathcal{Y}(\Bbbk)$, the kernel $\ker(G_x \to G_y)$ of the induced map of stabilizer groups is finite (resp. discrete); see Exercise 3.2.7. In particular, if $\mathcal{Y}$ is a scheme, the diagonal is quasi-finite if and only if all stabilizers of $\mathcal{Y}$ are finite. For instance, if G is a finite group scheme over a field $\Bbbk$ (e.g. μ_p), then $BG \to \operatorname{Spec} \Bbbk$ is quasi-finite. On the other hand, $B\mathbb{G}_m \to \operatorname{Spec} \Bbbk$ is not quasi-finite despite that $|B\mathbb{G}_m|$ is a single point.

We will later establish that every separated, quasi-finite, and representable morphism is quasi-affine (Proposition 4.4.5).

3.3.6 Étale and unramified morphisms

We will require that étale and unramified morphisms are relatively Deligne–Mumford.

Definition 3.3.33. A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ over $\mathrm{Sch}_{\text{ét}}$ is *relatively Deligne–Mumford* (or simply *DM*) if for every morphism $T \to \mathcal{Y}$ from a scheme, the fiber product $\mathcal{X} \times_{\mathcal{Y}} T$ is a Deligne–Mumford stack.

We will see in Corollary 3.6.5 that relatively Deligne–Mumford morphisms are characterized by the unramifiedness of the diagonal. A morphism $\mathcal{X} \to \mathcal{Y}$ satisfying the weaker condition that the diagonal $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is locally quasi-finite is called *quasi-DM* [SP, Tag 04YW].

For morphisms of schemes, étaleness and unramifiedness are étale local on the source and smooth local on the target. These notions thus extend to relatively Deligne–Mumford morphisms.

Definition 3.3.34. A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *étale* (resp. *unramified*) if it is relatively Deligne–Mumford and for every smooth presentation $V \to \mathcal{Y}$ and étale presentation $U \to \mathcal{X} \times_{\mathcal{Y}} V$, the induced morphism $U \to V$ of schemes is étale (resp. unramified).

A morphism is étale if and only if it is smooth and unramified, and a morphism is unramified if and only if the diagonal is étale. These follows from the analogous facts for morphisms of schemes (Theorems A.3.2 and A.3.3), noting that for a morphism of schemes, the diagonal is étale if and only if it is an open immersion.

While étale morphisms are smooth and locally quasi-finite, the converse is not true, e.g. over a characteristic p field $\Bbbk$, the map $B_{\Bbbk}\boldsymbol{\mu}_p \to \mathrm{Spec}\,\Bbbk$ is smooth and quasi-finite but is not étale as $B_{\Bbbk}\boldsymbol{\mu}_p$ is not Deligne–Mumford (see Exercise 6.2.12). Similarly, étale morphisms are smooth of relative dimension 0, but again the converse doesn't hold, e.g. $B_{\Bbbk}\boldsymbol{\mu}_p \to \mathrm{Spec}\,\Bbbk$ in characteristic p or $[\mathbb{A}_{\Bbbk}^1/\mathbb{G}_m] \to \mathrm{Spec}\,\Bbbk$ in any characteristic.

Exercise 3.3.35 (technical)**.** For a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, show that the following are equivalent:

(1) the diagonal $\Delta_f\colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is unramified (resp. separated, quasi-separated),

(2) the relative inertia $I_{\mathcal{X}/\mathcal{Y}} \to \mathcal{X}$ is unramified (resp. separated, quasi-separated), and

(3) the double diagonal (or equivalently the identity section of the relative inertia) $\mathcal{X} \to I_{\mathcal{X}/\mathcal{Y}} = \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ is an open immersion (resp. closed immersion, quasi-compact).

Hint: Extend the characterization of unramified (resp. separated, quasi-separated) group schemes of Proposition B.2.9(4) to group algebraic spaces.

3.4 Equivalence relations and groupoids

Definition 3.4.1. An *étale (resp. smooth) groupoid of schemes* is a pair of schemes U and R together with étale (resp. smooth) morphisms $s\colon R \to U$ called the *source* and $t\colon R \to U$ called the *target*, and a *composition* morphism $c\colon R \times_{s,U,t} R \to R$ satisfying:

(1) (associativity) the following diagram commutes

$$
\begin{array}{ccc}
R \times_{s,U,t} R \times_{s,U,t} R & \xrightarrow{c \times \mathrm{id}} & R \times_{s,U,t} R \\
{\scriptstyle \mathrm{id} \times c}\downarrow & & \downarrow{\scriptstyle c} \\
R \times_{s,U,t} R & \xrightarrow{\quad c \quad} & R,
\end{array}
$$

(2) (identity) there exists a morphism $e\colon U \to R$ (called the *identity*) such that the following diagrams commute

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(3) (inverse) there exists a morphism $i\colon R \to R$ (called the *inverse*) such that the following diagrams commute

$$R \xrightarrow{\ i\ } R \xrightarrow{\ i\ } R \qquad\qquad R \xrightarrow{\ s\ } U \qquad\qquad R \xrightarrow{\ t\ } U$$

We will often denote this data as $s,t\colon R \rightrightarrows U$.

If $(s,t)\colon R \to U \times U$ is a monomorphism, then we say that $s,t\colon R \rightrightarrows U$ is an *étale (resp. smooth) equivalence relation*.

If U and R are algebraic spaces, and the source, target, and composition are morphisms of algebraic spaces, we obtain the notion of an *étale (resp. smooth) groupoid of algebraic spaces* and similarly an *étale (resp. smooth) equivalence relation of algebraic spaces*.

An étale (resp. smooth) group scheme $p\colon R \to U$ is an example of an étale (resp. smooth) groupoid with $s = t = p$. Group actions give additional examples (Example 3.4.5).

We can view R as a *scheme of relations on U*: a point $r \in R$ specifies a relation on the points $s(r), t(r) \in U$, which we sometimes write as $s(r) \xrightarrow{r} t(r)$. For every scheme T, the morphisms $R(T) \rightrightarrows U(T)$ define a groupoid of sets, i.e. there is composition morphism $R(T) \times_{s,U(T),t} R(T) \to R(T)$ satisfying axioms analogous to (1)–(3). We can think of an element $r \in R(T)$ as specifying a relation $u \xrightarrow{r} v$ between elements $u, v \in U(T)$. The composition morphism composes relations $u \xrightarrow{r} v$ and $v \xrightarrow{r'} w$ to the relation $u \xrightarrow{r \circ r'} w$ while the identity morphism takes $u \in U(T)$ to $u \xrightarrow{\mathrm{id}} u$ and the inverse morphism takes $u \xrightarrow{r} v$ to $v \xrightarrow{r^{-1}} u$. When $R \rightrightarrows U$ is an equivalence relation, the morphism $R(T) \to U(T) \times U(T)$ is injective, and there is thus at most one relation between any two elements of $U(T)$.

Definition 3.4.2 (Orbits and stabilizers). Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces, and let $x\colon \operatorname{Spec}\Bbbk \to U$ be a field-valued point. The *stabilizer G_x of x* is defined as the fiber product of $(s,t)\colon R \to U \times U$ by $(x,x)\colon \operatorname{Spec}\Bbbk \to U \times U$. The *orbit $O_R(x)$* is defined as the *set* $s(t^{-1}(x)) \subset U$.

Remark 3.4.3. Assuming that U is defined over $\Bbbk$ and that $x \in U(\Bbbk)$, then the $\Bbbk$-points of G_x are relations $\rho\colon x \xrightarrow{\sim} x$ in $R(\Bbbk)$ while the orbit $O_R(x)$ consists of points $y \in U$ such that there exists a relation $x \xrightarrow{\sim} y$ in R.

Exercise 3.4.4. In Definition 3.4.1, show that the identity and inverse morphism are uniquely determined.

Example 3.4.5 (Group actions). If $G \to S$ is an étale (resp. smooth) group scheme with multiplication $\mu\colon G \times_S G \to G$ acting on a scheme U over S via multiplication $\sigma\colon G \times U \to U$, then

$$p_2, \sigma\colon G \times_S U \rightrightarrows U$$

is an étale (resp. smooth) groupoid of schemes. The inverse $G \times_S U \to G \times_S U$ is given by $(g,u) \mapsto (g^{-1}, gu)$ and the composition is

$$(G \times_S U) \times_{\sigma,U,p_2} (G \times_S U) \to G \times_S U, \quad \big((g',u'),(g,u)\big) \mapsto (g'g, u).$$

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where $u' = gu$. Here (g, u) is a T-valued point of $G \times_S U$ and can be viewed as the relation $u \to gu$.

The following identifies projections in the groupoid with maps arising from the group action:

$$
\begin{array}{ccc}
U \times_{[U/G]} U \times_{[U/G]} U & \xrightarrow{\;\sim\;} & G \times_S G \times_S U \\
\Big\downarrow{\scriptstyle p_{12}}\;\;\Big\downarrow{\scriptstyle p_{13}}\;\;\Big\downarrow{\scriptstyle p_{23}} & \;\;\mathrm{id}\times\sigma\Big\downarrow\;\;\Big\downarrow{\scriptstyle \mu\times\mathrm{id}}\;\;\Big\downarrow{\scriptstyle p_{23}} \\
U \times_{[U/G]} U & \xrightarrow{\;\sim\;} & G \times_S U \\
\Big\downarrow{\scriptstyle p_1}\;\;\Big\downarrow{\scriptstyle p_2} & & \Big\downarrow{\scriptstyle p_2}\;\;\Big\downarrow{\scriptstyle \sigma} \\
U & =\!=\!=\!=\!=\!= & U \\
\Big\downarrow & & \\
[U/G]. & &
\end{array}
$$

The identification $U \times_{[U/G]} U \xrightarrow{\sim} G \times_S U$ is given by $u_2 \times_g u_1 \mapsto (g, u_1)$ where $u_2 \times_g u_1$ is shorthand notation for the triple (u_2, u_1, g) (with $u_2 = gu_1$) defining an element of the fiber product. Similarly $U \times_{[U/G]} U \times_{[U/G]} U \xrightarrow{\sim} G \times_S G \times_S U$ is given by $u_3 \times_{g_2} u_2 \times_{g_1} u_1 \mapsto (g_2, g_1, u_1)$.

More generally, the n-fold fiber product $(U/[U/G])^n$ of U over $[U/G]$ is identified with $G^{n-1} \times U$ via $u_n \times_{g_{n-1}} u_{n-1} \cdots \times_{g_1} u_1 \mapsto (g_{n-1}, \ldots, g_1, u_1)$. Under these identifications, the projection $p_{\widehat{k}} \colon (U/[U/G])^{n+1} \to (U/[U/G])^n$ forgetting the kth term is identified with that map $G^n \times U \to G^{n-1} \times U$ taking an element $(g_n, \ldots, g_1, u_1)$ to $(g_{n-1}, \ldots, g_1, u_1)$ for $k = 1$, to $(g_n, \ldots, g_{k+1}, g_k g_{k-1}, g_{k-2}, \ldots, g_1, u_1)$ for $k = 2, \ldots, n$ and to $(g_n, \ldots, g_2, g_1 u_1)$ for $k = n+1$.

Example 3.4.6. Let $\mathcal{X}$ be a Deligne–Mumford stack (resp. algebraic stack) and $U \to \mathcal{X}$ be an étale (resp. smooth) presentation which we assume is not only representable but representable by schemes. Define the scheme $R := U \times_{\mathcal{X}} U$, the source morphism $s = p_1 \colon R \to U$, the target morphism $t = p_2 \colon R \to U$ and the composition morphism $(s \circ p_1, t \circ p_2) \colon R \times_{s,U,t} R \to R := U \times_{\mathcal{X}} U$. This gives the structure of an étale (resp. smooth) groupoid $R \rightrightarrows U$. If X is an algebraic space, then $R \rightrightarrows U$ is an étale equivalence relation.

Choosing different presentations yields different groupoids which are equivalent under a notion called *Morita equivalence*; we will not use this notion in these notes.

3.4.1 Algebraicity of groupoid quotients

Definition 3.4.7 (Quotient stack of a smooth groupoid)**.** Let $s, t \colon R \rightrightarrows U$ be a smooth groupoid of algebraic spaces. Define $[U/R]^{\mathrm{pre}}$ as the prestack whose objects are morphisms $T \to U$ from a scheme T. A morphism $(S \xrightarrow{a} U) \to (T \xrightarrow{b} U)$ is the data of a morphism of schemes $f \colon S \to T$ and an element $r \in R(S)$ such that $s(r) = a$ and $t(r) = f \circ b$.

Define $[U/R]$ to be the stackification of $[U/R]^{\mathrm{pre}}$ in the big étale topology $\mathrm{Sch}_{\text{ét}}$.

If in addition $R \rightrightarrows U$ is an equivalence relation, then $[U/R]$ is isomorphic to a sheaf (Exercise 3.4.8) and we denote it as U/R.

The fiber category $[U/R]^{\mathrm{pre}}(T)$ is the groupoid whose objects are $U(T)$ and morphisms are $R(T)$. The identity morphism $\mathrm{id} \colon U \to U$ defines a map $U \to [U/R]^{\mathrm{pre}}$ and therefore a map $p \colon U \to [U/R]$.

Exercise 3.4.8. Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces. Show that $[U/R]$ is equivalent to a sheaf if and only if $R \rightrightarrows U$ is an equivalence relation.

Exercise 3.4.9. Extend Exercise 2.3.35 to show that if $s, t \colon R \rightrightarrows U$ is a smooth groupoid of algebraic spaces, the following diagrams are cartesian:

$$
\begin{array}{ccc}
R & \xrightarrow{\ s\ } & U \\
\downarrow{\scriptstyle t} & \square & \downarrow{\scriptstyle p} \\
U & \xrightarrow{\ p\ } & [U/R]
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
R & \xrightarrow{\ (s,t)\ } & U \times U \\
\downarrow & \square & \downarrow{\scriptstyle p \times p} \\
[U/R] & \xrightarrow{\ \Delta\ } & [U/R] \times [U/R].
\end{array}
$$

Exercise 3.4.10. Let $R \rightrightarrows U$ be a smooth groupoid of algebraic spaces and $x \colon \operatorname{Spec} \Bbbk \to U$ be a field-valued point. Show that the stabilizer of x as defined in Definition 3.4.2 is identified with the stabilizer of $\operatorname{Spec} \Bbbk \to [U/R]$ as defined in Definition 3.2.5.

Theorem 3.4.11 (Algebraicity of Quotients by Groupoids)**.**

  (1) If $R \rightrightarrows U$ is an étale (resp. smooth) groupoid of algebraic spaces. Then $[U/R]$ is a Deligne–Mumford stack (resp. algebraic stack) and $U \to [U/R]$ is an étale (resp. smooth) presentation.

  (2) If $R \rightrightarrows U$ be an étale equivalence relation of schemes, then U/R is an algebraic space and $U \to U/R$ is an étale presentation.

Remark 3.4.12. In Corollary 4.4.12, we show that in fact the quotient U/R of an étale equivalence relation of *algebraic spaces* is an algebraic space, establishing that one doesn't obtain new algebro-geometric objects by considering sheaves which are étale locally algebraic spaces. This result is delayed until §4.4 as it takes more work to show that the diagonal of U/R is representable by *schemes*.

More generally, if $R \rightrightarrows U$ is an fppf groupoid (resp. fppf equivalence relation) of algebraic spaces, then $[U/R]$ is an algebraic stack [SP, Tag 06FI] (resp. U/R is an algebraic space [SP, Tag 04S6]). See also [Art74, Thm. 6.1] and [LMB00, Thm. 10.1].

Proof. For (1), we will show that $U \to \mathcal{X} := [U/R]$ is surjective, smooth, and representable. Let $T \to \mathcal{X}$ be a morphism from a scheme T. It follows from the definition of $[U/R]$ as the stackification of $[U/R]^{\mathrm{pre}}$ that there exists an étale cover $T' \to T$ and a commutative diagram

$$
\begin{array}{ccc}
T' & \longrightarrow & U \\
\downarrow & & \downarrow \\
T & \longrightarrow & \mathcal{X}.
\end{array}
$$

In the commutative cube

(3.4.13)

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the front, back, top and bottom squares are cartesian where U_T is a sheaf and $U_{T'}$ is a scheme. Since $T' \to T$ is a surjective étale morphism representable by schemes, so is $U_{T'} \to U_T$. This establishes that U_T is an algebraic space. By descent $U_{T'}$ is surjective and étale over T.

For (2), it suffices to show that the diagonal of the quotient sheaf $X := U/R$ is representable by schemes. Indeed, this implies that $U \to X$ is representable by schemes via the argument of Corollary 3.2.3 and étale descent implies that $U \to X$ is étale and surjective. Let $T \to X \times X$ be a morphism from a scheme and consider the cartesian cube

$$
\begin{array}{ccc}
Q_{T'} & \longrightarrow & T' \\
& R \longrightarrow U \times U & \\
& & \\
& Q_T \longrightarrow T & \\
X & \longrightarrow X \times X,
\end{array}
\tag{3.4.14}
$$

as in (3.2.2). Since $R \to U \times U$ is separated and locally quasi-finite, so is $Q_{T'} \to T'$. Effective Descent (Proposition B.1.8) implies that sheaf Q_T is a scheme. □

As a consequence, we see that in Theorem 3.1.10, the hypothesis that the group scheme $G \to S$ is affine is not necessary for the quotient stack $[X/G]$ and classifying stack BG to be algebraic.

Exercise 3.4.15. Show that if $\mathcal{X}$ is an algebraic stack (resp. algebraic space) and $U \to \mathcal{X}$ is a smooth presentation, then $\mathcal{X}$ is isomorphic to the quotient stack $[U/R]$ (resp. quotient sheaf U/R) of the étale groupoid (resp. equivalence relation) $R \rightrightarrows U$ where $R = U \times_{\mathcal{X}} U$.

3.4.2 Inducing and slicing presentations

We provide here two useful techniques to build new presentations from given ones.

First, let $\mathcal{X} = [X/H]$ be a quotient stack of a smooth algebraic group H acting on a scheme X over $\Bbbk$ and $H \subset G$ be an inclusion of algebraic groups. Then H acts freely on $G \times X$ via $h \cdot (g, x) = (gh^{-1}, hx)$ and we let $G \times^H X$ be the algebraic space quotient $(G \times X)/H$. When H is finite, this quotient exists by definition of an algebraic space and is affine (resp. quasi-projective, projective) when X is by Theorem 4.3.6 (resp. Exercise 4.2.9). In the non-finite case, it follows from Corollary 3.6.8 $G \times^H X$ is an algebraic space if X is noetherian. There is an action of G on $G \times^H X$ via $g \cdot (g', x) = (gg', x)$.

Exercise 3.4.16. Show that $[X/H] \cong [(G \times^H X)/G]$.

The second method is sometimes referred to as *slicing a groupoid*. Let $U \to \mathcal{X}$ be a smooth presentation of an algebraic stack with the corresponding groupoid $s, t \colon R = U \times_{\mathcal{X}} U \rightrightarrows U$. If $g \colon U' \to U$ is a morphism, we define *the restriction of*

$R \rightrightarrows U$ *along* $U' \to U$ to be the groupoid $R|_{U'} \rightrightarrows U'$ defined by the fiber product

$$
\begin{array}{ccc}
R|_{U'} & \xrightarrow{(t',s')} & U' \times U' \\
\downarrow & \square & \downarrow \\
R & \xrightarrow{(t,s)} & U \times U
\end{array}
$$

Exercise 3.4.17.

(a) Show that $R|_{U'}$ fits into a cartesian diagram

$$
\begin{array}{ccccc}
R|_{U'} & \longrightarrow & R \times_{s,U} U' & \longrightarrow & U' \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\
U' \times_{U,t} R & \longrightarrow & R & \xrightarrow{\ s\ } & U \\
\downarrow & & \downarrow{\scriptstyle t} & & \downarrow \\
U' & \xrightarrow{\ g\ } & U & \longrightarrow & [U/R]
\end{array}
$$

Assume in addition that $U' \times_{U,t} R \to R \xrightarrow{s} U$ is étale (resp. smooth).

(2) Show that $R|_{U'} \rightrightarrows U'$ is an étale (resp. smooth) groupoid.

(3) Show that there is an open immersion $[U'/R|_{U'}] \to [U/R]$.

(4) Show that $[U'/R|_{U'}] \to [U/R]$ is an isomorphism if and only if for every point $u \in U$, there exists a point $u' \in U$ and a relation $u \to g(u')$ in R.

3.5 Dimension, tangent spaces, and residual gerbes

3.5.1 Dimension

Recall that the dimension $\dim X$ of a scheme X is the Krull dimension of the underlying topological space while the dimension $\dim_x X$ at a point $x \in X$ is the minimum dimension of open subsets containing x.

We now extend these definitions to algebraic spaces and stacks.

Definition 3.5.1.

(1) Let X be a noetherian algebraic space and $x \in |X|$. We define the *dimension of X at x* to be
$$
\dim_x X = \dim_u U \in \mathbb{Z}_{\geq 0} \cup \infty
$$
where $U \to X$ is an étale presentation and $u \in U$ is a preimage of x.

(2) Let $\mathcal{X}$ be a noetherian algebraic stack with smooth presentation $U \to \mathcal{X}$ and corresponding smooth groupoid $s, t \colon R \rightrightarrows U$, and let $u \in U$ be a preimage of $x \in |\mathcal{X}|$. We define the *dimension of $\mathcal{X}$ at x* to be
$$
\dim_x \mathcal{X} = \dim_u U - \dim_{e(u)} R_u \in \mathbb{Z} \cup \infty
$$
where R_u is the fiber of $s \colon R \to U$ over u and $e \colon U \to R$ denotes the identity morphism in the groupoid.

(3) If $\mathcal{X}$ is a noetherian algebraic space or stack, we define the *dimension of $\mathcal{X}$* to be
$$
\dim \mathcal{X} = \sup_{x \in |\mathcal{X}|} \dim_x \mathcal{X} \in \mathbb{Z} \cup \infty.
$$

Proposition 3.5.2. *The definition of the dimension $\dim_x \mathcal{X}$ of a noetherian algebraic stack $\mathcal{X}$ at a point $x \in |\mathcal{X}|$ is independent of the presentation $U \to \mathcal{X}$ and of the choice of preimage u of x.*

Proof. The dimension of an algebraic space at a point is well defined as étale morphisms have relative dimension 0.

If $U \to \mathcal{X}$ is a smooth presentation (with U a scheme) and $u \in U$ is a preimage of x with residue field $\kappa(u)$, then the fiber R_u is identified with the fiber product

$$
\begin{array}{ccccc}
R_u & \longrightarrow & R & \overset{t}{\longrightarrow} & U \\
\downarrow & & \downarrow{\scriptstyle s} & & \downarrow \\
\operatorname{Spec}\kappa(u) & \longrightarrow & U & \longrightarrow & \mathcal{X},
\end{array}
$$

and is a smooth algebraic space over $\kappa(u)$.

If $U' \to \mathcal{X}$ is a second presentation and $u' \in U'$ a preimage of x, then define the algebraic space $U'' := U \times_{\mathcal{X}} U'$. Observe that there is a cartesian diagram

$$
\begin{array}{ccccc}
U''_u & \longrightarrow & U'' & \longrightarrow & U' \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec}\kappa(u) & \longrightarrow & U & \longrightarrow & \mathcal{X}
\end{array}
\tag{3.5.3}
$$

where the fiber U''_u is identified with $R'_{u'}$. By Exercise 3.5.5 applied to $U'' \to U$, we have the identity

$$
\dim_{u''} U'' = \dim_u U + \dim_{u''} U''_u = \dim_u U + \dim_{e'(u')} R'_{u'}. \tag{3.5.4}
$$

Choose a representative $\operatorname{Spec} L \to U''$ in $|U''|$ mapping to u and u'. Note that the compositions $\operatorname{Spec}\kappa(u) \to U \to \mathcal{X}$, $\operatorname{Spec}\kappa(u') \to U' \to \mathcal{X}$ and $\operatorname{Spec} L \to U'' \to \mathcal{X}$ all define the same point $x \in |\mathcal{X}|$. Let $R \rightrightarrows U$ and $R' \rightrightarrows U'$ be the corresponding smooth groupoids, and set $R''_{u''} = U'' \times_{\mathcal{X}} \operatorname{Spec} L$.

We need to show that

$$
\dim_u U - \dim_{e(u)} R_u = \dim_{u'} U' - \dim_{e'(u')} R'_{u'}
$$

and by symmetry between U and U', it suffices to show that

$$
\dim_u U - \dim_{e(u)} R_u = \dim_{u''} U'' - \dim_{e''(u'')} R''_{u''}
$$

where $e''(u'') \in |R''_{u''}|$ is the image of the map $\operatorname{Spec} L \to R''_{u''} = U'' \times_{\mathcal{X}} \operatorname{Spec} L$ defined by the identity automorphism of u''. By (3.5.4), this is in turn equivalent to

$$
\dim_{e''(u'')} R''_{u''} = \dim_{e(u)} R_u + \dim_{e'(u')} R'_{u'}
$$

This last fact follows from the cartesian cube

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and properties of dimension (see Exercise 3.5.5). □

Exercise 3.5.5.

(a) Show that the analogue of Proposition A.3.10 holds for algebraic spaces; that is, if $X \to Y$ is a smooth morphism of noetherian algebraic spaces, and if $x \in |X|$ is a point with image $y \in |Y|$, then

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y).$$

(b) If X and X' are noetherian algebraic spaces over a field $\Bbbk$ with $\Bbbk$-points x and x', show that

$$\dim_{(x,x')} X \times_\Bbbk X' = \dim_x X + \dim_{x'} X'.$$

(c) Let $\mathcal{X}$ be a noetherian algebraic space over a field $\Bbbk$ and $\Bbbk \to L$ be a field extension. Set $\mathcal{X}_L = \mathcal{X} \times_\Bbbk L$. If $x' \in |\mathcal{X}_L|$ is a point with image $x \in |\mathcal{X}|$, show that $\dim_{x'} \mathcal{X} \times_\Bbbk L = \dim_x \mathcal{X}$.

Example 3.5.6. If U is a scheme of pure dimension with an action of an affine algebraic group G (which is necessarily of pure dimension) over a field $\Bbbk$, then

$$\dim[U/G] = \dim U - \dim G.$$

In particular, the classifying stack has dimension $\dim BG = -\dim G$ and we see that the dimension may be negative!

3.5.2 Tangent spaces

The *dual numbers* is the ring $\Bbbk[\epsilon] := \Bbbk[\epsilon]/\epsilon^2$ defined over a field $\Bbbk$.

Definition 3.5.7. For an algebraic stack $\mathcal{X}$ and a point $x \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$, we define the *Zariski tangent space* or simply the *tangent space* of $\mathcal{X}$ at x as the set

$$T_{\mathcal{X},x} := \left\{ \text{2-commutative diagrams} \quad \begin{array}{c} \operatorname{Spec} \Bbbk \\ \downarrow \quad \searrow^x \\ {}_{\alpha} \\ \operatorname{Spec} \Bbbk[\epsilon] \xrightarrow{\tau} \mathcal{X} \end{array} \right\} \Big/ \sim$$

or in other words the set of pairs (τ, α) where $\tau \colon \operatorname{Spec} \Bbbk[\epsilon] \to \mathcal{X}$ and $\alpha \colon x \xrightarrow{\sim} \tau|_\Bbbk$. Two pairs are equivalent $(\tau, \alpha) \sim (\tau', \alpha')$ if there is an isomorphism $\beta \colon \tau \xrightarrow{\sim} \tau'$ in $\mathcal{X}(k[\epsilon])$ compatible with α and α', i.e. $\alpha' = \beta|_{\operatorname{Spec} \Bbbk} \circ \alpha$

Proposition 3.5.8. *If $\mathcal{X}$ is an algebraic stack with affine diagonal and $x \in \mathcal{X}(\Bbbk)$, then $T_{\mathcal{X},x}$ is naturally a $\Bbbk$-vector space.*

Proof. Scalar multiplication of $c \in \Bbbk$ on $(\tau, \alpha) \in T_{\mathcal{X},x}$ is defined as the composition $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec} \Bbbk[\epsilon] \xrightarrow{\tau} \mathcal{X}$ where the first map is defined by $\epsilon \mapsto c\epsilon$ and with the same 2-isomorphism α. To define addition, we will show that there is an equivalence of categories

$$\mathcal{X}(\Bbbk[\epsilon_1] \times_\Bbbk \Bbbk[\epsilon_2]) \to \mathcal{X}(\Bbbk[\epsilon_1]) \times_{\mathcal{X}(\Bbbk)} \mathcal{X}(\Bbbk[\epsilon_2]) \tag{3.5.9}$$

or in other words that

$$\begin{array}{ccc} \operatorname{Spec} \Bbbk & \hookrightarrow & \operatorname{Spec} \Bbbk[\epsilon_1] \\ \downarrow & & \downarrow \\ \operatorname{Spec} \Bbbk[\epsilon_2] & \hookrightarrow & \operatorname{Spec}(\Bbbk[\epsilon_1] \times_\Bbbk \Bbbk[\epsilon_2]) \end{array}$$

is a pushout among algebraic stacks with affine diagonal (see §B.4). Once this is established, we define addition of (τ_1, α_1) and (τ_2, α_2) by the composition $\operatorname{Spec} \Bbbk[\epsilon] \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_\Bbbk \Bbbk[\epsilon_2]) \to \mathcal{X}$ where the first map is defined sending both $(\epsilon_1, 0)$ and $(0, \epsilon_2)$ to ϵ.

Choose a smooth morphism $(U, u) \to (\mathcal{X}, x)$ from an affine scheme U. Since $\mathcal{X}$ has affine diagonal $U \to \mathcal{X}$ is an affine morphism. Let $\operatorname{Spec} A_0 = \operatorname{Spec} \Bbbk \times_\mathcal{X} U$, $\operatorname{Spec} A_1 = \operatorname{Spec} \Bbbk[\epsilon_1] \times_\mathcal{X} U$ and $\operatorname{Spec} A_2 = \operatorname{Spec} \Bbbk[\epsilon_2] \times_\mathcal{X} U$. Since $\operatorname{Spec}(A_1 \times_A A_2)$ is clearly the pushout of $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_1$ and $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_2$ in the category of affine schemes, there are unique morphisms $\operatorname{Spec}(A_1 \times_A A_2) \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_\Bbbk \Bbbk[\epsilon_2])$ and $\operatorname{Spec}(A_1 \times_A A_2) \to U$ completing the diagram

By the Flatness Criterion over Artinian Rings (A.2.3), we see that the map $\operatorname{Spec}(A_1 \times_A A_2) \to \operatorname{Spec}(\Bbbk[\epsilon_1] \times_\Bbbk \Bbbk[\epsilon_2])$ is faithfully flat. By repeating this argument on $U \times_\mathcal{X} U$, one argues that the $\operatorname{Spec}(A_1 \times_A A_2) \to U$ descends uniquely and provides the desired dotted arrow. $\qquad\square$

Exercise 3.5.10. Show that $T_{\mathcal{X}, x}$ is naturally a representation of G_x which is given set-theoretically by: $g \cdot (\tau, \alpha) = (\tau, g \circ \alpha)$ for $g \in G_x$ and $(\tau, \alpha) \in T_{\mathcal{X}, x}$.

Example 3.5.11. Consider a smooth, connected, and projective curve $[C] \in \mathcal{M}_g(\Bbbk)$ defined over $\Bbbk$ of genus $g \geq 2$. Deformation Theory (C.1.12) implies that $T_{\mathcal{M}_g, [C]} = H^1(C, T_C)$. Since $\deg T_C < 0$, $\mathrm{H}^0(C, T_C) = 0$ and Riemann–Roch implies

$$\dim T_{\mathcal{M}_g, [C]} = \dim \mathrm{H}^1(C, T_C) = -\chi(T_C) = -(\deg T_C + (1-g)) = 3g - 3.$$

Example 3.5.12. Let C be a smooth, connected, and projective curve over $\Bbbk$ and $E \in \underline{\operatorname{Bun}}_{r,d}(C)(\Bbbk)$ be a vector bundle on C of rank r and degree d. Deformation Theory (C.1.17) implies that $T_{\underline{\operatorname{Bun}}_{r,d}(C), [E]} = \operatorname{Ext}^1_{\mathcal{O}_C}(E, E) = H^1(C, E \otimes E^\vee)$. By Riemann–Roch, $\chi(E \otimes E^\vee) = r^2(1-g)$. Since $\dim \operatorname{Aut}(E) = \dim_\Bbbk \operatorname{Hom}_{\mathcal{O}_C}(E, E) = \mathrm{H}^0(C, E \otimes E^\vee)$, we compute that

$$\dim T_{\underline{\operatorname{Bun}}_{r,d}(C), [E]} = \dim \operatorname{Ext}^1_{\mathcal{O}_C}(F, F) = \dim \operatorname{Aut}(F) + r^2(g-1).$$

Remark 3.5.13. Suppose that $A \to A'$ and $A \to A''$ are homomorphisms of artinian local rings such that $A \twoheadrightarrow A'$ is surjective. If $\mathcal{X}$ is an algebraic stack, then the above argument extends to show that

$$\mathcal{X}(A_1 \times_{A_0} A_2) \to \mathcal{X}(A_1) \times_{\mathcal{X}(A_0)} \mathcal{X}(A_2) \qquad (3.5.14)$$

is an equivalence of categories. This condition is usually referred to as *homogeneity*. The conditions (RS$_1$)–(RS$_2$) in Rim–Schlessinger's Criteria (C.3.12) are weaker

versions of homogeneity which ensure the existence of a formal miniversal deformation space, and also appear in Artin's Axioms for Algebraicity (Theorem C.7.4).

More generally, (3.5.14) holds if $A \to A'$ and $A \to A''$ are arbitrary ring homomorphisms with $A \twoheadrightarrow A'$ surjective which shows that the Ferrand Pushout (B.4.1) $\operatorname{Spec} A' \times_A A''$ is a pushout in the category of algebraic stacks.

3.5.3 Residual gerbes

Attached to every point $x \in X$ of a scheme is a residue field $\kappa(x)$ and a monomorphism $\operatorname{Spec} \kappa(x) \to X$ with image x. The residual gerbe will provide us with an analogous property for algebraic stacks. Note that non-trivial stabilizers prevent field-valued points from being monomorphisms, e.g. BG for a finite abstract group G.

Definition 3.5.15. Let $\mathcal{X}$ be an algebraic stack and $x \in |\mathcal{X}|$ be a point. We say that the *residual gerbe at x exists* if there is a reduced, locally noetherian algebraic stack $\mathcal{G}_x$ and a monomorphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}_x|$ is a point mapping to x. The algebraic stack $\mathcal{G}_x$ is called the *residual gerbe at x*.

We define gerbes later in Definition 6.2.18 and show that $\mathcal{G}_x$ is a gerbe over a field $\kappa(x)$, called the *residue field* (Proposition 6.2.36). While not apparent from the definition, we will also show that residual gerbes are in fact unique. This will justify our terminology of calling $\mathcal{G}_x$ *the* residual *gerbe*.

In the meantime, we will be content with verifying the existence of residual gerbes at *finite type points* (see Definition 3.5.16 and Proposition 3.5.19). This statement will suffice for most of our purposes, but we will later prove that residual gerbes in fact exist for every point (as long as the stack is quasi-separated) in Proposition 6.2.36—this result is postponed as we will utilize the Fppf Criterion for Algebraicity (Theorem 6.2.1). In §6.2.6, we also provide other characterizations of residual gerbes and fields.

Definition 3.5.16. A point $x \in |\mathcal{X}|$ in an algebraic stack is *of finite type* if there exists a representative $\operatorname{Spec} \Bbbk \to \mathcal{X}$ of x that is locally of finite type.[2]

Remark 3.5.17. If X is a noetherian scheme, a point $x \in X$ is of finite type if and only if $x \in X$ is locally closed, and, in fact, a morphism $\operatorname{Spec} \Bbbk \to X$ from a field with image x is of finite type if and only if the image $x \in X$ is locally closed and $\kappa(x)/\Bbbk$ is a finite extension. Indeed, to see the nontrivial implication ($\Rightarrow$), we replace X with $\overline{\{x\}}$, and since $\operatorname{Spec} \Bbbk \to X$ is of finite type with dense image, Generic Flatness (A.2.13) implies that $\operatorname{Spec} \Bbbk \to X$ is fppf and thus its image is open.

An example of a finite type point of a scheme that is not closed is the generic point of a DVR. On the other hand, if X is a scheme of finite type over a field $\Bbbk$, then every finite type point is a closed point. The analogous fact is *not* true for algebraic stacks of finite type over $\Bbbk$, e.g. $1 \colon \operatorname{Spec} \Bbbk \to [\mathbb{A}^1/\mathbb{G}_m]$ is an open finite type point.

Exercise 3.5.18. Let $\mathcal{X}$ be an algebraic stack.

(a) Show that a point $x \in |\mathcal{X}|$ is of finite type if and only if there exists a scheme U, a closed point $u \in U$, and a smooth morphism $(U, u) \to (\mathcal{X}, x)$.

(b) Show that any algebraic stack (resp. quasi-compact algebraic stack) has a finite type point (resp. closed point).

---

[2]If $\mathcal{X}$ has quasi-compact diagonal, e.g. $\mathcal{X}$ is quasi-separated (Definition 3.3.10), then every field-valued point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is automatically quasi-compact, and thus the locally of finite typeness of $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is equivalent to finite typeness.

Proposition 3.5.19 (Existence of Residual Gerbes I)**.** *If $\mathcal{X}$ is a noetherian algebraic stack and $x \in |\mathcal{X}|$ is a finite type point, then the residual gerbe $\mathcal{G}_x$ exists at x and is unique, and moreover satisfies the following:*

(1) The algebraic stack $\mathcal{G}_x$ is regular and the morphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is a locally closed immersion.

(2) If in addition $\mathcal{X}$ is of finite type over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$ has a smooth affine stabilizer G_x, then $\mathcal{G}_x \cong BG_x$.

(3) If in addition $\mathcal{X}$ is a noetherian algebraic space, then $\mathcal{G}_x \cong \operatorname{Spec} \kappa(x)$ for a field $\kappa(x)$, called the residue field of x.

Proof. We first show the existence. After replacing $\mathcal{X}$ with $\overline{\{x\}}$, we may assume that $\mathcal{X}$ is reduced and $x \in |\mathcal{X}|$ is dense. Let $\operatorname{Spec} \Bbbk \to \mathcal{X}$ be a finite type representative of x. By Generic Flatness (3.3.30), $\operatorname{Spec} \Bbbk \to \mathcal{X}$ is flat. Therefore the image of $\operatorname{Spec} \Bbbk \to \mathcal{X}$—which is $\{x\} \subset |\mathcal{X}|$—is open (Exercise 3.3.25). The corresponding open substack $\mathcal{G}_x \subset \mathcal{X}$ satisfies the properties of being a residual gerbe. Since $\operatorname{Spec} \Bbbk \to \mathcal{G}_x$ is fppf and the property of being regular descends under fppf morphisms (Proposition B.1.12), $\mathcal{G}_x$ is regular.

For the uniqueness, suppose that $\mathcal{G}$ and $\mathcal{G}'$ are reduced, locally noetherian algebraic stacks with monomorphisms $\mathcal{G} \hookrightarrow \mathcal{X}$ and $\mathcal{G}' \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}|$ and $|\mathcal{G}'|$ are singletons mapping to x. Then $\mathcal{G}'' := \mathcal{G} \times_{\mathcal{X}} \mathcal{G}'$ is a nonempty algebraic stack with monomorphisms $\mathcal{G}'' \to \mathcal{G}$ and $\mathcal{G}'' \to \mathcal{G}'$. Let $\operatorname{Spec} \Bbbk \to \mathcal{G}$ be a finite type morphism from a field (which exists by Exercise 3.5.18); by Generic Flatness (3.3.30) $\operatorname{Spec} \Bbbk \to \mathcal{G}$ is fppf. The base change $\mathcal{G}'' \times_{\mathcal{G}} \operatorname{Spec} \Bbbk$ is a nonempty algebraic space equipped with a monomorphism to $\operatorname{Spec} \Bbbk$. This implies that $\mathcal{G}'' \times_{\mathcal{G}} \operatorname{Spec} \Bbbk \to \operatorname{Spec} \Bbbk$ is an isomorphism, and by fppf descent $\mathcal{G}'' \to \mathcal{G}$ is also an isomorphism. Similarly, $\mathcal{G}'' \to \mathcal{G}'$ is an isomorphism.

Suppose now that $\mathcal{X}$ is of finite type over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$ has a smooth affine stabilizer G_x. There is a monomorphism of prestacks $BG_x^{\mathrm{pre}} \to \mathcal{X}$: for a $\Bbbk$-scheme T, there is a unique object of BG_x^{pre} over T, and this object gets mapped to the composition $T \to \operatorname{Spec} \Bbbk \xrightarrow{x} \mathcal{X}$. Similarly, a morphism over $T' \to T$ corresponds to a map $T' \to G_x$, and this gets mapped to the corresponding morphism in $\mathcal{X}$. Under stackification, this induces a monomorphism $BG_x \to \mathcal{X}$, and thus BG_x satisfies the properties of a residual gerbe. $\square$

Exercise 3.5.20. Let $\mathcal{X}$ be a (possibly non-noetherian) algebraic stack and $x \in \mathcal{X}$ be a finite type point such that the stabilizer is unramified (i.e. the stabilizer group scheme of any representative is unramified). Show that the residual gerbe exists and is unique. See also [SP, Tag 06G3].

Corollary 3.5.21. *Let $x \in |\mathcal{X}|$ be a finite type point of a noetherian algebraic stack $\mathcal{X}$. If $(U, u) \to (\mathcal{X}, x)$ is a smooth morphism from a scheme U with $u \in U$ a finite type point, then there is a cartesian diagram*

$$
\begin{array}{ccc}
O(u) & \hookrightarrow & U \\
\downarrow & \square & \downarrow \\
\mathcal{G}_x & \hookrightarrow & \mathcal{X}
\end{array}
\tag{3.5.22}
$$

where $O(u)$ is identified set-theoretically with the orbit $s(t^{-1}(u))$ of the induced groupoid $s, t \colon R := U \times_{\mathcal{X}} U \rightrightarrows U$. $\square$

Remark 3.5.23. If $\mathcal{X} = [U/G]$ is the quotient stack of a smooth affine algebraic group over a field $\Bbbk$ acting on a noetherian $\Bbbk$-scheme U and $u \in U(\Bbbk)$, there is a cartesian diagram

$$
\begin{array}{ccc}
Gu & \lhook\joinrel\longrightarrow & U \\
\downarrow & \square & \downarrow \\
BG_x & \lhook\joinrel\longrightarrow & [U/G].
\end{array}
$$

We recover the familiar fact that orbit $Gu \hookrightarrow U$ is locally closed (B.2.42(6)).

Corollary 3.5.24. *A finite type point $x \in |X|$ of a noetherian algebraic space has a residue field $\kappa(x)$, i.e. there is a field $\kappa(x)$ and a locally closed immersion $\operatorname{Spec} \kappa(x) \hookrightarrow X$ with image x.* $\qquad\square$

Exercise 3.5.25. Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $\overline{x} \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ be a representative of x. Show that $\dim \mathcal{G}_x = -\dim G_{\overline{x}}$.

3.6 Characterization of Deligne–Mumford stacks

3.6.1 Existence of minimal presentations

Theorem 3.6.1 (Existence of Minimal Presentations)**.** *Let $\mathcal{X}$ be a noetherian algebraic stack and let $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer G_x. Then there exists a scheme U, a closed point $u \in U$, and a smooth morphism $(U, u) \to (\mathcal{X}, x)$ of relative dimension $\dim G_x$ from a scheme U such that the diagram*

$$
\begin{array}{ccc}
\operatorname{Spec} \kappa(u) & \lhook\joinrel\longrightarrow & U \\
\downarrow & \square & \downarrow \\
\mathcal{G}_x & \lhook\joinrel\longrightarrow & \mathcal{X}
\end{array}
$$

is cartesian. In particular, if G_x is finite and reduced, there is an étale morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme.

Proof. Let $(U, u) \to (\mathcal{X}, x)$ be a smooth morphism of relative dimension n from a scheme U such that $u \in U$ is a finite type point. By Proposition 3.5.19, the residual gerbe $\mathcal{G}_x$ at x exists, the inclusion $\mathcal{G}_x \hookrightarrow \mathcal{X}$ is locally closed, and $\mathcal{G}_x$ is regular of dimension $-\dim G_x$ (Exercise 3.5.25). We obtain a cartesian diagram

$$
\begin{array}{ccc}
O(u) & \lhook\joinrel\longrightarrow & U \\
\downarrow & \square & \downarrow \\
\mathcal{G}_x & \lhook\joinrel\longrightarrow & \mathcal{X}.
\end{array}
$$

It follows that $O(u)$ is a regular scheme of dimension $c := n - \dim G_x$. Let $f_1, \ldots, f_c \in \mathcal{O}_{O(u),u}$ be a regular sequence generating the maximal ideal at u. After replacing U with an open affine neighborhood of u, we may assume that each f_i is a global function on U. We can consider the closed subscheme $W := V(f_1, \ldots, f_c)$ which by design intersects $O(u)$ transversely at U, i.e. $W \cap O(u) = \operatorname{Spec} \kappa(u)$ scheme-theoretically.

By inductively applying the Slicing Criterion for Flatness (A.2.9) to the smooth groupoid $U \times_{\mathcal{X}} U \rightrightarrows U$ at a preimage of u and the applying smooth descent, we

conclude that the composition $W \hookrightarrow U \to \mathcal{X}$ is flat at u. Since G_x is smooth, so is $\operatorname{Spec} \kappa(u) \to \mathcal{G}_x$. For flat morphisms, smoothness is a property that can be checked on fibers and thus (again arguing on $R \rightrightarrows U$ and using descent) $W \to \mathcal{X}$ is smooth at u. The statement follows after replacing W with an open neighborhood of u. $\quad\square$

Remark 3.6.2. A smooth presentation $p \colon U \to \mathcal{X}$ is called a *miniversal at* $u \in U(\Bbbk)$ if $T_{U,u} \to T_{\mathcal{X},p(u)}$ is an isomorphism of $\Bbbk$-vector spaces. We will see that the above presentations are miniversal in Proposition 3.7.5.

If the stabilizer G_x is not smooth, there are two candidates for 'minimal presentations.' There still exists a miniversal presentation $(U, u) \to (\mathcal{X}, x)$, but its relative dimension is equal to the dimension of the Lie algebra of G_x (rather than $\dim G_x$) and the fiber product $\mathcal{G}_x \times_{\mathcal{X}} U$ may thus be positive dimensional. For example, $B\boldsymbol{\mu}_p$ is an algebraic stack in characteristic p (Proposition 6.2.10) and it can be realized as the quotient of $\mathbb{G}_m$ acting on itself via $t \cdot x = t^p x$; here $\mathbb{G}_m \to B\boldsymbol{\mu}_p$ is a miniversal presentation. On the other hand, there is an fppf (but not smooth) morphism $(U, u) \to (\mathcal{X}, x)$ such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$. In particular, if $\mathcal{X}$ has quasi-finite diagonal, then there is an fppf and quasi-finite morphism $(U, u) \to (\mathcal{X}, x)$. In our example, the map $\operatorname{Spec} \Bbbk \to B\boldsymbol{\mu}_p$ is such a presentation.

Exercise 3.6.3. If $\mathcal{X}$ is a (possibly non-noetherian) algebraic stack and $x \in \mathcal{X}$ is a finite type point with unramified stabilizer, show that there is an étale morphism $(U, u) \to (\mathcal{X}, x)$ from a scheme U where $u \in U$ is a closed point.

Hint: Replicate the argument above using Exercise 3.5.20.

3.6.2 Equivalent characterizations

Theorem 3.6.4 (Characterization of Deligne–Mumford Stacks)**.** *Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:*

(1) *the stack $\mathcal{X}$ is a Deligne–Mumford;*

(2) *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified; and*

(3) *every point of $\mathcal{X}$ has a finite and reduced stabilizer group.*

Proof. The equivalence (2) $\iff$ (3) is essentially the definition of unramified: since the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is always locally of finite type (Exercise 3.3.3), it is unramified if and only if every geometric fiber (which is either empty or isomorphic to a stabilizer) is discrete and reduced. It is not hard to see that a Deligne–Mumford stack has unramified diagonal (Exercise 3.2.8). For the converse, Existence of Minimal Presentations (Theorem 3.6.1 and Exercise 3.6.3) shows that for every finite type point $x \in \mathcal{X}$, there is an étale morphism $U \to \mathcal{X}$ from a scheme whose image contains x. Thus $\mathcal{X}$ is Deligne–Mumford. See also [LMB00, Thm 8.1] and [SP, Tag 06N3]. $\quad\square$

Corollary 3.6.5. *A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is relatively Deligne–Mumford (Definition 3.3.33) if and only if $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is unramified.*

Theorem 3.6.6 (Characterization of Algebraic Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack whose diagonal is representable by schemes. The following are equivalent:*

(1) *the stack $\mathcal{X}$ is an algebraic space;*

(2) *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism; and*

(3) *every point of $\mathcal{X}$ has a trivial stabilizer.*

Remark 3.6.7. We will remove the pesky hypothesis that $\Delta_{\mathcal{X}}$ is representable by schemes in Theorem 4.4.10.

Proof. Condition (2) is equivalent to the condition that $\mathcal{X}$ is a sheaf. The implication $(1) \Rightarrow (2)$ follows from the definition of an algebraic space. For the converse, if $\mathcal{X}$ is a sheaf, then Theorem 3.6.1 implies that there exists a surjective, étale, and representable morphism $U \to \mathcal{X}$ from a scheme. Since $\Delta_{\mathcal{X}}$ is representable by schemes, so is $U \to \mathcal{X}$. The equivalence $(2) \iff (3)$ follows from the fact that a group scheme of finite type is trivial if and only if every fiber is trivial (Proposition B.2.9). $\square$

Corollary 3.6.8. *Let $G \to S$ be a smooth and affine group scheme over a scheme S. Let U be an algebraic space over S with an action of G. Then*

(1) *$[U/G]$ is Deligne–Mumford if and only if every point of U has a discrete and reduced stabilizer group, or equivalently if and only if the action map $G \times U \to U \times U$ is unramified.*

(2) *$[U/G]$ is an algebraic space if and only if every point of U has a trivial stabilizer group, or equivalently if and only if the action map $G \times U \to U \times U$ is a monomorphism.* $\square$

Corollary 3.6.9 (Characterization of Representable Morphisms)**.** *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of noetherian algebraic stacks whose diagonal is representable by schemes. Then $\mathcal{X} \to \mathcal{Y}$ is representable if and only if for every geometric point $x \in \mathcal{X}(\Bbbk)$, the map $G_x \to G_{f(x)}$ on automorphism groups is injective.* $\square$

Corollary 3.6.10. *If $g \geq 2$, $\mathcal{M}_g$ is a Deligne–Mumford stack of finite type over $\mathbb{Z}$ with affine diagonal.*

Proof. It only remains to show that $\mathcal{M}_g$ is Deligne–Mumford, and by the Characterization of Deligne–Mumford Stacks (3.6.4), it suffices to show that for every smooth, connected, and projective curve C over $\Bbbk$ that $\mathrm{Aut}(C)$ is discrete and reduced, or in other words that the dimension of the Lie algebra $\dim T_{\mathrm{Aut}(C),e} = 0$. Consider the diagrams

$$
\begin{array}{ccccc}
\mathrm{Spec}\,\Bbbk & \xrightarrow{\ e\ } & \mathrm{Aut}(C) & \longrightarrow & \mathcal{M}_g \\
\downarrow & {\scriptstyle \tau}\nearrow & \downarrow & & \downarrow \\
\mathrm{Spec}\,\Bbbk[\epsilon] & \longrightarrow & \mathrm{Spec}\,\Bbbk & \xrightarrow{([C],[C])} & \mathcal{M}_g \times \mathcal{M}_g
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathrm{Spec}\,\Bbbk & \xrightarrow{[C]} & \mathcal{M}_g \\
\downarrow & {\scriptstyle \tau}\Downarrow\nearrow & \downarrow \\
\mathrm{Spec}\,\Bbbk[\epsilon] & \xrightarrow{\ p\ } & \mathrm{Spec}\,\Bbbk.
\end{array}
$$

A lifting $\tau \colon \mathrm{Spec}\,\Bbbk[\epsilon] \to \mathrm{Aut}(C)$ of the left diagram, i.e. a tangent vector $\tau \in T_{\mathrm{Aut}(C),e}$, translates into an automorphism τ of the trivial first order deformation $[C] \circ p$ is trivial. By Deformation Theory (C.2.7), the automorphism group of the trivial first order deformation is identified with $\mathrm{H}^0(C, T_C)$, but this vector space is zero since the degree of $T_C = \Omega_C^\vee$ is $2 - 2g < 0$. $\square$

3.7 Smoothness and the Infinitesimal Lifting Criteria

We prove the Infinitesimal Lifting Criteria (3.7.1) providing functorial criteria to verify that moduli stacks are smooth, étale, and unramified. As an application, we

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show that the moduli stacks $\mathcal{M}_g$ of smooth curves and $\underline{\mathrm{Bun}}_{r,d}(C)$ of vector bundles are smooth (Propositions 3.7.6 and 3.7.8).

3.7.1 Infinitesimal Lifting Criteria

As the property of smoothness for a morphism of schemes is smooth-local on the source and target, this property extends to morphisms of algebraic stacks using Definition 3.3.2. On the other hand, étaleness and unramifiedness are smooth-local on the target, but only étale-local on the source. Recall that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *relatively Deligne–Mumford* if every base change by a map $T \to \mathcal{Y}$ from a scheme is Deligne–Mumford (Definition 3.3.33); this is equivalent to the unramifiedness of the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (Corollary 3.6.5). A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is étale (resp. unramified) if it is relatively Deligne–Mumford and every base change is étale (resp. unramified); see Definition 3.3.34.

Theorem 3.7.1 (Infinitesimal Lifting Criteria for Smoothness/Étaleness/Unramifedness)**.**
Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a locally of finite type morphism of locally noetherian algebraic stacks. Considera 2-commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A_0 & \longrightarrow & \mathcal{X} \\
\cap \downarrow & \nearrow & \downarrow f \\
\operatorname{Spec} A & \longrightarrow & \mathcal{Y},
\end{array}
\tag{3.7.2}
$$

of solid arrows where $A \to A_0$ is a surjection of artinian local rings with residue field $\Bbbk$ such that $\ker(A \to A_0) \cong \Bbbk$. Then

(1) *f is smooth if and only if there exists a lifting of every diagram (3.7.2);*

(2) *f is étale if and only if there exists a lifting, which is unique up to unique isomorphism, of every diagram (3.7.2),*

(3) *f is unramified if and only if every two liftings of a diagram (3.7.2) are uniquely isomorphic; and*

(4) *f has unramified diagonal if and only if every automorphism of a lifting of a diagram (3.7.2) is trivial.*

Remark 3.7.3. To be explicit, a *lifting* of a 2-commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\ x\ } & \mathcal{X} \\
g \downarrow & \nearrow_{\alpha} & \downarrow f \\
T & \xrightarrow{\ y\ } & \mathcal{Y},
\end{array}
\tag{3.7.4}
$$

is a triple $(\widetilde{x}, \beta, \gamma)$ where $\widetilde{x}\colon T \to \mathcal{X}$ is a map and $\beta\colon x \xrightarrow{\sim} \widetilde{x} \circ g$ and $\gamma\colon f \circ \widetilde{x} \xrightarrow{\sim} y$ are 2-isomorphisms such that

$$
\begin{array}{ccc}
 & f \circ x & \\
{\scriptstyle f(\beta)} \swarrow & & \searrow {\scriptstyle \alpha} \\
f \circ \widetilde{x} \circ g & \xrightarrow[\ g^{*}\gamma\ ]{} & y \circ g
\end{array}
$$

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commutes. We may view a lifting as

$$
\begin{array}{ccc}
S & \xrightarrow{\ x\ } & \mathcal{X} \\
\Big\downarrow{\scriptstyle g} & {\scriptstyle \beta}\ \Swarrow\ {\scriptstyle \widetilde{x}}\quad\nearrow & \Big\downarrow{\scriptstyle f} \\
& \Swarrow\ {\scriptstyle \gamma} & \\
T & \xrightarrow[\ y\ ]{} & \mathcal{X}.
\end{array}
$$

A morphism $(\widetilde{x}, \beta, \gamma) \to (\widetilde{x}', \beta', \gamma')$ of liftings is a 2-isomorphism $\Theta \colon \widetilde{x} \xrightarrow{\sim} \widetilde{x}'$ such that $\beta = g^*\Theta \circ \beta'$ and $\gamma = \gamma' \circ f(\Theta)$.

We can also interpret liftings using the map

$$\Psi \colon \mathcal{X}(T) \to \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$$

of groupoids. A diagram (3.7.4) defines an object $(x, y, \alpha) \in \mathcal{X}(S) \times_{\mathcal{Y}(S)} \mathcal{Y}(T)$ and the category of liftings is the fiber category over this object. The criteria above states that $f \colon \mathcal{X} \to \mathcal{Y}$ is smooth (resp. is étale, is unramified, has unramified diagonal) if and only if $\mathcal{X}(A) \to \mathcal{X}(A_0) \times_{\mathcal{Y}(A_0)} \mathcal{Y}(A)$ is essentially surjective (resp. an equivalence, fully faithful, faithful).

Proof. We first handle the criterion for smoothness. Suppose that $f \colon \mathcal{X} \to \mathcal{Y}$ is smooth and that we are given a diagram (3.7.2). By replacing $\mathcal{Y}$ with $\operatorname{Spec} A$ and $\mathcal{X}$ with $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} A$, we may assume that $\mathcal{Y} = \operatorname{Spec} A$ is affine, and we need to show that a section over $\operatorname{Spec} A_0$

$$
\begin{array}{ccc}
& & \mathcal{X} \\
& \nearrow & \Big\downarrow \\
\operatorname{Spec} A_0 & \hookrightarrow & \operatorname{Spec} A
\end{array}
$$

extends to a section over $\operatorname{Spec} A$. If $\mathcal{X}$ is a scheme, the existence of a lifting is provided by the Infinitesimal Lifting Criterion for Smoothness (A.3.1) for schemes. If $\mathcal{X}$ is an algebraic space, we may apply Proposition 4.2.15 (proven later but independently) to construct an étale presentation $U \to \mathcal{X}$ from a scheme and a lifting $\operatorname{Spec} \Bbbk \to U$ of $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} A_0 \to \mathcal{X}$. This gives a diagram

$$
\begin{array}{ccccc}
& & & & U \\
& & & \nearrow & \Big\downarrow \\
& & & & \mathcal{X} \\
& \nearrow & & & \Big\downarrow \\
\operatorname{Spec} \Bbbk & \hookrightarrow & \operatorname{Spec} A_0 & \hookrightarrow & \operatorname{Spec} A.
\end{array}
$$

Since $U \to \mathcal{X}$ is representable by schemes, the Infinitesimal Lifting Criterion (A.3.1) gives a lifting $\operatorname{Spec} A_0 \to U$ of $\operatorname{Spec} A_0 \to \mathcal{X}$. Since $U \to \operatorname{Spec} A$ is a smooth morphism of schemes, another application of the Infinitesimal Lifting Criterion gives a lifting $\operatorname{Spec} A \to U$ of $\operatorname{Spec} A_0 \to U$, and the composition $\operatorname{Spec} A \to U \to \mathcal{X}$ gives the desired extension. Finally, if $\mathcal{X}$ is an algebraic stack, we can repeat the above argument by applying Proposition 4.2.15 to construct a smooth presentation $U \to \mathcal{X}$ with a lifting $\operatorname{Spec} \Bbbk \to U$, where we use the representability of $U \to \mathcal{X}$ and the algebraic space case to construct the lifting $\operatorname{Spec} A_0 \to U$.

Conversely, suppose that every diagram (3.7.2) for $f\colon \mathcal{X} \to \mathcal{Y}$ has a lifting. Choose smooth presentations $V \to \mathcal{Y}$ and $U \to \mathcal{X}_V$ giving a commutative diagram

$$
\begin{array}{ccccc}
U & \longrightarrow & \mathcal{X}_V & \longrightarrow & \mathcal{X} \\
& \searrow & \downarrow & \square & \downarrow f \\
& & V & \longrightarrow & \mathcal{Y}.
\end{array}
$$

Since $U \to \mathcal{X}_V$ is smooth, by the implication already proven, every diagram (3.7.2) for $U \to \mathcal{X}_V$ has a lifting. Since every diagram (3.7.2) for $\mathcal{X}_V \to V$ also has a lifting (as it is the base change of $\mathcal{X} \to \mathcal{Y}$), so does the composition $U \to V$. As $U \to V$ is a morphism of schemes, the Infinitesimal Lifting Criterion (A.3.1) implies that it is smooth. Since smoothness is a smooth-local property on the source and target, $\mathcal{X} \to \mathcal{Y}$ is smooth.

Second, we handle the criterion for unramified diagonal. Let $\Delta_f\colon \mathcal{X} \to \mathcal{X} \times_\mathcal{Y} \mathcal{X}$ be the diagonal and $\Delta_{\Delta_f}\colon \mathcal{X} \to I_{\mathcal{X}/Y} = \mathcal{X} \times_{\mathcal{X} \times_\mathcal{Y} \mathcal{X}} \mathcal{X}$ be the double diagonal taking an object a to $(a, \mathrm{id}_a) \in I_{\mathcal{X}/\mathcal{Y}}$. By Exercise 3.3.35, Δ_f is unramified if and only Δ_{Δ_f} is an open immersion. Consider 2-commutative diagrams of the form

$$
\begin{array}{ccc}
\operatorname{Spec} A_0 & \xrightarrow{\ x\ } & \mathcal{X} \\
\downarrow i & \overset{\Nearrow_\alpha}{} & \downarrow f \\
\operatorname{Spec} A & \xrightarrow{\ y\ } & \mathcal{Y}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\operatorname{Spec} A_0 & \xrightarrow{\ x\ } & \mathcal{X} \\
\downarrow i & \overset{\Nearrow_\beta}{} & \downarrow e \\
\operatorname{Spec} A & \xrightarrow{(\widetilde{x},\Theta)} & I_{\mathcal{X}/\mathcal{Y}}.
\end{array}
$$

Assume that Δ_f is unramified. Let $(\widetilde{x}, \beta, \gamma)$ be a lifting of the left diagram, where $\beta\colon x \xrightarrow{\sim} \widetilde{x} \circ i$ and $\gamma\colon f \circ \widetilde{x} \xrightarrow{\sim} y$ are 2-isomorphisms satisfying $f(\beta) = i^*\gamma \circ \alpha$. Let Θ be an automorphism of $(\widetilde{x}, \beta, \gamma)$, which means that there is a 2-isomorphism $\Theta\colon \widetilde{x} \xrightarrow{\sim} \widetilde{x}$ such that $\beta = i^*\Theta \circ \beta$ and $\gamma = \gamma \circ f(\Theta)$. These relations imply that $i^*\Theta = \mathrm{id}_{\widetilde{x} \circ i}$ and $f(\Theta) = \mathrm{id}_{f \circ \widetilde{x}}$. Then $(\widetilde{x}, \Theta)$ defines an object of $I_{\mathcal{X}/\mathcal{Y}}$ over $\operatorname{Spec} A$, and induces the diagram on the right where β expresses the 2-commutativity. Since $e\colon \mathcal{X} \to I_{\mathcal{X}/\mathcal{Y}}$ is an open immersion, there is a lifting $\operatorname{Spec} A \to \mathcal{X}$ of the right diagram, which implies that $\Theta = \mathrm{id}_{\widetilde{x}}$. Conversely, assume the every automorphism of a lifting of a diagram on the left is trivial. Since $e\colon \mathcal{X} \to I_{\mathcal{X}/\mathcal{Y}}$ is a quasi-compact monomorphism representable by schemes, we may apply the Infinitesimal Lifting Criterion (A.3.2) for schemes to show that it is étale and thus an open immersion, which in turn implies that Δ_f is unramified. Given a diagram as in the right diagram, we set $y = f(\widetilde{x}) \in \mathcal{Y}$ and $\gamma = \mathrm{id}_y$. Since the 2-isomorphism $\beta\colon x \to \widetilde{x} \circ i$ expressing the commutativity of the right diagram satisfies $\beta = i^*\Theta \circ \beta$, it follows that $i^*\Theta = \mathrm{id}_{\widetilde{x} \circ i}$. Then the 2-isomorphism $\alpha := f(\beta)\colon f \circ x \xrightarrow{\sim} y \circ i$ defines a commutative diagram as in the left, $(\widetilde{x}, \beta, \gamma)$ defines a lifting, and Θ defines an automorphism of $(\widetilde{x}, \beta, \gamma)$. Thus, Θ is the identity and the right diagram has a lifting.

Since étaleness is equivalent to smoothness and unramifiedness and since we've already established the criterion for smoothness, the criterion for étaleness reduces to the criterion for unramifiedness. By Corollary 3.6.5, $f\colon \mathcal{X} \to \mathcal{Y}$ is relatively Deligne–Mumford if and only if Δ_f is unramified, which we've shown is equivalent to the triviality of any isomorphism of a lift of a diagram (3.7.2). Therefore, it suffices to assume that $f\colon \mathcal{X} \to \mathcal{Y}$ is relatively Deligne–Mumford, and show that f is unramified if and only if every two liftings of a diagram (3.7.2) are isomorphic. Suppose that f is unramified, and that $\widetilde{x}_1, \widetilde{x}_2\colon \operatorname{Spec} A \to \mathcal{X}$ are two liftings. By base changing by the given map $\operatorname{Spec} A \to \mathcal{Y}$, we can reduce to the case that $\mathcal{Y} = \operatorname{Spec} A$ and $\mathcal{X}$ is Deligne–Mumford. Apply Proposition 4.2.15 to construct

an étale presentation $U \to \mathcal{X}$ and a lifting $\operatorname{Spec} \Bbbk \to U$ of $\operatorname{Spec} \Bbbk \hookrightarrow \operatorname{Spec} A_0 \to \mathcal{X}$. Since $U \to \mathcal{X}$ is smooth and we've shown the criterion for smoothness, there exists a lifting $\operatorname{Spec} A_0 \to U$ of $\operatorname{Spec} A_0 \to \mathcal{X}$ and $\operatorname{Spec} \Bbbk \to U$. We may also choose liftings $\widetilde{u}_1, \widetilde{u}_2 \colon \operatorname{Spec} A \to U$ of $\widetilde{x}_1$ and $\widetilde{x}_2$. Since $U \to \operatorname{Spec} A$ is unramified, the Infinitesimal Lifting Criterion (A.3.3) for schemes implies that there is an isomorphism $\widetilde{u}_1 \xrightarrow{\sim} \widetilde{u}_2$. This induces the desired isomorphism $\widetilde{x}_1 \xrightarrow{\sim} \widetilde{x}_2$. Conversely, suppose that every two liftings of a diagram (3.7.2) for $\mathcal{X} \to \mathcal{Y}$ are isomorphic. Then the same holds for the base change $\mathcal{X}_V \to V$ by a smooth presentation $V \to \mathcal{Y}$, and the same also holds for étale presentation $U \to \mathcal{X}_V$ by the implication just proven. By the Infinitesimal Lifting Criterion (A.3.3) for schemes, $U \to V$ is unramified and thus $\mathcal{X} \to \mathcal{Y}$ is unramified. See also [LMB00, Prop. 4.15] and [SP, Tag 0DP0]. $\qquad\square$

As a first application, we see that the presentation produced by the Existence of Minimal Presentations (Theorem 3.6.1) is miniversal, i.e. induces an isomorphism on tangent spaces at the chosen preimage, and we can also express the dimension of a smooth algebraic stack in terms of its tangent space and stabilizer.

Proposition 3.7.5. *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $f \colon (U, u) \to (\mathcal{X}, x)$ be a smooth morphism from a scheme such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$. Then $U \to \mathcal{X}$ is miniversal at u, i.e. $T_{U,u} \to T_{\mathcal{X}, f(u)}$ is an isomorphism of $\kappa(u)$-vector spaces. In particular, if $\mathcal{X}$ is smooth over a field $\Bbbk$ and $x \in \mathcal{X}(\mathbb{F})$ is a point with smooth stabilizer over a finite extension $\mathbb{F}/\Bbbk$, then*

$$\dim_x \mathcal{X} = \dim T_{\mathcal{X}, x} - \dim G_x.$$

Proof. Surjectivity of $T_{U,u} \to T_{\mathcal{X}, f(u)}$ follows from the Infinitesimal Lifting Criterion (3.7.1). Let $\Bbbk = \kappa(u)$. Injectivity follows from the fact that

$$
\begin{array}{ccc}
\operatorname{Spec} \Bbbk & \lhook\joinrel\longrightarrow & U \\
\downarrow & \square & \downarrow \\
\mathcal{G}_x & \lhook\joinrel\longrightarrow & \mathcal{X}
\end{array}
$$

is cartesian. Indeed, if $\tau \colon \operatorname{Spec} \Bbbk[\epsilon] \to U$ is an element of $T_{U,u}$ mapping to $0 \in T_{\mathcal{X}, f(u)}$, then the composition $\operatorname{Spec} \Bbbk[\epsilon] \to U \to \mathcal{X}$ factors through the residual gerbe $\mathcal{G}_x$ and therefore also factors through the fiber product $\operatorname{Spec} \Bbbk$. We conclude that $\tau = 0$. For the final statement, Existence of Minimal Presentations (3.6.1) produces a smooth morphism $f \colon (U, u) \to (\mathcal{X}, x)$ of relative dimension $\dim G_x$ such that $\mathcal{G}_x \times_{\mathcal{X}} U \cong \operatorname{Spec} \kappa(u)$, and we have just shown that $T_{U,u} \cong T_{\mathcal{X}, f(u)}$. Therefore, $\dim_x \mathcal{X} = \dim_u U - \dim G_x$, and since U is smooth at u, $\dim_u U = \dim T_{U,u} = \dim T_{\mathcal{X}, x}$. $\qquad\square$

3.7.2 Smoothness of moduli stacks

Combining the Infinitesimal Lifting Criterion for Smoothness (3.7.1) with deformation theory allows for the verification of smoothness of a moduli stack and the computation of its dimension.

Proposition 3.7.6. *For $g \geq 2$, the Deligne–Mumford stack $\mathcal{M}_g$ is smooth over $\operatorname{Spec} \mathbb{Z}$ of relative dimension $3g - 3$.*

Proof. Let $\operatorname{Spec}\Bbbk \to \mathcal{M}_g$ be a morphism from a field $\Bbbk$ corresponding to smooth, connected, and projective curve C over $\Bbbk$. Consider a diagram

$$(3.7.7)$$

where $A \to A_0$ is surjection of artinian local rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A \to A_0)$. The map $\operatorname{Spec} A_0 \to \mathcal{M}_g$ corresponds to a family of curves $\mathcal{C}_0 \to \operatorname{Spec} A_0$, and filling in (3.7.7) translates to filling in the cartesian diagram

of solid arrows. By Deformation Theory (C.2.7), there is cohomology class $\mathrm{ob}_C \in \mathrm{H}^2(C, T_C)$ such that there exists a lifting if and only if $\mathrm{ob}_C = 0$. Since C is a curve, $\mathrm{H}^2(C, T_C) = 0$, and thus $\mathrm{ob}_C = 0$ and there is a lifting. Finally, Deformation Theory (C.2.7) also gives the identification $T_{\mathcal{M}_g, [C]} = \mathrm{H}^1(C, T_C)$, which has dimension $3g - 3$ by a Riemann–Roch computation (see Example 3.5.11). Since $\dim \operatorname{Aut}(C) = 0$, we conclude using Proposition 3.7.5 that $\dim_{[C]} \mathcal{M}_g = 3g - 3$. $\qquad\square$

Proposition 3.7.8. *The algebraic stack $\underline{\operatorname{Bun}}_{r,d}(C)$ is smooth over $\operatorname{Spec}\Bbbk$ of dimension $r^2(g - 1)$.*

Proof. Let $[F] \in \underline{\operatorname{Bun}}_{r,d}(C)(\mathbb{F})$ be a vector bundle on $C_{\mathbb{F}} = C \times_{\Bbbk} \mathbb{F}$ of rank r and degree d. Let $A \to A_0$ be a surjection of artinian local rings with residue field $\mathbb{F}$ such that $\mathbb{F} = \ker(A \to A_0)$. We need to check that every vector bundle $\mathcal{F}_0$ on C_{A_0} that restricts to F extends to a vector bundle $\mathcal{F}$ on C_A. By deformation theory (Proposition C.2.17), there is an element $\mathrm{ob}_F \in \operatorname{Ext}^2_{\mathcal{O}_C}(F, F)$ such that $\mathrm{ob}_F = 0$ if and only if there exists an extension. Since C is a smooth curve, $\operatorname{Ext}^2_{\mathcal{O}_C}(F, F) = \mathrm{H}^2(C_{\mathbb{F}}, F \otimes F^\vee) = 0$. Deformation theory also provides an identification $T_{\underline{\operatorname{Bun}}_{r,d}(C), [F]} = \operatorname{Ext}^1_{\mathcal{O}_C}(F, F)$ and a Riemann–Roch calculation yields $\dim \operatorname{Ext}^1_{\mathcal{O}_C}(F, F) = \dim \operatorname{Aut}(F) + r^2(g - 1)$ (see Example 3.5.12). Therefore $\dim_{[F]} \underline{\operatorname{Bun}}_{r,d}(C) = \dim \operatorname{Ext}^1_{\mathcal{O}_C}(F, F) - \dim \operatorname{Aut}(F) = r^2(g - 1)$. $\qquad\square$

3.8 Properness and the Valuative Criterion

After defining universally closed, separated, and proper morphisms of algebraic stacks, we prove Valuative Criteria for Universally Closed/Separated/Proper Morphisms (3.8.2), providing a generalization of the usual valuative criteria for schemes (Theorem A.4.5). These valuative criteria are essential in moduli theory and applied in this book to show that $\overline{\mathcal{M}}_g$ is proper (Theorem 5.5.16) and $\underline{\operatorname{Bun}}^{\mathrm{ss}}_{r,d}(C)$ is universally closed.

3.8.1 Definitions

With some care, we define separatedness and properness for morphisms of algebraic stacks. Recall from Definition 3.3.10 that we say a representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the diagonal $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is representable by schemes) is proper.

Definition 3.8.1.

(1) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *universally closed* if for every morphism $\mathcal{Y}' \to \mathcal{Y}$ of algebraic stacks, the morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ induces a closed map $|\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'| \to |\mathcal{Y}'|$.

(2) A representable morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated, and of finite type.

(3) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *separated* if the representable morphism $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is proper.

(4) A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *proper* if it is universally closed, separated, and of finite type.

As universal closedness is a smooth-local property on the target, $\mathcal{X} \to \mathcal{Y}$ is universally closed if and only if $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is closed for all maps $T \to \mathcal{Y}$ from schemes, or equivalently $\mathcal{X} \times_{\mathcal{Y}} V$ is universally closed for a smooth presentation $V \to \mathcal{Y}$.

For a morphism of schemes, properness is equivalent to the diagonal being a closed immersion. This is also true for algebraic spaces as proper monomorphisms of algebraic spaces are closed immersions (Corollary 4.4.14). This fails for morphisms of algebraic stacks as the diagonal need not be a monomorphism. Recall that the stabilizer G_x of a field-valued point $x \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ is given by the cartesian diagram

$$
\begin{array}{ccc}
G_x & \longrightarrow & \operatorname{Spec} \Bbbk \\
\downarrow & \square & \downarrow {\scriptstyle (x,x)} \\
\mathcal{X} & \longrightarrow & \mathcal{X} \times \mathcal{X}.
\end{array}
$$

If $\mathcal{X}$ is a separated algebraic stack over a scheme S, then G_x is a proper group algebraic space over $\Bbbk$, and even a group *scheme* by Theorem 4.4.28. If $\mathcal{X}$ is separated and has affine diagonal, then G_x is proper and affine, thus finite. Since $\underline{\operatorname{Bun}}_{r,d}(C)$ has affine diagonal (Example 3.3.12) and infinite automorphism groups, we see that $\underline{\operatorname{Bun}}_{r,d}(C)$ is *not* separated.

3.8.2 Valuative Criteria

For moduli problems, the valuative criterion for properness translates to the geometric question of whether an object over $\operatorname{Spec} K$ curve extends uniquely to a family over $\operatorname{Spec} R$. We will use the notions of liftings of 2-commutative diagrams and their morphisms as defined formally in Remark 3.7.3.

Theorem 3.8.2 (Valuative Criteria for Properness/Universal Closedness/Separatedness)**.** *Let* $f \colon \mathcal{X} \to \mathcal{Y}$ *be a quasi-compact and quasi-separated morphism of locally noetherian algebraic stacks. Consider a 2-commutative diagram*

$$
\begin{array}{ccc}
\operatorname{Spec} K & \longrightarrow & \mathcal{X} \\
\downarrow & {\scriptstyle \Swarrow \alpha} & \downarrow {\scriptstyle f} \\
\operatorname{Spec} R & \longrightarrow & \mathcal{Y}
\end{array}
\tag{3.8.3}
$$

where R is a DVR with fraction field K. Then

(1) *f is universally closed if for every diagram* (3.8.3)*, there exists an extension $R \to R'$ of DVRs with $K \to K' = \mathrm{Frac}(R)$ of finite transcendence degree and a lifting*

$$
\begin{array}{ccc}
\mathrm{Spec}\, K' \longrightarrow \mathrm{Spec}\, K \longrightarrow & \mathcal{X} \\
\downarrow \qquad\qquad \downarrow \qquad\qquad & \downarrow f \\
\mathrm{Spec}\, R' \longrightarrow \mathrm{Spec}\, R \longrightarrow & \mathcal{Y};
\end{array}
\tag{3.8.4}
$$

(2) *f is proper if and only if f is finite type and for every diagram* (3.8.3)*, there exists an extension $R \to R'$ of DVRs with the map $K \to K'$ on fraction fields having finite transcendence degree and a lifting as in* (3.8.4)*, which is unique up to unique isomorphism;*

(3) *f is separated if and only if every two liftings of a diagram* (3.8.3) *are uniquely isomorphic; and*

(4) *f has separated diagonal if and only if every automorphism of a lifting of a diagram* (3.8.3) *is trivial.*

Exercise 3.8.5 (good practice)**.**

(a) If G is an abstract finite group, show that $BG \to \mathrm{Spec}\,\mathbb{Z}$ is proper.

(b) Show that $B\mathbb{G}_m \to \mathrm{Spec}\,\mathbb{Z}$ is universally closed but not separated.

Try to give two arguments for each part—one using the definitions and the other using the valuative criterion.

Exercise 3.8.6 (hard)**.** Show that the stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves introduced in Exercise 3.1.19(c) is proper over $\mathrm{Spec}\,\mathbb{Z}$.

Are base changes necessary? For a morphism $X \to Y$ of schemes, for the valuative criterion of properness (Theorem A.4.5), it is not necessary to allow extensions of the DVR, i.e. there exists a unique lift

$$
\begin{array}{ccc}
\mathrm{Spec}\, K & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\mathrm{Spec}\, R & \longrightarrow & Y.
\end{array}
$$

The same holds for morphisms of algebraic spaces [SP, Tag 0A40]. For general morphisms of algebraic stacks, it is necessary to allow extensions.

Example 3.8.7. Consider the structure morphism $\mathcal{X} = B\boldsymbol{\mu}_n \to \mathrm{Spec}\,\Bbbk$ where $\Bbbk$ is a field whose characteristic is prime to n. Let $R = \Bbbk[x]_{(x)}$ with fraction field $K = \Bbbk(x)$. Then $\mathrm{Spec}\, K \to \mathrm{Spec}\, K$, given by $x \mapsto x^n$, is a principal $\boldsymbol{\mu}_n$-torsor, and the classifying map $\mathrm{Spec}\, K \to B\boldsymbol{\mu}_n$ does not extend to a map $\mathrm{Spec}\, R \to B\boldsymbol{\mu}_n$. Note however the principal $\boldsymbol{\mu}_n$-bundle $\mathrm{Spec}\, K \to \mathrm{Spec}\, K$ becomes trivial after base changing by the field extension $K(x^{1/n})$ of K, and therefore the composition $\mathrm{Spec}\, K(x^{1/n}) \to B\boldsymbol{\mu}_n$ trivially extends to a map $\mathrm{Spec}\, R[x^{1/n}] \to B\boldsymbol{\mu}_n$.

Example 3.8.8. Let $X = \mathrm{Spec}\, R$ be the spectrum of a DVR and $\mathcal{X}$ be the nth root stack $\sqrt[n]{X/\mathcal{O}_X/\pi}$ (Example 3.9.13). The morphism $\mathcal{X} \to X$ is an isomorphism over the generic point, but the section $\mathrm{Spec}\, K \to \mathcal{X}$ not extend to a global section $X \to \mathcal{X}$.

If G is a special algebraic group over a field (i.e. every principal G-bundle is Zariski-locally trivial) such as SL_n or GL_n, then BG satisfies the valuative criterion for universal closedness without a base change: any map $\mathrm{Spec}\, K \to BG$ corresponds to the trivial principal G-bundle and thus extends to a map $\mathrm{Spec}\, R \to BG$. On the other hand, base changes are necessary for $B\,\mathrm{PGL}_n$.

Exercise 3.8.9 (hard). Show that there is a principal PGL_n-bundle over the fraction field of a DVR that does not extend to the DVR.

For the valuative criterion of properness for $\overline{\mathcal{M}}_g$, extensions of the DVR are necessary (see Example 5.5.6). On the other hand, the valuative criterion for universal closedness for $\underline{\mathrm{Bun}}_{r,d}(C)$ holds without extensions.

3.8.3 Proof of the Valuative Criteria

We modify the proof of the valuative criterion for schemes given in §A.4. The starting point is a lifting criterion for closed morphisms generalizing Lemma A.4.1.

Lemma 3.8.10. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact morphism of algebraic stacks. Then f is closed if and only for every point $x \in |\mathcal{X}|$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$.*

Proof. The statement is equivalent to the equality that $f(\overline{\{x\}}) = \overline{\{f(x)\}}$ (Exercise 3.3.29(a)). $\square$

Specializations are induced by maps from DVRs, just as in the case of schemes (Proposition A.4.2).

Proposition 3.8.11. *If $f \colon \mathcal{X} \to \mathcal{Y}$ is a finite type morphism of noetherian algebraic stacks, $x \in |\mathcal{X}|$ and $f(x) \rightsquigarrow y_0$ is a specialization, then there exists a diagram*

$$
\begin{array}{ccc}
\mathrm{Spec}\, K \longrightarrow \mathcal{X} & \qquad & x \\
\downarrow \qquad\quad \downarrow {\scriptstyle f} & & \downarrow \\
\mathrm{Spec}\, R \longrightarrow \mathcal{Y} & \qquad & f(x) \rightsquigarrow y_0.
\end{array}
$$

where R is a DVR with fraction field K, the image of $\mathrm{Spec}\, K \to \mathcal{X}$ is x and $\mathrm{Spec}\, R \to \mathcal{Y}$ realizes the specialization $f(x) \rightsquigarrow y_0$. In particular, every specialization $x \rightsquigarrow x_0$ in a noetherian algebraic stack is realized by a map $\mathrm{Spec}\, R \to \mathcal{X}$ from a DVR.

Proof. Let $V \to \mathcal{Y}$ be a smooth presentation and $v_0 \in V$ be a preimage of y_0. Since $V \to \mathcal{Y}$ is smooth, it is an open morphism (Exercise 3.3.25), and thus there exists a specialization $v \rightsquigarrow v_0$ over $f(x) \rightsquigarrow y_0$ (Exercise 3.3.29(c)). Let $x' \in |\mathcal{X}_V|$ be a preimage of $v \in V$ and $x \in |\mathcal{X}|$. Let $U \to \mathcal{X}_V$ be a smooth presentation and $u \in U$ be a preimage of x'. Applying Proposition A.4.4 to the morphism $U \to V$ of schemes and $u \mapsto v \rightsquigarrow v_0$ gives the desired diagram. $\square$

Proof of Theorem 3.8.2. The quasi-separatedness of $f \colon \mathcal{X} \to \mathcal{Y}$ means that the diagonal $\Delta_f \colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and the double diagonal of $\Delta_{\Delta_f} \colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ are quasi-compact. The quasi-compactness of f, Δ_f, and Δ_{Δ_f} are needed in the proof for the implication that the valuative criterion implies universally closedness, separatedness, and separated diagonal, respectively.

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We first establish the criterion for universal closedness. Suppose that the valuative criterion holds and that $f\colon \mathcal{X} \to \mathcal{Y}$ is not universally closed. Since universal closedness is a smooth-local property on the target, we may assume that $\mathcal{Y} = Y$ is a noetherian scheme and there is a map $T \to Y$ of schemes such that $f_T \colon \mathcal{X}_T \to T$ is not closed. We will reduce to the case that $T \to Y$ is a *finite type* morphism. By Lemma 3.8.10, there exists $z \in |\mathcal{X}_T|$ and a specialization $f_T(z) \rightsquigarrow t_0$ which doesn't lift to a specialization $z \rightsquigarrow z_0$. This implies that $\mathcal{Z} = \overline{\{z\}} \subset \mathcal{X}_T$ has trivial intersection with the fiber $(\mathcal{X}_T)_{t_0}$. If $p\colon X \to \mathcal{X}$ is a smooth presentation, then the preimage Z of $\mathcal{Z}$ under $X_T \to \mathcal{X}_T$ does not meet the fiber $(X_T)_{t_0}$. Lemma A.4.8 shows that after replacing T with an open neighborhood of t_0, there exists a factorization $T \xrightarrow{g} T' \to Y$ and a closed subscheme $Z' \subset X_{T'}$ such that $T' \to Y$ is of finite type, $Z \cap (X_{T'})_{g(t_0)} = \emptyset$, and $\mathrm{im}(Z \hookrightarrow X_T \to X_{T'}) \subset Z'$. Letting $z' \in |\mathcal{X}_{T'}|$ be the image of $z \in |\mathcal{X}_T|$, we have that z' maps to $g(f_T(z)) \in T'$ and that there is a specialization $g(f_T(z)) \rightsquigarrow g(t_0)$ which does not lift to a specialization of z'. By Lemma 3.8.10, $\mathcal{X}_{T'} \to T'$ is not closed.

If $T \to Y$ is a finite type morphism, the base change $\mathcal{X}_T \to T$ is a finite type morphism of noetherian algebraic stacks which also satisfies the valuative criterion for universal closedness. It therefore suffices to show that $f\colon \mathcal{X} \to Y$ is closed. By Lemma 3.8.10, we need to show that given a point $x \in |\mathcal{X}|$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$. By Proposition 3.8.11, there exists a diagram

$$
\begin{array}{ccc}
\operatorname{Spec} K \longrightarrow \mathcal{X} & \quad & x \\
\downarrow \qquad\quad \downarrow{\scriptstyle f} & \quad & \vdots \\
\operatorname{Spec} R \longrightarrow \mathcal{Y} & \quad & f(x) \rightsquigarrow\rightsquigarrow y_0.
\end{array}
\tag{3.8.12}
$$

The valuative criterion implies the existence of a lift $\operatorname{Spec} R \to \mathcal{X}$, which in turn yields a specialization $x \rightsquigarrow x_0$ lifting $f(x) \rightsquigarrow y_0$.

Conversely, assume that $f\colon \mathcal{X} \to \mathcal{Y}$ is universally closed and that we are given a diagram as in (3.8.12). By replacing $\mathcal{Y}$ with $\operatorname{Spec} R$ and $\mathcal{X}$ with $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} R$, we may assume that $\mathcal{Y} = \operatorname{Spec} R$ and that we have a diagram

$$
\begin{array}{ccc}
\operatorname{Spec} K & \longrightarrow & \mathcal{X} \\
& \searrow & \downarrow{\scriptstyle f} \\
& & \operatorname{Spec} R.
\end{array}
$$

By replacing $\mathcal{X}$ with $\overline{\{x\}}$, we may assume that $\mathcal{X}$ is integral with generic point x. Since $\mathcal{X} \to \operatorname{Spec} R$ is closed, there exists a specialization $x \rightsquigarrow x_0$ mapping to the specialization of the generic point to the closed point in $\operatorname{Spec} R$. Since f is quasi-separated, $\operatorname{Spec} K \to \mathcal{X}$ is quasi-compact. Applying Proposition 3.8.11 to $\operatorname{Spec} K \to \mathcal{X}$ yields a DVR R' with fraction field K' and a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} K' & \longrightarrow & \operatorname{Spec} K \\
\downarrow & & \downarrow \\
\operatorname{Spec} R' & \longrightarrow & \mathcal{X}
\end{array}
$$

such that $\operatorname{Spec} R' \to \mathcal{X}$ realizes the specialization $x \rightsquigarrow x_0$. As $\operatorname{Spec} R' \to \operatorname{Spec} R$ is surjective, $R \to R'$ is an extension of DVRs and $\operatorname{Spec} R' \to \mathcal{X}$ provides a lift of (3.8.12).

The criterion for separated diagonal follows exactly as the argument for the Infinitesimal Lifting Criterion for Unramified Diagonal (3.7.1(4)), using instead that Δ_f is separated if and only if Δ_{Δ_f} is a closed immersion (Exercise 3.3.35), and noting that since Δ_{Δ_f} a quasi-compact monomorphism representable by schemes, the Valuative Criterion (A.4.5) for schemes applies to verify that it is a closed immersion. Assuming that f has separated diagonal, the valuative criterion for the separatedness of $f \colon \mathcal{X} \to \mathcal{Y}$ translates to the valuative criterion for the universal closedness of the diagonal $\Delta_f \colon \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Thus, the valuative criteria for properness and separatedness follow from the valuative criterion for universal closedness. See also [LMB00, Thm. 7.10], [Fal03, §4], and [SP, Tags 0CLQ, 0CLS, 0CLV, and 0CLY]. $\qquad\square$

3.9   Further examples

This section provides examples of algebraic spaces, Deligne–Mumford stacks, and algebraic stacks.

3.9.1   Examples of algebraic spaces

Example 3.9.1. As discussed in Example 0.5.5, there exists a smooth proper complex 3-fold U with a free action of $\mathbb{Z}/2$-action such that there is an orbit not contained in an affine open subscheme. The quotient sheaf $U/(\mathbb{Z}/2)$ is an algebraic space (Corollary 3.1.14) which is not a scheme.

Example 3.9.2 (The bug-eyed cover)**.** Let $\Bbbk$ be field of $\mathrm{char}(k) \neq 2$. Let $\mathbb{Z}/2 = \{\pm 1\}$ act on the non-separated affine line $U = \mathbb{A}^1 \bigcup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ over $\Bbbk$ by swapping the origins and by $(-1) \cdot x = -x$ for $x \neq 0$. Since the orbit of an origin is not contained in an affine, the quotient sheaf $U/(\mathbb{Z}/2)$ is not representable by a scheme; it is however an algebraic space (Corollary 3.1.14).

For an alternative description, let $\mathbb{Z}/2 = \{\pm 1\}$ act on $\mathbb{A}^1$ with multiplication $\sigma \colon \mathbb{Z}/2 \times \mathbb{A}^1 \to \mathbb{A}^1$ defined by $-1 \cdot x = -x$. If we remove the non-identity element of the stabilizer of the origin, we obtain a scheme $R = (\mathbb{Z}/2 \times \mathbb{A}^1) \setminus \{(-1, 0)\}$ and an equivalence relation $\sigma, p_2 \colon R \rightrightarrows \mathbb{A}^1$. The algebraic space quotient $\mathbb{A}^1/R$ is isomorphic to $U/(\mathbb{Z}/2)$ (Exercise 3.9.3(a)) For another way to see that $X = \mathbb{A}^1/R$ is not a scheme, observe that the diagonal $X \to X \times X$ is not a locally closed immersion as there is a cartesian diagram

$$
\begin{array}{ccc}
(\mathbb{A}^1 \setminus 0) \amalg \{0\} & \longrightarrow & \mathbb{A}^1 \\
\downarrow & & \downarrow \\
R & \xrightarrow{(\sigma, p_2)} & \mathbb{A}^1 \times \mathbb{A}^1 \\
\downarrow & & \downarrow \\
X & \longrightarrow & X \times X.
\end{array}
\qquad
\begin{array}{c}
x \\
\downarrow \\
(x, -x)
\end{array}
$$

Exercise 3.9.3.

(a) Show that $X = \mathbb{A}^1/R$ is isomorphic to $U/(\mathbb{Z}/2)$.

(b) Show that there is a universal homeomorphism $X \to \mathbb{A}^1$ which is ramified over the origin.

(c) Show that every map to a scheme $X \to Z$ factors through $X \to \mathbb{A}^1$. (In other words, while $\mathbb{A}^1$ may be the categorical quotient of U by $\mathbb{Z}/2$ (or equivalently

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the category quotient of $R \rightrightarrows \mathbb{A}^1$) in the category of schemes, it is distinct from the algebraic space quotient.

(d) Consider the SL_2 action on $V_d = \mathrm{Sym}^d \Bbbk^2$, the space of homogeneous polynomials in x and y of degree d. Let $W \subset V_1 \times V_4$ be the reduced locally closed subscheme defined as the set (L, F) such that $L \neq 0$ and F is the square of a homogeneous quadratic with discriminant 1. Show that the induced SL_2-action on W is free (i.e. $\mathrm{SL}_2 \times W \to W \times W$ is a monomorphism) and that quotient sheaf W/SL_2 is an algebraic space isomorphic to $\mathbb{A}^1/R$ and $U/(\mathbb{Z}/2)$.

While the descriptions of X as $\mathbb{A}^1/R$ and $U/(\mathbb{Z}/2)$ may seem pathological, this exercise shows that in fact this algebraic space also arises as a quotient of a quasi-affine variety by SL_2.

Example 3.9.4. Let $\mathbb{Z}/2 = \{\pm 1\}$ act on $\mathbb{A}^1_{\mathbb{C}}$ via conjugation over $\mathrm{Spec}\,\mathbb{R}$. Note that the action defined over $\mathbb{R}$ of $\mathbb{Z}/2$ on $\mathrm{Spec}\,\mathbb{C}$ is free, and therefore the product action of $\mathbb{Z}/2$ on $\mathbb{A}^1_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{R}} \times_{\mathbb{R}} \mathbb{C}$ (which is trivial on the first factor) is also free. Defining $R = (\mathbb{Z}/2 \times \mathbb{A}^1_{\mathbb{C}}) \setminus \{(-1, 0)\}$, show that there is an equivalence relation $\sigma, p_2 \colon R \rightrightarrows U$ such that the algebraic space $X = \mathbb{A}^1_{\mathbb{C}}/R$ is not a scheme. (The quotient X looks like $\mathbb{A}^1_{\mathbb{R}}$ except that the origin has residue field $\mathbb{C}$.)

3.9.2 Examples of stacks with finite stabilizers

In characteristic 0, the following examples are Deligne–Mumford stacks.

Example 3.9.5 (Classifying stacks)**.** If G is an finite abstract group scheme over a field $\Bbbk$, then the *classifying stack BG of G* is the stack defined as the category of pairs (T, P) where T is a scheme and $P \to T$ is a G-torsor (Definition 2.3.14). Then BG is a smooth and proper algebraic stack over $\Bbbk$ of dimension 0. Properness follows from the fact the base change of $BG \to BG \times BG$ by the smooth presentation $\mathrm{Spec}\,\Bbbk \to BG \times BG$ is the finite morphism $G \to \mathrm{Spec}\,\Bbbk$, and smoothness follows because smoothness is a smooth-local property on the source and $S \to BG$ is a smooth presentation).

Example 3.9.6 (Weighted projective stacks)**.** For a tuple of positive integers $(d_0, \ldots, d_n)$, let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ via $t \cdot (x_0, \ldots, x_n) = (t^{d_0} x_0, \ldots, t^{d_n} x_n)$. We define the *weighted projective stack* as

$$\mathcal{P}(d_0, \ldots, d_n) = [(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m].$$

If the d_i are all 1, then we recover projective space $\mathbb{P}^n$; otherwise, $\mathcal{P}(d_0, \ldots, d_n)$ is not an algebraic space.

More generally, if R is a finitely generated positively graded $\Bbbk$-algebra, we can define *stacky proj* as $\mathcal{P}\mathrm{roj}\,R = [(\mathrm{Spec}(R) \setminus 0)/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts such that the weight of x_i is the same as its degree.

For example, over $\mathbb{Z}[1/6]$ the stack of stable elliptic curves $\overline{\mathcal{M}}_{1,1}$ is isomorphic to $\mathcal{P}(4, 6)$ by Exercise 3.1.19(c).

Exercise 3.9.7.

(a) If $\Bbbk$ is a field of characteristic p, show that $\mathcal{P}(d_0, \ldots, d_n)$ is a Deligne–Mumford stack if and only if p doesn't divide each d_i.

(b) Classify all the points of $\mathcal{P}(3, 3, 4, 6)$ that have non-trivial stabilizers.

(c) We say that an algebraic stack $\mathcal{X}$ has *generically trivial stabilizer* if there exists a dense open substack $U \subset \mathcal{X}$ which is an algebraic space. Provide conditions for when $\mathcal{P}(d_0, \ldots, d_n)$ has generically trivial stabilizer.

(d) Show that there is a bijective morphism $\mathcal{P}(d_0, \ldots, d_n)$ to weighted projective space $\operatorname{Proj} \Bbbk[x_0, \ldots, x_n]$, where x_i has degree d_i. (This is an example of a coarse moduli space.)

Example 3.9.8. Suppose $\operatorname{char}(k) \neq 2$. Let $\mathbb{Z}/2$ act on $\mathbb{A}_{\Bbbk}^2$ via $-1 \cdot (x, y) = (-x, -y)$. Show that $[\mathbb{A}_{\Bbbk}^2/(\mathbb{Z}/2)]$ is a smooth algebraic stack over a field $\Bbbk$ and that there is a proper and bijective morphism $[\mathbb{A}_{\Bbbk}^2/(\mathbb{Z}/2)] \to Y$ where Y is the singular variety $\operatorname{Spec} \Bbbk[x^2, xy, y^2]$ defined by the $\mathbb{Z}/2$-invariants of $\Gamma(\mathbb{A}_{\Bbbk}^2, \mathcal{O}_{\mathbb{A}_{\Bbbk}^2})$.

Example 3.9.9 (Stacky curves). A *stacky curve* is a one-dimensional Deligne—Mumford stack of finite type over a field $\Bbbk$.

Exercise 3.9.10. If d_1 and d_2 are relatively prime positive integers, show that $\mathbb{P}(d_1, d_2)$ is a smooth and proper stacky curve with generically trivial stabilizer.

Exercise 3.9.11. We say that a stacky curve $\mathcal{X}$ over $\Bbbk$ is *nodal* if there exists a étale presentation $U \to \mathcal{X}$ from a nodal curve (equivalently every étale presentation is a nodal curve); see Definition 5.2.1. Show that a nodal stacky curve has abelian stabilizers.

Example 3.9.12 (Root gerbes). Let X be a scheme and L be a line bundle. This data determines a morphism $[L]\colon X \to B\mathbb{G}_m$. Let $r\colon B\mathbb{G}_m \to B\mathbb{G}_m$ be the morphism induced from the rth power map $r\colon \mathbb{G}_m \to \mathbb{G}_m$, where $t \mapsto t^r$; alternatively $r\colon B\mathbb{G}_m \to B\mathbb{G}_m$ is defined functorially on objects by the assignment $L \mapsto L^{\otimes r}$. For a positive integer r, define the *rth root gerbe $X(\sqrt[r]{L})$ of X and L* (sometimes denoted as $\sqrt[r]{L/X}$) as the fiber product

$$
\begin{array}{ccc}
X(\sqrt[r]{L}) & \longrightarrow & B\mathbb{G}_m \\
\downarrow & \square & \downarrow r \\
X & \xrightarrow{\ [L]\ } & B\mathbb{G}_m.
\end{array}
$$

Example 3.9.13 (Root stacks). Let X be a scheme, L be a line bundle, and $s \in \Gamma(X, L)$ be a section. This data determines a morphism $[L, s]\colon X \to [\mathbb{A}^1/\mathbb{G}_m]$ (see Example 3.9.16). Let $r\colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ be the morphism induced from the rth power map $r\colon \mathbb{A}^1 \to \mathbb{A}^1$, given by $x \mapsto x^r$, which is equivariant under the rth power map $r\colon \mathbb{G}_m \to \mathbb{G}_m$; alternatively $r\colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ is defined functorially by $(L, s) \mapsto (L^{\otimes r}, s^r)$. For a positive integer r, define the *rth root stack $X(\sqrt[r]{L, s})$ of X and L along s* (sometimes denoted as $\sqrt[r]{(L, s)/X}$) as the fiber product

$$
\begin{array}{ccc}
X(\sqrt[r]{L, s}) & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\
\downarrow & \square & \downarrow r \\
X & \xrightarrow{\ [L, s]\ } & [\mathbb{A}^1/\mathbb{G}_m].
\end{array}
$$

Observe that if $s = 0$ is the zero section, then there is an identification of $X(\sqrt[r]{L, 0})$ with the root gerbe $X(\sqrt[r]{L})$ of the previous example.

Exercise 3.9.14. Let S be a scheme and r be an integer invertible in $\Gamma(S, \mathcal{O}_S)$. (This hypothesis ensures that $\boldsymbol{\mu}_{r,S} \to S$ is an étale group scheme; it will be removed in Exercise 6.2.34.)

(a) Show that $X(\sqrt[r]{L})$ and $X(\sqrt[r]{L, s})$ are Deligne–Mumford stacks.

(b) Show that $X(\sqrt[r]{L})$ has the equivalent description as the category of tuples $(T \xrightarrow{f} X, M, \alpha)$ where $f \colon T \to X$ is a morphism from a scheme, M is a line bundle on T and $\alpha \colon M^{\otimes r} \xrightarrow{\sim} f^*L$ is an isomorphism. In particular, there is a line bundle $L^{1/r}$ on $X(\sqrt[r]{L})$ and an isomorphism $(L^{1/r})^{\otimes r} \xrightarrow{\sim} \pi^*L$.

(c) Show that $X(\sqrt[r]{L, s})$ has the equivalent description as the category of triples $(T \xrightarrow{f} X, M, \alpha, t)$ where $f \colon T \to X$ is a morphism from a scheme, M is a line bundle on T, $\alpha \colon M^{\otimes r} \to f^*L$ is an isomorphism, and $t \in \Gamma(T, M)$ is a section such that $\alpha(t^{\otimes r}) = f^*s$. In particular, there is a line bundle $L^{1/r}$ on $X(\sqrt[r]{L, s})$ with a section $s^{1/r}$ together with an isomorphism $(L^{1/r})^{\otimes r} \xrightarrow{\sim} \pi^*L$ identifying $(s^{1/r})^{\otimes r}$ with π^*s.

(d) If $X = \operatorname{Spec} A$ is an affine scheme over S and $L = \mathcal{O}_X$ is trivial, show that

$$X(\sqrt[r]{L}) \cong [X/\boldsymbol{\mu}_r] \quad \text{and} \quad X(\sqrt[r]{L, s}) \cong [\operatorname{Spec}(A[x]/(x^r - s))/\boldsymbol{\mu}_r]$$

where $\boldsymbol{\mu}_r$ acts trivially on X and acts on $\operatorname{Spec}(A[x]/(x^r - s))$ via $t \cdot x = tx$.

(e) Show that the fiber of $X(\sqrt[r]{L}) \to X$ at a point $x \in X$ is isomorphic to $B\boldsymbol{\mu}_{r,\kappa(x)}$.

(f) Show that $X(\sqrt[r]{L, s}) \to X$ is an isomorphism over $X_s = \{s \neq 0\}$ and that it restricts to the root gerbe $V(s)(\sqrt[r]{(L|_{V(s)})})$ over $V(s)$. In particular, the fiber over a point $x \in X$ is either the point x itself if $x \notin V(s)$ and $B\boldsymbol{\mu}_{r,\kappa(x)}$ if $x \in V(s)$.

(You will show later in Exercise 6.2.34 that $X(\sqrt[r]{L}) \to X$ and the restriction of $X(\sqrt[r]{L, s}) \to X$ along $V(s)$ are banded $\boldsymbol{\mu}_r$-gerbes.)

3.9.3 Examples of algebraic stacks

Example 3.9.15. The classifying stack $B\operatorname{GL}_n$ over $\operatorname{Spec}\mathbb{Z}$ classifies vector bundles of rank n. When $n = 1$, $B\mathbb{G}_m = B\operatorname{GL}_1$ classifies line bundles. The stack $B\operatorname{GL}_n$ is a universally closed and smooth algebraic stack over $\operatorname{Spec}\mathbb{Z}$ of relative dimension $-n^2$ with affine diagonal. However, $B\operatorname{GL}_n$ is not separated nor Deligne–Mumford.

Example 3.9.16. If $\mathbb{G}_m$ acts on $\mathbb{A}^1$ over $\mathbb{Z}$ via scaling, the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$ whose objects over a scheme T are pairs (L, s) where L is a line bundle on T and $s \in \Gamma(T, L)$. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ is an algebraic stack universally closed and smooth over $\operatorname{Spec}\mathbb{Z}$ of relative dimension 0 with affine diagonal. The stack $[\mathbb{A}^1/\mathbb{G}_m]$ is not separated nor Deligne–Mumford

Over a field $\mathbb{k}$, $[\mathbb{A}^1/\mathbb{G}_m]$ has two points—one open and one closed—corresponding to the two $\mathbb{G}_m$-orbits (see Figure 0.13). There is an open immersion and closed immersion

$$\operatorname{Spec}\mathbb{k} \hookrightarrow [\mathbb{A}^1/\mathbb{G}_m] \hookleftarrow B\mathbb{G}_m.$$

The morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \operatorname{Spec}\mathbb{k}$ identifies the two orbits and is an example of a good moduli space.

Example 3.9.17. Working over a field $\mathbb{k}$, let $\mathbb{G}_m$ act on $\mathbb{A}^2$ via $t \cdot (x, y) = (tx, t^{-1}y)$. The quotient stack $\mathcal{X} = [\mathbb{A}^2/\mathbb{G}_m]$ is a smooth algebraic stack. An object of $\mathcal{X}$ over a scheme T is a triple (L, s, t) where L is a line bundle on T, $s \in \Gamma(T, L)$ and $t \in \Gamma(T, L^{-1})$. The complement $\mathcal{X} \setminus 0$ of the origin is isomorphic to the non-separated affine line. There is a morphism $\mathcal{X} \to \mathbb{A}^1$ defined by $(x, y) \mapsto xy$, which is an isomorphism over $\mathbb{A}^1 \setminus 0$ and identifies the three orbits defined by $xy = 0$.

Example 3.9.18 (Toric stacks)**.** A fan Σ on a lattice $L = \mathbb{Z}^n$ defines a toric variety $X(\Sigma)$, i.e. a normal separated variety with an action of $\mathbb{G}_m^n$ such that there is a dense orbit with trivial stabilizer; see [Ful93].

Meanwhile, a *stacky fan* is a pair (Σ, β) where Σ is a fan on a lattice L and $\beta \colon L \to N$ is a homomorphism of lattices. As L and N are lattices (i.e. finitely generated free abelian groups), the $\mathbb{Z}$-linear duals define tori $T_L := D(L^\vee)$ and $T_N := D(N^\vee)$ (Example B.2.7) where T_L is a torus for the toric variety $X(\Sigma)$. The map β induces a homomorphism $T_\beta \colon T_L \to T_N$, naturally identifying β with the induced map on lattices of 1-parameter subgroups. We can then define $G_\beta = \ker(T_\beta)$ and the *toric stack*

$$X(\Sigma, \beta) := [X(\Sigma)/G_\beta].$$

Example 3.9.19 (Picard schemes and stacks)**.** If X is a scheme over a field $\mathbb{k}$, the *Picard functor of X* and *Picard stack of X* are defined as the sheaf $\underline{\mathrm{Pic}}(X)$ and stack $\underline{\mathcal{P}\mathrm{ic}}(X)$ on $\mathrm{Sch}_{\text{ét}}$ by

$$\underline{\mathrm{Pic}}(X) = \text{sheafification of } T \mapsto \mathrm{Pic}(X_T)$$
$$\underline{\mathcal{P}\mathrm{ic}}(X)(T) = \{\text{groupoid of line bundles } L \text{ on } X_T\}$$

A morphism $(T, L) \to (T', L')$ in $\underline{\mathcal{P}\mathrm{ic}}(X)$ is the data of a morphism $f \colon T \to T'$ of schemes and an isomorphism $\alpha \colon L \to f^* L'$ (or more precisely a morphism $f_* L \to L'$ whose adjoint is an isomorphism).

If X is proper over a field $\mathbb{k}$, then $\underline{\mathrm{Pic}}(X)$ is a proper scheme and the tensor product of line bundles provides it with the structure of a group scheme, hence an abelian variety. Moreover, $\underline{\mathcal{P}\mathrm{ic}}(X)$ is a smooth algebraic stack over $\mathbb{k}$ and there is a morphism $\underline{\mathcal{P}\mathrm{ic}}(X) \to \underline{\mathrm{Pic}}(X)$ such that the fiber over a line bundle L is isomorphic to $B\mathbb{G}_m$. The tensor product of line bundles provides $\underline{\mathcal{P}\mathrm{ic}}(X)$ with the structure of a group stack, a notion which we will not spell out precisely.

Gerbes provide important examples of algebraic stacks, but we postpone our treatment until §6.2.5.

3.9.4   Pathological examples

Example 3.9.20. If $\underline{\mathbb{Z}}$ denotes the constant group scheme over $\mathrm{Spec}\,\mathbb{Z}$ associated to the abstract discrete group $\mathbb{Z}$, the classifying stack $B\underline{\mathbb{Z}}$ is a smooth algebraic stack of dimension 0 whose diagonal is not quasi-compact (i.e. $B\underline{\mathbb{Z}}$ is not quasi-separated).

Example 3.9.21. The non-separated affine line $G := \mathbb{A}^1 \bigcup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ is a group scheme over $\mathbb{A}^1$, where every fiber is trivial except over the origin. The classifying stack BG is a Deligne–Mumford stack whose diagonal is non-separated.

Example 3.9.22. We give an example of a non-quasi-separated étale group algebraic space G over a field $\mathbb{k}$ of characteristic 0 that is not a scheme. The classifying stack BG is Deligne–Mumford whose diagonal is not representable by schemes. Both G and BG provide counterexamples to many results that hold for schemes or quasi-separated algebraic spaces and Deligne–Mumford stacks, but fail in general. Let $\underline{\mathbb{Z}}$ act on $\mathbb{A}^1$ over $\mathbb{k}$ via $n \cdot x = x + n$ for $x \in \mathbb{A}^1$ and $n \in \underline{\mathbb{Z}}$, and define the algebraic space $G := \mathbb{A}^1/\underline{\mathbb{Z}}$ is an algebraic space. As the action map $\underline{\mathbb{Z}} \times \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1$ is not quasi-compact, the diagonal of G is not quasi-compact. If G were a scheme, then there would exist a non-empty open affine subscheme $U = \mathrm{Spec}\,A \subset G$. Since $p \colon \mathbb{A}^1 \to G$ is an étale presentation, we can compute A as the subring of $\underline{\mathbb{Z}}$-invariants $\Gamma(p^{-1}(U), \mathcal{O}_{\mathbb{A}^1})^{\underline{\mathbb{Z}}}$, which the reader can check consists of only the constant functions,

i.e. $A = \Bbbk$. As G is obtained by gluing such affine schemes, it follows that $G = \operatorname{Spec} \Bbbk$, a contradiction.

Similarly, one can consider the algebraic space quotient $\mathbb{A}^1_{\mathbb{C}}/\underline{\mathbb{Z}}^2$ where $(a, b) \cdot x = x + a + ib$. While the analytic quotient $\mathbb{C}/\underline{\mathbb{Z}}^2$ of this action is an elliptic curve over $\mathbb{C}$, the algebraic space quotient is a non-quasi-separated algebraic space that is not a scheme.

Exercise 3.9.23. Let $X = \mathbb{A}^1/\underline{\mathbb{Z}}$ be the algebraic space defined above.

(a) Show that X is locally noetherian and quasi-compact but not noetherian.

(b) Show that the generic point $\operatorname{Spec} \Bbbk(x) \to \mathbb{A}^1 \to X$ is fixed under the $\underline{\mathbb{Z}}$-action.

(c) Show that $\operatorname{Spec} \Bbbk(x) \to X$ does not factor through a monomorphism $\operatorname{Spec} L \to X$ for a field L. (In other words, the generic point of X does not have a residue field.)

Example 3.9.24 (Deligne–Mumford stacks with non-separated diagonal)**.** Let $G \to S$ be a finite étale group scheme. If $H \subset G$ is a subgroup scheme over S, then G/H is separated if and only if $H \subset G$ is closed. For instance, taking $G = \mathbb{Z}/2 \times \mathbb{A}^1 \to \mathbb{A}^1$ and the subgroup $H = G \setminus \{-1, 0\}$, the quotient $Q = G/H$ is the non-separated affine line and is a group scheme over $\mathbb{A}^1$ which is trivial away from the origin and where the fiber over 0 is $\mathbb{Z}/2$. In this case, BQ is a Deligne–Mumford stack with non-separated diagonal; however, $\mathcal{X}$ is quasi-compact and quasi-separated (i.e. BG, the first diagonal Δ_{BG} and second diagonal $\Delta_{\Delta_{BG}}$ are quasi-compact).

Chapter 4

Geometry of Deligne–Mumford stacks

4.1 Quasi-coherent sheaves and cohomology

4.1.1 Sheaves

The small étale site of a Deligne–Mumford stack can be defined analogously to the small étale site of a scheme (Example 2.1.4).

Definition 4.1.1. If $\mathcal{X}$ is a Deligne–Mumford stack, the *small étale site of $\mathcal{X}$* is the category $\mathcal{X}_{\text{ét}}$ of schemes étale over $\mathcal{X}$. A covering of an $\mathcal{X}$-scheme U is a collection of étale morphisms $\{U_i \to U\}$ over $\mathcal{X}$ such that $\coprod_i U_i \to U$ is surjective.

We can therefore discuss sheaves of abelian groups on $\mathcal{X}_{\text{ét}}$ and their morphisms. We denote $\text{Ab}(\mathcal{X}_{\text{ét}})$ as the category of abelian sheaves on $\mathcal{X}_{\text{ét}}$. For an abelian sheaf F on $\mathcal{X}_{\text{ét}}$, the sections over an étale $\mathcal{X}$-scheme U are denoted by $F(U)$ or $\Gamma(U, F)$; you should remember that this group depends not only on U but the structure morphism $U \to \mathcal{X}$.

Example 4.1.2 (Structure sheaf). The structure sheaf $\mathcal{O}_{\mathcal{X}}$ on a Deligne–Mumford stack is defined by $\mathcal{O}_{\mathcal{X}}(U) = \Gamma(U, \mathcal{O}_U)$ on an étale $\mathcal{X}$-scheme U.

Example 4.1.3 (Differentials). If $\mathcal{X}$ is a Deligne–Mumford stack over a scheme S, the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ is defined by $\Omega_{\mathcal{X}/S}(U) = \Gamma(U, \Omega_{U/S})$.

Example 4.1.4 (Hodge bundle). Define the sheaf $\mathcal{H}$ on $\mathcal{M}_g$ (for $g \geq 2$) as follows: for every étale morphism $U \to \mathcal{M}_g$ from a scheme corresponding to a family $\mathcal{C} \to U$ of smooth curves, we set $\mathcal{H}(U) = \Gamma(\mathcal{C}, \Omega_{\mathcal{C}/U})$. We will see later that $\mathcal{H}$ is a coherent $\mathcal{O}_{\mathcal{M}_g}$-module which is locally free of rank g, i.e. a vector bundle.

While a sheaf F on $\mathcal{X}_{\text{ét}}$ by definition only has sections defined on étale $\mathcal{X}$-schemes, one can extend the definition to a Deligne–Mumford stack $\mathcal{U}$ étale over $\mathcal{X}$. Choose étale presentations $U \to \mathcal{U}$ and $R \to U \times_{\mathcal{U}} U$ by schemes and define

$$F(\mathcal{U}) := \text{Eq}(F(U) \rightrightarrows F(R)).$$

One checks that this is independent of the choice of presentation. In particular, it makes sense to discuss global sections $\Gamma(\mathcal{X}, F) := F(\mathcal{X})$ over the identity $\text{id} \colon \mathcal{X} \to \mathcal{X}$.

Exercise 4.1.5. If F is an abelian sheaf on a Deligne–Mumford stack $\mathcal{X}$, show that $\Gamma(\mathcal{X}, F) = \mathrm{Hom}_{\mathrm{Ab}(\mathcal{X}_{\text{ét}})}(\underline{\mathbb{Z}}, F)$ where $\underline{\mathbb{Z}}$ is the constant sheaf. If F is an $\mathcal{O}_{\mathcal{X}}$-module, show that $\Gamma(\mathcal{X}, F) = \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}, F)$.

Given a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, there are functors

$$\mathrm{Ab}(\mathcal{X}_{\text{ét}}) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathrm{Ab}(\mathcal{Y}_{\text{ét}})$$

where $f_* F(V) := F(V \times_{\mathcal{Y}} \mathcal{X})$ and $f^{-1} G$ is the sheafification of the presheaf

$$U \mapsto \varinjlim_{V \to \mathcal{Y}, U \to V \times_{\mathcal{Y}} \mathcal{X}} G(V),$$

with the limit is taken over the category of pairs of étale morphisms $V \to \mathcal{Y}$ and $U \to V \times_{\mathcal{Y}} \mathcal{X}$ (i.e. étale morphisms $V \to \mathcal{Y}$ and a choice of factorization of $U \to \mathcal{X} \to \mathcal{Y}$ through $V \to \mathcal{Y}$). Note that when $f \colon \mathcal{X} \to \mathcal{Y}$ is étale, then $f^{-1} G(U) = G(U)$ for an étale $\mathcal{X}$-scheme.

Exercise 4.1.6. Show that f^{-1} is left adjoint to f_*.

Exercise 4.1.7. If $\mathcal{X}$ is a Deligne–Mumford stack, define instead the site $\mathcal{X}_{\text{ét}'}$ as the category of *algebraic spaces* over $\mathcal{X}$ where coverings are étale coverings. Show that the categories of sheaves on $\mathcal{X}_{\text{ét}}$ and $\mathcal{X}_{\text{ét}'}$ are equivalent.

4.1.2 $\mathcal{O}_{\mathcal{X}}$-modules

On a Deligne–Mumford stack $\mathcal{X}$, the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is a ring object in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ and we define:

Definition 4.1.8. If $\mathcal{X}$ is a Deligne–Mumford stack, a *sheaf of $\mathcal{O}_{\mathcal{X}}$-modules* (or simply an *$\mathcal{O}_{\mathcal{X}}$-module*) is a sheaf F on $\mathcal{X}_{\text{ét}}$ which is a module object for $\mathcal{O}_{\mathcal{X}}$ in the category of sheaves, i.e. for every étale $\mathcal{X}$-scheme U, $F(U)$ is an $\mathcal{O}_{\mathcal{X}}(U)$-module and the module structure is compatible with respect to restriction along étale morphisms $V \to U$ of $\mathcal{X}$-schemes.

We denote $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ for the category of $\mathcal{O}_{\mathcal{X}}$-modules. Given two $\mathcal{O}_{\mathcal{X}}$-modules F and G, we can define the *tensor product* $F \otimes G := F \otimes_{\mathcal{O}_{\mathcal{X}}} G$ as the sheafification of the $\mathcal{O}_{\mathcal{X}}$-module given by $(U \to \mathcal{X}) \mapsto F(U \to \mathcal{X}) \otimes_{\mathcal{O}_{\mathcal{X}}(U \to \mathcal{X})} G(U \to \mathcal{X})$. The *Hom sheaf* $\mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(F, G)$ has sections $\mathrm{Hom}_{\mathcal{O}_U}(F|_U, G|_U)$ over an étale morphism $f \colon U \to \mathcal{X}$ from scheme, where $F|_U = f^{-1} F$ denotes the restriction of F to $U_{\text{ét}}$.

Given a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, there are functors

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}})$$

where for an $\mathcal{O}_{\mathcal{X}}$-module F, $f_* F$ is the pushforward as sheaves and is naturally an $\mathcal{O}_{\mathcal{Y}}$-module. For an $\mathcal{O}_{\mathcal{Y}}$-module G, since there is a morphism $f^{-1} \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ of sheaves of rings in $\mathcal{X}_{\text{ét}}$ and $f^{-1} G$ is a $f^{-1} \mathcal{O}_{\mathcal{Y}}$-module, it makes sense to define the pullback $\mathcal{O}_{\mathcal{X}}$-module

$$f^* G := f^{-1} G \otimes_{f^{-1} \mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}}.$$

Exercise 4.1.9. Show that f^* is left adjoint to f_*.

Exercise 4.1.10. Show that $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ is an abelian category.

4.1.3 Quasi-coherent sheaves

Let F be an $\mathcal{O}_{\mathcal{X}}$-module on a Deligne–Mumford stack $\mathcal{X}$. For an étale $\mathcal{X}$-scheme U, we have the restriction $F|_U$ to the étale site of U and the further restriction $F|_{U_{\mathrm{Zar}}}$ restricted to the Zariski topology of U. Note that when X is a scheme, $\mathcal{O}_X$ could refer to the structure sheaf either in $X_{\text{ét}}$ or X_{Zar}. If there is a possibility for confusion, we write either $\mathcal{O}_{X_{\mathrm{Zar}}}$ or $\mathcal{O}_{X_{\text{ét}}}$.

Definition 4.1.11. Let $\mathcal{X}$ be a Deligne–Mumford stack. An $\mathcal{O}_{\mathcal{X}}$-module F is *quasi-coherent* if

(1) for every étale $\mathcal{X}$-scheme U, the restriction $F|_{U_{\mathrm{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\mathrm{Zar}}}$-module, and

(2) for every étale morphism $f\colon U \to V$ of étale $\mathcal{X}$-schemes, the natural morphism $f^*(F|_{V_{\mathrm{Zar}}}) \to F|_{U_{\mathrm{Zar}}}$ is an isomorphism.

A quasi-coherent F on $\mathcal{X}$ is a *vector bundle* (resp. *vector bundle of rank r*, *line bundle*) if $F|_{U_{\mathrm{Zar}}}$ is for every morphism $U \to \mathcal{X}$ from a scheme.

If in addition $\mathcal{X}$ is locally noetherian, we say F is *coherent* if $F|_{U_{\mathrm{Zar}}}$ is coherent for every morphism $U \to \mathcal{X}$ from a scheme.

We denote by $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{X})$ (in the noetherian setting) the categories of quasi-coherent and coherent sheaves. The property that a quasi-coherent sheaf is a vector bundle, line bundle, or coherent (in the noetherian setting) is étale local (Proposition B.1.11), and thus it suffices to check the condition on an étale presentation.

Example 4.1.12. The structure sheaf $\mathcal{O}_{\mathcal{X}}$ is always a line bundle, which is coherent when $\mathcal{X}$ is locally noetherian.

Example 4.1.13. For a Deligne–Mumford stack $\mathcal{X}$ over a scheme S, the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ of Example 4.1.3 is quasi-coherent since for an étale morphisms $f\colon U \to V$ of étale $\mathcal{X}$-schemes, $f^*\Omega_{V/\mathcal{S}} \to \Omega_{U/S}$ is an isomorphism; it is a vector bundle when $\mathcal{X} \to S$ is smooth.

Example 4.1.14. For $\mathcal{M}_g$ (with $g \geq 2$), the Hodge bundle $\mathcal{H}$ of Example 4.1.4 is a vector bundle of rank g. This follows from Proposition 5.1.15(2): for a smooth family $\pi\colon \mathcal{C} \to V$ of genus g curves corresponding to a $\mathcal{M}_g$-scheme V, the construction of $\pi_*\Omega_{\mathcal{C}/V}$ commutes with the base change along a map $f\colon U \to V$, i.e. $f^*(\pi_*\Omega_{\mathcal{C}/V}) \xrightarrow{\sim} \pi_{U,*}\Omega_{\mathcal{C}_U/U})$, which shows quasi-coherence of $\mathcal{H}$. Moreover, $\pi_*\Omega_{\mathcal{C}/V}$ is a vector bundle on V of rank g, which shows that $\mathcal{H}$ is also a vector bundle of rank g.

Example 4.1.15. If G is a finite abstract group viewed as a group scheme over a field $\Bbbk$, a quasi-coherent sheaf on BG corresponds to a representation V of G. If G acts on an affine $\Bbbk$-scheme $\mathrm{Spec}\, A$, a quasi-coherent sheaf on $[\mathrm{Spec}\, A/G]$ is the data of an A-module M equipped with a group homomorphism $G \to \mathrm{End}_A(M)$. These descriptions follow from Exercise 4.1.18(1).

Exercise 4.1.16 (Equivalent definition)**.** There is a general definition of a quasi-coherent module on a site $\mathcal{S}$ with a sheaf of rings $\mathcal{O}$ (see [SGA4$\frac{1}{2}$] and [SP, Tag 03DL]): an $\mathcal{O}$-module F is quasi-coherent if for every object $U \in \mathcal{S}$, there is a covering $\{U_i \to U\}$ such that the restriction $F|_{U_i}$ to the localized site $\mathcal{S}/U_i$ has a free presentation

$$\mathcal{O}_{U_i}^{\oplus J} \to \mathcal{O}_{U_i}^{\oplus I} \to F|_{U_i} \to 0.$$

Show the definition of quasi-coherence above for a Deligne–Mumford stack $\mathcal{X}$ agrees with this general definition on the ringed site $(\mathcal{X}_{\text{ét}}, \mathcal{O}_{\mathcal{X}})$.

The following exercise tells us that quasi-coherence is consistent with the usual one when $\mathcal{X}$ is a scheme.

Exercise 4.1.17. Let X be a scheme and F be an $\mathcal{O}_{X_{\mathrm{Zar}}}$-module.

(a) Define a presheaf $F_{\mathrm{\acute{e}t}}$ on $X_{\mathrm{\acute{e}t}}$ as follows: for an étale map $f \colon U \to \mathcal{X}$ from a scheme, set $F_{\mathrm{\acute{e}t}}(U) = \Gamma(U, f^*F)$. Show that $F_{\mathrm{\acute{e}t}}$ is a sheaf of $\mathcal{O}_{X_{\mathrm{\acute{e}t}}}$-modules and that the assignment $F \mapsto F_{\mathrm{\acute{e}t}}$ defines an exact functor $\mathrm{Mod}(\mathcal{O}_{X_{\mathrm{Zar}}}) \to \mathrm{Mod}(\mathcal{O}_{X_{\mathrm{\acute{e}t}}})$.

(b) Show that if F is a quasi-coherent $\mathcal{O}_{X_{\mathrm{Zar}}}$-module, then $F_{\mathrm{\acute{e}t}}$ is a quasi-coherent $\mathcal{O}_{X_{\mathrm{\acute{e}t}}}$-module, and that $F \mapsto F_{\mathrm{\acute{e}t}}$ is an equivalence of categories between quasi-coherent $\mathcal{O}_{X_{\mathrm{Zar}}}$-modules and quasi-coherent $\mathcal{O}_{X_{\mathrm{\acute{e}t}}}$-modules. See also [SP, Tag 03DX].

Exercise 4.1.18 (Groupoid and functorial perspectives)**.** Let $\mathcal{X}$ be a Deligne–Mumford stack.

(1) Let $U \to \mathcal{X}$ be an étale presentation from a scheme U. If G is a quasi-coherent sheaf on U and $\alpha \colon p_1^*G \xrightarrow{\sim} p_2^*G$ is an isomorphism on $R := U \times_{\mathcal{X}} U$ satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$, show that G descends to a unique quasi-coherent sheaf on $\mathcal{X}$.

(2) If F is a quasi-coherent sheaf on $\mathcal{X}$ and $f \colon S \to \mathcal{X}$ is a morphism from a scheme, then show that $(f^*F)|_{S_{\mathrm{Zar}}}$ is a quasi-coherent sheaf on S.

Given a groupoid presentation $R \rightrightarrows U$ of $\mathcal{X}$, (1) gives an equivalence between quasi-coherent sheaves on $\mathcal{X}$ and quasi-coherent sheaves on U with descent datum. Meanwhile, (2) above allows us to think of a quasi-coherent sheaf F on $\mathcal{X}$ as the data of a quasi-coherent sheaf F_S for every map $S \to \mathcal{X}$ and compatible isomorphisms $f^*F_T \to F_S$ for every map $f \colon S \to T$ over $\mathcal{X}$. For instance, the Hodge bundle on $\mathcal{M}_g$ is the data of the sheaf $\pi_*\Omega_{\mathcal{C}/S}$ for every smooth family of curves $\pi \colon \mathcal{C} \to S$

4.1.4 Pushforwards and pullbacks

Exercise 4.1.19 (Pushforward–Pullback Adjunction)**.** Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks.

(a) Show that if G is a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module, then f^*G is quasi-coherent.

Assume in addition that f is quasi-compact and quasi-separated.

(b) Show that if F is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module, then f_*F is quasi-coherent.

(c) Show that the functors

$$\mathrm{QCoh}(\mathcal{X}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{QCoh}(\mathcal{Y})$$

are adjoints (with f_* the right adjoint).

Exercise 4.1.20. Let G be a finite group and $\Bbbk$ be a field.

(a) Under the composition $\mathrm{Spec}\,\Bbbk \xrightarrow{p} \mathbb{B}_{\Bbbk}G \xrightarrow{\pi} \mathrm{Spec}\,\Bbbk$, show that for a G-representation V, $\pi_*V = V^G$ where V^G is the subspace of G-invariants and $p^*V = V$ forgetting the G-action, and that for a $\Bbbk$-vector space W, $\pi^*W = W$ with the trivial G-action and $p_*W = W \otimes p_*\Bbbk$ where $p_*\Bbbk$ is the regular representation $\Gamma(G, \mathcal{O}_G)$.

(b) Given an action of G an an affine $\Bbbk$-scheme $\operatorname{Spec} A$, consider the diagram

$$\operatorname{Spec} A \xrightarrow{\ p\ } [\operatorname{Spec} A/G] \xrightarrow{\ \pi\ } \operatorname{Spec} A^G$$
$$\downarrow{\scriptstyle q}$$
$$BG$$

and recall from Example 4.1.15 that a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is an A-module M with a group homomorphism $G \to \operatorname{End}_A(M)$. Provide explicit descriptions of the functors $p_*, p^*, \pi_*, \pi^*, q_*$ and q^* on quasi-coherent sheaves.

Exercise 4.1.21. Let $\mathcal{X}$ be a noetherian Deligne–Mumford stack. Prove the following two statements:

(a) Every quasi-coherent sheaf on $\mathcal{X}$ is a directed colimit of its coherent subsheaves.

(b) If $\mathcal{U} \subset \mathcal{X}$ is an open substack, then every coherent sheaf on $\mathcal{U}$ extends to a coherent sheaf on $\mathcal{X}$.

This exercise extends [Har77, Exer II.5.15] from schemes to Deligne–Mumford stacks; see also [LMB00, Prop. 15.4], [Ols16, Prop. 7.1.11] and [SP, Tag 01PD].

4.1.5 Quasi-coherent constructions

A *quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra* on a Deligne–Mumford stack is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module with the compatible structure of a ring object in $\operatorname{Ab}(\mathcal{X}_{\text{ét}})$. We define the *relative spectrum* $\mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A}$ as the stack whose objects over a scheme S consists of a morphism $f \colon S \to \mathcal{X}$ and a morphism $f^*\mathcal{A} \to \mathcal{O}_S$ of $\mathcal{O}_S$-algebras.

Exercise 4.1.22. Show that $\mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A}$ is an algebraic stack affine over $\mathcal{X}$.

Example 4.1.23 (Reduction). Let $\mathcal{X}$ be a Deligne-Mumford stack and let $\mathcal{O}_{\mathcal{X}}^{\text{red}}$ be the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras where $\mathcal{O}_{\mathcal{X}}^{\text{red}}(U) = \Gamma(U, \mathcal{O}_U)_{\text{red}}$ for an étale $\mathcal{X}$-scheme U. Then $\mathcal{O}_{\mathcal{X}}^{\text{red}}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra and $\mathcal{X}_{\text{red}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{O}_{\mathcal{X}}^{\text{red}}$ defines the *reduction of $\mathcal{X}$.*

Example 4.1.24 (Normalization). Let $\mathcal{X}$ be an integral Deligne-Mumford stack and let $\mathcal{A}$ be the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras whose sections over an étale morphism $U \to \mathcal{X}$ from a scheme is the normalization of $\Gamma(U, \mathcal{O}_U)$. Since normalization commutes with étale extensions (Proposition A.7.4), $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra. The *normalization of $\mathcal{X}$* is defined as $\widetilde{\mathcal{X}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A}$.

Exercise 4.1.25. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of Deligne–Mumford stacks.

(a) Show that there is factorization $f \colon \mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$.

(b) Show that f is affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

(c) Show that f is quasi-affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\, f_*\mathcal{O}_{\mathcal{X}}$ is an open immersion.

Exercise 4.1.26. Use Exercise 4.1.21 to show that every quasi-coherent sheaf of algebras on a noetherian Deligne–Mumford stack is a directed colimit of finite type subalgebras.

4.1.6 Cohomology

We develop a cohomology theory for abelian sheaves on Deligne–Mumford stacks. Despite utilizing the cohomology of quasi-coherent sheaves on schemes throughout these notes, we surprisingly have little need for cohomology on algebraic spaces and Deligne–Mumford stacks, and many of the results here are included only for completeness.

The existence of enough injective objects is shown analogously to the case of schemes [Har77, Prop. 2.2].

Lemma 4.1.27. *If $\mathcal{X}$ is a Deligne–Mumford stack, the categories $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$ and $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ have enough injectives. If in addition $\mathcal{X}$ is quasi-separated, then $\mathrm{QCoh}(\mathcal{X})$ has enough injectives.*

Proof. Recall that a functor $R\colon \mathcal{A} \to \mathcal{B}$ between abelian categories with an exact left adjoint L preserves injectives: for an injective I in $\mathcal{A}$, we have that $\mathrm{Hom}_{\mathcal{B}}(-, R(I)) = \mathrm{Hom}_{\mathcal{A}}(L(-), I)$ is exact.

By taking Λ to be the constant sheaf $\underline{\mathbb{Z}}$ or the structure sheaf $\mathcal{O}_{\mathcal{X}}$, the first statement will follow if we show that the category $\mathrm{Mod}(\Lambda)$ of Λ-modules has enough injectives for every sheaf of rings Λ on $\mathcal{X}_{\text{ét}}$. Let F be a Λ-module and let $U \to \mathcal{X}$ be an étale presentation. For each $u \in U$, we have a map $j_u\colon \{u\} \hookrightarrow U \to \mathcal{X}$ from a point and the stalk $F_u = j_u^{-1}F$ is an Λ_u-module. Choose an inclusion $F_u \hookrightarrow I_u$ into an injective Λ_u-module. Adjunction gives a map $F \to j_{u,*}I_u$, where $j_{u,*}$ is injective since j_u^{-1} is exact. By taking the product, we obtain an injection $F \to \prod_{u \in U} j_{u,*}I_u$ into an injective Λ-module.

For the final statement, let $F \in \mathrm{QCoh}(\mathcal{X})$ and let $p\colon U = \coprod_i \mathrm{Spec}\, A_i \to \mathcal{X}$ be an étale presentation. Choose an injection $p^*F \hookrightarrow I$ into an injective quasi-coherent $\mathcal{O}_U$-module. The composition $F \hookrightarrow p_*p^*F \hookrightarrow p_*I$ is injective and since p^* is exact, p_*I is injective. $\qquad\square$

Remark 4.1.28. The above argument for the existence of enough injectives in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ extends to the category of $\mathcal{O}$-modules in any ringed site with enough points (see [Ols16, Thm. 2.3.2]) and is even true in any ringed site [SP, Tag 01DP]. The category of quasi-coherent sheaves on an arbitrary Deligne–Mumford stack (or even algebraic stack) is a Grothendieck abelian category [SP, Tag 0781] and any such category has enough injectives [Gro57], [SP, Tag 079H].

Definition 4.1.29 (Cohomology). Let $\mathcal{X}$ be a Deligne–Mumford stack and F a sheaf of abelian groups on $\mathcal{X}_{\text{ét}}$. The *cohomology group* $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$ is defined as the ith right derived functor of the global sections functor $\Gamma\colon \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \to \mathrm{Ab}$.

Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, the *higher direct image* R^if_*F is defined as the ith right derived functor of $f_*\colon \mathrm{Ab}(\mathcal{X}_{\text{ét}}) \to \mathrm{Ab}(\mathcal{Y}_{\text{ét}})$.

The following is a key input to the development of quasi-coherent cohomology.

Theorem 4.1.30. *For a quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module F on an affine scheme X, $\mathrm{H}^i(X_{\text{ét}}, F) = 0$ for all $i > 0$.*

We will prove this using Čech cohomology. Čech cohomology in the étale topology is defined similarly to the case of the Zariski topology [Har77, III.4] replacing intersections $U_{i_0} \cap \cdots \cap U_{i_n}$ with fiber products $U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ and considering all (possibly non-distinct) indices $i_0, \ldots, i_n$ in any order.

Definition 4.1.31 (Čech cohomology)**.** Given an étale covering $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i \in I}$ of a Deligne–Mumford stack and an abelian sheaf F on $\mathcal{X}_{\text{ét}}$, the *Čech complex of F with respect to $\mathcal{U}$* is $\check{\mathcal{C}}^\bullet(\mathcal{U}, F)$ where

$$\check{\mathcal{C}}^n(\mathcal{U}, F) = \prod_{(i_0, \dots, i_n) \in I^{n+1}} F(U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n})$$

with differential

$$d^n : \check{\mathcal{C}}^n(\mathcal{U}, F) \to \check{\mathcal{C}}^{n+1}(\mathcal{U}, F), \qquad (s_{i_0, \dots, i_n}) \mapsto \left(\sum_{k=0}^{n+1} (-1)^k p_{\widehat{k}}^* s_{i_0, \dots, \widehat{i_k}, \dots, i_n} \right)_{(i_0, \dots, i_{n+1})}$$

where $p_{\widehat{k}} : U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n} \to U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \widehat{U_{i_k}} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ is the map forgetting the kth component (with indexing starting at 0). The *Čech cohomology of F with respect to $\mathcal{U}$* is

$$\check{\mathrm{H}}^i(\mathcal{U}, F) := \mathrm{H}^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, F)).$$

The following is a standard result in Čech cohomology whose proof for sites is analogous to topological spaces. It is often referred to as Cartan's criterion; see [God58, II.5.9.2], [Mil80, Prop. 2.12], [SP, Tag 03F9] or [Ols16, Prop. 2.3.15].

Lemma 4.1.32. *Let $\mathcal{X}$ be a Deligne–Mumford stack and let F be an abelian sheaf on $\mathcal{X}_{\text{ét}}$. Suppose $\mathrm{Cov}'(\mathcal{X}) \subset \mathrm{Cov}(\mathcal{X})$ is a subset of coverings of $\mathcal{X}$ such that every covering of $\mathcal{X}$ has a refinement in $\mathrm{Cov}'(\mathcal{X})$. If for every covering $\mathcal{U} \in \mathrm{Cov}'$, $\check{\mathrm{H}}^i(\mathcal{U}, F) = 0$ for $i > 0$, then $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F) = 0$.*

With these preliminaries, we can prove Theorem 4.1.30.

Proof of Theorem 4.1.30. Let $X = \mathrm{Spec}\, A$, $F = \widetilde{M}$ be a quasi-coherent $\mathcal{O}_X$-module and $F_{\text{ét}}$ be the corresponding quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module (Exercise 4.1.17). The set of étale coverings of the form $\mathcal{U} = \{\mathrm{Spec}\, B \to \mathrm{Spec}\, A\}$ is sufficient to refine any other covering. For the covering $\mathcal{U}$, faithful flat descent (Exercise B.1.3) implies that there is a long exact sequence

$$0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A B \otimes_A B \to \cdots,$$

which is identified with the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, F)$. This shows that $\check{\mathrm{H}}^i(\mathcal{U}, F) = 0$ for $i > 0$ and thus Lemma 4.1.32 implies that $\mathrm{H}^i(X_{\text{ét}}, F_{\text{ét}}) = 0$. $\qquad\square$

As with ordinary topological spaces [Har77, Exer. III.4.11], Čech cohomology can be computed using a covering with vanishing cohomology; see for instance [SP, Tag 03F7].

Lemma 4.1.33. *Let F be an abelian sheaf on $\mathcal{X}_{\text{ét}}$ and $(U_i \to \mathcal{X})_{i \in I}$ an étale covering. If $\mathrm{H}^i(U_{j_0} \times_U \cdots \times_U U_{j_n}, F) = 0$ for all $i > 0$, $n \geq 0$ and $j_0, \dots, j_n \in I$, then $\check{\mathrm{H}}^i(\mathcal{U}, F) = \mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$.*

On a scheme with affine diagonal, both the étale and Zariski cohomology of a quasi-coherent sheaf can be computed on every affine open covering. We thus obtain:

Proposition 4.1.34. *Let X be a scheme with affine diagonal. Let F be a quasi-coherent $\mathcal{O}_X$-module and let $F_{\text{ét}}$ denote the corresponding quasi-coherent $\mathcal{O}_{X_{\text{ét}}}$-module (see Exercise 4.1.17). Then $\mathrm{H}^i(X, F) = \mathrm{H}^i(X_{\text{ét}}, F_{\text{ét}})$ for all i.*

Remark 4.1.35. The same result holds in the lisse-étale or fppf topology and without the affine diagonal hypothesis; see [SP, Tag 03DW] and [Mil80, Prop. 3.7].

Of course, in addition to being convenient to develop the theory of cohomology, Čech cohomology is also an extremely effective tool to compute cohomology groups. We have the following consequence of Theorem 4.1.30 and Lemma 4.1.33.

Proposition 4.1.36. *Let $\mathcal{X}$ be a Deligne–Mumford stack with affine diagonal and F be a quasi-coherent sheaf. If $\mathcal{U} = \{U_i \to \mathcal{X}\}$ is an étale covering with each U_i affine, then $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F) = \check{\mathrm{H}}^i(\mathcal{U}, F)$.*

To compare cohomologies computed in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$, $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ and $\mathrm{QCoh}(\mathcal{X})$, we have.

Proposition 4.1.37. *Let $\mathcal{X}$ be a Deligne–Mumford stack.*

(1) If F is an $\mathcal{O}_{\mathcal{X}}$-module, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$ of F as an abelian sheaf agrees with the ith right derived functor of $\Gamma\colon \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Ab}$.

(2) If $\mathcal{X}$ has affine diagonal and F is a quasi-coherent sheaf on $\mathcal{X}$, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F)$ of F as an abelian sheaf agrees with the ith right derived functor of $\Gamma\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.

For a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks (resp. quasi-compact morphism of Deligne–Mumford stacks with affine diagonals), then (1) (resp. (2)) holds also for the higher direct images $R^i f_* F$ of an $\mathcal{O}_{\mathcal{X}}$-module (resp. quasi-coherent sheaf): it can be computed as the ith right derived functor of $f_*\colon \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{Y}})$ (resp. $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$).

Proof. For (1), we need to show that an injective object in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ is acyclic in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$. This uses a standard technique in Čech cohomology. We need some notation: given an étale covering $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i \in I}$, we set $U_i := U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ with structure morphism $j_i\colon U_i \to \mathcal{X}$. There is a chain complex $\underline{\mathbb{Z}}_{\mathcal{U},\bullet}$ of presheaves on $\mathcal{X}$ defined by

$$\underline{\mathbb{Z}}_{\mathcal{U},n} := \bigoplus_{i \in I^{n+1}} j_{i,!}\underline{\mathbb{Z}}$$

where $\underline{\mathbb{Z}}$ denotes the constant presheaf and $j_{i,!}\underline{\mathbb{Z}}$ is the presheaf whose sections over an $\mathcal{X}$-scheme V are $\bigoplus_{\mathrm{Mor}_{\mathcal{X}}(V,U_i)} \mathbb{Z}$. The differentials of $\underline{\mathbb{Z}}_{\mathcal{U},\bullet}$ are the alternating sums of the natural maps. This complex of presheaves is exact in positive degrees and has the property that for every presheaf F

$$\check{\mathcal{C}}(\mathcal{U}, F) = \mathrm{Mor}_{\mathrm{PAb}(\mathcal{X}_{\text{ét}})}(\underline{\mathbb{Z}}_{\mathcal{U},\bullet}, F) = \mathrm{Mor}_{\mathrm{PMod}(\mathcal{O}_{\mathcal{X}})}(\underline{\mathbb{Z}}_{\mathcal{U},\bullet} \otimes_{\underline{\mathbb{Z}}} \mathcal{O}_{\mathcal{X}}, F),$$

where morphisms are computed in the categories $\mathrm{PAb}(\mathcal{X}_{\text{ét}})$ and $\mathrm{PMod}(\mathcal{O}_{\mathcal{X}})$ of presheaves. If $F \in \mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ is injective, then it is also injective as a presheaf of $\mathcal{O}_{\mathcal{X}}$-modules. It follows that $\check{\mathcal{C}}(\mathcal{U}, F)$ is exact in positive degrees and thus $\check{\mathrm{H}}^i(\mathcal{U}, F) = 0$ for $i > 0$. Therefore Lemma 4.1.32 implies that $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, F) = 0$. For more details, see [SP, Tag 03FD] or [Ols16, Cor. 2.3.16].

For (2), let $F \in \mathrm{QCoh}(\mathcal{X})$ be an injective object. Let $p\colon U = \coprod_i \mathrm{Spec}\, A_i \to \mathcal{X}$ be an étale presentation and choose an injection $p^*F \hookrightarrow G$ into an injective object $G \in \mathrm{QCoh}(U)$. Then pushforward p_*G is injective (as the right adjoint p^* is exact) and we have an inclusion $F \hookrightarrow p_*p^*F \hookrightarrow p_*G$ of injectives which splits. It thus suffices to show that p_*G is acyclic in $\mathrm{Ab}(\mathcal{X}_{\text{ét}})$. Since $\mathcal{X}$ has affine diagonal, $p\colon U \to \mathcal{X}$ is an affine morphism. By descent and Flat Base Change (Proposition A.2.12), p_* is exact on the category of quasi-coherent sheaves. It follows that $\mathrm{H}^i(\mathcal{X}_{\text{ét}}, p_*G) = \mathrm{H}^i(U_{\text{ét}}, G) = 0$ by Theorem 4.1.30. $\qquad\square$

It follows from (2) that for a scheme X with affine diagonal and for a quasi-coherent sheaf F, we have that $\mathrm{H}^i(X, F) = \mathrm{H}^i(X_{\text{ét}}, F_{\text{ét}})$.

Exercise 4.1.38 (Flat Base Change). Consider a cartesian diagram

$$
\begin{CD}
\mathcal{X}' @>{g'}>> \mathcal{X} \\
@V{f'}VV \square @VV{f}V \\
\mathcal{Y}' @>{g}>> \mathcal{Y}
\end{CD}
$$

of Deligne–Mumford stacks, and let F be a quasi-coherent sheaf on $\mathcal{X}$. If $g \colon \mathcal{Y}' \to \mathcal{Y}$ is flat and $f \colon \mathcal{X} \to \mathcal{Y}$ is quasi-compact and quasi-separated, the natural adjunction map

$$
g^* \mathrm{R}^i f_* F \to \mathrm{R}^i f'_* g'^* F
$$

is an isomorphism for all $i \geq 0$.

Exercise 4.1.39. If $\mathcal{X}$ is a Deligne–Mumford stack and F_i is a directed system of abelian sheaves in $\mathcal{X}_{\text{ét}}$, show that $\operatorname{colim}_i \mathrm{H}^i(\mathcal{X}_{\text{ét}}, F_i) \to \mathrm{H}^i(\mathcal{X}_{\text{ét}}, \operatorname{colim}_i F_i)$ is an isomorphism.

Example 4.1.40 (Group cohomology). Let G be a finite abstract group viewed as a group scheme over a field $\Bbbk$, and let V be a G-representation. The *group cohomology* $\mathrm{H}^i(G, V)$ is defined as the ith right derived functor of $\mathrm{Rep}(G) \to \mathrm{Vect}_{\Bbbk}, V \mapsto V^G$. Since $\mathrm{H}^i(BG_{\text{ét}}, \widetilde{V})$ can be computed in $\mathrm{QCoh}(BG)$ (Proposition 4.1.37(2)) where $\widetilde{V}$ is the corresponding quasi-coherent sheaf on BG and there is an equivalence $\mathrm{Rep}_{\Bbbk}(G) \cong \mathrm{QCoh}(BG)$, we have the identification

$$
\mathrm{H}^i(G, V) \cong \mathrm{H}^i(BG_{\text{ét}}, \widetilde{V}).
$$

The Čech complex of $\widetilde{V}$ on BG corresponding to V with respect to the étale cover $\mathcal{U} = \{\operatorname{Spec} \Bbbk \to BG\}$ has terms

$$
\check{C}^n(\mathcal{U}, V) := \widetilde{V}((\operatorname{Spec} \Bbbk / BG)^{n+1}) \cong \Gamma(G, \mathcal{O}_G)^{\otimes n} \otimes V.
$$

To describe the differentials, let $\mu_k \colon G^{n+1} \to G^n$ for $k = 0, \ldots, n$ be defined by sending $(g_1, \ldots, g_{n+1})$ to $(g_1, \ldots, g_n)$ for $k = 0$ and to $(g_1, \ldots, g_{k-1}, g_k g_{k+1}, g_{k+2}, \ldots, g_{n+1})$ for $k = 1, \ldots, n$. Let $\sigma \colon V \to \Gamma(G, \mathcal{O}_G) \otimes V$ be the coaction map. The projection $p_{\hat{k}} \colon (\operatorname{Spec} \Bbbk / BG)^{n+2} \to (\operatorname{Spec} \Bbbk / BG)^{n+1}$ is identified with $\mu_{n+1-k} \otimes \mathrm{id}$ for $k = 0, \ldots n$ and $\mathrm{id} \otimes \sigma$ for $k = n+1$ (see Example 3.4.5). Thus the differentials in $\check{C}^\bullet(\mathcal{U}, V)$ are described by

$$
d^n \colon \Gamma(G, \mathcal{O}_G)^{\otimes n} \otimes V \to \Gamma(G, \mathcal{O}_G)^{\otimes (n+1)} \otimes V
$$

$$
f \otimes v \mapsto \sum_{k=0}^{n} (-1)^k \mu_{n+1-k}^*(f) \otimes v + (-1)^{n+1} f \otimes \sigma(v)
$$

In low degrees, we have $d^0(v) = v - \sigma(v)$ and $d^1(f_1, v) = f_1 \otimes 1 \otimes v - \mu^*(f_1) \otimes v + f_1 \otimes \sigma(v)$ where $\mu = \mu_1$ is group multiplication $G \times G \to G$.

Since G is finite, there is an identification $\Gamma(G^n, \mathcal{O}_{G^n}) \otimes V \cong \mathrm{Map}(G^n, V)$ with set-theoretic maps, where a map $\phi \colon G^n \to V$ is identified with $\sum_{g \in G^n} e_g \phi(g)$ where e_g denotes the function which is 1 on g but otherwise 0. Thus the Čech complex $\check{C}^\bullet(\mathcal{U}, V)$ can be equivalently described as

$$
0 \to V \xrightarrow{d^0} \mathrm{Map}(G, V) \xrightarrow{d^1} \mathrm{Map}(G^2, V) \xrightarrow{d^2} \cdots \tag{4.1.41}
$$

where the differential d^n is defined by the formula

$$(d^n\phi)(g_1,\ldots,g_{n+1}) = \phi(g_1,\ldots,g_n)+$$
$$\sum_{k=1}^{n}(-1)^{n+1-k}\phi(g_1,\ldots,g_{k-1},g_kg_{k+1},\ldots,g_{n+1}) + (-1)^{n+1}g_1\phi(g_2,\ldots,g_n)$$

for $\phi \in \mathrm{Map}(G^n, V)$. The complex (4.1.41) is sometimes referred to as the bar resolution (except that the differential d^n is usually multiplied by $(-1)^{n+1}$), and is an effective means to compute group cohomology. In low degrees, $d^0(v)(g) = v - gv$ and $d^1(\phi)(g_1, g_2) = \phi(g_1) - \phi(g_1g_2) + g_1\phi(g_2)$.

The following example illustrates that coherent sheaf cohomology on a Deligne–Mumford stack can be nonzero in arbitrary high degrees. This doesn't happen in characteristic 0 or for *tame* Deligne–Mumford stacks (see Exercise 4.3.25).

Exercise 4.1.42. For a prime p, how that $\mathrm{H}^i(B(\mathbb{Z}/p\mathbb{Z}), \mathcal{O}_{B(\mathbb{Z}/p\mathbb{Z})}) = \mathbb{F}_p$ for each i.

Remark 4.1.43 (Comparison of topologies)**.** One can also define the fppf cohomology groups $\mathrm{H}^i(\mathcal{X}_{\mathrm{fppf}}, F)$ of an abelian sheaf on the small fppf site of $\mathcal{X}$. There are some cases when this agrees with the small étale cohomology. For instance, if $G \to S$ is a *smooth, commutative*, and quasi-projective group scheme, then $\mathrm{H}^i(S_{\text{ét}}, G) = \mathrm{H}^i(S_{\mathrm{fppf}}, G)$ [Mil80, Thm. 3.9]. For $\mathbb{G}_m$, there are identifications $\mathrm{Pic}(X) = \mathrm{H}^1(X_{\mathrm{Zar}}, \mathcal{O}_X^*) = \mathrm{H}^1(X_{\text{ét}}, \mathbb{G}_m) = \mathrm{H}^1(X_{\mathrm{fppf}}, \mathbb{G}_m)$ for a scheme X (Hilbert's Theorem 90, [Mil80, Prop. 4.9]).

On the other hand, if X is a smooth scheme over $\mathbb{C}$ and G is a finite abelian group, then the classical complex cohomology $\mathrm{H}^i(X(\mathbb{C}), G)$ agrees with the étale cohomology $\mathrm{H}^i(X_{\text{ét}}, G)$ of the constant sheaf associated to G [Mil80, Thm. 3.12].

Exercise 4.1.44 (Forms of group schemes)**.** Let G be an algebraic group over a field $\mathbb{k}$. We say that a group scheme $H \to \mathrm{Spec}\,\mathbb{k}$ is a *form of G* if there is an isomorphism $G_{\overline{\mathbb{k}}} \cong H_{\overline{\mathbb{k}}}$. We call G the *trivial form of G*.

(a) Show the algebraic group $H = \mathrm{Spec}\,\mathbb{R}[x,y]/(x^2 + y^2 - 1)$ over $\mathbb{R}$, with the group structure induced from the embedding $H \subset \mathrm{SL}_2$ given by

$$(x,y) \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

is a non-trivial form of $\mathbb{G}_{m,\mathbb{R}}$.

(b) Assume that $\mathrm{char}(\mathbb{k}) \neq 2$. Recall the orthogonal groups $O(q)$ defined in Exercise B.2.33 for a non-degenerate quadratic form q on an n-dimensional vector space V. Show that every $O(q)$ is a form of the subgroup $O_n \subset \mathrm{GL}_n$ of orthogonal matrices.

(c) If G is smooth and commutative, show that forms of G are classified by $\mathrm{H}^1((\mathrm{Sch}/\mathbb{k})_{\text{ét}}, \mathrm{Aut}(G))$.

Remark 4.1.45 (Other cohomology theories)**.** See §6.1.6 for the development of sheaf cohomology on an algebraic stack. See §6.1.7 for a discussion of the Chow group of an algebraic stack, and §6.1.8 for a discussion of de Rham and singular cohomology.

4.2 Quotients by finite groups and the local structure of Deligne–Mumford stacks

Quotient stacks $[\operatorname{Spec} A/G]$ of affine schemes by finite abstract groups are a particularly nice class of Deligne–Mumford stacks. Their geometry is the G-equivariant geometry of $\operatorname{Spec} A$. In this section, we show that the natural map $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is universal for maps to algebraic spaces (Theorem 4.3.6) and that every Deligne–Mumford stack is étale locally isomorphic to a quotient stack of the form $[\operatorname{Spec} A/G]$ (Theorem 4.2.12).

4.2.1 Quotients by finite groups

Definition 4.2.1 (Geometric quotients). If G is a finite abstract group (viewed as a group scheme over $\mathbb{Z}$) acting on an algebraic space U, a G-invariant morphism $U \to X$ to an algebraic space is a *geometric quotient* if

(1) for every algebraically closed field $\Bbbk$, the map $U \to X$ induces a bijection $U(\Bbbk)/G \overset{\sim}{\to} X(\Bbbk)$, and

(2) $U \to X$ is universal for G-invariant maps to algebraic spaces, i.e. , every G-invariant map $U \to Y$ to an algebraic space factors uniquely as

$$
\begin{array}{ccc}
U & & \\
\downarrow & \searrow & \\
X & \dashrightarrow & Y.
\end{array}
$$

If $\pi \colon U \to X$ is a geometric quotient, we often write $X = U/G$. In the case that G acts freely on U (i.e. the action map $G \times U \to U \times U$ is a monomorphism), then we have already defined the algebraic space quotient U/G and the map $U \to U/G$ is a geometric quotient.

If a finite abstract group G acts on an affine scheme $\operatorname{Spec} A$, then G also acts on the ring A. We define the *invariant ring* as

$$A^G = \{f \in A \mid g \cdot f = f \text{ for all } g \in G\}.$$

We will show shortly that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a geometric quotient (Theorem 4.2.6).

Example 4.2.2. Assume $\operatorname{char}(\Bbbk) \neq 2$. Let $G = \mathbb{Z}/2$ acts on $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[x]$ via $-1 \cdot x = -x$, then $\Bbbk[x]^G = \Bbbk[x^2]$. The geometric quotient is the map $\mathbb{A}^1 = \operatorname{Spec} \Bbbk[x] \to \operatorname{Spec} \Bbbk[x^2] = \mathbb{A}^1$ sending p to p^2.

Let $G = \mathbb{Z}/2$ acts on $\mathbb{A}^2 = \operatorname{Spec} \Bbbk[x,y]$ via $-1 \cdot (x,y) = (-x,-y)$. Then $\Bbbk[x,y]^G = \Bbbk[x^2, xy, y^2]$ and the geometric quotient is $\mathbb{A}^2 \to \mathbb{A}^2/G = \operatorname{Spec} \Bbbk[x^2, xy, y^2]$. By setting $A = x^2, B = xy$ and $C = y^2$, the invariant ring can be identified with $\Bbbk[A, B, C]/(B^2 - AC)$ so the quotient $\mathbb{A}^2/G$ is a cone over a conic and in particular singular.

Lemma 4.2.3. *If G is a finite abstract group acting on a R-algebra A via R-algebra automorphisms, then $A^G \to A$ is integral. If R is noetherian and A is finitely generated over R, then $A^G \to A$ is finite and A^G is finitely generated over R.*

Proof. To see that $A^G \to A$ is integral, for every element $a \in A$ the product $\prod_{g \in G}(x - ga) \in A^G[x]$ is polynomial with invariant coefficients which has a as a root. If R is noetherian and $R \to A$ is of finite type, then $A^G \to A$ is also of finite type.

As $A^G \to A$ is integral, it is finite (c.f. [AM69, Cor. 5.2]). Since R is noetherian, we may conclude by the Artin–Tate Lemma (c.f. [AM69, Prop. 7.8]) that $R \to A^G$ is of finite type. □

The invariant ring is compatible with flat base change.

Lemma 4.2.4. *Let G be a finite abstract group acting on an affine scheme $\mathrm{Spec}\, A$. If $A^G \to B$ is a flat ring homomorphism, then G acts on the affine scheme $\mathrm{Spec}(B \otimes_{A^G} A)$ and $B = (B \otimes_{A^G} A)^G$.*

Proof. By definition, the invariant ring is the equalizer

$$0 \to A^G \to A \mathrel{\mathop{\rightrightarrows}^{p_1}_{p_2}} \prod_{g \in G} A$$

where $p_1(f) = (f)_{g \in G}$ and $p_2(f) = (gf)_{g \in G}$. Since $A^G \to B$ is flat, we have that

$$0 \to B \to A \otimes_{A^G} B \mathrel{\mathop{\rightrightarrows}^{p_1}_{p_2}} \prod_{g \in G} A \otimes_{A^G} B$$

is also exact and we conclude that $B = (B \otimes_{A^G} A)^G$. □

Exercise 4.2.5 (Base Change)**.** Let $A^G \to B$ be a ring homomorphism and consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}\, B \otimes_{A^G} A & \longrightarrow & \mathrm{Spec}\, A \\
\downarrow & \square & \downarrow \\
\mathrm{Spec}(B \otimes_{A^G} A)^G \longrightarrow \mathrm{Spec}\, B & \longrightarrow & \mathrm{Spec}\, A^G.
\end{array}
$$

(a) Show that $\mathrm{Spec}(B \otimes_{A^G} A)^G \to \mathrm{Spec}\, B$ is an integral homeomorphism.

(b) If $|G|$ is invertible in A, show that $B \to (B \otimes_{A^G} A)^G$ is an isomorphism.

(c) Provide an example where $B \to (B \otimes_{A^G} A)^G$ is not an isomorphism.

We now turn to the main theorem: $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^G$ is a geometric quotient (Definition 4.2.1).

Theorem 4.2.6. *If G is a finite abstract group acting on an affine scheme $\mathrm{Spec}\, A$, then $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^G$ is a geometric quotient. If A is finitely generated over a noetherian ring R, then A^G is also finitely generated over R.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
U = \mathrm{Spec}\, A & & \\
\downarrow & \searrow^{\widetilde{\pi}} & \\
\mathcal{X} = [U/G] & \xrightarrow{\pi} & X = \mathrm{Spec}\, A^G.
\end{array}
$$

Since $\widetilde{\pi}$ is integral and dominant, it is surjective. To see that $\widetilde{\pi}$ is injective on G-orbits of geometric points, let $\Bbbk$ be an algebraically closed field and $x, x' \in U(\Bbbk)$ with $\widetilde{\pi}(x) = \widetilde{\pi}(x') \in X(\Bbbk)$. The base change $U \times_X \mathrm{Spec}\, \Bbbk = \mathrm{Spec}(A \otimes_{A^G} \Bbbk)$ inherits a G-action and the G-orbits $Gx, Gx' \subset U \times_X A^G$ are closed subschemes. If $Gx \neq Gx'$, then the orbits are disjoint and there exists a function $f \in A \otimes_R \Bbbk$ with $f|_{Gx} = 0$

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and $f|_{Gx'} = 1$. Then $\widetilde{f} = \prod_{g \in G} gf \in (A \otimes_{A^G} \Bbbk)^G$ is a G-invariant function with $f'(x) = 0$ and $f'(x') = 1$. But this implies that $\widetilde{\pi}(x) \neq \widetilde{\pi}(x') \in X(\Bbbk)$, which is a contradiction.

The map $\widetilde{\pi} \colon U \to X$ is universal for G-invariant maps to algebraic spaces if and only if $\pi \colon \mathcal{X} = [U/G] \to X$ is universal for maps to algebraic spaces. In other words, we need to show that if Y is an algebraic space, then the natural map

$$\mathrm{Map}(X, Y) \to \mathrm{Map}(\mathcal{X}, Y) \tag{4.2.7}$$

is bijective. We note that this is immediate when Y is affine as $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = \Gamma(X, \mathcal{O}_X)$ and the case when Y is a scheme can be reduced to this case without much effort: if $g \colon \mathcal{X} \to Y$ is a map, an affine covering Y_i of Y induces an open covering $X_i = X \setminus \pi(\mathcal{X} \setminus g^{-1}(Y_i))$ of X, and g restricts to a map $\pi^{-1}(X_i) \to Y_i$ which factors uniquely through X_i since $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$; see also [GIT, §0.6]. We need to work harder to handle the case that Y is an algebraic space.

For the injectivity of (4.2.7), let $h_1, h_2 \colon X \to Y$ be two maps such that $h_1 \circ \pi = h_2 \circ \pi$. Let $E \to X$ be the equalizer of h_1 and h_2, i.e. the pullback of the diagonal $Y \to Y \times Y$ along $(h_1, h_2) \colon X \to Y \times Y$. The equalizer $E \to X$ is a monomorphism and locally of finite type. By construction $\pi \colon \mathcal{X} \to X$ factors through $E \to X$ and since π is universally closed and schematically dominant (i.e. $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is injective), so is $E \to X$. As every universally closed and locally of finite type monomorphism is a closed immersion (see Corollary A.7.5 and Remark A.7.6), we conclude that $E \to X$ is an isomorphism.

For the surjectivity of (4.2.7), let $g \colon \mathcal{X} \to Y$ be a map. We claim that the question is étale-local on X. Indeed, if $V \to X$ is an étale cover and $h \colon V \to Y$ is a morphism such that the two compositions $V \times_X \mathcal{X} \to V \xrightarrow{h} Y$ and $V \times_X \mathcal{X} \to \mathcal{X} \xrightarrow{g} Y$ agree, then by the injectivity of (4.2.7) applied to the good moduli space $V \times_X V \times_X \mathcal{X} \to V \times_X V$, the two compositions $V \times_X V \rightrightarrows V \xrightarrow{h} Y$ agree and thus $h \colon V \to Y$ descends to a morphism $\overline{h} \colon X \to Y$. Étale descent also implies the commutativity of $g = \overline{h} \circ \pi$.

Since $\mathcal{X}$ is quasi-compact, we may assume that Y is quasi-compact as $g \colon \mathcal{X} \to Y$ factors through a quasi-compact open algebraic subspace of Y. Let $Y' \to Y$ be an étale presentation from an affine scheme and let $\mathcal{X}' := \mathcal{X} \times_Y Y'$. We claim that after replacing X with an étale cover $V \to X$ and $\mathcal{X}$ with the base change $\mathcal{X} \times_X V$, there is a section $s \colon \mathcal{X} \to \mathcal{X}'$ of $\mathcal{X}' \to \mathcal{X}$ in the commutative diagram

$$
\begin{array}{ccc}
\mathcal{X}' \xleftarrow{\;\;s\;\;} \mathcal{X} & \xrightarrow{\;\pi\;} & X \\
\downarrow{g'} \;\;\square\;\; \downarrow{g} & & \\
Y' \longrightarrow Y. & &
\end{array}
$$

The surjectivity of (4.2.7) follows from this claim: since X and Y' are affine, the equality $\Gamma(X, \mathcal{O}_X) = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ implies that $\mathcal{X} \xrightarrow{s} \mathcal{X}' \xrightarrow{g'} Y'$ factors through $\pi \colon \mathcal{X} \to X$ via a morphism $X \to Y'$. The composition $X \to Y' \to Y$ yields the desired dotted arrow above.

We claim that limit methods allow us to reduce to the case that $X = \mathrm{Spec}\, A^G$ is the spectrum of a strictly henselian local ring. Indeed, for a closed point u of $U := \mathrm{Spec}\, A$ over $x \in |\mathcal{X}|$, the strict henselization $X^{\mathrm{sh}} := \mathcal{O}_{X, \pi(x)}^{\mathrm{sh}}$ is the limit $\lim_i X_i$ over all affine étale neighborhoods $X_i \to X$ of $\pi(x)$. The base change $U^{\mathrm{sh}} := U \times_X X^{\mathrm{sh}}$ is the limit of the affine schemes $U_i := U \times_X X_i$. We also set

$\mathcal{X}^{\mathrm{sh}} := \mathcal{X} \times_X X^{\mathrm{sh}} = [U^{\mathrm{sh}}/G]$ and $\mathcal{X}_i := \mathcal{X} \times_X X_i = [U_i/G]$. Since $\mathcal{X}' \to \mathcal{X}$ is locally of finite presentation, the natural map

$$\mathrm{colim}_i \, \mathrm{Mor}_\mathcal{X}(\mathcal{X}_i, \mathcal{X}') \to \mathrm{Mor}_\mathcal{X}(\mathcal{X}^{\mathrm{sh}}, \mathcal{X}')$$

is an equivalence; this follows from Exercise 3.3.31 using that $\mathrm{Mor}_\mathcal{X}(\mathcal{X}^{\mathrm{sh}}, \mathcal{X}')$ is the equalizer of $\mathrm{Mor}_\mathcal{X}(U^{\mathrm{sh}}, \mathcal{X}') \rightrightarrows \mathrm{Mor}_\mathcal{X}(G \times U^{\mathrm{sh}}, \mathcal{X}')$ and similarly for the left-hand side. A section of $\mathcal{X}' \times_\mathcal{X} \mathcal{X}^{\mathrm{sh}} \to \mathcal{X}^{\mathrm{sh}}$ is determined by a map $\mathcal{X}^{\mathrm{sh}} \to \mathcal{X}'$ over $\mathcal{X}$. This map extends to a morphism $\mathcal{X}_i \to \mathcal{X}'$ for some i, giving us the desired section.

Let κ be the residue field of A^G. As $A^G \to A$ is finite, $A = A_1 \times \cdots \times A_r$ is a product of strictly henselian local rings, each finite over A^G (Proposition B.5.13). If $u \in \mathrm{Spec}\, A_1 \subset \mathrm{Spec}\, A$ is a closed point, then $\mathrm{Spec}\, A_1$ is G_u-invariant and the orbit Gu is in bijection with the r connected components of $\mathrm{Spec}\, A$. There is an isomorphism $\mathcal{X} \cong [\mathrm{Spec}\, A_1/G_u]$; this can be verified directly by for instance slicing the groupoid $G \times \mathrm{Spec}\, A \rightrightarrows \mathrm{Spec}\, A$ by $\mathrm{Spec}\, A_1 \hookrightarrow \mathrm{Spec}\, A$ (as in Exercise 3.4.17). We may thus replace $\mathcal{X} = [\mathrm{Spec}\, A/G]$ with $[\mathrm{Spec}\, A_1/G_u]$, and we can assume that there is a unique closed point $u \in \mathrm{Spec}\, A$ which is set-theoretically fixed by G. As $Y' \to Y$ is representable by schemes, we can write $\mathcal{X}' = [U'/G]$ for a scheme U'. Let $u' \in U'$ be a preimage of $u \in \mathrm{Spec}\, A$. As A is strictly henselian and the G-equivariant morphism $U' \to U$ is the base change of the étale morphism $Y' \to Y$, we see that $\kappa(u') = \kappa(u)$ and $G_{u'} = G_u = G$, and moreover the stabilizers act trivially on the residue fields. Again using that A is strictly henselian, there is a unique section $s \colon \mathrm{Spec}\, A \to U'$ with $s(u) = u'$ (Proposition B.5.10). This section is G-invariant because for every $g \in G$, both $s \circ g$ and $g \circ s$ are sections of $U' \to \mathrm{Spec}\, A \xrightarrow{g^{-1}} \mathrm{Spec}\, A$ with $u' \mapsto u$ and thus the sections agree. It follows that s descends to a section $\mathcal{X} = [\mathrm{Spec}\, A/G] \to [U'/G] = \mathcal{X}'$ of $\mathcal{X}' \to \mathcal{X}$. This finishes the proof that $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^G$ is a geometric quotient.

The final statement follows from Lemma 4.2.3. $\qquad\square$

Corollary 4.2.8. *Let G be a finite abstract group acting freely on an affine scheme $U = \mathrm{Spec}\, A$, then the algebraic space quotient U/G is isomorphic to $\mathrm{Spec}\, A^G$.* $\quad\square$

Exercise 4.2.9. Let R be a noetherian ring. Let G be a finite abstract group acting on a scheme U projective (resp. quasi-projective, quasi-affine) over a ring R. Show that there exists a geometric quotient $U \to U/G$ such that U/G is a projective (resp. quasi-projective, quasi-affine) scheme over R.

Exercise 4.2.10. Suppose that G is a finite abstract group acting on an affine scheme $\mathrm{Spec}\, A$ of finite type over a noetherian ring R. If $x \in \mathrm{Spec}\, A$ is a closed point, show that there is an isomorphism

$$\widehat{A}^{G_x} \cong \widehat{A^G}$$

between the G_x-invariants of the completion at $\mathrm{Spec}\, A$ at x and the completion of $\mathrm{Spec}\, A^G$ at the image of x.

The following exercise generalizes Theorem 4.3.6 from quotients of finite groups to quotients of finite flat groupoids.

Exercise 4.2.11. Let $s, t \colon R \rightrightarrows U$ be a finite flat groupoid of affine schemes, and define $A^R \subset A$ as the subring of R-invariants, i.e. the subring of elements $a \in A$ such that $s^*a = t^*a \in \Gamma(R, \mathcal{O}_R)$. Show that $U \to X := \mathrm{Spec}\, A^R$ induces a bijection $U(\Bbbk)/R(\Bbbk) \xrightarrow{\sim} X(\Bbbk)$ for every algebraically closed field $\Bbbk$ and that $U \to X$ is universal for R-invariant maps to algebraic spaces. Moreover, show that if A is finitely generated over a noetherian ring, then so is A^R.

4.2.2 The Local Structure Theorem

We show that a Deligne–Mumford stack $\mathcal{X}$ near a point x is étale locally the quotient stack $[\operatorname{Spec} A/G_x]$ of an affine scheme by the stabilizer group scheme. Conceptually, this tells us that just as schemes (resp. algebraic spaces) are obtained by gluing affine schemes in the Zariski topology (resp. étale topology), Deligne–Mumford stacks are obtained by gluing quotient stacks $[\operatorname{Spec} A/G]$ in the étale topology.[1] This has the practical application of allowing one to reduce many properties of Deligne–Mumford stacks to quotient stacks $[\operatorname{Spec} A/G]$. We will take advantage of this local structure to construct a coarse moduli space (Theorem 4.3.12).

The *geometric stabilizer* of a point x of a Deligne–Mumford stack $\mathcal{X}$ is the abstract group defined as the stabilizer of any geometric point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ with image x.

Theorem 4.2.12 (Local Structure Theorem of Deligne–Mumford Stacks). *Let $\mathcal{X}$ be a separated Deligne–Mumford stack and $x \in \mathcal{X}$ be a finite type point with geometric stabilizer G_x. There exists an affine étale morphism*

$$f \colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

where $w \in [\operatorname{Spec} A/G_x]$ such that f induces an isomorphism of geometric stabilizer groups at w.

Proof. Choose a geometric point $\operatorname{Spec} \Bbbk \to \mathcal{X}$ of representing x, and let d be the cardinality of G_x. Viewing G_x has a group scheme over $\operatorname{Spec} \Bbbk$, let $BG_x \to \mathcal{X}$ be the induced map. Let $(U, u) \to (\mathcal{X}, x)$ be an étale representable morphism from an affine scheme. Since $\mathcal{X}$ is separated, $U \to \mathcal{X}$ is affine. Define the quasi-affine scheme

$$\operatorname{SEC}_d := \underbrace{U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U}_{d \text{ times}} \backslash \Delta,$$

where Δ is the union of all pairwise diagonals. A map $S \to \operatorname{SEC}_d$ from a scheme is classified by a morphism $S \to \mathcal{X}$ and d sections $s_1, \ldots, s_d$ of $U_S := U \times_{\mathcal{X}} S \to S$ which are disjoint (i.e. the intersection of s_i and s_j is empty for $i \neq j$. There is an action of S_d on SEC_d given by permuting the sections and we define the quotient stack

$$\operatorname{ET}_d := [\operatorname{SEC}_d / S_d].$$

By the correspondence between principal S_d-bundles and finite étale covers of degree d (Exercise B.2.23), an object of ET_d over a scheme S corresponds to a diagram

$$
\begin{array}{ccc}
Z \lhook\joinrel\longrightarrow U_S \longrightarrow & U \\
\searrow \quad \downarrow \quad \square & \downarrow \\
S \longrightarrow & \mathcal{X}
\end{array}
$$

where $Z \hookrightarrow U_S$ is a closed subscheme and $Z \to S$ is finite étale of degree d. The fiber product $BG_x \times_{\mathcal{X}} U$ is a finite disjoint union of $\operatorname{Spec} \Bbbk$'s, and choosing one of them leads to a diagram

$$
\begin{array}{ccc}
\operatorname{Spec} \Bbbk \lhook\joinrel\longrightarrow \coprod \operatorname{Spec} \Bbbk \longrightarrow & U \\
\searrow \quad \downarrow \quad \square & \downarrow \\
BG_x \longrightarrow & \mathcal{X}
\end{array}
$$

---

[1] Of course, Deligne–Mumford stacks are also étale locally schemes but the étale neighborhoods $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ produced by Theorem 4.2.12 preserve the stabilizer group at w.

with $\operatorname{Spec} \Bbbk \to BG_x$ finite étale of degree d. The further base change by $\operatorname{Spec} \Bbbk \to BG_x$ defines a point $w \in \mathrm{ET}_d(\Bbbk)$.

There is an induced morphism $\mathrm{ET}_d \to \mathcal{X}$ and a commutative diagram

$$
\begin{array}{ccc}
\mathrm{SEC}_d & \longrightarrow & \mathrm{ET}_d \\
\cup \big\downarrow & & \big\downarrow \\
U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U & \longrightarrow U \longrightarrow & \mathcal{X}
\end{array}
$$

We claim that $\mathrm{ET}_d \to \mathcal{X}$ is an étale morphism that induces an isomorphism of stabilizer groups at w. The étaleness follows from étale descent as $U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U \to \mathcal{X}$ is étale. To see that the induced map on stabilizers at w is an isomorphism, it suffices to assume that $\mathcal{X} = BG_x$ and $U = \operatorname{Spec} \Bbbk$. In this case, there is an isomorphism $U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U \cong G_x^{d-1}$, which we can further identify with the quotient G_x^d / G_x of the diagonal action. Then $\mathrm{SEC}_d \subset U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U$ is identified with the quotient $(G_x^d \setminus \Delta)/G_x$. The permutation S_d-action is transitive with stabilizer isomorphic to G_x, and in fact SEC_d is S_d-equivariantly isomorphic to the quotient S_d/G of the regular representation $G \subset S_d$. We thus see that $\mathrm{ET}_d \cong BG_x$.

Since $\mathrm{ET}_d \to \mathcal{X}$ is separated, the relative inertia stack $I_{\mathrm{ET}_d / \mathcal{X}} \to \mathrm{ET}_d$ is finite and thus an isomorphism in an open substack $[W/S_d] \subset \mathrm{ET}_d$ around w, where $W \subset \mathrm{SEC}_d$ is a quasi-affine scheme. It follows that $[W/S_d] \to \mathcal{X}$ is an étale representable morphism inducing an isomorphism on stabilizer groups at w. By quotienting out by $G_x \subset S_d$ instead, the morphism $[W/G_x] \to \mathcal{X}$ is also étale representable inducing an isomorphism of stabilizer groups at w. Letting $W' \subset W$ be an affine open subscheme containing w, we may replace W with the G_x-invariant affine open subscheme $\bigcap_{g \in G_x} g \cdot W'$.

It remains to show that $[W/G_x] \to \mathcal{X}$ is affine. Since $\mathcal{X}$ is separated, its diagonal is affine and the morphism $W \to \mathcal{X}$ from the affine scheme W is affine. The fiber product

$$
\begin{array}{ccc}
[W/G_x] \times_{\mathcal{X}} W & \longrightarrow & W \\
\big\downarrow & \square & \big\downarrow \\
[W/G_x] & \longrightarrow & \mathcal{X}
\end{array}
$$

is affine over $[W/G_x]$ and thus isomorphic to a quotient stack $[\operatorname{Spec} B/G_x]$. On the other hand, since $[W/G_x] \to \mathcal{X}$ is representable, the quotient stack $[\operatorname{Spec} B/G_x]$ is an algebraic space and the action of G_x on $\operatorname{Spec} B$ is free. By Corollary 4.2.8, $[\operatorname{Spec} B/G_x]$ is isomorphic to the affine scheme $\operatorname{Spec} B^{G_x}$. By étale descent $[W/G_x] \to \mathcal{X}$ is affine.

See also [LMB00, Thm. 6.2]. $\qquad \square$

Exercise 4.2.13. Let $\mathcal{X}$ be a Deligne–Mumford stack. Show that $\mathcal{X}$ is isomorphic to a quotient stack $[U/G]$ where U is an affine scheme (resp. scheme, algebraic space) and G is a finite abstract group if and only if there exists a finite étale morphism $V \to \mathcal{X}$ from an affine scheme (resp. scheme, algebraic space).

Hint: If $V \to \mathcal{X}$ is a finite étale cover of degree d, consider the associated principal S_d-torsor $\underbrace{V \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} V}_{d \ times} \setminus \Delta \to \mathcal{X}$; see Exercise B.2.23.

Proposition 4.2.14. *If $R \rightrightarrows U$ is a finite étale equivalence relation of affine schemes, then the algebraic space quotient U/R is an affine scheme.*

Proof. By Exercise 4.2.13, the algebraic space U/R is isomorphism to V/G for the free action of a finite group G on an affine scheme $V = \operatorname{Spec} B$. Theorem 4.3.6 shows that $V/G \to \operatorname{Spec} B^G$ is universal for maps to algebraic spaces and thus an isomorphism. Alternatively, this follows from Exercise 4.2.11: if $U = \operatorname{Spec} A$, then $U/R \to \operatorname{Spec} A^R$ is universal for maps to algebraic spaces and thus an isomorphism. $\square$

With a similar technique to the proof of Theorem 4.2.12, we can prove the following useful result asserting the existence of presentations with a lift of a given field-valued point.

Proposition 4.2.15. *If $\mathcal{X}$ is an algebraic stack with quasi-separated diagonal (resp. quasi-separated algebraic space) and $x \in \mathcal{X}(\Bbbk)$ is a field-valued point, then there exists a smooth (resp. étale) morphism $U \to \mathcal{X}$ from an affine scheme and a point $u \in U(\Bbbk)$ over x.*

Proof. Let $U \to \mathcal{X}$ be a smooth morphism from an affine scheme such that x is contained in its image. Consider the fiber product

$$
\begin{array}{ccc}
U_x & \longrightarrow & U \\
\downarrow & \square & \downarrow \\
\operatorname{Spec} \Bbbk & \xrightarrow{\ x\ } & \mathcal{X}.
\end{array}
$$

Since $\mathcal{X}$ has quasi-separated diagonal, the morphism $U \to \mathcal{X}$ is quasi-separated. As U_x is finite type and quasi-separated over $\operatorname{Spec} \Bbbk$, it is a noetherian algebraic space. Choosing a closed point $u \in U_x$, by Existence of Residual Gerbes (3.5.19), there is a closed immersion $\operatorname{Spec} \kappa(u) \to U_x$ from the residue field. The field extension $\Bbbk \to \kappa(u)$ is finite and separable, and let d be its degree.

Following the notation of the proof of Theorem 4.2.12, we consider the open subspace $\mathrm{SEC}_d \subset U \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U$ parameterizing d disjoint sections of $U \to \mathcal{X}$ and the morphism $\mathrm{ET}_d = [\mathrm{SEC}_d / S_d] \to \mathcal{X}$. As $U \to \mathcal{X}$ is quasi-separated, SEC_d and ET_d are also quasi-separated. As $\operatorname{Spec} \kappa(u) \to \operatorname{Spec} \Bbbk$ is finite étale of degree d, the closed immersion $\operatorname{Spec} \kappa(u) \hookrightarrow U_x$ defines a $\Bbbk$-point v of ET_d. This gives a commutative diagram

$$
\begin{array}{ccc}
& & \operatorname{Spec} \Bbbk \\
& \swarrow{\scriptstyle v} & \downarrow{\scriptstyle x} \\
\mathrm{SEC}_d \longrightarrow \mathrm{ET}_d & \longrightarrow & \mathcal{X}.
\end{array}
$$

The point $v \in \mathrm{ET}_d(\Bbbk)$ does not necessarily lift to SEC_d, but we can use the following trick to choose a different quotient presentation of ET_d where v does lift. Namely, choose a faithful representation $S_d \subset \mathrm{GL}_n$ and write $\mathrm{ET}_d \cong [V/\mathrm{GL}_n]$ where $V = \mathrm{SEC}_d \times^{S_d} \mathrm{GL}_n$ is a quasi-separated algebraic space. Then $v \colon \operatorname{Spec} \Bbbk \to [V/\mathrm{GL}_n]$ corresponds to a principal GL_n-bundle $P \to \operatorname{Spec} \Bbbk$ and a GL_n-equivariant map $P \to V$. Since principal GL_n-bundles are in bijection to vector bundles (Exercise B.2.27), P is the trivial principal GL_n-bundle and there is a section $\operatorname{Spec} \Bbbk \to P$. The composition $V \to [V/\mathrm{GL}_n] \to \mathcal{X}$ is smooth and the composition $\operatorname{Spec} \Bbbk \to P \to V$ is a lift of x.

It remains to show that a $\Bbbk$-point of a quasi-separated algebraic space V lifts to an étale presentation by a scheme. We will use the fact that a quasi-separated algebraic space has quasi-affine diagonal; this is proved in Corollary 4.4.8 using the result Proposition 4.2.14 above and the theory of quasi-coherent sheaves (§4.1). We

repeat the above argument by choosing an étale map $U \to V$ from an affine scheme such that the image contains x. Then the space SEC_d of d disjoint sections with respect to $U \to V$ is a quasi-affine scheme with a free action of S_d. The quotient $\mathrm{ET}_d = \mathrm{SEC}_d / S_d$ is also quasi-affine (Exercise 4.2.9). The induced map $\mathrm{ET}_d \to \mathcal{X}$ is étale and by construction the $\Bbbk$-point x lifts to a $\Bbbk$-point of ET_d.

See also [LMB00, Thm. 6.3]. $\qquad\square$

4.3 Coarse moduli spaces and the Keel–Mori Theorem

The goal of this section is to establish the Keel–Mori Theorem: every separated Deligne–Mumford stack $\mathcal{X}$ of finite type over a noetherian scheme admits a separated coarse moduli space $\pi\colon \mathcal{X} \to X$ (see Theorem 4.3.12). One can view this theorem as a way to remove the stackiness of a Deligne–Mumford stack; at the expense of sacrificing universal properties of $\mathcal{X}$ (e.g. existence of a universal family), one can replace $\mathcal{X}$ with an algebraic space without changing the underlying topological space.

We will later apply this theorem to show that the Deligne–Mumford stack $\overline{\mathcal{M}}_g$ parameterizing stable curves admits a coarse moduli space $\pi\colon \overline{\mathcal{M}}_g \to \overline{M}_g$ where $\overline{M}_g$ is a separated algebraic space, which we later show to be proper and then finally projective.

To prove Theorem 4.3.12, we will apply the Local Structure Theorem (4.2.12) to construct étale neighborhoods $[\mathrm{Spec}(A_i)/G] \to \mathcal{X}$ and show that the geometric quotients $\mathrm{Spec}(A_i^G)$ glue in the étale topology to a coarse moduli space of $\mathcal{X}$.

4.3.1 Coarse moduli spaces

We begin with the definition:

Definition 4.3.1. A morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a *coarse moduli space* if

(1) for every algebraically closed field $\Bbbk$, the induced map $\mathcal{X}(\Bbbk)/\!\!\sim \; \to X(\Bbbk)$, from the set of isomorphism classes of objects of $\mathcal{X}$ over $\Bbbk$, is bijective, and

(2) π is universal for maps to algebraic spaces, i.e. every map $\mathcal{X} \to Y$ to an algebraic space factors uniquely as

$$
\begin{array}{ccc}
 & \mathcal{X} & \\
{\scriptstyle\pi}\downarrow & & \searrow \\
X & \dashrightarrow & Y.
\end{array}
$$

Remark 4.3.2. If G is a finite abstract group acting on an algebraic space U, then $[U/G] \to X$ is a coarse moduli space if and only if $U \to X$ is a geometric quotient (Definition 4.2.1).

Remark 4.3.3. In practice, we desire coarse moduli spaces with additional properties of $\pi\colon \mathcal{X} \to X$ as otherwise it is difficult to work with this notion. For instance, it is not true that this notion is stable under étale base change (or even open immersions) or that $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$. However, we emphasize that the Keel–Mori Theorem produces a coarse moduli space $\pi\colon \mathcal{X} \to X$ with the additional properties: (a) it is stable under flat base change, (b) $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$, (c) π is proper (and in particular separated!) and (d) π is a universal homeomorphism.

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Lemma 4.3.4. *Let $\pi\colon \mathcal{X} \to X$ be a coarse moduli space such that for every étale morphism $X' \to X$ from an affine scheme, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space. Then the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.*

Proof. As π is universal for maps to algebraic spaces, we have that $\mathrm{Map}(X, \mathbb{A}^1) \to \mathrm{Map}(\mathcal{X}, \mathbb{A}^1)$ is bijective or in other words $\Gamma(X, \mathcal{O}_X) \cong \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For every étale map $X' \to X$, the base change $\mathcal{X}' = \mathcal{X} \times_X X' \to X'$ is also a coarse moduli space and thus $\Gamma(X', \mathcal{O}_{X'}) \cong \Gamma(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. This shows that $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is isomorphism. $\square$

The property that a given map is a coarse moduli space can be checked étale locally.

Lemma 4.3.5. *Let $\pi\colon \mathcal{X} \to X$ be a morphism to an algebraic space. Suppose that there is an étale covering $\{X_i \to X\}$ such that $\mathcal{X} \times_X X_i \to X_i$ is a coarse moduli space for each i. Then $\pi\colon \mathcal{X} \to X$ is a coarse moduli space.* $\square$

Proof. Axiom (1) of a coarse moduli space is a condition on geometric fibers and can thus be checked étale locally, while Axiom (2) follows from the fact that algebraic spaces are sheaves in the étale topology. $\square$

Theorem 4.3.6. *If G is a finite abstract group acting on an affine scheme $\mathrm{Spec}\, A$, then $\pi\colon [\mathrm{Spec}\, A/G] \to \mathrm{Spec}\, A^G$ is a coarse moduli space. Moreover,*

(1) the base change of π along a flat morphism $X' \to \mathrm{Spec}\, A^G$ of algebraic spaces is a coarse moduli space,

(2) the natural map $X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism, and

(3) if A is finitely generated over a noetherian ring R, then A^G is finitely generated over R and π is a proper universal homeomorphism.

Proof. In Theorem 4.2.6, we showed that $\pi\colon [\mathrm{Spec}\, A/G] \to \mathrm{Spec}\, A^G$ is a coarse moduli space and that $R \to A^G$ is of finite type if R is noetherian and $R \to A$ is of finite type. To see (1), it suffices by Lemma 4.3.5 to consider flat morphisms $Y' \to Y$ from an affine scheme. But in this case, the base change $\mathcal{X} \times_Y Y'$ is isomorphic to a quotient stack $[\mathrm{Spec}\, B/G]$ and Lemma 4.2.4 implies that $Y' \cong \mathrm{Spec}\, B^G$. It follows that $\mathcal{X} \times_Y Y' \to Y'$ is a coarse moduli space. Part (2) follows directly from (1) by Lemma 4.3.4. For (3), it remains to show that π is a proper universal homeomorphism. Since π is bijective and universally closed, its set-theoretic inverse is continuous, and thus π is a homeomorphism. The base change of π along a morphism $\mathrm{Spec}\, B \to \mathrm{Spec}\, A^G$ factors as $[\mathrm{Spec}(B \otimes_{A^G} A)/G] \to \mathrm{Spec}(B \otimes_{A^G} A)^G \to \mathrm{Spec}\, B$ where the first map is a homeomorphism by the above argument and the second is a homeomorphism by Exercise 4.2.5. We conclude that π is a universal homeomorphism. $\square$

4.3.2 Descending étale morphisms to quotients

Proposition 4.3.7. *Let G be a finite abstract group and $f\colon \mathrm{Spec}\, A \to \mathrm{Spec}\, B$ be a G-equivariant morphism of affine schemes of finite type over a noetherian ring R. Let $x \in \mathrm{Spec}\, A$ be a closed point. Assume that*

(a) f is étale at x and

(b) the induced map $G_x \to G_{f(x)}$ of stabilizer group schemes is bijective.

Then there is an open affine neighborhood $W \subset \mathrm{Spec}\, A^G$ of the image of x such that $W \to \mathrm{Spec}\, A^G \to \mathrm{Spec}\, B^G$ is étale and $\pi_A^{-1}(W) \cong W \times_{\mathrm{Spec}\, B^G} [\mathrm{Spec}\, B/G]$, where $\pi_A\colon [\mathrm{Spec}\, A/G] \to \mathrm{Spec}\, A^G$.

Remark 4.3.8. In other words, after replacing $\operatorname{Spec} A^G$ with an affine neighborhood W of $\pi_A(x)$ and $\operatorname{Spec} A$ with $\pi_A^{-1}(W)$, it can be arranged that the diagram

$$
\begin{array}{ccc}
[\operatorname{Spec} A/G] & \xrightarrow{\ f\ } & [\operatorname{Spec} B/G] \\
\downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\
\operatorname{Spec} A^G & \longrightarrow & \operatorname{Spec} B^G
\end{array}
\tag{4.3.9}
$$

is cartesian where both horizontal maps are étale.

Condition (b) can be tested on a field-valued point $\operatorname{Spec} \Bbbk \to \operatorname{Spec} A$ representing x (e.g. the inclusion of the residue field).

The above proposition will be applied in the following form in the proof of the Keel–Mori Theorem (Theorem 4.3.12).

Corollary 4.3.10. *Let G be a finite abstract group and $f \colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a G-equivariant morphism of affine schemes of finite type over a noetherian ring R. Assume that for every closed point $x \in \operatorname{Spec} A$,*

(a) f is étale at x and

(b) the induced map $G_x \to G_{f(x)}$ of stabilizer group schemes is bijective.

Then $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale and (4.3.9) is cartesian. $\qquad\square$

Proof of Proposition 4.3.7. Set $y = f(x)$. We first claim that the question is étale local around $\pi_B(y) \in \operatorname{Spec} B^G$. Indeed, if $Y' \to Y := \operatorname{Spec} B^G$ is an affine étale neighborhood of $\pi_B(y)$, we let $X', \mathcal{X}'$ and $\mathcal{Y}'$ denote the base changes of $X := \operatorname{Spec} A^G$, $\mathcal{X} := [\operatorname{Spec} A/G]$, and $\mathcal{Y} := [\operatorname{Spec} B/G]$. By Lemma 4.2.4, we know that $\mathcal{Y}' \cong [\operatorname{Spec} B'/G]$ with $Y' \cong \operatorname{Spec} B'^G$ and similarly for $\mathcal{X}'$ and X'. If the result holds after this base change, there is an open neighborhood $W' \subset X'$ containing a preimage of $\pi_A(x)$ such that $W' \hookrightarrow X' \to Y'$ is étale and such that the preimage of W' in $\mathcal{X}'$ is isomorphic to $W' \times_{Y'} \mathcal{Y}'$. Taking W as the image of W' under $X' \to \operatorname{Spec} A^G$ and applying étale descent yields the desired claim.

We now claim that this allows us to assume that B^G is strictly henselian. To see this, let $Y^{\mathrm{sh}} = \operatorname{Spec} \mathcal{O}_{Y, \pi_B(y)}^{\mathrm{sh}}$ and $X^{\mathrm{sh}}, \mathcal{X}^{\mathrm{sh}}$ and $\mathcal{Y}^{\mathrm{sh}}$ be the base changes of X, $\mathcal{X}$ and $\mathcal{Y}$ along $Y^{\mathrm{sh}} \to Y$. Suppose $U^{\mathrm{sh}} \subset X^{\mathrm{sh}}$ is an open affine subscheme of the unique point in X^{sh} over x and the closed point of Y^{sh} such that $U^{\mathrm{sh}} \to Y^{\mathrm{sh}}$ is étale with $\pi_{\mathcal{X}^{\mathrm{sh}}}^{-1}(U^{\mathrm{sh}}) \cong U^{\mathrm{sh}} \times_{Y^{\mathrm{sh}}} \mathcal{Y}^{\mathrm{sh}}$. Then $Y^{\mathrm{sh}} = \lim_\lambda Y_\lambda$ is the limit of affine étale neighborhood $Y_\lambda \to Y$ of y and we set $X_\lambda, \mathcal{X}_\lambda$, and $\mathcal{Y}_\lambda$ to be the base changes of X, $\mathcal{X}$, and $\mathcal{Y}$ along $Y_\lambda \to Y$. By Proposition B.3.4, the morphism $U^{\mathrm{sh}} \to X^{\mathrm{sh}}$ descends to $U_\eta \to X_\eta$ for some η. Setting $U_\lambda = U_\eta \times_{X_\eta} X_\lambda$ for $\lambda > \eta$, it follows from Proposition B.3.7 that for $\lambda \gg 0$ (a) $U_\lambda \to X_\lambda$ is an open immersion, (b) the composition $U_\lambda \to X_\lambda \to Y_\lambda$ is étale, and (c) $\pi_{\mathcal{X}_\lambda}^{-1}(U_\lambda) \cong U_\lambda \times_{Y_\lambda} \mathcal{Y}_\lambda$ (by arguing on the étale presentations of $\mathcal{X}$ and $\mathcal{Y}$).

Finally, As $B^G \to B$ is finite (Lemma 4.2.3), $B = B_1 \times \cdots \times B_r$ is a product of strictly henselian local rings (Proposition B.5.13). As in the proof of Theorem 4.3.6, we may replace $[\operatorname{Spec} B/G]$ with $[\operatorname{Spec} B_1/G_y]$ and $[\operatorname{Spec} A/G]$ with $[f^{-1}(\operatorname{Spec} B_1)/G]$ to assume that G fixes x and y while acting trivially on the residue fields $\kappa(x) = \kappa(y)$. Thus $\operatorname{Spec} A \to \operatorname{Spec} B$ has a unique section $s \colon \operatorname{Spec} B \to \operatorname{Spec} A$ taking y to x. The section s is necessarily G-invariant (as in the proof Theorem 4.3.6). Thus s descends to a section of $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ which gives our desired open and closed subscheme $W \subset \operatorname{Spec} A^G$. $\qquad\square$

Remark 4.3.11. Here's a conceptual reason why we should expect the induced map of quotients to be étale. For simplicity, assume that $R = \Bbbk$ is an algebraically closed field. Let $\widehat{A}$ and $\widehat{B}$ be the completions of the local rings at x and $f(x)$. The stabilizers G_x and $G_{f(x)}$ act on $\operatorname{Spec}\widehat{A}$ and $\operatorname{Spec}\widehat{B}$, respectively, and the map $\operatorname{Spec}\widehat{A} \to \operatorname{Spec}\widehat{B}$ is equivariant with respect to the map $G_x \to G_{f(x)}$. The completion $\widehat{A^G}$ of A^G at the image of x is isomorphic to $\widehat{A}^{G_x}$ (Exercise 4.2.10) and similarly $\widehat{B^G} = \widehat{B}^{G_{f(x)}}$. Since f is étale at x, $\widehat{B} \to \widehat{A}$ is an isomorphism and since $G_x \to G_{f(x)}$ is bijective, the induced map $\widehat{B^G} \to \widehat{A^G}$ is an isomorphism which shows that $\operatorname{Spec} A^G \to \operatorname{Spec} B^G$ is étale at the image of x.

4.3.3 The Keel–Mori Theorem

We now state and prove the Keel–Mori Theorem.

Theorem 4.3.12. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_*\mathcal{O}_\mathcal{X}$ such that*

- *(1) X is separated and of finite type over S,*
- *(2) π is a proper universal homeomorphism, and*
- *(3) for every flat morphism $X' \to X$ of algebraic spaces, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space.*

Remark 4.3.13. The Keel–Mori Theorem [KM97] holds more generally with the 'separated' condition on $\mathcal{X} \to S$ by the finiteness of the inertia $I_\mathcal{X} \to \mathcal{X}$; see Remark 4.3.15. In particular, it holds for algebraic stacks with finite but *non-reduced* automorphism groups. The theorem also holds without any noetherian or finiteness conditions; see [Con05b, Ryd13] and [SP, Tag 0DUK].

Proof. We first handle the case when $S = \operatorname{Spec} R$ is affine. The question is Zariski-local on $\mathcal{X}$: if $\{\mathcal{X}_i\}$ is a Zariski open covering of $\mathcal{X}$ with coarse moduli spaces $\mathcal{X}_i \to X_i$, then since coarse moduli spaces are unique (Definition 4.3.1(2)), the X_i's glue to form an algebraic space X and a map $\mathcal{X} \to X$, which is a coarse moduli space by Lemma 4.3.5. It thus suffices to show that every closed point $x \in |\mathcal{X}|$ has an open neighborhood that admits a coarse moduli space.

By the Local Structure Theorem of Deligne–Mumford Stacks (4.2.12), there exists an affine étale morphism

$$f\colon \big(\mathcal{W} = [\operatorname{Spec} A/G_x], w\big) \to (\mathcal{X}, x)$$

such that f induces an isomorphism of geometric stabilizer groups at w.

We claim that since $\mathcal{X}$ is separated, the locus $\mathcal{U}$ consisting of points $z \in |\mathcal{W}|$, such that f induces an isomorphism of geometric stabilizer groups at z, is open. To establish this, we will analyze the natural morphism $I_\mathcal{W} \to I_\mathcal{X} \times_\mathcal{X} \mathcal{W}$ of relative group schemes over $\mathcal{W}$ as the fiber of this morphism over $z \in \mathcal{W}(\Bbbk)$ is precisely the morphism $G_z \to G_{f(z)}$ of stabilizers. We will exploit the cartesian diagram

$$
\begin{array}{ccc}
I_\mathcal{W} & \overset{\Psi}{\longrightarrow} & I_\mathcal{X} \times_\mathcal{X} \mathcal{W} \\
\downarrow & \square & \downarrow \\
\mathcal{W} & \longrightarrow & \mathcal{W} \times_\mathcal{X} \mathcal{W};
\end{array}
$$

see Exercise 3.2.14. Since $\mathcal{W} \to \mathcal{X}$ is representable, étale and separated, the diagonal $\mathcal{W} \to \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ is an open and closed immersion and thus so is Ψ. Since $I_{\mathcal{X}} \to \mathcal{X}$ is finite, so is $p_2 \colon I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W} \to \mathcal{W}$. Thus $p_2(|I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}| \setminus |I_{\mathcal{W}}|) \subset |\mathcal{W}|$ is closed and its complement, which is identified with the locus $\mathcal{U}$, is open.

Let $\pi_{\mathcal{W}} \colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ be the coarse moduli space (Theorem 4.3.6). Choose an affine open subscheme $X_1 \subset W$ containing $\pi_{\mathcal{W}}(w)$. Then $\mathcal{X}_1 = \pi_{\mathcal{W}}^{-1}(X_1)$ is isomorphic to a quotient stack $[\operatorname{Spec} A_1 / G_x]$ such that $X_1 = \operatorname{Spec} A_1^{G_x}$. This provides an affine étale morphism

$$g \colon (\mathcal{X}_1 = [\operatorname{Spec} A_1 / G_x], w) \to (\mathcal{X}, x)$$

which induces a bijection on all geometric stabilizer groups.

We now show that the open substack $\mathcal{X}_0 := \operatorname{im}(f)$ admits a coarse moduli space. Since g is affine and $\mathcal{X}_1 = [\operatorname{Spec} A_1 / G_x]$, the algebraic stack $\mathcal{X}_2 := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$ is isomorphic to $[\operatorname{Spec} A_2 / G_x]$ for an A_1-algebra A_2, and there is a coarse moduli space $\pi_2 \colon \mathcal{X}_2 \to X_2 = \operatorname{Spec} A_2^{G_x}$. By universality of coarse moduli spaces, there is a diagram

$$
\begin{array}{ccccc}
\mathcal{X}_2 & \rightrightarrows & \mathcal{X}_1 & \xrightarrow{\; g \;} & \mathcal{X}_0 = \operatorname{im}(g) \\
\big\downarrow{\scriptstyle \pi_2} & & \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_0} \\
X_2 & \rightrightarrows & X_1 & \dashrightarrow & X_0
\end{array}
\qquad (4.3.14)
$$

where the natural squares commute. Since g induces bijections of geometric stabilizer groups at all points, the same is true for each projection $\mathcal{X}_2 \to \mathcal{X}_1$. Corollary 4.3.10 implies that each map $X_2 \to X_1$ is étale, and the natural squares of solid arrows in (4.3.14) are cartesian.

The universality of coarse moduli spaces induces an étale groupoid structure $X_2 \rightrightarrows X_1$. To check that this is an étale equivalence relation, it suffices to check that $X_2 \to X_1 \times X_1$ is injective on geometric points, but this follows from the observation the $|\mathcal{X}_2| \to |\mathcal{X}_1| \times |\mathcal{X}_1|$ is injective on closed points. Therefore there is an algebraic space quotient $X_0 := X_1 / X_2$ and a map $X_1 \to X_0$. By étale descent along $\mathcal{X}_1 \to \mathcal{X}_0$, there is a map $\pi_0 \colon \mathcal{X}_0 \to X_0$ making the right square in (4.3.14) commute.

To argue that $\pi \colon \mathcal{X}_0 \to X_0$ is a coarse moduli space, we will use the commutative cube

where the top, left, and bottom faces are cartesian. It follows from étale descent along $\mathcal{X}_1 \to \mathcal{X}_0$ that the right face is also cartesian and since being a coarse moduli space is étale local on X_0 (Lemma 4.3.5), we conclude that $\mathcal{X}_0 \to X_0$ is a coarse moduli space. Except for the separatedness, the additional properties in the statement are étale-local on X_0, so they follow from the analogous properties of the coarse moduli

space $[\mathrm{Spec}(A_1)/G_x] \to \mathrm{Spec}(A_1^{G_x})$ from Theorem 4.3.6. As $\mathcal{X}_0 \to X_0$ is proper, the separatedness of $\mathcal{X}_0$ is equivalent to the separatedness of X_0.

Finally, the case when S is a noetherian algebraic space can be reduced to the affine case by imitating the above argument to étale locally construct the coarse moduli space of $\mathcal{X}$. $\qquad\square$

Remark 4.3.15. The more general case when $\mathcal{X}$ is an algebraic stack with finite inertia $I_{\mathcal{X}} \to \mathcal{X}$ (see Remark 4.3.13) is proven in an analogous but more technical manner. Namely, the use of the Local Structure Theorem for Deligne–Mumford stacks (Theorem 4.2.12) is replaced by the existence of an étale neighborhood $\mathcal{W} \to \mathcal{X}$ around every closed point such that $\mathcal{W}$ admits a finite flat presentation $V \to \mathcal{W}$ from an affine scheme and the corresponding groupoid $R := V \times_{\mathcal{W}} V \rightrightarrows V$ is a finite flat groupoid of affine schemes. This in turn is proven in an analogous way to Theorem 4.2.12 where one chooses a quasi-finite and flat surjection $U \to \mathcal{X}$ and one replaces the use of $[(U/\mathcal{X})_0^d/S_d])$ with a Hilbert stack $\mathcal{H}$ whose objects over a scheme S consists of a morphism $S \to \mathcal{X}$ and a closed subscheme $Z \hookrightarrow U_S$ finite and flat (rather than finite and étale) over S. (Aside: it is also possible to prove this without reference to a Hilbert scheme by using étale localization of groupoidsand splitting for groupoids; see [KM97, §4] or [SP, Tags 0DU4 and 04RJ]. Finally, the existence of a coarse moduli space for quotients $[V/R]$ is proven analogously to Theorem 4.3.6 (see Exercise 4.2.11).

The Local Structure Theorem of Deligne–Mumford Stacks (Theorem 4.2.12) can also be formulated étale locally on a coarse moduli space:

Corollary 4.3.16 (Local Structure of Coarse Moduli Spaces)**.** *Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space S, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. For every closed point $x \in |\mathcal{X}|$ with geometric stabilizer group G_x, there exists a cartesian diagram*

$$
\begin{array}{ccc}
[\mathrm{Spec}\, A/G_x] & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathrm{Spec}\, A^{G_x} & \longrightarrow & X
\end{array}
$$

such that $\mathrm{Spec}\, A^{G_x} \to X$ is an étale neighborhood of $\pi(x) \in |X|$.

Proof. This follows from the construction of the coarse moduli space in the proof of Theorem 4.3.12. Alternatively, it follows from the Local Structure Theorem of Deligne–Mumford stacks (Theorem 4.2.12) and Exercise 4.3.17. $\qquad\square$

Exercise 4.3.17. Establish the following generalization of Proposition 4.3.7: Let S be a noetherian algebraic space. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks separated and of finite type over S and

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\ f\ } & \mathcal{Y} \\
\downarrow{\scriptstyle \pi_{\mathcal{X}}} & & \downarrow{\scriptstyle \pi_{\mathcal{Y}}} \\
X & \longrightarrow & Y
\end{array}
$$

be a commutative diagram where $\pi_{\mathcal{X}} \colon \mathcal{X} \to X$ and $\pi_{\mathcal{Y}} \colon \mathcal{Y} \to Y$ are coarse moduli spaces. Let $x \in |\mathcal{X}|$ be a closed point such that

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(1) f is étale at x and

(2) the induced map $G_x \to G_{f(x)}$ of geometric stabilizer groups is bijective.

Then there exists an open neighborhood $U \subset X$ of $\pi_{\mathcal{X}}(x)$ such that $U \to X \to Y$ is étale and $\pi_{\mathcal{X}}(U) \cong U \times_Y \mathcal{Y}$.

4.3.4 Examples

Example 4.3.18. Consider the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves over a field $\Bbbk$ with $\operatorname{char}(\Bbbk) \neq 2, 3$. The Weierstrass form $y^2 = x(x-1)(x-\lambda)$ gives an isomorphism $\mathcal{M}_{1,1} \cong [(\mathbb{A}^1 \setminus \{0,1\})/S_3]$ (see Exercise 3.1.19) where the S_3-orbit of λ is $\{\lambda, 1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(\lambda-1), (\lambda-1)/\lambda\}$. The coarse moduli space is given by j-invariant

$$j \colon \mathcal{M}_{1,1} \to \mathbb{A}^1, \quad \lambda \mapsto 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^3}.$$

Indeed, one can verify $\Bbbk[\lambda]_{\lambda(\lambda-1)}^{S_3} = \Bbbk[j(\lambda)]$.

Alternatively, the Weierstrass form $y^2 = x^3 + ax + b$ gives an isomorphism $\mathcal{M}_{1,1} \cong [\mathbb{A}^2_\Delta/\mathbb{G}_m]$ (see Exercise 3.1.19(b)) where the action is given by $t \cdot (a,b) = (t^4 a, t^6 b)$ and Δ is the discriminant $4a^3 + 27b^2$. As $\Bbbk[a,b]_\Delta^{\mathbb{G}_m} = \Bbbk[a^3/\Delta]$ (noting that $\beta := b^2/\Delta$ is generated by $\alpha := a^3/\Delta$ under the relation $4\alpha + 27\beta = 1$), the coarse moduli space $\mathcal{M}_{1,1} \to \mathbb{A}^1$ is given by $(a,b) \mapsto a^3/\Delta$.

Exercise 4.3.19. (hard) Let $\operatorname{char}(\Bbbk) \neq 2$ and $G = \mathbb{Z}/2$.

(a) Let G act on the non-separated union $X = \mathbb{A}^1 \bigcup_{x \neq 0} \mathbb{A}^1$ by exchanging the copies of $\mathbb{A}^1$. The quotient $[X/G]$ is a Deligne–Mumford stack with quasi-finite but not finite inertia, and in particular non-separated. Show nevertheless that there is a coarse moduli space $[X/G] \to \mathbb{A}^1$.

(b) Let X be the non-separated union $\mathbb{A}^2 \bigcup_{x \neq 0} \mathbb{A}^2$. Let $G = \mathbb{Z}/2$ act on X by simultaneously exchanging the copies of $\mathbb{A}^2$ and by acting via the involution $y \mapsto -y$ on each copy. Show that $[X/G]$ does not admit a coarse moduli space.

Example 4.3.20. Consider the action of PGL_2 on the scheme $\operatorname{Sym}^4 \mathbb{P}^1 \cong (\mathbb{P}^1)^4/S_4$ (which is the coarse moduli space of $[(\mathbb{P}^1)^4/S_4]$) parameterizing four unordered points in $\mathbb{P}^1$. Let $\mathcal{X} \subset [\operatorname{Sym}^4 \mathbb{P}^1 / \operatorname{PGL}_2]$ be the open substack parameterizing tuples (p_1, p_2, p_3, p_4) where at least three points are distinct. Consider the family $(0, 1, \lambda, \infty)$ with $\lambda \in \mathbb{P}^1$. If $\lambda \notin \{0, 1\infty\}$, then we claim that $\operatorname{Aut}(0, 1, \lambda, \infty) = \mathbb{Z}/2 \times \mathbb{Z}/2$. To see this, there is a unique element $\sigma \in \operatorname{PGL}_2$ such that $\sigma(0) = \infty$, $\sigma(\infty) = 0$ and $\sigma(1) = \lambda$ which acts on $\mathbb{P}^1$ via $\sigma([x,y]) = [y, \lambda, x]$ and thus $\sigma(\lambda) = 1$. Similarly, there is an element interchanging 0 with 1 and λ with ∞ and an element interchanging 0 with λ and 1 with ∞. However, if $\lambda \in \{0, 1\infty\}$, then $\operatorname{Aut}(0, 1, \lambda, \infty) = \mathbb{Z}/2$. We therefore see that the inertia $I_{\mathcal{X}} \to \mathcal{X}$ while quasi-finite is not finite and that $\mathcal{X}$ is not separated. Nevertheless, the map $\mathcal{X} \to \mathbb{P}^1$ taking (p_1, p_2, p_3, p_4) to its cross-ratio is a coarse moduli space.

Exercise 4.3.21. Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space S, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space.

(a) Show that if $\mathcal{X}$ is normal, then so is X.

(b) If in addition $\mathcal{X}$ is regular, show that $\mathcal{X} \to X$ is flat if and only if X is regular.

(c) Provide an example of a coarse moduli space $\mathcal{X} \to X$ that is *not* flat.

4.3.5 Tame coarse moduli spaces

Definition 4.3.22 (Tame stacks)**.** A Deligne–Mumford stack $\mathcal{X}$ is *tame* if for every geometric point $x \in \mathcal{X}(\Bbbk)$, the order of $\mathrm{Aut}_{\mathcal{X}(\Bbbk)}(x)$ is invertible in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If in addition $\mathcal{X}$ admits a coarse moduli space $\mathcal{X} \to X$, then we say that $\mathcal{X} \to X$ is a *tame coarse moduli space*.

Remark 4.3.23. If $\mathcal{X}$ is defined over a field $\Bbbk$, then this means that the order of every geometric stabilizer group is prime to the characteristic of $\Bbbk$.

Lemma 4.3.24. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. If $\pi \colon \mathcal{X} \to X$ is a tame coarse moduli space, then π_* is exact.*

Proof. The question is étale-local on X: if $g \colon X' \to X$ is an étale cover inducing a cartesian diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\ g'\ } & \mathcal{X} \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\ g\ ]{} & X \end{array}$$

then by Flat Base Change (4.1.38) there is an identification $g^* \pi_* = \pi'_* g'^*$ of functors on quasi-coherent sheaves. Since g^* is faithfully exact, we see that π_* is exact if and only π'_* is. We can therefore use Corollary 4.3.16 to reduce to the case that $\mathcal{X} = [\mathrm{Spec}\, A/G]$ and $X = \mathrm{Spec}\, A^G$, and this case follows from Exercise 4.2.5. $\qquad\square$

Exercise 4.3.25. If $\mathcal{X}$ is a tame and proper Deligne–Mumford stack over a noetherian ring A, show that $\mathrm{H}^i(\mathcal{X}, F) = 0$ for all $i > \dim \mathcal{X}$ and coherent sheaves F.

Exercise 4.3.26 (Base change)**.** Let $\mathcal{X}$ be a Deligne–Mumford stack of finite type and separated over a noetherian algebraic space S. If $\pi \colon \mathcal{X} \to X$ is a *tame* coarse moduli space and $X' \to X$ is a morphism of algebraic spaces, show that $\mathcal{X} \times_X X' \to X'$ is also a coarse moduli space.

On the other hand, Exercise 4.2.5(c) provides an example of a coarse moduli space $\mathcal{X} \to X$ and a map $X' \to X$ such that $\mathcal{X} \times_X X' \to X'$ is *not* a coarse moduli space. In particular, it is does not automatically follow that $\overline{M}_g \times_{\mathbb{Z}} \mathbb{F}_p$ is the coarse moduli space of $\overline{\mathcal{M}}_g \times_{\mathbb{Z}} \mathbb{F}_p$; see Question 5.5.18.

4.3.6 Descending vector bundles to the coarse moduli space

We begin with a Nakayama lemma for coherent sheaves.

Lemma 4.3.27. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. Let $x \in |\mathcal{X}|$ be a closed point.*

(1) If F is a coherent sheaf on $\mathcal{X}$ such that $F|_{\mathcal{G}_x} = 0$, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $F|_{\pi^{-1}(U)} = 0$.

(2) If $\phi \colon F \to G$ is a morphism of coherent sheaves (resp. vector bundles of the same rank) on $\mathcal{X}$ such that $\phi|_{\mathcal{G}_x}$ is surjective, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is surjective (resp. an isomorphism).

Proof. For (1), the support $\mathrm{Supp}(F) \subset |\mathcal{X}|$ of F is a closed subset (which follows from using descent along a presentation) and the open set $U = X \setminus \pi(\mathrm{Supp}(F))$ satisfies the conclusion. For (2), apply (1) to the coherent sheaf $\mathrm{coker}(\phi)$ noting that a surjection of vector bundles of the same rank is an isomorphism. $\qquad\square$

We say that a vector bundle F on $\mathcal{X}$ *descends to its coarse moduli space* $\pi\colon \mathcal{X} \to X$ if there exists a vector bundle $\overline{F}$ on X and an isomorphism $F \cong \pi^*\overline{F}$. Observe that one necessary condition is that for every field-valued point $x\colon \mathrm{Spec}\,\Bbbk \to \mathcal{X}$, which induces a commutative diagram

$$
\begin{array}{ccc}
BG_x & \overset{i_x}{\hookrightarrow} & \mathcal{X} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathrm{Spec}\,\Bbbk & \hookrightarrow & X,
\end{array}
$$

the pullback $i_x^* F = p^*(\overline{F} \otimes \Bbbk)$ is trivial or in other words G_x acts trivially on the fiber $F \otimes \Bbbk$.

Proposition 4.3.28. *Let $\mathcal{X}$ be a tame Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi\colon \mathcal{X} \to X$ be its coarse moduli space. A vector bundle F on $\mathcal{X}$ descends to a vector bundle on X if and only if for every field-valued point $x\colon \mathrm{Spec}\,\Bbbk \to \mathcal{X}$ with closed image, the action of G_x on the fiber $F \otimes \Bbbk$ is trivial. In this case, $\pi_* F$ is a vector bundle and the adjunction map $\pi^* \pi_* F \to F$ is an isomorphism.*

Remark 4.3.29. The above condition is insensitive to field extensions and equivalent to the condition that the restriction of F to the residual gerbe is trivial.

Proof. To see that the condition is sufficient, consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{G}_x & \hookrightarrow & \mathcal{X} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathrm{Spec}\,\kappa(x) & \hookrightarrow & X.
\end{array}
$$

We break down the proof into three steps.

Step 1: $\pi^* \pi_* F \to F$ is surjective. It suffices by Lemma 4.3.27 to show that $(\pi^* \pi_* F)|_{\mathcal{G}_x} \to F|_{\mathcal{G}_x}$ is surjective for every closed point $x \in |\mathcal{X}|$. Since $F \to F|_{\mathcal{G}_x}$ is surjective and π_* is exact (Lemma 4.3.24), $(\pi^* \pi_* F)|_{\mathcal{G}_x} \to \pi^*(\pi_*(F|_{\mathcal{G}_x}))|_{\mathcal{G}_x} \cong p^* p_*(F|_{\mathcal{G}_x})$ is surjective. The hypotheses imply that the adjunction $p^* p_*(F|_{\mathcal{G}_x}) \to F|_{\mathcal{G}_x}$ is an isomorphism and it follows that the composition $(\pi^* \pi_* F)|_{\mathcal{G}_x} \to p^* p_*(F|_{\mathcal{G}_x}) \overset{\sim}{\to} F|_{\mathcal{G}_x}$ is surjective.

Step 2: $\pi_* F$ is a vector bundle. We can assume that the rank r of F is constant. Since being a vector bundle is an étale-local property, we can assume that $X = \mathrm{Spec}\,A$. The surjection $\bigoplus_{s \in \Gamma(X, \pi_* F)} A \to \pi_* F$ pulls back to a surjection $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^* \pi_* F$ and by Step 1, the composition $\bigoplus_{s \in \Gamma(\mathcal{X}, F)} \mathcal{O}_{\mathcal{X}} \to \pi^* \pi_* F \to F$ is surjective. As $F|_{\mathcal{G}_x} \cong \mathcal{O}_{\mathcal{G}_x}^r$ is trivial, for each closed point $x \in |\mathcal{X}|$, we can find r sections $\phi\colon \mathcal{O}_{\mathcal{X}}^r \to F$ such that $\phi|_{\mathcal{G}_x}$ is an isomorphism. By Lemma 4.3.27, there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is an isomorphism. Thus $\pi_* \phi\colon \mathcal{O}_X^r \to \pi_* F$ is an isomorphism over U and we conclude that $\pi_* F$ is a vector bundle of the same rank as F.

Step 3: $\pi^* \pi_* F \to F$ is an isomorphism. Since $\pi^* \pi_* F \to F$ is a surjection of vector bundles of the same rank, it is an isomorphism. $\qquad\square$

Remark 4.3.30. The analogous statement for coherent sheaves is not true. For example, if the characteristic is not 2, then letting $\mathbb{Z}/2$ on $\mathbb{A}^1$ via $x \mapsto -x$, we have a tame coarse moduli space $[\mathbb{A}^1/(\mathbb{Z}/2)] \to \mathbb{A}^1 = \operatorname{Spec} \operatorname{Spec} k[x^2]$. The inclusion $B\mathbb{Z}/2 \hookrightarrow [\mathbb{A}^1/(\mathbb{Z}/2)]$ of 0 is a closed substack and $\mathcal{O}_{B\mathbb{Z}/2}$ is a coherent sheaf which does not descend. Observe that in this case, the pullback of the residue field of $0 \in \mathbb{A}^1$ is $k[x]/x^2$. This example also illustrated that the fibers of a coarse moduli space $\mathcal{X} \to X$ can be non-reduced and larger than the residual gerbe.

When $\mathcal{X}$ is not tame, we have the following variant for descending line bundles.

Proposition 4.3.31. *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S, and let $\pi \colon \mathcal{X} \to X$ be its coarse moduli space. If $\mathcal{L}$ is a line bundle on $\mathcal{X}$, then for N sufficiently divisible $\mathcal{L}^{\otimes N}$ descends to X.*

Proof. To be added. $\qquad\square$

Example 4.3.32. Show $\operatorname{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$ generated by the Hodge bundle (see Example 4.1.4).

4.4 When are algebraic spaces schemes?

We prove various results providing conditions for an algebraic space to be a scheme. We show:

- a quasi-separated algebraic space is a scheme on a dense open subspace (Theorem 4.4.1);

- Zariski's Main Theorem for algebraic spaces (Theorem 4.4.9);

- an algebraic space separated and locally quasi-finite over a scheme is a scheme (Corollary 4.4.7);

- if the diagonal of a Deligne–Mumford stack is separated and quasi-compact diagonal, then the diagonal is quasi-affine (and in particular representable by schemes) (Corollary 4.4.8);

- an algebraic stack with trivial stabilizers is an algebraic space (Theorem 4.4.10) generalizing Theorem 3.6.6;

- Serre's and Chevalley's Criteria for Affineness (Theorems 4.4.16 and 4.4.20) for algebraic spaces;

- if X is a quasi-separated algebraic space locally of finite type over a field $\mathbb{k}$ such that $X_{\bar{\mathbb{k}}}$ has the property that every finite set of points is contained in an affine (e.g. $X_{\bar{\mathbb{k}}}$ is quasi-projective), then X is a scheme (Proposition 4.4.27);

- quasi-separated group algebraic spaces locally of finite type over a field are schemes (Theorem 4.4.28); and

- separated one-dimensional algebraic spaces are schemes (Theorem 4.4.32)

We also give applications to the algebraicity of quotients of étale and smooth equivalence relations (Corollary 4.4.12).

4.4.1 Algebraic spaces are schemes over a dense open

Theorem 4.4.1. *Every quasi-separated algebraic space has a dense open subspace which is a scheme.*

Proof. We may assume that X is quasi-compact. Let $f\colon V \to X$ be an étale presentation with V an affine scheme. Since X is quasi-separated, $f\colon V \to X$ is quasi-compact and there exists an open algebraic subspace $U \subset X$ such that $f^{-1}(U) \to U$ is finite. By Exercise 4.2.13, U is isomorphic to a quotient stack $[V/G]$ for the free action of a finite abstract group G on a scheme V. If $V_1 \subset V$ is a dense affine open subscheme, then $V_2 = \bigcap_{g \in G} gV_1$ is a G-invariant quasi-affine open subscheme of V and in particular separated. Repeating this argument, we can choose a dense affine open subscheme $V_3 \subset V_2$ and now $V_4 = \bigcap_{g \in G} gV_3$ is a G-invariant affine open subscheme. Proposition 4.2.14 implies that $V_4/G \cong \operatorname{Spec} A^G$ is a dense affine open algebraic subspace of U. $\qquad\square$

Remark 4.4.2. See also [Knu71, II.6.7] and [SP, Tag 06NN]. The above result is not necessarily true if X is not quasi-separated, e.g. $\mathbb{A}^1/\underline{\mathbb{Z}}$ (Example 3.9.22).

Corollary 4.4.3. *An integral quasi-separated algebraic space has a well-defined fraction field.* $\qquad\square$

Exercise 4.4.4. Let G be a finite abstract group acting on a quasi-separated algebraic space U. Show that there is a G-invariant affine open subscheme of U.

4.4.2 Zariski's Main Theorem for algebraic spaces

We prove Zariski's Main Theorem (4.4.9) for algebraic spaces and Deligne–Mumford stacks, which has the important application that separated and locally quasi-finite morphisms of algebraic spaces are representable by schemes (Corollary 4.4.7). Its proof relies on the theory of quasi-coherent sheaves, and, specifically, on the factorization

$$f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$$

of a quasi-compact and quasi-separated morphism, which relies on the quasi-coherence of $f_*\mathcal{O}_X$ (Exercise 4.1.19). See Definition 3.3.32 for the definition of a quasi-finite morphism of algebraic spaces and §A.7 for a discussion of Zariski's Main Theorem for schemes.

Proposition 4.4.5. *A separated, quasi-finite, and representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks factors as the composition of an open immersion $\mathcal{X} \hookrightarrow \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}}$ and an affine morphism $\mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. In particular, f is quasi-affine.*

Proof. Since the construction of $f_*\mathcal{O}_{\mathcal{X}}$ commutes with flat base change on $\mathcal{Y}$ (Exercise 4.1.38), so does the formation of the factorization $f\colon \mathcal{X} \to \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\, f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. The statement is thus étale-local on $\mathcal{Y}$. In particular, we can assume that $\mathcal{Y} = Y$ is an affine scheme and that $\mathcal{X} = X$ is an algebraic space. After replacing Y with $\mathcal{S}\mathrm{pec}_Y\, f_*\mathcal{O}_X$, we can assume that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and we must show that $f\colon X \to Y$ is an open immersion.

Since X is quasi-compact, there is an étale presentation $\pi\colon U \to X$ from an affine scheme. Since X is separated, $U \to X$ is also separated. As the composition

$$U \xrightarrow{\pi} X \xrightarrow{f} Y$$

is a quasi-finite morphism of schemes, we can apply Étale Localization of Quasi-finite Morphisms (Theorem A.7.1) around every point $y \in Y$: after replacing Y with an étale neighborhood, we can assume that $U = U_1 \amalg U_2$ with $U_1 \to Y$ finite and $(U_2)_y = \emptyset$. Then $\pi(U_1)$ is open (as π is étale) and closed (as $U_1 \to Y$ is finite and $X \to Y$ is separated). Thus $X = X_1 \amalg X_2$ with $X_1 = \pi(U_1)$ and $(X_2)_y = \emptyset$. This shows that $\mathcal{O}_Y = f_*\mathcal{O}_X$ is the product $\mathcal{A}_1 \times \mathcal{A}_2$ of quasi-coherent $\mathcal{O}_X$-algebras, and thus we can also decompose Y as $Y_1 \amalg Y_2$ such that $y \in Y_1$ and $f(Y_i) \subset X_i$ for $i = 1, 2$. After replacing Y with Y_1, the composition $U \to X \to Y$ is finite and Lemma 4.4.6 implies that X is affine. Thus $X = Y = \mathcal{S}pec_Y f_*\mathcal{O}_X$. $\qquad\square$

Lemma 4.4.6. *Suppose that $U \to X$ is a surjective étale morphism of algebraic spaces and that $X \to Y$ is a separated morphism of algebraic spaces. If the composition $U \to X \to Y$ is finite, so is $X \to Y$.*

Proof. The statement is étale-local on Y so we can assume that Y and U are affine. As $X \to Y$ is separated, $U \to X$ is also finite. Since X is identified with the quotient U/R of the finite étale equivalence relation $R := U \times_X U \rightrightarrows U$ of affine schemes, Proposition 4.2.14 implies that X is affine. As $U \to Y$ is proper, so is $X \to Y$. As $X \to Y$ is a proper and quasi-finite morphism of schemes, it is finite (Corollary A.7.5). $\qquad\square$

Corollary 4.4.7. *A morphism of algebraic spaces which is separated and locally quasi-finite is representable by schemes. In particular, an algebraic space separated and locally quasi-finite over a scheme is a scheme.*

Proof. It suffices to show that if $X \to Y = \operatorname{Spec} A$ is a separated and locally quasi-finite, then X is a scheme. Since being a scheme is a Zariski-local property, we can assume that X is quasi-compact. In this case, Proposition 4.4.5 applies. $\qquad\square$

Corollary 4.4.8. *The diagonal of a Deligne–Mumford stack with separated and quasi-compact diagonal is quasi-affine. In particular, a quasi-separated algebraic space has quasi-affine diagonal.*

Proof. Since the diagonal is separated, quasi-finite, and representable, Proposition 4.4.5 applies. Note that the diagonal of an algebraic space is a monomorphism, hence separated. $\qquad\square$

As in the case for schemes, we can refine Proposition 4.4.5 to obtain Zariski's Main Theorem.

Theorem 4.4.9 (Zariski's Main Theorem)**.** *A separated, quasi-finite, and representable morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of noetherian Deligne–Mumford stacks factors as the composition of a dense open immersion $\mathcal{X} \hookrightarrow \widetilde{\mathcal{Y}}$ and a finite morphism $\widetilde{\mathcal{Y}} \to \mathcal{X}$.*

Proof. Let $\mathcal{A} \subset f_*\mathcal{O}_\mathcal{X}$ be the integral closure of $\mathcal{O}_\mathcal{Y} \to f_*\mathcal{O}_\mathcal{X}$: for a map $T \to \mathcal{Y}$ from a scheme, $\Gamma(T \to \mathcal{Y}, \mathcal{A})$ is the integral closure of $\Gamma(T, \mathcal{O}_T) \to \Gamma(\mathcal{X} \times_\mathcal{Y} T, \mathcal{O}_{\mathcal{X} \times_\mathcal{Y} T})$. Since the integral closure is compatible with étale extensions (Proposition A.7.4), $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_\mathcal{Y}$-algebras. Using Exercise 4.1.26, write $\mathcal{A} = \operatorname{colim} \mathcal{A}_\lambda$ as the colimit of finite type $\mathcal{O}_\mathcal{Y}$-algebras. As $\mathcal{Y}$ is quasi-compact, there exists an étale presentation $p \colon U \to \mathcal{Y}$ from an affine scheme. Then the base change $\mathcal{X}_U \to U$ is a separated and quasi-finite morphism of algebraic spaces, thus a morphism of schemes by Corollary 4.4.7. Since $p^*\mathcal{A} = \operatorname{colim} p^*\mathcal{A}_\lambda$ is identified with the integral closure of $\mathcal{O}_U \to f_{U,*}\mathcal{O}_{\mathcal{X}_U}$, by Zariski's Main Theorem (A.7) for schemes, it follows that $\mathcal{X}_U \to \mathcal{S}pec_U p^*\mathcal{A}_\lambda$ is an open immersion and $\mathcal{S}pec_U p^*\mathcal{A}_\lambda \to U$ is finite for

$\lambda \gg 0$. By étale descent, $\mathcal{X} \to \mathcal{S}\mathrm{pec}_{\mathcal{Y}}\,\mathcal{A}_{\lambda}$ is an open immersion and $\mathcal{S}\mathrm{pec}_{\mathcal{Y}}\,\mathcal{A}_{\lambda} \to \mathcal{Y}$ is finite. See also [Knu71, II.6.15], [LMB00, Thm. A.2], [Ols16, Thm. 7.2.10], and [SP, Tag 05W7]. □

4.4.3 Characterization of algebraic spaces

We now can remove the hypothesis in Theorem 3.6.6 that the diagonal is representable by schemes.

Theorem 4.4.10 (Characterization of Algebraic Spaces II). *For an algebraic stack $\mathcal{X}$, the following are equivalent:*

(1) *the stack $\mathcal{X}$ is an algebraic space,*

(2) *the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism, and*

(3) *every point of $\mathcal{X}$ has a trivial stabilizer.*

Proof. We only need to show (2) $\Rightarrow$ (1). As the diagonal of $\mathcal{X}$ is a monomorphism, it is separated and locally quasi-finite. Corollary 4.4.7 implies that the diagonal $\mathcal{X}$ is representable by schemes and thus Theorem 3.6.6 applies. □

This allows us to prove a more general version of Corollary 3.6.9.

Corollary 4.4.11 (Characterization of Representable Morphisms II). *A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is representable if and only if for every geometric point $x \in \mathcal{X}(\Bbbk)$, the map $G_x \to G_{f(x)}$ on automorphism groups is injective.* □

Corollary 4.4.12.

(1) *If X is a sheaf on $\mathrm{Sch}_{\text{ét}}$ such that there exists a surjective, étale (resp. smooth), and representable morphism $U \to X$ from an algebraic space, then X is an algebraic space.*

(2) *If $R \rightrightarrows U$ is an étale (resp. smooth) equivalence relation of algebraic spaces, then the quotient U/R is an algebraic space.*

Remark 4.4.13. The above statement holds with with 'étale' replaced with 'fppf'; see Theorem 6.2.1 and Corollary 6.2.4.

Proof. We first handle the étale case. For (1), by taking an étale presentation of U by a scheme, we may assume that U is a scheme. Let $T \to X$ be a morphism from a scheme, and we must show that the algebraic space $U \times_X T$ is a scheme. Since $U \times_X T \to U \times T$ is the base change of $X \to X \times X$, it is a monomorphism, thus separated and locally quasi-finite. By Corollary 4.4.7, $U \times_X T$ is a scheme. For (2), let $X = U/R$ be the quotient sheaf. By copying the argument of Theorem 3.4.11(1), we see that $U \to X$ is representable. The statement then follows from (1). Alternatively, Theorem 3.4.11(1) implies that U/R is an algebraic stack and the statement follows from Theorem 4.4.10.

In the noetherian and smooth case, the sheaf X in (1) is an algebraic stack by definiton and the quotient stack $[U/R]$ is an algebraic stack by Theorem 3.4.11. Theorem 4.4.10 implies that X and $[U/R]$ are algebraic spaces. □

Corollary 4.4.14. *A proper and quasi-finite morphism (resp. proper monomorphism) of algebraic spaces is finite (resp. a closed immersion).*

Proof. Proper and quasi-finite morphisms are representable by schemes. Thus the statement follows from the corresponding result for schemes (Corollary A.7.5) and étale descent. □

Exercise 4.4.15. Consider the prestack $\underline{\mathrm{AlgSp}}$ over $\mathrm{Sch}_{\text{ét}}$ whose objects over a scheme T are algebraic spaces over T and where morphisms correspond to cartesian diagrams of algebraic spaces. Show that $\underline{\mathrm{AlgSp}}$ is a stack.

4.4.4 Affineness criteria

Theorem 4.4.16 (Serre's Criterion for Affineness)**.** *Let X be a quasi-compact and quasi-separated (resp. noetherian) algebraic space. If the functor $\Gamma(X, -)$ is exact on the category of quasi-coherent (resp. coherent) sheaves, then X is an affine scheme.*

Proof. If X is noetherian, then every quasi-coherent sheaf is a colimit of coherent sheaves (Exercise 4.1.21) and $\Gamma(X, -)$ commutes with colimits. Assume that $\Gamma(X, -)$ is exact on coherent sheaves. Given a surjection $p\colon F \twoheadrightarrow G$ of quasi-coherent sheaves on X, write $G = \mathrm{colim}_i\, G_i$ as a colimit of coherent sheaves and choose coherent subsheaves $F_i \subset p^{-1}(G_i)$ surjecting onto G_i. Then $\Gamma(X, F_i) \twoheadrightarrow \Gamma(X, G_i)$ and the composition $\mathrm{colim}_i \Gamma(X, F_i) \to \Gamma(X, F) \to \Gamma(X, G) = \mathrm{colim}_i \Gamma(X, G_i)$ is surjective. Thus $\Gamma(X, F) \to \Gamma(X, G)$ is surjective and we conclude that $\Gamma(X, -)$ is exact on quasi-coherent sheaves.

We show that the canonical morphism $\pi\colon X \to Y := \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ is a proper monomorphism. This gives the result as Corollary 4.4.14 implies that $X \to Y$ is a closed immersion so that that X is affine and $X \to Y$ is an isomorphism. As a first step, we establish:

Claim: If $g\colon Y' \to Y$ is a morphism of algebraic spaces, then the base change $\pi'\colon X' := X \times_Y Y' \to Y'$ has the following properties:

(a) *π'_* induces an equivalence of the categories of quasi-coherent sheaves on X' and Y'.*

(b) *$\mathcal{O}_{Y'} \to \pi'_*\mathcal{O}_{X'}$ is an isomorphism.*

(c) *$X' \to Y'$ is a homeomorphism.*

By Flat Base Change (Exercise 4.1.38), properties (a) and (b) are étale local on Y' so we may assume $Y' = \operatorname{Spec} B$. We will show that the adjunction morphisms $G \to \pi'_*\pi'^*G$ and $\pi'^*\pi'_*F \to F$ are isomorphisms for quasi-coherent sheaves G and F and Y' and X', respectively. For the first adjunction, choose a free presentation $\mathcal{O}_Y^{\oplus J} \to \mathcal{O}_Y^{\oplus J} \to g_*G \to 0$ of G as an $\mathcal{O}_Y$-module. As π_* is exact, we have a morphism of right exact sequences

$$
\begin{array}{ccccccc}
\mathcal{O}_Y^{\oplus J} & \longrightarrow & \mathcal{O}_Y^{\oplus I} & \longrightarrow & g_*G & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_*\pi^*(\mathcal{O}_Y^{\oplus J}) & \longrightarrow & \pi_*\pi^*(\mathcal{O}_Y^{\oplus I}) & \longrightarrow & \pi_*\pi^*g_*G & \longrightarrow & 0
\end{array}
$$

The two left vertical arrows are isomorphisms since $\pi_*\mathcal{O}_X = \mathcal{O}_Y$. Therefore $g_*G \to g_*\pi_*\pi^*G \cong g_*\pi'_*\pi'^*G$ is an isomorphism. Since g_* is faithfully exact, $G \to \pi'_*\pi'^*G$ is also an isomorphism. We note that property (b) already follows from this fact by taking $G = \mathcal{O}_{Y'}$ and the fact that affine morphisms are faithfully exact on quasi-coherent sheaves.

To see the second adjunction, let K and Q be the kernel and cokernel of $\pi'^* \pi'_* F \to F$. As π_* is exact and g_* is faithfully exact, we see that π'_* is exact. Since $\pi'_* \pi'^* \pi'_* F \to \pi'_* F$ is an isomorphism (using that the first adjunction is an isomorphism), we see that $\pi'_* K = \pi'_* Q = 0$. It thus suffices to show that for a quasi-coherent sheaf F' on X', then $F' \neq 0$ implies $\pi'_* F' \neq 0$. If $x \colon \operatorname{Spec} \Bbbk \to X'$ is a geometric point such that $x^* F \neq 0$, then by base changing by the composition $\pi' \circ x \colon \operatorname{Spec} \Bbbk \to Y'$, we may assume that $Y' = \operatorname{Spec} \Bbbk$ and that $x \colon \operatorname{Spec} \Bbbk \to X'$ is a section of π'. Since every $\Bbbk$-point of an algebraic space defined over $\Bbbk$ is a closed point, $x \colon \operatorname{Spec} \Bbbk \to X'$ is a closed immersion, and hence $F \to x_* x^* F = F \otimes \Bbbk$ is surjective. It follows from the exactness of π'_* that $\pi'_* F \to F \otimes \Bbbk$ is surjective and hence $\pi'_* F \neq 0$. This finishes the proof of (a) and (b).

To see (c), if $y \colon \operatorname{Spec} \Bbbk \to Y$ is a geometric point, then by (b) $\Gamma(X_y, \mathcal{O}_{X_y}) = \Bbbk$ as thus the fiber X_y is non-empty. On the other hand, if $x, x' \in X_y(\Bbbk)$ were distinct points each necessarily closed, then $\mathcal{O}_{X_y} \to \mathcal{O}_{\{x, x'\}}$ is surjective. Since π_* is exact, we also get a surjection $\Bbbk = \Gamma(X_y, \mathcal{O}_{X_y}) \to \Bbbk \oplus \Bbbk$, a contradiction. To see that π' is closed, let $Z \subset X'$ be a closed subspace and $q \colon Z \to \operatorname{im}(Z)$ denote the morphism to its scheme-theoretic image. Then $\mathcal{O}_Z \to q_* \mathcal{O}_{\operatorname{im}(Z)}$ is an isomorphism and q_* is exact. Applying the surjectivity result above to q, we see that q is surjective, and hence $\pi'(Z)$ is closed.

With the claim established, we now show that $X \to Y$ is a monomorphism and in particular separated. To see that the diagonal $\Delta \colon X \to X \times_Y X$ is an isomorphism, observe that the pushforward of $\mathcal{O}_{X \times_Y X} \to \Delta_* X$ along the first projection $p_1 \colon X \times_Y X \to X$ is an isomorphism. Thus (a) applied to p_1 shows that $\mathcal{O}_{X \times_Y X} \to \Delta_* X$ is an isomorphism. Zariski's Main Theorem (A.7.3) implies that Δ is an open immersion. Applying (c) to p_1 shows that $p_1 \colon |X \times_Y X| \to |X|$ is bijective. Hence Δ must be an isomorphism.

It remains to show that $X \to Y$ is of finite type. Let $U = \operatorname{Spec} A \to X$ be an étale presentation. Since X is separated, $R := U \times_X U$ is a closed subscheme of $U \times_Y U = \operatorname{Spec} A \otimes_{\Gamma(X, \mathcal{O}_X)} A$. Hence $R = \operatorname{Spec} B$ is affine. Letting s and t denote the two maps $A \rightrightarrows B$, we have a commutative diagram

$$
\begin{array}{c}
A \\
\Gamma(X, \mathcal{O}_X) \quad A \otimes_{\Gamma(X, \mathcal{O}_X)} A \xrightarrow{\substack{s \\ t}} B. \\
A
\end{array}
$$

Since $U \to X$ is étale, $t \colon A \to B$ is of finite type and there are generators $b_1, \ldots, b_n \in B$ over t. For each i, choose a preimage $\sum_j a_{ij} \otimes a'_{ij} \in A \otimes_{\Gamma(X, \mathcal{O}_X)} A$ of b_i. Viewing B as an A-algebra via t, then $\sum_j a'_{ij} s(a_{ij}) = b_i$ and thus we have elements $a_{ij} \in A$ such that $s(a_{ij})$ generate B over t. Then $a_{ij} \in \Gamma(X, p_* \mathcal{O}_U) = A$ define a homomorphism $\mathcal{O}_X[z_{ij}] \to p_* \mathcal{O}_U$ of $\mathcal{O}_X$-algebras taking z_{ij} to a_{ij}. Its pullback via p is identified with $\mathcal{O}_U[z_{ij}] \to p^* p_* \mathcal{O}_U \cong t_* \mathcal{O}_R$, where the last equivalence comes from Flat Base Change (Exercise 4.1.38), and this map is surjective precisely because $s(a_{ij})$ generate B over t. By étale descent, $\mathcal{O}_X[z_{ij}] \to p_* \mathcal{O}_U$ is surjective and therefore so is $\Gamma(X, \mathcal{O}_X)[z_{ij}] \to A$. Thus $\Gamma(X, \mathcal{O}_X) \to A$ is of finite type and by étale descent $X \to Y$ is also of finite type.

See also [Knu71, Thm. III.2.5], [Ryd15, Thm. 8.7] and [SP, Tag 07V6]. $\qquad\square$

Corollary 4.4.17. *Let X be a quasi-compact and quasi-separated (resp. noetherian) algebraic space. Then X is an affine scheme if and only if $\mathrm{H}^i(X, F) = 0$ for every quasi-coherent (resp. coherent) sheaf F and $i > 0$.*

Proof. If X is affine, then Theorem 4.1.30 establishes the vanishing of quasi-coherent cohomology. Conversely, the vanishing of quasi-coherent (resp. coherently) cohomology implies that $\Gamma(X, -)$ is exact on the category of quasi-coherent (resp. coherent) sheaves: if $0 \to F_1 \to F_2 \to F_3 \to 0$ is exact, then $\Gamma(X, F_2) \to \Gamma(X, F_3)$ is surjective as $\mathrm{H}^1(X, F_1) = 0$. $\square$

Remark 4.4.18. Given a quasi-compact and quasi-separated morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of Deligne–Mumford stacks, the condition that $f_* \colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact is fppf local on $\mathcal{Y}$ (see Lemma 6.3.16). Since $R^i f_* F$ can be computed in $\mathrm{QCoh}(\mathcal{X})$, the relative versions of Theorem 4.4.16 and Corollary 4.4.17 also hold: f is affine if and only if $f_* \colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact if and only if $R^i f_* F = 0$ for all $i > 0$ and $F \in \mathrm{QCoh}(\mathcal{X})$.

Proposition 4.4.19. *Let X be a noetherian algebraic space. If X_{red} is a scheme (resp. quasi-affine, affine), then so is X.*

Proof. If X_{red} is affine, then one uses Corollary 4.4.17 to show that X is affine exactly as in [Har77, Exer. III.3.1]: if F is a coherent sheaf on X and $I \subset \mathcal{O}_X$ denotes the nilpotent ideal defining X_{red}, then one shows the vanishing of $\mathrm{H}^i(X, F)$ using the filtration $0 = I^N F \subset I^{N-1} F \subset \cdots \subset IF \subset F$, whose factors $I^k F / I^{k+1} F$ are supported on X_{red}.

If X_{red} is quasi-affine, then $X_{\mathrm{red}} \to \mathrm{Spec}\,\Gamma(X, \mathcal{O}_X)_{\mathrm{red}}$ is an open immersion. Thus $X \to \mathrm{Spec}\,\Gamma(X, \mathcal{O}_X)$ is an open immersion and X is quasi-affine. If X_{red} is a scheme, then every point $x \in |X|$ has an open neighborhood U such that U_{red} is affine. Thus U is affine and X is a scheme. $\square$

Theorem 4.4.20 (Chevalley's Criterion for Affineness)**.** *Let Y be a noetherian algebraic space and $X \to Y$ be a finite surjective morphism of algebraic spaces. If X is affine, then so is X.*

Proof. One can argue as in [Har77, Exer. 4.1] using Corollary 4.4.17. $\square$

There is also a cohomological criterion for ampleness generalizing [Har77, Prop. 5.3]:

Exercise 4.4.21. Let X be a proper algebraic space over a noetherian ring. For a line bundle L on X, show that the following are equivalent:

(1) X is a scheme and L is ample;

(2) for every coherent sheaf F on X, there is an integer n_0 such that $\mathrm{H}^i(X, F \otimes L^n) = 0$ for $i > 0$ and $n \geq n_0$.

See also [SP, Tag 0D2W].

The following generalizes [Har77, Exer. III.5.7].

Exercise 4.4.22. Let X be a proper algebraic space over a noetherian ring and L be a line bundle on X. If $f \colon X' \to X$ is a finite surjective morphism, L is ample if and only if $f^* L$ is.

This generalizes Proposition B.6.9 to algebraic spaces. See also [SP, Tag 0GFB].

Exercise 4.4.23. A noetherian scheme has the *Chevalley–Kleiman property* if every finite set of points is contained in an affine. Show that if $X \to Y$ is a finite surjective morphism of noetherian algebraic spaces such that X has the Chevalley-Kleiman property, then Y also has the Chevalley-Kleiman property. See [Kol12, Cor. 48].

The following generalizes Proposition B.6.10 to algebraic spaces.

Proposition 4.4.24 (Openness of Ampleness)**.** *Let X be an algebraic space proper, flat, and finitely presented over a scheme S, and L be a line bundle on X. If for some geometric point $s\colon \operatorname{Spec} \Bbbk \to S$, the restriction L_s of L to the fiber X_s is ample, then there exists an open neighborhood $U \subset S$ of s such that X_U is a scheme and the restriction L_U to X_U is relatively ample over U. In particular, for all $u \in U$, L_u is ample on X_u.*

Proof. TO ADD. $\qquad\qquad\square$

4.4.5   Effective descent along field extensions

Lemma 4.4.25. *Let X be a quasi-separated algebraic space locally of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is an affine scheme, then so is X.*

Proof. By Chevalley's Criterion for Affineness (Theorem 4.4.20), it suffices to show that there is a finite field extension $\Bbbk \to K$ such that X_K is affine. (Note that the lemma follows directly from the strengthening of Chevelley's Criterion to integral surjective morphisms.)

The algebraic space X is necessarily quasi-compact and we choose an étale presentation $U \to X$ be an affine scheme. We write $\overline{\Bbbk} = \operatorname{colim} \Bbbk_\lambda$ as the colimit of finite field extensions $\Bbbk_\lambda/\Bbbk$. Set $X_\lambda := X_{\Bbbk_\lambda}$ and $U_\lambda = U_{\Bbbk_\lambda}$. By Flat Base Change (Exercise 4.1.38), $\Gamma(X, \mathcal{O}_X) \otimes_{\Bbbk} \Bbbk_\lambda = \Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ and $\Gamma(X, \mathcal{O}_X) \otimes_{\Bbbk} \overline{\Bbbk} = \Gamma(X_{\overline{\Bbbk}}, \mathcal{O}_{X_{\overline{\Bbbk}}})$. We have a cartesian diagram

$$
\begin{array}{ccccc}
U_{\overline{\Bbbk}} & \longrightarrow & U_\lambda & \longrightarrow & U \\
\downarrow & & \downarrow & & \downarrow \\
X_{\overline{\Bbbk}} & \longrightarrow & X_\lambda & \longrightarrow & X \\
\wr\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec}\Gamma(X_{\overline{\Bbbk}}, \mathcal{O}_{X_{\overline{\Bbbk}}}) & \longrightarrow & \operatorname{Spec}\Gamma(X_\lambda, \mathcal{O}_{X_\lambda}) & \longrightarrow & \operatorname{Spec}\Gamma(X, \mathcal{O}_X)
\end{array}
$$

Since $U_{\Bbbk} \to \operatorname{Spec}\Gamma(X_{\overline{\Bbbk}}, \mathcal{O}_{X_{\overline{\Bbbk}}})$ is an étale morphism of schemes, so is $U_\lambda \to \operatorname{Spec}\Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ for $\lambda \gg 0$ (Proposition B.3.7). Thus $X_\lambda \to \operatorname{Spec}\Gamma(X_\lambda, \mathcal{O}_{X_\lambda})$ is étale for $\lambda \gg 0$. Let $R = U \times_X U$ with base changes $R_\lambda := R_{\Bbbk_\lambda}$ and $R_{\overline{\Bbbk}}$. Since $R_{\overline{\Bbbk}} \to U_{\overline{\Bbbk}} \times_{\overline{\Bbbk}} U_{\overline{\Bbbk}}$ is a closed immersion, so is $R_\lambda \to U_\lambda \times_{\Bbbk_\lambda} U_\lambda$ for $\lambda \gg 0$ (Proposition B.3.7) and in particular X_λ are separated for $\lambda \gg 0$. For $\lambda \gg 0$, since X_λ is étale and separated over a scheme, X_λ is a scheme (Corollary 4.4.7). We may therefore apply Proposition B.3.3 to X (or Proposition B.3.7 to $X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$) to conclude that X_λ is affine for $\lambda \gg 0$. $\qquad\square$

Proposition 4.4.26. *Let X be a quasi-separated algebraic space of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is a scheme, then there exists a finite separable field extension $\Bbbk \to K$ such that X_K is a scheme.*

Proof. Choose an étale presentation $U \to X$ be an affine scheme and set $R = U \times_X U$. As in the proof of the previous lemma, we write $\overline{\Bbbk} = \operatorname{colim} \Bbbk_\lambda$ with $\Bbbk_\lambda/\Bbbk$ finite, and set $X_\lambda := X_{\Bbbk_\lambda}$, $U_\lambda = U_{\Bbbk_\lambda}$ and $R_\lambda = R_{\Bbbk_\lambda}$.

Let $\overline{V} \subset X_{\overline{\Bbbk}}$ be an open affine subscheme. We claim that for $\lambda \gg 0$, there exists an open subscheme $V_\lambda \subset X_\lambda$ such that $\overline{Z} = U_\lambda \times_{\Bbbk_\lambda} \overline{\Bbbk}$. Indeed, the preimage $V' \subset U_{\overline{\Bbbk}}$ of $\overline{V}$ has the property that its two preimages in $R_{\overline{\Bbbk}}$ are equal. Using Proposition B.3.4 and Proposition B.3.7, for $\lambda \gg 0$ there is an open subscheme $V'_\lambda \subset U_\lambda$ with $V' = V'_\lambda \times_{\Bbbk_\lambda} \overline{\Bbbk}$ such that the two preimages of V'_λ in R_λ are equal. By étale descent, V'_λ descends to the desired closed subscheme $V_\lambda \subset U_\lambda$.

Lemma 4.4.25 implies that V_λ is a scheme. By covering $X_{\overline{\Bbbk}}$ with finitely many affines and choosing λ sufficiently large, we obtain a finite field extension $K = \Bbbk_\lambda$ of $\Bbbk$ such that X_λ is a scheme. If $\Bbbk^s \subset K$ be the separable closure of $\Bbbk$, then $X_K \to X_{\Bbbk^s}$ is a finite universal homeomorphism and by Chevalley's Theorem for Affineness (Theorem 4.4.20), the image of an affine subscheme X_K in $X_{\Bbbk^s}$ is also affine. We conclude that $X_{\Bbbk^s}$ is a scheme. $\qquad\square$

With an additional condition on $X_{\overline{\Bbbk}}$, we can conclude that X is a scheme.

Proposition 4.4.27. *Let X be a quasi-separated algebraic space locally of finite type over a field $\Bbbk$. If $X_{\overline{\Bbbk}}$ is a scheme such that every finite set of $\overline{\Bbbk}$-points is contained in an affine (e.g. $X_{\overline{\Bbbk}}$ is quasi-projective), then X is a scheme.*

Proof. We may assume that X is quasi-compact. We will show that every closed point $x \in X$ has an affine open neighborhood. Let $\operatorname{Spec} l \hookrightarrow X$ be the inclusion of the residue field of x (Corollary 3.5.24) and let $\Bbbk^s$ be the separable closure of the finite field extension $\Bbbk \to l$. We have a cartesian diagram

$$\begin{array}{ccccc}
\coprod_{i=1}^n \operatorname{Spec} A_i & \hookrightarrow & X_{\overline{\Bbbk}} & \longrightarrow & \operatorname{Spec} \overline{\Bbbk} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} l & \stackrel{x}{\hookrightarrow} & X & \longrightarrow & \operatorname{Spec} \Bbbk
\end{array}$$

where A_i is a artinian local $\Bbbk$-algebra where n is the degree of the separable closure $\kappa^s \subset l$ of $\Bbbk$; here we are using that $\Bbbk^s \otimes_\Bbbk \overline{\Bbbk} = \prod_{i=1}^n \overline{\Bbbk}$ and that $\operatorname{Spec} l \to \operatorname{Spec} \Bbbk^s$ is a finite universal homeomorphism. The hypotheses on $X_{\overline{\Bbbk}}$ ensure that there is an affine open subscheme $\overline{U} \subset X_{\overline{\Bbbk}}$ containing the images of each $\operatorname{Spec} A_i$.

By Proposition 4.4.26, there is a finite field extension $\Bbbk \to K$ such that X_K is a scheme. After enlarging K, we can arrange that $\overline{U}$ descends to an affine open subscheme $U' \subset U_K$ by using Proposition B.3.4 to descend the morphism $\overline{U} \to X$, Proposition B.3.7 to arrange that it is an open immersion, and Proposition B.3.3 to arrange affineness. Observe that U' contains all preimages of x under $X_K \to X$. By taking the normal closure of K, we can assume K is normal over $\Bbbk$. Let $G = \operatorname{Aut}(K/\Bbbk)$ so that K^G is a purely inseparable field extension of $\Bbbk$. Then G acts on X_K freely such that $X_K/G = X_{K^G}$.

The intersection of the translates of U' by elements of G is a G-invariant quasi-affine variety U''. Choosing an affine in U'' containing all of the preimages of x and intersecting again the translates of G, we obtain a G-invariant affine $V \subset X_K$ containing the preimages of x. Then the quotient V/G is an affine subscheme of X_{K^G} containing the unique preimage of x (Theorem 4.3.6). Letting W be the image of V/H under the finite universal homeomorphism $X_{K^G} \to X$, Chevelley's Criterion for Affineness (Theorem 4.4.20) implies that W is an affine neighborhood of x. $\quad\square$

4.4.6 Group algebraic spaces are schemes

Every quasi-separated group algebraic space over a field $\Bbbk$ is a scheme. When $\Bbbk$ is algebraically closed, this follows easily from Theorem 4.4.1 as we know there is a dense open that is a scheme, and we can translate this around by rational points. The general case relies on Proposition 4.4.27.

Theorem 4.4.28. *A quasi-separated group algebraic space G locally of finite type over a field $\Bbbk$ is a separated scheme. The connected component of the identity G^0 is quasi-projective.*

Remark 4.4.29. If G is not quasi-separated, then the above corollary does not hold, e.g. $G = \mathbb{G}_a/\mathbb{Z}$ over $\Bbbk$ (Example 3.9.22).

Note that Proposition 4.4.19 implies that the result also holds over an Artinian base. Over a general base scheme, the statement is not true; see [Ray70, Lem. X.14].

If in addition G is quasi-compact, then G is quasi-projective by Proposition B.2.40(7).

Proof. It suffices to show that G is a scheme. Indeed, a group scheme locally of finite type over a field is necessarily separated (B.2.40(1)) and the identity component G^0 is quasi-compact (B.2.40(4)), thus quasi-projective (B.2.40(7)).

Assume first that $\Bbbk$ is algebraically closed. There is a non-empty open subscheme U of G (Theorem 4.4.1) with a point $h \in U(\Bbbk)$. For every $g \in G(\Bbbk)$, left multiplication by gh^{-1} defines an isomorphism $G \xrightarrow{\sim} G$ and the image $gh^{-1}U$ of U is a scheme containing g.

The general case follows from Proposition 4.4.27 using that $G_{\bar{\Bbbk}}$ is a scheme with the property that every finite set of points is contained in an affine (Lemma 4.4.30).

See also [Art69b, Lem. 4.2] and [SP, Tag 0B8D]. $\qquad\square$

Lemma 4.4.30. *Every group scheme G locally of finite type over an algebraically closed field $\Bbbk$ has the property that every finite set of $\Bbbk$-points is contained in an affine open subscheme.*

Proof. Let $g_1, \ldots, g_n \in G(\Bbbk)$. We first use induction on n to assume that all of the elements g_i are in the same connected component. If not, we can write $G = W_1 \amalg W_2$ with r points in W_1 and $n - r$ points in W_2 for $0 < r < n$. By induction, there are affine opens $U_1 \subset W_1$ and $U_2 \subset W_2$ containing the r and $n - r$ points, respectively. Then $U_1 \amalg U_2$ is an affine containing each g_i.

By translating by g_1^{-1}, we may assume that $g_1, \ldots, g_n \in G^0(\Bbbk)$. Let $U \subset G^0$ be an affine open neighborhood of the identity. Since G^0 is irreducible (B.2.40(4)), $Ug_1^{-1} \cap \cdots \cap Ug_n^{-1}$ is non-empty and contains a closed point h. Since $h \in Ug_i^{-1}$, each g_i is contained in the affine open $h^{-1}U$.

See also [SP, Tag 0B7S]. It is also true that every group scheme of *finite type* over a field is quasi-projective [SP, Tag 0BF7]. $\qquad\square$

Corollary 4.4.31. *Let $\mathcal{X}$ be an algebraic stack with quasi-separated diagonal. Then the stabilizer of every field-valued point is a group scheme locally of finite type.*

Proof. By Exercise 3.3.3, the diagonal of $\mathcal{X}$ is locally of finite type. As the stabilizer is the base change of the diagonal, the statement follows from Theorem 4.4.28. $\quad\square$

4.4.7 Separated one-dimensional algebraic spaces are schemes

Theorem 4.4.32. *A separated noetherian algebraic space X with $\dim X \leq 1$ is a scheme.*

Proof. TO ADD. See also [SP, Tags 0ADD and 09NZ]. $\qquad\square$

4.5 Finite covers of Deligne–Mumford stacks

4.5.1 Le Lemme de Gabber

This section aims to prove the following theorem asserting that Deligne–Mumford stacks have finite covers by schemes.

Theorem 4.5.1 (Le Lemme de Gabber). *Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian scheme S. Then there exists a finite, generically étale, and surjective morphism $Z \to \mathcal{X}$ from a scheme Z.*

By applying Chow's Lemma (c.f [Har77, Exer. II.4.10]) to Z, we obtain:

Corollary 4.5.2. *There exists a projective, generically étale, and surjective morphism $Z \to \mathcal{X}$ from a scheme Z quasi-projective over S.* $\qquad\square$

See also [LMB00, Thm. 16.6 and Cor. 16.6.1], [Del85], [Vis89, Prop. 2.6], [Ols16, Thm. 11.4.1] and [SP, Tag 09YC] (for the case of algebraic spaces). More generally, every separated algebraic stack of finite type over S has a proper cover by a quasi-projective scheme [Ols05].

We provide two arguments, each of which uses normalization to construct a finite cover.

Proof 1 (following [LMB00, Thm. 16.6]):

By replacing $\mathcal{X}$ with the disjoint union of the irreducible components with their reduced stack structure, we may assume that $\mathcal{X}$ is irreducible and reduced. Every étale presentation $U \to \mathcal{X}$ is separated, quasi-finite and representable and thus factors as the composition of an open immersion $U \hookrightarrow \widetilde{\mathcal{X}}$ and a finite morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ by Zariski's Main Theorem (4.4.9). After replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may assume that $\mathcal{X}$ has a dense open subscheme. If $p\colon U \to \mathcal{X}$ is an étale presentation, there is therefore a dense open subscheme $V \subset \mathcal{X}$ such that $p^{-1}(V) \to V$ is finite étale of degree d. We may choose a finite étale covering $V' \to V$ such that $p^{-1}(V) \times_V V' \to V'$ is a trivial étale covering; indeed as in Proposition A.3.11, we may take V' to be the complement of all pairwise diagonals in

$$(V'/V)^d = \underbrace{V' \times_V \cdots \times_V V'}_{d}.$$

Applying Zariski's Main Theorem (Theorem 4.4.9) to the composition $V' \to V \hookrightarrow \mathcal{X}$ gives a finite surjective morphism $\widetilde{\mathcal{X}} \to \mathcal{X}$ restricting to $V' \to V$. Thus after replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may assume that there is an étale presentation $U \to \mathcal{X}$ which over a dense open subscheme $j\colon V \hookrightarrow \mathcal{X}$ is a trivial étale covering, i.e. there is a cartesian diagram

$$
\begin{array}{ccc}
\coprod_{i=1}^{d} V & \overset{j'}{\hookrightarrow} & U \\
\downarrow & & \downarrow{\scriptstyle p} \\
V & \overset{j}{\hookrightarrow} & \mathcal{X}.
\end{array}
$$

We will construct a finite surjective morphism $Z \to \mathcal{X}$ from a scheme that is an isomorphism over V. Let $\mathcal{A} \subset j_*\mathcal{O}_\mathcal{X}$ be integral closure of $\mathcal{O}_\mathcal{X} \to j_*\mathcal{O}_\mathcal{X}$. Then $\pi^*\mathcal{A}$ is the integral closure of $\mathcal{O}_U$ in $j'_*\mathcal{O}_{\amalg V} = p^*j_*\mathcal{O}_V$ (Proposition A.7.4). The idempotent $e_i \in \Gamma(U, j'_*\mathcal{O}_{\amalg V}) = \Gamma(\amalg V, \mathcal{O}_{\amalg V})$, defining the ith copy of V, is integral over $\mathcal{O}_U$ and thus defines a global section $e_i \in \Gamma(U, p^*\mathcal{A})$. Now write $\mathcal{A} = \operatorname{colim}_\lambda \mathcal{C}_\lambda$

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as a filtered colimit of finite type $\mathcal{O}_{\mathcal{X}}$ algebra (Exercise 4.1.26). Since $\mathcal{A}$ is integral over $\mathcal{O}_{\mathcal{X}}$, each $\mathcal{C}_\lambda$ is a finite $\mathcal{O}_{\mathcal{X}}$-algebra. For $\lambda \gg 0$, we have $e_i \in \Gamma(U, p^*\mathcal{A}_\lambda)$. The Deligne–Mumford stack $Z := \mathcal{S}\mathrm{pec}_{\mathcal{X}} \mathcal{A}_\lambda$ is finite over $\mathcal{X}$ and we claim that Z is a scheme. To see this, consider the cartesian diagram

$$
\begin{array}{ccc}
Z' & \longrightarrow & U \\
\downarrow & & \downarrow {\scriptstyle p} \\
Z & \longrightarrow & \mathcal{X}
\end{array}
$$

noting that Z' is a scheme since it is finite over U. Each idempotent e_i defines a global section of Z' and thus yields a decomposition $Z' = \coprod_{i=1}^d Z_i'$. Each morphism $Z_i' \to Z$ is étale, separated and birational, thus an open immersion. Since $Z' \to Z$ is surjective, the collection of Z_i' defines an open covering of Z, and it follows that Z is a scheme.

Proof 2 (following [Vis89, Prop. 2.6]):

We first use limit methods to reduce to the case that S is of finite type over $\mathbb{Z}$ to ensure that normalizations are finite. By Noetherian Approximation (Proposition B.3.2), we may write $S = \lim_\lambda S_\lambda$ as the limit of schemes with affine transition maps where each S_i is of finite type over $\mathbb{Z}$. Let $U \to \mathcal{X}$ be an étale presentation and set $R = U \times_{\mathcal{X}} U \rightrightarrows U$ be the corresponding étale groupoid equipped with source, target, identity and compositions morphisms s,t,i and c. There exists an index 0 and schemes U_0 and R_0 of finite type over S_0 such that $U = U_0 \times_{S_0} S$ and $R = R_0 \times_{R_0} S$ (Proposition B.3.4(1)). For $\lambda \geq 0$, set $U_\lambda = U_0 \times_{S_0} S_\lambda$ and $R_\lambda = R_0 \times_{S_0} S_\lambda$. For $\lambda \gg 0$, there are morphisms $s_\lambda, t_\lambda \colon R_\lambda \to U_\lambda$, $i_\lambda \colon R_\lambda \to R_\lambda$ and $c_\lambda \colon R_\lambda \times_{t_\lambda, U_\lambda, s_\lambda} R_\lambda \to R_\lambda$ that base change to s,t,i and c (Proposition B.3.4(2)). Finally, for $\lambda \gg 0$, the morphisms s_λ and t_λ are étale, and $R_\lambda \to U_\lambda \times_{S_\lambda} R_\lambda$ is finite (Proposition B.3.7). It follows that $R_\lambda \rightrightarrows U_\lambda$ defines an étale groupoid of schemes and that the quotient stack $\mathcal{X}_\lambda := [U_\lambda/R_\lambda]$ is a Deligne–Mumford stack separated and of finite type over S_λ such that $\mathcal{X} \cong \mathcal{X}_\lambda \times_{S_\lambda} S$. A finite, generically étale cover of $\mathcal{X}_\lambda$ by a scheme will pull back to a finite, generically étale cover of $\mathcal{X}$ by a scheme. This finishes the reduction.

By replacing $\mathcal{X}$ with the disjoint union of the irreducible components with their reduced stack structure, we may assume that $\mathcal{X}$ is irreducible and reduced. Let $\widetilde{\mathcal{X}}$ be the normalization of $\mathcal{X}$ (Example 4.1.24). Then $\widetilde{\mathcal{X}} \to \mathcal{X}$ is finite and so after replacing $\mathcal{X}$ with $\widetilde{\mathcal{X}}$, we may assume that $\mathcal{X}$ is also normal.

Let $\mathcal{X} \to X$ be the coarse moduli space (Theorem 4.3.12) and let $U \to \mathcal{X}$ be an étale presentation. As $\mathcal{X}$ is normal, so is X (Exercise 4.3.21(a). We can write $U = \coprod_i U_i$ as the disjoint union of integral affine schemes U_i; each morphism $U_i \to \mathcal{X}$ is étale and in particular quasi-finite and dominant.

Each field extension $\mathrm{Frac}(X) \to \mathrm{Frac}(U_i)$ of fraction fields is finite, and we let F be a finite normal extension of $\mathrm{Frac}(X)$ containing each $\mathrm{Frac}(U_i)$. The normalization $Y \to X$ of X in F is finite; here X is an algebraic space and the normalization is well-defined by Proposition A.7.4. Meanwhile, by the universal property of the normalization $Y \to X$, the normalization Y_i of U_i in F admits a morphism $Y_i \to Y$ over X. As $Y_i \to Y$ is separated, quasi-finite, and birational, it is an open immersion.

The automorphism group $G = \mathrm{Aut}(F/\mathrm{Frac}(X))$ acts on Y over X and for each pair $\alpha = (i,\sigma)$ of an integer i and $\sigma \in G$, we set $Y_\alpha = \sigma(Y_i)$. We claim that $Y = \bigcup_\alpha Y_\alpha$. To see this, we first show that G acts transitively on the fibers of $Y \to X$. The fixed field F^G is a purely inseparable field extension of $\mathrm{Frac}(X)$ and the normalization $X' \to X$ of X in F^G is a universal homeomorphism. Thus

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to see that G acts transitively on the fibers, we may assume that $\mathrm{Frac}(X) \to F$ is a Galois extension. We may also assume that $X = \mathrm{Spec}\, A$ and $Y = \mathrm{Spec}\, B$ with B the integral closure of A in F. Then G acts on B and we have inclusions $A \subset B^G \subset F^G = \mathrm{Frac}(X)$. Since A is normal and B^G is integral over A, we see that $A = B^G$. By Theorem 4.3.6, $[\mathrm{Spec}\, A/G] \to \mathrm{Spec}\, B$ is a coarse moduli space, and it follows that G acts transitively on the fibers of $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$. To prove the claim, observe that since $\coprod_i Y_i \to \coprod_i U_i \to X$ is surjective, each point $x \in X$ has a preimage $y \in Y_i$ for some i. Since G acts transitively on the fibers, $\bigcup Y_\alpha$ contains the fiber of $Y \to X$ over x.

The claim implies that Y is a scheme and that $Y \to X$ factors through $\mathcal{X}$ Zariski-locally on Y. Indeed, each Y_α is separated and quasi-finite over U_i and thus a scheme by Corollary 4.4.7. Each $Y_\alpha \to X$ factors via $s_\alpha \colon Y_\alpha \to U_i \to \mathcal{X}$. After replacing X with Y and $\mathcal{X}$ with $\mathcal{X} \times_X Y$, we may assume that we have a coarse moduli space $\mathcal{X} \to X$ with X a scheme and an open covering $X = \bigcup X_\alpha$ together with a commutative diagram

$$
\begin{array}{ccc}
 & & \mathcal{X} \\
 & \overset{s_\alpha}{\nearrow} & \downarrow \\
X_\alpha & \hookrightarrow & X
\end{array}
$$

for each α. We will show that after replacing X with a finite cover, the sections s_α glue to a global section s. Such a section is necessarily finite since $\mathcal{X}$ is Deligne–Mumford, and this then finishes the proof as $X \to \mathcal{X}$ is a finite surjective morphism from a scheme.

To show that the sections glue, we first claim that the diagonal $\Delta_\mathcal{X} \colon \mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is étale. This is a Zariski local question on X, so we may assume that there is a section $s \colon X \to \mathcal{X}$ of $\pi \colon \mathcal{X} \to X$. Then $s \colon X \to \mathcal{X}$ is a dominant and unramified (since $\Delta_\mathcal{X}$ is unramified) morphism of normal Deligne–Mumford stacks and thus étale (Proposition A.3.12). It follows that $\Delta_\mathcal{X} \colon \mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is étale (and also that $\pi \colon \mathcal{X} \to X$ is étale).

Since the diagonal $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is finite and étale, the scheme $J_{\alpha,\beta} := \underline{\mathrm{Isom}}_{X_{\alpha,\beta}}(s_\alpha|_{X_{\alpha,\beta}}, s_\alpha|_{X_{\alpha,\beta}})$ of isomorphisms is finite and étale over $X_{\alpha,\beta} := X_\alpha \cap X_\beta$. We may choose a finite étale cover $V_{\alpha,\beta} \to X_{\alpha,\beta}$ trivializing $J_{\alpha,\beta} \to X_{\alpha,\beta}$ (see Proposition A.3.11). By Zariski's Main Theorem (A.7.3), $V_{\alpha,\beta} \to X$ factors as an open immersion $V_{\alpha,\beta} \hookrightarrow \widetilde{X}$ and a finite morphism $\widetilde{X} \to X$. After replacing X with $\widetilde{X}$, we may assume that $J_{\alpha,\beta} \to X_{\alpha,\beta}$ is trivial.

The intersection $\bigcap_\alpha X_\alpha$ is non-empty and we may choose a geometric point $x \colon \mathrm{Spec}\, \Bbbk \to \bigcap_\alpha X_\alpha$. All objects in the fiber $\mathcal{X}_x(\Bbbk)$ of $\mathcal{X} \to X$ over x are isomorphic. We may therefore choose an object $t \in \mathcal{X}_x(\Bbbk)$ and isomorphisms $\mu_\alpha \colon t \overset{\sim}{\to} x^* s_\alpha$ for each α. This allows us to define isomorphisms $\phi_{\alpha,\beta} \colon x^* s_\alpha \xrightarrow{\mu_\alpha^{-1}} t \xrightarrow{\mu_\beta} x^* s_\beta$. It is readily checked that the isomorphisms $\phi_{\alpha,\beta}$ satisfy the cocycle $\phi_{\alpha,\gamma} = \phi_{\beta,\gamma} \circ \phi_{\alpha,\beta}$. Each $\phi_{\alpha,\beta}$ defines a lift $\mathrm{Spec}\, \Bbbk \to J_{\alpha,\beta}$ of $x \colon \mathrm{Spec}\, \Bbbk \to X_{\alpha,\beta}$ which extends uniquely to a section $\lambda_{\alpha,\beta} \colon X_{\alpha,\beta} \to J_{\alpha,\beta}$. The triple intersections $X_\alpha \cap X_\beta \cap X_\gamma$ are connected and since the $\phi_{\alpha,\beta}$ satisfy the cocycle condition, so does $\lambda_{\alpha,\beta}$. The isomorphisms $\lambda_{\alpha,\beta}$ between $s_\alpha|_{X_{\alpha,\beta}}$ and $s_\alpha|_{X_{\alpha,\beta}}$ therefore glue to a global section of $\mathcal{X} \to X$. $\quad\square$

Exercise 4.5.3 (good practice)**.** Let X be a normal algebraic space of finite type over a noetherian scheme S. Show that there is a normal scheme U with an action of a finite abstract group G such that X is the quotient of U by G, i.e. $[U/G] \to X$ is a coarse moduli space.

Hint: After reducing to the case that X is integral, choose a finite, generically étale and surjective morphism $U \to X$ from a scheme. Let K be the Galois closure of the finite separable field extension $\mathrm{Frac}(U)/\mathrm{Frac}(X)$. Then take U to be the integral closure of X in K (which is finite over X as $K/\mathrm{Frac}(X)$ is separable) and take $G = \mathrm{Gal}(K/\mathrm{Frac}(X))$. See also [LMB00, Cor. 16.6.2].

Exercise 4.5.4 (Valuative criteria can be checked on dense opens)**.**

(a) Let $\mathcal{X}$ be a Deligne–Mumford stack separated and of finite type over a noetherian scheme S, and let $\mathcal{U} \subset \mathcal{X}$ be a dense open substack. Show that $\mathcal{X} \to S$ is proper if and only if for every DVR R defined over S with fraction field K, every map $\mathrm{Spec}\, K \to \mathcal{U}$ extends to a map $\mathrm{Spec}\, R \to \mathcal{X}$ over S, after replacing R with an extension of DVRs.

   Hint: Use Corollary 4.5.2 to reduce to the case that $\mathcal{X} \to S$ is quasi-projective.

(b) Let $\mathcal{X}$ be a Deligne–Mumford stack with separated and quasi-compact diagonal. Show that a finite type morphism $\mathcal{X} \to S$ to a noetherian scheme is separated if and only if for every DVR R over S with fraction field K and every pair $h, g \colon \mathrm{Spec}\, R \to \mathcal{X}$ of morphisms over S, any isomorphism $h_K \xrightarrow{\sim} g_K$ of their generic fibers extends to a unique isomorphism $h \xrightarrow{\sim} g$.

   Hint: Reduce to Part (a).

4.5.2   Applications to cohomology

Most of the foundational theorems for coherent sheaf cohomology that hold for proper schemes—Finiteness of Cohomology (A.5.3), Cohomology and Base Change (A.6.2), Semicontinuity (A.6.4), Formal Functions (A.5.4), Grothendieck's Existence Theorem (C.4.4), and Stein Factorization [Har77, Cor. III.11.5]— also hold for proper Deligne–Mumford stacks. They are necessary to prove Zariski's Connectedness Theorem (4.5.11), which is an essential ingredient in the proof of the irreducibility of $\mathcal{M}_g$ in positive characteristic (Theorem 5.7.25).

Theorem 4.5.5 (Finiteness of Cohomology)**.** *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a proper morphism of noetherian Deligne–Mumford stacks. For any coherent sheaf F on $\mathcal{X}$ and any $i \geq 0$, $\mathrm{R}^i f_* F$ is coherent.*

Proof. By Flat Base Change (4.1.38), we can assume that $\mathcal{Y} = \mathrm{Spec}\, A$ is an affine scheme, and we need to show that $\mathrm{H}^i(\mathcal{X}, F)$ is a finite A-module for $i \geq 0$. We will use the following generalization of Dévissage (A.5.1), which is proved in the same way: if $\mathcal{P}$ is a property of coherent sheaves on $\mathcal{X}$ such that (a) in a short exact sequence in $\mathrm{Coh}(\mathcal{X})$, if two out of the three satisfies $\mathcal{P}$, then the third does too, and (b) for all integral closed substacks $\mathcal{Z} \subset \mathcal{X}$ with ideal sheaf $I_{\mathcal{Z}} \subset \mathcal{O}_{\mathcal{X}}$, there exists a coherent sheaf G on $\mathcal{X}$ satisfying $\mathcal{P}$ with $I_{\mathcal{Z}} G = 0$ and a morphism $F \to G$ which is an isomorphism over an open substack of $\mathcal{Z}$, then every coherent sheaf on $\mathcal{X}$ satisfies $\mathcal{P}$. Taking $\mathcal{P}$ to be the property that $\mathrm{H}^i(\mathcal{X}, F)$ is a finite A-module for $i \geq 0$, we see that (a) holds. For (b), using Le Lemme de Gabber (4.5.1), let $g_1 \colon V_0 \to \mathcal{Z}$ be a finite surjective morphism from a scheme. Letting $V_1 = V_0 \times_{\mathcal{Z}} V_0$ with projection $g_2 \colon V_1 \to \mathcal{Z}$, there is a complex $0 \to F \to g_{0,*} g_0^* F \to g_{1,*} g_1^* F$, which is exact if $V_0 \to \mathcal{X}$ is flat (Proposition B.1.1). Defining $G := \ker(g_{0,*} g_0^* F \to g_{1,*} g_1^* F)$, there is a map $F \to G$. By Generic Flatness (3.3.30), $V_0 \to \mathcal{X}$ is flat over a nonempty open substack $\mathcal{U} \subset \mathcal{Z}$, and it follows that $F \to G$ is an isomorphism over $\mathcal{U}$. Since V_0 and V_1 are proper over $\mathrm{Spec}\, A$, $g_{0,*} g_0^* F$ and $g_{1,*} g_1^* F$ have finite cohomology, and thus so does G. See also [Fal03, Thm. 1], [Ols05, Thm. 1.2], [LMB00, Thm. 15.6], and [Ols16, Thm. 11.6.1]. □

Remark 4.5.6. Unlike schemes, $\mathrm{H}^i(\mathcal{X}, F)$ can be nonzero for arbitrary large i; see Exercise 4.1.42.

Proposition 4.5.7 (Semicontinuity). *Let $\mathcal{X} \to \mathcal{Y}$ be a proper morphism of noetherian Deligne–Mumford stacks, and let F be a coherent sheaf on $\mathcal{X}$ which is flat over $\mathcal{Y}$. For each $i \geq 0$, the function*

$$|\mathcal{Y}| \to \mathbb{Z}, \qquad y \mapsto \dim_K \mathrm{H}^i(\mathcal{X}_K, F_K) \tag{4.5.8}$$

where $\operatorname{Spec} K \to \mathcal{Y}$ is any point representing y, is upper semicontinuous.

Proof. We leave the reader to check that the function (4.5.12) is constructible. It therefore suffices to check that it is upper semicontinuous with respect to a specialization $y \rightsquigarrow y_0$ in $|\mathcal{Y}|$. By Proposition 3.8.11, this specialization is realized by a morphism $\operatorname{Spec} R \to \mathcal{Y}$ from a DVR, and we may therefore assume that $\mathcal{Y} = \operatorname{Spec} R$. Letting K and κ be the fraction and residual field of R, we need to show that $\dim_K \mathrm{H}^i(\mathcal{X}_K, F_K) \leq \dim_\kappa \mathrm{H}^i(\mathcal{X}_\kappa, F_\kappa)$. If $\pi \in R$ denotes a uniformizer, then $\pi \colon F \to F$ is injective since F is flat over R. Apply $\mathrm{H}^i(\mathcal{X}, -)$ to the short exact sequence $0 \to F \to F \to F_\kappa \to 0$, we obtain an exact sequence

$$\cdots \to \mathrm{H}^i(\mathcal{X}, F) \xrightarrow{\pi} \mathrm{H}^i(\mathcal{X}, F) \to \mathrm{H}^i(\mathcal{X}_\kappa, F_\kappa) \to \cdots .$$

This gives an injection $\mathrm{H}^i(\mathcal{X}, F)/\pi \mathrm{H}^i(\mathcal{X}, F) \hookrightarrow \mathrm{H}^i(\mathcal{X}_\kappa, F_\kappa)$. On the other hand, $\mathrm{H}^i(\mathcal{X}, F)$ is a finite R-module by Finiteness of Cohomology (4.5.5) with $\mathrm{H}^i(\mathcal{X}, F) \otimes_R K = \mathrm{H}^i(\mathcal{X}, F_K)$ by Flat Base Change (4.1.38). This gives

$$\dim_K \mathrm{H}^i(\mathcal{X}_K, F_K) \leq \dim_\kappa \mathrm{H}^i(\mathcal{X}, F)/\pi \mathrm{H}^i(\mathcal{X}, F) \leq \dim_\kappa \mathrm{H}^i(\mathcal{X}_\kappa, F_\kappa).$$

$\square$

Exercise 4.5.9 (hard). Let $\mathcal{X}$ be a noetherian Deligne–Mumford stack proper over a complete local noetherian ring $(A, \mathfrak{m})$. Let $\mathcal{X}_n = \mathcal{X} \times_A A/\mathfrak{m}^{n+1}$.

(a) (Formal Functions) If F is a coherent sheaf on $\mathcal{X}$, there is a natural isomorphism

$$\mathrm{H}^i(\mathcal{X}, \mathcal{F}) \xrightarrow{\sim} \varprojlim_n \mathrm{H}^i(\mathcal{X}_n, F|_{\mathcal{X}_n})$$

for every $i \geq 0$.

(b) (Grothendieck's Existence Theorem) There is an equivalence of categories

$$\operatorname{Coh}(\mathcal{X}) \to \varprojlim \operatorname{Coh}(\mathcal{X}_n), \qquad F \mapsto \{F|_{\mathcal{X}_n}\}.$$

See also [Ols05, Thm. 1.4] and [Con05a, Cor. 3.2, Thm. 4.1].

Theorem 4.5.10 (Stein Factorization/Zariski's Connectedness Theorem). *A proper morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of noetherian Deligne–Mumford stacks factors as*

$$f \colon \mathcal{X} \xrightarrow{f'} \mathcal{Y}' = \mathcal{S}\mathrm{pec}_\mathcal{Y}\, f_*\mathcal{O}_\mathcal{X} \xrightarrow{g} \mathcal{Y},$$

where f' has geometrically connected fibers and g is finite. In particular, if $f_*\mathcal{O}_\mathcal{X} = \mathcal{Y}$, then f has geometrically connected fibers.

Proof. This is proved just as in the case of schemes—[Har77, Cor. III.11.5], [EGA, III$_1$.4.3.1], and [SP, Tag 03H2]—using Formal Functions. As $f_*\mathcal{O}_\mathcal{X}$ is coherent by Finiteness of Cohomology (4.5.5), g is finite. We are thus reduced to showing that

f has geometrically connected fibers if $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{Y}$. Since the question is étale local, we can further assume that $\mathcal{Y} = S$ is affine. Moreover, since geometric connectedness of a fiber $\mathcal{X}_s$ can be checked on separable finite field extensions of $\kappa(s)$, it suffices to verify that for every étale morphism $S' \to S$, the fibers of $f' \colon \mathcal{X}_{S'} \to S'$ are connected. By Flat Base Change (4.1.38), $f'_*\mathcal{O}_{\mathcal{X}_{S'}} = \mathcal{O}_{S'}$, so it is enough to verify that f has connected fibers. If a fiber $\mathcal{X}_s$ is disconnected, then setting $S_n = \operatorname{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$, there are compatible isomorphisms $\mathrm{H}^0(\mathcal{X} \times_S S_n, \mathcal{O}_{\mathcal{X} \times_S S_n}) \cong A_n \times B_n$ for nonzero coherent $\mathcal{O}_{S_n}$-modules A_n and B_n. By Formal Functions (4.5.9(b)), $\widehat{\mathcal{O}}_{S,s} \cong \varprojlim A_n \times \varprojlim B_n$, contradicting that $\widehat{\mathcal{O}}_{S,s}$ is a local ring. $\qquad\square$

The following result will be applied to the morphism $\overline{\mathcal{M}}_g \to \operatorname{Spec}\mathbb{Z}$ to show that $\mathcal{M}_g \times_{\mathbb{Z}} \Bbbk$ is connected for any field $\Bbbk$ (see Theorem 5.7.25). This application requires only the lower semincontinuity below, which relies on Stein Factorization (4.5.10) but not Semicontinuity (4.5.7).

Corollary 4.5.11 (Zariski's Connectedness Theorem II). *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a proper flat morphism of noetherian Deligne–Mumford stacks. The function*

$$|\mathcal{Y}| \to \mathbb{Z}, \qquad y \mapsto \#\textit{conn. cpts. of } \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec} K, \qquad (4.5.12)$$

where $\operatorname{Spec} K \to \mathcal{Y}$ is any geometric point representing y, is lower semicontinuous. If in addition f has geometrically reduced fibers, then (4.5.12) is locally constant.

Proof. By étale descent, we can assume that $\mathcal{Y} = S$ is a scheme. Let $\mathcal{X} \xrightarrow{f'} S' \to S$ be the Stein Factorization (4.5.10). As $S' \to S$ is finite, there are only finitely many points $s'_1, \ldots, s'_m$ over a point $s \in S$. By Étale Localization of Quasi-finite Morphisms (A.7.1), after replacing S with an étale neighborhood of s, we can arrange that $S' = S'_1 \amalg \cdots \amalg S'_m$ with $s'_i \in S'_i$ and $\kappa(s'_i)/\kappa(s)$ purely inseparable. Since $\mathcal{X} \to S'$ has geometrically connected fibers, each fiber $\mathcal{X}_{s'_i}$ over $\kappa(s'_i)$ is geometrically connected, and thus $\mathcal{X} \to S$ has precisely m geometric fibers over $s \in S$. As $\mathcal{X} \to S$ is flat, the image U_i of $f'^{-1}(S'_i)$ is open in S. In the open neighborhood $\bigcap_i U_i$ of s, every geometric fiber has at least m connected components.

Assuming now that every fiber is geometrically reduced, the number of geometric components over $s \in S$ is precisely $\dim_{\kappa(s)} \mathrm{H}^0(\mathcal{X}_s, \mathcal{O}_{\mathcal{X}_s})$. By Semicontinuity (4.5.7), (4.5.12) is also upper semicontinuous. This was stated but unproven in [DM69, Thm. 4.17] for Deligne–Mumford stacks. See [EGA, IV$_3$.15.5.3-7] and [SP, Tags 0BUI and 0E0N] for schemes, and [SP, Tags 0E1D and 0E1E] for algebraic spaces. $\qquad\square$

Chapter 5

Moduli of stable curves

This chapter proves Theorem A: for $g \geq 2$, $\overline{\mathcal{M}}_g$ is a smooth, proper, and irreducible Deligne–Mumford stack of dimension $3g - 3$ which admits a projective coarse moduli space $\overline{\mathcal{M}}_g \to \overline{M}_g$. In terms of the six-step strategy to construct projective moduli spaces outlined in §0.7.2, we proceed as follows.

① (Algebraicity) We express $\overline{\mathcal{M}}_g$ as a substack of the stack of all curves $\mathcal{M}_g^{\mathrm{all}}$, and we prove that $\mathcal{M}_g^{\mathrm{all}}$ is an algebraic stack locally of finite type over $\mathbb{Z}$ (Theorem 5.4.6).

② (Openness of Stability) $\overline{\mathcal{M}}_g \subset \mathcal{M}_g^{\mathrm{all}}$ is an open substack (Proposition 5.3.22).

③ (Boundedness) $\overline{\mathcal{M}}_g$ is finite type (Theorem 5.4.14).

④ (Stable reduction) $\overline{\mathcal{M}}_g$ is proper (Theorem 5.5.16).

⑤ (Existence of a moduli space) Applying Keel–Mori (4.3.12) gives a coarse moduli space $\overline{\mathcal{M}}_g \to \overline{M}_g$ with $\overline{M}_g$ a proper algebraic space.

⑥ (Projectivity) Using Kollár's Criterion (5.8.7), we show that $\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ is ample for $k \gg 0$, where $\pi \colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ is the universal family (Theorem 5.8.18).

Along the way, we develop the foundational properties of nodal curves (Section 5.2) and stable curves (Sections 5.3 and 5.6). For the application to moduli theory, this necessitates proving most properties for *families of curves*, which is one of the most technically challenging facets of the exposition. The irreducibility of $\overline{\mathcal{M}}_g$ is established in Section 5.7, and an alternative GIT construction of $\overline{M}_g$ is given in §5.8.4.

5.1 Review of smooth curves

We review some basic properties of smooth curves, which we will generalize to nodal curves in the section.

5.1.1 Curves

A *curve over a field* $\Bbbk$ is a one-dimensional scheme C of finite type over $\Bbbk$. Proper curves are projective; this can be reduced to the case of smooth curves [Har77, Prop. I.6.7]. More generally, every one-dimensional separated algebraic space is a quasi-projective scheme (Theorem 4.4.32).

If C is a proper curve over a field $\Bbbk$, we define the *arithmetic genus of C* or simply the *genus of C* as
$$g(C) = 1 - \chi(C, \mathcal{O}_C),$$
which is equal to $\mathrm{h}^1(C, \mathcal{O}_C)$ if C is geometrically connected and reduced. The *geometric genus* of a reduced proper curve C is the (arithmetic) genus of the normalization $\widetilde{C}$.

For a connected, reduced, and projective curve C over an algebraically closed field $\Bbbk$, the degree of a *very ample* line bundle L on C is defined as the number of zeros (counted with multiplicity) of any section of L. In other words, if $C \hookrightarrow \mathbb{P}^n$ is the projective embedding defined by L, then $\deg L = \dim_{\Bbbk} \Gamma(C \cap H, \mathcal{O}_{C \cap H})$, where H is any hyperplane and $C \cap H$ is the scheme-theoretic intersection. Any line bundle on C can be written as the difference of two very ample line bundles: if M is very ample on C, then $M' := L \otimes M^n$ is very ample for $n \gg 0$, and $L \cong M' \otimes (M^{\otimes n})^{\vee}$. In this way, we also see that $L = \mathcal{O}_C(D)$ for a divisor $D = \sum_i n_i p_i$ supported on the smooth locus of C, i.e. each $p_i \in C$ is a smooth point. Note that $\deg(L \otimes M) = \deg L + \deg M$, and that if $C = \bigcup_i C_i$ denotes the irreducible decomposition, then $\deg L = \sum_i \deg L|_{C_i}$.

Theorem 5.1.1 (Riemann–Roch). *Let C be a connected, reduced, and projective curve of genus g over an algebraically closed field $\Bbbk$. If L is a line bundle on C, then*
$$\chi(C, L) = \deg L + 1 - g.$$

Proof. We can write $L = \mathcal{O}_C(D)$ for a divisor D supported on the smooth locus. Since Riemann–Roch holds for $\mathcal{O}_C$, it suffices by adding and subtracting points to show that Riemann–Roch holds for $\mathcal{O}_C(D)$ if and only if it holds for $\mathcal{O}_C(D + p)$ for a smooth point $p \in C(\Bbbk)$. This follows by considering the short exact sequence
$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D + p) \to \kappa(p) \to 0$$
and the identity $\chi(C, \mathcal{O}_C(D + p)) = \chi(C, \mathcal{O}_C(D)) + 1$. See also [Har77, Thm IV.1.3, Exer. IV.1.9] and [Vak17, Exers. 18.4.B and S]. $\qquad\square$

5.1.2 Smooth curves

If C is a smooth curve, then the sheaf of differentials Ω_C is a line bundle. Serre Duality states that Ω_C is a dualizing sheaf on C; this is a deep result that is indispensable in the study of curves.

Theorem 5.1.2 (Serre Duality for Smooth Curves). *If C is a smooth projective curve over a field $\Bbbk$, then Ω_C is a dualizing sheaf, i.e. there is a linear map* $\mathrm{tr} \colon \mathrm{H}^1(C, \Omega_C) \to \Bbbk$ *such that for every coherent sheaf $\mathcal{F}$, the natural pairing*
$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \Omega_C) \times \mathrm{H}^1(C, \mathcal{F}) \to \mathrm{H}^1(C, \Omega_C) \xrightarrow{\mathrm{tr}} \Bbbk$$
is perfect.

Proof. See [Ser88, Thm. II.8.2] and [Har77, Cor. III.7.12]. $\qquad\square$

Remark 5.1.3. The pairing being perfect means that the $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \Omega_C)$ is identified with the dual $\mathrm{H}^1(C, \mathcal{F})^{\vee}$. If $\mathcal{F}$ is a vector bundle, $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{F}, \Omega_C) \cong \mathrm{H}^0(C, \mathcal{F}^{\vee} \otimes \Omega_C)$ and Serre Duality gives an isomorphism
$$\mathrm{H}^0(C, \mathcal{F}^{\vee} \otimes \Omega_C) \cong \mathrm{H}^1(C, \mathcal{F})^{\vee}.$$

Taking $\mathcal{F} = \Omega_C$, we see that $\mathrm{H}^1(C, \Omega_C) \cong \mathrm{H}^0(C, \mathcal{O}_C)^{\vee}$ and in particular that the trace map $\mathrm{tr} \colon \mathrm{H}^1(C, \Omega_C) \to \Bbbk$ is an isomorphism if C is geometrically connected and reduced.

Combining the above version of Riemann–Roch (5.1.1) with Serre Duality leads to a more powerful version of Riemann–Roch.

Theorem 5.1.4 (Riemann–Roch II)**.** *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field. If L is a line bundle on C, then*

$$\mathrm{h}^0(C, L) - \mathrm{h}^0(C, \Omega_C \otimes L^\vee) = \deg L + 1 - g.$$

Remark 5.1.5. This is often written in divisor form as $\mathrm{h}^0(C, L) - \mathrm{h}^0(C, K - L) = \deg L + 1 - g$ where K denotes a canonical divisor, i.e. $\Omega_C = \mathcal{O}_C(K)$.

The theorem of Riemann–Hurwitz (5.7.3), akin to Riemann–Roch and Serre–Duality, plays an indispensable role in the study of smooth curves, describing how the sheaf of differentials behaves under finite covers. Our treatment is deferred until we discuss branched covers in §5.7.1.

Exercise 5.1.6 (easy)**.** For any $g \geq 0$, show that there exists a smooth, connected, and projective curve of genus g.

Automorphisms. If C is a smooth curve over a field $\Bbbk$, we let $\mathrm{Aut}(C)$ denote the abstract group of automorphisms of C over $\Bbbk$. If $p_1, \ldots, p_n$ are distinct $\Bbbk$-points of a smooth curve C over $\Bbbk$, an *automorphism of* (C, p_i) is an automorphism $\alpha \colon C \xrightarrow{\sim} C$ such that $\alpha(p_i) = p_i$ for all i, and we let $\mathrm{Aut}(C, p_i)$ denote the abstract group of automorphisms.

Proposition 5.1.7. *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field $\Bbbk$, and let $p_1, \ldots, p_n \in C(\Bbbk)$ be distinct points. The automorphism group $\mathrm{Aut}(C, p_i)$ is finite if $2g - 2 + n > 0$, i.e. either $g \geq 2$, or $g = 1$ and $n \geq 1$, or $g = 0$ and $n \geq 3$. Moreover, if $g \geq 2$ and $\mathrm{char}(\Bbbk) = 0$, then $|\mathrm{Aut}(C)| \leq 84(g - 1)$.*

Proof. This is a classical result of Hurwitz [Hur92]. See also [Har77, Exer. IV.5.2], [ACGH85, Exer. I.F.4], and [Mir95, Thm. 4.18] for the finiteness and [Har77, Exer. IV.2.5], [ACGH85, Exer. I.F.10], and [Mir95, Thm.3.9] for the explicit bound. □

Exercise 5.1.8. Let C be a smooth, connected, and projective curve over an algebraically closed field $\Bbbk$ of genus $g \geq 2$. Show that there is no non-trivial automorphism of C fixing more than $2g + 2$ points.

While every hyperelliptic curve and thus every curve of genus 2 has a non-trivial hyperelliptic involution, the following exercise will be used later to show that general curves of higher genus are automorphism free.

Exercise 5.1.9. Show that if $g \geq 3$ and $\Bbbk$ is an algebraically closed field, there exists a smooth, connected, and projective curve C over $\Bbbk$ with trivial automorphism group.

For further background on smooth curves, we recommend [Har77, §IV], [Vak17, §20], [Liu02, §7], [ACGH85], and [Mir95].

5.1.3 Positivity of divisors on smooth curves

The following consequence of Riemann–Roch provides useful criteria to determine whether a given line bundle is base point free (equivalently globally generated), ample, or very ample.

Corollary 5.1.10. *Let C be a connected, smooth, and projective curve over an algebraically closed field $\Bbbk$, and let L be a line bundle on C.*

(1) if $\deg L < 0$, then $\mathrm{h}^0(C, L) = 0$;

(2) if $\deg L > 0$, then L is ample;

(3) if $\deg L \geq 2g$, then L is base point free; and

(4) if $\deg L \geq 2g + 1$, then L is very ample.

Proof. See [Har77, Cor. IV.3.2]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.1.11. If $g > 1$, we can use Riemann–Roch and Serre Duality to compute that: (a) $\mathrm{h}^0(C, \Omega_C) = \mathrm{h}^1(C, \mathcal{O}_C) = g$, (b) $\mathrm{h}^1(C, \Omega_C) = \mathrm{h}^0(C, \mathcal{O}_C) = 1$, and (c) Ω_C has degree $2g - 2$ and is thus ample on C. Similarly, if $k > 1$, we have: (a) $\mathrm{h}^0(C, \Omega_C^{\otimes k}) = (2k - 1)(g - 1)$, (b) $\mathrm{h}^1(C, \Omega_C^{\otimes k}) = 0$, and (c) $\Omega_C^{\otimes k}$ has degree $2k(g - 1)$ and is very ample if $k \geq 3$. Note that Ω_C is not very ample precisely when C is *hyperelliptic*. On the other hand, if $g = 1$ then $\Omega_C \cong \mathcal{O}_C$, and if $g = 0$ then $C = \mathbb{P}^1$ and $\Omega_C = \mathcal{O}(-2)$.

5.1.4 Classification of rational curves

Over an algebraically closed field $\Bbbk$, every connected, smooth, and proper curve of genus 0 is isomorphic to $\mathbb{P}^1_\Bbbk$. Over an arbitrary field, the classification is slightly more involved.

Exercise 5.1.12 (Classificiation of the projective line). Show that the following are equivalent for a proper curve C over a field $\Bbbk$:

(a) $C \cong \mathbb{P}^1_\Bbbk$;

(b) C is smooth and geometrically irreducible over $\Bbbk$, C has genus 0, and $C(\Bbbk) \neq \emptyset$.

(c) C is Gorenstein and geometrically integral over $\Bbbk$, C has genus 0, and C has a line bundle of odd degree;

(d) $\mathrm{H}^0(C, \mathcal{O}_C) = \Bbbk$, C has genus 0, and C has a line bundle of degree 1;

(e) $\mathrm{H}^1(C, \mathcal{O}_C) = 0$ and C has a line bundle of degree 1.

See also [SP, Tag 0C6U].

There is also a classification of singular rational curves that will be convenient to understand nodal rational curves as well as the classification of rational tails and bridges in prestable curves.

Exercise 5.1.13 (Classification of singular rational curves). Let C be a singular proper curve of genus 0 over a field $\Bbbk$ with $\mathrm{H}^0(C, \mathcal{O}_C) = \Bbbk$, and let $\pi \colon C' \to C$ be the normalization. Show the following:

(a) C has a unique singular point p, which is a $\Bbbk$-rational point;

(b) $\pi \colon C' \to C$ is an isomorphism over $C \setminus p$ and $\pi^{-1}(p) = \{p'\}$ for a $\Bbbk'$-rational point $p' \in C'$;

(c) $C' \cong \mathbb{P}^1_{\Bbbk'}$ for a finite field extension $\Bbbk'/\Bbbk$; and

(d) C is identified with the Ferrand Pushout (B.4.1)

$$
\begin{array}{ccc}
\operatorname{Spec} \Bbbk' & \xrightarrow{p'} & C' \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
\operatorname{Spec} \Bbbk & \xrightarrow{p} & C.
\end{array}
$$

See also [SP, Tag 0DJB].

5.1.5 Families of smooth curves

Definition 5.1.14. A *family of smooth curves (of genus g)* over a scheme S is a smooth and proper morphism $\mathcal{C} \to S$ of schemes such that every geometric fiber is a connected curve (of genus g).

Recall that the relative sheaf of differentials $\Omega_{\mathcal{C}/S}$ is a line bundle on $\mathcal{C}$ such that for every geometric point $s \colon \operatorname{Spec} \Bbbk \to S$, the restriction $\Omega_{\mathcal{C}/S}|_{\mathcal{C}_s}$ is identified with $\Omega_{\mathcal{C}_s}$. More generally, for every morphism $T \to S$ of schemes, the pullback of $\Omega_{\mathcal{C}/S}$ to $\mathcal{C}_T := \mathcal{C} \times_S T$ is canonically isomorphic to $\Omega_{\mathcal{C}_T/T}$. Generalizing Corollary 5.1.10, we show that $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample for $k \geq 3$, and that its pushforward is a vector bundle on S.

Proposition 5.1.15 (Properties of Families of Smooth Curves). *Let $\pi \colon \mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$.*

(1) $\pi_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_S$;

(2) *The pushforward $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank*

$$
r(k) := \begin{cases} g & \text{if } k = 1 \\ (2k-1)(g-1) & \text{if } k > 1. \end{cases}
$$

whose construction commutes with base change (i.e. for a morphism $f \colon T \to S$ of schemes, $f^* \pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) \cong \pi_{T,*}(\Omega_{\mathcal{C}_T/T}^{\otimes k})$);

(3) $R^1 \pi_* \Omega_{\mathcal{C}/S}^{\otimes k}$ *is isomorphic to $\mathcal{O}_S$ if $k = 1$ and zero otherwise; and*

(4) *For $k \geq 3$, $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample. In particular, $\mathcal{C} \to S$ is projective.*

Proof. Items (1)–(3) follows from Cohomology and Base Change (A.6.8) as detailed in Proposition A.6.10. For (4), observe that for every point $s \in S$, the fiber $\Omega_{\mathcal{C}/S}^{\otimes k} \otimes \kappa(s) = \Omega_{\mathcal{C}_s}^{\otimes k}$ is very ample by Corollary 5.1.10 as $\deg \Omega_{\mathcal{C}_s}^{\otimes k} = k(2g-2) > 0$. Since $\mathrm{H}^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s}^{\otimes k}) = 0$, we may apply Proposition B.6.10 to conclude that $\Omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample. $\square$

It is also true that the relative sheaf of differentials $\Omega_{\mathcal{C}/S}$ is a relative dualizing sheaf, i.e. satisfies a relative version of Serre Duality; see [Liu02, §6.4].

5.2 Nodal curves

After providing a Characterization of Nodes (5.2.3), we discuss the Genus Formula (5.2.11), the dualizing sheaf (Definition 5.2.16), and the Local Structure of Nodal Families (5.2.23).

5.2.1 Nodes

Definition 5.2.1 (Nodes)**.** Let C be a curve over a field $\Bbbk$.

- We say that $p \in C(\Bbbk)$ is a *split node* if there is an isomorphism $\widehat{\mathcal{O}}_{C,p} \cong \Bbbk[\![x,y]\!]/(xy)$.

- We say that a closed point $p \in C$ is a *node* if there exists a split node $\overline{p} \in C_{\overline{\Bbbk}}$ over p.

We say that a curve C is a *nodal* (or has *at-worst nodal singularities*) if C is pure one-dimensional and every closed point is either smooth or nodal.
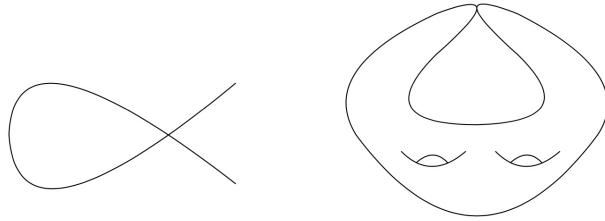
Figure 5.1: A node of a curve over $\mathbb{C}$ viewed algebraically (left-hand side) or analytically (right-hand side).

Example 5.2.2.

(1) The curve $C = \operatorname{Spec}\Bbbk[x,y]/(xy)$ has a split node at 0. The normalization $\widetilde{C} \cong \mathbb{A}^1\amalg\mathbb{A}^1$ has coordinate ring $\Bbbk[x]\times\Bbbk[y]$ and $\Gamma(C,\mathcal{O}_C) = \{(f,g) \in \Gamma(\widetilde{C},\mathcal{O}_{\widetilde{C}}) \mid f(0) = g(0)\}$, or in other words C is the Ferrand Pushout (B.4.1)

$$
\begin{array}{ccc}
\operatorname{Spec}\Bbbk \amalg \operatorname{Spec}\Bbbk & \xrightarrow{\ 0\amalg 0\ } & \widetilde{C} \\
\downarrow & & \downarrow \\
\operatorname{Spec}\Bbbk & \xrightarrow{\ \ 0\ \ } & C.
\end{array}
$$

(2) The nodal cubic $C = \operatorname{Spec}\Bbbk[x,y]/(y^2 - x^2(x+1))$ has a split node at 0. The normalization $\widetilde{C} \cong \mathbb{A}^1$ has a coordinate $t = y/x$ with $x = t^2 - 1$ and $y = t^3 - t$, and C is again the Ferrand Pushout obtained from $\widetilde{C}$ by gluing the two preimages of 0.

(3) The curve $C = \operatorname{Spec}\mathbb{R}[x,y]/(x^2 + y^2)$ has a node at 0, but it is not a split nodes as the quadratic form $x^2 + y^2$ does not split into linear factors.

(4) The curve $\operatorname{Spec}\mathbb{Q}[x,y]/(x^2 - 2)(y^2 - 3)$ has a non-split node at the point p defined by the maximal ideal $(x^2 - 2, y^2 - 3)$. Note that unlike the previous examples where the nodes are rational points, the node p is not a rational point and the field extension $\mathbb{Q} \to \kappa(p)$ has degree 4.

(5) Let $\Bbbk \to \Bbbk'$ be a separable field extension of degree 2, and let C be the affine curve over $\Bbbk$ defined by the $\Bbbk$-algebra $\{f \in \Bbbk'[x] \mid f(0) \in \Bbbk\}$. In other words, C is the Ferrand Pushout of the inclusion of the origin $0\colon \operatorname{Spec}\Bbbk' \hookrightarrow \mathbb{A}^1_{\Bbbk'}$ along $\operatorname{Spec}\Bbbk' \to \operatorname{Spec}\Bbbk$. The curve C has a non-split node at a $\Bbbk$-rational point.

5.2.2 Equivalent characterizations of nodes

Recall that the singular locus $\mathrm{Sing}(C)$ of a curve C is defined scheme-theoretically as the vanishing of the first fitting ideal sheaf of Ω_C: locally if $C = V(f_1, \ldots, f_m) \subset \mathbb{A}^n$, then $\mathrm{Sing}(C)$ is defined by the vanishing of all $(n-1) \times (n-1)$ minors of the Jacobian matrix $J = (\frac{\partial f_j}{\partial x_i})$; note that if $C = V(f) \subset \mathbb{A}^2$ is a plane affine curve, then $\mathrm{Sing}(C) = V(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ (see §A.3.4). We will also use properties of local complete intersections as discussed in §A.3.5.

Proposition 5.2.3 (Characterization of Nodes)**.** *Let C be a pure one-dimensional curve over a field $\mathbb{k}$, and let $p \in C$ be a closed point with maximal ideal $\mathfrak{m} \subset \mathcal{O}_{C,p}$. The following are equivalent:*

(1) $p \in C$ is a node;

(2) C is a local complete intersection at p, and $\mathrm{Sing}(C)$ is unramified over $\mathbb{k}$ at $p \in \mathrm{Sing}(C)$;

(3) $\mathbb{k} \to \kappa(p)$ is separable, $\mathcal{O}_{C,p}$ is reduced, $\dim \mathfrak{m}/\mathfrak{m}^2 = 2$, and there is a nondegenerate quadratic form $q \in \mathrm{Sym}^2 \mathfrak{m}/\mathfrak{m}^2$ mapping to 0 in $\mathfrak{m}^2/\mathfrak{m}^3$;

(4) $\mathbb{k} \to \kappa(p)$ is separable and $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[\![x,y]\!]/(q)$ where q is a nondegenerate quadratic form; and

(5) there exists a finite separable field extension $\mathbb{k} \to \mathbb{k}'$ and a point $p' \in C_{\mathbb{k}'}$ such that $\widehat{\mathcal{O}}_{C_{\mathbb{k}'},p'} \cong \mathbb{k}'[\![x,y]\!]/(xy)$.

Proof. Assuming (1), let $\overline{p} \in C_{\overline{\mathbb{k}}}$ be a node over p and let $\mathrm{Sing}(C) \subset C$ be the scheme-theoretic singular locus. Then $\mathrm{Sing}(C) \times_{\mathbb{k}} \overline{\mathbb{k}} = \mathrm{Sing}(C_{\overline{\mathbb{k}}})$ and the preimage of $\mathrm{Sing}(C_{\overline{\mathbb{k}}})$ under $\mathrm{Spec}\, \widehat{\mathcal{O}}_{C_{\overline{\mathbb{k}}},\overline{p}} \to C_{\overline{\mathbb{k}}}$ is $\mathrm{Sing}(\mathrm{Spec}\, \widehat{\mathcal{O}}_{C_{\overline{\mathbb{k}}},\overline{p}})$ by properties of fitting ideals (see §A.3.4). Since $\widehat{\mathcal{O}}_{C_{\overline{\mathbb{k}}},\overline{p}} \cong \overline{\mathbb{k}}[\![x,y]\!]/(xy)$, $\mathrm{Sing}(\mathrm{Spec}\, \widehat{\mathcal{O}}_{C_{\overline{\mathbb{k}}},\overline{p}}) = V(x,y) = \mathrm{Spec}\, \overline{\mathbb{k}}$. Therefore $\mathrm{Sing}(C) \to \mathrm{Spec}\, \mathbb{k}$ is unramified at p. Since $\widehat{\mathcal{O}}_{C_{\overline{\mathbb{k}}},\overline{p}}$ is a complete intersection, C is a local complete intersection at p (Proposition A.3.16). This gives (2).

Assuming (2), since $\mathrm{Sing}(C)$ is unramified at p, the field extension $\mathbb{k} \to \kappa(p)$ is separable and there is an open neighborhood $U \subset C$ of p such that $\mathrm{Sing}(U) = \mathrm{Sing}(C) \cap U = \{p\}$. In particular, C and $\mathcal{O}_{C,p}$ are generically reduced. On the other hand, since C is a local complete intersection, $\mathcal{O}_{C,p}$ is a one-dimensional Cohen–Macaulay local ring and thus has no embedded primes. It follows that $\mathcal{O}_{C,p}$ is reduced. Using that C is a local complete intersection, we can write $\widehat{\mathcal{O}}_{C,p} = R/(f_1, \ldots, f_{n-1})$ where R is a regular complete local ring. As R contains the base field $\mathbb{k}$, the Cohen Structure Theorem (B.5.7) implies that $R \cong \kappa(p)[\![x_1, \ldots, x_n]\!]$. Since $\mathrm{Sing}(C)$ is unramified at p, the $(n-1) \times (n-1)$ minors of the Jacobian matrix $\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j}$ generate the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n) \subset \widehat{\mathcal{O}}_{C,p}$. If $\frac{\partial f_j}{\partial x_i} \in R$ is a unit for some i and j, then the sequence $x_1, \ldots, \widehat{x_i}, x_n, f_j$ also generates $\mathfrak{m}/\mathfrak{m}^2$. We may use Complete Nakayama's Lemma (B.5.6(2)) to change coordinates by replacing the generators $x_1, \ldots, x_n$ with $x_1, \ldots, \widehat{x_i}, x_n, f_j$. Eliminating f_j allows us to write $R = \kappa(p)[\![x_1, \ldots, \widehat{x_i}, x_n]\!]/(f_1, \ldots, \widehat{f_j}, \ldots, f_{n-1})$. After finitely many such replacements, we can assume that $\frac{\partial f_j}{\partial x_i} \in \mathfrak{m}$ for every i,j. This implies that every $(n-1) \times (n-1)$ minor is in $\mathfrak{m}^{n-1}$, but since these minors generate $\mathfrak{m}$, we must have that $n = 2$. Therefore, $\widehat{\mathcal{O}}_{C,p} = \kappa(p)[\![x,y]\!]/(f)$ with $f = f_2 + f_3 + \cdots$ and each f_i homogeneous of degree i. Since the partials f_x and f_y generate (x,y), the quadratic form $q := f_2 \in \mathrm{Sym}^2 \mathfrak{m}/\mathfrak{m}^2$ must be nondegenerate. This gives (3).

Assuming (3), we have that $\dim_{\kappa(p)} \mathfrak{m}^d/\mathfrak{m}^{d+1} = 2$ for every $d \geq 1$ since q maps to 0 in $\mathfrak{m}^2/\mathfrak{m}^3$. A choice of elements $x_0, y_0 \in \mathfrak{m}$ mapping to a basis in $\mathfrak{m}/\mathfrak{m}^2$ induces a

surjection $\kappa(p)[\![x,y]\!] \to \widehat{\mathcal{O}}_{C,p}$ by Complete Nakayama's Lemma (B.5.6(2)). Since $\mathcal{O}_{C,p}$ is reduced, Descent of Properties of Schemes (B.1.12), $\widehat{\mathcal{O}}_{C,p}$ is also reduced. Since $\kappa(p)[\![x,y]\!]$ is a UFD, we can write $\widehat{\mathcal{O}}_{C,p}/\kappa(p)[\![x,y]\!]/(f)$ for an element f expressed as a product of distinct irreducible elements such that the quadratic component $q = ax^2 + bxy + cy^2$ of f is a nondegenerate quadratic form. We claim that we can modify our choice of coordinates $x_0, y_0 \in \mathfrak{m}$ so that $f = q$, i.e. $q(x_0, y_0) = 0 \in \widehat{\mathcal{O}}_{C,p}$. We will show inductively that for each N, there exists elements $x_i, y_i \in \mathfrak{m}^{i+1}$ for $i = 0, \ldots, N$ such that $q(x_0 + \cdots + x_N, y_0 + \cdots + y_N) \in \mathfrak{m}^{N+3}$. Since $\widehat{\mathcal{O}}_{C,p}$ is complete, this would enable us to replace x_0 and y_0 with $\sum_i x_i$ and $\sum_i y_i$ and conclude that $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[\![x,y]\!]/(q)$. Supposing that we've already chosen $x' = x_0 + \cdots + x_{N-1}$ and $y' = y_0 + \cdots + y_{N-1}$, then for every x_N and $y_N \in \mathfrak{m}^{N+1}$, we have that

$$q(x' + x_N, y' + y_N) = q(x', y') + (2ax_0 + by_0)x_N + (bx_0 + 2cy_0)y_N \mod \mathfrak{m}^{N+3}$$

The nondegeneracy of $q = ax^2 + bxy + y^2$ implies that $2ax_0 + by_0$ and $bx_0 + 2cy_0$ are linearly independent. Since $\dim_{\kappa(p)} \mathfrak{m}^{N+2}/\mathfrak{m}^{N+3} = 2$, we may choose x_N and y_N such that $Q(x' + x_N, y' + y_N) \in \mathfrak{m}^{N+3}$. This completes (4).

Assuming (4) and using that q is nondegenerate, we may choose a degree 2 separable field extension $\kappa(p) \to \Bbbk'$ such that q splits as a product of linear forms. Thus $\widehat{\mathcal{O}}_{C,p} \otimes_{\Bbbk} \Bbbk' \cong \Bbbk'[\![x,y]\!]/(xy)$, yielding (5). Finally, (5) implies that p is a node.

See also [SP, Tags 0C49, 0C4D, and 0C4E]. $\qquad\square$

Exercise 5.2.4. (easy) Show that $\operatorname{Spec} \Bbbk[x,y]/(f)$ has a node at 0 if and only if $f(0) = f_x(0) = f_y(0) = 0$ and $\det \begin{pmatrix} f_{xx}(0) & f_{xy}(0) \\ f_{yx}(0) & f_{yy}(0) \end{pmatrix}$ is nonzero.

Exercise 5.2.5. (practice) Let C be a pure one-dimensional reduced curve over a field $\Bbbk$ with normalization $\pi \colon \widetilde{C} \to C$. Show that $p \in C$ is a node if and only if $\Bbbk \to \kappa(p)$ is separable, $\dim_{\kappa(p)}(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p) = 1$, and $\sum_{\pi(q)=p}[\kappa(q) : \kappa(p)]_{\mathrm{sep}} = 2$.

Hint: Identify $(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p)$ with the quotient $\widetilde{A}/A$ where $A = \widehat{\mathcal{O}}_{C,p}$ and $\widetilde{A}$ is its normalization, using that normalization commutes with completion (see Remark B.1.15). To show $(\Leftarrow)$, use that $\widetilde{A}$ is a product of complete DVRs to derive the structure of A. See also [SP, Tag 0C4A].

Remark 5.2.6. The quantity $\dim_{\kappa(p)}(\pi_* \mathcal{O}_{\widetilde{C}}/\mathcal{O}_C) \otimes \kappa(p)$ is referred to as the *δ-invariant of p*, and the sum $\sum_{\pi(q)=p}[\kappa(q) : \kappa(p)]_{\mathrm{sep}}$ is referred to as the *number of geometric branches over p*. A cusp $\Bbbk[\![x,y]\!]/(y^2 - x^3)$ has δ-invariant one but has only one geometric branch.

Proposition 5.2.7 (Local Structure of Nodes)**.** *Let C be a curve over a field $\Bbbk$. If $p \in C$ is a node, then there exists a finite separable field extension $\Bbbk \to \Bbbk'$ and étale neighborhoods*

$$(U, u)$$

(5.2.8)

$$(C, p) \qquad\qquad (\operatorname{Spec} \Bbbk'[x,y]/(xy), 0).$$

Proof. Using the Characterization of Nodes (5.2.3(5)), there is a finite separable field extension $\Bbbk \to \Bbbk'$ such that $\widehat{\mathcal{O}}_{C,p} \otimes_{\Bbbk} \Bbbk' \cong \Bbbk'[\![x,y]\!]/(xy)$. The result is now a consequence of Artin Approximation (B.5.31). $\qquad\square$

We will shortly generalize this to families of nodal curves (Theorem 5.2.23).

Example 5.2.9. Let's construct an explicit étale neighborhood for the nodal cubic $C = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2(x - 1))$. In fact, in Example 0.5.2 we essentially already showed how do this: if we add a square root $t = \sqrt{x - 1}$, then $y^2 - x^3 + x^2 = (y - xt)(y + xt)$. Therefore, we can take the elementary étale neighborhood $U = \operatorname{Spec} \mathbb{C}[x, y, t]_t/(y^2 - x^3 + x^2, t^2 - x + 1) \to C$ given by $(x, y, t) \mapsto (x, y)$ and $U \to \operatorname{Spec} \mathbb{C}[x, y]/(xy)$ by $(x, y, t) \mapsto (y - xt, y + xt)$.

Exercise 5.2.10. Provide a proof of the Local Structure of Nodes (5.2.7) without appealing to Artin Approximation.

Hint: Use that the normalization of a strict henselization $\mathcal{O}_{C,p}^{\mathrm{sh}}$ has two components to find an affine étale neighborhood $(\operatorname{Spec} R, u) \to (C, p)$ with $\widetilde{R} = R_1 \times R_2$. Use the exact sequence $0 \to R \to R_1 \times R_2 \to \kappa(u) \to 0$ to construct elements $x, y \in R$ mapping to $(1, 0), (0, 1) \in R_1 \times R_2$, and argue that $\kappa(u)[x, y]/(xy) \to R$ is étale.

5.2.3 Genus formula

Proposition 5.2.11 (Genus Formula)**.** *Let C be a connected, nodal, and projective curve over an algebraically closed field $\mathbb{k}$ with δ nodes and ν irreducible components. Let $C = \bigcup_i C_i$ be the irreducible decomposition and let $\widetilde{C}_i$ be the normalization of C_i with genus $g(\widetilde{C}_i)$. The genus g of C satisfies*

$$g = \sum_{i=1}^{\nu} g(\widetilde{C}_i) + \delta - \nu + 1.$$

Proof. Let $p_1, \ldots, p_\delta \in C$ denote the nodes of C. We claim that the normalization $\pi \colon \widetilde{C} \to C$ induces a short exact sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\widetilde{C}} \to \bigoplus_{j=1}^{\delta} \kappa(p_j) \to 0.$$

It suffices to verify this étale-locally around a node $p_i \in C$, and so by the Local Structure of Nodes (5.2.7), we can assume that $C = \operatorname{Spec} \mathbb{k}[x, y]/(xy)$. In this case, $\widetilde{C} = \operatorname{Spec}(\mathbb{k}[x] \times \mathbb{k}[y])$ and the sequence above corresponds to $0 \to \mathbb{k}[x, y]/(xy) \to \mathbb{k}[x] \times \mathbb{k}[y] \to \mathbb{k} \to 0$. Alternatively, normalization commutes with completion and a direct calculation as above shows that if $A := \widehat{\mathcal{O}}_{C,p} \cong \mathbb{k}[\![x, y]\!]/(xy)$, then $\widetilde{A}/A \cong \mathbb{k}$.

The short exact sequence induces a long exact sequence on cohomology

$$0 \to \underbrace{\mathrm{H}^0(C, \mathcal{O}_C)}_{1} \to \underbrace{\mathrm{H}^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}})}_{\nu} \to \underbrace{\bigoplus_j \kappa(p_j)}_{\delta} \to \underbrace{\mathrm{H}^1(C, \mathcal{O}_C)}_{g} \to \underbrace{\mathrm{H}^1(\widetilde{C}, \mathcal{O}_{\widetilde{C}})}_{\sum_i g(\widetilde{C}_i)} \to 0$$

where the labels underneath indicate the dimension. The statement follows. $\qquad\square$
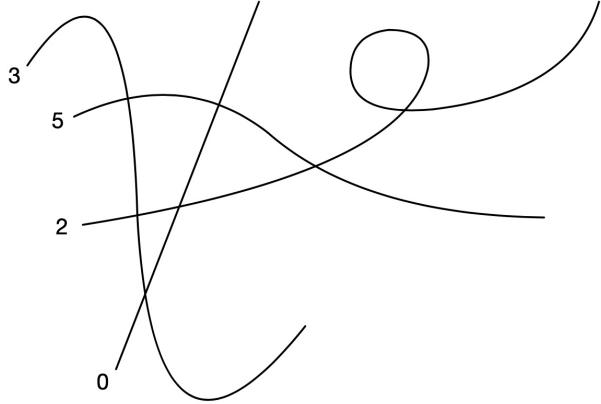
Figure 5.2: An example of a nodal curve of genus 14.

Remark 5.2.12. Notice that $\delta - \nu + 1$ is precisely the number of connected regions bounded by the curve C as in Figure 5.2. Thus, the genus of a nodal curve can be easily computed from the picture by summing the geometric genera of the irreducible components and adding the number of bounded regions. See Definition 5.3.7 for another interpretation using dual graphs.

5.2.4 The dualizing sheaf

Since a nodal curve C over a field $\Bbbk$ is a locally a complete intersection, C is Gorenstein and there is a *dualizing line bundle* ω_C with a trace map $\mathrm{tr}_C \colon \mathrm{H}^1(C, \omega_C) \xrightarrow{\sim} \Bbbk$; see [Har77, III.7.11] or [Ser88, §IV]. In other words, for every coherent sheaf F, the natural pairing

$$\mathrm{Hom}_{\mathcal{O}_C}(F, \omega_C) \times \mathrm{H}^1(C, F) \to \mathrm{H}^1(C, \omega_C) \xrightarrow{\mathrm{tr}} \Bbbk$$

is perfect.

Due to its importance in the study of stable curves, we now provide an explicit description of ω_C below in the case that $\Bbbk$ is algebraically closed. Let $\Sigma := C^{\mathrm{sing}}$ be the singular locus and $U = C \setminus \Sigma$. Let $\pi \colon \widetilde{C} \to C$ be the normalization of C, and let $\widetilde{\Sigma}$ and $\widetilde{U}$ be the preimages of Σ and U as in the diagram

$$\begin{array}{ccc}
\widetilde{U} \hookrightarrow & \widetilde{C} & \hookleftarrow \widetilde{\Sigma} \\
\downarrow \wr & \downarrow \pi & \downarrow \\
U \hookrightarrow & C & \hookleftarrow \Sigma.
\end{array} \tag{5.2.13}$$

Let $\Sigma = \{z_1, \ldots, z_n\}$ be an ordering of the points and $\pi^{-1}(z_i) = \{p_i, q_i\}$. Since $\widetilde{C}$ is smooth, the sheaf of differentials $\Omega_{\widetilde{C}}$ is a dualizing line bundle. There is a short exact sequence

$$0 \to \Omega_{\widetilde{C}} \to \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \to \mathcal{O}_{\widetilde{\Sigma}} \to 0 \tag{5.2.14}$$

obtained by tensoring the sequence $0 \to \mathcal{O}_{\widetilde{C}}(-\widetilde{\Sigma}) \to \mathcal{O}_{\widetilde{C}} \to \mathcal{O}_{\widetilde{\Sigma}} \to 0$ with $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$. As $\Omega_{\widetilde{C}}(\widetilde{\Sigma})|_{\widetilde{U}} = \Omega_{\widetilde{U}}$, we can interpret sections of $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ as rational sections of $\Omega_{\widetilde{C}}$

with at worst simple poles along $\widetilde{\Sigma}$. Evaluating (5.2.14) on an open $\widetilde{V} \subset \widetilde{C}$ yields

$$0 \longrightarrow \Gamma(\widetilde{V}, \Omega_{\widetilde{C}}) \longrightarrow \Gamma(\widetilde{V}, \Omega_{\widetilde{C}}(\widetilde{\Sigma})) \longrightarrow \bigoplus_{y \in \widetilde{V} \cap \widetilde{\Sigma}} \kappa(y), \qquad (5.2.15)$$
$$s \mapsto (\mathrm{res}_y(s))$$

where the last map takes a rational section $s \in \Gamma(\widetilde{V} \cap \widetilde{U}, \Omega_{\widetilde{C}})$ to the tuple whose coordinate at $y \in \widetilde{V} \cap \widetilde{\Sigma}$ is the *residue* $\mathrm{res}_y(s)$ *of s at y.*

Definition 5.2.16. Let C be a nodal curve over an algebraically closed field $\Bbbk$. Using the notation of (5.2.13) and (5.2.15), we define the subsheaf $\omega_C \subset \pi_* \Omega_{\widetilde{C}}(\widetilde{\Sigma})$ by declaring that sections along $V \subset C$ consist of rational sections s of $\Omega_{\widetilde{C}}$ along $\pi^{-1}(V)$ with at worst simple poles along $\widetilde{\Sigma}$ such that for every node $z_i \in V \cap \Sigma$ with preimages $p_i, q_i \in \pi^{-1}(V)$,

$$\mathrm{res}_{p_i}(s) + \mathrm{res}_{q_i}(s) = 0.$$

The definition implies that ω_C sits in the following two exact sequences:

$$0 \longrightarrow \omega_C \longrightarrow \pi_* \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \longrightarrow \bigoplus_{z_i \in \Sigma} \Bbbk \longrightarrow 0$$
$$s \mapsto (\mathrm{res}_{p_i}(s) + \mathrm{res}_{q_i}(s))$$

$$0 \longrightarrow \pi_* \Omega_{\widetilde{C}} \longrightarrow \omega_C \longrightarrow \bigoplus_{z_i \in \Sigma} \Bbbk \longrightarrow 0$$
$$s \mapsto (\mathrm{res}_{p_i}(s))$$

Example 5.2.17 (Local calculation)**.** Let $C = \mathrm{Spec}\, \Bbbk[x,y]/(xy)$. Then $\widetilde{C} = \mathbb{A}^1 \amalg \mathbb{A}^1$ with coordinates x and y respectively. The singular locus of C is $\Sigma = \{0\}$ with preimage $\widetilde{\Sigma} = \{p, q\}$ consisting of the two origins. Then $\Gamma(\widetilde{C}, \Omega_{\widetilde{C}}) = \Gamma(\mathbb{A}^1, \Omega_{\mathbb{A}^1}) \times \Gamma(\mathbb{A}^1, \Omega_{\mathbb{A}^1})$ and $(\frac{dx}{x}, -\frac{dy}{y})$ is a rational section of $\Omega_{\widetilde{C}}$ with opposite residues at p and q. In fact, every section of $\Gamma(C, \omega_C)$ is of the form

$$\left(f(x) \frac{dx}{x}, g(y) \frac{-dy}{y} \right) = (f(x) + g(y) - f(0)) \cdot (\frac{dx}{x}, \frac{-dy}{y})$$

for polynomials $f(x)$ and $g(y)$ such that $f(0) = g(0)$, which is precisely the condition for $(f, g) \in \Gamma(\widetilde{C}, \mathcal{O}_{\widetilde{C}})$ to descend to a global function $f(x) + g(y) - f(0) \in \Gamma(C, \mathcal{O}_C)$. In other words, $\omega_C \cong \mathcal{O}_C$ with generator $(\frac{dx}{x}, -\frac{dy}{y})$.

Example 5.2.18. Let C be the nodal projective plane cubic and $\mathbb{P}^1 \to C$ be the normalization with coordinates $[x : y]$ such that 0 and ∞ are the fibers of the node. Observe that the rational differential $\eta := \frac{dx}{x} = -\frac{dy}{y}$ on $\mathbb{P}^1$ satisfies $\mathrm{res}_0\, \eta + \mathrm{res}_\infty\, \eta = 0$. It is easy to see that every local section of ω_C is a multiple of η, so $\omega_C \cong \mathcal{O}_C$ with generator η.

Exercise 5.2.19 (details)**.** Let C be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$.

(a) Show that if $\pi \colon C' \to C$ is an étale morphism, then $\pi^* \omega_C \cong \omega_{C'}$.

   Hint: Use the fact that normalization commutes with étale base change.

(b) Conclude that ω_C is a line bundle.

(c) Show that ω_C is a dualizing sheaf.

  Hint: Reduce to the case of a smooth curve by considering the normalization.

(d) If $T \subset C$ is a subcurve with complement $T^c := \overline{C \setminus T}$, show that

$$\omega_C|_T = \omega_T(T \cap T^c).$$

Exercise 5.2.20 (good practice). Let C be a connected, nodal, and projective curve over an algebraically closed field $\Bbbk$. Let $\widetilde{C} \to C$ be the normalization and $\widetilde{\Sigma} \subset \widetilde{C}$ the set of preimages of nodes. Show that there is an identification

$$\mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) \cong \mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C),$$

or in other words that regular vector fields on C correspond to regular vector fields on $\widetilde{C}$ vanishing at the preimages of nodes.

5.2.5 Nodal families

Recall that the relative singular locus $\mathrm{Sing}(\mathcal{C}/S)$ of a morphism $\mathcal{C} \to S$ with one-dimensional fibers is defined by the first fitting ideal sheaf of $\Omega_{\mathcal{C}/S}$; see Definition A.3.14. Syntomic morphisms are fppf morphisms whose fibers are local complete intersections; see §A.3.5.

Proposition 5.2.21. *Let $\mathcal{C} \to S$ be an fppf morphism of schemes and $s \in S$ a point such that the fiber $\mathcal{C}_s$ is pure one-dimensional. A point $p \in \mathcal{C}_s$ is a node if and only if $\mathcal{C} \to S$ is syntomic at p and the relative singular locus $\mathrm{Sing}(\mathcal{C}/S) \to S$ is unramified at p.*

Proof. The conditions that $\mathcal{C} \to S$ is syntomic at p and $\mathrm{Sing}(\mathcal{C}) \to S$ is unramified at p are both conditions on the fibers over s. Since $\mathrm{Sing}(\mathcal{C}/S)_s = \mathrm{Sing}(\mathcal{C}_s)$, the result follows from the equivalence of (1) $\iff$ (4) of the Characterization of Nodes (5.2.3). $\square$

The above characterization shows that the property of being a nodal family descends under limits (Definition B.3.6).

Lemma 5.2.22. *The following property of morphisms of schemes descends under limits: an fppf morphism such that every fiber is a pure one-dimensional nodal curve.*

Proof. From Descent of Properties of Morphisms under Limits (B.3.7), we know that the properties of being fppf, syntomic, unramified, and having connected pure one-dimensional fibers descend under limits. Since the relative singular locus commutes with base change, the result follows from Proposition 5.2.21. $\square$

Later in Definition 5.3.17, we define a *family of nodal curves* to be a proper fppf morphism $\mathcal{C} \to S$ of schemes such that every geometric fiber $\mathcal{C}_s$ is a connected nodal curve.

5.2.6 Local structure of nodal families

By the Local Structure of Smooth Morphisms (A.3.4), if $\mathcal{C} \to S$ is a family of smooth curves, then every point $p \in \mathcal{C}$ over $s \in S$ is étale locally isomorphic to relative

one-dimensional affine space. More precisely, there is a commutative diagram

$$(\mathcal{C}, p) \xleftarrow{\text{op}} (\mathcal{C}', p') \xrightarrow{\text{ét}} (S' \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}, (s', 0))$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \swarrow$$
$$(S, s) \xleftarrow{\text{op}} (S', s')$$

where the left horizontal maps are open immersions, the right-hand map is étale map, and $S' \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} = \mathbb{A}^1_{S'} \to S'$ is the base change of $\mathbb{A}^1_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$. We give a local structure of a family of nodal curves generalizing the Local Structure of Nodes (5.2.7).

Theorem 5.2.23 (Local Structure of Nodal Families). *Let $\pi \colon \mathcal{C} \to S$ be an fppf morphism such that every geometric fiber is a curve. Let $p \in \mathcal{C}$ be a node in the fiber $\mathcal{C}_s$ over a point $s \in S$. There is a commutative diagram*

$$(\mathcal{C}, p) \xleftarrow{\text{ét}} (\mathcal{C}', p') \xrightarrow{\text{ét}} (\operatorname{Spec} A[x, y]/(xy - f), (s', 0))$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \swarrow \qquad\qquad\qquad\qquad (5.2.24)$$
$$(S, s) \xleftarrow{\text{ét}} (\operatorname{Spec} A, s'),$$

where each horizontal map is étale and $f \in A$ is a function vanishing at s'.

Remark 5.2.25. In other words, every family of nodal curves étale locally looks like the top horizontal arrow in the fiber product

$$\operatorname{Spec} A[x, y]/(xy - f) \longrightarrow \operatorname{Spec} A$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow f$$
$$\operatorname{Spec} \mathbb{Z}[x, y, t]/(xy - t) \longrightarrow \operatorname{Spec} \mathbb{Z}[t]$$

induced by a function $f \in A$.

Proof 1 (local-to-global).

Step 1: Reduce to the case where S is of finite type over $\mathbb{Z}$. Use limit methods and Lemma 5.2.22.

Step 2: Reduce to the case where $\widehat{\mathcal{O}}_{\mathcal{C}_s, p} \cong \kappa(s)[\![x, y]\!]/(xy)$. By Proposition 5.2.3, there is a finite separable field extension $\kappa(s) \to \Bbbk'$ and a point $p' \in \mathcal{C}_s \times_{\kappa(s)} \Bbbk'$ whose completion is isomorphic to $\Bbbk'[\![x, y]\!]/(xy)$. Letting $(S', s') \to (S, s)$ be an étale morphism such that there is an isomorphism $\kappa(s') \cong \Bbbk'$ over $\kappa(s)$, we replace S with S'.

Step 3: Show that $\widehat{\mathcal{O}}_{\mathcal{C}, p} \cong \widehat{\mathcal{O}}_{S,s}[\![x, y]\!]/(xy - \widehat{f})$ for a function $\widehat{f} \in \widehat{\mathfrak{m}}_s \subset \widehat{\mathcal{O}}_{S,s}$. We claim that there exists elements $\widehat{x}, \widehat{y} \in \widehat{\mathcal{O}}_{\mathcal{C}, p}$ and $\widehat{f} \in \widehat{\mathfrak{m}}_s \subset \widehat{\mathcal{O}}_{S,s}$ such that $\widehat{x}\widehat{y} - \widehat{f} = 0$. To achieve this, we will inductively construct elements $x_n, y_n \in \widehat{\mathcal{O}}_{\mathcal{C}, p}$ and $f_n \in \widehat{\mathfrak{m}}_s$ for $n \geq 0$ which are compatible (i.e. $x_{n+1} \equiv x_n \pmod{\mathfrak{m}^{n+1}}$, etc.) such that

$$x_n y_n - f_n \in \widehat{\mathfrak{m}}_s^{n+1} \widehat{\mathcal{O}}_{\mathcal{C}, p}. \qquad\qquad (5.2.26)$$

The claim follows by defining $\widehat{x} = \lim_n x_n$, $\widehat{y} = \lim_n y_n$, and $\widehat{f} = \lim_n f_n$.

The base case $n = 0$ is handled by Step 2: letting $\overline{x}, \overline{y} \in \widehat{\mathcal{O}}_{\mathcal{C}_s,p}$ be the images of x and y under the isomorphism $\widehat{\mathcal{O}}_{\mathcal{C}_s,p} \cong \kappa(s)[\![x,y]\!]/(xy)$, choose $x_0, y_0 \in \widehat{\mathcal{O}}_{\mathcal{C},p}$ to be any lifts of $\overline{x}$ and $\overline{y}$ under the surjection $\widehat{\mathcal{O}}_{\mathcal{C},p} \to \widehat{\mathcal{O}}_{\mathcal{C}_s,p}$ and set $f_0 = 0$.

Assuming that we've constructed x_n, y_n, and f_n satisfying (5.2.26), write

$$x_n y_n - f_n = \sum_i a_i b_i, \qquad \text{where } a_i \in \widehat{\mathfrak{m}}_s^{n+1} \text{ and } b_i \in \widehat{\mathcal{O}}_{\mathcal{C},p}.$$

Since x_n and y_n generate the maximal ideal of p in the fiber $\mathcal{C}_s$, we can use the identity $\kappa(s) = \kappa(p)$ on residue fields to find $a_i' \in \widehat{\mathcal{O}}_{S,s}$ and $b_i', b_i'' \in \widehat{\mathcal{O}}_{\mathcal{C},p}$ such that

$$b_i - (x_n b_i' + y_n b_i'' + a_i') \in \widehat{\mathfrak{m}}_s \widehat{\mathcal{O}}_{\mathcal{C},p}.$$

We then define

$$x_{n+1} = x_n - \sum_i a_i b_i'', \quad y_{n+1} = y_n - \sum_i a_i b_i', \quad \text{and} \quad f_{n+1} = f_n + \sum_i a_i a_i',$$

and check that

$$
\begin{aligned}
x_{n+1} y_{n+1} - f_{n+1} &= (x_n - \sum_i a_i b_i'')(y_n - \sum_i a_i b_i') - (f_n + \sum_i a_i a_i') \\
&= (x_n y_n - f_n) - x_n \sum_i a_i b_i' - y_n \sum_i a_i b_i'' - \sum_i a_i a_i' + \sum_{i,j} a_i a_j b_i'' b_j' \\
&= \sum_i a_i b_i - x_n \sum_i a_i b_i' - y_n \sum_i a_i b_i'' - \sum_i a_i a_i' + \sum_{i,j} a_i a_j b_i'' b_j' \\
&= \sum_i \underbrace{a_i}_{\widehat{\mathfrak{m}}_s^{n+1}} \underbrace{(b_i - x_n b_i' - y_n b_i'' - a_i')}_{\widehat{\mathfrak{m}}_s \widehat{\mathcal{O}}_{C,p}} + \sum_{i,j} \underbrace{a_i a_j}_{\widehat{\mathfrak{m}}_s^{2(n+1)}} b_i'' b_j'
\end{aligned}
$$

is an element of $\widehat{\mathfrak{m}}_s^{n+2} \widehat{\mathcal{O}}_{C,p}$.

With the claim established, we have a well-defined $\widehat{\mathcal{O}}_{S,s}$-algebra homomorphism

$$\widehat{\mathcal{O}}_{S,s}[\![x,y]\!]/(xy - \widehat{f}) \to \widehat{\mathcal{O}}_{C,p},$$

defined by $x \mapsto \widehat{x}$ and $y \mapsto \widehat{y}$. This map is surjective by Complete Nakayama's Lemma (B.5.6(3)) as it is surjective modulo $\widehat{\mathfrak{m}}_s$, and it is injective by a version of the local criterion for flatness (Lemma A.2.8); it is thus an isomorphism.

Step 4 (formal-to-étale): Extend the isomorphism in Step 3 on formal neighorhoods to étale neighborhoods. From Step 3, we have a diagram

$$
\begin{array}{ccc}
\mathcal{C} \times_S \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} & & \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f}) \\
& \searrow \qquad \swarrow & \\
& \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} &
\end{array}
$$

such that the points $(p,s) \in \mathcal{C} \times_S \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ and $(s,0) \in \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f})$ have isomorphic completion. A consequence of Artin Approximation (Corollary B.5.31) implies that there are étale morphisms

$$
\begin{array}{ccc}
& (U, u) & \\
\swarrow & & \searrow \\
(\mathcal{C} \times_S \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}, (p,s)) & & (\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}[x,y]/(xy - \widehat{f}), (s,0))
\end{array}
$$

$$\tag{5.2.27}$$

defined over $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$. After replacing S with an open affine neighborhood of s, we can assume that $S = \operatorname{Spec} A$ is affine. By Neron–Popescu (B.5.21), we may write $\widehat{\mathcal{O}}_{S,s} = \operatorname{colim} B_\lambda$ as a directed colimit of smooth A-algebras. Set $S_\lambda = \operatorname{Spec} B_\lambda$, $\mathcal{C}_\lambda = \mathcal{C} \times_S S_\lambda$, and $U_\lambda = U \times_S S_\lambda$. For $\lambda \gg 0$, $\widehat{f} \in \widehat{\mathcal{O}}_{S,s}$ is the image of an element $f_\lambda \in B_\lambda$, and the pullbacks of x and y to $\Gamma(U, \mathcal{O}_U)$ are the pullbacks of elements in $\Gamma(U_\lambda, \mathcal{O}_{U_\lambda})$ under $U \to U_\lambda$. This yields a commutative diagram

$$
\begin{array}{ccc}
& U_\lambda & \\
\swarrow & & \searrow \\
\mathcal{C}_\lambda & & \operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda)
\end{array}
$$

over S_λ which base changes to (5.2.27) under $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to S_\lambda$. By Descent of Properties of Morphisms under Limits(B.3.7), étaleness descends under limits and so the maps $U_\lambda \to \mathcal{C}_\lambda$ and $U_\lambda \to \operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda)$ are étale for $\lambda \gg 0$. Letting $u_\lambda = (u, s_\lambda) \in U_\lambda$, we have a commutative diagram

$$
\begin{array}{ccc}
(\mathcal{C}, p) \xleftarrow{\ \mathrm{sm}\ } (U_\lambda, u_\lambda) \xrightarrow{\ \mathrm{\acute{e}t}\ } (\operatorname{Spec} B_\lambda[x,y]/(xy - f_\lambda), (s_\lambda, 0)) \\
\downarrow \qquad\qquad\quad \downarrow \qquad\qquad\qquad \swarrow \\
(S, s) \xleftarrow{\ \mathrm{sm}\ } (\operatorname{Spec} B_\lambda, s_\lambda)
\end{array}
$$

This gives our desired diagram (5.2.24) except that the left horizontal arrows are smooth rather than étale. Since smooth maps étale locally have sections (Corollary A.3.5), there is an étale map $(\operatorname{Spec} A, s') \to (S, s)$ and a map $(\operatorname{Spec} A, s') \to (S_\lambda, s_\lambda)$ over S. The result follows from setting $\mathcal{C}' := U_\lambda \times_{S_\lambda} \operatorname{Spec} A$ and $p' = (u_\lambda, s')$.

See also [SP, Tag 0CBY]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Proof 2 (avoiding Artin Approximation/Neron–Popescu). We can reduce to the case where S is of finite type over $\mathbb{Z}$ by Lemma 5.2.22. By Proposition 5.2.21, we may replace $\mathcal{C}$ with an open neighborhood of p such that $\mathcal{C} \to S$ is syntomic and $\operatorname{Sing}(\mathcal{C}/S) \to S$ is unramified. After replacing $\mathcal{C}$ and S with open neighborhoods, we may also assume that $\mathcal{C}$ and S are affine and that the geometric fibers of $\mathcal{C} \to S$ are connected with at most two irreducible components. We can choose a closed immersion $S \hookrightarrow \mathbb{A}_{\mathbb{Z}}^n$, and we can apply Proposition A.3.7 to find a syntomic morphism $\mathcal{C}' \to \mathbb{A}_{\mathbb{Z}}^n$ extending $\mathcal{C} \to S$. The fiber $\mathcal{C}'_s$ has a node at p, and after replacing $\mathcal{C} \to S$ with $\mathcal{C}' \to \mathbb{A}_{\mathbb{Z}}^n$, we may assume that the base S is regular. By the étale local structure of unramified morphisms (Proposition A.3.6), after replacing $\mathcal{C}$ and S with étale neighborhoods, we can arrange that $\operatorname{Sing}(\mathcal{C}/S) \hookrightarrow S$ is a closed immersion.

We claim that after replacing S with an open neighborhood of s, we can arrange that $\operatorname{Sing}(\mathcal{C}/S) = S$ or that $\operatorname{Sing}(\mathcal{C}/S) \subsetneq S$ is defined by a nonzerodivisor $f \in \Gamma(S, \mathcal{O}_S)$. This holds over the completion of S at s by Step 3 in the first proof above: since $\widehat{\mathcal{O}}_{\mathcal{C},p} \cong \widehat{\mathcal{O}}_{S,s}[\![x,y]\!]/(xy - \widehat{f})$ where $\widehat{f} \in \widehat{\mathfrak{m}}_s$, $\operatorname{Sing}(\mathcal{C}/S) \times_S \operatorname{Spec} \widehat{\mathcal{O}}_{S,s} = V(\widehat{f})$. The claim then follows from using fppf descent along $\operatorname{Spec} \widehat{\mathcal{O}}_{S,s} \to \operatorname{Spec} \mathcal{O}_{S,s}$ and properties of the ideal sheaf $\mathcal{I}$ defining $\operatorname{Sing}(\mathcal{C}/S)$. Indeed, if $\widehat{f} = 0$, then $\mathcal{I}_s = 0$ and hence $\mathcal{I}$ is zero in an open neighborhood of s. If $\widehat{f}$ is a nonzerodivisor, then $\mathcal{I}_s$ is a line bundle (by Proposition B.1.11) and hence $\mathcal{I}$ is defined by a nonzerodivisor in an open neighborhood of s.

If $\operatorname{Sing}(\mathcal{C}/S) = S$, we first claim that after replacing $\mathcal{C}$ with an étale neighborhood, we can arrange that $\mathcal{C}$ is the scheme-theoretic union $\mathcal{C}_1 \cup \mathcal{C}_1$ of closed subschemes such

that $\mathrm{Sing}(\mathcal{C}/S) = \mathcal{C}_1 \cap \mathcal{C}_2$. The normalization $\widetilde{Z} \to \mathrm{Spec}\,\mathcal{O}_{\mathcal{C},p}^{\mathrm{h}}$ of the henselization is a finite morphism, and since normalization commutes with completion (see Remark B.1.15), there are two preimages in $\widetilde{Z}$ of the unique closed point. By properties of the henselization (Proposition B.5.13), $\widetilde{Z}$ can be written as a disjoint union $\widetilde{Z} = \widetilde{Z}_1 \amalg \widetilde{Z}_2$ of closed subschemes. Therefore, $\mathrm{Spec}\,\mathcal{O}_{\mathcal{C},p}^h$ is the union of the (closed) images of $\widetilde{Z}_1$ and $\widetilde{Z}_2$. This establishes the claim. After replacing $\mathcal{C}$ with an open neighborhood, we can arrange that $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined by global functions $g_1, g_2 \in B := \Gamma(\mathcal{C}, \mathcal{O}_\mathcal{C})$ on $\mathcal{C}$ with $g_1 g_2 = 0$. Letting $S = \mathrm{Spec}\,A$, the ring map $A[x,y]/(xy) \to B$, defined by $x \mapsto g_1$ and $y \mapsto g_2$, induces a morphism $\mathcal{C} \to \mathrm{Spec}\,A[x,y]/(xy)$ over S. This map is étale at p since it induces an isomorphism of completions at p (using Step 3 of the first proof).

If $\mathrm{Sing}(\mathcal{C}/S) = V(f)$ with $f \in A := \Gamma(S, \mathcal{O}_S)$ a nonzerodivisor, then the argument above shows that $\mathcal{C} \times_A (A/f)$ is the scheme-theoretic union $Z_1 \cup Z_1$ of effective Cartier divisors such that $\mathrm{Sing}(\mathcal{C}/S) = Z_1 \cap Z_2$. After replacing $\mathcal{C}$ with an open neighborhood, we can write each $Z_i = V(g_i)$ for global functions $g_i \in B := \Gamma(\mathcal{C}, \mathcal{O}_\mathcal{C})$. As the restrictions of $g_1 g_2$ and f define the same closed subscheme of $\mathrm{Spec}\,\widehat{\mathcal{O}}_{\mathcal{C},p}$, we have that $f = u g_1 g_2$ for a unit $u \in B$ after replacing $\mathcal{C}$ with an open neighborhood. The ring map $A[x,y]/(xy - f) \to B$, defined by $x \mapsto u g_1$ and $y \mapsto g_2$, induces a morphism $\mathcal{C} \to \mathrm{Spec}\,A[x,y]/(xy)$ over S; this map is étale at p since it induces an isomorphism of completions at p. $\qquad\square$

5.3 Stable curves

Stable curves were introduced in unpublished joint work by Mayer and Mumford [MM64].

5.3.1 Definition and equivalences

An *n-pointed curve* is a curve C over an algebraically closed field $\Bbbk$ together with an ordered collection of $\Bbbk$-points $p_1, \ldots, p_n \in C$; we call the points $p_i \in C$ *marked points*. A point $q \in C$ of an n-pointed curve is called *special* if q is a node or a marked point.

Definition 5.3.1 (Stable, semistable, and prestable curves)**.** An n-pointed geometrically connected, nodal, and projective curve $(C, p_1, \ldots, p_n)$ of genus g over an algebraically closed field $\Bbbk$ is *stable* if

(1) $p_1, \ldots, p_n \in C$ are distinct smooth points,

(2) C is not of genus 1 without marked points, i.e. $(g, n) \neq (1, 0)$, and

(3) every smooth irreducible rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special points.

If (1)-(2) hold, and (3) is replaced with the condition that every smooth rational subcurve contains at least 2 (rather than 3) special points, we say that (C, p_i) is *semistable*. If only (1)-(2) hold, we say that (C, p_i) is *prestable*.

We have the implications:

$$\boxed{\text{stable} \implies \text{semistable} \implies \text{prestable} \implies n\text{-pointed nodal.}}$$

In the unpointed case, a curve is a prestable curve if it is nodal. When $\Bbbk$ is not algebraically closed, the definition of stability needs to be amended; see Proposition 5.3.12

Remark 5.3.2. Note that there are no n-pointed stable curves of genus g if $(g, n) \in \{(0, 0), (0, 1), (0, 2), (1, 0)\}$, or equivalently $2g - 2 + n \leq 0$. We often impose the condition that $2g - 2 + n > 0$ to exclude these special cases.
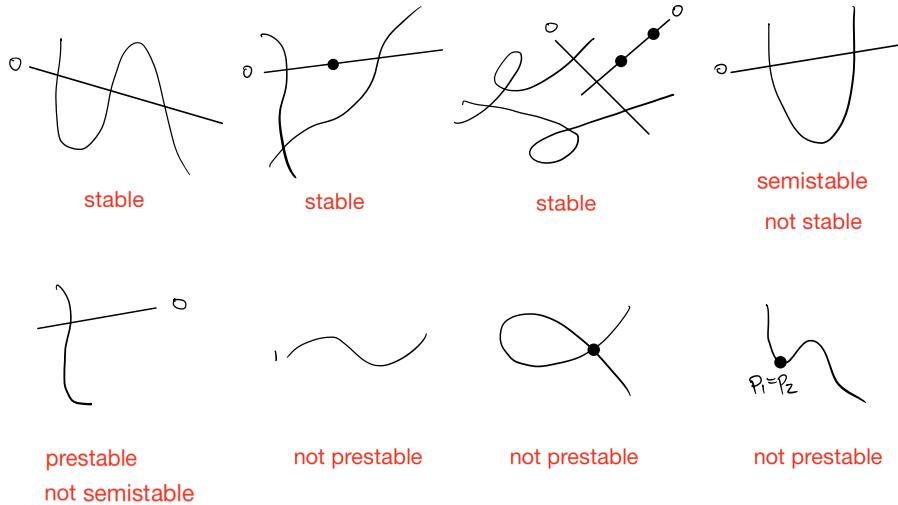
Figure 5.3: Examples of stable, semistable, and prestable curves

An automorphism of a stable curve $(C, p_1, \ldots, p_n)$ is an automorphism $\alpha \colon C \xrightarrow{\sim} C$ such that $\alpha(p_i) = p_i$. By $\mathrm{Aut}(C, p_1, \ldots, p_n)$, we denote the (abstract) group of automorphisms. Recall also that if C is a geometrically connected, smooth, and projective curve of genus $g \geq 2$, then $\mathrm{Aut}(C)$ is finite [Har77, Exer. III.2.5].

Proposition 5.3.3. *Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve over an algebraically closed field $\Bbbk$. The following are equivalent:*

(1) $(C, p_1, \ldots, p_n)$ is stable,

(2) $\mathrm{Aut}(C, p_1, \ldots, p_n)$ is finite, and

(3) $\omega_C(p_1 + \cdots + p_n)$ is ample.

Proof. The equivalence (1) $\iff$ (2) follows from Exercise 5.3.5 and the fact that the only way that a smooth prestable n-pointed curves (C, p_i) can have a positive dimensional automorphism group is if $C = \mathbb{P}^1$ with $n \leq 2$ or if C is a genus 1 curve with $n = 0$ (see Proposition 5.1.7).

To see the equivalence with (3), we will use the fact that for a subcurve $T \subset C$, we have $\omega_C|_T = \omega_T(T \cap T^c)$ (Exercise 5.2.19). The line bundle $\omega_C(p_1 + \cdots + p_n)$ is ample if and only if its restriction to each irreducible component $T \subset C$

$$\omega_C(p_1 + \cdots + p_n)|_T = \omega_T(\sum_{p_i \in T} p_i + (T \cap T^c)) \tag{5.3.4}$$

is ample. If the genus $g(T)$ of T is at least two, then ω_T is ample and thus so is (5.3.4). If $g(T) = 1$, then (5.3.4) is ample if and only if $n \geq 1$ or T must meet the complement T^C. If $g(T) = 0$, then (5.3.4) is ample if and only if T contains at least three special points. $\qquad\square$

Exercise 5.3.5 (Pointed normalization). Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve. Let $\pi \colon \widetilde{C} \to C$ be the normalization of C, $\widetilde{p}_i \in \widetilde{C}$ be the unique preimage of p_i, and $\widetilde{q}_1, \ldots, \widetilde{q}_m \in \widetilde{C}$ be an ordering of the preimages of nodes. We call $(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$ the *pointed normalization of* (C, p_i). Show that (C, p_i) is stable if and only if every pointed connected component of $(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$ (i.e. each connected component of $\widetilde{C}$ together with the points $\widetilde{p}_i$ and $\widetilde{q}_j$ lying on it) is stable.

Exercise 5.3.6. Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve.

(a) Show that the automorphism group scheme $\underline{\mathrm{Aut}}(C, p_i)$ is an algebraic group.

(b) Show that $\underline{\mathrm{Aut}}(C, p_i)$ is naturally a closed subgroup of the automorphism group $\underline{\mathrm{Aut}}(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$ of the pointed normalization $(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$.

(c) Show that $\underline{\mathrm{Aut}}(C, p_i)^0 = \underline{\mathrm{Aut}}(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)^0$ but that in general $\underline{\mathrm{Aut}}(C, p_i) \neq \underline{\mathrm{Aut}}(\widetilde{C}, \widetilde{p}_i, \widetilde{q}_j)$.

Dual graphs. A *vertex-weighted, n-marked graph* $\Gamma = (G, w, m)$ is the data of a finite, connected, and undirected graph G with vertices $V(G)$ and edges $E(G)$ (where loops and parallel edges are allowed), a weight function $w \colon V(G) \to \mathbb{Z}_{\geq 0}$, and an n-marking $m \colon \{1, \ldots, n\} \to V(G)$. Each vertex $m(i) \in V(G)$ is viewed as a *half-edge*. The *genus of* Γ is

$$g(\Gamma) = g(G) + \sum_{v \in V(G)} w(v),$$

where

$$g(G) = |E(G)| - |V(G)| + 1$$

denotes the genus of G, i.e. the first Betti number of G considered as a 1-dimensional CW complex. We say that a vertex-weighted, n-marked graph Γ is *stable* if for every vertex $v \in V(G)$

$$2w(v) - 2 + \mathrm{val}(v) + |m^{-1}(v)| > 0,$$

where $\mathrm{val}(v)$ is the valence of v, defined as the number of edges containing v with loops counted twice. In other words, stability means that for every vertex v either (a) $w(v) \geq 2$, (b) $w(v) = 1$ and v is contained in an edge or half-edge, or (c) $w(v) = 0$ and $\mathrm{val}(v) + |m^{-1}(v)| \geq 3$.
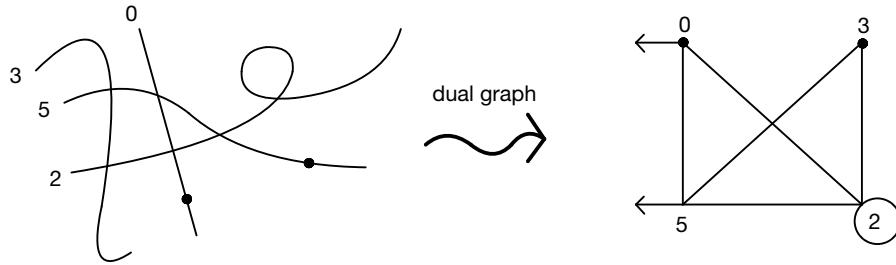
Figure 5.4: A genus 13 curve and its dual graph.

Definition 5.3.7 (Dual graph). The *dual graph* of an n-pointed prestable curve $(C, p_1, \ldots, p_n)$ is defined as $\Gamma = (G, w, m)$ where the vertices v_i of G correspond to irreducible components C_i of C, the weight $w(v_i)$ is the geometric genus of C_i, for every node q of C lying on components C_i and C_j there is an edge e_q between v_i and v_j, and the marking $m(i)$ is the vertex v_j with $p_i \in C_j$.

Exercise 5.3.8 (easy). Show that an n-pointed prestable curve (C, p_i) is stable if and only if its dual graph Γ is stable, and that the genus of C is equal to the genus of Γ.

The stack $\overline{\mathcal{M}}_{g,n}$ of stable curves admits a stratification according to the dual graph (Exercise 5.6.17).
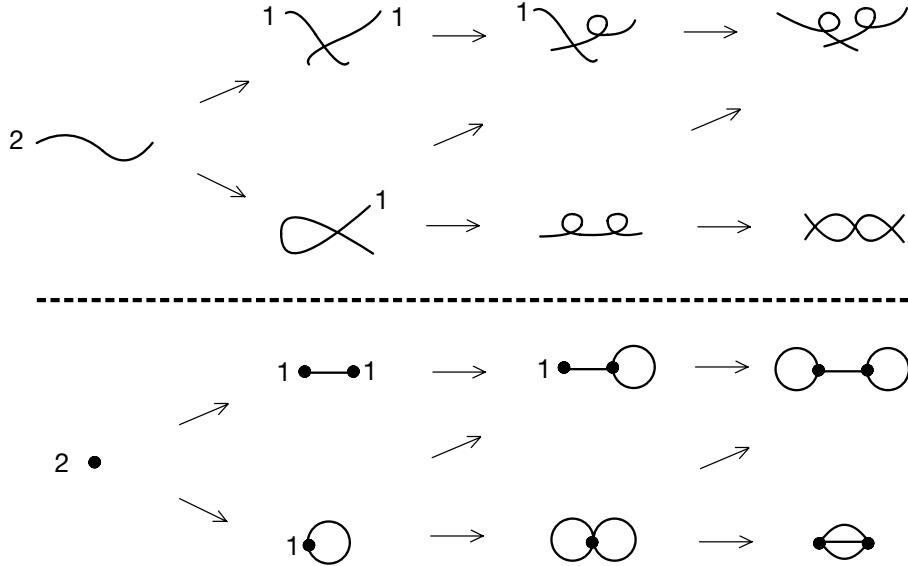
Figure 5.5: There are seven genus 2 vertex-weighted graphs with no marked points. Curves are displayed on the top while their dual graphs are displayed below. The arrows indicate specialization. The dimension of the substacks of $\overline{\mathcal{M}}_2$ parameterizing curves with a given dual graph are 3, 2, 1, and 0 in the columns going from left to right.

Exercise 5.3.9.

(a) Classify stable, vertex-weighted 2-marked graphs of genus 1.

(b) Determine the number of isomorphism classes of stable, vertex-weighted 0-marked graphs of genus 3.

5.3.2 Rational tails and bridges

Definition 5.3.10 (Rational tails and bridges). Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve over a field $\Bbbk$. We say that an irreducible smooth subcurve $E \subset C$ of genus 0 with nonempty complement $E^c = \overline{C \setminus E}$ is

- a *rational tail* if the scheme-theoretic intersection $E \cap E^c$ is a single reduced point x, $\mathrm{H}^0(E, \mathcal{O}_E) \cong \kappa(x)$, and E contains no marked points, and

- a *rational bridge* if the scheme-theoretic intersection $E \cap E^c$ has degree 2 over $\mathrm{H}^0(E, \mathcal{O}_E)$ and E contains no marked points, or if E contains a single marked point p_j and E is a rational tail of $(C, p_1, \ldots, \widehat{p}_j, \ldots, p_n)$.

Over an algebraically closed field, every irreducible smooth rational subcurve E is isomorphic to $\mathbb{P}^1$. The subcurve E is a rational tail if $E \cdot E^c = 1$ and E has no

marked points, and is a rational bridge if either $E \cdot E^c = 2$ and E has no marked points or if $E \cdot E^c = 1$ and E has precisely one marked point.
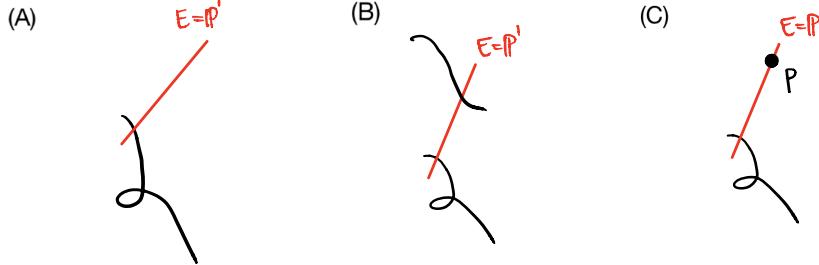
Figure 5.6: (A) features a rational tail while (B) and (C) feature rational bridges.

Over an arbitrary field, the classification of rational tails and bridges is more subtle, which is unsurprising given that the classification of rational curves is also more involved (see Exercises 5.1.12 and 5.1.13).

Lemma 5.3.11. *Let (C, p_i) be an n-pointed prestable curve over a field $\Bbbk$.*

(1) *If E is a rational tail with $\Bbbk' := \mathrm{H}^0(E, \mathcal{O}_E) = \kappa(x)$, then $\Bbbk'$ is a finite separable field extension of $\Bbbk$ and $E \cong \mathbb{P}^1_{\Bbbk'}$.*

(2) *If E is a rational bridge with no marked points, then $\Bbbk' := \mathrm{H}^0(E, \mathcal{O}_E)$ is a finite separable field extension of $\Bbbk$ and $\Bbbk'' := \mathrm{H}^0(E \cap E^c, \mathcal{O}_{E \cap E^c})$ is either a degree 2 separable field extension of $\Bbbk'$ or $\Bbbk'' = \Bbbk' \times \Bbbk'$.*

If $K/\Bbbk$ is a field extension, (C, p_i) has a rational tail (resp. rational bridge) if and only if $(C \times_\Bbbk K, p_i \times_\Bbbk K)$ has a rational tail (resp. rational bridge).

Proof. In both cases, since E is reduced and connected, $\Bbbk' = \mathrm{H}^0(E, \mathcal{O}_E)$ is a field. For (1), the intersection $x = E \cap E^c$ is a node and hence $\Bbbk' = \kappa(x)$ is a separable field extension of $\Bbbk$ (Proposition 5.2.3). Viewing E as a scheme over $\Bbbk'$ via structure map $E \to \mathrm{Spec}\,\Gamma(E, \mathcal{O}_E) = \mathrm{Spec}\,\Bbbk'$, we see that $E \cong \mathbb{P}^1_{\Bbbk'}$ since it is irreducible, smooth, genus 0, and contains a rational point. For (2), the points of $E \cap E^c$ are nodes in C but smooth points in both E and E^c. As $\dim_{\Bbbk'} \Bbbk'' = 2$, we see that either $\Bbbk''$ is a field, in which case it is separable over $\Bbbk$ by Proposition 5.2.3, or $\Bbbk'' = \Bbbk' \times \Bbbk'$.

If E is a rational tail (resp. bridge) of (C, p_i), then the above characterization implies that $E_{\Bbbk'}$ is a disjoint union of rational tails (resp. bridges). Conversely, if $E' \subset C_{\Bbbk'}$ is a rational tail (resp. bridge) of the base change, we claim that the image $E \subset C$ of E' is also a rational tail (resp. bridge). As E' is irreducible, so is E. Since the complement $(E')^c$ is nonempty, so is E^c. We observe that the base change E_K is a disjoint union of curves isomorphic to E': to see this, it suffices to assume either that $K/\Bbbk$ is Galois in which case $E_K = \amalg_{\sigma \in \mathrm{Gal}(K/\Bbbk)} \sigma \cdot E'$, or $K/\Bbbk$ is purely inseparable in which case $E_K = E'$. It follows from descent that E is smooth over $\Bbbk$ of genus 0. By Flat Base Change (A.2.12) $\mathrm{H}^0(E_K, \mathcal{O}_{E_K}) = \mathrm{H}^0(E, \mathcal{O}_E) \otimes_\Bbbk K$ and it follows that the natural map $E \to \mathrm{Spec}\,\mathrm{H}^0(E, \mathcal{O}_E)$ pulls back to $E_K \to \mathrm{Spec}\,\mathrm{H}^0(E', \mathcal{O}_{E'})$. It follows from descent that the composition $E \cap E^c \hookrightarrow E \to \mathrm{Spec}\,\mathrm{H}^0(E, \mathcal{O}_E)$ is an isomorphism (resp. finite étale of degree 2) as the base change over K is. If E' doesn't have marked points (resp. E' is a rational tail with a marked point), the same holds for E. $\square$

Stability is equivalent to not containing a rational tail or bridge.

Proposition 5.3.12. *Let (C, p_i) be an n-pointed prestable curve of genus g over a field $\Bbbk$, and assume that $(g, n) \neq (1, 0)$. If K is an algebraically closed field containing $\Bbbk$, then $(C \times_\Bbbk K, p_i \times_\Bbbk K)$ is stable (resp. semistable) if and only if (C, p_i) contains no rational tails or rational bridges (resp. no rational tails).*

Proof. Over an algebraically closed field, stability (resp. semistability) is equivalent to not containing a rational tail or bridge (resp. rational tail). Therefore, the statement follows from Lemma 5.3.11 as the condition of having a rational tail or bridge is insensitive to field extensions. $\square$

Remark 5.3.13 (Relationship to -1 and -2 curves). Suppose that $\mathcal{C} \to \Delta = \operatorname{Spec} R$ is a family of nodal curves over a DVR R with algebraically closed residue field $\Bbbk$ such that the generic fiber $\mathcal{C}^*$ is smooth. If $E \cong \mathbb{P}^1 \subset \mathcal{C}'_0$ is a smooth rational subcurve in the central fiber, then $E^2 = -E \cdot E^c$ (which follows from the identity $0 = E \cdot \mathcal{C}_0 = E \cdot E + E \cdot E^c$). Thus if E is a rational tail (resp. rational bridge without a marked point), then $E^2 = -1$ (resp. $E^2 = -2$).
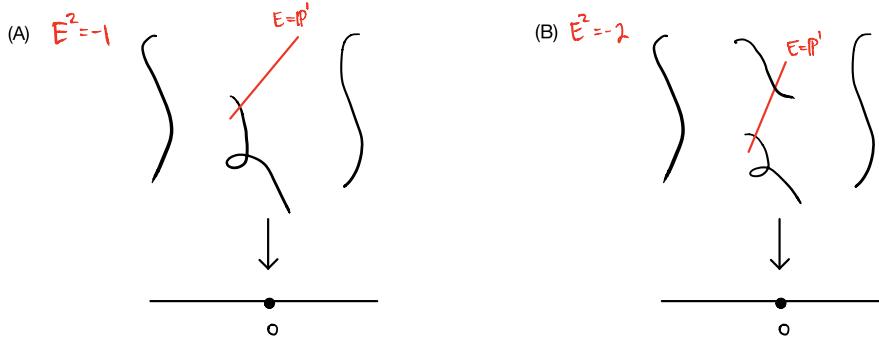
Figure 5.7: In (A) (resp. (B)), the exceptional component E meets the rest of the curve at one point (resp. two points) and $E^2 = -1$ (resp. $E^2 = -2$).

The following exercise will be generalized later by the Stable Contraction of a Prestable Family (5.6.4).

Exercise 5.3.14 (hard). Let $\mathcal{C} \to \Delta = \operatorname{Spec} R$ be a family of nodal curves over a DVR R with smooth generic fiber. For any rational tail or bridge $E \subset \mathcal{C}_0$ in the central fiber, show that there is a contraction $\mathcal{C} \to \mathcal{C}'$ of E where $\mathcal{C}' \to \Delta$ is a family of nodal curves.

Hint: Castelnuovo's Contraction Theorem (B.6.6) gives the existence of the contraction $\mathcal{C} \to \mathcal{C}'$. The challenge here is to show that the central fiber $\mathcal{C}'_0$ is nodal. You may want to appeal to a fact in the minimal model program that $\mathcal{C}'_0$ is nodal if and only if the pair $(\mathcal{C}', \mathcal{C}'_0)$ is log canonical [Kol13, Cor. 2.32, Thm. 4.9(2)], and show that the latter property holds because $(\mathcal{C}, \mathcal{C}_0)$ (resp. $(\mathcal{C}, \mathcal{C}_0 - E)$) is log canonical.

5.3.3 Positivity of the dualizing sheaf

Exercise 5.3.15 (moderate, details). Let $(C, p_1, \ldots, p_n)$ be an n-pointed prestable curve over an algebraically closed field $\Bbbk$, and let $L := \omega_C(p_1 + \cdots + p_n)$.

(a) If (C, p_i) is stable, show that $L^{\otimes k}$ is very ample for $k \geq 3$ and that $\mathrm{H}^1(C, (\omega_C(p_1 + \cdots + p_n))^{\otimes k}) = 0$ for $k \geq 2$.

(b) For $k \geq 2$, show that (C, p_i) is semistable if and only if $L^{\otimes k}$ is base point free.

Hint: For (a), show that the global sections of $L^{\otimes k}$ separate points and tangent vectors. In other words, show that the maps

$$\mathrm{H}^0(C, L^{\otimes k}) \to \left(L^{\otimes k} \otimes \kappa(x)\right) \oplus \left(L^{\otimes k} \otimes \kappa(y)\right) \qquad \mathrm{H}^0(C, L^{\otimes k}) \to L^{\otimes k} \otimes \mathcal{O}_{C,x}/\mathfrak{m}_x^2$$

are surjective. Establish this using Serre Duality and a case analysis on whether x, y are smooth or nodal. See also [DM69, Thm. 2], [ACG11, Lem. 10.6.1], [SP, Tag 0E8X], and [Ols16, Prop. 13.2.17].

Exercise 5.3.16 (good practice)**.** If C is the nodal union $C_1 \cup C_2$ of genus i and $g - i$ curves along a single node $p = C_1 \cap C_2$, show that ω_C has a base point at p.

5.3.4   Families of stable curves

Definition 5.3.17. A *family of n-pointed curves* over a scheme S is a proper, flat, and finitely presented morphism $\mathcal{C} \to S$ of *algebraic spaces* together with n sections $\sigma_1, \ldots, \sigma_n \colon S \to \mathcal{C}$ such that every geometric fiber $\mathcal{C}_s$ is a curve over $\mathrm{Spec}\,\kappa(s)$ (e.g. a finite type $\kappa(s)$-scheme of dimension 1).

A *family of n-pointed stable curves* (resp. semistable, prestable, nodal) curves is a family of n-pointed curves such that every geometric fiber $(\mathcal{C}_s, \sigma_1(s), \ldots, \sigma_n(s))$ is stable (resp. semistable, prestable, nodal).

Caution 5.3.18. If $(\mathcal{C} \to S, \sigma_i)$ is a stable family over a scheme S, we will show that $\mathcal{C} \to S$ is necessarily projective (Proposition 5.3.19), hence $\mathcal{C}$ is a scheme. While every one-dimensional separated algebraic space of finite type over a field is a scheme (Theorem 4.4.32), in the relative setting, the total family $\mathcal{C}$ may not be a scheme. There exists a families of prestable genus 0 curves [Ful10, Ex. 2.3] and smooth genus 1 curves [Ray70, XIII 3.2] where the total families are not schemes. In the next section, we show that the stack of all curves (resp. nodal curves, semistable curves, prestable curves) are algebraic and in particular satisfies the descent condition for families of curves, and this result requires that the total family is an algebraic space.

For a family of n-pointed curves (resp. nodal curves), there is no condition on whether the marked points are distinct or land in the relative smooth locus of $\mathcal{C} \to S$. For stable, semistable, and prestable families, the marked points are distinct and avoid the nodes.

If $(\mathcal{C} \to S, \sigma_i)$ is a family of n-pointed prestable curves, then $\mathcal{C} \to S$ is local complete intersection morphism. Thus there is a relative dualizing line bundle $\omega_{\mathcal{C}/S}$ that is compatible with base change $T \to S$ and in particular restricts to the dualizing line bundle $\omega_{\mathcal{C}_s}$ on every fiber of $\mathcal{C} \to S$; see [Har66c] or [Liu02, §6.4]. The image of each section σ_i is a divisor contained in the smooth locus, and we can form the line bundle $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots + \sigma_n)$.

The following statement extends Proposition 5.3.3 to families.

Proposition 5.3.19. *Let $(\mathcal{C} \to S, \sigma_i)$ be a family of n-pointed prestable curves of genus g over a scheme S. The following are equivalent:*

(1) $(\mathcal{C} \to S, \sigma_i)$ is a family of stable curves, and

(2) $\omega_{\mathcal{C}/S}(\sigma_1 + \cdots + \sigma_n)$ is relatively ample over S.

In particular, a family of n-pointed stable curves is a projective morphism of schemes.

Proof. Relative ampleness can be checked on geometric fibers by Proposition 4.4.24. Since stability is also a condition on geometric fibers, the statement thus follows from Proposition 5.3.3. □

Exercise 5.3.20 (not essential)**.** Let $(\pi\colon \mathcal{C} \to S, \sigma_i)$ be a family of n-pointed prestable curves and let $L = \omega_{\mathcal{C}/S}(\sigma_1 + \cdots + \sigma_n)$. For $k \geq 2$, show that $(\mathcal{C} \to S, \sigma_i)$ is semistable if and only if $\pi^*\pi_*L \to L$ is surjective.

The following generalization of Properties of Families of Smooth Curves (5.1.15) is proven in the same way using the very ampleness of $\omega_C(p_1 + \cdots + p_n)^{\otimes 3}$ for a stable n-pointed curve (C, p_i) over an algebraically closed field (Exercise 5.3.15).

Proposition 5.3.21 (Properties of Families of Stable Curves)**.** *Let $(\mathcal{C} \to S, \sigma_i)$ be a family of n-pointed stable curves of genus g, and set $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample and $\pi_*(L^{\otimes k})$ is a vector bundle of rank $(2k - 1)(g - 1) + kn$.* $\qquad\square$

Proposition 5.3.22 (Openness of Stability)**.** *Let $(\pi\colon \mathcal{C} \to S, \sigma_i)$ be a family of n-pointed curves. The locus of points $s \in S$ such that $(\mathcal{C}_s, \sigma_i(s))$ is stable (resp. semistable, prestable, nodal) is open.*

Proof. We first claim that the nodal locus is open. When $\mathcal{C}$ is a scheme, this follows from the Local Structure of Nodes (5.2.23): the locus $\mathcal{C}^{\leq \mathrm{nod}} \subset \mathcal{C}$ of points which are smooth or nodal in their fiber is open, and therefore the locus of points $s \in S$ such that $\mathcal{C}_s$ has worse than nodal singularities is identified with the closed locus $\pi(\mathcal{C} \setminus \mathcal{C}^{\leq \mathrm{nod}}) \subset S$. In general, we can choose an étale cover $\mathcal{C}' \to \mathcal{C}$ by a scheme and use the fact that a point $p \in \mathcal{C}$ is a node in its fiber $\mathcal{C}_{\pi(p)}$ if and only if a preimage $p' \in \mathcal{C}'$ of p is a node in its fiber $\mathcal{C}'_{\pi(p)}$.

Since the locus in S where $\sigma_1(s), \ldots, \sigma_n(s)$ are distinct and smooth points in $\mathcal{C}_s$ is open, the condition that $\mathcal{C}_s$ is prestable is open. For an n-pointed prestable family $(\pi\colon \mathcal{C} \to S, \sigma_i)$, the locus of points $s \in S$ where $\mathcal{C}_s$ is semistable is identified with the open locus over which $\pi^*\pi_*L \to L$ is surjective, where $L := \omega_{\mathcal{C}/S}(\sum_i \sigma_i)$ (see Exercise 5.3.20).

To see that stability is an open condition in an n-pointed prestable (e.g. semistable) family, we provide two arguments. First, observe that $\underline{\mathrm{Aut}}(\mathcal{C}/S, \sigma_1, \ldots, \sigma_n) \to S$ is a finite type group scheme and that fiber dimension is upper semicontinuous (B.2.9). Therefore, the locus of points $s \in S$ such that $\mathrm{Aut}(\mathcal{C}_s, \sigma_1(s), \ldots, \sigma_n(s))$ is finite is open. As this condition categorizes stability (Proposition 5.3.3(2)), this open subset is identified with the stable locus. Second, by Openness of Ampleness (Proposition 4.4.24), the locus of points $s \in S$ such that $\omega_{\mathcal{C}/S}(\sum_i \sigma_i)|_{\mathcal{C}_s} \cong \omega_{\mathcal{C}_s}(\sum_i \sigma_i(s))$ is ample is open, and this condition also categorizes stability (Proposition 5.3.3(3)). $\qquad\square$

5.3.5 Deformation theory of stable curves

If C is a smooth curve over a field $\Bbbk$, then every first-order deformation is locally trivial (Proposition C.1.9) and the set $\mathrm{Def}(C)$ of isomorphism classes of first-order deformations is naturally in bijection with $\mathrm{H}^1(C, T_C)$ (Proposition C.1.12). Moreover, automorphisms, deformations, and obstructions of higher-order deformations are classified by $\mathrm{H}^i(C, T_C)$ for $i = 0, 1, 2$ (Proposition C.2.7).

Nodal singularities, on the other hand, have first-order deformations that are not locally trivial, e.g. $\mathrm{Spec}\,\Bbbk[x, y, \epsilon]/(xy - \epsilon) \to \mathrm{Spec}\,\Bbbk[\epsilon]$. In this section, we classify automorphisms, deformations, and obstructions of nodal curves (Proposition 5.3.23), describe first-order deformations of a prestable curve in terms of the pointed normalization and the singularities (Proposition 5.3.24), and then compute the dimensions of the groups classifying automorphisms, deformations, and obstructions of a stable curve (Proposition 5.3.27).

Proposition 5.3.23. *Let $A' \twoheadrightarrow A$ be a surjection of noetherian local rings with residue field $\Bbbk$. Suppose that $J = \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$. Let $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ be a family of prestable curves over A, and let (C, p_i) be its base change to the residue field $\Bbbk$.*

(1) The group of automorphisms of a deformation of $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over A' is bijective to $\operatorname{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_{\Bbbk} J)$.

(2) If there exists a deformation of $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over A', then the set of isomorphism classes of all such deformations is a torsor under $\operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_{\Bbbk} J)$.

(3) There is an element $\operatorname{ob}_{\mathcal{C}} \in \operatorname{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C \otimes_{\Bbbk} J)$ with the property that there exists a deformation of $(\mathcal{C} \to \operatorname{Spec} A, \sigma_i)$ over A' if and only if $\operatorname{ob}_{\mathcal{C}} = 0$.

Proof. Since nodal curves are generically smooth and local complete intersections, this is a consequence of Proposition C.2.12 (unpointed case) and Exercise C.2.13(a) (pointed case). $\qquad\square$

We denote by $\operatorname{Def}(C, p_i)$ (resp. $\operatorname{Def}^{\mathrm{lt}}(C, p_i)$) the vector space of first-order deformations (resp. locally trivial first-order deformations) of (C, p_i).

Proposition 5.3.24 (Local-to-global Deformation Sequence). *Let (C, p_i) be an n-pointed prestable curve over an algebraically closed field $\Bbbk$ with nodes $q_1, \ldots, q_s \in C$. Let $(\widetilde{C}, p_i, q'_j, q''_j)$ be the pointed normalization where $\pi \colon \widetilde{C} \to C$ is the normalization and $\pi^{-1}(q_j) = \{q'_j, q''_j\}$. There is an exact sequence*

$$0 \to \operatorname{Def}^{\mathrm{lt}}(C, p_i) \to \operatorname{Def}(C, p_i) \to \bigoplus_j \operatorname{Def}(\widehat{\mathcal{O}}_{C, q_j}) \to 0 \qquad (5.3.25)$$

and identifications

$$\begin{aligned} \operatorname{Def}^{\mathrm{lt}}(C, p_i) &\cong \operatorname{Def}(\widetilde{C}, p_i, q'_j, q''_j) \cong \operatorname{H}^1\big(\widetilde{C}, T_{\widetilde{C}}(-\textstyle\sum_i p_i - \sum_j (q'_j + q''_j))\big) \\ &\cong \operatorname{H}^1(C, \mathscr{H}om_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C)) \\ \operatorname{Def}(C, p_i) &\cong \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \\ \operatorname{Def}(\widehat{\mathcal{O}}_{C, q_j}) &\cong \operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C, q_j}}(\Omega^1_{\widehat{\mathcal{O}}_{C, q_j}}, \widehat{\mathcal{O}}_{C, q_j}) \cong \Bbbk. \end{aligned}$$

Under these identifications, (5.3.25) corresponds to short exact sequence

$$0 \to \operatorname{H}^1(C, \mathscr{H}om_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C)) \to \operatorname{Ext}^1_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C) \to$$
$$\bigoplus_j \operatorname{Ext}^1_{\widehat{\mathcal{O}}_{C, q_j}}(\Omega_{\widehat{\mathcal{O}}_{C, q_j}}, \widehat{\mathcal{O}}_{C, q_j}) \to 0, \quad (5.3.26)$$

arising as low-degree terms of the convergent spectral sequence
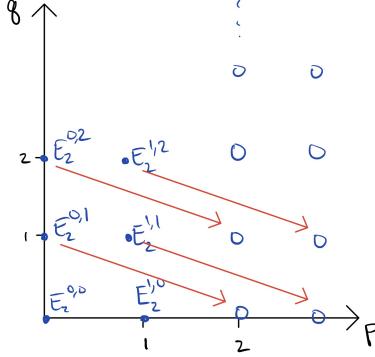
$$\operatorname{H}^p(C, \mathscr{E}xt^q_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C)) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}_C}(\Omega_C(\textstyle\sum_i p_i), \mathcal{O}_C).$$

Moreover, $\operatorname{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$.

Proof. There's a lot to unpack here. For simplicity, we handle only the case without marked points, i.e. $n = 0$. First, since $\operatorname{Hom}_{\mathcal{O}_C}(\Omega_C, -)$ is the composition $\Gamma \circ \mathscr{H}om_{\mathcal{O}_C}(\Omega_C, -)$ of left exact functors, there is a Grothendieck spectral sequence with E_2-page

$$E_2^{p,q} = \operatorname{H}^p(C, \mathscr{E}xt^q_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C))$$

which converges to $\mathrm{Ext}^{p+q}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ (c.f. [Wei94, Thm. 5.8.3]). Since $\dim C = 1$, we have that $E_2^{p,q} = 0$ if $p \geq 2$. We can thus draw the E_2 page as:

The associated exact sequence of low-degree terms is

$$0 \to E_2^{1,0} \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to E_2^{0,1} \to E_2^{2,0} = 0.$$

As Ω_C is locally free away from the nodes, $\mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ is a zero-dimensional sheaf supported only at the nodes of C. It follows that that $E_2^{1,1} = \mathrm{H}^1(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = 0$ and

$$E_2^{0,1} = \mathrm{H}^0(C, \mathscr{E}xt^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C))$$
$$= \bigoplus_j \mathrm{Ext}^1_{\mathcal{O}_{C,q_j}}(\Omega_{C,q_j}, \mathcal{O}_{C,q_j}) = \bigoplus_j \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$$

where we've used that $\widehat{\Omega}_{C,q_j} \cong \Omega_{\widehat{\mathcal{O}}_{C,q_j}}$. This gives the exact sequence (5.3.26).

The identifications

$$\mathrm{Def}^{\mathrm{lt}}(C) \cong \mathrm{H}^1(C, T_C) \quad \text{and} \quad \mathrm{Def}(\widetilde{C}, q'_j, q''_j) \cong \mathrm{H}^1\big(\widetilde{C}, T_{\widetilde{C}}(-\textstyle\sum_j(q'_j + q''_j))\big)$$

were given in Proposition C.1.12 and Exercise C.1.15. To define an equivalence $\mathrm{Def}^{\mathrm{lt}}(C) \xrightarrow{\sim} \mathrm{Def}(\widetilde{C}, q'_j, q''_j)$, let $C' \to \mathrm{Spec}\,\Bbbk[\epsilon]$ be a locally trivial first-order deformation of C. Each node $q_j \colon \mathrm{Spec}\,\Bbbk \to C$ extends to a section $\widetilde{q}_j \colon \mathrm{Spec}\,\Bbbk[\epsilon] \to C$ whose image is contained in the relative singular locus of C over $\Bbbk[\epsilon]$. The pointed normalization of C along the sections $\widetilde{q}_j$ is a first-order deformation of the (possibly disconnected) pointed normalization $(\widetilde{C}, q'_j, q''_j)$. This defines a map $\mathrm{Def}^{\mathrm{lt}}(C) \to \mathrm{Def}(\widetilde{C}, q'_j, q''_j)$. The inverse is provided by gluing the sections of a first-order deformation $(\widetilde{C}', \sigma'_j, \sigma''_j)$ of $(\widetilde{C}, q'_j, q''_j)$ along nodes; more precisely, the deformation C' is obtained as the Ferrand Pushout (B.4.1)

$$
\begin{array}{ccc}
\coprod_j(\mathrm{Spec}\,\Bbbk[\epsilon] \amalg \mathrm{Spec}\,\Bbbk[\epsilon]) & \xrightarrow{q'_j \amalg q''_j} & \widetilde{C}' \\
\downarrow & & \downarrow \\
\coprod_j \mathrm{Spec}\,\Bbbk[\epsilon] & \longrightarrow & C'.
\end{array}
$$

The bijection $\mathrm{Def}(C, p_i) \cong \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C)$ follows from Proposition 5.3.23, while $\mathrm{Def}(\widehat{\mathcal{O}}_{C,q_j}) \cong \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega^1_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$ follows from Item (b) .

To see that $\mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega^1_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j})$ is one-dimensional and that $\mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$, observe that $\mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ is a zero-dimensional sheaf supported only at the nodes. We have identifications

$$E_2^{0,2} = \mathrm{H}^0(C, \mathscr{E}xt^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) = \bigoplus_j \mathrm{Ext}^2_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}).$$

Write $\widehat{\mathcal{O}}_{C,q_j} = \Bbbk[\![x,y]\!]/(xy)$ and consider the locally free resolution

$$0 \to \widehat{\mathcal{O}}_{C,q_j} \xrightarrow{\binom{y}{x}} \widehat{\mathcal{O}}_{C,q_j}^{\oplus 2} \xrightarrow{(dx,dy)} \Omega_{\widehat{\mathcal{O}}_{C,q_j}} \to 0.$$

This allows us to compute that $\mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = \mathrm{coker}(\widehat{\mathcal{O}}_{C,q_j}^{\oplus 2} \xrightarrow{(x,y)} \widehat{\mathcal{O}}_{C,q_j}) = \Bbbk$ and $\mathrm{Ext}^2_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = 0$. As $E_2^{0,2} = E_2^{1,1} = E_2^{2,0} = 0$, we have $\mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) = 0$. See also [DM69, Prop. 1.5] and [ACG11, §11.3]. $\qquad\square$

Recall that automorphisms, deformations, and obstructions of a nodal curve (C, p_i) are classified by $\mathrm{Ext}^i_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C)$ for $i = 0, 1, 2$ (Proposition 5.3.23).

Proposition 5.3.27. *Let $(C, p_1, \ldots, p_n)$ be an n-pointed stable curve of genus g over $\Bbbk$. Then*

$$\dim_\Bbbk \mathrm{Ext}^i_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = \left\{ \begin{array}{rl} 0 & \text{if } i = 0, 2 \\ 3g - 3 + n & \text{if } i = 1 \end{array} \right.$$

Proof. We may assume $\Bbbk = \overline{\Bbbk}$, and for simplicity we handle the case that there are no marked points. Let $\pi\colon \widetilde{C} \to C$ be the normalization, $\Sigma = \{q_1, \ldots, q_s\} \subset C$ be the set of nodes of C, and $\widetilde{\Sigma} = \pi^{-1}(\Sigma)$. The vanishing of Ext^2 was established in Proposition 5.3.24. For Ext^0, we will use the identification $\mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) \cong \mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ of Exercise 5.2.20. Since the pointed normalization $(\widetilde{C}, \widetilde{\Sigma})$ is smooth and each pointed connected component is stable (Exercise 5.3.5), the degree of the restriction of $T_{\widetilde{C}}(-\widetilde{\Sigma})$ to each connected component of $\widetilde{C}$ is strictly negative. Thus, $\mathrm{Hom}_{\mathcal{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathcal{O}_{\widetilde{C}}) = \mathrm{H}^0(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) = 0$.

To compute Ext^1, Proposition 5.3.24 implies that there is an exact sequence

$$0 \to \mathrm{H}^1(\widetilde{C}, T_C(-\widetilde{\Sigma})) \to \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \to \bigoplus_j \mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) \to 0$$

and an identification $\mathrm{Ext}^1_{\widehat{\mathcal{O}}_{C,q_j}}(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = \Bbbk$. We write $\widetilde{C} = \coprod_{i=1}^\nu \widetilde{C}_i$ as a union of its connected components and define $\widetilde{\Sigma}_i = \widetilde{C}_i \cap \widetilde{\Sigma}$. Using that $\Omega_{\widetilde{C}_i}$ is a line bundle, we compute using Serre Duality and Riemann–Roch that

$$\mathrm{h}^1(\widetilde{C}_i, T_{\widetilde{C}_i}(-\widetilde{\Sigma}_i)) = \mathrm{h}^0(\widetilde{C}_i, \Omega_{\widetilde{C}_i}^{\otimes 2}(\widetilde{\Sigma}_i)) = 3g(\widetilde{C}_i) - 3 + |\widetilde{\Sigma}_i|.$$

Thus

$$\begin{aligned} \dim_\Bbbk \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) &= \mathrm{h}^1(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) + |\Sigma| \\ &= \sum_{i=1}^\nu \big(3g(\widetilde{C}_i) - 3 + |\widetilde{\Sigma}_i|\big) + |\Sigma| \\ &= 3\big(\sum_{i=1}^\nu g(\widetilde{C}_i) - \nu + |\Sigma|\big) \\ &= 3g - 3, \end{aligned}$$

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where we've used the Genus Formula (5.2.11) $g = \sum_{i=1}^{\nu} g(\widetilde{C}_i) - \nu + |\Sigma| + 1$. $\qquad\square$

Remark 5.3.28 (Consequences of deformation theory). In Theorem 5.4.14, we will use deformation theory to argue that $\overline{\mathcal{M}}_{g,n}$ is a smooth Deligne–Mumford stack of dimension $3g - 3 + n$. Here's the central idea:

- Ext^0: We've already seen that a stable curve (C, p_i) has finitely many automorphism (Proposition 5.3.3). The vanishing of Ext^0 implies that an n-pointed stable curve (C, p_i) has no infinitesimal automorphisms, i.e. the Lie algebra of the automorphism group scheme $\underline{\mathrm{Aut}}(C, p_i)$ is trivial. Since $\underline{\mathrm{Aut}}(C, p_i)$ is of finite type, it must be finite and discrete. Once we know that the algebraicity of the stack $\overline{\mathcal{M}}_{g,n}$, we use the Characterization of Deligne–Mumford Stacks (3.6.4) to conclude that $\overline{\mathcal{M}}_{g,n}$ is Deligne–Mumford.

- Ext^1: Since Ext^1 parametrizes isomorphism classes of deformations of a stable curve (C, p_i) over a field $\Bbbk$, it is identified with the Zariski tangent space of $\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \Bbbk$ at the $\Bbbk$-point corresponding to (C, p_i). The computation of Ext^1 therefore implies that $\overline{\mathcal{M}}_{g,n}$ has relative dimension $3g - 3 + n$ over $\mathrm{Spec}\,\mathbb{Z}$.

- Ext^2: The vanishing of Ext^2 implies that there are no obstructions to deforming (C, p_i), and thus the Infinitesimal Lifting Criterion (Theorem 3.7.1) implies that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\mathrm{Spec}\,\mathbb{Z}$.

5.4 The stack of all curves

We show that the stack $\mathcal{M}_{g,n}^{\mathrm{all}}$ of all n-pointed proper curves is algebraic (Theorem 5.4.6) and that the open substack $\overline{\mathcal{M}}_{g,n} \subset \mathcal{M}_{g,n}^{\mathrm{all}}$ of stable curves is a quasi-compact Deligne–Mumford stack smooth over $\mathrm{Spec}\,\mathbb{Z}$ of dimension $3g - 3 + n$ (Theorem 5.4.14).

5.4.1 Families of arbitrary curves

Recall that a *curve over a field* $\Bbbk$ is a one-dimensional scheme of finite type over $\Bbbk$, and that the *genus* of a projective curve C is

$$g(C) = 1 - \chi(C, \mathcal{O}_C).$$

Curves may be non-pure dimensional, non-connected, and arbitrarily singular (e.g. non-reduced). We need to allow such curves as otherwise the stack of curves would fail to be algebraic. For instance, a rational normal curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ degenerates in a flat family a non-reduced curve C_0, supported along a plane nodal cubic with an embedded point at the node [Har77, Ex. 9.8.4], while C_0 deforms in a flat family to the disjoint union of a plane nodal cubic and a point in $\mathbb{P}^3$.

A *family of n-pointed curves over a scheme* S is a proper, flat, and finitely presented morphism $\mathcal{C} \to S$ of algebraic spaces together with n sections $\sigma_1, \dots, \sigma_n \colon S \to \mathcal{C}$ such that every fiber $\mathcal{C}_s$ is a curve. We say that a family $(\mathcal{C} \to S, \sigma_i)$ has *genus g* if every fiber $\mathcal{C}_s$ has genus g. Note that the marked points $\sigma_i(s)$ may be non-distinct and singular in a fiber $\mathcal{C}_s$. For the stack of curves to be algebraic, it is necessary that we allow the total family $\mathcal{C}$ to be an algebraic space (see Caution 5.3.18).

Proposition 5.4.1. *If $\mathcal{C} \to S$ is a family of curves over a scheme S, there exists an étale cover $S' \to S$ such that $\mathcal{C}_{S'} \to S'$ is projective.*

Proof 1 (Local-to-global). We first reduce to the case that S is of finite type over $\operatorname{Spec}\mathbb{Z}$. By Noetherian Approximation (B.3.2), we can write $S = \lim_{\lambda \in \Lambda} S_\lambda$ as a limit of finitely presented $\mathbb{Z}$-schemes with affine transition maps. By Descent of Morphisms under Limits (B.3.4), there is an index $0 \in \Lambda$ and a finitely presented morphism $\mathcal{C}_0 \to S_0$ such that $\mathcal{C} \cong \mathcal{C}_0 \times_{S_0} S$, and moreover if we set $\mathcal{C}_\lambda = \mathcal{C}_0 \times_{S_0} S$ for $\lambda \geq 0$, then $\mathcal{C} = \lim_{\lambda \geq 0} \mathcal{C}_\lambda$. By Descent of Properties of Morphisms under Limits (B.3.7), $\mathcal{C}_\lambda \to S_\lambda$ is a family of curves for $\lambda \gg 0$. Since projectivity is stable under base change, after replacing S with S_λ, we can assume that S is of finite type over $\operatorname{Spec}\mathbb{Z}$.

For a point $s \in S$, define $S_n = \operatorname{Spec}\mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ and $\widehat{S} = \operatorname{Spec}\widehat{\mathcal{O}}_{S,s}$. Consider the cartesian diagram

$$
\begin{array}{ccccccccc}
\mathcal{C}_s = \mathcal{C}_0 & \hookrightarrow & \mathcal{C}_1 & \hookrightarrow & \cdots & \hookrightarrow & \widehat{\mathcal{C}} & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\operatorname{Spec}\kappa(s) = S_0 & \hookrightarrow & S_1 & \hookrightarrow & \cdots & \hookrightarrow & \widehat{S} & \longrightarrow & S.
\end{array}
$$

Since separated one-dimensional algebraic spaces are schemes (Theorem 4.4.32) and proper one-dimensional schemes are projective, there exists an ample line bundle L_0 on $\mathcal{C}_0$. By Proposition C.2.17, the obstruction to deforming a line bundle L_n on $\mathcal{C}_n$ to L_{n+1} on $\mathcal{C}_{n+1}$ lives in $H^2(\mathcal{C}_0, \mathcal{O}_{\mathcal{C}_0} \otimes_{\kappa(s)} \mathfrak{m}^n)$. The obstruction vanishes since $\dim \mathcal{C}_0 = 1$, and there is a compatible sequence of line bundles L_n on $\mathcal{C}_n$. By Grothendieck's Existence Theorem (C.4.4), there exists a line bundle $\widehat{L}$ on $\widehat{\mathcal{C}}$ extending L_n.

Applying Artin Approximation (Theorem B.5.26) to the functor

$$
\operatorname{Sch}/S \to \operatorname{Sets}, \qquad (T \to S) \mapsto \operatorname{Pic}(\mathcal{C}_T),
$$

we obtain an étale neighborhood $(S', s') \to (S, s)$ of s and a line bundle L' on $\mathcal{C}_{S'}$ extending L_0. Since L_0 is ample and ampleness is an open condition in families (Proposition B.6.10), after replacing S' with an open neighborhood of s', we can arrange that L' is relatively ample over S'. $\qquad\square$

Proof 2 (Explicitly extend an ample line bundle). Given an ample line bundle L_s on a fiber $\mathcal{C}_s$ over a point $s \in S$, the idea here is to use geometric methods to find an étale neighborhood $S' \to S$ of s and a line bundle L on $\mathcal{C}_{S'}$ extending L_s. By Openness of Ampleness (B.6.10), L will be ample after replacing S' with an open neighborhood. Using limit methods as above, one can first reduce to the case where S is the spectrum of an strictly henselian local ring R. In this case, it suffices to show that

$$
\operatorname{Pic}(\mathcal{C}) \to \operatorname{Pic}(\mathcal{C}_s) \tag{5.4.2}
$$

is surjective.

Under the additional assumption that every fiber of $\mathcal{C} \to S$ is generically reduced (e.g. $\mathcal{C} \to S$ is a prestable family), there is a straightforward argument. Choose smooth points $p_1, \ldots, p_n \in \mathcal{C}_s$ such that every irreducible one-dimensional component of $\mathcal{C}_s$ contains at least one of the p_i's. Since the relative smooth locus $\mathcal{C}^{\mathrm{sm}}$ of $\mathcal{C} \to S$ surjects onto S, there are sections $\sigma_i \colon S \to \mathcal{C}^{\mathrm{sm}}$ extending p_i and the line bundle $L = \mathcal{O}_{\mathcal{C}_{S'}}(\sigma_1 + \cdots + \sigma_n)$ extends the ample line bundle $L_s := \mathcal{O}_{\mathcal{C}_s}(p_1 + \cdots + p_n)$. See also [Ols16, Cor. 13.2.5].

In general, one can argue as follows. By applying Le Lemma de Gabber (4.5.1), there exists a finite surjection $\mathcal{C}' \to \mathcal{C}$ from a scheme. If we assume that (5.4.2) is

surjective, then $\mathcal{C}' \to S$ is projective and in particular $\mathcal{C}'$ satisfies the Chevalley–Kleiman property, i.e. every finite set of points is contained in an affine. By Exercise 4.4.23, $\mathcal{C}$ also has the Chevalley–Kleiman property and in particular is a scheme. We may therefore assume that $\mathcal{C}$ is a scheme.

Assuming that (5.4.2) is surjective when $\mathcal{C}$ is a reduced scheme, let L_0 be an ample line bundle on $\mathcal{C}_{\mathrm{red}}$. If I denotes the ideal sheaf defining $\mathcal{C}_{\mathrm{red}} \hookrightarrow \mathcal{C}$, then I is nilpotent. By deformation theory, the obstruction to deformation a line bundle from the nth nilpotent thickening to the $n+1$st thickening lies in $\mathrm{H}^2(\mathcal{C}_{\mathrm{red}}, I^n)$ which vanishes since $\mathcal{C}_{\mathrm{red}} \to \operatorname{Spec} R$ is a curve. Therefore, there is a line bundle L on $\mathcal{C}$ extending L_0. We may therefore assume that $\mathcal{C}$ is a reduced scheme.

It suffices to show that a line bundle $L_s = \mathcal{O}_{\mathcal{C}_s}(-x_s)$, for a closed point $x_s \in \mathcal{C}_s$, extends to $\mathcal{C}$. There is an affine open subscheme U_s of x_s and a nonzerodivisor $f_s \in \Gamma(U_s, \mathcal{O}_{U_s})$ with $V(f_s) = \{x_s\}$. Choose an open affine neighborhood $U \subset \mathcal{C}$ such that $U \cap \mathcal{C}_s = U_s$, and choose a global function $f \in \Gamma(U, \mathcal{O}_U)$ extending f_s. Then $V(f) \to S$ is quasi-finite and separated, and since S is the spectrum of a henselian local ring, there is a decomposition $V(f) = V_1 \amalg V_2$ such that $V_1 \to S$ finite and $(V_2)_s = \emptyset$ (Proposition B.5.10). After shrinking U, we can assume that $V(f) \to S$ is finite. Therefore $V(f)$ is also closed in $\mathcal{C}$ and defines a cartier divisor $D \subset \mathcal{C}$ such that $\mathcal{O}_{\mathcal{C}}(-D)|_{\mathcal{C}_s} = \mathcal{O}_{\mathcal{C}_s}(-x_s)$. See also [SGA4$\frac{1}{2}$, IV.4.1] and [Hal13, Lem. 1.2]. $\qquad\square$

Remark 5.4.3. Raynaud gives an example of a family of smooth $g = 1$ curves over an affine curve which is Zariski-locally projective but not projective [Ray70, XIII 3.1]. The examples in Caution 5.3.18 are not even Zariski-locally projective.

5.4.2 Algebraicity of the stack of all curves

Definition 5.4.4. Let $\mathcal{M}_{g,n}^{\mathrm{all}}$ be the prestack over $\mathrm{Sch}_{\text{ét}}$, where an object over a scheme S is a family of curves $\mathcal{C} \to S$ of genus g with n sections $\sigma_1, \dots, \sigma_n \colon S \to \mathcal{C}$. A morphism $(\mathcal{C}' \to S', \sigma_1', \dots, \sigma_n') \to (\mathcal{C} \to S, \sigma_1, \dots, \sigma_n)$ is the data of a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{C}' & \xrightarrow{\ g\ } & \mathcal{C} \\
\sigma_i' \big\uparrow \ \Big\downarrow & \ \sigma_i\ & \sigma_i \big\uparrow \ \Big\downarrow \\
S' & \xrightarrow{\ f\ } & S
\end{array}
$$

such that $g \circ \sigma_i' = \sigma_i \circ f$.

As a stepping stone to the algebraicity of $\mathcal{M}_{g,n}^{\mathrm{all}}$, we show that the diagonal is representable.

Lemma 5.4.5. *The diagonal* $\mathcal{M}_{g,n}^{\mathrm{all}} \to \mathcal{M}_{g,n}^{\mathrm{all}} \times \mathcal{M}_{g,n}^{\mathrm{all}}$ *is representable.*

Proof. For simplicity, we handle the case when $n = 0$. Let S be a scheme and $S \to \mathcal{M}_g^{\mathrm{all}} \times \mathcal{M}_g^{\mathrm{all}}$ be a morphism corresponding to families of curves $\mathcal{C}_1 \to S$ and $\mathcal{C}_2 \to S$. Considering the cartesian diagram

$$
\begin{array}{ccc}
\underline{\operatorname{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) & \longrightarrow & S \\
\big\downarrow & \square & \big\downarrow \\
\mathcal{M}_g^{\mathrm{all}} & \longrightarrow & \mathcal{M}_g^{\mathrm{all}} \times \mathcal{M}_g^{\mathrm{all}},
\end{array}
$$

we need to show that $\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2)$ is an algebraic space. By Proposition 5.4.1, there exists an étale cover $S' \to S$ such that $\mathcal{C}_{S'} \to S'$ is projective. Since $\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \times_S S' = \underline{\mathrm{Isom}}_{S'}(\mathcal{C}_{1,S'}, \mathcal{C}_{2,S'})$, the morphism $\underline{\mathrm{Isom}}_{S'}(\mathcal{C}_{1,S'}, \mathcal{C}_{2,S'}) \to \underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2)$ is surjective, étale, and representable. We may thus assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are projective over S.

Consider the inclusion of functors:

$$\underline{\mathrm{Isom}}_S(\mathcal{C}_1, \mathcal{C}_2) \subset \underline{\mathrm{Mor}}_S(\mathcal{C}_1, \mathcal{C}_2) \subset \mathrm{Hilb}_S(\mathcal{C}_1 \times_S \mathcal{C}_2)$$

where the second inclusion assigns a morphism $\mathcal{C}_1 \xrightarrow{\alpha} \mathcal{C}_2$ to the graph $\mathcal{C}_1 \xrightarrow{\Gamma_\alpha} \mathcal{C}_1 \times_S \mathcal{C}_2$ (and is similarly defined on T-valued points). Since the subfunctor of $\mathrm{Mor}(-, S)$, parameterizing maps $T \to S$ where $(\mathcal{C}_1)_S \to (\mathcal{C}_2)_S$ is an isomorphism, is open (Exercise 0.3.33), the first inclusion is a representable open immersion. Analyzing the second inclusion, we see that a subscheme $[\mathcal{Z} \subset \mathcal{C}_1 \times_S \mathcal{C}_2] \in \mathrm{Hilb}_S(\mathcal{C}_1 \times_S \mathcal{C}_2)(S)$ is contained in the image of $\underline{\mathrm{Mor}}(\mathcal{C}_1, \mathcal{C}_2)(S)$ if and only if the composition $Z \hookrightarrow \mathcal{C}_1 \times_S \mathcal{C}_2 \xrightarrow{p_1} \mathcal{C}_1$ is an isomorphism (and similarly for T-valued points). Therefore, Exercise 0.3.33 also establishes that the second inclusion is a representable open immersion. $\qquad\square$

Theorem 5.4.6. $\mathcal{M}_{g,n}^{\mathrm{all}}$ *is an algebraic stack locally of finite type over* $\mathrm{Spec}\,\mathbb{Z}$.

Proof. To see that $\mathcal{M}_{g,n}^{\mathrm{all}}$ is a stack over $\mathrm{Sch}_{\text{ét}}$, suppose that $\{S_i \to S\}$ is an étale cover of schemes, $(\mathcal{C}_i \to S_i, \sigma_{i,1}, \ldots, \sigma_{i,n})$ is a family of n-pointed curves for each i, and $\alpha_{ij} \colon \mathcal{C}_i|_{S_{ij}} \to \mathcal{C}_j|_{S_{ij}}$ is an isomorphism over $S_{ij} = S_i \times_S S_j$ compatible with the sections and satisfying the cocycle condition. The quotient of the étale equivalence relation

$$\coprod_{i,j} \mathcal{C}_{i,j} \rightrightarrows \coprod_i \mathcal{C}_i$$

is an algebraic space $\mathcal{C}$. Moreover, by étale descent of morphisms, there are sections $\sigma_1, \ldots, \sigma_n \colon S \to \mathcal{C}$ such that $\sigma_{k,i} = \sigma_k|_{S_i}$. Thus $(\mathcal{C} \to S, \sigma_k)$ is a n-pointed family of curves that restricts to $(\mathcal{C}_i \to S_i, \sigma_{i,k})$ for each i.

To see algebraicity, we claim that it suffices to handle the $n = 0$ case. Observe that the map $\mathcal{M}_{g,n+1}^{\mathrm{all}} \to \mathcal{M}_{g,n}^{\mathrm{all}}$, defined by forgetting the last marked point, is well-defined and representable: if $S \to \mathcal{M}_{g,n}^{\mathrm{all}}$ is a map corresponding to a n-pointed family $(\mathcal{C} \to S, \sigma_k)$, then $S \times_{\mathcal{M}_{g,n}^{\mathrm{all}}} \mathcal{M}_{g,n+1}^{\mathrm{all}} \cong \mathcal{C}$. (In fact, $\mathcal{M}_{g,n+1}^{\mathrm{all}} \to \mathcal{M}_{g,n}^{\mathrm{all}}$ is identified with the universal family; see Exercise 5.4.8.) Therefore, if $U \to \mathcal{M}_{g,n}^{\mathrm{all}}$ is a smooth presentation by a scheme, then $U' := U \times_{\mathcal{M}_{g,n}^{\mathrm{all}}} \mathcal{M}_{g,n+1}^{\mathrm{all}} \to \mathcal{M}_{g,n+1}^{\mathrm{all}}$ is a surjective, étale, and representable map from an algebraic space U'. Choosing an étale presentation $V \to U'$ by a scheme, the composition $V \to U' \to \mathcal{M}_{g,n+1}^{\mathrm{all}}$ is a smooth presentation.

To show algebraicity of $\mathcal{M}_g^{\mathrm{all}}$, it suffices show that for every projective curve X over a field $\mathbb{k}$, there exists a smooth representable morphism $U \to \mathcal{M}_g^{\mathrm{all}}$ from a scheme U of finite type over $\mathbb{Z}$ with $[X]$ in the image. Choose an embedding $X \hookrightarrow \mathbb{P}^N$ such that $\mathrm{H}^1(X, \mathcal{O}_X(1)) = 0$, and let $P(t)$ be its Hilbert polynomial. Let $H := \mathrm{Hilb}^P(\mathbb{P}_{\mathbb{Z}}^N / \mathbb{Z})$ be the Hilbert scheme, which is projective over $\mathbb{Z}$ by Theorem 1.1.2. Considering the universal family

there is a point $h \in H(\Bbbk)$ such that $\mathcal{C}_h = X$ as closed subschemes of $\mathbb{P}^N_{\Bbbk}$. Cohomology and Base Change (A.6.8) implies that there exists an open neighborhood $H' \subset H$ of h such that for all $s \in H'$, $\mathrm{H}^1(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}(1)) = 0$. Consider the morphism

$$H' \to \mathcal{M}_g^{\mathrm{all}}, \qquad [C \hookrightarrow \mathbb{P}^n] \mapsto [C],$$

defined by forgetting the embedding. The representability of the diagonal of $\mathcal{M}_g^{\mathrm{all}}$ (Lemma 5.4.5) implies that $H' \to \mathcal{M}_g^{\mathrm{all}}$ is representable as every morphism from a scheme is representable (Corollary 3.2.3).

We claim that $H' \to \mathcal{M}_g^{\mathrm{all}}$ is smooth. Even though we haven't yet established the algebraicity of $\mathcal{M}_g^{\mathrm{all}}$, we do know that $H' \to \mathcal{M}_g^{\mathrm{all}}$ is representable and this is suffices to apply the Infinitesimal Lifting Criterion (3.7.1) to verify smoothness. To this end, let $A' \to A$ be a surjection of artinian local rings with residue field $\Bbbk$ such that $\Bbbk = \ker(A' \to A)$. For every embedded curve $[C \subset \mathbb{P}^N] \in H'$ and diagram

$$
\begin{array}{c}
\xymatrix{
& & [C \subset \mathbb{P}^N_{\Bbbk}] & \\
\operatorname{Spec} \Bbbk \ar[r] & \operatorname{Spec} A \ar[r]^{[\mathcal{C} \subset \mathbb{P}^N_A]} & H' \\
& \operatorname{Spec} A' \ar[r]_{[\mathcal{C}']} & \mathcal{M}_g^{\mathrm{all}},
}
\end{array} \tag{5.4.7}
$$

we need to show that there is a dotted arrow extending the diagram. This translates to the existence of a dotted arrow in the diagram

The closed immersion $\mathcal{C} \hookrightarrow \mathbb{P}^N_A$ is defined by a very ample line bundle L on $\mathcal{C}$ and sections $s_0, \ldots, s_N \in \Gamma(\mathcal{C}, L)$. As the obstruction to deforming L to $\mathcal{C}'$ lives in $\mathrm{H}^2(C, \mathcal{O}_C) = 0$ (Proposition C.2.17), we may extend L to a line bundle L' on $\mathcal{C}'$. We now argue that the sections s_i deform to sections $s_i' \in \Gamma(\mathcal{C}', L')$. As $\ker(A' \to A) = \Bbbk$, the ideal sheaf defining $\mathcal{C} \hookrightarrow \mathcal{C}'$ is isomorphic to $\mathcal{O}_C$, and we have a short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{\mathcal{C}'} \to \mathcal{O}_{\mathcal{C}} \to 0.$$

Tensoring with L' yields a short exact sequence

$$0 \to L|_C \to L' \to L \to 0.$$

Since $[C \subset \mathbb{P}^N] \in H'$, we have that $\mathrm{H}^1(C, L|_C) = 0$. Thus, $\mathrm{H}^0(\mathcal{C}', L') \to \mathrm{H}^0(\mathcal{C}, L)$ is surjective and we may lift the sections s_i to sections s_i'. The sections s_i are base point free, and Nakayama's Lemma implies that so are the sections s_i'. This gives a morphism $j' \colon \mathcal{C}' \to \mathbb{P}^N_{A'}$ restricting to $\mathcal{C} \hookrightarrow \mathbb{P}^N_A$. The map $j' \colon \mathcal{C}' \to \mathbb{P}^N_{A'}$ is proper and quasi-finite, thus finite. To see that it is a closed immersion, it suffices to show

that $\mathcal{O}_{\mathbb{P}^N_{A'}} \to j'_* \mathcal{O}_{\mathcal{C}'}$ is surjective. The cokernel is a coherent sheaf whose restriction to A vanishes, and thus Nakayama's lemma implies that the cokernel vanishes.

See also [Hal13, Thm. 1.1] and [JHS11, Prop. 3.3]. This can also be established using Artin's Axioms; see Exercise C.7.10 and [SP, Tag 0D5A]. $\qquad\square$

Exercise 5.4.8 (easy)**.** Show that map $\mathcal{M}^{\mathrm{all}}_{g,n+1} \to \mathcal{M}^{\mathrm{all}}_{g,n}$ forgetting the last marked point is the universal family as defined in Definition 3.1.26. (We will see in Proposition 5.6.8 that $\overline{\mathcal{M}}_{g,n+1}$ is also the universal family of $\overline{\mathcal{M}}_{g,n}$, but this is a more remarkable fact since an n-pointed stable curve can become unstable if a marked point is forgotten.)

Exercise 5.4.9 (technical)**.** Show that $\mathcal{M}^{\mathrm{all}}_{g,n}$ has quasi-compact and separated diagonal, and in particular $\mathcal{M}^{\mathrm{all}}_{g,n}$ is quasi-separated.

Remark 5.4.10. The moduli stacks of varieties of higher dimension are not algebraic. For instance, the stack parameterizing abstract K3 surfaces is not algebraic (see Remark C.4.15); on the other hand, there is an analytic stack parameterizing K3 surfaces (see [BHPV04, §VII.12]) and an algebraic stack parameterizing polarized K3 surfaces (i.e. K3 surfaces with a primitive ample line bundle).

Exercise 5.4.11 (good practice)**.** Show that the stack $\mathcal{P}$, parameterizing a family of curves $\mathcal{C} \to S$ and a line bundle L on $\mathcal{C}$ relatively ample over S, is algebraic, and that the natural map $\mathcal{P} \to \mathcal{M}^{\mathrm{all}}_g$ is smooth and representable.

5.4.3 Algebraicity and boundedness of $\overline{\mathcal{M}}_{g,n}$

Consider the inclusions of prestacks

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n} \subset \mathcal{M}^{\mathrm{ss}}_{g,n} \subset \mathcal{M}^{\mathrm{pre}}_{g,n} \subset \mathcal{M}^{\leq\mathrm{nodal}}_{g,n} \subset \mathcal{M}^{\mathrm{all}}_{g,n}, \qquad (5.4.12)$$

where $\overline{\mathcal{M}}_{g,n}$ (resp. $\mathcal{M}^{\mathrm{ss}}_{g,n}$, $\mathcal{M}^{\mathrm{pre}}_{g,n}$, and $\mathcal{M}^{\leq\mathrm{nodal}}_{g,n}$) denotes the full subcategory of $\mathcal{M}^{\mathrm{all}}_{g,n}$ consisting of n-pointed families $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ of stable curves (resp. semistable, prestable, and nodal curves).

Corollary 5.4.13. *The sequence of inclusions in* (5.4.12) *are open immersions, and each prestack is an algebraic stack locally of finite type over* $\mathrm{Spec}\,\mathbb{Z}$.

Proof. Theorem 5.4.6 establishes that $\mathcal{M}^{\mathrm{all}}_{g,n}$ is algebraic and locally of finite type. Openness of the stable (resp. semistabile, prestable, nodal) locus was proved in Proposition 5.3.22, and this implies that each inclusion is an open immersion. $\qquad\square$

At this point, we know the following properties of $\overline{\mathcal{M}}_{g,n}$.

Theorem 5.4.14. *Assuming that* $2g - 2 + n > 0$*, the stack* $\overline{\mathcal{M}}_{g,n}$ *is a non-empty, quasi-compact, and Deligne–Mumford stack smooth over* $\mathrm{Spec}\,\mathbb{Z}$ *such that* $\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \Bbbk$ *has pure dimension* $3g - 3 + n$ *for every field* $\Bbbk$.

Proof. By Corollary 5.4.13, $\overline{\mathcal{M}}_{g,n}$ is an algebraic stack of finite type over $\mathrm{Spec}\,\mathbb{Z}$. As there are smooth curves of any genus (Exercise 5.1.6), $\overline{\mathcal{M}}_{g,n}$ is non-empty. For the boundedness of $\overline{\mathcal{M}}_{g,n}$ (i.e. finite typeness or equivalently quasi-compactness), we will appeal to the fact that if $(C, p_1, \ldots, p_n)$ is an n-pointed stable curve over a field $\Bbbk$, then $L := \left(\omega_{C/\Bbbk}(p_1 + \cdots p_n)\right)^{\otimes 3}$ is very ample (Exercise 5.3.15). Let $P(t)$ be the Hilbert polynomial of $C \hookrightarrow \mathbb{P}^N_{\Bbbk}$ embedded via L; this is independent of $[C, p_i] \in \overline{\mathcal{M}}_{g,n}$. Consider the closed subscheme

$$H \subset \mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}) \times (\mathbb{P}^N)^n$$

parameterizing pairs $(C \hookrightarrow \mathbb{P}^N, p_1, \ldots, p_n)$ such that $p_i \in C$. Since $\mathrm{Hilb}^P(\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z})$ is a projective scheme (Theorem 1.1.2) and in particular quasi-compact, so is H. The image of the forgetful morphism

$$H \to \mathcal{M}^{\mathrm{all}}_{g,n} \qquad [C \hookrightarrow \mathbb{P}^N, p_1, \ldots, p_n] \mapsto [C, p_1, \ldots, p_n]$$

contains $\overline{\mathcal{M}}_{g,n}$, and we conclude that $\overline{\mathcal{M}}_{g,n}$ is quasi-compact.

To see the final assertions, we invoke each part of Proposition 5.3.27 characterizing automorphisms, deformations, and obstructions of stable curve. This is analogous to our proof of Proposition 3.7.6 asserting the same properties for the stack $\mathcal{M}_g$ of smooth curves. Let (C, p_i) be an n-pointed stable curve of genus g. Since $\mathrm{Ext}^0_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$, the Lie algebra of $\underline{\mathrm{Aut}}(C, p_i)$ is trivial and $\underline{\mathrm{Aut}}(C, p_i)$ is a finite and reduced group scheme. By the Characterization of Deligne–Mumford stacks (3.6.4), $\overline{\mathcal{M}}_{g,n}$ is Deligne–Mumford. Since $\mathrm{Ext}^2_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C) = 0$, there are no obstructions to deforming stable curves, and the Infinitesimal Lifting Criterion (3.7.1) implies that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\mathrm{Spec}\,\mathbb{Z}$. Finally, since the Zariski tangent space of $[C, p_i] \in \overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \Bbbk$ is bijective to $\mathrm{Ext}^1_{\mathcal{O}_C}(\Omega_C(\sum_i p_i), \mathcal{O}_C)$ and this vector space has dimension $3g - 3 + n$, we conclude that $\overline{\mathcal{M}}_{g,n} \to \mathrm{Spec}\,\mathbb{Z}$ has relative dimension $3g - 3 + n$. $\qquad\square$

Exercise 5.4.15. Show that $\overline{\mathcal{M}}_{g,n}$ is algebraic by explicitly presenting it as a quotient stack of a locally closed subscheme of the Hilbert scheme.

Hint: Follow the proof of Theorem 3.1.17.

Exercise 5.4.16. Show that each of the stacks $\mathcal{M}^{\mathrm{ss}}_{g,n}$, $\mathcal{M}^{\mathrm{pre}}_{g,n}$, $\mathcal{M}^{\leq \mathrm{nodal}}_{g,n}$, and $\mathcal{M}^{\mathrm{all}}_{g,n}$ are not quasi-compact.

Hint: Use the presence of rational bridges of arbitrary length.

Remark 5.4.17. There are various other open substacks of $\mathcal{M}^{\mathrm{all}}_g$ parameterizing particular classes of curves such as

$$
\begin{aligned}
\mathcal{M}^{\mathrm{DM}}_g &= \{\text{curves with finite, reduced automorphism group}\} & [\mathrm{SP}, \mathrm{Tag}\ 0DST] \\
\mathcal{M}^{\mathrm{CM}}_g &= \{\text{Cohen–Macaulay curves}\} & [\mathrm{SP}, \mathrm{Tag}\ 0E0H] \\
\mathcal{M}^{\mathrm{geom\text{-}red}}_g &= \{\text{geometrically reduced curves}\} & [\mathrm{SP}, \mathrm{Tag}\ 0E0F] \\
\mathcal{M}^{\mathrm{Gor}}_g &= \{\text{Gorenstein curves}\} & [\mathrm{SP}, \mathrm{Tag}\ 0E1L] \\
\mathcal{M}^{\mathrm{lci}}_g &= \{\text{local complete intersection curves}\} & [\mathrm{SP}, \mathrm{Tag}\ 0E0J] \\
\mathcal{M}^{\mathrm{iso}}_g &= \{\text{curves with isolated singularities}\} & [\mathrm{SP}, \mathrm{Tag}\ 0E0K].
\end{aligned}
$$

The open substack $\mathcal{M}^{\mathrm{DM}}_g$ is identified with the maximal open Deligne–Mumford substack of $\mathcal{M}^{\mathrm{all}}_g$. For any g, there are curves in $\mathcal{M}^{\mathrm{all}}_g \backslash \mathcal{M}^{\mathrm{DM}}_g$ with positive dimensional and non-reduced automorphism group.

While $\mathcal{M}^{\mathrm{all}}_{g,n}$ is not smooth over $\mathrm{Spec}\,\mathbb{Z}$, the open substack $\mathcal{M}^{\mathrm{lci}}_g \cap \mathcal{M}^{\mathrm{iso}}_g$ is smooth [SP, Tag 0DZX]. The open substack $\mathcal{M}_g \subset \mathcal{M}^{\mathrm{all}}_g$ is not dense: Mumford provided examples of reduced, irreducible proper curves that do not deform to a smooth curve [Mum75b]. However, we show in Proposition 5.7.21 that $\mathcal{M}_g$ is dense in $\overline{\mathcal{M}}_g$, and it is also true that $\mathcal{M}_g$ is dense in $\mathcal{M}^{\mathrm{lci}}_g \cap \mathcal{M}^{\mathrm{iso}}_g$ [SP, Tag 0E86].

Exercise 5.4.18 (not important)**.** Show that $\mathcal{M}^{\mathrm{all}}_g \to \mathrm{Spec}\,\mathbb{Z}$ satisfies the existence part of the valuative criterion for properness but that it is not universally closed.

Hint: Consider a closed substack of $\mathcal{M}^{\mathrm{all}}_g \times \mathbb{A}^1_{\mathbb{Z}}$ consisting of the disjoint union of $([C_n], n)$ over positive integers n, where C_n is the nodal union of a genus g curve and a tree of $\mathbb{P}^1$ with n nodes. The argument should also show that $\mathcal{M}^{\mathrm{pre}}_g$ is not universally closed over any field.

5.5 Stable reduction and the properness of $\overline{\mathcal{M}}_{g,n}$

In this section, we discuss Deligne and Mumford's celebrated theorem that $\overline{\mathcal{M}}_{g,n}$ is proper [DM69]. The key ingredient is the stable reduction theorem.

Theorem 5.5.1 (Stable Reduction). *Let R be a DVR with $K = \mathrm{Frac}(R)$, and set $\Delta = \mathrm{Spec}\,R$ and $\Delta^* = \mathrm{Spec}\,K$. If $(\mathcal{C}^* \to \Delta^*, \sigma_1^*, \ldots, \sigma_n^*)$ is a family of n-pointed stable curves, then there exists an extension of DVRs $R \to R'$ and an n-pointed family $(\mathcal{C} \to \Delta' = \mathrm{Spec}\,R', \sigma_1, \ldots, \sigma_n)$ of stable curves extending the base change of $(\mathcal{C}^* \to \Delta^*, \sigma_1^*, \ldots, \sigma_n^*)$ to $K' = \mathrm{Frac}\,R'$.*

While the theorem holds for any DVR, we give a complete proof in this section only in *characteristic 0*, following [KKMS73, Ch. II], [HM98, §3.C], and [ACG11, §X.4].

Remark 5.5.2. In characteristic 0, the stable reduction theorem was first stated in [MM64, Lem. A] with a proof given in [May69]. The general version first appeared in Deligne and Mumford's seminar paper [DM69]. Their proof relied on semistable reduction for abelian varieties, which had been established in [SGA7-I, SGA7-II], by embedding the generic fiber $\mathcal{C}^*$ into its Jacobian. Gieseker offerered a different proof using GIT [Gie82]; as expressed in [SP, Tag 0C2Q], "this is quite an amazing feat: it seems somewhat counterintuitive that one can prove such a result without ever truly studying families of curves over a positive dimensional base." Later arguments due to Artin–Winters [AW71] and Saito [Sai87] follow roughly the same strategy as we provide but require additional techniques in positive characteristic; see Remark 5.5.5.

5.5.1 Proof of Stable Reduction

Throughout, we use the notation that $\Delta = \mathrm{Spec}\,R$ for a DVR R defined over $\mathbb{Q}$, $\Delta^* = \mathrm{Spec}\,K$ where $K = \mathrm{Frac}(R)$, $t \in R$ is a uniformizer, and $0 = (t) \in \mathrm{Spec}\,R$ is the unique closed point. We are given an n-pointed stable family $(\mathcal{C}^* \to \Delta^*, \sigma_1^*, \ldots, \sigma_n^*)$ of curves of genus g.

Proof strategy

① Reduce to the case where $\mathcal{C}^* \to \Delta^*$ is smooth.

② Construct a flat extension $\mathcal{C} \to \Delta$.

③ Replace $\mathcal{C}$ with a resolution of singularities such that the reduced central fiber $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal.

④ Take a ramified base extension $\Delta' \to \Delta$ such that the normalization of $\mathcal{C} \times_\Delta \Delta$ has reduced central fiber.

⑤ Arrange that the marked points are smooth and distinct.

⑥ Contract rational tails and bridges in the central fiber.

It may be useful to keep examples in mind while reading the proof. For a very simple example, consider the family of elliptic curves $(\mathcal{C}^* \to \Delta^*, \sigma^*)$ defined by the equation

$$y^2 z = x(x-z)(x-tz) \tag{5.5.3}$$

in $\mathbb{P}^2 \times \Delta^*$ and the section $\sigma^*(t) = [0,1,0]$. In this case, the stable limit is transparent from our description of $\mathcal{C}^*$: the family $(\mathcal{C} \to \Delta, \sigma)$, where $\mathcal{C} \subset \mathbb{P}^2 \times \Delta$ is defined by (5.5.3) and $\sigma(t) = [0,1,0]$, is a stable family extending $(\mathcal{C}^* \to \Delta^*, \sigma^*)$; see Figure 0.18. The stable limit $\mathcal{C}_0$ is the nodal cubic $y^2 z = x^2(x-z)$. Additional examples are given in the proof, and more involved examples are described in §5.5.2.

Proof of Stable Reduction (5.5.1) in characteristic 0.

Step 1: Reduce to the case where $\mathcal{C}^* \to \Delta^*$ *is smooth.* If $\mathcal{C}^*$ has δ nodes, then after replacing K and R with extensions, we can arrange that the jth node is given by a K-point $n_j^* \in \mathcal{C}^*$ whose preimage under the normalization $\widetilde{\mathcal{C}}^* \to \mathcal{C}^*$ consists of two K-points q_j^* and r_j^*. We call $(\widetilde{\mathcal{C}}^* \to \Delta^*, \sigma_i^*, q_j^*, r_j^*)$ the *pointed normalization*, and we let $(\widetilde{\mathcal{C}}_k^* \to \Delta^*, q_{kl}^*)$ be the pointed connected components, where $\{q_{kl}^*\} = \{\sigma_i^*, q_j^*, r_j^*\} \cap \widetilde{\mathcal{C}}_k^*$. Since $\mathcal{C}^*$ is stable, each $(\widetilde{\mathcal{C}}_k^* \to \Delta^*, q_{kl}^*)$ is also stable (Exercise 5.3.5).

By induction on the genus g, we can apply stable reduction to each $(\widetilde{\mathcal{C}}_k^* \to \Delta^*, q_{kl}^*)$. After replacing K and R with extensions, this gives stable families $(\widetilde{\mathcal{C}}_k \to \Delta, q_{kl})$ extending $(\widetilde{\mathcal{C}}_k^* \to \Delta^*, q_{kl}^*)$. For each $j = 1, \ldots, \delta$, we use a Ferrand Pushout (B.4.1) to glue the sections q_{il} and $q_{i'l'}$ corresponding to q_j^* and r_j^*. By the étale local structure of smooth morphisms (Proposition A.3.4) and the étale local nature of pushouts (Proposition B.4.10), there is an étale neighborhood of the pushout of the form

$$
\begin{array}{ccc}
\operatorname{Spec} R \amalg \operatorname{Spec} R & \xrightarrow{\ 0 \amalg 0\ } & \mathbb{A}_R^1 \amalg \mathbb{A}_R^1 \\
\downarrow & & \downarrow \\
\operatorname{Spec} R & \dashrightarrow & \operatorname{Spec} A,
\end{array}
\qquad \text{where} \qquad
\begin{aligned}
A &= R \coprod_{R \times R} R[x] \times R[y] \\
&= \{(f, g) \in R[x] \times R[y] \mid f(0) = g(0)\} \\
&= R[x, y]/(xy).
\end{aligned}
$$

The sections q_{il} and $q_{i'l'}$ are glued to a node. This produces an n-pointed family $(\mathcal{C} \to \Delta, \sigma_i)$ of nodal curves with δ additional sections picking out the nodes. The pointed normalization of the central fiber $\mathcal{C}_0$ is the disjoint union of the central fibers $(\widetilde{\mathcal{C}}_k)_0$, and it follows from Exercise 5.3.5 that $(\mathcal{C}_0, \sigma_i(0))$ is stable. We conclude that $(\mathcal{C} \to \Delta, \sigma_i)$ is a stable family.
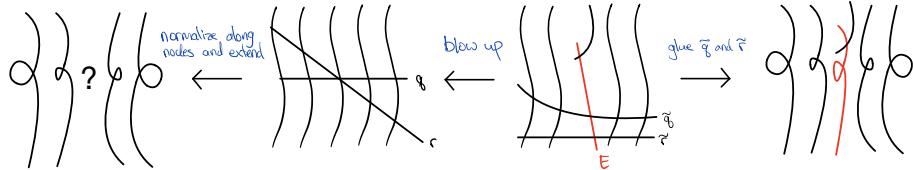
Figure 5.8: (Step 1) Consider a family $\mathcal{C}^* \to \Delta^*$ of stable curves with a single node, e.g. a node degenerating to a cusp locally defined by $y^2 = x^3 + tx^2$. The stable limit is obtained by normalizing along the nodes, extending the 2-pointed family, blowing up where the two sections q and r intersect, and then gluing the proper transforms $\widetilde{q}$ and $\widetilde{r}$. If the normalization $\widetilde{\mathcal{C}}^* = C \times \Delta^*$ is a constant family, then the stable limit is the nodal union of C and a rational nodal curve.

Alternatively, using that $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is dense (Theorem 5.7.23) Exercise 4.5.4 asserts that it suffices to check the valuative criterion for properness under the condition that $\Delta^* \to \overline{\mathcal{M}}_{g,n}$ factors through $\mathcal{M}_{g,n}$.

Step 2: Find a flat extension $(\mathcal{C} \to \Delta, \sigma_i)$.

Using that $\left(\omega_{\mathcal{C}^*/\Delta^*}(\sum_i \sigma_i^*) \right)^{\otimes 3}$ is very ample (Proposition 5.3.21), we may embed $\mathcal{C}^*$ as a closed subscheme of $\mathbb{P}^N \times \Delta^*$. By the Flatness Criterion over Smooth Curves (A.2.2), the scheme-theoretic image $\mathcal{C}$ of $\mathcal{C}^* \hookrightarrow \mathbb{P}^N \times \Delta$ is flat over Δ since the closure doesn't introduce any embedded points in the central fiber. This gives a family of curves $\mathcal{C} \to \Delta$ extending $\mathcal{C}^* \to \Delta^*$. (This is the same argument we used in Proposition 1.4.2 to show the properness of the Hilbert scheme using the valuative

criterion.) The sections $\sigma_i^* \colon \Delta^* \to \mathcal{C}^*$ extend to sections $\sigma_i \colon \Delta \to \mathcal{C}$ by the Valuative Criterion of Properness (A.4.5).
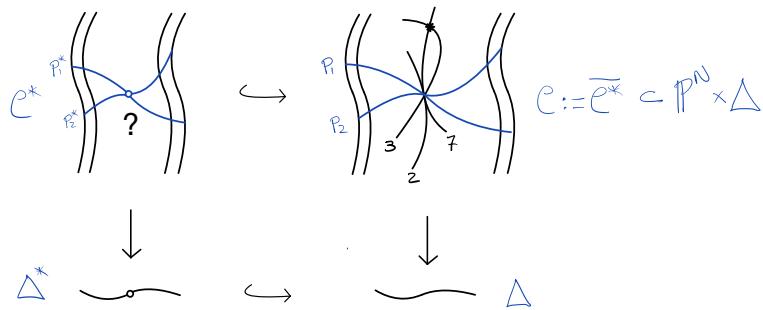
Figure 5.9: (Step 2) The closure $\mathcal{C} := \overline{\mathcal{C}^*}$ is a flat family over Δ. The central fiber $\mathcal{C}_0$ may be generically non-reduced with embedded points; the numbers 3, 2, and 7 indicate the multiplicity of the irreducible components. The marked points $\sigma_i(0) \in \mathcal{C}_0$ may be singular and non-distinct.

Step 3: Replace $\mathcal{C}$ with a resolution of singularities to arrange that the reduced central fiber $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal.

By combining Existence of Resolutions (B.6.1) and Existence of Embedded Resolutions (B.6.3), there is a projective birational morphism $\widetilde{\mathcal{C}} \to \mathcal{C}$ from a regular scheme such that $\widetilde{\mathcal{C}} \to \mathcal{C}$ is an isomorphism over Δ^* and such that the central fiber $\widetilde{\mathcal{C}}_0$ has set-theoretic normal crossings, i.e. $(\widetilde{\mathcal{C}}_0)_{\mathrm{red}}$ is nodal. By the Flatness Criterion over Smooth Curves (A.2.2), $\widetilde{\mathcal{C}} \to \Delta$ is flat. We then replace $\mathcal{C}$ with $\widetilde{\mathcal{C}}$ and the sections σ_i with their strict transform.

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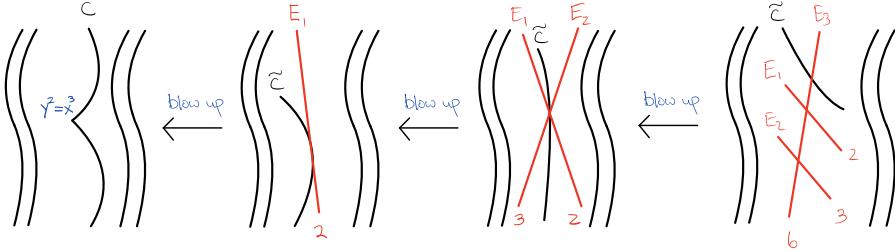
Figure 5.10: (Step 3) Suppose $\mathcal{C} \to \Delta = \Bbbk[t]_{(t)}$ is a generically smooth family degenerating to a cusp $y^2 = x^3$ in the central fiber such that the local equation in $\mathcal{C}$ around the singular point is $y^2 = x^3 + t$. We repeatedly blow up the singular point in the central fiber using local coordinates x, y on the original surface and $\widetilde{x}, \widetilde{y}$ on the new surface, where in the first chart of the blowup $\widetilde{x} = x, \widetilde{y} = y/x$ with exceptional divisor $\widetilde{x} = 0$, while in the second chart, $\widetilde{x} = x/y, \widetilde{y} = y$ with exceptional divisor $\widetilde{y} = 0$.

→ For the first blowup, the preimage of the singularity in the first chart is given by $\widetilde{x}^2(\widetilde{y}^2 - \widetilde{x})$ and in the second chart by $\widetilde{y}^2(1 - \widetilde{x}^3\widetilde{y})$. The exceptional divisor E_1 has multiplicity 2. The normalization $\widetilde{C}$ of C has genus $g - 1$.

→ The second blowup has charts defined by $\widetilde{x}^3(\widetilde{x}\widetilde{y}^3 - 1)$ and $\widetilde{x}^2\widetilde{y}^3(\widetilde{y} - \widetilde{x})$, and the new exceptional divisor E_2 has multiplicity 3.

→ The final blowup has charts $\widetilde{x}^6\widetilde{y}^3(\widetilde{y} - 1)$ and $\widetilde{x}^2\widetilde{y}^6(1 - \widetilde{x})$, and the new exceptional divisor E_3 has multiplicity 6. The central fiber is non-reduced and its reduction is nodal.

Step 4: Take a ramified base extension $\Delta' = \operatorname{Spec} R' \to \operatorname{Spec} R = \Delta$ such that the central fiber of the normalization of $\mathcal{C} \times_\Delta \Delta'$ becomes <u>reduced</u> and nodal. This is the most difficult step and is where the characteristic 0 assumption is used. The argument can be viewed as a version of Abhyankar's lemma on tame ramification (see [SGA1, §XIII.5] and [SP, Tag 0EXT]). Since $\mathcal{C}$ is regular and $t \in R$ is a nonzerodivisor, the central fiber $\mathcal{C}_0$ is Cohen–Macaulay, and thus has no embedded points. On the other hand, $(\mathcal{C}_0)_{\mathrm{red}}$ is an effective Cartier divisor and thus is locally cut out by a single equation. For every $p \in \mathcal{C}_0$, we can find an étale neighborhood and local coordinates x, y such that the map $\mathcal{C} \to \Delta$ is given explicitly by:

– if $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a smooth point, then $(x, y) \mapsto x^a$ and the multiplicity of the irreducible component of $\mathcal{C}_0$ containing p is a, and

– if $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a node , then $(x, y) \mapsto x^a y^b$ and the two components of $\mathcal{C}_0$ containing p have multiplicities a and b. If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a non-separating node (i.e. $\mathcal{C}_0 \setminus p$ is connected), then $a = b$.

Let N be the least common multiple of the multiplicities of the irreducible components of $\mathcal{C}_0$. After replacing R with an extension, we can assume that R contains a primitive Nth root of unity ρ. Let $\Delta' = \operatorname{Spec} R' \to \operatorname{Spec} R = \Delta$ be a totally ramified extension of DVRs of degree N (i.e. the image of the uniformizer $t \in R$ is t'^N for uniformizer $t' \in R'$). Let $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ and $p' \in \mathcal{C}'$ be the unique preimage of p, and let $\widetilde{\mathcal{C}}'$ be the normalization of $\mathcal{C}'$.

– If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is smooth, then $\mathcal{C}'$ is defined étale locally by $x^a = t^N$ near p'.

Since R is characteristic 0, there is a factorization

$$x^a - t^N = \prod_{i=0}^{a-1} (x - (\rho^i t)^{N/a})$$

into distinct factors. (If the characteristic were positive and divided N, then $\mathcal{C}'$ would be non-reduced.) In the normalization $\widetilde{\mathcal{C}'}$, the point p has a preimages, each etale locally defined by $x = \rho^i t^{N/a}$. Each preimage is a smooth point in both the total family $\widetilde{\mathcal{C}'}$ and the central fiber $\widetilde{\mathcal{C}'_0}$.

– If $p \in (\mathcal{C}_0)_{\mathrm{red}}$ is a node, then $\mathcal{C}'$ is defined étale locally by $x^a y^b = t^N$ near p'. Let $d = \gcd(a,b)$. If $d > 1$, there is a factorization (again using the characteristic 0 hypothesis)

$$x^a y^b - t^N = \prod_{i=0}^{d-1} (x^{a/d} y^{b/d} - (\rho^i t)^{N/d}).$$

Étale locally near p, the normalization factors as $\widetilde{\mathcal{C}'} \to \mathcal{C}'' \to \mathcal{C}'$ such that p has d preimages in $\mathcal{C}''$, each described by $x^{a/d} y^{b/d} - (\rho^i t)^{N/d}$. Unless $d = a = b$, the central fiber $\mathcal{C}''_0$ is still non-reduced. The reduction $(\mathcal{C}''_0)_{\mathrm{red}}$ is nodal at each preimage of p but now the multiplicities of the two branches are relatively prime.

Therefore, we can assume that $\gcd(a,b) = 1$. After possibly exchanging x and y, we can write $1 = \alpha a - \beta b$ for positive integers α and β. The normalization of the integral domain $R[x,y]/(x^a y^b - t^N)$ is given by

$$R[x,y]/(x^a y^b - t^N) \hookrightarrow R[u,v]/(uv - t^{N/(ab)}), \qquad \begin{aligned} x &\mapsto u^b \\ y &\mapsto v^a, \end{aligned}$$

where $u = t^{\alpha N/b}/(x^\beta y^\alpha)$ and $v = x^\beta y^\alpha / t^{\beta N/a}$. The central fiber $\widetilde{\mathcal{C}'_0}$ is reduced with a node at a unique preimage of p. If $N = ab$, $\widetilde{\mathcal{C}'}$ is smooth at this preimage; otherwise $\widetilde{\mathcal{C}'}$ has an A_{n-1}-singularity where $n = N/(ab)$.

We have arranged the central fiber to be reduced and nodal, but we have also introduced A_{n-1}-singularities (i.e. $xy - t^n$) into the total family. Repeatedly blowing up each A_{n-1}-singularity replaces the singularity with the nodal union of $\lfloor \frac{n}{2} \rfloor$ smooth rational curves; this explicitly describes the minimal resolution of $\mathcal{C}$ as in Theorem B.6.2. We now replace $\mathcal{C}$ with the minimal resolution of $\widetilde{\mathcal{C}'}$ so that the total family $\mathcal{C}$ is regular and the central fiber $\mathcal{C}_0$ is reduced and nodal.
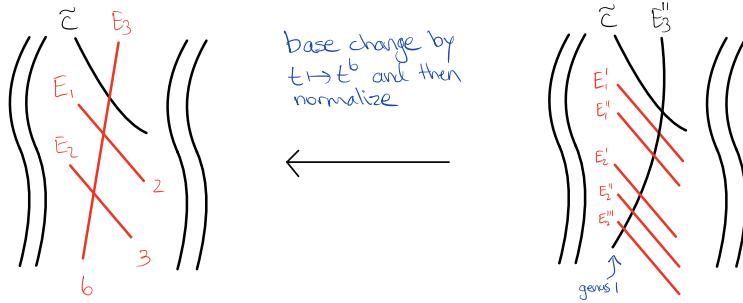
Figure 5.11: (Step 4) Continuing from Figure 5.10, we base change $\mathcal{C} \to \Delta = \operatorname{Spec} \Bbbk[t]_{(t)}$ by $\Delta' = \operatorname{Spec} \Bbbk[t]_{(t)} \to \operatorname{Spec} \Bbbk[t]_{(t)} = \Delta$, given by $t \mapsto t^6$, and then normalize. The central fiber is now reduced and nodal. Each preimage of E_1 and E_2 are smooth rational curves, while the preimage E_3'' of E_3 is a genus 1 curve. To understand this description, it is convenient to break the base change into the composition of the normalized base change by $t \mapsto t^2$ and the normalized base change by $t \mapsto t^3$, which we describe in Example 5.5.8.

Step 5: Arrange that the marked points $\sigma_i(0) \in \mathcal{C}_0$ are smooth and distinct.

After repeatedly blowing up closed points in the central fiber $\mathcal{C}_0$ where the marked points $\sigma_i(0)$ are singular or collide, the strict transform of the sections become distinct and smooth points of the central fiber. After replacing $\mathcal{C}$ with the blowup and the sections σ_i with their strict transform, we have a prestable family $(\mathcal{C} \to \Delta, \sigma_i)$ with regular total family.
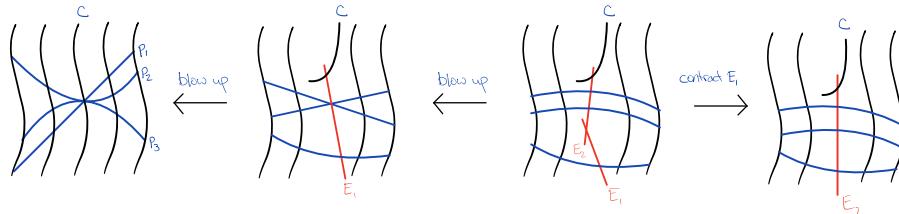
Figure 5.12: (Step 5) Consider a constant family $C \times \Delta \to \Delta$ with sections locally defined by $(\sigma_1, \sigma_2, \sigma_3) = (t^2, -t^2, 4t)$. After blowing up twice, the sections become disjoint but the central fiber is not stable as the exceptional component $E_1 \cong \mathbb{P}^1$ only has one node and one marked point. As explained in the next step, the stable limit is obtained by contracting the rational bridge E_1.

Step 6: Contract rational tails and bridges in the central fiber.

The central fiber $\mathcal{C}_0$ of the prestable family $(\mathcal{C} \to \Delta, \sigma_i)$ will be stable, unless there are rational tails and bridges (see Definition 5.3.10). If $E \subset \mathcal{C}_0$ is a rational tail or rational bridge with a marked point, then $E^2 = -1$, while if E is a rational bridge without a marked point, then $E^2 = -2$ (see Remark 5.3.13). By Exercise 5.3.14, contracting each rational tail and bridge yields a morphism $\mathcal{C} \to \mathcal{C}'$ of families of nodal curves over Δ. Letting $\sigma_i' \colon \Delta \xrightarrow{\sigma_i} \mathcal{C} \to \mathcal{C}'$, then n-pointed family $(\mathcal{C}' \to \Delta, \sigma_i')$ is now stable! Alternatively, we can construct the stable family $(\mathcal{C}' \to \Delta, \sigma_i')$ using the Stable Contraction of a Prestable Family (5.6.4). □
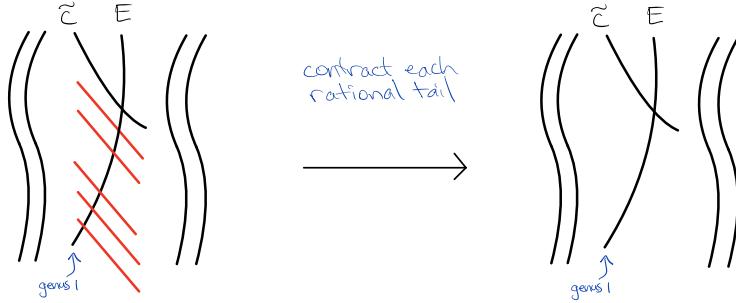
Figure 5.13: (Step 6) Continuing from Figure 5.11, the contraction of each rational tail in red produces a stable family. The central fiber is the nodal union of the normalization $\widetilde{C}$ of the original central fiber (with a cusp) and an elliptic curve E.

If we proceed with the six-step procedure above, but stop in Step 6 after contracting only rational tails (but not raitonal bridges), then Castelnuovo's Contraction Theorem (B.6.6) implies that the total family is still regular. This important variant is called *semistable reduction*.

Theorem 5.5.4 (Semistable Reduction). *Let R be a DVR with $K = \text{Frac}(R)$, and set $\Delta = \text{Spec } R$ and $\Delta^* = \text{Spec } K$. If $(\mathcal{C}^* \to \Delta^*, s_1^*, \ldots, s_n^*)$ is a family of n-pointed* <u>smooth</u> *curves, then there exists an extension of DVRs $R \to R'$ and an n-pointed family $(\mathcal{C} \to \Delta' = \text{Spec } R', s_1, \ldots, s_n)$ of semistable curves with* <u>regular</u> *total family $\mathcal{C}$ extending the base change of $(\mathcal{C}^* \to \Delta^*, s_1^*, \ldots, s_n^*)$ to $K' = \overline{\text{Frac } R'}.$* $\qquad\square$

Remark 5.5.5 (Proof in characteristic p). Our proof of stable reduction fails in Step 4 if the residue field of R has characteristic $p > 0$ and any of the multiplicities of the components of the central fiber are divisible by p. A different approach is needed in positive characteristic. After resolving the singularities of $\mathcal{C}$ and arranging that $(\mathcal{C}_0)_{\text{red}}$ is nodal (as we do in Step 3 above), Artin and Winters arrange that the l-torsion of $\text{Pic}(\mathcal{C}_{K'}^*)$ is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^{2g}$ for a sufficiently large prime $l \neq p$, and they show that this magically forces the central fiber to be reduced and nodal! See [AW71], [Liu02, §10.4], and or [SP, Tag 0C2P].

Example 5.5.6 (Base changes are necessary). The stable reduction of the cusp worked out in Figures 5.10, 5.11 and 5.13 demonstrates the necessity of allowing for extensions of the DVR. Indeed, after Step 3, we have a generically smooth family $\mathcal{C} \to \Delta$ with a regular total family $\mathcal{C}$ and with a non-reduced central fiber $\mathcal{C}_0$. Suppose that there is a stable family $\mathcal{C}' \to \Delta$ such that $\mathcal{C}|_{\Delta^*} \cong \mathcal{C}'|_{\Delta^*}$. After resolving the A_n-singularities in $\mathcal{C}'$, there is semistable family $\mathcal{C}'' \to \Delta$ with $\mathcal{C}''$ regular and $\mathcal{C}_0''$ semistable (and in particular reduced). Since $\mathcal{C}$ and $\mathcal{C}''$ are birational, by taking a resolution of singularities of the closure of the graph of a rational map $\mathcal{C} \dashrightarrow \mathcal{C}'$, there is a regular two-dimensional scheme $\mathcal{D}$ and birational morphisms $\mathcal{D} \to \mathcal{C}$ and $\mathcal{D} \to \mathcal{C}''$ over Δ, each which factors as a composition of blowups by Factorization of Birational Maps (B.6.4). However, any blowup of $\mathcal{C}$ will have a non-reduced central fiber, while any blowup of $\mathcal{C}''$ will have a reduced central fiber, and this yields a contradiction.

5.5.2 Explicit stable reduction

In applications to the geometry of curves, it is often essential to explicitly describe the stable limit. While the proof of Stable Reduction offers a strategy, additional care is needed to get an explicit handle. The main challenge is determining the normalization $\widetilde{\mathcal{C}}'$ of the base change $\mathcal{C}' = \mathcal{C} \times_\Delta \Delta'$ by a totally ramified extension $\Delta' \to \Delta$ in Step 4. It is often simpler to factor $\Delta' \to \Delta$ as a composition of prime order base changes, because it is straightforward to determine the ramification locus of prime order normalized base changes $\widetilde{\mathcal{C}}' \to \mathcal{C}$.

Proposition 5.5.7. *Let p be a prime integer. Let $\mathcal{C} \to \Delta = \operatorname{Spec} R$ be a generically smooth family of curves, where R is a DVR of characteristic 0 containing a pth root of unity. Assume that the reduced central fiber $(\mathcal{C}_0)_{\mathrm{red}}$ is nodal. As a divisor on $\mathcal{C}$, we may write $\mathcal{C}_0 = \sum m_i D_i$ where m_i is the multiplicity of the irreducible component D_i. Let $\Delta' = \operatorname{Spec} R' \to \operatorname{Spec} R = \Delta$ be a totally unramified extension of degree p, and set $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$ with normalization $\widetilde{\mathcal{C}}'$. Then $\widetilde{\mathcal{C}}' \to \mathcal{C}$ is ramified over a divisor D_i if and only if m_i is relatively prime to p. If $q' \in \widetilde{\mathcal{C}}'_0$ is a preimage of a point $q \in D_i$ which is a smooth point of $(\mathcal{C}_0)_{\mathrm{red}}$, then the multiplicity of $\widetilde{\mathcal{C}}'_0$ around q' is m_i/p if p divides m_i and m_i otherwise.*

Proof. The point $q \in \mathcal{C}$ has an étale neighborhood with local coordinates x and y such that $\mathcal{C} \to \Delta$ is described as $(x, y) \mapsto x^{m_i}$. The base change $\mathcal{C}'$ is described étale locally by $\operatorname{Spec} \Bbbk[x, y, t]/(x^{m_i} - t^p)$ near q'. If p divides m_i,

$$x^{m_i} - t^p = \prod_{i=0}^{p-1} (x^{m_i/p} - \rho^i t), \qquad \text{where } \rho \text{ is a } p\text{th root of unity},$$

and its normalization has p smooth components. The central fiber $\widetilde{\mathcal{C}}'_0$ has multiplicity m_i/p near each of the p preimages of q. If $m_i = pm_i' + r$ with $r \neq 0$, then $\Bbbk[x, y, t]/(x^{m_i} - t^p)$ is an integral domain with normalization

$$\Bbbk[x, y, t']/(x^r - t'^p), \qquad \text{where } t' = t/x^{m_i'}.$$

Thus, $\widetilde{\mathcal{C}}' \to \mathcal{C}$ is ramified at q. The central fiber $\widetilde{\mathcal{C}}' \to \Delta$ is defined by $t = t'x^{m_i'}$ and the multiplicity at the unique preimage of q can be computed as $\dim_{\Bbbk} \Bbbk[x, t']/(x^r - t'^p, t'x^{m_i'})$. Since the union of $t'^i x^j$ for $i = 0, \ldots, p-1$ and $j = 0, \ldots, m_i' - 1$ and $x^{m_i'}, \ldots, x^{m_i'+r-1}$ forms a basis, the multiplicity is m_i. $\square$

Example 5.5.8 (Stable reduction of a cusp). Let $\mathcal{C} \to \Delta = \operatorname{Spec} \Bbbk[t]_{(t)}$ be a generically smooth family degenerating to a cusp $y^2 = x^3$ in the central fiber $C := \mathcal{C}_0$ such that the local equation in $\mathcal{C}$ around the singular point is $y^2 = x^3 + t$. As described in Figure 5.10, after three blowups of $\mathcal{C}$, we obtain a family $\mathcal{C}' \to \Delta$ such that $(\mathcal{C}'_0)_{\mathrm{red}}$ is the nodal union of the normalization $\widetilde{C}$, which has multiplicity 1, and three exceptional components E_1, E_2, and E_3, which have multiplicities 2, 3, and 6.
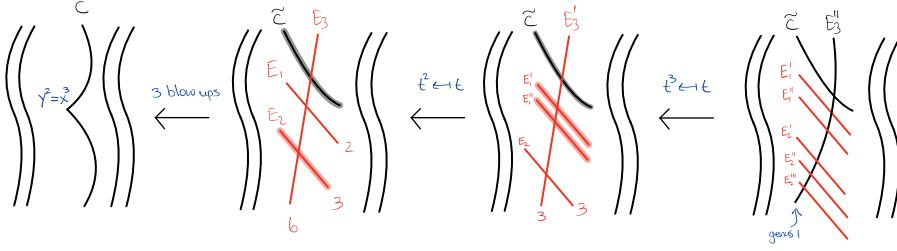
Figure 5.14: The numbers indicate the multiplicity of a component; if no number is given, the multiplicity is 1. The thickened components indicate the ramification locus under the normalized base change.

Using Proposition 5.5.7, the ramification locus of the first normalized base change $t \mapsto t^2$ is the union of $\widetilde{C}$ and E_2. The preimage of E_1 is the disjoint union $E_1' \amalg E_1''$ of two smooth rational curves. The preimage E_3' of E_3 is a curve which is a two-to-one cover of $E_3 = \mathbb{P}^1$ ramified at two points, and thus $E_3 = \mathbb{P}^1$ but now with multiplicity 2. For the second normalized base change, the ramification locus is $\widetilde{C} \amalg E_1' \amalg E_1''$ (again by Proposition 5.5.7). The preimage of E_2 is the union of three smooth rational curves, while the preimage E_3'' of E_3' is a smooth curve which is a three-to-one cover of $E_3' = \mathbb{P}^1$ ramified at three points, each with ramification index two. The genus $g_{E_3''}$ of E_3'' can be computed to be 1 using Riemann–Hurwitz (5.7.3): $2g_{E_3''} - 2 = 3(2(0) - 2) + 3(2) = 0$. The final step contracts the five rational tails as pictured in Figure 5.13, and the stable limit is the nodal union $\widetilde{C} \cup E_3''$ of a smooth genus $g - 1$ curve $\widetilde{C}$ (the normalization of the original central fiber) and a genus 1 curve.

Remark 5.5.9. Precisely which elliptic curve E_3'' appears in the stable limit and how does the stable limit depend on the choice of degeneration? The deformation space of a cusp is $y^2 = x^3 + a_1(t)x + a_0(t)$ and, in other words, we are asking how the stable limit depends on $a_i(t)$. For instance, what happens when the total family of the surface is singular (e.g. $y^2 = x^{2k+1} + t^2$)? These questions are addressed in detail in [HM98, §3.C].

Exercise 5.5.10 (good practice).
  (a) Find the stable limit of a generically smooth family degenerating to a tacnode $y^2 = x^4$.
  (b) More generally, show that the stable limit of a generically smooth family degenerating to a A_{2k} singularity $y^2 = x^{2k+1}$ (resp. $y^2 = x^{2k+2}$) is the nodal union of a genus $g - k$ curve and a genus k hyperelliptic curve attached at a Weierstrass point (resp. at two Weierstrass conjugate points). Here a Weierstrass point of a hyperelliptic curve H is a ramification point under the double cover $H \to \mathbb{P}^1$, while two points are Weierstrass conjugate points if their union is a fiber of $H \to \mathbb{P}^1$.

Example 5.5.11 (Stable reduction of a double conic). Consider a generically smooth family $\mathcal{C} \to \Delta = \operatorname{Spec} \Bbbk[t]_{(t)}$, where $\mathcal{C} \subset \mathbb{P}^2 \times \Delta$ is defined by $F^2 + tG$ where F is a smooth conic and G is a smooth quartic. The central fiber is the double conic defined by F^2. The total space has an A_1-singularity with local equation $x^2 + yt$ at each of the 8 intersection points $p_1, \ldots, p_8$ of $F \cap G$. Each σ_i is resolved with a single blowup with an exceptional divisor $E_i = \mathbb{P}^1$ of multiplicity 1. This gives a family $\mathcal{C}_2 \to \Delta$ where the central fiber is $2C + \sum_i E_i$ as a divisor. We then take the

normalization $\mathcal{C}_3$ of the base change $\mathcal{C}_2 \times_\Delta \Delta'$ by the ramified cover $\Delta' \to \Delta$, $t \mapsto t^2$. By Proposition 5.5.7, $\mathcal{C}_3 \to \mathcal{C}_2$ is ramified over the disjoint union of the E_i's. The preimage of C is a two-to-one cover of $\mathbb{P}^1$ branched over the 8 points σ_i, and hence is *smooth* hyperelliptic genus 3 curve.
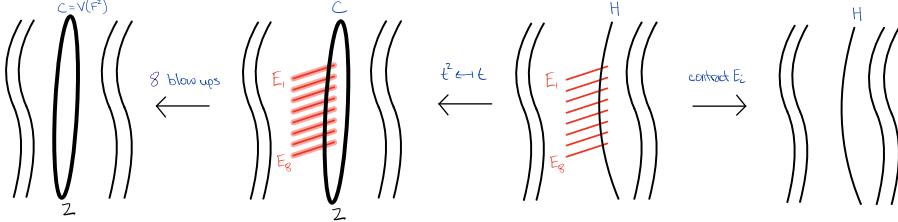
Figure 5.15: Stable reduction of a generically smooth family $F^2 + tG$ degenerating to a double conic $C = V(F^2)$.

Meta-exercise 5.5.12. Read the exposition of [HM98, §3.C] and do the exercises.

5.5.3 Properness of $\overline{\mathcal{M}}_{g,n}$

Stable Reduction (5.5.1) implies the existence part of the valuative criterion for properness for $\overline{\mathcal{M}}_{g,n}$. We must also show that the stable limit is unique, i.e. $\overline{\mathcal{M}}_{g,n}$ is separated.

Proposition 5.5.13. *Let R be a DVR with $K = \mathrm{Frac}(R)$, and set $\Delta = \mathrm{Spec}\, R$ and $\Delta^* = \mathrm{Spec}\, K$. If $(\mathcal{C} \to \Delta, \sigma_1, \ldots, \sigma_n)$ and $(\mathcal{D} \to \Delta, \tau_1, \ldots, \tau_n)$ are families of n-pointed stable curves, then every isomorphism $\alpha^* \colon \mathcal{C} \times_\Delta \Delta^* \to \mathcal{D} \times_\Delta \Delta^*$ over Δ^* compatible with the restriction of the sections (i.e. $\tau_i^* = \alpha^* \circ \sigma_i^*$) extends to a unique isomorphism $\alpha \colon \mathcal{C} \to \mathcal{D}$ over Δ with $\tau_i = \alpha \circ \sigma_i$.*

Proof. For simplicity, we handle the case without marked points ($n = 0$). We claim that we can reduce to the case where the generic fiber $\mathcal{C}^* \cong \mathcal{D}^*$ is smooth over Δ^*, where $\mathcal{C}^* := \mathcal{C} \times_\Delta \Delta^*$. We will employ the same strategy as in the reduction to the generically smooth case in Step 1 of the proof of Stable Reduction (5.5.1). By flat descent, it suffices to construct $\alpha \colon \mathcal{C} \to \mathcal{D}$ after an extension of DVRs $\Delta' \to \Delta$; indeed, an isomorphism $\alpha' \colon \mathcal{C} \times_\Delta \Delta' \to \mathcal{D} \times_\Delta \Delta'$ will satisfy the cocycle condition (by the separatedness of $\mathcal{D} \to \Delta$) and thus descend to an isomorphism α. Therefore, we may assume that each node of $\mathcal{C}^* \cong \mathcal{D}^*$ is given by a K-point whose preimage under the normalization consists of two K-points. Each node extends to sections $\Delta \to \mathcal{C}$ and $\Delta \to \mathcal{D}$. The pointed normalizations $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ are each disjoint unions of pointed stable families, and we can use induction on the genus to extend the generic isomorphism to an isomorphism $\widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}}$ which descends to an isomorphism $\mathcal{C} \to \mathcal{D}$. Alternatively, using that $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is dense (proven later in Theorem 5.7.23), it suffices to check the valuative criterion for separatedness under the condition that $\Delta^* \to \overline{\mathcal{M}}_{g,n}$ factors through $\mathcal{M}_{g,n}$ (Exercise 4.5.4).

Let $\widetilde{\mathcal{C}} \to \mathcal{C}$ and $\widetilde{\mathcal{D}} \to \mathcal{D}$ be the minimal resolutions (Theorem B.6.2). Let Γ be the closure of the image of $(\mathrm{id}, \alpha^*) \colon \mathcal{C}^* \to \widetilde{\mathcal{C}} \times_\Delta \widetilde{\mathcal{D}}$, and let $\widetilde{\Gamma} \to \Gamma$ be its minimal

resolution. This gives a commutative diagram

$$
\begin{array}{ccc}
& \widetilde{\Gamma} & \\
\swarrow & & \searrow \\
\widetilde{\mathcal{C}} & & \widetilde{\mathcal{D}} \\
\downarrow & & \downarrow \\
\mathcal{C} & & \mathcal{D} \\
\searrow & & \swarrow \\
& \Delta. &
\end{array}
\qquad (5.5.14)
$$

We will give two arguments that there is an isomorphism $\alpha\colon \mathcal{C} \to \mathcal{D}$ extending α^*. First, the local structure of $\mathcal{C}$ (resp. $\mathcal{D}$) around a node in the central fiber is an A_n singularity of the form $xy = t^{n+1}$, where $t \in R$ is a uniformizer. The preimage of each node under $\widetilde{\mathcal{C}} \to \mathcal{C}$ (resp. $\widetilde{\mathcal{D}} \to \mathcal{D}$) is a chain $E_1 \cup \cdots \cup E_n$ of rational bridges with $E_i^2 = -2$. Since $\mathcal{C}$ and $\mathcal{D}$ are families of stable curves, they each have no smooth rational -1 curves and thus neither do $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$. By the Factorization of Birational Maps (Theorem B.6.4), both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ are the compositions of finite sequences of blowups at closed points. Since $\widetilde{\Gamma} \to \Gamma$ is a minimal resolution, $\widetilde{\Gamma}$ has no smooth rational -1 curves that get contracted under both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$. We claim that both $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ are isomorphisms. This would finish the proof as both $\mathcal{C}$ and $\mathcal{D}$ are obtained by contracting the smooth rational -2 curves of $\widetilde{\mathcal{C}} \cong \widetilde{\mathcal{D}}$.

To see the claim, suppose for instance that $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ is not an isomorphism. Then there is a smooth rational -1 curve $E \subset \widetilde{\Gamma}$ not contracted under $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$. Let $E_{\widetilde{\mathcal{D}}} \subset \widetilde{\mathcal{D}}$ be its image. Since blowing up only decreases the self-intersection number (indeed, if we write the pre-image of $E_{\widetilde{\mathcal{D}}}$ in $\widetilde{\Gamma}$ as $E + F$, then the projection formula implies that $E_{\widetilde{\mathcal{D}}}^2 = E \cdot (E + F) = E^2 + E \cdot F$), we have that $E_{\widetilde{\mathcal{D}}}^2 \geq E^2 = -1$. On the other hand, the Hodge Index Theorem for Exceptional Curves (B.6.5) implies that $E_{\widetilde{\mathcal{D}}}^2 \leq -1$. Hence $E_{\widetilde{\mathcal{D}}}^2 = -1$. But $E_{\widetilde{\mathcal{D}}}$ is not a smooth rational -1 curve so it must be singular, and one of the blowups in the composition $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ must be at a singular point of $E_{\widetilde{\mathcal{D}}}$. But this implies that exceptional locus F of $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ intersects E non-trivially, and thus $E_{\widetilde{\mathcal{D}}}^2 \geq E^2 + 1$, a contradiction.

Alternatively, we could argue as follows. The birational maps $\widetilde{\Gamma} \to \widetilde{\mathcal{C}}$ and $\widetilde{\Gamma} \to \widetilde{\mathcal{D}}$ of smooth projective surfaces are isomorphisms in codimension 2. As the relative dualizing sheaves are line bundles, there are identifications of the pluricanonical sections

$$
\Gamma(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/\Delta}^{\otimes k}) \cong \Gamma(\widetilde{\Gamma}, \omega_{\widetilde{\Gamma}/\Delta}^{\otimes k}) \cong \Gamma(\widetilde{\mathcal{D}}, \omega_{\widetilde{\mathcal{D}}/\Delta}^{\otimes k})
$$

for each non-negative integer k (c.f. [Har77, Thm. II.8.19]). Using that $\mathcal{C}$ and $\mathcal{D}$ are the stable contraction of the families $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ over Δ (Theorem 5.6.4) arising as the Proj of the graded ring of pluricanonical sections, we obtain an isomorphism

$$
\mathcal{C} \xrightarrow{\sim} \operatorname{Proj} \bigoplus_k \Gamma(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/\Delta}^{\otimes k}) \xrightarrow{\sim} \operatorname{Proj} \bigoplus_k \Gamma(\widetilde{\mathcal{D}}, \omega_{\widetilde{\mathcal{D}}/\Delta}^{\otimes k}) \xrightarrow{\sim} \mathcal{D}
$$

extending $\alpha^*\colon \mathcal{C}^* \to \mathcal{D}^*$.

See also [DM69, Lem. 1.12], [ACG11, Lem. 10.5.1], and [SP, Tag 0E97]. $\qquad \square$

Exercise 5.5.15 (details)**.** Extend the proof above to the case of marked points.

Even though we've only proved Stable Reduction (5.5.1) in characteristic 0, we state the next result over $\mathbb{Z}$.

Theorem 5.5.16. *If $2g - 2 + n > 0$, the Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}$ is proper over $\operatorname{Spec}\mathbb{Z}$. Moreover, there is a coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$, where $\overline{M}_{g,n}$ is a proper algebraic space over $\operatorname{Spec}\mathbb{Z}$.*

Proof. Using the Valuative Criterion (3.8.2), Stable Reduction (5.5.1) gives the existence of limits, while Proposition 5.5.13 gives the uniqueness of a limit. As $\overline{\mathcal{M}}_{g,n}$ is separated, the Keel–Mori Theorem (4.3.12) gives a coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$, where $\overline{M}_{g,n}$ is an algebraic space separated and of finite type over $\operatorname{Spec}\mathbb{Z}$. As $\overline{\mathcal{M}}_{g,n}$ is proper over $\operatorname{Spec}\mathbb{Z}$, so is $\overline{M}_{g,n}$. $\qquad\square$

Exercise 5.5.17. Show that the coarse moduli space of $\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \mathbb{F}_p$ is the normalization of $\overline{M}_{g,n} \times_{\mathbb{Z}} \mathbb{F}_p$.

Question 5.5.18 (Open)**.** Is $\overline{M}_{g,n} \times_{\mathbb{Z}} \mathbb{F}_p$ the coarse moduli space of $\overline{\mathcal{M}}_{g,n} \times_{\mathbb{Z}} \mathbb{F}_p$?[1]

5.5.4   Aside: semistable reduction in higher dimension

We attempt a brief survey of generalizations of semistable reduction, providing answers to the following question.

Question 5.5.19 (Semistable Reduction Problem)**.** Given a scheme S and a flat, finite type, and generically smooth morphism $X \to S$, when can we find a commutative diagram

$$
\begin{array}{ccc}
X' \longrightarrow X \times_S S' \longrightarrow X \\
\searrow \quad \downarrow \qquad\qquad \downarrow \\
S' \longrightarrow S
\end{array}
\tag{5.5.20}
$$

with $X' \to S'$ flat such that the morphisms $X' \to S'$, $S' \to S$, and $X' \to X \times_S S'$ satisfy additional 'nice' properties?

By a '*semistable reduction theorem*', one usually requires that

- X' and S' are regular, and the fibers of $X' \to S'$ are reduced normal crossings divisors,
- $S' \to S$ an *alteration*, i.e. proper, generically quasi-finite, and dominant, and
- $X' \to X \times_S S'$ is a *modification*, i.e. proper and birational.

Kempf, Knudsen, Mumford, and Saint-Donat prove a semistable reduction theorem in characteristic 0 when S is the spectrum of a DVR R and the generic fiber X_K is smooth over $K = \operatorname{Frac}(R)$ [KKMS73, p. 53]. Akin to our proof above, their strategy is to first use Hironaka's resolution of singularities to arrange that X' is smooth over R and that $(X'_0)_{\mathrm{red}}$ has normal crossings, and then to perform normalized base changes to arrange that the fibers are reduced. We also have the related question:

---

[1] See https://mathoverflow.net/questions/9981/coarse-moduli-spaces-over-z-and-f-p and https://mathoverflow.net/questions/72903/what-is-m-g-over-a-finite-field-really.

Question 5.5.21 (Extension Problem)**.** When does a flat family $X \to U$ over an open subscheme $U \subset S$ extends to a flat family $\widetilde{X} \to S$ after an alteration $S' \to S$, and what addition properties do the fibers of $\widetilde{X} \to S$ satisfy?

Is it possible to further simplify the special fiber? For families of smooth curves, Stable Reduction (5.6.4) arranges that that the special fiber of $X' \to \operatorname{Spec} R'$ is not just reduced and nodal, but stable. Is there a stable reduction theory in higher dimension for a suitably defined notion of stability? This question is essentially equivalent to the long-standing problem of compactifying moduli of higher dimensional varieties. The last several decades has seen great progress in this direction due to advances in singularity theory and the minimal model program. Kollár and Shepherd-Barron introduce a stability condition for surfaces of general type, called *KSB stability*, that has now been extended to any dimension. See §8.1 for a quick overview or [Kol23] for an exhaustive account.

Does semistable reduction hold in characteristic p? In other words, can [KKMS73, p. 53] be extended to positive characteristic? While both resolution of singularities and semistable reduction are both open problems in positive characteristic, de Jong's theory of alterations provides an alternative. First, for any variety X over a field $\Bbbk$, there is an alteration (i.e. a proper, generically quasi-finite, and dominant morphism) $\widetilde{X} \to X$ from a regular variety [dJ96, Thm. 4.1], When S is the spectrum of a complete DVR, the result of [KKMS73, p. 53] holds where $X' \to X \times_S S'$ is an alteration (instead of a proper birational map) [dJ96, Thm. 6.5].

Does semistable reduction hold over higher dimensional bases? As already raised in [KKMS73, p. vii], it is natural to seek semistable reductions when the base S has $\dim(S) > 1$? Many interesting results were shown in [dJ96] and [dJ97], including strong versions of semistable reduction for a family of curves $X \to S$. It was pointed out in [Kar99] that it is not possible to arrange that all fibers are normal crossing divisors when the base S and the fibers $X \to S$ have dimension at least 2. In characteristic 0, Abramovich and Karu prove that that there is a diagram as in (5.5.20), where $X' \to X \times_S S'$ is birational (but with X' possibly singular) and $X' \to S'$ *toroidal* with reduced fibers (but possibly not normal crossings divisors) [AK00, Thm. 0.3].

   Already with the techniques we've developed, we can show that Stable Reduction (5.5.1) has consequences for the Extension Problem (5.5.21) over any base scheme S. If $X \to U$ is stable family of curves over an open subscheme, let $U \to \overline{\mathcal{M}}_g$ be the corresponding map. Choose a finite cover $V \to \overline{\mathcal{M}}_g$ by a scheme using Le Lemme de Gabber (4.5.1). The base change $U \times_{\overline{\mathcal{M}}_g} V$ is a scheme finite over U, and by Zariski's Main Theorem there is a scheme S' containing $U \times_{\overline{\mathcal{M}}_g} V$ as a dense open subscheme and a morphism $S' \to S$ extending $U \times_{\overline{\mathcal{M}}_g} V \to U$. By Stable Reduction (5.5.1), V is proper. Thus, there is a projective birational map $S'' \to S'$ and a map $S'' \to V$ extending the rational map $S' \dashrightarrow V$. The picture is:

$$
\begin{array}{ccccc}
S'' & \xrightarrow{\ \text{brtl}\ } S' \xleftarrow{\ \text{op}\ } U \times_{\overline{\mathcal{M}}_g} V & \longrightarrow & V \\
& \ \downarrow{\scriptstyle\text{fin}} \qquad\qquad \downarrow{\scriptstyle\text{fin}} & & \downarrow{\scriptstyle\text{fin}} \\
& S \xleftarrow{\ \text{op}\ } U & \longrightarrow & \overline{\mathcal{M}}_g.
\end{array}
$$

The composition $S'' \to V \to \overline{\mathcal{M}}_g$ corresponds to a stable family of curves $\widetilde{X} \to S''$ extending the base change of $X_U \to U$. This also holds for families of KSB stable

varieties for the same reason: the stack parameterizing KSB varieties of fixed dimension and volume is proper.

Is a base change necessary? For the Extension Problem (Question 5.5.21), when is it possible to find an extension of a family $X \to U$ over an open immersion $U \hookrightarrow S$ without performing a base change $S' \to S$? For Stable Reduction (5.5.1) (i.e. S is the spectrum of a DVR and $X \to U$ is a family of smooth curves), base changes may be necessary due to the non-triviality of automorphism groups of smooth curves (see Example 5.5.6). Nevertheless, there are conditions that imply a base change is not necessary. Deligne and Mumford showed that a family of smooth curves extends to a family of stable curves if and only if the associated Jacobian family extends to a family of semi-abelian varieties in [DM69, §2], while Grothendieck equated the latter to the unipotence of the monodromy representation on the first homology of the fibers. On the other hand, de Jong and Oort proved the following [dJO97, Thm. 5.1]: if S is a regular scheme and $U \subset S$ is an open subscheme whose complement $S \setminus U$ is a normal crossings divisor, then every family $\mathcal{C} \to U$ of smooth curves (or more generally stable curves of locally constant topological type) extends to a family $\widetilde{\mathcal{C}} \to S$ of stable curves as long as it extends over an open subset $V \subset S$ containing each generic point of Δ. This result built on ideas of Moret–Bailly, who prove a simplified version [MB85]. There is also an analogous version for abelian varieties due to Faltings and Chai [FC90, Thm. 6.7].

5.6 Contraction, forgetful, and gluing morphisms

In this section, we construct several important morphisms between moduli spaces of curves.

- (Stable Contraction) There is a morphism $\mathcal{M}_{g,n}^{\mathrm{pre}} \to \overline{\mathcal{M}}_{g,n}$, which maps a prestable family $(\mathcal{C} \to S, \sigma_i)$ to a stable family $(\mathcal{C}^{\mathrm{st}}, \sigma_i^{\mathrm{st}})$ by contracting all rational tails and bridges (Theorem 5.6.4).

- (Forgetful) There is a morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ assign an $n+1$-pointed stable family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_{n+1})$ to the stable contraction of the n-pointed prestable family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ (Proposition 5.6.7). Moreover, we identify $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ with the universal family (Proposition 5.6.8).

- (Gluing) There is a morphism $\overline{\mathcal{M}}_{i,k} \times \overline{\mathcal{M}}_{g-i,n-k+2} \to \overline{\mathcal{M}}_{g,n}$, gluing a k-pointed stable family of genus i curves to a $n-k+2$-pointed stable family of genus $g-i$ curves along the final sections, and a morphism $\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$, gluing the final two sections of an $n+2$-pointed stable family of genus $g-1$ (Corollary 5.6.11).

As with the Local Structure of Nodal Families (5.2.23), the biggest challenge is ensuring that the constructions hold for families over an arbitrary base. We conclude this section in §5.6.4 with a discussion of boundary divisors and line bundles on $\overline{\mathcal{M}}_{g,n}$.

5.6.1 Contracting rational tails and bridges

Rational tails and bridges of a prestable curve over a field were defined in Definition 5.3.10 and characterized in Lemma 5.3.11. We first show that rational tails and bridges can be contracted to a stable curve over a field (Corollary 5.6.2), and then we extend the construction to families (Theorem 5.6.4).

Proposition 5.6.1 (Contracting a rational tail or bridge). *Let (C, p_i) be an n-pointed prestable curve over a field $\Bbbk$, and E be a rational tail or rational bridge. Then there is a canonical morphism*

$$c \colon C \to C'$$

contacting E to a point such that $c_* \mathcal{O}_C = \mathcal{O}_{C'}$ and $\mathrm{R}^1 c_* \mathcal{O}_C = 0$. Moreover, C' is identified with the pushout $\operatorname{Spec} \Gamma(E, \mathcal{O}_E) \amalg_E C$, and the formation of $c \colon C \to C'$ commutes with field extensions of $\Bbbk$.

Proof. In both cases, we construct C' as the Ferrand Pushout (B.4.1)

$$
\begin{array}{ccc}
E \cap E^c & \longrightarrow & E^c \\
\downarrow & & \downarrow \\
\operatorname{Spec} \Bbbk' & \longrightarrow & C',
\end{array}
$$

where $\Bbbk' = \Gamma^0(E, \mathcal{O}_E)$. Since scheme-theoretic unions are pushouts (Exercise B.4.6), we have that $C = E \amalg_{E \cap E^c} E^c$. This induces a larger commutative diagram

$$
\begin{array}{ccc}
E \cap E^c & \longrightarrow & E^c \\
\downarrow & & \downarrow \\
E & \longrightarrow & C \\
\downarrow & & \downarrow c \\
\operatorname{Spec} \Bbbk' & \longrightarrow & C',
\end{array}
$$

where $c \colon C \to C'$ is the unique map making the diagram commute. Since the top and outer squares are pushouts, so is the bottom square, i.e. $C' = E \amalg_{\operatorname{Spec} \Bbbk'} C$. Since Ferrand pushouts commute with field extension (Proposition B.4.10), so does the construction of $c \colon C \to C'$.

If E is a rational tail or a rational bridge with a marked point, then $E \cap E^c = \operatorname{Spec} \Bbbk' = \operatorname{Spec} \kappa(x)$ for a point $x \in E \cap E^c$ (see Lemma 5.3.11), and thus $C' = E^c$. The map $c \colon C \to C' = E^c$ is the identity on E^c and contracts E to the point $x \in E^c$ via the structure morphism $E \to \operatorname{Spec} \Gamma(E, \mathcal{O}_E) = \operatorname{Spec} \kappa(x)$. Consider the short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_E \oplus \mathcal{O}_{E^c} \to \kappa(x) \to 0.$$

Applying c_* yields a long exact sequence

$$0 \to c_* \mathcal{O}_C \to i_{x,*} \mathrm{H}^0 \widetilde{(E, \mathcal{O}_E)} \oplus \mathcal{O}_{C'} \to \kappa(x) \to \mathrm{R}^1 c_* \mathcal{O}_C \to i_{x,*} \mathrm{H}^1 \widetilde{(E, \mathcal{O}_E)} \to 0,$$

where $i_x \colon \operatorname{Spec} \kappa(x) \hookrightarrow C'$ is the inclusion of x. Since $\mathrm{H}^0(E, \mathcal{O}_E) = \kappa(x)$ and $\mathrm{H}^1(E, \mathcal{O}_E) = 0$, we see that $c_* \mathcal{O}_C = \mathcal{O}_{C'}$ and $\mathrm{R}^1 c_* \mathcal{O}_C = 0$.

If E is a rational bridge without marked points, set $\Bbbk'' = \mathrm{H}^0(E \cap E^c, \mathcal{O}_{E \cap E^c})$. By Lemma 5.3.11, $E \cap E^c$ is either a single point with $\Bbbk''$ a degree 2 separable extension of $\Bbbk'$ or two points with $\Bbbk'' = \Bbbk' \times \Bbbk'$. Since the construction of the pushout is étale local (Proposition B.4.10), the affine pushouts

$$
\begin{array}{ccc}
\operatorname{Spec} \Bbbk'' & \xrightarrow{\ \ 0\ \ } & \operatorname{Spec} \Bbbk''[y] \\
\downarrow & & \downarrow \\
\operatorname{Spec} \Bbbk' & \longrightarrow & \operatorname{Spec}\{f \in \Bbbk''[y] \mid f(0) \in \Bbbk'\},
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\operatorname{Spec} \Bbbk' \times \Bbbk' & \xrightarrow{\ 0 \amalg 0\ } & \operatorname{Spec} \Bbbk'[y] \times \Bbbk'[z] \\
\downarrow & & \downarrow \\
\operatorname{Spec} \Bbbk' & \longrightarrow & \operatorname{Spec} \Bbbk'[y, z]/(yz)
\end{array}
$$

are étale neighborhoods of C at the image of E, depending on whether $E \cap E^c$ is one or two points. In both cases, the image of E is a node. One shows that $c_* \mathcal{O}_C = \mathcal{O}_{C'}$ and $\mathrm{R}^1 c_* \mathcal{O}_C = 0$ as above using the short exact sequence $0 \to \mathcal{O}_C \to \mathcal{O}_E \oplus \mathcal{O}_{E^c} \to \mathcal{O}_{E \cap E^c} \to 0$. See also [SP, Tags 0E3H and 0E3M]. $\qquad\square$

A curve can also contain a chain of rational tails and bridges of arbitrary length.
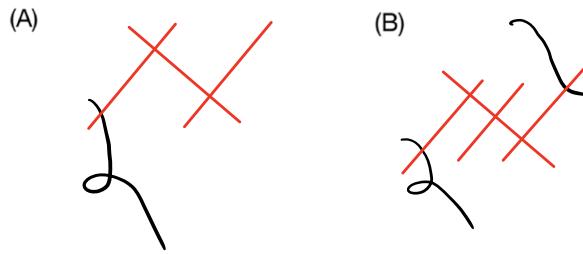
Figure 5.16: Chains of rational tails and bridges

Corollary 5.6.2 (Stable Contraction). *Let (C, p_i) be an n-pointed prestable curve of genus g over a field $\Bbbk$ such that $2g - 2 + n > 0$. Then there is a canonical morphism*

$$c \colon C \to C^{\mathrm{st}},$$

called the __stable contraction__, contacting all rational tails and rational bridges to points, such that $(C^{\mathrm{st}}, c(p_i))$ is an n-pointed stable curve of genus g, $c_* \mathcal{O}_C = \mathcal{O}_{C^{\mathrm{st}}}$, and $\mathrm{R}^1 c_* \mathcal{O}_C = 0$. Moreover, the formation of c commutes with field extensions of $\Bbbk$.

Proof. If E denotes the scheme-theoretic union of all rational tails and bridges, then $\mathrm{H}^0(E, \mathcal{O}_E)$ is a finite product of fields. Iteratively applying Proposition 5.6.1 yields a pushout diagram

$$
\begin{array}{ccc}
E & \longrightarrow & C \\
\downarrow & & \downarrow{\scriptstyle c} \\
\operatorname{Spec} \mathrm{H}^0(E, \mathcal{O}_E) & \longrightarrow & C^{\mathrm{st}},
\end{array}
$$

such that $c_* \mathcal{O}_C = \mathcal{O}_{C^{\mathrm{st}}}$ and $\mathrm{R}^1 c_* \mathcal{O}_C = 0$, with the construction of c commuting with field extensions. Since $(C^{\mathrm{st}}, c(p_i))$ has no rational tails and bridges, it is stable (Proposition 5.3.12). See also [SP, Tag 0E7Q]. $\qquad\square$
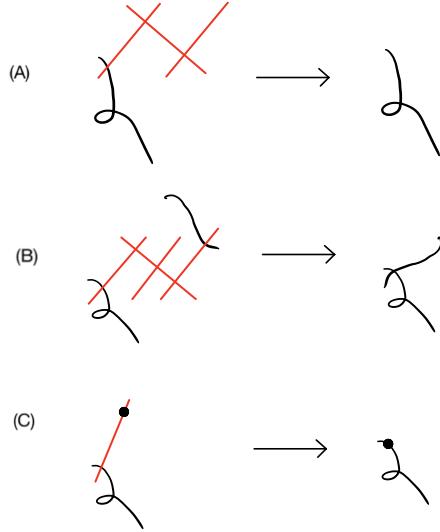
Figure 5.17: Stable contractions

Exercise 5.6.3. Let (C, p_i) be an n-pointed prestable curve over a field $\Bbbk$.

(a) Show that the stabilization morphism $c \colon C \to C^{\mathrm{st}}$ is the unique morphism such that $(C^{\mathrm{st}}, c(p_i))$ is stable, $c_* \mathcal{O}_C = \mathcal{O}_{C^{\mathrm{st}}}$, and $\mathrm{R}^1 c_* \mathcal{O}_C = 0$.

(b) If (C, p_i) is semistable and $L := \omega_C(\sum_i p_i)$, show that $C^{\mathrm{st}} \cong \mathrm{Proj} \bigoplus_{d \geq 0} \mathrm{H}^0(C, L^{\otimes d}))$ and that $L \cong c^* \omega_{C^{\mathrm{st}}}(\sum_i c(p_i))$.

The construction of the stable contraction extends to families of prestable curves.

Theorem 5.6.4 (Stable Contraction of a Prestable Family). *If $(\mathcal{C} \to S, \sigma_i)$ is a family of n-pointed prestable curves of genus g such that $2g - 2 + n > 0$, then there exists a unique morphism $c \colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ over S such that*

(1) *$(\mathcal{C}^{\mathrm{st}} \to S, \sigma_i^{\mathrm{st}})$ is an n-pointed family of stable curves of genus g where $\sigma_i^{\mathrm{st}} = c \circ \sigma_i$;*

(2) *$\mathcal{O}_{\mathcal{C}^{\mathrm{st}}} = c_* \mathcal{O}_{\mathcal{C}}$ and $R^1 c_* \mathcal{O}_{\mathcal{C}} = 0$;*

(3) *the construction of $c \colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ is compatible with base change $S' \to S$; and*

(4) *for each $s \in S$, $(\mathcal{C}_s, \sigma_i(s)) \to (\mathcal{C}_s^{\mathrm{st}}, \sigma_i^{\mathrm{st}}(s))$ is the stable contraction of rational bridges and tails as in* Corollary 5.6.2.

Moreover, if $(\mathcal{C} \xrightarrow{\pi} S, \sigma_i)$ is a semistable family, then $\omega_{\mathcal{C}/S}(\sum_i \sigma_i)$ is the pullback of the relatively ample line bundle $L := \omega_{\mathcal{C}^{\mathrm{st}}/S}(\sum_i \sigma_i^{\mathrm{st}})$; in particular, $\mathcal{C}^{\mathrm{st}} \cong \mathcal{P}\mathrm{roj}_S \bigoplus_{d \geq 0} \pi_*(L^{\otimes d})$.

Proof. This will be a local-to-global argument. For any $s \in S$, Corollary 5.6.2 yields the stable contraction $c_s \colon \mathcal{C}_s \to Y_0$ over $\kappa(s)$, and satisfies the uniqueness property by Exercise 5.6.3(a). The key idea of the proof is the following: since $c_{s,*} \mathcal{O}_{\mathcal{C}_s} = \mathcal{O}_{Y_0}$ and $\mathrm{R}^1 c_{s,*} \mathcal{O}_{\mathcal{C}_s} = 0$, any infinitesimal deformation of $\mathcal{C}_s$ extends uniquely to a deformation of $\mathcal{C}_s \to Y_0$ (Exercise C.2.15). Setting $S_n = \mathrm{Spec}\, \mathcal{O}_{S,s}/\mathfrak{m}^{n+1}$, this yields compatible morphisms

$$\mathcal{C} \times_S S_n \to Y_n$$

over S_n. Artin Approximation (B.5.26) then yields a morphism $\mathcal{C} \times_S S' \to Y'$ after

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an étale cover $S' \to S$, which, by the uniqueness properties, descends to a morphism $\mathcal{C} \to \mathcal{C}^{\mathrm{st}}$.

To make this argument work, we first reduce to the case that S is finite type over $\mathbb{Z}$ using Limit Methods (§B.3). As in the first paragraph, we find compatible morphisms $\mathcal{C} \times_S S_n \to Y_n$ over S_n. The central fibers $\mathcal{C}_s$ and Y_0 are schemes by Proposition 4.4.19, thus so are $\mathcal{C} \times_S S_n$ and Y_n. Since $\widehat{S} := \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ is noetherian, Grothendieck's Existence Theorem (Corollary C.4.9) yields a projective morphism $\widehat{Y} \to \widehat{S} := \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ extending $Y_n \to S_n$. By the Infinitesimal Criterion for Flatness (A.2.5), $\widehat{Y} \to \widehat{S}$ is flat. A consequence of Grothendieck's Existence Theorem (C.4.13) asserts that there is a morphism $\mathcal{C} \times_S \widehat{S} \to \widehat{Y}$ over $\widehat{S}$ extending $\mathcal{C} \times_S S_n \to Y_n$. Letting $\widehat{\tau}_i$ be the composition $\widehat{S} \xrightarrow{\sigma_i \times_S \widehat{S}} \mathcal{C} \times_S \widehat{S} \to \widehat{Y}$, the $(\widehat{Y} \to \widehat{S}, \widehat{\tau}_i)$ is an n-pointed family of curves. Since the central fiber Y_0 is stable, Openness of Stability (5.3.22) implies that $(\widehat{Y} \to \widehat{S}, \widehat{\tau}_i)$ is a family of stable curves.

Since S is finite type over $\mathbb{Z}$, Artin Approximation (B.5.26) gives an étale neighborhood $(S', s') \to (S, s)$, a family of stable curves $(Y' \to S', \tau_i)$, and a morphism $c' \colon \mathcal{C} \times_S S' \to Y'$ of algebraic spaces over S' whose fiber at s' corresponds to $\mathcal{C}_s \to Y_0$. By Exercise 5.6.6, after replacing S' with an open neighborhood of s', we can assume that $c_* \mathcal{O}_{\mathcal{C} \times_S S'} = \mathcal{O}_{Y'}$ and $\mathrm{R}^1 c_* \mathcal{O}_{\mathcal{C} \times_S S'} = 0$, and that this holds after base change. Therefore, the existence of the desired family of stable curves will follow from étale descent once we establish uniqueness.

To show uniqueness, let $(Y \to S, \tau_i)$ and $(Y' \to S, \tau_i')$ be families of stable curves over a noetherian scheme, let $c \colon \mathcal{C} \to Y$ and $c' \colon \mathcal{C} \to Y'$ be morphisms over S, and suppose that there is an isomorphism $\alpha_s \colon Y_s \xrightarrow{\sim} Y_s'$ compatible with c_s and c_s' such that $c_{s,*} \mathcal{O}_{\mathcal{C}_s} = \mathcal{O}_{Y_s}$ and $\mathrm{R}^1 c_{s,*} \mathcal{O}_{\mathcal{C}_s} = 0$. We need to show that there exists an open neighborhood $U \subset S$ of s and an isomorphism $\alpha \colon Y_U \to Y_U'$ extending α_s, which is compatible with c_U and c_U'. By Limit Methods (§B.3), we may assume that $S = \operatorname{Spec} A$ is the spectrum of a noetherian local ring. Letting Y'' be the scheme-theoretic image of $\mathcal{C} \to Y \times_S Y'$, it suffices to show that the projections $Y'' \to Y$ and $Y'' \to Y'$ are isomorphisms. Since the scheme-theoretic image commutes with flat base change, we may further assume that A is complete. By Exercise C.2.15, the restrictions $Y_n'' \to Y_n$ and $Y_n'' \to Y_n'$ to the base change to $\operatorname{Spec} A/\mathfrak{m}^{n+1}$ are isomorphisms. A consequence of Grothendieck's Existence Theorem (C.4.13) yields the desired isomorphism $Y \to Y'$. See also [SP, Tag 0E8A].

If $(\mathcal{C} \to S, \sigma_i)$ is a family of semistable curves, then we claim that the natural map $c^* \omega_{\mathcal{C}^{\mathrm{st}}/S} \to \omega_{\mathcal{C}/S}$ is an isomorphism. Indeed, since the relative dualizing sheaves are line bundles, it suffices to show that this map is surjective. Since the relative dualizing sheaves and the cokernel of the map are compatible with base change, by Nakayama's lemma, it suffices to show that the claim holds when S is the spectrum of a field, which is the assertion in Exercise 5.6.3(b). By the projection formula, $\omega_{\mathcal{C}^{\mathrm{st}}/S} \cong c_* c^* \omega_{\mathcal{C}^{\mathrm{st}}/S}$, and thus the composition $\omega_{\mathcal{C}^{\mathrm{st}}/S} \cong c_* c^* \omega_{\mathcal{C}^{\mathrm{st}}/S} \to c_* \omega_{\mathcal{C}/S}$ is an isomorphism. It follows that $\omega_{\mathcal{C}^{\mathrm{st}}/S}(\sum_i \sigma_i^{\mathrm{st}}) \cong c_*(\omega_{\mathcal{C}/S}(\sum_i \sigma_i))$, which implies the statement as $\omega_{\mathcal{C}^{\mathrm{st}}/S}$ is relatively very ample (Proposition 5.3.19). Alternatively, one can explicitly construct the stable contraction globally by showing that the natural maps $\pi_*(L^{\otimes 2}) \otimes \pi_*(L^{\otimes d}) \to \pi_*(L^{\otimes d+2})$ are surjective for $d \geq 4$. Thus, $\bigoplus_{d \geq 0} \pi_*(L^{\otimes 4d})$ is a finite type $\mathcal{O}_S$-algebra, and we can define $\mathcal{C}^{\mathrm{st}} := \mathcal{P}\mathrm{roj}_S \bigoplus_{d \geq 0} \pi_*(L^{\otimes 4d})$. Since for all $d \geq 0$ the pushforward $\pi_*(L^{\otimes 4d})$ is a vector bundle and its construction commutes with base change (Proposition 5.3.21), it follows that $\mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ is well-defined and $\mathcal{C}^{\mathrm{st}} \to S$ is a family of stable curves. See [Knu83, Prop. 2.1] and [ACG11, Prop. 10.6.7]. $\square$

Corollary 5.6.5 (Stable Contraction Morphism)**.** *There is a morphism of algebraic*

stacks

$$\mathcal{M}^{\mathrm{pre}}_{g,n} \to \overline{\mathcal{M}}_{g,n}, \qquad (C, p_i) \mapsto (C^{\mathrm{st}}, c(p_i)),$$

which is the identity on the open substack $\overline{\mathcal{M}}_{g,n} \subset \mathcal{M}^{\mathrm{pre}}_{g,n}$. □

Exercise 5.6.6 (details). Let $f \colon X \to Y$ be a morphism of families of curves over a noetherian scheme S. Suppose that for a point $s \in S$, the morphism $f_s \colon X_s \to Y_s$ satisfies $f_{s,*}\mathcal{O}_{X_s} = \mathcal{O}_{Y_s}$ and $\mathrm{R}^1 f_{s,*}\mathcal{O}_{X_s} = 0$. Show that after replacing S with an open neighborhood of s, $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $\mathrm{R}^1 f_*\mathcal{O}_X = 0$, and that this remains true after base change by a morphism $S' \to S$. See also [SP, Tag 0E88].

5.6.2 The forgetful morphism and the universal family of $\overline{\mathcal{M}}_{g,n}$

We give two consequences of the Stable Contraction Morphism (5.6.5): we show that there is a map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ which is the stable contraction of forgetting the last marked point, and that this map is identified with the universal family.
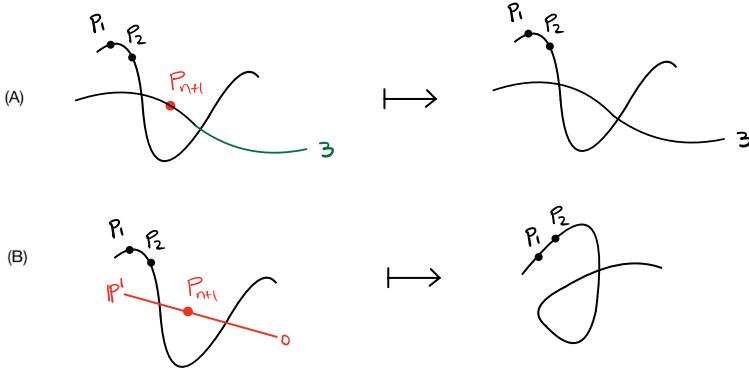
Figure 5.18: In (A), the $n+1$th point is simply forgotten. In (B), if p_{n+1} is forgotten, the curve is no longer stable, and we must contract the rational bridge.

Proposition 5.6.7 (Forgetful Morphism). *There is a morphism of algebraic stacks*

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}, \qquad (C, p_1, \ldots, p_{n+1}) \mapsto (C^{\mathrm{st}}, c(p_1), \ldots, c(p_n)),$$

where $c \colon C \to C^{\mathrm{st}}$ *is the stable contraction of* $(C, p_1, \ldots, p_n)$.

Proof. The desired morphism is constructed as the composition

$$\overline{\mathcal{M}}_{g,n+1} \to \mathcal{M}^{\mathrm{pre}}_{g,n} \to \overline{\mathcal{M}}_{g,n},$$

where $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{M}^{\mathrm{pre}}_{g,n}$ is the morphism taking an $n+1$-pointed stable family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_{n+1})$ to the n-pointed *prestable* family $(\mathcal{C} \to S, \sigma_1, \ldots, \sigma_n)$ and $\mathcal{M}^{\mathrm{pre}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the Stable Contraction Morphism (5.6.5). □

By the Generalized 2-Yoneda Lemma (3.1.24), the identity morphism $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ corresponds to an object of $\overline{\mathcal{M}}_{g,n}$ over the algebraic stack $\overline{\mathcal{M}}_{g,n}$, which in turn corresponds via descent to an n-pointed family

$$(\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}, \sigma_1^{\mathrm{univ}}, \ldots, \sigma_n^{\mathrm{univ}})$$

of stable curves, called the *universal family*. An object of $\mathcal{U}_{g,n}$ over a scheme S is an n-pointed family of stable curves $(\mathcal{C} \to S, \sigma_i)$ with an additional section $\tau \colon S \to \mathcal{C}$ (that may land in the nodal locus $\mathcal{C} \to S$ and may intersection non-trivially with the sections σ_i). Given an n-pointed family of stable curves $(\mathcal{C} \to S, \sigma_i)$, there is a morphism $S \to \overline{\mathcal{M}}_{g,n}$, unique up to unique isomorphism, and a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{U}_{g,n} \\
\Big\downarrow \Big\uparrow \sigma_i & & \Big\downarrow \Big\uparrow \sigma_i^{\mathrm{univ}} \\
S & \longrightarrow & \overline{\mathcal{M}}_{g,n}.
\end{array}
$$

On the other hand, there is the Forgetful Morphism (5.6.7) $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ and a morphism $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ of algebraic stacks taking an $n+1$-pointed stable family $(\mathcal{C} \to S, \sigma_i)$ to the $n+1$-pointed family $(\mathcal{C}^{\mathrm{st}}, \sigma_1^{\mathrm{st}}, \ldots, \sigma_n^{\mathrm{st}}, c \circ \sigma_{n+1})$, arising as the Stable Contraction (5.6.4) $c \colon \mathcal{C} \to \mathcal{C}^{\mathrm{st}}$ of the *prestable* family $(\mathcal{C}, \sigma_1, \ldots, \sigma_n)$. This yields a commutative diagram

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_{g,n+1} & \longrightarrow & \mathcal{U}_{g,n} \\
& \searrow & \Big\downarrow \\
& & \overline{\mathcal{M}}_{g,n}.
\end{array}
$$
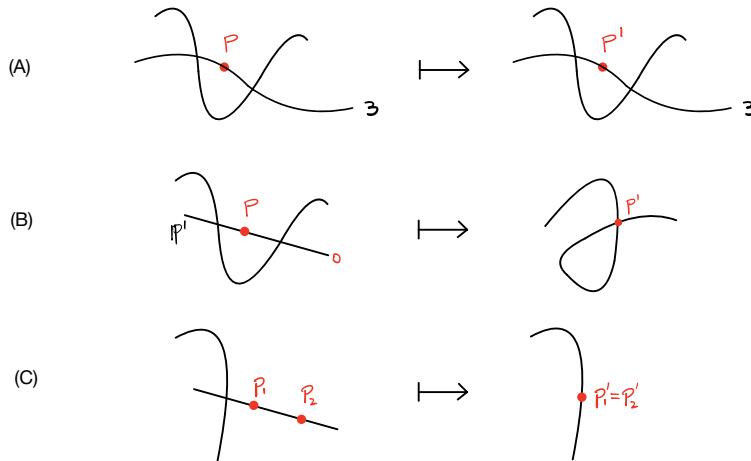
Figure 5.19: Examples of the map $\overline{\mathcal{M}}_{g,n} \to \mathcal{U}_{g,n}$.

Proposition 5.6.8. *The morphism $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ is an isomorphism over $\overline{\mathcal{M}}_{g,n}$. In other words, the forgetful morphism $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family.*

Proof. The stacks $\overline{\mathcal{M}}_{g,n+1}$ and $\mathcal{U}_{g,n}$ are both proper and representable over $\overline{\mathcal{M}}_{g,n}$ by Stable Reduction (5.5.1). Hence, the morphism $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ is proper and representable. For every algebraically closed field $\Bbbk$, the induced map $\overline{\mathcal{M}}_{g,n+1}(\Bbbk)/\!\sim \to \overline{\mathcal{U}}_{g,n}(\Bbbk)/\!\sim$ on isomorphism classes is bijective. Hence, $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ is proper and quasi-finite, hence finite. On the other hand, $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ is birational as it induces an isomorphism between the open substack of $\overline{\mathcal{M}}_{g,n+1}$ parameterizing pointed curves $(C, p_1, \ldots, p_{n+1})$ such that $(C, p_1, \ldots, p_n)$ does *not* contain a rational

bridge and the open substack of $\mathcal{U}_{g,n}$ parameterizing pointed curves $(C, p_1, \ldots, p_{n+1})$ such that $p_{n+1} \in C$ is a smooth point that doesn't coincide with p_i for $i = 1, \ldots, n$. By Theorem 5.4.14, $\overline{\mathcal{M}}_{g,n+1}$ is smooth. Since $\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ has Cohen–Macaulay fibers and $\overline{\mathcal{M}}_{g,n}$ is smooth, $\mathcal{U}_{g,n}$ is Cohen–Macaulay. The stack $\mathcal{U}_{g,n}$ is regular in the open locus where $p_{n+1} \in C$ is a smooth point, and since the complement of this locus is codimension 2, Serre's criterion for normality implies that $\mathcal{U}_{g,n}$ is normal. We thus have a finite birational morphism $\overline{\mathcal{M}}_{g,n+1} \to \mathcal{U}_{g,n}$ between normal Deligne–Mumford stacks, which is necessarily an isomorphism. $\qquad\square$

Remark 5.6.9. One can also explicitly construct the inverse morphism $\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$. Let $(\mathcal{C} \to S, \sigma_i)$ be an n-pointed family of stable curves and $\tau \colon S \to \mathcal{C}$ be an additional section defined by an ideal sheaf $\mathcal{I}_\tau$. Define the coherent $\mathcal{O}_\mathcal{C}$-module $\mathcal{K}$ by

$$0 \to \mathcal{O}_\mathcal{C} \xrightarrow{(\alpha, \beta)} \mathcal{I}_\tau^\vee \oplus \mathcal{O}_\mathcal{C}(\sigma_1 + \cdots + \sigma_n) \to \mathcal{K} \to 0,$$

where α is the dual of the inclusion $\mathcal{I}_\tau \hookrightarrow \mathcal{O}_\mathcal{C}$ and β is the natural inclusion. Define

$$\widetilde{\mathcal{C}} = \mathcal{P}\mathrm{roj}_S \operatorname{Sym} \mathcal{K} \to \mathcal{C}.$$

With some work, one can prove that $\tau^*(\mathcal{I}_\tau^\vee/\mathcal{O}_\mathcal{C})$ is a line bundle [Knu83, Lem. 2.2]. The surjection $\tau^*\mathcal{K} \to \tau^*(\mathcal{K}/\mathcal{O}_\mathcal{C}) \cong \tau^*(\mathcal{I}_\tau^\vee/\mathcal{O}_\mathcal{C})$ defines a section $\widetilde{\tau} \colon S \to \widetilde{\mathcal{C}}$. The cokernel of the injection $\mathcal{I}_\tau^\vee \hookrightarrow \mathcal{K}$ is identified with $\mathcal{O}_\mathcal{C}(\sigma_1 + \cdots + \sigma_n)|_{\bigcup_i s_i}$, and the surjections $\sigma_i^*\mathcal{K} \to \sigma_i^*\mathcal{O}_\mathcal{C}(\sigma_1 + \cdots + \sigma_n)$ defines sections $\widetilde{\sigma}_i \colon S \to \widetilde{\mathcal{C}}$. One checks that $(\widetilde{\mathcal{C}} \to S, \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n, \widetilde{\tau})$ is an $n + 1$-pointed family of stable curves such that $c \colon \widetilde{\mathcal{C}} \to \mathcal{C}$ is the stable contraction of the n-pointed prestable family $(\widetilde{\mathcal{C}}, \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n)$ with $\sigma_i = c \circ \widetilde{\sigma}_i$ and $\tau = c \circ \widetilde{\tau}$. In many cases, $\widetilde{\mathcal{C}} \to \mathcal{C}$ can be constructed more directly as a blow up: for instance, if $\mathcal{C} \to S$ is a generically smooth family of stable curves over the spectrum of a DVR such that $\mathcal{C}$ is regular and $\tau \colon S \to \mathcal{C}$ is a section such that $\tau(0) \in \mathcal{C}_0$ is a node, then $\widetilde{\mathcal{C}} \to \mathcal{C}$ is simply the blow up at $\tau(0)$ and $\widetilde{\tau}$ is the strict transform of τ. See [Knu83, Thm. 2.4] and [ACG11, §X.8].

5.6.3 Gluing morphisms

After showing how sections of families of curves can be glued to nodal families (Proposition 5.6.10), we show that there are well-defined finite morphisms $\overline{\mathcal{M}}_{i,k} \times \overline{\mathcal{M}}_{g-i,n-k+2} \to \overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ Corollary 5.6.11, called *gluing morphisms* (or sometimes *clutching morphisms*).
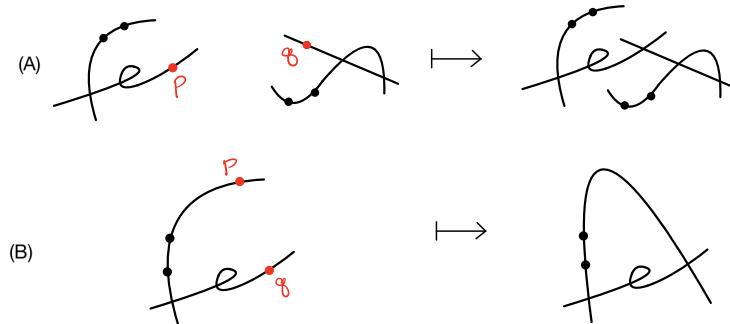
Figure 5.20: The nodal gluing of marked points p and q.

Proposition 5.6.10 (Gluing Families Along Sections). *Let $(\mathcal{C} \to S, \sigma, \tau)$ be a 2-pointed projective family of (possibly disconnected) curves over a scheme S such that for every point $s \in S$, $\sigma(s)$ and $\tau(s)$ are distinct smooth points of $\mathcal{C}_s$. Then there is a canonical finite morphism*

$$g \colon \mathcal{C} \to \mathcal{C}'$$

of schemes over S such that

(1) $\mathcal{C}' \to S$ is a family of curves with a section $\nu \colon S \to \mathcal{C}'$ such that $\nu = g \circ \sigma = g \circ \tau$ and $\nu(s) \in \mathcal{C}_s$ is a node for all $s \in S$;

(2) the morphism g restricts to an isomorphism $\mathcal{C} \setminus (\sigma(S) \cup \tau(S)) \xrightarrow{\sim} \mathcal{C}' \setminus \nu(S)$, and there is an identification of the topological space $|\mathcal{C}'|$ with the quotient of $|\mathcal{C}|$ under the equivalence relation $\sigma(s) \sim \tau(s)$ for $s \in S$;

(3) $\mathcal{C}'$ is identified with the Ferrand Pushout (B.4.1) of $\sigma \amalg \tau \colon S \amalg S \to \mathcal{C}$ and the projection $S \amalg S \to S$; in particular, for every open subset $U \subset \mathcal{C}'$,

$$\Gamma(U, \mathcal{C}') = \{ f \in \Gamma(g^{-1}(U), \mathcal{C}) \mid \sigma^* f = \tau^* f \};$$

and

(4) the construction is compatible with base change $T \to S$.

Proof. Since $\mathcal{C} \to S$ is projective, every two points of a fiber $\mathcal{C}_s$ are contained in an affine open subscheme of $\mathcal{C}$. Therefore, the Ferrand Pushout (B.4.1)

$$
\begin{array}{ccc}
S \amalg S & \xrightarrow{\sigma \amalg \tau} & \mathcal{C} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\nu} & \mathcal{C}'
\end{array}
$$

exists as a scheme $\mathcal{C}'$. By the universal property of pushouts, there is a natural map $\mathcal{C}' \to S$. Since $S \amalg S \to S$ is finite, so is $\mathcal{C} \to \mathcal{C}'$. Since $S \amalg S \to S$ is flat, the construction is compatible with arbitrary base change by Properties of Pushouts (B.4.10(3)). This gives (2)-(4).

To see (1), first observe that $\mathcal{C}' \to S$ is proper since $\mathcal{C} \to \mathcal{C}'$ is finite. By the Local Structure of Smooth Morphisms (A.3.4), for every $s \in S$, there is an affine open neighborhood $\operatorname{Spec} A \subset S$ and étale neighborhoods $\mathcal{C} \to \operatorname{Spec} A[x]$ and $\mathcal{C} \to \operatorname{Spec} A[y]$ of $\sigma(s)$ and $\tau(s)$. By Properties of Pushouts (B.4.10(4)), the pushout $\mathcal{C}'$ has an étale neighborhood of $\nu(s)$ isomorphic to the pushout

$$
\begin{array}{ccc}
\operatorname{Spec} A \times A & \xrightarrow{0 \amalg 0} & \operatorname{Spec} A[x] \times A[y] \\
\downarrow & & \downarrow \\
\operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A[x,y]/(xy).
\end{array}
$$

It follows that $\mathcal{C}' \to S$ is flat and $\nu(s) \in \mathcal{C}_s$ is a node.

Alternatively, one can construct $\mathcal{C}'$ using a global Proj construction in the case that $\omega_{\mathcal{C}/S}(\sigma + \tau)$ is a relatively very ample line bundle (as it will be in our application to stable curves). Namely, there are well-defined morphisms $q_\sigma \colon \omega_{\mathcal{C}/S}(\sigma + \tau) \to \mathcal{O}_{\sigma(S)}$ and $q_\tau \colon \omega_{\mathcal{C}/S}(\sigma + \tau) \to \mathcal{O}_{\tau(S)}$. We then define $\mathcal{C}' := \mathcal{P}\mathrm{roj}_S \bigoplus_{d \geq 0} \mathcal{A}_d$, where $\mathcal{A}_d$ is

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the fiber product

$$
\begin{array}{ccc}
\mathcal{A}_d & \longrightarrow & \mathcal{O}_S \\
\downarrow & & \downarrow{\scriptstyle\Delta} \\
\pi_*(\omega_{\mathcal{C}/S}(\sigma+\tau)^{\otimes d}) \times \pi_*(\omega_{\mathcal{C}/S}(\sigma+\tau)^{\otimes d}) & \xrightarrow{\pi_*(q_\sigma^{\otimes d}) \times \pi_*(q_\tau^{\otimes d})} & \mathcal{O}_S \times \mathcal{O}_S
\end{array}
$$

and $\pi\colon \mathcal{C} \to S$ denotes the structure morphism. One can check $\mathcal{C}' \to S$ satisfies the desired properties by essentially the same argument as above. See [Knu83, Thm. 3.4] and [ACG11, §X.7]. $\qquad\square$

Corollary 5.6.11 (Gluing Morphisms). *Assume that $2g - 2 + n > 0$.*

(1) If $2g_1 - 2 + n_1 > 0$ and we set $g_2 = g - g_1$ and $n_2 = n - n_1 + 2$, there is a finite morphism of algebraic stacks

$$
\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g,n}
$$
$$
\big((C,p_i),(D,q_i)\big) \mapsto (C \amalg_{p_{n_1} \sim q_{n_2}} D, p_1, \dots, p_{n_1-1}, q_1, \dots, q_{n_2-1}).
$$

(2) If $g \geq 1$, there is a finite morphism of algebraic stacks

$$
\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}
$$
$$
(C,p_i) \mapsto (C/_{p_{n+1} \sim p_{n+2}}, p_1, \dots, p_{n-2}).
$$

Proof. In both cases, Gluing Families Along Sections (5.6.10) directly yields morphisms of algebraic stacks to $\mathcal{M}_{g,n}^{\mathrm{pre}}$. Since stability can be checked on geometric fibers, it suffices to show that the nodal gluing $C \amalg_{p_{n_1} \sim q_{n_2}} D$ and $C/_{p_{n+1} \sim p_{n+2}} D$ are stable, but this is clear as stability is detected under pointed normalization (Exercise 5.3.5). Since both maps are quasi-finite and representable morphisms of proper Deligne–Mumford stacks, they are finite. $\qquad\square$

More generally, for a finite index set I, we can define the notion of an I-pointed stable curve of genus g, and we denote $\overline{\mathcal{M}}_{g,I}$ as the stack of such curves. A bijection $I \cong \{1, \dots, n\}$ induces an isomorphism $\overline{\mathcal{M}}_{g,I} \cong \overline{\mathcal{M}}_{g,n}$. With this notation, we have the obvious generalization of Corollary 5.6.11: there are finite morphisms

$$
\overline{\mathcal{M}}_{g_1,I_1} \times \overline{\mathcal{M}}_{g_2,I_2} \to \overline{\mathcal{M}}_{g_1+g_2,(I_1 \cup I_2)\setminus\{i_1,i_2\}} \quad \text{and} \quad \overline{\mathcal{M}}_{g-1,J} \to \overline{\mathcal{M}}_{g,J\setminus\{j_1,j_2\}},
$$

gluing the marked points $i_1 \in I_1$ to $i_2 \in I_2$ (resp. $j_1, j_2 \in J$), subject to the numerical conditions ensuring that each stack is non-empty.

5.6.4 Boundary divisors and line bundles on $\overline{\mathcal{M}}_{g,n}$

We define the boundary divisors $\delta_{i,I}$ of $\overline{\mathcal{M}}_{g,n}$ and show that the total boundary divisor δ is a normal crossings divisor (Proposition 5.6.13).

Definition 5.6.12 (Boundary Divisors). Suppose $2g - 2 + n > 0$. Let $0 \leq i \leq g$ be an integer and $I \subset \{1, \dots, n\}$ be a subset such that $1 - 2i < |I| < 2(g-i) + n - 1 > 0$ (which ensures that both $\overline{\mathcal{M}}_{i,I\cup\{p\}}$ and $\overline{\mathcal{M}}_{g-i,I^c\cup\{q\}}$ are nonempty). The *boundary divisors* are defined as the closed substacks of $\overline{\mathcal{M}}_{g,n}$

$$
\delta_{i,I} = \operatorname{im}\big(\overline{\mathcal{M}}_{i,I\cup\{p\}} \times \overline{\mathcal{M}}_{g-i,I^c\cup\{q\}} \to \overline{\mathcal{M}}_{g,I\cup I^c} \cong \overline{\mathcal{M}}_{g,n}\big)
$$
$$
\delta_0 = \operatorname{im}\big(\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}\big)
$$
$$
\delta = \delta_0 \cup \bigcup_{i,I} \delta_{i,I}
$$

with the convention that δ_0 is empty if $g = 0$.

Note that $\delta_{i,I} = \delta_{g-i,I^c}$. If $n = 0$, then $\delta = \delta_0 \cup \cdots \cup \delta_{\lfloor \frac{g}{2} \rfloor}$. If $g = 0$, then δ is the union of δ_I over subsets I of size $2 \le |I| \le n - 2$.
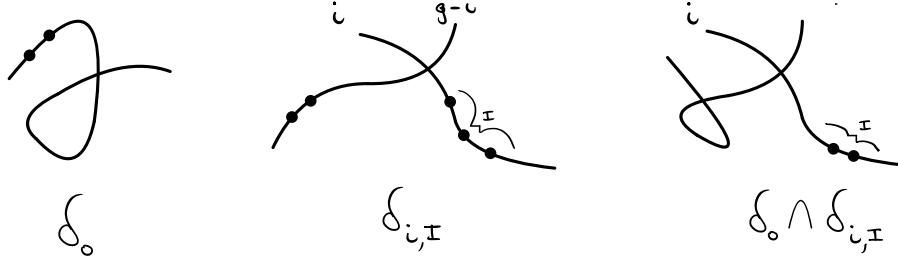
Figure 5.21: Examples of stable curves in the boundary.

A closed substack $\mathcal{Z} \subset \mathcal{X}$ of a Deligne–Mumford stack is called a *divisor* (resp. *normal crossings divisor*) if there is an étale presentation $U \to \mathcal{X}$ such that $\mathcal{Z} \times_{\mathcal{X}} U \subset U$ is a divisor (resp. normal crossings divisor). Recall that a divisor $U \subset X$ of a regular scheme has normal crossings if for every $u \in U$, $\widehat{\mathcal{O}}_{U,u} \cong \widehat{\mathcal{O}}_{X,x}/(f_1 \cdots f_k)$ where the sequence $f_1, \ldots, f_k \in \widehat{\mathcal{O}}_{X,x}$ extends to a regular system of parameters $f_1, \ldots, f_n$.

Proposition 5.6.13. *Over a field* $\Bbbk$, *$\delta \subset \overline{\mathcal{M}}_{g,n}$ is a normal crossings divisor.*

Proof. An easy dimension count shows that δ has pure dimension $3g - 2 + n$. To see that δ has normal crossings, we may assume that $\Bbbk$ is algebraically closed. Let $(C, p_i) \in \overline{\mathcal{M}}_{g,n}$ be a stable curve with nodes $q_1, \ldots, q_s \in C$, and let $(U, u) \to (\overline{\mathcal{M}}_{g,n}, [C, p_i])$ be an étale neighborhood where U is a scheme. By the Local-to-global Deformation Sequence (5.3.24), there is a surjection

$$\mathrm{Def}(C, p_i) \to \bigoplus_j \mathrm{Def}(\widehat{\mathcal{O}}_{C,q_j}) \tag{5.6.14}$$

of first order deformations spaces with $\mathrm{Def}(\widehat{\mathcal{O}}_{C,q_j}) \cong \Bbbk$.

Letting $\mathrm{Art}_{\Bbbk}$ be the category of local artinian $\Bbbk$-algebras with residue field $\Bbbk$, define the functors

$$F, G_j \colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets},$$

by setting $F(A)$ to be the set of isomorphism classes of pairs $(\mathcal{C} \to \mathrm{Spec}\, A, \sigma_i, \alpha)$ where $(\mathcal{C} \to \mathrm{Spec}\, A, \sigma_i)$ is a family of n-pointed stable curves and $\alpha\colon (C, p_i) \xrightarrow{\sim} (\mathcal{C} \times_A \Bbbk, \sigma_i \times_A \Bbbk)$ is an isomorphism, and by setting $G_j(A)$ to be the set of isomorphism classes of pairs (B, β) where B is a flat A-algebra and $\beta\colon \widehat{\mathcal{O}}_{C,q_j} \to B \otimes_A A/\mathfrak{m}_A$ is an isomorphism. The map $\mathrm{Spec}\, \widehat{\mathcal{O}}_{U,u} \to \overline{\mathcal{M}}_{g,n}$ induces a miniversal formal deformation of F over $\Bbbk[\![x_1, \ldots, x_{3g-3+n}]\!]$. By Rim–Schlessinger's Criterion (C.3.12), G_j admits a miniversal formal deformation over $\Bbbk[\![z_j]\!]$ (see Exercise C.3.21). Since (5.6.14) is surjective, the natural morphism of functors $F \to \bigoplus_j G_j$ is versal (or formally smooth), or in other words the natural map

$$\widehat{\mathcal{O}}_{U,u} \cong \Bbbk[\![x_1, \ldots, x_{3g-3+n}]\!] \to \Bbbk[\![z_1, \ldots, z_s]\!] \tag{5.6.15}$$

is surjective. We may therefore write $\widehat{\mathcal{O}}_{U,u} \cong \Bbbk[\![z_1, \ldots, z_s, x'_{s+1}, \ldots, x'_{3g-3+n}]\!]$ so that (5.6.15) maps z_i to itself. Letting $\delta_U = \delta \times_{\overline{\mathcal{M}}_{g,n}} U$, we conclude that $\widehat{\mathcal{O}}_{\delta_U, u} \cong$

$\widehat{\mathcal{O}}_{U,u}/(z_1 \cdots z_s)$. See also [DM69, Thm. 5.2], where it is shown more generally that δ is a relative normal crossings divisor over $\mathbb{Z}$. $\qquad\square$

Exercise 5.6.16. Show that $\overline{\mathcal{M}}_{0,n}$ has $2^{n-1} - n - 1$ irreducible boundary divisors.

Exercise 5.6.17. The dual graph $\Gamma = (G, w, m)$ of a stable curve C was defined in Definition 5.3.7.

(a) Show that for every dual graph Γ, there is a locally closed substack $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{g,n}$ parameterizing stable curves with dual graph Γ, and conclude that there is a stratification

$$\overline{\mathcal{M}}_{g,n} = \coprod_\Gamma \overline{\mathcal{M}}_\Gamma,$$

called the *dual graph stratification*.

(b) Given two dual graphs Γ and Γ', provide a combinatorial condition for when $\mathcal{M}_{\Gamma'} \subset \overline{\mathcal{M}}_\Gamma$.

(c) The dual graph stratification for $\overline{\mathcal{M}}_2$ was described in Figure 5.5. Describe the stratifications for $\overline{\mathcal{M}}_{2,1}$ and $\overline{\mathcal{M}}_3$.

Definition 5.6.18 (Hodge line bundle). Letting $\pi \colon \mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ denote the universal family, the *Hodge vector bundle* is defined as the pushforward $E := \pi_* \omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}}$. It is a vector bundle by Properties of Families of Stable Curves (5.3.21). The *Hodge line bundle* is defined as the determinant of the Hodge vector bundle $\lambda := \det E$.

Exercise 5.6.19. Show that under every Forgetful Morphism (5.6.7) and Gluing Morphism (5.6.11), λ pulls back to λ.

Definition 5.6.20 (The Psi line bundles). Let $\sigma_i \colon \overline{\mathcal{M}}_{g,n} \to \mathcal{U}_{g,n}$ be the ith section of the universal family. Define the *line bundle* ψ_i on $\overline{\mathcal{M}}_{g,n}$ as the conormal bundle of σ_i; namely, $\psi_i = \mathcal{I}_{\sigma_i}/\mathcal{I}_{\sigma_i}^2$ where $\mathcal{I}_{\sigma_i} \subset \mathcal{O}_{\mathcal{U}_{g,n}}$ is the ideal sheaf defining σ_i. The *line bundle* ψ is defined (using additive notation) as

$$\psi = \psi_1 + \cdots + \psi_n.$$

There are identifications

$$\psi_i \cong \sigma_i^* \Omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}} \cong \sigma_i^* \omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}} \cong \sigma_i^* \mathcal{O}_{\mathcal{U}_{g,n}}(-\sigma_i(\overline{\mathcal{M}}_{g,n})).$$

The *Psi classes* are defined as $c_1(\psi_i) \in \mathrm{CH}^1(\overline{\mathcal{M}}_{g,n})$; see §6.1.7 for the definition of the Chow group.

Exercise 5.6.21.

(a) If C is the nodal union $C_1 \cup C_2$ of two smooth curves along $p_1 \in C_1$ and $p_2 \in C_2$, show that there is a natural identification of vector spaces

$$\delta \otimes \kappa([C]) \cong T_{p_1} C_1 \otimes T_{p_2} C_2.$$

(b) Show that δ pulls back under the gluing morphism $\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n}$ to $\delta - \psi_{n+1} - \psi_{n+2}$ and pulls back under $\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g,n}$ to $(\delta - \psi_{n_1}) \boxtimes (\delta - \psi_{n_2})$.

(c) Conclude that $\delta - \psi$ pulls back to itself under every gluing morphism.

See also [HM98, Prop. 3.32].

Definition 5.6.22 (Kappa classes). The *ith kappa class* is defined as

$$\kappa_i := \pi_*(\psi_{n+1}^{i+1}) \in \mathrm{CH}^i(\overline{\mathcal{M}}_{g,n}).$$

The first kappa class $\kappa := \kappa_1$ is the class of a line bundle.

Definition 5.6.23 (Canonical line bundle). Over a field $\Bbbk$, let $\Omega_{\overline{\mathcal{M}}_{g,n}/\Bbbk}$ be the sheaf of differentials (Example 4.1.3), which is a vector bundle since $\overline{\mathcal{M}}_{g,n}$ is smooth. Define the *canonical line bundle* as

$$K = K_{\overline{\mathcal{M}}_{g,n}} := \det \Omega_{\overline{\mathcal{M}}_{g,n}/\Bbbk}.$$

Remark 5.6.24 (Mumford's Formula). An application of Grothendeick–Riemann–Roch gives the relations

$$\kappa = 12\lambda - \delta \quad \text{and} \quad K = 13\lambda - 2\delta;$$

see [Mum83, §5] and [HM98, §3.E].

Exercise 5.6.25 (Comparison with coarse moduli space). Let $\Delta_i, \Delta \subset \overline{M}_{g,n}$ be the image of $\delta_i, \delta \subset \overline{\mathcal{M}}_{g,n}$ under the coarse moduli space map $\pi \colon \overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$. Show that

$$\pi^* \Delta_i = \begin{cases} \delta_i & \text{if } i \neq 1 \\ 2\delta_1 & \text{if } i = 1 \end{cases}$$
$$\pi^* \Delta = \delta + \delta_1$$
$$\pi^* K_{\overline{M}_{g,n}} = K_{\overline{\mathcal{M}}_{g,n}} - \delta_1.$$

5.7 Irreducibility

> *In your "appendix", you refer to a result of Matsusaka I did not hear of before, namely the connectedness or irreducibility of the variety of moduli for curves of genus g, in any characteristic. I did not know there was any algebraic proof for this (whatever way you state it). Yet I have some hope to prove the connectedness of the $M_{g,n}$ (arbitrary levels) using the transcendental result in char. 0 and the connectedness theorem; but first one should get a natural "compactification" of $M_{g,n}$ which should be simple over $\mathbb{Z}$.*
>
> Grothendieck, letter to Mumford, 1961 [Mum10, p. 638]

We provide several arguments that $\overline{\mathcal{M}}_{g,n}$ is irreducible:

- the classical topological argument due to Clebsch, Lüroth, and Hurwitz (Theorem 5.7.16),

- a purely algebraic argument in characteristic 0 using degenerations of smooth curves and the inductive nature of the boundary $\delta = \overline{\mathcal{M}}_g \backslash \mathcal{M}_g$ (Theorem 5.7.23), and

- an proof in characteristic $p > 0$ by reduction to characteristic 0 (Theorem 5.7.25)

These proofs rely fundamentally on the theory of branched covering as discussed in §5.7.1.

Equivalences. We begin with a few remarks regarding the equivalences between the connectedness/irreducibility of $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}_g$, and their coarse moduli spaces. Since $\overline{\mathcal{M}}_{g,n}$ is a smooth algebraic stack over a field $\Bbbk$, its irreducibility is equivalent to its connectedness. Moreover, since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.8) and has connected fibers, it suffices to verify the connectedness of $\overline{\mathcal{M}}_g$. We thus have equivalences

$$\overline{\mathcal{M}}_{g,n} \text{ irreducible over } \Bbbk \iff \overline{\mathcal{M}}_{g,n} \text{ connected over } \Bbbk$$
$$\iff \overline{\mathcal{M}}_g \text{ connected over } \Bbbk$$
$$\iff \mathcal{M}_g \text{ connected over } \Bbbk \text{ and dense in } \overline{\mathcal{M}}_g$$

Finally, we note that since the coarse moduli space $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ induces a homeomorphism $|\overline{\mathcal{M}}_{g,n}| \xrightarrow{\sim} |\overline{M}_{g,n}|$ on topological spaces, each statement above can be equivalently stated in terms of the coarse moduli space.

5.7.1 Branched coverings

Let $f\colon C \to D$ be a finite morphism of smooth, connected, and projective curves over an algebraically closed field $\Bbbk$. Assume that f is separable, i.e. the induced map $K(D) \to K(C)$ of functions fields is a separable extension. For a point $p \in C(\Bbbk)$ with image $q \in D(\Bbbk)$, the *ramification index at p* is the integer e_p such that $s \mapsto ut^{e_p}$ under the map $\mathcal{O}_{D,Q} \to \mathcal{O}_{C,p}$, where s and t are uniformizers and u is a unit. We say that f is

$$\begin{cases} \textit{ramified at } p & \text{if } e_p > 1 \\ \textit{tamely ramified at } p & \text{if } e_p > 1 \text{ and either } \mathrm{char}(\Bbbk) = 0 \text{ or } \mathrm{char}(\Bbbk) \nmid e_p \\ \textit{unramified at } p & \text{if } e_p = 1 \end{cases}$$

If f is unramified at p, then the scheme-theoretic fiber over $f(p)$ at p is isomorphic to $\mathrm{Spec}\,\kappa(p)$, and thus this agrees with the usual definition of unramified by Unramified Equivalences (A.3.3). Moreover, since f is flat, f is unramified at p if and only if f is étale at p.

There is a short exact sequence of differentials

$$0 \to f^*\Omega_D \to \Omega_C \to \Omega_{C/D} \to 0. \tag{5.7.1}$$

Indeed, the sequence above is always right exact. Since $f^*\Omega_D$ and Ω_C are line bundles, the left map is injective if and only if it is nonzero. However, $K(D) \to K(C)$ is separable so $\Omega_{C/D} \otimes K(C) = \Omega_{K(C)/K(D)} = 0$, and thus $f^*\Omega_D \to \Omega_C$ is nonzero at the generic point. Examining the sequence above at the stalks at a point $p \in C(\Bbbk)$, the differential dt maps to $d(us^{e_p}) = eus^{e_p-1}ds + s^{e_p}du$. If f is tamely ramified at p, then $(\Omega_{C/D})_p \cong \mathcal{O}_{C,p}\langle ds\rangle/(s^{e_p-1}ds)$ and $\mathrm{length}(\Omega_{C/D})_p = \dim \Omega_{C/D} \otimes \kappa(p) = e_p - 1$.

Definition 5.7.2. Let $\Bbbk$ be an algebraically closed field.

(1) A *branched covering* is a finite separable morphism $f\colon C \to D$ of smooth, connected, and projective curves over $\Bbbk$.

(2) A *simply branched covering* is a branched covering such that there is at most one ramification point in every fiber and every ramification point $p \in C(\Bbbk)$ is tamely ramified with ramification index $e_p = 2$.
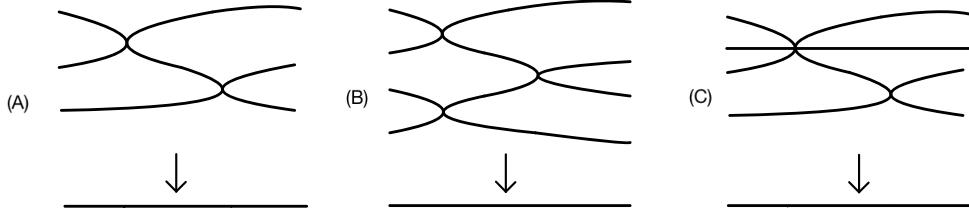
Figure 5.22: Examples of branched coverings over $\mathbb{P}^1$: (A) is simply branched while (B) and (C) are not. While the picture may suggest that the source curve C is not smooth, C is, in fact, smooth over the base field $\Bbbk$. However, the map $C \to \mathbb{P}^1$ is not smooth, and the pictures above are designed to reflect the singularities of C over $\mathbb{P}^1$.

If $f\colon C \to D$ is a branched covering, the *ramification divisor* is defined as

$$R = \sum_{p \in C(\Bbbk)} \operatorname{length}(\Omega_{C/D})_p \cdot p \overset{\substack{\text{if } f \text{ is tamely} \\ \text{ramified}}}{=} \sum_{p \in C(\Bbbk)} (e_p - 1).$$

Theorem 5.7.3 (Riemann–Hurwitz)**.** *If $f\colon C \to D$ is a branched covering with ramification divisor R, then $\Omega_C \cong f^*\Omega_D \otimes \mathcal{O}_C(R)$ and*

$$2g(C) - 2 = \deg(f)(2g(D) - 2) + \deg R.$$

In particular, $f\colon C \to \mathbb{P}^1$ is simply branched, then it is ramified over $2g + 2d - 2$ distinct points.

Proof. This follows directly from the exact sequence (5.7.1). See also [Har77, Prop. IV.2.3] $\qquad\square$

Example 5.7.4. For a local model of a branched cover, consider the map $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ defined by $x \mapsto x^n$. The relative sheaf of differentials is $\Omega_{\mathbb{A}^1/\mathbb{A}^1} = k[x]\langle dx\rangle/(nx^{n-1}dx)$. Thus, if $\operatorname{char}(\Bbbk)$ does not divide n, f is a branched cover étale over $\mathbb{A}^1 \setminus 0$ and ramified at 0 with index $n - 1$.

Exercise 5.7.5. Show that every branched covering is étale locally isomorphic to $\mathbb{A}^1 \to \mathbb{A}^1, x \mapsto x^n$ around a branched point of index $n - 1$.

Algebraic vs. holomomorphic vs. topological branched covers. *A holomorphic (resp. topological) branched covering of $\mathbb{P}^1$ is a non-constant and holomorphic (resp. continuous) morphism $f\colon C \to \mathbb{P}^1$ of connected and compact Riemann surfaces (resp. connected, Hausdorff, and compact topological spaces such that f is a covering space over the complement of finitely many points of $\mathbb{P}^1$).*

Proposition 5.7.6. *Over $\mathbb{C}$, there are natural bijections*

$$\{C \to \mathbb{P}^1 \text{ algebraic branched coverings}\} \longleftrightarrow \{C \to \mathbb{P}^1 \text{ holomorphic branched coverings}\}$$
$$\longleftrightarrow \{C \to \mathbb{P}^1 \text{ topological branched coverings}\}$$

Proof. An algebraic branched covering is holomorphic and a holomorphic branched covering is topological. Conversely, if $f\colon C \to \mathbb{P}^1$ is a topological covering, then the holomorphic structure on $\mathbb{P}^1$ induces naturally a holomorphic structure on C such that $f\colon C \to \mathbb{P}^1$ is analytic. It is a classical fact relying on the Implicit Function Theorem that there are local charts of C and $\mathbb{P}^1$ such that f is described by $z \mapsto z^k$ [Mir95, Prop. II.4.1], which implies that C is algebraic. $\qquad\square$

Monodromy actions. Let $f \colon C \to \mathbb{P}^1$ be a (topological) branched covering of degree d over $\mathbb{C}$ and $B \subset \mathbb{P}^1$ its ramification locus, i.e. the smallest set of points such that $f^{-1}(\mathbb{P}^1 \setminus B) \to \mathbb{P}^1 \setminus B$ is a covering space. Choose a base point $q \in \mathbb{P}^1 \setminus B$. The *monodromy action of $\pi_1(\mathbb{P}^1 \setminus B, q)$ on the fiber $f^{-1}(q)$* is defined as follows: for $\gamma \in \pi_1(\mathbb{P}^1 \setminus B, q)$ and $p \in f^{-1}(q)$, then the path $\gamma \colon [0,1] \to \mathbb{P}^1$ lifts uniquely to a path $\widetilde{\gamma} \colon [0,1] \to C$ such that $\widetilde{\gamma}(0) = p$, and the action is defined by $\gamma \cdot p = \widetilde{\gamma}(1)$.
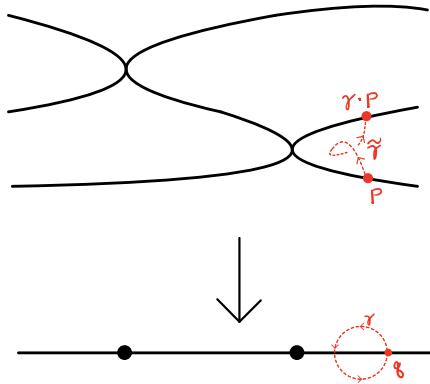
Figure 5.23: Monodromy action

A choice of a bijection $f^{-1}(p) \cong \{1, \ldots, d\}$ defines a group homomorphism

$$\rho \colon \pi_1(\mathbb{P}^1 \setminus B, q) \to S_d,$$

which we call the *monodromy representation*. The converse is also true: every such group homomorphism is induced by a branched covering.

Proposition 5.7.7. *Let $B \subset \mathbb{P}^1$ be a finite subset of size b, $q \in \mathbb{P}^1 \setminus B$ be a point, and $d > 0$ a positive integer. The fundamental group $\pi_1(\mathbb{P}^1 \setminus B, q)$ is identified with the free group generated by the simple loops σ_i for $i = 1, \ldots, b$ around the points of B. There is a natural bijection of isomorphism classes*

$$\left\{ \begin{array}{l} \textit{simply branched} \\ \textit{coverings } C \to \mathbb{P}^1 \textit{ of} \\ \textit{of degree } d \\ \textit{branched over } B \end{array} \right\} \Big/ \sim \; \longleftrightarrow \; \left\{ \begin{array}{l} \textit{group homomorphisms} \\ \rho \colon \pi_1(\mathbb{P}^1 \setminus B, q) \to S_d \textit{ such that} \\ \textit{im}(\rho) \subset S_d \textit{ is a transitive subgroup} \\ \textit{and each } \rho(\sigma_i) \textit{ is a transposition} \end{array} \right\} \Big/ \sim,$$

where two branched covers are equivalent if they are isomorphic over $\mathbb{P}^1$, and two homomorphisms ρ and ρ' are equivalent if they differ by an inner automorphism of S_d, i.e. $\exists h \in S_d$ such that $\rho' = \mathrm{h}^{-1} \rho h$.

Proof. We've already explained a natural map from left to right. Conversely, given a group homomorphism $\rho \colon \pi_1(\mathbb{P}^1 \setminus B, q) \to S_d$, we let $H \subset \pi_1(\mathbb{P}^1 \setminus B, q)$ be the subgroup containing elements γ fixing 1. Since this subgroup has index d, it corresponds to a covering space $C^* \to \mathbb{P}^1 \setminus B$, which one can show extends to a finite morphism $C \to \mathbb{P}^1$ of degree d. The connectedness of C translates into the condition that $\mathrm{im}(\rho) \subset S_d$ is transitive, and the cover $C \to \mathbb{P}^1$ being simply branched translates into each $\rho(\sigma_i)$ being a transposition. See [Mir95, Prop. III.4.9]. $\square$

5.7.2 Hurwitz moduli spaces

For positive integers g and b, we define and study the *Hurwitz moduli space*

$$\text{Hur}_{g,b} := \{\text{simply branched coverings } C \to \mathbb{P}^1 \text{ of genus } g \text{ over } b \text{ ordered points}\}$$

of simply branched covers and its relation to $\mathcal{M}_g$. By Riemann–Hurwitz (5.7.3), if $C \to \mathbb{P}^1$ has degree d, then $b = 2g + 2d - 2$. In the literature, Hurwitz spaces are sometimes indexed as $\text{Hur}_{d,b}$ or $\text{Hur}_{d,g}$. The key diagram is

$$
\begin{array}{ccc}
 & \text{Hur}_{g,b} & \\
 \swarrow & & \searrow \\
\mathcal{M}_g & & M_{0,b}
\end{array}
\tag{5.7.8}
$$

where a simply branched covering $(C \to \mathbb{P}^1) \in \text{Hur}_{g,b}$ gets mapped to $C \in \mathcal{M}_g$ and the b ordered branch points in $M_{0,b}$.

The moduli space $\text{Hur}_{g,b}$ over $\mathbb{C}$ can be viewed analytically as the topological space with the coarsest topology such that $\text{Hur}_{g,b} \to M_{0,b}$ is continuous. Algebraically, we define it using, as usual, our functorial approach. To this end, we define a *family* $(\mathcal{C} \to \mathcal{D} \to S, \sigma_i)$ *of coverings of $\mathbb{P}^1$ of genus g simply branched over b ordered points over a scheme S* as a family $\mathcal{C} \to S$ of smooth genus g curves and a b-pointed family $(\mathcal{D} \to S, \sigma_i)$ of smooth genus 0 curves together with a finite, flat morphism $\mathcal{C} \to \mathcal{D}$ of schemes over S of degree d such that for every geometric point $s \in S(\mathbb{k})$, $\mathcal{C}_s \to \mathcal{D}_s \cong \mathbb{P}^1_{\mathbb{k}}$ is simply branched over $\sigma_1(s), \ldots, \sigma_b(s)$.

Definition 5.7.9 (Hurwitz functor)**.** For positive integers g and d, set $b = 2g + 2d - 2$. The *Hurwitz functor* is defined as the functor

$$\text{Hur}_{g,b} \colon \text{Sch} \to \text{Sets}$$

$$S \mapsto \left\{ \begin{array}{l} \text{families } (\mathcal{C} \to \mathcal{D} \to S, \sigma_i) \text{ of simply branched coverings} \\ \text{of } \mathbb{P}^1 \text{ of genus } g \text{ and degree } d \end{array} \right\} \Big/ \sim,$$

where two families $(\mathcal{C} \to \mathcal{D} \to S, \sigma_i)$ and $(\mathcal{C}' \to \mathcal{D}' \to S, \sigma_i')$ are equivalent if there are isomorphisms $\mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ and $\mathcal{D}' \xrightarrow{\sim} \mathcal{D}'$ compatible with the sections and the structure morphisms $\mathcal{C} \to \mathcal{D}$ and $\mathcal{C}' \to \mathcal{D}'$.

There are morphisms $\text{Hur}_{g,b} \to \mathcal{M}_g$ and $\text{Hur}_{g,b} \to M_{0,b}$ of stacks taking a family of coverings $(\mathcal{C} \to \mathcal{D} \to S, \sigma_i)$ to $\mathcal{C} \to S$ and $(\mathcal{D} \to S, \sigma_i)$ giving Diagram (5.7.8).

Remark 5.7.10 (Variants)**.** There are a few variants of the above definition that are also referred to as Hurwitz moduli spaces in the literature. First, one can rigidify the moduli problem by considering the functor $\text{Hur}_{g,b}^{\text{rig}} \colon \text{Sch} \to S$, where an object over S is a finite, flat morphism $\mathcal{C} \to \mathbb{P}^1_S$ over S from a family of smooth curves of genus g together with disjoint ordered sections $\sigma_1, \ldots, \sigma_b \colon S \to \mathbb{P}^1_S$ such that for every geometric point $s \in S(\mathbb{k})$, $\mathcal{C}_s \to \mathbb{P}^1_{\mathbb{k}}$ is a covering simply branched over $\sigma_1(s), \ldots, \sigma_b(s)$. Here two families are equivalent if they are isomorphic over $\mathbb{P}^1_S$ and the sections are equal. There is a morphism $\text{Hur}_{g,b}^{\text{rig}} \to (\mathbb{P}^1)^b \setminus \Delta$, where Δ is the union of all pairwise diagonals. The algebraic group PGL_2 acts freely on $\text{Hur}_{g,b}^{\text{rig}}$ and the quotient is identified with $\text{Hur}_{g,b}$, while the quotient of the PGL_2-equivariant morphism $\text{Hur}_{g,b}^{\text{rig}} \to (\mathbb{P}^1)^b \setminus \Delta$ is identified with $\text{Hur}_{g,b} \to M_{0,b}$

Another common variant, which was the version originally considered by Hurtwitz in [Hur91], is to consider the branch locus as a set of *unordered* points. Let

$\mathrm{Hur}_{g,b}^{\mathrm{rig,unord}} \colon \mathrm{Sch} \to \mathrm{Sets}$ be the functor, where an object over S is a finite, flat morphism $f \colon \mathcal{C} \to \mathbb{P}_S^1$ over S from a family of smooth curves of genus g such that for every geometric point $s \in S(\Bbbk)$, $\mathcal{C}_s \to \mathbb{P}_{\Bbbk}^1$ is a covering simply branched over b points. We claim that there is a morphism $\mathrm{Hur}_{g,b}^{\mathrm{rig,unord}} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$ to the space parameterizing b unordered distinct points. To see this, observe that the *ramification divisor* $R_f \subset \mathcal{C}$, defined as the relative singular locus of $f \colon \mathcal{C} \to \mathbb{P}_S^1$, is finite and étale over S of degree $b = 2g + 2d - 2$. The complement $R_f \times_S \cdots \times_S R_f \setminus \Delta$ of the pairwise diagonals is a principal S_b-torsor over S (see Exercise B.2.23) and the map $R_f \times_S \cdots \times_S R_f \setminus \Delta \to \mathbb{P}_S^1 \times_S \cdots \times_S \mathbb{P}_S^1 \setminus \Delta$ is an S_b-equivariant morphism. This defines an S-valued point of the quotient stack $[((\mathbb{P}^1)^b \setminus \Delta)/S_b]$ over S, which in turn is identified with the scheme $\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$. Note also that $\mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta \cong \mathbb{P}^b \setminus \Delta$, where the latter Δ is the discriminant hypersurface.

The symmetric group S_b acts freely on $\mathrm{Hur}_{g,b}^{\mathrm{rig}}$ with quotient $\mathrm{Hur}_{g,b}^{\mathrm{rig,unord}}$, and the S_b-quotient of $\mathrm{Hur}_{g,b}^{\mathrm{rig}} \to (\mathbb{P}^1)^b \setminus \Delta$ is identified with $\mathrm{Hur}_{g,b}^{\mathrm{rig,unord}} \to \mathrm{Sym}^b \mathbb{P}^1 \setminus \Delta$.

The following exercise allows us to conclude that the map $\mathrm{Hur}_{g,b} \to M_{0,b}$ is a finite, étale, and surjective morphism, i.e. an algebraic covering space. When working over a field $\Bbbk$, we will abuse notation by also referring to $\mathrm{Hur}_{g,b}$ as the base change $\mathrm{Hur}_{g,b} \times_{\mathbb{Z}} \Bbbk$.

Exercise 5.7.11. Let $\Bbbk$ be an algebraically closed field of characteristic 0. Let g, b, and d be positive integers with $b = 2g + 2d - 2$. Assume that $d > 2$.

(a) Show that the diagonal of $\mathrm{Hur}_{g,b}$ over $\Bbbk$ is representable by schemes.

(b) If $\mathcal{C} \to \mathbb{P}_{\Bbbk}^1$ is a simply branched covering of degree d, show that every automorphism of $\mathcal{C}$ over $\mathbb{P}^1$ is trivial. Conclude that the automorphism group scheme of a family of simply branched coverings of $\mathbb{P}^1$ of degree d is also trivial.

   Hint: Use the fact that there are no non-trivial automorphisms of a smooth curve fixing more than $2g + 2$ points (Exercise 5.1.8).

(c) (hard) Show that $\mathrm{Hur}_{g,b} \to M_{0,b}$ is a finite, étale, and representable morphism. In particular, the functor $\mathrm{Hur}_{g,b}$ is represented by a scheme. See also [Ful69, Thm. 7.2], where it is shown more generally that $\mathrm{Hur}_{g,b}$ is representable over $\mathrm{Spec}\,\mathbb{Z}$, and the map $\mathrm{Hur}_{g,b} \to M_{0,b}$ is étale (resp. finite étale over bases of characteristic greater than d).

(d) Verify that the functors $\mathrm{Hur}_{g,b}^{\mathrm{rig}}$ and $\mathrm{Hur}_{g,b}^{\mathrm{rig,unord}}$ defined in Remark 5.7.10 are also representable by schemes.

Remark 5.7.12. For the proof of the Clebsch–Hurwitz–Lüroth Theorem (5.7.16) in the next section, we only need to know that $\mathrm{Hur}_{g,b} \to M_{0,b}$ is a topological covering space, and this fact has an elementary argument. Consulting Figure 5.24, given a simply branched covering $f \colon C \to \mathbb{P}^1$ and a branched point $p \in C$, we can choose an open neighborhood $U \subset \mathbb{P}^1$ around $f(p)$ such that $f^{-1}(U) \to U$ is isomorphic to an open neighborhood of the map $\mathbb{C} \to \mathbb{C}$ given by $x \mapsto x^n$. For every other point $q' \in U$, we can construct a branched cover $C' \to \mathbb{P}^1$ which outside U is the same as $C \to \mathbb{P}^1$ and over U is locally isomorphic to $x \mapsto x^n$ but centered over q' (rather than $f(p)$).
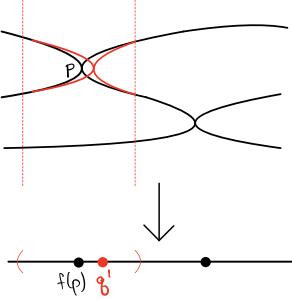
Figure 5.24

We turn now to studying the other map: $\mathrm{Hur}_{g,b} \to \mathcal{M}_g$.

Lemma 5.7.13. *Let C be a smooth, connected, and projective curve of genus g over an algebraically closed field $\Bbbk$ of characteristic 0. If L is a line bundle of degree $d \geq 2g + 3$ and $V \subset \mathrm{H}^0(C, L)$ is a linear system of dimension 2, then a choice of basis of V induces a simply branched covering $C \to \mathbb{P}^1$.*

Proof. We proceed with a dimension count. Since $\mathrm{h}^1(C, L) = \mathrm{h}^0(C, \omega_C \otimes L^\vee) = 0$ as $\deg(\omega_C \otimes L^\vee) < 0$, Riemann–Roch implies that $\mathrm{h}^0(C, L) = d + 1 - g$, and it follows that the dimension of the Grassmannian $\mathrm{Gr}(2, \mathrm{H}^0(L))$ of 2-dimensional subspaces is $2(d - g - 1)$. Since $\mathrm{char}(\Bbbk) = 0$, every finite morphism $C \to \mathbb{P}^1$ is automatically separable. Thus, if V does *not* induce a simply branched covering $C \to \mathbb{P}^1$, then one of the following three conditions must hold:

(a) V has a base point,

(b) there exists a ramification point with index greater than 2, or

(c) there exists two ramification points in the same fiber.

We claim that each condition is closed of codimension at least one. For condition (a), if $p \in C(\Bbbk)$ is a base point, then $V \subset \mathrm{H}^0(C, L(-p))$. Since $d \geq 2g$, we have that $\deg(\omega_C \otimes (L(-p))^\vee) < 0$, and thus $\mathrm{h}^0(C, L(-p)) = d - g$ and $\dim \mathrm{Gr}(2, \mathrm{H}^0(C, L(-p))) = 2(d - g - 2)$. Varying $p \in C$, we see that the locus defining (a) has dimension $2(d - g - 2) + 1 = \dim \mathrm{Gr}(2, \mathrm{H}^0(L)) - 1$. We leave cases (b) and (c) for the reader. See also [Sev21] and [Ful69, Prop. 8.1]. $\qquad\square$

Exercise 5.7.14 (details)**.**

(a) Verify that conditions (b) and (c) above are closed of codimension at least one.

(b) Show that the argument holds as long as $\mathrm{char}(\Bbbk) \neq 2$ and explain why it fails if $\mathrm{char}(\Bbbk) = 2$.

Corollary 5.7.15. *Let $\Bbbk$ be an algebraically closed field of characteristic 0. If $g \geq 2$ and $d \geq 2g + 3$ with $b = 2g + 2d - 2$, the morphism*

$$\mathrm{Hur}_{g,b} \to \mathcal{M}_g, \qquad (C \to \mathbb{P}^1, \sigma_i) \mapsto C$$

is surjective. $\qquad\square$

5.7.3 The Clebsch–Hurwitz–Lüroth Theorem

We provide the classical argument due to Clebsch [Cle73], Hurwitz [Hur91], and Lüroth [Lür71] that the Hurwitz moduli space $\mathrm{Hur}_{g,b}$ is connected over $\mathbb{C}$. From

the surjectivity of the map $\mathrm{Hur}_{g,b} \to \mathcal{M}_g$, this allows us to conclude that $\mathcal{M}_g$ is irreducible, a classical result of Klein [Kle82, §19] and Severi [Sev21, §F]. For a modern treatment, see [Ful69, Prop. 1.5].

Theorem 5.7.16 (The Clebsch–Hurwitz–Lüroth Theorem). *For $g \geq 2$ and $d \geq 2$ with $b = 2g + 2d - 2$, the Hurwitz moduli space $\mathrm{Hur}_{g,b}$ over $\mathbb{C}$ is connected.*

Proof. We utilize the morphism

$$\beta \colon \mathrm{Hur}_{g,b} \to M_{0,b},$$

sending a simply branched cover to the ordered branch locus. By Exercise 5.7.11(c), this map is a covering space. For every finite ordered set $B = \{q_1, \ldots, q_b\} \subset \mathbb{P}^1$ of b distinct points and $q \in \mathbb{P}^1 \setminus B$, the fundamental group

$$\pi_1(\mathbb{P}^1 \setminus B, q) = \langle \sigma_i \,|\, \sigma_1 \cdots \sigma_b = 1 \rangle,$$

where σ_i is a simple loop around q_i, acts on the fiber $f^{-1}(p)$ of a simply branched covering $f \colon C \to \mathbb{P}^1$. Similarly, since β is a covering space, $\pi_1(M_{0,b}, B)$ acts on the fiber

$$\mathrm{Hur}_{d,B} := \beta^{-1}(B) \subset \mathrm{Hur}_{g,b}$$

over $B \in M_{0,b}$. (Note the distinction between the use of the uppercase subscript 'B' and the lowercase 'b'.) Using Proposition 5.7.7, we have bijections

$$\mathrm{Hur}_{g,B} = \{\text{genus } g \text{ coverings } C \to \mathbb{P}^1 \text{ simply branched over } B\}/\sim$$

$$\cong \left\{ \begin{array}{l} \text{group homomorphisms } \rho \colon \pi_1(\mathbb{P}^1 \setminus B, q) \to S_d \\ \text{such that } \mathrm{im}(\rho) \subset S_d \text{ is a transitive subgroup} \\ \text{and each } \rho(\sigma_i) \text{ is a transposition} \end{array} \right\} \Big/ \text{conjugation by } S_d$$

$$\cong \left\{ \begin{array}{l} \text{sequences } (\tau_1, \ldots, \tau_b) \in (S_d)^b \text{ of transpositions} \\ \text{with product 1 generating a transitive subgroup} \end{array} \right\} \Big/ \text{conjugation by } S_d.$$

The connectedness of $\mathrm{Hur}_{g,b}$ is equivalent to the transitivity of the action of $\pi_1(M_{0,b}, B)$ on the fiber $\mathrm{Hur}_{d,B} = \beta^{-1}(B)$. Consider the sequence

$$\tau := \Big(\underbrace{(12), (12), (13), (13), \ldots, (1\,d-1), (1\,d-1)}_{2(d-2)}, \underbrace{(1d), (1d), \ldots, (1d)}_{2g+2} \Big) \in \mathrm{Hur}_{d,B}.$$

It suffices to show that every orbit of the action of $\pi_1(M_{0,b}, B)$ on the fiber $\mathrm{Hur}_{d,B} = \beta^{-1}(B)$ contains τ, and, to this end, we define the loop

$$\Gamma_i \colon [0,1] \to M_{0,b}$$
$$t \mapsto (q_1, \ldots, q_{i-1}, \gamma_i(t), \gamma_i'(t), q_{i+2}, \ldots, q_b),$$

where γ_i and γ_i' are paths as in Figure 5.25.
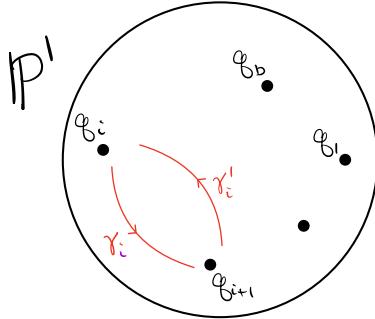
Figure 5.25: Paths in $\mathbb{P}^1$

For an element $(\lambda_1, \ldots, \lambda_b) \in \mathrm{Hur}_{g,b}$, the action of Γ_i is given by

$$\Gamma_i \cdot (\lambda_1, \ldots, \lambda_b) = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i^{-1}\lambda_{i+1}\lambda_i, \lambda_i, \lambda_{i+2}, \ldots, \lambda_b).$$

It is now a combinatorial problem that we leave to the reader to show that there exists a sequence $\Gamma_{i_1}, \ldots, \Gamma_{i_k}$ of loops such that $\tau = \Gamma_{i_1}\Gamma_{i_2}\cdots\Gamma_{i_k} \cdot (\lambda_1, \ldots, \lambda_b)$. $\qquad \square$

Exercise 5.7.17. Solve the combinatorial problem at the end of the proof.

Corollary 5.7.18 (Irreducibility of $\mathcal{M}_g$)**.** *For any field $\Bbbk$ of characteristic 0, $\mathcal{M}_g$ is irreducible.*

Proof. Since the morphsim $\mathrm{Hur}_{g,b} \to \mathcal{M}_g$ is surjective (Corollary 5.7.15), the connectedness of $\mathrm{Hur}_{g,b}$ over $\mathbb{C}$ implies the connectedness of $\mathcal{M}_g$ over $\mathbb{C}$. This, in turn, implies that $\mathcal{M}_g$ is geometrically connected over $\mathbb{Q}$, and thus connected over any field $\Bbbk$ of characteristic 0. Since $\mathcal{M}_g$ is smooth over $\Bbbk$ (Theorem 5.4.14), its connectedness is equivalent to its irreducibility. $\qquad \square$

The irreducibility of $\mathcal{M}_g$ can also be established using the Teichmüller space $\mathcal{T}_g$ parameterizing complex structures on a topological surface Σ_g of genus g. It was first proved in [Ber60b] that $\mathcal{T}_g$ is homeomorphic to a ball in $\mathbb{C}^{3g-3}$, and that $\mathcal{M}_g$ is identified with the quotient $\mathcal{T}_g/\Gamma_g$ by the mapping class group Γ_g.

5.7.4 Irreducibility via degeneration

We now give a completely algebraic argument for the irreducibility of $\overline{\mathcal{M}}_g$ in characteristic 0. The key idea is to show every smooth curve degenerates to a singular stable curve (Proposition 5.7.21)—this is the most challenging part of the argument and is achieved with a technique inspired by the theory of admissible covers. This reduces the connectedness of $\overline{\mathcal{M}}_g$ to the connectedness of the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$, which we show using the inductive structure of the boundary and the Gluing Morphisms (5.6.11). This argument is similar in spirit to Deligne and Mumford's first proof of the irreducibility of $\overline{M}_g$ in positive characteristic (see Remark 5.7.26). We follow the treatment in Fulton's appendix of Harris and Mumford's paper [HM82].

We begin with a warmup—the genus 0 case.

Proposition 5.7.19. *For every algebraically field $\Bbbk$ and every integer $n \geq 3$, $\overline{M}_{0,n}$ is irreducible.*

Proof. As $\overline{M}_{0,n}$ is smooth, it suffices to show it is connected. In this case, $M_{0,n}$ is obviously connected as it is identified with $(\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \Delta$, where Δ is the union of the pairwise diagonals (see Exercise 3.1.20). Therefore, it suffices to show that the boundary $\delta = \overline{M}_{0,n} \setminus M_{0,n}$ does not have a connected component consisting entirely of singular curves. Given a singular curve $(C, p_i) \in \overline{M}_{0,n}(\mathbb{k})$, it is easy to directly construct a stable family of curves $(\mathcal{C} \to \operatorname{Spec} R, \sigma_i)$ over a DVR R whose generic fiber is smooth and whose special fiber is (C, p_i): indeed, every node of C has a Zariski-open neighborhood of the form $\operatorname{Spec} \mathbb{k}[x,y]/(xy)$ and we can glue the deformations $\operatorname{Spec} \mathbb{k}[x,y]/(xy - \pi)$, where $\pi \in R$ is a uniformizer, to a family $\mathcal{C} \to R$. Alternatively, by deformation theory Proposition 5.3.23, there are compatible stable families $(\mathcal{C}_n \to \operatorname{Spec} \mathbb{k}[t]/(t^{n+1}), \sigma_{n,i})$ with $(\mathcal{C}_0, \sigma_{0,i}) = (C, p_i)$, which by Grothendieck's Existence Theorem (C.4.9) effectives to a stable family $(\mathcal{C} \to \operatorname{Spec} \mathbb{k}[\![t]\!]), \sigma_i)$ with smooth generic fiber.

We now give an alternative proof, which uses the identical strategy that we will use shortly to show the connectedness of $\overline{\mathcal{M}}_{g,n}$. First, we claim that an n-pointed curve $(\mathbb{P}^1, p_i)$ degenerates in a one-parameter family to a singular n-pointed stable curve. Indeed, if we let the points p_1 and p_2 approach each other at different rates and blow up the limit in the central fiber where they intersect, then the total family together with the strict transform of the sections defines a family of stable n-pointed genus 0 curves with singular central fiber. It therefore suffices to show that the boundary $\delta = \overline{M}_{0,n} \setminus M_{0,n}$ is connected. The boundary divisor δ decomposes as a union

$$\delta = \bigcup_I \delta_{0,I}, \qquad \text{where } \delta_{0,I} = \operatorname{im}\left(\overline{\mathcal{M}}_{0,I \cup \{p\}} \times \overline{\mathcal{M}}_{0,I^c \cup \{q\}} \to \overline{\mathcal{M}}_{0,n}\right)$$

over subsets $I \subset \{1,\dots,n\}$ of size $2 \leq |I| \leq n-2$, subject to the relation $\delta_{0,I} = \delta_{0,I^c}$. By induction, we can assume that $\overline{M}_{0,k}$ is irreducible for $k < n$, where the base case holds as $\overline{M}_{0,3} = \operatorname{Spec} \mathbb{k}$. This implies that each $\delta_{0,I}$ is connected. Let $I, J \subset \{1,\dots,n\}$ be two distinct subsets. After possibly replacing I with I^c, we can assume that $K := I \cap J$ has size at least 2. If $K \subsetneq I$ and $K \subsetneq J$, then by consulting Figure 5.26, we see that $\delta_{0,I} \cap \delta_{0,K} \neq \emptyset$ and $\delta_{0,J} \cap \delta_{0,K} \neq \emptyset$. If $I \subset J$ or $J \subset I$, then $\delta_{0,I} \cap \delta_{0,J} \neq \emptyset$ for the same reason. The connectedness of δ follows. $\square$
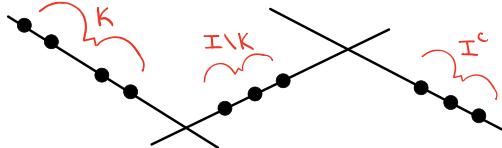
Figure 5.26: A curve in $\delta_{0,I} \cap \delta_{0,K}$ as long as $K \subsetneq I$.

Exercise 5.7.20. Show that every stable curve C over an algebraically closed field $\mathbb{k}$ deforms a smooth curve. More precisely, show that there is a family of stable curves $\pi \colon \mathcal{C} \to T$ over a connected curve T over $\mathbb{k}$ and a point $t \in T(\mathbb{k})$ such that $\mathcal{C}_t \cong C$ and $\pi^{-1}(T \setminus \{t\}) \to T \setminus \{t\}$ is smooth.

The converse to the above exercise, i.e. that every smooth curve *degenerates* to a stable curve, is more difficult.

Proposition 5.7.21. *Let C be a smooth, connected, and projective curve of genus $g \geq 2$ over an algebraically closed field $\mathbb{k}$ of characteristic 0. There exists a family*

$\mathcal{C} \to T$ *of stable curves over a connected smooth curve T over $\Bbbk$ with points $s, t \in T(\Bbbk)$ such that $\mathcal{C}_s \cong C$ and $\mathcal{C}_t$ is a singular stable curve.*

Proof. The essential strategy is as follows:

① For $d \gg 0$, choose a simply branched covering $C \to \mathbb{P}^1$ of degree d branched over distinct points $q_1, \ldots, q_b \in \mathbb{P}^1$ where $b = 2g + 2d - 2$.

② Deform the covering $C \to \mathbb{P}^1$ branched over q_i to a covering $C' \to \mathbb{P}^1$ branched over *general* points q_i'.

③ Degenerate $(\mathbb{P}^1, q_i') \in M_{0,b}$ to the stable curve $(D, d_i) \in \overline{M}_{0,b}$ featured in Figure 5.27.
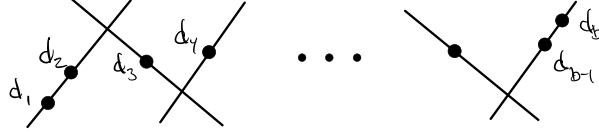
Figure 5.27: The degenerate curve $(D, d_i) \in \overline{M}_{0,b}$

④ Degenerate the cover $C' \to \mathbb{P}^1$ to a cover $C'' \to D$ branched over d_i such that C'' is singular and stable.

For Step 1, Lemma 5.7.13 guarantees the existence of a simply branched covering $C \to \mathbb{P}^1$ of degree $d \gg 0$ simply branched over $b = 2g + 2d - 2$ distinct points q_i. This defines a b-pointed stable curve $(\mathbb{P}^1, q_i) \in M_{0,b}$. For the remaining steps, we first claim that it suffices to find a family $\mathcal{C} \to T$ over a *connected* scheme T of finite type over $\Bbbk$. Indeed, this would show that C is in the same connected (hence irreducible) component $\mathcal{M}' \subset \overline{\mathcal{M}}_g$ as a singular stable curve C''. By Le Lemme de Gabber (4.5.1), there is a finite cover $Z \to \overline{\mathcal{M}}_g$ by a scheme, and Z must have an irreducible component Z' surjecting onto $\mathcal{M}'$. As any two $\Bbbk$-points of an irreducible scheme Z' of finite type over $\Bbbk$ are contained in an integral curve $Q \subset Z'$, the claim follows by considering the map $\widetilde{Q} \to Z' \to \overline{\mathcal{M}}_g$ from the normalization of Q.

For Step 2, let $H \subset \mathrm{Hur}_{g,b}$ be the connected component containing $C \to \mathbb{P}^1$. Since $\mathrm{Hur}_{g,b} \to M_{0,b}$ is finite and étale (Exercise 5.7.11(c)) and thus both an open and closed morphism, the composition $H \hookrightarrow \mathrm{Hur}_{d,b} \to M_{0,b}$ is surjective. As H is connected, it suffices to find a *single* simply branched cover $(C' \to \mathbb{P}^1) \in H$ and a family of stable curves $\mathcal{C} \to T$ over a connected base T connecting C' to a singular stable curve. This brings us to Step 3.

We claim that there is a commutative diagram

$$
\begin{array}{ccc}
T \setminus \{t\} & \longrightarrow & H \overset{\text{op \& cl}}{\hookrightarrow} \mathrm{Hur}_{g,b} \\
\big\downarrow & & \big\downarrow {\scriptstyle \text{fin ét}} \\
T \underset{t \mapsto (D, d_i)}{\longrightarrow} \overline{M}_{0,b} & \longleftarrow & M_{0,b}
\end{array}
\tag{5.7.22}
$$

where T is a connected smooth curve and $t \in T(\Bbbk)$ maps to the stable curve (D, d_i) of Figure 5.27. To see this, observe that since $\overline{M}_{0,b}$ is irreducible (Proposition 5.7.19), $M_{0,n} \subset \overline{M}_{0,n}$ is dense. We can thus choose a map $Y \to \overline{M}_{0,n}$ from a connected reduced curve and a point $y \in Y(\Bbbk)$ such that $y \mapsto (D, d_i)$ and such that the image of every $y' \neq y$ is smooth. Let Y' be a connected component of the base

change $(Y \setminus \{y\}) \times_{M_{0,b}} H$, and let T be the normalization of the closure of the image $Y' \to \overline{M}_{0,b}$. There is a point $t \in T$ mapping to $(D, d_i) \in \overline{M}_{0,b}$, and after replacing T with an open neighborhood of t, we can arrange that $T \setminus \{t\} \to \overline{M}_{0,b}$ factors through H.

Diagram 5.7.22 gives a family $(\mathcal{D} \to T, \tau_i)$ of b-pointed stable curves of genus 0 together with a simply branched covering $\mathcal{C}^* \to \mathcal{D}^*$ over $T^* := T \setminus \{t\}$ (where $\mathcal{D}^* \subset \mathcal{D}$ is the preimage of T^*) such that the sections $\tau_i^* : T^* \to \mathcal{D}^*$ pick out the branch locus. We now claim that, after replacing (T, t) by a ramified cover, there are dotted arrows completing the cartesian diagram

$$
\begin{array}{ccccc}
\mathcal{C}^* & \longrightarrow & \mathcal{D}^* & \longrightarrow & T^* \\
\vert & & \downarrow & & \downarrow \\
\vert & \square & \downarrow & \square & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C} & \overset{f}{-\!-\!\rightarrow} & \mathcal{D} & \longrightarrow & T,
\end{array}
$$

where $\mathcal{C} \to T$ is a family of nodal curves. If we define $\mathcal{C}$ as the integral closure of $\mathcal{O}_{\mathcal{D}}$ in $K(\mathcal{C}^*)$, then $f : \mathcal{C} \to \mathcal{D}$ is a finite morphism. Letting $\mathcal{D}^{\mathrm{sm}} = \mathcal{D} \setminus \{d_i\}$ be the relative smooth locus of $\mathcal{D} \to T$, then Purity of the Branch Locus (A.3.13) implies that the ramification locus of $f^{-1}(\mathcal{D}^{\mathrm{sm}}) \to \mathcal{D}^{\mathrm{sm}}$ is a divisor. Therefore, $\mathcal{C} \to \mathcal{D}$ is ramified only over the sections $\tau_1, \ldots, \tau_b$ and possibly over the nodes d_i of $D = \mathcal{D}_t$. To see that $\mathcal{C} \to S$ is nodal, observe that étale locally around each $d_i \in \mathcal{D}$, $\mathcal{C} \to \mathcal{D}$ has the form of a finite morphism $U \to V := \mathrm{Spec}\,\Bbbk[x,y]/(xy - \pi^k)$ where $k \leq d$ and π is a local coordinate on an étale neighborhood of $t \in T$. The map $U \to V$ is finite étale over $V \setminus 0$. Since $\mathrm{Spec}\,\Bbbk[x', y']/(x'^k y'^k - \pi^k) \setminus 0 \to V \setminus 0$, defined by $(x', y') \mapsto (x^k, y^k)$, is the universal cover, we see that the preimage of $V \setminus 0$ in U must be isomorphic to $\mathrm{Spec}\,\Bbbk[x'', y'']/(x''^l y''^l - \pi^k) \setminus 0$ for some integer l dividing k. Thus, $U = \mathrm{Spec}\,\Bbbk[x'', y'']/(x''^l y''^l - \pi^k)$, and we see that the reduced scheme structure of the fiber U_t, defined by $\pi = 0$, is nodal. It follows that $(\mathcal{C}_t)_{\mathrm{red}}$ is nodal. By the same argument as in Step 4 of the proof of Stable Reduction (5.5.1), there is ramified cover $T' \to T$ such that the central fiber of the normalization of $\widetilde{\mathcal{C}'}$ of $\mathcal{C}' = \mathcal{C} \times_T T'$ is *reduced* and nodal. After replacing $\mathcal{C}$ with the normalization of $\widetilde{\mathcal{C}'}$ and T with T', we have arranged that $\mathcal{C} \to T$ is a family of nodal curves.
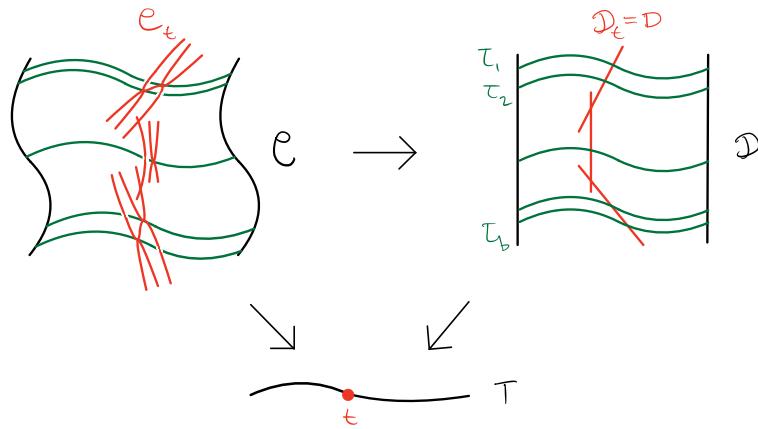
Figure 5.28: Picture of $\mathcal{C} \to \mathcal{D} \to T$.

The family $\mathcal{C} \to \mathcal{D}$ may not be stable, but using the Stable Contraction of a Prestable Family (Theorem 5.6.4), we can contract rational tails and bridges to

obtain a stable family $\mathcal{C}^{\mathrm{st}} \to T$. To show that $\mathcal{C}_0^{\mathrm{st}}$ is singular, it suffices to show that every smooth irreducible component of $\mathcal{C}_0$ has genus 0. Letting $X \subset \mathcal{C}_0$ be such a component, the image of X under $\mathcal{C}_0 \to \mathcal{D}_0 = D$ is one of the $\mathbb{P}^1$'s. Let d' be the degree of the induced map $X \to \mathbb{P}^1$. We know that $X \to \mathbb{P}^1$ is ramified over the marked points of the $\mathbb{P}^1$ and possibly ramified over the nodes of D contained in the $\mathbb{P}^1$ with ramification index at most $d' - 1$. If the $\mathbb{P}^1$ is at the end of the chain in D, it contains two marked points and one node, and Riemann–Hurwitz (5.7.3) implies that $2g_X - 2 \le -2d' + 2 + (d' - 1)$, which implies that $g_X = 0$. If the $\mathbb{P}^1$ is in the middle, it contains one marked point and two nodes, and Riemann–Hurwitz implies that $2g_X - 2 \le -2d' + 1 + 2(d' - 1)$, which also implies that $g_X = 0$. $\qquad\square$

Theorem 5.7.23 (Irreducibility of $\overline{\mathcal{M}}_{g,n}$)**.** *Let g and n be nonnegative integers satisfying $2g - 2 + n > 0$. For any field $\Bbbk$ of characteristic 0, $\overline{\mathcal{M}}_{g,n}$ is irreducible and contains $\mathcal{M}_{g,n}$ as a nonempty dense open substack.*

Proof. We may assume that $\Bbbk$ is algebraically closed. There exists smooth curves of every genus (Exercise 5.1.6). As $\mathbb{P}^1$ with $n \ge 3$ distinct points is stable and a smooth, connected, and projective genus 1 curve with $n \ge 1$ distinct points is stable, $\mathcal{M}_{g,n}$ is nonempty as long as $2g - 2 + n > 0$. As $\overline{\mathcal{M}}_{g,n}$ is smooth (Theorem 5.4.14), it suffices to show that $\overline{\mathcal{M}}_{g,n}$ is connected. As $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.8), the connectedness of $\overline{\mathcal{M}}_{g,n}$ implies the connectedness of $\overline{\mathcal{M}}_{g,n'}$ for $n' > n$. We already know that $\overline{\mathcal{M}}_{0,n}$ is connected for $n \ge 3$ (Proposition 5.7.19). We also know that $\overline{\mathcal{M}}_{1,1}$ is isomorphic to $[(\mathbb{A}^2 \setminus 0)/\mathbb{G}_m]$ and thus is connected Exercise 3.1.19(c), and it follows that $\overline{\mathcal{M}}_{1,n}$ is connected for $n \ge 1$.

We are thus reduced to show that $\overline{\mathcal{M}}_g$ is connected for $g \ge 2$, and by induction we can assume that $\overline{\mathcal{M}}_{g'}$ is connected for $g' < g$. Since every smooth curve degenerates to a singular stable curve in the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ (Proposition 5.7.21), we are further reduced to showing that the boundary δ is connected. We write $\delta = \delta_0 \cup \cdots \cup \delta_{\lfloor g/2 \rfloor}$ where $\delta_0 = \mathrm{im}(\overline{\mathcal{M}}_{g-1,2} \to \overline{\mathcal{M}}_g)$ and $\delta_i = \mathrm{im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \to \overline{\mathcal{M}}_g)$, using the Gluing Morphisms (5.6.11). By the inductive hypotheses, $\overline{\mathcal{M}}_{g-1,2}$ and $\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1}$ are connected, and thus so is each δ_i. But on the other hand, the boundary divisors δ_i intersect! Namely, for every $i, j = 0, \ldots, \lfloor g/2 \rfloor$, the intersection $\delta_i \cap \delta_j$ contains curves as in Figure 5.29. See also Fulton's appendix [HM82]. $\qquad\square$
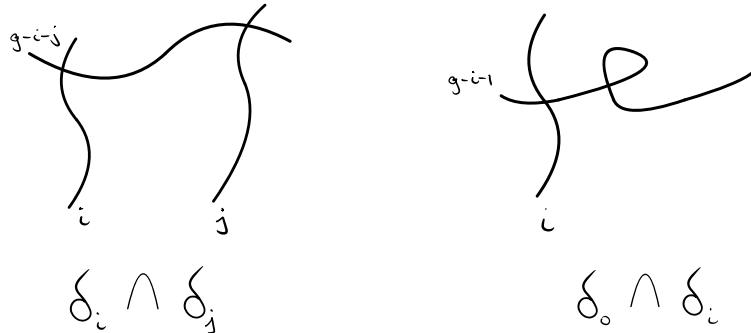
Figure 5.29: The boundary divisors δ_i and δ_j have a nonempty intersection.

Remark 5.7.24 (Admissible Covers)**.** The above argument was motivated by the theory of admissible covers introduced by Harris and Mumford [HM82] to compute the Kodaira dimension of $\overline{M}_g$. See also the exposition in [HM98, §3.G]. Admissible

covers are a generalization of simply branched covers $C \to \mathbb{P}^1$ where the source *and* target curves may have nodal singularities. The inspiration behind admissible covers is to define a moduli stack $\overline{\mathcal{H}\mathrm{ur}}_{g,b}$ fitting into the diagram

$$
\begin{array}{ccccc}
\overline{\mathcal{H}\mathrm{ur}}_{g,b} & & & & \mathrm{Hur}_{g,b} \\
\swarrow \qquad \searrow & & \text{compactifying} & & \swarrow \qquad \searrow \\
\overline{\mathcal{M}}_g \qquad\qquad \overline{M}_{0,b} & & & & \mathcal{M}_g \qquad\qquad M_{0,b}.
\end{array}
$$

An *admissible cover of genus g* is a surjective finite morphism $f\colon C \to B$ from a prestable genus g curve to a stable b-pointed genus 0 curve $(B, p_1, \ldots, p_b)$ such that

(a) the preimage of the smooth locus B^{sm} under f is the smooth locus C^{sm} and the induced morphism $C^{\mathrm{sm}} \to B^{\mathrm{sm}}$ is simply branched of degree d over the points p_i, i.e. each ramification index is 2 and there is at most one ramification point in every fiber; and

(b) for every node $q \in B$ and every node $r \in C$ over q, the étale local structure of $C \to B$ at r is of the form $\mathbb{k}[x,y]/(xy) \to \mathbb{k}[x,y]/(xy)$ defined by $(x,y) \mapsto (x^m, y^m)$ for some m, i.e. the two branches have equal ramification indices at the node.

This definition extends to families resulting in a proper Deligne–Mumford stack $\overline{\mathcal{H}\mathrm{ur}}_{g,b}$. (Note that admissible covers may have non-trivial automorphism groups unlike simply branched coverings, and consequently $\overline{\mathcal{H}\mathrm{ur}}_{g,b}$ is not a scheme.) The assignment $(C \to B) \mapsto (B, p_i)$ defines a morphism $\overline{\mathcal{H}\mathrm{ur}}_{g,b} \to \overline{M}_{0,b}$. On the other hand, the assignment $(C \to B) \mapsto C^{\mathrm{st}}$ defines $\overline{\mathcal{H}\mathrm{ur}}_{g,b} \to \mathcal{M}_g^{\mathrm{pre}}$, noting that the source curve C of an admissible cover $C \to B$ need not be stable. Composing with the Stable Contraction Morphism (5.6.5) $\mathcal{M}^{\mathrm{pre}} \to \overline{\mathcal{M}}_g$ gives a morphism $\overline{\mathcal{H}\mathrm{ur}}_{g,b} \to \overline{\mathcal{M}}_g$.

The argument degenerating a smooth curve to a singular stable curve (Proposition 5.7.21) can be rewritten in this language. For $d \gg 0$, given a smooth curve $C \in \mathcal{M}_g$, we choose a preimage $(C \to \mathbb{P}^1) \in \mathrm{Hur}_{g,b}$ (Lemma 5.7.13). Let $H \subset \mathrm{Hur}_{g,b}$ be the connected component containing $(C \to \mathbb{P}^1)$, and let $\overline{\mathcal{H}} \subset \overline{\mathcal{H}\mathrm{ur}}_{g,b}$ be its closure. Since $\mathrm{Hur}_{g,b} \to M_{0,b}$ is finite and étale (Exercise 5.7.11(c)), $\overline{\mathcal{H}}$ surjects onto $\overline{M}_{0,b}$. Choose a commutative diagram

$$
\begin{array}{ccc}
T \setminus \{t\} & \longrightarrow & \overline{\mathcal{H}} \\
\downarrow & \nearrow & \downarrow {\scriptstyle \text{proper}} \\
T & \xrightarrow[t \mapsto (D, d_i)]{} & \overline{M}_{0,b},
\end{array}
$$

as in the proof of Proposition 5.7.21, where T is a connected smooth curve, $t \in T(\mathbb{k})$ maps to the stable curve $(D, d_i) \in \overline{M}_{0,b}$ of Figure 5.27, and there is some $t' \neq t \in T(\mathbb{k})$ mapping to a simply branched covering $(C' \to \mathbb{P}^1) \in H$. By the valuative criterion of properness, after replacing T with a ramified cover, there exists a lift $T \to \overline{\mathcal{H}}$ where t maps to an admissible cover $(C'' \to D)$. One shows that every smooth irreducible component of C'' has genus 0 as in the proof of Proposition 5.7.21. Thus C' degenerates to the singular stable curve $(C'')^{\mathrm{st}}$, and since $(C \to \mathbb{P}^1)$ and $(C' \to \mathbb{P}^1)$ are in the same connected component in $\mathrm{Hur}_{g,b}$, the original smooth curve C also degenerates to $(C'')^{\mathrm{st}}$. (Note that the argument constructing the family $\mathcal{C} \to \mathcal{D} \to T$ of covers in the proof of Proposition 5.7.21 is one of the essential ingredients in the proof of the properness of $\overline{\mathcal{H}\mathrm{ur}}_{g,b}$.)

5.7.5 Irreducibility in positive characteristic

We prove that $\overline{\mathcal{M}}_g$ is irreducible in positive characteristic following the historical arguments of Deligne–Mumford [DM69] and Fulton [Ful69]. A central ingredient in each argument is Zariski's Connectedness Theorem (4.5.11): for a flat, proper morphism of noetherian Deligne–Mumford stacks, the number of geometrically connected components of a fiber is lower semicontinuous. This connectedness theorem, stated but unproven in [DM69, Thm. 4.17], requires a surprisingly large amount of the theory of Deligne–Mumford stacks.

Theorem 5.7.25 (Irreducibility of $\overline{\mathcal{M}}_{g,n}$). *For g and n satisfying $2g - 2 + n > 0$, $\overline{\mathcal{M}}_{g,n}$ is irreducible over any field.*

Proof. As we already know that $\overline{\mathcal{M}}_{g,n}$ is irreducible in characteristic 0 by Theorem 5.7.16 (transcendental proof) or Theorem 5.7.23 (algebraic proof), it suffices to show irreducibility over $\overline{\mathbb{F}}_p$. The morphism $\overline{\mathcal{M}}_{g,n} \to \operatorname{Spec} \mathbb{Z}$ is smooth (Theorem 5.4.14) and proper (Theorem 5.5.16), and thus Zariski's Connectedness Theorem (4.5.11) implies that all geometric fibers have the same number of connected components. As $\overline{\mathcal{M}}_{g,n}$ is connected over $\overline{\mathbb{Q}}$, it is also connected, hence irreducible, over $\overline{\mathbb{F}}_p$. $\qquad\square$

Remark 5.7.26 (Deligne and Mumford's proofs). The above proof was the *second* argument given by Deligne and Mumford [DM69, §5]. Keep in mind that this is the same paper that introduced stacks (now called Deligne–Mumford stacks), introduced stable curves, and proved Stable Reduction (5.5.1). It was quite a remarkable paper!

Their first irreducibility argument was very similar in spirit to the proof of Theorem 5.7.23 using the degeneration of a smooth curve to a singular stable curve and the inductive nature of the boundary, but their proof still relied on the topological irreducibility of $\mathcal{M}_g$ over $\mathbb{C}$. For historical completeness, we sketch the argument in [DM69, §3] for the irreducibility of $\overline{\mathcal{M}}_g$ over any field $\Bbbk$. At the time, it was known from [GIT] that there exists a quasi-projective coarse moduli space $\mathcal{M}_g \to M_g$ over $\operatorname{Spec} \mathbb{Z}$.

Step 1: No connected component of $M_g \times_{\mathbb{Z}} \Bbbk$ is proper over $\Bbbk$. Let $W(\Bbbk)$ be the Witt vectors for $\Bbbk$; this is a noetherian complete local ring whose generic point η has characteristic 0 and whose closed point 0 has residue field $\Bbbk$. (For example, $W(\mathbb{F}_p) = \mathbb{Z}_p$ is the ring of p-adics.) Choose a projective variety X over $W(\Bbbk)$ containing $M_g \times_{\mathbb{Z}} W(\Bbbk)$ as a dense open subscheme. By the characteristic 0 result, we know that the generic fiber X_η is connected. Since X is flat and proper over $W(\Bbbk)$, Zariski's Connectedness Theorem (4.5.11) implies that the special fiber X_0 is also connected.

Suppose $Y \subset M_g \times_{\mathbb{Z}} \Bbbk$ is a connected component proper over $\Bbbk$. Then Y is an open subscheme of X_0 but also a closed subscheme since Y is proper. Since X_0 is connected, we conclude that Y must be all of $M_g \times_{\mathbb{Z}} \Bbbk$, hence $M_g \times_{\mathbb{Z}} \Bbbk$ is proper and irreducible. To obtain a contradiction, denote by $A_{g,\Bbbk}$ the moduli space of principally polarized g-dimensional abelian varieties over $\Bbbk$ and consider the morphism

$$\Theta \colon M_g \times_{\mathbb{Z}} \Bbbk \to A_{g,\Bbbk}, \qquad C \mapsto \operatorname{Jac}(C)$$

assigning to a smooth curve C its Jacobian $\operatorname{Jac}(C)$. The properness of $M_g \times_{\mathbb{Z}} \Bbbk$ implies that the image is closed, but it was well known at the time that the closure of the image of Θ contains products of lower dimensional Jacobians.

Step 2: There is no connected component of $\overline{M}_g \times_{\mathbb{Z}} \Bbbk$ consisting entirely of smooth curves. Let $\overline{M}_{g,1}, \ldots, \overline{M}_{g,r}$ be the connected components of $\overline{M}_g \times_{\mathbb{Z}} \Bbbk$. Step 1 implies

that $M_{g,i} := \overline{M}_{g,i} \cap (M_g \times_{\mathbb{Z}} \Bbbk)$ is not proper for each i. Therefore, there is a morphism $\Delta^* = \operatorname{Spec} \Bbbk((t)) \to M_{g,i}$ that does not extend to Δ. By Stable Reduction (5.5.1), after possibly replacing Δ with a ramified extension, $\Delta^* \to M_{g,i}$ extends to a morphism $\Delta \to \overline{M}_g$. This shows that $\overline{M}_{g,i} \setminus M_{g,i}$ is non-empty.

Step 3: The boundary $\delta \times_{\mathbb{Z}} \Bbbk$ is connected, where $\delta = \overline{M}_g \setminus M_g$. Step 2 shows that every smooth curve degenerates to a singular stable curve (giving a proof of Proposition 5.7.21 in positive characteristic). Therefore, the connectedness of $\overline{M}_g \times_{\mathbb{Z}} \Bbbk$ follows from the connectedness of $\delta \times_{\mathbb{Z}} \Bbbk$. It suffices to show the connectedness of $\delta_i \times_{\mathbb{Z}} \Bbbk$ for each i and the nonemptyness of the intersection $(\delta_i \cap \delta_j) \times_{\mathbb{Z}} \Bbbk$ for each i and j. This is precisely what we showed in the proof of Theorem 5.7.23 using induction on the genera, the Gluing Morphisms (5.6.11), and the curves in Figure 5.29 in $(\delta_i \cap \delta_j) \times_{\mathbb{Z}} \Bbbk$. Deligne and Mumford gave essentially the same argument using ad hoc techniques rather than the formalism of the moduli space $\overline{M}_{g,n}$ of n-pointed stable curves and the Gluing Morphisms (5.6.11) (these were only introduced later by Knudsen).

Interestingly, neither of Deligne and Mumford's proofs relies essentially on the theory of the stacks. Their second proof (see the proof of Theorem 5.7.25) is valid as long as one had the existence of a proper and flat coarse moduli space $\overline{M}_{g,n} \to \operatorname{Spec} \mathbb{Z}$. However, both proofs rely fundamentally on the compactification $\overline{\mathcal{M}}_g$ and Stable Reduction (5.5.1).

Remark 5.7.27 (Fulton's proof). In [Ful69], Fulton studied the variant of the Hurwitz moduli space $\operatorname{Hur}_{g,b}^{\mathrm{rig,unord}}$ parameterizing families $\mathcal{C} \to \mathbb{P}_S^1$ of simply branched coverings of degree d over b unordered points (see Remark 5.7.10), and showed that the diagram

$$
\begin{array}{ccc}
 & \operatorname{Hur}_{g,b}^{\mathrm{rig,unord}} & \\
 \swarrow & & \searrow{\scriptstyle \text{ét}} \\
 \mathcal{M}_g & & \operatorname{Sym}^b \mathbb{P}^1 \setminus D = \mathbb{P}^b \setminus D
\end{array}
$$

is defined over $\mathbb{Z}$ such that $\operatorname{Hur}_{g,b}^{\mathrm{rig,unord}} \to \mathcal{M}_g$ surjective if $d \geq g+1$ (which is a better bound than we proved in Corollary 5.7.15) and that $\operatorname{Hur}_{g,b}^{\mathrm{rig,unord}} \to M_{0,n}$ is finite étale over $\mathbb{Z}_{d!}$, i.e. after inverting all primes $p \leq d$. Fulton then establishes a "reduction theorem": if X is a smooth and projective scheme over a complete DVR R with algebraically closed residue field $\Bbbk$ and characteristic 0 fraction field K and $D \subset X$ is relative divisor over R which is *simple* (i.e. each fiber of D has no multiple components), and if $Y \to X \setminus D$ is a finite étale covering such that $Y_{\overline{K}}$ is irreducible, then $Y_{\Bbbk}$ is also irreducible. Using the irreducibility of the Hurwitz moduli space in characteristic 0, Fulton's reduction theorem applied to $\operatorname{Hur}_{g,b}^{\mathrm{rig,unord}} \to M_{0,n}$ gives the irreducibility of $\operatorname{Hur}_{g,b}^{\mathrm{rig,unord}}$ over fields of characteristic $p > d$. Taking $d = g+1$ gives the irreducibility of $\mathcal{M}_g$ over fields of characteristic $p > g+1$.

5.8   Projectivity

We finally prove that the coarse moduli space $\overline{M}_{g,n}$ is projective (Theorem 5.8.18), completing the proof of Theorem A. We follow the approach introduced by Kollár in [Kol90] building on ideas of Viehweg [Vie95].

To introduce the general strategy for projectivity, we need some terminology. Let $\pi \colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ be the universal family. For each integer $k \geq 1$, define the *kth pluricanonical bundle* as

$$\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$$

on $\overline{\mathcal{M}}_g$, which by Proposition 5.3.21 is a vector bundle of rank g if $k = 1$ or rank $(2k - 1)(g - 1)$ if $k > 1$. Their determinants

$$\lambda_k := \det \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$$

are line bundles on $\overline{\mathcal{M}}_g$. For $k = 1$, this is the *Hodge line bundle* from Definition 5.6.18.

> **Strategy for projectivity:** Show that for $k \gg 0$, a positive power of λ_k descends to an *ample* line bundle on the coarse moduli space $\overline{M}_g$.

Outline of this section: After discussing positivity and ampleness criteria for algebraic spaces (§5.8.1), we prove Kollar's Criterion for ampleness (Theorem 5.8.7), which we use to prove the projectivity of $\overline{M}_g$. This requires establishing the nefness of the pluricanonical bundles (Theorem 5.8.17). Finally, in §5.8.4, we sketch an alternative projectivity argument using GIT.

5.8.1   Positivity and ampleness criteria

After extending notions of positivity from §B.6.2 to algebraic spaces, we extend the Nakai–Moishezon Criterion to algebraic spaces (Theorem 5.8.4), which will be used in the proof of Kollár's Criterion (5.8.7).

Definition 5.8.1. Let X be a proper algebraic space over a field $\Bbbk$. We say that a line bundle L is:

(1) *ample* if X is a scheme and L is ample in the usual sense;

(2) *nef* if for every integral closed curve $C \subset X$, $c_1(L) \cdot C \geq 0$;

(3) *base point free* if for every $x \in X$, there exists $s \in \Gamma(X, L)$ with $s(x) \neq 0$; and

(4) *big* if there is a constant C such that $\mathrm{h}^0(Z, L|_Z^{\otimes m}) \geq C m^{\dim(Z)}$.

We also say that a vector bundle E on X is *nef* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E)$.

If $f \colon X' \to X$ is a finite surjective map of algebraic spaces, then a line bundle L on X is ample if and only if f^*L is (Exercise 4.4.22). The following generalizes analogous properties of nefness and bigness (Proposition B.6.14(1) and Proposition B.6.21(1)) to algebraic spaces.

Exercise 5.8.2. Let X be a proper algebraic space over a field $\Bbbk$ and L be a line bundle.

(a) Show that if $f \colon X' \to X$ is a surjective proper morphism of algebraic spaces, then L is nef if and only if f^*L is.

(b) Show that if $f \colon X' \to X$ is a surjective, generically quasi-finite, and proper morphism of algebraic spaces, then L is big if and only if f^*L is big.

The above exercises allow us to extend the Characterizations of Nefness and Bigness (B.6.13, B.6.20, B.6.23, and B.6.34) to algebraic spaces. We will, however, only need the following property.

(c) If E is a nef vector bundle on X, then $\det Q$ is nef.

We now extend the ampleness criteria of §B.6.3 to algebraic spaces and Deligne–Mumford stacks. The Nakai–Moishezon Criterion (5.8.4) will be used in Kollár's Criterion . We begin with a conceptually simple criterion generalizing Lemma B.6.25, which is useful to keep in mind, even if we don't apply it.

Lemma 5.8.3. *Let $\mathcal{X}$ be a proper Deligne-Mumford stack over a field $\Bbbk$ with coarse moduli space $\mathcal{X} \to X$. Suppose that L is a line bundle on $\mathcal{X}$ satisfying*

(a) L is semiample (i.e. $L^{\otimes N}$ is base point free for some $N > 0$), and

(b) for every map $f\colon T \to \mathcal{X}$ from an integral proper curve over $\Bbbk$ such that $f(T) \subset |\mathcal{X}|$ is not a single point, $\deg f^*L > 0$.

Then $L^{\otimes N}$ descends to an ample line bundle on X for some $N > 0$, and X is projective.

Proof. For N sufficiently divisible, $L^{\otimes N}$ descends to a line bundle M on X (Proposition 4.3.31) and there is commutative diagram

$$
\begin{array}{ccc}
\mathcal{X} & & \\
\downarrow & \searrow & \\
X & \longrightarrow & \mathbb{P}(H^0(\mathcal{X}, L^N)).
\end{array}
$$

Property (a) ensures that $\mathcal{X} \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is well-defined and the existence of $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ follows from the universal property of coarse moduli spaces. Property (b) implies that $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ doesn't contract curves. Thus $X \to \mathbb{P}(H^0(\mathcal{X}, L^N))$ is quasi-finite and proper, hence finite. As M is the pullback of $\mathcal{O}(1)$, M is ample. $\qquad\square$

The semiampleness condition in (a) can be very challenging to verify in practice. In the GIT approach, semiampleness is hard-coded into the definition of semistability (Definition 6.7.5). If G is a reductive group acting linearly on projective space $\mathbb{P}(V)$ and $L = \mathcal{O}(1)$ is the corresponding line bundle on $[\mathbb{P}(V)/G]$, then $v \neq 0 \in V$ is in the stable base locus of L (i.e. $s(v) = 0$ for all $s \in \Gamma([\mathbb{P}(V)/G], L^{\otimes d})$ and $d > 0$) if and only if $0 \in \overline{Gv} \subset V$, which by the Hilbert–Mumford Criterion (6.7.15) holds if and only if there exists a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot v = 0$. This latter condition can sometimes be verified combinatorially.

In the Nakai–Moishezon Criterion (B.6.27) for a scheme X, the semiampleness of a line bundle $L = \mathcal{O}_X(D)$ is obtained from the nefness of L and the bigness of the restriction $L|_Z$ to any integral closed subspace Z. The Nakai–Moishezon Criterion holds for algebraic spaces.

Theorem 5.8.4 (Nakai–Moishezon Criterion)**.** *If X is a proper algebraic space, a line bundle L is ample if and only if for all integral closed subspaces $Z \subset X$, $c_1(L)^{\dim Z} \cdot Z > 0$.*

Remark 5.8.5. Equivalently, L is ample if and only if L is nef and for every integral closed subspace $Z \subset X$, $L|_Z$ is big.

Proof. By Le Lemme de Gabber (4.5.1), there exists a finite surjection $f\colon X' \to X$ from a scheme X, and L is ample if and only if f^*L is ample (Exercise 4.4.22). By the projection formula, $c_1(f^*L)^{\dim Z'} \cdot Z' > 0$ for all integral closed subspace $Z' \subset X'$. The statement thus follows from the case for schemes (Theorem B.6.27). $\qquad\square$

Seshadri's Criterion (B.6.31) holds for proper algebraic spaces [Cor93], while Kleiman's Criterion (B.6.29) is unknown in general for proper algebraic spaces (and even proper schemes).

5.8.2 Kollár's Criterion

We prove Kollár's Criterion [Kol90, Lem. 3.9] for projectivity with improvements from [KP17, Thm. 4.1]. See also the exposition in [CLM22]. Let X be a proper algebraic space over a field $\Bbbk$. Let $W \to Q$ be a surjection of vector bundles of rank w and q. In our application to $\overline{\mathcal{M}}_g$, we will use the multiplication map $\mathrm{Sym}^d \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}) \to \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes dk})$ on pluricanonical bundles, where $\mathcal{U}_g \to \overline{\mathcal{M}}_g$ is the universal family.

We will assume that W has a *reduction of structure group of W by $G \to \mathrm{GL}_w$* (Definition B.2.36) where G is a reductive group. This means the following: the frame bundle $\mathrm{Fr}_W \to W$ is the principal GL_w-bundle corresponding to W under the bijection between vector bundles and principal GL-bundles (see Exercise B.2.27) and we require that there exists a principal G-bundle $F \to T$ and an isomorphism $F \times^G \mathrm{GL}_w \xrightarrow{\sim} \mathrm{Fr}_W$ of principal GL_w-bundles. Recall that the frame bundle Fr_W represents $\mathrm{Sch}/X \to \mathrm{Sets}$ assigning $f \colon T \to X$ to the set of trivializations $\mathcal{O}_T^w \xrightarrow{\sim} f^*W$ (Exercise B.2.26). Note in our application to $\overline{\mathcal{M}}_g$, the vector bundle $\mathrm{Sym}^d \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ has a reduction of structure group to GL_v where v is the rank of the vector bundle $\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$. The reader may want to keep this case in mind while reading the proof as it simplifies several of the details (in addition to being one of the hypotheses in characteristic p!).

The *classifying map* is the morphism

$$\begin{aligned} X &\to [\mathrm{Gr}(q, \Bbbk^w)/G] \\ x &\mapsto [W \otimes \kappa(x) \twoheadrightarrow Q \otimes \kappa(x)]. \end{aligned}$$

It is well-defined because a choice of isomorphism $W \otimes \kappa(x) \cong \kappa(x)^w$ of the fiber of W over x is well-defined up to the structure group G. The above definition extends to functorial maps $X(S) \to [\mathrm{Gr}(q, \Bbbk^w)/G](S)$ for $\Bbbk$-schemes S. Alternatively, we may construct the classifying map using the GL_w-equivariant morphism $\mathrm{Fr}_W \to \mathrm{Gr}(q, \Bbbk^w)$, where over $f \colon T \to W$, a trivialization $\mathcal{O}_T^w \xrightarrow{\sim} f^*W$ maps to the quotient $\mathcal{O}_T^w \xrightarrow{\sim} f^*W \twoheadrightarrow f^*Q$. The principal G-bundle $F \to X$ corresponding to the reduction of structure group of W to G and the G-equivariant map

$$F \xrightarrow{f \mapsto (f, \mathrm{id})} F \times^G \mathrm{GL}_w \xrightarrow{\sim} \mathrm{Fr}_W \to \mathrm{Gr}(q, \Bbbk^w)$$

defines a map $X \to [\mathrm{Gr}(q, \Bbbk^w)/G]$ and a cartesian diagram

$$\begin{CD} F @>>> \mathrm{Gr}(q, \Bbbk^w) \\ @VVV @VVV \\ X @>>> [\mathrm{Gr}(q, \Bbbk^w)/G]. \end{CD} \qquad (5.8.6)$$

Kollár's Criterion deduces the projectivity of X from the projectivity of $\mathrm{Gr}(q, \Bbbk^w)$ and is a generalization of the fact a proper scheme that is quasi-finite over $\mathrm{Gr}(q, \Bbbk^w)$ is projective. Indeed, when W is the trivial vector bundle, there is a reduction of structure group to the trivial group $G = \{1\}$ and the quasi-finiteness of the classifying map $X \to \mathrm{Gr}(q, \Bbbk^w)$ implies that it is finite, and hence the pullback $\det(Q)$ of the Plücker line bundle is ample.

Theorem 5.8.7 (Kollár's Criterion). *Let X be a proper algebraic space over a field $\Bbbk$. Let $W \twoheadrightarrow Q$ be a surjection of vector bundles of rank w and q. Assume that W has a reduction of structure group by a homomorphism $G \to \mathrm{GL}_w$ from a reductive group G. Suppose that*

(a) *W has a reduction of structure group by a homomorphism $G \to \mathrm{GL}_w$ such that the closure of the image of G in the projective space $\mathbb{P}(\mathrm{Mat}_{w,w})$ of $w \times w$ matrices is normal.*

(b) *W is nef,*

(c) *the classifying map $X(\overline{\mathbb{k}}) \to \mathrm{Gr}(q, \mathbb{k}^w)(\overline{\mathbb{k}})/G(\overline{\mathbb{k}})$ has finite fibers, and*

(d) *either G is linearly reductive (e.g. $\mathrm{char}(\mathbb{k}) = 0$ and G is reductive), or $G = \mathrm{GL}_v$ and $W = \mathrm{Sym}^d V$ for a nef vector bundle V of rank v.*

Then $\det Q$ is ample.

Remark 5.8.8 (Unwinding quasi-finiteness)**.** Condition (c) is equivalent to the map $|X| \to |[\mathrm{Gr}(q, \mathbb{k}^w)/G]|$ on topological spaces having finite fibers. It is weaker than the quasi-finiteness[2] of $X \to [\mathrm{Gr}(q, \mathbb{k}^w)/G]$, which also requires that for every $x \in X(\overline{\mathbb{k}})$ only finitely many elements of $G(\overline{\mathbb{k}})$ leave $\ker(W \otimes \overline{\mathbb{k}} \to Q \otimes \overline{\mathbb{k}})$ invariant. The original statement [Kol90, Lem. 3.9] assumed the quasi-finiteness of $X \to [\mathrm{Gr}(q, \mathbb{k}^w)/G]$, and this was weakened in [KP17, Thm. 5.1] to condition (c). Letting $\Gamma \colon X \to X \times [\mathrm{Gr}(q, \mathbb{k}^w)/G]$ be the graph of the classifying map, condition (c) is equivalent to the quasi-finiteness of $\mathrm{im}(\Gamma) \to [\mathrm{Gr}(q, \mathbb{k}^w)/G]$, which is what we use in the proof.

Remark 5.8.9 (Hypotheses (a) and thm:kollars-criterion-char)**.** In our application to $\overline{M}_g$, W is the symmetric product $\mathrm{Sym}^d V$ of a vector bundle V of rank v, and thus W has a reduction of structure group by $\mathrm{GL}_v \to \mathrm{GL}_w$. In this case, the closure of GL_v in $\mathbb{P}(\mathrm{Mat}_{w \times w})$ is identified with $\mathbb{P}(\mathrm{Mat}_{v,v})$ and is hence normal. This shows that hypothesis (a) holds. In other applications, (a) can be verified by appealing to results on the normality of orbit closures; see [Tim03, DC04, BGMR11]. Hypothesis (d) in positive characteristic can be relaxed to assuming that $G = \mathrm{GL}_{v_1} \times \cdots \times \mathrm{GL}_{v_d}$ with $G \to \mathrm{GL}_w$ a semipositive representation (i.e. it extends to a map $\mathrm{Mat}_{v_1,v_1} \times \cdots \times \mathrm{Mat}_{v_d,v_d} \to \mathrm{Mat}_{w,w}$); see [Kol90, Condition Δ in Prop. 3.6].

Remark 5.8.10 (Stability)**.** The above theorem does not require that the image of X lands in the G-stable locus of $\mathrm{Gr}(q, \mathbb{k}^w)$. However, if this happens, then the composition $X \to [\mathrm{Gr}(q, \mathbb{k}^w)^{\mathrm{s}}/G] \to \mathrm{Gr}(q, \mathbb{k}^w)/\!\!/G$ is a quasi-finite morphism of proper algebraic spaces. It is hence finite and $\det Q$ is ample is it is the pullback of an ample line bundle on $\mathrm{Gr}(q, \mathbb{k}^w)/\!\!/G$.

The proof will proceed by reducing the ampleness of $\det(Q)$ to its bigness, which we establish in Proposition 5.8.11.

Proof of Theorem 5.8.7. First note that the nefness of W implies the nefness of Q, which in turn implies the nefness of $\det(Q)$ by Exercise 5.8.2(c). Therefore, by the Nakai–Moishezon Criterion (5.8.4), it suffices to verify that $\det(Q)|_Z$ is big for each integral subspace $Z \subset X$. Since the hypotheses also hold for Z and the restrictions $W|_Z \twoheadrightarrow Q|_Z$, it suffices to verify that if X is an integral algebraic space with $W \twoheadrightarrow Q$ satisfying (a)–(d), then $\det(Q)$ is big. By Le Lemme de Gabber (4.5.2), there exists a projective, generically quasi-finite, and surjective morphism $f \colon Y \to X$ from a projective integral scheme. By taking the normalization, we can assume that Y is normal. As $\det Q$ is big if and only if $f^*(\det Q)$ is big (Exercise 5.8.2(b)), the theorem therefore follows from the bigness of $f^*(\det Q)$, which is the conclusion of the following proposition. $\qquad\square$

---

[2]A finite type morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *quasi-finite* if $|\mathcal{X}| \to |\mathcal{Y}|$ has finite fibers and for every field-valued point $x \in \mathcal{X}(\mathbb{k})$, the map $\mathrm{Aut}_{\mathcal{X}(\mathbb{k})}(x) \to \mathrm{Aut}_{\mathcal{Y}(\mathbb{k})}(f(x))$ has finite cokernel; see Definition 3.3.32.

Assuming X is normal and projective, we establish the weaker conclusion that $\det Q$ is big under the weaker hypothesis that the classifying map to have *generically* finite fibers. When W is trivial with a reduction of structure group to $G = \{1\}$, this follows from the invariance of bigness under a generically quasi-finite morphism of proper schemes (Proposition B.6.21). The main idea of the proof is to apply the invariance of bigness to a morphism from a blowup of a compactification of the projectivized frame bundle $\mathbb{P}\mathrm{Fr}_W$ to $\mathrm{Gr}(q, \mathbb{k}^w)$.

Proposition 5.8.11. *Let X be a normal projective integral scheme over a field $\mathbb{k}$. Let $W \twoheadrightarrow Q$ be a surjection of vector bundles of rank w and q. Suppose that*

(a) *W has a reduction of structure group by a homomorphism $G \to \mathrm{GL}_w$ such that the closure of the image of G in the projective space $\mathbb{P}(\mathrm{Mat}_{w,w})$ of $w \times w$ matrices is normal.*

(b) *W is nef, and*

(c′) *the classifying map $X(\overline{\mathbb{k}}) \to \mathrm{Gr}(q, \mathbb{k}^w)(\overline{\mathbb{k}})/G(\overline{\mathbb{k}})$ generically has finite fibers.*

(d) *either G is linearly reductive (e.g. $\mathrm{char}(\mathbb{k}) = 0$ and G is reductive), or $G = \mathrm{GL}_v$ and $W = \mathrm{Sym}^d V$ for a nef vector bundle V of rank v.*

Then $\det Q$ is big.

Proof. The projectivized frame bundle (Exercise B.2.26) is a principal PGL_w-bundle $\mathbb{P}\mathrm{Fr}_W \to X$, where $\mathbb{P}\mathrm{Fr}_W \subset \mathbb{P} := \mathbb{P}(\mathscr{H}om_{\mathcal{O}_X}(W, \mathcal{O}_X^{\oplus w}))$ is the open subscheme defined by the non-vanishing of the determinant of the natural map

$$\mathcal{O}_{\mathbb{P}}^{\oplus w} \to W_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(1),$$

induced from the universal quotient $\mathscr{H}om_{\mathcal{O}_{\mathbb{P}}}(W_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}^{\oplus w}) \to \mathcal{O}_{\mathbb{P}}(1)$, where $W_{\mathbb{P}}$ is the pullback of W via $\mathbb{P} \to X$. The composition

$$\mathcal{O}_{\mathbb{P}}^{\oplus w} \to W_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(1) \to Q_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(1) \tag{5.8.12}$$

is surjective over $\mathbb{P}\mathrm{Fr}_W$ and defines a morphism $\mathbb{P}\mathrm{Fr}_W \to \mathrm{Gr}(q, \mathbb{k}^w)$. In summary, we have a commutative diagram

$$
\begin{array}{ccccc}
\mathbb{P}\mathrm{Fr}_W & \hookrightarrow & \mathbb{P} & \dashrightarrow & \mathrm{Gr}(q, \mathbb{k}^w) \\
& & \downarrow & & \downarrow \\
& & X & \longrightarrow & [\mathrm{Gr}(q, \mathbb{k}^w)/\mathrm{PGL}_w].
\end{array}
$$

Step 1: Take a blowup $\mathbb{P}' \to \mathbb{P}$ which is an isomorphism over $\mathbb{P}\mathrm{Fr}_W$ such that $\mathbb{P}\mathrm{Fr}_W \to \mathrm{Gr}(q, \mathbb{k}^w)$ extends to a morphism $\mathbb{P}' \to \mathrm{Gr}(q, \mathbb{k}^w)$. Note that there is no need to blow up if the map (5.8.12) is surjective. Otherwise, taking the qth wedge power of (5.8.12) gives a map $\bigwedge^q \mathcal{O}_{\mathbb{P}}^{\oplus w} \to \det(Q_{\mathbb{P}}) \otimes \mathcal{O}_{\mathbb{P}}(q)$, and we let I be the image subsheaf. Let $\mathbb{P}' \to \mathbb{P}$ be the blowup of the ideal sheaf $I \otimes \det(Q_{\mathbb{P}})^{\vee} \otimes \mathcal{O}_{\mathbb{P}}(-q) \subset \mathcal{O}_{\mathbb{P}}$. Then $\mathbb{P}' \to \mathbb{P}$ is an isomorphism over $\mathbb{P}\mathrm{Fr}_W$ and there is a map $\mathbb{P}' \to \mathrm{Gr}(q, \mathbb{k}^w)$ extending $\mathbb{P}\mathrm{Fr}_W \to \mathrm{Gr}(q, \mathbb{k}^w)$ yielding a commutative diagram

$$
\begin{array}{ccccc}
E & H & & A & \\
| & | & & | & \\
\mathbb{P}' & \longrightarrow & \mathbb{P} & \dashrightarrow & \mathrm{Gr}(q, \mathbb{k}^w) \\
& & \downarrow & & \downarrow \\
& & X & \longrightarrow & [\mathrm{Gr}(q, \mathbb{k}^w)/\mathrm{PGL}_w],
\end{array}
$$

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where $E \subset \mathbb{P}'$ is the exceptional divisor, $H \subset \mathbb{P}$ is the hyperplane class on the projective space $\mathbb{P}$, and A is the ample divisor on $\mathrm{Gr}(q, \Bbbk^w)$ corresponding to the Plücker embedding. Abusing notation, we also view $\det(Q)$ as a divisor on X. By using subscripts to denote the pullback of divisors, we have the formula

$$\det(Q)_{\mathbb{P}'} + qH_{\mathbb{P}'} = A_{\mathbb{P}'} + E. \tag{5.8.13}$$

Step 2: Relate the PGL_w-action on $\mathrm{Gr}(q, \Bbbk^w)$ to the action by the image $PG \subset \mathrm{PGL}_w$ of G. Setting $PG := \mathrm{im}(G \to \mathrm{GL}_w \to \mathrm{PGL}_w) \subset \mathrm{PGL}_w$, the quotient PGL_w / PG is quasi-projective (Proposition B.2.40(7)) and thus separated. This implies that the morphism $[\mathrm{Gr}(q, \Bbbk^w)/PG] \to [\mathrm{Gr}(q, \Bbbk^w)/\mathrm{PGL}_w]$ is separated. We have a large commutative diagram

with cartesian squares, where $\mathbb{Q}^*$ is defined as the cartesian product, $\mathbb{Q}^* \hookrightarrow \mathbb{P}\mathrm{Fr}_W$ is a closed immersion, $\mathbb{Q}$ is defined as the closure of $\mathbb{Q}^*$ in $\mathbb{P}$, and $\mathbb{Q}'$ is defined as $\mathbb{P}' \times_{\mathbb{P}} \mathbb{Q}$. We will utilize the induced morphism $\mathbb{Q}' \to \mathrm{Gr}(q, \Bbbk^w)$. Since X is normal and the fibers of $\mathbb{Q} \to X$ are identified with the closure of the image of G in $\mathbb{P}(\mathrm{Mat}_{w,w})$, which is normal by hypothesis (a), it follows that $\mathbb{Q}$ is normal (see Remark B.1.14).

Step 3: Use the generic quasi-finiteness to show that for any ample divisor D on X, the line bundle $\mathcal{O}_{\mathbb{Q}'}(mA_{\mathbb{Q}'} - D_{\mathbb{Q}'})$ is effective for some $m > 0$. As $G(\overline{\Bbbk}) \to PG(\overline{\Bbbk})$ is surjective, the map $X(\overline{\Bbbk}) \to \mathrm{Gr}(q, \Bbbk^w)(\overline{\Bbbk})/PG(\overline{\Bbbk})$ is also generically quasi-finite. (Note that if $X \to [\mathrm{Gr}(q, \Bbbk^w)/PG]$ is generically quasi-finite, then $\mathbb{Q}' \to \mathrm{Gr}(q, \Bbbk^w)$ is as well. This implies that $A_{\mathbb{P}'}$ is big so Kodaira's Lemma (B.6.19) gives the claim.)

The set-theoretic quasi-finiteness of $X \to [\mathrm{Gr}(q, \Bbbk^w)/PG]$ implies that the projection morphism from $\mathrm{im}(X \to X \times [\mathrm{Gr}(q, \Bbbk^w)/PG])$ to $[\mathrm{Gr}(q, \Bbbk^w)/PG]$ is generically quasi-finite. Therefore, letting $Z := \mathrm{im}(\mathbb{Q}' \to \mathbb{Q}' \times \mathrm{Gr}(q, \Bbbk^w))$, the morphism $Z \to \mathrm{Gr}(q, \Bbbk^w)$ is also generically quasi-finite, and we have a factorization

$$\mathbb{Q}' \to Z \to \mathrm{Gr}(q, \Bbbk^w).$$

It follows that the pullback A_Z is big. Viewing Z as a scheme over X via the composition $Z \hookrightarrow \mathbb{Q}' \times \mathrm{Gr}(q, \Bbbk^w) \to \mathbb{Q}' \to X$, Kodaira's Lemma (B.6.19) implies that $mA_Z - D_Z$ is effective on Z for some $m > 0$. Therefore, the pullback $mA_{\mathbb{Q}'} - D_{\mathbb{Q}'}$ to $\mathbb{Q}'$ is also effective for some $m > 0$.

Step 4: Construct a nonzero map $q_* \mathcal{O}_{\mathbb{Q}}(mq)^\vee \to \mathcal{O}_X(m \det(Q) - D)$ by pushing forward a section of $mA_{\mathbb{Q}'} - D_{\mathbb{Q}'}$ under the composition

$$q' \colon \mathbb{Q}' \xrightarrow{\pi} \mathbb{Q} \xrightarrow{q} X.$$

Restricting the identity (5.8.13) via $\mathbb{Q}' \hookrightarrow \mathbb{P}'$, we obtain an inclusion of line bundles

$$
\begin{aligned}
\mathcal{O}_{\mathbb{Q}'}(mA_{\mathbb{Q}'} - D_{\mathbb{Q}'}) &\cong \mathcal{O}_{\mathbb{Q}'}(m \det(Q)_{\mathbb{Q}'} + mqH_{\mathbb{Q}'} - mE_{\mathbb{Q}'} - D_{\mathbb{Q}'}) \\
&\subset \mathcal{O}_{\mathbb{Q}'}(m \det(Q)_{\mathbb{Q}'} + mqH_{\mathbb{Q}'} - D_{\mathbb{Q}'}) \\
&\cong \pi^* \big(q^* \mathcal{O}_X(m \det(Q) - D) \otimes \mathcal{O}_{\mathbb{Q}}(mq) \big)
\end{aligned}
$$

Choose a nonzero section $\mathcal{O}_{\mathbb{Q}'} \to \pi^*\big(q^*\mathcal{O}_X(m\det(Q) - D) \otimes \mathcal{O}_{\mathbb{Q}}(mq)\big)$. Since $\mathbb{Q}$ is normal and $\pi\colon \mathbb{Q}' \to \mathbb{Q}$ is birational, $\pi_*\mathcal{O}_{\mathbb{Q}'} = \mathcal{O}_{\mathbb{P}}$. Therefore, applying π_* to the chosen section gives a nonzero section

$$\mathcal{O}_{\mathbb{Q}} \to q^*\mathcal{O}_X(m\det(Q) - D) \otimes \mathcal{O}_{\mathbb{Q}}(mq).$$

and pushing forward again under q gives a nonzero section $\mathcal{O}_X \to \mathcal{O}_X(m\det(Q) - D) \otimes q_*\mathcal{O}_{\mathbb{Q}}(mq)$ which we rearrange as

$$(q_*\mathcal{O}_{\mathbb{Q}}(mq))^\vee \to \mathcal{O}_X(m\det(Q) - D). \tag{5.8.14}$$

Step 5: Show that the nefness of W implies the nefness of $(q_*\mathcal{O}_{\mathbb{Q}}(mq))^\vee$, and conclude that $\det(Q)$ is big. Recall that $\mathbb{Q}$ is a closed subschemes of $\mathbb{P} := \mathbb{P}(\mathscr{H}om_{\mathcal{O}_X}(W, \mathcal{O}_X^{\oplus w})) = \mathbb{P}((W^\vee)^{\oplus w})$ so that q factors as $\mathbb{Q} \hookrightarrow \mathbb{P} \xrightarrow{p} X$. There is an identification $p_*\mathcal{O}_{\mathbb{P}}(mq) \cong \mathrm{Sym}^{mq}((W^\vee)^{\oplus w})$. For $m \gg 0$, the map $p_*\mathcal{O}_{\mathbb{P}}(mq) \to q_*\mathcal{O}_{\mathbb{Q}}(mq)$ is surjective and dualizes to an inclusion

$$(q_*\mathcal{O}_{\mathbb{Q}}(mq))^\vee \hookrightarrow (p_*\mathcal{O}_{\mathbb{P}}(mq))^\vee \cong \mathrm{Sym}^{mq}((W^\vee)^{\oplus w})^\vee \tag{5.8.15}$$

of vector bundles on X. Since W is nef, so is $\mathrm{Sym}^{mq}((W^\vee)^{\oplus w})^\vee$ (Proposition B.6.35(3)). We now use hypotheses (d) to conclude that $q_*\mathcal{O}_{\mathbb{Q}}(mq))^\vee$ is nef. Since (5.8.15) is an inclusion of vector bundles with reduction of structure groups to G corresponding to a G-invariant subspace of vector spaces. If G is linearly reductive, this subspace has a G-invariant complement, and thus $(q_*\mathcal{O}_{\mathbb{Q}}(mq))^\vee$ is a quotient of $(p_*\mathcal{O}_{\mathbb{P}}(mq))^\vee \cong \mathrm{Sym}^{mq}(W^{\oplus w})$, hence nef. If $W = \mathrm{Sym}^d V$ for a nef vector bundle V of rank v, then $\mathbb{P} := \mathbb{P}(\mathscr{H}om_{\mathcal{O}_X}(V, \mathcal{O}_X^{\oplus v}))$ is the projectivized frame bundle of V, and $(q_*\mathcal{O}_{\mathbb{Q}}(mq))^\vee \cong \mathrm{Sym}^{mq}((V^\vee)^{\oplus v})^\vee$ is nef by Proposition B.6.35(3).

If (5.8.14) is surjective, we are done: $m\det(Q) - D$ is nef and thus $m\det(Q)$ is the sum of an ample and nef divisor, which is not only big but even ample. Otherwise, if I is the image subsheaf of (5.8.14), we blow up the ideal sheaf $I \otimes \mathcal{O}_X(m\det(Q) - D)^\vee \subset \mathcal{O}_X$ to obtain a birational morphism $f\colon X' \to X$ with exceptional divisor E' and a surjection

$$f^*(q_*\mathcal{O}_{\mathbb{P}}(mq))^\vee \twoheadrightarrow N \cong f^*\big(\mathcal{O}_X(m\det(Q) - D)\big) \otimes \mathcal{O}_{X'}(-E').$$

Since $f^*(q_*\mathcal{O}_{\mathbb{P}}(mq))^\vee$ is nef, so is N. This expresses $f^*\big(\mathcal{O}_X(m\det(Q))\big)$ as the sum of a nef, a big, and an effective divisor, which is big. This in turn implies that $\det(Q)$ is big. $\qquad\square$

5.8.3 Nefness of pluricanonical bundles and the ampleness of λ_k on $\overline{M}_g$

To apply Kollár's Criterion to $\overline{M}_g$, we will make use of multiplication maps between pluricanonical bundles and their symmetric products. The construction will depend on positive integers k and d, the same two integers on which the GIT construction depends. Given a family of stable curves $\pi\colon \mathcal{C} \to S$, we consider the *multiplication map*

$$\mathrm{Sym}^d \pi_*(\omega_{\mathcal{C}/S}^{\otimes k}) \to \pi_*(\omega_{\mathcal{C}/S}^{\otimes dk}). \tag{5.8.16}$$

For a stable curve C over a field $\Bbbk$, this is the map $\mathrm{Sym}^d H^0(C, \omega_C^{\otimes k}) \to H^0(C, \omega_C^{\otimes dk})$, whose kernel consists of degree d equations cutting out the image of $C \xrightarrow{|\omega_C^{\otimes k}|} \mathbb{P}^{r(k)-1}$. If $k \geq 3$, $\omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and $\mathcal{C} \to S$ can be recovered from the kernel of the multiplication map.

Theorem 5.8.17. *If T is a smooth, connected, and projective curve over a field $\Bbbk$ and $\pi\colon \mathcal{C} \to T$ is a family of stable curves, then $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq 2$.*

Proof.

Step 1: Reduction to characteristic p. Assume that $\mathrm{char}(\Bbbk) = 0$. Since T and $\mathcal{C}$ are of finite type over $\Bbbk$, their defining equations involve finitely many coefficients of $\Bbbk$. Thus there exists a finitely generated $\mathbb{Z}$-subalgebra $A \subset \Bbbk$ and a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \widetilde{\mathcal{C}} \\
\downarrow & & \downarrow \\
T & \longrightarrow & \widetilde{T} \\
\downarrow & & \downarrow \\
\mathrm{Spec}\,\Bbbk & \longrightarrow & \mathrm{Spec}\,A
\end{array}
$$

where $\widetilde{C}$ and $\widetilde{T}$ are schemes of finite type over A. By possibly enlarging A, we can arrange that $\widetilde{T} \to \mathrm{Spec}\,A$ is a smooth and projective family of curves and that $\widetilde{C} \to \widetilde{T}$ is a family of stable curves. Finally, by restricting along a map $\mathrm{Spec}\,R \to \mathrm{Spec}\,A$, we may assume that A is a DVR whose closed and generic points have characteristic p and 0, respectively. Since Nefness for Bundles is Stable under Generization (B.6.36), it suffices to prove the theorem when $\mathrm{char}(\Bbbk) = p > 0$.

Step 2: Further reductions. We reduce to the case where

(a) $\mathcal{C}$ is a smooth and minimal surface;

(b) $\mathcal{C} \to T$ is generically smooth; and

(c) the genus of T is at least 2

(details to be added). These conditions imply that $\mathcal{C}$ is of general type.

Step 3: Positive characteristic case. If $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is not nef, then there exists a quotient line bundle $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee$ where $d = \deg M > 0$. Consider the absolute Frobenius morphisms $F\colon \mathcal{C} \to \mathcal{C}$ and $F\colon T \to T$ which fit into a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F\ } & \mathcal{C} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
T & \xrightarrow{\ F\ } & T.
\end{array}
$$

By properties of the dualizing sheaf, we have $F^*\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) = \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$. Since $\deg F^*M = pd$, we can apply the Frobenius repeatedly to arrange that d, the degree of M, is as large as we want. Specifically, we can arrange that $M \cong \omega_T^{\otimes k} \otimes L$ where L is a very ample line bundle on T. (This was the entire point of reducing to characteristic p: to repeatedly apply the Frobenius to arbitrarily increase the degree.)

The surjection $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee \cong (\omega_T^{\otimes k} \otimes L)^\vee$ yields a surjection

$$
\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L \twoheadrightarrow \mathcal{O}_T
$$

Since the genus $\mathrm{h}^1(T, \mathcal{O}_T)$ of T is at least 2, we have $\mathrm{h}^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \geq 2$. Using the Leray spectral sequence to relate $H^1(\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L)$ to

$H^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L)$, this implies that $h^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L) \geq 2$ (details to be added). This however contradicts a consequence of Bombieri–Ekedahl Vanishing (B.6.38) with $D = \pi^* L$. $\qquad \square$

Theorem 5.8.18. *For $g \geq 2$, $k \gg 0$, and N sufficiently divisible, the line bundle $\lambda_k^{\otimes N}$ on $\overline{\mathcal{M}}_g$ descends to a line bundle on $\overline{M}_g$ which is relatively ample over $\operatorname{Spec}\mathbb{Z}$. For integers g and n with $2g - 2 + n > 0$, $\overline{M}_{g,n}$ is projective over $\operatorname{Spec}\mathbb{Z}$.*

Proof. We first reduce the second statement to the first: we already know that $\overline{M}_{1,1} = \mathbb{P}^1$ and $\overline{M}_{0,3} = \operatorname{Spec}\mathbb{Z}$, and since $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal family (Proposition 5.6.8) and is a projective morphism (Proposition 5.3.21), the projectivity of $\overline{M}_{g,n}$ implies theh projectivity of $\overline{M}_{g,n+1}$.

By Keel–Mori (4.3.12) and Stable Reduction (5.5.16), there is a coarse moduli space $\overline{\mathcal{M}}_g \to \overline{M}_g$ where $\overline{M}_g$ is an algebraic space proper over $\operatorname{Spec}\mathbb{Z}$. By Proposition 4.3.31, for each k, $\lambda_k^{\otimes N}$ descends to a line bundle L on $\overline{M}_g$ for N sufficiently divisible. Since Ampleness is Open (Proposition 4.4.24), it suffices show the result in the case that $\overline{\mathcal{M}}_g$ is defined over a field $\Bbbk$.

Let $\pi \colon \mathcal{U}_g \to \overline{\mathcal{M}}_g$ denote the universal family. For $k \geq 3$, $\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}$ is relatively very ample and $\pi_* \omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k} = 0$ is a vector bundle of rank $r(k) := (2k-1)(g-1) + kn$ (Proposition 5.3.21). Choose k and d large enough such that every stable curve C over $\Bbbk$, the closed immersion $|\omega_C^{\otimes k}| \colon C \hookrightarrow \mathbb{P}^{r(k)-1}$ is cut out by equations of degree d. This implies that the multiplication map

$$W := \operatorname{Sym}^d \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}) \twoheadrightarrow \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes dk}) =: Q$$

is a surjection of vector bundles on $\overline{\mathcal{M}}_g$ where W has rank $w = \binom{r(k)+d-1}{d}$ and Q has rank $q = r(dk)$. We will verify hypotheses (a)–(d) of Kollár's Criterion (5.8.7). For (a), W has a reduction of structure group by $G := \operatorname{GL}_{r(k)} \to \operatorname{GL}_w$ and, as we pointed out in Remark 5.8.9, the closure of the image of G in $\mathbb{P}(\operatorname{Mat}_{w,w})$ is identified with $\mathbb{P}(\operatorname{Mat}_{r(k),r(k)})$ and is hence normal. For (b), the nefness of W follows from the nefness of $\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ Theorem 5.8.17 and by the preservation of nefness under symmetric products (Proposition B.6.35(3)). For (c), the classifying map

$$\overline{\mathcal{M}}_g \to [\operatorname{Gr}(q, \Bbbk^w)/G]$$
$$[C] \mapsto \big[\underbrace{\operatorname{Sym}^d H^0(C, \omega_C^{\otimes k})}_{\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))} \twoheadrightarrow \underbrace{H^0(C, \omega_C^{\otimes dk})}_{\Gamma(C, \mathcal{O}(d))} \big]$$

is injective as the conditions on d and k imply that the kernel of the multiplication map uniquely determines C. Finally, (b) holds as W is a symmetric product.

By Le Lemme de Gabber (4.5.1), we may choose a finite cover $Z \to \overline{\mathcal{M}}_g$ be a finite cover where Z is a scheme (Theorem 4.5.1). By Kollár's Criterion (5.8.7), the pullback of λ_k to Z is ample for $k \gg 0$. Since the pullback of L under the finite morphism $Z \to \overline{\mathcal{M}}_g \to \overline{M}_g$ is ample, Exercise 4.4.22 implies that L is ample on $\overline{M}_g$. $\qquad \square$

Remark 5.8.19. It also possible to show the projectivity of $\overline{M}_{g,n}$ directly using Kollár's Criterion applied to the determinant of $\pi_*(L^k)$ where $L := \omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sigma_1 + \cdots + \sigma_n)$ and $\mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the universal family with sections $\sigma_1, \ldots, \sigma_n$.

Remark 5.8.20. In recent years, Kollár's Criterion has been applied in more and more general settings, e.g. Hassett's moduli space of weighted pointed curves [Has03], the moduli of stable varieties of any dimension [KP17], and the moduli of K-polystable Fano varieties [CP21, XZ20, LXZ21].

5.8.4 Projectivity via Geometric Invariant Theory

TO ADD SKETCH

Chapter 6

Geometry of algebraic stacks

6.1 Quasi-coherent sheaves and quotient stacks

We will define quasi-coherent sheaves on an algebraic stack in the same way that we did for Deligne–Mumford stacks in §4.1 but using the lisse-étale site on $\mathcal{X}$ instead of the small étale site. The entirety of §4.1 on sheaves, $\mathcal{O}_{\mathcal{X}}$-modules, and quasi-coherent sheaves remains valid for algebraic stacks (with the same affine diagonal hypotheses).

6.1.1 Sheaves and $\mathcal{O}_{\mathcal{X}}$-modules

To develop abelian sheaf theory on an algebraic stack, we use the lisse-étale site.

Definition 6.1.1 (Lisse-étale site)**.** The *lisse-étale site* $\mathcal{X}_{\text{lis-ét}}$ on an algebraic stack $\mathcal{X}$ is the category of schemes smooth over $\mathcal{X}$ where morphisms are arbitrary maps of schemes over $\mathcal{X}$. A covering $\{U_i \to U\}$ is a collection of morphisms such that $\coprod_i U_i \to U$ is surjective and étale.

This allows us to discuss sheaves of abelian groups on $\mathcal{X}_{\text{lis-ét}}$ and their morphisms. Extending §4.1.1, we can define sections $\Gamma(\mathcal{U}, F)$ or $F(\mathcal{U})$ of an abelian sheaf on an algebraic stack $\mathcal{U}$ smooth over $\mathcal{X}$. The structure sheaf $\mathcal{O}_{\mathcal{X}}$, defined as $\mathcal{O}_{\mathcal{X}}(U) = \Gamma(U, \mathcal{O}_U)$, is a ring object in the abelian category $\text{Ab}(\mathcal{X}_{\text{lis-ét}})$. We can therefore define $\mathcal{O}_{\mathcal{X}}$-modules as in Definition 4.1.8 and the abelian category $\text{Mod}(\mathcal{O}_{\mathcal{X}})$ of $\mathcal{O}_{\mathcal{X}}$-modules. Given a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, there are adjoint functors

$$\text{Ab}(\mathcal{X}_{\text{lis-ét}}) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \text{Ab}(\mathcal{Y}_{\text{lis-ét}}) \qquad \text{Mod}(\mathcal{O}_{\mathcal{X}}) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \text{Mod}(\mathcal{O}_{\mathcal{Y}}).$$

Given two $\mathcal{O}_{\mathcal{X}}$-modules F and G, the *tensor product* $F \otimes G := F \otimes_{\mathcal{O}_{\mathcal{X}}} G$ is the sheafification of $U \mapsto F(U) \otimes_{\mathcal{O}_{\mathcal{X}}(U)} G(U)$, and the *Hom sheaf* $\mathscr{H}om_{\mathcal{O}_{\mathcal{X}}}(F, G)$ is the sheaf given by $U \mapsto \text{Hom}_{\mathcal{O}_U}(F|_U, G|_U)$, where $F|_U$ denotes the restriction of F to $U_{\text{lis-ét}}$.

6.1.2 Quasi-coherent sheaves

Following §4.1.3, given an $\mathcal{O}_{\mathcal{X}}$-module F on an algebraic stack $\mathcal{X}$ and a smooth $\mathcal{X}$-scheme U, we let $F|_U$ be the restriction of F to the lisse-étale site of U and $F|_{U_{\text{Zar}}}$ the further restriction to the small Zariski site.

Definition 6.1.2. Let $\mathcal{X}$ be an algebraic stack. An $\mathcal{O}_{\mathcal{X}}$-module F is *quasi-coherent* if

(1) for every smooth $\mathcal{X}$-scheme U, the restriction $F|_{U_{\mathrm{Zar}}}$ is a quasi-coherent $\mathcal{O}_{U_{\mathrm{Zar}}}$-module, and

(2) for every morphism $f\colon U \to V$ of smooth $\mathcal{X}$-schemes, the natural morphism $f^*(F|_{V_{\mathrm{Zar}}}) \to F|_{U_{\mathrm{Zar}}}$ is an isomorphism.

A quasi-coherent sheaf F on $\mathcal{X}$ is a *vector bundle* (resp. *vector bundle of rank r, line bundle*) if $F|_{U_{\mathrm{Zar}}}$ is for every smooth $\mathcal{X}$-scheme U.

If in addition $\mathcal{X}$ is locally noetherian, we say F is *coherent* if $F|_{U_{\mathrm{Zar}}}$ is coherent for every smooth $\mathcal{X}$-scheme U

We denote by $\mathrm{QCoh}(\mathcal{X})$ and $\mathrm{Coh}(\mathcal{X})$ (in the noetherian setting) the categories of quasi-coherent and coherent sheaves. We encourage the reader to check that the equivalent formulations of quasi-coherent given in Exercises 4.1.16 to 4.1.18 still hold, and that the above definition of quasi-coherence is consistent with the definition of quasi-coherence on a Deligne–Mumford stack (Definition 4.1.11) and with the usual definition on a scheme. For a quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, f_* and f^* preserve quasi-coherence (by the same argument as for Exercise 4.1.19).

Exercise 6.1.3. Let G be an affine algebraic group over a field $\Bbbk$. Recall that a G-representation is a $\Bbbk$-vector space with a dual action $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} V$ satisfying two natural compatibility conditions (see §B.2.12).

(a) Show that $\mathrm{QCoh}(BG)$ is equivalent to the category $\mathrm{Rep}(G)$ of G-representations.

(b) If $\mathrm{Spec}\,A$ is an affine $\Bbbk$-scheme with a G-action, show that a quasi-coherent sheaf on $[\mathrm{Spec}\,A/G]$ is the data of an A-module M together with a coaction $\sigma\colon M \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} M$ over $\Bbbk$ (i.e. a map of $\Bbbk$-vector spaces giving M the structure of a G-representation) such that multiplication $A \otimes_{\Bbbk} M \to M$ is a map of G-representations. This extends Example 4.1.15 where G is finite.

(c) Considering the diagram

$$\mathrm{Spec}\,A \xrightarrow{\ p\ } [\mathrm{Spec}\,A/G] \xrightarrow{\ \pi\ } \mathrm{Spec}\,A^G$$
$$\downarrow{\scriptstyle q}$$
$$BG,$$

extend Exercise 4.1.20 by providing descriptions of the functors $p_*, p^*, \pi_*, \pi^*, q_*$ and q^* on quasi-coherent sheaves.

(d) If U is a $\Bbbk$-scheme with an action of G, then a *line bundle with a G-action* is a line bundle L on U together with an isomorphism $\alpha\colon \sigma^*L \xrightarrow{\sim} p_2^*L$ satisfying a cocycle condition $p_{23}^*\alpha \circ (\mathrm{id}_G \times \sigma)^*\alpha = (\mu \times \mathrm{id}_U)^*\alpha$; see B.2.7. Show that a line bundle with a G-action is the same as a line bundle on the quotient stack $[U/G]$.

Example 6.1.4. If G and H are affine algebraic groups over a field $\Bbbk$ such that $BG \cong BH$, then G and H have equivalent categories of representations. For example, if $O(q)$ and $O(q')$ are orthogonal groups with respect to non-degenerate quadratic forms q and q' on an n-dimensional $\Bbbk$-vector space V, then $BO(q) \cong BO(q')$ (see Exercise 3.1.13), and thus $O(q)$ and $O(Q')$ have equivalent categories of representations.

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Recall that one of the first examples we gave of a quasi-coherent sheaf on a Deligne–Mumford stack was the Hodge line bundle on $\mathfrak{M}_g$ (Example 4.1.4). The more general pluricanonical line bundles λ_k on $\overline{\mathfrak{M}}_g$ played an important role in the projectivity of $\overline{M}_g$ (see §5.8) and are equally essential in the study of its geometry. Determinantal line bundles play a similar role in the study of the moduli stack of vector bundles.

Example 6.1.5 (Determinantal line bundles). Consider the stack $\mathcal{M} := \underline{\mathrm{Bun}}_{C,r,d}$ of vector bundles on a smooth, connected, and projective curve C over $\Bbbk$. Consider the diagram

$$
\begin{array}{ccc}
 & C \times \mathcal{M} & \\
{}^{p_1}\swarrow & & \searrow^{p_2} \\
C & & \mathcal{M}.
\end{array}
$$

The projection $p_2\colon C \times \mathcal{M} \to \mathcal{M}$ is representable, projective, and smooth of relative dimension 1. For every vector bundle F on $C \times \mathcal{M}$, the cohomology $\mathrm{R}^i p_{2,*} F$ (as defined below) is computed as a 2-term complex $[K^0 \to K^1]$ of vector bundles and that the line bundle

$$
\det \mathrm{R} p_{2,*} F := \det(K_0) \otimes \det(K_1)^\vee
$$

is well-defined on $\mathcal{M}$. Note that if $\mathrm{rk}\, K_0 = \mathrm{rk}\, K_1$, i.e. $\mathrm{rk}\, \mathrm{R} p_{2,*} F = 0$, then we have a map $\det K^0 \to \det K^1$ of line bundles and the corresponding map $\mathcal{O}_\mathcal{M} \to \det(K^0)^\vee \otimes \det(K_1)$ defines a section of the dual $(\det \mathrm{R} p_{2,*} F)^\vee$.

Let $\mathcal{E}_{\mathrm{univ}}$ be the universal vector bundle on $C \times \mathcal{M}$. For every vector bundle V on C, we define the *determinantal line bundle*

$$
\mathcal{L}_V := \big(\det \mathrm{R} p_{2,*} (\mathcal{E}_{\mathrm{univ}} \otimes p_1^* V) \big)^\vee.
$$

associated to V.

Example 6.1.6. If $\mathcal{X}$ is an algebraic stack of finite presentation over a scheme S, then the relative sheaf of differentials $\Omega_{\mathcal{X}/S}$ on $\mathcal{X}_{\mathrm{lis\text{-}\acute{e}t}}$, defined on a smooth $\mathcal{X}$-scheme U by $\Omega_{\mathcal{X}/S}(U) = \Omega_{U/S}$, is not quasi-coherent. This is because for a non-étale map $f\colon U \to V$ of smooth $\mathcal{X}$-schemes, $f^* \Omega_{V/S} \to \Omega_{U/S}$ is not an isomorphism. This differs from the Deligne–Mumford case where the sheaf $\Omega_{\mathcal{X}/S}$ on $\mathcal{X}_{\mathrm{\acute{e}t}}$ is quasi-coherent (Example 4.1.13). When $\mathcal{X}$ is Deligne–Mumford, $\Omega_{\mathcal{X}/S}$ extends to a quasi-coherent sheaf on $\mathcal{X}_{\mathrm{lis\text{-}\acute{e}t}}$ by defining $\Omega_{\mathcal{X}/S}(U)$, for a smooth map $f\colon U \to \mathcal{X}$ from a scheme, to be the global sections of the sheaf $f^* \Omega_{\mathcal{X}/S}$ on $U_{\mathrm{lis\text{-}\acute{e}t}}$.

Exercises 4.1.21 and 4.1.38 generalize to algebraic stacks.

Proposition 6.1.7 (Flat Base Change). *Consider a cartesian diagram*

$$
\begin{array}{ccc}
X' & \xrightarrow{\;g'\;} & X \\
{\scriptstyle f'}\big\downarrow & \square & \big\downarrow{\scriptstyle f} \\
Y' & \xrightarrow{\;g\;} & Y
\end{array}
$$

of algebraic stacks, and let F be a quasi-coherent sheaf on X. If $g\colon Y' \to Y$ is flat and $f\colon X \to Y$ is quasi-compact and quasi-separated, the natural adjunction map

$$
g^* f_* F \to f'_* g'^* F
$$

is an isomorphism.

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Proposition 6.1.8. *Let $\mathcal{X}$ be a noetherian algebraic stack. Every quasi-coherent sheaf on $\mathcal{X}$ is a directed colimit of its coherent subsheaves. If $\mathcal{U} \subset \mathcal{X}$ is an open substack, then every coherent sheaf on $\mathcal{U}$ extends to a coherent sheaf on $\mathcal{X}$.*

Exercise 6.1.9. Let $\mathcal{X} \to \mathcal{Y}$ be a smooth affine morphism of noetherian algebraic stacks with affine diagonal.

(1) Show that there is a vector bundle $\Omega_{\mathcal{X}/\mathcal{Y}}$ on $\mathcal{X}$ with the property that if $V \to \mathcal{Y}$ is a morphism from a scheme, the pullback of $\Omega_{\mathcal{X}/\mathcal{Y}}$ to $X_V := \mathcal{X} \times_{\mathcal{Y}} V$ is $\Omega_{\mathcal{X}_V/V}$.

(2) Given a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \xrightarrow{\;f_0\;} & \mathcal{X} \\ \big\uparrow & \nearrow & \big\downarrow \\ \operatorname{Spec} A & \longrightarrow & \mathcal{Y} \end{array}$$

where $A \twoheadrightarrow A_0$ is a surjection of noetherian rings with square-zero kernel J, show that the set of liftings is a torsor under $\operatorname{Hom}_{A_0}(f_0^*\Omega_{\mathcal{X}/\mathcal{Y}}, J)$ and in particular is non-empty.

(3) Can you weaken the hypotheses?

6.1.3 Quasi-coherent constructions

Extending the constructions of §4.1.5 on a Deligne–Mumford stack to an algebraic stack $\mathcal{X}$, a *quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra* is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{A}$ with a compatible structure as a ring object. The *relative spectrum* $\mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{A}$, defined as the stack of pairs (f, α) where $f : S \to \mathcal{X}$ is a morphism from a scheme and $\alpha : f^*\mathcal{A} \to \mathcal{O}_S$ is a map of $\mathcal{O}_S$-algebras, is an algebraic stack affine over $\mathcal{X}$. On a noetherian algebraic stack, every quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra is a directed colimit of finite type subalgebras.

The *reduction of $\mathcal{X}$* is $\mathcal{X}_{\mathrm{red}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ where $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}$ is the sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras defined by $\mathcal{O}_{\mathcal{X}}^{\mathrm{red}}(U) = \Gamma(U, \mathcal{O}_U)_{\mathrm{red}}$ for a smooth $\mathcal{X}$-scheme U. If $\mathcal{X}$ is integral, the *normalization of $\mathcal{X}$* is defined as $\widetilde{\mathcal{X}} := \mathcal{S}\mathrm{pec}_{\mathcal{X}}\,\mathcal{A}$, where $\mathcal{A}$ is the $\mathcal{O}_{\mathcal{X}}$-algebra whose ring of sections over a smooth $\mathcal{X}$-scheme U is the normalization of $\Gamma(U, \mathcal{O}_U)$; this is well-defined since normalization commutes with smooth base change (Proposition A.7.4). For a quasi-compact and quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, there is a factorization $f : \mathcal{X} \to \mathcal{S}\mathrm{pec}\,f_*\mathcal{O}_{\mathcal{X}} \to \mathcal{Y}$. The morphism f is affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\,f_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism, and quasi-affine if and only if $\mathcal{X} \to \mathcal{S}\mathrm{pec}\,f_*\mathcal{O}_{\mathcal{X}}$ is an open immersion. The proof of Zariski's Main Theorem (4.4.9) for Deligne–Mumford stacks extends to algebraic stacks.

Theorem 6.1.10 (Zariski's Main Theorem). *A separated, quasi-finite, and representable morphism $f : \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks factors as the composition of a dense open immersion $\mathcal{X} \hookrightarrow \widetilde{\mathcal{Y}}$ and a finite morphism $\widetilde{\mathcal{Y}} \to \mathcal{X}$.* $\qquad\square$

6.1.4 Picard groups

If $\mathcal{X}$ is an algebraic stack, we let $\operatorname{Pic}(\mathcal{X})$ denote the set of isomorphism classes of line bundles on $\mathcal{X}$. It is an abelian group under tensor product.

Example 6.1.11. If G is an affine algebraic group over a field $\Bbbk$, then $\mathrm{Pic}(BG)$ is equivalent to the group of characters $G \to \mathbb{G}_m$. For example, $\mathrm{Pic}(B\mathbb{G}_m) = \mathbb{Z}$, $\mathrm{Pic}(B\,\mathrm{GL}_n) = \mathbb{Z}$, and $\mathrm{Pic}(\mathrm{PGL}_n) = \{0\}$.

Exercise 6.1.12. Let $\mathcal{X}$ be a smooth and irreducible algebraic stack over a field $\Bbbk$.

(a) If $\mathcal{D} \subset \mathcal{X}$ is a reduced substack with complement $\mathcal{U}$, show that there is a naturally defined line bundle $\mathcal{O}(\mathcal{D})$ (generalizing the usual construction for schemes) such that $\mathcal{O}(\mathcal{D})|_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}$.

(b) If V is a vector bundle on $\mathcal{X}$, show that

$$\mathrm{Pic}(\mathbb{A}(V)) = \mathrm{Pic}(\mathcal{X}) \quad \text{and} \quad \mathrm{Pic}(\mathbb{P}(V)) = \mathrm{Pic}(\mathcal{X}) \times \mathbb{Z}.$$

Exercise 6.1.13. Let $\mathbb{G}_m$ acts on $\mathbb{A}^n$ over a field $\Bbbk$ with weights $d_1, \dots, d_n$. Let $\mathcal{O}(1)$ be the line bundle on $[\mathbb{A}^n/\mathbb{G}_m]$ corresponding to the projection $[\mathbb{A}^n/\mathbb{G}_m] \to B\mathbb{G}_m$.

(a) Show that $\mathrm{Pic}([\mathbb{A}^n/\mathbb{G}_m]) \cong \mathbb{Z}$ generated by $\mathcal{O}(1)$.

(b) Show that the restriction $\mathrm{Pic}([\mathbb{A}^n/\mathbb{G}_m]) \to \mathrm{Pic}(\mathcal{P}(d_1, \dots, d_n))$ is an isomorphism, where $\mathcal{P}(d_1, \dots, d_n)$ is the weighted projective stack (see Example 3.9.6).

(c) If $f \in \Gamma(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$ is a homogenous polynomial of degree d such that $V(f) \subset \mathbb{A}^n$ is reduced, show that $\mathcal{O}(V(f)) \cong \mathcal{O}(d)$.

Exercise 6.1.14. Let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) \neq 2, 3$.

(a) Show that $\mathrm{Pic}(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}$.

Hint: Use the description $\overline{\mathcal{M}}_{1,1} = [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ of Exercise 3.1.19(c) where $\mathbb{G}_m$ acts with weights 4 and 6. Show that the restriction $\mathrm{Pic}(\mathbb{A}^2/\mathbb{G}_m]) \to \mathrm{Pic}(\overline{\mathcal{M}}_{1,1})$ is an equivalence.

(b) Show that $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$.

Hint: Show that the restriction $\mathrm{Pic}(\overline{\mathcal{M}}_{1,1}) \to \mathrm{Pic}(\mathcal{M}_{1,1})$ is surjective and that the image of $\mathcal{O}(\Delta) = \mathcal{O}(12)$ is trivial. Show that the images of $\mathcal{O}(4)$ and $\mathcal{O}(6)$ are non-trivial by considering their restrictions to the residual gerbes of the unique elliptic curves with $\mathbb{Z}/4$ and $\mathbb{Z}/6$ automorphism groups. See also [Mum65].

6.1.5 Global quotient stacks and the resolution property

Definition 6.1.15. An algebraic stack $\mathcal{X}$ is a *global quotient stack* if there exists an isomorphism $\mathcal{X} \cong [U/\,\mathrm{GL}_n]$ where U is an algebraic space.

In other words, $\mathcal{X}$ is a global quotient stack if and only if there is a principal GL_n-bundle $U \to \mathcal{X}$ from an algebraic space, or equivalently a representable morphism $\mathcal{X} \to B\,\mathrm{GL}_n$.

Exercise 6.1.16. Show that a noetherian algebraic stack $\mathcal{X}$ is a global quotient stack if and only if there exists a vector bundle E on $\mathcal{X}$ such that for every geometric point $x \colon \mathrm{Spec}\,\Bbbk \to \mathcal{X}$ with closed image, the stabilizer G_x acts faithfully on the fiber $E \otimes \Bbbk$.

Hint: Use the correspondence between principal GL_n-bundles and vector bundles from Exercise B.2.27.

Exercise 6.1.17. Let $\mathcal{X} \to \mathcal{Y}$ be a surjective, flat, and projective morphism of noetherian algebraic stacks. If $\mathcal{X}$ is a quotient stack, show that $\mathcal{Y}$ is a quotient stack.

Being a quotient stack is also related to the following notion:

Definition 6.1.18. A noetherian algebraic stack has the *resolution property* if every coherent sheaf is the quotient of a vector bundle.

A smooth or quasi-projective scheme over a field has the resolution property. More generally, a scheme admitting an "ample family" of line bundles has the resolution property, and this implies that every noetherian normal $\mathbb{Q}$-factorial scheme with affine diagonal has the resolution property [BS03].

Proposition 6.1.19. *Let G be an affine algebraic group over a field $\Bbbk$ acting on a quasi-projective $\Bbbk$-scheme U. Assume that there is an ample line bundle L with an action of G (e.g. U is quasi-affine and $L = \mathcal{O}_U$). Then $[U/G]$ has the resolution property.*

Remark 6.1.20. It is a general fact that every line bundle on a normal scheme over $\Bbbk$ has a positive power that has a G-action.

Proof. The line bundle L corresponds to a line bundle $\mathcal{L}$ on $[U/G]$ which is relatively ample with respect to the morphism $p \colon [U/G] \to BG$. For a coherent sheaf F on $[U/G]$, the natural map

$$\mathcal{L}^{-N} \otimes p^* p_*(\mathcal{L}^N \otimes F) \twoheadrightarrow F$$

is surjective for $N \gg 0$. The pushforward $p_*(\mathcal{L}^N \otimes F)$ is a quasi-coherent sheaf on BG, i.e. a G-representation, which we can write as a union of finite dimensional G-representations V_i (B.2.42(1)). We therefore obtain a surjection $\operatorname{colim}_i(\mathcal{L}^{-N} \otimes p^* V_i) \twoheadrightarrow F$. Since F is coherent, $\mathcal{L}^{-N} \otimes p^* V_i \twoheadrightarrow F$ is surjective for $i \gg 0$. $\qquad\square$

Totaro established an interesting converse [Tot04], which was later generalized by Gross [Gro17].

Theorem 6.1.21. *Let $\mathcal{X}$ be a quasi-separated normal algebraic stack of finite type over a field $\Bbbk$. Assume that the stabilizer group at every closed point is smooth and affine. Then the following are equivalent:*

(1) $\mathcal{X}$ has the resolution property,

(2) $\mathcal{X} \cong [U/\operatorname{GL}_n]$ with U quasi-affine, and

(3) $\mathcal{X} \cong [\operatorname{Spec} A/G]$ with G an affine algebraic group.

In particular, $\mathcal{X}$ has affine diagonal.

Remark 6.1.22. While the normal hypothesis on $\mathcal{X}$ and smoothness hypothesis on the stabilizers are unnecessary, the affineness hypothesis on the stabilizers is necessary, e.g. the classifying stack BE of an elliptic curve has the resolution property.

Proof. The implications that (2) and (3) imply (1) were established in Proposition 6.1.19.

To see (3) $\Rightarrow$ (2), it suffices to find a faithful representation $G \hookrightarrow \operatorname{GL}_N$ such that GL_N/G is quasi-affine. Indeed, in this case, $[\operatorname{Spec} A/G] \cong [(\operatorname{Spec} A \times^G \operatorname{GL}_N)/\operatorname{GL}_N]$ (Exercise 3.4.16) and $\operatorname{Spec} A \times^G \operatorname{GL}_N$ is affine over GL_N/G. We begin by choosing a faithful representation $G \subset \operatorname{GL}_n$. By B.2.42B.2.40, there is a GL_n-representation V and a $\Bbbk$-point $x \in \mathbb{P}(V)$ with stabilizer G. Under the action of $\operatorname{GL}_n \times \mathbb{G}_m$ on $\mathbb{A}(V)$ (where $\mathbb{G}_m$ acts via scaling), the stabilizer of a lift $\widetilde{x} \in \mathbb{A}(V)$ of x is G. The map $(\operatorname{GL}_n \times \mathbb{G}_m)/G \hookrightarrow \mathbb{A}(V)$, defined by $g \mapsto g\widetilde{x}$, is a locally closed immersion and thus

$(\mathrm{GL}_n \times \mathbb{G}_m)/G$ is quasi-affine. Under the natural inclusion $\mathrm{GL}_n \times \mathbb{G}_m \hookrightarrow \mathrm{GL}_{n+1}$, the quotient $\mathrm{GL}_{n+1}/(\mathrm{GL}_n \times \mathbb{G}_m)$ is affine (and is sometimes called the "Steifel manifold"). The composition $BG \to B(\mathrm{GL}_n \times \mathbb{G}_m) \to B\,\mathrm{GL}_{n+1}$ is quasi-affine and therefore so is GL_{n+1}/G.

Conversely for (2) $\Rightarrow$ (3), we may choose a GL_n-equivariant open immersion $U \hookrightarrow \operatorname{Spec} A$ into an affine scheme of finite type over $\Bbbk$. Indeed, the morphism $p \colon [U/\mathrm{GL}_n] \to B\,\mathrm{GL}_n$ is quasi-affine and $[U/\mathrm{GL}_n] \to \operatorname{Spec}_{B\,\mathrm{GL}_n} p_* \mathcal{O}_{[U/\mathrm{GL}_n]}$ is an open immersion. By writing $p_* \mathcal{O}_{[U/\mathrm{GL}_n]} = \operatorname{colim}_\lambda \mathcal{A}_\lambda$ as a colimit of finite type $\mathcal{O}_{B\,\mathrm{GL}_n}$-algebras, then limit methods imply that $[U/\mathrm{GL}_n] \to \operatorname{Spec}_{B\,\mathrm{GL}_n} \mathcal{A}_\lambda$ is an open immersion for $\lambda \gg 0$. Let $Z \subset \operatorname{Spec} A$ be the reduced complement of U. By Proposition B.2.43(2), there is a GL_n-equivariant morphism $f \colon \operatorname{Spec} A \to \mathbb{A}^r$ such that $f^{-1}(0) = Z$. This induces an affine morphism $U \to \mathbb{A}^r \smallsetminus 0$. The complement $\mathbb{A}^r \smallsetminus 0$ can be realized as the quotient GL_r/H where $H \subset \mathrm{GL}_r$ is the subgroup consisting of matrices whose last row is $(0, \dots, 0, 1)$; H is identified with the semi-direct product $\mathbb{G}_a^{r-1} \rtimes \mathrm{GL}_{r-1}$. In the GL_n-equivariant cartesian diagram

$$
\begin{array}{ccc}
P & \longrightarrow & \mathrm{GL}_r \\
\downarrow & & \downarrow \\
U & \longrightarrow & \mathbb{A}^r \smallsetminus 0,
\end{array}
$$

P is affine over GL_r, thus affine. We conclude using the equivalent $[U/\mathrm{GL}_n] \cong [P/(\mathrm{GL}_n \times H)]$.

It remains to show (1) $\Rightarrow$ (2). We first show that $\mathcal{X} \cong [U/\mathrm{GL}_n]$ with U an algebraic space. Given a vector bundle E on $\mathcal{X}$ of rank n, the frame bundle $\mathrm{Fr}(E)$ is a principal GL_n-torsor and $\mathcal{X} \cong [\mathrm{Fr}(E)/\mathrm{GL}_n]$ (Exercise 6.1.16).

For every closed point $x \in \mathcal{X}$, let $i_x \colon \mathcal{G}_x \hookrightarrow \mathcal{X}$ be the inclusion of the residual gerbe (Proposition 3.5.19). Let $\kappa(x) \to \Bbbk$ be a finite field extension trivializing $\mathcal{G}_x$, i.e. there is a map $\widetilde{x} \colon \operatorname{Spec} \Bbbk \to \mathcal{X}$ representing x inducing a finite cover $p \colon BG_{\widetilde{x}} \to \mathcal{G}_x$. Since $G_{\widetilde{x}}$ is affine, we can choose a faithful representation W. Using the resolution property, there is a vector bundle E and a surjection $E \twoheadrightarrow (i_x \circ p)_* W$. The associated frame bundle $\mathrm{Fr}(E) \to \mathcal{X}$ has trivial stabilizers over x. In other words, the kernel subgroup $S_E \subset I_{\mathcal{X}}$ of E (i.e. the subgroup stack of the inertia stack parameterizing elements acting trivially on E) is trivial over x. If F is another vector bundle, then $S_{E \oplus F} \subset S_E$ is a closed subgroup. Since $I_{\mathcal{X}}$ is noetherian, we can inductively enlarge the vector bundle E so that $U := \mathrm{Fr}(E)$ is an algebraic space and $\mathcal{X} \cong [U/\mathrm{GL}_n]$.

Since $\mathcal{X}$ is normal, U is also normal and we may apply Exercise 4.5.3 to conclude that U is the coarse moduli space of the action of a finite group H acting on a normal scheme U'. Let $p \colon U' \to U$ be the quotient morphism, and let $U'_1, \dots, U'_r$ be an affine covering of U' with reduced complements $Z'_1, \dots, Z'_r$. Then $F := p_*(\bigoplus_i I_{Z'_i})$ is a coherent sheaf on U. Moreover, since $q \colon U \to [U/\mathrm{GL}_n]$ is affine, $q^* q_* F \twoheadrightarrow F$ is surjective, and by writing $q_* F$ as a colimit of coherent sheaves, we may find a coherent sheaf G on $\mathcal{X} \cong [U/\mathrm{GL}_n]$ and a surjection $q^* G \to F$. Since $\mathcal{X}$ has the resolution property, we see that there is even a vector bundle G and a surjection $q^* G \to F$. Since $p \colon U' \to U$ is affine, we have a surjection $p^* q^* G \twoheadrightarrow p^* F \twoheadrightarrow \bigoplus_i I_{Z'_i}$. Let $V = \mathrm{Fr}(G)$ and consider the cartesian diagram

$$
\begin{array}{ccccc}
U'_V & \longrightarrow & U_V & \longrightarrow & V \\
\downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow \\
U' & \xrightarrow{\ p\ } & U & \xrightarrow{\ q\ } & \mathcal{X}
\end{array}
$$

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where the horizontal arrows are principal GL_n-bundles and the vertical arrows are GL_m-bundles where $m = \mathrm{rk}(G)$. Since the pullback of G to V is trivial, the pullback $\beta^*(\bigoplus_i I_{Z_i'})$ is globally generated. This implies that $\beta^{-1}(Z_i')$ is defined by global functions on U_V' and that the complement $\beta^{-1}(U_i')$ is covered by affine opens of the form $\{f \neq 0\}$ for $f \in \Gamma(U_V', \mathcal{O}_{U_V'})$. This implies that $\mathcal{O}_{U_V'}$ is ample and that U_V' is a quasi-affine scheme. Since $\beta \colon U_V' \to U_V$ is the quotient by a finite group, U_V is also quasi-affine (Exercise 4.2.9). We have thus shown that $\mathcal{X} \cong [U_V/(\mathrm{GL}_n \times \mathrm{GL}_m)]$. Under the embedding $\mathrm{GL}_n \times \mathrm{GL}_m \hookrightarrow \mathrm{GL}_{n+m}$, the quotient $\mathrm{GL}_{n+m}/(\mathrm{GL}_n \times \mathrm{GL}_m)$ is quasi-affine. Setting $W = U_V \times^{(\mathrm{GL}_n \times \mathrm{GL}_m)} \mathrm{GL}_{n+m}$, we conclude that $\mathcal{X} \cong [W/\mathrm{GL}_{n+m}]$. $\qquad\square$

6.1.6 Sheaf cohomology

Abelian sheaf cohomology for algebraic stacks can be developed using essentially the same approach as we used in §4.1.6 for Deligne–Mumford stacks.

Lemma 6.1.23. *If $\mathcal{X}$ is an algebraic stack, the categories $\mathrm{Ab}(\mathcal{X}_{\text{lis-ét}})$ and $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}})$ have enough injectives. If in addition $\mathcal{X}$ is quasi-separated, then $\mathrm{QCoh}(\mathcal{X})$ has enough injectives.*

Proof. The argument of Lemma 4.1.27 generalizes. $\qquad\square$

Definition 6.1.24 (Cohomology). Let $\mathcal{X}$ be an algebraic stack and F a sheaf of abelian groups on $\mathcal{X}_{\text{lis-ét}}$. The *cohomology group* $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ is defined as the ith right derived functor of the global sections functor $\Gamma \colon \mathrm{Ab}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{Ab}$.

Given a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, the *higher direct image* $R^i f_* F$ is defined as the ith right derived functor of $f_* \colon \mathrm{Ab}(\mathcal{X}_{\text{lis-ét}}) \to \mathrm{Ab}(\mathcal{Y}_{\text{lis-ét}})$.

Definition 6.1.25 (Čech cohomology). Given a smooth covering $\mathcal{U} = \{U_i \to \mathcal{X}\}_{i \in I}$ of algebraic stacks and an abelian sheaf F on $\mathcal{X}_{\text{lis-ét}}$, the *Čech complex of F with respect to $\mathcal{U}$* is $\check{\mathcal{C}}^\bullet(\mathcal{U}, F)$ where

$$\check{\mathcal{C}}^n(\mathcal{U}, F) = \prod_{(i_0, \ldots, i_n) \in I^{n+1}} F(U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n})$$

with differential

$$d^n \colon \check{\mathcal{C}}^n(\mathcal{U}, F) \to \check{\mathcal{C}}^{n+1}(\mathcal{U}, F), \qquad (s_{i_0, \ldots, i_n}) \mapsto \left(\sum_{k=0}^{n+1} (-1)^k p_{\widehat{k}}^* s_{i_0, \ldots, \widehat{i_k}, \ldots, i_n} \right)_{(i_0, \ldots, i_{n+1})}$$

where $p_{\widehat{k}} \colon U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n} \to U_{i_0} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \widehat{U_{i_k}} \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_{i_n}$ is the map forgetting the kth component (with indexing starting at 0). The *Čech cohomology of F with respect to $\mathcal{U}$* is

$$\check{\mathrm{H}}^i(\mathcal{U}, F) := \mathrm{H}^i(\check{\mathcal{C}}^\bullet(\mathcal{U}, F)).$$

The arguments of Theorem 4.1.30 and Propositions 4.1.34, 4.1.36 and 4.1.37 as well as Exercise 4.1.39 extend.

Theorem 6.1.26. *For a quasi-coherent $\mathcal{O}_{X_{\text{lis-ét}}}$-module F on an affine scheme X, $\mathrm{H}^i(X_{\text{lis-ét}}, F) = 0$ for all $i > 0$.* $\qquad\square$

Proposition 6.1.27. *Let $\mathcal{X}$ be an algebraic stack with affine diagonal and F be a quasi-coherent sheaf. If $\mathcal{U} = \{U_i \to \mathcal{X}\}$ is an étale covering with each U_i affine, then $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F) = \check{\mathrm{H}}^i(\mathcal{U}, F)$.* $\qquad\square$

Proposition 6.1.28. *If X is a scheme with affine diagonal and F be a quasi-coherent sheaf, then $\mathrm{H}^i(X, F) = \mathrm{H}^i(X_{\text{lis-ét}}, F_{\text{lis-ét}})$ for all i, where $F_{\text{lis-ét}}$ is the sheaf of $\mathcal{O}_{X_{\text{lis-ét}}}$-module defined by $F_{\text{ét}}(U) = \Gamma(U, f^*F)$ for a smooth map $f \colon U \to \mathcal{X}$ from a scheme.*

Similarly, if $\mathcal{X}$ is a Deligne–Mumford stack with affine diagonal and F is a quasi-coherent sheaf, then $\mathrm{H}^i(\mathcal{X}, F) = \mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F_{\text{lis-ét}})$ for all i. $\qquad\square$

Proposition 6.1.29. *Let $\mathcal{X}$ be an algebraic stack.*

(1) *If F is an $\mathcal{O}_\mathcal{X}$-module, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ of F as an abelian sheaf agrees with the ith right derived functor of $\Gamma \colon \mathrm{Mod}(\mathcal{O}_\mathcal{X}) \to \mathrm{Ab}$.*

(2) *If $\mathcal{X}$ has affine diagonal and F is a quasi-coherent sheaf on $\mathcal{X}$, then the cohomology $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ of F as an abelian sheaf agrees with the ith right derived functor of $\Gamma \colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.*

For a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks (resp. quasi-compact morphism of algebraic stacks with affine diagonals), then (1) (resp. (2)) holds also for the higher direct images $R^i f_* F$ of an $\mathcal{O}_\mathcal{X}$-module (resp. quasi-coherent sheaf): it can be computed as the ith right derived functor of $f_* \colon \mathrm{Mod}(\mathcal{O}_\mathcal{X}) \to \mathrm{Mod}(\mathcal{O}_\mathcal{Y})$ (resp. $f_* \colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$). $\qquad\square$

Remark 6.1.30. If $\mathcal{X}$ does not have affine diagonal, then the sheaf cohomology $\mathrm{H}^i(\mathcal{X}_{\text{lis-ét}}, F)$ of a quasi-coherent sheaf may differ from the ith right derived functor of $\Gamma(\mathcal{X}, -) \colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Ab}$.

Proposition 6.1.31. *If $\mathcal{X}$ is an algebraic stack and F_i is a directed system of abelian sheaves in $\mathcal{X}_{\text{lis-ét}}$, then $\operatorname{colim}_i \mathrm{H}^i(\mathcal{X}, F_i) \to \mathrm{H}^i(\mathcal{X}, \operatorname{colim}_i F_i)$ is an isomorphism.* $\quad\square$

6.1.7 Chow groups

Following [Tot99] and [EG98], we introduce the Chow groups of a quotient stack. Let G be a smooth affine algebraic group over an algebraically closed field $\Bbbk$ of dimension g, and let X be an n-dimensional scheme of finite type over $\Bbbk$. For each i, choose an r-dimensional G-representation V such that there is a nonempty open subscheme $U \subset \mathbb{A}(V)$ such that (a) G acts freely on U, (b) the quotient U/G is a scheme, and (c) $\operatorname{codim} \mathbb{A}(V) \setminus U > n - i - g$. Such representations exist. We define the $(i-g)$*th equivariant Chow group of X* or equivariantly the i*th Chow group of* $[X/G]$ as

$$\mathrm{CH}_{i-g}^G(X) = \mathrm{CH}_i([X/G]) := \mathrm{CH}_{i+r}(X \times^G U).$$

This definition is independent of the choice of representation. The definition is forced upon us if we desire invariance of Chow groups under vector bundles and open immersions of high codimension:

$$
\begin{array}{ccc}
X \times^G U \overset{\text{open}}{\lhook\joinrel\longrightarrow} [(X \times \mathbb{A}(V))/G] & \qquad & \mathrm{CH}_{i+r}(X \times^G U) \overset{\sim}{\longleftarrow} \mathrm{CH}_{i+r}([(X \times \mathbb{A}(V))/G]) \\
\Big\downarrow {\scriptstyle \text{vect bdl}} & & \Big\uparrow {\scriptstyle \wr} \\
[X/G] & & \mathrm{CH}_i([X/G])
\end{array}
$$

If $[X/G]$ is smooth of pure dimension $d = n - g$, then we define

$$\mathrm{CH}_G^i(X) = \mathrm{CH}^i([X/G]) := \mathrm{CH}_{d-i}([X/G])$$
$$\mathrm{CH}_G^*(X) = \mathrm{CH}^*([X/G]) := \bigoplus_i \mathrm{CH}^i((X/G)).$$

The intersection product gives a ring structure, and we call $\mathrm{CH}^*_G(X)$ the *equivariant Chow ring of* X and $\mathrm{CH}^*([X/G])$ the *Chow ring of* $[X/G]$.

Example 6.1.32 ($\mathrm{CH}^*(B\mathbb{G}_m)$)**.** Let V be the r-dimensional $\mathbb{G}_m$-representation with equal weights 1. Then $\mathbb{G}_m$ acts freely on $\mathbb{A}^n \setminus 0$, and for $-1 \geq i > -r-1$, we have that $\mathrm{CH}_i(B\mathbb{G}_m) = \mathrm{CH}_{i+r}(\mathbb{P}^{r-1}) = \mathbb{Z}$. It follows that $\mathrm{CH}_i(B\mathbb{G}_m) = \mathbb{Z}$ for $i \leq -1$ and is 0 otherwise. Therefore $\mathrm{CH}^j(B\mathbb{G}_m) = \mathbb{Z}$ for $j \geq 0$, and $\mathrm{CH}^*(B\mathbb{G}_m) = \mathbb{Z}[x]$. More generally, if $T \cong \mathbb{G}_m^r$ is a rank r torus, then $\mathrm{CH}^*(BT)$ is isomorphic to the character ring $\mathbb{Z}[x_1, \ldots, x_r]$ of T.

We summarize some of the important properties of equivariant Chow groups.

Properties 6.1.33.

(1) *(Independent of quotient presentation) If* $[X/G] \cong [X'/G']$, *then* $\mathrm{CH}_i^G(X) \cong \mathrm{CH}_i^{G'}(X')$, *and in particular the definition of* $\mathrm{CH}_i([X/G])$ *is independent of the quotient presentation. The definition of Chow groups can be extended to finite type algebraic stacks over* $\Bbbk$; *see [Kre99].*

(2) *(Vector bundle invariance) If* $Y \to X$ *is* G-*equivariant and a Zariski-local affine fibration of relative dimension* r *(e.g. the total space of a rank* r *vector bundle), then* $\mathrm{CH}_*^G(X) \cong \mathrm{CH}_{*+r}^G(Y)$.

(3) *(Excision sequence) If* $\mathcal{Z} \subset \mathcal{X} = [X/G]$ *is a closed substack with complement* $\mathcal{U}$, *then there is a right exact sequence*

$$\mathrm{CH}_*(\mathcal{Z}) \to \mathrm{CH}_*(\mathcal{X}) \to \mathrm{CH}_*(\mathcal{U}) \to 0.$$

(4) *(Comparison with coarse moduli space) If* $\mathcal{X} \cong [U/G]$ *is a separated Deligne–Mumford stack with coarse moduli space* X, *then* $\mathrm{CH}_*(\mathcal{X}) \otimes \mathbb{Q} \cong \mathrm{CH}_*(X) \otimes \mathbb{Q}$.

(5) *(Functoriality and self-intersection) Flat morphisms induce pullback maps on Chow groups while proper morphisms induce pushforward maps. If* $\mathcal{X} = [X/G]$ *is smooth and* $i \colon \mathcal{Z} \hookrightarrow \mathcal{X}$ *is a smooth substack of pure codimension* d, *then there is pullback* $i^* \colon \mathrm{CH}^*(\mathcal{X}) \to \mathrm{CH}^*(\mathcal{Z})$ *given by intersection with* $\mathcal{Z}$ *such that* $i^* i_* \alpha = c_d(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}) \cap \alpha$ *for* $\alpha \in \mathrm{CH}^*(\mathcal{Z})$, *where* c_d *is the top Chern class of the normal bundle.*

(6) *Let* T *be a torus acting on a smooth scheme* X *such that* $T = T_1 \times T_2$ *is a product of two tori with* T_2 *acting trivially. Then* $\mathrm{CH}_T^*(X) \cong \mathrm{CH}_{T_1}^*(X) \otimes \mathrm{CH}^*(BT_2)$.

(7) *If* G *is a connected reductive group with maximal torus* T, *and* X *is a smooth scheme with a* G-*action, then the Weyl group* $W = N_G(T)/T$ *acts on* $\mathrm{CH}_T^*(X)$ *and* $\mathrm{CH}_G^*(X)_\mathbb{Q} = \mathrm{CH}_T^*(X)_\mathbb{Q}^W$.

Exercise 6.1.34.

(a) Let $\mathcal{P}(d_0, \ldots, d_n)$ be the weighted projective stack of Example 3.9.6. Show that $\mathrm{CH}^*(\mathcal{P}(d_0, \ldots, d_n)) \cong \mathbb{Z}[x]/(d_1 \cdots d_n x^{n+1})$.

(b) If $\mathrm{char}(\Bbbk) \neq 2, 3$, show that $\mathrm{CH}^*(\mathcal{M}_{1,1}) \cong \mathbb{Z}[x]/(12x)$ and $\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) \cong \mathbb{Z}[x]/(24x^2)$. (Compare with Exercise 6.1.14).

(c) Let $\mathbb{G}_m$ act on $\mathbb{P}^n$ with weights $d_0, \ldots, d_n$. Show that $A^*([\mathbb{P}^n/\mathbb{G}_m]) = \mathbb{Z}[h,t]/p(h,t)$ where $p(h,t) = \sum_{i=0}^n h^i e_i(a_0 t, \ldots, a_n t)$ and e_i is the ith symmetric polynomial.

6.1.8 de Rham and singular cohomology

We quickly discuss the de Rham and singular cohomology of an algebraic stack following [Beh04].

Analyticification. If $\mathcal{X}$ is a smooth algebraic stack over $\mathbb{C}$ with affine diagonal, there is an analyticification $\mathcal{X}^{\mathrm{an}}$, analogous to the analyticification of a finite type $\mathbb{C}$-scheme, such that $\mathcal{X}^{\mathrm{an}}$ is a *differentiable stack*. If $U_0 \to \mathcal{X}$ is a smooth presentation by a scheme so that $\mathcal{X}$ is the quotient of the smooth groupoid $U_1 := U_0 \times_{\mathcal{X}} U_0 \rightrightarrows U_0$, then $U_0^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}}$ is a smooth presentation and $\mathcal{X}^{\mathrm{an}}$ is the quotient of the *Lie groupoid* $U_1^{\mathrm{an}} \rightrightarrows U_0^{\mathrm{an}}$.

De Rham cohomology of a differential stack. Given a differentiable stack $\mathcal{X}$ with a smooth presentation $U_0 \to \mathcal{X}$, we can define a *simplicial manifold* $U_\bullet$

$$\cdots U_3 \mathrel{\substack{\longrightarrow\\[-0.6em]\longrightarrow\\[-0.6em]\longrightarrow}} U_2 \mathrel{\substack{\longrightarrow\\[-0.6em]\longrightarrow}} U_1 \longrightarrow U_0, \quad \text{where } U_p := \underbrace{U_0 \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} U_0}_{p \text{ times}} \quad (6.1.35)$$

with maps $\partial_i \colon U_p \to U_{p-1}$ forgetting the ith term along with degeneracy maps $s_i \colon U_{p-1} \to U_p$ inserting an identity morphism in the ith term. This defines a double complex $\Omega^q(U_p)$ with differentials given by exterior differentiation $d \colon \Omega^{q-1}(U_p) \to \Omega^q(U_p)$ and $\partial := \sum_{i=0}^p (-1)^i \partial_i^* \colon \Omega^q(U_{p-1}) \to \Omega^q(U_p)$. We define the *de Rham complex* $C_{\mathrm{dR}}^\bullet(\mathcal{X})$ as the total complex

$$C_{\mathrm{dR}}^k(\mathcal{X}) := \bigoplus_{p+q=k} \Omega^q(U_p),$$

with differential $\delta \colon C_{\mathrm{dR}}^k(\mathcal{X}) \to C_{\mathrm{dR}}^{k+1}(\mathcal{X})$ defined by $\delta(\omega) = \partial(\omega) + (-1)^p d(\omega)$ for $\omega \in \Omega^p(U_q)$. The *de Rham cohomology* is

$$\mathrm{H}_{\mathrm{dR}}^n(\mathcal{X}) := \mathrm{H}^n(C_{\mathrm{dR}}^\bullet(\mathcal{X})),$$

and is independent of the choice of presentation. As with the case of smooth manifolds, there is an identification of $\mathrm{H}_{\mathrm{dR}}^n(\mathcal{X})$ with the sheaf cohomology of the constant sheaf $\mathbb{R}$ on the big smooth site of smooth manifolds over $\mathcal{X}$.

Singular homology/cohomology of a topological stack. For a *topological stack* $\mathcal{X}$, one can replicate the constructions of singular homology and cohomology. Let $U_0 \to \mathcal{X}$ be a presentation and $U_\bullet$ be the simplicial topological space as in (6.1.35). For each p, we have the singular chain complex $C_\bullet(U_p)$ with differentials $d \colon C_q(U_p) \to C_{q-1}(U_p)$. This defines a double complex $C_q(U_p)$ using the differential $\partial = \sum_{i=0}^p (-1)^i \partial_j \colon C_q(U_p) \to C_q(U_{p-1})$ induced by the maps $\partial_i \colon U_p \to U_{p-1}$. We define the *singular chain complex* $C_\bullet(\mathcal{X})$ *of* $\mathcal{X}$ as the total complex

$$C_k(\mathcal{X}) := \bigoplus_{p+q=k} C_q(U_p)$$

with the differential $\delta \colon C_k(\mathcal{X}) \to C_{k-1}(\mathcal{X})$ given by $\delta(\gamma) = (-1)^{p+q} \partial(\gamma) + (-1)^q d(\gamma)$ for $\gamma \in C_q(U_p)$. For an abelian group A, we can therefore define the *singular homology groups of $\mathcal{X}$ with coefficients in A* as

$$\mathrm{H}_n(\mathcal{X}, A) := \mathrm{H}_n(C_\bullet(\mathcal{X}) \otimes_{\mathbb{Z}} A).$$

Dualizing, we define the *singular cochain complex* $C^\bullet(\mathcal{X})$ by $C^n(\mathcal{X}) := \mathrm{Hom}(C_n(\mathcal{X}), \mathbb{Z})$ and the *singular cohomology groups of $\mathcal{X}$ with coefficients in A* as

$$\mathrm{H}^n(\mathcal{X}, A) := \mathrm{H}^n(C^\bullet(\mathcal{X}) \otimes_{\mathbb{Z}} A).$$

Comparisons.

- There are pairings $\mathrm{H}_k(\mathcal{X}, \mathbb{Z}) \otimes \mathrm{H}^k(\mathcal{X}, \mathbb{Z}) \to \mathbb{Z}$ which after tensoring with $\mathbb{Q}$ gives identifications $\mathrm{H}^k(\mathcal{X}, \mathbb{Q}) \cong \mathrm{H}_k(\mathcal{X}, \mathbb{Q})^\vee$.

- If G is a topological group acting on a space U, then the *equivariant cohomology* is defined as $\mathrm{H}^*_G(U, A) := \mathrm{H}^*(EG \times^G U, A)$, where EG is a contractible space with a free action of G, and there is an identification $\mathrm{H}^*([U/G], A) = \mathrm{H}^*_G(U, A)$.

- For a differential stack $\mathcal{X}$, there is an identification $\mathrm{H}^*_{\mathrm{dR}}(\mathcal{X}) = \mathrm{H}^*(\mathcal{X}, \mathbb{R})$.

- If $\mathcal{X}$ is a *topological Deligne–Mumford stack* (e.g. the topological stack associated to a separated Deligne–Mumford stack of finite type over $\mathbb{C}$) with coarse moduli space $\mathcal{X} \to X$, then $\mathrm{H}^*(\mathcal{X}, \mathbb{Q}) = \mathrm{H}^*(X, \mathbb{Q})$.

6.2 The fppf topology and gerbes

This section is not essential for the proofs of the two main theorems of this book and is included for completeness. We prove that algebraic spaces/stacks are sheaves/stacks in the fppf topology and that quotients by fppf groupoids/equivalence relations are algebraic. One upshot is that BG is an algebraic stack for any (non-necessarily smooth) algebraic group, e.g. $\boldsymbol{\mu}_p$ in characteristic p.

We also introduce gerbes, a central topic in the theory of stacks. For us, we want to know that residual gerbes are gerbes (justifying the terminology) and later that the moduli stack $\underline{\mathrm{Bun}}^{\mathrm{s}}(C)_{r,d}$ of stable vector bundles is a $\mathbb{G}_m$-gerbe over its coarse moduli space.

6.2.1 Fppf criterion for algebraicity

Theorem 6.2.1 (Fppf Criterion for Algebraicity)**.**

(1) If X is a sheaf on $\mathrm{Sch}_{\mathrm{fppf}}$ such that there exists an fppf representable morphism $U \to X$ from a scheme, then X is an algebraic space.

(2) If $\mathcal{X}$ is a stack over $\mathrm{Sch}_{\mathrm{fppf}}$ such that there exists an fppf representable morphism $U \to \mathcal{X}$ from a scheme, then $\mathcal{X}$ is an algebraic stack.

Proof. To add. $\qquad\square$

Algebraic spaces are by definition sheaves in the big étale topology but it turns out they are also sheaves in the big fppf topology.

Proposition 6.2.2.

(1) An algebraic space X over a scheme S is a sheaf on $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

(2) An algebraic stack $\mathcal{X}$ over a scheme S is a stack over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$.

Proof. To add.

$\qquad\square$

This allows us to finally prove that many properties of representable morphisms of algebraic stacks descend in the fppf topology, generalizing the case of smooth descent from Proposition 3.3.4.

Proposition 6.2.3. *Let $\mathcal{P}$ be one of the following properties of morphisms of algebraic stacks: representable, isomorphism, open immersion, closed immersion,*

locally closed immersion, affine, or quasi-affine. Consider a cartesian diagram

$$
\begin{array}{ccc}
\mathcal{X}' & \longrightarrow & \mathcal{Y}' \\
\downarrow & \square & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{Y}
\end{array}
$$

of algebraic stacks where $\mathcal{Y}' \to \mathcal{Y}$ is fppf. Then $\mathcal{X} \to \mathcal{Y}$ has $\mathcal{P}$ if and only if $\mathcal{X}' \to \mathcal{Y}'$ has $\mathcal{P}$.

6.2.2   Fppf groupoids and quotient stacks

If $R \rightrightarrows U$ is an fppf equivalence relation of algebraic spaces, we define U/R as the sheafification in big fppf topology $\mathrm{Sch}_{\mathrm{fppf}}$ of the presheaf $T \mapsto U(T)/R(T)$. Likewise, if $s, t\colon R \rightrightarrows U$ is an fppf groupoid of algebraic spaces, we define $[U/R]$ as the stackification in $\mathrm{Sch}_{\mathrm{fppf}}$ of the prestack $[U/R]^{\mathrm{pre}}$, whose fiber category over a scheme T is the category of T-points of U where a morphism from $a \in U(T)$ to $b \in U(T)$ is an element $r \in R(R)$ such that $s(r) = a$ and $t(r) = b$.

The definitions of U/R and $[U/R]$ are consistent with the quotient of a smooth equivalence relation or groupoid as defined in Definition 3.4.7 using in the big étale topology $\mathrm{Sch}_{\mathrm{ét}}$. This is because the sheafification U/R in $\mathrm{Sch}_{\mathrm{ét}}$ is an an algebraic space by Corollary 4.4.12 and thus a sheaf in the fppf topology by Proposition 6.2.2. Similarly, the stackification $[U/R]$ over $\mathrm{Sch}_{\mathrm{ét}}$ is an algebraic stack by Theorem 3.4.11 and thus a stack in the fppf topology by Proposition 6.2.2.

Corollary 6.2.4.

(1) *If $R \rightrightarrows U$ is an fppf equivalence relation of algebraic spaces, then the quotient U/R is an algebraic space.*

(2) *If $R \rightrightarrows U$ is an fppf groupoid of algebraic spaces, then the quotient $[U/R]$ is an algebraic stack.*

Proof. To add. $\qquad\square$

We will now show that a quotient stack arising from the action of an fppf group algebraic space is an algebraic stack; this was shown for the action of a smooth affine group scheme in Theorem 3.1.10. We first need to generalize the definition of a principal G-bundle given in Definition B.2.16 for an action by an fppf group algebraic space.

Definition 6.2.5 (Principal G-bundles)**.** If $G \to S$ is an fppf group algebraic space, then a *principal G-bundle over an S-scheme T* is an algebraic space P with an action of G via $\sigma\colon G \times_S P \to P$ such that $P \to X$ is a G-invariant fppf morphism and $(\sigma, p_2)\colon G \times_S P \to P \times_T P$ is an isomorphism. *Morphisms of principal G-bundles* are G-equivariant morphisms of schemes. We say that a principal G-bundle $P \to T$ is *trivial* if there is a G-equivariant isomorphism $P \cong G \times T$.

When $G \to S$ is smooth, then every principal G-bundle $P \to T$ is trivialized by the smooth cover $P \to T$ and since smooth morphisms étale locally have sections, there is an étale cover $T' \to T$ such that $P_{T'}$ is trivial.

Remark 6.2.6. It is important to require that P is an algebraic space and not a scheme since we want principal G-bundles to satisfy descent and to be equivalent to the notion of a G-torsor (Definition 6.2.13). If $G \to S$ is affine, then P is

automatically a scheme and the above definition thus agrees with Definition B.2.16. Indeed, P is a sheaf in the fppf topology (Proposition 6.2.2) and if $U \to P$ is an étale presentation, then $P \to T$ pulls back under the fppf composition $U \to P \to T$ to the affine morphism $G \times_S U \to U$. By Effective Descent (Proposition 2.2.15), $P \to T$ is affine and in particular that P is a scheme.

Raynaud provides an example of an abelian variety G and a principal G-bundle that is not scheme [Ray70, XIII 3.2].

Definition 6.2.7 (Quotient stacks)**.** Let $G \to S$ be an fppf group algebraic space acting on an algebraic space U over S. We define the *quotient stack* $[U/G]$ as the category over Sch/S whose objects over an S-scheme T are diagrams

$$
\begin{array}{ccc}
P & \longrightarrow & U \\
\downarrow & & \\
T & &
\end{array}
\tag{6.2.8}
$$

where $P \to T$ is a principal G-bundle and $P \to U$ is a G-equivariant morphism of schemes. A morphism $(P' \to T', P' \to U) \to (P \to T, P \to U)$ consists of a morphism $T' \to T$ and a G-equivariant morphism $P' \to P$ of schemes such that the diagram

$$
\begin{array}{ccc}
P' \longrightarrow P \longrightarrow U \\
\downarrow \quad \square \quad \downarrow \\
T' \longrightarrow T
\end{array}
$$

is commutative and the left square is cartesian.

Definition 6.2.9 (Classifying stacks)**.** Let $G \to S$ be an fppf group algebraic space. The *classifying stack* BG of G is defined as the quotient stack $[S/G]$. It classifies principal G-bundles $P \to T$.

Proposition 6.2.10. *If $G \to S$ is an fppf group algebraic space acting on an algebraic space U over S, then the quotient stack $[U/G]$ is an algebraic stack. In particular, the classifying stack BG is algebraic.*

Proof. Given a map $T \to [U/G]$ corresponding to an object (6.2.8), there is a cartesian diagram

$$
\begin{array}{ccc}
P & \longrightarrow & U \\
\downarrow & \square & \downarrow \\
T & \longrightarrow & [U/G]
\end{array}
$$

of stacks over $\mathrm{Sch}_{\mathrm{fppf}}$; this extends Exercise 2.3.35. As $P \to T$ is an fppf morphism of algebraic spaces, $U \to [U/G]$ is an fppf representable morphism. It follows from Theorem 6.2.1 that $[U/G]$ is an algebraic stack. $\square$

Exercise 6.2.11. Let $G \to S$ be an fppf group algebraic space acting on an algebraic space U over S.

(a) Generalize Exercise 2.4.20 by showing that the stackification of the prestack $[U/G]^{\mathrm{pre}}$ in the fppf topology is $[U/G]$.

(b) Provide an example where the stackification of $[U/G]^{\mathrm{pre}}$ in the étale topology is not isomorphic to $[U/G]$.

Recalling that $\boldsymbol{\mu}_{n,\mathbb{Z}}$ is the subgroup of $\mathbb{G}_{m,\mathbb{Z}}$ defined by $\operatorname{Spec}\mathbb{Z}[x]/(x^n - 1)$, we can now deduce that $B\boldsymbol{\mu}_{n,\mathbb{Z}}$ is an algebraic stack. If $\mathbb{k}$ is a field of characteristic p, then $\boldsymbol{\mu}_n := \boldsymbol{\mu}_{n,\mathbb{k}}$ is smooth if and only if p doesn't divide n.

Exercise 6.2.12. Let $\mathbb{k}$ be a field.

(a) Exhibit an explicit smooth presentation of $B\boldsymbol{\mu}_n$.

(b) Show that $B\boldsymbol{\mu}_n$ is equivalent to the stack over $(\operatorname{Sch}/\mathbb{k})_{\text{ét}}$ whose objects over a scheme T are pairs (L, α) consisting of a line bundle L on T and a trivialization $\alpha\colon \mathcal{O}_T \xrightarrow{\sim} L^{\otimes n}$.

(c) Show that $B\boldsymbol{\mu}_n$ is a smooth and proper algebraic stack of dimension 0.

(d) Show that $B\boldsymbol{\mu}_n$ is a Deligne–Mumford stack if and only if n is prime to the characteristic.

(e) If $x\colon \operatorname{Spec}\mathbb{k} \to B\boldsymbol{\mu}_n$ denotes the canonical presentation, compute the tangent space $T_{B\boldsymbol{\mu}_n,x}$.

6.2.3 Torsors

If G is a sheaf of groups, then a G-torsor is a sheaf of sets locally isomorphic to G.

Definition 6.2.13 (Torsors). Let $\mathcal{S}$ be a site and G a sheaf of (not necessarily abelian) groups on $\mathcal{S}$. A *G-torsor on $\mathcal{S}$* is a sheaf P of sets on $\mathcal{S}$ with a left action $\sigma\colon G \times P \to P$ of G such that

(1) for every object $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ such that $P(T_i) \neq 0$ for each i, and

(2) the action map $(\sigma, p_2)\colon G \times P \to P \times P$ is an isomorphism.

If $T \in \mathcal{S}$ is an object and G is a sheaf of groups on the localized site $\mathcal{S}/T$ (Example 2.1.11), then a *G-torsor over T* is by definition a G-torsor on the $\mathcal{S}/T$.

Morphisms of G-torsors are G-equivariant morphisms of sheaves. We say that a G-torsor P is *trivial* if P is G-equivalently isomorphic to G.

Exercise 6.2.14. Show that Any morphism of G-torsors is an isomorphism.

Example 6.2.15. Let $\mathcal{X}$ be a stack over a site $\mathcal{S}$, and let $a, b \in \mathcal{X}$ be objects over $S \in \mathcal{S}$. The sheaf $\underline{\operatorname{Isom}}_S(a, b)$ of isomorphisms is a torsor for $\underline{\operatorname{Aut}}(a)$ under the action given by precomposition.

Given a morphism $f\colon T' \to T$ and a G-torsor P over T, the restriction $P|_{T'}$ is the sheaf on $\mathcal{S}/T'$ whose whose sections over a T'-scheme S are $P(S)$; the restriction $P|_{T'}$ is naturally a G-torsor over T'.

Exercise 6.2.16. Let $\mathcal{S}$ be a site with a final object S and G be a sheaf of groups on $\mathcal{S}$.

(a) Show that Axiom (1) is equivalent to $P \to S$ being an epimorphism of sheaves.

(b) If P is a G-torsor, show that S is isomorphic to the quotient sheaf P/G.

(c) Show that a G-torsor P is trivial if and only if there exists a section $s\colon S \to P$ of the structure morphism $P \to S$.

(d) Show that a sheaf P of sets on $\mathcal{S}$ with a left action by G is a G-torsor if and only if there exists a covering $\{S_i \to S\}$ and isomorphisms $P|_{S_i} \cong G|_{S_i}$ of $G|_{S_i}$-torsors.

Example 6.2.17 (Principal G-bundles). If $G \to S$ is an fppf group scheme, then there is an equivalence of categories between G-torsors in the fppf topology and principal G-bundles (as defined in Definition 6.2.5). To see this, first suppose that $P \to T$ is a principal G-bundle over an S-scheme T, i.e. $P \to T$ is an fppf morphism of algebraic spaces where G is equipped with a free and transitive action of $G \times_S T$. Since algebraic spaces are sheaves in the fppf topology (Proposition 6.2.2), we may view $G \times_S T$ as a sheaf of groups on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$ and P as a sheaf of sets on $(\mathrm{Sch}/T)_{\mathrm{\acute{e}t}}$. Since every principal G-bundle is locally trivial in the fppf topology (Proposition B.2.19), P is a $G \times_S T$-torsor on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$. Conversely, given a $G \times_S T$-torsor P on $(\mathrm{Sch}/T)_{\mathrm{fppf}}$, then by Exercise 6.2.16 there is an fppf cover $T' \to T$ such that $P \times_T T' \cong G \times_T T'$. Therefore, $P \times_T T' \to P$ is an fppf morphism from an algebraic space, and Corollary 6.2.4 implies that P is an algebraic space. It follows that $P \to T$ is a principal G-bundle.

If in addition $G \to S$ is smooth, then there is an equivalence of categories between G-torsors in the étale topology and principal G-bundles. This holds because every principal G-bundle $P \to T$ is étale locally trivial and therefore P is a $G \times_S T$-torsor on $(\mathrm{Sch}/T)_{\mathrm{\acute{e}t}}$.

6.2.4   Gerbes

Gerbes are a 2-categorical generalization of torsors. While torsors are locally isomorphic to a sheaf of groups G, gerbes are locally isomorphic to classifying stacks BG. Gerbes are central figures in moduli theory; for the purposes of this book, our main examples are residual gerbes (Proposition 6.2.36) and banded $\mathbb{G}_m$-gerbes such as the map $\underline{\mathrm{Pic}}_X \to \underline{\mathrm{Pic}}_X$ from the Picard stack to Picard scheme (Theorem 6.2.58) and the map $\mathrm{Bun}^s_{r,d}(C) \to M^s_{r,d}(C)$ from the stack of stable vector bundles to its coarse moduli space.

Definition 6.2.18 (Gerbes). A stack $\mathcal{X}$ over a site $\mathcal{S}$ is called a *gerbe* if

(1) for every object $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ in $\mathcal{S}$ such that each fiber category $\mathcal{X}(T_i)$ is non-empty; and

(2) for objects $x, y \in \mathcal{X}$ over $T \in \mathcal{S}$, there exists a covering $\{T_i \to T\}$ and isomorphisms $x|_{T_i} \xrightarrow{\sim} y|_{T_i}$ for each i.

We say that a gerbe $\mathcal{X}$ is *trivial* if there is a section $\mathcal{S} \to \mathcal{X}$ of $\mathcal{X} \to \mathcal{S}$. When $\mathcal{S}$ has a final object S, then the triviality of a gerbe $\mathcal{X}$ is equivalent to the existence of an element of $\mathcal{X}(S)$.

Example 6.2.19. If G is a sheaf of groups on a site $\mathcal{S}$, then we extend Definition 6.2.9 by defining the *classifying prestack of G* as the category BG over $\mathcal{S}$ consisting of pairs (P, T) where $T \in \mathcal{S}$ and P is G-torsor over $\mathcal{S}/T$ (Definition 6.2.13). A morphism $(P', T') \to (P, T)$ is the data of a morphism $T' \to T$ in $\mathcal{S}$ and an isomorphism $P' \to P|_{T'}$ of G-torsors, where $P|_{T'}$ denotes the restriction of P along $T' \to T$.

The classifying stack BG is a gerbe over $\mathcal{S}$ because every G-torsor over T is locally isomorphic to the trivial G-torsor $G \times T$.

Exercise 6.2.20 (Gerbes are locally classifying stacks). Let $\mathcal{S}$ be a site with a final object $S \in \mathcal{S}$, and let $\mathcal{X}$ be a stack over $\mathcal{S}$. Show that $\mathcal{X}$ is a gerbe if and only if there exists a covering $\{S_i \to S\}$ and sheaves of groups G_i on the localized site $\mathcal{S}/S_i$ (Example 2.1.11) such that there is an isomorphism $\mathcal{X} \times_{\mathcal{S}} \mathcal{S}/S_i \cong BG_i$ over $\mathcal{S}/S_i$.

Exercise 6.2.21. Let S be a scheme and let $\mathcal{X}$ be a gerbe over $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. If the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, show that $\mathcal{X}$ is an algebraic stack.

Important examples of gerbes $\mathcal{X}$ are those *banded* by a sheaf of groups. This means that $\mathcal{X}$ is equipped with the additional data of a natural isomorphism $G(T) \to \mathrm{Aut}_T(x)$ for every object $x \in \mathcal{X}(t)$.

Definition 6.2.22 (Banded G-gerbes). Let G be an *abelian* sheaf on a site $\mathcal{S}$. A stack $\mathcal{X}$ over $\mathcal{S}$ is a *gerbe banded by G* (or a *banded G-gerbe* or simply a *G-gerbe*) is a gerbe together with the data of isomorphisms $\psi_x \colon G|_T \to \underline{\mathrm{Aut}}_T(x)$ of sheaves for each object $x \in \mathcal{X}(T)$. We require that for each isomorphism $\alpha \colon x \overset{\sim}{\to} y$ over T, the diagram

$$
\begin{array}{ccc}
& G|_T & \\
{\scriptstyle \psi_x} \swarrow & & \searrow {\scriptstyle \psi_y} \\
\underline{\mathrm{Aut}}_T(x) & \xrightarrow{\quad \mathrm{Inn}_\alpha \quad} & \underline{\mathrm{Aut}}_T(y).
\end{array}
\tag{6.2.23}
$$

commutes, where $\mathrm{Inn}_\alpha(\tau) = \alpha \tau \alpha^{-1}$. The data of the isomorphisms ψ_x is called the *band* of $\mathcal{X}$.

A *morphism of banded G-gerbes* is a morphism of stacks compatible with the bands.

Remark 6.2.24. Here's another way to think about a band of a gerbe. Let $\mathcal{X}_\mathcal{S}$ be the *restricted site* whose underlying category is $\mathcal{X}$ and where a covering of $a \in \mathcal{X}(S)$ is a covering of S. Then the inertia stack $I_\mathcal{X} = \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is a sheaf of groups on $\mathcal{X}_\mathcal{S}$: for $a \in \mathcal{X}(S)$, we have $I_\mathcal{X}(a) = \mathrm{Isom}_S(a)$. The compatibility condition (6.2.23) ensures that there is an isomorphism $\psi \colon G|_\mathcal{X} \to I_\mathcal{X}$ of sheaves on $\mathcal{X}_\mathcal{S}$.

Example 6.2.25 (The trivial banded gerbe). If G is an abelian sheaf on a site $\mathcal{S}$ with a final object S, then the classifying stack BG of Example 6.2.19 is a banded G-gerbe and is trivial (i.e. $BG(S) \neq \emptyset$). A banded G-gerbe $\mathcal{X}$ over $\mathcal{S}$ is trivial if and only if $\mathcal{X} \cong BG$. We refer to BG as the *trivial banded G-gerbe*.

Exercise 6.2.26 (Band associated to a gerbe). Let $\mathcal{S}$ be a site with a final object S. Let $\mathcal{X}$ be an *abelian gerbe* over $\mathcal{S}$, i.e. a gerbe $\mathcal{X}$ such that $\mathrm{Aut}_T(a)$ is abelian for every object $a \in \mathcal{X}(T)$. Show that there is a sheaf of groups G on S such that $\mathcal{X}$ is banded by G.

Hint: Use Axiom (1) of a gerbe to find a covering $\{X_i \to X\}$ and elements $a_i \in \mathcal{X}(X_i)$. Use Axiom (2) to glue the sheaves $G_i := \underline{\mathrm{Aut}}_{X_i}(a_i)$ to a sheaf G.

6.2.5 Algebraic gerbes

Attached to any algebraic stack $\mathcal{Y}$ is the big fppf site $(\mathrm{Sch}\,/\mathcal{Y})_{\mathrm{fppf}}$ of schemes over $\mathcal{Y}$: the underlying category of $(\mathrm{Sch}\,/\mathcal{Y})_{\mathrm{fppf}}$ is $\mathcal{Y}$ and a covering of an object $y \in \mathcal{Y}(T)$ is a covering of T. Moreover, if $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, then $\mathcal{X}$ is a stack over $(\mathrm{Sch}\,/\mathcal{Y})_{\mathrm{fppf}}$ thanks to Proposition 6.2.2.

Definition 6.2.27 (Gerbes and banded G-gerbes). A morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a *gerbe* if $\mathcal{X}$ is a gerbe over the big fppf site $(\mathrm{Sch}\,/\mathcal{Y})_{\mathrm{fppf}}$.

If $G \to S$ is a commutative fppf group scheme, a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks over S is called a *banded G-gerbe* (or simply *G-gerbe*) if $\mathcal{X}$ is a gerbe over $(\mathrm{Sch}\,/\mathcal{Y})_{\mathrm{fppf}}$ banded by the sheaf of groups $G \times_S \mathcal{Y}$.

We say that an algebraic stack $\mathcal{X}$ is a *gerbe* (resp. *banded G-gerbe*) if there exists a morphism $\mathcal{X} \to X$ to an algebraic space which is a gerbe (resp. banded G-gerbe).

If $G \to S$ is a commutative fppf group scheme, the classifying stack $BG \to S$ is a banded G-gerbe. Note that a banded G-gerbe $\mathcal{X} \to X$ over an algebraic space X is trivial if and only if $\mathcal{X} \cong BG \times_S X$.

Proposition 6.2.28. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks.*

(1) *The morphism $\mathcal{X} \to \mathcal{Y}$ is a gerbe if and only if there exists an fppf morphism $V \to \mathcal{Y}$ from a scheme and an fppf group algebraic space $G \to V$ such that $\mathcal{X} \times_{\mathcal{Y}} V \cong BG$.*

(2) *If $\mathcal{X} \to \mathcal{Y}$ is a gerbe, then $\mathcal{X} \to \mathcal{Y}$ is a smooth morphism.*

(3) *If $G \to S$ is an fppf group scheme and $\mathcal{X} \to X$ is a banded G-gerbe over an algebraic space X, then there exists an étale cover $X' \to X$ such that $\mathcal{X} \times_X X' \cong X' \times_S BG$.*

Proof. We first prove (1). For ($\Rightarrow$), Exercise 6.2.20 implies that there is an fppf morphism $V \to \mathcal{Y}$ and a sheaf of groups G on V such that $\mathcal{X} \times_{\mathcal{Y}} V \cong BG$. Since $\mathcal{X} \times_{\mathcal{Y}} V$ is an algebraic stack, its diagonal is representable, and thus G is an algebraic space. Conversely, suppose that $\mathcal{X} \times_{\mathcal{Y}} V \cong BG$ for an fppf morphism $V \to \mathcal{Y}$ and an fppf group algebraic space $G \to V$. To see Axiom (1) of a gerbe, if $a \in (\mathrm{Sch}\,/\mathcal{Y})$ is an object over a scheme T, then $V_T := V \times_{\mathcal{Y}} T \to T$ is an fppf covering and since $\mathcal{X} \times_{\mathcal{Y}} V_T \cong BG_{V_T}$, there is an object of $\mathcal{X}$ over V_T. Similarly for Axiom (2), if $x_1, x_2 \in \mathcal{X}$ are objects over $y \in \mathcal{Y}(T)$, then pull backs of x_1 and x_2 become isomorphic under the fppf covering $V_T \to T$.

Part (2) follows from (1) since $BG \to V$ is a smooth morphism; indeed smoothness is an fppf local property on the source (Proposition B.1.10). For part (3), $\mathcal{X} \to X$ has a section after base changing by the *smooth* and surjective morphism $\mathcal{X} \to X$. Choosing a smooth presentation $U \to \mathcal{X}$, then the composition $U \to \mathcal{X} \to X$ is smooth and the base change $\mathcal{X} \times_X U \to U$ has a section. Since smooth morphisms étale locally have sections (Corollary A.3.5), there is an étale cover $X' \to X$ factoring through $U \to X$. It follows that $\mathcal{X} \times_X X' \to X'$ has a section or in other words that $\mathcal{X} \times_X X' \cong X' \times_S BG$. $\qquad\square$

Exercise 6.2.29. Show that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a gerbe if and only if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ are fppf.

We will later show that an algebraic stack $\mathcal{X}$ is a gerbe if and only if $I_{\mathcal{X}} \to \mathcal{X}$ is fppf (Proposition 6.2.47).

Exercise 6.2.30. Show that there is a non-trivial isomorphism

$$\alpha \colon B(\mathbb{Z}/2) \times (\mathbb{A}^1 \smallsetminus 0) \to B(\mathbb{Z}/2) \times (\mathbb{A}^1 \smallsetminus 0)$$

of trivial banded $\mathbb{Z}/2$-gerbes over $\mathbb{A}^1$ which glues to a non-trivial banded $\mathbb{Z}/2$-gerbe over $\mathbb{P}^1$.

Exercise 6.2.31. Let $1 \to K \to G \to Q \to 1$ be a short exact sequence of affine algebraic groups over $\Bbbk$ such that K is commutative. Show that $BG \to BQ$ is a banded K-gerbe which is trivial if and only if the sequence splits.

Exercise 6.2.32. Assume that $\mathrm{char}(\Bbbk) \neq 2, 3$. Recall from Exercise 3.1.19(c) that the moduli stack of stable elliptic curves has a quotient description $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4, 6) := [(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts with weights 4 and 6.

(a) Show that the j-line $\pi \colon \mathcal{M}_{1,1} \to \mathbb{A}^1$ is a trival banded $\mathbb{Z}/2$-gerbe over $\mathbb{A}^1 \setminus \{0, 1728\}$.

Hint: Construct a family of elliptic curves over $\mathbb{A}^1_{\Bbbk}\setminus\{0,1728\}$ via the Weierstrass equation

$$y^2 z + xyz = x^3 - \frac{36}{t-1728}xz^2 - \frac{1}{t-1728}z^3,$$

where t is the coordinate on $\mathbb{A}^1$, where the discriminant $\Delta = t^2/(t-1728)^3$. See [Sil09, Prop. III.1.4(c)].

(b) Consider the map $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4,6) \to \mathcal{P}(2,3)$ induced the homomorphism $\mathbb{G}_m \to \mathbb{G}_m, t \mapsto t^2$; note that $\mathcal{P}(2,3)$ is the banded $\mathbb{Z}/2$-gerbe obtained by rigidifying along the hyperelliptic involution (see Proposition 6.2.47), and the restriction along $\mathcal{P}(2,3) \setminus \{0,1728,\infty\}$ is tje gerbe from (a). Show that $\overline{\mathcal{M}}_{1,1} \to \mathcal{Y}$ is non-trivial.

   Hint: If it's trivial, show that there are torsion line bundles contradicting that $\mathrm{Pic}(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}$ from Exercise 6.1.14(a).

(c) Show that the rigidification $\mathcal{M}_{1,1} \to \mathcal{P}(2,3) \setminus \infty$ is also non-trivial.

   Hint: If it's trivial, show that $\mathcal{M}_{1,1}$ has three 2-torsion line bundles contradicting that $\mathrm{Pic}(\mathcal{M}_{1,1}) = \mathbb{Z}/12$ from Exercise 6.1.14(b).

(d) For $g \geq 2$, let $\mathcal{H}_g \subset \mathcal{M}_g$ be the closed substack classifying hyperelliptic curves. Show that the rigidification $\mathcal{H}_g \to \mathcal{Y}$ along the hyperelliptic involution is a non-trivial banded $\mathbb{Z}/2$-gerbe.

Exercise 6.2.33.

(a) Show that every gerbe $\mathcal{X}$ over an algebraic space that is étale locally isomorphic to $B\mathbb{Z}/2$ is in fact banded by $\mathbb{Z}/2$.

(b) Give an example of a gerbe over an algebraic space that is étale locally isomorphic to $B\mathbb{G}_m$ but not banded by $\mathbb{G}_m$.

   Hint: Consider the classifying stack of a form of $\mathbb{G}_m$ (see Exercise 4.1.44).

Exercise 6.2.34 (Root gerbes and root stacks revisited)**.** Recall that root gerbes and stacks were introduced in Examples 3.9.12 and 3.9.13.

(a) Since we now know how to construct quotient stacks by actions of $\boldsymbol{\mu}_r$ over any base scheme S, show that Exercise 3.9.14 still holds without the condition that r is invertible in $\Gamma(S, \mathcal{O}_S)$.

(b) Given a scheme X, a line bundle L, and a section $s \in \Gamma(X, L)$, show that $X(\sqrt[r]{L}) \to X$ and the restriction of $X(\sqrt[r]{L,s}) \to X$ along $V(s)$ are banded $\boldsymbol{\mu}_r$-gerbes.

(c) Show that $X(\sqrt[r]{L}) \to X$ is the trivial $\boldsymbol{\mu}_r$-gerbe if and only if L has an rth root.

(d) Consider an exact sequence $1 \to \mu_r \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ and a $\mathbb{G}_m$-torsor P'' corresponding to a line bundle L''. Show that $X(\sqrt[r]{L})$ is isomorphic to the gerbe of trivializations $\mathcal{G}_{P''}$ defined in Exercise 6.2.40(b).

6.2.6 Residual gerbes revisited

Given an algebraic stack $\mathcal{X}$ and $x \in |\mathcal{X}|$, recall from Definition 3.5.15 that *the residual gerbe at x* (if it exists) is a reduced, locally noetherian algebraic stack $\mathcal{G}_x$ with a monomorphism $\mathcal{G}_x \hookrightarrow \mathcal{X}$ such that $|\mathcal{G}_x|$ is a point mapping to x. We've already shown that the residual gerbe at a finite type point exists (Proposition 3.5.19).

Residual gerbes are unique.

Lemma 6.2.35. *Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a point. If the residual gerbe at x exists, it is unique.*

Proof. To add. $\qquad\square$

We now establish the existence of residual gerbes at all points and moreover show that they are in fact gerbes.

Proposition 6.2.36 (Existence of Residual Gerbes II)**.** *If $\mathcal{X}$ is a noetherian algebraic stack and $x \in |\mathcal{X}|$ is a point, then the residual gerbe $\mathcal{G}_x$ exists and is a gerbe over a field $\kappa(x)$, called the* residue field *of x.*

Proof. To add. $\qquad\square$

If $\mathcal{X}$ is a quasi-separated algebraic stack of finite type over a field $\Bbbk$ and $x \in \mathcal{X}(\Bbbk)$, then $\mathcal{G}_x = BG_x$ (Proposition 3.5.19). More generally, we have:

Exercise 6.2.37. Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point.

(1) For any representative $\overline{x}\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$ of x, there is a cartesian diagram

$$
\begin{array}{ccc}
B_{\Bbbk}G_{\overline{x}} \longrightarrow \mathcal{G}_x \lhook\joinrel\longrightarrow \mathcal{X} \\
\downarrow \qquad \square \qquad \downarrow \\
\operatorname{Spec}\Bbbk \longrightarrow \operatorname{Spec}\kappa(x).
\end{array}
$$

(2) If the stabilizer of x is smooth, show that there is a finite separable extension $\kappa(x) \to \Bbbk$ and a representative of x over $\Bbbk$.

Exercise 6.2.38. Let $C \subset \mathbb{P}^2_{\Bbbk}$ be a non-split quadric over a field $\Bbbk$, and let $\Bbbk \to \Bbbk'$ be a quadratic extension such that $C \times_{\Bbbk} \Bbbk' \cong \mathbb{P}^1_{\Bbbk'}$. Let $D \subset C$ be a divisor of degree 6 and let $X \to \mathbb{P}^1_{\Bbbk'}$ be the double cover ramified over $D \times_{\Bbbk} \Bbbk'$. Show that the residual gerbe of $[X] \in \mathcal{M}_2$ is non-trivial and has residue field $\Bbbk$.

To give some context for the above exercise, consider the rigidification $\mathcal{M}_2 \to \mathcal{Y}$ of the hyperelliptic involution is a non-trivial banded $\mathbb{Z}/2$-gerbe (see Exercise 6.2.32(d)). The restriction to the locus $\mathcal{M}_2^\circ \subset \mathcal{M}_2$ of curves whose only non-trivial automorphism is the hyperelliptic involution is the coarse moduli space $\mathcal{M}_2^\circ \to M_2^\circ$. This is a non-trivial banded $\mathbb{Z}/2$-gerbe and the generic fiber (over the residue field of the generic point of M_2) is also a non-trivial gerbe. Exercise 6.2.38 on the other hand shows that the reduced fibers of $\mathcal{M}_2^\circ \to M_2^\circ$ over closed points can be non-trivial gerbes. $=$

6.2.7 Cohomological characterization

The following exercises provide cohomological characterizations of torsors and gerbes for an abelian sheaf G on the small fppf site S_{fppf} of a scheme S. If G is represented by a smooth, commutative, and quasi-projective group scheme, then it turns out that $\mathrm{H}^i((\mathrm{Sch}/S)_{\mathrm{fppf}}, G) = \mathrm{H}^i((\mathrm{Sch}/S)_{\text{ét}}, G)$ (see Remark 4.1.43) and thus in this case we can use étale cohomology. For an extra challenge, try to prove these statements for abelian sheaves over any site. The reader may consult [Gir71] and [Ols16, §12] for detailed proofs.

Exercise 6.2.39 (Torsors)**.** Let S be a scheme.

(a) If G is an abelian sheaf on S_{fppf}, show that $\text{H}^1(S_{\text{fppf}}, G)$ is in bijective correspondence with isomorphism classes of G-torsors.

Hint: Imitate the proof using Čech cohomology that $\text{H}^1(X, \mathcal{O}_X^*) = \text{Pic}(X)$ for a scheme X.

(b) Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of abelian sheaves on S_{fppf}, and let

$$0 \to \text{H}^0(S_{\text{fppf}}, G') \to \text{H}^0(S_{\text{fppf}}, G) \to \text{H}^0(S_{\text{fppf}}, G'') \xrightarrow{\delta}$$
$$\to \text{H}^1(S_{\text{fppf}}, G') \xrightarrow{\alpha} \text{H}^1(S_{\text{fppf}}, G) \xrightarrow{\beta} \text{H}^1(S_{\text{fppf}}, G'') \to \cdots$$

be the corresponding long exact sequence. Show that under the bijection in (a), the boundary map δ assigns a section $S \to G''$ to the G'-torsor defined by the fiber product $G \times_{G''} S$. Show also that α assigns a G'-torsor P' to the quotient $P' \times^{G'} G := (P' \times G)/G'$ while β assigns a G-torsor P to $P \times^G G''$.

Exercise 6.2.40 (Gerbes)**.** Let S be a scheme.

(a) If G is an abelian sheaf on S_{fppf}, show that $\text{H}^2(\mathcal{S}, G)$ is in bijective correspondence with isomorphism classes of G-banded gerbes.

Hint: Let $0 \to G \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \cdots$ be an injective resolution. For a cohomology class $\alpha \in \text{H}^2(\mathcal{S}, G)$, define a stack $\mathcal{G}_\alpha$ over $\mathcal{S}$ as follows. Choose $\tau \in \Gamma(\mathcal{S}, I^2)$ with $d^2(\tau) = 0$ such that the image of τ in $\text{H}^2(\mathcal{S}, G)$ is α. Define $\mathcal{G}_\alpha$ as the category of pairs (S, σ) consisting of an object $S \in \mathcal{S}$ and a section $\sigma \in \Gamma(S, I^1)$ with $d^1(\sigma) = \tau|_S$. A morphism $(S', \sigma') \to (S, \sigma)$ is the data of a morphism $f : S' \to S$ and an element $\rho \in \Gamma(S', I^0)$ with boundary $d^0(\rho) = \sigma' - f^*(\sigma)$. Show that $\mathcal{G}_\alpha$ is a G-banded gerbe and that the assignment $\alpha \mapsto \mathcal{G}_\alpha$ gives the stated bijection.

Let $0 \to G' \to G \to G'' \to 0$ be an exact sequence of abelian sheaves on S_{fppf}, and let

$$\cdots \to \text{H}^1(S_{\text{fppf}}, G'') \xrightarrow{\delta} \text{H}^2(S_{\text{fppf}}, G') \xrightarrow{\alpha} \text{H}^2(S_{\text{fppf}}, G) \xrightarrow{\beta} \text{H}^2(S_{\text{fppf}}, G'') \to \cdots$$

be the corresponding long exact sequence.

(b) Show that under the bijection in (a), the boundary map δ assigns a G''-torsor $P'' \to S$ to the *gerbe of trivializations* $\mathcal{G}_{P''}$. The objects of the prestack $\mathcal{G}_{P''}$ over an S-scheme T is a pair (P, α) consisting of a G-torsor $P \to T$ and a trivialization $\alpha : P \times^G G'' \cong P'' \times_S T$ of G''-torsors. Morphisms in $\mathcal{G}_{P''}$ are morphisms of G-torsors compatible with the trivializations.

(c) Suppose that G', G, and G'' are represented by commutative and affine algebraic groups over a field $\Bbbk$. Show that if $P'' \to S$ is a G''-torsor, then $\mathcal{G}_{P''}$ is identified with both the quotient stack $[P''/G]$ and the fiber product of $BG \to BG''$ and the map $S \to BG''$ corresponding the G''-torsor P''.

Exercise 6.2.41 (Group structure)**.** If G is an abelian sheaf on the small fppf site S_{fppf} of a scheme S, show that the group laws of $\text{H}^1(S_{\text{fppf}}, G)$ and $\text{H}^2(S_{\text{fppf}}, G)$ can be described geometrically as follows:

(a) The product of two G-torsors P_1 and P_2 is the *contracted product* $P_1 \wedge^G P_2$ defined as the sheaf quotient $(P_1 \times P_2)/G$ where $h \cdot (p_1, p_2) = (h^{-1}p_1, hp_2)$ with the G-action specified by $g \cdot (p_1, p_2) = (gp_1, p_2) = (p_1, gp_2)$. The inverse of a G-torsor P is the sheaf P with the inverted G-action: $g \cdot p = g^{-1}p$.

(b) The product of two banded G-gerbes $(\mathcal{X}_1, \psi_{1,x})$ and $(\mathcal{X}_2, \psi_{2,x})$ is the *contracted product* $\mathcal{X}_1 \wedge^G P_2$, which is defined as the rigidification $(\mathcal{X}_1 \times \mathcal{X}_2) /\!\!/ G$ (see Proposition 6.2.47) of the product $\mathcal{X}_1 \times \mathcal{X}_2$ along the subgroup $(\psi_1, \psi_2) \colon G|_{\mathcal{X}_1 \times \mathcal{X}_2} \to I_{\mathcal{X}_1 \times \mathcal{X}_2}$ defined by the bands ψ_1 and ψ_2. The inverse $(\mathcal{X}, \psi_x)^{-1} = (\mathcal{X}, \psi_x^{-1})$ inverts the band.

Remark 6.2.42 (Banded μ_n-gerbes over $\mathbb{P}^1$). Over an algebraically closed field $\Bbbk$, isomorphism classes of banded μ_n-gerbes over $\mathbb{P}^1$ are in bijection with $\mathbb{Z}/n\mathbb{Z}$. To see this, observe that the exact sequence $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$ induces an exact sequence on cohomology

$$\mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) \xrightarrow{n} \mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) \to \mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mu_n) \to \mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m).$$

Since $\mathrm{H}^1(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) = \mathrm{Pic}(\mathbb{P}^1_{\text{ét}}) = \mathbb{Z}$, we can use the fact that $\mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mathbb{G}_m) = 0$ to conclude that $\mathrm{H}^2(\mathbb{P}^1_{\text{ét}}, \mu_n) = \mathbb{Z}/n\mathbb{Z}$. The image of a line bundle $\mathcal{O}(d)$ is equivalent to the root stack $\mathbb{P}^1(\sqrt[n]{\mathcal{O}(d)})$, and this gerbe is trivial if and only if n divides d. The gerbe $\mathbb{P}^1(\sqrt[n]{\mathcal{O}(1)})$ is isomorphic to the quotient stack $[(\mathbb{A}^2 \smallsetminus 0)/\mathbb{G}_m]$ where $t \cdot (x,y) = (t^n x, t^n y)$.

Exercise 6.2.43 ($\mathbb{G}_m$-gerbes and twisted sheaves). Let $\mathcal{X} \to X$ be a $\mathbb{G}_m$-gerbe over an algebraic space X. We say that a coherent sheaf F on $\mathcal{X}$ is *1-twisted* if for every field-valued point $\mathrm{Spec}\,\Bbbk \to \mathcal{X}$, the $\mathbb{G}_m$-representation corresponding to the pullback of E under $B\mathbb{G}_m = BG_x \to \mathcal{X}$ decomposes as a direct sum of one-dimensional representations of *weight one*. Show that the $\mathbb{G}_m$-gerbe $\mathcal{X} \to X$ is trivial if and only if there exists a 1-twisted line bundle on $\mathcal{X}$.

Exercise 6.2.44 (Azumaya algebras). An *Azumaya algebra of rank r^2* over a noetherian scheme X is a (possibly non-commutative) associative $\mathcal{O}_X$-algebra A which is coherent as an $\mathcal{O}_X$-module and such that there is an étale covering $X' \to X$ where $A \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ is isomorphic to the matrix algebra $M_r(\mathcal{O}_X)$. We say that A is *trivial* if it is isomorphic to $M_r(\mathcal{O}_X)$. By Exercise B.2.31, Azumaya algebras are in bijection with principal PGL_r-bundles (which are also in bijection with Brauer–Severi schemes).

Let A be an Azumaya algebra over a noetherian scheme X of rank r^2.

(a) Define the *gerbe of trivializations of A* as the stack $\mathcal{G}_A$ over $(\mathrm{Sch}/X)_{\text{ét}}$ where an object over a X-scheme T is a pair (E, α) consisting of a vector bundle E on T of rank r and a trivialization $\alpha \colon \mathrm{End}_{\mathcal{O}_X}(E) \xrightarrow{\sim} A \otimes_{\mathcal{O}_X} \mathcal{O}_T$. Morphisms in $\mathcal{G}_A(T)$ are isomorphisms of vector bundles compatible with the trivializations. Show that $\mathcal{G}_A \to X$ is a banded $\mathbb{G}_m$-gerbe.

(b) Let P_A be the principal PGL_r-bundle corresponding to A. Identify $\mathcal{G}_A$ with the gerbe of trivializations $\mathcal{G}_{P_A}$ defined in Exercise 6.2.40(b) with respect to the PGL_r-torsor P_A and the surjection $\mathrm{GL}_r \to \mathrm{PGL}_r$.

(c) The exact sequence $1 \to \mathbb{G}_{m,X} \to \mathrm{GL}_{r,X} \to \mathrm{PGL}_{r,X} \to 1$ of sheaves on $X_{\text{ét}}$ induces a boundary map

$$\delta \colon \mathrm{H}^1(X_{\text{ét}}, \mathrm{PGL}_r) \to \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m).$$

Show that $\delta(P_A) = [\mathcal{G}_A] \in \mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)$ and that this is an r-torsion element.

(d) Show that the Azumaya algebra A is trivial if and only if $\mathcal{G}_A$ is trivial.

(e) Use the quarternions to construct a non-trivial $\mathbb{G}_m$-gerbe over $\mathrm{Spec}\,\mathbb{R}$.

Remark 6.2.45 (Brauer groups)**.** Two Azumaya algebras A and A' on a noetherian scheme X are *similar* if there exists vector bundles E and E' on X such that $A \otimes_{\mathcal{O}_X} \operatorname{End}_{\mathcal{O}_X}(E) \cong A \otimes_{\mathcal{O}_X} \operatorname{End}_{\mathcal{O}_X}(A')$. This defines an equivalence relation, and the *Brauer group of X* is the set $\operatorname{Br}(X)$ of Azumaya algebras up to similarity. The set $\operatorname{Br}(X)$ becomes a group under the operators $[A] \cdot [A'] = [A \otimes A']$ and $[A]^{-1} = [A^{\mathrm{op}}]$ (where A^{op} is the opposite algebra with same elements and addition as A but with multiplication reversed: $a \cdot^{A^{\mathrm{op}}} b = b \cdot^A a$).

The exact sequence $1 \to \mathbb{G}_{m,X} \to \operatorname{GL}_{r,X} \to \operatorname{PGL}_{r,X} \to 1$ induces a boundary map $\operatorname{H}^1(X_{\mathrm{\acute{e}t}}, \operatorname{PGL}_r) \to \operatorname{H}^2(X_{\mathrm{\acute{e}t}}, \mathbb{G}_m)$. Viewing $\operatorname{H}^1(X_{\mathrm{\acute{e}t}}, \operatorname{PGL}_r)$ as the set of isomorphism classes of Azumaya algebras of rank r^2 and $\operatorname{H}^2(X_{\mathrm{\acute{e}t}}, \mathbb{G}_m)$ as the set of isomorphism classes of banded $\mathbb{G}_m$-gerbes, the boundary map assigns an Azumaya algebra A to the gerbe of trivializations $\mathcal{G}_A$, which is an r-torsion element (see Exercise 6.2.44). Two Azumaya algebras A and A' (of possibly different rank) are similar if and only if $\mathcal{G}_A \cong \mathcal{G}_{A'}$, and thus there is an injective map

$$\operatorname{Br}(X) \hookrightarrow \operatorname{Br}'(X) := \operatorname{H}^2(X_{\mathrm{\acute{e}t}}, \mathbb{G}_m)_{\mathrm{tors}}, \qquad A \mapsto \mathcal{G}_A,$$

into the *cohomological Brauer group* $\operatorname{Br}'(X)$. See [Gro68] and [Mil80, §IV.2] for additional background.

Grothendieck asked whether $\operatorname{Br}(X) \hookrightarrow \operatorname{Br}'(X)$ is surjective. This is known in some cases. The strongest result is due to Gabber: $\operatorname{Br}(X) = \operatorname{Br}'(X)$ if X admits an ample line bundle (see [dJ03]). It is however open in general, even for smooth separated schemes over a field.

Exercise 6.2.46. Let X be a noetherian scheme and $\mathcal{X} \to X$ be a banded $\mathbb{G}_m$-gerbe corresponding to a cohomology class $[\mathcal{X}] \in \operatorname{H}^2(X_{\mathrm{\acute{e}t}}, \mathbb{G}_m)$.

(a) Show that the following are equivalent:

    (i) There exists an Azumaya algebra A on X such that $\mathcal{X} \cong \mathcal{G}_A$, i.e. $[\mathcal{X}]$ is in the image of $\operatorname{Br}'(X) \to \operatorname{Br}(X)$,

    (ii) $\mathcal{X}$ is a global quotient stack, and

    (iii) there exists a 1-twisted vector bundle E on $\mathcal{X}$ (see Exercise 6.2.43).

(b) Let X be a normal separated surface over $\mathbb{C}$ such that $\operatorname{H}^2(X, \mathbb{G}_m)$ contains a non-torsion element α; for an example, see [Gro68, II.1.11.b]. Conclude that the banded $\mathbb{G}_m$-gerbe corresponding to α is not a global quotient stack.

(c) Let $Y = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^2)$. Show that there is a non-trivial involution α of $(Y \smallsetminus 0) \times B(\mathbb{Z}/2)$ such that the stack $\mathcal{X}$, obtained by gluing the trivial banded $\mathbb{Z}/2$-gerbes over Y along α, is a banded $\mathbb{Z}/2$-gerbe over the non-separated union $Y \bigcup_{Y \smallsetminus 0} Y$ which is not a global quotient stack.

See also [EHKV01].

6.2.8 Rigidification

Proposition 6.2.47. *Let $\mathcal{X}$ be an algebraic stack such that $I_{\mathcal{X}} \to \mathcal{X}$ is fppf. Let X be the sheaf on $\operatorname{Sch}_{\mathrm{fppf}}$ defined by the sheafification of the functor assigning a scheme S to the set of isomorphism classes $\mathcal{X}(S)/\sim$ of objects. Then X is an algebraic space and $\mathcal{X} \to X$ is a gerbe.*

Proof. To show that X is an algebraic space, it suffices to show that $\mathcal{X} \to X$ is a smooth representable morphism. In this case, a smooth presentation $U \to \mathcal{X}$ induces a smooth presentation $U \to X$, and it follows from Corollary 4.4.12 (or

) that X is an algebraic space. As gerbes are smooth morphisms, it suffices to show that for every morphism $S \to X$ from a scheme, the fiber product $\mathcal{X} \times_X S \to S$ is a gerbe. By construction, there is an fppf cover $S' \to S$ and a morphism $a' : S' \to \mathcal{X}$ lifting the composition $S' \to S \to X$. Since the property of being a gerbe is fppf local, after replacing S with S', we may assume that $S \to X$ lifts to a map $a : S \to \mathcal{X}$. We claim that there is an isomorphism

$$\Psi : \mathcal{X} \times_X S \to B\underline{\mathrm{Aut}}_S(a).$$

An object of the fiber product $\mathcal{X} \times_X S$ consists of a pair (f, a') where $f : T \to S$ is a map of schemes and $b \in \mathcal{X}(T)$ such that $T \to S \to X$ and $T \xrightarrow{b} \mathcal{X} \to X$ agree. Define $\Psi(f, b)$ as the principal $\underline{\mathrm{Aut}}_S(a)$-bundle $\underline{\mathrm{Isom}}_T(f^*a, b)$. Observe that $\Psi(f, f^*a)$ maps to the trivial bundle.

Since $\mathcal{X} \times_X S$ and $B\underline{\mathrm{Aut}}_S(a)$ are both stacks in the fppf topology, we may verify that Ψ is essentially surjective fppf locally: if $P \to T$ is a principal $\underline{\mathrm{Aut}}_S(a)$-bundle, then there is an fppf cover $T' \to T$ such that $P \times_T T'$ is the trivial bundle, which we've seen is in the essential image. Similarly, we may verify that Ψ is fully faithful fppf locally. Let $(f, b), (f', b') \in (\mathcal{X} \times_X S)(T)$. Since the objects $f^*a, b, b' \in \mathcal{X}(T)$ map to the same T-valued point of X, by the construction of X, there is an fppf cover $T' \to T$ such that their pullbacks become isomorphic. By replacing T with T', we may assume that $f^*a \simeq b \simeq b'$ are isomorphic. In this case, the full faithfulness claim is clear as both $\Psi(f, b)$ and $\Psi(f, b')$ are trivial bundles.

Alternatively, we may construct X directly. Let $U \to \mathcal{X}$ be a smooth presentation and $R \rightrightarrows U$ the corresponding smooth groupoid. The stabilizer groupoid scheme $S_U = R \times_{U \times U} U = I_{\mathcal{X}} \times_{\mathcal{X}} U$ is fppf over U. There is an fppf equivalence relation $S_U \times_U R \rightrightarrows R$ where one arrow is given by composition and the other is projection. By , the fppf quotient $R' := R/(S_U \times_U R)$ is an algebraic space. There is an induced fppf equivalence relation $R' \rightrightarrows U$ and X is isomorphic to the fppf quotient U/R'.

See also [LMB00, Cor. 10.8] and [SP, Tags 06QD and 06QJ]. $\qquad\square$

We now consider a more general situation. If $\mathcal{X}$ is an algebraic stack, then the inertia stack $I_{\mathcal{X}}$ can be viewed as a group scheme over the big étale site $(\mathrm{Sch}\,/\mathcal{X})_{\text{ét}}$ of $\mathcal{X}$. As a group functor, $I_{\mathcal{X}}$ assigns an object $a \in \mathcal{X}(S)$ to the group $\mathrm{Aut}_S(a)$, and a morphism $\alpha : a' \to a$ over $S' \to S$ to the natural pullback map $\alpha^* : \mathrm{Aut}_S(a) \to \mathrm{Aut}_{S'}(a')$ (see (3.2.10)). Given $a : S \to \mathcal{X}$, there is a canonical isomorphism $I_{\mathcal{X}} \times_{\mathcal{X}} S \cong \underline{\mathrm{Aut}}_S(a)$ of group schemes over S.

Suppose that $\mathcal{H} \subset I_{\mathcal{X}}$ is a closed subgroup scheme over $\mathcal{X}$ such that $\mathcal{H} \to \mathcal{X}$ is fppf. This is equivalent to requiring that for every $a \in \mathcal{X}(S)$, there is a closed fppf subgroup scheme $\mathcal{H}_a \subset \underline{\mathrm{Aut}}_S(a)$ over S such that if $a' \to a$ is a morphism over $S' \to S$, the canonical isomorphism $\underline{\mathrm{Aut}}_{S'}(a') \cong \underline{\mathrm{Aut}}_S(a) \times_S S'$ restricts to an isomorphism $\mathcal{H}_{a'} \cong \mathcal{H}_a \times_S S'$. If $\alpha : a \xrightarrow{\sim} a$ is an automorphism over the identity, then the canonical isomorphism $\alpha^* : \mathrm{Aut}_S(a) \to \mathrm{Aut}_S(a)$ is conjugation by α. In particular, $\mathcal{H}_a \subset \underline{\mathrm{Aut}}_S(a)$ is a *normal* group scheme.

Frequently in applications when $\mathcal{X}$ is defined over scheme S, the closed subgroup $\mathcal{H} \subset I_{\mathcal{X}}$ is obtained by the pullback of a fppf group scheme $H \to S$, i.e. $\mathcal{H} = H \times_S \mathcal{X}$.

Definition 6.2.48 (Rigidification)**.** Let $\mathcal{X}$ be an algebraic stack and $\mathcal{H} \subset I_{\mathcal{X}}$ be an fppf closed subgroup scheme over $\mathcal{X}$. The *rigidification* $\mathcal{X} /\!\!/ \mathcal{H}$ is defined as the stackification in $\mathrm{Sch}_{\text{fppf}}$ of the prestack with the same objects as $\mathcal{X}$ and where the set of morphisms beteen $b \in \mathcal{X}(T)$ and $a \in \mathcal{X}(S)$ over $f : T \to S$ is defined as $\mathrm{Mor}(b, a) = \mathrm{Mor}_{\mathcal{X}(T)}(b, f^*a)/\mathcal{H}(T)$.

If $\mathcal{X}$ is defined over S and $\mathcal{H} = H \times_S \mathcal{X}$ is the base change of an fppf group scheme $H \to S$, then we write $\mathcal{X} /\!\!/ H := \mathcal{X} /\!\!/ \mathcal{H}$.

One can think of the subgroup $\mathcal{H}$ as giving an action of $B\mathcal{H}$ on $\mathcal{X}$ and the rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ as the quotient $\mathcal{X}/B\mathcal{H}$.

Example 6.2.49. If $I_{\mathcal{X}} \to \mathcal{X}$ is fppf, then we can take $\mathcal{H} = I_{\mathcal{X}}$ and the rigidification $\mathcal{X} /\!\!/ I_{\mathcal{X}}$ is the algebraic space X constructed in Proposition 6.2.47.

Proposition 6.2.50. *Let $\mathcal{X}$ be an algebraic stack and $\mathcal{H} \subset I_{\mathcal{X}}$ be an fppf closed subgroup scheme over $\mathcal{X}$. The rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ is an algebraic stack such that*

(1) the natural morphism $\pi \colon \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is a gerbe;

(2) for every object $a \in \mathcal{X}(S)$, the natural map $\mathrm{Aut}_S(a) \to \mathrm{Aut}_S(\pi(a))$ is surjective with kernel $\mathcal{H}(S)$;

(3) a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ factors uniquely through $\mathcal{X} /\!\!/ \mathcal{H}$ if and only if for every object $a \in \mathcal{X}(S)$, the composition $\mathcal{H}(S) \subset \ker(\mathrm{Aut}_{\mathcal{X}(S)}(a) \to \mathrm{Aut}_{\mathcal{Y}(S)}(f(a)))$; and

(4) if $\mathcal{H}$ is a commutative group scheme, then $\mathcal{H}$ descends to an fppf group scheme $H \to X$ such that $\mathcal{X} \to X$ is banded H-gerbe. If in addition $\mathcal{X}$ is defined over a scheme S and $\mathcal{H} = H \times_S \mathcal{X}$ is the pullback of a commutative fppf group scheme $H \to S$, then $\mathcal{X} \to X$ is a banded H-gerbe.

Proof. To show that $\mathcal{X}$ is algebraic, it suffices to show that $\pi \colon \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is a smooth representable morphism: if $U \to \mathcal{X}$ is a smooth presentation, then so is the composition $U \to \mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$. If $g \colon S \to \mathcal{X} /\!\!/ \mathcal{H}$, then by the definition of $\mathcal{X} /\!\!/ \mathcal{H}$ as the stackification, there is an fppf cover $S' \to S$ such that $S' \to S \to \mathcal{X} /\!\!/ \mathcal{H}$ lifts to a map $a' \colon S' \to \mathcal{X}$. By replacing S with S', we may assume that $g \colon S \to \mathcal{X} /\!\!/ \mathcal{H}$ lifts to a morphism $a \colon S \to \mathcal{X}$.

We claim that there is an isomorphism

$$\Psi \colon \mathcal{X} \times_{\mathcal{X} /\!\!/ \mathcal{H}} S \to B\mathcal{H}_a.$$

Since $\mathcal{H}_a \to S$ is fppf, the classifying stack $B\mathcal{H}_a$ is algebraic (Proposition 6.2.10) and smooth over S' (Proposition B.1.10), and the isomorphism Ψ would imply that $\mathcal{X} \to \mathcal{X} /\!\!/ \mathcal{H}$ is smooth and representable. An object of $\mathcal{X} \times_{\mathcal{X} /\!\!/ \mathcal{H}} S$ consists of a triple (f, b, α) where $f \colon T \to S$, $b \in \mathbb{X}(T)$, and $\alpha \colon g \circ f \xrightarrow{\sim} \pi \circ b$. Define $\Psi(f, b, \alpha)$ as the principal $\mathcal{H}_a$-bundle $T \times_{\underline{\mathrm{Isom}}_T(f^*a, b)/\mathcal{H}_a} \underline{\mathrm{Isom}}_T(f^a, b)$. Noting that $\Psi(f, f^*a, \mathrm{id})$ is the trivial bundle, the proof that Ψ is an isomorphism follows exactly as in Proposition 6.2.47. The remaining statements are left to the reader.

See also [ACV03, Thm. 5.1.5], [AGV08, §C], [Rom05, §5], and [AOV08, §A]. $\square$

Example 6.2.51 (Ridigification of $\mathrm{Bun}_{r,d}(C)$). Moduli stacks of sheaves provide interesting examples of rigidification since there is a canonical scaling $\mathbb{G}_m$-action on sheaves. Recall that $\mathrm{Bun}_{r,d}(C)$ is the moduli stack of vector bundles of rank r and degree d on a fixed smooth, connected, and projective curve C over an algebraically closed field $\Bbbk$. For any vector bundle $\mathcal{E}$ on $C \times S$ where S is a $\Bbbk$-scheme, there is a canonical closed immersion $i_{\mathcal{E}} \colon \mathbb{G}_{m,S} \to \underline{\mathrm{Aut}}(\mathcal{E})$ of group schemes over S. Thus, $\mathbb{G}_m := \mathbb{G}_{m,\mathrm{Bun}_{r,d}} \subset I_{\mathrm{Bun}_{r,d}}$ is a closed fppf group scheme of the inertia stack, and we can construct the rigidification

$$\mathrm{Bun}_{r,d}(C) /\!\!/ \mathbb{G}_m.$$

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Over the substack $\mathrm{Bun}^{\mathrm{simple}}_{r,d}(C)$ of simple bundles (i.e. vector bundles E with $\mathrm{Aut}(E) = \Bbbk^*$), the rigidification $\mathrm{Bun}^{\mathrm{simple}}_{r,d}(C) /\!\!/$ is an algebraic space and $\mathrm{Bun}^{\mathrm{simple}}_{r,d}(C) \to \mathrm{Bun}^{\mathrm{simple}}_{r,d}(C) /\!\!/$ is a banded $\mathbb{G}_m$-gerbe.

Exercise 6.2.52.

(a) If H is a commutative fppf group scheme over S, show that $BH /\!\!/ H \cong S$. More generally, show that if $\mathcal{X} \to X$ is a banded H-gerbe, then $X \cong \mathcal{X} /\!\!/ H$.

(b) Let $G \to S$ be an fppf group scheme acting on a S-scheme U. Suppose that $H \subset G$ is a central commutative fppf subgroup scheme acting trivially on U. Show that $[U/G] /\!\!/ H \cong [U/(G/H)]$.

Exercise 6.2.53. Let $\mathcal{X} \to S$ be a smooth, integral, and separated Deligne–Mumford stack over a scheme S. Let $\mathrm{Spec}\, K \to \mathcal{X}$ be a representative of the generic point. Show that the closure $\mathcal{H} \subset I_{\mathcal{X}}$ of generic fiber $I_{\mathcal{X}} \times_{\mathcal{X}} K$ of the inertia is a closed étale subgroup scheme and that the rigidification $\mathcal{X} /\!\!/ \mathcal{H}$ is a smooth, integral, and separated Deligne–Mumford stack over S with generically trivial inertia.

Exercise 6.2.54. Let $\mathcal{X}$ be an algebraic stack over a scheme S, $H \to S$ be an fppf group scheme, and $H \times_S \mathcal{X} \subset I_{\mathcal{X}}$ a closed subgroup scheme. Show that the rigidification $\mathcal{X} /\!\!/ H$ can be given the moduli interpretation where an object over a scheme S is a pair $(\mathcal{G}, f)$ where $\mathcal{G} \to S$ is a banded H-gerbe and $f \colon \mathcal{G} \to \mathcal{X}$ is an H-equivariant morphism (i.e. for every object $a \in \mathcal{G}(T)$ over an S-scheme T, the composition $H(T) \xrightarrow{\sim} \mathrm{Aut}_T(a) \to \mathrm{Aut}_T(f(a))$ agrees with the inclusion $H(T) \hookrightarrow \mathrm{Aut}_T(f(a))$ given by the subgroup $H \times_S \mathcal{X} \subset I_{\mathcal{X}}$.

6.2.9 Picard stacks and spaces

If $X \to S$ is a proper flat morphism of noetherian schemes, define the *Picard stack*

$$\underline{\mathcal{P}\mathrm{ic}}_{X/S}$$

as the stack over $(\mathrm{Sch}/S)_{\mathrm{\acute{e}t}}$ whose objects over an S-scheme T are line bundles on $X_T = X \times_S T$ and whose morphisms are isomorphisms of line bundles. This is an open substack of $\underline{\mathrm{Coh}}(X/S)$ whose objects over an S-scheme T are coherent sheaves on X_T flat over T. The stack $\underline{\mathrm{Coh}}(X/S)$ is an algebraic stack locally of finite type over S; see Exercise 3.1.23 for the case when $X \to S$ is strongly projective and Exercise C.7.9 in general. Therefore $\underline{\mathcal{P}\mathrm{ic}}_{X/S}$ is also algebraic and locally of finite type over S.

On the other hand, there are several candidates for Picard functors:

(1) The *naive Picard functor* (or *absolute Picard functor*) is

$$\underline{\mathrm{Pic}}^{\mathrm{naive}}_{X/S} \colon \mathrm{Sch}/S \to \mathrm{Gps}, \qquad (T \to S) \mapsto \mathrm{Pic}(X_T).$$

(2) The *Picard functor*

$$\underline{\mathrm{Pic}}_{X/S} \colon \mathrm{Sch}/S \to \mathrm{Gps},$$

is the fppf sheafification of $\underline{\mathrm{Pic}}^{\mathrm{naive}}_{X/S}$.

(3) The *relative Picard functor* is

$$\underline{\mathrm{Pic}}^{\mathrm{rel}}_{X/S} \colon \mathrm{Sch}/S \to \mathrm{Gps}, \qquad (T \to S) \mapsto \mathrm{Pic}(X_T)/\mathrm{Pic}(T),$$

where an object over an S-scheme T is a line bundle L on X_T, and two line bundles L and L' on X_T are identified if there exists a line bundle $M \in \mathrm{Pic}(T)$ such that $L \cong L' \otimes f_T^* M$.

(4) We can define the *rigidification of the Picard stack*

$$\underline{\mathcal{P}\text{ic}}_{X/S} \,/\!\!/\, \mathbb{G}_m$$

under the hypothesis that $\mathcal{O}_T \xrightarrow{\sim} f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for any map $T \to S$. This hypothesis implies that for a line bundle L on X_T, there is a canonical isomorphism

$$\mathbb{G}_m(T) = \Gamma(T, \mathcal{O}_T)^* \xrightarrow{\sim} \Gamma(X_T, \mathcal{O}_{X_T})^* = \mathrm{Aut}(L),$$

which further implies that the inertia stack $I_{\underline{\mathcal{P}\text{ic}}_{X/S}}$ is isomorphic to fppf group scheme $\mathbb{G}_m := \mathbb{G}_{m,\underline{\mathcal{P}\text{ic}}_{X/S}}$ over $\underline{\mathcal{P}\text{ic}}_{X/S}$.

While the Picard functor $\underline{\mathrm{Pic}}_{X/S}$ and the rigidification $\underline{\mathcal{P}\text{ic}}_{X/S} \,/\!\!/\, \mathbb{G}_m$ are sheaves in the big fppf topology by definition, it may seem surprising that $\underline{\mathrm{Pic}}_{X/S}^{\mathrm{rel}}$ is also a sheaf under relatively mild hypotheses.

Proposition 6.2.55. *Let $f\colon X \to S$ be a proper flat morphism of noetherian schemes such that $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ is an isomorphism and this holds after base change, i.e. for every map $g\colon T \to S$, the map $\mathcal{O}_T \xrightarrow{\sim} f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism. Then*

(1) $\underline{\mathrm{Pic}}_{X/S}$ is representable by an algebraic space locally of finite type over S,

(2) the map $\underline{\mathcal{P}\text{ic}}_{X/S} \to \underline{\mathrm{Pic}}_{X/S}$ from the Picard stack is a banded $\mathbb{G}_m$-gerbe and $\underline{\mathrm{Pic}}_{X/S} \cong \underline{\mathcal{P}\text{ic}}_{X/S} \,/\!\!/\, \mathbb{G}_m$, and

(3) if in addition there is a section $s\colon S \to X$, then $\underline{\mathrm{Pic}}_{X/S} \cong \underline{\mathrm{Pic}}_{X/S}^{\mathrm{rel}}$.

If in addition the geometric fibers of f are integral, then $\underline{\mathrm{Pic}}_{X/S}$ is separated over S.

Remark 6.2.56. If $f\colon X \to S$ is a proper flat proper morphism with geometrically connected and reduced fibers, then $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ is an isomorphism (and remains so after base chagne) by Lemma A.6.12. In [FGA$_{\mathrm{V}}$, Thm. 3.1], Grothendieck proved that $\underline{\mathrm{Pic}}_{X/S}$ is a scheme in the case that $X \to S$ is projective. See [Mum66, §20-21], [AK80], and [?, §9.4] for alternative expositions and various generalizations. The representability as an algebraic space above was first established by Artin [Art69b, Thm. 7.3], and this holds with the slighly weaker hypothesis that f is *cohomologically flat in dimension* 0, i.e. the formation of $f_*\mathcal{O}_X$ commutes with base change.

Proof. As pointed out above, the condition that $\mathcal{O}_S \xrightarrow{\sim} f_*\mathcal{O}_X$ holds after base change implies that the inertia stack of $\underline{\mathcal{P}\text{ic}}_{X/S}$ is isomorphic to the fppf group scheme $\mathbb{G}_m := \mathbb{G}_{m,\underline{\mathcal{P}\text{ic}}_{X/S}}$. Therefore Proposition 6.2.47 (or alternatively Proposition 6.2.50) implies that the rigidification $\underline{\mathcal{P}\text{ic}}_{X/S} \,/\!\!/\, \mathbb{G}_m$ is an algebraic space locally of finite type over S. Moreover, $\underline{\mathcal{P}\text{ic}}_{X/S} \,/\!\!/\, \mathbb{G}_m$ is identified with the Picard functor $\underline{\mathrm{Pic}}_{X/S}$ by the definition of rigidification. This gives both (1) and (2). When the fibers are geometrically integral, the separatedness of $\underline{\mathrm{Pic}}_{X/S}$ over S follows from Proposition A.6.17 and Remark A.6.19.

To identify $\underline{\mathrm{Pic}}_{X/S}^{\mathrm{rel}}$ with $\underline{\mathrm{Pic}}_{X/S}$, it suffices to prove that $\underline{\mathrm{Pic}}_{X/S}^{\mathrm{rel}}$ is a sheaf in the big fppf topology. To this end, it will be convenient to identify $\underline{\mathrm{Pic}}_{X/S}^{\mathrm{rel}}$ with the prestack $\mathcal{P}^{s-\mathrm{rig}}$, called the *s-rigidification*, whose fiber category over an S-scheme T is

$$\mathcal{P}^{s-\mathrm{rig}}(T) = \big\{ (L, \alpha) \mid L \in \mathrm{Pic}(X_T) \text{ and } \alpha\colon \mathcal{O}_T \xrightarrow{\sim} s_T^* L \big\},$$

where a morphism $(L, \alpha) \sim (L', \alpha')$ is the data of an isomorphism $\beta\colon L \to L'$ of line bundles such that $\alpha' = s_T^* \beta \circ \alpha$.

The advantage of considering $\mathcal{P}^{s-\mathrm{rig}}$ is that it is a straightforward application of fppf descent of quasi-coherent sheaves to check that $\mathcal{P}^{s-\mathrm{rig}}$ is a stack over the big

fppf topology $(\mathrm{Sch}/S)_{\mathrm{fppf}}$. To show that $\underline{\mathrm{Pic}}_{X/S}^{\mathrm{rel}}$ is a sheaf, we can therefore verify that the natural map

$$\mathcal{P}^{s-\mathrm{rig}} \to \underline{\mathrm{Pic}}_{X/S}^{\mathrm{rel}}, \qquad (L, \alpha) \mapsto L \qquad (6.2.57)$$

is an equivalence.

We first check that $\mathcal{P}^{s-\mathrm{rig}}$ is equivalent to a functor, i.e. the functor (6.2.57) is faithful. We must show that if $\beta \colon L \xrightarrow{\sim} L$ is an automorphism with $s_T^* \beta = \mathrm{id}_{s_T^* L}$, then $\beta = \mathrm{id}_L$. Since $\mathcal{O}_T \to f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism, the pullback map $f_T^* \colon \mathrm{H}^0(T, \mathcal{O}_T) \to \mathrm{H}^0(X_T, \mathcal{O}_{X_T})$ is an isomorphism; as s_T is a section, its inverse is given by s_T^*. The composition

$$s_T^* \colon \ \mathrm{Hom}_{\mathcal{O}_{X_T}}(L, L) \cong \mathrm{H}^0(X_T, \mathcal{O}_{X_T}) \xrightarrow{s_T^*} \mathrm{H}^0(T, \mathcal{O}_T) \cong \mathrm{Hom}_{\mathcal{O}_T}(s_T^* L, s_T^* L)$$

is an isomorphism of groups, and thus $\beta = \mathrm{id}_L$.

To see that (6.2.57) is full (i.e. the induced map on the functors of isomorphism classes is injective), let $(L, \alpha) \in \mathcal{P}^{s-\mathrm{rig}}(T)$ be an element such that there is a line bundle M on T and an isomorphism $\beta \colon L \xrightarrow{\sim} f_T^* M$. The isomorphism

$$\gamma \colon \mathcal{O}_{X_T} = f_T^* \mathcal{O}_T \xrightarrow{f_T^* \alpha} f_T^* s_T^* L \xrightarrow{f_T^* s_T^* \beta} f_T^* s_T^* f_T^* M = f_T^* M$$

satisfies $s_T^* \gamma = s_T^* \beta \circ \alpha$ and shows that (L, α) is isomorphic to $(\mathcal{O}_{X_T}, \mathrm{id}) \in \mathcal{P}^{s-\mathrm{rig}}(T)$.

Finally, to see that the functor (6.2.57) is essentially surjective, let $L \in \mathrm{Pic}(X_T)$ and define

$$L' = L \otimes (f_T^* s_T^* L)^{\vee}.$$

The images of L and L' are equal in $\underline{\mathrm{Pic}}_{X/S}^{\mathrm{rel}}(T)$, and $s_T^* L' \cong s_T^* L \otimes (s_T^* f_T^* s_T^* L)^{\vee} \cong \mathcal{O}_T$ defines an isomorphism $\alpha' \colon \mathcal{O}_T \xrightarrow{\sim} s_T^* L'$ such that $L \in \underline{\mathrm{Pic}}_{X/S}^{\mathrm{rel}}(T)$ is the image of $(L', \alpha') \in \mathcal{P}^{s-\mathrm{rig}}(T)$. $\qquad\square$

Over an algebraically closed field $\Bbbk$, it is remarkably easy to verify that the Picard functor $\underline{\mathrm{Pic}}_X := \underline{\mathrm{Pic}}_{X/\Bbbk}$ is a scheme.

Theorem 6.2.58. *Let X be a proper integral scheme over an algebraically closed field $\Bbbk$.*

- *(1) $\underline{\mathrm{Pic}}_X$ is an algebraic group over $\Bbbk$, and in particular a disjoint union of quasi-projective schemes.*
- *(2) $\underline{\mathrm{Pic}}_X \cong \underline{\mathrm{Pic}}_X^{\mathrm{rel}}$ and $\underline{\mathcal{P}ic}_X \to \underline{\mathrm{Pic}}_X$ is a banded $\mathbb{G}_m$-gerbe,*
- *(3) If X is smooth, the connected component of the identity $\underline{\mathrm{Pic}}_X^0$ is projective.*
- *(4) If $\mathrm{char}(\Bbbk) = 0$, then $\underline{\mathrm{Pic}}_X$ is smooth of dimension $\mathrm{h}^0(X, \mathcal{O}_X)$. In particular, $\underline{\mathrm{Pic}}_X^0$ is an abelian variety.*

Proof. As $\Bbbk$ is algebraically closed and X is integral, the structure map $f \colon X \to \mathrm{Spec}\,\Bbbk$ has a section and $\mathcal{O}_T \xrightarrow{\sim} f_{T,*}\mathcal{O}_{X_T}$ is an isomorphism for any $\Bbbk$-scheme T. Proposition 6.2.55 implies that $\underline{\mathrm{Pic}}_X$ is an algebraic space locally of finite type over $\Bbbk$, and that (2) holds. The Picard stack $\underline{\mathcal{P}ic}_X$ is quasi-separated (it even has affine diagonal) and it follows that $\underline{\mathrm{Pic}}_X$ is also quasi-separated. This is enough to show that $\underline{\mathrm{Pic}}_X$ is a separated scheme and that $\underline{\mathrm{Pic}}_X^0$ is quasi-projective. It is a direct consequence of Theorem 4.4.28, but it's worth recalling the argument: $\underline{\mathrm{Pic}}_X$ has a dense open subspace which is a scheme (Theorem 4.4.1), the group structure $\underline{\mathrm{Pic}}_X$ allows us to translate this open to show that $\underline{\mathrm{Pic}}_X$ is an algebraic

group, which is automatically separated with quasi-projective connected components (Proposition B.2.40). This gives (1).

For (3), it suffices to show that $\underline{\mathrm{Pic}}^0_X$ is proper. As we already know it is separated, we only need to verify the existence part of the valuative criterion for properness: let R be a DVR over $\Bbbk$ with fraction field K and L be a line bundle on X_K. As X_R is regular, the line bundle L extends to a line bundle $\widetilde{L}$ on X_R (for instance, if $L = \mathcal{O}(D)$ for a divisor $D \subset X_K$, then take $\widetilde{L} = \mathcal{O}(\overline{D})$).

The smoothness in (4) follows from the fact that algebraic groups are smooth in characteristic 0 (Proposition B.2.40). If L is a line bundle on X, then the Zariski tangent spaces of the Picard stack and Picard scheme agree, and deformation theory (Proposition C.1.17) implies that $T_{\underline{\mathrm{Pic}}_X, L} \cong \mathrm{H}^1(X, \mathcal{O}_X)$. $\qquad\square$

Remark 6.2.59. As a consequence of the representability of $\underline{\mathrm{Pic}}_X \cong \underline{\mathrm{Pic}}^{\mathrm{rel}}_X$, there is a *universal family* (or *Poincaré family*) $\mathcal{P}$ on $X \times \underline{\mathrm{Pic}}_X$ that satisfies the following: for any $\Bbbk$-scheme T and any line bundle L on X_T, there is a unique morphism $T \to \underline{\mathrm{Pic}}_X$ such that

$$L \cong \mathcal{P}|_{X \times T} \otimes p_2^* M$$

for some line bundle M on T.

The connected component of the identity $\underline{\mathrm{Pic}}^0_X$ has the functorial description of parameterizing line bundles L *algebraically equivalent* to $\mathcal{O}_X$ (i.e. there is a connected $\Bbbk$-scheme T with points $t_0, t_1 \in T(\Bbbk)$ and a family of line bundles $\mathcal{L}$ on X_T such that $L_{t_0} \cong L$ and $L_{t_1} \cong \mathcal{O}_X$). When X is a smooth curve, $\underline{\mathrm{Pic}}^0_X$ parameterizes degree 0 line bundles.

Remark 6.2.60. The theorem's conclusion holds if X is an integral proper *algebraic space*. The only missing ingredient is the algebraicity of the Picard stack, and this can be shown using Artin's Axioms similar to Exercise C.7.9.

In characteristic p, Igusa showed that Pic_X may fail to be reduced [Igu55]. We also note that when X is not normal (e.g. a nodal or cuspidal curve), then $\underline{\mathrm{Pic}}^0_X$ is not projective. Altman and Kleiman [AK80] provide a compactification of $\underline{\mathrm{Pic}}^0_X$ by classifying rank 1 torsion free sheaves.

Picard functors and schemes have a fascinating history as they were one of the first examples of moduli spaces constructed in algebraic geometry. See Kleiman's article [Kle05] for a beautiful account of the history and a broader discussion of the properties of Picard schemes.

6.3 Affine Geometric Invariant Theory and good moduli spaces

Good moduli spaces capture the stack-intrinsic properties of quotients that appear in Geometric Invariant Theory (GIT). In the affine case, GIT concerns the action of a linearly reductive group on an affine scheme. Recall that an affine algebraic group G over a field $\Bbbk$ is *linearly reductive* if the functor $\mathrm{Rep}(G) \to \mathrm{Vect}_\Bbbk$, taking a G-representation V to its G-invariants V^G, is exact. Examples include:

- finite discrete groups G whose order is not divisible by $\mathrm{char}(\Bbbk)$ (Maschke's Theorem (B.2.57));
- tori $\mathbb{G}^n_m$ and diagonalizable group schemes (Proposition B.2.15); and
- reductive algebraic groups (e.g. GL_n, SL_n and PGL_n) in $\mathrm{char}(\Bbbk) = 0$ (Theorem B.2.61).

See §B.2.8 for further equivalences, properties, and a discussion of linearly reductive groups.

Given an action of G on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, the inclusion $A^G \hookrightarrow A$ induces a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A & & \\
\downarrow & \searrow^{\widetilde{\pi}} & \\
[\operatorname{Spec} A/G] & \xrightarrow{\ \pi\ } & \operatorname{Spec} A^G.
\end{array}
$$

Let's observe the following two properties of $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$:

(1) $\Gamma([\operatorname{Spec} A/G], \mathcal{O}_{[\operatorname{Spec} A/G]}) = A^G$; this follows from the definition of global sections.

(2) The functor $\pi_*\colon \operatorname{QCoh}([\operatorname{Spec} A/G]) \to \operatorname{QCoh}(\operatorname{Spec} A^G)$ is exact. This holds because functor π_* takes a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module $\widetilde{M}$, corresponding to an A-module M with a G-action, to $\widetilde{M^G}$ (Exercise 6.1.3) and is therefore exact by the defining property of linear reductivity.

In this case, following the terminology of Mumford and Seshadri, we say that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is a *good quotient* or *GIT quotient*, and $\operatorname{Spec} A^G$ is sometimes denoted as $(\operatorname{Spec} A)/\!\!/G$. See §6.7 for a more general discussion of good quotients and the projective case of GIT.

6.3.1 Good moduli spaces

The definition of a good moduli space is inspired by properties of GIT quotients and specifically properties of the morphisms $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ and $\pi\colon [X^{\mathrm{ss}}/G] \to X^{\mathrm{ss}}/\!\!/G := \operatorname{Proj} \bigoplus_{d \geq 0} \Gamma(X, \mathcal{O}_X(d))^G$, where G is linearly reductive and $X \subset \mathbb{P}(V)$ is a G-invariant closed subscheme of a projectivized G-representation.

Definition 6.3.1 (Good moduli spaces). A quasi-compact and quasi-separated morphism $\pi\colon \mathcal{X} \to X$ from an algebraic stack $\mathcal{X}$ to an algebraic space X is a *good moduli space* if

(1) $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism, and

(2) $\pi_*\colon \operatorname{QCoh}(\mathcal{X}) \to \operatorname{QCoh}(X)$ is exact.

Example 6.3.2 (Basic example: affine GIT). If G is a linearly reductive group over a field $\Bbbk$ acting on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, then $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space.

Example 6.3.3 (Concrete examples). If $\mathbb{G}_m$ acts on $\mathbb{A}^n$ over a field $\Bbbk$ via $t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$, then $[\mathbb{A}^n/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ is a good moduli space. Observe that a nonzero $\Bbbk$-point $[\mathbb{A}^n/\mathbb{G}_m]$ is not closed and contains 0 in its closures, or in other words every $\mathbb{G}_m$-orbit contains 0 in its closure. Note that $[\mathbb{A}^n/\mathbb{G}_m] \setminus 0 = \mathbb{P}^{n-1}$.

If $\mathbb{G}_m$ acts on $\mathbb{A}^2$ via $t \cdot (x, y) = (tx, t^{-1}y)$, then $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[xy] = \mathbb{A}^1$ is a good moduli space. The fiber over $a \neq 0 \in \mathbb{A}^1$ under the good quotient $\mathbb{A}^2 \to \mathbb{A}^1$ is the hyperbola $xy = a$ in $\mathbb{A}^2$ and the fiber under the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \mathbb{A}^1$ is the point $\operatorname{Spec} \Bbbk \cong [V(xy - a)/\mathbb{G}_m)$. The fiber over the origin is the union of the three orbits $\{(x,0) | x \neq 0\} \cup \{(0,y) | y \neq 0\} \cup \{0,0\}$ in $\mathbb{A}^2$. Note that $[\mathbb{A}^2/\mathbb{G}_m] \setminus 0 = \mathbb{A}^1 \bigcup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ is the non-separated affine line.

Example 6.3.4 (Tame coarse moduli spaces)**.** If $\mathcal{X}$ is a separated Deligne–Mumford stack of finite type over a noetherian scheme S, then the Keel–Mori Theorem (4.3.12) implies that there exists a coarse moduli space $\pi\colon \mathcal{X} \to X$. We say that the coarse moduli space $\mathcal{X} \to X$ is *tame* if every automorphism group has order prime to the characteristic, i.e. invertible in $\Gamma(S, \mathcal{O}_S)$. A tame coarse moduli space is a good moduli space. Indeed, this will follow from the fact that the property of being a good moduli space is local on the base in the étale topology (Lemma 6.3.21) and the Local Structure of Coarse Moduli Spaces (4.3.16). If $\mathcal{X}$ has quasi-finite stabilizers, then in fact every good moduli space $\pi\colon \mathcal{X} \to X$ is a coarse moduli space and π is separated; see Proposition 6.3.32.

The goal of this section is to establish the following theorem.

Theorem 6.3.5. *Let $\pi\colon \mathcal{X} \to X$ be a good moduli space where $\mathcal{X}$ is a quasi-separated algebraic stack defined over an algebraic space S. Then*

(1) π is surjective and universally closed;

(2) For closed substacks $\mathcal{Z}_1, \mathcal{Z}_2 \subset \mathcal{X}$, $\mathrm{im}(\mathcal{Z}_1 \cap \mathcal{Z}_2) = \mathrm{im}(\mathcal{Z}_1) \cap \mathrm{im}(\mathcal{Z}_2)$. For geometric points $x_1, x_2 \in \mathcal{X}(\Bbbk)$, $\pi(x_1) = \pi(x_2) \in X(\Bbbk)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$ in $|\mathcal{X} \times_S \Bbbk|$. In particular, π induces a bijection between closed points in $\mathcal{X}$ and closed points in X;

(3) If $\mathcal{X}$ is noetherian, so is X. If $\mathcal{X}$ is of finite type over S and S is noetherian, then X is of finite type over S and π_* preserves coherence, i.e. for $F \in \mathrm{Coh}(\mathcal{X})$, $\pi_* F \in \mathrm{Coh}(X)$; and

(4) If $\mathcal{X}$ is noetherian, then π is universal for maps to algebraic spaces.

Remark 6.3.6. In (2), the images and intersections are taken scheme-theoretically. Note that since π is closed, the set-theoretic image of a closed substack $\mathcal{Z}$ is identified with the topological space of its scheme-theoretic image $\mathrm{im}(\mathcal{Z})$. If $I \subset \mathcal{O}_{\mathcal{X}}$ is the sheaf of ideals defining $\mathcal{Z}$, the image $\mathrm{im}(\mathcal{Z})$ is defined by $\pi_* I \subset \pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$.

In the case of affine GIT where we have a good moduli space $\pi\colon [\mathrm{Spec}\, A / G] \to \mathrm{Spec}\, A^G$ and a good quotient $\widetilde{\pi}\colon \mathrm{Spec}\, A \to \mathrm{Spec}\, A^G$, this theorem translates to:

Corollary 6.3.7 (Affine GIT)**.** *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$. Then $\widetilde{\pi}\colon U = \mathrm{Spec}\, A \to U /\!\!/ G := \mathrm{Spec}\, A^G$ satisfies: Then*

(1) $\widetilde{\pi}$ is surjective and for every G-invariant closed subscheme $Z \subset U$, $\mathrm{im}(Z) \subset U /\!\!/ G$ is closed. The same holds for the base change $T \to U /\!\!/ G$ by a morphism from a scheme;

(2) For closed G-invariant closed subschemes $Z_1, Z_2 \subset U$, $\mathrm{im}(Z_1 \cap Z_2) = \mathrm{im}(Z_1) \cap \mathrm{im}(Z_2)$. In particular, for $x_1, x_2 \in X(\Bbbk)$, $\widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)$ if and only if $\overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset$ and $\widetilde{\pi}$ induces a bijection between closed G-orbits of $\Bbbk$-points in U and $\Bbbk$-points of $U /\!\!/ G$.

(3) If A is noetherian, so is A^G. If A is finitely generated over $\Bbbk$, then A^G is also finitely generated over $\Bbbk$ and for every finitely generated A-module M with a G-action, M^G is a finitely generated A^G-module; and

(4) If A is noetherian, then $\widetilde{\pi}$ is universal for G-invariant maps to algebraic spaces. □

Remark 6.3.8. If $Z \subset U = \mathrm{Spec}\, A$ is defined by a G-invariant ideal I, then (1) implies that $\pi(Z)$ is defined by $I^G \subset A^G$. If Z_1, Z_2 are defined by G-invariant ideals

$I_1, I_2 \subset A$, then (2) implies that $(I_1 + I_2)^G = I_1^G + I_2^G$. In particular, if Z_1 and Z_2 are disjoint, then so are $\operatorname{im}(Z_1)$ and $\operatorname{im}(Z_2)$ and we can write $1 = f_1 + f_2$ with $f_1 \in I_1^G$ and $f_2 \in I_2^G$; the function f_1 restricts to 0 on Z_1 and 1 on Z_2. We see that G-invariant functions separate disjoint G-invariant closed subschemes.

Remark 6.3.9 (Hilbert's 14th problem). Hilbert's 14th problem asks when the invariant ring A^G is finitely generated. While it is not true for every group G, Hilbert showed it is true when G is linearly reductive—this is what (3) above asserts. Hilbert's original argument in [Hil90] is so elegant and played such an important role in the development of modern algebra that we reproduce it here. Our proof of Theorem 6.3.5(3)—while similar in spirit—will not be as explicit.

Let $f_1, \ldots, f_n$ be $\Bbbk$-algebra generators of A and let $V \subset A$ be a finite dimensional G-invariant subspace containing each f_i (Proposition B.2.42(1)). Then we have a surjection $\operatorname{Sym}^* V = \Bbbk[x_1, \ldots, x_m] \to A$ of $\Bbbk$-algebras with G-actions and we set $I = \ker(\Bbbk[x_1, \ldots, x_m] \to A)$. Since G is linearly reductive, $A^G = (\Bbbk[x_1, \ldots, x_m]/I)^G = \Bbbk[x_1, \ldots, x_m]^G/I^G$ and we can assume that $A = \Bbbk[x_1, \ldots, x_m]$ is the polynomial ring so that A^G is a graded $\Bbbk$-algebra whose degree 0 component is $\Bbbk$. It therefore suffices to show that the ideal $J_+ := \sum_{d > 0} A_d^G \subset A^G$ is finitely generated since its generators would then generate A^G as a $\Bbbk$-algebra.

Hilbert first showed that every ideal in $A = \Bbbk[x_1, \ldots, x_n]$ is finitely generated—this is what is referred to today as Hilbert's Basis Theorem and was developed by Hilbert precisely to make this argument. It follows that $J_+ A \subset A$ is finitely generated by homogenous invariants $f_1, \ldots, f_n \in A^G$. We will show that they also generate J_+ as an ideal in A^G. For $f \in A_d^G$, we can write

$$f = \sum_{i=1}^n f_i g_i \tag{6.3.10}$$

with $g_i \in A$ a homogeneous (not necessarily invariant) function of degree $d - \deg f_i$ (with $g_i = 0$ if $\deg f_i > d$). Since G is linearly reductive, there is a $\Bbbk$-linear map $R \colon A \to A^G$ called the *Reynolds operator* (see Remark B.2.60), which is the identity on A^G, respects the grading, and satisfies $R(xy) = xR(y)$ for $x \in A^G$ and $y \in A$. Applying R to (6.3.10) shows that $f = R(f) = \sum_i f_i R(g_i)$ with $R(g_i) \in A^G$ and thus f lies in the ideal in A^G generated by the f_i.[1]

Hilbert gave a constructive proof of this theorem in [Hil93], which required the development of the Syzygy Theorem, the Nullstellensatz, a version of Noether normalization, and a version of the Hilbert–Mumford criterion. We strongly encourage you to read [Hil90] and [Hil93] (or Hilbert's translated lecture notes [Hil93]).

Remark 6.3.11 (Reductivity in positive characteristic). In characteristic p, every smooth linear reductive group is an extension of a torus by a finite étale group scheme prime to the characteristic. In particular, GL_n is not linearly reductive (see Example B.2.62). In characteristic p, there are the following variant notions for an affine algebraic group G over an algebraically closed field $\Bbbk$:

(1) *G is reductive* if G is smooth and every smooth, connected, unipotent, and normal subgroup of G is trivial, and

(2) *G is geometrically reductive* if for every surjection $V \to W$ of G-representations and $w \in W^G$, there exists $n > 0$ such that w^{p^n} is in the image of $\operatorname{Sym}^{p^n} V \to \operatorname{Sym}^{p^n} W$.

---

[1] For an alternative argument that A^G is noetherian, linear reductivity can be used to show that $JA \cap A^G = J$ for every ideal $J \subset A^G$ (see Lemma 6.3.24(5)). If $J_1 \subset J_2 \subset \cdots \subset A^G$ is an ascending chain of ideals, then the ascending chain $J_1 A \subset J_2 A \subset \cdots \subset A$ terminates, which implies that the original sequence $J_1 = J_1 A \cap A^G \subset J_2 = J_2 A \cap A^G \subset \cdots \subset A^G$ also terminates.

It is a deep theorem due to Haboush [Hab75] that these notions are equivalent when G is smooth. See also §B.2.8 for further properties, equivalences, and discussion.

Geometric reductivity (sometimes called semi-reductivity) was introduced by Mumford in [GIT, preface] in an effort to extend GIT—originally developed for linearly reductive groups—to reductive groups in positive characteristic. Indeed, it is precisely the geometric reductivity property that yield the same geometric properties that we saw for affine GIT quotients by linearly reductive groups: if G is geometrically reductive acting on an affine $\Bbbk$-scheme $\operatorname{Spec} A$, then $\widetilde{\pi}\colon \operatorname{Spec} A \to \operatorname{Spec} A^G$ satisfies Corollary 6.3.7(1)-(4) (with the exception that the noetherianess of A does not necessarily imply the noetherianess of A^G). The arguments are not substantially more complicated than the linearly reductive case. See [Nag64], [MFK94, App. 1.C], [New78, §3], [Dol03, §3.4], [Spr77, §2] and [DC71, §2].

Likewise, the notion of a good moduli space can be extended to characterize quotients by geometrically reductive groups: in [Alp14], a quasi-compact and quasi-separated morphism $\pi\colon \mathcal{X} \to X$, from an algebraic stack to an algebraic space, is called an *adequate moduli space* if (1) $\mathcal{O}_X \to \pi_*\mathcal{O}_\mathcal{X}$ is an isomorphism and (2) for every surjection $\mathcal{A} \to \mathcal{B}$ of quasi-coherent $\mathcal{O}_\mathcal{X}$-algebras, then every section s of $\pi_*(\mathcal{B})$ over a smooth morphism $\operatorname{Spec} A \to \mathcal{Y}$ has a positive power that lifts to a section of $\pi_*(\mathcal{A})$). An adequate moduli space satisfies Theorem 6.3.5(1)-(4) (except again for the noetherian implication). If G is geometrically reductive, then $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is an adequate moduli space. In characteristic 0, an adequate moduli space is necessarily good.

In this book, we restrict to linearly reductive groups and good moduli spaces since the proofs of the basic properties are more elementary in this case and probably best seen first. In addition, there is currently no analogue of the Local Structure Theorem for Algebraic Stacks (6.5.1) around points with reductive stabilizers.

6.3.2 Cohomologically affine morphisms

The exactness condition on the pushforward π_* in the definition of a good moduli space (Definition 6.3.1(2)) is a non-representable analogue of affineness.

Definition 6.3.12 (Cohomologically affine). A quasi-compact and quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is *cohomologically affine* if

$$f_*\colon \operatorname{QCoh}(\mathcal{X}) \to \operatorname{QCoh}(\mathcal{Y})$$

is exact. A quasi-compact and quasi-separated algebraic stack $\mathcal{X}$ is *cohomologically affine* if $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ is.

Example 6.3.13. An affine algebraic group G over a field $\Bbbk$ is linearly reductive (Definition B.2.54) if and only if BG is cohomologically affine.

Remark 6.3.14. By Serre's Criterion for Affineness (4.4.16), an algebraic space is cohomologically affine if and only if it is an affine scheme. An algebraic stack $\mathcal{X}$ with affine diagonal is cohomologically affine if and only if $\operatorname{H}^i(\mathcal{X}, F) = 0$ for all $i > 0$ and every quasi-coherent sheaf F; this follows because the cohomology $\operatorname{H}^i(\mathcal{X}, F)$ can be computed in $\operatorname{QCoh}(\mathcal{X})$ for such stacks $\mathcal{X}$ by Proposition 6.1.29(2). This is not true for algebraic stacks with non-affine diagonal, e.g. BE for an elliptic curve E.

Likewise, a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks, with both $\mathcal{X}$ and $\mathcal{Y}$ having affine diagonal, is cohomologically affine if and only if $\operatorname{R}^i f_*(F) = 0$ for all $i > 0$ and every quasi-coherent sheaf F. If in addition f is representable, then f is cohomologically affine if and only if it is affine (see Corollary 6.3.17 below).

Remark 6.3.15 (Noetherian case)**.** If $\mathcal{X}$ is noetherian, then a quasi-compact, quasi-separated morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is cohomologically affine if and only if $f_*\colon \mathrm{Coh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ is exact. This holds because every quasi-coherent sheaf is a colimit of coherent sheaves (Proposition 6.1.8) and f_* commutes with colimits. Since cohomology also commutes with colimits (Proposition 6.1.31), a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks, both with affine diagonal, is cohomologically affine if and only if $\mathrm{R}^i f_*(F) = 0$ for all $i > 0$ and every *coherent* sheaf F.

Lemma 6.3.16. *Consider a cartesian diagram*

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\ g'\ } & \mathcal{X} \\
{\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\mathcal{Y}' & \xrightarrow{\ g\ } & \mathcal{Y}.
\end{array}
$$

of algebraic stacks.

   (1) If g is faithfully flat and π' is cohomologically affine, then π is cohomologically affine.

   (2) If $\mathcal{Y}$ has quasi-affine diagonal (e.g. a quasi-separated algebraic space) and π is cohomologically affine, then π' is cohomologically affine.

Proof. For (1), by Flat Base Change (6.1.7) there is an equivalence $g^* \pi_* \simeq \pi'_* g'^*$ of functors defined on categories of quasi-coherent sheaves. Since π'_* and g'^* are exact and g^* is faithfully exact, π_* is exact.

For (2), we first show that if g is quasi-affine and π is cohomologically affine, then π' is also cohomologically affine. It suffices to handle the cases that g is an open immersion and g is affine. If g is an open immersion and $F' \twoheadrightarrow G'$ is a surjection in $\mathrm{QCoh}(\mathcal{X}')$, we define $G = \mathrm{im}(g'_* F' \to g'_* G')$. Note that $g'^* G \cong G'$. Since π_* is exact, $\pi_* g'_* F \twoheadrightarrow \pi_* G$. If we apply g^* and use the identifies $g^* \pi_* \simeq \pi'_* g'^*$ and $g'^* g'_* \simeq \mathrm{id}$, we obtain a surjection $\pi'_* F' \twoheadrightarrow \pi'_* g'^* G \cong \pi'_* G'$. On the other hand, if g is affine then g_* is faithfully exact. Since π_* and g'_* are are exact, the identity $g_* \pi'_* \simeq \pi_* g'_*$ implies that π'_* is also exact. To show (2), we may assume that $\mathcal{Y}$ and $\mathcal{Y}'$ are quasi-compact and we can choose a smooth presentation $Y = \mathrm{Spec}\, A \to \mathcal{Y}$, which will be quasi-affine (since $\mathcal{Y}$ has quasi-affine diagonal). Then the base change $\mathcal{X}_Y \to Y$ of π along $Y \to \mathcal{Y}$ is cohomologically affine. To check that the base change $\mathcal{X}'_Y \to \mathcal{Y}'_Y$ is cohomologically affine, it suffices by (1) to check this after base changing by a smooth presentation $Y' = \mathrm{Spec}\, A' \to \mathcal{Y}' \times_{\mathcal{Y}} Y$ but this holds as $Y' \to Y$ is affine. Since $\mathcal{X}'_Y \to \mathcal{Y}'_Y$ is cohomologically affine so is $\pi'\colon \mathcal{X}' \to \mathcal{Y}'$ by invoking (1) again. $\square$

Corollary 6.3.17. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a representable and cohomologically affine morphism of algebraic stacks where $\mathcal{Y}$ has quasi-affine diagonal, then f is affine.*

Proof. Under the hypotheses, both affine and cohomologically affine morphisms descend under faithfully flat morphisms, and we can reduce to the case where $\mathcal{X}$ is an algebraic space and $\mathcal{Y}$ is an affine scheme which is Serre's Criterion for Affineness (4.4.16). $\square$

6.3.3   Properties of linearly reductive groups

Recall that an affine algebraic group G over a field $\Bbbk$ is *linearly reductive* if the functor $\mathrm{Rep}(G) \to \mathrm{Vect}_\Bbbk$, defined by $V \mapsto V^G$, is exact (Definition B.2.54). This is equivalent to the map $BG \to \mathrm{Spec}\, \Bbbk$ being cohomologically affine.

Proposition 6.3.18. *Let $1 \to K \to G \to Q \to 1$ be an exact sequence of affine algebraic groups over a field $\Bbbk$. Then G is linearly reductive if and only if both K and Q are.*

Proof. We will use the cartesian diagram

$$
\begin{array}{ccccc}
Q & \longrightarrow & BK & \longrightarrow & \operatorname{Spec}\Bbbk \\
\downarrow & \square & \downarrow & \square & \downarrow \\
\operatorname{Spec}\Bbbk & \longrightarrow & BG & \longrightarrow & BQ
\end{array}
$$

of Exercise 2.3.38(c). To see ($\Rightarrow$), note that $BK \to BG$ is affine by descent since Q is affine. Therefore the composition $BK \to BG \to \operatorname{Spec}\Bbbk$ is cohomologically affine and K is linearly reductive. If V is a Q-representation, then its pullback under $q \colon BG \to BQ$ is the G-representation induced by the projection $G \to Q$ and in particular K acts trivially. On the other hand, the pushforward of a G-representation W under $q \colon BG \to BQ$ is the Q-representation W^K. Thus, the adjunction $V \to q_* q^* V$ is an isomorphism and $\Gamma(BQ, -) = \Gamma(BG, q^* -)$ is exact.

For the converse, descent (Lemma 6.3.16(2)) implies that $BG \to BQ$ is cohomologically affine and thus so is the composition $BG \to BQ \to \operatorname{Spec}\Bbbk$. $\qquad\square$

Proposition 6.3.19. *Let H be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$. If H acts freely on an affine scheme U over $\Bbbk$, then the algebraic space quotient U/H is affine.*

Proof. The algebraic space U/H and the good quotient $\operatorname{Spec} A^H$ are both universal for maps to algebraic spaces Theorem 6.3.5(4). Alternatively, the composition $U/H \to BH \to \operatorname{Spec}\Bbbk$ is an affine morphism followed by a cohomologically affine morphism. It follows from Serre's Criterion for Affineness (4.4.16) that U/H is affine. $\qquad\square$

In particular, if H is a linearly reductive subgroup of an affine algebraic group G, then the quotient G/H is affine. Matsushima's Theorem provides a converse.

Proposition 6.3.20 (Matsushima's Theorem). *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$.*

(1) *A subgroup H of G is linearly reductive if and only if G/H is affine.*

(2) *Given an action of G on an algebraic space U of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in U$ with stabilizer G_u, then G_u is linearly reductive if and only if the orbit Gu is affine.*

Proof. Part (2) follows from (1) since $Gu = G/G_u$. For (1), the ($\Rightarrow$) implication follows from Proposition 6.3.19. For the converse, consider the cartesian diagram

$$
\begin{array}{ccc}
G/H & \longrightarrow & \operatorname{Spec}\Bbbk \\
\downarrow & \square & \downarrow \\
BH & \longrightarrow & BG.
\end{array}
$$

If G/H is affine, then by smooth descent $BH \to BG$ is affine and therefore $BH \to BG \to \operatorname{Spec} k$ is cohomologically affine, i.e. H is linearly reductive. $\qquad\square$

6.3.4 First properties of good moduli spaces

Lemma 6.3.21. *Consider a cartesian diagram*

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\;g'\;} & \mathcal{X} \\
{\scriptstyle \pi'}\downarrow & \square & \downarrow{\scriptstyle \pi} \\
X' & \xrightarrow{\;g\;} & X
\end{array}
$$

of algebraic stacks where X and X' are quasi-separated algebraic spaces.

(1) *If g is faithfully flat and π' is a good moduli space, then π is a good moduli space.*

For the remaining statements, assume in addition that π is a good moduli space.

(2) *The morphism π' is a good moduli space.*

(3) *For $F \in \mathrm{QCoh}(\mathcal{X})$ and $G \in \mathrm{QCoh}(X)$, the adjunction map $\pi_* F \otimes G \to \pi_*(F \otimes \pi^* G)$ is an isomorphism. In particular, the adjunction map $G \xrightarrow{\sim} \pi_* \pi^* G$ is an isomorphism.*

(4) *For $F \in \mathrm{QCoh}(\mathcal{X})$, then the adjunction map $g^* \pi_* F \xrightarrow{\sim} \pi'_* g'^* F$ is an isomorphism.*

(5) *For a quasi-coherent sheaf of ideals $J \subset \mathcal{O}_X$, the natural map $J \to \pi_*(\pi^{-1} J \cdot \mathcal{O}_\mathcal{X})$ is an isomorphism.*

Proof. If $g \colon X' \to X$ is flat, then the pullback of the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_\mathcal{X}$ under g is the map $\mathcal{O}_{X'} \to \pi'_* \mathcal{O}_{\mathcal{X}'}$. Thus (1) and the case of (2) when g is flat follows from Lemma 6.3.16 and descent. Note that since X' is quasi-separated, it has quasi-affine diagonal (Corollary 4.4.8).

Before proving the general case of (2), we first prove (3). Choose an étale presentation $U \to X$ with U the disjoint union of affine schemes. Since the base change $\pi_U \colon \mathcal{X}_U \to U$ is a good moduli space (by the flat case of (2)) and the adjunction map $\mathrm{id} \to \pi_* \pi^*$ pulls back to the adjunction map $\mathrm{id} \to \pi_{U,*} \pi_U^*$, we may assume that $X = \mathrm{Spec}\, A$ is affine. If $G_2 \to G_1 \to G \to 0$ is a free presentation, then the projection maps $\pi_* F \otimes G_i \to \pi_*(F \otimes \pi^* G_i)$ are isomorphisms. Since $\pi_* F \otimes -$ and $\pi_*(F \otimes \pi^* -)$ are right exact, we have a commutative diagram

$$
\begin{array}{ccccccc}
\pi_* F \otimes G_2 & \longrightarrow & \pi_* F \otimes G_1 & \longrightarrow & \pi_* F \otimes G & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_*(F \otimes \pi^* G_2) & \longrightarrow & \pi_*(F \otimes \pi^* G_1) & \longrightarrow & \pi_*(F \otimes \pi^* G) & \longrightarrow & 0
\end{array}
$$

Since the left two vertical maps are isomorphisms, so is the right one.

For (2), we must show that $\mathcal{O}_{X'} \to \pi'_* \mathcal{O}_{\mathcal{X}'}$ is an isomorphism as Lemma 6.3.16(2) already established that π'_* is exact. We can assume that X and X' are affine. In this case, g_* is faithfully exact so it suffices to show that

$$
g_* \mathcal{O}_{X'} \to g_* \pi'_* \mathcal{O}_{\mathcal{X}'} \cong \pi_* g'_* \mathcal{O}_{\mathcal{X}'} \cong \pi_* \pi^* g_* \mathcal{O}_{X'} \tag{6.3.22}
$$

is an isomorphism, where the last equivalence uses the identity $g'_* \pi'^* \mathcal{O}_{X'} \cong \pi^* g_* \mathcal{O}_{X'}$ following from the affineness of g. Thus the composition (6.3.22) is the adjunction isomorphism of (3) applied to $F = g_* \mathcal{O}_{X'}$.

For (4), we know by Flat Base Change (6.1.7) that (4) is fppf local on X and X' and that it holds when g is flat. We may therefore reduce to when $X' \to X$

is a morphism of affine schemes. By factoring $X' \to X$ as a closed immersion followed by a flat morphism, we can further reduce to the case that $X' \hookrightarrow X$ is a closed immersion defined by a quasi-coherent sheaf of ideals $J \subset \mathcal{O}_X$. We aim to show that $\pi_* F / J \pi_* F \cong \pi_* (F/(\pi^{-1} J \cdot \mathcal{O}_{\mathcal{X}})F)$. Using the exactness of π_*, this is equivalent to the inclusion $J \pi_* F \hookrightarrow \pi_* ((\pi^{-1} J \cdot \mathcal{O}_{\mathcal{X}})F)$ being surjective. The sheaf $(\pi^{-1} J \cdot \mathcal{O}_{\mathcal{X}})F$ is the image of $\pi^* J \otimes F \to F$. By the exactness of π_*, the pushforward $\pi_* ((\pi^{-1} J \cdot \mathcal{O}_{\mathcal{X}})F)$ is the image of $\pi_* (\pi^* J \otimes F) \to \pi_* F$, but by (3) this is identified with the image of $J \otimes \pi_* F \to \pi_* F$.

For (5), if $Z \subset X$ is the closed subspace defined by J, then the preimage ideal sheaf $\pi^{-1} J \cdot \mathcal{O}_{\mathcal{X}}$ defines the preimage $\pi^{-1}(Z)$. The exactness of π_* implies that there is a commutative diagram of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_*(\pi^{-1}J \cdot \mathcal{O}_{\mathcal{X}}) & \longrightarrow & \pi_* \mathcal{O}_{\mathcal{X}} & \longrightarrow & \pi_* \mathcal{O}_{\pi^{-1}(Z)} & \longrightarrow & 0.
\end{array}
$$

As $\mathcal{X} \to X$ and $\pi^{-1}(Z) \to Z$ are good moduli spaces, the right two vertical arrows are isomorphisms and so is the left arrow. $\square$

Remark 6.3.23. The isomorphism $\pi_* F \otimes G \to \pi_* (F \otimes \pi^* G)$ in (3) is similar to the projection formula but holds even if G is not locally free. It holds as long as π is cohomologically affine.

Lemma 6.3.24. *Let $\pi \colon \mathcal{X} \to X$ be a good moduli space with X quasi-separated.*

(1) *If $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras, then $\mathcal{S}pec_{\mathcal{X}} \mathcal{A} \to \mathcal{S}pec_X \pi_* \mathcal{A}$ is a good moduli space.*

(2) *If $\mathcal{Z} \subset \mathcal{X}$ is a closed substack defined by a sheaf of ideals I and $\mathrm{im}\,\mathcal{Z} \subset X$ is the scheme-theoretic image, i.e. the closed subspace defined by $\pi_* I \subset \mathcal{O}_X$, then $\mathcal{Z} \to \mathrm{im}\,\mathcal{Z}$ is a good moduli space.*

Proof. For (1), since $\mathcal{X} \times_X \mathcal{S}pec_X \pi_* \mathcal{A} \to \mathcal{S}pec_X \pi_* \mathcal{A}$ is cohomologically affine by Lemma 6.3.16 and $\mathcal{S}pec_{\mathcal{X}} \mathcal{A} \to \mathcal{X} \times_X \mathcal{S}pec_X \pi_* \mathcal{A}$ is affine, it follows that $\mathcal{S}pec_{\mathcal{X}} \mathcal{A} \to \mathcal{S}pec_X \pi_* \mathcal{A}$ is cohomologically affine and therefore a good moduli space as the push forward of $\mathcal{O}_{\mathcal{S}pec_{\mathcal{X}} \mathcal{A}}$ is $\mathcal{O}_{\mathcal{S}pec_X \pi_* \mathcal{A}}$ by construction. Applying (1) to $\mathcal{Z} = \mathcal{S}pec_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}/I)$ recovers (2) using that $\pi_*(\mathcal{O}_{\mathcal{X}}/I) = \mathcal{O}_X/\pi_* I$. $\square$

The above lemmas allow us to give quick proofs of the first two parts of Theorem 6.3.5.

Proof of Theorem 6.3.5(1). As $\mathcal{X}$ is quasi-separated, so is X. For every field-valued point $x \in X(\Bbbk)$, consider the base change $\mathcal{X} \times_X \mathrm{Spec}\,\Bbbk$. By Lemma 6.3.21(2), $\mathcal{X}_x \to \mathrm{Spec}\,\Bbbk$ is a good moduli space and in particular $\Gamma(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}) = \Bbbk$. It follows that $\mathcal{X}_x$ is non-empty and that $\pi \colon \mathcal{X} \to X$ is surjective. For a closed substack $\mathcal{Z} \subset \mathcal{X}$, Lemma 6.3.24(2) implies that $\mathcal{Z} \to \mathrm{im}\,\mathcal{Z}$ is a good moduli space and therefore also surjective. Thus, the set-theoretic image $\pi(\mathcal{Z})$ is identified with the scheme-theoretic image $\mathrm{im}\,\mathcal{Z}$ and is therefore closed. Since good moduli spaces are stable under base change, they are universally closed. $\square$

Proof of Theorem 6.3.5(2). For two substacks $\mathcal{Z}_1, \mathcal{Z}_2 \subset X$ defined by ideal sheaves $I_1, I_2 \subset \mathcal{O}_X$, we apply the exact functor π_* to the short exact sequence $0 \to I_1 \to$

$I_1 + I_2 \to I_2/I_1 \cap I_2 \to 0$ and surjection $I_2 \to I_2/I_1 \cap I_2$ to obtain a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & \pi_* I_2 & & & & \\
 & & & & \downarrow & \searrow & & & \\
0 & \longrightarrow & \pi_* I_1 & \longrightarrow & \pi_*(I_1 + I_2) & \twoheadrightarrow & \pi_* I_2/\pi_*(I_1 \cap I_2) & \longrightarrow & 0.
\end{array}
$$

It follows that the natural inclusion $\pi_* I_1 + \pi_* I_2 \to \pi_*(I_1 + I_2)$ is surjective. $\qquad\square$

6.3.5 Finite typeness of good moduli spaces

We will use the fact that a good moduli space $\mathcal{X} \to X$ is universally submersive and show that finite typeness descends under universally submersive morphisms. Recall from §A.4.2 that a morphism $f \colon X \to Y$ of schemes is *universally submersive* if f is surjective and Y has the quotient topology, and these properties are stable under base change. This notion extends to morphisms of algebraic stacks. Examples of universally submersive morphisms include fppf morphisms and universally closed morphisms.

Proposition 6.3.25. *Consider a commutative diagram*

$$
\begin{array}{ccc}
X & & \\
\downarrow & \searrow & \\
Y & \longrightarrow & S
\end{array}
$$

of noetherian schemes where $X \to Y$ is universally submersive. If $X \to S$ is of finite type, then so is $Y \to S$.

Proof. We can assume that $S = \operatorname{Spec} R$ and $Y = \operatorname{Spec} B$ are affine. Since a noetherian ring B is of finite type over R if and only if the reductions of the irreducible components of $\operatorname{Spec} B$ are of finite type over R, we can assume that B is an integral domain.

By Generic Flatness (A.2.13) and Raynaud-Gruson Flatification (A.2.18), there is a commutative diagram

$$
\begin{array}{ccc}
\widetilde{X} & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle f} \quad \searrow \\
Y' & \overset{g}{\longrightarrow} Y & \longrightarrow S
\end{array}
$$

where $\widetilde{X} \to Y'$ is flat, $Y' = \operatorname{Bl}_I Y \to Y$ is the blowup along an ideal $I \subset B$ and $\widetilde{X}$ is the strict transform of X, i.e. the closure of $(Y' \setminus g^{-1}(V(I))) \times_Y X$ in the base change $Y' \times_Y X$. We claim that $\widetilde{X} \to Y'$ is surjective. As $g \colon Y' \to Y$ is an isomorphism over $U = Y \setminus V(I)$ and $f \colon X \to Y$ is surjective, we know that $g^{-1}(U) \subset Y'$ is contained in the image. If $y' \in Y'$ is a point, we can choose a map $\operatorname{Spec} R \to Y'$ from a DVR whose generic point maps to $g^{-1}(U)$ and whose special point maps to y'. Since $X \to Y$ is universally submersive, there exists an extension of DVRs $R \to R'$ and a lift $\operatorname{Spec} R' \to X$ (see Exercise A.4.9). The induced map $\operatorname{Spec} R' \to X \times_Y Y'$ factors through $\widetilde{X}$, and we see that y' is thus in the image of $\widetilde{X}$.

Since X is of finite type over S, so is $\widetilde{X}$. Using faithfully flat descent, $Y' \to S$ is also of finite type. To show that $Y \to S$ is of finite type, we may choose generators

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$a_1, \ldots, a_n \in I$ so that $Y' = \bigcup_i \operatorname{Spec} B_i$ where $B_i = B\langle f_j/f_i \rangle \subset K = \operatorname{Frac}(B)$ is the subalgebra generated by B and the elements f_j/f_i for $j \neq i$. Write $B = \bigcup_\lambda B_\lambda$ as a union of its finitely generated R-subalgebras. For $\lambda \gg 0$, each $f_i \in B_\lambda$ and we set $I_\lambda = (f_1, \ldots, f_n) \subset B_\lambda$. Since Y' is finite type over B, each B_i is finitely generated over B and thus for $\lambda \gg 0$, we see that in the diagram

$$
\begin{array}{ccc}
B_{\lambda,i} & \!\!=\!\!\!=\!\! B_\lambda\langle f_j/f_i \rangle \hookrightarrow & \operatorname{Frac}(B_\lambda) \\
\cup\big\downarrow & & \cup\big\downarrow \\
B_i & \!\!=\!\!\!=\!\! B\langle f_j/f_i \rangle \hookrightarrow & \operatorname{Frac}(B)
\end{array}
$$

the inclusion $B_{\lambda,i} \hookrightarrow B_i$ is surjective. It follows that $Y' = \operatorname{Bl}_I \operatorname{Spec} B = \operatorname{Bl}_{I_\lambda} \operatorname{Spec} B_\lambda$ for $\lambda \gg 0$. Considering the composition

$$ g_\lambda \colon Y' \xrightarrow{g} Y = \operatorname{Spec} B \xrightarrow{p_\lambda} \operatorname{Spec} B_\lambda, $$

the push forward of the injection $\mathcal{O}_Y \hookrightarrow g_* \mathcal{O}_{Y'}$ along p_λ yields an inclusion $p_{\lambda,*} \mathcal{O}_Y \hookrightarrow g_{\lambda,*} \mathcal{O}_{Y'}$. But $g_{\lambda,*} \mathcal{O}_{Y'}$ is a coherent module on $\operatorname{Spec} B_\lambda$ and thus so is $p_{\lambda,*} \mathcal{O}_Y$. This shows that B is a finite B_λ-module and thus finitely generated as an R-algebra. $\quad\square$

We apply this proposition to show that good moduli spaces are of finite type.

Proof of Theorem 6.3.5(3). If $\mathcal{X}$ is noetherian, if $I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideal sheaves of $\mathcal{O}_X$, then $\pi^{-1} I_1 \cdot \mathcal{O}_\mathcal{X} \subset \pi^{-1} I_2 \cdot \mathcal{O}_\mathcal{X} \subset$ is an ascending chain of ideal sheaves of $\mathcal{O}_\mathcal{X}$ which terminates. By Lemma 6.3.24(5), $I_n = \pi_*(\pi^{-1} I_n \cdot \mathcal{O}_\mathcal{X})$ and therefore the chain $I_1 \subset I_2 \subset \cdots$ terminates and X is noetherian.

Assume now that S is noetherian and $\mathcal{X}$ is of finite type over S. As $\mathcal{X} \to X$ is universally closed (Theorem 6.3.5(1)), it is also universally submersive. Choose a smooth presentation $U \to \mathcal{X}$ from a scheme. Since $U \to \mathcal{X}$ is universally submersive, so is the composition $U \to \mathcal{X} \to X$. Since $U \to S$ is of finite type and X is noetherian, Proposition 6.3.25 implies that $X \to S$ is also of finite type.

Given a coherent sheaf F on $\mathcal{X}$, to show that the pushforward $\pi_* F$ is coherent, we may assume that $X = \operatorname{Spec} A$ is affine and that $\mathcal{X}$ is irreducible. We first handle the case when $\mathcal{X}$ is reduced. By noetherian induction, we can assume that $\pi_* F$ is coherent if $\operatorname{Supp}(F) \subsetneq \mathcal{X}$. The maximal torsion subsheaf $F_{\mathrm{tors}} \subset F$ has support strictly contained in $\mathcal{X}$. Using the exact sequence $0 \to F_{\mathrm{tors}} \to F \to F/F_{\mathrm{tors}} \to 0$ and the exactness of π_*, we see the coherence of $\pi_*(F/F_{\mathrm{tors}})$ implies the coherence of $\pi_* F$. In other words, we can assume that F is torsion free. In this case, every section $s \colon \mathcal{O}_\mathcal{X} \to F$ is injective. We now argue by induction on the dimension of the vector space $\xi^* F$ where $\xi \colon \operatorname{Spec} K \to \mathcal{X}$ is a field-valued point whose image is the generic point. If F has no sections, then $\pi_* F = 0$ is coherent. Otherwise, a section induces a short exact sequence $0 \to \mathcal{O}_\mathcal{X} \to F \to F/\mathcal{O}_\mathcal{X} \to 0$ and $\xi^*(F/\mathcal{O}_\mathcal{X})$ has strictly smaller dimension. By again appealing to the exactness of π_*, we see that the coherence of $\pi_*(F/\mathcal{O}_\mathcal{X})$ implies the coherence of $\pi_* F$. Finally, to reduce to the reduced case, let $I \subset \mathcal{O}_\mathcal{X}$ be the ideal sheaf defining $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$. Then for some $N > 0$, we have that $I^N = 0$. By examining the exact sequences $0 \to \pi_*(I^{k+1} F) \to \pi_*(I^k F) \to \pi_*(I^k F/I^{k+1} F) \to 0$ and using that $\pi_*(I^k F/I^{k+1} F)$ is coherent (since $I^k F/I^{k+1} F$ is supported on $\mathcal{X}_{\mathrm{red}}$), we conclude by induction that $\pi_* F$ is coherent. $\quad\square$

6.3.6 Universality of good moduli spaces

We now complete the proof of Theorem 6.3.5 by showing that $\pi\colon \mathcal{X} \to X$ is universal for maps to algebraic spaces. Our argument follows the same logic for coarse moduli spaces in Theorem 4.3.6.

Proof of Theorem 6.3.5(4). We need to show that every diagram

$$\begin{array}{ccc} & \mathcal{X} & \\ \pi \downarrow & & \searrow f \\ X & \text{-- -- } & Y \end{array} \tag{6.3.26}$$

has a unique filling, or in other words that the natural map $\mathrm{Mor}(X, Y) \to \mathrm{Mor}(\mathcal{X}, Y)$ is bijective.

The uniqueness follows as in the proof of Theorem 4.3.6 and uses only that $\pi\colon \mathcal{X} \to X$ is universally closed, schematically dominant and surjective: if $h_1, h_2\colon X \to Y$ are two fillings of (6.3.26), then $\pi\colon \mathcal{X} \to X$ factors through the equalizer $E \to X$ of h_1 and h_2. Since $E \to X$ is universally closed, locally of finite type, surjective, and a monomorphism, it is an isomorphism.

For existence, the case when Y is affine is easy:

$$\mathrm{Mor}(X, Y) = \mathrm{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X)) =$$
$$\mathrm{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})) = \mathrm{Mor}(\mathcal{X}, Y).$$

(Although unnecessary for the argument below, the case when Y is a scheme is also straightforward: if $\{Y_i\}$ is an affine cover of Y and we set $\mathcal{X}_i := f^{-1}(Y_i) \subset \mathcal{X}$ with complement $\mathcal{Z}_i$, then $\mathcal{X} \setminus \pi^{-1}(\pi(\mathcal{Z}_i)) \to X \setminus \pi(\mathcal{Z}_i)$ is a good moduli space and $\mathcal{X} \setminus \pi^{-1}(\pi(\mathcal{Z}_i)) \subset \mathcal{X}_i$. By the affine case, we have unique factorizations $X \setminus \pi(\mathcal{Z}_i) \to Y_i$ and since $\bigcap_i \pi(\mathcal{Z}_i) = \emptyset$, these maps glue to the desired map $X \to Y$; see also [GIT, §0.6].)

For the general case, since $\mathcal{X}$ is quasi-compact, the map $\mathcal{X} \to Y$ factors through a quasi-compact subspace, so we can further assume that Y is quasi-compact. We can also use étale descent and limit methods to reduce to the case that $X = \mathrm{Spec}\, A$ where A is a strictly henselian local ring. This reduction works just as in the case of coarse moduli spaces (Theorem 4.3.6). Since A is local, there is a unique closed point $x \in \mathcal{X}$; let $\mathcal{G}_x \hookrightarrow \mathcal{X}$ be the closed immersion of the residual gerbe (Proposition 3.5.19).

Let $(Y' = \mathrm{Spec}\, B, y') \to (Y, f(x))$ be an étale presentation. The base change $\mathcal{X}' := \mathcal{X} \times_Y Y' \to \mathcal{X}$ is an étale, separated, surjective and representable morphism. Let $x' \in \mathcal{X}'$ be a preimage of $x \in \mathcal{X}$ and $\mathcal{U}' \subset \mathcal{X}'$ be a quasi-compact open substack containing x'.

$$\begin{array}{ccccc} \mathcal{U}' & \hookrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{X} \\ & & & & \downarrow \pi \\ & & & & X \\ & & & & \\ \downarrow & & & & \searrow f \\ Y' & \longrightarrow & & & Y \end{array}$$

Then $\mathcal{U}' \to \mathcal{X}$ is a quasi-finite, separated, and representable morphism, and Zariski's Main Theorem (6.1.10) implies that there is a factorization $\mathcal{U}' \to \widetilde{\mathcal{X}} \to \mathcal{X}$ with $\mathcal{U}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism. Writing $\widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A}$ for a

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coherent sheaf of algebras $\mathcal{A}$, Lemma 6.3.24(1) implies that $\widetilde{\pi} \colon \widetilde{\mathcal{X}} \to \widetilde{X} := \operatorname{Spec}_X \pi_* \mathcal{A}$ is a good moduli space and we know from Theorem 6.3.5(3) that $\pi_* \mathcal{A}$ is coherent. As $\widetilde{X} \to X = \operatorname{Spec} A$ is finite with A henselian, we can write $\widetilde{X} = \coprod_i \operatorname{Spec} A_i$ with each A_i a henselian local ring (Proposition B.5.10). Replace $\widetilde{\mathcal{X}}$ with the copy of $\widetilde{\mathcal{X}}_i := \widetilde{\pi}^{-1}(\operatorname{Spec} A_i)$ containing x' and replace $\mathcal{U}'$ with $\widetilde{\mathcal{X}}_i \cap \mathcal{U}'$. Then $\widetilde{\mathcal{X}}$ has a unique closed point which is the point $x' \in \mathcal{U}'$ and thus the complement $\widetilde{\mathcal{X}} \setminus \mathcal{U}'$ is empty, i.e. $\mathcal{U}' = \widetilde{\mathcal{X}}$. We conclude that $\mathcal{U}' \to \mathcal{X}$ is a finite étale morphism, and since it induces an isomorphism of residual gerbes at x', the map has degree one; it follows that $\mathcal{U}' \to \mathcal{X}$ is an isomorphism. Since Y' is an affine, the morphism $\mathcal{X} \cong \mathcal{U}' \to Y'$ factors through a map $X \to Y'$, and thus $f \colon \mathcal{X} \to Y$ factors through the composition $X \to Y' \to Y$. $\qquad\square$

6.3.7 Luna's Fundamental Lemma

We will apply the following result in our construction of good moduli spaces (Theorem 6.8.1), in the refinements of the Local Structure Theorem for Algebraic Stacks (6.5.1), and in the proof of Luna's Étale Slice Theorem (6.5.4), but it appears in many other arguments as well.

Theorem 6.3.27 (Luna's Fundamental Lemma). *Consider a commutative diagram*

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\ f\ } & \mathcal{X} \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
X' & \xrightarrow{\ g\ } & X
\end{array}
\tag{6.3.28}
$$

where $f \colon \mathcal{X}' \to \mathcal{X}$ is a separated and representable morphism of noetherian algebraic stacks, each with affine diagonal, and where π and π' are good moduli spaces. Let $x' \in \mathcal{X}'$ be a point such that

(a) f is étale at x',

(b) f induces an isomorphism of stabilizer groups at x', and

(c) $x' \in \mathcal{X}'$ and $x = f(x') \in \mathcal{X}$ are closed points.

Then there is an open neighborhood $U' \subset X'$ of $\pi'(x')$ such that $U' \to X$ is étale and such that $U' \times_X \mathcal{X} \cong \pi'^{-1}(U')$.

Remark 6.3.29. This result is really saying two things: (1) g is étale at $\pi'(x')$ and (2) after replacing X' with an open neighborhood of $\pi'(x')$ the diagram (6.3.28) is cartesian. In the case of quotients by finite groups, this was established in Proposition 4.3.7. Luna's original formulation [Lun73, p. 94] was the case when $\mathcal{X}' \cong [\operatorname{Spec} A'/G]$ and $\mathcal{X} \cong [\operatorname{Spec} A/G]$ with G linearly reductive and where $\mathcal{X}' \to \mathcal{X}$ is induced by a G-equivariant map $\operatorname{Spec} A' \to \operatorname{Spec} A$.

Proof. We will adapt the argument of Theorem 6.3.5(4). Since the question is étale local on X, limit methods (see the proof of Proposition 4.3.7) allow us to assume that $X = \operatorname{Spec} A$ with A a strictly henselian local ring. If $\mathcal{U}' \subset \mathcal{X}'$ is the étale locus of f, then $\mathcal{X}' \setminus \pi'^{-1}(\pi'(\mathcal{X}' \setminus \mathcal{U}'))$ contains x' since $\pi'(x')$ and $\pi'(\mathcal{X}' \setminus \mathcal{U}')$ are disjoint by Theorem 6.3.5(2). We can therefore replace $\mathcal{X}'$ with $\mathcal{X}' \setminus \pi'^{-1}(\pi'(\mathcal{X}' \setminus \mathcal{U}'))$ and assume that f is étale.

By Zariski's Main Theorem (6.1.10), we may choose a factorization $\mathcal{X}' \to \widetilde{\mathcal{X}} = \mathcal{S}pec_{\mathcal{X}} \mathcal{A} \to \mathcal{X}$ with $\mathcal{X}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism.

Then $\widetilde{\mathcal{X}} \to \widetilde{X} := \mathrm{Spec}_X\, \pi_* \mathcal{A}$ is a good moduli space and $\widetilde{X} \to X$ is finite. As A is henselian, we can write $\widetilde{X} = \coprod_i \mathrm{Spec}\, A_i$ with each A_i a henselian local ring. If $U' = \mathrm{Spec}\, A_i$ denotes the connected component containing the image of x', then $\widetilde{\pi}^{-1}(U') \subset \widetilde{\mathcal{X}}$ is an open substack containing a unique closed point, which is necessarily x'; it follows that $\mathcal{X}' = \pi^{-1}(U')$. Since $\mathcal{X}' \to \mathcal{X}$ is a finite étale morphism of degree one (as it preserves residual gerbes at x'), we see that $f\colon \mathcal{X}' \to \mathcal{X}$ is an isomorphism and thus so is $g\colon X' \to X$. $\qquad\square$

Corollary 6.3.30. *With the same hypotheses as* Theorem 6.3.27, *suppose that f is étale and that for all closed points $x' \in \mathcal{X}'$*

(a) *$f(x') \in \mathcal{X}$ is closed, and*

(b) *f induces an isomorphism of stabilizer groups at x'.*

Then $g\colon X' \to X$ is étale and (6.3.28) *is cartesian.* $\qquad\square$

6.3.8 Finite covers of good moduli spaces

Proposition 6.3.31. *Consider a commutative diagram*

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{\ f\ } & \mathcal{X} \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
X' & \xrightarrow{\ g\ } & X
\end{array}
$$

where $\mathcal{X}$ and $\mathcal{X}'$ are noetherian algebraic stacks with affine diagonal, and π and π' are good moduli spaces. Assume that

(a) *$f\colon \mathcal{X}' \to \mathcal{X}$ is quasi-finite, separated and representable,*

(b) *f maps closed points to closed points, and*

(c) *g is finite.*

Then f is finite.

Proof. By Zariski's Main Theorem (6.1.10), there is a factorization $\mathcal{X}' \to \widetilde{\mathcal{X}} = \mathcal{S}\mathrm{pec}_{\mathcal{X}}\, \mathcal{A} \to \mathcal{X}$ with $\mathcal{X}' \hookrightarrow \widetilde{\mathcal{X}}$ an open immersion and $\widetilde{\mathcal{X}} \to \mathcal{X}$ a finite morphism. Then $\widetilde{X} = \mathrm{Spec}_X\, \pi_* \mathcal{A}$ is a finite over X and $\widetilde{\mathcal{X}} \to \widetilde{X}$ is a good moduli space. By replacing $\mathcal{X} \to X$ with $\widetilde{\mathcal{X}} \to \widetilde{X}$, we can assume that f is an open immersion. By replacing $\mathcal{X}$ with the fiber product $X' \times_X \mathcal{X}$, we can further reduce to the case that $X' = X$. For every closed point $x \in X$, let $x' \in \mathcal{X}'$ be the unique closed point over x. By (b), $f(x') \in \mathcal{X}$ is the unique closed point over x. Since $\mathcal{X}'$ contains all the closed points of $\mathcal{X}$, $f\colon \mathcal{X}' \to c\mathcal{X}$ is an isomorphism. $\qquad\square$

Proposition 6.3.32. *Suppose $\mathcal{X}$ is a noetherian algebraic stack with affine diagonal and a good moduli space $\pi\colon \mathcal{X} \to X$. If the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-finite, then it is finite (i.e. $\pi\colon \mathcal{X} \to X$ is separated).*

Proof. We claim that $\mathcal{X} \times_X \mathcal{X} \to X$ is a good moduli space. By Lemma 6.3.16, the projection $p_1\colon \mathcal{X} \times_X \mathcal{X} \to \mathcal{X}$ is cohomologically affine and therefore so is the composition $\mathcal{X} \times_X \mathcal{X} \xrightarrow{p_1} \mathcal{X} \xrightarrow{\pi} X$. On the other hand, if $U \to \mathcal{X}$ is a smooth presentation, then $p_1\colon U \times_X \mathcal{X} \to U$ is a good moduli space (Lemma 6.3.21) and in particular $\mathcal{O}_U \xrightarrow{\sim} p_{1,*} \mathcal{O}_{U \times_X \mathcal{X}}$. It follows from descent that $\mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} p_{1,*} \mathcal{O}_{\mathcal{X} \times_X \mathcal{X}}$ and thus $\mathcal{O}_X \xrightarrow{\sim} (\pi \circ p_1)_* \mathcal{O}_{\mathcal{X} \times_X \mathcal{X}}$; the claim follows.

The diagonal $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is a quasi-finite, separated, and representable morphism that sends closed points to closed points and induces an isomorphism on good moduli spaces. Proposition 6.3.31 implies that $\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ is finite. Note that since $\mathcal{X}$ has affine diagonal, the finiteness of the diagonal is equivalent to its properness. $\square$

6.3.9 Descending vector bundles

Proposition 6.3.33. *Let $\mathcal{X}$ be a noetherian algebraic stack and $\pi\colon \mathcal{X} \to X$ be a good moduli space. A vector bundle F on $\mathcal{X}$ descends to a vector bundle on X if and only if for every field-valued point $x\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$ with closed image, the action of G_x on the fiber $F \otimes \Bbbk$ is trivial. In this case, π_*F is a vector bundle and the adjunction map $\pi^*\pi_*F \to F$ is an isomorphism.*

Proof. We follow the argument in the case of a tame coarse moduli space (Proposition 4.3.28). The condition is clearly necessary. To see that the condition is sufficient, consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{G}_x & \hookrightarrow & \mathcal{X} \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\
\operatorname{Spec}\kappa(x) & \hookrightarrow & X.
\end{array}
$$

We first claim that $\pi^*\pi_*F \to F$ is surjective. For every closed point $x \in \mathcal{X}$, the hypotheses imply that $p^*p_*(F|_{\mathcal{G}_x}) \cong F|_{\mathcal{G}_x}$. Applying $\pi^*\pi_*(-)|_{\mathcal{G}_x}$ to the surjection $F \to F|_{\mathcal{G}_x}$ and using the exactness of π_*, we obtain that $(\pi^*\pi_*F)|_{\mathcal{G}_x} \to \pi^*(\pi_*(F|_{\mathcal{G}_x}))|_{\mathcal{G}_x} \cong p^*p_*(F|_{\mathcal{G}_x}) \cong F|_{\mathcal{G}_x}$ is surjective. The claim now follows from Lemma 6.3.34.

To show that π_*F is a vector bundle, we may assume that $X = \operatorname{Spec}A$ is affine and that the rank r of F is constant. The surjection $\bigoplus_{s \in \Gamma(X,\pi_*F)} A \to \pi_*F$ pulls back to a surjection $\bigoplus_{s \in \Gamma(\mathcal{X},F)} \mathcal{O}_\mathcal{X} \to \pi^*\pi_*F$ and by the above claim, the composition $\bigoplus_{s \in \Gamma(\mathcal{X},F)} \mathcal{O}_\mathcal{X} \to \pi^*\pi_*F \to F$ is surjective. As $F|_{\mathcal{G}_x} \cong \mathcal{O}_{\mathcal{G}_x}^r$ is trivial, for each closed point $x \in |\mathcal{X}|$, we can find r sections $\phi\colon \mathcal{O}_\mathcal{X}^r \to F$ such that $\phi|_{\mathcal{G}_x}$ is an isomorphism. By Lemma 6.3.34, there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is an isomorphism. Thus $\pi_*\phi\colon \mathcal{O}_X^r \to \pi_*F$ is an isomorphism over U and we conclude that π_*F is a vector bundle of the same rank as F. Finally, since $\pi^*\pi_*F \to F$ is a surjection of vector bundles of the same rank, it is an isomorphism.

The case of a good quotient is due to Kempf. See also [KKV89, Prop. 4.2], [Alp13, Thm. 10.3] and [Ryd20, Thm. B]. $\square$

Lemma 6.3.34. *Let $\mathcal{X}$ be a noetherian algebraic stack and $\pi\colon \mathcal{X} \to X$ be a good moduli space. Let $x \in |\mathcal{X}|$ be a closed point.*

(1) If F is a coherent sheaf on $\mathcal{X}$ such that $F|_{\mathcal{G}_x} = 0$, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $F|_{\pi^{-1}(U)} = 0$.

(2) If $\phi\colon F \to G$ is a morphism of coherent sheaves (resp. vector bundles of the same rank) on $\mathcal{X}$ such that $\phi|_{\mathcal{G}_x}$ is surjective, then there exists an open neighborhood $U \subset X$ of $\pi(x)$ such that $\phi|_{\pi^{-1}(U)}$ is surjective (resp. an isomorphism).

Proof. The argument of Lemma 4.3.27 applies. $\square$

6.4 Coherent Tannaka duality and coherent completeness

We prove a version of Tannaka duality for noetherian algebraic stacks with affine diagonal (Theorem 6.4.1). We also introduce the notion of an algebraic stack $\mathcal{X}$ being coherently complete along a closed substack $\mathcal{X}_0$ (Definition 6.4.5) and show that certain quotient stacks with a unique closed point are coherently complete (Theorem 6.4.12). This includes the important examples of $[\mathbb{A}^1/\mathbb{G}_m]_R$ and ϕ_R defined in §6.8.2 where R is a complete DVR.

The combined power of Tannaka duality and coherent completeness allows us to extend compatible maps $\mathcal{X}_n \to \mathcal{Y}$ from the nth nilpotent thickenings of $\mathcal{X}_0$ to a morphism $\mathcal{X} \to \mathcal{Y}$ (Corollary 6.4.9). This technique is used in an essential way in the proof of the Local Structure Theorem for Algebraic Stacks (6.5.1) and also appears in many other arguments—it becomes a powerful new addition in our algebraic stack toolkit.

6.4.1 Coherent Tannaka Duality

A classical theorem of Gabriel [Gab62] states that two noetherian schemes X and Y are isomorphic if and only if their abstract categories $\mathrm{Coh}(X)$ and $\mathrm{Coh}(Y)$ of coherent sheaves are equivalent, or in other words that a scheme X can be recovered from the category $\mathrm{Coh}(X)$. In representation theory, classical Tannaka duality by Saavedra Rivano [SR72] (see also Deligne and Milne's article [DMOS82, Ch. II]) asserts that an affine group scheme G over a field $\Bbbk$ can be recovered from the tensor category $\mathrm{Rep}^{\mathrm{fd}}(G)$ of finite dimensional representations and its forgetful functor $\mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_\Bbbk$.

Combining these two facts, one might hope that an algebraic stack $\mathcal{X}$ is recovered by the tensor category $\mathrm{Coh}(\mathcal{X})$.[2] Following a brilliant observation of Lurie [Lur04], we will not only confirm this expectation, but we will show that in fact a tensor functor $\mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ is enough to recover a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks.

Theorem 6.4.1 (Coherent Tannaka Duality). *For noetherian algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$ with affine diagonal, the functor*

$$\mathrm{Mor}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Mor}^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X})), \qquad f \mapsto f^* \tag{6.4.2}$$

is an equivalence of categories, where $\mathrm{Mor}^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}))$ denotes the category of right exact additive tensor functors $\mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ of symmetric monoidal abelian categories where morphisms are tensor natural transformations.

Remark 6.4.3. A *symmetric monoidal category* is a category $\mathcal{A}$ endowed with a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and a unit $1 \in \mathcal{A}$ together with associativity isomorphisms $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C$, left and right unit isomorphisms $l_A : 1 \otimes A \xrightarrow{\sim} A \xrightarrow{\sim} A$ and $r_A : A \otimes 1 \xrightarrow{\sim} A$, and commutativity isomorphisms $s_{A,B} : A \otimes B \cong B \otimes A$ (with $s_{A,B} \circ s_{B,A} = \mathrm{id}$) satisfying certain coherence conditions [Mac71, §XI.1]. A *tensor functor* $F : \mathcal{A} \to \mathcal{B}$ between symmetric monoidal abelian categories is a functor equipped with isomorphisms $\Phi_{A,B} : F(A) \otimes F(B) \xrightarrow{\sim} F(A \otimes B)$ and $\varphi : 1_\mathcal{B} \xrightarrow{\sim} F(1_\mathcal{A})$ compatible with the isomorphisms $\alpha_{A,B,C}$, l_A, r_A and $s_{A,B}$ [Mac71, §XI.2]. A

---

[2]The structure as an abelian category is not enough, e.g. $\mathrm{Coh}(B\mathbb{Z}/2) \cong \mathrm{Coh}(\mathrm{Spec}\,\Bbbk \coprod \mathrm{Spec}\,\Bbbk)$ in $\mathrm{char}(\Bbbk) \neq 2$.

tensor natural transformation between tensor functors is a natural transformation of functors compatible with the isomorphisms $\Phi_{A,B}$ and φ [Mac71, §XI.2].

A *symmetric monoidal abelian category* (resp. *symmetric monoidal R-linear abelian category* for a ring R) is a symmetric monoidal (resp. R-linear) abelian category $\mathcal{A}$ such that $\otimes \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is additive (resp. R-linear) in each variable. A tensor functor is *additive* or *R-linear* if the underlying functor is. When $\mathcal{X}$ and $\mathcal{Y}$ are defined over a noetherian ring R, then Theorem 6.4.1 induces an equivalence

$$\mathrm{Mor}_R(\mathcal{X}, \mathcal{Y}) \xrightarrow{\sim} \mathrm{Mor}_R^{\otimes}(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X}))$$

between morphisms over R and right exact R-linear tensor functors.

Proof. Since every quasi-coherent sheaf on a noetherian algebraic stack is a colimit of its coherent subsheaves (Proposition 6.1.8), every right exact tensor functor $F \colon \mathrm{Coh}(\mathcal{Y}) \to \mathrm{Coh}(\mathcal{X})$ extends to a tensor functor $F \colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ preserving colimits. Likewise every tensor natural transformation between functors of coherent sheaves extends uniquely to one defined on quasi-coherent sheaves.

Fully faithfulness: Let $f, g \colon \mathcal{X} \to \mathcal{Y}$. Choose a smooth presentation $p \colon U \to \mathcal{Y}$ where U is an affine scheme. Since the question is smooth-local on $\mathcal{X}$, after replacing $\mathcal{X}$ with $\mathcal{X} \times_{f, \mathcal{Y}, p} U$, we may assume there is a factorization $f \colon \mathcal{X} \xrightarrow{\tilde{f}} U \xrightarrow{p} \mathcal{Y}$. Likewise, we may assume there is a factorization $g \colon \mathcal{X} \xrightarrow{\tilde{g}} V \xrightarrow{q} \mathcal{Y}$ where V is an affine scheme. Since $\mathcal{Y}$ has affine diagonal, $p \colon U \to \mathcal{Y}$ is affine and we have identifications

$$\mathrm{Mor}_{\mathcal{Y}}(\mathcal{X}, U) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(p_* \mathcal{O}_U, f_* \mathcal{O}_{\mathcal{X}}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}-\mathrm{alg}}(f^* p_* \mathcal{O}_U, \mathcal{O}_{\mathcal{X}})$$

Therefore $\tilde{f}$ and $\tilde{g}$ correspond to sections $s_{\tilde{f}} \colon f^* p_* \mathcal{O}_U \to \mathcal{O}_{\mathcal{X}}$ and $s_{\tilde{g}} \colon g^* q_* \mathcal{O}_V \to \mathcal{O}_{\mathcal{X}}$. A 2-isomorphism $\alpha \colon f \to g$ (i.e. a morphism in $\mathrm{Mor}(\mathcal{X}, \mathcal{Y})$) is identified with a factorization

$$\mathcal{X} \xrightarrow{(\tilde{f}, \tilde{g}, \alpha)} U \times_{\mathcal{X}} V$$

with a map id from $\mathcal{X}$ and π to $\mathcal{X}$.

which is the same data as a section s_{α} of $\mathcal{O}_{\mathcal{X}} \to f^* \pi_* \mathcal{O}_{U \times_{\mathcal{X}} V}$. Letting $\alpha^* \colon f^* \to g^*$ be the image of α under (6.4.2), i.e. the pullback tensor natural transformation, the section s_{α} can be written as

$$f^* \pi_* \mathcal{O}_{U \times_{\mathcal{X}} V} \cong f^*(p_* \mathcal{O}_U) \otimes f^*(q_* \mathcal{O}_V) \xrightarrow{\mathrm{id} \otimes \alpha^*_{q_* \mathcal{O}_V}} f^*(p_* \mathcal{O}_U) \otimes g^*(q_* \mathcal{O}_V) \xrightarrow{s_{\tilde{f}} \otimes s_{\tilde{g}}} \mathcal{O}_S.$$

To see the faithfulness of (6.4.2), if $\alpha, \alpha' \colon f \to g$ are 2-isomorphisms with $\alpha^* = \alpha'^*$, then $\alpha^*_{q_* \mathcal{O}_V} = \alpha'^*_{q_* \mathcal{O}_V}$ and therefore the two sections s_{α} and $s_{\alpha'}$ are equal and $\alpha = \alpha'$. For the fullness of (6.4.2), let $\beta \colon f^* \to g^*$ be a tensor natural transformation. Then $\mathrm{id} \otimes \beta_{q_* \mathcal{O}_V}$ defines a section $f^* \pi_* \mathcal{O}_{U \times_{\mathcal{X}} V} \to \mathcal{O}_S$ and thus a 2-isomorphism $\alpha \colon f \to g$ such that $\beta_{q_* \mathcal{O}_V} = \alpha^*_{q_* \mathcal{O}_V}$. To see that $\beta_E = \alpha^*_E$ for every $E \in \mathrm{QCoh}(\mathcal{Y})$, note that the factorization $g = q \circ \tilde{g}$ yields a splitting of $g^* E \to g^*(q_* q^* E)$. Since f^* and g^* commute with direct sums, it suffices to assume that $E = q_* G$ for $G \in \mathrm{QCoh}(V)$. Writing $G = \mathrm{colim}(\mathcal{O}_V^{\oplus I} \to \mathcal{O}_V^{\oplus J})$ as a colimit of free $\mathcal{O}_V$-modules, we can conclude that $\beta_{q_* G} = \alpha^*_{q_* G}$ since f^* and g^* commute with colimits and q_* is exact.

Essential surjectivity: Let $F \colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$ be a tensor functor preserving colimits.

The affine case: If $\mathcal{X} = \operatorname{Spec} A$ and $\mathcal{Y} = \operatorname{Spec} B$ are noetherian affine schemes, then we have a map

$$\phi\colon B \cong \operatorname{End}(\mathcal{O}_\mathcal{Y}) \xrightarrow{F} \operatorname{End}(\mathcal{O}_\mathcal{X}) = A.$$

We claim that ϕ is a ring homomorphism and that there is a functorial isomorphism $F(N) = N \otimes_B A$ for $N \in \operatorname{Mod}_B$. For $b, b' \in B$, consider the commutative diagrams

$$
\begin{array}{ccc}
\mathcal{O}_\mathcal{Y} \otimes \mathcal{O}_\mathcal{Y} \longrightarrow \mathcal{O}_\mathcal{Y} & & A \otimes A \longrightarrow A \\
\Big\downarrow{\scriptstyle b \otimes b'} \quad \Big\downarrow{\scriptstyle bb'} & \xmapsto{\ F\ } & \Big\downarrow{\scriptstyle \phi(b) \otimes \phi(b')} \Big\downarrow{\scriptstyle \phi(bb')} \\
\mathcal{O}_\mathcal{Y} \otimes \mathcal{O}_\mathcal{Y} \longrightarrow \mathcal{O}_\mathcal{Y} & & A \otimes A \longrightarrow A
\end{array}
$$

where the horizontal maps correspond to multiplication. The commutativity of the right square is implied by the fact that F preserves tensor products. This shows that $\phi(b)\phi(b') = \phi(bb')$. For a B-module N, choose a free presentation $B^{\oplus J} \to B^{\oplus I} \to N \to 0$. Since both F and $- \otimes_B A$ are right exact and preserve direct sums, applying them to the free presentation yields an identification $F(N) \cong N \otimes_B A$ as both are cokernels of $A^{\oplus J} \to A^{\oplus I}$. One checks similarly that this identification is functorial.

Reduction to the case that $\mathcal{X}$ is affine: Choose a smooth presentation $g\colon U \to \mathcal{X}$ from an affine scheme and consider the diagram

$$
\begin{array}{c}
U \times_X U \\
{\scriptstyle p_2}\Big\downarrow\Big\downarrow{\scriptstyle p_1} \\
U \\
{\scriptstyle g}\Big\downarrow \qquad {\scriptstyle g^* \circ F} \\
\mathcal{X} \dashrightarrow[F] \mathcal{Y}
\end{array}
$$

where the dashed arrow $\mathcal{X} \dashrightarrow \mathcal{Y}$ is denoting that we have a tensor functor $\operatorname{QCoh}(\mathcal{Y}) \to \operatorname{QCoh}(\mathcal{X})$ in the other direction. Assuming that the result holds when U is affine, there is a morphism $h\colon U \to \mathcal{Y}$ and an isomorphism $h \xrightarrow{\sim} g^* \circ F$ of functors. By full faithfulness, there is an isomorphism $p_1 \circ h \xrightarrow{\sim} p_2 \circ h$ satisfying the cocycle condition, and thus smooth descent implies that there is a unique morphism $f\colon \mathcal{X} \to \mathcal{Y}$ with $F \simeq f^*$.

Reduction to the case that $\mathcal{Y}$ is affine: Let $\mathcal{X} = \operatorname{Spec} A$ and choose a smooth presentation $q\colon V = \operatorname{Spec} C \to \mathcal{Y}$. Since $\mathcal{Y}$ has affine diagonal, q is an affine morphism. Define $B := F(q_* \mathcal{O}_V)$ which is an A-algebra since $q_* \mathcal{O}_V$ is an $\mathcal{O}_\mathcal{Y}$-algebra. Consider the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} B & \dashrightarrow[F'] & V = \operatorname{Spec} C \\
\Big\downarrow & & \Big\downarrow{\scriptstyle q} \\
\operatorname{Spec} A = \mathcal{X} & \dashrightarrow[F] & \mathcal{Y}
\end{array}
$$

where $F'\colon \operatorname{Mod}_C \to \operatorname{Mod}_B$ is the right exact tensor functor sending M to $F(q_* \widetilde{M})$ (which is a module over $B = F(q_* \mathcal{O}_V)$ because $q_* \widetilde{M}$ is a $q_* \mathcal{O}_V$-module). By the

affine case, F' is induced by a morphism $f' \colon \operatorname{Spec} B \to \operatorname{Spec} C$. We can extend the above diagram into

$$
\begin{array}{ccc}
\operatorname{Spec} B \otimes_A B & \xrightarrow{\;f''\;} & V \times_{\mathcal{Y}} V \\
\downarrow\downarrow & & \downarrow\downarrow \\
\operatorname{Spec} B & \xrightarrow{\;f'\;} & V = \operatorname{Spec} C \\
\downarrow & & \downarrow{\scriptstyle q} \\
\operatorname{Spec} A = \mathcal{X} & \xdashrightarrow{\;F\;} & \mathcal{Y}.
\end{array}
$$

Since q is affine, $V \times_{\mathcal{Y}} V$ is affine and the top square (under either set of projections) is cartesian.

If we could show that $A \to B$ is faithfully flat, we would be done as the full faithfulness in the affine case would imply that f' descends to our desired morphism $f \colon \mathcal{X} \to \mathcal{Y}$. This seems hard to directly check, but we do know already that the maps $B \rightrightarrows B \otimes_A B$ are faithfully flat as they correspond to base changes of the smooth maps $V \times_{\mathcal{Y}} V \rightrightarrows V$. We will show instead that $A \to B$ is universally injective. Since faithful flatness descends under universal injectivity maps (Proposition A.2.24(4)), the faithful flatness of $A \to B$ follows from the universal injectivity.

Universal injectivity of $A \to B$: Recall from Definition A.2.22 that an injective map of A-modules is called universally injective if it remains injective after tensoring by every A-module. By Proposition A.2.24(3), this notion is local under faithfully flat morphisms and thus extends to morphisms $F \to G$ of quasi-coherent sheaves on an algebraic stack.

Since $q \colon V \to \mathcal{Y}$ is faithfully flat, $\mathcal{O}_{\mathcal{Y}} \to q_*\mathcal{O}_V$ is universally injective (Proposition A.2.24(1)). We write $q_*\mathcal{O}_V = \operatorname{colim} Q_i$ as a colimit of coherent subsheaves (Proposition 6.1.8) and we may assume that each Q_i contains the image of $\mathcal{O}_{\mathcal{Y}} \to q_*\mathcal{O}_V$. Then $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is also universally injective and since Q_i is coherent, $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is a split injection smooth-locally on $\mathcal{Y}$ (Proposition A.2.24(2)). Applying F to $\mathcal{O}_{\mathcal{Y}} \to q_*\mathcal{O}_V = \operatorname{colim} Q_i$ and using that it preserves colimits, we have a factorization

$$
\begin{array}{ccc}
& & F(Q_i) \\
& \nearrow & \downarrow \\
A = F(\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & B = F(q_*\mathcal{O}_V) = \operatorname{colim} F(Q_i).
\end{array}
$$

It suffices to show that $A \to F(Q_i)$ is universally injective. We will show in fact that it is a split injection. As $\mathcal{O}_{\mathcal{Y}} \to Q_i$ is smooth-locally split, the map on duals $Q_i^\vee \to \mathcal{O}_{\mathcal{Y}}^\vee = \mathcal{O}_{\mathcal{Y}}$ is surjective. Applying F, we have a surjection $F(Q_i^\vee) \to F(\mathcal{O}_{\mathcal{Y}}) = A$ (using right exactness) and we can choose an element $\lambda \in F(Q_i^\vee)$ mapping to 1. Under the natural map $F(Q_i^\vee) \to F(Q_i)^\vee$, the element λ is sent to a map $F(Q_i) \to A$, which one checks to be a section of the given map $A \to F(Q_i)$.

See also [Lur04], [HR19b], [BHL17] and [SP, Tag 0GRR]. $\qquad\square$

Remark 6.4.4 (Relation to classical Tannaka duality)**.** If G is an affine group scheme over a field $\Bbbk$, then the category $\mathcal{C} = \operatorname{Rep}^{\mathrm{fd}}(G)$ of finite dimensional representations is a symmetric monoidal $\Bbbk$-linear category and there is a tensor functor $\omega \colon \operatorname{Rep}^{\mathrm{fd}}(G) \to \operatorname{Vect}_{\Bbbk}$. For $\Bbbk$-algebra R, let ω_R denote the composition $\operatorname{Rep}^{\mathrm{fd}}(G) \to \operatorname{Vect}_{\Bbbk} \xrightarrow{-\otimes_{\Bbbk} R} \operatorname{Mod}_R$ and let $\operatorname{Aut}^{\otimes}(\omega_R)$ denote the group of tensor natural isomorphisms of ω_R.

Then G is recovered as the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ on affine $\Bbbk$-schemes assigning R to $\mathrm{Aut}^{\otimes}(\omega_R)$ [DMOS82, II.2.8].

On the other hand, Coherent Tannaka Duality for Algebraic Stacks (Theorem 6.4.1) implies that for every noetherian $\Bbbk$-algebra R, there is an equivalence of categories

$$\mathrm{Mor}_{\Bbbk}(\mathrm{Spec}\,R, BG) \xrightarrow{\sim} \mathrm{Mor}^{\otimes}(\mathrm{Rep}(G)^{\mathrm{fd}}, \mathrm{Mod}_R).$$

In this way, we see that $\mathrm{Rep}(G)^{\mathrm{fd}}$ determines BG. To recover G, the fiber functor $\omega\colon \mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_{\Bbbk}$ corresponds to a morphism $p\colon \mathrm{Spec}\,\Bbbk \to BG$ and $G = \mathrm{Aut}_{\Bbbk}(p)$. For example, if $O(q)$ and $O(q')$ are orthogonal groups with respect to non-degenerate quadratic forms q and q' of the same dimension, then $\mathrm{Rep}(O(q)) \cong \mathrm{Rep}(O(q'))$ even though $O(q)$ and $O(q')$ may not be isomorphic; in this case the two maps $\mathrm{Spec}\,\Bbbk \to BO(q)$ and $\mathrm{Spec}\,\Bbbk \to BO(q')$ define two different fiber functors on the same category.

The classical version also provides conditions when the data of $(\mathcal{C}, \omega)$ is isomorphic to the category of representations of a group scheme. Namely, we say $\mathcal{C}$ is *rigid* if for every object of $X \in \mathcal{C}$, there is a 'dual' $X^{\vee} \in \mathcal{C}$, i.e. an object $X^{\vee}$ such that $X^{\vee} \otimes -\colon \mathcal{C} \to \mathcal{C}$ is right adjoint to $X \otimes -\colon \mathcal{C} \to \mathcal{C}$. If $\mathcal{C}$ is a rigid symmetric monoidal $\Bbbk$-linear abelian category with $\mathrm{End}(1) = \Bbbk$ and $\omega\colon \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$ is an exact faithful $\Bbbk$-linear tensor functor, then $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ is represented by an affine group scheme G over $\Bbbk$ and there is a tensor equivalence $\mathcal{C} \cong \mathrm{Rep}^{\mathrm{fd}}(G)$ under which ω corresponds to the forgetful functor [DMOS82, II.2.11]. Moreover, G is of finite type over $\Bbbk$ if and only if $\mathcal{C}$ has a tensor generator.

6.4.2 Coherent completeness

Coherent Tannaka Duality becomes especially powerful when combined with coherent completeness.

Definition 6.4.5. A noetherian algebraic stack $\mathcal{X}$ is *coherently complete along a closed substack* $\mathcal{X}_0$ if the natural functor

$$\mathrm{Coh}(\mathcal{X}) \to \varprojlim \mathrm{Coh}(\mathcal{X}_n), \quad F \mapsto (F_n)$$

is an equivalence of categories, where $\mathcal{X}_n$ denotes the nth nilpotent thickening of $\mathcal{X}_0$ and F_n is the pullback of F to $\mathcal{X}_n$.

Remark 6.4.6. If $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ is the coherent sheaf of ideals defining $\mathcal{X}_0$, then $\mathcal{X}_n$ is defined by $\mathcal{I}^{n+1}$. Letting $i_n\colon \mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$ denote the natural inclusion, an object in $\varprojlim \mathrm{Coh}(\mathcal{X}_n)$ corresponds to a sequence $F_n \in \mathrm{Coh}(\mathcal{X}_n)$ of coherent sheaves together with maps $\alpha_n\colon i_{n,*}F_n \to F_{n+1}$ inducing isomorphism $F_n \to i_n^* F_{n+1}$. A morphism $(F_n, \alpha_n) \to (F_n', \alpha_n')$ is a sequence of maps $\phi_n\colon F_n \to F_n'$ such that $\phi_{n+1} \circ \alpha_n = \alpha_{n+1} \circ i_{n,*}\phi_n$.

Example 6.4.7. If $(R, \mathfrak{m})$ is a complete noetherian local ring, then the Artin–Rees Lemma (B.5.4) implies that $\mathrm{Spec}\,R$ is coherently complete along $\mathrm{Spec}\,R/\mathfrak{m}$. The same is true if $R = \lim R/I^n$ is a noetherian I-adically complete ring.

Example 6.4.8. Grothendieck's Existence Theorem (C.4.4) asserts that if X is a proper scheme over a complete local ring $(R, \mathfrak{m})$ and $X_0 = X \times_R R/\mathfrak{m}$, then X is coherently complete along X_0. If $\mathcal{X}$ is a proper Deligne–Mumford stack or more generally a proper algebraic stack over $\mathrm{Spec}\,R$, the same is true. The result also holds if R is an I-adically complete noetherian ring. See [Ols05, Thm. 1.4] or [Con05a, Thm. 4.1].

Corollary 6.4.9 (Coherent Tannaka Duality). *Let $\mathcal{X}$ and $\mathcal{Y}$ be noetherian algebraic stacks with affine diagonal. Suppose that $\mathcal{X}$ is coherently complete along $\mathcal{X}_0$. Then there is an equivalence of categories*

$$\mathrm{Mor}(\mathcal{X}, \mathcal{Y}) \to \varprojlim \mathrm{Mor}(\mathcal{X}_n, \mathcal{Y}), \quad f \mapsto (f_n),$$

where $f_n \colon \mathcal{X}_n \to Y$ denotes the restriction of f to the nth nilpotent thickening $\mathcal{X}_n$ of $\mathcal{X}_0$.

Proof. This follows from the equivalences

$$
\begin{aligned}
\mathrm{Mor}(\mathcal{X}, \mathcal{Y}) &\simeq \mathrm{Mor}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}(\mathcal{X})\big) && \text{(Coherent Tannaka Duality)}\\
&\simeq \mathrm{Mor}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}), \varprojlim \mathrm{Coh}\big(\mathcal{X}_n\big)\big) && \text{(coherent completeness)}\\
&\simeq \varprojlim \mathrm{Mor}^{\otimes}\big(\mathrm{Coh}(\mathcal{Y}), \mathrm{Coh}\big(\mathcal{X}_n\big)\big)\\
&\simeq \varprojlim \mathrm{Mor}\big(\mathcal{X}_n, \mathcal{Y}\big) && \text{(Coherent Tannaka Duality).}
\end{aligned}
$$

$\square$

Remark 6.4.10. If $\mathcal{X}$ and $\mathcal{Y}$ are defined over a noetherian ring R, then there is an equivalence $\mathrm{Mor}_R(\mathcal{X}, \mathcal{Y}) \to \varprojlim \mathrm{Mor}_R(\mathcal{X}_n \mathcal{Y})$. This follows in the same way using the Tannaka duality equivalence between the category of morphisms $\mathcal{X} \to \mathcal{Y}$ over R and the category of right exact R-linear tensor functors (Remark 6.4.3).

For example, to show that there is map $\mathrm{Spec}\, A \to \mathcal{Y}$ from the spectrum of a noetherian I-adically complete ring A, it suffices to construct compatible maps $\mathrm{Spec}\, A/I^n \to \mathcal{Y}$. This is only easy to see directly if A is local.

Exercise 6.4.11. Let G be an affine algebraic group acting on a separated noetherian algebraic space W over $\Bbbk$. Let $W_0 \subset W$ be a G-invariant closed subspace and let W_n be its nth nilpotent thickenings. Suppose that $[W/G]$ is coherently complete along a closed substack $[W_0/G]$. For every noetherian algebraic space X over $\Bbbk$ with affine diagonal equipped with an action of G, the natural map on equivariant maps

$$\mathrm{Mor}^G(W, X) \to \varprojlim_n \mathrm{Mor}^G\big(W_n, X\big)$$

is bijective.

Hint: reduce to Corollary 6.4.9 by using that a G-equivariant map $W \to X$ corresponds to a morphism $[W/G] \to [X/G]$ over BG, i.e $\mathrm{Mor}^G(W, X) = \{*\} \times_{\mathrm{Mor}([W/G], BG)} \mathrm{Mor}([W/G], [X/G])$.

6.4.3 Coherent completeness of quotient stacks

The coherent completeness result that we will exploit through the rest of the book and in particular in the proof of the Local Structure Theorem for Algebraic Stacks (6.5.1) is the following:

Theorem 6.4.12. *Let $\Bbbk$ be an algebraically closed field and R be a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$. Let G be a linearly reductive group over $\Bbbk$ acting on an affine scheme $\mathrm{Spec}\, A$ of finite type over R. Suppose that $A^G = R$ and that there is a G-fixed $\Bbbk$-point $x \in \mathrm{Spec}\, A$. Then $[\mathrm{Spec}\, A/G]$ is coherently complete along the closed substack BG defined by x.*

Example 6.4.13. If $\mathbb{G}_m$ acts diagonally on $\mathbb{A}^r$, then $[\mathbb{A}^r/\mathbb{G}_m]$ is coherently complete along the origin $B\mathbb{G}_m$. In other words a $\mathbb{G}_m$-equivariant coherent sheaf on $\mathbb{A}^r$ is equivalent to a compatible family of $\mathbb{G}_m$-equivariant modules over $\Bbbk[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^{n+1}$.

Remark 6.4.14. We have a commutative diagram

$$
\begin{array}{ccccc}
BG & \hookrightarrow & [\operatorname{Spec} A/G] \times_{A^G} \Bbbk & \hookrightarrow & [\operatorname{Spec} A/G] \\
& & \downarrow & \square & \downarrow \\
& & \operatorname{Spec} \Bbbk & \hookrightarrow & \operatorname{Spec} A^G.
\end{array}
$$

A formal consequence of the above theorem is that $[\operatorname{Spec} A/G]$ is also coherently complete with respect to the fiber $[\operatorname{Spec} A/G] \times_{A^G} \Bbbk$. This version is analogous to Grothendieck's Existence Theorem (6.4.8), but the coherent completeness along BG is a substantially stronger statement, e.g. for $[\mathbb{A}^n/\mathbb{G}_m]$ where the fiber of $[\mathbb{A}^n/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk$ is everything.

Proof of Theorem 6.4.12. We need to show that $\operatorname{Coh}(\mathcal{X}) \to \varprojlim \operatorname{Coh}(\mathcal{X}_n)$ is an equivalence of categories, where $\mathcal{X} = [\operatorname{Spec} A/G]$ and $\mathcal{X}_n$ is the nth nilpotent thickening of $BG \hookrightarrow \mathcal{X}$ of the inclusion of the residual gerbe at x.

Full faithfulness: Suppose that F and F' are coherent $\mathcal{O}_\mathcal{X}$-modules, and let F_n and F'_n denote the restrictions to $\mathcal{X}_n$, respectively. We need to show that

$$
\operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(F, F') \to \varprojlim \operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(F_n, F'_n)
$$

is bijective. Since $\mathcal{X}$ has the resolution property (Proposition 6.1.19), we can find a resolution $F_2 \to F_1 \to F \to 0$. by vector bundles. This induces a diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(F, F') & \longrightarrow & \operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(F_1, F') & \longrightarrow & \operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(F_2, F') \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \varprojlim \operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(F_n, F'_n) & \longrightarrow & \varprojlim \operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(F_{1,n}, F'_n) & \longrightarrow & \varprojlim \operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(F_{2,n}, F'_n)
\end{array}
$$

with exact rows. We may therefore assume that F is a vector bundle. In this case,

$$
\operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(F, F') = \operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(\mathcal{O}_\mathcal{X}, F^\vee \otimes F')
$$
$$
\operatorname{Hom}_{\mathcal{O}_\mathcal{X}}(F_n, F'_n) = \operatorname{Hom}_{\mathcal{O}_\mathcal{X}}\big(\mathcal{O}_{\mathcal{X}_n}, (F_n^\vee \otimes F'_n)\big).
$$

Therefore, we can also assume that $F = \mathcal{O}_\mathcal{X}$ and we are reduced to showing that

$$
\Gamma(\mathcal{X}, F') \to \varprojlim \Gamma\big(\mathcal{X}_n, F'_n\big) \tag{6.4.15}
$$

is an isomorphism. Writing $F' = \widetilde{M}$ where M is a finitely generated A-module with an action of G and letting $\mathfrak{m} \subset A$ be the maximal ideal for x, then $\Gamma(\mathcal{X}_n, F'_n) = M^G/(\mathfrak{m}^n M)^G$ since G is linearly reductive. We must therefore verify that

$$
M^G \to \varprojlim M^G/(\mathfrak{m}^n M)^G \tag{6.4.16}
$$

is an isomorphism. To this end, we first show that

$$
\bigcap_{n \geq 0} \big(\mathfrak{m}^n M\big)^G = 0. \tag{6.4.17}
$$

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or in other words that (6.4.16) is injective. Let $N := \bigcap_{n \geq 0} \mathfrak{m}^n M$. The Artin–Rees Lemma (B.5.4) applied to $N \subset M$ implies that there exists an integer c such that $\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-c}(\mathfrak{m}^c M \cap N)$ for all $n \geq c$. Taking $n = c + 1$, we see that $N = \mathfrak{m}N$ so $N \otimes_A A/\mathfrak{m} = 0$. Since the support of N is a closed G-invariant subscheme of $\operatorname{Spec} A$ which does not contain x, it follows that $N = 0$.

Note also that since G is linearly reductive, M^G is a finitely generated A^G-module (Corollary 6.3.7(3)). We next establish that (6.4.16) is an isomorphism if A^G is artinian. In this case, $\{(\mathfrak{m}^n M)^G\}$ automatically satisfies the Mittag–Leffler condition (it is a sequence of artinian A^G-modules). Therefore, taking the inverse limit of the exact sequences $0 \to (\mathfrak{m}^n M)^G \to M^G \to M^G/(\mathfrak{m}^n M)^G \to 0$ and applying (6.4.17) yields an exact sequence

$$0 \to 0 \to M^G \to \varprojlim M^G/(\mathfrak{m}^n M)^G \to 0$$

and shows that (6.4.16) is an isomorphism. To establish (6.4.16) in the general case, let $J = (\mathfrak{m}^G)A \subset A$ and observe that

$$M^G \cong \varprojlim M^G/(\mathfrak{m}^G)^n M^G \cong \varprojlim (M/J^n M)^G, \qquad (6.4.18)$$

since G is linearly reductive. For each n, we know that

$$(M/J^n M)^G \cong \varprojlim_l M^G/((J^n + \mathfrak{m}^l)M)^G \qquad (6.4.19)$$

using the artinian case proved above. Finally, combining (6.4.18) and (6.4.19) together with the observation that $J^n \subset \mathfrak{m}^l$ for $n \geq l$, we conclude that

$$
\begin{aligned}
M^G &\cong \varprojlim_n (M/J^n M)^G \\
&\cong \varprojlim_n \varprojlim_l M^G/((J^n + \mathfrak{m}^l)M)^G \\
&\cong \varprojlim_l M^G/(\mathfrak{m}^l M)^G.
\end{aligned}
$$

Essential surjectivity: The linear reductivity of G implies that every coherent sheaf $F = \widetilde{M}$ on $[\operatorname{Spec} A/G]$ decomposes as a direct sum

$$M = \bigoplus_{\rho \in \Gamma} M^{(\rho)}, \qquad (6.4.20)$$

where Γ denotes the set of isomorphism classes of irreducible representations of G and $M^{(\rho)}$ is the isotypic component corresponding to ρ; explicitly if W_ρ denotes the irreducible representation corresponding to ρ, then $M^{(\rho)} = \operatorname{Hom}_{\Bbbk}^G(W_\rho, M) \otimes W_\rho$. Moreover, the decomposition (6.4.20) is compatible with the A-module structure of M and the decomposition $A = \bigoplus_{\rho \in \Gamma} A^{(\rho)}$.

Let us also note that if $F = \widetilde{M} \in \operatorname{Coh}(\mathcal{X})$ with restrictions $M_n = M/\mathfrak{m}^{n+1}M$, then applying (6.4.16) to $M \otimes W_\rho^\vee$ shows that $M^{(\rho)} = \varprojlim M_n^{(\rho)}$. By Theorem 6.3.5(3), we also know that $M^{(\rho)}$ is a finitely generated A^G-module. In particular, $A^{(\rho)} = \varprojlim (A/\mathfrak{m}^{n+1})^{(\rho)}$ is a finitely generated A^G-module.

This suggests that if $F_n = \widetilde{M}_n$ is a compatible system of coherent $\mathcal{O}_{\mathcal{X}_n}$-modules with $M_n = \bigoplus_\rho M_n^{(\rho)}$, we define

$$M^{(\rho)} := \varprojlim M_n^{(\rho)} \qquad \text{and} \qquad M := \bigoplus_{\rho \in \Gamma} M^{(\rho)}. \qquad (6.4.21)$$

To see that M is an A-module with a G-action, let $\rho, \gamma \in \Gamma$ be irreducible representations and let $\Lambda \subset \Gamma$ denote the finite set of nonzero irreducible representations appearing $W_\rho \otimes W_\gamma$. Taking limits of the maps $A_n^{(\rho)} \otimes_{(A/\mathfrak{m}^{n+1})^G} M_n^{(\gamma)} \to \bigoplus_{\lambda \in \Lambda} M_n^{(\lambda)}$, defines multiplication

$$A^{(\rho)} \otimes_A M^{(\gamma)} \to \varprojlim \left(A_n^{(\rho)} \otimes_{(A/\mathfrak{m}^{n+1})^G} M_n^{(\gamma)} \right) \to \varprojlim \left(\bigoplus_{\lambda \in \Lambda} M_n^{(\lambda)} \right) \cong \bigoplus_{\lambda \in \Lambda} M^{(\lambda)}.$$

Note that we also have $M/\mathfrak{m}^{n+1}M \cong M_n$ by construction.

We need to show that the A-module M of (6.4.21) is finitely generated. The coherent sheaf $F_0 = \widetilde{M_0}$ on $\mathcal{X}_0 = BG$ is a finite dimensional G-representation and we can consider the coherent $\mathcal{O}_\mathcal{X}$-module $F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_\mathcal{X}$ or equivalently the A-module $M_0 \otimes_\Bbbk A$ with its natural G-action. Since $\mathcal{X}$ is cohomologically affine, the functor

$$\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_\mathcal{X}, -) = \Gamma(\mathcal{X}, (F_0^\vee \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_\mathcal{X}) \otimes_{\mathcal{O}_\mathcal{X}} -)$$

is exact. Apply the functor to the surjection $M \twoheadrightarrow M_0$ induces a map

$$M_0 \otimes_\Bbbk A \to M \tag{6.4.22}$$

which we would like to show is surjective. We do know that the restrictions $M_0 \otimes_\Bbbk (A/\mathfrak{m}^{n+1}) \to M_n$ are surjective as its cokernel is a coherent module on $\mathcal{X}_n$ not supported at the unique closed point.

As above, we first handle the case that A^G is artinian. Since $A^{(\rho)} \xrightarrow{\sim} \varprojlim (A/\mathfrak{m}^n)^{(\rho)}$ is a finitely generated A^G-module, it follows that $(A/\mathfrak{m}^n)^{(\rho)}$ stabilizes to $A^{(\rho)}$ for $n \gg 0$. Since (6.4.22) induces surjections $M_0 \otimes_\Bbbk (A/\mathfrak{m}^{n+1}) \to M_n$, it follows that the modules $M_n^{(\rho)}$ stabilize to $M_\infty^{(\rho)}$ for $n \gg 0$ and that $M = \bigoplus_\rho M_\infty^{(\rho)}$ is finitely generated. In the general case, let $X_m = \mathrm{Spec}\, A^G/(\mathfrak{m} \cap A^G)^{m+1}$ and consider the cartesian diagram

$$
\begin{array}{ccccccc}
\mathcal{X} \times_X X_m & \xhookrightarrow{i_m} & \mathcal{X} \times_X X_{m+1} & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{X} \\
\downarrow{\scriptstyle \pi_m} & & \downarrow{\scriptstyle \pi_{m+1}} & & & & \downarrow{\scriptstyle \pi} \\
X_m & \xhookrightarrow{j_m} & X_{m+1} & \hookrightarrow & \cdots & \hookrightarrow & X.
\end{array}
$$

For each m, we may consider the nth nilpotent thickenings $\mathcal{Z}_{m,n}$ of $\mathcal{X}_0 \hookrightarrow \mathcal{X} \times_X X_m$ which are closed substacks $\mathcal{X}_n$. Since X_m is the spectrum of artinian ring, the restrictions $F_n|_{\mathcal{Z}_{m,n}}$ extend to a coherent sheaf $H_m = \widetilde{N_m}$ on $\mathcal{X} \times_X X_m$. Moreover, there is a canonical isomorphism between H_m and the restriction of H_{m+1} to $\mathcal{X} \times_X X_m$. By [Lemma 6.3.21(4)](#), the adjunction morphism $j_m^* \pi_{m+1,*} \xrightarrow{\sim} \pi_{m,*} i_m^*$ is an isomorphism on quasi-coherent sheaves. This implies that $N_{m+1}^{(\rho)} = \Gamma(\mathcal{X} \times_X X_{m+1}, H_{m+1} \otimes W_\rho^\vee)$ restricts to $N_m^{(\rho)}$ and that $M^{(\rho)} = \varprojlim N_m^{(\rho)}$ is a finitely generated A^G-module. By Nakayama's lemma, the map (6.4.22) is surjective on each ρ-isotypical component. Thus (6.4.22) is surjective and M is finitely generated.

For an alternative (but similar) argument for essential surjectivity, we first choose a surjection $E \twoheadrightarrow F_0$ from a vector bundle E on $\mathcal{X}$. For this we can either apply the resolution property of $\mathcal{X}$ ([Proposition 6.1.19](#)) or take $E = F_0 \otimes_{\mathcal{O}_{\mathcal{X}_0}} \mathcal{O}_\mathcal{X}$ as above. Since each $F_{n+1} \to F_n$ is surjective and $\mathrm{Hom}_{\mathcal{O}_\mathcal{X}}(E, -) = \Gamma(\mathcal{X}, E^\vee \otimes_{\mathcal{O}_\mathcal{X}} -)$ is exact, we can lift $E \twoheadrightarrow F_0$ to compatible maps $E \to F_n$, each which is surjective (Nakayama's lemma). The sequence $(\ker(E_n \to F_n))$ not necessarily an adic sytem of coherent sheaves on $\mathcal{X}_n$ as the restriction $\ker(E_{n+1} \to F_{n+1})$ to $\mathcal{X}_n$ may not be

$\ker(E_n \to F_n)$. But we can modify it as follows: For each $l \geq m \geq n$, the images of $\ker(E_l \to F_l)$ in E_m stabilize to K'_m for $l \gg m$ and $K'_m/\mathfrak{m}^{n+1}K'_m$ stabilize to K_n for $m \gg n$ (see also [SP, Tag 087X]). Then $(K_n) \in \varprojlim \mathrm{Coh}(\mathcal{X}_n)$ is an adic sequence. Repeating the construction, we can find a vector bundle E' on $\mathcal{X}$ and compatible surjections $E' \to K_n$. By full faithfulness, there is a morphism $E' \to E$ extending the maps $E'_n \to E_n$. Then $\mathrm{coker}(E' \to E)$ is a coherent $\mathcal{O}_{\mathcal{X}}$ extending (F_n).

See also [AHR20, Thm. 1.3] and [AHR19, Thm. 1.6]. $\qquad\square$

Exercise 6.4.23. If S is a noetherian affine scheme, show that $[\mathbb{A}^1/\mathbb{G}_m]_S$ is coherently complete along $BG_{m,S}$.

6.5 Local structure of algebraic stacks

We establish a local structure theorem for algebraic stack around points with linearly reductive stabilizer. The main theorem (Theorem 6.5.1) implies that quotient stacks of the form $[\mathrm{Spec}\, A/G]$, where G is linearly reductive, are the building blocks of algebraic stacks near points with linearly reductive stabilizers in a similar way to how affine schemes are the building blocks of schemes and algebraic spaces. When $\mathcal{X}$ is Deligne–Mumford, we've already seen an analogous Local Structure Theorem for Deligne–Mumford Stacks (4.2.12). The local structure theorem will be applied to construct good moduli spaces in a similar way to how the result for Deligne–Mumford stacks was used to prove the Keel–Mori Theorem (4.3.12) on the existence of coarse moduli spaces.

Theorem 6.5.1 (Local Structure Theorem for Algebraic Stacks). *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. For every point $x \in \mathcal{X}(\Bbbk)$ with linearly reductive stabilizer G_x, there exists an affine étale morphism*

$$f \colon ([\mathrm{Spec}\, A/G_x], w) \to (\mathcal{X}, x)$$

which induces an isomorphism of stabilizer groups at w.

Remark 6.5.2. In the case that $x \in \mathcal{X}$ is a smooth point, then one can say more: there is also an étale morphism

$$([\mathrm{Spec}\, A/G_x], w) \to ([T_{\mathcal{X},x}/G_x], 0)$$

where $T_{\mathcal{X},x}$ is the Zariski tangent space equipped as a G_x-representation. This addendum follows from the proof but also follows from applying Luna's Étale Slice Theorem (6.5.4) to $[\mathrm{Spec}\, A/G_x]$. The upshot is that we can reduce étale local properties of $\mathcal{X}$ to G_x-equivariant properties of $T_{\mathcal{X},x}$; for moduli problems, this translates into studying the first-order deformation space as a representation under the automorphism group.

By combining this theorem with Luna's Fundamental Lemma (6.3.27), we obtain the following result.

Corollary 6.5.3 (Local Structure for Good Moduli Spaces). *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose that there exists a good moduli space $\pi \colon \mathcal{X} \to X$. Then for every closed point $x \in \mathcal{X}$,*

there exists an étale neighborhood $W \to X$ of $\pi(x)$ and a cartesian diagram

$$
\begin{array}{ccc}
[\operatorname{Spec} A/G_x] & \longrightarrow & \mathcal{X} \\
\downarrow & \square & \downarrow{\scriptstyle \pi} \\
W = \operatorname{Spec} A^{G_x} & \longrightarrow & X.
\end{array}
$$

$\square$

Section outline: We first discuss Luna's Étale Slice Theorem (6.5.4), a beautiful argument providing an explicit construction of an étale neighborhood in the case that $\mathcal{X}$ is already known to have the form $[\operatorname{Spec} B/G]$ with G reductive. The proof of the Local Structure Theorem (6.5.1) is far less explicit requiring: (1) deformation theory, (2) coherent completeness, (3) Coherent Tannaka Duality and (4) Artin Approximation or Equivariant Artin Algebraization.

Letting $\mathcal{T} = [T_{\mathcal{X},x}/G_x]$, deformation theory produces an embedding $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$ of the nth nilpotent thickenings of x and 0. The key step in the proof is to show that the system of closed morphisms $\{\mathcal{X}_n \to \mathcal{X}\}$ algebraizes. The first step is effectivization: the fiber product $\widehat{\mathcal{T}} := \mathcal{T} \times_T \operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)}$, where $\pi \colon \mathcal{T} \to T := T_{\mathcal{X},x} /\!\!/ G_x$, is coherently complete (Theorem 6.4.12). We can thus construct a closed substack $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{T}}$ extending $\mathcal{X}_n \hookrightarrow \mathcal{T}$ and then apply Coherent Tannaka Duality (6.4.9) to construct a morphism $\widehat{\mathcal{X}} \to \mathcal{X}$ extending $\mathcal{X}_n \to \mathcal{X}$.

If $x \in \mathcal{X}$ is smooth, Artin Approximation over the GIT quotient $T_{\mathcal{X},x} /\!\!/ G_x$ produces an étale neighborhood $U \to T_{\mathcal{X},x} /\!\!/ G_x$ such that $\pi^{-1}(U) \to \mathcal{X}$ algebraizes $\widehat{\mathcal{T}} \to \mathcal{X}$. In the general case, Artin Approximation cannot handle this final step and we need to establish an equivariant version of Artin Algebraization (Theorem 6.5.17).

6.5.1   Luna's Étale Slice Theorem

The local structure theorem was inspired by Luna's étale slice theorem in equivariant geometry.

Theorem 6.5.4 (Luna's Étale Slice Theorem)**.** *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ and let X be an affine scheme of finite type over $\Bbbk$ with an action of G. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, then there exists a G_x-invariant, locally closed, and affine subscheme $W \subset X$ such that the induced map*

$$
[W/G_x] \to [X/G] \tag{6.5.5}
$$

is affine étale. If in addition the orbit $Gx \subset X$ is closed, then there is a cartesian diagram

$$
\begin{array}{ccc}
[W/G_x] & \longrightarrow & [X/G] \\
\downarrow & \square & \downarrow \\
W /\!\!/ G_x & \longrightarrow & X /\!\!/ G
\end{array}
$$

where $W /\!\!/ G_x \to X /\!\!/ G$ is also étale.

Moreover, if $x \in X$ is a smooth point and if we denote by $N_x = T_{X,x}/T_{Gx,x}$ the normal space to the orbit, then it can be arranged that there is an G_x-invariant étale morphism $W \to N_x$ which is the pullback of an étale map $W /\!\!/ G_x \to N_x /\!\!/ G_x$ of GIT quotients.

Remark 6.5.6. One can also formulate the statement G-equivariantly: G acts naturally on the quotient $G \times^{G_x} W := (G \times W)/G_x$ and there is an identification $[W/G_x] \cong [(G \times^{G_x} W)/G]$ and likewise $W /\!\!/ G_x \cong (G \times^{G_x} W) /\!\!/ G$ (see Exercise 3.4.16). The morphism (6.5.5) corresponds to an étale G-equivariant morphism $G \times^{G_x} W \to X$.

If the orbit Gx is closed, then Matsushima's Theorem (6.3.20) implies that the stabilizer G_x is linearly reductive.

The proof will rely on the existence of a G_x-invariant morphism $X \to T_{X,x}$, which we refer to as the *Luna map*. The use of $T_{x,x}$ here is an abuse of notation—$T_{X,x}$ is a vector space over $\Bbbk$ and we view it as a scheme via $\mathrm{Spec}(\mathrm{Sym}^* T_{X,x}^\vee)$.

Lemma 6.5.7 (Luna map). *Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ and let X be an affine scheme of finite type over $\Bbbk$ with an action of G. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, there exists a G_x-equivariant morphism*

$$f \colon X \to T_{X,x} \tag{6.5.8}$$

sending x to the origin. If X is smooth at x, then f is étale at x.

Proof. Letting $X = \mathrm{Spec}\, A$ and $\mathfrak{m} \subset A$ be the maximal ideal of x, then $\mathfrak{m}$ and $\mathfrak{m}/\mathfrak{m}^2$ are G_x-representations and we see that G_x acts naturally on the tangent space $T_{X,x} := \mathrm{Spec}(\mathrm{Sym}^* \mathfrak{m}/\mathfrak{m}^2)$. Since G_x is linearly reductive, the surjection $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ of G_x-representations has a section $\mathfrak{m}/\mathfrak{m}^2 \hookrightarrow \mathfrak{m}$. This induces a G_x-equivariant ring map $\mathrm{Sym}^* \mathfrak{m}/\mathfrak{m}^2 \to A$ and thus a G_x-equivariant morphism $f \colon \mathrm{Spec}\, A \to T_{X,x}$ sending x to the origin. If $x \in X$ is smooth, then since f induces an isomorphism of tangent spaces at x, we conclude that f is étale at x (Theorem A.3.2). $\square$

Proof of Theorem 6.5.4. Since X is affine and of finite type, we can choose a finite dimensional G-representation V and a G-equivariant closed immersion $X \hookrightarrow \mathbb{A}(V)$ (Proposition B.2.43). If $W \subset \mathbb{A}(V)$ is an affine G_x-invariant locally closed subscheme such that $[W/G_x] \to [\mathbb{A}(V)/G]$ is étale, then the same is true for $W' := W \cap X \subset X$ and $[W'/G_x] \to [X/G]$. We can therefore immediately reduce to the case that $x \in X$ is smooth. In this case, there is a Luna map $f \colon X \to T_{X,x}$ (see (6.5.7)) which is G_x-invariant, étale at x, and with $f(x) = 0$. The subspace $T_{Gx,x} \subset T_{X,x}$ is G_x-invariant and again since G_x is linearly reductive, the surjection $T_{X,x} \to N_x = T_{X,x}/T_{Gx,x}$ has a section $N_x \hookrightarrow T_{X,x}$. We define W as the preimage of N_x under f:

$$
\begin{array}{ccc}
W & \longrightarrow & N_x \\
\cap\downarrow & \square & \downarrow \\
X & \xrightarrow{\ f\ } & T_{X,x}.
\end{array}
$$

Since the maps $f \colon [W/G_x] \to [X/G]$ and $g \colon [W/G_x] \to [N_x/G_x]$ induce an isomorphism of tangent spaces and stabilizer groups at w, they are both étale at $x \in W$ (or equivalently the G-equivariant maps $G \times^{G_x} W \to X$ and $G \times^{G_x} W \to G \times^{G_x} N_x$ are étale at (id, x)). We have a commutative diagram

$$
\begin{array}{ccccc}
[N_x/G_x] & \xleftarrow{\ g\ } & [W/G_x] & \xrightarrow{\ f\ } & [X/G] \\
\downarrow & & \downarrow & & \downarrow \\
N_x /\!\!/ G_x & \longleftarrow & W /\!\!/ G_x & \longrightarrow & X /\!\!/ G_x
\end{array}
$$

where both f and g are étale at x, preserve stabilizer groups at x and map x to closed points. We can therefore apply Luna's Fundamental Lemma (6.3.27) to replace W with a G_x-equivariant, open, and affine neighborhood of x so that the above squares are cartesian. $\qquad \square$

When $\mathcal{X}$ is already known to be a quotient stack of a normal quasi-projective scheme, the Local Structure Theorem follows from a direct argument. This case is sufficient to handle many moduli problems, e.g. $\underline{\mathrm{Bun}}_{r,d}^{\mathrm{ss}}(C)$ in characteristic 0.

Exercise 6.5.9. If G is a connected affine algebraic group over an algebraically closed field $\Bbbk$ acting on a normal finite type $\Bbbk$-scheme X, and $x \in X(\Bbbk)$ has linearly reductive stabilizer, show that there is a G_x-invariant, locally closed, and affine subscheme $W \hookrightarrow X$ such that $[W/G_x] \to [X/G]$ is étale.

Hint: Sumihiro's Theorem on Linearizations (B.2.51) to reduce to the case that $X = \mathbb{P}(V)$. Choose a homogenous polynomial f not vanishing at x such that $\mathbb{P}(V)_f$ is G_x-invariant and then argue as in the proof of Luna's Étale Slice Theorem by considering the G_x-equivariant étale map $\mathbb{P}(V)_f \to T_x\mathbb{P}(V)$.

6.5.2 Deformation theory

In our proof of the Local Structure Theorem (6.5.1), we will need some deformation theory of algebraic stacks in the form of the following two propositions.

Proposition 6.5.10. *Consider a commutative diagram*

of noetherian algebraic stacks with affine diagonal where $\mathcal{X} \to \mathcal{Y}$ is smooth and affine and $\mathcal{W}' \hookrightarrow \mathcal{W}'$ is a closed immersion defined by a square-zero sheaf of ideals J. If $\mathcal{W}$ is cohomologically affine, there exists a lift in the above diagram.

Proof. When $\mathcal{W}$ is affine, the statement follows from the Infinitesimal Lifting Criterion (A.3.1). To reduce to this case, let $U' \to \mathcal{W}'$ be a smooth presentation with U' an affine scheme and set $U = U' \times_{\mathcal{W}'} \mathcal{W}$. Since $\mathcal{W}$ has affine diagonal, each n-fold fiber product $(U/\mathcal{W})^n := U \times_{\mathcal{W}} \cdots \times_{\mathcal{W}} U$ is affine. We have a commutative diagram

where we have chosen a lift $f'_U : U' \to \mathcal{X}$. Defining the coherent sheaf $F = f^*(\Omega_{\mathcal{X}/\mathcal{Y}}^\vee) \otimes J$ on $\mathcal{W}$, we know by Exercise 6.1.9 that the set of lifts $U' \to \mathcal{X}$ is a torsor under $\Gamma(U, q_1^* F)$ so that any other lift differs from f'_U by an element of $\Gamma(U, q_1^* F)$. Because $\mathcal{X} \to \mathcal{Y}$ is representable, to check that f'_U descends to a morphism $f' : \mathcal{W}' \to \mathcal{X}$, we need to arrange that $f'_U \circ p_1 = f'_U \circ p_2$. Let $q_n : (U/\mathcal{W})^n \to \mathcal{W}$. The difference $f'_U \circ p_1 - f'_U \circ p_2$ can be viewed an element of $\Gamma((U/\mathcal{W})^2, q_2^* F)$.

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Since $q_1 \colon U \to \mathcal{W}$ is a surjective, smooth, and affine morphism, there is an exact sequence of quasi-coherent sheaves

$$0 \to F \to q_{1,*}q_1^*F \to q_{2,*}q_2^*F \to q_{3,*}q_3^*F \to \cdots;$$

see Exercise B.1.3. Since $\mathcal{W}$ is cohomologically affine, taking global sections yields an exact sequence

$$\Gamma(U, q_1^*F) \xrightarrow{\; d_0 \;} \Gamma((U/\mathcal{W})^2, q_2^*F) \xrightarrow{\; d_1 \;} \Gamma((U/\mathcal{W})^3, q_3^*F)$$

$$s \longmapsto p_1^*s - p_2^*s$$

$$s \longmapsto p_{12}^*s - p_{13}^*s + p_{23}^*s.$$

One checks that $d_1(f_U' \circ p_1 - f_U' \circ p_2) = 0$ so there exists an element $s \in \Gamma(U, q_1^*F)$ with $d_0(s) = f_U' \circ p_1 - f_U' \circ p_2$. After modifying the lift f_U' by s, we see that $f_U' \circ p_1 - f_U' \circ p_2 = 0$ so that f_U' descends to $f' \colon \mathcal{W}' \to \mathcal{X}$. $\qquad \square$

Remark 6.5.11. Alternatively, one can show that the obstruction to this deformation problem lies in $\operatorname{Ext}^1_{\mathcal{O}_{\mathcal{W}}}(f^*\Omega_{\mathcal{X}/\mathcal{Y}}, J) = \operatorname{H}^1(\mathcal{W}, f^*(\Omega^\vee_{\mathcal{X}/\mathcal{Y}}) \otimes J)$, which vanishes since $\mathcal{W}$ is cohomologically affine. The above result holds more generally [Ols06, Thm. 1.5].

Proposition 6.5.12. *Let $\mathcal{W} \hookrightarrow \mathcal{W}'$ be a closed immersion of algebraic stacks of finite type over $\Bbbk$ with affine diagonal defined by a square-zero sheaf of ideals J. Let G be an affine algebraic group over $\Bbbk$. If $\mathcal{W}$ is cohomologically affine, then every principal G-bundle $\mathcal{P} \to \mathcal{W}$ extends to a principal G-bundle $\mathcal{P}' \to \mathcal{W}'$.*

Proof. Our proof will use smooth descent and the deformation theory of principal G-bundles over schemes (Exercise C.2.10). Let $U' \to \mathcal{W}'$ be a smooth presentation from an affine scheme and let $U := \mathcal{W} \times_{\mathcal{W}'} U'$. Since $\mathcal{W}$ has affine diagonal, each n-fold fiber products $(U/\mathcal{W})^n = U \times_{\mathcal{W}} \cdots \times_{\mathcal{W}} U$ is affine and we denote the projection by $q_n \colon (U/\mathcal{W})^n \to \mathcal{W}$. By descent theory, the principal G-bundle $\mathcal{P} \to \mathcal{W}$ corresponds to a principal G-bundle $P \to U$ together with an isomorphism $\alpha \colon p_1^*P \xrightarrow{\sim} p_2^*P$ on $(U/\mathcal{W})^2$ satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ on $(U/\mathcal{W})^3$. Letting $F = \mathfrak{g} \otimes J$ be the coherent sheaf on $\mathcal{W}$ where $\mathfrak{g}$ denotes the Lie algebra of G (viewed as an $\mathcal{O}_{\mathcal{W}}$-module by pulling back along $\mathcal{W} \to \operatorname{Spec} \Bbbk$), we know by Exercise C.2.10 that the deformation theory of $q_n^*\mathcal{P} \to (U/\mathcal{W})^n$ with respect to the closed immersion $(U/\mathcal{W})^n \hookrightarrow (U'/\mathcal{W}')^n$ is controlled by q_n^*F.

Since U is affine, we can choose a deformation $P' \to U'$ of $P \to U$. We can also choose an isomorphism $\alpha' \colon p_1^*P' \xrightarrow{\sim} p_2^*P'$ on $(U'/\mathcal{W}')^2$ lifting α where any other choice of an isomorphism differs by an element of $\Gamma((U/\mathcal{W})^2, q_2^*F)$. The isomorphism $(p_{13}^*\alpha')^{-1} \circ p_{23}^*\alpha' \circ p_{12}^*\alpha'$ restricts to the identity on $(U/\mathcal{W})^3$ and thus corresponds to an element $\Psi \in \Gamma((U/\mathcal{W})^3, q_3^*F)$. If $\Psi = 0$, then descent theory implies that $P' \to U'$ descends to the desired principal G-bundle $\mathcal{P}' \to \mathcal{W}'$. Since $0 \to F \to q_{1,*}q_1^*F \to q_{2,*}q_2^*F \to \cdots$ is an exact sequence and $\mathcal{W}$ is cohomologically affine, taking global sections gives an exact sequence

$$\Gamma((U/\mathcal{W})^2, q_2^*F) \xrightarrow{\; d_2 \;} \Gamma((U/\mathcal{W})^3, q_3^*F) \xrightarrow{\; d_3 \;} \Gamma((U/\mathcal{W})^4, q_4^*F)$$

$$s \longmapsto p_{12}^*s - p_{13}^*s + p_{23}^*s$$

$$s \longmapsto p_{123}^*s - p_{134}^*s + p_{124}^*s - p_{234}^*s.$$

$$\tag{6.5.13}$$

While Ψ may be nonzero, one can check that $d_3(\Psi) = 0$ and thus there exists an element $s \in \Gamma((U/\mathcal{W})^2, q_2^* F)$ such that $d_2(s) = \Psi$. Thus modifying the isomorphism α' by s, we see that we can arrange the cocycle condition to hold. $\qquad\square$

Remark 6.5.14. The deformation question is equivalent to deforming the morphism $f \colon \mathcal{W} \to BG$ classified by $P \to \mathcal{W}$ to a morphism $\mathcal{W}' \to BG$, which is analogous to Proposition 6.5.10 except that $\mathcal{X} = BG \to \mathcal{Y} = \operatorname{Spec}\Bbbk$ is not affine. The obstruction to deforming a principal G-bundle lies in the group $\mathrm{H}^2(\mathcal{W}, \mathfrak{g} \otimes J)$. When $\mathcal{W} \to BG$ is representable, one can see this as a consequence of [Ols06, Thm. 1.5] (see Remarks C.5.14 and C.7.6): the obstruction lies in $\operatorname{Ext}^1_{\mathcal{O}_{\mathcal{W}}}(Lf^* L_{BG/\Bbbk}, J)$. Under the composition $\operatorname{Spec}\Bbbk \xrightarrow{p} BG \to \operatorname{Spec}\Bbbk$, we have an exact triangle $p^* L_{BG/\Bbbk} \to L_{\Bbbk/\Bbbk} \to L_{\Bbbk/BG}$. Since $L_{\Bbbk/\Bbbk} = 0$, we obtain that $p^* L_{BG/\Bbbk} = L_{\Bbbk/BG}[-1] \cong \mathfrak{g}^\vee[-1]$ and $L_{BG/\Bbbk} \cong \mathfrak{g}^\vee[-1]$, where the Lie algebra $\mathfrak{g}$ is equipped with the adjoint representation. Thus $\operatorname{Ext}^1_{\mathcal{O}_{\mathcal{W}}}(Lf^* L_{BG/\Bbbk}, J) = \mathrm{H}^1(\mathcal{W}, f^* \mathfrak{g}[1] \otimes J) = \mathrm{H}^2(\mathcal{W}, \mathfrak{g} \otimes J)$. Since $\mathcal{W}$ is cohomologically affine with affine diagonal, this cohomology group is 0 and the obstruction vanishes.

Here's a third approach in the case that $\mathcal{W} = [\operatorname{Spec} A/G]$ where G is linearly reductive and A^G is an artinian $\Bbbk$-algebra. Since $\mathcal{W}$ is a global quotient stack, there exists a vector bundle E on $\mathcal{W}$ such that the stabilizer groups act faithfully on the fibers (Exercise 6.1.16). Generalizing the deformation theory of vector bundles on schemes (Proposition C.2.17), the obstruction to deforming E to a vector bundle E' lies in $\mathrm{H}^2(\mathcal{W}, \mathscr{E}nd_{\mathcal{O}_{\mathcal{W}}}(E) \otimes J)$ which vanishes as $\mathcal{W}$ is cohomologically affine. Since the stabilizer groups also act faithfully on the fibers of E', we have that $\mathcal{W}' \cong [V'/\operatorname{GL}_n]$ where V' is an algebraic space. Then $\mathcal{W} \cong [V/\operatorname{GL}_n]$ with $V_{\mathrm{red}} = V'_{\mathrm{red}}$. Since $\mathcal{W}$ is cohomologically affine and $V \to \mathcal{W}$ is affine, V is cohomologically affine and thus affine by Serre's Criterion for Affineness (4.4.16). It follows that V' is also affine (Proposition 4.4.19). Since $\Gamma(\mathcal{W}', \mathcal{O}_{\mathcal{W}'})$ is an artinian $\Bbbk$-algebra and has no non-trivial affine étale covers, Luna's Étale Slice Theorem (6.5.4) implies that we can arrange that $\mathcal{W}' \cong [\operatorname{Spec} A'/G]$.

We will also need the following criteria for morphisms to be closed immersions or isomorphisms.

Lemma 6.5.15. *Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of algebraic stacks of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that $|\mathcal{X}| = \{x\}$ and $|\mathcal{Y}| = \{y\}$ consist of a single point and that f induces an isomorphism $\mathcal{X}_0 := BG_x$ with $\mathcal{Y}_0 := BG_y$. Let $\mathfrak{m}_x \subset \mathcal{O}_{\mathcal{X}}$ and $\mathfrak{m}_y \subset \mathcal{O}_{\mathcal{Y}}$ be the ideal sheaves defining $\mathcal{X}_0$ and $\mathcal{Y}_0$, and let $f_1 \colon \mathcal{X}_1 \to \mathcal{Y}_1$ be the induced morphism between the first nilpotent thickenings of $\mathcal{X}_0$ and $\mathcal{Y}_0$.*

(1) If f_1 is a closed immersion, then so is f.

(2) If f_1 is a closed immersion and there is an isomorphism $\bigoplus_{n \geq 0} \mathfrak{m}_y^n / \mathfrak{m}_y^{n+1} \cong \bigoplus_{n \geq 0} \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1}$ of graded $\mathcal{O}_{\mathcal{X}_0}$-modules, then f is an isomorphism.

Proof. Choose a smooth presentation $V = \operatorname{Spec} B \to \mathcal{Y}$ from an affine scheme such that $V \times_{\mathcal{Y}} \mathcal{Y}_0 \cong \operatorname{Spec}\Bbbk$ (Theorem 3.6.1). Then B is a local artinian $\Bbbk$-algebra as $\mathcal{Y}$ consists of only one point. The base change $U = V \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space and since $U_{\mathrm{red}} = V_{\mathrm{red}}$ is a point, it follows from Proposition 4.4.19 that $U = \operatorname{Spec} A$ with A a local artinian $\Bbbk$-algebra. We can therefore assume that $f \colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of local artinian schemes.

For (1), we need to show that if $B/\mathfrak{m}_B^2 \to A/\mathfrak{m}_A^2$ is surjective, so is $B \to A$. We first claim that the inclusion $\mathfrak{m}_B A \hookrightarrow \mathfrak{m}_A$ is surjective. By Nakayama's Lemma, it suffices to show that $\mathfrak{m}_B A/\mathfrak{m}_A \mathfrak{m}_B A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective, but this follows

from the hypothesis that the composition $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B A/\mathfrak{m}_A \mathfrak{m}_B A \to \mathfrak{m}_A/\mathfrak{m}_A^2$ is surjective. Since $B/\mathfrak{m}_B \to A/\mathfrak{m}_B A = A/\mathfrak{m}_A$ is surjective, another application of Nakayama's Lemma shows that $B \to A$ is surjective. See also [Har77, Lem. II.7.4] for a related criterion.

For (2), since $\dim_{\Bbbk} \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1} = \dim_{\Bbbk} \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$, the surjections $\mathfrak{m}_B^n/\mathfrak{m}_B^{n+1} \to \mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}$ are isomorphisms and it follows that f is an isomorphism. $\qquad\square$

6.5.3 Proof of the Local Structure Theorem—smooth case

Proof of Theorem 6.5.1—smooth case. Since the $\Bbbk$-point $x \in \mathcal{X}$ is locally closed (Proposition 3.5.19), by replacing $\mathcal{X}$ by an open substack we may assume that $x \in \mathcal{X}$ is a closed point. Let $\mathcal{I}$ be the coherent sheaf of ideals defining $\mathcal{X}_0 := BG_x \hookrightarrow \mathcal{X}$ and set $\mathcal{X}_n$ to be the nth nilpotent thickening defined by $\mathcal{I}^{n+1}$. The Zariski tangent space $T_{\mathcal{X},x}$ can be identified with the normal space $(\mathcal{I}/\mathcal{I}^2)^\vee$ to the orbit, viewed as a G_x-representation. (Note that when $\mathcal{X} = [X/G]$ with G a smooth affine algebraic group, then $T_{\mathcal{X},x}$ is identified with the normal space to the orbit $T_{X,\widetilde{x}}/T_{Gx,\widetilde{x}}$ for a point $\widetilde{x} \in X(\Bbbk)$ over x.)

Define the quotient stack $\mathcal{T} = [T_{\mathcal{X},x}/G_x]$ and let $\mathcal{T}_0 = BG_x$ be the closed substack supported at the origin and $\mathcal{T}_n$ its nth nilpotent thickenings. We claim that there are compatible isomorphisms $\mathcal{X}_n \cong \mathcal{T}_n$. Since G_x is linearly reductive, $\mathcal{X}_0 = BG_x$ is cohomologically affine. By the deformation theory of principal G_x-bundles (Proposition 6.5.12), we can inductively extending the principal G_x-bundle $\operatorname{Spec}\Bbbk \to \mathcal{X}_0$ to principal G_x-bundles $\operatorname{Spec} A_n \to \mathcal{X}_n$. This yields isomorphisms $\mathcal{X}_n \cong [\operatorname{Spec} A_n/G_x]$ and affine morphisms $\mathcal{X}_n \to BG_x$. We have a closed immersion $\mathcal{X}_0 \hookrightarrow \mathcal{T}$ and we can inductively find lifts

$$
\begin{array}{ccc}
\mathcal{X}_n & \longrightarrow & \mathcal{T} \\
\downarrow & \nearrow & \downarrow \\
\mathcal{X}_{n+1} & \longrightarrow & BG_x
\end{array}
$$

since $\mathcal{T} \to BG_x$ is smooth and affine (Proposition 6.5.10). The induced morphism $\mathcal{X}_1 \to \mathcal{T}_1$ is an isomorphism since it is a morphism between deformations $BG_x \hookrightarrow \mathcal{X}_1$ and $BG_x \hookrightarrow \mathcal{T}_1$ of the coherent sheaf $\mathcal{I}/\mathcal{I}^2$ and any such morphism is an isomorphism (by reducing to Lemma C.1.8 by smooth descent). (In fact, both $\mathcal{X}_1$ and $\mathcal{T}_1$ are trivial deformations as they admit retractions to BG_x.) Lemma 6.5.15(2) now implies that the maps $\mathcal{X}_n \to \mathcal{T}_n$ are isomorphisms.

Let $\pi\colon \mathcal{T} \to T = T_{\mathcal{X},x}/\!\!/ G_x$ be the morphism to the GIT quotient. The fiber product $\widehat{\mathcal{T}} := \operatorname{Spec}\widehat{\mathcal{O}}_{T,\pi(0)} \times_T \mathcal{T}$ is a quotient stack of the form $[\operatorname{Spec} B/G]$ where B is of finite type over the noetherian complete local $\Bbbk$-algebra $B^G = \widehat{\mathcal{O}}_{T,\pi(0)}$. Therefore $\widehat{\mathcal{T}}$ is coherently complete along $\mathcal{T}_0$ (Theorem 6.4.12) and $\operatorname{Mor}(\mathcal{T},\mathcal{X}) \xrightarrow{\sim} \varprojlim \operatorname{Mor}(\mathcal{T}_n,\mathcal{X})$ is an equivalence by Coherent Tannaka Duality (6.4.9). It follows that the morphisms $\mathcal{X}_n \cong \mathcal{T}_n \hookrightarrow \mathcal{X}$ extend to a morphism $\widehat{\mathcal{T}} \to \mathcal{X}$ filling in the diagram

$$
\begin{array}{ccccc}
\mathcal{X}_n \cong \mathcal{T}_n & \longrightarrow & \widehat{\mathcal{T}} & \dashrightarrow & \mathcal{T} \longrightarrow \mathcal{X} \\
& & \downarrow & \square & \downarrow \\
& & \operatorname{Spec}\widehat{\mathcal{O}}_{T,\pi(0)} & \longrightarrow & T.
\end{array}
$$

The functor parameterizing isomorphism classes of morphisms

$$F \colon \operatorname{Sch}/T \to \operatorname{Sets}, \qquad (T' \to T) \mapsto \{T' \times_T \mathcal{T} \to \mathcal{X}\}/\sim$$

is limit preserving as $\mathcal{X}$ is of finite type over $\Bbbk$ (see Exercise 3.3.31). The morphism $\widehat{\mathcal{T}} \to \mathcal{X}$ yields an element of F over $\operatorname{Spec} \widehat{\mathcal{O}}_{T,\pi(0)}$. By Artin Approximation (B.5.26), there exists an étale morphism $(U, u) \to (T, 0)$ where U is an affine scheme with a $\Bbbk$-point $u \in U$ and a morphism $(U \times_T \mathcal{T}, (u, 0)) \to (\mathcal{X}, x)$ agreeing with $(\widehat{\mathcal{T}}, 0) \to (\mathcal{X}, x)$ to first order. Since $U \times_T \mathcal{T}$ is smooth at $(u, 0)$ and $\mathcal{X}$ is smooth at x, and since $U \times_T \mathcal{T} \to \mathcal{X}$ induces an isomorphism of tangent spaces and stabilizer groups at $(u, 0)$, the morphism $U \times_T \mathcal{T} \to \mathcal{X}$ is étale at $(u, 0)$. Observe that $U \times_T \mathcal{T}$ is of the form $[\operatorname{Spec} A/G_x]$ for a finitely generated $\Bbbk$-algebra A such that $U = \operatorname{Spec} A^{G_x}$. We can arrange that $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ is étale everywhere after replacing U with an open affine subscheme and $\operatorname{Spec} A$ with its preimage. That $[\operatorname{Spec} A/G_x] \to \mathcal{X}$ can be arranged to be affine follows from Proposition 6.5.16. $\qquad\square$

Proposition 6.5.16. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field with affine diagonal. Let $f \colon [\operatorname{Spec} A/G] \to \mathcal{X}$ be a finite type morphism with G linearly reductive. If $w \in \operatorname{Spec} A$ has closed G-orbit and f induces an isomorphism of stabilizer groups at w, then there exists a G-invariant, affine, and open subscheme $U \subset \operatorname{Spec} A$ containing w such that $f|_{[U/G]}$ is affine.*

Proof. Set $\mathcal{W} = [\operatorname{Spec} A/G]$ with $\pi \colon \mathcal{W} \to \operatorname{Spec} A^G$. Since $f \colon \mathcal{W} \to \mathcal{X}$ is quasi-finite on an open subset $\mathcal{U}$, then $\{\pi(w)\}$ and $\pi(\mathcal{W} \setminus \mathcal{U})$ are disjoint closed subspaces and choosing an affine open $V \subset \operatorname{Spec} A^G \setminus \pi(\mathcal{W} \setminus \mathcal{U})$ containing $\pi(w)$, we may replace $\mathcal{W}$ with $\pi^{-1}(V)$ and we can assume that $f \colon \mathcal{W} \to \mathcal{X}$ is quasi-finite.

Choose a smooth presentation $V = \operatorname{Spec} B \to \mathcal{X}$ and consider the fiber product

$$\begin{array}{ccc}
\mathcal{W}_V & \longrightarrow & V = \operatorname{Spec} B \\
\downarrow & \square & \downarrow \\
\mathcal{W} = [\operatorname{Spec} A/G] & \longrightarrow & \mathcal{X}.
\end{array}$$

Since $\mathcal{X}$ has affine diagonal, $\operatorname{Spec} B \to \mathcal{X}$ is affine and therefore $\mathcal{W}_V$ is cohomologically affine. As $\mathcal{W}_V$ has quasi-finite diagonal, Proposition 6.3.32 implies that $\mathcal{W}_V \to V$ is separated, and it follows from descent that $\mathcal{W} \to \mathcal{X}$ is also separated and that the relative inertia $I_{\mathcal{W}/\mathcal{X}} \to \mathcal{W}$ is finite. Since the fiber over $w \in \mathcal{W}$ is trivial, there is an open neighborhood $\mathcal{U}$ over which the relative inertia is trivial. As in the first paragraph, we may replace $\mathcal{U}$ with an open substack of the form $[\operatorname{Spec} C/G]$ containing w. Since $f|_{\mathcal{U}} \colon \mathcal{U} \to \mathcal{X}$ is a representable and cohomologically affine morphism, Serre's Criterion for Affineness (6.3.17) implies that $f|_{\mathcal{U}}$ is affine. $\qquad\square$

6.5.4 Equivariant Artin Algebraization

The smoothness hypothesis of $x \in \mathcal{X}$ was used above to establish that $\mathcal{T}_n \cong \mathcal{X}_n$ and that $U \times_T \mathcal{T} \to \mathcal{X}$ is étale. More critically, it implied that $\varprojlim \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$, which is identified with the G_x-invariants of a miniversal deformation space, is the completion of a finitely generated $\Bbbk$-algebra, namely $\widehat{\mathcal{O}}_{T,0}$. If $x \in \mathcal{X}$ is not smooth, it seems difficult to directly establish that $\varprojlim \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ is the completion of a finitely generated $\Bbbk$-algebra. Recall that we encountered a similar issue when discussing Artin Algebraization (C.6.8). When the complete local ring R is known to be the completion of a finitely generated algebra, then Artin Algebraization is an easy

consequence of Artin Approximation (see Remark C.6.10). To circumvent this issue in our general proof of Artin Algebraization, we wrote $R = \widehat{\mathcal{O}}_{V,v}/I$ where V is a finite type $\Bbbk$-scheme and used Artin Approximation to simultaneously approximate both the given object over R and the equations defining I. We follow a similar strategy but proceed G-equivariantly this time.

We will use the following extension of the notion of formal versality introduced in Definition C.3.5: for an algebraic stack $\widehat{\mathcal{T}}$ with a unique closed point t, a morphism $\widehat{\xi} \colon \widehat{\mathcal{T}} \to \mathcal{X}$ of prestacks over Sch is *formally versal at t* if every commutative diagram

$$
\begin{array}{ccc}
\mathcal{Z} & \longrightarrow & \widehat{\mathcal{T}} \\
\big\uparrow & \nearrow & \big\downarrow{\scriptstyle\widehat{\xi}} \\
\mathcal{Z}' & \longrightarrow & \mathcal{X}
\end{array}
$$

has a lift, where $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ is a closed immersion of noetherian algebraic stacks with affine diagonal, $|\mathcal{Z}| = |\mathcal{Z}'|$ consists of a single point and the image of $\mathcal{Z} \to \widehat{\mathcal{T}}$ is t.

Theorem 6.5.17 (Equivariant Artin Algebraization)**.** *Let $\Bbbk$ be an algebraically closed field and R be a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$. Let $\widehat{\mathcal{T}} = [\operatorname{Spec} B/G]$ be an algebraic stack of finite type over $R = B^G$, where G is linearly reductive. Assume that the unique closed point $t \in \widehat{\mathcal{T}}$ has stabilizer equal to G. If $\mathcal{X}$ is a limit preserving prestack over $\mathrm{Sch}/\Bbbk$ and $\eta \colon \widehat{\mathcal{T}} \to \mathcal{X}$ is a morphism of prestacks formally versal at t, then there exists*

(1) an algebraic stack $\mathcal{W} = [\operatorname{Spec} A/G]$ of finite type over $\Bbbk$ and a closed point $w \in \mathcal{W}$;

(2) morphisms $f \colon \mathcal{W} \to \mathcal{X}$ and $\varphi \colon \widehat{\mathcal{T}} \to \mathcal{W}$ such that in the diagram

$$
\begin{array}{ccc}
 & \widehat{\mathcal{T}} & \\
{\scriptstyle\varphi}\big\downarrow & \searrow^{\eta} & \\
\mathcal{W} & \xrightarrow{\ f\ } & \mathcal{X}
\end{array}
\tag{6.5.18}
$$

the induced morphisms $\varphi_n \colon \widehat{\mathcal{T}}_n \to \mathcal{W}_n$ between the nth nilpotent thickenings of t and w are isomorphisms, and there exists compatible 2-isomorphisms $\eta_n \xrightarrow{\sim} f_n \circ \varphi_n$.

Moreover, if $\mathcal{X}$ is an algebraic stack of finite type over $\Bbbk$ with affine diagonal, then it can be arranged that (6.5.18) is commutative and that φ induces an isomorphism $\widehat{\mathcal{T}} \to \widehat{\mathcal{W}} := \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$, where $\pi \colon \mathcal{W} \to W = \operatorname{Spec} A^G$.

Remark 6.5.19. If one takes G to be the trivial group, one recovers the classical version of Artin Algebraization (C.6.8).

As in the proof of Theorem C.6.8, we will apply Artin Approximation to a well-chosen integer N to construct $\mathcal{W}$ such that there are isomorphisms $\mathcal{W}_n \cong \widehat{\mathcal{T}}_n$ for $n \leq N$ and such that the Artin–Rees Lemma (B.5.4) implies that there are also isomorphisms $\mathcal{W}_n \cong \widehat{\mathcal{T}}_n$ for $n > N$. To get control over the constant in the Artin–Rees Lemma, we need to generalize Definition C.6.3: for a noetherian algebraic stack $\mathcal{X}$ with a closed point x defined by a a sheaf of ideals $\mathfrak{m}_x$ and an integer $c \geq 0$, we say that $(\mathsf{AR})_c$ *holds at x* for a map $\varphi \colon E \to F$ of coherent sheaves on $\mathcal{X}$ if

$$
\varphi(E) \cap \mathfrak{m}_x^n F \subset \varphi(\mathfrak{m}_x^{n-c} E), \quad \forall n \geq c.
$$

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When $\mathcal{X}$ is a scheme, $(\mathsf{AR})_c$ holds for all sufficiently large c by the Artin–Rees Lemma, and it even holds replacing $\{x\}$ with a closed subscheme. By smooth descent, $(\mathsf{AR})_c$ also holds for algebraic stacks for $c \gg 0$.

Proof. The morphism $\eta \colon \widehat{\mathcal{T}} \to \mathcal{X}$ and the structure morphism $\widehat{\mathcal{T}} \to BG$ induce a morphism $\widehat{\mathcal{T}} \to \mathcal{X} \times BG$. We let $\widehat{T} = \operatorname{Spec} R$ be the GIT quotient of $\widehat{\mathcal{T}} = [\operatorname{Spec} B/G]$. Since R is the colimit of its finitely generated $\Bbbk$-subalgebras and $\mathcal{X} \times BG$ is limit preserving, limit methods (§B.3) imply that there is a commutative diagram

$$
\begin{array}{ccccc}
\widehat{\mathcal{T}} & \overset{\longrightarrow}{\longrightarrow} \mathcal{S} & \longrightarrow & \mathcal{X} \times BG \\
\downarrow & \square \quad \downarrow & & \downarrow \\
\widehat{T} & \longrightarrow S & \longrightarrow & \operatorname{Spec} \Bbbk
\end{array}
$$

where $S = \operatorname{Spec} R'$ is an affine scheme of finite type over $\Bbbk$, $\mathcal{S}$ is an algebraic stack of finite type over S with affine diagonal such that $\widehat{\mathcal{T}} = \widehat{T} \times_S \mathcal{S}$, and $\widehat{\mathcal{T}} \to \mathcal{X} \times BG$ factors as $\widehat{\mathcal{T}} \to \mathcal{S} \to \mathcal{X} \times BG$. Moreover, we can arrange that $\mathcal{S} \to BG$ is affine. Let $\widetilde{s} \in \mathcal{S}$ and $s \in S$ be the images of t. By possibly adding generators to R' so that $R' \to R \to R/\mathfrak{m}_R^2$ is surjective, we can arrange that $\widehat{\mathcal{O}}_{S,s} \to R$ is surjective by Complete Nakayama's Lemma (B.5.6(2)), or in other words that $\widehat{T} \to \widehat{S} := \operatorname{Spec} \widehat{\mathcal{O}}_{S,s}$ is a closed immersion.

Note that $\widehat{\mathcal{T}}$ is a closed substack of $\mathcal{S} \times_S \widehat{S}$. By choosing a resolution $\mathcal{O}_{\widehat{S}}^{\oplus r} \to \mathcal{O}_{\widehat{S}} \twoheadrightarrow R$ and pulling it back $\mathcal{S} \times_S \widehat{S}$, we obtain a resolution

$$
\ker(\beta) \overset{\alpha}{\hookrightarrow} \mathcal{O}_{\mathcal{S} \times_S \widehat{S}}^{\oplus r} \overset{\beta}{\longrightarrow} \mathcal{O}_{\mathcal{S} \times_S \widehat{S}} \twoheadrightarrow \mathcal{O}_{\widehat{\mathcal{T}}}. \tag{6.5.20}
$$

Consider the functor $F \colon \operatorname{Sch}/S \to \operatorname{Sets}$ assigning $(U \to S)$ to the set of isomorphism classes of complexes

$$
L \overset{\alpha}{\to} \mathcal{O}_{\mathcal{S} \times_S U}^{\oplus r} \overset{\beta}{\to} \mathcal{O}_{\mathcal{S} \times_S U}
$$

of finitely presented, quasi-coherent $\mathcal{O}_{\mathcal{S} \times_S U}$-modules. By standard limit arguments, F is limit preserving. The complex (6.5.20) defines an element $(\alpha, \beta) \in F(\widehat{S})$ such that $\operatorname{coker}(\beta) = \mathcal{O}_{\widehat{T}}$. Let N be an integer such that $(\mathsf{AR})_N$ holds for α and β at $(\widetilde{s}, s)$.

Artin Approximation (B.5.26) gives an étale neighborhood $(S', s') \to (S, s)$ and an element $(\alpha', \beta') \in F(S')$ such that $(\alpha, \beta) = (\alpha', \beta')$ in $F(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$. We let $\mathcal{W} \hookrightarrow \mathcal{S} \times_S S'$ be the closed substack defined by $\operatorname{coker}(\beta')$ and set $w = (\widetilde{s}, s') \in \mathcal{W}$. Letting S_n, S'_n and $\widehat{T}_n$ be the nth nilpotent thickenings of S, S' and $\widehat{T}$ at the images of $t \in \mathcal{T}$, we have that $\widehat{\mathcal{T}} \times_{\widehat{T}} \widehat{T}_N$ and $\mathcal{W} \times_{S'} S'_N$ are equal as closed substacks of $\mathcal{S} \times_S S_N$. This gives (1)-(2) for $n \leq N$. In particular, we have an isomorphism $\varphi_N \colon \widehat{\mathcal{T}}_N \to \mathcal{W}_N$ and we let $\psi_N \colon \mathcal{W}_N \to \widehat{\mathcal{T}}_N$ be its inverse.

Using that $\eta \colon \widehat{\mathcal{T}} \to \mathcal{X}$ is formally versal, we can inductively find compatible lifts for $n \geq N$

$$
\begin{array}{ccc}
\mathcal{W}_n & \overset{\psi_n}{\longrightarrow} & \widehat{\mathcal{T}} \\
\downarrow & \overset{\psi_{n+1}}{\nearrow} & \downarrow \eta \\
\mathcal{W}_{n+1} & \longrightarrow & \mathcal{X}.
\end{array}
$$

On the other hand, applying Lemma C.6.4 (generalized to stacks by smooth descent) on $\mathcal{S} \times_S \widehat{S}$ with $c = N$ to the complex (6.5.20) and the restriction of the complex

defined by (α', β'), we obtain an isomorphism $\mathrm{Gr}_{\mathfrak{m}_t} \mathcal{O}_{\widehat{\mathcal{T}}} \cong \mathrm{Gr}_{\mathfrak{m}_w} \mathcal{O}_{\mathcal{W}}$ of graded $\mathcal{O}_{BG}$-modules. By Lemma 6.5.15, the induced morphisms $\psi_n \colon \mathcal{W}_n \to \widehat{\mathcal{T}}_n$ are isomorphisms for all n. As $\widehat{\mathcal{T}}$ is coherently complete (Theorem 6.4.12), Coherent Tannaka Duality (6.4.9) implies that the inverses $\varphi_n = \psi_n^{-1} \colon \widehat{\mathcal{T}}_n \to \mathcal{W}_n$ effectivize to a morphism $\varphi \colon \widehat{\mathcal{T}} \to \mathcal{W}$. This completes (1)-(2).

For the final statement, when $\mathcal{X}$ is algebraic, we again apply Coherent Tannaka Duality, using the coherent completeness of both $\widehat{\mathcal{T}}$ and $\mathcal{W}$. By applying Corollary 6.4.9 to the inverses $\psi_n = \varphi_n^{-1}$, we can construct an inverse $\psi \colon \widehat{\mathcal{W}} \to \widehat{\mathcal{T}}$ of φ. Thus $\varphi \colon \widehat{\mathcal{T}} \to \widehat{\mathcal{W}}$ is an isomorphism. Using the fully faithfulness of Corollary 6.4.9, there is a 2-isomorphism $\eta \to f \circ \varphi$ extending the given 2-isomorphisms $\eta_n \xrightarrow{\sim} f_n \circ \varphi_n$ and thus $\widehat{\mathcal{T}} \to \widehat{\mathcal{W}}$ is a morphism over $\mathcal{X}$. $\qquad\square$

6.5.5 Proof of the Local Structure Theorem—general case

Proof of Theorem 6.5.1. We may assume that $x \in \mathcal{X}$ is a closed point. Let $\mathcal{T} := [T_{\mathcal{X},x}/G_x]$, let $\pi \colon \mathcal{T} \to T = T_{\mathcal{X},x} /\!\!/ G_x$ be the morphism to the GIT quotient, and let $\widehat{\mathcal{T}} := \mathrm{Spec}\, \widehat{\mathcal{O}}_{T,\pi(0)} \times_T \mathcal{T}$. Let $\mathcal{T}_0 = BG_x$ be the closed substack supported at the origin and $\mathcal{T}_n$ its nth nilpotent thickenings.

We will construct compatible closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$. Since G_x is linearly reductive, $\mathcal{X}_0 = BG_x$ is cohomologically affine. By deforming the principal G_x-bundle $\mathrm{Spec}\, \Bbbk \to \mathcal{X}_0$ using Proposition 6.5.12, we can inductively construct isomorphisms $\mathcal{X}_n \cong [\mathrm{Spec}\, A_n/G_x]$. By the deformation theory of the smooth and affine morphism $\widehat{\mathcal{T}} \to BG_x$ (Proposition 6.5.10), we can inductively find lifts

$$
\begin{array}{ccc}
\mathcal{X}_n & \longrightarrow & \mathcal{T} \\
\downarrow & \nearrow & \downarrow \\
\mathcal{X}_{n+1} & \longrightarrow & BG_x.
\end{array}
$$

As in the smooth case, $\mathcal{X}_1 \to \mathcal{T}_1$ is an isomorphism. By Lemma 6.5.15(1), each morphism $\mathcal{X}_n \to \mathcal{T}_n$ is a closed immersion.

If I_n denotes the ideal sheaf defining $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$, then $\mathcal{O}_{\mathcal{T}_n}/I_n$ is a system of coherent $\mathcal{O}_{\mathcal{T}_n}$-modules. Since $\widehat{\mathcal{T}}$ is coherently complete (Theorem 6.4.12), there exists a coherent sheaf of ideals $I \subset \mathcal{O}_{\widehat{\mathcal{T}}}$ such that the surjection $\mathcal{O}_{\widehat{\mathcal{T}}} \to \mathcal{O}_{\widehat{\mathcal{T}}}/I$ extends the surjections $\mathcal{O}_{\mathcal{T}_n} \to \mathcal{O}_{\mathcal{X}_n}$. The closed immersion $\widehat{\mathcal{X}} \hookrightarrow \widehat{\mathcal{T}}$ defined by I extends the given closed immersions $\mathcal{X}_n \hookrightarrow \mathcal{T}_n$ yielding a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{X}_n & \lhook\joinrel\longrightarrow & \widehat{\mathcal{X}} & \dashrightarrow^{\ \eta} & \mathcal{X} \\
\downarrow & & \downarrow & & \\
\mathcal{T}_n & \lhook\joinrel\longrightarrow & \widehat{\mathcal{T}} & \longrightarrow & \mathcal{T} \\
& & \downarrow & \square & \downarrow \\
& & \mathrm{Spec}\, \widehat{\mathcal{O}}_{T,0} & \longrightarrow & T.
\end{array}
$$

of solid arrows. Since $\widehat{\mathcal{X}}$ is also coherently complete, Coherent Tannaka Duality (6.4.9) gives a morphism $\eta \colon \widehat{\mathcal{X}} \to \mathcal{X}$ extending the above diagram. Since $\widehat{\mathcal{X}}$ has the same nilpotent thickenings of $\widehat{\mathcal{X}}$, the morphism $\eta \colon \widehat{\mathcal{X}} \to \mathcal{X}$ is formally versal at 0.

By Equivariant Artin Algebraization (6.5.17) with $G = G_x$, we obtain a morphism $f \colon \mathcal{W} = [\operatorname{Spec} B / G_x] \to \mathcal{X}$ from an algebraic stack $\mathcal{W}$ of finite type over $\Bbbk$ with a closed point $w \in \mathcal{W}$ and a morphism $\varphi \colon \widehat{\mathcal{X}} \to \mathcal{W}$ over $\mathcal{X}$ inducing an isomorphism $\widehat{\mathcal{X}} \to \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W, \pi(w)}$ where $\pi \colon \mathcal{W} \to \operatorname{Spec} B^{G_x}$. Since $f \colon \mathcal{W} \to \mathcal{X}$ induces isomorphisms $\mathcal{W}_n \to \mathcal{X}_n$, f is étale at w. After replacing $\mathcal{W}$ with an open substack, we can arrange that f is étale everywhere. By Proposition 6.5.16, we can also arrange that f is affine.

See also [AHR20, AHR19, AHLHR22]. $\qquad\square$

6.5.6 The coherent completion at a point

We say that $(\mathcal{X}, x)$ is a *complete local stack* if $\mathcal{X}$ is a noetherian algebraic stack with affine stabilizers and with a unique closed point x such that $\mathcal{X}$ is coherently complete along the residual gerbe $\mathcal{G}_x$. An important example is $([\operatorname{Spec} A/G], x)$ where G is linearly reductive over an algebraically closed field $\Bbbk$, A^G is a complete noetherian local $\Bbbk$-algebra with residue field $\Bbbk$, A is of finite type over R, and the unique closed point x is fixed by G (Theorem 6.4.12). For instance, $([\mathbb{A}^n/\mathbb{G}_m], 0)$ is complete local.

The *coherent completion* of a noetherian algebraic stack $\mathcal{X}$ at a point x is a complete local stack $(\widehat{\mathcal{X}}_x, \widehat{x})$ together with a morphism $\eta \colon (\widehat{\mathcal{X}}_x, \widehat{x}) \to (\mathcal{X}, x)$ inducing isomorphisms of nth infinitesimal neighborhoods of $\widehat{x}$ and x. If $\mathcal{X}$ has affine stabilizers, then the pair $(\widehat{\mathcal{X}}_x, \eta)$ is unique up to unique 2-isomorphism by Coherent Tannaka Duality (6.4.9).

Theorem 6.5.21. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. For every point $x \in \mathcal{X}(k)$ with linearly reductive stabilizer G_x, the coherent completion $\widehat{\mathcal{X}}_x$ exists. Moreover,*

(1) The coherent completion is a quotient stack $\widehat{\mathcal{X}}_x = [\operatorname{Spec} B/G_x]$ such that the invariant ring B^{G_x} is the completion of a finite type $\Bbbk$-algebra and $B^{G_x} \to B$ is of finite type.

(2) Let $f \colon (\mathcal{W}, w) \to (\mathcal{X}, x)$ be an étale morphism where $\mathcal{W} = [\operatorname{Spec} A/G_x]$, the point $w \in |\mathcal{W}|$ is closed, and f induces an isomorphism of stabilizer groups at w. Then $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W, \pi(w)}$, where $\pi \colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ is the morphism to the GIT quotient.

(3) If $\pi \colon \mathcal{X} \to X$ is a good moduli space, then $\widehat{\mathcal{X}}_x = \mathcal{X} \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X, \pi(x)}$.

Proof. The Local Structure Theorem (6.5.1) gives an étale morphism $f \colon (\mathcal{W}, w) \to (\mathcal{X}, x)$, where $\mathcal{W} = [\operatorname{Spec} A/G_x]$ and f induces an isomorphism of stabilizer groups at the closed point w. The main statement and Parts (1) and (2) follow by taking $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W, \pi(w)}$ and $B = A \otimes_{A^{G_x}} \widehat{A^{G_x}}$. Indeed, $\widehat{\mathcal{X}}_x = [\operatorname{Spec} B/G_x]$ is coherently complete by Theorem 6.4.12. Part (3) follows from (2) using Corollary 6.5.3. $\qquad\square$

We have the following stacky generalization of the fact that completions determine the étale local structure of finite type schemes (Corollary B.5.31).

Theorem 6.5.22. *Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are $\Bbbk$-points with linearly reductive stabilizer group schemes G_x and G_y, respectively. Then the following are equivalent:*

(1) There exist compatible isomorphisms $\mathcal{X}_n \to \mathcal{Y}_n$.

(2) There exists an isomorphism $\widehat{\mathcal{X}}_x \to \widehat{\mathcal{Y}}_y$.

(3) There exist an affine scheme $\operatorname{Spec} A$ *with an action of* G_x, *a point* $w \in \operatorname{Spec} A$ *fixed by* G_x, *and a diagram of étale morphisms*

$$
\begin{array}{ccc}
 & [\operatorname{Spec} A/G_x] & \\
 & \swarrow^{f} \qquad \searrow^{g} & \\
\mathcal{X} & & \mathcal{Y}
\end{array}
$$

such that $f(w) = x$ *and* $g(w) = y$, *and both* f *and* g *induce isomorphisms of stabilizer groups at* w.

If, in addition, the points $x \in \mathcal{X}$ *and* $y \in \mathcal{Y}$ *are smooth, then the conditions above are equivalent to the existence of an isomorphism* $G_x \to G_y$ *of group schemes and an isomorphism* $T_{\mathcal{X},x} \to T_{\mathcal{Y},y}$ *of tangent spaces which is equivariant under* $G_x \to G_y$.

Proof. The implications (3) $\implies$ (2) $\implies$ (1) are immediate. We also have (1) $\implies$ (2) by Coherent Tannaka Duality (6.4.9) To show that (2) $\implies$ (3), let $(\mathcal{W} = [\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an étale neighborhood as given by the Local Structure Theorem (6.5.1). Let $\pi \colon \mathcal{W} \to W = \operatorname{Spec} A^{G_x}$ denote the good moduli space. Then $\widehat{\mathcal{X}}_x = \mathcal{W} \times_W \operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)}$. The functor

$$
F \colon \operatorname{Sch}/W \to \operatorname{Sets}, \quad (T \to W) \mapsto \operatorname{Hom}(\mathcal{W} \times_W T, \mathcal{Y})
$$

is locally of finite presentation. Artin Approximation (B.5.26) applied to F and $\alpha \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{W,\pi(w)})$ provides an étale morphism $(W', w') \to (W, w)$ and a morphism $\varphi \colon \mathcal{W}' := \mathcal{W} \times_W W' \to \mathcal{Y}$ such that $\varphi|_{\mathcal{W}'_1} \colon \mathcal{W}'_1 \to \mathcal{Y}_1$ is an isomorphism. Since $\widehat{\mathcal{W}'}_{w'} \cong \widehat{\mathcal{X}}_x \cong \widehat{\mathcal{Y}}_y$, it follows that φ induces an isomorphism $\widehat{\mathcal{W}'} \to \widehat{\mathcal{Y}}$ by Lemma 6.5.15. After replacing W' with an open neighborhood, we thus obtain an étale morphism $(\mathcal{W}', w') \to (\mathcal{Y}, y)$. The final statement is clear from Luna's Etale Slice Theorem (6.5.4). $\qquad\square$

6.5.7 Applications to equivariant geometry

Sumihiro's Theorem on Torus Action (B.2.52) asserts that for a normal scheme of finite type over $\Bbbk$ with the action of a torus T, every $\Bbbk$-point has a T-invariant affine open neighborhood. If X is not normal, there are not necessarily T-invariant affine open neighborhoods, e.g. consider nodal cubic X equipped with a $\mathbb{G}_m$-action near its node $x \in X$. However, there is always a T-equivariant affine *étale* neighborhood.

Theorem 6.5.23. *Let X be an algebraic space locally of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Suppose that X has an action of an affine algebraic group G. If $x \in X(\Bbbk)$ has linearly reductive stabilizer, then there exists a G-equivariant étale neighborhood $(\operatorname{Spec} A, u) \to (X, x)$ inducing an isomorphism of stabilizer groups at u.*

If G is a torus, then every point has a G-invariant étale neighborhood $(\operatorname{Spec} A, u) \to (X, x)$ inducing an isomorphism of stabilizer groups at u.

Proof. By the Local Structure Theorem (6.5.1), there is an étale neighborhood $([\operatorname{Spec} A/G_x], u) \to ([X/G], x)$ such that w is a closed point and f induces an isomorphism of stabilizer groups at w. By Proposition 6.5.16, after replacing $\operatorname{Spec} A$ with a G_x-invariant open affine neighborhood of w, we can arrange that the composition $[\operatorname{Spec} A/G_x] \to [X/G] \to BG$ is affine. Therefore, $W := [\operatorname{Spec} A/G_x] \times_{[X/G]} X$ is an affine scheme and $W \to X$ is a G-equivariant étale neighborhood of x.

When G is a torus, then any subgroup of G and in particular each stabilizer group is linearly reductive. $\qquad\square$

6.6 $\mathbb{G}_m$-actions, one-parameter subgroups, and filtrations

We show that the fixed locus of a linearly reductive group action on a smooth variety is smooth (Theorem 6.6.2) and prove the representability and properties of the attractor locus with respect to a $\mathbb{G}_m$-action (Theorem 6.6.8). After establishing a general version of the Białynicki-Birula Stratification (Theorem 6.6.13), we discuss applications to computing cohomology (§6.6.18). The Cartan Decomposition (Theorem 6.6.30) is used to prove the Destabilization Theorem (Theorem 6.6.35), which will be used to prove the Hilbert–Mumford Criterion in the next section. Finally, we interpret maps from $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ into quotient stacks as the data of a one-parameter subgroup and a point whose limit exists (Proposition 6.6.39) and into a stack of coherent sheaves as a filtration of a vector bundle (Proposition 6.6.43). Many of these results will be useful for the development of both Geometric Invariant Theory in §6.7 and the existence of good moduli spaces in §6.8.

6.6.1 Fixed loci

Definition 6.6.1 (Fixed locus)**.** If X is an algebraic space over a field $\Bbbk$ equipped with an action of an affine algebraic group G, we define the *fixed locus* as the functor

$$X^G := \underline{\mathrm{Mor}}^G(\mathrm{Spec}\,\Bbbk, X) \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$$

assigning a $\Bbbk$-scheme S to the set of G-equivariant maps from S to X, where S is endowed with the trivial G action.

Theorem 6.6.2. *Let X be an algebraic space of finite type over an algebraically closed field $\Bbbk$ with affine diagonal equipped with an action of a linearly reductive algebraic group G. Then*

(1) The fixed locus X^G is represented by a subscheme of X;

(2) If G is a torus, then X^G is a closed subscheme.

(3) If X is smooth, so is X^G.

Proof. If G is connected and $U \to X$ is a G-invariant étale morphism, we claim that

$$
\begin{array}{ccc}
U^G & \lhook\joinrel\longrightarrow & U \\
\downarrow & \square & \downarrow \\
X^G & \lhook\joinrel\longrightarrow & X
\end{array}
\tag{6.6.3}
$$

is cartesian. Indeed, suppose $S \to U$ is a map such that $S \to U \to X$ is G-invariant. Let $U_S \to S$ be the base change of $U \to X$ by $S \to X$. Since $U_S \to S$ is G-invariant, it suffices to show that the section $j \colon S \to U_S$ is G-invariant. As $U \to X$ is étale, $j \colon S \to U_S$ is an open immersion. Because G is connected, for each point $s \in S$, the G-orbit $Gj(s) \subset U_S$ is connected and thus contained in S.

For (1), given a fixed point $x \in X^G(\Bbbk)$, Theorem 6.5.23 produces a G-invariant étale neighborhood $(U, u) \to (X, x)$ with U affine and $u \in U^G(\Bbbk)$. If G is connected, then $U^G \to X^G$ is étale and representable by (6.6.3). Thus it suffices to show that U^G is representable. Since U is affine, we can choose a G-equivariant embedding $U \hookrightarrow \mathbb{A}(V)$ into a finite dimensional G-representation. In this case, $\mathbb{A}(V)^G = \mathbb{A}(V^G)$ and thus $U^G = U \cap \mathbb{A}(V)^G$ is representable. In general, let $G^0 \subset G$ be the connected

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component of the identity, and let $g_1, \ldots, g_n \in G(\Bbbk)$ be representatives of the finitely many cosets $G(\Bbbk)/G^0(\Bbbk)$. Then G/G_0 acts on X^{G_0} and $X^G = \bigcap_i (X^{G_0})^{g_i}$, where $(X^{G_0})^{g_i}$ is identified with the fiber product of the diagonal $X^G \to X^G \times X^G$ and the map $X^G \to X^G \times X^G$ given by $x \mapsto (x, gx)$.

For (2), every subgroup of G is linearly reductive and Theorem 6.5.23 therefore produces a G-invariant étale surjective morphism $U \to X$ from an affine scheme. As G is connected, the argument above shows that $U^G \hookrightarrow U$ is a closed immersion and thus by étale descent so is $X^G \hookrightarrow X$.

For (3), if $x \in X^G(\Bbbk)$, there is a G-invariant étale morphism $(U, u) \to (X, x)$ from an affine scheme and a G-invariant étale morphism $U \to T_{U,u}$ as in the proof of Luna's Étale Slice Theorem (see (6.5.8)). Since $T_{U,u}^G$ is a linear subspace, it is smooth. Since $U^G \to X^G$ and $U^G \to T_{U,u}^G$ are étale at u, the statement follows from étale descent. See also [Ive72, Prop. 1.3] and [Mil17, Thm. 13.1]. $\qquad\square$

6.6.2 Limits under $\mathbb{G}_m$-actions and attractor loci

Definition 6.6.4 (Limits). Given a $\mathbb{G}_m$-action on an algebraic space X over a field $\Bbbk$ and a point $u \in U(\Bbbk)$, we say that the *limit* $\lim_{t \to 0} t \cdot u$ *exists* if there exists an extension of the diagram

$$
\begin{array}{ccc}
\mathbb{G}_m & \xrightarrow{\ t \mapsto t \cdot u\ } & U \\
\cap \downarrow & \nearrow & \\
\mathbb{A}^1 & &
\end{array}
$$

The valuative criteria for separatedness and properness imply that the limit is unique if X is separated and that there is always a unique limit if X is proper.

Definition 6.6.5 (Attractor locus). Let X be a separated algebraic space of finite type over $\Bbbk$ equipped with an action of $\mathbb{G}_m$. Define the *attractor locus* as the functor

$$
X^+ := \underline{\mathrm{Mor}}^{\mathbb{G}_m}(\mathbb{A}^1, X) \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}
$$

assigning a $\Bbbk$-scheme S to the set of $\mathbb{G}_m$-equivariant maps from $S \times \mathbb{A}^1$ to X, where $\mathbb{G}_m$ acts trivially on S and with the usual scaling action on $\mathbb{A}^1$

Evaluation at 0 defines a morphism of functors

$$
\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}.
$$

On $\Bbbk$-points, $X^+(\Bbbk)$ is the set of points $x \in X(\Bbbk)$ such that $\lim_{t \to 0} t \cdot x$ exists, and $\mathrm{ev}_0(x)$ is this limit. Since X is separated, the limit is unique if it exists. If X is proper, the limit always exists and $X^+(\Bbbk) = X(\Bbbk)$. The functorial definition of X^+ endows it with an interesting scheme-structure, e.g. when $\mathbb{G}_m$ acts on $X = \mathbb{P}^1$ via $t \cdot [x : y] = [tx : y]$, then $X^+ = \mathbb{A}^1 \coprod \{\infty\}$.

Exercise 6.6.6 (Affine case). If $X = \mathrm{Spec}\, A$ is affine, then the $\mathbb{G}_m$-action induces a grading $A = \bigoplus_{d \in \mathbb{A}} A_d$. Show that the functors $X^{\mathbb{G}_m}$ and X^+ are representable by the closed subschemes of X defined by the ideals $\sum_{d \neq 0} A_d$ and $\sum_{d < 0} A_d$.

Example 6.6.7 (Centralizers and parabolics). Let G be an affine algebraic group over an algebraically closed field $\Bbbk$. A one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ induces a $\mathbb{G}_m$-action on G via conjugation $t \cdot g := \lambda(t) g \lambda(t)^{-1}$. Under this action, the fixed locus $G^{\mathbb{G}_m} = C_\lambda$ is identified with the centralizer of λ and the attractor locus

$G_\lambda^+ = P_\lambda$ is identified with the subgroup consisting of elements $g \in G$ such that $\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1}$ exists. The unipotent subgroup U_λ is identified with kernel of $\mathrm{ev}_0 \colon P_\lambda \to C_\lambda$. When G is reductive, $P_\lambda \subset G$ is a parabolic subgroup or in other words G/P_λ is projective. See §B.2.6 for more properties of these subgroups.

We say that a map $X \to Y$ is an *affine fibration* (resp. *Zariski-local affine fibration*) if there exists an étale (resp. Zariski) cover $\{Y_i \to Y\}$ such that $X \times_Y Y_i \cong \mathbb{A}_{Y_i}^n$ over Y_i. Since the transition functions are not required to be linear, this notion is more general than a vector bundle.

Theorem 6.6.8. *Let X be a separated algebraic space of finite type over an algebraically closed field $\Bbbk$ equipped with an action of $\mathbb{G}_m$. The functor X^+ is representable by an algebraic space of finite type over $\Bbbk$ and $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is an affine morphism.*

Assume in addition that X is smooth (resp. smooth scheme). Then $X^{\mathbb{G}_m}$ is also smooth and $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is an affine fibration (resp. Zariski-local affine fibration). If $x \in X^{\mathbb{G}_m}$ and $T_{X,x} = T_{>0} \oplus T_0 \oplus T_{<0}$ is the $\mathbb{G}_m$-equivariant decomposition into nonnegative, zero, and positive weights, then $T_{X_i,x} = T_0 \oplus T_{>0}$, $T_{F_i,x} = T_0$, and $X_i \to F_i$ has relative dimension $\dim T_{>0}$.

Proof. If $X = \operatorname{Spec} A$ is affine, then $X^{\mathbb{G}_m}$ and X^+ are closed subschemes of X (Exercise 6.6.6). In the special case that $X = \mathbb{A}(V)$ where V is a finite dimensional G-representation, then $X^{\mathbb{G}_m} = \mathbb{A}(V^G)$ and $X^+ = \mathbb{A}(V_{\geq 0})$ where $V_{\geq 0}$ is the direct sum of the non-negative isotypic components, and moreover $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ is a relative affine space.

We claim that if $U \to X$ is a $\mathbb{G}_m$-invariant étale morphism, then the diagram

$$
\begin{array}{ccccc}
U^+ & \xrightarrow{\;\mathrm{ev}_0\;} & U^{\mathbb{G}_m} & \hookrightarrow & U \\
\downarrow & \square & \downarrow & \square & \downarrow \\
X^+ & \xrightarrow{\;\mathrm{ev}_0\;} & X^{\mathbb{G}_m} & \hookrightarrow & X
\end{array}
\tag{6.6.9}
$$

is cartesian. The right square was verified in the proof of Theorem 6.6.2. For the left square, we need to show that there exists a unique $\mathbb{G}_m$-equivariant morphism filling in a $\mathbb{G}_m$-equivariant diagram

$$
\begin{array}{ccc}
\operatorname{Spec}\Bbbk \times S & \longrightarrow & U \\
\downarrow & \nearrow & \downarrow \\
\mathbb{A}^1 \times S & \longrightarrow & X
\end{array}
\tag{6.6.10}
$$

where S is an affine scheme of finite type over k, and the vertical left arrow is the inclusion of the origin. For each $n \geq 1$, the formal lifting property of étaleness yields a unique $\mathbb{G}_m$-equivariant map $\operatorname{Spec} k[x]/x^n \times S \to U$ such that

$$
\begin{array}{ccc}
\operatorname{Spec}\Bbbk \times S & \longrightarrow & U \\
\downarrow & \nearrow & \downarrow \\
\operatorname{Spec}(\Bbbk[x]/x^n) \times S & \longrightarrow & X
\end{array}
$$

commutes. As $[\mathbb{A}^1/\mathbb{G}_m] \times S$ is coherently complete along $B\mathbb{G}_{m,S}$ (Exercise 6.4.23), Coherent Tannaka Duality in the form of Exercise 6.4.11 yields a unique $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \times S \to U$ such that (6.6.10) commutes.

Choose a $\mathbb{G}_m$-invariant étale surjective morphism $U \to X$ from an affine scheme (Theorem 6.5.23). Then (6.6.9) implies that $U^+ \to X^+$ is etale and representable, and since U^+ is an affine scheme of finite type, it follows that X^+ is an algebraic space of finite type. Since $U^+ \to U^{\mathbb{G}_m}$ is affine, étale descent implies that $X^+ \to X^{\mathbb{G}_m}$ is also affine.

If X is smooth, then $X^{\mathbb{G}_m}$ is smooth by Theorem 6.6.2. As U is also smooth, for each $u \in U^{\mathbb{G}_m}(\mathbb{k})$, there is a $\mathbb{G}_m$-equivariant morphism $U \to T_{U,u}$ étale at u with $f(u) = 0$ (Lemma 6.5.7). Then $U^+ \to T_{U,u}^+$ is also étale at u. Let $V \subset U$ be the open locus where $U^+ \to T_{U,u}^+$ is étale. Since V is $\mathbb{G}_m$-equivariant, if $v \in V^{\mathbb{G}_m}$, then $\mathrm{ev}_0^{-1}(v) \subset V$. Choosing an affine subscheme $V' \subset V^{\mathbb{G}_m}$ containing u and replacing U^+ with $\mathrm{ev}_0^{-1}(V')$, we may assume that $U^+ \to T_{U,u}^+$ is everywhere étale. By (6.6.9), we have a cartesian diagram

$$
\begin{array}{ccccc}
T_{U,u}^+ & \longleftarrow & U^+ & \longrightarrow & X^+ \\
\downarrow & \square & \downarrow & \square & \downarrow \\
T_{U,u}^{\mathbb{G}_m} & \longleftarrow & U^{\mathbb{G}_m} & \longrightarrow & X^{\mathbb{G}_m}
\end{array}
\tag{6.6.11}
$$

where the horizontal arrows are étale. With $T_{X,x} = T_{>0} \oplus T_0 \oplus T_{<0}$, there are identifications $T_{X,x}^{\mathbb{G}_m} = T_0$ and $T_{X,x}^+ = T_{>0} \oplus T_0$. Since $T_{U,u}^+ \to T_{U,u}^{\mathbb{G}_m}$ a surjection of vector spaces, $U \to U^{\mathbb{G}_m}$ is a Zariski-local affine fibration. By étale descent, $X \to X^{\mathbb{G}_m}$ is an affine fibration of relative dimension $\dim T_{>0}$.

If X is a smooth scheme, then by Sumihiro's Theorem on Torus Actions (B.2.52), we may choose $U = \coprod_i U_i \to X$ such that $\{U_i\}$ is a $\mathbb{G}_m$-invariant affine open covering. Then (6.6.11) implies that $X^+ \to X^{\mathbb{G}_m}$ is a Zariski-local affine fibration.

See also [Dri13, Prop. 1.2.2, Thm. 1.4.2] and [AHR20, Thm. 5.16]. $\qquad \square$

Remark 6.6.12. Another approach to establish the algebraicity of X^+ in Theorem 6.6.8 is to show that the stack $\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X})$, whose objects over a $\mathbb{k}$-scheme S are morphisms $[\mathbb{A}^1/\mathbb{G}_m]_S \to \mathcal{X}$, is algebraic when $\mathcal{X}$ has affine diagonal. This can be shown by verifying Artin's Axioms (C.7.4) where the crucial step is to verify the effectivity condition (AA_5): this follows from the coherent completeness of $[\mathbb{A}^1/\mathbb{G}_m]_R$, where R is a noetherian local $\mathbb{k}$-algebra, along the unique closed point (Theorem 6.4.12) together with Coherent Tannaka Duality (6.4.9).

When $\mathcal{X} = [X/\mathbb{G}_m]$, then a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \to X$ corresponds to a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/\mathbb{G}_m]$ over $B\mathbb{G}_m$ (Exercise 3.1.16), and there is a cartesian diagram

$$
\begin{array}{ccc}
\underline{\mathrm{Mor}}^{\mathbb{G}_m}(\mathbb{A}^1, X) & \longrightarrow & \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [X/\mathbb{G}_m]) \\
\downarrow & \square & \downarrow \\
\mathrm{Spec}\,\mathbb{k} & \longrightarrow & \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], B\mathbb{G}_m).
\end{array}
$$

The algebraicity of the stacks of morphisms implies that $\underline{\mathrm{Mor}}^{\mathbb{G}_m}(\mathbb{A}^1, X)$ is an algebraic space.

6.6.3 The Białynicki-Birula Stratification

Theorem 6.6.13 (Białynicki-Birula Stratification[3])**.** *Let X be a separated algebraic space of finite type over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$. Let $X^{\mathbb{G}_m} = \coprod_{i=1}^n F_i$ be the fixed locus with connected components F_i. There exists an affine morphism $X_i \to F_i$ for each i and a monomorphism $\coprod_i X_i \to X$. Moreover,*

(1) *If X is proper, then $\coprod_i X_i \to X$ is surjective.*

(2) *If X is smooth (resp. smooth scheme), then F_i is smooth and $X_i \to F_i$ is a (resp. Zariski-local) affine fibration. If $x \in F_i$ and $T_{X,x} = T_{x,>0} \oplus T_{x,0} \oplus T_{x,<0}$ is the $\mathbb{G}_m$-equivariant decomposition into nonnegative, zero, and positive weights, then $T_{X_i,x} = T_{x,>0} \oplus T_{x,0}$, $T_{F_i,x} = T_{x,0}$, and $X_i \to F_i$ has relative dimension $\dim T_{x,>0}$.*

(3) *The map $X_i \hookrightarrow X$ is a locally closed immersion under any of the following conditions:*

    (a) *X is affine,*

    (b) *X is a smooth scheme, or*

    (c) *there exists a $\mathbb{G}_m$-equivariant locally closed immersion $X \hookrightarrow \mathbb{P}(V)$ where V is a $\mathbb{G}_m$-representation (e.g., X is a normal quasi-projective variety).*

(4) *If X is smooth, irreducible, and quasi-projective, then the stratification $X^+ = \coprod_i X_i$ is* filterable, *i.e. there is an ordering of the indices such that $X_{\geq i} := \bigcup_{j \geq i} X_j$ is closed for each i. If in addition there are finitely many fixed points $\{x_1, \ldots, x_n\}$, then $T_{x_i,0} = 0$ and $X_i = \mathbb{A}(T_{x_i,>0})$ is an affine space; in particular,*

$$X^+ = X_{\geq 1} \supset X_{\geq 2} \supset \cdots \supset X_{\geq n} \supset \emptyset$$

is a cell decomposition, *i.e. each $X_{\geq i} \setminus X_{\geq i-1} = X_i$ is an affine space.*

Proof. By Theorem 6.6.8, X^+ is representable and there is affine morphism $\mathrm{ev}_0 \colon X^+ \to X^{\mathbb{G}_m}$ of finite type. We define X_i as the preimage $\mathrm{ev}_0^{-1}(F_i)$. Since X is separated, the inclusion $X^+ \hookrightarrow X$ is a monomorphism. This gives the main statement. If X is proper, then $X^+ \to X$ is surjective (i.e. (1) holds) as $\lim_{t \to 0} t \cdot x$ exists for every $x \in X(\Bbbk)$. Statement (2) follows directly from Theorem 6.6.8.

For (3), if $X = \operatorname{Spec} A$ and $A = \bigoplus_d A_d$ is the grading induced by the $\mathbb{G}_m$-action, then X^+ is the closed subscheme defined by the ideal $\sum_{d<0} A_d$ (Exercise 6.6.6) and in particular affine. If X is a smooth scheme, then there exists a $\mathbb{G}_m$-invariant affine open cover (Theorem B.2.51). For any point $x \in X^+$, let x_0 be the image of x under $\mathrm{ev}_0 \colon X^+ \to X^0$, and choose a $\mathbb{G}_m$-invariant affine open neighborhood $U \subset X$ of x_0. This induces a diagram

$$
\begin{array}{ccc}
U^+ \longhookrightarrow \mathrm{ev}_1^{-1}(U) \longrightarrow U \\
\\
X^+ \xrightarrow{\ \mathrm{ev}_1\ } X.
\end{array}
\tag{6.6.14}
$$

Since $U^+ \to U$ is a closed immersion (as U is affine) and $X^+ \to X$ is separated (it is a monomorphism), $U^+ \to \mathrm{ev}_1^{-1}(U)$ is a closed immersion. Since $U^+ = X^+ \times_{X^0} U^0$ (see (6.6.9)), $x \in U^+$ and $U^+ \to X^+$ is an open immersion. In particular, $U^+ \subset \mathrm{ev}_1^{-1}(U)$

---

[3]This is frequently referred to as the 'Białynicki-Birula Decomposition' as some authors prefer to reserve the term 'stratification' to a decomposition where each stratum has a neighborhood which is topologically locally trivial.

is an open and closed subscheme containing x. On the other hand, X_i is smooth and connected (as $X_i \to F_i$ is an affine fibration), thus irreducible. It follows that $X_i \cap U^+ = X_i \cap \mathrm{ev}_1^{-1}(U)$ and that $X_i \cap \mathrm{ev}_1^{-1}(U) \to U$ is a closed immersion which in turn implies that $X_i \to X$ is a locally closed immersion. The final case (3)(c) easily reduces to the case of $X = \mathbb{P}(V)$ in which a direct calculation shows that each X_i is of the form $\mathbb{P}(W) \setminus \mathbb{P}(W')$ for linear subspaces $W' \subset W \subset V$. See also [BB73, Thm. 4.1], [Hes81, Thm. 4.5,p. 69], [Dri13, Thm. B.0.3], [AHR20, Thm. 5.27], and [JS21, Thm. 1.5].

For (4), by Sumihiro's Theorem on Linearizations (B.2.51), we can choose a G-equivariant locally closed immersion $X \hookrightarrow \mathbb{P}^n$, where $\mathbb{G}_m$ acts on $\mathbb{P}^n$ via $t \cdot [x_0 : \ldots : x_n] = [t^{d_0} x_0 : \ldots : t^{d_n} x_n]$ with $d_0 \le \cdots \le d_n$. Let $D_1, \ldots, D_s$ be the distinct weights and set $J_i = \{j \mid d_j = D_i\}$ so that $J_1 \cup \cdots \cup J_s$ is a partition of $\{0, 1, \ldots, n\}$. Then $(\mathbb{P}^n)^{\mathbb{G}_m} = \coprod_{i=1}^s F_i$ where $F_i = V(x_j \mid j \in J_i)$. The preimage of F_i under the morphism $\mathrm{ev}_0 \colon \mathbb{P}^n \to (\mathbb{P}^n)^{\mathbb{G}_m}$, given by $p \mapsto \lim_{t \to 0} t \cdot p$, is

$$P_i := \mathrm{ev}_0^{-1}(F_i) = \left\{ [x_0 : \ldots : x_n] \, \middle| \, \begin{array}{l} x_j = 0 \text{ for all } j \in J_1 \cup \cdots \cup J_{i-1} \\ x_k \ne 0 \text{ for some } k \in J_i \end{array} \right\},$$

Moreover, the union

$$P_{\ge i} := \bigcup_{j \ge i} P_j = V(x_k \mid k \in J_1 \cup \cdots \cup J_{i-1}) \subset \mathbb{P}^n$$

is closed. The fixed locus for X is $X^{\mathbb{G}_m} = (\mathbb{P}^n)^{\mathbb{G}_m} \cap X = \coprod_i F_i \cap X$. For each i, we write $F_i \cap X = \coprod_{j=1}^{l_i} F_{ij}$ and $P_i \cap X = \coprod_{j=1}^{l_i} X_{ij}$ as the irreducible decompositions. Then $\mathrm{ev}_0 \colon \mathbb{P}^n \to (\mathbb{P}^n)^{\mathbb{G}_m}$ restricts to morphisms $\mathrm{ev}_0 \colon X_{ij} \to F_{ij}$. For $j \ne k$, the strata X_{ij} and X_{ik} are disjoint, and thus $\overline{X}_{ij} \cap \overline{X}_{ik} \subset P_{\ge i+1} \cap X$. It follows that

$$(P_{\ge i+1} \cap X) \cup X_{i1} \cup \cdots \cup X_{ij} \subset X$$

is closed for each $j = 1, \ldots, s$. Ordering the strata as $X_{11}, \ldots, X_{1l_1}, \ldots, X_{s1}, \ldots, X_{sl}$ establishes the claim. See also [Bir76, Thm. 3]. $\qquad\square$

Remark 6.6.15. It is not true in general that $X_i \hookrightarrow X$ is a locally closed immersion. Based on Hironaka's example of a proper, non-projective, smooth 3-fold, Sommese constructed a smooth algebraic space X such that $X_i \hookrightarrow X$ is not a locally closed immersion [Som82]. On the other hand, Konarski provided an example of a normal proper toric variety X such that $X_i \hookrightarrow X$ is not a locally closed immersion [Kon82].

Remark 6.6.16 (Morse stratifications)**.** The Białynicki-Birula stratification of X can be obtained as the Morse stratification corresponding to the non-degenerate Morse function $\mu \colon X \to \mathrm{Lie}(S^1)^\vee = \mathbb{R}$: a point $x \in X$ lies in X_i if only if the limit of its forward trajectory under the gradient flow of μ lies in F_i. See [CS79].

Example 6.6.17. Suppose $\mathbb{G}_m$ acts on $X = \mathbb{P}^2$ via $t \cdot [x : y : z] = [x : ty : t^2]$. Then $X^{\mathbb{G}_m} = F_1 \coprod F_2 \coprod F_3$ where $F_1 = \{[1 : 0 : 0]\}$, $F_2 = \{[0 : 1 : 0]\}$, and $F_3 = \{[0 : 0 : 1]\}$, and $X_1 = \{x \ne 0\} = \mathbb{A}^2$, $X_2 = \{[0 : y : z] \mid y \ne 0\} = \mathbb{A}^1$ and $X_3 = F_3$.

Let $\widetilde{X}$ be the blowup $\mathrm{Bl}_p X$ at the fixed point $p = [0 : 1 : 0]$. Then $\mathbb{G}_m$ acts on the exceptional divisor $E \cong \mathbb{P}^1$ via $t \cdot [u : v] = [u : t^2 v]$ with fixed points $q_1 = [1 : 0]$ and $q_2 = [0 : 1]$. The fixed locus $\widetilde{X}^{\mathbb{G}_m}$ contains four points $\widetilde{F}_1 = \{[1 : 0 : 0]\}$, $\widetilde{F}_2 = \{q_1\}$, $\widetilde{F}_3 = \{q_2\}$, and $\widetilde{F}_4 = \{[0 : 0 : 1]\}$. We have that $\widetilde{X}_1 = X_1 \cong \mathbb{A}^2$, $\widetilde{X}_2 = X_2 \cong \mathbb{A}^1$,

$X_3 = E \setminus \{q_2\} \cong \mathbb{A}^1$, and $\widetilde{X}_4 = X_4 = \widetilde{F}_4$ as illustrated in Figure 6.1. Observe that $\overline{X}_3 \setminus X_3 = \{q_2\}$ is not the union of other strata.
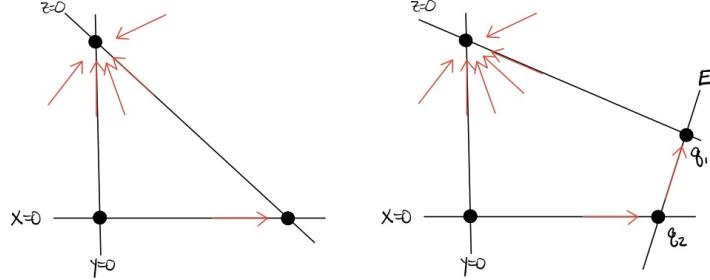
Figure 6.1: Białynicki-Birula stratifications for $\mathbb{P}^2$ (left) and $\mathrm{Bl}_p \, \mathbb{P}^2$ (right).

6.6.4 Applications of the Białynicki-Birula Stratification to cohomology

Proposition 6.6.18. *Let X be a smooth, irreducible, and quasi-projective scheme over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$ such that there are only finitely many fixed points. Then $A_i(X)$ is a free $\mathbb{Z}$-module generated by the closures of the i-dimensional cells. If in addition $\Bbbk = \mathbb{C}$, then the cycle map $\mathrm{CH}_i(X) \to \mathrm{H}^{\mathrm{BM}}_{2i}(X, \mathbb{Z})$ to Borel–Moore homology is an isomorphism and $\mathrm{H}^{\mathrm{BM}}_{2i+1}(X, \mathbb{Z}) = 0$.*

Remark 6.6.19. When X is compact (e.g. projective), then $H^{\mathrm{BM}}_{2i}(X, \mathbb{Z})$ is ordinary integral singular homology.

Proof. The Białynicki-Birula Stratification (6.6.13(4)) implies that X has a cell decomposition, and the statement follows from [Ful98, Ex. 19.1.11]. See also [Bri97, §3.2]. $\qquad\square$

Example 6.6.20 (Chow groups of $\mathrm{Hilb}_n(\mathbb{A}^2)$). Let $X = \mathrm{Hilb}_n(\mathbb{A}^2)$ be the Hilbert scheme of n points; this is a smooth irreducible scheme (see 1.5.3). The natural action of $T = \mathbb{G}_m^2$ induces a T-action on X. Under the $\mathbb{G}_m$-action induced by a one-parameter subgroup $\mathbb{G}_m \to T$ given by positive weights, the evaluation map $\mathrm{ev}_0 \colon X^+ \to X$ is surjective, and the $\mathbb{G}_m$-fixed points correspond to subschemes $Z = V(I) \subset \mathbb{A}^2$ supported at the origin where I is a monomial ideal. We see that there are only finitely many $\mathbb{G}_m$-fixed points. We may therefore use Proposition 6.6.18 to compute $\mathrm{CH}^*(X)$.

For a monomial ideal $I \subset R := \Bbbk[x, y]$, for each integer i, define

$$a_i := \min\{j \mid x^i y^j \in I\}$$

and let r be the largest integer such that $a_r > 0$. Then $a_0 \geq \ldots \geq a_r$ is a partition of n and $I = (y^{a_0}, xy^{a_1}, \ldots, x^{r+1})$. We need to compute the dimension of the positive weight space $T_{I, >0}$ of the $\mathbb{G}_m$-action on the tangent space

$$T_I = \mathrm{Hom}_R(I, R/I)$$

of X at the monomial ideal I; see Exercise 1.5.6 for the identification of the tangent space. To accomplish this, we first argue that

$$T_I = \sum_{0 \leq i \leq j \leq r} \sum_{s = a_{j+1}}^{a_j - 1} (\chi_1^{i-j-1} \chi_2^{a_i - s - 1} + \chi_1^{j-i} \chi_2^{s - a_i}), \qquad (6.6.21)$$

as $T = \mathbb{G}_m^2$ representations, where $\chi_i \colon T \to \mathbb{G}_m$ denotes the one-dimensional representation giving by $(t_1, t_2) \mapsto t_i^{-1}$. There are

$$\sum_{0 \le i \le j \le r} 2(a_j - a_{j+1}) = 2 \sum_{0 \le i \le r} a_i = 2n = \dim T_I$$

one-dimensional representations appearing on the right-hand side, and they are linearly independent. It thus suffices to show that each of them occurs in T_I. An R-module map $\phi \colon I \to R/I$ is given by the values $\phi(x^i y^{a_i})$ subject to the relations

$$\phi(x^{i+1} y^{a_i}) = x\phi(x^i a_i) \quad \text{and} \quad \phi(x^i y^{a_{i-1}}) = y^{a_{i-1} - a_i} \phi(x^i a_i).$$

Let $0 \le i \le j \le r$ and $a_{j+1} \le s < a_j$. Defining

$$\phi_{i,j,s} \colon I \to R/I, \quad x^l y^{a_l} \mapsto \begin{cases} x^{l+j-i} y^{a_l + s - a_i} & \text{if } l \le i \\ 0 & \text{otherwise} \end{cases}$$
$$\psi_{i,j,s} \colon I \to R/I, \quad x^l y^{a_l} \mapsto \begin{cases} x^{l+i-j-1} y^{a_l + s - a_i} & \text{if } l \ge j+1 \\ 0 & \text{otherwise,} \end{cases}$$

one checks that $\phi_{i,j,s}$ and $\psi_{i,j,s}$ are R-module maps that are eigenvectors for $\chi_1^{j-i} \chi_2^{s-a_i}$ and $\chi_1^{i-j-1} \chi_2^{a_i - s - 1}$. Thus (6.6.21) holds.

Choose $\lambda = (\lambda_1, \lambda_2) \colon \mathbb{G}_m \to T$ with $\lambda_1 \gg \lambda_2$. Under our sign conventions, a character $\chi_1^a \chi_2^b$ appearing in (6.6.21) has positive weight with respect to λ if $a < 0$, or if $a = 0$ and $b < 0$. Thus

$$T_{I,>0} = \sum_{0 \le i \le j \le r} \sum_{s=a_{j+1}}^{a_j - 1} \chi_1^{i-j-1} \chi_2^{a_i - s - 1} + \sum_{j=0}^{r} \sum_{s=a_{j+1}}^{a_j - 1} \chi_1^{j-i} \chi_2^{s-a_i}$$

and

$$\dim T_{I,>0} = \left(\sum_{i=0}^{r} \sum_{j=i}^{r} (a_j - a_{j+1}) \right) + \left(\sum_{j=0}^{r} (a_j - a_{j+1}) \right)$$
$$= \left(\sum_{i=0}^{r} a_i \right) + a_0 = n + a_0$$

Since there is a bijection between monomial ideals $I \subset R = \mathbb{k}[x,y]$ with $\dim_{\mathbb{k}} R/I = n$ and partitions $a_0 \ge \cdots \ge a_r$ of n, for every $d \ge 0$, the number of monomial ideals I such that $\dim T_{I,>0} = d$ is equal to

$$P(2n-d, d-n) := \# \{\text{partitions } a_1 \ge \cdots \ge a_r \text{ of } 2n-d \text{ with each } a_i \le d-n\}. \tag{6.6.22}$$

It follows from [Proposition 6.6.18](#) that

$$\dim \mathrm{CH}_d(\mathrm{Hilb}_n(\mathbb{A}^2))_{\mathbb{Q}} = P(2n-d, d-n).$$

See also [ESm87, Thm. 1.1] and [Göt94, §2.2].

Exercise 6.6.23 (Chow groups of $\mathrm{Hilb}_n(\mathbb{P}^2)$)**.** Follow the above strategy to show that the dth Betti number b_d of $\mathrm{Hilb}_n(\mathbb{P}^2)$ (or equivalently $\dim \mathrm{CH}_d(\mathrm{Hilb}_n(\mathbb{P}^2))$) is equal to

$$b_d = \sum_{n_0 + n_1 + n_2 = n} \sum_{p + r = d - n_1} P(p, n_0 - p) P(n_1) P(2n_2 - r, r - n_2),$$

where $P(a)$ is the number of partitions of a and $P(a,b)$ is defined by (6.6.22).

Remark 6.6.24. Göttsche used the Weil conjectures in [Göt90, Thm. 0.1] (see also [Göt94, Thm. 2.3.10]) to show that for any smooth projective surface S over $\mathbb{C}$ or $\overline{\mathbb{F}}_q$ that the Poincaré polynomial $p(S^{[n]}, z) = \sum_i b_i(S^{[n]})z^n$ of $S^{[n]} := \mathrm{Hilb}_n(S)$ satisfies

$$\sum_{n=0}^{\infty} p(S^{[n]}, z)t^n = \prod_{m=1}^{\infty} \frac{(1 + z^{2m-1}t^m)^{b_1(S)}(1 + z^{2m+1}t^m)^{b_3(S)}}{(1 - z^{2m-2}t^m)^{b_0(S)}(1 - z^{2m}t^m)^{b_2(S)}(1 - z^{2m+2}t^m)^{b_4(S)}}$$

In particular, the Betti numbers of $S^{[n]}$ only depend on the Betti numbers of S. While each term $p(S^{[n]}, z)$ does not admit a particularly nice expression, the generating function involving *all n* does.

On the other hand, Nakajima constructed an action of the Heisenberg algebra on $\mathrm{H}_*(S^{[n]})$, which can be used to recover the Betti number formula above as well as additional properties of the cohomology ring [Nak97] (see also [Nak99b]).

We can also use the Białynicki-Birula Stratification to compute equivariant Chow rings $\mathrm{CH}_G^*(X)$ (or equivalently the Chow ring $\mathrm{CH}^*([X/G])$ of the quotient stack) as introduced in §6.1.7. The following statements can also be made in de Rham or singular cohomology (§6.1.8) where instead of the excision sequence above, one uses the Thom–Gysin long exact sequence.

We will use the following two lemmas, which we state in a generality that we can also apply to the HKKN stratification of §6.7.6.

Lemma 6.6.25. *Let X be a smooth irreducible scheme over an algebraically closed $\mathbb{k}$ with an action of a smooth affine algebraic group G. Let $S_1, \ldots, S_r \subset X$ be nonempty, disjoint, smooth, irreducible, and locally closed G-invariant subschemes such that $X = \coprod_i S_i$ and such that $S_{\geq i} := \bigcup_{j \geq i} S_j$ is closed for each i. Let d_i be the codimension of S_i in X. If the top Chern class $c_{d_i}^G(N_{S_i/X}) \in \mathrm{CH}_G^*(S_i)_{\mathbb{Q}}$ is a nonzerodivisor for each i, then*

$$\dim \mathrm{CH}_G^k(X)_{\mathbb{Q}} = \sum_{i=1}^{r} \dim \mathrm{CH}_G^{k-d_i}(S_i)_{\mathbb{Q}}$$

for each k.

Proof. By assumption, $S_{\leq i} = \bigcup_{j \leq i} S_j$ is open for each i, and $S_i \subset S_{\leq i}$ is a closed subscheme with open complement $S_{<i}$. We have a commutative diagram

$$\mathrm{CH}_G^{k-d_i}(S_i) \longrightarrow \mathrm{CH}_G^k(S_{\leq i}) \longrightarrow \mathrm{CH}_G^k(S_{<i}) \longrightarrow 0$$
$$\searrow \qquad \downarrow$$
$$\mathrm{CH}_G^k(S_i)$$

where the top row is the right exact excision sequence (6.1.33(3)) and the vertical downward arrow is given by intersecting with S_i. By the self-intersection formula (6.1.33(5)), the composition $\mathrm{CH}_G^{k-d_i}(S_i) \to \mathrm{CH}_G^k(S_i)$ is multiplication by $c_{d_i}^G(N_{S_i/X})$. By hypothesis, this map is injective after tensoring with $\mathbb{Q}$. It follows that the top row is an exact sequence after tensoring with $\mathbb{Q}$, and that

$$\dim \mathrm{CH}_G^k(S_{\leq i})_{\mathbb{Q}} = \dim \mathrm{CH}_G^{k-d_i}(S_i)_{\mathbb{Q}} + \dim \mathrm{CH}_G^k(S_{<i})_{\mathbb{Q}}.$$

The formula follows from induction. See [AB83, Prop. 1.9]. $\square$

Remark 6.6.26. If $[S_i/G]$ is Deligne–Mumford, then $\mathrm{CH}_G^k(S_i)$ vanishes for $k \gg 0$ and $c_{d_i}^G(N_{S_i/X})$ is a zero divisor.

The following gives a condition for the top Chern class to be a nonzerodivisor.

Lemma 6.6.27. *Let X be a smooth irreducible scheme over an algebraically closed $\Bbbk$ with an action of a connected, smooth, and affine algebraic group G, and let N be a G-equivariant vector bundle of rank d on X. Suppose that there is a subgroup $\mathbb{G}_m \subset G$ acting trivially on X and a point $x \in X(\Bbbk)$ such that $N \otimes \kappa(x)$ contains no $\mathbb{G}_m$-invariant vectors. Then $c_d^G(N) \in \mathrm{CH}_G^*(X)_{\mathbb{Q}}$ is a nonzerodivisor.*

Proof. Choose a maximal torus T containing $\mathbb{G}_m$ and a character $T \to \mathbb{G}_m$ such that the composition $\mathbb{G}_m \hookrightarrow T \to \mathbb{G}_m$ is given by $t \mapsto t^d$ for $d > 0$. By (6.1.33(7)), $\mathrm{CH}_G^*(X)_{\mathbb{Q}} = \mathrm{CH}_T^*(X)_{\mathbb{Q}}^W$ where W is the Weyl group. Since $\mathrm{CH}_G^*(X)_{\mathbb{Q}}$ is a subring of $\mathrm{CH}_T^*(X)_{\mathbb{Q}}$, we are reduced to show that $c_d^T(N) \in \mathrm{CH}_T^*(X)_{\mathbb{Q}}$ is a nonzerodivisor. If we write T as the product of the given $\mathbb{G}_m$ and a subtorus T', then

$$\mathrm{CH}_T^*(X) \cong \mathrm{CH}_{T'}^*(X) \otimes \mathrm{CH}^*(B\mathbb{G}_m) \cong \mathrm{CH}_{T'}^*(X)[z]$$

by (6.1.33(6)). For $x \in X(\Bbbk)$, we can write

$$c_d^T(N) = \sum c_i^{T'}(N) \otimes c_{d-i}^{\mathbb{G}_m}(N \otimes \kappa(x))$$
$$= 1 \otimes c_d^{\mathbb{G}_m}(N \otimes \kappa(x)) + \text{higher degree terms}.$$

If $a_1, \ldots, a_d$ denote the $\mathbb{G}_m$-weights of $N \otimes \kappa(x)$, then by hypothesis each $a_i \neq 0$ and

$$c_d^{\mathbb{G}_m}(N \otimes \kappa(x)) = \left(\prod_i a_i \right) z^d \in \mathrm{CH}^*(B\mathbb{G}_m)_{\mathbb{Q}} \cong \mathbb{Q}[z]$$

is a nonzerodivisor, and therefore $c_d^T(N)$ is also a nonzerodivisor.

See also [AB83, Prop. 13.4] and [Bri97, §3.2]. $\qquad\qquad\square$

We define the *G-equivariant Chow–Poincaré polynomial* of a G-equivariant scheme X as

$$p_G(X,t) = \sum_{d=0}^{\infty} \left(\dim \mathrm{CH}_G^d(X)_{\mathbb{Q}} \right) t^d.$$

We also denote $p(X,t) = \sum_{d=0}^{\infty} \left(\dim \mathrm{CH}^d(X)_{\mathbb{Q}} \right) t^d$ as the (non-equivariant) *Chow–Poincaré polynomial*.

Proposition 6.6.28. *Let X be a smooth, irreducible, and quasi-projective scheme over an algebraically closed field $\Bbbk$ with an action of $\mathbb{G}_m$ such that $X^+ \to X$ is surjective (i.e. X is projective). Let $X = \coprod_{i=1}^r X_i$ and $X^{\mathbb{G}_m} = \coprod_{i=1}^r F_i$ be the Białynicki-Birula Stratification (6.6.13), and let d_i be the codimension of X_i in X. Then*

$$p_{\mathbb{G}_m}(X,t) = \sum_{i=1}^r p(F_i) \cdot t^{d_i}(1-t)^{-1}.$$

Proof. Since each F_i is smooth and $X_i \to F_i$ is a Zariski-local affine fibration (Theorem 6.6.8), the pullback map $\mathrm{CH}_{\mathbb{G}_m}^*(F_i) \xrightarrow{\sim} \mathrm{CH}_{\mathbb{G}_m}^*(X_i)$ is an isomorphism (6.1.33(2)). Under this isomorphism, $N_{X_i/X}$ is the image of its restriction $(N_{X_i/X})|_{F_i}$. For $x \in F_i$, $N_{X_i/X} \otimes \kappa(x) = T_{x,<0}$ has no $\mathbb{G}_m$-invariant vectors and thus Lemma 6.6.27

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implies that $c_{d_i}^{\mathbb{G}_m}((N_{X_i/X})|_{F_i})$ is a nonzerodivisor. Lemma 6.6.25 therefore implies that $p_{\mathbb{G}_m}(X,t) = \sum_i p_{\mathbb{G}_m}(X_i,t)$. Since

$$\mathrm{CH}^*_{\mathbb{G}_m}(X_i) \cong \mathrm{CH}^*_{\mathbb{G}_m}(F_i) \cong \mathrm{CH}^*(F_i) \otimes \mathrm{CH}^*(B\mathbb{G}_m) \cong \mathrm{CH}^*(F_i)[z],$$

where the second equality uses (6.1.33(6)), we have the identity $p_{\mathbb{G}_m}(F_i,t) = p(F_i)(1-t)^{-1}$ and the statement follows. $\qquad\square$

6.6.5 One-parameter subgroups and the Cartan Decomposition

If G is an algebraic group over a field $\Bbbk$, a *one-parameter subgroup* is a homomorphism $\lambda\colon \mathbb{G}_m \to G$ of algebraic groups. Despite the terminology, we do not require that a one-parameter subgroup is injective, e.g. $\mathbb{G}_m \to \mathbb{G}_m, t \mapsto t^2$ is a one-parameter subgroup. See §B.2.6 for more background and examples.

Given an action of G on an algebraic space X, the limit $\lim_{t\to 0} \lambda(t) \cdot x$ exists if the multiplication map $\mathbb{G}_m \to X$, defined by $t \mapsto \lambda(t) \cdot x$ extends to a map $\mathbb{A}^1 \to X$. If X is separated, then the limit is unique if it exists, while if X is proper, then there is always a unique limit.

Example 6.6.29. If $X = \mathbb{P}(V)$ where V is a finite dimensional representation of G and $\lambda\colon \mathbb{G}_m \to G$ is a one-parameter subgroup, then we can choose a basis of V such that $\lambda(t) \cdot (x_1,\ldots,x_n) = (t^{d_1}x_1,\ldots,t^{d_n}x_n)$ with $d_1 \leq \cdots \leq d_n$. If $d = \min\{d_i \mid x_i \neq 0\}$, then $\lim_{t\to 0} \lambda(t) \cdot [x_0 : \ldots : x_n] = [x'_0 : \ldots : x'_n]$ where $x'_i = x_i$ for all i such that $d_i = d$ and is 0 otherwise.

We now state and prove the Cartan Decomposition: an element $g \in G(K)$ over the fraction field K of a DVR R can be multiplied on the left and right by elements of $G(R)$ such that it is induced from a one-parameter subgroup. For a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, we denote by $\lambda|_K \in G(K)$ the image of the composition

$$\operatorname{Spec} K \to \mathbb{G}_m \xrightarrow{\lambda} G$$

where the first map is defined by the $\Bbbk$-algebra map $\Bbbk[t]_t \to K$ taking t to a uniformizer in R.

The Cartan Decomposition will provide the key algebraic input in the proof of the Hilbert–Mumford Criterion (6.7.15).

Theorem 6.6.30 (Cartan Decomposition[4])**.** *Let G be a reductive algebraic group. Let R be a complete DVR over $\Bbbk$ with residue field $\Bbbk$ and fraction field K. Then for every element $g \in G(K)$, there exists $h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that*

$$g = h_1 \lambda|_K h_2.$$

Proof. As our proof will utilize that BG is S-complete (Definition 6.8.13), a concept developed §6.8, we postpone the proof until Proposition 6.8.57. In fact, we show not only that the theorem holds for reductive groups but that it characterizes reductivity. See also [IM65, Cor. 2.17], [Ses72, Thm. 2.1] and [BT72, §4]. $\qquad\square$

Remark 6.6.31 (Equivalent formulation)**.** Let $T \subset G$ be a maximal torus. The above theorem is equivalent to the identity

$$G(K) = G(R)T(K)G(R).$$

---

[4]This is sometimes also referred to as the Iwahori Decomposition or the Cartan–Iwahori–Matsumoto Decomposition

To see how the theorem implies the above identity, choose $h \in G(R)$ such that $h\lambda|_K h^{-1} \in T(K)$. Then

$$g = h_1\lambda|_K h_2 = \underbrace{(h_1 h^{-1})}_{\in G(R)}\underbrace{(h\lambda|_K h^{-1})}_{\in T(K)}\underbrace{(hh_2)}_{\in G(R)}.$$

Conversely, suppose $g = h_1 t h_2$ for $h_1, h_2 \in G(R)$ and $t \in T(K)$. If we write $T \cong \mathbb{G}_m^r$ and $\pi \in R$ as the uniformizing parameter, then $t = (u_1\pi^{d_1}, \ldots, u_r\pi^{d_r})$ for units $u_i \in R^\times$ and integers $d_i \in \mathbb{Z}$. After replacing h_1 with $h_1 \cdot (u_1, \ldots, u_r)$, we can write $g = h_1\lambda|_K h_2$ where $\lambda \colon \mathbb{G}_m \to T \subset G$ is the one-parameter subgroup given by $t \mapsto (t^{d_1}, \ldots, t^{d_r})$.

Remark 6.6.32 (Case of GL_n). The Cartan Decomposition for GL_n can be established by an elementary linear algebra argument. Let $g = (g_{ij}) \in \mathrm{GL}_n(K)$. After performing row and column operations, we can assume that $g_{1,1} = \pi^d$ has minimal valuation among the g_{ij}, where $\pi \in R$ is a uniformizer. For each $k \geq 2$, we write $g_{k,1} = u\pi^e$. Now perform the row operations where the nth row r_n is exchanged for $r_n - u\pi^{e-d}r_1$. In this way, we can arrange that $g_{k,1} = 0$ for $k \geq 2$. By performing analogous column operations, we can also arrange that $g_{1,k} = 0$ for $k \geq 2$. The statement is thus established by induction.

Exercise 6.6.33. Let $\Bbbk$ be a field.

(a) Let $X \subset \mathbb{P}(V)$ be a $\mathbb{G}_m$-equivariant locally closed subscheme where V is a finite dimensional $\mathbb{G}_m$-representation. Show that $[X/\mathbb{G}_m]$ is separated if and only if X has no $\mathbb{G}_m$-fixed points, or in other words that the diagonal $[X/\mathbb{G}_m] \to [X/\mathbb{G}_m] \times [X/\mathbb{G}_m]$ is finite if and only if it is quasi-finite.

(b) Let G be a reductive algebraic group acting on an algebraic space X over $\Bbbk$. Show that $[X/G]$ is separated if and only if for every one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, the corresponding quotient stack $[X/\mathbb{G}_m]$ is separated.

Hint: Verify the valuative criterion by applying the Cartan Decomposition.

Remark 6.6.34. Unlike the case of $\mathbb{G}_m$ in (a), it is not true $[X/G]$ is separated for an action of an affine algebraic group G acting linearly on a quasi-projective scheme X with finite stabilizers. See Exercise 3.9.3(d) for such an example by a free action of SL_2 on a quasi-affine variety.

6.6.6 The Destabilization Theorem

Theorem 6.6.35 (Destabilization Theorem). *Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ acting on an affine scheme X of finite type over $\Bbbk$. Given $x \in X(\Bbbk)$, there exists a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $x_0 := \lim_{t \to 0} \lambda(t) \cdot x$ exists and has closed G-orbit.*

Proof. Let $R = \Bbbk[\![t]\!]$ with fraction field $K = \Bbbk(\!(t)\!)$. We can choose an element $g \in G(K)$ and a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & Gx \\ \big\uparrow & & \big\uparrow \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} R & \xrightarrow{\;\widetilde{g}\;} & X \end{array}$$

where the top map is given by the composition $\operatorname{Spec} K \xrightarrow{g} G \to Gx$ and such that $y := \widetilde{g}(0)$ has closed G-orbit. By the Cartan decomposition, there exists

$h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $h_1 g = \lambda|_K h_2$. By applying the general fact that for $a \in G(R)$ and $b \in X(R)$, $(a \cdot b)(0) = a(0) \cdot b(0)$ to $h_1 \in G(R)$ and $\widetilde{g} \in X(R)$, we obtain that

$$\lim_{t \to 0} \lambda(t) h_2(t) \cdot x = \lim_{t \to 0} h_1(t) g(t) \cdot x = h_1(0) \cdot \widetilde{g}(0) = h_1(0) \cdot y \in Gy. \qquad (6.6.36)$$

We claim that the related but *possibly different* limit $\lim_{t \to 0} \lambda(t) h_2(0) \cdot x$ exists and is also contained in the closed orbit Gy. Once this is established, the theorem would be established by using the one-parameter subgroup $h_2(0)^{-1} \lambda h_2(0)$:

$$\lim_{t \to 0} (h_2(0)^{-1} \lambda h_2(0))(t) \cdot x = h_2^{-1}(0) \cdot \lim_{t \to 0} \lambda(t) h_2(0) \cdot x \in Gy.$$

First, to see that $\lim_{t \to 0} \lambda(t) h_2(0) \cdot x$ exists, we may apply Proposition B.2.43(1) below to reduce to the case that $X = \mathbb{A}(V)$ is a G-representation. We may choose a basis of $V \cong \mathbb{k}^n$ such that the λ-action has weights $\lambda_1, \dots, \lambda_n$. We may also write $h_2 \cdot x = (a_1, \dots, a_n) \in X(R)$ with each $a_i \in \mathbb{k}[\![t]\!]$ and further decompose $a_i = a_i(0) + a_i'$ with $a_i' \in (t)$. Since

$$\lim_{t \to 0} \lambda(t) h_2(t) \cdot x = \lim_{t \to 0} (t^{\lambda_1}(a_1(0) + a_1'), \dots, t^{\lambda_n}(a_n(0) + a_n')) \qquad (6.6.37)$$

exists, we see that for each i with $\lambda_i < 0$, we must have that $a_i(0) = 0$, which in turn implies that $\lim_{t \to 0} \lambda(t) h_2(0) \cdot x$ exists.

Finally, to see that this limit lies in Gy, we may apply Proposition B.2.43(2) to obtain a G-equivariant map $f \colon X \to \mathbb{A}(W)$ such that $f^{-1}(0) = Gy$. We are thus reduced to showing that $\lim_{t \to 0} \lambda(t) h_2(0) \cdot f(x) = 0$. By computing the limit $\lim_{t \to 0} \lambda(t) h_2(t) \cdot f(x)$ as in (6.6.37), the same argument shows that since $\lim_{t \to 0} \lambda(t) h_2(t) \cdot f(x) = 0$, we must also have that $\lim_{t \to 0} \lambda(t) h_2(0) \cdot f(x) = 0$. See also [GIT, p. 53] and [Kem78, Thm. 1.4]. $\qquad \square$

Corollary 6.6.38 (Destabilization Theorem II). *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\mathbb{k}$ with affine diagonal. Let $x \rightsquigarrow x_0$ be a specialization of $\mathbb{k}$-points such that the stabilizer G_{x_0} is linearly reductive. Then there exists a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \mathcal{X}$ representing the specialization $x \rightsquigarrow x_0$.* $\qquad \square$

Proof. The Local Structure Theorem (6.5.1) yields an étale morphism $[\operatorname{Spec} A/G_{x_0}] \to \mathcal{X}$ and a point w_0 mapping to x_0. After possibly replacing $\operatorname{Spec} A$ with a G_{x_0}-invariant affine subscheme, we can assume that w_0 is a closed point. The specialization $x \rightsquigarrow x_0$ lifts a specialization $w \rightsquigarrow w_0$ in $[\operatorname{Spec} A/G_{x_0}]$, and we can choose a representative $\widetilde{w} \in \operatorname{Spec} A$ of the orbit corresponding to w. The Destabilization Theorem gives a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $\widetilde{w}_0 = \lim_{t \to 0} \lambda(t) \cdot \widetilde{w}$ exists and has closed orbit. By Affine GIT (6.3.7), there is a unique closed orbit in $\overline{G\widetilde{w}}$ and thus $\widetilde{w}_0 \in \operatorname{Spec} A$ maps to w_0. The $\mathbb{G}_m$-equivariant extension $\mathbb{A}^1 \to X$ of $t \mapsto \lambda(t) \cdot \widetilde{w}$ defines a morphism of algebraic stacks $[\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec} A/G_{x_0}]$ such that the image of the specialization $1 \rightsquigarrow 0$ is $w \rightsquigarrow w_0$. The composition $[\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec} A/G_{x_0}] \to \mathcal{X}$ yields the desired map. $\qquad \square$

6.6.7 Maps from $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$

We define the quotient stack

$$\Theta = [\mathbb{A}^1/\mathbb{G}_m]$$

over $\operatorname{Spec}\mathbb{Z}$; when we are working over a field $\mathbb{k}$, we will abuse notation by also using Θ to denote $\Theta_\mathbb{k} = [\mathbb{A}^1_\mathbb{k}/\mathbb{G}_{m,\mathbb{k}}]$.

While a map $\mathcal{X} \to \Theta$ from an algebraic stack is classified by a line bundle and a section (Example 3.9.16), maps $\Theta \to \mathcal{X}$ *from* Θ often also have geometric significance. We provide such descriptions for maps from Θ to quotient stacks, stacks of coherent sheaves, and the stack of all curves. These descriptions will be useful to interpret the valuative criteria of Θ and S-completeness introduced in §6.8.2. In fact, we will provide descriptions of maps from $\Theta_R := \Theta \times_k R$ where R is a $\mathbb{k}$-algebra; the reader is encouraged to consider the case that $R = \mathbb{k}$ on a first reading.

Quotient stacks

Given a quotient stack $[X/G]$, a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, and a point $x \in X(\mathbb{k})$ such that $\lim_{t \to 0} \lambda(t) \cdot x \in X$ exists, the $\mathbb{G}_m$-equivariant extension $\mathbb{A}^1 \to X$ induces a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ of algebraic stacks. The next proposition asserts that the converse is also true, i.e. any map $[\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ is induced by a one-parameter subgroup λ and a point $x \in X(\mathbb{k})$.

Recall from §B.2.6 that if $\lambda \colon \mathbb{G}_m \to G$ is a one-parameter subgroup, then $P_\lambda \subset G$ denotes the subgroup of elements $g \in G$ such that $\lim_{t \to 0} \lambda(t) g \lambda(t)^{-1}$ exists. If G is reductive, then P_λ is a parabolic .

Proposition 6.6.39. *Let G be a smooth affine algebraic group over an algebraically closed field $\mathbb{k}$, and let X be a separated algebraic space of finite type over $\mathbb{k}$. For every complete noetherian local $\mathbb{k}$-algebra R with algebraically closed residue field $\mathbb{F}$, there is an equivalence of groupoids*

$$\operatorname{Mor}_\mathbb{k}(\Theta_R, [X/G]) \xrightarrow{\sim} \big\{ (x \in X(R), \lambda \colon \mathbb{G}_m \to G) \mid \lim_{t \to 0} \lambda(t) \cdot x \in X(R) \text{ exists} \big\};$$

a morphism $(x, \lambda) \to (x', \lambda')$ is an isomorphism class of a pair (g, h) with $g \in P_\lambda(R)$ and $h \in G(\mathbb{k})$ such that $x' = hgx$ and $\lambda' = h\lambda h^{-1}$ where $(g, h) \sim (c^{-1}g, hc)$ for $c \in C_\lambda$.

Under this correspondence, the morphism $\Theta_R \to [X/G]$ sends 1 to x and 0 to $\lim_{t \to 0} \lambda(t) \cdot x$.

Remark 6.6.40. Observe that over $R = \mathbb{k}$ an isomorphism $(x, \lambda) \xrightarrow{\sim} (x', \lambda)$ is given by an element $g \in P_\lambda(\mathbb{k})$ such that $x' = gx$. In particular, the automorphism group of (x, λ) is $P_\lambda \cap G_x$.

Proof. Given (x, λ), the $\mathbb{G}_m$-equivariant map $m_{x,\lambda} \colon \mathbb{G}_{m,R} \to X$ defined by $t \mapsto \lambda(t) \cdot x$ extends to a commutative diagram

$$
\begin{array}{ccc}
\mathbb{G}_{m,R} & \xrightarrow{\ m_{x,\lambda}\ } & X \\
\cap \big\downarrow & {\nearrow} \atop {\widetilde{m}_{x,\lambda}} & \\
\mathbb{A}^1_R & &
\end{array}
$$

The extension is $\mathbb{G}_m$-equivariant and induces a morphism of quotient stacks $f_{x,\lambda} \colon \Theta_R \to [X/G]$. We will show that this defines a functor

$$\{(x, \lambda) \mid \lim_{t \to 0} \lambda(t) \cdot x \text{ exists}\} \to \operatorname{Mor}_\mathbb{k}(\Theta_R, [X/G])$$

$$(x, \lambda) \mapsto f_{x,\lambda}. \tag{6.6.41}$$

Given a morphism $(g, h): (x, \lambda) \to (x', \lambda')$, we need to define a 2-morphism $f_{x,\lambda} \overset{\sim}{\to} f_{x',\lambda'}$. Since h determines a canonical isomorphism $f_{x',\lambda'} \overset{\sim}{\to} f_{h^{-1}x',\lambda}$, it suffices to define a 2-morphism $f_{x,\lambda} \overset{\sim}{\to} f_{h^{-1}x',\lambda}$. Since $g \in P_\lambda(R)$, the map $t \mapsto \lambda(t)g\lambda(t)^{-1}$ extends to a map $\widetilde{g}: \mathbb{A}^1_R \to G$ such that $\widetilde{m}_{h^{-1}x',\lambda} = \widetilde{g} \cdot \widetilde{m}_{x,\lambda}$ (as $h^{-1}x' = gx$). The element $\widetilde{g}$ defines an isomorphism $f_{x,\lambda} \overset{\sim}{\to} f_{h^{-1}x',\lambda}$. For $c \in C_\lambda$, the pairs (g, h) and $(c^{-1}g, hc)$ define the same isomorphism: indeed this follows from the observation that if $c \in G_x$, then (c^{-1}, c) defines the identify automorphism of $f_{x,\lambda}$. Conversely, any isomorphism $f_{x,\lambda} \overset{\sim}{\to} f_{x',\lambda}$ is an induced by element $\widetilde{g} \in G(\mathbb{A}^1)$ satisfying $\widetilde{m}_{x',\lambda} = \widetilde{g} \cdot \widetilde{m}_{x,\lambda}$, and we see that (6.6.41) is a fully faithful functor.

To see essential surjectivity of (6.6.41), let $f: \Theta_R \to [X/G]$ be a morphism. In the fiber diagram

$$
\begin{array}{ccc}
\mathcal{P} & \longrightarrow & X \\
\downarrow & \square & \downarrow \\
\Theta_R & \overset{f}{\longrightarrow} & [X/G]
\end{array}
$$

$\mathcal{P} \to \Theta_R$ is a principal G-bundle. The restriction $\mathcal{P}|_{B\mathbb{G}_{m,\mathbb{F}}}$ along the unique closed point $0: B\mathbb{G}_{m,\mathbb{F}} \to \Theta_R$, corresponds to a $\mathbb{G}_m$-equivariant principal G-bundle P on $\operatorname{Spec} \mathbb{F}$. After choosing an isomorphism $P \cong G$, we see that $\mathcal{P}$ corresponds to a one-parameter subgroup $\lambda': \mathbb{G}_{m,\mathbb{F}} \to G_\mathbb{F}$.

Choose a maximal torus $T \subset G$ over $\Bbbk$. Since all maximal tori of $G_\mathbb{F}$ are conjugate (Proposition B.2.44), there exists an element $q \in G(\mathbb{F})$ such that the image of $q\lambda'q^{-1}$ is contained in $T_\mathbb{F}$. Letting $n = \dim T$, there are equivalences

$$
\operatorname{Hom}_\Bbbk(\mathbb{G}_m, T) \cong \mathbb{Z}^n \cong \operatorname{Hom}_\mathbb{F}(\mathbb{G}_{m,\mathbb{F}}, T_\mathbb{F}),
$$

where the composition is given by $\lambda \mapsto \lambda \times_\Bbbk \mathbb{F}$. It follows that there is a one-parameter subgroup $\lambda: \mathbb{G}_m \to T$ whose base change $\lambda \times_\Bbbk \mathbb{F}$ is conjugate to λ'. On the other hand, every one-parameter subgroup λ induces a $\mathbb{G}_m$-action on the product $\mathbb{A}^1_R \times G$ by $t \cdot (x, g) = (tx, g\lambda(t)^{-1})$ and thus a principal G-bundle $\mathcal{P}_\lambda := [(\mathbb{A}^1_R \times G)/\mathbb{G}_m]$ over Θ_R. We claim that there is an isomorphism $\alpha: \mathcal{P} \to \mathcal{P}_\lambda$ of principal G-bundles. By construction, we have an isomorphism $\alpha_0: \mathcal{P}|_{B\mathbb{G}_{m,\mathbb{F}}} \to \mathcal{P}_\lambda|_{B\mathbb{G}_{m,\mathbb{F}}}$. Since $\underline{\operatorname{Isom}}_{\Theta_R}(\mathcal{P}, \mathcal{P}_\lambda) \to \Theta_R$ is smooth (as it's a principal G-bundle and G is smooth), we may use deformation theory (Proposition 6.5.10) to construct compatible isomorphisms $\alpha_n: \mathcal{P}|_{\mathcal{X}_n} \to \mathcal{P}_\lambda|_{\mathcal{X}_n}$ over the nilpotent thickenings $\mathcal{X}_n$ of $0: B\mathbb{G}_{m,\mathbb{F}} \hookrightarrow \Theta_R$. Coherent Tannaka Duality (6.4.9) coupled with the coherent completeness of Θ_R along $B\mathbb{G}_{m,\mathbb{F}}$ (Theorem 6.4.12) implies that the isomorphisms α_n extend to an isomorphism $\alpha: \mathcal{P} \to \mathcal{P}_\lambda$. Restricting the composition $\mathbb{A}^1_R \times G \to \mathcal{P}_\lambda \xrightarrow{\alpha^{-1}} \mathcal{P} \to X$ to the identity in G yields a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1_R \to X$. One checks that this corresponds to the given map $f: \Theta_R \to [X/G]$ on quotient stacks. Letting $x \in X(R)$ be the image of 1, we see that $f_{x,\lambda}$ is 2-isomorphic to f. $\qquad\square$

Remark 6.6.42. Proposition 6.6.39 can be upgraded to a description of the stack of morphisms from $[\mathbb{A}^1/\mathbb{G}_m]$ to $[X/G]$. Namely, there is a decomposition

$$
\underline{\operatorname{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [X/G]) \cong \coprod_\lambda [X_\lambda^+/P_\lambda]
$$

where λ varies over conjugacy classes of one-parameter subgroups. The loci X_λ^+ are often locally closed subschemes (see Theorem 6.6.13(3)), and $P_\lambda \subset G$ is a parabolic subgroup if G is reductive.

The algebraicity of this stack was discussed already in Remark 6.6.12. See also [HL14, Thm. 1.37].

Stacks of coherent sheaves

Given a projective scheme X, let $\underline{\mathrm{Coh}}(X)$ denote the algebraic stack of coherent sheaves on X (see Exercise 3.1.23).

Proposition 6.6.43. *Let X be a projective scheme over an algebraically closed field $\Bbbk$. For a noetherian $\Bbbk$-algebra R, $\mathrm{Mor}_{\Bbbk}(\Theta_R, \underline{\mathrm{Coh}}(X))$ is equivalent to the groupoid of pairs $(E, E_\bullet)$ where E is a coherent sheaf on X_R flat over R and*

$$E_\bullet : 0 \subset \cdots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \cdots \subset E$$

is a $\mathbb{Z}$-graded filtration such that $E_i = 0$ for $i \ll 0$, $E_i = E$ for $i \gg 0$, and each factor E_i/E_{i-1} is flat over R. A morphism $(E, E_\bullet) \to (E', E'_\bullet)$ is an isomorphism $E \to E'$ of coherent sheaves compatible with the filtration.

Under this correspondence, the morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ sends 1 to E and 0 to the associated graded $\mathrm{gr}\, E_\bullet := \bigoplus_i E_i/E_{i-1}$, and factors through $\underline{\mathrm{Bun}}(X) \subset \underline{\mathrm{Coh}}(X)$ if and only if E and each factor E_i/E_{i-1} is a vector bundle.

Proof. A morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ corresponds to a coherent sheaf $\mathcal{F}$ on $C \times \Theta_R$ flat over Θ_R. By smooth descent, this corresponds to a coherent sheaf on $C \times \mathbb{A}^1_R$ flat over $\mathbb{A}^1_R$ together with a $\mathbb{G}_m$-action. Pushing forward $\mathcal{F}$ along the affine morphism $C \times \Theta_R \to C \times B\mathbb{G}_{m,R}$, we see that $\mathcal{F}$ also corresponds to a graded $\mathcal{O}_{C_R}[x]$-module flat over $R[x]$. Writing $\mathcal{F} = \bigoplus_i E_i$ with each E_i a coherent sheaf on C_R, then multiplication by x induces maps $x \colon E_i \to E_{i+1}$ which are necessarily injective as $\mathcal{F}$ is flat over $R[x]$, hence torsion free. Since $\mathcal{F}$ is finitely generated as a graded $R[x]$-module, there exists finitely many homogeneous generators with bounded degree. Thus $E_i = E$ for $i \gg 0$. On the other hand, considering the $\mathcal{O}_{C_R}[x]$-submodule $E_{\geq d} := \bigoplus_{i \geq n} E_i \subset \mathcal{F}$, the ascending chain $\cdots \subset E_{\geq d} \subset E_{\geq d-1} \subset \cdots \subset \mathcal{F}$ must terminate as $\mathcal{F}$ is noetherian. It follows that $E_i = 0$ for $i \ll 0$. Since $\mathcal{F}$ is flat as an $R[x]$-module, the quotient $\mathcal{F}/x\mathcal{F} = \bigoplus_i E_i/E_{i-1}$ is flat as an R-module and thus each factor E_i/E_{i-1} is flat over R.

Conversely, given E and a filtration $E_\bullet$ satisfying the above conditions, consider the graded $\mathcal{O}_{C_R}[x]$-module $\mathcal{F} := \bigoplus_i E_i$; this is frequently referred to as the 'Rees construction'. We will show by induction that $E_{\geq d} := \bigoplus_{i \geq d} E_i$ is flat and finitely generated over $R[x]$; this implies that $\mathcal{F}$ is flat and finitely generated over $R[x]$ since $E_i = 0$ for $i \ll 0$. For $d \gg 0$, $E_{\geq d}$ is isomorphic to the graded $R[x]$-module $(E \otimes_R R[x])\langle d \rangle$, where $\langle d \rangle$ denotes the grading shift, and is thus flat and finitely generated. For every d, we have an exact sequence

$$0 \to (E_d \otimes_R R[x])\langle d \rangle \to E_{\geq d} \to ((E_{d+1}/E_d) \otimes_R R[x])\langle d+1 \rangle \to 0.$$

The flatness of E and the quotients E_{d+1}/E_d implies the flatness of each E_d. Thus the left and right term above are flat and finitely generated as $R[x]$-modules, and thus so is the middle term. $\qquad\square$

Stack of all curves

Proposition 6.6.44. *Let $\mathcal{M}_g^{\mathrm{all}}$ be the algebraic stack of all proper curves (Theorem 5.4.6) over an algebraically closed field $\Bbbk$. For every $\Bbbk$-algebra R, $\mathrm{Mor}_{\Bbbk}(\Theta_R, \mathcal{M}_g^{\mathrm{all}})$ is the groupoid whose objects are $\mathbb{G}_m$-equivariant families of proper curves $\mathcal{C} \to \mathbb{A}^1_R$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1_R$ with the usual scaling action. Morphisms are $\mathbb{G}_m$-equivariant morphisms.*

Proof. The statement follows from smooth descent applied to $\mathbb{A}^1_R \to \Theta_R$. $\qquad\square$

Remark 6.6.45. A similar description holds for other moduli stacks of varieties. Such $\mathbb{G}_m$-equivariant maps are often called 'test configurations' in the literature.

6.7 Geometric Invariant Theory (GIT)

Geometric Invariant Theory (GIT) was developed by Mumford in [GIT] as a means to construct quotients and moduli spaces in algebraic geometry. For other expository accounts, we recommend [New78], [Kra84], [Dol03], [Muk03] and [Stu08].

6.7.1 Good quotients

Let G be an affine algebraic group over an algebraically closed field $\Bbbk$ acting on an algebraic space U of finite type over $\Bbbk$. In the following cases, we've already established the existence of a *geometric quotient* U/G (Definition 4.3.1), i.e. a G-invariant map $U \to U/G$ inducing a bijection $U(\Bbbk)/G(\Bbbk) \to (U/G)(\Bbbk)$ and universal for G-invariant maps to algebraic spaces; in other words $[U/G] \to U/G$ is a coarse moduli space.

- If G is the group scheme corresponding to a finite abstract group and the action is free (i.e. the action map $G \times U \to U \times U$ is a monomorphism), then $U/G :=$ $[U/G]$ exists as an algebraic space of finite type over $\Bbbk$ (Corollary 3.1.14). This also holds in the non-finite case: if G is an algebraic group and the action is free, then $[U/G]$ is an algebraic stack (Proposition 6.2.10) such that $[U/G] \to [U/G] \times [U/G]$ is a monomorphism and therefore $U/G := [U/G]$ is an algebraic space (Theorem 4.4.10).

- If G is finite and $U = \operatorname{Spec} A$ is affine, then $U/G := \operatorname{Spec} A^G$ is a geometric quotient (Theorem 4.3.6).

- If G is finite and U is projective (resp. quasi-projective, quasi-affine), then the quotient U/G exists as a projective (resp. quasi-projective, quasi-affine) $\Bbbk$-scheme (Exercise 4.2.9).

- If G is finite and U is separated, then U/G exists as a separated algebraic space as a consequence of the Keel–Mori Theorem (4.3.12). This also holds in the non-finite case: if G is an affine algebraic group, the stabilizers of the action are finite and reduced, and the action map $G \times U \to U \times U$ is proper, then $[U/G]$ is a separated Deligne–Mumford stack (Theorem 3.6.4) and the existence of a geometric quotient follows from the Keel–Mori Theorem.

GIT studies the case where G is linearly reductive[5] but not necessarily finite. GIT allows for the possibility of points $u \in U$ where the stabilizer G_u may not be finite and the orbit Gu may not be closed, e.g. $\mathbb{G}_m$ acting on $\mathbb{A}^1$.

In Corollary 6.3.7, we've already considered the affine case of GIT where G is a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on an affine $\Bbbk$-scheme $\operatorname{Spec} A$. In this case, we have a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A & & \\
\downarrow & \searrow^{\widetilde{\pi}} & \\
[\operatorname{Spec} A/G] & \xrightarrow{\ \pi\ } & (\operatorname{Spec} A)/\!/G := \operatorname{Spec} A^G
\end{array}
$$

---

[5]GIT can be developed in the more general setting of actions by *reductive* algebraic groups; see Remark 6.3.11.

where $\pi\colon [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space and $\widetilde{\pi}\colon \operatorname{Spec} A \to \operatorname{Spec} A^G$ is a *good quotient*.

Definition 6.7.1 (Good quotients)**.** Given an action of a linearly reductive algebraic group G over an algebraically closed field $\Bbbk$ on an algebraic space U over $\Bbbk$, a G-invariant map $\widetilde{\pi}\colon U \to X$ is a *good quotient* if

(1) $\mathcal{O}_X \to (\pi_*\mathcal{O}_U)^G$ is an isomorphism (where $(\pi_*\mathcal{O}_U)^G(V) = \Gamma(U_V, \mathcal{O}_{U_V})^G$ for an étale X-scheme V) and

(2) $\widetilde{\pi}$ is affine.[6]

The good quotient of U by G is often denoted as $U /\!/ G = X$.

Remark 6.7.2. The map $\widetilde{\pi}\colon U \to X$ is a good quotient if and only if $\pi\colon [U/G] \to X$ is a good moduli space. To see the equivalence, we may assume that $X = \operatorname{Spec} B$ is affine since both properties are étale local (Lemma 6.3.21(1)). For $(\Rightarrow)$, $U = \operatorname{Spec} A$ is also affine and $B = A^G$, and thus $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space. To see $(\Leftarrow)$, observe that since $U \to [U/G]$ is affine and π_* is exact on quasi-coherent sheaves, the pushforward $\widetilde{\pi}_*$ is exact on quasi-coherent sheaves and thus $\widetilde{\pi}$ is affine by Serre's Criterion for Affineness (4.4.16).

Proposition 6.7.3. *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on an algebraic space U over $\Bbbk$. If $\widetilde{\pi}\colon U \to X$ is a good quotient, then*

(1) $\widetilde{\pi}$ is surjective and the image of a closed G-invariant subscheme is closed. The same holds for the base change $T \to X$ by a morphism from a scheme;

(2) for closed G-invariant closed subschemes $Z_1, Z_2 \subset U$, $\operatorname{im}(Z_1 \cap Z_2) = \operatorname{im}(Z_1) \cap \operatorname{im}(Z_2)$. In particular, for $x_1, x_2 \in X(\Bbbk)$, $\widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)$ if and only if $\overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset$, and $\widetilde{\pi}$ induces a bijection between closed G-orbits in U and $\Bbbk$-points of X;

(3) if U is noetherian, so is X. If U is finite type over $\Bbbk$, then so is X, and for every coherent $\mathcal{O}_U$-module F with a G-action, $(\pi_*F)^G$ is coherent; and

(4) $\widetilde{\pi}$ is universal for G-invariant maps to algebraic spaces.

Proof. This follows from Theorem 6.3.5 as $[U/G] \to X$ is a good moduli space. $\qquad\square$

Remark 6.7.4 (Semistable reduction in GIT)**.** Since $[U/G] \to X$ is universally closed (Theorem 6.3.5(1)), it satisfies the valuative criterion for universal closedness (Theorem 3.8.2). This translates into the following: for every DVR R over $\Bbbk$ with fraction field K and every map $\operatorname{Spec} R \to X$ with a lift $\eta\colon \operatorname{Spec} K \to U$, there exists an extension $R \to R'$ of DVRs, an element $g' \in G(K')$ over the fraction field of R', and a lift in the commutative diagram

$$
\begin{array}{ccc}
 & \xrightarrow{\quad g'\cdot\eta|_{K'}\quad} & \\
\operatorname{Spec} K' & \operatorname{Spec} K \xrightarrow{\ \eta\ } & U \\
\downarrow & \downarrow & \downarrow{\scriptstyle\widetilde{\pi}} \\
\operatorname{Spec} R' \longrightarrow & \operatorname{Spec} R \longrightarrow & X.
\end{array}
$$

In fact, if $R = \Bbbk[\![x]\!]$, it can be arranged that $R \to R'$ is finite; see [Mum77, Lem. 5.3] and [AHLH18, Thm. A.8].

---

[6] A *good quotient* is sometimes defined as an affine G-invariant morphism $\widetilde{\pi}\colon U \to X$ such that $\mathcal{O}_X \xrightarrow{\sim} (\pi_*\mathcal{O}_U)^G$ and properties Proposition 6.7.3(1)-(2) holds, c.f. [Ses72, Def. 1.5].

6.7.2 Projective GIT

Let U be a projective scheme over an algebraically closed field $\Bbbk$ with an action of a linearly reductive algebraic group G. Suppose that there is a G-equivariant embedding $U \hookrightarrow \mathbb{P}(V)$, where V is a finite dimensional G-representation; this is equivalent to giving a very ample line bundle $\mathcal{O}_U(1)$ with a G-action, i.e. a very ample G-linearization (see §B.2.7).

Definition 6.7.5. We define the *semistable* and *stable* locus as

$$U^{\mathrm{ss}} := \{u \in U \mid \text{there exists } f \in \Gamma(U, \mathcal{O}_U(d))^G \text{ with } d > 0 \text{ such that } f(u) \neq 0\},$$

$$U^{\mathrm{s}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in \Gamma(U, \mathcal{O}_U(d))^G \text{ with } d > 0 \text{ such that} \\ \quad -f(u) \neq 0, \\ \quad -\text{the orbit } Gu \subset U_f \text{ is closed, and} \\ \quad -\text{the function } U \to \mathbb{Z},\, x \mapsto \dim G_x \text{ is constant in an open} \\ \qquad \text{neighborhood of } u^7 \end{array} \right\}.$$

A point $u \in U$ is called *semistable* (resp. *stable*) if $u \in U^{\mathrm{ss}}$ (resp. $u \in U^{\mathrm{s}}$).[8] The *nullcone* $\widehat{N} \subset \mathbb{A}(V)$ is by definition the affine cone over $U \setminus U^{\mathrm{ss}}$: it is set of points u in the affine cone $\widehat{U} \subset \mathbb{A}(V)$ such that $f(u) = 0$ for every non-constant G-invariant polynomial on $\mathbb{A}(V)$.

We stress that the stable and semistable loci depend on the choice of G-equivariant embedding $U \hookrightarrow \mathbb{P}(V)$. When U is a normal projective variety, then every line bundle L has a positive tensor power $L^{\otimes n}$ that has a G-linearization by Sumihiro's Theorem on Linearizations (B.2.51). For example, $\mathcal{O}(1)$ on $\mathbb{P}^n$ does not have a PGL_{n+1}-linearization, but $\mathcal{O}(n+1)$ does.

Let $R = \bigoplus_{d \geq 0} \Gamma(U, \mathcal{O}_U(d))$ be the projective coordinate ring. We consider the map

$$\widetilde{\pi} \colon U^{\mathrm{ss}} \to U^{\mathrm{ss}} /\!\!/ G := \operatorname{Proj} R^G. \tag{6.7.6}$$

Note that U^{ss} may be empty in which case $\operatorname{Proj} R^G$ is the empty scheme. If U^{ss} is non-empty, it is precisely the locus where the rational map $\operatorname{Proj} R \dashrightarrow \operatorname{Proj} R^G$ is defined.

Theorem 6.7.7. *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$. Let $U \subset \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimensional G-representation. Then there is a cartesian diagram*

$$
\begin{array}{ccccc}
U^{\mathrm{s}} & \hookrightarrow & U^{\mathrm{ss}} & \hookrightarrow & U \\
\downarrow & & \downarrow{\scriptstyle\widetilde{\pi}} & & \\
U^{\mathrm{s}}/G & \hookrightarrow & U^{\mathrm{ss}} /\!\!/ G & &
\end{array}
$$

where $U^{\mathrm{s}}/G \subset U^{\mathrm{ss}} /\!\!/ G$ is an open subscheme, the map $\widetilde{\pi}$ of (6.7.6) is a good quotient, and the restriction $\widetilde{\pi}|_{U^{\mathrm{s}}} \colon U^{\mathrm{s}} \to U^{\mathrm{s}}/G$ is a geometric quotient. Moreover, $U^{\mathrm{ss}} /\!\!/ G$ is projective with an ample line bundle L such that $\widetilde{\pi}^* L \cong \mathcal{O}_U(N)$ for some N.

If in addition the action of G on U has generically finite stabilizers, then the action of G on U^{s} is proper (i.e. the action map $G \times U^{\mathrm{s}} \to U^{\mathrm{s}} \times U^{\mathrm{s}}$ is proper) or in other words $[U^{\mathrm{s}}/G]$ is separated.

---

[7] Since the function $x \mapsto \dim G_x$ is upper semi-continuous, this condition is automatic if $\dim G_u = 0$.

[8] In the literature, a point $u \in U$ is sometimes called 'unstable' if it is not semistable; we avoid this potentially misleading terminology.

Proof. Since U is projective, $R = \bigoplus_{d \geq 0} \Gamma(U, \mathcal{O}_U(d))$ is finitely generated over $\Bbbk$. Thus by Corollary 6.3.7(3), R^G is also finitely generated over $\Bbbk$ and $U^{\mathrm{ss}} /\!/ G = \operatorname{Proj} R^G$ is projective. As localization commutes with taking invariants, $(R^G)_{(f)} = (R_{(f)})^G$ for every homogeneous element $f \in R^G$ of positive degree. We thus have a cartesian diagram

$$
\begin{array}{ccc}
U_f = \operatorname{Spec} R_{(f)} \lhook\joinrel\longrightarrow & U^{\mathrm{ss}} \lhook\joinrel\longrightarrow & U \\
\downarrow \qquad\qquad & \square \qquad & \downarrow \widetilde{\pi} \\
U_f /\!/ G = (U^{\mathrm{ss}} /\!/ G)_f \lhook\joinrel\longrightarrow & U^{\mathrm{ss}} /\!/ G. &
\end{array}
$$

Since the property of being a good quotient is Zariski local and since the loci $(U^{\mathrm{ss}} /\!/ G)_f$ cover $U^{\mathrm{ss}} /\!/ G$, we conclude that $\widetilde{\pi} \colon U^{\mathrm{ss}} \to U^{\mathrm{ss}} /\!/ G$ is a good quotient. By construction, $U^{\mathrm{ss}} /\!/ G$ is projective and there is an integer N such that $L := \mathcal{O}_{U^{\mathrm{ss}} /\!/ G}(N)$ is an ample line bundle which pulls back to $\mathcal{O}_U(N)|_{U^{\mathrm{ss}}}$.

To show that $U^{\mathrm{s}} \to U^{\mathrm{s}}/G$ is a geometric quotient, it suffices to show that every G-orbit in U^{s} is closed. Since the dimension of the stabilizer increases under orbit degeneration, it in fact suffices to show that the dimension of the stabilizers in U^{s} is locally constant. Every point $u \in U^{\mathrm{s}}$ has by definition an open neighborhood $V \subset U$ such that $\dim G_v = \dim G_u$ for all $v \in V$. Since $\dim G = \dim G_v + \dim Gv$, we see that the dimension of the orbit is constant on V. Finally, if there is a dense open subset of U which has dimension 0 stabilizers, then it follows from the definition of stability that every $u \in U^{\mathrm{s}}$ has a finite (possibly non-reduced) stabilizer. Since $[U^{\mathrm{s}}/G] \to U^{\mathrm{s}}/G$ is also a good moduli space and $[U^{\mathrm{s}}/G]$ has quasi-finite diagonal, it follows from Proposition 6.3.32 that $[U^{\mathrm{s}}/G]$ is separated. $\square$

Example 6.7.8. Given $\mathbb{G}_m$ acting on $\mathbb{P}^2$ via $t \cdot [x : y : z] = [tx : t^{-1}y : z]$, the semistable locus is the complement of $V(xy, z) = \{[0 : 1 : 0], [1 : 0 : 0]\}$ and the good quotient is $(\mathbb{P}^2)^{\mathrm{ss}} \to \operatorname{Proj} \Bbbk[xy, z] = \mathbb{P}^1$. The fiber over $xy = 0$ is the union of three orbits and its complement is the stable locus. Observe that the restriction to $z \neq 0$ is the good quotient $\mathbb{A}^2 \to \mathbb{A}^1$, given by $(x, y) \mapsto xy$, while the fiber over $z = 0$ is the line at infinity with $[0 : 1 : 0]$ and $[1 : 0 : 0]$ removed.

Example 6.7.9. Consider the diagonal action of SL_2 on $X = (\mathbb{P}^1)^4$ and the SL_2-equivariant Segre embedding

$$
(\mathbb{P}^1)^4 \to \mathbb{P}^{15}, \quad ([x_1 : y_1], \ldots, [x_4 : y_4]) \mapsto [x_1 x_2 x_3 x_4, \ldots, y_1 y_2 y_3 y_4].
$$

This corresponds to the SL_2-linearization of $L := \mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)$. The invariant ring $\bigoplus_{d \geq 0} \Gamma(X, L^{\otimes d})$ is generated in degree 1 by the *generalized cross ratios*

$$
\begin{aligned}
I_1 &= (x_1 y_2 - x_2 y_1)(x_3 y_4 - x_4 y_3) \\
I_2 &= (x_1 y_3 - x_3 y_1)(x_2 y_4 - x_4 y_2) \\
I_3 &= (x_1 y_4 - x_4 y_1)(x_2 y_3 - x_3 y_2)
\end{aligned}
$$

with the linear relation $I_1 - I_2 + I_3 = 0$. The invariant ring is $\Bbbk[I_1, I_2]$ and the quotient $X^{\mathrm{ss}} /\!/ \mathrm{SL}_2 = \mathbb{P}^1$. The semistable locus X^{ss} consists of tuples where at most two points are equal, while the stable locus consists of tuples of distinct points.
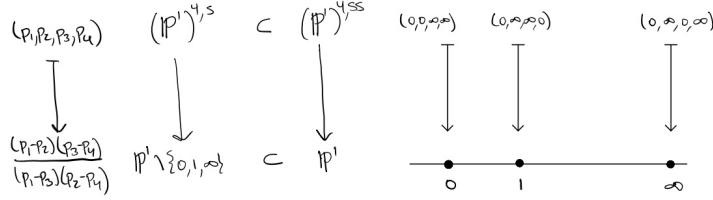
Figure 6.2: 4 unordered points up to projective equivalence

An ordered tuple $(p_1, \ldots, p_4)$ of distinct points is mapped to the *cross ratio*

$$\frac{(p_1 - p_2)(p_3 - p_4)}{(p_1 - p_3)(p_2 - p_4)}.$$

In particular, two stable tuples are projectively equivalent (i.e. in the same SL_2 orbit) if and only if they have the same cross ratio. The complement $X^{\mathrm{ss}} \setminus X^{\mathrm{s}}$ contains 3 closed orbits: the SL_2-orbits of $(0, 0, \infty, \infty)$, $(0, \infty, 0, \infty)$, and $(0, \infty, \infty, 0)$. Tuples such as $(0, 0, 1, \infty)$ or $(1, \infty, 0, 0)$ have non-closed SL_2-orbits in X^{ss} with $\mathrm{SL}_2 \cdot (0, 0, \infty, \infty)$ in the orbit closure. See Example 6.7.19 to see the computations of the stable and semistable locus for the more general case of n ordered points in $\mathbb{P}^1$.

Remark 6.7.10 (Symplectic reduction)**.** There is an interesting connection between GIT and symplectic geometry. Let G be a reductive algebraic group over $\mathbb{C}$ acting on a smooth projective variety $U \subset \mathbb{P}(V)$ where V is an $n + 1$ dimensional G-representation. Let ω be a symplectic form on U, and let $K \subset G$ be a maximal compact subgroup K and $\mathfrak{k}$ its Lie algebra. There is a *moment map*

$$\mu \colon U \to \mathfrak{k}^\vee$$

which is K-equivariant with respect to the coadjoint action on $\mathfrak{k}^\vee$ and satisfies $d\mu(x)(\xi) \cdot a = \omega_x(\xi, v_x)$ for $u \in U$, $\xi \in T_x U$, and $a \in \mathfrak{k}$, where v_x is the vector field on U obtained by the infinitesimal action of K on U. Then

$$u \in U \text{ is semistable} \iff \overline{Gu} \cap \mu^{-1}(0) \neq \emptyset$$

and the inclusion $\mu^{-1}(0) \hookrightarrow U$ induces a homeomorphism $\mu^{-1}(0)/K \to U^{\mathrm{ss}}/\!\!/ G$. See [MFK94, §8].

Exercise 6.7.11 (Affine GIT with respect to a character)**.** Let $U = \operatorname{Spec} A$ be a finite type scheme over an algebraically closed field $\Bbbk$ with an action of an affine algebraic group G specified by a coaction $\sigma \colon A \to \Gamma(G, \mathcal{O}_G) \otimes A$. Let $\chi \colon G \to \mathbb{G}_m = \operatorname{Spec} \Bbbk[t]_t$ be a character. Define the *semistable* and *stable* locus as

$$U^{\mathrm{ss}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in A \text{ such that } f(u) \neq 0 \\ \text{and } \sigma(f) = \chi^*(t)^d \otimes f \text{ for } d > 0 \end{array} \right\}$$

$$U^{\mathrm{s}} := \left\{ u \in U \;\middle|\; \begin{array}{l} \text{there exists } f \in A \text{ such that } f(u) \neq 0, \, \sigma(f) = \chi^*(t)^d \otimes f \\ \text{for } d > 0, \text{ the orbit } Gu \subset U_f \text{ is closed, and the function} \\ x \mapsto \dim G_x \text{ is constant in an open neighborhood of } u \end{array} \right\}.$$

Defining $U^{\mathrm{ss}}/\!\!/ G := \operatorname{Proj} \bigoplus_{d \geq 0} A_d$ where $A_d = \{f \in A \mid \sigma(f) = \chi^*(t)^d \otimes f\}$, show that the conclusion of Theorem 6.7.7 holds except that $U^{\mathrm{ss}}/\!\!/ G$ is projective over $A^G = A_0$ (rather than $\Bbbk$).

For example, under the scaling $\mathbb{G}_m$-action on $U = \mathbb{A}^n$ and with respect to the the identity character $\chi = \mathrm{id}$, then $U^{\mathrm{ss}} = U^{\mathrm{s}} = \mathbb{A}^n \setminus 0$ and the quotient is $\mathbb{P}^{n-1}$.

Exercise 6.7.12 (Projective GIT over an affine)**.** Let U be a projective scheme over a finitely generated $\mathbb{k}$-algebra B, where $\mathbb{k}$ is an algebraically closed field, and let G be an affine algebraic group acting on U. Suppose that there is a G-equivariant embedding $U \hookrightarrow \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle with a G-action. Defining the semistable locus U^{ss} and stable locus U^{s} exactly as in Definition 6.7.5, show that the conclusion of Theorem 6.7.7 holds except that $U^{\mathrm{ss}} /\!\!/ G$ is projective over B^G (rather than $\mathbb{k}$).

6.7.3 Hilbert–Mumford Criterion

The stable and semistable locus can often be effectively computed using the Hilbert–Mumford Criterion. To set up the formulation, let $U \subset \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimensional G-representation, and let $u \in U$ be a $\mathbb{k}$-point with a lift $\widetilde{u} \in \mathbb{A}(V)$. Given a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, we can choose a basis $V \cong \mathbb{k}^n$ such that $\lambda(t) \cdot (v_1, \ldots, v_n) = (t^{d_1} v_1, \ldots, t^{d_n} v_n)$. Define the *Hilbert–Mumford index* as

$$\mu(u, \lambda) := \max_{i, \widetilde{u}_i \neq 0} -d_i. \tag{6.7.13}$$

Equivalently, if $u_0 = \lim_{t \to 0} \lambda(t) \cdot u \in \mathbb{P}(V)$ (which exists since $\mathbb{P}(V)$ is proper), then $\mathbb{G}_m$ fixes u_0 and $\mu(u, \lambda)$ is the opposite of the weight of the induced $\mathbb{G}_m$-action on the line $L_{u_0} \subset V$ classified by u_0.

Remark 6.7.14. From the definition of the Hilbert–Mumford index, we see that

(a) $\lim_{t \to 0} \lambda(t) \cdot \widetilde{u}$ exists if and only if $\mu(u, \lambda) \leq 0$,

(b) $\lim_{t \to 0} t \cdot \widetilde{u} = 0$ if and only if $\mu(u, \lambda) < 0$, and

(c) $\mu(gx, g\lambda g^{-1}) = \mu(x, \lambda)$.

Theorem 6.7.15 (Hilbert–Mumford Criterion)**.** *Let G be a linearly reductive algebraic group over an algebraically closed field $\mathbb{k}$ acting on a G-equivariant closed subscheme $U \subset \mathbb{P}(V)$, where V is a finite dimensional G-representation. Let $u \in \mathbb{P}(V)$ be a $\mathbb{k}$-point with a lift $\widetilde{u} \in \mathbb{A}(V)$. Then*

$$
\begin{aligned}
u \in U^{\mathrm{ss}} &\iff 0 \notin \overline{G\widetilde{u}} \\
&\iff \lim_{t \to 0} \lambda(t) \cdot \widetilde{u} \neq 0 \text{ for all } \lambda \colon \mathbb{G}_m \to G \\
&\iff \mu(u, \lambda) \geq 0 \text{ for all } \lambda \colon \mathbb{G}_m \to G.
\end{aligned}
$$

If in addition the action of G on U has generically finite stabilizers, then

$$
\begin{aligned}
u \in U^{\mathrm{s}} &\iff G\widetilde{u} \subset \mathbb{A}(V) \text{ is closed} \\
&\iff \mu(u, \lambda) > 0 \text{ for all non-trivial } \lambda \colon \mathbb{G}_m \to G.
\end{aligned}
$$

Remark 6.7.16. The criterion that is now referred to as the "Hilbert–Mumford Criterion" was first developed by Hilbert in [Hil93, § 15-16] and then adapted by Mumford in [GIT, p. 53]. It holds more generally when G is reductive.

Proof. For semistability, the first ($\Rightarrow$) implication is clear: if $0 \in \overline{G\widetilde{u}}$, then for every non-constant invariant function, we have that $f(\widetilde{u}) = f(0) = 0$; hence $u \notin U^{\mathrm{ss}}$. For the converse, if $0 \notin \overline{G\widetilde{u}}$, then 0 and $\overline{G\widetilde{u}}$ are disjoint closed G-invariant

subschemes of $\mathbb{A}(V)$. Therefore their images in $\mathbb{A}(V)/\!\!/G = \mathrm{Spec}(\mathrm{Sym}^* V^\vee)^G$ are disjoint (Corollary 6.3.7(2)). We may thus find an invariant function $f \in (\mathrm{Sym}^* V^\vee)^G$ with $f(0) = 0$ and $f(\widetilde{u}) \neq 0$ which we may assume to be homogeneous of positive degree, i.e $f \in \mathrm{Sym}^d V^\vee = \Gamma(\mathbb{P}(V), \mathcal{O}(d))$ for $d > 0$. In the second equivalence, ($\Rightarrow$) is again clear: if there is a λ such that $\lim_{t\to 0} \lambda(t) \cdot \widetilde{u} = 0$, then $0 \in \overline{G\widetilde{u}}$. Conversely, if $0 \in \overline{G\widetilde{u}}$, Theorem 6.6.35 provides a one-parameter subgroup λ such that the limit of u under λ is 0. The third equivalence follows from the definition of the Hilbert–Mumford index (see Remark 6.7.14).

For stability, we may assume that $u \in U^{\mathrm{ss}}$; otherwise 0 is in the closure of $G\widetilde{u}$ and thus $G\widetilde{u}$ is not closed. By definition, there is an invariant section $f \in \Gamma(U, \mathcal{O}(d))^G$ of positive degree not vanishing at u. After possibly increasing d, we can arrange that f extends to an invariant section $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(d))^G$: this follows from the exact sequence $0 \to I_U \to \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}_U \to 0$ using the vanishing of $\mathrm{H}^1(\mathbb{P}(V), I_U(N))$ for $N \gg 0$ and the exactness of taking invariants (i.e. the linear reductivity of G). We may thus view f as a homogeneous polynomial of degree d on $\mathbb{A}(V)$. Letting $\alpha = f(\widetilde{u})$, we have a commutative diagram

$$
\begin{array}{ccc}
G \xrightarrow{\ \Psi_{\widetilde{u}}\ } & V(f - \alpha) \lhook\joinrel\longrightarrow & \mathbb{A}(V) \\
\ \ \searrow{\scriptstyle \Psi_u} & \downarrow & \\
& \mathbb{P}(V)_f &
\end{array}
$$

where $\Psi_u(g) = g \cdot u$ and $\Psi_{\widetilde{u}}(g) = g \cdot \widetilde{u}$. By assumption, we have that $\dim G_u = \dim G_{\widetilde{u}} = 0$ so both stabilizers are finite, thus proper. By Exercise 3.3.15(b), $Gu \subset \mathbb{P}(V)_f$ is closed if and only if Ψ_u is proper, and $G\widetilde{u} \subset \mathbb{A}(V)$ is closed if and only if $\Psi_{\widetilde{u}}$ is proper. On the other hand, $V(f - \alpha) \to \mathbb{P}(V)_f$ is proper, and thus Ψ_u is proper if and only if $\Psi_{\widetilde{u}}$ is. Thus $Gu \subset U_f$ is closed if and only if $G\widetilde{u} \subset \mathbb{A}(V)$ is closed giving the first equivalence. For the second equivalence, if $G\widetilde{u}$ is not closed, then there exists a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot \widetilde{u}$ exists and is not contained in $G\widetilde{u}$. This gives a non-trivial λ with $\mu(u, \lambda) \leq 0$. Conversely, if $G\widetilde{u}$ is closed, then $\Psi_{\widetilde{u}}$ is proper and therefore for every non-trivial λ, the map $\mathbb{G}_m \to \mathbb{A}(V)$, defined by $t \mapsto \lambda(t) \cdot \widetilde{u}$, is also proper. This implies that $\lim_{t\to 0} \lambda(t)\widetilde{u}$ does not exist as otherwise the limit would define an extension $\mathbb{A}^1 \to \mathbb{A}(V)$ of $\mathbb{G}_m \to \mathbb{A}(V)$ and applying the valuative criterion

$$
\begin{array}{ccc}
\mathbb{G}_m & =\!=\!= & \mathbb{G}_m \\
\cap \downarrow & \nearrow & \downarrow \\
\mathbb{A}^1 & \longrightarrow & \mathbb{A}(V)
\end{array}
$$

would yield a contradiction. Since the limit doesn't exist, $\mu(u, \lambda) > 0$. $\qquad \square$

We also provide a stack-theoretic criterion for a point $u \in [U/G]$ to be *semistable*, i.e. u is contained in the open substack $[U^{\mathrm{ss}}/G]$. The data of a G-equivariant embedding $U \subset \mathbb{P}(V)$ is classified by a line bundle L on $[U/G]$ such that the pullback of L under $U \to [U/G]$ is very ample. Since the stable and semistable locus are G-invariant, they define open substacks of $[U/G]$. The data of a point $u \in U(\Bbbk)$ and a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ up to conjugation is classified by a map $f_{u,\lambda} \colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G]$ such that the induced map

$$
B\mathbb{G}_m \xhookrightarrow{\ 0\ } [\mathbb{A}^1/\mathbb{G}_m] \xrightarrow{\ f_{u,\lambda}\ } [U/G] \to BG
$$

corresponds λ. The Hilbert–Mumford index is $\mu(u, \lambda) = -\operatorname{wt}(f^*_{u,\lambda}L)|_{B\mathbb{G}_m}$.

Corollary 6.7.17 (Hilbert–Mumford Criterion). *Let G be a linearly reductive algebraic group over an algebraically closed field $\Bbbk$ acting on a projective $\Bbbk$-scheme U. Let L be a line bundle on $[U/G]$ corresponding to a very ample G-linearization. Then $u \in [U/G]$ is semistable if and only if $\operatorname{wt}((f^*L)|_{B\mathbb{G}_m}) \geq 0$ for all maps*

$$f \colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G], \quad \text{with } f(1) \simeq u.$$

If in addition the action of G on U has generically finite stabilizers, then u is stable if and only if $\operatorname{wt}((f^*L)|_{B\mathbb{G}_m}) > 0$ for all maps $f \colon [\mathbb{A}^1/\mathbb{G}_m] \to [U/G]$ such that $f(1) \simeq u$ and the induced map $\mathbb{G}_m \to G_{f(0)}$ on stabilizers is non-trivial. $\qquad\square$

Exercise 6.7.18 (Affine Hilbert-Mumford Criterion). Let G be a linearly reductive group over an algebraically closed field $\Bbbk$ acting on an affine scheme $U = \operatorname{Spec} A$ of finite type. Let $\chi \colon G \to \mathbb{G}_m$ be a character, and let U^{ss} and U^{s} be the semistable and stable locus with respect to χ as defined in Exercise 6.7.11. For $u \in U(\Bbbk)$, show that

$$u \in U^{\mathrm{ss}} \iff \text{for all one-parameter subgroups } \lambda \colon \mathbb{G}_m \to G \text{ such that}$$
$$\lim_{t \to 0} \lambda(t) \cdot u \text{ exists, } \langle \chi, \lambda \rangle \geq 0$$

where $\langle -, - \rangle$ is the natural pairing of characters and one-parameter subgroups. If in addition the action of G on U has generically finite stabilizers, show that $u \in U^{\mathrm{s}}$ if and only if the same condition holds with strict inequality $\langle \chi, \lambda \rangle > 0$.

Hint: Consider the action of G on $U \times \mathbb{A}^1$ induced by χ defined by $g \cdot (u, z) = (g \cdot u, \chi(g)^{-1} \cdot z)$, and show that $u \notin U^{\mathrm{ss}}$ if and only if $\overline{G \cdot (u, 1)} \cap (U \times \{0\}) \neq \emptyset$. Use the Destabilization Theorem (6.6.35) to show that this is equivalent to the existence of a one-parameter subgroup λ such that

$$\lim_{t \to 0} \lambda(t) \cdot (u, 1) = \lim_{t \to 0} (\lambda(t) \cdot u, t^{-\langle \chi, \lambda \rangle}) \in U \times \{0\}.$$

6.7.4 Examples

Example 6.7.19. Consider the diagonal action of SL_2 on $X = (\mathbb{P}^1)^n$, and consider the SL_2-equivariant Segre embedding $(\mathbb{P}^1)^n \to \mathbb{P}^{2^n-1}$ (or equivalently the SL_2-linearization $\mathcal{O}(1) \boxtimes \cdots \boxtimes \mathcal{O}(1)$). We claim that

$$X^{\mathrm{s}} = \{(p_1, \ldots, p_n) \mid \text{for all } q \in \mathbb{P}^1, \#\{i \mid p_i = q\} < n/2\}$$
$$X^{\mathrm{ss}} = \{(p_1, \ldots, p_n) \mid \text{for all } q \in \mathbb{P}^1, \#\{i \mid p_i = q\} \leq n/2\}.$$

To see this, let $(p_1, \ldots, p_n) \in X(\Bbbk)$ and $\lambda \colon \mathbb{G}_m \to \mathrm{SL}_2$ be a one-parameter subgroup. There exists $g \in \mathrm{SL}_2(\Bbbk)$ such that $g\lambda g^{-1} = \lambda_0^d$ for some $d \in \mathbb{Z}$ where $\lambda_0(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$. We can assume $d \geq 0$ as the case $d < 0$ is handled similarly. Since $\mu(x, \lambda) = \mu(gx, \lambda_0^d) = d\mu(gx, \lambda_0)$, it suffices to compute $\mu(gx, \lambda_0)$. Since $\mu(-, \lambda_0)$ is symmetric with respect to the S_n-action, we can assume that $gx = (0, \ldots, 0, p_k, \ldots, p_n)$ with $p_k, \ldots, p_n \neq 0$. A coordinate of the Segre embedding is of the form $(\prod_{i \in \Sigma} x_i)(\prod_{i \notin \Sigma} y_i)$ for a subset $\Sigma \subset \{1, \ldots, n\}$, and its weight is $n - 2(\#\Sigma)$. The coordinate where gx is nonzero with the largest weight is $y_1 \cdots y_k x_{k+1} \cdots x_n$ with weight $2k - n$. Thus $\mu(gx, \lambda_0) = n - 2k$. Therefore, if no

more than (resp. less than) $n/2$ of the points p_i are the same, then x is semistable (resp. stable) if and only if $n \geq 2k$ (resp. $n > 2k$). Conversely, if more than (resp. at least) $n/2$ of the same, then after translating by an element of SL_2 and using the symmetry of the S_n-action, we can write $u = (0, \ldots, 0, p_k, \ldots, p_n)$ with $k > n/2$ (resp. $k \geq n/2$) and $\lambda_0 = \mathrm{diag}(t^{-1}, t)$ destabilizes u.

If n is odd, then $X^{\mathrm{ss}} = X^{\mathrm{s}}$ and $X^{\mathrm{ss}} \to X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2$ is a geometric quotient. If n is even, the map $X^{\mathrm{ss}} \to X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2$ identifies $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ if there is a subset $\Sigma \subset \{1, \ldots, n\}$ of size $n/2$ such that $p_i = p_j$ and $q_i = q_j$ for all $i, j \in \Sigma$; in this case, the unique closed orbit in fiber is the orbit of the n-tuple with 0's in positions in Σ and ∞'s elsewhere. The complement $X^{\mathrm{ss}} \setminus X^{\mathrm{s}}$ has precisely $\frac{1}{2}\binom{n}{n/2}$ closed orbits.

A modification of the argument yields the same stable and semistable locus for the action of PGL_2 on $(\mathbb{P}^1)^n$ under the PGL_2-linearization $\mathcal{O}(2) \boxtimes \cdots \boxtimes \mathcal{O}(2)$. Since $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$, the quotient $X^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2 = X^{\mathrm{ss}} /\!\!/ \mathrm{PGL}_2$ can be viewed as a compactification of the moduli of n ordered points in $\mathbb{P}^1$ up to projective equivalence.

Exercise 6.7.20.

(a) Under the action of SL_2 on the projectivization $\mathbb{P}(\Gamma(\mathbb{P}^1, \mathcal{O}(n))) \cong \mathbb{P}^n$ of binary forms of degree n, show that the semistable (resp. stable) locus consists of binary forms $f(x, y)$ such that every linear factor has multiplicity less than or equal to (resp. less than) $n/2$.

(b) Under the SL_2-linearization $\mathcal{O}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}(a_n)$ on $(\mathbb{P}^1)^n$ with each $a_i > 0$, show that the semistable (resp. stable) locus consists of tuples $(p_1, \ldots, p_n)$ such that for all $q \in \mathbb{P}^1(\Bbbk)$,

$$\sum_{p_i = q} a_i \leq \left(\sum_{i=1}^n a_i\right)/2$$

(resp. strict inequality holds).

(c) Under the SL_{r+1} action on $(\mathbb{P}^r)^n$ and the SL_{r+1}-linearization $\mathcal{O}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}(a_n)$ with each $a_i > 0$, show that the semistable (resp. stable) locus consists of tuples $(p_1, \ldots, p_n)$ such that for every linear subspace $W \subsetneq \mathbb{P}^r$

$$\sum_{p_i \in W} a_i \leq \frac{\dim W + 1}{r + 1}\left(\sum_{i=1}^n a_i\right)$$

(resp. strict inequality holds).

Exercise 6.7.21 (Cubic curves). Consider the action of SL_3 on the projective space $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(3)))$ of cubic curves in $\mathbb{P}^2$. Show that the semistable locus consists of curves with at worst nodal singularities and that the stable locus consists of smooth curves.

Remark 6.7.22 (Quartic curves). A more involved calculation shows that under the SL_3 action on $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(4)))$, a quartic curve is semistable if and only if it doesn't contain a triple point and is not the union of a cubic curve and an inflection tangent line, and is stable if and only if it has at worst nodal and cuspidal singularities. See also [Mum77, §1.13].

Remark 6.7.23 (Cubic surfaces). Under the action of SL_4 on $\mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}(3)))$, a cubic surface is stable (resp. semistable) if and only if it has finitely many singular points and the singularities are ordinary double points (resp. ordinary double points

or rank two double points whose axes are not contained in the surface). See [Muk03, Thm. 7.14] and [Hil93].

Exercise 6.7.24 (Quiver GIT). A *quiver* $Q = (Q_0, Q_1)$ is a directed graph where Q_0 is a finite set of vertices and Q_1 is a finite set of arrows; there are source and target maps $s, t: Q_1 \to Q_0$. A $\Bbbk$-representation of Q consists of a vector space V_i for every $i \in Q_0$ together with linear maps $L_\alpha: V_i \to V_j$ for every arrow $\alpha: i \to j$. If each V_i is finite dimensional with $d_i = \dim V_i$, we say that $d = (d_i)$ is the dimension vector of V.

Fix $d = (d_i)$ and consider the space

$$R(Q, d) = \prod_{\alpha \in Q_1} \operatorname{Hom}(\Bbbk^{s(\alpha)}, \Bbbk^{t(\alpha)})$$

of representations with dimension vector d. This inherits an action of $\prod_i \operatorname{GL}_{d_i}$ via $(g_i) \cdot (L_\alpha) = (g_{t(\alpha)} L_\alpha g_{s(\alpha)}^{-1})$. The diagonal subgroup $\mathbb{G}_m \subset \prod_i \operatorname{GL}_{d_i}$ consisting of tuples $(t\operatorname{id}_{\Bbbk^{d_i}})$ of scalar matrices for $t \in \mathbb{G}_m$ is normal and acts trivially. Therefore the quotient $G := (\prod_i \operatorname{GL}_{d_i})/\mathbb{G}_m$ also acts on $R(Q, d)$.

For any tuple $a = (a_i)_{i \in Q_0}$ of integers such that $\sum_i a_i d_i = 0$, consider the character

$$\chi_a: G \to \mathbb{G}_m, \quad (g_i) \mapsto \prod_i \det(g_i)^{a_i}.$$

Use the Affine Hilbert–Mumford Criterion (6.7.18) to show that a representation $V \in R(Q, d)$ is semistable (resp. stable) with respect to χ if and only if for every subrepresentation $W \subset V$ (i.e. subspaces $W_i \subset V_i$ such that $L_\alpha(W_{s(\alpha)}) \subset W_{t(\alpha)}$),

$$\sum_i a_i \dim W_i \geq 0$$

(resp. strict inequality holds). See also [Kin94, Prop. 3.1].

Remark 6.7.25 (Cox construction of toric varieties). Let $X = X(\Sigma)$ be a proper toric variety with fan $\Sigma \subset N_{\mathbb{R}}$ and torus T_N, where N is a lattice with dual M. Letting $\Sigma(1)$ denote the rays of the fan, the divisors D_ρ associated to $\rho \in \Sigma(1)$ generate the class group. There is a short exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to \operatorname{Cl}(X) \to 0.$$

The algebraic group $G := \operatorname{Hom}(\operatorname{Cl}(X), \mathbb{G}_m)$ is diagonalizable (hence linearly reductive) and sits in a short exact sequence

$$1 \to G \to \mathbb{G}_m^{\Sigma(1)} \to T_N \to 1$$

obtained by applying $\operatorname{Hom}(-, \mathbb{G}_m)$ to the above sequence. The group G acts naturally on $\mathbb{A}^{\Sigma(1)}$.

For a cone $\sigma \in \Sigma$, let $x^\sigma := \prod_{\rho \in \sigma(1)} x_\rho$. Define the closed subset $Z \subset \mathbb{A}^{\Sigma(1)}$ by the vanishing of the ideal generated by the monomials x^σ as σ varies over maximal dimensional cones; this set can also be described as the union $\bigcup_C V(x_\rho \mid \rho \in C)$ where the union runs over primitive collections $C \subset \Sigma(1)$, i.e. subsets C such that C is not contained in $\sigma(1)$ for any $\sigma \in \Sigma$ and such that for any $C' \subsetneq C$, there exists $\sigma \in \Sigma$ with $C' \subset \sigma(1)$. This locus Z is G-invariant.

The main theorem here is that X is isomorphic to the good quotient $(\mathbb{A}^{\Sigma(1)} \setminus Z)/\!/G$. This is the so-called 'Cox construction of X', and it gives X homogeneous

coordinates in a similar fashion to how $\mathbb{A}^{n+1}$ gives homogeneous coordinates for $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m$. When Σ is a simplicial fan, X is a geometric quotient $(\mathbb{A}^{\Sigma(1)} \setminus Z)/G$. Moreover, the class group $\mathrm{Cl}(X)$ is identified with group of character $\mathbb{X}^*(G)$, and if L is an ample line bundle on X corresponding to a character χ, then $\mathbb{A}^{\Sigma(1)} \setminus Z$ is the semistable locus for the action of G on $\mathbb{A}^{\Sigma(1)}$ with respect to the character χ. See [Cox95] and [CLS11, §5].

Example 6.7.26 (Variation of GIT for $\mathbb{G}_m$-actions). Consider a $\mathbb{G}_m$-action on an affine scheme $X = \mathrm{Spec}\, A$ of finite type over $\Bbbk$. In this example, we will consider how the GIT quotients (with respect to a character in the sense Exercise 6.7.11) vary as we vary the character of $\mathbb{G}_m$. There is a bijection $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ and we write $\chi_d(t) = t^d$ as the character corresponding to $d \in \mathbb{Z}$.

Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be the induced grading. There are three cases for the semistable locus $X^{\mathrm{ss}}_{\chi_d}$ with respect to the character χ_d:

(1) $d = 0$: $X^{\mathrm{ss}}(0) := X^{\mathrm{ss}}_{\chi_0} = X$ and $X^{\mathrm{ss}}_{\chi_0} /\!\!/ \mathbb{G}_m = \mathrm{Spec}\, A_0$.

(2) $d > 0$: $X^{\mathrm{ss}}(+) := X^{\mathrm{ss}}_{\chi_d} = X \setminus V(\sum_{n<0} A_n)$ and $X^{\mathrm{ss}}_{\chi_d} = \mathrm{Proj}\bigoplus_{d \geq 0} A_{nd}$ is independent of d; moreover $X^{\mathrm{ss}}(0)$ is identified with $X^+_{\chi_d}$ with respect to the one-parameter subgroup χ_d (Exercise 6.6.6).

(3) $d < 0$: $X^{\mathrm{ss}}(-) := X^{\mathrm{ss}}_{\chi_d} = X \setminus V(\sum_{n>0} A_n) = X^+_{\chi_d}$ and $X^{\mathrm{ss}}_{\chi_d} = \mathrm{Proj}\bigoplus_{d \geq 0} A_{-nd}$ is independent of d.

There is a commutative diagram

$$
\begin{array}{ccccc}
X^{\mathrm{ss}}(+) & \hookrightarrow & X & \hookleftarrow & X^{\mathrm{ss}}(-) \\
\downarrow & & \downarrow & & \downarrow \\
X^{\mathrm{ss}}(+) /\!\!/ \mathbb{G}_m & \longrightarrow & X /\!\!/ \mathbb{G}_m & \longleftarrow & X^{\mathrm{ss}}(-) /\!\!/ \mathbb{G}_m
\end{array}
$$

where the vertical maps are good quotients, the top maps are open immersions, and the bottom maps are projective. The Affine Hilbert–Mumford Criterion (6.7.18) implies that there are identifications of the stable loci with respect to χ_0, χ_1, and χ_{-1}: $X^{\mathrm{s}}(0) = X \setminus (X^{\mathrm{ss}}(+) \cap X^{\mathrm{ss}}(-))$, $X^{\mathrm{s}}(+) = X^{\mathrm{ss}}(+) = X \setminus X^{\mathrm{ss}}(-)$, and $X^{\mathrm{s}}(-) = X^{\mathrm{ss}}(-) = X \setminus X^{\mathrm{ss}}(+)$. Therefore, we see that if both $X^{\mathrm{ss}}(+)$ and $X^{\mathrm{ss}}(-)$ are nonempty, then $X^{\mathrm{ss}}(+)/\mathbb{G}_m \to X /\!\!/ \mathbb{G}_m$ and $X^{\mathrm{ss}}(-)/\mathbb{G}_m \to X /\!\!/ \mathbb{G}_m$ are isomorphisms over $X^{\mathrm{s}}(0)/\mathbb{G}_m$, and in particular birational. We also see that if the complements of $X^{\mathrm{ss}}(+)$ and $X^{\mathrm{ss}}(-)$ in X each have codimension at least two, then the birational map $X^{\mathrm{ss}}(+) /\!\!/ \mathbb{G}_m \dashrightarrow X^{\mathrm{ss}}(-) /\!\!/ \mathbb{G}_m$ is an isomorphism in codimension 2 such that the divisor $\mathcal{O}(1)$ (which is relatively ample over $X /\!\!/ \mathbb{G}_m$) pushes forward to a divisor on $X^{\mathrm{ss}}(-) /\!\!/ \mathbb{G}_m$ whose dual is relatively ample, i.e. $X^{\mathrm{ss}}(+) /\!\!/ \mathbb{G}_m \dashrightarrow X^{\mathrm{ss}}(-) /\!\!/ \mathbb{G}_m$ is a flip with respect to $\mathcal{O}(1)$.

Remark 6.7.27 (Variation of GIT). Extending the previous example, consider a projective variety X over $\Bbbk$ with an action of a linearly reductive group G. Two line bundles (resp. G-linearizations) L_1 and L_2 on X are algebraically equivalent (resp. G-algebraically equivalent) if there is a connected variety T, points $t_1, t_2 \in T(\Bbbk)$, and a line bundle (resp. G-linearization) $\mathcal{L}$ on $X \times T$ such that $L_i = \mathcal{L}|_{X \times \{t_i\}}$. The Neron–Severi group $\mathrm{NS}(X)$ (resp. G-equivariant Neron–Severi group $\mathrm{NS}^G(X)$) of line bundles (resp. G-linearizations) on X up to $(G\text{-})$algebraic equivalence is finitely generated. The kernel of $\mathrm{NS}^G(X)_{\mathbb{R}} \to \mathrm{NS}(X)$ is identified with the rational character group $\mathbb{X}^*(G)_{\mathbb{R}}$. We let $\mathrm{Eff}^G(X) \subset \mathrm{NS}^G(X)_{\mathbb{R}}$ be the cone of G-effective linearizations, i.e. G-linearizations L such that there is a nonzero section of $L^{\otimes d}$) for

some $d > 0$ or in other words such that $X_L^{\mathrm{ss}} \neq \emptyset$. We also let $\mathrm{Amp}^G(X) \subset \mathrm{NS}^G(X)_{\mathbb{R}}$ be the cone of ample G-linearizations.

The main results of variation of GIT can be formulated as follows. The semistable locus X_L^{ss} only depends on the G-algebraic equivalence class of L. There is a polyhedral decomposition of the cone $\mathrm{Amp}^G(X) \cap \mathrm{Eff}^G(X)$ defined by codimension 1 walls such that the semistable locus is constant in any open chamber. If L_0 is on a wall while L_+ and L_- are on opposite adjacent chambers, then there is a commutative diagram

$$
\begin{array}{ccccc}
X_{L_+}^{\mathrm{ss}} & \hookrightarrow & X_{L_0}^{\mathrm{ss}} & \hookleftarrow & X_{L_-}^{\mathrm{ss}} \\
\downarrow & & \downarrow & & \downarrow \\
X_{L_+}^{\mathrm{ss}} /\!\!/ G & \longrightarrow & X_{L_0}^{\mathrm{ss}} /\!\!/ G & \longleftarrow & X_{L_-}^{\mathrm{ss}} /\!\!/ G
\end{array}
$$

where the vertical maps are good quotients, the top maps are open immersions, and the bottom maps are projective. If $X_{L_+}^{\mathrm{ss}}$ and $X_{L_-}^{\mathrm{ss}}$ are non-empty, the bottom maps are birational; when the bottom maps are isomorphisms in codimension 2, then $X_{L_+}^{\mathrm{ss}} /\!\!/ G \dashrightarrow X_{L_-}^{\mathrm{ss}} /\!\!/ G$ is a flip with respect to the line bundle $\mathcal{O}(1)$ on $X_{L_+}^{\mathrm{ss}} /\!\!/ G$, which is relatively ample over $X_{L_0}^{\mathrm{ss}} /\!\!/ G$ and which pulls back to $L_+|_{X_{L_+}^{\mathrm{ss}}}$.

See [Tha96] and [DH98].

Remark 6.7.28 (Mori Dream Spaces). There is an interesting connection between the Mori program and variation of GIT. A normal $\mathbb{Q}$-factorial projective variety X is a *Mori dream space* if (1) $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}}$, (2) the cone $\mathrm{Nef}(X)$ of nef line bundles is the affine hull of finitely many semiample line bundles, and (3) there are finitely many birational maps $f_i \colon X \dashrightarrow X_i$, which are isomorphisms in codimension 1, to a $\mathbb{Q}$-factorial normal projective variety X_i such that the movable cone $\mathrm{Mov}(X)$ is the union of $f_i^{-1}(\mathrm{Amp}(X_i)_{\mathbb{Q}})$; a line bundle is *movable* if its stable base locus has codimension at least 2. In other words, X is a Mori dream space if $\mathrm{Mov}(X)$ has a finite wall and chamber decomposition such that the projective variety determined by the line bundle is constant within an open chamber.

Equivalently, X is a Mori dream space if $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}}$ and the Cox ring

$$
\mathrm{Cox}(X) := \bigoplus_{(d_1, \ldots, d_n) \in \mathbb{N}^n} \Gamma(X, L_1^{d_1} \otimes \cdots \otimes L_n^{d_n})
$$

is finitely generated, where $L_1, \ldots, L_n$ is a basis for $\mathrm{Pic}(X)_{\mathbb{Q}}$ such that their affine hull contains $\mathrm{Eff}(X)_{\mathbb{Q}}$. If X is a Mori dream space, then X along with each birational model X_i is a GIT quotient of the semistable locus of $\mathrm{Spec}(\mathrm{Cox}(X))$ by the torus $\mathbb{G}_m^n$ with respect to a character. Moreover, there is an identification of the Mori chambers of $\mathrm{Mov}(X)$ with the variation of GIT chambers for the action of $\mathbb{G}_m^n$ on $\mathrm{Spec}(\mathrm{Cox}(X))$. See also [HK00].

Example 6.7.29 (Partial desingularization). If U is a smooth variety and $U \to X$ is a geometric quotient by a linearly reductive group, then X necessarily has finite quotient singularities; this is a consequence of the Local Structure Theorem (4.3.16). On the other hand, if $U \to X$ is a good quotient, then X can have worse singularities. Nevertheless, there is a canonical procedure to partially resolve the singularities of X so that they become finite quotient singularities.

Suppose that there is an open subset $X' \subset X$ such that $\pi_0(X') \to X'$ is a geometric quotient; this happens for example if $U = V^{\mathrm{ss}}$ is the semistable locus with

respect to the action of G on a projective variety $V \subset \mathbb{P}^N$ and the stable locus V^s is nonempty. Then there is a commutative diagram

$$
\begin{array}{ccccccc}
U_n & \longrightarrow & U_{n-1} & \longrightarrow & \cdots & \longrightarrow & U = U_0 \\
\downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n-1}} & & & & \downarrow{\scriptstyle \pi_0} \\
X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X = X_0
\end{array}
$$

such that:

- Each U_{i+1} is a G-invariant open subscheme of the blowup $\mathrm{Bl}_Z U_i$, where Z is a G-invariant smooth closed subscheme whose stabilizers are of maximal dimension, and $U_{i+1} \subset \mathrm{Bl}_Z U_i$ is the complement of the strict transform of $\pi_i^{-1}(\pi_i(Z))$. If $U = V^{ss}$ is the semistable locus of a projective variety with respect to a G-linearization L, then U_{i+1} is the semistable locus with respect to $(q^* L)^{\otimes n} \otimes \mathcal{O}(-E)$ for $n \gg 0$, where $q \colon \mathrm{Bl}_Z U_i \to U_i$ and E denotes the exceptional divisor.

- The maps $X_{i+1} \to X_i$ are projective birational.

- The maps $\pi_i \colon U_i \to X_i$ are good quotients by G, and the map $\pi_n \colon U_n \to X_n$ is a geometric quotient. In particular, X_n has finite quotient singularities.

For a simple example of this procedure in action, consider the $\mathbb{G}_m$-action on $\mathbb{A}^2$ with weights 1 and -1. In this case, the quotient $\mathbb{A}^2 /\!\!/ \mathbb{G}_m \cong \mathbb{A}^1$ is smooth but not a geometric quotient. The procedure tells us to take the blowup $\mathrm{Bl}_0 \mathbb{A}^2$ at the origin and the complement U_1 of the strict transform of $V(xy)$. Then $\mathbb{G}_m$-acts with finite stabilizers on U_1 and $U_1 \to \mathbb{A}^2$ is $\mathbb{G}_m$-invariant birational (but neither proper nor surjective) map inducing an isomorphism $U_1/\mathbb{G}_m \to \mathbb{A}^2 /\!\!/ \mathbb{G}_m$ on quotients.

See [Kir85], [Rei89], and [ER21].

6.7.5 Kempf's Optimal Destabilization Theorem

Given an algebraic group G over an algebraically closed field $\Bbbk$, we define $\mathbb{X}_*(G)$ as the set of one-parameter subgroups $\mathbb{G}_m \to G$. Recall that for a torus $T \cong \mathbb{G}_m^n$, $\mathbb{X}_*(T) \cong \mathbb{Z}^n$ (see Example B.2.45).

Definition 6.7.30. A *length* $\|-\|$ *on* $\mathbb{X}_*(G)$ is a non-negative real-valued function on $\mathbb{X}_*(G)$ which is conjugation invariant, i.e. $\left\| g \lambda g^{-1} \right\| = \|\lambda\|$ for $\lambda \in \mathbb{X}_*(G)$ and $g \in G(\Bbbk)$, and such that for every maximal torus $T \subset G$, there is a positive definite integral-valued bilinear form $(-,-)$ on $\mathbb{X}_*(T)$ with $(\lambda, \lambda) = \|\lambda\|^2$ for $\lambda \in \mathbb{X}_*(T)$.

Example 6.7.31. If $G = \mathrm{GL}_n$, then any one-parameter subgroup λ is conjugate to a one-parameter subgroup of the form $t \mapsto \mathrm{diag}(t^{d_1}, \ldots, t^{d_n})$ and we can define $\|\lambda\| = \sqrt{d_1^2 + \cdots + d_n^2}$.

Example 6.7.32. For every reductive algebraic group G, there is a length $\|-\|$ on $\mathbb{X}_*(G)$. To see this, let $T \subset G$ be a maximal torus and choose a positive definite integral-valued bilinear form $(-,-)$ on $\mathbb{X}_*(T)$, which is invariant under the conjugation action of the Weyl group $W := N(T)/T$. There is a bijection $\mathbb{X}_*(G)/G \cong \mathbb{X}_*(T)/W$ between conjugacy classes of $\mathbb{X}_*(G)$ under G and conjugacy classes of $\mathbb{X}_*(T)$ under W. In other words, for every $\lambda \in \mathbb{X}_*(G)$ there exists $g \in G(\Bbbk)$ such that $g\lambda g^{-1} \in \mathbb{X}_*(T)$, and moreover for any other element $g' \in G(\Bbbk)$ such that $g'\lambda g'^{-1} \in \mathbb{X}_*(T)$, then $g\lambda g^{-1}$ and $g'\lambda g'^{-1}$ are conjugate under W. It follows that $\|\lambda\|^2 := (g\lambda g^{-1}, g\lambda g^{-1})$ is well-defined.

Let $X = \operatorname{Spec} A$ be an affine $\mathbb{k}$-scheme with the action of G and let $x_0 \in X(\mathbb{k})$ be a point with closed orbit. For every point $x \in X(\mathbb{k})$ with $Gx_0 \subset \overline{Gx}$ and a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot x$ exists, we define the *Hilbert–Mumford index of x with respect to λ* as

$$\mu(x, \lambda) = -\deg f_{x,\lambda}^{-1}(Gx_0). \tag{6.7.33}$$

where $f_{x,\lambda} \colon \mathbb{A}^1 \to X$ is the map extending $\mathbb{G}_m \to X$, $t \mapsto \lambda(t) \cdot x$. Note that if $\lim_{t \to 0} \lambda(t) \cdot x \notin Gx_0$, then $\mu(x, \lambda) = 0$.

Since $\mu(x, \lambda^n) = n \cdot \mu(x, \lambda)$, it natural to consider the *normalized Hilbert–Mumford index*

$$\frac{\mu(x, \lambda)}{\|\lambda\|}$$

as a measure of how quickly $\lambda(t) \cdot x$ approaches the closed orbit Gx_0. The more negative the normalized Hilbert–Mumford index is, the faster $\lambda(t) \cdot x$ approaches Gx_0. Kempf proved that there is a one-parameter subgroup minimizing this index and that it is unique up to conjugation.

Theorem 6.7.34 (Kempf's Optimal Destabilization Theorem—affine version)**.** *Let G be a reductive algebraic group over an algebraically closed field $\mathbb{k}$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\mathbb{k}$ with an action of G. Let $x_0 \in X(\mathbb{k})$ be a point with a closed orbit. For every point $x \in X(\mathbb{k})$ with $Gx_0 \subset \overline{Gx}$, there exists a one-parameter subgroup $\lambda_0 \colon \mathbb{G}_m \to G$ such that $\mu(x, \lambda_0)/\|\lambda_0\|$ achieves a minimal value $M(x)$ over all $\lambda \in \mathbb{X}_*(G)$ such that $\lim_{t \to 0} \lambda(t) \cdot x \in Gx_0$.*

If λ_0' is another such one-parameter subgroup, then $P(\lambda_0) = P(\lambda_0')$ and $\lambda_0' = u\lambda_0 u^{-1}$ for a unique element $x \in X(\lambda_0)$. Every maximal torus $T \subset P_{\lambda_0}$ contains a unique indivisible element achieving this minimum value.

Remark 6.7.35. The subgroup $P_{\lambda_0} = \{g \in G \mid \lim_{t \to 0} \lambda_0(t) g \lambda_0(t)^{-1} \text{ exists}\}$ is the parabolic associated to λ_0 and $U_{\lambda_0} = \{g \in G \mid \lim_{t \to 0} \lambda_0(t) g \lambda_0(t)^{-1} = 1\}$ is the unipotent radical of P_{λ_0}; see §B.2.6.

In the projective case where there is a G-equivariant embedding $X \hookrightarrow \mathbb{P}(V)$, we have already defined the Hilbert–Mumford index $\mu(x, \lambda)$ in (6.7.13) as follows: choosing a basis of V such that $\mathbb{G}_m$ acts on $\mathbb{A}(V) = \mathbb{A}^n$ with weights $d_1, \ldots, d_n$ and a lift $\widehat{x} = (u_1, \ldots, u_n) \in \mathbb{A}(V)$ of x, then $-\mu(x, \lambda)$ is defined as the smallest d_i with $u_i \neq 0$. If $\lim_{t \to 0} \lambda(t) \cdot \widehat{x}$ exists, then this agrees with the definition in (6.7.33). To see this, observe that the extension $f_{\widehat{x}, \lambda} \colon \mathbb{A}^1 \to \mathbb{A}^n$ of the map $t \mapsto \lambda(t) \cdot \widehat{x}$ is the map $t \mapsto (t^{d_i} u_i)$ and $f_{\widehat{x}, \lambda}^{-1}(0) = \operatorname{Spec} \mathbb{k}[t]/(t^d)$ where d is the smallest d_i with $u_i \neq 0$.

The projective version below follows from applying the affine version (Theorem 6.7.34) to a lift $\widehat{x} \in \mathbb{A}(V)$ of a non-semistable point $x \in \mathbb{P}(V)$. In this case, the closed orbit in $\overline{G\widehat{x}}$ is the fixed point 0. The following theorem also holds for reductive groups, but we restrict to linearly reductive groups as we've only discussed semistability in that context.

Theorem 6.7.36 (Kempf's Optimal Destabilization Theorem—projective version)**.** *Let G be a linearly reductive algebraic group over an algebraically closed field $\mathbb{k}$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X \subset \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimensional G-representation. For every non-semistable point $x \in X(\mathbb{k})$, there exists a one-parameter subgroup $\lambda_0 \colon \mathbb{G}_m \to G$ such that $\mu(x, \lambda_0)/\|\lambda_0\|$ achieves a minimal value $M(x)$ over all $\lambda \in \mathbb{X}_*(G)$.*

If λ_0' is another such one-parameter subgroup, then $P_{\lambda_0} = P_{\lambda_0'}$ and $\lambda_0' = g\lambda_0 g^{-1}$ for a unique element $g \in X(\lambda_0)$. Every maximal torus $T \subset P_{\lambda_0}$ contains a unique indivisible element achieving this minimum value. □

Definition 6.7.37. We call any λ_0 satisfying Theorem 6.7.34 or Theorem 6.7.36 a *Kempf optimal destabilizing one-parameter subgroup for x*, and we call $M(x)$ the *optimal normalized Hilbert–Mumford index* for x.

Proof of Theorem 6.7.34. The proof is simpler when $x_0 \in \overline{Gx}$ is a fixed point, such as in the projective version when the closed orbit is $0 \in \mathbb{A}(V)$; the reader is encouraged to keep this case in mind. By Proposition B.2.43, we may choose finite dimensional G-representations V and W along with G-equivariant maps

$$
\begin{array}{ccc}
& & \mathbb{A}(V) \\
& \overset{i}{\nearrow} & \\
X & & \\
& \underset{f}{\searrow} & \\
& & \mathbb{A}(W),
\end{array}
\tag{6.7.38}
$$

where $i\colon X \hookrightarrow \mathbb{A}(V)$ is a closed immersion with $i(x_0) = 0$ and $f\colon X \to \mathbb{A}(W)$ is a morphism with $f^{-1}(0) = Gx_0$. When x_0 is a fixed point, we can take $f = i$ in (6.7.38).

A one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ induces $\mathbb{G}_m$-actions on V and W, and thus gradings $V = \bigoplus_{d \in \mathbb{Z}} V_d$ and $W = \bigoplus_{d \in \mathbb{Z}} W_d$. We define

$$m(i(x), \lambda) = \min\{d \mid \text{the projection of } i(x) \text{ to } V_d \text{ is nonzero}\},$$
$$m(f(x), \lambda) = \min\{d \mid \text{the projection of } f(x) \text{ to } W_d \text{ is nonzero}\}.$$

For any $g \in G$, we have the identities $m(i(v), \lambda) = m(i(g{\cdot}v), g\lambda g^{-1})$ and $m(f(v), \lambda) = m(f(g \cdot v), g\lambda g^{-1})$.

It is easy to see that if $\lim_{t\to 0} \lambda(t) \cdot x$ exists, then $\mu(x, \lambda) = -m(f(x), \lambda)$, and that

$$\lim_{t\to 0} \lambda(t) \cdot x \text{ exists} \iff m(i(x), \lambda) \geq 0,$$
$$\lim_{t\to 0} \lambda(t) \cdot x \in Gx_0 \iff m(i(x), \lambda) \geq 0 \text{ and } m(f(x), \lambda) > 0.$$

By the Destabilization Theorem (6.6.35), there exists $\lambda_x \in \mathbb{X}_*(G)$ such that $m(i(x), \lambda_x) \geq 0$ and $m(f(x), \lambda_x) > 0$.

Case of a torus: Let $T \subset G$ be a maximal torus containing λ_x. We can decompose $V = \bigoplus_{\chi \in \mathbb{X}^*(T)} V_\chi$ as a T-representation where $\mathbb{X}^*(T)$ denotes the set of characters of T. We define *the state of $i(x) \in V$ with respect to T* to be the set

$$\text{State}_T(i(x)) = \{\chi \in \mathbb{X}^*(T) \mid \text{the projection of } i(x) \text{ to } V_\chi \text{ is nonzero}\}.$$

Likewise, we have the state $\text{State}_T(f(x)) \subset \mathbb{X}^*(T)$ of $f(x) \in W$ with respect to T.

Let $\langle -, - \rangle$ be the natural pairing $\mathbb{X}^*(T) \times \mathbb{X}_*(T) \to \mathbb{Z}$. For a one-parameter subgroup $\lambda \in \mathbb{X}_*(T)$, we have identifications

$$m(i(x), \lambda) = \min_{\chi \in \text{State}_T(i(x))} \langle \chi, \lambda \rangle \quad \text{and} \quad m(f(x), \lambda) = \min_{\chi \in \text{State}_T(f(x))} \langle \chi, \lambda \rangle.$$

We claim that the function $\lambda \mapsto m(f(x), \lambda)/\|\lambda\|$ achieves a maximum value on the set $\{\lambda \neq 0 \in \mathbb{X}_*(T) \mid m(i(x), \lambda_T) \geq 0\}$ at a one-parameter subgroup λ_T, and that

any other one-parameter subgroup achieving this minimum is a positive multiple of λ_T. This is precisely the conclusion of Lemma 6.7.39 below applied to the lattice $L = \mathbb{X}_*(T) \cong \mathbb{Z}^r$ and the subsets of $\mathbb{X}^*(T) \cong \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ given by $F := \mathrm{State}_T(i(x))$ and $G := \mathrm{State}_T(f(x))$.

General case: If $T \subset G$ is a maximal torus and $g \in G(\Bbbk)$, then there is an identification $\mathbb{X}^*(T) \cong \mathbb{X}^*(gTg^{-1})$ given by identifying $\chi \in \mathbb{X}^*(T)$ with the character $gTg^{-1} \to \mathbb{G}_m$ defined by $gtg^{-1} \mapsto \chi(t)$. Under this identification, $\mathrm{State}_T(i(x)) = \mathrm{State}_{gTg^{-1}}(i(gx))$. Given a one-parameter subgroup $\lambda \in \mathbb{X}_*(G)$, we've seen that $m(f(x), \lambda) = m(f(gx), g\lambda g^{-1})$ for $g \in G(\Bbbk)$. We claim that in fact $m(f(x), \lambda) = m(f(x), p\lambda p^{-1})$ for $p \in P_\lambda$. By symmetry, it suffices to show that $m(f(x), \lambda) \le m(f(x), p\lambda p^{-1})$. Interpreting $-m(f(x), \lambda)$ as the smallest integer d such that $\lim_{t\to 0} t^d \lambda(t) \cdot f(x) \in \mathbb{A}(W)$ exists, we need to show that $\lim_{t\to 0} t^d p\lambda(t)p^{-1} \cdot f(x) \in \mathbb{A}(W)$ exists. This follows from the computation

$$\lim_{t\to 0} \left(t^d p\lambda(t)p^{-1} \cdot f(x) \right) = \lim_{t\to 0} \left(p \cdot \left(\lambda(t)p^{-1}\lambda(t)^{-1} \right) \cdot \left(t^d \lambda(t)f(x) \right) \right)$$
$$= p \cdot \left(\lim_{t\to 0} \lambda(t)p^{-1}\lambda(t)^{-1} \right) \cdot \left(\lim_{t\to 0} t^d \lambda(t)f(x) \right).$$

We now show that the function $\lambda \mapsto m(f(x), \lambda)/\|\lambda\|$ achieves a minimum value on

$$\Sigma := \{\lambda \in \mathbb{X}_*(G) \,|\, m(i(x), \lambda) \ge 0\}.$$

If T is a maximal torus, by the torus case, we know that for every $g \in G(\Bbbk)$ there is a minimum value on each *non-empty* set $\mathbb{X}_*(gTg^{-1}) \cap \Sigma$, and that the minimum is determined by the subsets of $\mathbb{X}_*(T)$ given by $\mathrm{State}_{gTg^{-1}}(i(x)) \cong \mathrm{State}_T(i(g^{-1}u))$ and $\mathrm{State}_{gTg^{-1}}(f(x)) \cong \mathrm{State}_T(f(g^{-1}u))$. Since these subsets are contained in the finite set of characters χ with $V_\chi \ne 0$ (resp. $W_\chi \ne 0$), there are only finitely many minimum values as g ranges over $G(\Bbbk)$. Since the image of any $\lambda \in \mathbb{X}_*(G)$ is contained in gTg^{-1} for some $g \in G(\Bbbk)$, it follows that there is a global minimum value achieved by a one-parameter subgroup $\lambda_0 \in \Sigma$. We may assume that λ_0 is indivisible, i.e. λ_0 cannot be written as a positive multiple of another one-parameter subgroup.

To establish the uniqueness, we choose a maximal torus $T \subset G$ containing λ_0. By the torus case, $\lambda_0 \in \mathbb{X}_*(T) \cap \Sigma$ is the unique indivisible one-parameter subgroup achieving the minimal value. For $p \in P_{\lambda_0}$, the conjugate one-parameter subgroup $p\lambda_0 p^{-1}$ also achieves this minimal value. Since any other maximal torus $T' \subset P_{\lambda_0}$ is pTp^{-1} for some $p \in P_{\lambda_0}$, we see that $\mathbb{X}_*(T') \cap \Sigma$ also contains a unique indivisible element achieving the minimum value. Finally, let $\lambda_1 \in \mathbb{X}_*(G)$ be another indivisible element achieving the minimum value. The intersection $P_{\lambda_0} \cap P_{\lambda_1}$ contains a maximal torus T of G (Proposition B.2.49(d)), and we can write $\lambda_T = p_0 \lambda_0 p_0^{-1} = p_1 \lambda_1 p_1^{-1}$ for $p_0, p_1 \in P_{\lambda_T}$. It follows that $P_{\lambda_0} = P_{\lambda_T} = P_{\lambda_1}$, and that λ_0 and λ_1 are conjugate by a unique element element of U_{λ_T} (Proposition B.2.49(c)).

See also [Kem78, Thm. 3.4]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The argument above used the following lemma in convex geometry.

Lemma 6.7.39. *Let Λ be a finite dimensional lattice, and let F and G be non-empty finite subsets of $\Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. Assume that $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ has a positive definite inner product which is integral valued on Λ. Define*

$$f_{\min} \colon \Lambda_{\mathbb{R}} \to \mathbb{R}, \; \lambda \mapsto \min_{f\in F} f(\lambda) \quad and \quad g_{\min} \colon \Lambda_{\mathbb{R}} \to \mathbb{R}, \; \lambda \mapsto \min_{g\in G} g(\lambda).$$

Suppose that there exists $\lambda \in \Lambda_{\mathbb{R}}$ such that $f_{\min}(\lambda) \geq 0$ and $g_{\min}(\lambda) > 0$. Then the function

$$C_F := \{\lambda \neq 0 \in \Lambda_{\mathbb{R}} \mid f_{\min}(\lambda) \geq 0\} \to \mathbb{R},$$
$$\lambda \mapsto g_{\min}(\lambda)/\|\lambda\|$$

obtains a maximum value M, and there exists a unique element $\lambda_0 \in C_F \cap \Lambda$ such that $M = g_{\min}(\lambda_0)/\|\lambda_0\|$ and such that any other element $\lambda \in C_F \cap \Lambda$ with $M = g_{\min}(\lambda)/\|\lambda\|$ is an integral multiple of λ_0.

Proof. The set $\{\lambda \in C_F \mid g_{\min}(\lambda) \geq 1\}$ is closed and convex, and therefore contains a unique point λ' closest to the origin. Since $g_{\min}(\alpha\lambda') = \alpha g_{\min}(\lambda')$ for $\alpha \in \mathbb{R}$, we must have that $g_{\min}(\lambda') = 1$ and that $\lambda' \in C_F$ is the unique point with $g_{\min}(\lambda') = 1$ and $g_{\min}(\lambda')/\|\lambda'\| = M$.

We now argue that the ray spanned by λ' contains an integral point. If λ' is in the interior of $\{\lambda \in C_F \mid g_{\min}(\lambda) \geq 1\}$, i.e. $f(\lambda') > 0$ for all $f \in F$ and there is a unique $g \in G$ with $g(\lambda') = 1$, then λ' is the closed point to the origin on the affine plane defined by $g = 1$. We claim that $\lambda' = g^*/\langle g^*, g^* \rangle$ where $g^* \in \Lambda_{\mathbb{R}}$ is the unique point such that $\langle g^*, \lambda \rangle = g(\lambda)$ for all $\lambda \in \Lambda_{\mathbb{R}}$. Indeed, the point λ' is contained in the plane $g = 1$, and for any other point λ on this plane, we have that $\langle \lambda', \lambda \rangle = 1/\langle g^*, g^* \rangle = \langle \lambda', \lambda' \rangle$ and the Cauchy–Schwarz inequality implies that $\langle \lambda', \lambda' \rangle^2 = \langle \lambda', \lambda \rangle^2 \leq \langle \lambda', \lambda' \rangle \langle \lambda, \lambda \rangle$ so that $\langle \lambda', \lambda' \rangle \leq \langle \lambda, \lambda \rangle$. Since the inner product and g take integral values, $g^* \in \Lambda$. We then take λ_0 to be the unique indivisible element in the ray spanned by g^*.

To reduce to this case, let $f_1, \ldots, f_t \in F$ be the functions satisfying $f_i(\lambda') = 0$, and let $g_1, \ldots, g_s \in G$ be the functions satisfying $g_i(\lambda') = g_{\min}(\lambda')$. Since each f_i and g_j take integral values, we may restrict to the subspace

$$W := \left\{ \lambda \in \Lambda_{\mathbb{R}} \;\middle|\; \begin{array}{l} f_1(\lambda) = \cdots = f_t(\lambda) = 0 \\ g_1(\lambda) = \cdots = g_s(\lambda) \end{array} \right\},$$

and the lattice $W \cap \Lambda$. Then λ' is in the interior of $\{\lambda \in C_F \cap W \mid g_{\min}(\lambda) \geq 1\}$ and thus is the closest point to the origin contained in the affine plane define by $g_1 = 1$. $\square$

Corollary 6.7.40. *In the setting of [Theorem 6.7.34](#) or [Theorem 6.7.36](#), there is a unique morphism $f : [\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ with $f(1) \simeq x$ and $f(0) \simeq x_0$.*

Proof. By [Proposition 6.6.39](#), a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to [X/G]$ is determined by a one-parameter subgroup λ such that $\lim_{t \to 0} \lambda(t)x \in Gx_0$, and that λ is unique up to conjugation by P_λ. Since any two of Kempf's worst one-parameter subgroups are conjugate under U_λ (and thus P_λ), the statement follows. $\square$

Example 6.7.41. We revisit the SL_2 action on $(\mathbb{P}^1)^n$ with the linearization given by the Segre embedding $(\mathbb{P}^1)^n \hookrightarrow \mathbb{P}^{2^n-1}$ ([Example 6.7.19](#)). The non-semistable consists of tuples $x = (p_1, \ldots, p_n)$ where more than $n/2$ points are equal. Suppose that precisely $k > n/2$ points are equal. Since the Hilbert–Mumford index is symmetric, we can assume that the first k are equal. If $\lambda : \mathbb{G}_m \to \mathrm{SL}_2$ is a one-parameter subgroup, we can choose $g \in \mathrm{SL}_2(\mathbb{k})$ with $g\lambda g^{-1} = \lambda_0^d$ where $d \in \mathbb{Z}$ and $\lambda_0(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$. After rescaling the norm, we can assume that $\|\lambda_0\| = 1$. We also assume that $d \geq 0$ as the $d < 0$ case can be handled similarly. Then

$$\frac{\mu(x, \lambda)}{\|\lambda\|} = \frac{\mu(gx, g\lambda g^{-1})}{\|g\lambda g^{-1}\|} = \mu(gx, \lambda_0)$$

This index is negative if and only if $gx = \{0, \ldots, 0, p_{k+1}, \ldots, p_n\}$ in which case $\mu(gx, \lambda_0) = n - 2k$. It follows that λ_0 (resp. $g^{-1}\lambda_0 g$) is a Kempf optimal destabilizing one parameter subgroup for gx (resp x). Observe that the parabolic $P_{\lambda_0} \subset \mathrm{SL}_2$ of lower triangular matrices is also the stabilizer of $0 \in \mathbb{P}^1$, and thus $G_{gx} \subset P_{\lambda_0}$. For any $h \in P_{\lambda_0}$, $h^{-1}\lambda_0 h$ (resp. $(hg)^{-1}\lambda_0 hg$) is also a Kempf optimal destabilizing subgroup for gx (resp. x).

Exercise 6.7.42. Let G be a reductive algebraic group over an algebraically closed field $\Bbbk$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \mathrm{Spec}\, A$ be an affine scheme of finite type over $\Bbbk$ with an action of G. Let $x_0 \in X(\Bbbk)$ have closed G-orbit. Let $x \in X(\Bbbk)$ be a point such that $Gx_0 \subset \overline{Gx}$, and let P_x be the parabolic determined by Kempf's Optimal Destabilization Theorem (6.7.34).

(a) Show that for all $g \in G(\Bbbk)$ that $gP_x g^{-1} = P_{gx}$.

   Hint: Show that if $P_x = P_\lambda$ for a one-parameter subgroup λ, then $P_{gx} = P_{g\lambda g^{-1}}$.

(b) Show that $G_x \subset P_x$.

   Hint: Use that for a parabolic P, $N_G(P) = P$ (Proposition B.2.49).

The following criterion can sometimes be used to check stability/semistability by computing Hilbert–Mumford indices only for one-parameter subgroups in a fixed maximal torus.

Exercise 6.7.43 (Kempf–Morrison Criterion)**.** Let $G = \mathrm{GL}(W)$ or $\mathrm{SL}(W)$, where W is finite dimensional vector space over an algebraically closed field $\Bbbk$ of characteristic 0. Let $X \subset \mathbb{P}(V)$ be a G-invariant closed subscheme, where V is a finite dimensional G-representation. Let $x \in X(\Bbbk)$. Assume that there is a linearly reductive subgroup $H \subset G_x$ such that W decomposes as a direct sum of distinct H-representations. Let $T \subset G$ be a maximal torus compatible with this decomposition. Show that

$$x \in X^{\mathrm{ss}} \iff \mu(x, \lambda) \le 0 \text{ for all } \lambda \colon \mathbb{G}_m \to T,$$
$$x \in X^{\mathrm{s}} \iff \mu(x, \lambda) < 0 \text{ for all } \lambda \colon \mathbb{G}_m \to T.$$

Hint: If $u \notin X^{\mathrm{ss}}$, let $\lambda_0 \colon \mathbb{G}_m \to G$ be a Kempf optimal destabilization one-parameter subgroup and $0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$ be the filtration induced by the parabolic P_{λ_0}. Use Exercise 6.7.42 to conclude that each V_i is H-invariant, and use the hypothesis on the H-representation V to show that each V_i is T-invariant; thus $T \subset P_{\lambda_0}$. Apply Kempf's Optimal Destabilization Theorem again to find λ in T with $\mu(x, \lambda) < 0$. If $u \notin X^{\mathrm{s}}$, letting $\widehat{x} \in \mathbb{A}(V)$ be a lift of x and $\widehat{x}_0 \in \overline{G\widehat{x}}$ be a point with closed orbit, repeat the above argument using the affine version of Kempf's Optimal Destabilization Theorem.

Exercise 6.7.44 (Existence of destabilizing one-parameter subgroups over a perfect field)**.** Let X be an affine scheme of finite type over a perfect field $\Bbbk$, and let G be a reductive algebraic group over $\Bbbk$ acting on X. This exercise will show that for every point $x \in X(\Bbbk)$, there exists a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ *defined over $\Bbbk$* such that $\lim_{t \to 0} \lambda(t) \cdot x$ has closed G-orbit. See also [Kem78, §4].

(1) Show that if $\mathrm{Gal} := \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$ is the geometric Galois group, then Gal acts on the set $\mathbb{X}_*(G_{\overline{\Bbbk}})$ of one-parameters subgroups such that $\mathbb{X}_*(G) = \mathbb{X}_*(G_{\overline{\Bbbk}})^{\mathrm{Gal}}$.

(2) Show that there exists a length $\|-\|$ on $\mathbb{X}_*(G_{\overline{\Bbbk}})$ which is invariant under the action of Gal.

(3) Show that the subsets $\{\lambda \in \mathbb{X}_*(G_{\overline{\Bbbk}}) \mid \lim_{t \to 0} \lambda(t) \cdot x \in X(\overline{\Bbbk}) \text{ exists}\}$ and $\{\lambda \in \mathbb{X}_*(G_{\overline{\Bbbk}}) \mid \lim_{t \to 0} \lambda(t) \cdot x \in G_{\overline{\Bbbk}}x_0\}$ are Gal-invariant where $G_{\overline{\Bbbk}}x_0$ is

the unique closed orbit in $\overline{G_{\overline{\mathbb{k}}}u}$. Moreover, show that if V and W are G-representations as in (6.7.38), then the functions $m(i(x), \lambda)$ and $m(f(x), \lambda)$ are Gal-invariant.

(4) Generalize Theorem 6.7.34 and Theorem 6.7.36 to the case when $\mathbb{k}$ is a perfect field and $x \in X(\mathbb{k})$.

In particular, if G has no non-trivial one-parameter subgroups defined over $\mathbb{k}$, then the G-orbit of any $\mathbb{k}$-point is closed.

Finally, we record the following consequence of *the proof* of Kempf's Optimal Destabilization Theorem (6.7.34). This will play a key role in the proof of the HKKN Stratification (6.7.46).

Proposition 6.7.45. *Let G be a reductive algebraic group over an algebraically closed field $\mathbb{k}$ with a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over $\mathbb{k}$ with an action of G with a unique closed orbit Gx_0. Fix a maximal torus $T \subset G$. There are finitely many one-parameter subgroups $\lambda_1, \ldots, \lambda_n \in \mathbb{X}_*(T)$ and numbers $M_1, \ldots, M_n \in \mathbb{R}_{<0}$ such that for every point $x \in X(\mathbb{k})$, there exists a unique $i = 1, \ldots, n$ such that λ_i is an optimal Kempf one-parameter subgroup for gx for some $g \in G$, and such that $M_i = \mu(x, \lambda_i)/\|\lambda_i\|$.*

Proof. We will use the notation of the proof of Theorem 6.7.34. For $x \in X(\mathbb{k})$, the unique parabolic subgroup of a Kempf optimal destabilization one-parameter subgroup is determined by the subsets $\operatorname{State}_{gTg^{-1}}(i(x)) \cong \operatorname{State}_T(i(gx)) \subset \mathbb{X}_*(T)$ and $\operatorname{State}_{gTg^{-1}}(f(x)) \cong \operatorname{State}_T(f(gx)) \subset \mathbb{X}_*(T)$ as g ranges over $G(\mathbb{k})$. These subsets are contained in the finite subset of characters $\chi \in \mathbb{X}_*(T)$ with $V_\chi \neq 0$ or $W_\chi \neq 0$. Thus there are only finitely many possibilities for an optimal destabilizing subgroup of T, and the statement follows. $\square$

6.7.6 The Hesselink–Kempf–Kirwan–Ness Stratification

For an action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$, we show that the non-semistable locus admits a stratification into locally closed subschemes according to the normalized Hilbert–Mumford index

$$M(x) := \mu(x, \lambda)/\|\lambda\| \in \mathbb{R}_{<0}$$

of a Kempf optimal destabilizing one-parameter subgroup λ of a point $x \in X \setminus X^{\mathrm{ss}}$. The more negative the index $M(x)$ is, the more non-semistable (or 'unstable') the point x is. The strata will be indexed by pairs (λ, M) where $\lambda \in \mathbb{X}_*(G)$ and $M \in \mathbb{R}_{<0}$.

Recall from the Białynicki-Birula decomposition that for a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$, the attractor locus $X_\lambda^+ = \operatorname{Mor}^{\mathbb{G}_m}(\mathbb{A}^1, X)$ for the induced $\mathbb{G}_m$-action is a disjoint union of locally closed subschemes.

Theorem 6.7.46 (The HKKN Statification)**.** *Let G be a linearly reductive algebraic group over an algebraically closed field $\mathbb{k}$ with a maximal torus T and a length $\|-\|$ on $\mathbb{X}_*(G)$. Let $X \subset \mathbb{P}(V)$ be a G-equivariant closed subscheme where V is a finite dimensional G-representation. There is a finite subset $\Sigma \subset \mathbb{X}_*(T) \times \mathbb{R}_{<0}$ and a stratification of the non-semistable locus into G-invariant locally closed subschemes*

$$X \setminus X^{\mathrm{ss}} = \coprod_{(\lambda, M) \in \Sigma} S_{\lambda, M}$$

such that for each $(\lambda, M) \in \Sigma$,

(1) $X^+_{\lambda,M} := \{x \in X^+_\lambda \mid M(x) = M\}$ *is a* P_λ-*invariant locally closed subscheme of* X *consisting of points* x *such that* λ *is a Kempf optimal destabilizing one-parameter subgroup for* x, *and* $S_{\lambda,M} = G \cdot X^+_{\lambda,M}$;

(2) *a point* $x \in X^+_{\lambda,M}$ *if and only if* $\mathrm{ev}_0(x) = \lim_{t\to 0} \lambda(t) \cdot x \in X^+_{\lambda,M} \cap X^\lambda$; *thus* $Z_{\lambda,M} := \{x \in X^\lambda \mid M(x) = M\}$ *is a* C_λ-*invariant closed subscheme of* $X^+_{\lambda,M}$ *such that* $X^+_{\lambda,M} = \mathrm{ev}_0^{-1}(Z_{\lambda,M})$.

(3) *the natural map* $G \times^{P_\lambda} X^+_{\lambda,M} \to S_{\lambda,M}$ *is finite, surjective, and universally injective; if* $\mathrm{char}(\Bbbk) = 0$, *then* $G \times^{P_\lambda} X^+_{\lambda,M} \to S_{\lambda,M}$ *is an isomorphism.*

(4) *the locus*

$$\bigcup_{(\lambda',M')\in\Sigma, M'\leq M} S_{\lambda',M'}$$

is closed and in particular contains $\overline{S}_{\lambda,M}$;

(5) *if* X *is smooth, then so is each* $X^+_{\lambda,M}$; *in* $\mathrm{char}(\Bbbk) = 0$, *the strata* $S_{\lambda,M}$ *is also smooth.*

Remark 6.7.47. The locus $S_{\lambda,M}$ is called a *stratum* while $X^+_{\lambda,M}$ and $Z_{\lambda,M}$ are sometimes called a *blade* and *center* of the stratum. In characteristic 0, we have stack-theoretic equivalences $[X^+_{\lambda,M}/P_\lambda] \cong [S_{\lambda,M}/G]$ and a stratification

$$[(X \setminus X^{\mathrm{ss}})/G] = \coprod_{(\lambda,M)\in\Sigma} [S_{\lambda,M}/G].$$

For each (λ, M), there is a diagram

$$[Z_{\lambda,M}/C_\lambda] \underset{i}{\overset{\mathrm{ev}_0}{\longleftrightarrow}} [X^+_{\lambda,M}/P_\lambda] \cong [S_{\lambda,M}/G] \lhook\joinrel\longrightarrow [X/G] \qquad (6.7.48)$$

such that $\mathrm{ev}_0 \circ i = \mathrm{id}$.

Proof. Let $\widehat{X} \subset \mathbb{A}(V)$ be the affine cone of X, and let $\widehat{N} \subset \widehat{X}$ be the nullcone, i.e. the affine cone of $X \setminus X^{\mathrm{ss}}$. Then $0 \in \widehat{N}$ is the unique closed G-orbit. Applying Proposition 6.7.45 to the nullcone $\widehat{N} \subset \mathbb{A}(V)$, there is a finite subset $\Sigma \subset \mathbb{X}_*(T) \times \mathbb{R}_{<0}$ such that for every point $\widehat{x} \in \widehat{N} \setminus 0$, there is a unique $(\lambda, M) \in \Sigma$ such that λ is a Kempf optimal destabilizing one-parameter subgroup for $\widehat{x}$ with $M = \mu(\widehat{x}, \lambda)/\|\lambda\|$.

Since $\widehat{N}$ is affine, the locus $\widehat{N}^+_\lambda \subset \widehat{N}$ is a closed subscheme for each $(\lambda, M) \in \Sigma$ (Exercise 6.6.6). Since G is reductive, $P_\lambda \subset G$ is parabolic and

$$[\widehat{N}^+_\lambda/P_\lambda] \cong [G \times^{P_\lambda} \widehat{N}^+_\lambda/G] \to [\widehat{N}/G]$$

is projective. The image of this morphism is a closed substack corresponding to a closed G-invariant subscheme $\widehat{S}_\lambda$ such that $\widehat{S}_\lambda = G \cdot \widehat{N}^+_\lambda$. The loci $\widehat{N}^+_\lambda$ and $\widehat{S}_\lambda$ are invariant under scaling and are thus the affine cones over closed subschemes N_λ and S_λ of $X \setminus X^{\mathrm{ss}}$ such that $S_\lambda = G \cdot N_\lambda$.

The locus $X^+_{\lambda,M} := \{x \in X^+_\lambda \mid M(x) = M\}$ is identified with the points $x \in N_\lambda$ with $M(x) = M$. Moreover, $S_{\lambda,M} := \{x \in S_\lambda \mid M(x) = M\}$ is identified with $G \cdot X^+_{\lambda,M}$. There are identifications

$$X^+_{\lambda,M} = X^+_\lambda \setminus \bigcup_{(\lambda',M'), M'<M} X^+_{\lambda',M'} \quad \text{and} \quad S_\lambda = S_{\lambda,M} \setminus \bigcup_{(\lambda',M'), M'<M} S_{\lambda',M'}.$$

Thus $X_{\lambda,M}^+$ and $S_{\lambda,M}$ are open in X_λ^+ and S_λ, and each are locally closed in $U \setminus U^{\mathrm{ss}}$. From the conclusion of Proposition 6.7.45, the loci $S_{\lambda,M}$ are disjoint and cover $U \setminus U^{\mathrm{ss}}$. This gives (1).

For (2), if $x \in X \subset \mathbb{P}(V)$, then the limit $x_0 = \lim_{t \to 0} \lambda(t) \cdot x$ is the projection onto the subspace $W = \oplus V_\chi$ ranging over characters $\chi \in \mathbb{X}^*(T)$ such that the projection $\mathrm{proj}_\chi(x)$ of x to V_χ is nonzero and $\langle \chi, \lambda \rangle = -\mu(x, \lambda)$. By Lemma 6.7.39, λ lies on the ray spanned by the unique point closest to the origin in the closed convex set of $C_x = \{\lambda \in \mathbb{X}_*(T)_\mathbb{R} \mid \langle \chi, \lambda \rangle \geq 1, \mathrm{proj}_\chi(x) \neq 0\}$. It follows that λ is also the closest point to the origin in the analogously defined set C_{x_0}. Alternatively, one can check that if $\lambda_0 \in \mathbb{X}_*(T)$ is a optimal destabilizing one-parameter subgroup for x_0, then $\mu(x_0, \lambda_0)/\|\lambda_0\| \leq \mu(x, \lambda)/\|\lambda\|$ (giving the implication $x_0 \in X_{\lambda,M}^+ \Rightarrow x \in X_{\lambda,M}^+$) and $\mu(x, \lambda^N \lambda_0)/\|\lambda^N \lambda_0\| \leq \mu(x, \lambda)/\|\lambda\|$ for $N \gg 0$ (giving the implication $x \in X_{\lambda,M}^+ \Rightarrow x_0 \in X_{\lambda,M}^+$).

For (3), for $x \in X_{\lambda,M}^+$ we claim that

$$P_\lambda = \{g \in G(\Bbbk) \mid gx \in X_{\lambda,M}^+\}. \tag{6.7.49}$$

Since $X_{\lambda,M}^+$ is P_λ-invariant, we have the inclusion '$\subset$'. Conversely, if $gx \in X_{\lambda,M}^+$, then both λ and $g\lambda g^{-1}$ are optimal destabilization one-parameter subgroups for x. By Kempf's Optimal Destabilization Theorem (6.7.34), the parabolics P_λ and $P_{g\lambda g^{-1}} = gP_\lambda g^{-1}$ are equal. Since $N_G(P_\lambda) = P_\lambda$ (Proposition B.2.49), we conclude that $g \in P_\lambda$. Since $[X_\lambda^+/P_\lambda] \to [X/G]$ is proper, so is $[X_{\lambda,m}^+/P_\lambda] \to [S_{\lambda,M}/G]$. The map $[X_{\lambda,m}^+/P_\lambda] \to [S_{\lambda,M}/G]$ is surjective by construction, and injective on $\Bbbk$-points by (6.7.49); it is thus finite, surjective, and universally injective, and moreover an isomorphism if $\mathrm{char}(\Bbbk) = 0$.

For (4), given $M < 0$, assume by induction that $\bigcup_{(\lambda',M') \in \Sigma, M' < M} S_{\lambda',M'}$ is closed. Then for each (λ, M), we have that

$$S_{\lambda,M} = S_\lambda \setminus \bigcup_{(\lambda',M') \in \Sigma, M' < M} S_{\lambda',M'}$$

and it follows that $\bigcup_{(\lambda',M') \in \Sigma, M' \leq M} S_{\lambda',M'}$ is closed.

For (5), if X is smooth, then each X_λ^+ is smooth (Theorem 6.6.8). Since $X_{\lambda,M}^+ \subset X_\lambda^+$ is open, $X_{\lambda,M}^+$ is also smooth. In $\mathrm{char}(\Bbbk) = 0$, $S_{\lambda,M} = G \times^{P_\lambda} X_{\lambda,M}^+$ by Part (3) and thus also smooth.

See also [Hes81, §3], [Hes79, §4] and [Kir84, §12-13]. $\qquad\square$

Remark 6.7.50. When X is a smooth projective variety over $\mathbb{C}$, the HKKN stratification coincides with the Morse stratification of the square-norm of the moment map $\|-\|^2 : X \to \mathbb{R}$. Given $x \in X$, the optimal destabilizing one-parameter subgroup corresponds to the path of steepest descent starting from x. The centers $Z_{\lambda,M}$ correspond to the set of critical values of $\|-\|^2$ while the strata $S_{\lambda,M}$ are the locally closed submanifolds consisting of points which flow to $Z_{\lambda,M}$. See [Kir84, §6] and [Nes84].

Example 6.7.51. Let $\mathbb{G}_m$ act linearly on $X = \mathbb{P}^2$ with weights $-1, 2, 3$. Letting $\lambda = \mathrm{id}$ be the identity one-parameter subgroup, the non-semistable locus is $V(x^2 y, x^3 z)$ has the stratification $S_{\lambda^{-1},-1} \cup S_{\lambda,-2} \cup S_{\lambda,-3}$ where $S_{\lambda^{-1},-1} = \{[1:0:0]\}$, $S_{\lambda,-2} = \{[0:y:z] \mid y \neq 0\}$, and $S_{\lambda,-3} = \{[0:0:1]\}$.

Example 6.7.52. Revisiting the action of SL_2 on $X = (\mathbb{P}^1)^n$ with the Segre linearization (Example 6.7.41), let $\lambda_0 \colon \mathbb{G}_m \to \mathrm{SL}_2$ be the one-parameter subgroup defined by $\lambda_0(t) = \mathrm{diag}(t^{-1}, t)$. The strata are indexed by $(\lambda_0, -1), (\lambda_0, -3), \ldots, (\lambda_0, -n)$ if n is odd and by $(\lambda_0, -2), (\lambda_0, -4), \ldots, (\lambda_0, -n)$ if n is even. The strata $S_{\lambda_0, n-2k}$ consists of tuples with precisely $k > n/2$ points in common and has codimension $k - 1$. The blade $X^+_{\lambda_0, n-2k}$ consists of tuples where precisely k points are 0 while the center $Z_{\lambda_0, n-2k}$ is the set of $\mathbb{G}_m$-fixed points where k points are 0 and $n - k$ points are ∞.

Remark 6.7.53 (Θ-stratifications)**.** As indicated in Remark 6.6.42, there is an identification
$$\underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [X/G]) = \coprod_{\lambda \in \mathbb{X}_*(G)/\sim} [X^+_\lambda / P_\lambda],$$

where $\mathbb{X}_*(G)/\sim$ represents the set of one-parameter subgroups up to conjugation.

A Θ-*stratification* of an algebraic stack $\mathcal{X}$ locally of finite type over $\Bbbk$ is the data of a totally ordered set Σ with a minimal element $0 \in \Sigma$ and a stratification into locally closed substacks
$$\mathcal{X} = \coprod_{\lambda \in \Sigma} \mathcal{S}_\lambda$$

such that:

(1) for each $\lambda \in \Sigma$, $\mathcal{X}_{\leq \lambda} := \bigcup_{\rho \leq \lambda} \mathcal{S}_\rho$ is an open substack of $\mathcal{X}$,

(2) for each $\lambda \in \Sigma$, there is a union of connected components (called a Θ-*stratum* of $\mathcal{X}_{\leq \lambda}$)
$$\mathcal{S}'_\lambda \subset \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X}_{\leq \lambda})$$

such that $\mathrm{ev}_0 \colon \mathcal{S}'_\lambda \to \mathcal{X}_{\leq \lambda}$ is a closed immersion mapping isomorphically onto $\mathcal{S}_\lambda$, and

(3) for every $x \in |\mathcal{X}|$, the set $\{\lambda \in \Sigma \,|\, x \in |\mathcal{X}_{\leq \lambda}|\}$ has a minimal element.

See [HL14]. The semistable locus $\mathcal{X}^{\mathrm{ss}}$ is by definition the open substack $\mathcal{X}_{\leq 0} = \mathcal{S}_0$. Let $\mathcal{Z}'_\lambda$ be the preimage of $\mathcal{S}'_\lambda$ under the map
$$i \colon \underline{\mathrm{Mor}}(B\mathbb{G}_m, [X/G]) \to \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], \mathcal{X}).$$

The map $\mathrm{ev}_0 \colon \underline{\mathrm{Mor}}([\mathbb{A}^1/\mathbb{G}_m], [X/G]) \to \underline{\mathrm{Mor}}(B\mathbb{G}_m, \mathcal{X})$ obtained by restricting to 0 is a section of i, and there is a diagram analogous to (6.7.48)

$$\mathcal{Z}'_\lambda \xrightarrow[\;i\;]{\overset{\mathrm{ev}_0}{\curvearrowleft}} \mathcal{S}'_\lambda \lhook\joinrel\longrightarrow \mathcal{X}.$$

In characteristic 0, the HKKN stratification is an example of a Θ-stratification, where one orders the indices (λ, M) first by $-M$ and then arbitrarily by λ. In the next chapter, we will see that the moduli stack $\underline{\mathrm{Bun}}_{r,d}(C)$ has a Θ-stratification called the Harder–Narasimhan–Shatz stratification.

Recall that the *Chow–Poincare polynomial* of a G-equivariant scheme X is $p_G(X, t) = \sum_{d=0}^\infty \big(\dim \mathrm{CH}^d_G(U)_{\mathbb{Q}}\big) t^d$.

Proposition 6.7.54 (Kirwan Surjectivity)**.** *Under the hypotheses of Theorem 6.7.46, assume further assume that X is smooth and irreducible, and that* $\mathrm{char}(\Bbbk) = 0$.

Suppose that for all (λ, M), the stratum $S_{\lambda,M}$ is equidimensional of codimension $d_{\lambda,M}$. Then

$$\dim \mathrm{CH}_G^k(X)_{\mathbb{Q}} = \dim \mathrm{CH}_G^k(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}_{C_\lambda}^{k-d_{\lambda,M}}(Z_{\lambda,M})_{\mathbb{Q}}$$

and

$$p_G(X,t) = p_G(X^{\mathrm{ss}},t) + \sum_{(\lambda,M)} p_{C_\lambda}(Z_{\lambda,M},t)t^{d_{\lambda,M}}.$$

Proof. From Theorem 6.7.46, we know that $[S_{\lambda,M}/G] \cong [X_{\lambda,M}^+/P_\lambda]$. From Theorem 6.6.8, we know that $\mathrm{ev}_0 \colon X_{\lambda,M}^+ \to Z_{\lambda,M}$ sending a point to its limit is a Zariski-local affine fibration and equivariant with respect to $P_\lambda \to C_\lambda$. We claim that $[X_{\lambda,M}^+/P_\lambda] \to [Z_{\lambda,M}/C_\lambda]$ induces an isomorphism

$$\mathrm{CH}_{C_\lambda}^*(Z_{\lambda,M}) \to \mathrm{CH}_{P_\lambda}^*(X_{\lambda,M}^+). \tag{6.7.55}$$

By the definition of the equivariant Chow groups, $\mathrm{CH}_{P_\lambda}^i(X_{\lambda,M}^+)$ is identified by $\mathrm{CH}^i(X_{\lambda,M}^+ \times^{P_\lambda} V)$ where V is an open subspace $\mathbb{A}(W)$ of a P_λ-representation such that P_λ acts freely on V and $\mathbb{A}(W) \setminus V$ has sufficiently high codimension. On the other hand, $\mathrm{CH}_{C_\lambda}^i(Z_{\lambda,M})$ is identified with $\mathrm{CH}^i(Z_{\lambda,M} \times^{C_\lambda} V)$ and the map (6.7.55) corresponds to the pullback map on Chow induced from the composition

$$X_{\lambda,M}^+ \times^{P_\lambda} V \to Z_{\lambda,M} \times^{P_\lambda} V \to Z_{\lambda,M} \times^{C_\lambda} V.$$

The first map is a Zariski local affine fibration and the second map is a principal bundle under $U_\lambda = \ker(P_\lambda \to C_\lambda)$. Since U_λ is unipotent, U_λ is isomorphic to affine space and principal U_λ-bundles are locally trivial in the Zariski topology (see §B.2.5). We conclude that (6.7.55) is an isomorphism.

We also claim that $c_{d_{\lambda,M}}(N_{S_{\lambda,M}/X}) \in \mathrm{CH}_G^*(S_{\lambda,M})$ is a nonzerodivisor. Since $N_{S_{\lambda,M}/X}|_{Z_{\lambda,M}}$ is identified with $N_{S_{\lambda,M}/X}$ under $\mathrm{CH}_{C_\lambda}^*(Z_{\lambda,M}) \cong \mathrm{CH}_G^*(S_{\lambda,M})$, it suffices to show that $c_{d_{\lambda,M}}((N_{S_{\lambda,M}/X})|_{Z_{\lambda,M}}) \in \mathrm{CH}_{C_\lambda}^*(Z_{\lambda,M})_{\mathbb{Q}}$ is a nonzerodivisor where $d = d_{\lambda,M}$. By Theorem 6.6.8, λ acts on a fiber of the normal bundle with nonzero weights. Thus Lemma 6.6.27 implies that $c_{d_{\lambda,M}}((N_{S_{\lambda,M}/X})|_{Z_{\lambda,M}})$ is a nonzerodivisor.

We therefore can apply Lemma 6.6.25 with the strata $S_{\lambda,M}$ ordered first by $-M$ and then with any ordering of the λ's; the semistable locus U^{ss} is viewed as a stratum with the smallest index. This yields

$$\dim \mathrm{CH}_G^k(X)_{\mathbb{Q}} = \dim \mathrm{CH}_G^k(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}_G^{k-d_{\lambda,M}}(S_{\lambda,M})_{\mathbb{Q}}$$
$$= \dim \mathrm{CH}_G^k(X^{\mathrm{ss}})_{\mathbb{Q}} + \sum_{(\lambda,M)} \dim \mathrm{CH}_{C_{\lambda,M}}^{k-d_{\lambda,M}}(Z_{\lambda,M})_{\mathbb{Q}}.$$

$\square$

Remark 6.7.56. This formula was established for de Rham cohomology in [Kir84, Thm. 5.4]. Instead of the excision sequence

$$\mathrm{CH}_G^{k-d_{\lambda,M}}(S_{\lambda,M}) \to \mathrm{CH}_G^k(S_{\leq(\lambda,M)}) \to \mathrm{CH}_G^k(S_{<(\lambda,M)}) \to 0,$$

one uses the Thom–Gysin long exact sequence

$$\cdots \to \mathrm{H}_G^{k-d_{\lambda,M}}(S_{\lambda,M}) \to \mathrm{H}_G^k(S_{\leq(\lambda,M)}) \to \mathrm{H}_G^k(S_{<(\lambda,M)}) \to \cdots.$$

In this case, the surjectivity of the right map for all (λ, M) is equivalent to the injectivity of the left map for all (λ, M), and the latter condition is verified as above by the showing the top Chern class of the normal bundle is a nonzerodivisor.

Example 6.7.57. As an application, we can compute the dimension of the rational Chow groups of $[(\mathbb{P}^1)^{n,\mathrm{ss}}/\mathrm{SL}_2]$ using the computation of the stratification in Example 6.7.52. When n is odd, this also gives the dimension of the rational Chow groups of the GIT quotient $(\mathbb{P}^1)^{n,\mathrm{ss}}/\mathrm{SL}_2$ by Properties 6.1.33(4).

Since $[(\mathbb{P}^1)^n/\mathrm{SL}_2] \to B\,\mathrm{SL}_2$ is an iterated $\mathbb{P}^1$-bundle and $\mathrm{CH}^*(\mathrm{SL}_2) \cong \mathbb{Z}[T]$ generated in degree 2,

$$\mathrm{CH}^*([(\mathbb{P}^1)^n/\mathrm{SL}_2]) \cong \mathrm{CH}^*((\mathbb{P}^1)^n) \otimes \mathrm{CH}^*(B\,\mathrm{SL}_2)$$
$$\cong \mathbb{Z}[H_1, \ldots, H_n]/(H_1, \ldots, H_n)^2 \otimes \mathbb{Z}[T]$$

and the Chow–Poincare polynomial is $p_{\mathrm{SL}_2}((\mathbb{P}^1)^n, t) = (1+t)^n (1-t^2)^{-1}$. On the other hand, the strata $S_{\lambda, n-2k}$ where precisely k points are the same has codimension $k-1$ and its center $Z_{\lambda, n-2k}$ consists of $\binom{n}{k}$ $\mathbb{G}_m$-fixed points. Thus $p_{\mathbb{G}_m}(Z_{\lambda, n-2k}, t) = \binom{n}{k}(1-t)^{-1}$ and

$$p_G((\mathbb{P}^1)^{\mathrm{ss}}, t) = (1+t)^n (1-t)^{-1} - \sum_{k>n/2} \binom{n}{k} t^{k-1}(1-t)^{-1}$$

$$= 1 + nt + \cdots + \left(1 + (n-1) + \binom{n-1}{2} + \cdots + \binom{n-1}{\min(d, n-3-d)}\right) t^d$$

$$+ \cdots + nt^{n-4} + t^{n-3}.$$

See also [Kir84, §16.1].

6.8 Existence of good moduli spaces

In this section, we provide necessary and sufficient conditions for the existence of a separated good moduli space in characteristic 0.

Theorem 6.8.1 (Existence Theorem of Good Moduli Spaces)**.** *Let $\mathcal{X}$ be an algebraic stack, of finite type over an algebraic closed field $\Bbbk$ of characteristic 0, with affine diagonal. There exists a good moduli space $\pi\colon \mathcal{X} \to X$ with X a separated algebraic space if and only if $\mathcal{X}$ is Θ-complete (Definition 6.8.8) and S-complete (Definition 6.8.13).*

Moreover, X is proper if and only if $\mathcal{X}$ satisfies the existence part of the valuative criterion for properness.

The conditions of Θ-completeness and S-completeness are defined and discussed in detail in §6.8.2.

6.8.1 Strategy for constructing good moduli spaces

We first explain how the Local Structure Theorem for Algebraic Stacks (6.5.1) gives us a natural strategy to construct the good moduli space X. Namely, for each closed point $x \in \mathcal{X}$, we have an étale quotient presentation

$$\mathcal{W} = [\operatorname{Spec} A/G_x] \xrightarrow{\ f\ } \mathcal{X}$$
$$\downarrow$$
$$W = \operatorname{Spec} A^{G_x}$$

where f is affine étale, and there is a preimage $w \in \mathcal{W}$ of x such that f induces an isomorphism of stabilizer groups at w. We want to show that the GIT quotients $W = \operatorname{Spec} A^{G_x}$ as x ranges over closed points provide étale models that can be glued to a good moduli space of $\mathcal{X}$. To this end, we need to construct an étale equivalence relation on W. Since f is affine, the fiber product $\mathcal{R} := \mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ is isomorphic to a quotient stack $[\operatorname{Spec} B / G_x]$ and we have a diagram

$$
\begin{array}{ccc}
\mathcal{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{W} & \overset{f}{\longrightarrow} & \mathcal{X} \\
\downarrow \qquad \downarrow & & \\
\downarrow \qquad \downarrow & & \\
R \underset{q_2}{\overset{q_1}{\rightrightarrows}} W & &
\end{array}
$$

where $R = \operatorname{Spec} B^{G_x}$. If $q_1, q_2 \colon R \rightrightarrows W$ defines an étale equivalence relation, the algebraic space quotient W/R gives a candidate for a good moduli space of $f(\mathcal{W}) \subset \mathcal{X}$.

Luna's Fundamental Lemma (6.3.30) provides condition on when $q_1, q_2 \colon R \rightrightarrows W$ are étale: we need that for all closed points $r \in \mathcal{R}$ that

(a) $p_1(r), p_2(r) \in \mathcal{W}$ are closed points; and

(b) p_1 and p_2 induce isomorphisms of stabilizer groups at r.

On the other hand, we know that $f(w) \in \mathcal{X}$ is closed and f induces an isomorphism of stabilizer groups at the given preimage w of x. We want to show that there is an open neighborhood $\mathcal{U}$ of w such that the restriction $f|_{\mathcal{U}}$ satisfies: (a) $f|_{\mathcal{U}}$ sends closed points map to closed points and (b) $f|_{\mathcal{U}}$ induces isomorphisms of stabilizer groups at closed points, and moreover that these conditions are stable under base change. While property (a) is stable under base change, property (b) is not, and we will introduce a stronger condition below—called Θ-surjectivity (Definition 6.8.35)—which is stable under base change and implies (b).

The role of Θ-completeness and S-completeness in the construction of the good moduli space is the following: the Θ-completeness of $\mathcal{X}$ implies that Θ-surjectivity holds (and thus condition (a) and its base changes hold) in an open neighborhood of w (Proposition 6.8.40) while S-completeness implies that condition (b) holds in an open neighborhood of w (Proposition 6.8.47).

Counterexamples

The following examples do not admit good moduli spaces. We will explain why the approach outlined above fails and then later explain how they violate the conditions of Θ-completeness and S-completeness. We work over an algebraically closed field $\Bbbk$.

Example 6.8.2. Consider the action of $\mathbb{G}_m$ on $\mathbb{P}^1$ given by $t \cdot [x : y] = [tx : y]$. The quotient stack $\mathcal{X} = [\mathbb{P}^1 / \mathbb{G}_m]$ does not admit a good moduli space. Note that Theorem 6.3.5(2) implies that every $\Bbbk$-point has a unique closed point in its closure. Here we see that $[1 : 1]$ specializes to two closed points $[1 : 0]$ and $[0 : 1]$. Alternatively, if there were a good moduli space, it would have to be $\mathcal{X} \to \operatorname{Spec} \Bbbk$ (which is universal for maps to algebraic spaces), but then the composition $\mathbb{P}^1 \to \mathcal{X} \to \operatorname{Spec} \Bbbk$ would be affine by Serre's Criterion for Affineness (4.4.16), a contradiction.

There are two open substacks $\mathcal{U}_1, \mathcal{U}_2 \subset [\mathbb{P}^1 / \mathbb{G}_m]$ isomorphic to $[\mathbb{A}^1 / \mathbb{G}_m]$ each which admits a good moduli space $\pi_i \colon \mathcal{U}_i \to \operatorname{Spec} \Bbbk$ but they do not glue to a good moduli space of $\mathcal{X}$: the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is the open point in both $\mathcal{U}_1$ and $\mathcal{U}_2$ and not the preimage of an open subscheme under π_i. To see how the approach above

fails, observe that the étale presentation $f \colon \mathcal{W} := \mathcal{U}_1 \coprod \mathcal{U}_2 \to \mathcal{X}$ satisfies (a) and (b) but the base changes $p_1, p_2 \colon \mathcal{W} \times_{\mathcal{X}} \mathcal{W} = \mathcal{U}_1 \coprod \mathcal{U}_2 \coprod \mathcal{U}_1 \cap \mathcal{U}_2 \to \mathcal{W}$ fails (b), i.e. the closed point in $\mathcal{U}_1 \cap \mathcal{U}_2$ is mapped to a non-closed point under either projection.

Example 6.8.3. For a related example, let C be the projective nodal cubic with its $\mathbb{G}_m$-action. The quotient $\mathcal{X} = [C/\mathbb{G}_m]$ has two points—one open and one closed—but while there is no topological obstruction as above, $\mathcal{X}$ again does not admit a good moduli space because C is projective, not affine. Viewing the nodal cubic as the quotient of nodal union X' of two $\mathbb{P}^1$'s along 0 and ∞ modulo the rotation action of $\mathbb{Z}/2$, we have a finite étale cover $[X'/\mathbb{G}_m] \to [X/\mathbb{G}_m]$. Removing one of the origins, we have an affine étale cover $\mathcal{W} = [\operatorname{Spec}(k[x,y]/xy)/\mathbb{G}_m] \to \mathcal{X}$ where $\mathbb{G}_m$ acts via $t \cdot (x,y) = (tx, t^{-1}y)$. Again, this map sends closed points to closed points, but the projections $\mathcal{W} \times_{\mathcal{X}} \mathcal{W} \rightrightarrows \mathcal{W}$ do not.

Example 6.8.4. Let $\mathbb{G}_m$ act on $\mathbb{A}^2$ via $t \cdot (x,y) = (tx, y)$ and set $\mathcal{X} = [\mathbb{A}^2/\mathbb{G}_m] \smallsetminus 0$. The point $p = (1,0) \in \mathcal{X}$ is closed with trivial stabilizer, and the open immersion $f \colon \mathbb{A}^1 \hookrightarrow \mathcal{X}$, sending z to $(z,1)$, is an étale quotient presentation. Note that while $f(0)$ is closed, the image $f(z)$ is not closed for $z \neq 0$. The map $\mathcal{X} \to \mathbb{A}^1$ defined by $(x,y) \mapsto y$ is not a good moduli space as $\mathbb{A}^2 \smallsetminus 0$ is not affine.

We will see in the next section that the previous examples violate Θ-completeness. Similar phenomena can naturally occur in moduli, e.g. by removing a single polystable but not stable vector bundle from $\underline{\operatorname{Bun}}_{r,d}(C)^{\mathrm{ss}}$. The next examples violate S-completeness.

Example 6.8.5. Suppose $\operatorname{char}(\Bbbk) \neq 2$ and let $G = \mathbb{Z}/2$ act on the non-separated union $U = \mathbb{A}^1 \bigcup_{x \neq 0} \mathbb{A}^1$ by exchanging the copies of $\mathbb{A}^1$. The quotient stack $[U/G]$ has a $\mathbb{Z}/2$ stabilizer everywhere except at the origin. This is a Deligne–Mumford stack with quasi-finite but not finite inertia; in fact we've seen this before in Exercise 4.3.19 to illustrate the necessity of the separatedness condition in the Keel–Mori Theorem (4.3.12). By precomposing by the inclusion of one of the $\mathbb{A}^1$'s, we have an affine étale morphism $\mathbb{A}^1 \to [U/G]$ which is stabilizer preserving at 0 but not in any open neighborhood of 0.

For a related example, the Deligne–Mumford locus $\mathcal{X}^{\mathrm{DM}}$ in the moduli stack $\mathcal{X} = [\operatorname{Sym}^4 \mathbb{P}^1 / \operatorname{PGL}_2]$ of four unordered points in $\mathbb{P}^1$ is not separated (see Example 4.3.20). Note however that the stable locus $\mathcal{X}^{\mathrm{s}}$ consisting of four distinct points is separated and the semistable locus $\mathcal{X}^{\mathrm{ss}} = \mathcal{X}^{\mathrm{DM}} \cup \{[0:0:\infty:\infty]\}$ has a projective good moduli space.

Example 6.8.6. Consider the action of $G = \mathbb{G}_m \rtimes \mathbb{Z}/2$ on $X = \mathbb{A}^2 \smallsetminus 0$ via $t \cdot (a,b) = (ta, t^{-1}b)$ and $-1 \cdot (a,b) = (b,a)$. Note that every point $(a,b) \in X$ with $ab \neq 0$ is fixed by the order 2 element $(a/b, -1) \in G$. The quotient stack $[X/G]$ is a non-separated Deligne–Mumford stack which does not admit a good moduli space; note however that $[\mathbb{A}^2/G] \to \operatorname{Spec}\Bbbk[xy]$ is a good moduli space.

6.8.2 The valuative criteria: Θ- and S-completeness

We define the stack 'Theta' as

$$\Theta := [\mathbb{A}^1/\mathbb{G}_m]$$

over $\operatorname{Spec}\mathbb{Z}$.[9] If R is a DVR with fraction field K and residue field κ, we define $\Theta_R := \Theta \times \operatorname{Spec} R$ and set $0 \in \Theta_R$ to be the unique closed point. Observe that Θ_R

---

[9] The symbol Θ is used as it resembles the picture of the two orbits of $\mathbb{G}_m$ on the complex plane.

is a local model of the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ with weights $0,1$ as it is identified with the base change of the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[x]$ along the map $\operatorname{Spec} R \to \operatorname{Spec} \Bbbk[x]$ where x maps to a uniformizer π in R.

The following cartesian diagram gives a schematic picture of Θ_R (where x is the coordinate on $\mathbb{A}^1$ and $\pi \in R$ is the uniformizer).

$$(6.8.7)$$

where the maps to the left are open immersions and to the right are closed immersions.

In particular, a morphism $\Theta_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \Theta_K \to \mathcal{X}$ to an algebraic stack is the data of morphisms $\operatorname{Spec} R \to \mathcal{X}$ and $\Theta_K \to \mathcal{X}$ together with an isomorphism of their restrictions to $\operatorname{Spec} K$.

Definition 6.8.8. A noetherian algebraic stack $\mathcal{X}$ is Θ-*complete*[10] if for every DVR R, every commutative diagram

$$(6.8.9)$$

of solid arrows can be uniquely filled in.

Remark 6.8.10. We can state an equivalent formulation using the stack $\underline{\operatorname{Mor}}(\Theta, \mathcal{X})$ classifying morphisms $\Theta \to \mathcal{X}$. Evaluation at 1 gives a morphism

$$\operatorname{ev}_1 \colon \underline{\operatorname{Mor}}(\Theta, \mathcal{X}) \to \mathcal{X}, \quad f \mapsto f(1)$$

of stacks, and the Θ-completeness of $\mathcal{X}$ is equivalent to the morphism ev_1 satisfying the valuative criterion for properness. If $\mathcal{X}$ is of finite type over an algebraically closed field $\Bbbk$, then the stack $\underline{\operatorname{Mor}}(\Theta, \mathcal{X})$ is an algebraic stack *locally* of finite type over $\Bbbk$; see Remark 6.6.42 where an explicit description is given when $\mathcal{X}$ is a quotient stack. The stack $\underline{\operatorname{Mor}}(\Theta, \mathcal{X})$ is however rarely quasi-compact, e.g. for $\mathcal{X} = B\mathbb{G}_m$, and ev_1 is thus rarely proper.

For a DVR R with fraction field K, residue field κ, and uniformizer π, we define

$$\phi_R := [\operatorname{Spec}\big(R[s,t]/(st-\pi)\big)/\mathbb{G}_m], \qquad (6.8.11)$$

where s and t have $\mathbb{G}_m$-weights 1 and -1 respectively.[11] The quotient ϕ_R is a local model of the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$ with weights $1,-1$ as it is identified with the base change of the good moduli space $[\mathbb{A}^2/\mathbb{G}_m] \to \operatorname{Spec} \Bbbk[xy]$ along the map $\operatorname{Spec} R \to \operatorname{Spec} \Bbbk[xy]$ given by $xy \mapsto \pi$.

---

[10]In the literature, the term 'Θ-reductive' is often used.

[11]The symbol ϕ is used because it looks like the non-separated affine line with an additional origin. In the literature, $\overline{\operatorname{ST}}_R$ is used as it is a compactification of $\operatorname{ST}_R = \overline{\operatorname{ST}}_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R$, which is the 'standard test' scheme for separatedness.

The locus where $s \neq 0$ in Φ_R is isomorphic to $[\operatorname{Spec}\left(R[s,t]_s/(t - \pi/s)\right)/\mathbb{G}_m] \cong [\operatorname{Spec}(R[s]_s)/\mathbb{G}_m] \cong \operatorname{Spec} R$ and the locus where $t \neq 0$ has a similar description. We thus have cartesian diagrams analogous to (6.8.7)

$$
\begin{array}{ccccc}
& \operatorname{Spec} R & & \Theta_\kappa & \\
& \nearrow \quad \searrow {\scriptstyle s \neq 0} & {\scriptstyle s=0} \swarrow \quad \nwarrow & \\
\operatorname{Spec} K & & \Phi_R & & B\mathbb{G}_{m,\kappa} \\
& \searrow \quad \nearrow {\scriptstyle t \neq 0} & {\scriptstyle t=0} \nwarrow \quad \swarrow & \\
& \operatorname{Spec} R & & \Theta_\kappa &
\end{array}
\qquad (6.8.12)
$$

where the maps to the left are open immersions and to the right are closed immersions.

In particular, a morphism $\Phi_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R \to \mathcal{X}$ to an algebraic stack is the data of two morphisms $\xi, \xi' \colon \operatorname{Spec} R \to \mathcal{X}$ together with an isomorphism $\xi_K \simeq \xi'_K$ over $\operatorname{Spec} K$.

Definition 6.8.13. A noetherian algebraic stack $\mathcal{X}$ is S-*complete* if for every DVR R, every commutative diagram

$$
\begin{array}{ccc}
\Phi_R \smallsetminus 0 & \longrightarrow & \mathcal{X} \\
\downarrow & \nearrow & \\
\Phi_R & &
\end{array}
\qquad (6.8.14)
$$

of solid arrows can be uniquely filled in.[12]

Remark 6.8.15. There are obvious extensions of the definition of Θ-completeness and S-completeness to morphisms $f \colon \mathcal{X} \to \mathcal{Y}$ but we will not need such notions.

Lemma 6.8.16. *A noetherian algebraic stack with affine diagonal is Θ-complete (resp. S-complete), if and only if every diagram (6.8.9) (resp. (6.8.14)), there exists a lift after an extension of DVRs $R \subset R'$. In particular, Θ-completeness and S-completeness can be verified on complete DVRs with algebraically closed residue fields.*

Proof. We begin with the observation that if $\mathcal{X} \to \mathcal{Y}$ has affine diagonal and $j \colon \mathcal{U} \to \mathcal{T}$ is an open immersion of algebraic stacks over $\mathcal{Y}$ with $j_* \mathcal{O}_{\mathcal{U}} = \mathcal{O}_{\mathcal{T}}$, then two extensions $f_1, f_2 \colon \mathcal{T} \to \mathcal{X}$ of a $\mathcal{Y}$-morphism $\mathcal{U} \to \mathcal{X}$ are canonically 2-isomorphic. Indeed, since $\operatorname{Isom}_{\mathcal{T}}(f_1, f_2) \to \mathcal{T}$ is affine, the section over $\mathcal{U}$ induced by the 2-isomorphism $f_1|_{\mathcal{U}} \xrightarrow{\sim} f_2|_{\mathcal{U}}$ extends uniquely to a section of $\mathcal{T}$.

Consider a diagram (6.8.9), an extension of DVRs $R \subset R'$, and a lifting $\Theta_{R'} \to \mathcal{X}$. The open immersion $j \colon \Theta_R \smallsetminus 0 \to \Theta_R$ satisfies $j_* \mathcal{O}_{\Theta_R \smallsetminus 0} = \mathcal{O}_{\Theta_R}$ and by Flat Base Change (6.1.7) the same property holds for the morphisms obtained by base changing j along $\Theta_{R'} \to \Theta_R$, $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \to \Theta_R$, and $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \times_{\Theta_R} \Theta_{R'} \to \Theta_R$. By the above observation, there exists a canonical 2-isomorphism between the two extensions $\Theta_{R'} \times_{\Theta_R} \Theta_{R'} \rightrightarrows \Theta_{R'} \to \mathcal{X}$ which necessarily satisfies the cocycle condition. By fpqc descent, the lifting $\Theta_{R'} \to \mathcal{X}$ descends to a lifting $\Theta_R \to \mathcal{X}$. The same argument works for S-completeness. $\qquad \square$

---

[12]The 'S' stands for 'Seshadri' as S-completeness is a geometric property reminiscent of how the S-equivalence relation on sheaves implies separatedness of the moduli space.

Remark 6.8.17. It is even true that when $\mathcal{X}$ is of finite type over $\Bbbk$, these criteria can be verified on DVRs essentially of finite type over $\Bbbk$; see [AHLH18, §4]. We will not use this fact.

Lemma 6.8.18. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be an affine morphism of noetherian algebraic stacks. If $\mathcal{Y}$ is Θ-complete (resp. S-complete), so is $\mathcal{X}$.*

Proof. Since Θ_R is regular and $0 \in \Theta_R$ is codimension 2, the pushforward of the structure sheaf along $\Theta_R \smallsetminus 0 \to \Theta_R$ is the structure sheaf. We therefore have canonical equivalences

$$
\begin{aligned}
\mathrm{Mor}_{\mathcal{Y}}(\Theta_R \smallsetminus 0, \mathcal{X}) &\cong \mathrm{Mor}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(f_*\mathcal{O}_{\mathcal{X}}, (\Theta_R \smallsetminus 0 \to \mathcal{Y})_*\mathcal{O}_{\Theta_R \smallsetminus 0}) \\
&\cong \mathrm{Mor}_{\mathcal{O}_{\mathcal{Y}}-\mathrm{alg}}(f_*\mathcal{O}_{\mathcal{X}}, (\Theta_R \to \mathcal{Y})_*\mathcal{O}_{\Theta_R}) \\
&\cong \mathrm{Mor}_{\mathcal{Y}}(\Theta_R, \mathcal{X}).
\end{aligned}
$$

The case of S-completeness is identical. $\qquad\square$

Proposition 6.8.19. *If G is a reductive group over an algebraically closed field $\Bbbk$, then every quotient stack $[\mathrm{Spec}\, A/G]$ is Θ-complete and S-complete.*

Proof. We first show that $B\,\mathrm{GL}_n$ is Θ-complete. A morphism $\Theta_R \smallsetminus 0 \to \mathcal{X}$ corresponds to a vector bundle E on $\Theta_R \smallsetminus 0$. The algebraic stack Θ_R is regular and $0 \in \Theta_R$ is a codimension 2 point. If $\widetilde{E}$ is a coherent sheaf on Θ_R extending E, then the double dual $\widetilde{E}^{\vee\vee}$ is a vector bundle extending E. (In fact the pushforward of E along $\Theta_R \smallsetminus 0 \hookrightarrow \Theta_R$ is a vector bundle.) This provides the desired extension $\Theta_R \to \mathcal{X}$. As G is affine, we can choose a faithful representation $G \subset \mathrm{GL}_n$. As G is reductive, the quotient GL_n / G is affine by Matushima's Theorem (B.2.63). Using the cartesian diagram

$$
\begin{array}{ccc}
\mathrm{GL}_n / G & \longrightarrow & \mathrm{Spec}\,\Bbbk \\
\downarrow & \square & \downarrow \\
BG & \longrightarrow & B\,\mathrm{GL}_n
\end{array}
$$

and smooth descent, we see that $BG \to B\,\mathrm{GL}_n$ is affine. We conclude that BG and $[\mathrm{Spec}\,A/G]$ are Θ-complete by Lemma 6.8.18. $\qquad\square$

As a result, we see that the hypotheses of Θ-completeness and S-completeness in Theorem 6.8.1 are necessary.

Corollary 6.8.20. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. If $\pi\colon \mathcal{X} \to X$ be a good moduli space, then $\mathcal{X}$ is Θ-complete. Moreover, $\mathcal{X}$ is S-complete if and only if X is separated.*

Proof. For a $\Bbbk$-algebra A, the map $\Theta_A \to \mathrm{Spec}\,A$ is a good moduli space, and thus every map $\Theta_A \to X$ factors through $\mathrm{Spec}\,A$ by the universality of good moduli spaces (Theorem 6.3.5(4)). If R is a DVR with fraction field K, then every map $\Theta_R \to X$ (resp. $\Theta_K \to X$) factors through $\mathrm{Spec}\,R$ (resp. $\mathrm{Spec}\,K$). To see that $\mathcal{X}$ is Θ-complete, it therefore suffices to find a lift of every commutative diagram

$$
\begin{array}{ccc}
\Theta_R \smallsetminus 0 & \longrightarrow & \mathcal{X} \\
\downarrow & \nearrow & \downarrow \pi \\
\mathrm{Spec}\,R & \longrightarrow & X
\end{array}
$$

of solid arrows. By the Local Structure for Good Moduli Spaces (6.5.3), there exists an étale morphism $\operatorname{Spec} B \to X$ containing the image of $\operatorname{Spec} R$ such that $\mathcal{X} \times_X \operatorname{Spec} B \cong [\operatorname{Spec} A/G]$ with G linearly reductive and $B = A^G$. Since $\operatorname{Spec} R \to X$ lifts to $\operatorname{Spec} B$ after an extension of DVRs and since Θ-completeness can be checked after an extension (Lemma 6.8.16), we are reduced to the case of $[\operatorname{Spec} A/G]$. This is Proposition 6.8.19.

If X is separated, then X is S-complete as $\varphi_R \smallsetminus 0 = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \to X$ factors through $\operatorname{Spec} R$ by the valuative criterion for separatedness. The above argument can be repeated to show that $\mathcal{X}$ is S-complete. Conversely, suppose $f, g \colon \operatorname{Spec} R \to X$ are two maps such that $f|_K = g|_K$. After possibly an extension of R, we may choose a lift $\operatorname{Spec} K \to \mathcal{X}$ of $f|_K = g|_K$. Since $\mathcal{X} \to X$ is universally closed (Theorem 6.3.5(1)), after possibly further extensions of R, we may choose lifts $\widetilde{f}, \widetilde{g} \colon \operatorname{Spec} R \to \mathcal{X}$ of f, g such that $\widetilde{f}|_K \cong \widetilde{g}|_K$ by the Valuative Criterion for Universal Closedness (3.8.2). Since $\mathcal{X}$ is S-complete, we can extend $\widetilde{f}$ and $\widetilde{g}$ to a morphism $\varphi_R \to \mathcal{X}$. As $\varphi_R \to \operatorname{Spec} R$ is a good moduli space and hence universal for maps to algebraic spaces, the morphism $\varphi_R \to \mathcal{X}$ descends to a unique morphism $\operatorname{Spec} R \to X$ which necessarily must be equal to both f and g. We conclude that X is separated by the Valuative Criterion for Separatedness. □

Lemma 6.8.21. *Let $\mathcal{X}$ be a noetherian algebraic stack with affine and quasi-finite diagonal. If R is a complete DVR, every map $\Theta_R \to \mathcal{X}$ (resp. $\varphi_R \to \mathcal{X}$) factors through $\Theta_R \to \operatorname{Spec} R$ (resp. $\varphi_R \to \operatorname{Spec} R$).*

Proof. Since good moduli spaces are universal for maps to algebraic spaces, we already know the claim when $\mathcal{X}$ is an algebraic space. In fact, we will reduce to the case when $\mathcal{X}$ is affine, in which case the factorizations follow easily from the fact that $\Gamma(\Theta_R, \mathcal{O}_{\Theta_R}) = \Gamma(\varphi_R, \mathcal{O}_{\varphi_R}) = R$.

Let $x \in \mathcal{X}(\kappa)$ be the image of $0 \in \Theta_R$. Since $\mathbb{G}_m$ has no nontrivial finite quotients, the induced map $\mathbb{G}_m \to G_x$ on stabilizers is trivial. By Proposition 4.2.15, we may find a smooth presentation $U \to \mathcal{X}$ from an affine scheme together with a lift $u \in U(\kappa)$ of x. The map $B\mathbb{G}_{m,\kappa} \to \mathcal{X}$ factors through $u \colon \operatorname{Spec} \kappa \to U$ and thus lifts to a map $B\mathbb{G}_{m,\kappa} \to \operatorname{Spec} \kappa \xrightarrow{u} U$. Letting $\mathcal{T}_n$ be the nth nilpotent thickening of $B\mathbb{G}_{m,\kappa} \hookrightarrow \Theta_R$, deformation theory (Proposition 6.5.10) implies that we may find compatible lifts $\mathcal{T}_n \to U$ of $\mathcal{T}_n \hookrightarrow \Theta_R \to \mathcal{X}$. By Coherent Tannaka Duality (6.4.9), there is an extension $\Theta_R \to U$. Since $\Theta_R \to U$ factors through $\operatorname{Spec} R$, so does $\Theta_R \to \mathcal{X}$. □

Proposition 6.8.22. *Every noetherian algebraic stack $\mathcal{X}$ with affine and quasi-finite diagonal (e.g. a Deligne–Mumford stack with affine diagonal) is Θ-complete. Moreover, $\mathcal{X}$ is S-complete if and only if it is separated.*

Proof. By Lemma 6.8.16, Θ-completeness and S-completeness can be tested on a complete DVR R. Lemma 6.8.21 implies that that $\mathcal{X}$ is Θ-complete and also implies that $\mathcal{X}$ is S-complete if only if every diagram

$$
\begin{array}{ccc}
\operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R & \longrightarrow & \mathcal{X} \\
\downarrow & \nearrow & \\
\operatorname{Spec} R &&
\end{array}
$$

has a lift, which is the usual valuative criterion for separatedness. □

Example 6.8.23. The examples in Examples 6.8.5 and 6.8.6 of non-separated Deligne–Mumford stacks are not S-complete.

6.8.3 Examples of Θ- and S-completeness

We discuss the valuative criteria of Θ-completeness and S-completeness for quotient stacks, stacks of coherent sheaves, and the stack of all curves.

Quotient stacks

By Proposition 6.6.39, a map $\Theta \to [U/G]$ is classified by a point $u \in U$ and a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot u$ exists. We apply this to provide a geometric characterization of Θ-completeness for quotient stacks. Recall that the attractor locus U_λ^+ represents the functor $\mathrm{Mor}_{\Bbbk}^{\mathbb{G}_m}(\mathbb{A}^1, U)$ (Theorem 6.6.8). The evaluation map $\mathrm{ev}_1\colon U_\lambda^+ \to U$ is defined by sending $f\colon \mathbb{A}^1 \to U$ to $f(1)$.

Proposition 6.8.24. *Let G be a smooth linearly reductive group over an algebraically closed field $\Bbbk$, and U be a separated algebraic space of finite type over $\Bbbk$ with an action of G. Then*

$$
\begin{aligned}
[U/G] \text{ is } \Theta\text{-complete} \quad &\Longleftrightarrow \quad \text{for every map } u\colon \mathrm{Spec}\, R \to U \text{ from a complete} \\
&\qquad DVR \text{ over } \Bbbk \text{ with algebraically closed residue field} \\
&\qquad \text{and one-parameter subgroup } \lambda\colon \mathbb{G}_m \to G \text{ such that} \\
&\qquad \lim_{t\to 0} \lambda(t) \cdot u_K \in U(K) \text{ exists, then} \\
&\qquad \lim_{t\to 0} \lambda(t) \cdot u \in U(R) \text{ also exists;} \\
&\Longleftrightarrow \quad \text{for every one-parameter subgroup } \lambda\colon \mathbb{G}_m \to G, \\
&\qquad \text{the morphism } \mathrm{ev}_1\colon U_\lambda^+ \to U \text{ is a closed immersion.}
\end{aligned}
$$

Proof. Since G is linearly reductive, BG is Θ-complete (Proposition 6.8.19). Therefore Θ-completeness of $[U/G]$ is equivalent to the existence of a lift in every diagram

$$
\begin{array}{ccc}
\Theta_R \smallsetminus 0 & \longrightarrow & [U/G] \\
\Big\downarrow & \nearrow & \Big\downarrow \\
\Theta_R & \longrightarrow & BG
\end{array}
\tag{6.8.25}
$$

where R is a complete DVR with algebraically closed residue field (Lemma 6.8.16). By Proposition 6.6.39, the map $\Theta_R \to BG$ corresponds to a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ while $\Theta_R \smallsetminus 0 \to [U/G]$ corresponds to a map $u\colon \mathrm{Spec}\, R \to U$ such that $\lim_{t\to 0} \lambda(t) \cdot u_K \in U(K)$ exists. In other words, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}\, K & \longrightarrow & U_\lambda^+ \\
\Big\downarrow & \nearrow & \Big\downarrow {\scriptstyle \mathrm{ev}_1} \\
\mathrm{Spec}\, R & \xrightarrow{\ u\ } & U
\end{array}
\tag{6.8.26}
$$

of solid arrows. A lift of (6.8.25) corresponds to the existence of $\lim_{t\to 0} \lambda(t) \cdot u \in U(R)$ or equivalently to a lift of (6.8.26). Since $\mathrm{ev}_1\colon U_\lambda^+ \to U$ is a monomorphism of finite type, it is closed immersion if and only if it is proper or equivalently satisfies the existence part of the valuative criterion. $\square$

Example 6.8.27. When $U = \mathrm{Spec}\, A$ is affine, a one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$ induces a grading $A = \bigoplus_{d\in\mathbb{Z}} A_d$, and U_λ^+ is represented by $V(\sum_{d<0} A_d)$ (Exercise 6.6.6). We see thus that $\mathrm{ev}_1\colon U_\lambda^+ \hookrightarrow U$ is a closed immersion; this recovers the fact that $[U/G]$ is Θ-complete (Proposition 6.8.19).

Example 6.8.28. We can use this criterion to see that Examples 6.8.2 to 6.8.4 are not Θ-complete. For $[\mathbb{P}^1/\mathbb{G}_m]$ with action $t \cdot [x : y] = [tx : y]$, taking $\lambda = \mathrm{id}$ we have that $(\mathbb{P}^1)^+_\lambda = \mathbb{A}^1 \coprod \{\infty\}$. For the quotient $[C/\mathbb{G}_m]$ of the nodal cubic C with normalization $\mathbb{P}^1 \to C$ identifying 0 and ∞, then $C^+_\infty = \mathbb{P}^1 \setminus \infty$ for $\lambda = \mathrm{id}$. Finally, for $[X/\mathbb{G}_m]$ with $X = \mathbb{A}^2 \setminus 0$ with action $t \cdot (x, y) = (tx, y)$, then $X^+_\lambda = \{y \neq 0\}$ for $\lambda = \mathrm{id}$.

Example 6.8.29. We can also provide a interpretation using the algebraic stack $\underline{\mathrm{Mor}}(\Theta, [X/G])$ of morphisms which decomposes as a disjoint union $\coprod_\lambda [X^+_\lambda / P_\lambda]$ where λ varies over conjugation classes of one-parameter subgroups $\lambda \colon \mathbb{G}_m \to G$ (Remark 6.6.42). The evaluation morphism $\mathrm{ev}_0 \colon [X^+_\lambda / P_\lambda] \to [X/G]$ is induced by the inclusion $X^+_\lambda \to X$. The Θ-completeness of $[X/G]$ corresponds to the properness of the maps $[X^+_\lambda / P_\lambda] \to [X/G]$.

One can also give a criteria for when $[U/G]$ is S-complete in terms of one-parameter subgroups $\lambda \colon \mathbb{G}_m \to G$ and properties of the morphism $\mathrm{Mor}^{\mathbb{G}_m}_{\mathbb{A}^1}(\mathbb{A}^2, U \times \mathbb{A}^1) \to U \times U \times \mathbb{A}^1$, where the maps to U are obtained by restricting along the two maps $\mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x, 1)$ and $x \mapsto (1, x)$.

Stacks of coherent sheaves

Given a projective scheme X, let $\underline{\mathrm{Coh}}(X)$ denote the algebraic stack of coherent sheaves on X (see Exercise 3.1.23). Recall that maps $\Theta \to \underline{\mathrm{Coh}}(X)$ correspond to filtrations (Proposition 6.6.43).

Proposition 6.8.30. *For every projective scheme X over an algebraically closed field $\Bbbk$, the algebraic stack $\underline{\mathrm{Coh}}(X)$ is Θ-complete and S-complete.*

Proof. Given a DVR R, Proposition 6.6.43 implies that a map $\Theta_R \setminus 0 \to \underline{\mathrm{Coh}}(X)$ corresponds to a coherent sheaf E on X_R flat over R and a $\mathbb{Z}$-graded filtration $F_\bullet \colon \cdots F_{i-1} \subset F_i \subset \cdots \subset E_K$ such that $F_i = E_K$ for $i \gg 0$, $F_i = 0$ for $i \ll 0$, and F_i/F_{i-1} is flat over R. Viewing E is a subsheaf of E_K, we define $E_i := F_i \cap E$ as the intersection in E_K. Since E_i/E_{i-1} is a subsheaf of F_i/F_{i-1}, it is torsion free, hence flat as an R-module. The filtration $E_\bullet$ defines an extension $\Theta_R \to \underline{\mathrm{Coh}}(X)$. (Aside: this is exactly the argument for the valuative criterion of properness of the Quot scheme (Proposition 1.4.2). Note also that if we let $j \colon \Theta_R \setminus 0 \hookrightarrow \Theta_R$, let $j_x \colon \mathrm{Spec}\, R \hookrightarrow \Theta_R$ and let $j_\pi \colon \Theta_K \to \Theta_R$ denote the open immersions and we let $\mathcal{E}$ be the coherent sheaf on $C \times (\Theta_R \setminus 0)$ denoting the union of E and $F_\bullet$, then the extension is given by $(\mathrm{id} \times j)_* \mathcal{E} = j_{x,*} E \cap j_{\pi,*} F_\bullet = E[x^{\pm 1}] \cap F_\bullet = E_\bullet$, where $E[x^{\pm 1}]$ is the $\mathbb{Z}$-graded filtration given by placing E in every degree.)

For S-completeness, suppose we are given a map $\overline{\mathrm{ST}}_R \setminus 0 \to \underline{\mathrm{Coh}}(X)$ corresponding to coherent sheaves E and F flat over R and an isomorphism $\alpha \colon E_K \to F_K$. Recalling the quotient presentation $\overline{\mathrm{ST}}_R = [\mathrm{Spec}(R[s,t]/(st - \pi))/\mathbb{G}_m]$, we have several natural open immersions: $j \colon \overline{\mathrm{ST}}_R \setminus 0 \hookrightarrow \overline{\mathrm{ST}}_R$, $j_s, j_t \colon \mathrm{Spec}\, R \hookrightarrow \overline{\mathrm{ST}}_R$ (with $s \neq 0$ and $t \neq 0$), and $j_{st} \colon \mathrm{Spec}\, K \to \overline{\mathrm{ST}}_R$ (with $st \neq 0$). We compute the pushforward as the equalizer

$$0 \longrightarrow (\mathrm{id} \times j)_* \mathcal{E} \longrightarrow (\mathrm{id} \times j_s)_* E \oplus (\mathrm{id} \times j_t)_* F \longrightarrow (\mathrm{id} \times j_{st})_* F_K$$

$$(a, b) \longmapsto a - \alpha(b).$$

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The pushforwards can be computed as graded modules over $R[s,t]/(st-\pi)$:

$$(\mathrm{id}\times j_{st})_*F_K = F_K \otimes_R R[t^{\pm 1}] = \bigoplus_{n\in\mathbb{Z}} F_K t^n,$$

$$(\mathrm{id}\times j_s)_*E = E \otimes_R R[t^{\pm 1}] = \bigoplus_{n\in\mathbb{Z}} E t^n,$$

$$(\mathrm{id}\times j_t)_*F = F \otimes_R R[s^{\pm 1}] \cong \bigoplus_{n\in\mathbb{Z}} (\pi^{-n}\cdot F)t^n \subset (\mathrm{id}\times j_{st})_*F_K$$

where we've used that $s = t^{-1}\pi$. Thus

$$j_*\mathcal{E} \cong \bigoplus_{n\in\mathbb{Z}} \big(E\cap(\pi^{-n}\cdot F)\big)t^n \subset (\mathrm{id}\times j_{st})_*F_K.$$

Each R-module $E\cap(\pi^{-n}\cdot F)\subset E$ is finitely generated since E is. Moreover, the ascending chain $\cdots\subset E\cap(\pi^{-n}\cdot F)\subset E\cap(\pi^{-n-1}\cdot F)\subset\cdots$ terminates to E and it follows that $j_*\mathcal{E}$ is coherent. To show that $j_*\mathcal{E}$ is flat over ϕ_R, we only need to check that it is flat over 0. By the Local Criterion for Flatness (Theorem A.2.5), we need to show that $\mathrm{Tor}_1^A(A/\mathfrak{m}, j_*\mathcal{E}) = 0$ where $A = R[s,t]/(st-\pi)$ and $\mathfrak{m} = (s,t)$. The Koszul complex gives a resolution of the residue field $\kappa = A/\mathfrak{m} = R/\pi$:

$$0 \to A \xrightarrow{(t,-s)} A\oplus A \xrightarrow{(s,t)} A \to \kappa \to 0.$$

Tensoring with $j_*\mathcal{E}$ yields a complex

$$0 \to j_*\mathcal{E} \xrightarrow{(t,-s)} j_*\mathcal{E}\oplus j_*\mathcal{E} \xrightarrow{(s,t)} j_*\mathcal{E}. \tag{6.8.31}$$

The pushforward of the exact sequence

$$0 \to \mathcal{O}_{\phi_R\smallsetminus 0} \xrightarrow{(t,-s)} \mathcal{O}_{\phi_R\smallsetminus 0}\oplus\mathcal{O}_{\phi_R\smallsetminus 0} \xrightarrow{(s,t)} \mathcal{O}_{\phi_R\smallsetminus 0} \to 0$$

along $\mathrm{id}\times j\colon C\times\phi_R\smallsetminus 0 \hookrightarrow C\times\phi_R$ is a left exact sequence of vector bundles and tensoring with $j_*\mathcal{E}$ yields a left exact sequence which identified with (6.8.31). Thus $\mathrm{Tor}_1^A(A/\mathfrak{m}, j_*\mathcal{E}) = 0$. $\qquad\square$

The description in Proposition 6.6.43 interpreting maps from Θ as filtrations allows us to prove simple criteria for an open substack $\mathcal{U}\subset\underline{\mathrm{Coh}}(X)$ to be Θ-complete or S-complete. We call two $\mathbb{Z}$-graded filtrations

$$E_\bullet : 0 \subset \cdots \subset E_{i-1} \subset E_i \subset E_{i+1} \subset \cdots \subset E$$

and

$$F^\bullet : F \supset \cdots \supset F^{i-1} \supset F^i \supset F^{i+1} \supset \cdots \supset 0$$

are *opposite* if $E_i/E_{i-1} \cong F^i/F^{i+1}$ for all i. Observe that $F_\bullet$ defined by $F_i = F^{-i}$ is a $\mathbb{Z}$-graded filtration with the same indexing as $E_\bullet$ and being opposite means that $\mathrm{gr}\,E_\bullet$ is isomorphic as a $\mathbb{Z}$-graded sheaf to $\mathrm{gr}\,F_\bullet$ with the opposite grading. A map $[(\mathrm{Spec}\,\Bbbk[x,y]/xy)/\mathbb{G}_m] \to \underline{\mathrm{Coh}}(X)$, where $t\cdot(x,y) = (tx,t^{-1}y)$, is the same data as two opposite filtration $E_\bullet$ and $F^\bullet$ such that $E_i = 0$ and $F^i = F$ for $i \ll 0$, and $E_i = E$ and $F^i = 0$ for $i \gg 0$; in this case, under this map $(1,0)\mapsto E$, $(0,1)\mapsto F$, and $(0,0)\mapsto \mathrm{gr}\,E_\bullet$.

Proposition 6.8.32. *Let C be a smooth, connected, and projective scheme over an algebraically closed field $\Bbbk$, and let $\mathcal{U}\subset\underline{\mathrm{Coh}}(C)$ be an open substack.*

(1) *The substack $\mathcal{U}$ is Θ-complete if and only if for every DVR R (with fraction field K and residue field κ), coherent sheaf E on C_R flat over R, and $\mathbb{Z}$-graded filtration $E_\bullet$ with $E_i = 0$ for $i \ll 0$, $E_i = E$ for $i \gg 0$ and with each E_i/E_{i-1} flat over R, then if E and $\mathrm{gr}(E_\bullet|_K)$ are in $\mathcal{U}$, so is $\mathrm{gr}(E_\bullet|_\kappa)$.*

(2) *The substack is S-complete if and only if for every pair of opposite filtrations $E_\bullet$ and $F^\bullet$ of $E, F \in \mathcal{U}(\mathbb{k})$, the associated graded $\mathrm{gr}\, E_\bullet$ is in $\mathcal{U}$.*

Remark 6.8.33. For a projective scheme of arbitrary dimension, Part (1) and the ($\Leftarrow$) implication in (2) hold with the same proof.

Proof. Since we already know that $\underline{\mathrm{Coh}}(C)$ is S-complete and Θ-complete, the valuative criteria for $\mathcal{U}$ are equivalent to the existence of lifts for all commutative diagrams

$$
\begin{array}{ccc}
\Theta_R \smallsetminus 0 \longrightarrow \mathcal{U} & & \phi_R \smallsetminus 0 \longrightarrow \mathcal{U} \\
\downarrow \qquad \qquad \downarrow & \text{and} & \downarrow \qquad \qquad \downarrow \\
\Theta_R \longrightarrow \underline{\mathrm{Coh}}(C) & & \phi_R \longrightarrow \underline{\mathrm{Coh}}(C)
\end{array}
$$

where R is a DVR. In other words, we need to show that the images of 0 under the unique fillings $\Theta_R \to \underline{\mathrm{Coh}}(C)$ and $\phi_R \to \underline{\mathrm{Coh}}(C)$ are contained in $\mathcal{U}$. Therefore (1) holds as the image of 0 under $\Theta_R \to \underline{\mathrm{Coh}}(C)$ is $\mathrm{gr}(E_\bullet|_\kappa)$.

For the ($\Leftarrow$) implication in (2), the restriction of $\phi_R \to \underline{\mathrm{Coh}}(C)$ along $\pi = 0$ yields a map $[\mathrm{Spec}(\mathbb{k}[x,y]/xy)/\mathbb{G}_m] \to \underline{\mathrm{Coh}}(C)$ corresponding to opposite filtrations $E_\bullet$ and $F^\bullet$. If we $\mathrm{gr}\, E_\bullet \in \mathcal{U}(\mathbb{k})$, then the image of $\phi_R \to \underline{\mathrm{Coh}}(C)$ is contained in $\mathcal{U}$. Conversely, let $[\mathrm{Spec}(\mathbb{k}[x,y]/xy)/\mathbb{G}_m] \to \underline{\mathrm{Coh}}(X)$ be a map such that the images of $(1,0)$ and $(0,1)$ are in $\mathcal{U}$ but the image of $(0,0)$ is not in $\mathcal{U}$. Let $\mathcal{X}_n$ be the nth nilpotent thickening of the closed immersion $[\mathrm{Spec}(\mathbb{k}[x,y]/xy)/\mathbb{G}_m] \hookrightarrow \phi_R$. Since the obstruction to lifting a coherent sheaf E lies in the second coherent cohomology of $\mathcal{X}_0$ and since $\mathcal{X}_0$ is cohomologically affine, deformation theory and Coherent Tannaka Duality (6.4.9) yield an extension $\phi_R \to \underline{\mathrm{Coh}}(X)$ with the image of $\phi_R \smallsetminus 0$ contained in $\mathcal{U}$. $\square$

Remark 6.8.34. If the genus of C is at least 2, then the stack of vector bundles $\underline{\mathrm{Bun}}(C)$ is not Θ-complete nor S-complete. Let $p \in C$ be a point defined by the vanishing of a section $s \in \Gamma(C, \mathcal{O}(p))$, and let $I \subset \mathcal{O}_{C_R}$ be the ideal sheaf of $(p, 0) \in C \times \mathrm{Spec}\, R$. The injection $(s, -\pi)\colon \mathcal{O}_{C_R}(-p) \hookrightarrow \mathcal{O}_{C_R} \oplus \mathcal{O}_{C_R}(p)$ has quotient I, which is torsion free, hence flat over R, but is not a vector bundle. By Proposition 6.8.32, we see that $\underline{\mathrm{Bun}}(C)$ is not Θ-complete.

Let L and M be line bundles on C, and let $p \in C$ be a point such that $\mathrm{Ext}^1_{\mathcal{O}_C}(M, L(p))$ and $\mathrm{Ext}^1_{\mathcal{O}_C}(L, M(p))$ are nonzero; if L and M have the same degree, then a Riemann–Roch calculation shows that both Ext^1 groups are nonzero. Let Q (resp. Q') be a nontrivial extension of M by $L(p)$ (resp. L by $M(p)$). Then

$$E_\bullet : 0 \subset L \subset L(p) \subset Q \quad \text{and} \quad F^\bullet : Q' \supset M(p) \supset M \supset 0$$

define opposite filtrations where $E_0 = L$ and $F^0 = Q'$. The associated graded $\mathrm{gr}\, E_\bullet = L \oplus \kappa(p) \oplus M$ is not a vector bundle, and thus $\underline{\mathrm{Bun}}(C)$ is not S-complete by Proposition 6.8.32.

We will apply the above criteria later to verify that the stack $\underline{\mathrm{Bun}}^{\mathrm{ss}}_{r,d}(C)$ of semistable vector bundles on a smooth, connected, and projective curve is both Θ-complete and S-complete.

Stack of all curves

The stacks $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ of smooth and stable curves are both Θ-complete and S-complete as they are separated Deligne–Mumford stacks. While maps from Θ to the stack of all curves correspond to test configurations (Proposition 6.6.44), there is unfortunately no known simple criteria—similar to the above criteria for quotient stacks and stacks of coherent sheaves—to verify whether a given substack of the stack $\mathcal{M}_g^{\mathrm{all}}$ of all curves is Θ-complete or S-complete.

6.8.4 Θ-completeness and Θ-surjectivity

The property that a morphism $\mathcal{X} \to \mathcal{Y}$ sends closed points to closed points is not stable under base change (see Examples 6.8.2 and 6.8.3). We introduce a stronger and better behaved property called Θ-surjectivity. The main result of this section is that an étale quotient presentation $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ is Θ-surjective in an open neighborhood of w as long as $\mathcal{X}$ is Θ-complete (prop:theta-surjective-in-open-neighborhood). As motivated in §6.8.1, this result will be crucial in proving the main existence theorem (Theorem 6.8.1) of this section.

Definition 6.8.35. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of algebraic stacks and $x \in \mathcal{X}(\Bbbk)$ be a geometric point. We say that f is Θ-*surjective at* x if every diagram

$$
\begin{array}{ccc}
\operatorname{Spec} \Bbbk & \xrightarrow{\;x\;} & \mathcal{X} \\
{\scriptstyle 1}\big\downarrow\cap & \nearrow & \big\downarrow{\scriptstyle f} \\
\Theta_{\Bbbk} & \longrightarrow & \mathcal{Y}
\end{array}
\tag{6.8.36}
$$

has a lift. We say that f is Θ-*surjective* if it is Θ-surjective at every geometric point.

This notion is clearly stable under base change. Every morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of noetherian algebraic stacks where $\mathcal{Y}$ has affine and quasi-finite diagonal is Θ-surjective since in this case every map $\Theta_{\Bbbk} \to \mathcal{Y}$ factors through $\operatorname{Spec} \Bbbk$ (Lemma 6.8.21). The next lemma gives conditions for when the lift is unique and when the definition is independent of the choice of geometric point.

Lemma 6.8.37. *Let* $f \colon \mathcal{X} \to \mathcal{Y}$ *be a separated, representable, and finite type morphism of noetherian algebraic stacks.*

(1) Every lift of (6.8.36) *is unique.*

(2) If f *is* Θ-*surjective at a geometric point* $x \in \mathcal{X}(\Bbbk)$, *then* f *is* Θ-*surjective at every other geometric point* $x' \in \mathcal{X}(\Bbbk')$ *representing the same point in* $|\mathcal{X}|$ *as* x.

Proof. Part (1) follows from descent and the valuative criterion for separatedness. To show (2), it suffices to show that given an extension $\Bbbk \to \Bbbk'$ of algebraically closed fields, a lift $\Theta_{\Bbbk'} \to \mathcal{X}$ implies the existence of a lift $\Theta_{\Bbbk} \to \mathcal{X}$. We write $\Bbbk' = \bigcup_\lambda A_\lambda$ as a union of finitely generated $\Bbbk$-subalgebras. By Limit Methods (§B.3), there exists a lift $\Theta_{A_\lambda} \to \mathcal{X}$ of $\operatorname{Spec} A_\lambda \to \mathcal{X}$. Restricting along a closed point of $\operatorname{Spec} A_\lambda$ provides a lift over $\Bbbk$. $\qquad\square$

Proposition 6.8.38. *Let* $f \colon \mathcal{X} \to \mathcal{Y}$ *be a morphism of algebraic stacks, each of finite type over an algebraically closed field* $\Bbbk$ *with affine diagonal. Suppose that the closed points of* $\mathcal{Y}$ *have linearly reductive stabilizer. If* f *is* Θ-*surjective, then* f *sends closed points to closed points.*

Proof. Let $x \in \mathcal{X}$ be a closed point. Let $f(x) \rightsquigarrow y_0$ be a specialization to a closed point. By Corollary 6.6.38, this specialization can be realized by a map $\Theta \to \mathcal{Y}$. Since f is Θ-surjective, this can be lifted to a map $g: \Theta \to \mathcal{X}$ with $g(1) = x$. But $x \in \mathcal{X}$ is a closed point, so this lift must correspond to the trivial specialization $x \rightsquigarrow x$. It follows that $f(x) = y_0$ is a closed point. $\qquad\square$

Remark 6.8.39. The converse is not true. In Example 6.8.3, where C is the nodal cubic with $\mathbb{G}_m$-action, the étale morphism $[\operatorname{Spec}(\Bbbk[x,y]/(xy))/\mathbb{G}_m] \to [C/\mathbb{G}_m]$ sends closed points to closed points but is not Θ-surjective.

Proposition 6.8.40. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal such that the closed points of $\mathcal{X}$ have linearly reductive stabilizers. Let $x \in \mathcal{X}$ be a closed point, let $f: ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an affine étale morphism inducing an isomorphism of stabilizer groups at w, and let $\pi: [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. If $\mathcal{X}$ is Θ-complete, there exists an open affine neighborhood $U \subset \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)}: \pi^{-1}(U) \to \mathcal{X}$ is Θ-surjective.*

Proof. Let $\mathcal{W} = [\operatorname{Spec} A/G_x]$ and define $\Sigma_f \subset |\mathcal{W}|$ as the set of points $y \in \mathcal{W}$ such that f is Θ-surjective at y. We first show that $\Sigma_f \subset \mathcal{W}$ is open if $\mathcal{X} \cong [\operatorname{Spec} B/G]$ with G linearly reductive. Zariski's Main Theorem (6.1.10) provides a factorization

$$ f: \mathcal{W} \xhookrightarrow{j} \widetilde{\mathcal{X}} \xrightarrow{\nu} \mathcal{X} $$

where j is an open immersion and ν is a finite morphism. By Lemma 6.8.18, $[\operatorname{Spec} B/G]$ is Θ-complete, and by Proposition 6.8.19, $\widetilde{\mathcal{X}}$ is also Θ-complete. As ν is finite, $\Sigma_j = \Sigma_f$ and we may assume that f is an open immersion. Let $\mathcal{Z} \subset \mathcal{X}$ be the reduced complement of $\mathcal{W}$ and let $\pi: \mathcal{X} \to \operatorname{Spec} B^G$ denote the good moduli space. We claim that $|\mathcal{W}| \setminus \Sigma_f = \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{W}|$. The inclusion "$\subset$" is clear: the morphism $\mathcal{X} \setminus \pi^{-1}(\pi(|\mathcal{Z}|)) \hookrightarrow \mathcal{X}$ is the base change of the Θ-surjective morphism $X \setminus \pi(|\mathcal{Z}|) \hookrightarrow X$ of algebraic spaces. Conversely, let $y \in \pi^{-1}(\pi(|\mathcal{Z}|)) \cap |\mathcal{W}|$ represented by a geometric point $\operatorname{Spec} \mathbb{F} \to \mathcal{X}$. Let $z \in |\mathcal{Z}_{\mathbb{F}}|$ be the unique closed point in the closure of $y \in |\Theta_{\mathbb{F}}|$ and let $\Theta_{\mathbb{F}} \to \mathcal{X}_{\mathbb{F}}$ be a morphism representing the specialization $y \rightsquigarrow z$ (Corollary 6.6.38). Since $\Theta_{\mathbb{F}} \to \mathcal{X}$ does not lift to $\mathcal{W}$, $y \notin \Sigma_f$.

We now claim that $\Sigma_f \subset \mathcal{W}$ is constructible. Use the Local Structure Theorem (6.5.1) to choose an affine, étale, and surjective morphism $g: \mathcal{X}' = [\operatorname{Spec} B/G] \to \mathcal{X}$ with G linearly reductive. Let $\mathcal{W}' = \mathcal{W} \times_{\mathcal{X}} \mathcal{X}'$ with projections $g': \mathcal{W}' \to \mathcal{W}$ and $f': \mathcal{W}' \to \mathcal{X}'$. Since we already know that $\Sigma_{f'}$ is open, the claim follows from Chevalley's Theorem (3.3.29) once we show that $\mathcal{W} \setminus \Sigma_f = g'(\mathcal{W}' \setminus \Sigma_{f'})$. To see this, it suffices to show that for an algebraically closed field K, every map $h: \Theta_{\mathbb{F}} \to \mathcal{X}$ lifts to a map $h': \Theta_{\mathbb{F}} \to \mathcal{X}'$. Let $x' \in \mathcal{X}'(\mathbb{F})$ be a preimage of $h(0) \in \mathcal{X}(\mathbb{F})$. Since g is representable and étale, the induced map $G_{x'} \to G_{h(0)}$ on stabilizers is injective with finite cokernel. Thus the map $\mathbb{G}_{m,\mathbb{F}} \to G_{h(0)}$ on stabilizers induced by $h: \Theta_{\mathbb{F}} \to \mathcal{X}$ factors through $G_{x'}$. We may therefore lift the map $h|_{B\mathbb{G}_{m,\mathbb{F}}}$ to a map $B\mathbb{G}_{m,\mathbb{F}} \to \mathcal{X}'$. Letting $\mathcal{X}_n$ be the nth nilpotent thickening of $B\mathbb{G}_{m,\mathbb{F}} \hookrightarrow \Theta_{\mathbb{F}}$, there are compatible lifts $\mathcal{X}_n \to \mathcal{X}'$ of $\mathcal{X}_n \to \mathcal{X}$ by deformation theory (Proposition 6.5.10) which extends to a lift $\Theta_{\mathbb{F}} \to \mathcal{X}'$ by Coherent Tannaka Duality (6.4.9).

Since $\Sigma_f \subset \mathcal{W}$ is constructible and $w \in \Sigma_f$, to show that Σ_f is open, it suffices to show that for every generization $\xi \rightsquigarrow w$ of w is contained in Σ_f. Let $h: \operatorname{Spec} R \to \mathcal{W}$ be a morphism from a complete DVR representing the specialization $\xi \rightsquigarrow w$. Letting K and κ be the fraction and residue field of R, we claim that there exists a lift

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(necessarily unique as f is separated)

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\ h_{\mathbb{F}}\ } & \mathcal{W} \\ {\scriptstyle\cap}\Big\downarrow & {\scriptstyle\widetilde{g}}\ \nearrow & \Big\downarrow {\scriptstyle f} \\ \Theta_K & \xrightarrow{\ g\ } & \mathcal{X}. \end{array} \qquad (6.8.41)$$

This claim implies that f is Θ-surjective at ξ, i.e. $\xi \in \Sigma_f$. To show the claim, we first apply the Θ-completeness of $\mathcal{X}$ to construct a lift

$$\begin{array}{ccc} \Theta_R \smallsetminus 0 & \xrightarrow{(f \circ h) \cup g} & \mathcal{X} \\ {\scriptstyle\cap}\Big\downarrow & {\scriptstyle q}\ \nearrow & \\ \Theta_R. & & \end{array}$$

Since $\mathcal{W} \to \mathcal{X}$ is stabilizer preserving at w, we have a lift $B\mathbb{G}_{m,\kappa} \to \mathcal{W}$ of $q|_{B\mathbb{G}_{m,\kappa}}$. Since Θ_R is coherently complete along $B\mathbb{G}_{m,\kappa}$ (6.4.12), we may apply deformation theory (Proposition 6.5.10) and Coherent Tannaka Duality (6.4.9) to construct a lift

$$\begin{array}{ccc} B\mathbb{G}_{m,\kappa} & \xrightarrow{\quad\quad} & \mathcal{W} \\ {\scriptstyle\cap}\Big\downarrow & {\scriptstyle\widetilde{q}}\ \nearrow & \Big\downarrow {\scriptstyle f} \\ \Theta_R & \xrightarrow{\ q\ } & \mathcal{X} \end{array}$$

The restriction $\widetilde{q}|_{\operatorname{Spec} R}$ is 2-isomorphic to h since it agrees at the closed point and f is étale. It follows that $\widetilde{g} := \widetilde{q}|_{\Theta_K}$ is a lift of (6.8.41). $\qquad\square$

The topology of $\Bbbk$-points of Θ-complete stacks is analogous to the topology of quotient stacks arising from GIT.

Proposition 6.8.42. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that $\mathcal{X}$ is Θ-complete and that the closed points of $\mathcal{X}$ have linearly reductive stabilizer. Then the closure of every $\Bbbk$-point contains a unique closed point.*

Proof. Assume that x and x' are two closed points in the closure of $p \in \mathcal{X}(\Bbbk)$. By Corollary 6.6.38, there are maps $f, f' \colon \Theta \to \mathcal{X}$ realizing the specializations $p \rightsquigarrow x$ and $p \rightsquigarrow x'$. Under the action of $\mathbb{G}_m^2$ on $\mathbb{A}^2$ given by $(t_1, t_2) \cdot (y_1, y_2) = (t_1 y_1, t_2 y_2)$, the maps f and f' glue to define a map $[\mathbb{A}^2/\mathbb{G}_m^2] \smallsetminus 0 \to \mathcal{X}$. By considering only the diagonal $\mathbb{G}_m$-action, the map $[\mathbb{A}^2/\mathbb{G}_m] \smallsetminus 0 \to \mathcal{X}$ extends to $\Psi \colon [\mathbb{A}^2/\mathbb{G}_m] \to \mathcal{X}$ by the Θ-completeness of $\mathcal{X}$. Then $\Psi(0,0)$ is a common specialization of $x = \Psi(1,0)$ and $x' = \Psi(0,1)$. Since x and x' are closed points, we have that $x = \Psi(0,0) = x'$. $\qquad\square$

Exercise 6.8.43. With the hypotheses of Proposition 6.8.42, show that if in addition $\mathcal{X}$ has a unique closed point, then $\mathcal{X} \cong [\operatorname{Spec}(A)/G_x]$ such that A^{G_x} is an artinian local $\Bbbk$-algebra with residue field $\Bbbk$.

6.8.5 Unpunctured inertia

We prove that an S-complete stack $\mathcal{X}$ has 'unpunctured inertia' (Theorem 6.8.50) and the consequence that an étale quotient presentation $f \colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ is stabilizer preserving in an open neighborhood of w (Proposition 6.8.47).

Definition 6.8.44. We say that a noetherian algebraic stack has *unpunctured inertia* if for every closed point $x \in |\mathcal{X}|$ and every formally versal morphism $p \colon (T, t) \to (\mathcal{X}, x)$ where T is the spectrum of a local ring with closed point t, every connected component of the inertia group scheme $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ has non-empty intersection with the fiber over t.

Remark 6.8.45. Here $(T, t) \to (\mathcal{X}, x)$ is formally versal if the map $\widehat{T} \to \mathcal{X}$ from the completion is is formally versal at t as in Definition C.3.5.

Remark 6.8.46. Unpuncturedness is related to the purity of the morphism $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ as defined in [GR71, §3.3] (see also [SP, Tag 0CV5]). If T is the spectrum of a strictly henselian local ring, then purity requires that if $s \in T$ is an *arbitrary* point and γ is an associated point in the fiber $\underline{\mathrm{Aut}}_{\mathcal{X}}(p)_s$, then the closure of γ in $\underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ has non-empty intersection with the fiber over the closed point t of T.

Proposition 6.8.47. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let $x \in \mathcal{X}$ be a closed point with linearly reductive stabilizer. Let $f \colon ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ be an affine étale morphism inducing an isomorphism of stabilizer groups at w, and let $\pi \colon [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. If $\mathcal{X}$ has unpunctured inertia, there exists an open affine neighborhood $U \subset \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)} \colon \pi^{-1}(U) \to \mathcal{X}$ induces isomorphisms of stabilizer groups at all points.*

Proof. Set $\mathcal{W} = [\operatorname{Spec} A/G_x]$. It suffices to find an open neighborhood $\mathcal{U} \subset \mathcal{W}$ of w such that $f|_{\mathcal{U}} \colon \mathcal{U} \to \mathcal{X}$ induces an isomorphism $I_{\mathcal{U}} \to \mathcal{U} \times_{\mathcal{X}} I_{\mathcal{X}}$. Consider the cartesian diagram

$$
\begin{array}{ccc}
I_{\mathcal{W}} & \longrightarrow & \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \\
\downarrow & \square & \downarrow \\
\mathcal{W} & \longrightarrow & \mathcal{W} \times_{\mathcal{X}} \mathcal{W};
\end{array}
$$

see Exercise 3.2.14. Since f is separated and étale, the morphism $I_{\mathcal{W}} \to \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ is finite and étale. We set $\mathcal{Z} \subset \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ to be the open and closed substack over which $I_{\mathcal{W}} \to \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}}$ is *not* an isomorphism. Since f is stabilizer preserving at w, the point w is not contained in the image of $\mathcal{Z}$ under $p_1 \colon \mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \to \mathcal{W}$.

Consider a formally smooth morphism $(T, t) \to (\mathcal{X}, x)$ from the spectrum of a local ring with closed point t. Since $\mathcal{X}$ has unpunctured inertia, the preimage of $\mathcal{Z}$ in $\mathcal{W} \times_{\mathcal{X}} I_{\mathcal{X}} \times_{\mathcal{X}} T$ is empty; indeed, if there were a non-empty connected component of this preimage, it must intersect the fiber over t non-trivially contradicting that $w \notin p_1(\mathcal{Z})$. This in turn implies that $w \notin \overline{p_1(\mathcal{Z})}$. Therefore, if we set $\mathcal{U} = \mathcal{W} \setminus \overline{p_1(\mathcal{Z})}$, the induced morphism $I_{\mathcal{U}} \to \mathcal{U} \times_{\mathcal{X}} I_{\mathcal{X}}$ is an isomorphism. $\square$

Proposition 6.8.48. *Let $\mathcal{X}$ be a noetherian algebraic stack.*

(1) If $\mathcal{X}$ has quasi-finite inertia, then $\mathcal{X}$ has unpunctured inertia if and only if $\mathcal{X}$ has finite inertia.

(2) If $\mathcal{X}$ has connected stabilizer groups, then $\mathcal{X}$ has unpunctured inertia.

Proof. If $\mathcal{X}$ has finite inertia, then $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ is finite, so clearly the image of each connected component contains the unique closed point $t \in T$. For the converse, we may assume that T is the spectrum of a Henselian local ring, in which case $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) = G \coprod H$ where $G \to T$ finite and the fiber of $H \to T$ over t is empty (Proposition B.5.10). If T is nonempty (i.e. $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to T$ is not finite), then any

connected component of T doesn't meet the central fiber and thus $\mathcal{X}$ does not have unpunctured inertia.

For (2), by definition, all fibers of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to U$ are connected, so every connected component of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ intersects the component containing the identity section. $\square$

Remark 6.8.49. For algebraic stacks with connected stabilizer groups (e.g. the moduli stack $\underline{\mathrm{Bun}}_{r,d}^{\mathrm{ss}}(C)$ of semistable vector bundles on a curve), Proposition 6.8.48(2) implies unpunctured inertia. The deeper result below (Theorem 6.8.50) is therefore unneeded in the proof of the existence of a good moduli space of $\underline{\mathrm{Bun}}_{r,d}^{\mathrm{ss}}(C)$.

The rest of this section is dedicated to proving the following theorem.

Theorem 6.8.50. *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Assume that the closed points have linearly reductive stabilizers. If $\mathcal{X}$ is S-complete, then $\mathcal{X}$ has unpunctured inertia.*

Proof. Let $x \in |\mathcal{X}|$ be a closed point, let $p \colon (U, u) \to (\mathcal{X}, x)$ be a formally smooth morphism from the spectrum of a local ring, and let $H \subset \underline{\mathrm{Aut}}_{\mathcal{X}}(p)$ be a connected component. The image of the projection $H \to U$ is a constructible set whose closure contains u. It follows that we can find a DVR R with residue field $\Bbbk$ and a map $\mathrm{Spec}\,R \to U$ whose special point maps to u and whose generic point lies in the image of $H \to U$. Let $\xi \colon \mathrm{Spec}\,R \to U \xrightarrow{p} \mathcal{X}$ denote the composition. After a residually-trivial extension of DVRs, we may assume that the generic point $\mathrm{Spec}\,K \to U$ lifts to H. This gives a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}\,K & \longrightarrow & \mathrm{Spec}\,R \\
\downarrow & & \downarrow \quad \searrow{\scriptstyle \xi} \\
H & \longrightarrow U & \xrightarrow{\ p\ } \mathcal{X}.
\end{array}
$$

Let H_K be the base change of $H \to U$ along $\mathrm{Spec}\,K \to U$. We claim we can choose a finite type point $g \in H_K$ of finite order. If $g \in H_K$ is a finite type point, then after replacing K with a finite field extension, we can decompose $g = g_s g_u$ under the Jordan decomposition, where g_s is semisimple and g_u is unipotent (see §B.2.5). Now consider the reduced Zariski closed K-subgroup $H' \subset \underline{\mathrm{Aut}}_{\mathcal{X}}(p)_K$ generated by g_s. Because g_s is semisimple, H' is a diagonalizable group scheme over K, and we may replace g_s with a finite order element in H' which still commutes with g_u. If $\mathrm{char}(K) > 0$, then g_u has finite order and we are finished. If $\mathrm{char}(K) = 0$, then g_u lies in the identity component of G, so g lies on the same component as the finite order element g_s. This gives the desired element.

We claim that after replacing R with a residually-trivial extension, there is a map $\xi' \colon \mathrm{Spec}\,R \to \mathcal{X}$ such that $\xi_K' \simeq \xi_K$ and $g \in H_K$ extends to an automorphism of ξ'. This would finish the proof: since the closure of g meets the fiber of $\underline{\mathrm{Aut}}_{\mathcal{X}}(p) \to U$ over u, the component H must also meet the central fiber.

If $\mathcal{X} \cong [\mathrm{Spec}\,A/\mathrm{GL}_n]$, then this claim is precisely the content of Proposition 6.8.51 below. We will use the Local Structure Theorem (6.5.1) to reduce to this case: let $f \colon ([\mathrm{Spec}\,A/G_x], w) \to (\mathcal{X}, x)$ be an étale quotient presentation. After replacing R with a residually-trivial extension, we may lift ξ to a map $\widetilde{\xi} \colon \mathrm{Spec}\,R \to [\mathrm{Spec}\,A/G_x]$ such that $\widetilde{\xi}(0) = w$. To show that g lifts to an element $\widetilde{g} \in \mathrm{Aut}(\widetilde{\xi}_K)$, we will use S-completeness. We may glue ξ to itself along g to define a morphism

$$
\mathrm{Spec}\,R \bigcup_{\mathrm{Spec}\,K} \mathrm{Spec}\,R = \mathfrak{ST}_R \setminus 0 \to \mathcal{X}.
$$

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Since $\mathcal{X}$ is S-complete, this map extends to a morphism $h \colon \Theta_R \to \mathcal{X}$. Since $\xi(0) = x$ and x is a closed point, the image $h(0)$ of $0 \in \Theta_R$ is also x. Since f is stabilizer preserving at w, we may lift $h|_{B\mathbb{G}_m}$ to a map $\widetilde{h}_0 \colon B\mathbb{G}_m \to [\operatorname{Spec} A / G_x]$ with image w. By Deformation Theory (6.5.10), we may find compatible lifts to $[\operatorname{Spec} A / G_x]$ of the restrictions of h to the nilpotent thickenings of Θ_R along 0, and by Coherent Tannaka Duality (6.4.9), we may find construct a lift $\widetilde{h}$ below

$$
\begin{array}{ccc}
B\mathbb{G}_m & \xrightarrow{\ \widetilde{h}_0\ } & [\operatorname{Spec} A / G_x] \\
\Big\downarrow & \nearrow^{\widetilde{h}} & \Big\downarrow{f} \\
\Theta_R & \xrightarrow{\ h\ } & \mathcal{X}.
\end{array}
$$

Since f is affine and étale, both restrictions $\widetilde{h}|_{s \neq 0}$ and $\widetilde{h}|_{t \neq 0}$ to $\operatorname{Spec} R$ are isomorphic to $\widetilde{\xi}$ and thus $\widetilde{h}|_{\Theta_R \setminus 0}$ gives a lift $\widetilde{g} \in \operatorname{Aut}(\widetilde{\xi}_K)$ of g. Finally, we apply Proposition 6.8.51 to construct a map $\widetilde{\xi}' \colon \operatorname{Spec} R \to [\operatorname{Spec} A / G_x]$ with $\widetilde{\xi}'(0) = w$ such that $\widetilde{\xi}_K \simeq \widetilde{\xi}'_K$ and $\widetilde{g}$ extends to an automorphism of $\widetilde{\xi}'$. The composition $\xi'' := f \circ \widetilde{\xi}'' \colon \operatorname{Spec} R \to \mathcal{X}$ then satisfies the claim.

See also [AHLH18, Thm. 5.2]. $\qquad\square$

Our proof used the following valuative criterion for a quotient stack.

Proposition 6.8.51. *Let $\mathcal{X} = [\operatorname{Spec} A / G]$ where $\operatorname{Spec} A$ is an affine scheme of finite type over an algebraically closed field $\Bbbk$ equipped with an action by a linearly reductive group G. Let $x \in \mathcal{X}$ be a closed point. Then $\mathcal{X}$ satisfies the following property:*

($\star$) *For every DVR R with residue field $\Bbbk$ and fraction field K, for every morphism $\xi \colon \operatorname{Spec} R \to \mathcal{X}$ with $\xi(0) \simeq x$, and for every K-pint $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ of finite order, there is an extension $R \to R'$ of DVRs (with $K' = \operatorname{Frac}(R')$) and a morphism $\xi' \colon \operatorname{Spec} R' \to \mathcal{X}$ such that $\xi'(0) \simeq x$, $\xi'_{K'} \simeq \xi_{K'}$ and $g|_{K'}$ extends to an automorphism of ξ'.*

Remark 6.8.52. In other words, for every map $\xi \colon \operatorname{Spec} R \to \operatorname{Spec} A$ and element $g \in G_{\xi_K} \subset G(K)$ of finite order, there exists after an extension $R \subset R'$ of DVRs and an element $h \in G(K')$ such that $h \cdot \xi_{K'}$ extends to a map $\xi' \colon \operatorname{Spec} R' \to \operatorname{Spec} A$ with $\xi'(0) \in Gx$ and such that $h^{-1} g|_{K'} h$ extends to an R'-point of G.

To illustrate this criterion, consider the the action of $G = \mathbb{G}_m \rtimes \mathbb{Z}/2$ on $\mathbb{A}^2$ via $t \cdot (a, b) = (ta, t^{-1}b)$ and $-1 \cdot (a, b) = (b, a)$. Note that every point $(a, b) \in \mathbb{A}^2$ with $ab \neq 0$ is fixed by the order 2 element $(a/b, -1) \in G$. Consider $\xi \colon \operatorname{Spec} R = \Bbbk\llbracket z \rrbracket \to \mathbb{A}^2$ via $z \mapsto (z^2, z)$. The element $g = (z^{-1}, -1) \in G(\Bbbk(\!(z)\!))$ stabilizes ξ_K but does not extend to $G(\Bbbk\llbracket z \rrbracket)$. However, we may take the degree 2 ramified extension $\Bbbk\llbracket z \rrbracket \to \Bbbk\llbracket \sqrt{z} \rrbracket$ and define $\xi' \colon \operatorname{Spec} \Bbbk\llbracket \sqrt{z} \rrbracket \to \mathbb{A}^2$ by $\sqrt{z} \mapsto ((\sqrt{z})^3, (\sqrt{z})^3)$. Over the generic point, there is an isomorphism $\xi'_{\Bbbk(\!(\sqrt{z})\!)} \simeq \xi_{\Bbbk(\!(\sqrt{z})\!)}$ given by $h = (\sqrt{z}, -1) \in G(\Bbbk(\!(\sqrt{z})\!))$ and the element $g|_{\Bbbk(\!(\sqrt{z})\!)} = (\sqrt{z}, -1)^{-1} \cdot g|_{K'} \cdot (\sqrt{z}, -1) = (1, -1) \in G(\Bbbk(\!(\sqrt{z})\!))$ extends to an element of $G(\Bbbk\llbracket \sqrt{z} \rrbracket)$-point.

Proof. After choosing an embedding $G \hookrightarrow \operatorname{GL}_n$ and replacing $[\operatorname{Spec} A / G]$ with $[(\operatorname{Spec} A \times^{G_x} \operatorname{GL}_n) / \operatorname{GL}_n]$, we may assume that $G = \operatorname{GL}_n$.

We first verify ($\star$) for quotient stacks $[\operatorname{Spec} A / G] = \operatorname{Spec} A \times BG$ with a trivial action. As R is local and $G = \operatorname{GL}_n$, the composition $\operatorname{Spec} R \to [\operatorname{Spec} A / G] \to BG$ corresponds to the trivial G-bundle. We need to prove that every finite order element $g \in G(K)$ is conjugate to an element of $G(R)$ after passing to an extension of the

DVR R. We can conjugate g to its Jordan canonical form (after an extension of R). Since g has finite order, the diagonal entries of the resulting matrix are rth roots of unity for some r. Because the group μ_r of rth roots of unity is a finite group scheme over $\operatorname{Spec} R$, the entries of the Jordan canonical form must lie in R.

If $\mathcal{X} = [X/G]$ with X proper over $\Bbbk$, we show that $(\star)$ holds except that $\xi'(0)$ may not be isomorphic to x. Since $p \colon \mathcal{X} \to BG$ is proper and representable, for every morphism $\xi \colon \operatorname{Spec} R \to \mathcal{X}$ from a DVR, we have a closed immersion $\underline{\operatorname{Aut}}_{\mathcal{X}}(\xi) \hookrightarrow \underline{\operatorname{Aut}}_{BG}(p \circ \xi)$ of group schemes over $\operatorname{Spec} R$. Moreover, any lift of the generic point of a morphism $\operatorname{Spec} R \to BG$ to $[X/G]$ extends to a unique morphism $\operatorname{Spec} R \to BG$. Therefore, given an element $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$, we use that $(\star)$ holds for BG to find (after replacing R with an extension) a morphism $\eta \colon \operatorname{Spec} R \to BG$ such that $\eta_K \simeq (p \circ \xi)_K$ and $g|_K$ extends to a R-point of $\operatorname{Aut}_{BG}(\eta)$. If we lift η to a morphism $\xi' \colon \operatorname{Spec} R \to [X/G]$ such that $\xi'_K \simeq \xi_K$, then the element $g|_K$ extends to an automorphism of ξ'.

In verifying $(\star)$ for $[\operatorname{Spec} A/G]$, we may assume that A is reduced. Viewing $[\operatorname{Spec} A/G]$ as an algebraic stack which is affine and of finite type over $\operatorname{Spec} A^G \times BG$, we can choose a vector bundle $\mathcal{E}$ on $\operatorname{Spec} A^G \times BG$ and a G-equivariant embedding $\operatorname{Spec} A \hookrightarrow \mathbb{A}(\mathcal{E}|_{\operatorname{Spec} A^G})$ over A^G. Viewing $\mathbb{A}(\mathcal{E}|_{\operatorname{Spec} A^G})$ as an open subscheme of $\mathbb{P}(\mathcal{E}|_{\operatorname{Spec} A^G} \oplus \mathcal{O}_{\operatorname{Spec} A^G})$, we let X be the closure of $\operatorname{Spec} A$ in $\mathbb{P}(\mathcal{E}|_{\operatorname{Spec} A^G} \oplus \mathcal{O}_{\operatorname{Spec} A^G})$. This gives a G-equivariant diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A & \lhook\joinrel\longrightarrow & X \\
& \searrow & \downarrow \\
& & \operatorname{Spec} A^G
\end{array}
\qquad (6.8.53)
$$

where X is a reduced projective scheme and the complement $X \setminus \operatorname{Spec} A$ is the support of an ample G-invariant Cartier divisor E. We also claim that $\operatorname{Spec} A$ is precisely the semistable locus of X with respect to $\mathcal{O}_X(E)$ in the sense of Exercise 6.7.12. Indeed the tautological invariant section $s \colon \mathcal{O}_X \to \mathcal{O}_X(E)$ restricts to an isomorphism over $\operatorname{Spec} A$ and thus $\operatorname{Spec} A \subset X^{\mathrm{ss}}$. Conversely, s^n defines an isomorphism

$$
A^G \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X(nE))^G
$$

for all $n \geq 0$. Under this isomorphism, for every invariant global section $f \in \Gamma(X, \mathcal{O}_X(nE))^G$, the restriction $f|_{\operatorname{Spec} A}$ agrees with a section of the form $g s^n$, where g is the pullback of a function under the map $X \to \operatorname{Spec} A^G$. It follows that $f = g \cdot s^n$ because X is reduced. This shows that $X^{\mathrm{ss}} \subset \operatorname{Spec} A$.

We now verify that $(\star)$ holds for $[\operatorname{Spec} A/G]$. Let $\xi \colon \operatorname{Spec} R \to [\operatorname{Spec} A/G]$ be a map with $\xi(0) \simeq x$, and let $g \in \operatorname{Aut}_{\mathcal{X}}(\xi_K)$ be a finite order K-point. By applying the above result to $[X/G]$, there exists (after an extension of R) a map $\xi' \colon \operatorname{Spec} R \to [X/G]$ such that $\xi'_K \simeq \xi_K$ and g extends to an element of $\operatorname{Aut}_{\mathcal{X}}(\xi')$ but where $\xi'(0)$ may not be isomorphic to x. The stabilizer group scheme $\operatorname{Stab}_G(X) \subset X \times G$ is a closed subscheme equivariant with respect to the product action of G on $X \times G$ where G acts on itself via conjugation. The pair (ξ', g) defines a morphism

$$
\eta \colon \operatorname{Spec} R \to [\operatorname{Stab}_G(X)/G].
$$

We will show that after an extension of R, there is a map $\eta' \colon \operatorname{Spec} R \to [\operatorname{Stab}_G(\operatorname{Spec} A)/G]$

with $\eta'_K \simeq \eta_K$. Similar to (6.8.53), we have a G-equivariant diagram

$$\begin{array}{ccc}
\mathrm{Stab}_G(\mathrm{Spec}\,A) & \hookrightarrow & \mathrm{Stab}_G(X) \\
& \searrow & \downarrow \\
& & \mathrm{Spec}\,A^G \times G
\end{array}$$

with $\mathrm{Stab}_G(X)$ projective over $\mathrm{Spec}\,A^G \times G$. We claim that the semistable locus of $\mathrm{Stab}_G(X)$ for the action of G with respect to the pullback of $\mathcal{O}_X(E)$ is precisely $\mathrm{Stab}_G(\mathrm{Spec}\,A)$ in the sense of Exercise 6.7.12. The invariant section $s \in \Gamma(X, \mathcal{O}_X(E))^G$ pulls back to an invariant section on $\mathrm{Stab}_G(X)$ and thus $\mathrm{Stab}_G(\mathrm{Spec}\,A) \subset \mathrm{Stab}_G(X)^{\mathrm{ss}}$. To see the converse, suppose that $(y, h) \in \mathrm{Stab}_G(X)$ with $y \notin X^{\mathrm{ss}} = \mathrm{Spec}\,A$. Applying Kempf's Optimal Destabilizing Theorem (6.7.36) to a lift $\widehat{y}$ of y to the affine cone $\widehat{X} \to \mathrm{Spec}\,A^G$ of X yields a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot \widehat{y} \in \widehat{X}$ exists and is contained in the zero section $\mathrm{Spec}\,A^G$. Moreover, since $G_y \subset P_\lambda$ (Exercise 6.7.42), $\lim_{t \to 0} \lambda(t) \cdot (\widehat{y}, h)$ also exists and is contained in the zero section of the affine cone over of $\mathrm{Stab}_G(X)$; thus (y, h) is not semistable.

The induced morphism $\mathrm{Stab}_G(\mathrm{Spec}\,A)/\!\!/G \to (\mathrm{Spec}\,A^G \times G)/\!\!/G$ of GIT quotients is proper, and the good moduli space $[\mathrm{Stab}_G(\mathrm{Spec}\,A)/G] \to \mathrm{Stab}_G(\mathrm{Spec}\,A)/\!\!/G$ is universally closed. By the valuative criterion, the composition

$$\mathrm{Spec}\,R \xrightarrow{\eta} \mathrm{Stab}_G(X) \to \mathrm{Spec}\,A^G \times G \to (\mathrm{Spec}\,A^G \times G)/\!\!/G$$

lifts a morphism $\chi \colon \mathrm{Spec}\,R \to [\mathrm{Stab}_G(\mathrm{Spec}\,A)/G]$ such that $\chi_K \simeq \xi_K$ after an extension of R. The composition $\xi' \colon \mathrm{Spec}\,R \xrightarrow{\chi} [\mathrm{Stab}_G(\mathrm{Spec}\,A)/G] \to [\mathrm{Spec}\,A/G]$ has the property that $\xi'_K \simeq \xi_K$ and that g extends to an element of $\mathrm{Aut}_{\mathcal{X}}(\xi')$. To arrange that $\xi'(0) \simeq x$, we apply Lemma 6.8.54 below. $\qquad\square$

Lemma 6.8.54. *Let $\mathcal{X} = [\mathrm{Spec}\,A/G]$ where $\mathrm{Spec}\,A$ is an affine scheme of finite type over an algebraically closed field $\Bbbk$ equipped with an action of a reductive group G. Let $\xi, \xi' \colon \mathrm{Spec}\,R \to \mathcal{X}$ be morphisms from a DVR with residue field $\Bbbk$ such that $\xi_K \simeq \xi'_K$ and $\xi(0) \in \mathcal{X}$ is a closed point. For every element $g \in \mathrm{Aut}_{\mathcal{X}}(\xi')$, there exists (after replacing R with an extension) a morphism $\xi'' \colon \mathrm{Spec}\,R \to \mathcal{X}$ such that $\xi''_K \simeq \xi'_K$, $g|_K$ extends to an automorphism of ξ'', and $\xi''(0) \simeq \xi(0)$.*

Proof. Since $\xi(0)$ and $\xi'(0)$ lie in the same fiber of $\mathcal{X} \to \mathrm{Spec}\,A^G$, the closure of $\xi'(0)$ in $|\mathcal{X}|$ must contain $\xi(0)$. Kempf's Criterion (6.7.34) yields a canonical map $f \colon \Theta \to [\mathrm{Spec}\,A/G]$ with $f(1) \simeq \xi'(0)$ and $f(0) \simeq \xi(0)$. Since f is canonical, every automorphism of $f(1)$ extends to an automorphism of the map f. In particular the restriction of $g \in \mathrm{Aut}_{\mathcal{X}}(\xi')$ to $f(1) = \xi'(0)$ extends uniquely to an automorphism g_f of f.

We now apply the Strange Gluing Lemma (6.8.55), which after replacing R with $R[\pi^{1/N}]$ and precomposing f with the map $\Theta \to \Theta$ defined by $x \mapsto x^N$ for $N \gg 0$, yields a *unique* map $\gamma \colon \Phi_R \to \mathcal{X}$, such that $\gamma|_{s=0} \simeq f$ and $\gamma|_{t \neq 0} \simeq \xi'$. The uniqueness γ guarantees that the automorphism g of ξ' and g_f of f extends uniquely to an automorphism g_γ of γ. Finally, we construct the desired map ξ'' as the composition

$$\xi'' \colon \mathrm{Spec}(R[\sqrt{\pi}]) \xrightarrow{q} \Phi_R \xrightarrow{\gamma} \mathcal{X},$$

where in (s, t, π) coordinates the first map q is defined by $(\sqrt{\pi}, \sqrt{\pi}, \pi)$. Under q, the special point of $\mathrm{Spec}(R[\sqrt{\pi}])$ maps to the point $0 \in \Phi_R$. By construction,

$\xi''(0) \simeq \xi(0)$ and the automorphism g of γ restricts to an automorphism of ξ'' extending $g|_{K(\sqrt{\pi})}$. $\qquad\square$

Lemma 6.8.55 (Strange Gluing Lemma). *Let $\mathcal{X}$ be an algebraic stack of finite type over an algebraically closed field $\Bbbk$ with affine diagonal. Let R be a DVR with residue field $\Bbbk$. Let $f\colon \Theta \to \mathcal{X}$ and $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ be morphisms with an isomorphism $f(1) \simeq \xi(0)$. For $N \gg 0$, after replacing R with $R[\pi^{1/N}]$ and f with the composition $\Theta \xrightarrow{N} \Theta \xrightarrow{f} \mathcal{X}$, there is a unique morphism $\gamma\colon \phi_R \to \mathcal{X}$ such that $\gamma|_{s=0} \simeq f$ and $\gamma|_{t \neq 0} \simeq \xi$.*

Proof. For $n > 0$, define

$$\phi_R^{n,1} = [\operatorname{Spec}(R[s,t]/(st^n - \pi))/\mathbb{G}_m]$$

where the $\mathbb{G}_m$-acts with weight n on s and -1 on t. We have a closed immersion $\Theta \hookrightarrow \phi_R^{n,1}$ defined by $s = 0$ and an open immersion $\operatorname{Spec} R \hookrightarrow \phi_R^{n,1}$ defined by $t \neq 0$. Note that any morphism $\phi_R^{n,1} \to \mathcal{X}$ restricts to morphisms $f\colon \Theta \to \mathcal{X}$ and $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ along with an isomorphism $\xi(0) \simeq f(1)$. We will show conversely that for $n \gg 0$, any $f\colon \Theta \to \mathcal{X}$ and $\xi\colon \operatorname{Spec} R \to \mathcal{X}$ with $\xi(0) \simeq f(1)$ extends canonically to a map $\phi_R^{n,1} \to \mathcal{X}$.

Letting $C = R[t, \pi/t, \pi/t^2, \ldots] \subset R[t]_t$, the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} \Bbbk[t]_t & \longrightarrow & \operatorname{Spec} \Bbbk[t] \\
\downarrow & & \downarrow \\
\operatorname{Spec} R[t]_t & \longrightarrow & \operatorname{Spec} C
\end{array}
$$

is a pushout in the category of schemes (Theorem B.4.1). This diagram is $\mathbb{G}_m$-equivariant, and the diagram obtained by taking the fiber product with $\mathbb{G}_m$ is also a pushout. It follows from Proposition B.4.10 that taking quotients by $\mathbb{G}_m$ yields a diagram

$$
\begin{array}{ccc}
\operatorname{Spec} \Bbbk & \longrightarrow & \Theta \\
\downarrow & & \downarrow \\
\operatorname{Spec} R & \longrightarrow & [\operatorname{Spec}(C)/\mathbb{G}_m]
\end{array}
\qquad\qquad (6.8.56)
$$

with f from Θ, ξ from $\operatorname{Spec} R$, and dotted arrow Ψ to $\mathcal{X}$

where the square is a pushout in the category of algebraic stacks with affine diagonal; this induces the dotted arrow Ψ. We can write C as a union $C = \bigcup C_n$ where $C_n := R[t, \pi/t^n] \subset R[t]_t$. Note that $C_n \cong R[s,t]/(st^n - \pi)$ so in particular $[\operatorname{Spec}(C_n)/\mathbb{G}_m] \cong \phi_R^{n,1}$. As $\mathcal{X} \to S$ is locally of finite presentation, for $n \gg 0$ the morphism Ψ factors uniquely as $[\operatorname{Spec}(C)/\mathbb{G}_m] \to \phi_R^{n,1} \to \mathcal{X}$ (Exercise 3.3.31).

To finish the proof, compose the uniquely defined map $\phi_R^{n,1} \to \mathcal{X}$ with the canonical map $\phi_{R[\pi^{1/n}]} \to \mathcal{X}$ induced by the map of graded algebras $R[s,t]/(st^n - \pi) \to R[\pi^{1/n}][s^{1/n}, t]/(s^{1/n}t - \pi)$, where $s^{1/n}$ has weight 1. $\qquad\square$

6.8.6 S-completeness and reductivity

We've already seen that S-completeness characterizes separatedness (Proposition 6.8.22 and Corollary 6.8.20). We've also seen that it implies unpunctured inertia (Theorem 6.8.50) and therefore implies the existence of stabilizer preserving local quotient presentations (Proposition 6.8.47). We now prove a third remarkable property of S-completeness: it characterizes reductivity. More precisely, a smooth affine algebraic group G is reductive if and only if BG is S-complete if and only if G has Cartan Decompositions (Proposition 6.8.57). This also completes the proof of Theorem 6.6.30

Proposition 6.8.57. *Let G be a smooth affine algebraic group over an algebraically closed field $\Bbbk$. The following are equivalent:*

(1) G is reductive,

(2) BG is S-complete, and

(3) G satisfies the Cartan Decomposition: for every complete DVR R over $\Bbbk$ with residue field $\Bbbk$ and fraction field K and for every element $g \in G(K)$, there exists elements $h_1, h_2 \in G(R)$ and a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ such that

$$g = h_1 \lambda|_K h_2.$$

In particular, if $\mathcal{X}$ is an S-complete algebraic stack and $x \in \mathcal{X}$ is a closed point with smooth affine stabilizer G_x, then G_x is reductive.

Proof. For (2) $\Rightarrow$ (3), observe that since $\Phi_R \smallsetminus 0 = \operatorname{Spec} R \bigcup_{\operatorname{Spec} K} \operatorname{Spec} R$, an element $g \in G(K)$ determines a morphism

$$\rho_g \colon \Phi_R \smallsetminus 0 \to BG$$

by gluing two trivial G-torsors over $\operatorname{Spec} R$ via the isomorphism induced by g of their restrictions to $\operatorname{Spec} K$. Since BG is S-complete, we have a lift

$$
\begin{array}{ccc}
\Phi_R \smallsetminus 0 & \xrightarrow{\ \rho_g\ } & BG \\
\downarrow & \nearrow^{\!\!h} & \\
\Phi_R. & &
\end{array}
\tag{6.8.58}
$$

Restricting h to the origin gives a map $B\mathbb{G}_m \hookrightarrow \Phi_R \xrightarrow{h} BG$ which corresponds to a map $\lambda \colon \mathbb{G}_m \to G$ (up to conjugation); this provides us with our candidate one-parameter subgroup. We make two observations:

- If $g, g' \in G(K)$ are elements, the morphisms $\rho_g, \rho_{g'} \colon \Phi_R \smallsetminus 0 \to BG$ are isomorphic if and only if there are elements $h, h' \in G(R)$ such that $hg = g'h'$.

- If $\lambda \colon \mathbb{G}_m \to G$ is a one-parameter subgroup and $\lambda|_{\Phi_R \smallsetminus 0}$ denotes the composition $\Phi_R \smallsetminus 0 \hookrightarrow \Phi_R \to B\mathbb{G}_m \xrightarrow{\lambda} BG$, then $\lambda|_{\Phi_R \smallsetminus 0}$ and $\rho_{g'}$, where $g' = \lambda|_K$, are isomorphic.

It therefore suffices to show that the extension h in (6.8.58) is isomorphic to $\lambda|_{\Phi_R} \colon \Phi_R \to B\mathbb{G}_m \xrightarrow{\lambda} BG$. To see this, let $\mathcal{P}$ and $\mathcal{P}'$ denote the principal G-bundles over Φ_R classifying h and $\lambda|_{\Phi_R}$. Since G is smooth and affine, $\underline{\operatorname{Isom}}_{\Phi_R}(\mathcal{P}, \mathcal{P}') \to \Phi_R$ is smooth and affine. We have a section over the inclusion $\mathcal{X}_0 := B\mathbb{G}_m \hookrightarrow \Phi_R$ of

0. Letting $\mathcal{X}_n$ denote the nth nilpotent thickening, deformation theory (Proposition 6.5.10) and the cohomological affineness of $\mathcal{X}_n$ implies that we may find compatible sections over $\mathcal{X}_n$. Coherent Tannaka Duality (6.4.9) and the coherent completeness of ϕ_R along $B\mathbb{G}_m$ (Theorem 6.4.12) implies that the map

$$\mathrm{Mor}_{\phi_R}(\phi_R, \underline{\mathrm{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}')) \to \varprojlim \mathrm{Mor}_{\phi_R}(\mathcal{X}_n, \underline{\mathrm{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}'))$$

is an equivalence. We thus obtain a section of $\underline{\mathrm{Isom}}_{\phi_R}(\mathcal{P}, \mathcal{P}') \to \phi_R$, i.e. an isomorphism between $\mathcal{P}$ and $\mathcal{P}'$.

To see that (3) $\Rightarrow$ (2), it suffices to show that every map $\phi_R \smallsetminus 0 \to BG$ extends to a map $\phi_R \to BG$ where R is a complete DVR over $\Bbbk$ with residue field $\Bbbk$ (Lemma 6.8.16). Since every principal G-bundle over $\mathrm{Spec}\,R$ is trivial, the map $\phi_R \smallsetminus 0 \to BG$ is isomorphic to ρ_g for some element $g \in G(K)$. Writing $g = h_1\lambda|_K h_2$, the two observations above show that $\phi_R \to B\mathbb{G}_m \to BG$ is an extension.

We've already seen that (1) $\Rightarrow$ (2) in Proposition 6.8.19. Conversely, if G is not reductive, there is a normal subgroup $\mathbb{G}_a \lhd R_u(G)$ of the unipotent radical. As $G/R_u(G)$ and $R_u(G)/\mathbb{G}_a$ are both affine, the composition $B\mathbb{G}_a \to BR_uG \to BG$ is affine. By Lemma 6.8.18, this would imply that $B\mathbb{G}_a$ is S-complete but this is a contradiction: taking $R = \Bbbk[\![x]\!]$ and $K = \Bbbk(\!(x)\!)$ the element $x \in \mathbb{G}_a(K)$ cannot be written as $h_1\lambda|_K h_2$.

See also [AHHL21, Thm. A]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.8.7   Proof of the Existence Theorem of Good Moduli Spaces

The necessity of Θ-completeness and S-completeness for the existence of a good moduli space was established in Corollary 6.8.20. We now establish the sufficiency following the strategy outlined in §6.8.1.

Proof of Theorem 6.8.1. Since $\mathcal{X}$ is S-complete and $\mathrm{char}(\Bbbk) = 0$, the stabilizer G_x of every closed point $x \in \mathcal{X}$ is linearly reductive (Proposition 6.8.57). By the Local Structure Theorem (6.5.1), there exists an affine étale morphism $f\colon ([\mathrm{Spec}\,A/G_x], w) \to (\mathcal{X}, x)$ inducing an isomorphism of stabilizer groups at x. Since $\mathcal{X}$ is Θ-complete and S-complete, we may assume that f is Θ-surjective and stabilizer preserving at all points after replacing $[\mathrm{Spec}\,A/G_x]$ with an open neighborhood of x (Propositions 6.8.40 and 6.8.47). Since $\mathcal{X}$ is quasi-compact, there exists finitely many closed points $x_i \in \mathcal{X}$ and morphisms $f_i\colon [\mathrm{Spec}\,A_i/G_{x_i}] \to \mathcal{X}$ as above whose images cover $\mathcal{X}$. Choosing embeddings $G_{x_i} \hookrightarrow \mathrm{GL}_n$ for some n, there are equivalence $[\mathrm{Spec}\,A_i/G_{x_i}] \cong [(\mathrm{Spec}\,A_i \times^{G_{x_i}} \mathrm{GL}_n)/\mathrm{GL}_n]$. Setting $A = \prod_i (A_i \times^{G_{x_i}} \mathrm{GL}_n)$, there is an surjective, affine, and étale morphism

$$f\colon \mathcal{X}_1 := [\mathrm{Spec}\,A/\mathrm{GL}_n] \to \mathcal{X}$$

which is Θ-surjective and stabilizer preserving at all points. Since $\mathrm{char}(\Bbbk) = 0$, there is a good moduli space $\mathcal{X}_1 \to X_1 := \mathrm{Spec}\,A^{\mathrm{GL}_N}$.

Set $\mathcal{X}_2 = \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1$. The projections $p_1, p_2\colon \mathcal{X}_2 \to \mathcal{X}_1$ are also affine, étale, Θ-surjective, and stabilizer preserving. Since f is affine, $\mathcal{X}_2 \cong [\mathrm{Spec}\,B/\mathrm{GL}_n]$ and there is a good moduli space $\mathcal{X}_2 \to X_2 := \mathrm{Spec}\,B^{\mathrm{GL}_n}$. This provides a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{X}_2 & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & \mathcal{X}_1 & \overset{f}{\longrightarrow} & \mathcal{X} \\
\downarrow & & \downarrow & & \vdots \\
X_2 & \underset{q_2}{\overset{q_1}{\rightrightarrows}} & X_1 & \dashrightarrow & X
\end{array}
\tag{6.8.59}
$$

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which each square on the left is cartesian by Luna's Fundamental Lemma (6.3.30). Moreover, by the universality of good moduli spaces (Theorem 6.3.5(4)), the étale groupoid structure on $\mathcal{X}_2 \rightrightarrows \mathcal{X}_1$ induces a étale groupoid structure on $X_2 \rightrightarrows X_1$.

We claim that $X_2 \rightrightarrows X_1$ is an étale equivalence relation, i.e. that the quotient stack $[X_1/X_2]$ is an algebraic space. By the Characterization of Algebraic Spaces (3.6.6), it suffices to show that if $x_1 \in X_1$ is a $\Bbbk$-point, then (x_1, x_1) has a unique preimage under $(q_1, q_2) \colon X_2 \to X_1 \times X_1$. Let $x_2, x_2' \in X_2$ be two points mapping to $(x_1, x_1) \in X_1 \times X_1$, and let $\widetilde{x}_2, \widetilde{x}_2' \in \mathcal{X}_2$ be the unique closed points in their preimages. Since f is Θ-surjective, the images $p_1(\widetilde{x}_2)$, $p_2(\widetilde{x}_2)$, $p_1(\widetilde{x}_2')$, and $p_2(\widetilde{x}_2')$ are all closed points of $\mathcal{X}_1$ over x_1, and therefore they are all identified with the unique closed point $\widetilde{x}_1$ over x_1. On the other hand, since f is stabilizer preserving, the stabilizer groups of $\widetilde{x}_2$ and $\widetilde{x}_2'$ are the same as the stabilizer groups of $\widetilde{x}_1$ and of its image in $\mathcal{X}$. Let's denote this stabilizer group by G. It follows that the fiber product of $(p_1, p_2) \colon \mathcal{X}_2 \to \mathcal{X}_1 \times \mathcal{X}_1$ along the inclusion of the residual gerbe $\mathcal{G}_{(\widetilde{x}_1, \widetilde{x}_1)} = BG \times BG \to \mathcal{X}_1 \times \mathcal{X}_1$ is isomorphic to BG and thus identified with the residual gerbe of a *unique* closed point. Therefore $x_2 = x_2'$.

Since $X_2 \rightrightarrows X_1$ is an étale equivalence relation, the quotient $X = X_1/X_2$ is an algebraic space. From étale descent, there is a morphism $\mathcal{X} \to X$ which pulls back under $X_1 \to X$ to the good moduli space $\mathcal{X}_1 \to X_1$. By descent of good moduli spaces (Lemma 6.3.21(2)), $\mathcal{X} \to X$ is a good moduli space. Finally, we use that $\mathcal{X}$ is S-complete to conclude that X is separated (Corollary 6.8.20). $\qquad\square$

Chapter 7

Moduli of semistable vector bundles on a curve

TO BE WRITTEN

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7.1.1   Review of properties of coherent sheaves

7.1.2   Definition of semistability and basic properties

7.1.3   Families of vector bundles

7.1.4   Openness of semistability

7.1.5   Boundedness of semistability

7.1.6   Vector bundles with fixed determinant

7.1.7   Existence of stable bundles

7.2   Filtrations, stratifications, and existence of moduli spaces

7.2.1   Jordan–Hölder filtrations

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7.2.3   The Schatz Stratification

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7.2.5   Existence of a separated moduli space

7.3 Semistable reduction: Langton's Theorem

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7.4 Geometric properties: irreducibility, unirationality, and more

7.4.1 Smoothness and dimension

7.4.2 Properties of stable locus

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7.4.6 Picard groups

7.5 Projectivity

7.5.1 Projectivity of the Picard scheme

7.5.2 Determinantal line bundles

7.5.3 A characterization of semistability and the semiampleness of L_V

7.5.4 Ampleness of L_V

7.5.5 A GIT construction

Chapter 8

Glimpse of other moduli

TO BE WRITTEN

8.1 Moduli of varieties

8.1.1 Weighted pointed stable curves

8.1.2 Abelian varieties

8.1.3 Stable maps

8.1.4 Stable varieties

8.1.5 Stable pairs

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8.2 Moduli of sheaves and bundles

8.2.1 Higgs bundles

8.2.2 Sheaves on higher dimensional varieties

8.2.3 G-bundles

8.2.4 Complexes and Bridgeland stability

Appendix A

Morphisms of schemes

We recall definitions and properties of morphisms of schemes—locally of finite presentation, flat, smooth, étale, unramified, quasi-finite, and proper. As we intend to highlight results important in moduli theory, we pay close attention to *functorial* properties, i.e. properties of schemes and their morphisms characterized by their functors.

While first courses in algebraic geometry (e.g. [Har77]) often impose noetherian hypotheses, we try to state results in the non-noetherian setting when possible. Since we define moduli functors and stacks on the entire category of schemes, it is essential to work with non-noetherian schemes. Limit Methods (B.3) allow one to reduce properties of general schemes to noetherian schemes.

A.1   Morphisms locally of finite presentation

A morphism of schemes $f\colon X \to Y$ is *locally of finite type* (resp. *locally of finite presentation*) if for all affine open subschemes $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, there is surjection $A[x_1,\ldots,x_n] \to B$ of A-algebras (resp. a surjection $A[x_1,\ldots,x_n] \to B$ with finitely generated kernel). If in addition f is quasi-compact (resp. quasi-compact and quasi-separated), we say that f is *of finite type* (resp. *of finite presentation*). When Y is locally noetherian, being locally of finite type (resp. finite type) is equivalent to being locally of finite presentation (resp. finite presentation). In the non-noetherian setting, even closed immersions may not be locally of finite presentation, e.g. $\operatorname{Spec} \mathbb{C} \hookrightarrow \operatorname{Spec} \mathbb{C}[x_1, x_2, \ldots]$. Morphisms of finite presentation are better behaved than morphisms of finite type, so we often use the former condition. For example, when defining a family of stable curves $\pi\colon \mathcal{C} \to S$ (Definition 5.3.17), we require not only that π is proper and flat but also of finite presentation.

The functorial characterization of locally of finite presentation morphisms uses the notion of an *inverse system* in a category $\mathcal{C}$: a partially ordered set $(I, \geq)$ which is directed, i.e. for every $i, j \in I$ there exists $k \in I$ such that $k \geq i$ and $k \geq j$, together with a contravariant functor $I \to \mathcal{C}$.

Proposition A.1.1. *A morphism $f\colon X \to Y$ of schemes is locally of finite presentation if and only if for every inverse system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of affine schemes over Y, the natural map*

$$\operatorname{colim}_\lambda \operatorname{Mor}_Y(\operatorname{Spec} A_\lambda, X) \to \operatorname{Mor}_Y(\operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda), X)$$

is bijective.

Proof. See [EGA, IV.8.14.2] or [SP, Tag 01ZC]. $\qquad \square$

This is not a deep result: requiring that every map $\operatorname{Spec} A \to X$ over Y factors through some $\operatorname{Spec} A_\lambda \to \operatorname{Spec} A$ is essentially the condition that $\operatorname{Spec} A \to X$ depends on only finite data, i.e. there are only finitely many generators and relations for the ring maps locally defining $X \to Y$.

Exercise A.1.2. Verify Proposition A.1.1 in the case of a morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ of affine schemes.

Proposition A.1.1 is a *functorial* condition as it only depends on the functor $\operatorname{Mor}_Y(-, X)$, and therefore we can extend the definition of locally of finite presentation to functors.

Definition A.1.3. Let Y be a scheme. A contravariant $F \colon \operatorname{Sch}/Y \to \operatorname{Sets}$ is *locally of finite presentation* (or *limit preserving*) if for every inverse system $\{\operatorname{Spec} A_\lambda\}_{\lambda \in I}$ of affine schemes over Y, the natural map

$$\operatorname{colim}_\lambda F(A_\lambda) \to F(\operatorname{colim}_\lambda A_\lambda)$$

is bijective.

By Proposition A.1.1, a scheme X is locally of finite presentation over Y if and only if the functor $\operatorname{Mor}_Y(-, X)$ is locally of finite presentation.

A.2 Flatness

You can't get very far in moduli theory without flatness. While its definition is seemingly abstract and algebraic, it is a magical geometric property of a morphism $X \to Y$ that ensures that fibers X_y 'vary nicely' as $y \in Y$ varies. This principle is nicely evidenced by Flatness via the Hilbert Polynomial (A.2.4). It is the reason why we define objects of our moduli stacks as flat families.

A.2.1 Flatness criteria

A module M over a ring A is *flat* if the functor

$$- \otimes_A M \colon \operatorname{Mod}(A) \to \operatorname{Mod}(A)$$

is exact. We recall the following criteria:

(1) (Stalk Criterion) M is flat over A if and only if $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ for every prime (equivalently maximal) ideal $\mathfrak{p}$. More generally, if $A \to B$ is a ring map, a B-module N is flat if and only if for every prime $\mathfrak{q} \subset B$ with preimage $\mathfrak{p} \subset A$, $N_\mathfrak{q}$ is flat over $A_\mathfrak{p}$.

(2) (Ideal Criterion) M is flat if and only if for every finitely generated ideal $I \subset A$, the map $I \otimes_A M \to M$ is injective [Eis95, Prop. 6.1]. (When A is a PID, this implies that M is flat if and only if M is torsion free.)

(3) (Tor Criterion) M is flat if and only if $\operatorname{Tor}_1^A(A/I, M) = 0$ for all finitely generated ideals $I \subset A$ [Eis95, Prop. 6.1].

(4) (Finitely Presented Criterion) M is finitely presented and flat over A if and only if M is finite and projective if and only if M is finite and locally free (i.e. $M_{\mathfrak{p}}$ is finite and free for all prime—or equivalently maximal—ideals $\mathfrak{p}$); see [SP, Tag 00NX]. (Without the finitely presented hypothesis, Lazard's Theorem states that M is flat over A if and only if M can be written as a directed colimit $\operatorname{colim}_{i \in I} M_i$ of free finite A-modules M_i; see [Eis95, A6.6] or [SP, Tag 058G].)

(5) (Equational Criterion) M is flat if and only if for every relation $\sum_{i=1}^{n} a_i m_i = 0$ with $a_i \in A$ and $m_i \in M$, there exists $m'_j \in M$ for $j = 1, \ldots, r$ and $a'_{ij} \in A$ such that $\sum_{j=1}^{r} a'_{ij} m'_j = m_i$ for all i and $\sum_{i=1}^{n} a'_{ij} a_i = 0$ for all j [Eis95, Cor. 6.5].

If $f \colon X \to Y$ is a morphism of schemes, then a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if for all affine opens $\operatorname{Spec} B \subset Y$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, the B-module $\Gamma(\operatorname{Spec} A, \mathcal{F})$ is a flat.

Proposition A.2.1 (Flat Equivalences). *Let $f \colon X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The following are equivalent:*

(1) $\mathcal{F}$ is flat over Y;

(2) There exists a Zariski-cover $\{\operatorname{Spec} B_i\}$ of Y and $\{\operatorname{Spec} A_{ij}\}$ of $f^{-1}(\operatorname{Spec} B_i)$ such that $\Gamma(\operatorname{Spec} A_{ij}, \mathcal{F})$ is flat as an B_i-module under the ring map $B_i \to A_{ij}$;

(3) For all $x \in X$, the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$ is flat as an $\mathcal{O}_{Y,y}$-module.

(4) The functor

$$\operatorname{QCoh}(Y) \to \operatorname{QCoh}(X), \qquad \mathcal{G} \mapsto f^* \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

is exact.

Proof. See [Har77, §III.9] or [SP, Tag 01U2]. $\qquad\square$

We say that a morphism $f \colon X \to Y$ of schemes is *flat at $x \in X$* (resp. a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *flat at $x \in X$*) if there exists a Zariski open neighborhood $U \subset X$ containing x such that $f|_U$ (resp. $\mathcal{F}|_U$) is flat over Y. This is equivalent to the flatness of $\mathcal{O}_{X,x}$ (resp. $\mathcal{F}_x$) as an $\mathcal{O}_{Y,y}$-module.

Proposition A.2.2 (Flatness Criterion over Smooth Curves). *Let C be an integral and regular scheme of dimension 1 (e.g. the spectrum of a DVR or a smooth connected curve over a field), and let $X \to C$ be a morphism of schemes. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over C if and only if every associated point of $\mathcal{F}$ maps to the generic point of C.*

Proof. A short argument shows that this follows from the fact that a module over a DVR is flat if and only if it is torsion free; see [Har77, III.9.7]. $\qquad\square$

Over higher dimensional bases, it is sometimes possible to check flatness by reducing to the above criterion over a smooth curve. This is called the valuative criterion for flatness: if $f \colon X \to S$ is a finite type morphism of noetherian schemes over a reduced scheme S and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $\mathcal{F}$ is flat at $x \in X$ if and only if for every map $(\operatorname{Spec} R, 0) \to (S, f(x))$ from a DVR, the restriction $\mathcal{F}|_{X_R}$ is flat over R at all points in $X_R := X \times_S \operatorname{Spec} R$ over 0 and x [EGA, IV.11.8.1]. Despite providing a conceptual geometric criterion for flatness, it is surprisingly rarely used in moduli theory.

Proposition A.2.3 (Flatness Criterion over Artinian Rings). *A module over an artinian ring is flat if and only if it is free if and only if it is projective.*

Proof. See [SP, Tag 051E]. $\qquad\square$

Recall that if $X \subset \mathbb{P}_K^n$ is a subscheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, the *Hilbert polynomial* of $\mathcal{F}$ is $P_{\mathcal{F}}(z) = \chi(X, \mathcal{F}(z)) \in \mathbb{Q}[z]$.

Proposition A.2.4 (Flatness via the Hilbert Polynomial)**.** *Let S be a connected, reduced, and noetherian scheme, and let $X \subset \mathbb{P}_S^n$ be a closed subscheme. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is flat over S if and only if the function*

$$S \to \mathbb{Q}[z], \qquad s \mapsto P_{\mathcal{F}|_{X_s}}$$

assigning a point $s \in S$ to the Hilbert polynomial of the restriction $\mathcal{F}|_{X_s}$ to the fiber $X_s \subset \mathbb{P}_{\kappa(s)}^n$ is constant.

Proof. See [Har77, Thm. 9.9]. $\qquad\square$

Theorem A.2.5 (Local and Infinitesimal Criteria for Flatness)**.** *Let $A \to B$ be a local homomorphism of noetherian local rings, and let M be a finite B-module. The following are equivalent:*

(1) *M is flat over A,*

(2) *(Local Criterion) $\mathrm{Tor}_1^A(A/\mathfrak{m}_A, M) = 0$, and*

(3) *(Infinitesimal Criterion) $M/\mathfrak{m}_A^n M$ is flat over $A/\mathfrak{m}_A^n$ for every $n \geq 1$.*

Proof. See [Eis95, Thm. 6.8, Exer. 6.5] or [SP, Tag 00MK]. $\qquad\square$

The following consequence of the Local Criterion for Flatness is particularly useful in deformation theory.

Corollary A.2.6. *Let $A \twoheadrightarrow A_0$ be a surjective homomorphism of noetherian rings with kernel I such that $I^2 = 0$. An A-module M is flat over A if and only if*

(1) *$M_0 := M \otimes_A A_0$ is flat over A_0, and*

(2) *the map $M_0 \otimes_{A_0} I \to M$ is injective.*

Proof. For ($\Rightarrow$), condition (1) holds by base change and condition (2) holds by tensoring the exact sequence $0 \to I \to A \to A_0 \to 0$ with M and using the identification $M \otimes_A I \cong M_0 \otimes_{A_0} I$. For ($\Leftarrow$), by the Local Criterion for Flatness (A.2.5) it suffices to show that $\mathrm{Tor}_1^A(A/\mathfrak{p}, M) = 0$ for all prime ideals $\mathfrak{p} \subset A$. Let $\mathfrak{p}_0 := \mathfrak{p}/I \subset A$. Consider the following diagram which is obtained by tensoring the exact sequences $0 \to I \to \mathfrak{p} \to \mathfrak{p}_0 \to 0$ and $0 \to I \to A \to A_0 \to 0$ with M:

$$
\begin{array}{ccc}
\mathrm{Tor}_1^A(M, A/\mathfrak{p}) & \longrightarrow & \mathrm{Tor}_1^{A_0}(M_0, A_0/\mathfrak{p}_0) \\
\downarrow & & \downarrow \\
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M \otimes_A I & \longrightarrow & M \otimes_A \mathfrak{p} & \longrightarrow & M_0 \otimes_{A_0} \mathfrak{p}_0 & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_A I & \longrightarrow & M & \longrightarrow & M_0 & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & M \otimes_A A/\mathfrak{p} & \longrightarrow & M_0 \otimes_{A_0} A_0/\mathfrak{p}_0 & &
\end{array}
$$

Condition (2) implies that the second row is exact, and it follows that the first row is also exact, where we've used the identification $M \otimes_A \mathfrak{p} \cong M_0 \otimes_{A_0} \mathfrak{p}_0$. Condition

(1) implies that $\mathrm{Tor}_1^{A_0}(M_0, A_0/\mathfrak{p}_0) = 0$ and it follows from the snake lemma that $\mathrm{Tor}_1^A(M, A/\mathfrak{p}) = 0$. See also [Har10, Prop. 2.2]. $\qquad\square$

Remark A.2.7. Applying this with $A = \Bbbk[\epsilon]/(\epsilon^2)$ being the dual numbers and $A' = \Bbbk$, we recover the fact that an A-module M is flat if and only if $M \otimes_{\Bbbk[\epsilon]/(\epsilon^2)} \Bbbk \xrightarrow{\epsilon} M$ is injective. This also follows from the fact that a module N over a ring B is flat if and only if for every ideal $I \subset B$, the map $I \otimes_B M \to M$ is injective, and using that the only ideal in $\Bbbk[\epsilon]/(\epsilon^2)$ is (ϵ).

The following convenient facts are closely related to the Local Criterion of Flatness (A.2.5).

Lemma A.2.8. *Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local ring homomorphism of noetherian local rings.*

(1) Let M be a flat A-module and N be a finitely generated B-module. If $\phi\colon N \to M$ is a morphism of R-modules such that $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is injective, then $\phi\colon N \to M$ is injective and $M/\phi(N)$ is flat over A.

(2) If in addition $A \to B$ is flat and $f \in \mathfrak{m}_B$ is a nonzerodivisor in $B \otimes_A A/\mathfrak{m}_A$, then $A \to B/(f)$ is flat.

Proof. The proofs are elementary (see [Mat89, Thm. 22.5] or [SP, Tag 00ME]). Note that Part (2) follows directly from (1). $\qquad\square$

Part (2) can be viewed as a 'slicing criterion for flatness' and is often applied inductively to regular sequences. It has the following geometric interpretation.

Corollary A.2.9 (Slicing Criterion for Flatness)**.** *Let $f\colon X \to S$ be a morphism locally of finite presentation, and let $x \in X$ be a point with image $s \in S$. If f is flat at x and the image of $h \in \mathfrak{m}_x \subset \mathcal{O}_{X,x}$ in the local ring $\mathcal{O}_{X_s,x}$ of the fiber is a nonzerodivisor, then there exists an open neighborhood $U \subset X$ of x such that h extends to a global function on U and the composition $V(h) \hookrightarrow U \to S$ is locally of finite presentation and flat at x.*

Proof. The noetherian case is a direct consequence of Lemma A.2.8(2), and the general case can be reduced to the noetherian case using limit methods (B.3). See also [SP, Tag 056X]. $\qquad\square$

Theorem A.2.10 (Fibral Flatness Criterion)**.** *Consider a commutative diagram*

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
 & \searrow \quad \swarrow & \\
 & S &
\end{array}
$$

of schemes, and let F be a quasi-coherent $\mathcal{O}_X$-module of finite presentation. Assume that $X \to S$ is locally of finite presentation and $Y \to S$ is locally of finite type. Let $x \in X$ with images $y \in Y$ and $s \in S$. If the stalk $\mathcal{F}_x$ is nonzero, then the following are equivalent:

(1) F is flat over S at x, and $\mathcal{F}_s := \mathcal{F}|_{X_s}$ is flat over Y_s at x, and

(2) Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.

Proof. See [SP, Tag 039A]. $\qquad\square$

If $A \to B$ is a local ring map of noetherian local rings, then $\dim B = \dim A + \dim B/\mathfrak{m}_A B$. The following is a partial converse.

Theorem A.2.11 (Miracle Flatness). *Let $A \to B$ be a local homomorphism of noetherian local rings. Assume that*

1. *A is regular,*
2. *B is Cohen–Macaulay, and*
3. *$\dim B = \dim A + \dim B/\mathfrak{m}_A B$.*

Then $A \to B$ is flat.

Proof. See [Nag62, Thm. 25.16] or [SP, Tag 00R4]. $\qquad\square$

A.2.2 Properties of flatness

Proposition A.2.12 (Flat Base Change). *Consider a cartesian diagram*

$$
\begin{array}{ccc}
X' & \xrightarrow{\ g'\ } & X \\
\ \downarrow{\scriptstyle f'} & \square & \ \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{\ g\ } & Y
\end{array}
$$

of schemes, and let F be a quasi-coherent sheaf on X. If $g\colon Y' \to Y$ is flat and $f\colon X \to Y$ is quasi-compact and quasi-separated, the natural adjunction map

$$
g^* \mathrm{R}^i f_* F \to \mathrm{R}^i f'_* g'^* F
$$

is an isomorphism for all $i \geq 0$.

Proof. See [Har77, Prop. III.9.3] or [SP, Tag 02KH]. $\qquad\square$

Theorem A.2.13 (Generic Flatness). *Let $f\colon X \to S$ be a finite type morphism of schemes and $\mathcal{F}$ be a finite type quasi-coherent $\mathcal{O}_X$-module. If S is reduced, there exists a dense open subscheme $U \subset S$ such that $X_U \to U$ is flat and of presentation and such that $\mathcal{F}|_{X_U}$ is flat over U and of finite presentation as on $\mathcal{O}_{X_U}$-module.*

Proof. See [SP, Tag 052B]. $\qquad\square$

Proposition A.2.14 (Fppf Morphisms are Open). *Let $f\colon X \to Y$ be a morphism of schemes. If f is flat and locally of finite presentation, then for every open subset $U \subset X$, the image $f(U) \subset Y$ is open.*

Proof. See [SP, Tag 01UA]. $\qquad\square$

Proposition A.2.15. *A flat monomorphism locally of finite presentation (e.g. an étale monomorphism) is an open immersion.*

Theorem A.2.16 (Existence of Flattening Stratifications). *Let $X \to S$ be a projective morphism of noetherian schemes, $\mathcal{O}_X(1)$ be a relatively ample line bundle and $\mathcal{F}$ be a coherent sheaf on X. For each polynomial $P \in \mathbb{Q}[z]$, there exists a locally closed subscheme $S_P \hookrightarrow S$ such that a morphism $T \to S$ factors through S_P if and only if the pullback $\mathcal{F}_T$ of $\mathcal{F}$ to X_T is flat over T and for every $t \in T$, the pullback $\mathcal{F}_{\kappa(t)}$ to $X_{\kappa(t)}$ has Hilbert polynomial P.*

Moreover, there exists a finite indexing set I of polynomials such that $S = \coprod_{P \in I} S_P$ set-theoretically. The closure of S_P in S is contained set-theoretically in the union $\bigcup_{P \leq Q} S_Q$, where $P \leq Q$ if and only if $P(z) \leq Q(z)$ for $z \gg 0$.

Proof. See [FGA$_{\mathrm{IV}}$] or [Mum66, §8]. $\qquad\square$

Remark A.2.17. When $X \to S$ is only proper, there is a *universal flattening*, i.e. an algebraic space S' and a morphism $S' \to S$ such that a map $T \to S$ factors through $S' \to S$ if and only if the pullback $\mathcal{F}|_{X_T}$ to $X_T := X \times_S T$ is flat over T [SP, Tag 05UG]. In general, S' may not be a disjoint union of locally closed subschemes of S; see [Kre13].

Theorem A.2.18 (Raynaud-Gruson Flatification)**.** *Let Y be a quasi-compact and quasi-separated scheme and $X \to Y$ be a finitely presented morphism which is flat over a quasi-compact open subscheme $U \subset Y$. Then there is a commutative diagram*

$$
\begin{array}{ccc}
\widetilde{X} & \longrightarrow & X \\
\downarrow & & \downarrow f \\
Y' & \xrightarrow{\ p\ } & Y
\end{array}
$$

where $p\colon Y' \to Y$ is a blowup of a finitely presented closed subscheme $Z \subset Y$ disjoint from U and the strict transform $\widetilde{X}$ of X is flat over Y'.

The *strict transform* $\widetilde{X}$ above is by definition the closure of $(Y' \setminus p^{-1}(Z)) \times_Y X$ in the base change $Y' \times_Y X$.

Proof. See [GR71, Thm. I.5.2.2] or [SP, Tag 0815]. $\square$

A.2.3 Faithful flatness

A module M over a ring A is *faithfully flat* if the functor $- \otimes_A M\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ is faithfully exact, i.e. a sequence $N' \to N \to N''$ of A-modules is exact if and only if $N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M$ is exact

for every nonzero map $\phi\colon N \to N'$ of A-modules, the induced map $\phi \otimes_A M\colon N \otimes_A M \to N' \otimes_A M$ is also nonzero.

Proposition A.2.19 (Faithfully Flat Equivalences)**.** *Let A be a ring and M be a flat A-module. The following are equivalent:*

(1) *M is faithfully flat;*

(2) *for every nonzero map $\phi\colon N \to N'$ of A-modules, the induced map $\phi \otimes_A M\colon N \otimes_A M \to N' \otimes_A M$ is also nonzero;*

(3) *for every nonzero A-module N, the tensor product $N \otimes_A M$ is nonzero;*

(4) *for every prime ideal $\mathfrak{p} \subset A$, the tensor product $M \otimes_A \kappa(\mathfrak{p})$ is nonzero; and*

(5) *for every maximal ideal $\mathfrak{m} \subset A$, the tensor product $M \otimes_A \kappa(\mathfrak{m}) \cong M/\mathfrak{m}M$ is nonzero.*

Proof. See [SP, Tag 00H9]. $\square$

When $M = B$ is an A-algebra, then by (4) a flat ring map $A \to B$ is faithfully flat if $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is surjective, or equivalently by (5) every maximal ideal of A is in the image of $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$. The latter equivalence implies that any flat local ring map is faithfully flat.

A morphism $f\colon X \to Y$ of schemes is *faithfully flat* if f is flat and surjective. This is equivalent to the condition that $f^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is faithfully exact. It is also equivalent to the condition that a quasi-coherent $\mathcal{O}_Y$-module (resp. a morphism of quasi-coherent $\mathcal{O}_Y$-modules) is zero if and only if its pullback is. Faithfully flat morphisms play an important role in descent theory; see §B.1.

A.2.4 Fppf and fpqc morphisms

Fppf and fpqc morphisms are acronyms for 'fidèlement plate de présentation finie' and 'fidèlement plat et quasi-compact,' respectively. Despite this terminology, an fpqc morphism is more general than a faithfully flat and quasi-compact map.

Definition A.2.20. A morphism $f\colon X \to Y$ of schemes is:

(1) *fppf* if f is faithfully flat and locally of finite presentation, and

(2) *fpqc* if f is faithfully flat and every quasi-compact open subset of Y is the image of a quasi-compact open subset of X.

Remark A.2.21. A quasi-compact and faithfully flat morphism is fpqc. In addition, an open and faithfully flat morphism is fpqc: for a quasi-compact open subset $V \subset Y$, we can write $f^{-1}(V) = \bigcup_i U_i$ as a union of affines, and since each $f(U_i) \subset V$ is open and V is quasi-compact, we see that V is the image of finitely many of the U_i's. Fppf Morphisms are Open (A.2.14) gives the implication: fppf $\implies$ fpqc.

An fpqc morphism $f\colon X \to Y$ can be equivalently characterized by either requiring that there exists an affine covering $\{Y_i\}$ of Y such that each Y_i is the image of quasi-compact open subset of X, or by requiring that every point $x \in X$ has an open (resp. quasi-compact open) neighborhood U such that $f(U)$ is open and $U \to f(U)$ is quasi-compact; see [FGI$^+$05, Prop. 2.33].

An fppf (resp. fpqc) cover $\{X_i \to X\}$ is a collection of morphisms such that $\coprod_i X_i \to X$ is fppf (resp. fpqc).

A.2.5 Universally injective homomorphisms

The defining characteristic of a flat module is that it preserves every injection under tensoring. The dual notion of an injection of modules, which is preserved under tensoring by every module, is also a very useful property.

Definition A.2.22. A homomorphism $M \to N$ of A-modules is *universally injective* if for every A-module P, the map $M \otimes_A P \to N \otimes_A P$ is injective. A ring homomorphism $A \to B$ is *universally injective* if it is as a map of A-modules.

Remark A.2.23. This should not be confused with a *universally injective* or *radiciel* morphism of schemes $X \to Y$, i.e. an injective map that remains injective after any base change; see [SP, Tag 01S2].

We will use this notion in a fundamental way in our proof of Coherent Tannaka Duality (Theorem 6.4.1). To this end, the following properties will be used:

Proposition A.2.24.

(1) A faithfully flat ring homomorphism $A \to B$ is universally injective.

(2) A split injective $M \to N$ of A-modules is universally injective. The converse is true if N/M is finitely presented.

(3) If $A \to A'$ is faithfully flat, then a map $M \to N$ of A-modules is universally injective if and only if $M \otimes_A A' \to N \otimes_A A'$ is.

(4) If $A \to B$ is universally injective and $B \to B \otimes_A B$, defined by $b \mapsto b \otimes 1$, is faithfully flat, then $A \to B$ is faithfully flat.

Proof. For (1), (2), and (4), see [SP, Tags 08WP, 058L, and 08XD]. Part (3) follows directly from the faithful exactness of $- \otimes_A A'$. See also [Laz69] or [Lam99, §4J]. $\square$

Remarkably universally injective ring maps are precisely those maps that satisfy effective descent for modules; see Remark B.1.6.

A.3 Étale, smooth, and unramified morphisms

A morphism $f\colon X \to Y$ of schemes is

- *smooth* if f is locally of finite presentation, flat, and for every $y \in Y$, the geometric fiber $X_{\overline{\kappa(y)}} = X \times_Y \operatorname{Spec} \overline{\kappa(y)}$ is regular,
- *étale* if f is smooth of relative dimension 0, i.e. $\dim X_y = 0$ for all $y \in Y$, and
- *unramified* if f is locally of finite type[1] and every geometric fiber is discrete and reduced, i.e. for all $y \in Y$, the $X_y \cong \coprod_i \operatorname{Spec} K_i$ where each K_i is a separable field extension of $\kappa(y)$.

We say that a morphism $f\colon X \to Y$ of schemes is *smooth (resp. étale, unramified) at $x \in X$* if there exists an open neighborhood $U \subset X$ of x such that $f|_U\colon U \to Y$ is smooth (resp. étale, unramified). In §A.3.5, we discuss local complete intersections and syntomic morphisms. There are the following implications:

$$\text{unramified} \impliedby \text{étale} \implies \text{smooth} \implies \text{syntomic} \implies \text{fppf} \implies \text{fpqc}.$$

A.3.1 Equivalences

Smooth, étale, and unramified morphisms have many equivalent characterizations. These equivalences take considerable work to establish, and we recommend [Mil80, Ch. 1] and [Liu02, §4.3] for accessible accounts.

Theorem A.3.1 (Smooth Equivalences). *Let $f\colon X \to Y$ be a locally of finite presentation morphism of schemes (resp. locally noetherian schemes). The following are equivalent:*

(1) f is smooth,

(2) (Infinitesimal Lifting Criterion) for every surjection $A \to A_0$ of rings with nilpotent kernel (resp. surjection of local artinian rings with $\ker(A \to A_0) \cong A/\mathfrak{m}_A$) and every commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A_0 & \longrightarrow & X \\
\cap\ \downarrow & \nearrow & \downarrow f \\
\operatorname{Spec} A & \longrightarrow & Y
\end{array}
$$

of solid arrows, there exists a dotted arrow filling in the diagram,

(3) (Jacobian Criterion) for every point $x \in X$, there exists affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism

$$B \cong A[x_1, \ldots, x_n]/(h_1, \ldots, h_r)$$

for some $h_1, \ldots, h_r \in A[x_1, \ldots, x_n]$ with $r \le n$ such that the determinant $\det(\frac{\delta h_j}{\delta x_i})_{1 \le i,j \le r} \in B$ of the Jacobi matrix, defined by the partial derivatives with respect to the first r x_i's, is a unit. (The map $\operatorname{Spec} A \to \operatorname{Spec} B$ is called a standard smooth morphism.)

If in addition X and Y are locally noetherian and $x \in X$ has image $y \in Y$ with $\kappa(x) = \kappa(y)$, then $f\colon X \to Y$ is smooth at x if and only if

---

[1]We are following the conventions of [GR71] and [SP] rather than [EGA] as we only require that f is locally of finite type rather than locally of finite presentation.

(4) there is an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}[\![x_1, \ldots, x_r]\!]$ of $\widehat{\mathcal{O}}_{Y,y}$-algebras.

If in addition X and Y are smooth over an algebraically closed field $\Bbbk$, then f is smooth at $x \in X(\Bbbk)$ if and only if

(5) the induced map $T_{X,x} \to T_{Y,y}$ on tangent spaces is surjective.

Proof. See [Har77, Exer. II.8.6, Prop. III.10.4], [EGA, 0.22.6.1, IV$_4$.17.14.1-2, IV$_4$.17.5.3], and [SP, Tags 01V9, 02H6, and 02HX]. $\qquad\square$

We say that $f \colon X \to Y$ is *smooth of relative dimension n* if f is smooth and every fiber is equidimensional of dimension n. This is equivalent to f being flat, $\dim X' = \dim Y' + n$ for all irreducible components $X' \subset X$ and $Y' \subset Y$ with $f(X') \subset Y'$, and $\dim_{\kappa(x)}(\Omega_{X/Y} \otimes \kappa(x)) = n$ for all $x \in X$, which is the definition given in [Har77, §III.10].

Theorem A.3.2 (Étale Equivalences)**.** *Let $f \colon X \to Y$ be a locally of finite presentation morphism of schemes (resp. locally noetherian schemes). The following are equivalent:*

(1) f is étale;

(2) f is smooth and $\Omega_{X/Y} = 0$;

(3) f is smooth and unramified;

(4) f is flat and unramified;

(5) (Infinitesimal Lifting Criterion) *for every surjection $A \to A_0$ of rings with nilpotent kernel (resp. surjection of local artinian rings with $\ker(A \to A_0) \cong A/\mathfrak{m}_A$) and every commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} A_0 & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec} A & \longrightarrow & Y \end{array}$$

of solid arrows, there exists a unique dotted arrow filling in the diagram; and

(6) (Jacobi Criterion) *for every point $x \in X$, there exists affine open neighborhoods $\operatorname{Spec} B$ of $f(x)$ and $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$ of x and an A-algebra isomorphism*

$$B \cong A[x_1, \ldots, x_n]/(h_1, \ldots, h_n)$$

for some $h_1, \ldots, h_n \in A[x_1, \ldots, x_n]$ such that the determinant $\det(\frac{\delta h_j}{\delta x_i})_{1 \le i,j \le n} \in B$ is a unit. (The map $\operatorname{Spec} A \to \operatorname{Spec} B$ is called a standard étale morphism.*)*

If in addition X and Y are locally noetherian and $x \in X$ has image $y \in Y$ with $\kappa(x) = \kappa(y)$, then $f \colon X \to Y$ is smooth at x if and only if

(6) $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}$.

If in addition X and Y are smooth over an algebraically closed field $\Bbbk$, then f is étale at $x \in X(\Bbbk)$ if and only if

(7) the induced map $T_{X,x} \to T_{Y,y}$ on tangent spaces is an isomorphism.

Proof. See [Har77, Exer. III.10.3], [EGA, IV$_4$.17.14.1-2, IV$_4$.17.6.3], and [SP, Tags 02GH and 02HF]. $\qquad\square$

Theorem A.3.3 (Unramified Equivalences)**.** *Let $f \colon X \to Y$ be morphism of schemes locally of finite type. The following are equivalent:*

(1) f is unramified;

(2) $\Omega_{X/Y} = 0$;

(3) the diagonal $\Delta_f \colon X \to X \times_Y X$ is an open immersion;

(4) (Infinitesimal Lifting Criterion for Unramifiedness) *for every surjection $A \to A_0$ of rings with nilpotent kernel (resp. surjection of local artinian rings with $\ker(A \to A_0) \cong A/\mathfrak{m}_A$) and every commutative diagram*

$$
\begin{array}{ccc}
\operatorname{Spec} A_0 & \longrightarrow & X \\
\cap \downarrow & \nearrow \diagup & \downarrow f \\
\operatorname{Spec} A & \longrightarrow & Y
\end{array}
$$

of solid arrows, there exists at most one dotted arrow filling in the diagram.

If in addition X and Y are locally noetherian and $x \in X$ has image $y \in Y$ with $\kappa(x) = \kappa(y)$, then $f \colon X \to Y$ is smooth at x if and only if

(4) $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{X,x}$ is surjective.

Proof. See [EGA, IV$_4$.17.14.1-2, IV$_4$.17.6.3], and [SP, Tags 02G3, 02H7, and 02GE]. □

A.3.2 Étale-local structure of smooth, étale, and unramified morphisms

Every smooth morphism is étale-locally relative affine space.

Proposition A.3.4 (Local Structure of Smooth Morphisms)**.** *A morphism $X \to Y$ of schemes is smooth at $x \in X$ if and only if there exists affine open subschemes $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset Y$ with $x \in \operatorname{Spec} A$, and a commutative diagram*

$$
\begin{array}{ccc}
X & \xleftarrow{\text{op}} \operatorname{Spec} A & \xrightarrow{\text{ét}} \mathbb{A}_B^n \\
\downarrow & \downarrow & \diagup \\
Y & \xleftarrow{\text{op}} \operatorname{Spec} B &
\end{array}
$$

where $\operatorname{Spec} A \to \mathbb{A}_B^n$ is étale.

Proof. See [SP, Tag 039P] and [EGA, IV$_4$.17.11.4]. □

An important consequence is that smooth morphisms have sections étale locally.

Corollary A.3.5. *Let $f \colon X \to Y$ be a morphism of schemes smooth at $x \in X$. Then there exists an étale neighborhood $Y' \to Y$ of $f(x)$ such that $X \times_Y Y' \to Y'$ has a section.*

Proof. After applying the proposition, observe that the morphism $\mathbb{A}_B^n \to \operatorname{Spec} B$ admits the zero section $\operatorname{Spec} B \to \mathbb{A}_B^n$. The scheme $Y' := \operatorname{Spec} B \times_{\mathbb{A}_B^n} \operatorname{Spec} A$ is étale over Y, and the composition $Y' \to \operatorname{Spec} A \hookrightarrow X$ defines a section $Y' \to X \times_Y Y'$ of $X \times_Y Y' \to Y'$. □

Every étale (resp. unramified) morphism is étale-locally an isomorphism (resp. closed immersion).

Proposition A.3.6. *Let $f \colon X \to S$ be a separated morphism of schemes étale (resp. unramified) at $x \in X$. Then there exists an étale neighborhood $(U, u) \to (S, f(x))$ and a finite disjoint union decomposition*

$$X_U = W \amalg \coprod_i V_i$$

such that each $V_i \to U$ is an isomorphism (resp. closed immersion) and the the fiber W_u contains no point over x.

Proof. See [SP, Tags 04HM and 04HG]. $\qquad\square$

It is sometimes convenient to know that étale and smooth morphisms of affine schemes can be lifted along closed immersions. It is also holds for syntomic morphisms (see Definition A.3.17).

Proposition A.3.7. *Consider a diagram*

$$
\begin{array}{ccc}
\operatorname{Spec} A_0 & \dashrightarrow & \operatorname{Spec} A \\
\downarrow & \square & \vdots \\
\operatorname{Spec} B_0 & \hookrightarrow & \operatorname{Spec} B
\end{array}
$$

of solid arrows where $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} B_0$ is a closed immersion. If $\operatorname{Spec} A_0 \to \operatorname{Spec} B_0$ is étale (resp. smooth), then there exists an étale (resp. smooth) morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ making the above diagram cartesian.

Proof. See [SP, Tags 04D1 and 07M8]. $\qquad\square$

A.3.3 Further properties

Proposition A.3.8 (Fibral Étaleness/Smoothness/Unramifiedness Criteria)**.** *Consider a diagram*

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
& \searrow \quad \swarrow & \\
& S &
\end{array}
$$

of schemes where $X \to S$ and $Y \to S$ are locally of finite presentation. Let $x \in X$ with image $s \in S$. Then

(1) $X \to Y$ *is unramified at x if and only if $X_s \to Y_s$ is unramified at x, and*

(2) *if $X \to S$ is flat at x, then $X \to Y$ is étale (resp. smooth) at x if and only if $X_s \to Y_s$ is étale (resp. smooth) at x.*

Proof. Let $y \in Y$ be the image of $x \in X$. Part (1) holds since unramified is defined as a condition on the fiber and the fiber X_y is identified with the fiber of $X_s \to Y_s$ over $y \in Y_s$. For the nontrivial direction ($\Leftarrow$) of (2), the Fibral Flatness Criterion (A.2.10) implies that $X \to Y$ is flat at x. Therefore, the smoothness (resp. étaleness) of $X \to Y$ at x is equivalent to the smoothness (resp. étaleness) of $X_y \to \operatorname{Spec} \kappa(y)$ at x. The latter condition holds by the smoothness (resp. étaleness) of $X_s \to Y_s$ at x. $\qquad\square$

While smoothness is clearly an open condition on the *source*, it is also an open condition on the *target* when the morphism is proper.

Corollary A.3.9. *If $f\colon X \to Y$ is a proper, flat, and locally of finite presentation morphism, then the set of points $y \in Y$ where $X_y \to \operatorname{Spec} \kappa(y)$ is smooth is open.*

Proof. If $y \in Y$ is a point such that $X_y \to \operatorname{Spec} \kappa(y)$ is smooth, then $f\colon X \to Y$ is smooth in an open neighborhood of X_y. If $Z \subset X$ is the closed locus where $f\colon X \to Y$ is not smooth, then $f(Z) \subset Y$ is precisely the locus where the fibers of f are not smooth. Since f is proper, $f(Z)$ is closed. $\square$

Proposition A.3.10. *Let $X \to Y$ be a smooth morphism of noetherian schemes. For every point $x \in X$ with image $y \in Y$,*

$$\dim_x(X) = \dim_y(Y) + \dim_x(X_y).$$

Proof. See [SP, Tag 0AFF]. $\square$

Proposition A.3.11. *If $X \to Y$ is a finite étale morphism, there exists a finite étale cover $Y' \to Y$ such that $X \times_Y Y' \to Y'$ is a trivial covering, i.e. $X \times_Y Y'$ is isomorphism to $\coprod_i Y'$ over Y'.*

Proof. We may assume that the degree d of $X \to Y$ is constant. The scheme

$$(X/Y)^d = \underbrace{X \times_Y \cdots \times_Y X}_{d}$$

represents the functor $\operatorname{Sch}/Y \to \operatorname{Sets}$ assigning a Y-scheme T to the set of d sections of $X \times_Y T \to T$. Each pairwise diagonal $(X/Y)^{d-1} \to (X/Y)^d$ is an open and closed immersion, and we set $(X/Y)_0^d \subset (X/Y)^d$ to be the complement of all pairwise diagonals. The projection morphism $(X/Y)_0^d \to Y$ is finite étale and the functorial description gives d disjoint sections of $X \times_Y (X/Y)_0^d \to (X/Y)_0^d$. $\square$

Proposition A.3.12. *A dominant unramified morphism $X \to Y$ of schemes with Y normal and X connected is étale.*

Proof. See [SGA1, Cor. I.9.11]. $\square$

The following result is often called 'Nagata–Zariski Purity'.

Proposition A.3.13 (Purity of the Branch Locus)**.** *Let $f\colon X \to Y$ be a quasi-finite morphism of integral noetherian schemes such that X is normal and Y is regular. Then the locus of points in X where f is not étale is either empty or codimension 1.*

Proof. See [Zar58], [Nag59], [SGA1, Thm. X.3.1], and [SP, Tag 0BMB]. $\square$

A.3.4   Fitting ideals and the singular locus

Fitting ideals allows for a scheme-theoretic description of the singular locus of a scheme. We use fitting ideals in the Characterization of Nodes (5.2.3). For background references on fitting ideals, we recommend [SP, Tag 07Z6] and [Eis95, §20].

If R is a ring and M is a finitely generated R-module, the *kth fitting ideal* $\operatorname{Fit}_k(M)$ *of* M is the ideal generated by the $(n-k) \times (n-k)$ minors of a matrix A defining a presentation

$$\bigoplus_{i \in I} R \xrightarrow{A} R^n \to M \to 0.$$

Of course, when M is finitely presented (e.g. R is noetherian), then the left-hand term can be assumed to be a finite free module R^m, in which case A is an $m \times n$ matrix and $\mathrm{Fit}_k(M)$ is a finitely generated ideal. The fitting ideal is independent of the choice of presentation, and defines an increasing sequence of ideals

$$0 = \mathrm{Fit}_{-1}(M) \subset \mathrm{Fit}_0(M) \subset \mathrm{Fit}_1(M) \subset \cdots R$$

such that $\mathrm{Fit}_k(M) = R$ if M can be generated by k elements. The R-module M is locally free of rank r if and only if $\mathrm{Fit}_{r-1}(M) = 0$ and $\mathrm{Fit}_r(M) = R$, and in this case $\mathrm{Fit}_k(M) = 0$ for all $k < r$. There is an identification $\mathrm{Fit}_k(M \otimes_R S) = \mathrm{Fit}_k(M)S$ for a ring map $R \to S$. In particular, $\mathrm{Fit}_k(M_f) = \mathrm{Fit}_k(M)_f$ for $f \in R$, $\mathrm{Fit}_k(M_{\mathfrak{p}}) = \mathrm{Fit}_k(M)_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p} \subset R$, and $\mathrm{Fit}_k(M) \otimes_R \widehat{R} = \mathrm{Fit}_k(\widehat{M})$ if R is a noetherian local ring.

If X is a scheme and F is a finite type quasi-coherent sheaf on X, the *kth fitting ideal sheaf of F* is the quasi-coherent sheaf of ideals $\mathrm{Fit}_k(F) \subset \mathcal{O}_X$ defined by $\Gamma(U, \mathrm{Fit}_k(F)) = \mathrm{Fit}_k(\Gamma(F, U))$ for affine open subsets $U \subset X$. Fitting ideal sheaves give a scheme structure to the singular locus.

Definition A.3.14. If X is a noetherian scheme of pure dimension d over a field $\Bbbk$, we define the *singular locus of X* as the subscheme $\mathrm{Sing}(X) := V(\mathrm{Fit}_d(\Omega_{X/\Bbbk}))$ defined by the dth fitting ideal of of $\Omega_{X/\Bbbk}$. More generally, if $X \to S$ is an fppf morphism such that every fiber has pure dimension d, we define the *relative singular locus* as the subscheme $\mathrm{Sing}(X/S) := V(\mathrm{Fit}_d(\Omega_{X/S}))$.

For example, if $X = \mathrm{Spec}\, \Bbbk[x_1, \ldots, x_n]/I$ with $I = (f_1, \ldots, f_m)$, the exact sequence $I/I^2 \to \Omega_{\mathbb{A}^n/\Bbbk}|_X \to \Omega_{X/\Bbbk} \to 0$ induces a resolution

$$\mathcal{O}_X^m \xrightarrow{J} \mathcal{O}_X^n \to \Omega_{X/\Bbbk} \to 0 \quad \text{with } J = \left(\frac{\partial f_j}{\partial x_i} \right),$$

and $\mathrm{Sing}(X)$ is defined by all $(n-d) \times (n-d)$ minors of J.

A.3.5 Local complete intersections and syntomic morphisms

Definition A.3.15. A scheme X locally of finite type over a field $\Bbbk$ is a *local complete intersection at $p \in X$* (or *lci at p*) if there exists an affine open neighborhood $p \in \mathrm{Spec}\, A \subset A$ such that A is a global complete intersection over $\Bbbk$, i.e. $A \cong \Bbbk[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ with $\dim A = n - c$. The scheme X is a *local complete intersection* if it is at every point.

Proposition A.3.16. *For a scheme X locally of finite type over a field $\Bbbk$ and a point $p \in X$, the following are equivalent:*

(1) X is a local complete intersection at p,

(2) the local ring $\mathcal{O}_{X,x} \cong R/(f_1, \ldots, f_c)$ where R is a regular local ring and $f_1, \ldots, f_c \in R$ is a regular sequence, and

(3) the completion $\widehat{\mathcal{O}}_{X,x} \cong R/(f_1, \ldots, f_c)$ where R is a regular complete local ring and $f_1, \ldots, f_c \in R$ is a regular sequence.

Proof. See [SP, Tags 00S8 and 09PY]. $\square$

For a scheme locally of finite type over a field $\Bbbk$, there are implications:

smooth $\implies$ local complete intersection $\implies$ Gorenstein $\implies$ Cohen–Macaulay.

Here is the relative notion:

Definition A.3.17. A morphism of schemes $f\colon X \to S$ is *syntomic* (or a *flat local complete intersection morphism*) if f is fppf and every fiber is a local complete intersection. We say that $f\colon X \to S$ is *syntomic at* $x \in X$ if there is an open neighborhood U of x such that $f|_U$ is syntomic.

Syntomic morphisms have a local structure analogous to local complete intersections.

Proposition A.3.18. *A morphism $f\colon X \to S$ is syntomic at $x \in X$ if and only if there are affine open neighborhood $x \in \operatorname{Spec} A \subset X$ and $\operatorname{Spec} B \subset Y$ with $f(\operatorname{Spec} A) \subset \operatorname{Spec} B$ such that $A \cong B[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and every nonempty fiber of $\operatorname{Spec} A \to \operatorname{Spec} B$ has dimension $n - c$.*

Proof. See [EGA, IV$_4$ §19.3], [SGA6, VIII §1] and [SP, Tag 01UB]. $\qquad\square$

A.4 Properness and the Valuative Criterion

Properness, separatedness, and universal closedness can be verified using the Valuative Criteria (3.8.2). While the importance of valuative criteria may not be apparent after a first course in algebraic geometry, they becomes indispensable in moduli theory, as it provides a geometric strategy to verify separated, properness, and universal closedness. In this text, we apply the Valuative Criteria to show that $\overline{\mathcal{M}}_g$ is proper (Theorem 5.5.16) and $\underline{\operatorname{Bun}}_{r,d}^{\mathrm{ss}}(C)$ is universally closed.

A.4.1 The Valuative Criteria

As we generalize the criteria to algebraic stacks in Theorem 3.8.2, we quickly recap how the Valuative Criteria (A.4.5) are established for schemes. The starting point of the proof of is the following lifting criterion for quasi-compact morphisms to be closed.

Lemma A.4.1. *A quasi-compact morphism $f\colon X \to Y$ of schemes is closed if and only if for every point $x \in X$, every specialization $f(x) \rightsquigarrow y_0$ in Y lifts to a specialization $x \rightsquigarrow x_0$ in X:*

$$
\begin{array}{ccc}
X & & x \rightsquigarrow\ \sim\ \rightsquigarrow x_0 \\
\downarrow{\scriptstyle f} & & \quad\downarrow\qquad\ \ \vdots \\
Y & & f(x) \rightsquigarrow y_0.
\end{array}
$$

Proof. The implication $(\Rightarrow)$ is clear as $f(\overline{\{x\}}) \subset Y$ is closed. For the converse, after replacing X with a closed subscheme, it suffices to show that $f(X)$ is closed. We can assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine (since f is quasi-compact) and reduced (since the question is topological). The scheme-theoretic image of $\operatorname{Spec} A \to \operatorname{Spec} B$ is defined by $I := \ker(B \to A)$. By replacing B with B/I, we can assume that $B \to A$ is injective. For every minimal prime $\mathfrak{p} \in \operatorname{Spec} B$, the localization $B_{\mathfrak{p}}$ is a field and the map $B_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is injective. Thus, $A_{\mathfrak{p}} \neq 0$ and the fiber $f^{-1}(\mathfrak{p}) = \operatorname{Spec} A_{\mathfrak{p}}$ is non-empty. Since $f(X)$ contains all the minimal primes and is closed under specialization, $f(X) = Y$ is closed. $\qquad\square$

The noetherian valuative criterion depends on the following algebraic fact:

Proposition A.4.2. *Let $(A, \mathfrak{m}_A)$ be a noetherian local domain with fraction field K such that A is not a field. If $K \to L$ is a finitely generated field extension, then there exists a DVR R with fraction field L dominating A (i.e. $A \subset R$ and $\mathfrak{m}_A \cap K = \mathfrak{m}_R$).*

Proof. We reduce to the case that $K \to L$ is a finite field extension by choosing a transcendence basis $x_1, \ldots, x_n \in L$ over K and replacing A with $A[x_1, \ldots, x_n]_{\mathfrak{n}}$ where $\mathfrak{n} = \mathfrak{m}_A A[x_1, \ldots, x_n] + (x_1, \ldots, x_n)$. Let B be the blow of $\operatorname{Spec} A$ at $\mathfrak{m}_A$ and let $E \subset B$ be the exceptional divisor. If $\xi \in E$ is a generic point, then $\mathcal{O}_{B,\xi}$ is a noetherian domain of dimension 1 (by Krull's Hauptidealsatz) with fraction field K. We now let $R \subset L$ be the integral closure of $\mathcal{O}_{B,\xi}$ in L. By Krull–Akizuki (A.4.3), R is noetherian. Since R is also normal of dimension 1, it is a DVR. $\qquad\square$

Proposition A.4.3 (Krull–Akizuki)**.** *Let R be a noetherian domain of dimension 1 with fraction field K. If $K \to L$ is a finite extension of fields, then every ring A with $R \subset A \subset L$ is noetherian.*

Proof. See [Nag62, p. 115] or [SP, Tag 00PG]. $\qquad\square$

Proposition A.4.2 and Krull–Akizuki have the following geometric implication.

Proposition A.4.4. *If $f \colon X \to Y$ is a finite type morphism of noetherian schemes, $x \in X$, and $f(x) \rightsquigarrow y_0$ is a specialization, there exists a commutative diagram*

$$
\begin{array}{ccc}
\operatorname{Spec} K \longrightarrow X & \qquad & x \\
\Big\downarrow \qquad\quad \Big\downarrow f & & \Big\uparrow \\
\operatorname{Spec} R \longrightarrow Y & \qquad & f(x) \rightsquigarrow y_0.
\end{array}
$$

where R is a DVR with fraction field K, the image of $\operatorname{Spec} K \to X$ is x and $\operatorname{Spec} R \to Y$ realizes the specialization $f(x) \rightsquigarrow y_0$. In particular, every specialization $x \rightsquigarrow x_0$ in a noetherian scheme is realized by a map $\operatorname{Spec} R \to X$ from a DVR.

Proof. After replacing X with $\overline{\{f(x)\}}$ and Y with $\overline{\{x\}}$, we may assume that X and Y are integral with generic points x and $f(x)$. Then $\mathcal{O}_{Y, y_0}$ is a noetherian local domain with fraction field $\kappa(f(x))$. By applying Proposition A.4.2 to the field extension $\kappa(f(x)) \to \kappa(x)$, we obtain a DVR R with fraction field $\kappa(x)$ dominating $\mathcal{O}_{Y, y_0}$, yielding the desired diagram. $\qquad\square$

We only state a noetherian version of the Valuative Criteria.

Theorem A.4.5 (Valuative Criteria for Proper/Separated/Universally Closed Morphisms)**.** *Let $f \colon X \to Y$ be a quasi-compact morphism of noetherian schemes. Consider a commutative diagram*

$$
\begin{array}{ccc}
\operatorname{Spec} K & \longrightarrow & X \\
\Big\downarrow & \nearrow & \Big\downarrow f \\
\operatorname{Spec} R & \longrightarrow & Y
\end{array}
\tag{A.4.6}
$$

of solid arrows where R is a DVR with fraction field K. Then

(1) f is proper if and only if f is of finite type and for every diagram (A.4.6), there exists a unique lift,

(2) f is separated if and only if for every diagram (A.4.6), any two lifts are equal, and

(3) f is universally closed if and only if for every diagram (A.4.6), there exists a lift.

Proof. We first claim that it suffices to handle the universally closed case. Indeed, a morphism $X \to Y$ is separated if and only if the diagonal $X \to X \times_Y X$ is universally closed, and the equality of two lifts in the valuative criterion for $X \to Y$ corresponds to the existence of a lift in the valuative criterion for $X \to X \times_Y X$.

Suppose that $X \to Y$ satisfies the valuative criterion for universal closedness. To show that $X \to Y$ is universally closed, we claim that it suffices to check that the base change $X_T \to T$ is closed for every *finite type* morphism $T \to Y$. Indeed, suppose that $f_T \colon X_T \to T$ is not closed for some map $T \to Y$. By Lemma A.4.1, there exists $x \in X_T$ and a specialization $f_T(x) \rightsquigarrow t_0$ which doesn't lift to a specialization $x \rightsquigarrow x_0$. This implies that $Z = \overline{\{x\}} \subset X_T$ has trivial intersection with the fiber $(X_T)_{t_0}$. Applying Lemma A.4.8 shows that, after replacing T with an open neighborhood of t_0, there is a a commutative diagram

$$
\begin{array}{ccc}
x & \quad & X_T \longrightarrow X_{T'} \longrightarrow X \\
\Big\uparrow & \quad & \Big\downarrow{\scriptstyle f_T} \quad \Big\downarrow{\scriptstyle f_{T'}} \quad \Big\downarrow{\scriptstyle f} \\
f_T(x) \rightsquigarrow t_0 & \quad & T \xrightarrow{\;g\;} T' \longrightarrow Y
\end{array}
$$

where $T' \to Y$ is finite type and a closed subscheme $Z' \subset X_{T'}$ such that $f_{T'}(Z')$ contains $g(f_T(x))$ but not $g(t_0)$. This shows that $f_{T'} \colon X_{T'} \to T'$ is not closed.

Since the valuative criterion holds for $X \to Y$, it also holds for the morphism $X_T \to T$ of *noetherian* schemes. It therefore suffices to show that $X \to Y$ is closed. By Lemma A.4.1, it suffices to show that given $x \in X$, every specialization $f(x) \rightsquigarrow y_0$ lifts to a specialization $x \rightsquigarrow x_0$. By Proposition A.4.4, there exists a diagram (A.4.6) such that $\operatorname{Spec} R \to Y$ realizes $f(x) \rightsquigarrow y_0$ with a lift $\operatorname{Spec} K \to X$ whose image is x. The valuative criterion implies the existence of a lift $\operatorname{Spec} R \to X$, which in turn yields a specialization $x \rightsquigarrow x_0$ lifting $f(x) \rightsquigarrow y_0$.

Conversely, assume that $f \colon X \to Y$ is universally closed and that we are given a diagram (A.4.6). By replacing Y with $\operatorname{Spec} R$ and X with $X \times_Y \operatorname{Spec} R$, we may assume that $Y = \operatorname{Spec} R$ and that we have a diagram

$$
\begin{array}{ccc}
\operatorname{Spec} K & \xrightarrow{\;x\;} & X \\
\Big\downarrow & \swarrow & \\
\operatorname{Spec} R & &
\end{array}
$$

By replacing X with $\overline{\{x\}}$, we may assume that X is integral with generic point x. Since $X \to \operatorname{Spec} R$ is closed, there exists a specialization $x \rightsquigarrow x_0$ in X over the specialization of the generic point to the closed point in $\operatorname{Spec} R$. This gives an inclusion of local rings $R \hookrightarrow \mathcal{O}_{X,x_0}$ in K. Since R is a valuation ring with fraction field K (i.e. is maximal among local rings properly contained in K), we see that $R = \mathcal{O}_{X,x_0}$ and the inclusion $\operatorname{Spec} \mathcal{O}_{X,x_0} \to X$ gives the desired lift.

See also [Har77, Thm. 4.7, Exer. II.4.11], [EGA, §II.7], and [SP, Tags 0BX4 and 0CM1] $\qquad\qquad\square$

Remark A.4.7. The quasi-compactness of f (resp. Δ_f, Δ_{Δ_f}) are essential in the valuative criterion of universal closedness (resp. separatedness, separated diagonal). In fact, universally closed morphisms are necessarily quasi-compact [SP, Tag 04XU].

The lemma below was used in the proof and is also used in the proof of the Valuative Criteria (3.8.2) for algebraic stacks.

Lemma A.4.8. *Let $f\colon X \to Y$ be a quasi-compact morphism of schemes. Let $T \to Y$ be a morphism of schemes, $t_0 \in T$ be a point, and $Z \subset X_T$ a closed subscheme such that $Z \cap (X_T)_{t_0} = \emptyset$. Then after replacing T with an open neighborhood of t_0, there exists a finite type morphism $T' \to Y$ of schemes with a factorization $T \xrightarrow{g} T' \to Y$ and a closed subscheme $Z' \subset X_{T'}$ such that $Z' \cap (X_{T'})_{g(t_0)} = \emptyset$ and $\mathrm{im}(Z \hookrightarrow X_T \to X_{T'}) \subset Z'$.*

Proof. After reducing to the affine case $T = \operatorname{Spec} B \to Y = \operatorname{Spec} A$, we write B as a colimit of finite type A-algebras B_λ. Using techniques analogous to Limits of Schemes (§B.3), one shows that for $\lambda \gg 0$ there exists a subscheme $Z_\lambda \subset X_{B_\lambda}$ with the desired properties. The details are not hard, but also not inspiring. See [SP, Tag 05BD]. $\square$

A.4.2 Universally submersive morphisms

A morphism $f\colon X \to Y$ of schemes is *submersive* if f is surjective and Y has the quotient topology, i.e. a subset $U \subset Y$ is open if and only if $f^{-1}(U)$ is open, and $f\colon X \to Y$ is *universally submersive* if for every map $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is submersive.

Exercise A.4.9.

(1) Show that a morphism $f\colon X \to Y$ of noetherian schemes is universally submersive if and only if every map $\operatorname{Spec} R \to Y$ from a DVR has a lift

$$
\begin{array}{ccc}
\operatorname{Spec} R' & \dashrightarrow & X \\
\big\downarrow & & \big\downarrow f \\
\operatorname{Spec} R & \longrightarrow & Y,
\end{array}
$$

where $R \to R'$ is a local homomorphism of DVRs.

(2) Show that every fppf morphism or universally closed morphism of noetherian schemes is universally submersive.

A.5 Dévissage and finiteness of cohomology

Dévissage, or 'unscrewing' in French, is a specific type of noetherian induction designed to verify properties of coherent sheaves. We apply it to extend the theorem of Finiteness of Cohomology (A.5.3) from projective morphisms to proper morphisms.

A.5.1 Dévissage

Proposition A.5.1 (Dévissage)**.** *Let X be a noetherian scheme. Let $\mathcal{P}$ be a property of coherent sheaves on X satisfying*

(a) *if $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of coherent sheaves on X and two out of the three satisfy $\mathcal{P}$, then the third satisfies $\mathcal{P}$, and*

(b) *for every integral closed subscheme $Z \subset X$ with generic point ξ, there exists a coherent sheaf G satisfying $\mathcal{P}$ with $\mathfrak{m}_\xi G_\xi = 0$ and $\dim_{\kappa(\xi)} G_\xi = 1$ (so that $\operatorname{Supp}(G) = Z$).*

Then every coherent sheaf on X satisfies $\mathcal{P}$.

Proof. We say that the property $\mathcal{P}_Y$ holds for a closed subset $Y \subset X$ if every coherent sheaf F on X with $\mathrm{Supp}(F) \subset Y$ satisfies $\mathcal{P}$. We prove the proposition by noetherian induction: we need to show that if $\mathcal{P}_{Y'}$ holds for every closed subset $Y' \subsetneq Y$, then $\mathcal{P}_Y$ also holds. Let F be a coherent sheaf on X with $\mathrm{Supp}(F) \subset Y$. Giving $Y \subset X$ the reduced scheme structure, let $I \subset \mathcal{O}_X$ be its ideal sheaf. To show that F satisfies $\mathcal{P}$, we first claim that it suffices to assume that $IF = 0$. Since $I^n F = 0$ for some $n > 0$, we have a filtration $0 = I^n F \subset I^{n-1} F \subset \cdots \subset F$ and short exact sequences

$$0 \to I^{k-1}F/I^k F \to F/I^k F \to F/I^{k-1}F \to 0.$$

By induction and property (a), it suffices to show that $\mathcal{P}$ holds for $I^{k-1}F/I^k F$, which is annihilated by I. Second, we claim that we can assume that Y is irreducible, hence integral. Supposing that $Y = Y_1 \cup Y_2$ with $F_1 = F|_{Y_1}$ and $F_2 = F|_{Y_2}$, the map $\phi \colon F \to F_1 \oplus F_2$ has kernel and cokernel supported on $Y_1 \cap Y_2$. Applying property (a) to the exact sequence $0 \to \mathrm{im}\,\phi \to F_1 \oplus F_2 \to \mathrm{coker}\,\phi \to 0$ and then to $0 \to \ker\phi \to F \to \mathrm{im}\,\phi \to 0$, shows that $\mathcal{P}$ also holds for F. Finally, letting $\xi \in Y$ be the generic point, property (b) gives a coherent sheaf G satisfying $\mathcal{P}$ with $\mathrm{Supp}(G) \subset Y$ and $\dim_{\kappa(\xi)} G_\xi = 1$. Setting $d = \dim_{\kappa(\xi)} F_\xi$, since F_ξ and $G_\xi^{\oplus d}$ are isomorphic, there is an open subscheme $U \subset Y$ and an isomorphism $F|_U \to G^{\oplus d}|_U$. Let H be the image of the graph $F|_U \to F|_U \oplus G^{\oplus d}|_U$, and $\widetilde{H} \subset F \oplus G^{\oplus d}$ be a subsheaf on Y extending H [Har77, Exer. II.5.15]. The projections $H \to G^{\oplus d}$ and $H \to F$ induce isomorphisms over U and hence have kernels and cokernels supported on a closed subscheme $Y' \subsetneq Y$. Since $\mathcal{P}$ holds for G, $\mathcal{P}$ also holds for $G^{\oplus d}$ by property (a). It follows that $\mathcal{P}$ holds for H and thus also F. See also [EGA, III$_1$.3.1.2] and [SP, Tag 01YI]. $\square$

Remark A.5.2. The same proof also establishes some useful variants. First, if we assumed that $\mathcal{F}^{\oplus n} \in \mathcal{P}$ implies that $\mathcal{F} \in \mathcal{P}$, then condition (b) can be weakened to exhibiting a coherent sheaf G satisfying $\mathcal{P}$ with $\mathrm{Supp}(G) \subset Z$ and $G_\xi \neq 0$. Alternatively, (b) can be replaced with the condition that for every integral closed subschemes $Z \subset X$ with ideal sheaf I_Z and every coherent sheaf F on X with $I_Z F = 0$, there exists a coherent sheaf G on X with $I_Z G = 0$ satisfying $\mathcal{P}$ and a morphism $F \to G$ which is an isomorphism on a non-empty open subset of Z.

A.5.2 Dévissage

Theorem A.5.3 (Finiteness of Cohomology). *Let $f \colon X \to Y$ be a proper morphism of noetherian schemes. For any coherent sheaf F on X and any $i \geq 0$, $\mathrm{R}^i f_* F$ is coherent.*

Proof. By Flat Base Change (A.2.12), we can assume that $Y = \mathrm{Spec}\,A$ is the spectrum of a noetherian ring. We need to show that for every coherent sheaf F on X and any $i \geq 0$, $\mathrm{H}^i(X, F)$ is a finite A-module. When $X \to \mathrm{Spec}\,A$ is projective, this is [Har77, Thm. III.5.2]—this is a really nice argument exhibiting the power of cohomology, even if you are only interested in the H^0 case. In the proof, one quickly reduces to the case that $X = \mathbb{P}^n_A$. Choosing an exact sequence $0 \to K \to \mathcal{O}_{\mathbb{P}^n_A}(-m)^{\oplus d} \to F \to 0$, the statement holds for the middle term by a Céch cohomology computation, and the vanishing of cohomology in sufficiently high degree and a descending induction argument shows that it holds for F.

To apply Dévissage ($A.5.1$), we let $\mathcal{P}$ be the property of a coherent sheaf F on X that $\mathrm{H}^i(X, F)$ is a finite A-module for all i. This satisfies the two-out-of-three condition (a) of ($A.5.1$). To see that (b) holds, let $Z \subset X$ be an integral closed subscheme with generic point ξ. By Chow's Lemma [Har77, Exer. II.4.10], there exists a projective birational morphism $g \colon Z' \to Z$ such that $Z' \to Z \to Y$ is also projective. The ample sheaf $\mathcal{O}_{Z'}(1)$ is relatively ample over Z and thus for $d \gg 0$ and $i > 0$, $\mathrm{R}^i g_*(\mathcal{O}_{Z'}(d)) = 0$ and $\mathrm{H}^i(Z', \mathcal{O}_{Z'}(d)) = 0$. Taking $G := g_*\mathcal{O}_{Z'}(d)$ for $d \gg 0$, $\dim_{\kappa(\xi)} G_\xi = 1$. Using the vanishing of $\mathrm{R}^i g_*(\mathcal{O}_{Z'}(d))$, the Leray spectral sequence $\mathrm{H}^p(Z, \mathrm{R}^q g_* \mathcal{O}_{Z'}(d)) \Rightarrow \mathrm{H}^{p+q}(Z', \mathcal{O}_{Z'}(d))$ implies that $\mathrm{H}^i(Z, G) = \mathrm{H}^i(Z', \mathcal{O}_{Z'}(d))$ is a finite A-module (and in fact 0 for $i > 0$). See also [EGA, III$_1$.3.2.1] and [SP, Tag 02O5].

This argument can also be formulated using derived categories. Chow's Lemma gives a projective birational morphism $g \colon X' \to X$ with X' projective over $Y = \operatorname{Spec} A$, and consider the exact triangle $F \to \mathrm{R}g_* g^* F \to C$. Since g is projective, $\mathrm{R}g_* g^* F \in D^b_{\mathrm{Coh}}(X)$ and thus $C \in D^b_{\mathrm{Coh}}(X)$. Since g is birational $F \to \mathrm{R}g_* g^* F$ is an isomorphism over a dense open. Using the exact triangles $C^n \to C \to \tau_{>n}C$ arising from truncation and the fact that $C^n = 0$ for $n \ll 0$, an induction argument shows that $\mathrm{R}\Gamma(X, C) \in D^b_{\mathrm{Coh}}(Y)$. Since $\mathrm{R}\Gamma(X, g_* g^* F) \in D^b_{\mathrm{Coh}}(Y)$, we conclude that $\mathrm{R}\Gamma(X, F) \in D^b_{\mathrm{Coh}}(Y)$. Formalizing this argument leads to version of dévissage for derived categories (e.g. [LMB00, Lem. 15.7]). $\qquad\square$

The following version of Formal Functions is often applied over a complete *local* ring $(A, \mathfrak{m})$, but the non-local case is sometimes useful.

Theorem A.5.4 (Formal Functions). *Let X be a scheme proper over a noetherian ring A which is complete with respect to an ideal $I \subset A$. Let $X_n = X \times_A A/I^{n+1}$. If F is a coherent sheaf on X, there is a natural isomorphism*

$$\mathrm{H}^i(X, F) \xrightarrow{\sim} \varprojlim_n \mathrm{H}^i(X_n, F|_{X_n})$$

for every $i \geq 0$.

Proof. See [Har77, Thm. III.11.1] (projective over complete local), [Vak17, Thm. 30.8.1], [FGI$^+$05, Cor. 8.2.4], [EGA, III$_1$.4.1.7], and [SP, Tag 02OC]. $\qquad\square$

Exercise A.5.5. Show more generally that for coherent sheaves F and G on X, there is a natural isomorphism

$$\mathrm{Ext}^i_{\mathcal{O}_X}(F, G) \xrightarrow{\sim} \varprojlim_n \mathrm{Ext}^i_{\mathcal{O}_{X_n}}(F|_{X_n}, G|_{X_n})$$

for every $i \geq 0$.

Exercise A.5.6. Use dévissage to reduce the proper case of Formal Functions ($A.5.4$) to the projective case.

Hint: Reduce to showing that there is a universal d such that $\mathrm{R}^i g_* \mathcal{O}_{Z'_n}(d) = 0$ for all $n, i \geq 0$, where Z' is as in the proof of Finiteness of Cohomology ($A.5.3$) and $Z'_n = Z' \times_A A/\mathfrak{m}^{n+1}$. To show this, apply Serre's Vanishing Theorem [Har77, Thm. II.5.2] to the projective morphism $\mathcal{S}\mathrm{pec}_{Z'} \bigoplus_{i \geq 0} \mathfrak{m}^i \mathcal{O}_{Z'} \to \operatorname{Spec} \bigoplus_{i \geq 0} \mathfrak{m}^i$.

A.6 Cohomology and Base Change

If $f \colon X \to Y$ is a proper morphism of noetherian schemes and F is a coherent sheaf on X, then Finiteness of Cohomology (A.5.3) implies that $\mathrm{R}^i f_* F$ is coherent. We often want to know more:

(a) When is $\mathrm{R}^i f_* F$ a vector bundle on Y?

(b) When does the construction of $\mathrm{R}^i f_* F$ commute with base change, i.e. for a map $g \colon Y' \to Y$ of schemes inducing a cartesian diagram

$$
\begin{array}{ccc}
X_{Y'} & \xrightarrow{\ g'\ } & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{\ g\ } & Y,
\end{array}
$$

when is the comparison map

$$
\phi^i_{Y'} \colon g^* \mathrm{R}^i f_* F \to \mathrm{R}^i f'_* g'^* F \tag{A.6.1}
$$

an isomorphism?

When $f \colon X \to Y$ is flat, Flat Base Change (A.2.12) tells us that (A.6.1) is always an isomorphism. Cohomology and Base Change (A.6.8) provides an answer when F is flat over Y.

Cohomology and Base Change is an essential tool in moduli theory. It can be applied to verify properties of families of objects and construct vector bundles on moduli spaces. For instance, for a family $\pi \colon \mathcal{C} \to S$ of smooth curves, we can verify that $\pi_*(\Omega^{\otimes k}_{\mathcal{C}/S})$ is a vector bundle for $k > 0$ whose construction commutes with base change on S, and show that $\mathcal{C}$ embeds canonically into $\mathbb{P}(\pi_*(\Omega^{\otimes k}_{\mathcal{C}/S}))$ for $k \geq 3$ (Proposition 5.1.15). We apply Cohomology and Base Change in our study of $\mathcal{M}_g$. It is used to verify that it is an algebraic stack (Theorem 3.1.17) and to establish various geometric properties. Applied to the universal family $\pi \colon \mathcal{U}_g \to \mathcal{M}_g$, Cohomology and Base Change shows that $\pi_*(\Omega_{\mathcal{U}_g/\mathcal{M}_g})$ is a vector bundle of rank g on $\mathcal{M}_g$, called the *Hodge bundle* (Example 4.1.4).

A.6.1 Formulations of Cohomology and Base Change

We begin with the key algebraic version of Cohomology and Base Change, which is used to establish the other versions.

Theorem A.6.2 (Cohomology and Base Change I). *Let $X \to \operatorname{Spec} A$ be a proper morphism of noetherian schemes and F be a coherent sheaf on X which is flat over A. There is a complex*

$$
K^\bullet \colon 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0
$$

of finite and locally free A-modules such that $\mathrm{H}^i(X, F) = \mathrm{H}^i(K^\bullet)$ for all i. Moreover, for every A-module M, $\mathrm{H}^i(X, F \otimes_A M) = H^i(K^\bullet \otimes_A M)$. In particular, for a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of schemes, $\mathrm{H}^i(X_B, F_B) = \mathrm{H}^i(K^\bullet \otimes_A B)$ where $X_B := X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ and F_B is the pullback of F to X_B.

Proof. This is established by choosing a finite affine cover $\{U_i\}$ of X and considering the corresponding alternating Céch complex $C^\bullet$ on $\{U_i\}$ with coefficients in F.

Then $C^\bullet$ is a finite complex of flat (but not finitely generated) A-modules and $\mathrm{H}^i(X, F) = \mathrm{H}^i(C^\bullet)$. The result is then obtained by inductively refining $C^\bullet$ to build a finite complex $K^\bullet$ of finite and flat A-modules which is quasi-isomorphic to $C^\bullet$. See [Mum70, Thm. p.46], [SP, Tag 07VK], and [Vak17, 28.2.1]. $\qquad\square$

Remark A.6.3 (Perfect complexes). A bounded complex $K^\bullet$ of coherent sheaves on a noetherian scheme X is *perfect* if there is an affine cover $X = \bigcup_i U_i$ such that each $K^\bullet|_{U_i}$ is quasi-isomorphic to a bounded complex of vector bundles on U_i. If X is affine (or more generally has the resolution property, i.e. every coherent sheaf is the quotient of a vector bundle), then $K^\bullet$ is perfect if and only if it is quasi-isomorphic to a bounded complex of vector bundles on X [SP, Tag 0F8F]. Moreover, the compact objects in $D_{\mathrm{QCoh}}(X)$ are precisely the perfect complexes [SP, Tag 09M8].

With this terminology in place, Theorem A.6.2 has the following translation: $\mathrm{R}f_*F \in D^b_{\mathrm{Coh}}(\mathrm{Spec}\, A)$ is perfect. More generally, if $F^\bullet$ is a perfect complex on X, then $\mathrm{R}f_*F^\bullet$ is also perfect [SP, Tag 0A1H].

Theorem A.6.2 tells us that the cohomology $\mathrm{H}^i(X, F)$ can be computed as the cohomology of a bounded complex $K^\bullet$ of vector bundles on $\mathrm{Spec}\, A$, and thus reduces cohomological questions to linear algebra.

Theorem A.6.4 (Semicontinuity Theorem). *Let $X \to Y$ be a proper morphism of noetherian schemes and F be a coherent sheaf on X which is flat over Y.*

(1) For each $i \geq 0$, the function

$$Y \to \mathbb{Z}, \quad y \mapsto \mathrm{h}^i(X_y, F_y)$$

is upper semicontinuous.

(2) The function

$$Y \to \mathbb{Z}, \quad y \mapsto \chi(X_y, F_y) = \sum_{i=0}^{\infty}(-1)^i \mathrm{h}^i(X_y, F_y)$$

is locally constant.

Proof. We may assume that $Y = \mathrm{Spec}\, A$ so that Theorem A.6.2 applies: there is bounded complex $K^\bullet\colon \cdots \to K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \to \cdots$ of finite and locally free A-modules such that $\mathrm{H}^i(X_y, F_y) = H^i(K^\bullet \otimes_A \kappa(y))$ for all $y \in Y$. Using that $\mathrm{im}(d^i \otimes \kappa(y)) = (K^i \otimes_A \kappa(y))/\ker(d^i \otimes \kappa(y))$, we have

$$\mathrm{h}^i(X_y, F_y) = \dim_{\kappa(y)} \ker(d^i \otimes \kappa(y)) - \dim_{\kappa(y)} \mathrm{im}(d^{i-1} \otimes \kappa(y))$$
$$= \dim_{\kappa(y)} K^i \otimes \kappa(y) - \dim_{\kappa(y)} \mathrm{im}(d^i \otimes \kappa(y)) - \dim_{\kappa(y)} \mathrm{im}(d^{i-1} \otimes \kappa(y)).$$
$$(\text{A.6.5})$$

The statement follows as both $\dim_{\kappa(y)} \mathrm{im}(d^i \otimes \kappa(y))$ and $\dim_{\kappa(y)} \mathrm{im}(d^i \otimes \kappa(y))$ are lower semicontinuous. See also [Mum70, p. 47], [Har77, Thm. 12.8], or [Vak17, Thm. 25.1.1]. $\qquad\square$

To show more powerful results, we will need more sophisticated linear algebra. We follow an argument of Eric Larson, as also described in [Vak17, §25.2].

Proposition A.6.6. *Let X be a scheme and $\phi\colon E \to F$ be a map of vector bundles on X of rank e and f. For every point $x \in X$ and an integer $r \leq \min(e, f)$, the following are equivalent:*

(1) $\mathrm{coker}(\phi)$ *is a vector bundle of rank* $f - r$ *in an open neighborhood of* x,

(2) *there is an open neighborhood* U *of* x *and identifications* $E|_U \cong \mathcal{O}_U^{\oplus e}$ *and* $F|_U \cong \mathcal{O}_U^{\oplus f}$ *such that* $\phi|_U$ *corresponds to the composition of the projection* $\mathcal{O}_U^{\oplus e} \twoheadrightarrow \mathcal{O}_U^{\oplus r}$ *and inclusion* $\mathcal{O}_U^{\oplus r} \hookrightarrow \mathcal{O}_U^{\oplus f}$ *of the first* r *summands, and*

(3) $\ker(\phi) \otimes \kappa(x) \to \ker(\phi \otimes \kappa(x))$ *is surjective.*

If in addition X *is reduced, the above are equivalent to*

(4) *there is an open neighborhood* U *of* x *such that* $\phi \otimes \kappa(u)$ *has rank* r *for all* $u \in U$.

If these hold, then $\mathrm{coker}(\phi)$, $\ker(\phi)$, *and* $\mathrm{im}(\phi)$ *are vector bundles, and their construction commutes with base change.*

Proof. This is an exercise in linear algebra. $\qquad\square$

If the equivalent conditions above hold, we say that $\phi\colon E \to F$ is a *map of vector bundles (of rank* r*)*. We can now provide a quick proof of Grauert's Theorem.

Theorem A.6.7 (Grauert's Theorem). *Let* $f\colon X \to Y$ *be a proper morphism of noetherian schemes such that* Y *is reduced and connected. Let* F *be a coherent sheaf on* X *flat over* Y. *For each integer* i, *the following are equivalent:*

(1) the function $y \mapsto \mathrm{h}^i(X_y, F_y)$ *is constant; and*

(2) $\mathrm{R}^i f_* F$ *is a vector bundle and the comparison map*

$$\phi_y^i\colon \mathrm{R}^i f_* F \otimes \kappa(y) \to \mathrm{H}^i(X_y, F_y)$$

is an isomorphism for all $y \in Y$.

If these hold, then the construction of $\mathrm{R}^i f_* F$ *commutes with base change by an arbitrary map* $T \to Y$.

Proof. The direction (1) $\implies$ (2) is clear. For the converse, we can reduce to the case that $Y = \mathrm{Spec}\, A$ as the question is Zariski local. Let

$$K^\bullet\colon \cdots \to K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \to \cdots$$

be the complex of vector bundles on Y produced by Theorem A.6.2. As $y \mapsto \mathrm{h}^i(X_y, F_y)$ is constant, the identity (A.6.5) implies that $y \mapsto \dim_{\kappa(y)} \mathrm{im}(d^{i-1} \otimes \kappa(y))$ and $y \mapsto \dim_{\kappa(y)} \mathrm{im}(d^i \otimes \kappa(y))$ are also constant. As Y is reduced, Proposition A.6.6 implies that d^{i-1} and d^i are maps of vector bundles, that $\mathrm{im}(d^i)$ and $\mathrm{coker}(d^{i-1})$ are vector bundles, and that $\ker(d^i)$ commutes with base change. The cohomology $\mathrm{H}^i(K^\bullet) = \mathrm{im}(d^{i-1})/\ker(d^i)$ sits in a short exact sequence

$$0 \to \mathrm{H}^i(K^\bullet) \to \underbrace{K^i/\ker(d^i)}_{\mathrm{im}(d^i)} \to \mathrm{coker}(d^{i-1}) \to 0,$$

and thus $\mathrm{H}^i(K^\bullet)$ is also a vector bundle. As cokernels always compute with base change, so does $\mathrm{H}^i(K^\bullet) = \mathrm{coker}(K^{i-1} \to \ker(d^i))$. See also [Mum70, Cor. 2, p.48], [Har77, Cor. 12.9], or [Vak17, 28.1.5]. $\qquad\square$

The reducedness hypothesis in Grauert's Theorem is quite restrictive in applications to moduli theory, where we often need to establish properties of families of objects over an arbitrary base. Fortunately, with a little more linear algebra, we can establish the following criterion which holds over any base.

Theorem A.6.8 (Cohomology and Base Change II)**.** *Let $f\colon X \to Y$ be a proper and finitely presented morphism of schemes, and let F be a finitely presented quasi-coherent sheaf on X flat over Y. Suppose that for a point $y \in Y$ and integer i, the comparison map $\phi_y^i\colon \mathrm{R}^i f_* F \otimes \kappa(y) \to \mathrm{H}^i(X_y, F_y)$ is surjective. Then the following hold:*

(a) There is an open neighborhood $V \subset Y$ of y such that for every morphism $Y' \to V$ of schemes, the comparison map $\phi_{Y'}^i\colon g^* \mathrm{R}^p f_* F \to \mathrm{R}^i f'_* g'^* F$ is an isomorphism. In particular, ϕ_y^i is an isomorphism.

(b) ϕ_y^{i-1} is surjective if and only if $\mathrm{R}^i f_* F$ is a vector bundle in an open neighborhood of y.

Proof. Assuming that Y is noetherian, we reduce to the case that $Y = \operatorname{Spec} A$ is affine. Theorem A.6.2 constructs a complex $K^\bullet$ such that for each $y \in Y$, there is a morphism of complexes

$$
\begin{array}{ccccc}
K^{i-1} & \xrightarrow{d^{i-1}} & K^i & \xrightarrow{d^i} & K^{i+1} \\
\downarrow & & \downarrow & & \downarrow \\
K^{i-1} \otimes_A \kappa(y) & \xrightarrow{d^{i-1}\otimes\kappa(y)} & K^i \otimes_A \kappa(y) & \xrightarrow{d^i\otimes\kappa(y)} & K^{i+1} \otimes_A \kappa(y)
\end{array}
\qquad
\begin{array}{l}
\text{computing } \mathrm{H}^i(X, F) \\[2em]
\text{computing } \mathrm{H}^i(X_y, F_y)
\end{array}
$$

The following claim will be used twice in the proof: the surjectivity of $\mathrm{H}^i(X, F) \to \mathrm{H}^i(X_y, F_y)$ is equivalent to the surjectivity of $\ker(d^i) \to \ker(d^i \otimes \kappa(y))$. Since $K^{i-1} \to K^{i-1} \otimes_A \kappa(y)$ is surjective, so is $\operatorname{im}(d^{i-1}) \to \operatorname{im}(d^{i-1} \otimes \kappa(y))$. The snake lemma applied to

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{im}(d^{i-1}) & \longrightarrow & \ker(d^i) & \longrightarrow & \mathrm{H}^i(X, F) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{im}(d^{i-1} \otimes \kappa(y)) & \longrightarrow & \ker(d^i \otimes \kappa(y)) & \longrightarrow & \mathrm{H}^i(X_y, F_y) & \longrightarrow & 0
\end{array}
$$

establishes the claim.

For (a), our hypothesis is that $\mathrm{H}^i(X, F) \to \mathrm{H}^i(X_y, F_y)$ is surjective, which by the claim implies that $\ker(d^i) \to \ker(d^i \otimes \kappa(y))$ is surjective. Proposition A.6.6 implies that $d^i\colon K^i \to K^{i+1}$ is a map of vector bundles, and that after replacing Y with an open neighborhood of y, $\ker(d^i)$ is a vector bundle whose construction commutes with base change. Thus $\mathrm{H}^i(X, F) = \operatorname{coker}(K^{i-1} \to \ker(d^i))$ also commutes with base change. For (b), we use the equivalences:

$$
\begin{aligned}
\mathrm{H}^i(X, F) \text{ is a vector bundle } &\overset{A.6.6}{\Longleftrightarrow} K^{i-1} \to \ker(d^i) \text{ is a map of vector bundles} \\
&\Longleftrightarrow K^{i-1} \to K^i \text{ is a map of vector bundles} \\
&\overset{A.6.6}{\Longleftrightarrow} \ker(d^{i-1}) \otimes \kappa(y) \to \ker(d^{i-1} \otimes \kappa(y)) \text{ is surjective} \\
&\overset{\text{claim}}{\Longleftrightarrow} \mathrm{H}^{i-1}(X, F) \to \mathrm{H}^{i-1}(X_y, F_y) \text{ is surjective.}
\end{aligned}
$$

The first equivalence follows from Proposition A.6.6 as $\mathrm{H}^i(X, F)$ is the cokernel of $K^{i-1} \to \ker(d^i)$, the second follows from the observation that since d^i is a map of vector bundles, $\ker(d^i)$ and $\operatorname{im}(d^i)$ are vector bundles, and the map $\ker(d^i) \to K^i$ (whose cokernel is $\operatorname{im}(d^i)$) is also a map of vector bundles, the third also follows from Proposition A.6.6, and the fourth is the claim at the beginning of the proof.

Using the methods of Limits of Schemes (§B.3), it is not hard to see how the general statement follows from the noetherian version. Assuming Y is affine, write $Y = \lim_{\lambda \in \Lambda} Y_\lambda$ as a limit of affine schemes of finite type over $\mathbb{Z}$. Since $X \to Y$ is finitely presented, there exists an index $0 \in \Lambda$ and a finitely presented morphism $X_0 \to Y_0$ such that $X \cong X_0 \times_{Y_0} Y$ (Proposition B.3.4). For each $\lambda > 0$, we can define $X_\lambda = X_0 \times_{Y_0} Y_\lambda$ and we have $X \cong X_\lambda \times_{Y_\lambda} Y$. By Proposition B.3.7, $X_\lambda \to Y_\lambda$ is proper for $\lambda \gg 0$. By Proposition B.3.8(1), there exists an index $\mu \in \Lambda$ and a coherent sheaf F_μ on X_μ that pulls back to F under $X \to X_\mu$. For $\lambda > \mu$, set F_λ to be the pullback of F_μ under $X_\lambda \to X_\mu$. By Proposition B.3.8(3), F_λ is flat over Y_λ for $\lambda \gg 0$. We may now apply noetherian Cohomology and Base Change to the data of $X_\lambda \to Y_\lambda$ and F_λ for $\lambda \gg 0$, and we may deduce the same properties for $X \to Y$ and F under the base change $Y \to Y_\lambda$. See also [EGA, III$_2$.7.7.5, III$_2$.7.7.10, III$_2$.7.8.4], [Har77, Thm. 12.11], and [Vak17, Thm. 25.1.6]. □

The following exercise will give you some practice applying Cohomology and Base Change.

Exercise A.6.9. Let $f \colon X \to Y$ be a proper morphism of noetherian schemes. For a coherent sheaf F flat over Y, the following are equivalent:

(1) $\mathrm{H}^i(X_y, F_y) = 0$ for all $y \in Y$ and $i > 0$; and

(2) $\mathrm{R}^i f_* F = 0$ for all $i > 0$, and $f_* F$ is a vector bundle whose construction commutes with base change on Y.

A.6.2 Applications to moduli theory

Here is a typical moduli-theoretic application of Cohomology and Base Change establishing properties of smooth families of curves (Proposition 5.1.15), which is applied for instance in the Algebraicity of $\mathcal{M}_g$ (3.1.17). The argument below applies equally to families of stable curves (Proposition 5.3.21).

Proposition A.6.10. *Let $\pi \colon \mathcal{C} \to S$ be a family of smooth curves of genus $g \geq 2$ (i.e. $\mathcal{C} \to S$ is a smooth, proper morphism of schemes such that every geometric fiber is a connected curve of genus g). Then*

(1) $\pi_* \mathcal{O}_{\mathcal{C}} = \mathcal{O}_S$,

(2) $\pi_*(\Omega_{\mathcal{C}/S})$ is a vector bundle of rank g whose construction commutes with base change on S and $\mathrm{R}^1 \pi_*(\Omega_{\mathcal{C}/S}) \cong \mathcal{O}_S$ while $\mathrm{R}^i \pi_*(\Omega_{\mathcal{C}/S}) = 0$ for $i \geq 2$, and

(3) for $k > 1$, the pushforward $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$ whose construction commutes with base change on S and $\mathrm{R}^i \pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) = 0$ for $i > 0$.

Proof. To see (1), observe that $\mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = \kappa(s)$ for all $s \in S$ since $\mathcal{C}_s$ is proper and geometrically connected. It follows that $\phi_s^0 \colon \pi_* \mathcal{O}_{\mathcal{C}} \otimes \kappa(s) \to \mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s})$ is surjective. Cohomology and Base Change (A.6.8(a)–(b)) with $i = 0$ implies that ϕ_s^0 is an isomorphism and that $\pi_* \mathcal{O}_{\mathcal{C}}$ is a line bundle. On a fiber over $s \in S$, the natural map $\mathcal{O}_S \to \pi_* \mathcal{O}_{\mathcal{C}}$ induces a surjective map $\kappa(s) \to \pi_* \mathcal{O}_{\mathcal{C}} \otimes \kappa(s)$ (as post-composing with $\phi_s^0 \colon \pi_* \mathcal{O}_{\mathcal{C}} \otimes \kappa(s) \to \mathrm{H}^0(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}) = \kappa(s)$ is the identity). Thus $\mathcal{O}_S \to \pi_* \mathcal{O}_{\mathcal{C}}$ is a surjective morphism of line bundles, hence an isomorphism.

For (2), since $\Omega_{\mathcal{C}/S}$ is a relative dualizing sheaf (see [Liu02, §6.4]), Grothendieck–Serre Duality implies that $\mathrm{R}^1 \pi_* \Omega_{\mathcal{C}/S} \cong \pi_* \mathcal{O}_{\mathcal{C}}$ and this is identified with $\mathcal{O}_S$ by (1). For $i \geq 2$, $\mathrm{H}^i(\mathcal{C}_s, \Omega_{\mathcal{C}/S} \otimes \kappa(s)) = 0$ (as $\dim \mathcal{C}_s = 1$), and A.6.8(a) implies that $\mathrm{R}^i \pi_* \Omega_{\mathcal{C}/S} = 0$. Applying A.6.8(b) with $i = 2$ yields that $\phi_s^1 \colon \mathrm{R}^1 \pi_* \Omega_{\mathcal{C}/S} \otimes \kappa(s) \to$

$\mathrm{H}^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)})$ is surjective for every $s \in S$ and applying A.6.8(a) with $i = 1$ shows that ϕ_s^1 is an isomorphism. Since $\mathrm{R}^1\pi_*\Omega_{\mathcal{C}/S}$ is a line bundle, applying A.6.8(b) with $i = 1$ shows that ϕ_s^0 is surjective, and applying A.6.8(a)–(b) with $i = 0$ implies that $\pi_*\Omega_{\mathcal{C}/S}$ is a vector bundle of rank $\mathrm{h}^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}) = g$ whose construction commutes with base change.

For (3), since $k > 1$, we have that $\deg(\Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes(1-k)}) < 0$ for each $s \in S$, and Serre Duality (5.1.2) implies that $\mathrm{H}^1(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) = \mathrm{H}^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes(1-k)}) = 0$. Observe that $\mathrm{H}^i(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) = 0$ for $i \geq 2$ since $\dim \mathcal{C}_s = 1$. Cohomology and Base Change (A.6.8(a)) gives $\mathrm{R}^i\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k}) = 0$ for $i > 0$. On the other hand, $\mathrm{h}^0(\mathcal{C}_s, \Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) = \deg(\Omega_{\mathcal{C}_s/\kappa(s)}^{\otimes k}) + 1 - g = (2k-1)(g-1)$ by Riemann–Roch (5.1.1). Applying Cohomology and Base (A.6.8(b)) with $i = 0$ yields that $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle of rank $(2k-1)(g-1)$. $\qquad\square$

Similarly, we can apply Cohomology and Base Change to establish properties of families of coherent sheaves, which we will need for instance for the Algebraicity of $\underline{\mathrm{Bun}}(C)$ (3.1.21).

Proposition A.6.11. *Let $p\colon X \to S$ be a proper morphism of schemes and F be a finitely presented quasi-coherent sheaf on X flat over S. Suppose that $\dim X_s \leq d$ for all $s \in S$. The subset S' of points $s \in S$ such that $\mathrm{H}^j(X_s, F_s) = 0$ for all $j > 0$ is open. Denoting $X' = p^{-1}(S')$, $p' := p|_{X'}\colon X' \to S$, and $F' = F|_{X'}$, we have that $\mathrm{R}^j p'_* F' = 0$ for all $j > 0$ and that $p'_* F'$ is a vector bundle whose construction commutes with base change.*

Proof. For each $j = 1, \ldots, d$, A.6.8(a) implies that the locus of points $s \in S$ such that $\mathrm{H}^j(X_s, F_s) = 0$ is open and the comparison map $\phi_s^j \colon \mathrm{R}^j p_* F \otimes \kappa(s) \to \mathrm{H}^j(X_s, F_s)$ is an isomorphism. It follows that $\mathrm{R}^j p'_* F = 0$ which allows us to apply A.6.8(b) with $i = 1$ to conclude that $\phi_s^0 \colon p'_* F' \otimes \kappa(s) \to \mathrm{H}^0(X_s, F_s)$ is surjective. Applying A.6.8(a)-(b) with $i = 0$ gives the final statement. $\qquad\square$

A.6.3   Applications to line bundles

Given a proper flat morphism $f\colon X \to Y$, when is a line bundle L on X the pullback of a line bundle on Y? More generally, is there a largest subscheme $Z \subset Y$ where L_Z on $X_Z = X \times_Y Z$ is the pullback of a line bundle on Z? In this section, we provide three answers in increasing complexity. As the results depend on properties of the fibers X_y, we first discuss relationships between various conditions.

Lemma A.6.12. *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes. Consider the following conditions:*

(1) the geometric fibers of $f\colon X \to Y$ are non-empty, connected, and reduced;

(2) $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$; and

(3) $\mathcal{O}_Y = f_*\mathcal{O}_X$ and this holds after arbitrary base change (i.e. $\mathcal{O}_T = f_{T,*}\mathcal{O}_{X_T}$ for a morphism $T \to Y$ of schemes).

Then (1) $\Rightarrow$ (2) $\iff$ (3).

Proof. If (1) holds, then $\mathrm{H}^0(X_y, \mathcal{O}_{X_y}) \otimes_{\kappa(y)} \overline{\kappa(y)} = \mathrm{H}^0(X \times_Y \overline{\kappa(y)}, \mathcal{O}_{X \times_Y \overline{\kappa(y)}})$ by Flat Base Change (A.2.12, and since a connected, reduced, and proper scheme over an algebraically closed field has only constant functions, we conclude that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$. If (2) holds, then the comparison map $\phi_y^0\colon f_*\mathcal{O}_X \otimes \kappa(y) \to \mathrm{H}^0(X_y, \mathcal{O}_{X_y}) = \kappa(y)$

is necessarily surjective as there is a global section $1 \in \mathrm{H}^0(Y, f_*\mathcal{O}_X)$. Applying Theorem A.6.8 with $i = 0$ shows that $f_*\mathcal{O}_X$ is a line bundle. As $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a surjection of line bundles, it is an isomorphism. Since the same argument applies to the base change $X_T \to T$, this gives (3). The converse (3) $\Rightarrow$ (2) by considering the map $T = \mathrm{Spec}\,\kappa(y) \to Y$. $\qquad\square$

When the base is reduced, Grauert's Theorem provides a complete answer to when a line bundle is a pullback.

Proposition A.6.13 (Version 1)**.** *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes such that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$. Let L be a line bundle on X. If Y is reduced, then $L = f^*M$ for a line bundle M on Y if and only if L_y is trivial for all $y \in Y$. Moreover, if these conditions hold, then $M = f_*L$ and the adjunction morphism $f^*f_*L \to L$ is an isomorphism.*

Proof. The condition on geometric fibers implies that $\mathrm{h}^0(X_y, L_y) = 1$ and Grauert's Theorem (A.6.7) implies that f_*L is a line bundle and that $f_*L \otimes \kappa(y) \xrightarrow{\sim} \mathrm{H}^0(X_y, L_y)$ is an isomorphism. We claim that $f^*f_*L \to L$ is surjective. It suffices to show that $(f^*f_*L)|_{X_y} \to L|_{X_y}$ is surjective. Denoting $f_y\colon X_y \to \mathrm{Spec}\,\kappa(y)$, we have identifications $(f^*f_*L)|_{X_y} = f_y^*(f_*L \otimes \kappa(y)) = f_y^*(\mathcal{O}_{\mathrm{Spec}\,\kappa(y)}) = \mathcal{O}_{X_y}$ and the claim follows. Since $f^*f_*L \to L$ is a surjection of line bundles, it is an isomorphism. $\qquad\square$

Exercise A.6.14. Show that if Y is a connected and reduced noetherian scheme and E is a vector bundle on Y, then $\mathrm{Pic}(\mathbb{P}(E)) = \mathrm{Pic}(Y) \times \mathbb{Z}$. See also [Har77, Exer. III.12.5].

Proposition A.6.15 (Version 2)**.** *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes with integral geometric fibers. For a line bundle L on X, the locus $\{y \in Y \mid L_y$ is trivial on $X_y\}$ is a closed subset of Y.*

Proof. The important observation here is that for a geometrically integral and proper scheme Z over field $\Bbbk$, a line bundle M is trivial if and only if $\mathrm{h}^0(Z, M) > 0$ and $\mathrm{h}^0(Z, M^\vee) > 0$. To see that the latter condition is sufficient, observe that we have nonzero homomorphisms $\mathcal{O}_Z \to M$ and $\mathcal{O}_Z \to M^\vee$, the latter of which dualizes to a nonzero map $M \to \mathcal{O}_Z$. Since Z is integral, the composition $\mathcal{O}_Z \to M \to \mathcal{O}_Z$ is also nonzero and thus an isomorphism as it corresponds to a nonzero constant in $\mathrm{H}^0(Z, \mathcal{O}_Z) = \Bbbk$. It follows that $M \to \mathcal{O}_Z$ is a surjective map of line bundles, hence an isomorphism. By the Semicontinuity Theorem (A.6.4) the condition that $\mathrm{h}^0(X_y, L_y) > 0$ and $\mathrm{h}^0(X_y, L_y^\vee) > 0$ is closed, and the statement follows. See also [Mum70, Cor. 6, p. 54]. $\qquad\square$

Remark A.6.16. If the geometric fibers are only connected and reduced, the locus may fail to be closed. For example, giving a smooth family $f\colon X \to Y$ of curves over a smooth curve, and consider the blowup $\mathrm{Bl}_x X \to X$ at a closed point $x \in X$ with exceptional divisor E. Then $\mathrm{Bl}_x X \to Y$ is a proper flat morphism, and the fiber over $f(x) \in Y$ is connected and reduced but reducible. Setting $L = \mathcal{O}_{\mathrm{Bl}_x X}(E)$, the fiber L_y is trivial if and only if $y \neq f(x)$.

For moduli-theoretic applications, it is essential that we allow for the base Y to be non-reduced, and provide the locus $Z \subset Y$ with a functorial description. For many applications (e.g. to families of stable curves), it is also necessary to allow for reducible fibers X_y. Our final and strongest version incorporates both versions above and is proved using the algebraic formulation of Cohomology and Base Change (A.6.2). This result will be applied in the proof of Algebraicity of $\mathcal{M}_g$ (3.1.17) to

exhibit a locally closed subscheme of the Hilbert scheme parameterizing smooth curves that are tri-canonically embedded.

Proposition A.6.17 (Version 3). *Let $f\colon X \to Y$ be a proper flat morphism of noetherian schemes such that $\mathrm{h}^0(X_y, \mathcal{O}_{X_y}) = 1$ for all $y \in Y$ (resp. the geometric fibers are integral). For a line bundle L on X, there is a unique locally closed (resp. closed) subscheme $Z \subset Y$ such that*

(1) *L_Z on $X_Z = X \times_Y Z$ is the pullback of a line bundle on Z, and*

(2) *if $T \to Y$ is a morphism of schemes such that L_T on X_T is the pullback of a line bundle on T, then $T \to Y$ factors through Z.*

In other words, the functor

$$\mathrm{Sch}/Y \to \mathrm{Sets},$$

$$(T \to Y) \mapsto \left\{ \begin{array}{cl} \{*\} & \text{if } L_T \text{ is the pullback of a line bundle on } T \\ \emptyset & \text{otherwise} \end{array} \right.$$

is representable by a locally closed (resp. closed) subscheme of Y.

Proof. We begin with the observation that L is the pullback of a line bundle if and only if f_*L is a line bundle and the adjunction map $f_*f^*L \to L$ is an isomorphism. Indeed, if $L = f^*M$ for a line bundle M on Y, then the projection formula and the equality $\mathcal{O}_Y = f_*\mathcal{O}_X$ (Lemma A.6.12) shows that

$$f_*L \cong f_*f^*M \cong f_*\mathcal{O}_X \otimes M \cong M$$

is a line bundle and that $f^*f_*L \to L$ is an isomorphism. As the same holds for the base change $X_T \to T$, iwe see that the question is Zariski-local on Y and T. We will show that every point $y \in Y$ has an open neighborhood where the proposition holds.

By the Semicontinuity Theorem (A.6.4), the locus $V = \{y \in Y \mid \mathrm{h}^0(X_y, L_y) \leq 1\}$ is open. Since L_y is not trivialfor points $y \notin V$, we may replace Y with V and assume that $\mathrm{h}^0(X_y, L_y) \leq 1$ for all $y \in Y$. By Cohomology and Base Change (A.6.2) and after replacing Y with an open affine neighborhood of y, we may assume that there is a homomorphism $d\colon A^{r_0} \xrightarrow{d} A^{r_1}$ of finitely generated and free A-modules such that for every ring map $A \to B$, $\mathrm{H}^0(X_B, L_B) = \ker(d \otimes B)$. Using the dual $d^\vee$ of d, we define M as the cokernel in the sequence

$$A^{r_1} \xrightarrow{d^\vee} A^{r_0} \to M \to 0.$$

Tensoring over $A \to B$ yields a right exact sequence

$$B^{r_1} \xrightarrow{d^\vee \otimes B} B^{r_0} \to M \otimes_A B \to 0,$$

which after applying the contravariant left-exact functor $\mathrm{Hom}_B(-, B)$ becomes

$$0 \to \mathrm{Hom}_B(M \otimes_A B, B) \to B^{r_0} \xrightarrow{d \otimes_A B} B^{r_1}.$$

We conclude that

$$\mathrm{H}^0(X_B, L_B) = \mathrm{Hom}_B(M \otimes_A B, B) = \mathrm{Hom}_A(M, B). \qquad (\text{A.6.18})$$

Applying this to $A \to \kappa(y)$ for $y \in Y$, we obtain that $\mathrm{H}^0(X_y, L_y) = \mathrm{Hom}_A(M, \kappa(y)) = (M \otimes_A \kappa(y))^\vee$.

If $h^0(X_y, L_y) = 0$, then L_y is not trivial and since $M \otimes_A \kappa(y) = 0$, there is an open neighborhood U of y such that $\widetilde{M}|_U = 0$. The proposition holds over U. If $h^0(X_y, L_y) = 1$, then $M \otimes_A \kappa(y) = \kappa(y)$ and by Nakayama's lemma, after replacing Y with an open affine neighborhood of y, there is a surjection $A \to M$. Write $M = A/I$ for an ideal I and define the closed subscheme $Z = V(I) \subset Y$. Observe that $H^0(Z, L_Z) = \mathrm{Hom}_A(A/I, A/I) = A/I$ so that $f_{Z,*}L_Z$ is the trivial line bundle. To see that the construction of $f_{Z,*}L_Z$ commutes with base change, let B be an A/I-algebra and observe that $H^0(X_B, L_B) = \mathrm{Hom}_A(A/I, B) = B$ and that $H^0(X_Z, L_Z) \otimes_{A/I} B \to H^0(X_B, L_B)$ is an isomorphism.

We claim that $T \to Y$ factors through Z if and only if $f_{T,*}L_T$ is a line bundle. The ($\Rightarrow$) implication is clear: $f_{Z,*}L_Z$ is a line bundle and its construction commutes with base change. The converse is Zariski-local on T and we may assume that $T = \mathrm{Spec}\, B$ is affine and $f_{T,*}L_T = \mathcal{O}_T$. Then (A.6.18) implies that $B = \mathrm{Hom}_A(A/I, B)$. Thus, $I \subset \ker(A \to B)$ or, in other words, $A \to B$ factors as $A \to A/I \to B$. Finally, considering the adjunction morphism $\lambda\colon f_Z^* f_{Z,*}L_Z \to L_Z$ on X_Z, we claim that for $y \in Z$, L_y is trivial if and only if $\lambda|_{X_y}$ is surjective. If $\lambda|_{X_y}$ is surjective, then using that $f_{Z,*}L_Z = \mathcal{O}_Z$, we have a surjection $\mathcal{O}_{X_y} \to L_y$ of line bundles, hence an isomorphism. For converse, since $f_{Z,*}L_Z$ commutes with base change, the comparison map $f_{Z,*}L_Z \otimes \kappa(y) = H^0(X_y, L_y)$ is an isomorphism. Denoting $f_y\colon X_y \to \mathrm{Spec}\, \kappa(y)$, we have identifications $(f_Z^* f_{Z,*}L_Z)|_{X_y} = f_y^*(f_{Z,*}L_Z \otimes \kappa(y)) = f_y^* f_{y,*}L_y$ and $\lambda|_{X_y}$ corresponds to the adjunction map $f_y^* f_{y,*}L_y \to L_y$, which is an isomorphism (as L_y is trivial). Replacing Z with $Z \setminus \mathrm{Supp}(\mathrm{coker}(\lambda))$ establishes the proposition. If, in addition, the fibers X_y are geometrically integral, then Proposition A.6.15 implies that Z is closed. See also [Mum70, p. 90], [Vie95, Lem. 1.19], and [SP, Tags 0BEZ and 0BF0]. $\qquad\square$

Remark A.6.19. For a proper flat morphism $X \to S$, the *Picard functor* is defined as
$$\mathrm{Pic}_{X/S}\colon \mathrm{Sch}/S \to \mathrm{Sets}, \quad T \mapsto \mathrm{Pic}(X_T)/f_T^* \mathrm{Pic}(T);$$
see §6.2.9 for an exposition of Picard functors. If $f\colon X \to S$ has geometrically reduced (resp. integral) fibers, then the existence of a locally closed (resp. closed) subscheme $Z \subset Y$ characterized by Proposition A.6.17 is equivalent to the diagonal morphism $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S} \times_S \mathrm{Pic}_{X/S}$ of presheaves over Sch/S being representable by locally closed immersions (resp. closed immersions). In the case of geometrically integral fibers, this translates to the separatedness of $\mathrm{Pic}_{X/S} \to S$. In this language, the above result was established in [FGA$_\mathrm{V}$, Thm. 3.1].

A.7 Quasi-finite morphisms and Zariski's Main Theorem

A locally of finite type morphism $f\colon X \to Y$ of schemes is *locally quasi-finite at* $x \in X$ if x is isolated in the fiber $X_{f(x)} = X \times_Y \mathrm{Spec}\, \kappa(f(x))$. When $f\colon X \to Y$ is also quasi-compact, then this is equivalent to the finiteness of the set $f^{-1}(f(x))$, and we say that $f\colon X \to Y$ is *quasi-finite*.

A.7.1 Étale Localization of Quasi-finite Morphisms

Theorem A.7.1 (Étale Localization of Quasi-finite Morphisms)**.** *Let $f\colon X \to S$ be a separated and finite type morphism of schemes. Suppose that f is quasi-finite at every preimage of $s \in S$. There exists an étale neighborhood $(S', s') \to (S, s)$ with*

$\kappa(s') = \kappa(s)$ *and a decomposition $X \times_S S' = Z \amalg W$ into open and closed subschemes such that $Z \to S'$ is finite and the fiber $W_{s'}$ is empty. Moreover, it can be arranged that Z factors as $Z_1 \amalg \cdots \amalg Z_n$ where each Z_i contains precisely one point z_i over s with $\kappa(z_i)/\kappa(s)$ purely inseparable.*

Proof. See [EGA, IV.8.12.3] or [SP, Tag 04HF]. $\qquad\square$

The statement implies that any quasi-finite algebra A over a henselian local ring R is a product $A \cong B \times C$ with B finite over R and $C \otimes_R R/\mathfrak{m}_R = 0$, a property that, in fact, characterizes henselian local rings (Proposition B.5.10). It also provides the key technical input in factoring quasi-finite morphisms.

A.7.2 Factorizations of quasi-finite morphisms

Proposition A.7.2. *A separated and quasi-finite morphism $f \colon X \to Y$ of schemes factors as*

$$f \colon X \to \mathcal{S}\mathrm{pec}_Y\, f_* \mathcal{O}_X \to Y$$

where $X \hookrightarrow \mathcal{S}\mathrm{pec}_Y\, f_* \mathcal{O}_X$ is an open immersion and $\mathcal{S}\mathrm{pec}_Y\, f_* \mathcal{O}_X \to Y$ is affine.

Proof. As $f_* \mathcal{O}_X$ commutes with étale (even flat) base changes on Y, so does the factorization. Therefore, it suffices to show that every point $y \in Y$ has an étale neighborhood where the proposition holds. By Theorem A.7.1 we may assume that $X = X_1 \amalg X_2$ with X_1 finite over Y and $(X_2)_y = \emptyset$. After replacing Y with $\mathcal{S}\mathrm{pec}_Y\, f_* \mathcal{O}_X$, we may also assume that $f_* \mathcal{O}_X = \mathcal{O}_Y$. As $\mathcal{O}_X = \mathcal{A}_1 \times \mathcal{A}_2$ is the product of quasi-coherent $\mathcal{O}_X$-algebras, $\mathcal{O}_Y = f_* \mathcal{O}_X = f_* \mathcal{A}_1 \times f_* \mathcal{A}_2$ and thus Y decomposes as $Y_1 \amalg Y_2$ such that $y \in Y_1$ and $f(X_i) \subset Y_i$ for $i = 1, 2$. After replacing Y with Y_1, we see that $X \to Y$ is finite. Thus X is affine and $X = Y = \mathcal{S}\mathrm{pec}_Y\, f_* \mathcal{O}_X$. $\qquad\square$

In the above factorization, $f_* \mathcal{O}_Y$ may not be a finite type $\mathcal{O}_Y$-algebra; even if Y is a noetherian affine scheme, then $\Gamma(X, \mathcal{O}_X)$ may not be a noetherian ring (see [Ols16, Ex. 7.2.15]). However, we may modify the factorization to arrange that $X \to Y$ factors as an open immersion followed by a *finite* morphism.

Theorem A.7.3 (Zariski's Main Theorem)**.** *A separated and quasi-finite morphism $f \colon X \to Y$ of schemes factors as the composition of a dense open immersion $X \hookrightarrow \widetilde{Y}$ and a finite morphism $\widetilde{X} \to X$. In particular, f is quasi-affine.*

Proof. If $\mathcal{A} \subset f_* \mathcal{O}_X$ denotes the integral closure of $\mathcal{O}_Y \to f_* \mathcal{O}_X$, f factors as the composition of

$$f \colon X \xrightarrow{j} \mathcal{S}\mathrm{pec}_Y \mathcal{A} \to Y.$$

We claim that j is an open immersion. It suffices to show that for every point $x \in X$, there is an open neighborhood $V \subset \mathcal{S}\mathrm{pec}_Y \mathcal{A}$ of $j(x)$ such that $j^{-1}(V) \to V$ is an isomorphism. Since normalization commutes with étale base change (Proposition A.7.4) and since being an open immersion is an fpqc local property (Proposition B.1.9), we are free to replace Y by an étale neighborhood of $f(x)$. By Theorem A.7.1, we can assume that $X = F \amalg W$ with F finite over Y and $x \in F$. In this case, the normalization $\mathcal{S}\mathrm{pec}_Y \mathcal{A}$ of Y in X is $F \amalg \widetilde{W}$ where $\widetilde{W}$ is the normalization of Y in W. As $j^{-1}(F) \xrightarrow{\sim} F$, the claim follows. By construction, $\mathcal{S}\mathrm{pec}_Y \mathcal{A} \to Y$ is integral. We can write $\mathcal{A} = \mathrm{colim}\, \mathcal{A}_\lambda$, where each $\mathcal{A}_\lambda$ is a finite type $\mathcal{O}_Y$-algebra. Since open immersions descent under limits (Proposition B.3.7), $X \to \mathcal{S}\mathrm{pec}_Y \mathcal{A}_\lambda$ is an open immersion for $\lambda \gg 0$. Since $\mathcal{S}\mathrm{pec}_Y \mathcal{A}_\lambda \to Y$ is integral and of finite type, it is finite. See also [EGA, IV.8.12.6] or [SP, Tag 05K0]. $\qquad\square$

The following algebra result was used above and will also be used in the generalizations of Zariski's Main Theorem to algebraic spaces (Theorem 4.4.9) and stacks (Theorem 6.1.10).

Proposition A.7.4. *Let Y be a scheme, $\mathcal{B}$ be a quasi-coherent $\mathcal{O}_Y$-algebra and $\widetilde{\mathcal{B}}$ be the integral closure of $\mathcal{O}_Y$ in $\mathcal{B}$. If $f\colon X \to Y$ is a smooth morphism, then $f^*\widetilde{\mathcal{B}}$ is identified with the integral closure of $\mathcal{O}_X$ in $f^*\mathcal{B}$.*

Proof. See [SP, Tag 03GG] or [LMB00, Prop. 16.2]. □

Zariski's Main Theorem has some useful corollaries.

Corollary A.7.5. *A quasi-finite and proper morphism (resp. proper monomorphism) of schemes is finite (resp. a closed immersion).*

Proof. If $f\colon X \to Y$ is a quasi-finite and proper, Zariski's Main Theorem (A.7.3) gives a factorization $f\colon X \hookrightarrow \widetilde{X} \to Y$ and the dense open immersion $X \hookrightarrow \widetilde{X}$ is also closed, thus an isomorphism. On the other hand, if $f\colon X \to Y$ is a proper monomorphism, then it is also quasi-finite, thus finite. The statement reduces to the algebraic fact that a finite epimorphism of rings is surjective (c.f. [SP, Tag 04VT]). □

Remark A.7.6. As universally closed morphisms are necessarily quasi-compact [SP, Tag 04XU], every universally closed and locally of finite type monomorphism is a closed immersion; see also [SP, Tag 04XV].

Appendix B

Further topics in scheme theory

B.1 Descent theory

It is hard to overstate the importance of descent in moduli theory. The central idea of descent is as simple as it is powerful. While you already know that most properties of schemes and their morphisms can be checked on Zariski covers, descent theory implies that they can be checked on étale (or even fppf or fpqc) covers. For example, if $Y' \to Y$ is étale and surjective, then a morphism $X \to Y$ is proper if and only if $X \times_Y Y' \to Y'$ is. Similar to scheme theory where results are established by using Zariski covers to reduce to the affine case, we will use étale, smooth, or fppf descent to reduce properties of algebraic stacks to the case of schemes.

Applications of descent theory reach far beyond moduli theory. For instance, since field extensions are faithfully flat, one can reduce properties of a scheme over a field $\Bbbk$ to the case of an algebraically closed field. Similarly, since the map $A \to \widehat{A}$ of a local noetherian ring to its completion is faithfully flat, properties of A can be reduced to properties of $\widehat{A}$.

References: [BLR90, Ch.6], [Vis05], [Ols16, Ch. 4], [SP, Tag 0238], [EGA, §IV.2], and [SGA1, §VIII.7] (other descent results are scattered throughout EGA and SGA).

B.1.1 Descending quasi-coherent sheaves

Descent theory rests on the following algebraic fact.

Proposition B.1.1. *If $\phi\colon A \to B$ is a faithfully flat ring map, then the sequence*

$$0 \longrightarrow A \xrightarrow{\ \phi\ } B \underset{b \mapsto 1 \otimes b}{\overset{b \mapsto b \otimes 1}{\rightrightarrows}} B \otimes_A B$$

is exact. More generally, if M is an A-module, the sequence

$$0 \longrightarrow M \xrightarrow{\ m \mapsto m \otimes 1\ } M \otimes_A B \underset{m \otimes b \mapsto m \otimes 1 \otimes b}{\overset{m \otimes b \mapsto m \otimes b \otimes 1}{\rightrightarrows}} M \otimes_A B \otimes_A B \qquad (\text{B.1.2})$$

is exact.

Proof. Note that $A \to B$ and $M \to M \otimes_A B$ are necessarily injective by Proposition A.2.19. Since $A \to B$ is faithfully flat, the sequence (B.1.2) is exact if and only

if the sequence

$$M \otimes_A B \xrightarrow{m \otimes b' \mapsto m \otimes 1 \otimes b'} M \otimes_A B \otimes_A B \xrightarrow[m \otimes b \otimes b' \mapsto m \otimes 1 \otimes b \otimes b']{m \otimes b \otimes b' \mapsto m \otimes b \otimes 1 \otimes b'} M \otimes_A B \otimes_A B \otimes_A B$$

is exact. The above sequence can be rewritten as

$$M \otimes_A B \xrightarrow{x \mapsto x \otimes 1} (M \otimes_A B) \otimes_B (B \otimes_A B) \xrightarrow[x \otimes y \mapsto x \otimes 1 \otimes y]{x \otimes y \mapsto x \otimes y \otimes 1} (M \otimes_A B) \otimes_B (B \otimes_A B) \otimes_B (B \otimes_A B)$$

which is precisely sequence (B.1.2) applied to ring $B \to B \otimes_A B$ given by $b \mapsto 1 \otimes b$ and the B-module $M \otimes_A B$. Since this ring map has a section $B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$, we are reduced to proving the proposition when $\phi \colon A \to B$ has a section $s \colon B \to A$ with $s \circ \phi = \mathrm{id}_A$. Let $x = \sum_i m_i \otimes b_i \in M \otimes_A B$ such that $\sum_i m_i \otimes b_i \otimes 1 = \sum_i m_i \otimes 1 \otimes b_i$. Applying $\mathrm{id}_M \otimes \mathrm{id}_B \otimes s \colon M \otimes_A B \otimes_A B \to M \otimes_A B \otimes_A A \cong M \otimes_A B$ to this identity shows that $x = \sum_i m_i \otimes \phi(s(b_i)) = \sum_i \phi(s(b_i)) m_i \otimes 1$ is in the image of $M \to M \otimes_A B$. $\square$

Exercise B.1.3. Denoting $(B/A)^{\otimes n}$ as the n-fold tensor product $B \otimes_A \cdots \otimes_A B$, show that the short exact sequence (B.1.2) extends to a long exact sequence $0 \to M \to M \otimes_A (B/A)^{\otimes 1} \to M \otimes_A (B/A)^{\otimes 2} \to \cdots$ with differentials

$$d \colon M \otimes_A (B/A)^{\otimes n} \to M \otimes_A (B/A)^{\otimes (n+1)}$$

$$m \otimes b_1 \otimes \cdots b_n \mapsto \sum_{i=0}^{n+1} (-1)^i m \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_{i+1} \otimes \cdots b_n.$$

Proposition B.1.4 (Fpqc Descent of Quasi-coherent Sheaves)**.** *Let $f \colon X \to Y$ be an fpqc morphism of schemes.*

(1) Let G and G' be quasi-coherent $\mathcal{O}_Y$-modules. Let p_1, p_2 denote the two projections $X \times_Y X \to X$ and q denote the composition $X \times_Y X \xrightarrow{p_i} X \xrightarrow{f} Y$. Then the sequence

$$\mathrm{Hom}_{\mathcal{O}_Y}(G, G') \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{O}_X}(f^*G, f^*G') \xrightarrow[p_2^*]{p_1^*} \mathrm{Hom}_{\mathcal{O}_{X \times_Y X}}(q^*G, q^*G')$$

is exact.

(2) Let F be a quasi-coherent $\mathcal{O}_X$-module and $\alpha \colon p_1^*F \to p_2^*F$ an isomorphism of $\mathcal{O}_{X \times_Y X}$-modules satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ on $X \times_Y X \times_X Y$. Then there exists a quasi-coherent $\mathcal{O}_Y$-module G and an isomorphism $\phi \colon F \to f^*G$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$ on $X \times_Y X$. The data (F, ϕ) is unique up to unique isomorphism.

Remark B.1.5. The following diagram may be useful to internalize (2)

$$
\begin{array}{ccccccc}
p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha & & p_1^*F \xrightarrow{\alpha} p_2^*F & & F & & G \\
\big| & & \big| & & \big| & & \vdots \\
X \times_Y X \times_Y X \xrightarrow[\substack{-p_{12}\to \\ -p_{13}\to \\ -p_{23}\to}]{} X \times_Y X & \xrightarrow[-p_2\to]{-p_1\to} & X \xrightarrow{f} Y
\end{array}
$$

Keep in mind the special case that $\{U_i\}$ is an open covering of Y and $X = \coprod_i U_i$ in which case the above fiber products correspond to intersections.

The cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ and the condition $p_1^*\phi = p_2^*\phi \circ \alpha$ should be understood as the commutativity of

$$
\begin{array}{ccc}
& p_{12}^*p_1^*F \xrightarrow{\;p_{12}^*\alpha\;} p_{12}^*p_2^*F & \\
p_{13}^*p_1^*F \overset{p_{13}^*\alpha}{\longrightarrow} & & p_{23}^*p_1^*F \quad \text{and} \\
& p_{13}^*p_2^*F === p_{23}^*p_2^*F &
\end{array}
\qquad
\begin{array}{ccc}
p_1^*F & \xrightarrow{\;p_1^*\phi\;} & p_1^*f^*G \\
\downarrow{\scriptstyle\alpha} & & \Big\| \\
p_2^*F & \xrightarrow{\;p_2^*\phi\;} & p_2^*f^*G.
\end{array}
$$

Proposition B.1.4 can be reformulated as the statement that the category $\mathrm{QCoh}(Y)$ is equivalent to the *category of descent datum for* $X \to Y$, denoted by $\mathrm{QCoh}(X \to Y)$. Here the objects of $\mathrm{QCoh}(X \to Y)$ are pairs (F, α) consisting of a quasi-coherent $\mathcal{O}_X$-module F and an isomorphism $\alpha\colon p_1^*F \to p_2^*F$ satisfying the cocycle condition. A morphism $(F', \alpha') \to (F, \alpha)$ is a morphism $\beta\colon F' \to F$ such that $\alpha \circ p_1^*\beta = p_2^*\beta \circ \alpha'$.

Proof. If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine, write $G = \widetilde{M}$ and $G' = \widetilde{M}'$. Proposition B.1.1 implies that $0 \to M' \to M' \otimes_A B \rightrightarrows M' \otimes_A B \otimes_A B$ is exact. Applying $\operatorname{Hom}_A(M, -)$ and using tensor-hom adjunction yields (1). For (2), writing $F = \widetilde{M}$, one defines N as the equalizers of two maps $M \rightrightarrows M \otimes_A B$ defined by $m \mapsto m \otimes 1$ and $m \mapsto \alpha(m \otimes 1)$.

The general case is handled by first reducing to the case that Y is affine. Since f is fpqc, Y is the image of a quasi-compact open subset $U \subset X$. By choosing a finite affine cover $\{U_i\}$ of U and replacing X with the affine scheme $\coprod_i U_i$, we have reduced to the case that X is affine. We leave the details to the reader. $\qquad\square$

Remark B.1.6. It turns out that effective descent for modules holds for a class of ring maps $A \to B$ larger than just faithfully flat maps. Namely, it holds for universally injective maps (see Definition A.2.22), and moreover the converse is true! More precisely, $A \to B$ is universally injective if and only if the functor

$$
\begin{aligned}
\operatorname{Mod}_A &\to \left\{ (N, \alpha) \;\middle|\; \begin{array}{l} N \in \operatorname{Mod}_B, \alpha\colon N \otimes_{B,p_1} (B \otimes_A B) \overset{\sim}{\to} N \otimes_{B,p_2} (B \otimes_A B) \\ \text{satisfying the cocycle condition } p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha \end{array} \right\}. \\
M &\mapsto (M \otimes_A B, \mathrm{can})
\end{aligned}
$$

to the category of descent data is an equivalence of categories. See [Mes00] or [SP, Tag 08XA].

B.1.2 Descending morphisms

Proposition B.1.7 (Fpqc Descent for Morphisms). *Let $f\colon X \to Y$ be an fpqc morphism of schemes. If $g\colon X \to Z$ is a morphism to a scheme such that $p_1 \circ g = p_2 \circ g$ on $X \times_Y X$, then there exists a unique morphism $h\colon Y \to Z$ filling in the commutative diagram*

$$
\begin{array}{ccc}
X \times_Y X \; \overset{p_1}{\underset{p_2}{\rightrightarrows}} \; X & \xrightarrow{\;f\;} & Y \\
& {\scriptstyle g}\searrow & \Big\downarrow{\scriptstyle h} \\
& & Z
\end{array}
$$

of solid arrows.

This result implies that schemes are sheaves in the fpqc topology (Proposition 2.2.9).

B.1.3 Descending schemes

We will use the following notation: if $f\colon X \to Y$ and $W \to Y$ are morphisms of schemes, we denote f^*W as the fiber product $X \times_Y W$.

Proposition B.1.8 (Effective Descent). *Let $f\colon X \to Y$ be an fpqc morphism of schemes. Let $\mathcal{P}$ be one of the following properties of morphisms of schemes: open immersion, closed immersion, locally closed immersion, affine, quasi-affine, or separated and locally quasi-finite. If $Z \to X$ is $\mathcal{P}$-morphism of schemes and $\alpha\colon p_1{}^*(Z) \xrightarrow{\sim} p_2^*(Z)$ is an isomorphism over $X \times_Y X$ satisfying $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$, then there exists $\mathcal{P}$-morphism $W \to Y$ of schemes and an isomorphism $\phi\colon Z \to f^*(W)$ such that $p_1^*\phi = p_2^*\phi \circ \alpha$.*

In other words, there exists dotted arrows completing the diagram

$$
\begin{array}{ccccc}
p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha & \quad & p_1^*Z \xrightarrow{\alpha} p_2^*Z & \quad & Z \dashrightarrow W \\
\big\downarrow & & \big\downarrow & & \big\downarrow \qquad \big\downarrow \\
X \times_Y X \times_Y X \ \substack{\xrightarrow{\ p_{12}\ } \\ \xrightarrow{\ p_{13}\ } \\ \xrightarrow{\ p_{23}\ }}\ X \times_Y X & & \substack{\xrightarrow{\ p_1\ } \\ \xrightarrow{\ p_2\ }}\ X \xrightarrow{\ f\ } Y.
\end{array}
$$

When $Z \hookrightarrow X$ is an open or closed immersion $Z \hookrightarrow X$, the existence of α translates to the equality $p_1^{-1}(Z) = p_2^{-1}(Z)$ as subschemes of $X \times_Y X$, and there is no need for a cocycle condition.

Proof. If $Z \hookrightarrow X$ is a closed immersion defined by an ideal sheaf $I_Z \subset \mathcal{O}_X$, then one can apply Proposition B.1.4 to descend I_Z to a quasi-coherent sheaf I_W on Y and to descend the inclusion $I_X \hookrightarrow \mathcal{O}_X$ to an inclusion $I_W \hookrightarrow \mathcal{O}_Y$. Then $W = V(I_W) \hookrightarrow Y$ is the descended scheme. The case of an open immersion can be handled by considering the reduced complement. If $Z = \mathcal{S}\mathrm{pec}_X \mathcal{A}_X$ is affine over X, then Proposition B.1.4 allows us to first descend $\mathcal{A}_X$ to a quasi-coherent sheaf $\mathcal{A}_Y$ on Y, and then descend the the multiplication map $\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{A}_X \to \mathcal{A}_X$ to a morphism $\mathcal{A}_Y \otimes_{\mathcal{O}_Y} \mathcal{A}_Y \to \mathcal{A}_Y$, which by descent will necessarily satisfy the axioms making $\mathcal{A}_Y$ into a quasi-coherent $\mathcal{O}_Y$-algebra. Then one takes $W = \mathcal{S}\mathrm{pec}_Y \mathcal{A}_Y$. The case of quasi-affine morphisms is handled by combining the affine and open immersion cases. If $Z \to X$ is separated and locally quasi-finite, then working locally on Z one reduces to the quasi-compact case, in which case $Z \to X$ is quasi-affine by Proposition A.7.2. See also [GR71, Lem. 5.7.1] and [SP, Tag 02W7]. □

We will often apply Effective Descent in the form of Proposition 2.2.15 to show that a given sheaf in the big étale or fppf topology is representable by a scheme.

B.1.4 Descending properties

This is currently an incomplete list of the descent results needed.

B.1.5 Descent of Properties of Morphisms

Proposition B.1.9 (Fpqc Local Properties on the Target). *Let $Y' \to Y$ be an fpqc morphism of schemes. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) isomorphism;

(ii) closed immersion;

(iii) *open immersion;*

(iv) *surjective;*

(v) *proper;*

(vi) *flat;*

(vii) *smooth;*

(viii) *étale;*

(ix) *unramified;*

(x) *syntomic.*

Then $X \to Y$ has $\mathcal{P}$ if and only if $X \times_Y Y' \to Y'$ does.

Proposition B.1.10 (Fppf Local Properties on the Source)**.** *Let $X' \to X$ be an fppf morphism of schemes. Let $\mathcal{P}$ be one of the following properties of a morphism of schemes:*

(i) *surjective;*

(ii) *fppf;*

(iii) *smooth;*

Then $X \to Y$ has $\mathcal{P}$ if and only if $X' \to X \to Y$ does.

If $X' \to X$ is étale and surjective, then $X \to Y$ is étale if and only if $X' \to X \to Y$ is.

Proposition B.1.11 (Fpqc Descent of Properties of Quasi-coherent Sheaves)**.** *Let $f \colon X \to Y$ be an fpqc morphism of schemes. Let $\mathcal{P} \in \{$finite type, finite presentation, flat, vector bundle, line bundle$\}$ be a property of quasi-coherent sheaves. If G is a quasi-coherent $\mathcal{O}_Y$-module, then G has $\mathcal{P}$ if and only if f^*G does. If X and Y are noetherian, then the same holds for the property of coherence.*

Proof. This reduces to the algebra statement: if $A \to B$ is a faithfully flat ring map, then an A-module M is finitely generated (resp. finitely presented, flat, locally free of finite rank) if and only if $M \otimes_A B$ is. The ($\Rightarrow$) implications are clear. Conversely, if $M \otimes_A B$ is finitely generated, then let $y_1, \ldots, y_m \in M \otimes_A B$ be generators and write $y_i = \sum x_i \otimes b_i$. Since $(x_1, \ldots, x_n) \colon A^n \to M$ base changes to a surjective map, it is surjective. Repeating this argument to the kernel, we see that the property of being finite presentation descends. For flatness, suppose that $M \otimes_A B$ is flat. By faithful flatness, the exactness of $M \otimes_A -$ is equivalent to the exactness of $(M \otimes_A B) \otimes_B (- \otimes_A B)$, which follows from the flatness of $A \to B$ and the flatness of the B-module $M \otimes_A B$. As being locally free of finite rank is equivalent to being finitely presented and flat, the final statement also follows.

See also [SP, Tag 05AY]. $\square$

Proposition B.1.12 (Fpqc Descent of Properties of Schemes)**.** *Let $X \to Y$ be an fpqc morphism of schemes. Suppose X has one of the following properties: locally noetherian, quasi-compact, noetherian, integral, reduced, normal, and regular. Then Y has the same property.*

Proof. First, note that quasi-compactness descends under any surjective map. Therefore, this reduces to the following statements in algebra: if $A \to B$ is a faithfully flat ring map and B is noetherian (resp. a domain, reduced, normal, or regular), then so is A. The map $A \to B$ is injective and $I = IB \cap A$ for every ideal $I \subset A$. For noetherianess, if $I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideals, then since

$I_1 B \subset I_2 B \subset \cdots$ terminates, so does $I_1 = I_1 B \cap A \subset I_2 = I_2 B \cap A \subset \cdots$. By injectivity of $A \to B$, the 'domain' and 'reduced' cases are clear.

For normality and regularity, we can assume that $A \to B$ is a local ring map. If B is a normal domain and a/b is integral over A where $a, b \in A$, then $a/b \in B$ as B is normal. This implies that a is the ideal of B generated by b, and thus $a \colon B \to B/bB$ is the zero map. As this map is the base change of $a \colon A \to A/bA$, faithful flatness implies that $a \colon A \to A/bA$ is the zero map and thus $a \in (b)$ and $a/b \in A$. For regularity, we will appeal to the fact that a noetherian local ring A of dimension d is regular if and only if every finitely generated A-module M has a resolution $0 \to A^{k_d} \to \cdots \to A^{k_1} \to A^{k_0} \to M \to 0$, moreover if this holds and

$$0 \to K \to \cdots \to A^{k_1} \to A^{k_0} \to M \to 0 \tag{B.1.13}$$

is any exact sequence, then K is free; see [Eis95, Thm. 19.12] and [SP, Tag 00OC]. If M is a finitely generated A-module, choose an exact sequence (B.1.13). Since B is regular, $K \otimes_A B$ is free. By Proposition B.1.11, K is free and thus A is regular.

See also [SP, Tags 033D, 034B, and 06QL]. $\square$

Remark B.1.14. The property of being locally noetherian is fppf local, but the other properties are not. There are finite type $\Bbbk$-schemes that are non-reduced, non-normal, and non-regular, but any such scheme is flat over $\Bbbk$. On the other hand, if $X \to Y$ is a flat morphism of noetherian schemes such that Y is normal and every fiber X_y is normal, then X is normal; see [EGA, IV$_2$.6.5.4] or [SP, Tag 0C22].

Remark B.1.15. If A is a noetherian local ring, then the map $A \to \widehat{A}$ to its completion is faithfully flat. If the completion $\widehat{A}$ is reduced (resp. normal, regular), then the above result implies that the same holds for A. While the converse holds for regularity, it does not hold in general for reducedness and normality. However, if A is essentially of finite type over a field (or more generally excellent), then A is reduced (resp. normal) if and only if $\widehat{A}$ is, and moreover in this case the normalization commutes with completion. See [SP, Tags 07NZ and 0C23].

Proposition B.1.16 (Smooth Local Properties of Schemes)**.** *Let $X \to Y$ be a smooth and surjective morphism of schemes. Let $\mathcal{P}$ be one of the following properties: locally noetherian, reduced, normal, and regularity. Then X has $\mathcal{P}$ if and only if Y has $\mathcal{P}$.*

Proof. The ($\Rightarrow$) implications follows from Proposition B.1.12. If Y is locally noetherian, so is X by Hilbert's Basis Theorem. The remaining properties follow from the algebraic statement that if $A \to B$ is a smooth ring map and A is reduced (resp. normal, regular), then so is B [SP, Tags 033B, 033C, and 036D]. See also [SP, Tag 034D]. $\square$

The property of being a domain is however not smooth local (nor even étale local), e.g. there is a reducible étale neighborhood of the nodal cubic (see Example 0.5.2).

Proposition B.1.17 (Fpqc Descent of Ampleness)**.** *Let $f \colon X \to Y$ be a morphism of schemes and L be a line bundle on X. If $Y' \to Y$ is an fpqc morphism of schemes, then L is relatively ample over Y if and only if the pullback of L to $X \times_Y Y'$ is relatively ample over Y'.*

Proof. See [SP, Tag 0D3C]. $\square$

Proposition B.1.18 (Fppf Monomorphisms are Open Immersions)**.** *A flat monomorphism locally of finite presentation (e.g. an étale monomorphism) is an open immersion.*

B.2 Algebraic groups and their actions

B.2.1 Group schemes and their actions

Definition B.2.1. A *group scheme over a scheme S* is a morphism $\pi\colon G \to S$ of schemes together with a multiplication morphism $m\colon G \times_S G \to G$, an inverse morphism $\iota\colon G \to G$ and an identity morphism $e\colon S \to G$ (with each morphism over S) such that the following diagrams commute:

$$
\begin{array}{ccc}
G \times_S G \times_S G \xrightarrow{\mathrm{id}_G \times m} G \times_S G & \quad G \xrightarrow{(\mathrm{id}_G,\iota)} G \times_S G & \quad G \xrightarrow{(e\circ\pi,\mathrm{id}_G)} G \times_S G \\
\downarrow{\scriptstyle m\times\mathrm{id}_G} \qquad\qquad \downarrow{\scriptstyle m} & {\scriptstyle(\iota,\mathrm{id}_G)}\downarrow \ \searrow{\scriptstyle e\circ\pi}\ \downarrow{\scriptstyle m} & {\scriptstyle(\mathrm{id}_G,e\circ\pi)}\downarrow \ \searrow{\scriptstyle\mathrm{id}_G}\ \downarrow{\scriptstyle m} \\
G \times_S G \xrightarrow{\ m\ } G & \quad G \times_S G \xrightarrow{\ m\ } G & \quad G \times_S G \xrightarrow{\ m\ } G \\
\text{Associativity} & \text{Law of inverse} & \text{Law of identity}
\end{array}
$$

For group schemes H and G over S, a *morphism of group schemes* is a morphism $\phi\colon H \to G$ schemes over S such that $m_G \circ (\phi \times \phi) = \phi \circ m_H$. A *(closed) subgroup* of G is a (closed) subscheme $H \subset G$ such that $H \to G \xrightarrow{m_G} G \times G$ factors through $H \times H$.

Remark B.2.2. If G and S are affine, then by reversing the arrows above gives $\Gamma(G, \mathcal{O}_G)$ the structure of a *Hopf algebra* over $\Gamma(S, \mathcal{O}_S)$.

Exercise B.2.3. Show that a group scheme over S is equivalently defined as a scheme G over S together with a factorization

$$
\begin{array}{ccc}
\mathrm{Sch}/S & \dashrightarrow & \mathrm{Gps} \\
{\scriptstyle \mathrm{Mor}_S(-,G)}\searrow & & \downarrow \\
& \mathrm{Sets} &
\end{array}
$$

where $\mathrm{Gps} \to \mathrm{Sets}$ is the forgetful functor.
(*We are not requiring that there exists a factorization; the factorization is part of the data. Indeed, the same scheme can have multiple structures as a group scheme, e.g. $\mathbb{Z}/4$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$ over $\mathbb{C}$.*)

Example B.2.4. The following examples of group schemes are the most relevant for us. Let $S = \operatorname{Spec} R$.

(1) The *multiplicative group scheme over R* is $\mathbb{G}_{m,R} = \operatorname{Spec} R[t]_t$ with comultiplication $m^*\colon R[t]_t \to R[t]_t \otimes_R R[t']_{t'}$ given by $t \mapsto tt'$ while the *group scheme of nth roots of unity* is $\boldsymbol{\mu}_{n,R} = \operatorname{Spec} R[t]/(t^n - 1)$ with comultiplication also defined by $t \mapsto tt'$.

(2) The *additive group scheme over R* is $\mathbb{G}_{a,R} = \operatorname{Spec} R[t]$ with comultiplication $m^*\colon R[t] \to R[t] \otimes_R R[t']$ given by $t \mapsto t + t'$.

Let V be a free R-module of finite rank.

(3) The *general linear group on V* is

$$
\mathrm{GL}(V) = \operatorname{Spec}(\operatorname{Sym}^*(\operatorname{End}(V))_{\det})
$$

with the comultiplication $m^*\colon \operatorname{Sym}^*(\operatorname{End}(V)) \to \operatorname{Sym}^*(\operatorname{End}(V)) \otimes_R \operatorname{Sym}^*(\operatorname{End}(V))$ defined as following: for a basis $v_1, \ldots, v_n$ of V, then for $i, j = 1, \ldots, n$, the endomorphisms $x_{ij}\colon V \to V$ defined by $v_i \mapsto v_j$ and $v_k \mapsto 0$ if $k \neq i$ define a basis of $\operatorname{End}(V)$, and we define $m^*(x_{ij}) = x_{i1}x'_{1j} + \cdots + x_{in}x'_{nj}$.

(4) The *special linear group on V* is $\mathrm{SL}(V)$ is the closed subgroup of $\mathrm{GL}(V)$ defined by $\det = 1$.

(5) The *projective linear group* $\mathrm{PGL}(V)$ is the affine group scheme

$$\mathrm{Proj}(\mathrm{Sym}^*(\mathrm{End}(V)))_{\det}$$

with the comultiplication defined similarly to $\mathrm{GL}(V)$.

We write $\mathrm{GL}_{n,R} = \mathrm{GL}(R^n)$, $\mathrm{SL}_{n,R} = \mathrm{GL}(R^n)$ and $\mathrm{PGL}_{n,R} = \mathrm{PGL}(R^n)$. We often simply write $\mathbb{G}_m$, GL_n, SL_n and PGL_n when the base ring is understood.

Exercise B.2.5.

(a) Provide functorial descriptions of each of the group schemes above.

(b) Show that every abstract group G can be given the structure of a group scheme $\amalg_{g \in G} S$ over a base scheme S. Provide both explicit and functorial descriptions.

Example B.2.6 (Finite groups). If G is an finite abstract group and S is a scheme, then by Exercise B.2.5 we may view G as a finite group scheme over S. We will abuse notation by also denoting this as $G \to S$.

Example B.2.7 (Diagonalizable group schemes). Let R be a field and A be a finitely generated abelian group. We let $R[A]$ be the free R-module generated by elements of A. The R-module $R[A]$ has the structure of an R-algebra with multiplication on generators induced from multiplication in A. The comultiplication $R[A] \to R[A] \otimes_R R[A']$ defined by $a \mapsto a \otimes a'$ defines a group scheme $D_R(A) = \mathrm{Spec}\, R[A]$ over $\mathrm{Spec}\, R$. A group scheme G over $\mathrm{Spec}\, R$ is *diagonalizable* if $G \cong D_R(A)$ for some A.

The group scheme $D_R(\mathbb{Z}^r) = \mathbb{G}_{m,R}^r$ is the r-dimensional split torus while $D_R(\mathbb{Z}/n) = \boldsymbol{\mu}_{n,R} = \mathrm{Spec}\, R[t]/(t^n - 1)$ is the group of nth roots of unity. The classification of finitely generated abelian groups implies that every diagonalizable group scheme is a product of $\mathbb{G}_m^r \times \boldsymbol{\mu}_{n_1} \times \cdots \times \boldsymbol{\mu}_{n_k}$.

A group scheme $G \to S$ is of *multiplicative type* if it becomes diagonalizable after étale cover of S.

Exercise B.2.8. Describe $D_{\Bbbk}(A)$ as a functor $\mathrm{Sch}/R \to \mathrm{Gps}$.

We recall the following general properties of group schemes.

Proposition B.2.9. *Let $G \to S$ be a locally of finite type group scheme.*

(1) If $S = \mathrm{Spec}\,\Bbbk$, then $\dim G = \dim_e G$, where $e \in G(\Bbbk)$ denotes the identity.

(2) The function

$$S \to \mathbb{Z}, \quad s \mapsto \dim G_s$$

is upper semi-continuous.

(3) $G \to S$ is trivial if and only if the fiber G_s is trivial for each $s \in S$.

(4) $G \to S$ is unramified (resp. separated, quasi-separated) if and only if the identity section $e\colon S \to G$ is an open immersion (resp. a closed immersion, quasi-compact).

Proof. For (1), we can reduce to when $\Bbbk$ is algebraically closed. In this case, an element $g \in G(\Bbbk)$ defines an isomorphism $g\colon G \to G$, so that $\dim_e G = \dim_g G$. For (2), for any finite type morphism $\pi\colon G \to S$, the function $G \to \mathbb{Z}$, defined by $g \mapsto \dim G_{\pi(g)}$, is upper semi-continuous [EGA, IV.13.1.3]. As $G \to S$ is a group scheme, there is a section $S \to G$ and the composition $S \to G \to \mathbb{Z}$, defined

by $s \mapsto \dim_{e(s)} G_s = \dim G_s$, is also upper semi-continuous. For (3), $G \to S$ is unramified since every fiber is. Therefore $\Omega_{G/S} = 0$ and the diagonal $G \to G \times_S G$ is an open immersion. It follows that the identity section $S \to G$ is a surjective open immersion, thus an isomorphism. For (4), $G \to S$ is unramified (resp. separated, quasi-separated) if and only if $\Delta_{G/S} \colon G \to G \times_S G$ is an open immersion (resp. a closed immersion, quasi-compact), and the cartesian diagram

$$
\begin{array}{ccccc}
S & \xrightarrow{\ e\ } & G & \xrightarrow{\ \pi\ } & S \\
\downarrow{\scriptstyle e} & \square & \downarrow{\scriptstyle \Delta_{G/S}} \square & & \downarrow{\scriptstyle e} \\
G & \xrightarrow[\Delta_{G/S}]{} & G \times_S G & \xrightarrow[(g,h) \mapsto g^{-1}h]{} & G
\end{array}
$$

implies that this is equivalent to $e \colon S \to G$ being an open immersion (resp. a closed immersion, quasi-compact). $\qquad \square$

B.2.2 Actions and representations

Definition B.2.10. Let $G \to S$ be a group scheme with multiplication m and identity e. An *action* of G on a scheme $p \colon X \to S$ is a morphism $a \colon G \times_S X \to X$ over S such that the following diagrams commute:

$$
\begin{array}{ccc}
G \times_S G \times_S X & \xrightarrow{\ \mathrm{id}_G \times a\ } & G \times_S X \\
{\scriptstyle m \times \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle a} \\
G \times_S X & \xrightarrow{\qquad a \qquad} & G
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{(e \circ p, \mathrm{id}_X)} & G \times_S X \\
& {\scriptstyle \mathrm{id}_X}\searrow & \downarrow{\scriptstyle a} \\
& & X
\end{array}
$$

If $X \to S$ and $Y \to S$ are schemes with actions of $G \to S$, a morphism $f \colon X \to Y$ of schemes over S is *G-equivariant* if $a_Y \circ (\mathrm{id} \times f) = f \circ a_X$, and is *$G$-invariant* if G-equivariant and Y has the trivial G-action.

Exercise B.2.11. Show that giving a group action of $G \to S$ on $X \to S$ is the same as giving an action of the functor $\mathrm{Mor}_S(-, G) \colon \mathrm{Sch}/S \to \mathrm{Gps}$ on the functor $\mathrm{Mor}_S(-, X) \colon \mathrm{Sch}/S \to \mathrm{Sets}$.
(This requires first spelling out what it means for a functor to groups to act on a functor to sets.)

Definition B.2.12. Let $S = \mathrm{Spec}\, R$ be an affine scheme, and let $G \to S$ be a group scheme with multiplication m and identity e. A *representation* (or *comodule*) of G is an R-module V together with a homomorphism $\sigma \colon V \to \Gamma(G, \mathcal{O}_G) \otimes_R V$ of R-modules (referred to as a *coaction*) such that the following diagrams commute:

$$
\begin{array}{ccc}
V & \xrightarrow{\qquad\qquad \sigma \qquad\qquad} & \Gamma(G, \mathcal{O}_G) \otimes_R V \\
{\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_G \otimes \sigma} \\
\Gamma(G, \mathcal{O}_G) \otimes_R V & \xrightarrow{m^* \otimes \mathrm{id}_V} & \Gamma(G, \mathcal{O}_G) \otimes_R \Gamma(G, \mathcal{O}_G) \otimes_R V
\end{array}
\qquad
\begin{array}{ccc}
V & \xrightarrow{\ \sigma\ } & \Gamma(G, \mathcal{O}_G) \otimes_R V \\
& {\scriptstyle \mathrm{id}_V}\searrow & \downarrow{\scriptstyle e^* \otimes \mathrm{id}_V} \\
& & V
\end{array}
$$

Example B.2.13.

(1) Given an R-module V, the *trivial representation* on V is defined using the coaction $\sigma(v) = 1 \otimes v$.

(2) The *regular representation* on $\Gamma(G, \mathcal{O}_G)$ is defined using the comultiplication $m^* \colon \Gamma(G, \mathcal{O}_G) \to \Gamma(G, \mathcal{O}_G) \otimes_R \Gamma(G, \mathcal{O}_G)$.

(3) The *standard representation* of $\mathrm{GL}_{n,R} = \operatorname{Spec} R[x_{ij}]_{\det}$ (or a subgroup scheme of $\mathrm{GL}_{n,R}$) on $V = R^n$ is given by the coaction $\sigma\colon V \to \Gamma(\mathrm{GL}_{n,R}) \otimes_R V$ defined by $\sigma(e_i) = \sum_{j=1}^n x_{ij} \otimes e_j$ where $(e_1, \dots, e_n)$ is the standard basis of V.

A representation V of G induces an action of G on $\mathbb{A}(V) = \operatorname{Spec}(\operatorname{Sym}^* V)$, which we refer to as a *linear action*. Morphisms of representations and subrepresentations are defined in the obvious way.

Exercise B.2.14. If $R = \Bbbk$ is a field and V is a finite dimensional vector space, show that giving V the structure as a representation is the same as giving a homomorphism $G \to \mathrm{GL}(V)$ of group schemes.

Proposition B.2.15. *Let $G = D_{\Bbbk}(A)$ be a diagonalizable group scheme over a field $\Bbbk$. Every representation of G is a direct sum of one-dimensional representations.*

Proof. Let $G = D_{\Bbbk}(A)$ and let V be a free representation of G with coaction $\sigma\colon V \to \Bbbk[A] \otimes_{\Bbbk} V$. Each $a \in A$ defines a one-dimensional representation $W_a = A$ of $D_{\Bbbk}(A)$ defined by the coaction $W_a \to \Bbbk[A] \otimes_{\Bbbk} W_a$ defined by $1 \mapsto a \otimes 1$. For $a \in A$, the subspace

$$V_a := \{v \in V \mid \sigma(v) = a \otimes v\}$$

is isomorphic to $W_a \otimes V_a$ as G-representations, where V_a is viewed as the trivial representation (so that in the case V_a is finite dimensional, $V_a \cong W_a^{\dim V_a}$). (Note that when $a = 0$, W_a is the trivial one-dimensional representation and $V^G = V_0$.) We leave the reader to check that $V \cong \oplus_{a \in A} V_a$ as G-representations. See also [Mil17, Thm 12.30] and [SGA3$_{\mathrm{I}}$, 5.3.3]. $\qquad\square$

B.2.3   Principal G-bundles

The concept of a principal G-bundle is an algebraic formulation of a topological fiber bundle $P \to T$ where G acts freely and transitively on P with quotient $T = P/G$.

Definition B.2.16. Let $G \to S$ be an fppf affine group scheme. A *principal G-bundle over an S-scheme X* is a scheme P with an action of G via $\sigma\colon G \times_S P \to P$ such that $P \to X$ is a G-invariant fppf morphism and

$$(\sigma, p_2)\colon G \times_S P \to P \times_X P, \qquad (g, p) \mapsto (gp, p)$$

is an isomorphism.

Morphisms of principal G-bundles are G-equivariant morphisms of schemes. A principal G-bundle $P \to X$ is *trivial* if there is an G-equivariant isomorphism $P \cong G \times_S X$, where G acts on $G \times_S X$ via multiplication on the first factor.

A principal G-bundle $P \to X$ are examples of G-*torsors* (Definition 6.2.13) over $(\mathrm{Sch}/X)_{\mathrm{fppf}}$ by viewing P as a sheaf in the big fppf topology over X (see Example 6.2.17). In these notes, we will always distinguish between these two notions, but in conversation or the literature, they are often conflated.

Exercise B.2.17. Show that $P \to X$ is a principal G-bundle over an S-scheme X if and only if $P \to X$ is a principal $G \times_S X$-bundle.

Exercise B.2.18. Show that a morphism of principal G-bundles is necessarily an isomorphism.

Principal G-bundles can be trivialized fppf-locally, or even étale locally if G is smooth.

Proposition B.2.19. *Let $G \to S$ be an fppf affine group scheme and $P \to X$ be a G-equivariant morphism of S-schemes where X has the trivial action. Then $P \to X$ is a principal G-bundle if and only if there exists an fppf morphism $X' \to X$ such that $P \times_X X'$ is the trivial principal G-bundle over X'. Moreover, if $G \to S$ is smooth, we can arrange that $X' \to X$ is surjective and étale.*

Proof. The ($\Rightarrow$) direction follows from the definition by taking $X' = P \to X$. For ($\Leftarrow$), after base changing $G \to S$ by $X \to S$, we assume that G is defined over X (see Exercise B.2.17). Let $G_{X'}$ and $P_{X'}$ be the base changes of G and P along $X' \to X$. The base change of the action map $(\sigma, p_2)\colon G \times_X P \to P \times_X P$ along $X' \to X$ is the action map $G_{X'} \times_{X'} P_{X'} \to P_{X'} \times_{X'} P_{X'}$ of $G_{X'}$ acting on $P_{X'}$ over X'. Since $P_{X'}$ is trivial, this latter action map is an isomorphism. Since the property of being an isomorphism descends under fppf morphisms (Proposition B.1.9), we conclude that $(\sigma, p_2)\colon G \times_X P \to P \times_X P$ is an isomorphism.

If G is smooth, then $X' = P \to S$ is a surjective smooth morphism such that $P_{X'}$ is trivial. Since there is a section of $X' \to S$ after a surjective étale morphism $S' \to S$ (Corollary A.3.5), $P_{S'}$ is also trivial. $\qquad\square$

Proposition B.2.20 (Fpqc Descent of Principal G-bundles)**.** *Let $G \to S$ be an fppf affine group scheme. Let $f\colon X \to Y$ be an fpqc morphism of schemes over S. If $P \to X$ is a principal G-bundle and $\alpha\colon p_1^* P \xrightarrow{\sim} p_2^* P$ is an isomorphism of principal G-bundles over $X \times_Y X$ satisfying $p_{12}^* \alpha \circ p_{23}^* \alpha = p_{13}^* \alpha$, then there exists a principal G-bundle $Q \to Y$ and an isomorphism $\phi\colon P \to f^* Q$ of principal G-bundles such that $p_1^* \phi = p_2^* \phi \circ \alpha$.*

Proof. By Effective Descent (Proposition B.1.8) for affine morphisms, there is a scheme Q affine over Y and an isomorphism $\phi\colon P \to f^* Q$ of schemes. By applying descent for morphisms (Proposition B.1.7), we can descend the action $G \times_S P \to P$ to an action $G \times_S Q \to Q$ giving Q the structure of a principal G-bundle and making $\phi\colon P \to f^* Q$ a G-equivariant isomorphism. $\qquad\square$

B.2.4 Examples of principal G-bundles

Exercise B.2.21. Let L/K be a finite Galois extension and $G = \mathrm{Gal}(L/K)$ be its Galois group viewed as a finite group scheme over $\operatorname{Spec} K$. Show that $\operatorname{Spec} L \to \operatorname{Spec} K$ is a principal G-bundle.

Exercise B.2.22 (Line bundles vs $\mathbb{G}_m$-bundles)**.** If X is a scheme, show that there there is an equivalence of categories

$$\{\text{line bundles on } X\} \xrightarrow{\sim} \{\text{principal } \mathbb{G}_m\text{-bundles on } X\}$$
$$L \mapsto \mathbb{A}(L) \setminus X$$

between the *groupoids* of line bundles on X (where the only morphisms allowed are isomorphisms) and $\mathbb{G}_m$-torsors on X. If L is a line bundle (i.e. invertible $\mathcal{O}_X$-module), then $\mathbb{A}(L)$ denotes the total space $\operatorname{Spec}(\operatorname{Sym}^* L^\vee)$ of L and $X \subset \mathbb{A}(L)$ denotes the image of the zero section $X \to \mathbb{A}(L)$.

Exercise B.2.23 (S_d-bundles)**.** If X is a scheme and $d \geq 1$, show that there there is an equivalence of *groupoids*

$$\{\text{finite, étale, and degree } d \text{ covers of } X \} \xrightarrow{\sim} \{\text{principal } S_d\text{-bundles over } X\}$$
$$(Y \to X) \mapsto (\underbrace{Y \times_X \cdots \times_X Y}_{d \text{ times}} \setminus \Delta \to X)$$
$$(P/S_{d-1} \to X) \leftarrow\!\shortmid (P \to X).$$

For the rightward map, the symmetric group S_d acts on the d-fold fiber product $Y \times_X \cdots \times_X Y$ by permutation, and Δ denotes the S_d-equivariant closed locus of d-tuples where at least two points coincide. Alternatively, $Y \times_X \cdots \times_X Y \setminus \Delta$ can be identified with the scheme $\underline{\mathrm{Isom}}_X(X \times \{1, \ldots, d\}, Y)$ parameterizing isomorphisms of the trivial finite étale cover of degree d and Y. For the leftward map, $S_{d-1} \subset S_d$ denotes the subgroup of permutations fixing the dth index, and P/S_{d-1} denotes the quotient scheme of this free action (see Exercise 4.2.9).

Exercise B.2.24.

(a) Show that the standard projection $\mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ is a principal $\mathbb{G}_m$-bundle.

(b) For each line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$, explicitly determine the corresponding principal $\mathbb{G}_m$-bundle. In particular, which $\mathcal{O}(d)$ correspond to the principal $\mathbb{G}_m$-bundle of (a)?

Exercise B.2.25. Let $G \to S$ be an fppf affine group scheme, and let P and Q be principal G-bundles over an S-scheme X

(a) show that the functor

$$\underline{\mathrm{Isom}}_X(P, Q) \colon \mathrm{Sch}/X \to \mathrm{Sets},$$

assigning a X-scheme T to the set of isomorphisms of the principal G-bundles $P \times_X T$ and $Q \times_X T$, is representable by a principal G-bundle over X.

(b) Letting G act on itself via conjugation, show that $G \times^G P$, defined as the quotient $G \times P$ via $h \cdot (g, p) = (h^{-1}gh, h \cdot p)$, is a principal G-bundle over X isomorphic to $\underline{\mathrm{Aut}}_X(P) := \underline{\mathrm{Isom}}_X(P, P)$.

Exercise B.2.26 (Frame bundles). Let T be a scheme and E be a vector bundle over X of rank n.

(a) The *frame bundle* Fr_E is the functor $\underline{\mathrm{Isom}}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, E)$ on Sch/X, i.e

$$\mathrm{Fr}_E \colon \mathrm{Sch}/X \to \mathrm{Sets}$$
$$(T \to X) \mapsto \{\text{trivializations } \mathcal{O}_T^{\oplus n} \xrightarrow{\sim} E_T\}.$$

Show that Fr_E is representable by the open subscheme of $H := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, E) = \mathbb{A}(F^{\oplus n})^\vee)$ (see Exercise 0.3.15(a)) defined $\det(u) \neq 0$, where $u \colon \mathcal{O}_H^{\oplus n} \to E_H$ is the universal homomorphism. Moreover, show that $\mathrm{Fr}_E \to X$ is a principal GL_n-bundle.

(b) The *projectivized frame bundle* $\mathbb{P}\mathrm{Fr}_E$ is the functor

$$\mathbb{P}\mathrm{Fr}_E \colon \mathrm{Sch}/X \to \mathrm{Sets}$$
$$(T \to X) \mapsto \left\{ (L, \alpha) \;\middle|\; \begin{array}{l} L \text{ is a line bundle on } T \text{ and} \\ \alpha \colon \mathcal{O}_T^{\oplus n} \xrightarrow{\sim} E_T \otimes L \text{ is an isomorphism} \end{array} \right\} / \sim,$$

where $(L, \alpha) \sim (L', \alpha')$ if there is an isomorphism $\beta \colon L \to L'$ with $\alpha' = (\mathrm{id} \otimes \beta) \circ \alpha$. Show that $\mathbb{P}\mathrm{Fr}_E$ is representable by the open subscheme of $\mathbb{P} := \mathbb{P}(\mathcal{H}om(E, \mathcal{O}_X^{\oplus n})) = \mathcal{P}\mathrm{roj}_X(\mathrm{Sym}^*((E^\vee)^{\oplus n}))$ defined by $\det(u) \neq 0$, where

$$u \colon \mathcal{O}_{\mathbb{P}}^{\oplus n} \to E_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(1)$$

is the homomorphism corresponding to the universal quotient $\mathcal{H}om(E_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}^{\oplus n}) \to \mathcal{O}_{\mathbb{P}}(1)$. Moreover, show that $\mathbb{P}\mathrm{Fr}_E \to X$ is a principal PGL_n-bundle.

Exercise B.2.27 (Vector bundles vs GL-bundles). For a scheme X, show that the assignment of a vector bundle E to the Frame bundle Fr_E defines an equivalence of *groupoids*

$$\{\text{vector bundles over } X\} \overset{\sim}{\to} \{\text{principal } \mathrm{GL}_n\text{-bundles over } X\}$$
$$E \mapsto \mathrm{Fr}_W$$

with the inverse defined by assigning $P \to X$ to the quotient $(P \times^{\mathrm{GL}_n} \mathbb{A}^n)$ of the diagonal GL_n-action on $P \times \mathbb{A}^n$.

Exercise B.2.28 (SL-bundles). Show that the groupoid of principal SL_n-bundles over a scheme X is equivalent to the groupoid of pairs (V, α) where V is a vector bundle on X of rank n and $\alpha \colon \mathcal{O}_X \overset{\sim}{\to} \det V$ is a trivialization and a morphism $(V', \alpha') \to (V, \alpha)$ of pairs is an isomorphism $\phi \colon V' \to V$ such that $\alpha' = \alpha \circ \det \phi$.

Exercise B.2.29 (Brauer–Severi schemes). A morphism $P \to X$ of schemes is a *Brauer–Severi scheme of relative dimension r* if there exists an étale cover $X' \to X$ and an isomorphism $P \times_X X' \cong \mathbb{P}^r_{X'}$. An example of a non-trivial Brauer-Severi scheme is $\mathrm{Proj}\,\mathbb{R}[x,y,z]/(x^2+y^2+z^2) \to \mathrm{Spec}\,\mathbb{R}$. Show that

$$\{\text{Brauer–Severi schemes of rel. dim. } r \text{ over } X\} \to \{\text{principal } \mathrm{PGL}_r\text{-bundles over } X\}$$
$$X \mapsto \underline{\mathrm{Isom}}_X(\mathbb{P}^r_X, X)$$
$$(P \times \mathbb{P}^r)/\mathrm{PGL}_r \leftarrow\!\shortmid P$$

defines an equivalence of groupoids.

Exercise B.2.30. Let $P \to X$ be a proper, flat, and finitely presented morphism of schemes. Assume that for every geometric point $\mathrm{Spec}\,\Bbbk \to X$, the geometric fiber $P \times_X \Bbbk$ is isomorphic to $\mathbb{P}^1_\Bbbk$. Show that $P \to X$ is a Brauer–Severi scheme of relative dimension 1 following one of the approaches below.

Approach 1 (local-to-global): Show that for every point $x \in X$, there is a finite and separable field extension $\kappa(s) \to K$ such that $P \times_X K \cong \mathbb{P}^1_K$. Show that there an étale neighborhood $(X', x') \to (X, x)$ such that $K \cong \kappa(x')$ over $\kappa(x)$. Assuming now that $P \times_s \kappa(s) \cong \mathbb{P}^1_{\kappa(s)}$, use deformation theory (Proposition C.2.7) to show that there are compatible isomorphisms $P \times_X \mathcal{O}_{X,x}/\mathfrak{m}_x^n \cong \mathbb{P}^1_{\mathcal{O}_{X,x}/\mathfrak{m}_x^n}$ for $n > 0$. Use Grothendieck's Existence Theorem (C.4.4) to show that $P \times_X \widehat{\mathcal{O}}_{X,x} \cong \mathbb{P}^1_{\widehat{\mathcal{O}}_{X,x}}$. Apply Artin Approximation (B.5.26) to show that there is an étale neighborhood $(X', x') \to (X, x)$ such that $P \times_X X' \cong \mathbb{P}^1_{X'}$.

Approach 2 (direct): Assuming that there is a section $\sigma \colon X \to P$ of $\pi \colon P \to X$, show that every point $x \in X$ has an open neighborhood $U \subset X$ such that $P \times_X U \cong \mathbb{P}^1_U$. Letting L be the line bundle on P corresponding to the Cartier divisor σ, use Cohomology and Base Change (A.6.8) to show that $\mathcal{E} := \pi_* \mathcal{L}$ is a rank 2 vector bundle on X, that $\pi^* \mathcal{E} \twoheadrightarrow \mathcal{L}$ is surjective, and that $P \cong \mathbb{P}(\mathcal{E})$ over X. Conclude by choosing an open neighborhood of $x \in X$ where $\mathcal{E}$ is trivial. Returning to the general case, show that there is an effective divisor D associated to $\Omega^\vee_{P/X}$ such that $D \to X$ is étale. Reduce to the case where $P \to X$ has a section by base changing by $D \to X$. See also [Har77, Prop. 25.3 and Exer. 25.2].

Exercise B.2.31 (Azumaya algebras). An *Azumaya algebra of rank r^2* over a scheme X is a (possibly non-commutative) associative $\mathcal{O}_X$-algebra A which is coherent as an $\mathcal{O}_X$-module and such that there is an étale covering $X' \to X$ where $A \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$

is isomorphic to the matrix algebra $M_r(\mathcal{O}_X)$; see [Mil80, §IV.2]. An Azumaya algebra over a field $\Bbbk$ is a central simple algebra (i.e. a finite dimensional associative $\Bbbk$-algebra which is simple and whose center is $\Bbbk$); the quaternions is an example of a central simple algebra over $\mathbb{R}$.

Show that the assignment

$$A \mapsto \underline{\mathrm{Isom}}_X(M_r(\mathcal{O}_X), A)$$

defines a bijection between Azumaya algebras of rank r^2 over X and PGL_n-torsors over X.

Remark B.2.32. Exercises B.2.29 and B.2.31 provide bijections

$$\{\text{Azumaya algebras of rank } r^2\} \simeq \{\text{principal } \mathrm{PGL}_n\text{-torsors}\}$$
$$\simeq \{\text{Brauer–Severi schemes of relative dimension } r\}$$

on sets of isomorphism classes of objects over a scheme X. Xhe composition takes an Azumaya algebra $\mathcal{A}$ over X to the Brauer–Severi scheme defined as the closed subscheme $X \subset \mathrm{Gr}_X(r, \mathcal{A})$ classifying rank r right ideals. In Exercises 6.2.46 and 6.2.46 and remark 6.2.45, Azumaya algebras are used to define the *Brauer group* $\mathrm{Br}(X)$ and we discuss the relationship to $\mathbb{G}_m$-gerbes.

Exercise B.2.33 (Orthogonal group). Let $\Bbbk$ be a field with $\mathrm{char}(\Bbbk) \neq 2$, and let V be an n dimensional vector space with a non-degenerate quadratic form q. Let $O(q) \subset \mathrm{GL}(V)$ be the subgroup of invertible matrices preserving the quadratic form. If $q = x_1^2 + \cdots + x_n^2$ is the diagonalized quadratic form, then $O_n = O(q)$ is the set of orthogonal matrices A (i.e. $AA^\top = I$).

Show that there is a bijection between principal $O(q)$-bundles over a $\Bbbk$-scheme X and vector bundles of rank n on X with a non-degenerate quadratic form.

Balaji

Reduction of structure group.

Definition B.2.34. Let $H \to G$ be a homomorphism of fppf affine groups schemes over a scheme S. If P is a principal H-bundle over an S-scheme X, the *principal G-bundle induced by P via $H \to G$* is

$$G \times^H P \to X,$$

where $G \times^H P$ is the quotient $(G \times P)/H$ of the action $h \cdot (g, p) = (gh^{-1}, hp)$ and G acts on $G \times^H P$ via $g' \cdot (g, p) = (g'g, p)$.

Exercise B.2.35. Verify that $P \times^H G \to X$ is a principal G-bundle.

Definition B.2.36 (Reduction of structure group). Let $H \to G$ be a homomorphism of fppf affine groups schemes over a scheme S. If Q is a principal G-bundle over an S-scheme X, a *reduction of structure group of Q by $H \to G$* is a principal H-bundle $P \to X$ and an isomorphism $Q \xrightarrow{\sim} G \times^H P$ of principal G-bundles.

Lemma B.2.37. *Let $H \hookrightarrow G$ be a monomorphism of fppf affine groups schemes over a scheme S, and let Q be a principal G-bundle over an S-scheme X. A reduction of structure group of Q by $H \hookrightarrow G$ is equivalent to giving a section of Q/H over X.*

Proof. A section $s\colon X \to Q/H$ induces an principal H-bundle $P \to X$ via pullback

$$
\begin{array}{ccc}
P & \hookrightarrow & Q \\
\downarrow & \square & \downarrow \\
X & \stackrel{s}{\hookrightarrow} & H
\end{array}
$$

and the induced map $G \times^H P \to Q$ defined by $(g,p) \mapsto g \cdot p$ is an isomorphism of principal H-bundles. Conversely, by precomposing an isomorphism $G \times^H P \to Q$ with the H-equivariant inclusion $P \to G \times^H P$, defined by $p \mapsto (\mathrm{id}, p)$, defines an H-equivariant map $P \to Q$ which descends under the H-action to a section $X = P/H \to Q/H$. $\qquad\square$

Exercise B.2.38. Let $1 \to K \to G \to Q \to 1$ be an exact sequence of (abstract) abelian groups, which induces a short exact sequence

$$
\mathrm{H}^1(X_{\text{ét}}, K) \to \mathrm{H}^1(X_{\text{ét}}, G) \to \mathrm{H}^1(X_{\text{ét}}, Q)
$$

of étale cohomology groups over a scheme X. Show that a principal G-bundle $P \to X$ admits a reduction of structure group to K if and only if the class of $[P] \in P \in \mathrm{H}^1(X_{\text{ét}}, G)$ maps to $0 \in \mathrm{H}^1(X_{\text{ét}}, Q)$.

Exercise B.2.39. Let $G \to S$ be an fppf affine group scheme and P_1, P_2 be principal G-bundles over an S-scheme X. Show that giving an isomorphism $P_1 \to P_2$ of principal G-bundles is equivalent to giving a reduction of structure of $P_1 \times P_2 \to G \times G$ by the diagonal $G \to G \times G$.

B.2.5 Algebraic groups

An *algebraic group over a field* $\Bbbk$ is a group scheme G of finite type over $\Bbbk$.

Proposition B.2.40. *Let G be a group scheme locally of finite type over field $\Bbbk$ (e.g. G is an algebraic group). Let G^0 be the connected component containing the identity element.*

(1) G is separated.

(2) (Cartier's Theorem) If $\mathrm{char}(\Bbbk) = 0$, then G is smooth.

(3) If $\Bbbk$ is perfect, then G is smooth if and only if G is reduced if and only if G is geometrically reduced, and moreover $G_{\mathrm{red}} \subset G$ is a subgroup scheme.

(4) G^0 is an open and closed irreducible subgroup scheme of finite type that commutes with field extensions of $\Bbbk$, and G/G^0 is an etale algebraic group over $\Bbbk$.

(5) Any subgroup $H \subset G$ is closed.

(6) If G acts on a finite type $\Bbbk$-scheme X and $x \in X$ is a closed point, the orbit Gu, defined set-theoretically as the image of $G \to X, g \mapsto g \cdot x$, is open in its closure $\overline{Gx}$. Moreover,

$$
\dim G = \dim Gx + \dim G_x,
$$

and the function $x \mapsto \dim G_x$ is upper semicontinuous while $x \mapsto \dim Gx$ is lower semicontinuous.

(7) If G is of finite type and $H \subset G$ is a subgroup, then G/H is quasi-projective.

*(8) (Barsotti–Chevalley's Structure Theorem) If G is smooth and connected, then
there is a unique connected, affine, and normal subgroup $H \lhd G$, which is smooth
if $\Bbbk$ is perfect, such that G/H is an abelian variety.*

Remark B.2.41. In (7), the quotient G/H represents the sheafification of the
quotient presheaf

$$(G/H)^{\mathrm{pre}} \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}, \quad S \mapsto G(S)/H(S)$$

in the big étale topology $(\mathrm{Sch}/\Bbbk)_{\text{ét}}$. When we say that $1 \to K \to G \to Q \to 0$
is an *exact sequence of algebraic groups*, we mean that $K \subset G$ is a subgroup and
$Q \cong G/K$.

In (8), an *abelian variety* by definition is a smooth proper algebraic group over
a field. It is necessarily projective and has a commutative group law [Mum70, pgs.
39, 59]. An *elliptic curve* is an abelian variety of dimension 1.

Proof. Proposition B.2.9(4) implies (1) since any $\Bbbk$-point of a locally of finite type
$\Bbbk$-scheme is closed. For (2), see [Car62, §15], [Oor66], [Mum66, p.167], [Mil17,
Thm. 3.23 and Cor. 8.39], and [SP, Tag 047N]. For (3), see [Mil17, Prop. 1.26,
Cor. 1.39] and [SP, Tags 047P and 047R].

For (4), see [Mil17, Prop. 1.34] and [SP, Tag 0B7R]. What may seem surprising
here is that G^0 is automatically quasi-compact.[1] This follows from a simple argument:
reduce to the case that $\Bbbk$ is algebraically closed and choose a nonempty open
affine subscheme $U \subset G$. After shrinking, we may assume that U is closed under
taking inverses. The quasi-compactness of G follows from the surjectivity of the
multiplication map $U \times U \to G$ is surjective. If $g \in G(\Bbbk)$, then since U is dense, the
intersection $U \cap gU$ contains an element h. If we write $h = gu$, then $g = hu^{-1}$.

See [Mil17, Prop. 1.68] for (5) and [Mil17, Props. 1.65] for the first part of (6).
The identity $\dim G = \dim Gx + \dim G_x$ follows from the identification $Gx \cong G/G_x$,
while the semicontinuity statements follow from Proposition B.2.9(2) applied to the
stabilizer group scheme S_X over X, defined as the fiber product of the action map
$G \times X \to X \times X$ and the diagonal $X \to X \times X$.

For the existence of a projective embedding of G/H in (7), see [Ray70, Cor
VI.2.6], [Bri17, Thm. 5.2.2], and [Mil17, Thm. 8.4.4]. Chevalley announced a proof
of (8) in 1953, but a proof did not appear until [Che60]. In the meantime, Barsotti
provided an independent proof [Bar55a], [Bar55b]. Rosenlicht provided a more
elementary argument in [Ros56]. See [Con02], [Bri17, Thms. 1 and 2], and [Mil17,
Thm. 8.27] for modern expositions. $\qquad\square$

Properties of affine algebraic groups. We will mostly be concerned with
algebraic groups that are affine. These are sometimes also called *linear algebraic
groups*, justified by (2) below.

Proposition B.2.42. *Let G be an affine algebraic group over a field $\Bbbk$.*

(1) Every representation V of G is a union of its finite dimensional subrepresentations.

*(2) There exists a finite dimensional representation V and a closed immersion
$G \hookrightarrow \mathrm{GL}(V)$ of group schemes.*

Proof. For (1), let $\sigma \colon V \to \Gamma(G, \mathcal{O}_G) \otimes_{\Bbbk} V$ be the coaction. It suffices to show that
every finite dimensional subspace $W \subset V$ is contained in a finite dimensional sub-
G-representation $W' \subset V$. If $w_1, \ldots, w_n$ is a basis of W and $\sigma(w_i) = \sum_j f_{ij} \otimes v_{ij}$,

---

[1]We use this in the text to show the boundedness of $\underline{\mathrm{Pic}}^0_X$; see Theorem 6.2.58.

then one checks that the subspace generated by v_{ij} is G-invariant and contains W. For (2), we consider the regular representation $\Gamma(G, \mathcal{O}_G)$ of G and apply (1) to construct a finite dimensional subrepresentation V containing $\Bbbk$-algebra generators. One checks that this gives a closed immersion $G \hookrightarrow \mathrm{GL}(V)$. See [Mil17, Prop. 4.7, Cor. 4.10]. $\qquad\square$

We will repeatedly use the following simple consequence of Proposition B.2.42(1).

Proposition B.2.43. *Let G be an affine algebraic group over a field $\Bbbk$. Let X be an affine scheme of finite type over $\Bbbk$ with an action of G.*

(1) *There exists a G-equivariant closed immersion $X \hookrightarrow \mathbb{A}(V)$ where V is a finite dimensional G-representation.*

(2) *For every G-invariant closed subscheme $Z \subset X$, there exists a G-equivariant morphism $f \colon X \to \mathbb{A}(W)$, where W is a finite dimensional G-representation, such that $f^{-1}(0) = Z$.*

Proof. Write $X = \operatorname{Spec} A$ and let $f_1, \ldots, f_n$ be $\Bbbk$-algebra generators. By B.2.42(1) there is a finite dimensional G-invariant subspace $V \subset A$ containing each f_i. The surjection $\operatorname{Sym}^* V \to A$ induces a G-equivariant embedding $X \hookrightarrow \mathbb{A}(V)$. For (2), let $Z = \operatorname{Spec} A/I$ and let $g_1, \ldots, g_m \in I$ be generators. Letting $W \subset I$ be a finite dimensional G-invariant subspace containing each g_i, we see that $f \colon X \to \mathbb{A}(W)$ is a G-equivariant map with $f^{-1}(0) = Z$. $\qquad\square$

A subgroup $T \subset G$ of an affine algebraic group over a field $\Bbbk$ is called a *torus* if $T_{\bar{\Bbbk}} \cong \mathbb{G}_{m,\bar{\Bbbk}}^n$ and a *maximal torus* if it not contained in a larger torus. The set of diagonal matrices in GL_n is a maximal torus.

Proposition B.2.44. *Let G be an affine algebraic group over a field $\Bbbk$.*

(1) *G contains a maximal torus T such that $T_{\Bbbk'} \subset G_{\Bbbk'}$ is a maximal torus for every field extension $\Bbbk \to \Bbbk'$.*

(2) *If $\Bbbk$ is algebraically closed, all maximal tori are conjugate.*

Proof. See [Mil17, Thms. 17.82 and 17.105]. $\qquad\square$

There is of course much more to the theory of affine algebraic groups. We quickly mention a few facts that will be useful in the text.

Jordan decompositions. Recall that an element $g \in \mathrm{GL}_n(\Bbbk)$ is *semisimple* if it becomes diagonalizable after an extension of $\Bbbk$ and *unipotent* if $g - 1$ is nilpotent, i.e. $(1 - g)^n = 0$ for some n. If G is an affine algebraic group over a perfect field $\Bbbk$, then for every element $g \in G(\Bbbk)$, there are unique elements $g_s, g_u \in G(\Bbbk)$ such that $g = g_s g_u = g_u g_s$ such that the images of g_s and g_u under any representation $G \to \mathrm{GL}_n$ are semisimple and unipotent, respectively; see [Mil17, Thm. 9.17]. We call g_s and g_u the *semisimple* and *unipotent parts* of g.

Unipotent groups. An affine algebraic group G over a field $\Bbbk$ is *unipotent* if there is a faithful representation V and a basis $V \cong \Bbbk^n$ such that the image of the induced map $G \hookrightarrow \mathrm{GL}(V) \cong \mathrm{GL}_n$ is contained in the subgroup $\mathbb{U}_n$ of upper triangle matrices with 1's along the diagonal. An example of a unipotent group is $\mathbb{G}_a$. We have the following equivalences:

- G is unipotent,
- G has a filtration of normal subgroups $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ where $G_i/G_{i-1} \cong \mathbb{G}_a$,

- for every nonzero representation V of G, $V^G \neq 0$, and
- every element $g \in G$ is unipotent, i.e. $g = g_u$.

Special groups. An affine algebraic group G over a field $\Bbbk$ is *special* if every principal G-bundle $P \to T$ is Zariski-locally trivial. For example, GL_n (e.g. $\mathbb{G}_m = \mathrm{GL}_n$) is a special group as principal GL_n-bundles correspond to vector bundles (Exercise B.2.27). It is also true that $\mathbb{G}_a$ is special. One way to see this is to use that quasi-coherent cohomology can be computed either in the Zariski or étale-site (Proposition 4.1.34). Viewing the structure sheaf $\mathcal{O}_T$ as $\mathbb{G}_a$, this implies that $\mathrm{H}^1(T, \mathbb{G}_a) = \mathrm{H}^1(T_{\text{ét}}, \mathbb{G}_a)$ and there is thus a bijective correspondence between $\mathbb{G}_a$-torsors in the Zariski and small étale topology.

If $1 \to K \to G \to Q \to 1$ is an exact sequence of affine algebraic groups and both K and Q are special, it is not hard to see that G is also special. It follows from the characterization of unipotent groups as extensions of $\mathbb{G}_a$ that a unipotent group is special.

B.2.6 One-parameter subgroups, centralizers, and parabolics

Let G be a smooth affine algebraic group over an algebraically closed field $\Bbbk$. A *one-parameter subgroup* is a homomorphism $\lambda \colon \mathbb{G}_m \to G$ of algebraic groups (which is not necessarily a subgroup). We let $\mathbb{X}_*(G)$ be the set of one-parameter subgroups. On the other hand, a *character* is a homomorphism $\chi \colon G \to \mathbb{G}_m$ and we let $\mathbb{X}^*(G)$ be the group of characters. Since any character $\mathbb{G}_m \to \mathbb{G}_m$ is given by $t \mapsto t^d$ for some $d \in \mathbb{Z}$, there is a pairing

$$\langle -, - \rangle \colon \mathbb{X}_*(G) \times \mathbb{X}^*(G) \to \mathbb{X}^*(\mathbb{G}_m) \cong \mathbb{Z}, \quad (\lambda, \chi) \mapsto \chi \circ \lambda.$$

Example B.2.45 (Tori)**.** If $T \cong \mathbb{G}_m^n$ is an n-dimensional torus, then any one-parameter subgroup $\lambda \colon \mathbb{G}_m \to T$ is given by $t \mapsto (t^{\lambda_1}, \cdots, t^{\lambda_n})$ for integers λ_i while a character of T is given by $(t_1, \ldots, t_n) \mapsto t_1^{\chi_1} \cdots t_n^{\chi_n}$ for integers χ_i. We thus have bijections $\mathbb{X}_*(T) \cong \mathbb{Z}^n$ and $\mathbb{X}^*(T) \cong \mathbb{Z}^n$ where the pairing $\langle -, - \rangle \colon \mathbb{X}_*(T) \times \mathbb{X}^*(T) \to \mathbb{Z}$ is identified with the standard inner product.

Example B.2.46 (GL_n)**.** Every one-parameter subgroup λ is contained in a maximal torus, and since maximal tori are conjugate (Proposition B.2.44), there exists $g \in G(\Bbbk)$ such that $g\lambda g^{-1}$ is contained in the maximal torus consisting of diagonal matrices. The structure of $\mathbb{X}_*(\mathrm{GL}_n)$ is more complicated than in the torus case; see Remark B.2.50.

Definition B.2.47 (Centralizers, parabolics, and unipotents)**.** Given a one-parameter subgroup $\lambda \colon \mathbb{G}_m \to G$ of an algebraic group, we define the subgroups:

$$
\begin{aligned}
C_\lambda &= \{ g \in G \mid \lambda(t)g = g\lambda(t) \text{ for all } t \} && \text{(centralizer)} \\
P_\lambda &= \{ g \in G \mid \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists} \} && \text{(parabolic)} \\
U_\lambda &= \{ g \in G \mid \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} = 1 \} && \text{(unipotent)}.
\end{aligned}
$$

Functorially, for a $\Bbbk$-algebra R, $C_\lambda(R)$ (resp. $P_\lambda(R)$, $U_\lambda(R)$) consist of elements $g \in G(R)$ such that $\lambda_R = g^{-1}\lambda_R g$ (resp. $\lim_{t \to 0} \lambda(t)g\lambda(t)^{-1}$ exists, $\lim_{t \to 0} \lambda(t)g\lambda(t)^{-1} = 1$). Note that under the conjugation action of λ on G, C_λ is the fixed locus while while P_λ is the attractor locus G_λ^+ as defined in §6.6.1. There is a homomorphism $P_\lambda \to C_\lambda$ defined by $g \mapsto \lim_{t \to 0} \lambda(t)g\lambda(t)^{-1}$ which is the identity on C_λ yielding a split short exact sequence

$$1 \to U_\lambda \to P_\lambda \to C_\lambda \to 1.$$

Example B.2.48. Let $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_n$ be a one-parameter subgroup. After a change of basis, we can assume that $\lambda(t) = \mathrm{diag}(t^{\lambda_1}, \cdots, t^{\lambda_n})$ with $\lambda_1 \leq \cdots \leq \lambda_n$. Given $(g_{ij}) \in \mathrm{GL}_n$, one has that

$$\lambda(t)\big(g_{ij}\big)\lambda(t)^{-1} = \big(t^{\lambda_i - \lambda_j} g_{ij}\big).$$

If $n_1, \ldots, n_s$ are integers with $\sum_i n_i = n$ such that

$$\lambda_1 = \cdots = \lambda_{n_1} < \lambda_{n_1+1} = \cdots = \lambda_{n_1+n_2} < \cdots < \lambda_{n-n_s+1} = \cdots = \lambda_n,$$

then $C_\lambda = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_s}$ is the subgroup of block diagonal matrices while P_λ is the subgroup of block upper triangular matrices.

For example, if $\lambda(t) = (t^{-1}, t^2, t^2, t^7)$, then

$$U_\lambda = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_\lambda = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad C_\lambda = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

We record the following properties of parabolic subgroups. Reductive groups are discussed in more detail in §B.2.8.

Proposition B.2.49. *Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$, and let $\lambda\colon \mathbb{G}_m \to G$ be a one-parameter subgroup.*

(a) *The centralizer C_λ is connected and reductive.*

(b) *The subgroup P_λ is connected and parabolic, i.e. G/P_λ is projective, and $N_G(P_\lambda) = P_\lambda$.*

(c) *The subgroup U_λ is the unipotent radical of P_λ, and it acts freely and transitively on the set of one-parameter subgroups of P_λ which are conjugate (under P_λ) to λ.*

(d) *If $\lambda, \lambda'\colon \mathbb{G}_m \to G$ are one-parameter subgroups, the intersection $P_\lambda \cap P_{\lambda'}$ contains a maximal torus of G.*

Proof. See [Con14, Thm. 4.1.7, Cor. 5.2.8] for (a)-(c) and [Bor91, Prop. 20.7] for (d). $\square$

Remark B.2.50 (Spherical buildings)**.** The set of one-parameter subgroups of a reductive group can be given the structure of an "ungainly but remarkable metric space" (as described by Mumford in [GIT, p.55]), first introduced by J. Tits, after introducing the equivalence relation where $\lambda \sim \rho$ if there exists $g \in P_\lambda(\Bbbk)$ such that $\rho(t^m) = g^{-1}\lambda(t^m)g$ for integers n, m.

B.2.7 Line bundles with G-actions

If G is an algebraic group over a field $\Bbbk$ acting on a $\Bbbk$-scheme U via $\sigma\colon G \times U \to U$, then a *line bundle with a G-action* or a *G-linearization* is a line bundle L on U together with an isomorphism $\alpha\colon \sigma^* L \xrightarrow{\sim} p_2^* L$ satisfying the cocycle condition: the

diagram

$$
\begin{array}{ccc}
& (\sigma \circ (\mathrm{id}_G \times \sigma))^* L \xrightarrow{(\mathrm{id}_G \times \sigma)^* \alpha} (p_2 \circ (\mathrm{id}_G \times \sigma))^* L & \\
(\sigma \circ (\mu \times \mathrm{id}_U))^* L & & (\sigma \circ p_{23})^* L \\
& \quad \downarrow_{(\mu \times \mathrm{id}_U)^* \alpha} \qquad\qquad \downarrow_{p_{23}^* \alpha} & \\
& (p_2 \circ (\mu \times \mathrm{id}_U))^* L =\!=\!=\!= (p_2 \circ p_{23})^* L &
\end{array}
$$

commutes where $\mu \colon G \times G \to G$ denotes multiplication. When U is projective, a very ample line bundle L with a G-action corresponds to a finite dimensional G-representation $V = \mathrm{H}^0(U, L)$ and a G-equivariant closed immersion $U \hookrightarrow \mathbb{P}(V)$.

Theorem B.2.51 (Sumihiro's Theorem on Linearizations). *Let G be a connected, smooth, and affine algebraic group over an algebraically closed field. Let U be a normal scheme over $\Bbbk$ with an action of G.*

   (1) If L is a line bundle on U, then there exists an integer $n > 0$ such that $L^{\otimes n}$ admits a G-action.

   (2) If U is quasi-projective, there exists a locally closed embedding $U \hookrightarrow \mathbb{P}(V)$ where V is a finite dimensional G-representation.

   (3) Every point $u \in U$ has a G-invariant quasi-projective open neighborhood.

Proof. For (1), see [Sum74, Thm. 1], [Sum75, Lem. 1.2], and [KKLV89, Prop. 2.4]. Part (2) is a direct consequence of (1). For (3), see [Sum74, Lem. 8] and [Sum75, Thm. 3.8]. $\qquad\square$

When G is a torus, we have the stronger result that U has a G-invariant affine cover.

Theorem B.2.52 (Sumihiro's Theorem on Torus Actions). *Let U be a normal scheme over an algebraically closed field $\Bbbk$ with an action of a torus T. Then any point $u \in U$ has a T-invariant affine open neighborhood.*

Proof. See [Sum74, Cor. 2] and [Sum75, Cor. 3.11]. $\qquad\square$

Remark B.2.53. Theorems B.2.51 and B.2.52 can fail if U is not normal, e.g. the plane nodal cubic curve has a $\mathbb{G}_m$-action and no $\mathbb{G}_m$-invariant neighborhood of the origin can be embedded $\mathbb{G}_m$-equivariantly into projective space.

B.2.8   Reductivity

We denote by $\mathrm{Rep}(G)$ the category of representations of an algebraic group G. If V is a G-representation with coaction $\sigma \colon V \to \Gamma(G, \mathcal{O}_G) \otimes V$, then the *invariants* are $V^G := \{v \in V \mid \sigma(v) = 1 \otimes v\}$. A representation V of G is *irreducible* if every subrepresentation $W \subset V$ is either 0 or V.

Linear reductive groups. There are various notions of reductivity but the one most central to this book is linear reductivity.

Definition B.2.54. An affine algebraic group G over a field $\Bbbk$ is *linearly reductive* if the functor $\mathrm{Rep}(G) \to \mathrm{Vect}_\Bbbk$, taking a G-representation V to its G-invariants V^G, is exact.

Proposition B.2.55. *Let G be an affine algebraic group over a field $\Bbbk$. The following are equivalent:*

(1) *G is linearly reductive;*

(1′) *The functor $\mathrm{Rep}^{\mathrm{fd}}(G) \to \mathrm{Vect}_\Bbbk$, $V \mapsto V^G$, on the category of finite dimensional representations is exact;*

(2) *Every G-representation (resp. finite dimensional G-representation) is a direct sum of irreducible representations.*

(3) *Given a G-representation (resp. finite dimensional G-representation) V and a G-invariant subspace $W \subset V$, there exists a G-invariant subspace $W' \subset V$ such that $V = W \oplus W'$.*

(4) *For every finite dimensional representation V and fixed $\Bbbk$-point $x \in \mathbb{P}(V)^G$, there exists a G-invariant linear function $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(1))^G$ such that $f(x) \neq 0$.*

Proof. TO ADD ☐

Remark B.2.56. In the notation introduced in §6.3, G is linearly reductive if and only if $BG \to \mathrm{Spec}\, k$ is cohomologically affine or equivalently a good moduli space.

It is not hard to see that for a field extension $\Bbbk \to \Bbbk'$, G is linearly reductive if and only if $G_{\Bbbk'}$ is and that linearly reductive groups are closed under extension Linearly reductive algebraic group are also closed under extension. See Lemma 6.3.16 and Proposition 6.3.18.

Proposition B.2.57 (Maschke's Theorem)**.** *Let G be an finite abstract group viewed as a finite group scheme over a field $\Bbbk$. If the order of G is prime to $\mathrm{char}(\Bbbk)$, then G is linearly reductive.*

Proof. If V is a G-representation, averaging over translates gives a G-equivariant $\Bbbk$-linear

$$R_V : V \to V^G, \qquad v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v, \tag{B.2.58}$$

which is on the identity on V^G and compatible with a map $f \colon V \to W$ of G-representations, i.e. $R_W \circ f = f \circ R_V$. It follows that a surjection $V \to W$ of G-representations induces a surjection $V^G \to W^G$ on invariants. ☐

Example B.2.59. In characteristic p, there is a 2-dimensional representation V of $G = \mathbb{Z}/p$ where a generator acts via the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The surjection $V \to \Bbbk$ onto the first component is a surjection of G-representations. The induced map $V^G \to \Bbbk$ on invariants is the zero map. Note however that the element $e_1^p \in \mathrm{Sym}^p V$ is G-invariant and maps to 1. Geometrically, this gives an action of G on $\mathbb{A}^2 = \mathbb{A}(V)$ where $(1, 0)$ is G-invariant; the invariant hypersurface x^p doesn't contain p, but there is no such hyperplane.

Remark B.2.60 (Reynolds operator)**.** The map (B.2.58) is called a *Reynolds operator* for the action of G on V. If G is linearly reductive, the canonical projections $R_V \colon V \to V^G$ are Reynold operators, i.e. $\Bbbk$-linear maps which are the identity on V^G and compatible with maps of G-representations. For an action of G on a $\Bbbk$-scheme $\mathrm{Spec}\, A$ with dual action $A \to \Gamma(G, \mathcal{O}_G) \otimes A$, there is a projection $R_A \colon A \to A^G$. This

is not a ring map, but since multiplication $A^G \otimes A \to A$ is a map of G-representations commuting with the Reynold operators, we have that

$$R_A(xy) = xR_A(y) \quad \text{for } x \in A^G, \, y \in A.$$

This is called the *Reynolds identity* and shows that $A \to A^G$ is an A^G-algebra homomorphism.

In Remark 6.3.9, the Reynolds operator was applied to show that A^G is finitely generated whenever A is. It can also be used to show that $\operatorname{Spec} A \to \operatorname{Spec} A^G$ separates G-orbits and has the properties of affine GIT quotients (Corollary 6.3.7); see [GIT, §1.2]. As we will see in the proof of Theorem B.2.61, one technique to prove that a given group is linearly reductive is to construct Reynolds operators.

Reductive groups. An affine algebraic group G over an algebraically closed field $\Bbbk$ is called *reductive* if every smooth, connected, unipotent, and normal subgroup of G is trivial. [2] Over $\mathbb{C}$, an affine algebraic group is reductive if and only if it is the complexification of any maximal compact subgroup [Hoc65, XVII.5]. Reductive groups are a particularly nice class of algebraic groups appearing in many branches of mathematics and can be completely classified in terms of their root datum.

For a smooth affine algebraic group G, there are subgroups $R(G)$ and $R_u(G)$ of G, called the *radical* and *unipotent radical*, which are maximal among connected, normal and solvable (resp. connected normal unipotent) subgroups. Over an algebraically closed field $\Bbbk$, G is reductive if $R_u(G)$ is trivial and called *semisimple* if $R(G)$ is trivial. The center $Z(G)$ of a reductive group G is diagonalizable and contains the radical $R(G) \subset Z(G)$ as its largest subtorus, and the quotient $G/R(G)$ is semisimple. For a smooth affine algebraic group G, the quotient $G/R_u(G)$ is reductive. Over an arbitrary field $\Bbbk$, G is called reductive if $G_{\overline{\Bbbk}}$ is. The unipotent radical $R_u(G)$ commutes with separable field extensions, and so over a perfect field $\Bbbk$, G is reductive if and only if $R_u(G)$ is trivial. See [Bor91, Hum75, Spr98, Mil17].

The classical algebraic groups of GL_n, PGL_n, SL_n, or SP_{2n} are reductive in every characteristic. We develop GIT in this book for actions by linearly reductive groups, and it is therefore imperative to know that these classical groups are linearly reductive in characteristic 0.

Theorem B.2.61. *In characteristic 0, a reductive algebraic group is linearly reductive. The converse is true in every characteristic for smooth algebraic groups.*

Proof. In [Hil90], Hilbert established the linearly reductivity for $\mathrm{SL}_{n,\mathbb{C}}$ and $\mathrm{GL}_{n,\mathbb{C}}$ using a explicit differential operator well-known to 19th century invariant theorists: the Ω-*process*. We will sketch the argument for $G = \mathrm{GL}_n$ over $\mathbb{C}$. Write $\Gamma(\mathrm{GL}_{n,\mathbb{C}}, \mathcal{O}_{\mathrm{GL}_n}) = \mathbb{C}[X_{ij}]_{\det}$. Let V be a finite dimensional GL_n-representation such that the scalar matrices act with weight k, and let $\sigma \colon \mathbb{C}[V] \to \mathbb{C}[X_{ij}]_{\det} \otimes \mathbb{C}[V]$ be the dual action on the coordinate ring $\mathbb{C}[V]$ of $\mathbb{A}(V^\vee)$. The differential operator

$$\Omega := \det\left(\frac{\partial}{\partial X_{ij}}\right)$$

acts linearly on $\mathbb{C}[X_{ij}]_{\det}$ and also on $\mathbb{C}[X_{ij}]_{\det} \otimes \mathbb{C}[V]$. One checks that the map

$$V \to V^{\mathrm{GL}_n}, \quad f \mapsto \frac{1}{\Omega^k\left(\det(X_{ij})^k\right)} \Omega^k\left(\det(X_{ij})^k \sigma(f)\right)$$

[2]Sometimes G is also assumed to be connected. For a *reductive group scheme* $G \to S$, there is no such ambiguity in the literature: G is smooth and affine over S with connected and reductive geometric fibers [SGA3$_{\mathrm{III}}$, Exp. XIX, Defn. 2.7].

defines a Reynolds operator. As with the averaging operator (B.2.58) in Maschke's Theorem (B.2.57), this shows that GL_n is linearly reductive and a variant of the argument shows that SL_n is also linearly reductive. The argument is algebraic and works over every field of characteristic 0. See also [Stu08, §4.3], [Dol03, §2.1] and [DK15, §4.5.3].

Extending an integral procedure developed by Hurwitz and Schur and ideas of Cartan, Weyl [Wey26, Wey25] showed that every reductive algebraic group over $\mathbb{C}$ is linearly reductive. The technique is now referred to as 'Weyl's unitarian trick'. A Lie group G has a left G-invariant measure μ, called the *left Haar measure*. When G is compact, this measure is finite, and for a finite dimensional G-representation V, averaging gives a $\mathbb{k}$-linear map

$$V \to V^G, \quad v \mapsto \frac{1}{\int_G d\mu(g)} \int_G (g \cdot v) d\mu(g)$$

constant on V^G and compatible with maps of G-representations. This is a Reynolds operator (Remark B.2.60) exactly as the averaging map in Maschke's Theorem (B.2.57) and implies that $V \mapsto V^G$ is exact. For every reductive algebraic group G over $\mathbb{C}$, there is a real Lie subgroup $K \subset G(\mathbb{C})$ which is dense in the Zariski topology and compact in the Euclidean topology; for $\mathrm{GL}_{n,\mathbb{C}}$, $K = U_n$ is the subgroup of unitary matrices (hence the name 'unitarian trick'). Then for a finite dimensional G-representation V, there is an identification $V^K = V^G$, and since the functor taking K-invariant is exact, so is the functor taking G-invariants. See also [Dol03, §3.2], [Bum13, Thm. 14.3].[3]

There is also an algebraic argument using the *Casimir operator*. First, one reduces to the case that G is semisimple because every reductive group is an extension of a torus by a semisimple group. Given a representation $\rho \colon \mathfrak{g} \to V$ of the Lie algebra, there is a symmetric bilinear form on $\mathfrak{g}$ defined by $\langle x, y \rangle = \mathrm{Tr}(\rho(x) \circ \rho(y))$. Letting $\{e_i\}$ be a basis of $\mathfrak{g}$ and $\{e_i'\}$ be a dual basis with respect to $\langle -, - \rangle$, the *Casimir operator* is the $\mathfrak{g}$-endomorphism $c_V := \sum_{i=1}^n \rho(e_i) \circ \rho(e_i')$ on V. To show that G is linearly reductive, it suffices to find a complement of any codimension 1 irreducible subspace $W \subset V$. As G is semisimple, G acts trivial on V/W and therefore so does $\mathfrak{g}$. It follows that $\mathfrak{g}$ takes V into W and therefore so does c_V, i.e. $c_V(V) \subset W$. On the other hand, since W is irreducible, c_V acts on W by multiplication by a scalar (Schur's lemma). It follows that $\ker(c_V) \subset V$ is a complement of W. See also [Mil17, Thm. 22.42], [Muk03, §4.3], [Hum78, §6.2] and [DK15, §4.5.2].

For the converse, we need to show that for a linearly reductive group G, the unipotent radical $R_u(G)$ is trivial. Since $G/R_u(G)$ is affine, Matsushima's Theorem (6.3.20) implies that $R_u(G)$ is linearly reductive. However, a non-trivial unipotent group is not linearly reductive. Indeed, it suffices to show this for $\mathbb{G}_a$. Let $V = \mathbb{k}^2$ be the two-dimensional representation given by $t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The projection $V \to \mathbb{k}, (x, y) \mapsto x$ is a surjection of $\mathbb{G}_a$-representations with no complement, i.e. there is no invariant $(x, y) \in V^{\mathbb{G}_a}$ with $x \neq 0$ (in fact there is no invariant $f \in (\mathrm{Sym}^d V)^{\mathbb{G}_a}$ with $d > 0$ and $f(1, 0)$ nonzero. See also [NM64]. $\square$

---

[3]This analytic argument suffices to show the linear reductivity of a reductive group G over every characteristic 0 field by the limit methods of §B.3 (or by using the classification theorem of reductive groups): there is a subfield $\mathbb{k}' \subset \mathbb{k}$ of finite transcendence degree over $\mathbb{Q}$ and a group scheme $G' \to \mathrm{Spec}\,\mathbb{k}'$ such that $G'_{\mathbb{k}} = G$. Choosing an embedding $\mathbb{k}' \hookrightarrow \mathbb{C}$ and using the fact that both the notions of reductivity and linear reductivity are insensitive to separable field extensions, we see that the linear reductivity of $G'_{\mathbb{C}}$ implies the linear reductivity of G.

Example B.2.62. The algebraic groups such as GL_n, PGL_n, SL_n, or SP_{2n} are not linearly reductive in characteristic p. For example, in characteristic 2, consider the action of SL_2 acts on the space $V = \mathrm{Sym}^2(\Bbbk^\vee) = \{Ax^2 + Bxy + Cy^2\}$ of degree 2 binary forms. The subspace W consisting of squares L^2 of linear forms is a GL_2-invariant subspace with no complement; the quotient $V \to V/W = \mathbb{C}$ is given by $(A, B, C) \mapsto B$. While there is no invariant *linear* function not vanishing at $(0, 1, 0)$, the discriminant $\Delta = B^2 \in \mathrm{Sym}^2 V^\vee$ is an invariant function nonzero at $(0, 1, 0)$ (verifying the geometric reductivity condition given below).

Theorem B.2.63 (Matsushima's Theorem). *Let G be a reductive group over a field $\Bbbk$. Then a subgroup $H \subset G$ is reductive if and only if G/H is affine.*

Proof. See Proposition 6.3.20 for a proof when G is linearly reductive. The general case can be proven in a similar way relying on a generalization of Serre's Criterion for Affineness (4.4.16): an algebraic space U, satisfying the property that for a surjection $\mathcal{A} \to \mathcal{B}$ of $\mathcal{O}_U$-algebras every global section of $\mathcal{B}$ has a positive power that lifts, is affine. See also [Mat60], [BB63], [Ric77], [FS82], [Alp13, Thm. 12.5] and [Alp14, Thm 9.4.1]. $\square$

Geometrically reductive groups An affine algebraic group G over a field $\Bbbk$ of characteristic $p \geq 0$ is called *geometrically reductive* (or sometimes called *semi-reductive*) if for every surjection $V \to W$ of G-representations and $w \in W^G$, there exists $n > 0$ such that w^{p^n} is in the image of $\mathrm{Sym}^{p^n} V \to \mathrm{Sym}^{p^n} W$. When $p = 0$, the condition is equivalent to linear reductivity. It suffices to take V finite dimensional and W the trivial representation, in which case the condition translates to a geometric property that for a fixed $\Bbbk$-point $x \in \mathbb{P}(V)^G$, there is an invariant homogenous polynomial $f \in \Gamma(\mathbb{P}(V), \mathcal{O}(p^n))^G$ for $n > 0$ with $f(x) \neq 0$ (analogous to Proposition B.2.55(4) except that f need not be linear).

Geometrically reductive groups appear in the context of GIT (§6.7) as their defining property can be used to show that the quotient morphisms $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^G$ have desirable properties (e.g. separates closed orbits and A^G is finitely generated). In an effort to extend GIT to actions by reductive groups such as SL_n and GL_n in positive characteristic, Mumford conjectured in [GIT, preface] a reductive group is geometrically reductive. This conjecture was resolved by Haboush [Hab75]; see also [SS11], [Ses69] and [Oda64]. Conversely, a smooth geometrically reductive group is reductive. In fact, an affine algebraic group G is geometrically reductive if and only if G_{red} is reductive.

On the other hand, a linearly reductive group is clearly geometrically reductive. The converse is true in characteristic 0 [Alp14, Lem. 9.2.8].

We thus have the implications:

$$\text{linearly reductive} \underset{\text{char}=0}{\overset{}{\Longleftarrow\!\!\Longrightarrow}} \text{geometrically reductive} \underset{}{\overset{G \text{ smooth}}{\Longleftarrow}} \text{reductive}.$$

A smooth algebraic group G in characteristic p is linearly reductive if and only if the connected component G^0 is a torus and the order of G/G^0 is prime to p [Nag62]. Every finite (possibly non-reduced) group scheme G is geometrically reductive but is linearly reductive if and only if G^0 is diagonalizable and G/G^0 has order prime to p [HR15, Thm. 1.2]. A commutative algebraic group G is reductive if and only if it's diagonalizable.

We also point out that a smooth algebraic group G satisfying the property that A^G is finitely generated for every coaction on a finitely generated $\Bbbk$-algebra, then G is necessarily reductive.

B.3 Limits of schemes

In moduli theory, we often need to deal with non-noetherian rings for the simple reason that moduli functors and stacks are defined over the category Sch of all schemes. Working instead with the category of locally noetherian schemes has the limitation that it is not closed under fiber products while working instead with the category of schemes finite type over a field or $\mathbb{Z}$ doesn't contain local rings of schemes or their completions.

In any case, using the limit methods presented in this section, it is usually straightforward to reduce properties of schemes and their morphisms to the noetherian case.

B.3.1 Limits of schemes

The first result states that a limit exists for an inverse system $(S_\lambda, f_{\lambda\mu})_{\lambda\in\Lambda}$ of schemes over a directed set Λ where the transition map $f_{\lambda\mu}\colon S_\lambda \to S_\mu$ for every $\lambda \geq \mu$ is affine.

Proposition B.3.1 (Existence of Limits). *If $(S_\lambda, f_{\lambda\mu})_{\lambda\in\Lambda}$ is an inverse system of schemes with affine transition maps, then the limit $S = \lim_{\lambda\in\Lambda} S_\lambda$ exists in the category of schemes such that each morphism $f_\lambda\colon S \to S_\lambda$ is affine.*

Proof. If each $S_\lambda = \operatorname{Spec} A_\lambda$ is affine, one takes $S = \operatorname{Spec}(\operatorname{colim}_\lambda A_\lambda)$. In general, choose an element $0 \in I$ and set $S = \mathcal{S}pec_{S_0}(\operatorname{colim}_{\lambda \geq 0} f_{\lambda 0,*}\mathcal{O}_{S_\lambda})$. Details can be found in [EGA, IV.8.2] and [SP, Tag 01YX]. □

A morphism $f\colon X \to Y$ of schemes is locally of finite presentation if and only if for every inverse system $(S_\lambda, f_{\lambda\mu})$ of affine schemes over Y, the map $\operatorname{colim}_\lambda \operatorname{Mor}_Y(S_\lambda, X) \to \operatorname{Mor}_Y(\lim_\lambda S_\lambda, X)$ is bijective (Proposition A.1.1) The same holds for inverse systems of quasi-compact and quasi-separated schemes over S with affine transition maps; see [EGA, IV.8.14.1] and [SP, Tag 01ZC].

B.3.2 Noetherian approximation

Every affine scheme $\operatorname{Spec} A$ is the limit of affine schemes $\operatorname{Spec} A_\lambda$ of finite type over $\mathbb{Z}$. This follows from the fact that the ring A is the union of its finitely generated $\mathbb{Z}$-subalgebras. More generally, we have:

Proposition B.3.2 (Relative Noetherian Approximation). *Let $X \to S$ be a morphism of schemes with X quasi-compact and quasi-separated and with S quasi-separated. Then $X = \lim_{\lambda\in\Lambda} X_\lambda$ is a limit of an inverse system $(X_\lambda, f_{\lambda\mu})$ of schemes of finite presentation over S with affine transition maps over S.*

Proof. See [SP, Tag 09MV]. □

When $S = \operatorname{Spec} \mathbb{Z}$, this is often referred to as *Absolute Noetherian Approximation* and was first established in [TT90, Thm. C.9].

B.3.3 Descent of properties under limits

Proposition B.3.3 (Descent of Properties of Schemes under Limits). *Let $S = \lim_\lambda S_\lambda$ be a limit of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps. If S is affine (resp. quasi-affine, separated), then so is S_λ for $\lambda \gg 0$.*

Proof. See [SP, Tags 01Z6, 086Q, and 01Z5] and [TT90, Props C.6-7]. □

Proposition B.3.4 (Descent of Morphisms under Limits)**.** *Let $S = \lim_{\lambda \in \Lambda} S_\lambda$ be a limit of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps.*

(1) *For a finitely presented morphism $X \to S$ of schemes, there exists an index $0 \in \Lambda$ and a finitely presented morphism $X_0 \to S_0$ of schemes such that $X \cong X_0 \times_{S_0} S$. Moreover, if we define $X_\lambda := X_0 \times_{S_0} S_\lambda$ for $\lambda > 0$, then $X = \lim_{\lambda \geq 0} X_\lambda$ is the limit of the inverse system $(X_\lambda, f_{\lambda\mu})$ where the (affine) transition map $f_{\lambda\mu} \colon X_\lambda \to X_\mu$ is the base change of $S_\lambda \to S_\mu$ for $\lambda \geq \mu$.*

(2) *Let X_0 and Y_0 be finitely presented schemes over S_0 for some index $0 \in \Lambda$. For $\lambda > 0$, set $X_\lambda = X_0 \times_{S_0} S_\lambda$ and $Y_\lambda = Y_0 \times_{S_0} S_\lambda$, and let $X = \lim_\lambda X_\lambda$ and $Y = \lim_\lambda Y_\lambda$ be the limits (Proposition B.3.1). Then the natural map*

$$\operatorname{colim}_{\lambda \geq 0} \operatorname{Mor}_{S_\lambda}(X_\lambda, Y_\lambda) \to \operatorname{Mor}_S(X, Y)$$

is bijective.

Remark B.3.5. In other words, the category of schemes finitely presented over S is the colimit of the categories of schemes finitely presented over S_λ.

Proof. See [EGA, IV.8.8] and [SP, Tag 01ZM]. □

Definition B.3.6. We say that a property $\mathcal{P}$ of morphisms of schemes *descends under limits* if the following holds for every limit $S = \lim_{\lambda \in \Lambda} S_\lambda$ of an inverse system of quasi-compact and quasi-separated schemes with affine transition maps: for every index $0 \in \Lambda$, and for every morphism $g_0 \colon X_0 \to Y_0$ of quasi-compact and quasi-separated schemes with base changes $g_\lambda \colon X_\lambda \to Y_\lambda$ over S_λ and $g \colon X \to Y$ over S, we require that if g has $\mathcal{P}$, then g_λ has $\mathcal{P}$ for $\lambda \gg 0$.

Proposition B.3.7 (Descent of Properties of Morphisms under Limits)**.** *The following properties of morphisms of schemes descend under limits: isomorphism, closed immersion, open immersion, affine, quasi-affine, finite, quasi-finite, proper, projective, quasi-projective, separated, monomorphism, surjective, flat, locally of finite presentation, unramified, étale, smooth, syntomic, and for any integer d the property that every fiber is connected and has pure dimension d.*

Proof. See [EGA, IV 8.10.5] and [SP, Tags 081C and 05M5]. □

Suppose $S = \operatorname{colim} S_\lambda$ is the colimit of an inverse system $(S_\lambda, f_{\lambda\mu})$ of quasi-compact and quasi-separated schemes with affine transition maps. If F_0 is a quasi-coherent sheaf on S_0 for an index $0 \in \Lambda$, and $F = f_\lambda^* F$ and $F_\lambda = f_{\lambda 0}^* F_0$ are the pullbacks to S and S_λ for $\lambda \geq 0$, then $\Gamma(S, F) = \operatorname{colim}_{\lambda \geq 0} \Gamma(S_\lambda, F_\lambda)$ [SP, Tag 01Z0]. Moreover, a quasi-coherent sheaf on S and its properties often descend to some S_λ.

Proposition B.3.8 (Descent of Sheaves under Limits)**.** *Let $(S_\lambda, f_{\lambda\mu})$ be an inverse system of quasi-compact and quasi-separated schemes with affine transition maps and limit $S = \lim_{\lambda \in \Lambda} S_\lambda$. Denote the projection maps by $f_\lambda \colon S \to S_\lambda$.*

(1) *If F is an $\mathcal{O}_S$-module of finite presentation (resp. vector bundle, line bundle), then there exists an index $i \in \Lambda$ and an $\mathcal{O}_{S_\lambda}$ module F_λ of finite presentation (resp. vector bundle, line bundle) such that $F \cong f_\lambda^* F_\lambda$.*

(2) *For an index $0 \in \Lambda$, let F_0 and G_0 be $\mathcal{O}_{S_0}$-modules of finite presentation. The natural map*

$$\mathrm{colim}_{\lambda \geq 0} \mathrm{Hom}_{\mathcal{O}_{S_\lambda}}(f_{\lambda 0}^* F_0, f_{\lambda 0}^* G_0) \to \mathrm{Hom}_{\mathcal{O}_S}(f_0^* F_0, f_0^* G_0)$$

is bijective.

(3) *For an index $0 \in \Lambda$, let $f_0 \colon X_0 \to Y_0$ be a finitely presented morphism of schemes over S_0 and let F_0 be a quasi-coherent sheaf on X_0 of finite presentation. If the pullback of F_0 under $X_0 \times_{S_0} S \to X_0$ is flat over $Y_0 \times_{S_0} S$, then the pullback of F_0 under $X_0 \times_{S_0} S_\lambda \to X_0$ is flat over $Y_0 \times_{S_0} S_\lambda$ for $\lambda \gg 0$.*

Remark B.3.9. In other words, the category of finitely presented modules over S is the colimit of the categories of finitely presented modules over S_λ.

Proof. See [EGA, IV.8.5.2] and [SP, Tags 01ZR, 0B8W, and 05LY]. $\qquad\square$

B.3.4 Application

For a typical application of noetherian approximation in moduli theory, we illustrate here how properties of an arbitrary family of curves can be reduced to a family over a noetherian base.

Proposition B.3.10. *Let S be a quasi-compact and quasi-separated scheme (e.g. an affine scheme), and let $\mathcal{C} \to S$ be a proper flat and finitely presented morphism of schemes such that every geometric fiber has dimension at most 1. Then there exists a cartesian diagram*

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

where S' is a scheme of finite type over $\mathbb{Z}$ and $\mathcal{C}' \to S'$ is a proper flat morphism of schemes such that every geometric fiber has dimension at most 1. Moreover, if $\mathcal{C} \to S$ is smooth, then $\mathcal{C}' \to S'$ can also be arranged to be smooth.

Remark B.3.11. The upshot is that we can now establish properties of the morphism $\mathcal{C}_\lambda \to S_\lambda$ of *noetherian* schemes and then deduce properties of $\mathcal{C} \to S$ by base change. In Lemma 5.2.22, we show if $\mathcal{C} \to S$ is a nodal family of curves, then $\mathcal{C}' \to S'$ can be arranged to be nodal.

Proof. Write $S = \lim_{\lambda \in \Lambda} S_\lambda$ as a limit of an inverse system of schemes of finite type over $\mathbb{Z}$ (Proposition B.3.1). Note that each S_λ is quasi-compact and quasi-separated. Since $\mathcal{C} \to S$ is finitely presented, there exists an index $0 \in \Lambda$ and a finitely presented morphism $\mathcal{C}_0 \to S_0$ such that $\mathcal{C} \cong \mathcal{C}_0 \times_{S_0} S$ (Proposition B.3.4). For each $\lambda > 0$, we can define $\mathcal{C}_\lambda = \mathcal{C}_0 \times_{S_0} S_\lambda$ and we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}_\lambda \\ \downarrow & & \downarrow \\ S & \longrightarrow & S_\lambda. \end{array}$$

Since $\mathcal{C} \to S$ is flat and proper with fiber of dimension at most 1 (resp. smooth), then there exists $\lambda_0 \in I$ such that the same is true for $\mathcal{C}_\lambda \to S_\lambda$ for all $\lambda \geq \lambda_0$ (Proposition B.3.7). We now take $S' = S_\lambda$ and $\mathcal{C}' = \mathcal{C}_\lambda$ for every $\lambda \geq \lambda_0$. $\qquad\square$

B.4   Pushouts of schemes

B.4.1   Existence of pushouts

Pushouts are the dual notion of fiber products. Unlike fiber products, pushouts may not always exist. However, Ferrand showed that they often exist when one of the maps is a closed immersion and the other is an affine morphism.

Theorem B.4.1 (Ferrand Pushout). *Consider a diagram*

$$
\begin{array}{ccc}
X_0 & \overset{i}{\hookrightarrow} & X \\
{\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\
Y_0 & \overset{j}{\hookrightarrow} & Y
\end{array}
\tag{B.4.2}
$$

of schemes where $i\colon X_0 \hookrightarrow X$ is a closed immersion and $f_0\colon X_0 \to Y_0$ is affine. If

($\star$) *for every point $y_0 \in Y_0$, there exists an affine open subscheme $U \subset X$ with $f_0^{-1}(y) \subset U \cap X_0$,*

then there exists a scheme Y, a closed immersion $j\colon Y_0 \hookrightarrow Y$, and an affine morphism $f\colon X \to Y$ of schemes such that (B.4.2) is cocartesian (i.e. Y is identified with the pushout $X \coprod_{X_0} Y_0$). Moreover,

(a) *the square (B.4.2) is cartesian, $X \to Y$ restricts to an isomorphism $X \setminus X_0 \to Y \setminus Y_0$, and $|Y|$ is identified with the pushout $|X| \coprod_{|X_0|} |Y_0|$ as a topological space,*

(b) *the induced map*

$$
\mathcal{O}_Y \to j_*\mathcal{O}_{Y_0} \times_{(j \circ f_0)_*\mathcal{O}_{X_0}} f_*\mathcal{O}_X
$$

is an isomorphism of sheaves, and

(c) *if f_0 is finite, then so is f. In this case, if X_0, X, and Y_0 are locally of finite type over a noetherian scheme, then so is Y.*

Proof. See [Fer03, Thm. 5.4 and 7.1], [Art70, Thm. 6.1], and [SP, Tag 0ECH]. $\square$

Remark B.4.3 (Existence of pushouts in general). Condition ($\star$) does not always hold: for example, consider the diagram $f_0\colon X_0 \to \operatorname{Spec}\Bbbk$ and $i\colon X_0 \hookrightarrow X$, where X is a smooth proper (but not projective) 3-fold X over an algebraically closed field $\Bbbk$ and X_0 is the set of two $\Bbbk$-points not contained in an affine. However, the pushout always exists as an algebraic space and is a pushout in the category of algebraic spaces.

Example B.4.4 (Affine case). In the affine case where $X = \operatorname{Spec} A$, $X_0 = \operatorname{Spec} A_0$, $Y_0 = \operatorname{Spec} B_0$, then $\operatorname{Spec}(A \times_{A_0} B_0)$ is the pushout $X \amalg_{X_0} Y_0$.

Example B.4.5 (Gluing and pinching). If $X_0 \hookrightarrow X$ and $X_0 \hookrightarrow Y_0$ are closed immersions, the pushout $X \amalg_{X_0} Y_0$ can be viewed as the gluing of X and Y_0 along X_0. For example, the nodal curve $\operatorname{Spec} k[x,y]/xy$ is the union of $\mathbb{A}^1$ and $\mathbb{A}^1$ along their origins. If $X_0 = Z \amalg Z$ is the union of two isomorphic disjoint subschemes of X and $X_0 \to Z$ is the projection, then the pushout $X \amalg_{Z \amalg Z} Z$ can be viewed as the pinching of the two copies of Z in X. For example, the nodal cubic curve is the pinching of 0 and ∞ in $\mathbb{P}^1$.

Exercise B.4.6. If X is the scheme-theoretic union of two closed subschemes Z_1 and Z_2, show that $X = Z_1 \amalg_{Z_1 \cap Z_2} Z_2$.

Example B.4.7 (Non-noetherianess)**.** When $f_0\colon X_0 \to Y_0$ is affine but not finite, the pushout $X \amalg_{X_0} Y_0$ is often not noetherian. For example, if $X_0 = V(x) \subset X = \mathbb{A}^2_{\Bbbk}$ and $f_0\colon X_0 \to \operatorname{Spec}\Bbbk$, the pushout is the non-noetherian affine scheme defined by

$$k[x,y] \times_{k[x]} k = k[x, xy, xy^2, xy^3, \ldots] \subset k[x,y].$$

On the other hand, we wouldn't expect a finite type pushout as one cannot contract the y-axis in $\mathbb{A}^2_{\Bbbk}$.

B.4.2 Properties of pushouts

Consider a fiber product diagram of rings

$$
\begin{array}{ccc}
B & \longrightarrow & A \\
\downarrow & & \downarrow \\
B_0 & \longrightarrow & A_0
\end{array}
$$

where $A \twoheadrightarrow A_0$ surjective. The *fiber product* $\operatorname{Mod}(B_0) \times_{\operatorname{Mod}(A_0)} \operatorname{Mod}(A)$ is the category of triples (N_0, M, α) consisting of a B_0-module N_0, an A-module M and isomorphism $\alpha\colon N_0 \otimes_{B_0} A_0 \xrightarrow{\sim} M \otimes_A A_0$. Equivalently, an object is a diagram

where N_0, M_0 and M are modules over B_0, A_0 and A, and the maps $N_0 \to M_0$ and $M \to M_0$ are morphisms of B_0 and A-modules inducing isomorphisms $N_0 \otimes_{B_0} A_0 \to M_0$ and $M \otimes_A A_0 \to M_0$.

Lemma B.4.8. *Let $A \twoheadrightarrow A_0$ be a surjection of rings, $B_0 \to A_0$ be a ring map, and define $B := B_0 \times_{A_0} A$. The natural functor*

$$\operatorname{Mod}(B) \to \operatorname{Mod}(B_0) \times_{\operatorname{Mod}(A_0)} \operatorname{Mod}(A) \tag{B.4.9}$$

restricts to an equivalence on the full subcategories of flat (resp. flat and finitely presented) modules.

Proof. There is a natural functor $\operatorname{Mod}(B_0) \times_{\operatorname{Mod}(A_0)} \operatorname{Mod}(A) \to \operatorname{Mod}(B)$ defined by mapping an object (N_0, M, α) to the fiber product $N_0 \times_{M_0} M$. One proves this a right adjoint to (B.4.9), and that (B.4.9) restricts to equivalences on the full subcategories of flat (resp. flat and finitely presented) modules. See [Fer03, Thm. 2.2], [Sch68, Lem. 3.4], and [SP, Tag 0D2G]. $\qquad\square$

Proposition B.4.10 (Properties of Pushouts)**.** *Consider a commutative cube of schemes*

of schemes where $X_0 \hookrightarrow X$ is a closed immersion and $X_0 \to Y_0$ is affine.

(1) If $Y \cong X \coprod_{X_0} Y_0$, then the natural functor

$$\mathrm{QCoh}(Y') \to \mathrm{QCoh}(Y_0') \times_{\mathrm{QCoh}(X_0')} \mathrm{QCoh}(X'),$$

restricts to an equivalence on the full subcategories of flat (resp. finitely presented and flat $\mathcal{O}$-modules).

(2) Assume that $Y' \to Y$ is flat such that X_0', Y_0', and X' are the base changes. If $Y \cong X \coprod_{X_0} Y_0$, then $Y' \cong X' \coprod_{X_0'} Y_0'$. If $Y' \to Y$ is fppf, the converse is true.

(3) Assume that Y is defined over a scheme S and that $S' \to S$ is an arbitrary morphism of schemes such that Y', X_0', Y_0', and X' are the base changes. If $X_0 \to S$ is flat and $Y \cong X \coprod_{X_0} Y_0$, then $Y' \cong X' \coprod_{X_0'} Y_0'$.

(4) If the top and left faces are cartesian, and the front and back faces are pushouts, then all faces are cartesian. Moreover, if $Y_0' \to Y_0$ and $X' \to X$ are étale (resp. smooth, flat), so is $Y' \to Y$.

Proof. This follows from [Fer03, Thm. 2.2]. □

B.5 Completions, henselizations, and Artin Approximation

B.5.1 Complete local rings

Definition B.5.1.

(1) A local ring $(R, \mathfrak{m})$ is *complete* if the natural map

$$R \to \varprojlim_n R/\mathfrak{m}^n \tag{B.5.2}$$

is an isomorphism.

(2) The *completion* of a local ring $(R, \mathfrak{m})$ is defined as

$$\widehat{R} = \varprojlim_n R/\mathfrak{m}^n.$$

(3) More generally, we say that an R-module M over a local ring $(R, \mathfrak{m})$ is *complete* if $M \to \varprojlim_n M/\mathfrak{m}^n M$ is an isomorphism, and we define the *completion* of M as $\widehat{M} = \varprojlim_n M/\mathfrak{m}^n M$.

Caution B.5.3. The completion of a local ring may not be complete; see [SP, Tag 05JC]. However, if $\mathfrak{m}$ is finitely generated, then $\widehat{R}$ is complete and noetherian [SP, Tag 05GH].

In the literature, 'complete' sometimes only requires the surjectivity of (B.5.2), while 'separated' requires the injectivity, i.e. $\bigcap_n \mathfrak{m}^n = 0$.

The Artin–Rees Lemma plays an important role in establishing basic properties of noetherian complete local rings.

Lemma B.5.4 (Artin–Rees Lemma)**.** *Let R be a noetherian ring, $I \subset R$ an ideal, M a finitely generated A-module, and $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ a stable I-filtration*

(i.e. $IM_n = M_{n+1}$). If $M' \subset M$ is a submodule, then $M' = M' \cap M_0 \supset M' \cap M_1 \supset \cdots$ is a stable I-filtration. In particular, there exists an integer k such that

$$M' \cap (I^n M) = I^{n-k}(M' \cap (I^k M))$$

for all $n \geq k$.

Proof. See [AM69, Prop. 10.9, Cor. 10.10]. $\square$

Proposition B.5.5 (Properties of Noetherian Complete Local Rings)**.** *Let $(R, \mathfrak{m})$ is a noetherian local ring*

   (1) *$\widehat{R}$ is a noetherian complete local ring with maximal ideal $\widehat{\mathfrak{m}} = \widehat{R}\mathfrak{m} = \mathfrak{m} \otimes_R \widehat{R}$;*

   (2) *$\widehat{\mathfrak{m}^n} = \widehat{\mathfrak{m}}^n$ and $\mathfrak{m}^n / \mathfrak{m}^{n+1} = \widehat{\mathfrak{m}}^n / \widehat{\mathfrak{m}}^{n+1}$;*

   (3) *$R \to \widehat{R}$ is faithfully flat; and*

   (4) *For a finitely generated R-module M, $\widehat{M} = M \otimes_R \widehat{R}$.*

Proof. See [AM69, Prop. 10.13–10.16]. $\square$

The following provides a variants of Nakayama's lemma that hold for complete local rings without finite generated hypotheses. Like the usual Nakayama's lemma, there are several useful variants.

Lemma B.5.6 (Complete Nakayma's Lemma)**.**

   (1) *If $(A, \mathfrak{m})$ is a noetherian complete local rings and M is a (possibly not finitely generated) A-module such that $\bigcap_k \mathfrak{m}^k M = 0$ and $m_1, \ldots, m_n \in M$ generate $M/\mathfrak{m}M$, then $m_1, \ldots, m_n$ also generate M.*

   (2) *Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a local homomorphism of noetherian complete local rings. If $A/\mathfrak{m}_A^2 \to B/\mathfrak{m}_B^2$ is surjective, so is $A \to B$. If in addition $A = B$, then $A \to B$ is an isomorphism.*

   (3) *If $(A, \mathfrak{m}_A)$ is a local ring and $M \to N$ is a homomorphism of A-modules such that $M/\mathfrak{m}_A M \to N/\mathfrak{m}_A N$ is surjective, then $\widehat{M} \to \widehat{N}$ is surjective.*

Proof. See [Eis95, Exercise 7.2] and [SP, Tag 0315]. $\square$

Theorem B.5.7 (Cohen Structure Theorem)**.** *Let $(R, \mathfrak{m})$ be a complete local ring.*

   (1) *The ring R contains a coefficient ring, i.e. there is a subring $\Lambda \subset R$ such that Λ is a complete local ring with maximal ideal $\Lambda \cap \mathfrak{m}$, the map $\Lambda/(\Lambda \cap \mathfrak{m}) \to R/\mathfrak{m}$ on residue fields is an isomorphism, and $\Lambda \cap \mathfrak{m} = p\Lambda$ where p is the characteristic of $R/\mathfrak{m}$.*

   (2) *If $\mathfrak{m}$ is finitely generated, then*

$$R \cong \Lambda[\![x_1, \ldots, x_n]\!]/I$$

   where Λ is a field or Cohen ring (i.e. complete DVR with uniformizer p a prime number).

   (3) *If R is noetherian and contains a field (e.g. R is characteristic 0 or $pR = 0$ for $p = \mathrm{char}(R/\mathfrak{m})$), then Λ is field mapping isomorphically to $R/\mathfrak{m}$. Moreover, if R is regular, then $R \cong (R/\mathfrak{m})[\![y_1, \ldots, y_r]\!]$.*

Proof. See [Eis95, Thm. 7.7] and [SP, Tags 032A and 0C0S]. $\square$

B.5.2 Henselian and strictly henselian local rings

Let $(R, \mathfrak{m})$ be a local ring with residue field κ. We will denote the image of $a \in R$ (resp. $f \in R[x]$) as $\overline{a} \in \kappa$ (resp. $\overline{f} \in \kappa[x]$). If $f \in R[t]$, we denote its derivative by $f' \in R[t]$. Note that $\overline{f'} = \overline{f}'$.

Definition B.5.8. Let $(R, \mathfrak{m})$ be a local ring with residue field κ.

(1) We say that R is *henselian* if for a monic polynomial $f \in R[t]$, every root $\alpha_0 \in \kappa$ of $\overline{f}$ with $\overline{f'}(\alpha_0) \neq 0$ lifts to a root $\alpha \in R$ of f.

(2) We say that R is *strictly henselian* if R is henselian and κ is separably closed.

Remark B.5.9. Hensel's lemma states that a complete DVR R (e.g. $\mathbb{Z}_p$) is henselian.

Proposition B.5.10 (Henselian Equivalences)**.** *The following are equivalent for a local ring $(R, \mathfrak{m})$ with residue field κ:*

(1) R is henselian;

(2) for a polynomial $f \in R[t]$, every factorization $\overline{f} = g_0 h_0$ with $\gcd(g_0, h_0) = 1$ lifts to a factorization $f = gh$ with $\overline{g} = g_0$ and $\overline{h} = h_0$;

(3) every finite R-algebra is a finite product of local rings finite over R;

(4) every quasi-finite R-algebra A is isomorphic to a product $A \cong B \times C$ where B is a finite over R and $C \otimes_R \kappa = 0$;

(5) every étale ring homomorphism $\phi \colon R \to A$ and a prime $\mathfrak{p} \subset A$ with $\phi^{-1}(\mathfrak{p}) = \mathfrak{m}$ and $\kappa = \kappa(\mathfrak{p})$ has a unique section $s \colon A \to R$ with $s^{-1}(\mathfrak{m}) = \mathfrak{p}$.

Moreover R is strictly henselian if and only if for every étale ring homomorphism $\phi \colon R \to A$ and prime $\mathfrak{p} \subset A$ with $\phi^{-1}(\mathfrak{p}) = \mathfrak{m}$, there is a unique section $s \colon A \to R$ with $s^{-1}(\mathfrak{m}) = \mathfrak{p}$.

Proof. See [EGA, IV.18.5.11], [Mil80, Thm. I.4.2] and [SP, Tag 04GG]. $\qquad\square$

Remark B.5.11. The following stronger version of (4) holds for a henselian ring R: for every quasi-finite and separated morphism $X \to \operatorname{Spec} R$ of schemes, $X \cong F \amalg W$ with F finite over R and $W \times_R \kappa = \emptyset$. This is a reformulation of Étale Localization of Quasi-finite Morphisms (Theorem A.7.1); see Exercise B.5.17.

Remark B.5.12. Both property (5) and the analogous property of strictly henselian rings generalize to étale morphisms $X \to \operatorname{Spec} A$ of schemes: every section over $\operatorname{Spec} A/\mathfrak{m}$

extends to a global section.

Proposition B.5.13. *Let $(R, \mathfrak{m})$ be a henselian (resp. strictly henselian) local ring with residue field κ.*

(1) Every finite R-algebra is a product of finite henselian local (resp. strictly henselian) R-algebras.

(2) Every complete local ring is henselian.

(3) The functor $A \mapsto A \otimes_R \kappa$ gives an equivalence of categories between finite étale A-algebras and finite étale κ-algebras.

Proof. See [EGA, IV.18.5.10, IV.18.5.14-15], [Mil80, 4.3-4.5] and [SP, Tag 04GE]. □

Remark B.5.14. The more general notion of henselian pairs is sometimes useful (although it won't be used in these notes). A pair (X, X_0) consisting of a scheme X and a closed subscheme $X_0 \subset X$ is *henselian* if every finite morphism $f \colon U \to X$ induces a bijection $\mathrm{ClOpen}(U) \to \mathrm{ClOpen}(f^{-1}(X_0))$ between open and closed subschemes of U and those of $f^{-1}(X_0)$. If $(R, \mathfrak{m})$ is a henselian local ring, then $(\mathrm{Spec}\, R, \mathrm{Spec}(R/\mathfrak{m}))$ is a henselian pair by Proposition B.5.10(3). See [EGA, IV.18.5.5] or [SP, Tag 09XD] for further discussion and equivalences.

B.5.3 Henselizations and strict henselizations

Definition B.5.15. Let $(R, \mathfrak{m})$ be a local ring with residue field κ. The *henselization of R* is a local homomorphism $R \to R^{\mathrm{h}}$ into a henselian local ring R^{h} such that every other local homomorphism $R \to A$ into a henselian local ring factors uniquely through $R \to R^{\mathrm{h}}$.

Given a separable closure $\kappa \to \kappa^s$, the *strict henselization of R with respect to $\kappa \to \kappa^s$* is a local homomorphism $R \to R^{\mathrm{sh}}$ into a strictly henselian local ring $(R^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$ inducing $\kappa \to \kappa^s$ on residue fields such that every other local homomorphism $R \to A$ into a strictly henselian local ring $(A, \mathfrak{m}_A)$ factors through $R \to R^{\mathrm{sh}}$ and the factorization is uniquely determined by the inclusion $R^{\mathrm{sh}}/\mathfrak{m}^{\mathrm{sh}} \to A/\mathfrak{m}_A$ of residue fields.

Proposition B.5.16. *Let $(R, \mathfrak{m})$ be a local ring with residue field κ and let $\kappa \to \kappa^s$ be a separable closure. The henselization $R \to R^{\mathrm{h}}$ and strict henselization $R \to R^{\mathrm{sh}}$ exist and can be constructed as*

$$R^{\mathrm{h}} = \underset{(R,\mathfrak{m}) \xrightarrow{\text{ét}} (A,\mathfrak{m}_A),\ \kappa = A/\mathfrak{m}_A}{\mathrm{colim}} A$$

$$R^{\mathrm{sh}} = \underset{\substack{(R,\mathfrak{m}) \xrightarrow{\text{ét}} (A,\mathfrak{m}_A) \\ \downarrow \swarrow \\ \kappa^s}}{\mathrm{colim}} A$$

where the colimits are taken over the directed system of étale R-algebras A and maximal ideals $\mathfrak{m}_A$ over $\mathfrak{m}$; in the henselian case, we require that $\kappa = R/\mathfrak{m}_A$ while in the strictly henselian case, the data includes a homomorphism $A \to \kappa^s$ over R. Moreover,

(a) the residue fields of R^{h} and R^{sh} are κ and κ^s, respectively;

(b) the maps $R \to R^{\mathrm{h}}$ and $R \to R^{\mathrm{sh}}$ are faithfully flat; and

(c) if R is noetherian, then so is R^{h} and R^{sh}.

Proof. See [EGA, IV.18.5-8], [Mil80, I.4] and [SP, Tags 0BSK and 07QL]. □

For a scheme X and a point $x \in X$ with a choice of separable closure $\kappa(x) \to \kappa^s$, the henselization $\mathcal{O}_{X,x}^{\mathrm{h}}$ and strict henselization $\mathcal{O}_{X,x}^{\mathrm{h}}$ are the colimits of $\Gamma(U, \mathcal{O}_U)$ taken over diagrams

respectively, where $U \to X$ is an étale morphism of schemes and $\operatorname{Spec} \kappa^s \to U$ is a lift of $\operatorname{Spec} \kappa^s \to \operatorname{Spec} \kappa(x) \xrightarrow{x} X$. Both $\mathcal{O}_{X,x}^{\mathrm{h}}$ and $\mathcal{O}_{X,x}^{\mathrm{sh}}$ can be thought of as local rings in étale topology.

Exercise B.5.17. Show that Étale Localization of Quasi-finite Morphisms (Theorem A.7.1) follows from the case when S is the spectrum of a henselian ring (see Remark B.5.11).

Hint: Use limit methods (Propositions B.3.4 and B.3.7) to extend a decomposition $X \times_S \operatorname{Spec} \mathcal{O}_{S,s}^{\mathrm{h}} \cong F^{\mathrm{h}} \amalg W^{\mathrm{h}}$ to an étale neighborhood of s.

B.5.4 Artin Approximation

We now turn to the deep result of Artin Approximation (Theorem B.5.26) which can be vaguely expressed as the following principle:

algebraic properties that hold for the completion $\widehat{\mathcal{O}}_{S,s}$ of the local ring of a scheme S at a point s also hold in an étale neighborhood $(S', s') \to (S, s)$.

Artin Approximation is related to another equally deep and powerful result known as Néron–Popescu Desingularization (Theorem B.5.21). Both Artin Approximation and Néron–Popescu are difficult theorems that we will not attempt to prove here. However, we will show at least how Artin Approximation easily follows from Néron–Popescu Desingularization.

B.5.5 Néron–Popescu Desingularization

Definition B.5.18. A ring homomorphism $A \to B$ of noetherian rings is called *geometrically regular* if $A \to B$ is flat and for every prime ideal $\mathfrak{p} \subset A$ and every finite field extension $\kappa(\mathfrak{p}) \to \kappa'$ (where $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}$), the fiber $B \otimes_A \kappa'$ is regular.

Remark B.5.19. It is important to note that $A \to B$ is *not* assumed to be of finite type. If it is, then $A \to B$ is geometrically regular if and only if $A \to B$ is smooth.

Remark B.5.20. It can be shown that it is equivalent to require that each geometric fiber $B \otimes_A \overline{\kappa(\mathfrak{p})}$ is regular, or equivalently that $B \otimes_A \kappa'$ is regular for every *inseparable* finite field extensions $\kappa(\mathfrak{p}) \to \kappa'$. In particular, in characteristic 0, $A \to B$ is geometrically regular if it is flat, and every fiber $B \otimes_A \kappa(\mathfrak{p})$ is regular.

Theorem B.5.21 (Néron–Popescu Desingularization). *A homomorphism $A \to B$ of noetherian rings is geometrically regular if and only if there is a directed system B_λ of smooth A-algebras over a directed set Λ such that $B = \operatorname{colim} B_{\lambda \in \Lambda}$.*

Proof. This result was proved by Néron in [Nér64] in the case that A and B are DVRs and in general by Popescu in [Pop85], [Pop86] and [Pop90]. We recommend [Swa98] and [SP, Tag 07GC] for an exposition of this result. $\square$

Example B.5.22. A field extension $k \to l$ is geometrically regular if and only if it is separable. When $k \to l$ is algebraic, l is the colimit of separable finite (i.e. étale) extensions.

Definition B.5.23. A noetherian local ring A is called a *G-ring* if the homomorphism $A \to \widehat{A}$ is geometrically regular.

Remark B.5.24. One of the defining features of excellent schemes is that their local rings are G-rings.

To apply Néron–Popescu Desingularization, we will need that local rings of schemes are frequently G-rings.

Theorem B.5.25. *If A is the localization of a finitely generated algebra over a field or $\mathbb{Z}$, then A is a G-ring.*

Proof. While substantially easier than Néron–Popescu Desingularization, this result nevertheless requires some effort. See [EGA, IV.7.4.4] or [SP, Tag 07PX]. $\qquad\square$

B.5.6   Artin Approximation

Let S be a scheme and consider a contravariant functor

$$F \colon \mathrm{Sch}/S \to \mathrm{Sets}$$

where Sch/S denotes the category of schemes over S. We say that F is *limit preserving* (or *locally of finite presentation*) if for system of $\mathcal{O}_S$-algebras A_λ (i.e. each $\mathrm{Spec}\, A_\lambda$ is an S-scheme), the natural map

$$\mathrm{colim}\, F(B_\lambda) \to F(\mathrm{colim}\, B_\lambda)$$

is bijective. When F is a functor $\mathrm{Mor}_S(-, X)$ representable by a scheme X over S, then this equivalent to $X \to S$ being locally of finite presentation (Proposition A.1.1).

Theorem B.5.26 (Artin Approximation)**.** *Let S be a scheme and $s \in S$ a point such that $\mathcal{O}_{S,s}$ is a G-ring (Definition B.5.23), e.g. a scheme of finite type over a field or $\mathbb{Z}$. Let*

$$F \colon \mathrm{Sch}/S \to \mathrm{Sets}$$

be a limit preserving contravariant functor and $\widehat{\xi} \in F(\mathrm{Spec}\, \widehat{\mathcal{O}}_{S,s})$. For every integer $N \geq 0$, there exists an étale morphism

$$(S', s') \to (S, s) \qquad and \qquad \xi' \in F(S')$$

with $\kappa(s) = \kappa(s')$ such that the restrictions of $\widehat{\xi}$ and ξ' to $\mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.

Remark B.5.27. To make sense of the restriction ξ' to $\mathrm{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$, note that since $(S', s') \to (S, s)$ is a residually-trivial étale morphism, there are compatible identifications $\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \cong \mathcal{O}_{S',s'}/\mathfrak{m}_{s'}^{N+1}$.

Remark B.5.28. It is not possible in general to find $\xi' \in F(S')$ restricting to $\widehat{\xi}$ or even such that the restrictions of ξ' and $\widehat{\xi}$ to $\mathrm{Spec}\, \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$ agree for all $n \geq 0$. For instance, F could be the functor $\mathrm{Mor}(-, \mathbb{A}^1)$ representing the affine line $\mathbb{A}^1$ and $\widehat{\xi} \in \widehat{\mathcal{O}}_{S,s}$ could be a non-algebraic power series.

Proof. The theorem was originally proven in [Art69a, Cor. 2.2] in the case that S is of finite type over a field or an excellent dedekind domain. We also recommend [BLR90, §3.6] for an accessible account of the case of excellent and henselian DVRs. We will show how Artin Approximation follows from Néron–Popescu Desingularization (Theorem B.5.21).

Néron–Popescu Desingularization implies that $\widehat{\mathcal{O}}_{S,s} = \mathrm{colim}_{\lambda \in \Lambda} B_\lambda$ is a directed colimit of smooth $\mathcal{O}_{S,s}$-algebras. Since F is limit preserving, there exists $\lambda \in \Lambda$, a

factorization $\mathcal{O}_{S,s} \to B_\lambda \to \widehat{\mathcal{O}}_{S,s}$ and an element $\xi_\lambda \in F(\mathrm{Spec}\, B_\lambda)$ whose restriction to $F(\mathrm{Spec}\, \widehat{\mathcal{O}}_{S,s})$ is $\widehat{\xi}$. Letting $B = B_\lambda$ and $\xi = \xi_\lambda$, we have a commutative diagram

$$
\begin{array}{ccccc}
& & \overset{\widehat{\xi}}{\overbrace{}} & & \\
\mathrm{Spec}\,\widehat{\mathcal{O}}_{S,s} & \overset{g}{\longrightarrow} & \mathrm{Spec}\, B & \overset{\xi}{\longrightarrow} & F \\
& \searrow & \downarrow & & \\
& & \mathrm{Spec}\,\mathcal{O}_{S,s} & &
\end{array}
$$

where $\mathrm{Spec}\, B \to \mathrm{Spec}\,\mathcal{O}_{S,s}$ is smooth. We claim that we can find a commutative diagram

$$
\begin{array}{ccc}
S' & \hookrightarrow & \mathrm{Spec}\, B \\
& \searrow & \downarrow \\
& & \mathrm{Spec}\,\mathcal{O}_{S,s}
\end{array}
\tag{B.5.29}
$$

where $S' \hookrightarrow \mathrm{Spec}\, B$ is a closed immersion, $(S', s') \to (\mathrm{Spec}\,\mathcal{O}_{S,s}, s)$ is étale, and the composition $\mathrm{Spec}\,\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1} \to S' \to \mathrm{Spec}\, B$ agrees with the restriction of $g\colon \mathrm{Spec}\,\widehat{\mathcal{O}}_{S,s} \to \mathrm{Spec}\, B.$[4] To see this, since $\Omega_{B/\mathcal{O}_{S,s}}$ is a locally free B-module, after replacing $\mathrm{Spec}\, B$ with an affine open neighborhood of $g(s)$, we may assume that $\Omega_{B/\mathcal{O}_{S,s}}$ is free with basis $db_1, \ldots, db_n$. This induces a homomorphism $\mathcal{O}_{S,s}[x_1, \ldots, x_n] \to B$ defined by $x_i \mapsto b_i$ and provides a factorization

$$
\mathrm{Spec}\, B \to \mathbb{A}^n_{\mathcal{O}_{S,s}} \to \mathrm{Spec}\,\mathcal{O}_{S,s}
$$

where $\mathrm{Spec}\, B \to \mathbb{A}^n_{\mathcal{O}_{S,s}}$ is étale. Choosing a lift of the composition

$$
\begin{array}{ccccccc}
& & & & & & \mathcal{O}_{S,s} \\
& & & & & \nearrow & \downarrow \\
\mathcal{O}_{S,s}[x_1, \ldots, x_n] & \longrightarrow & B & \longrightarrow & \widehat{\mathcal{O}}_{S,s} & \longrightarrow & \mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1}
\end{array}
$$

defines a section $s\colon \mathrm{Spec}\,\mathcal{O}_{S,s} \to \mathbb{A}^n_{\mathcal{O}_{S,s}}$ and we define S' as the fibered product

$$
\begin{array}{ccc}
S' & \longrightarrow & \mathrm{Spec}\,\mathcal{O}_{S,s} \\
\uparrow & \square & \uparrow{\scriptstyle s} \\
\mathrm{Spec}\, B & \longrightarrow & \mathbb{A}^n_{\mathcal{O}_{S,s}}.
\end{array}
$$

This gives the desired diagram (B.5.29), and the composition $\xi'\colon S' \to \mathrm{Spec}\, B \overset{\xi}{\to} F$ is an element which agrees with $\widehat{\xi}$ up to order N.

Finally, we must explain how to "smear out" the étale morphism $(S', s') \to (\mathrm{Spec}\,\mathcal{O}_{S,s}, s)$ and the element $\xi' \in F(S')$ to an étale morphism $(S'', s'') \to (S, s)$ and an element $\xi'' \in F(S'')$. Writing $\mathcal{O}_{S,s} = \mathrm{colim}_{g \notin \mathfrak{m}_s} A_g$, we may apply Propositions B.3.3, B.3.4 and B.3.7 (or a direct argument) to find an element $g \notin \mathfrak{m}_s$ and an

---

[4]This is where the approximation occurs. It is not possible to find a morphism $S' \to \mathrm{Spec}\, B \to \mathrm{Spec}\,\mathcal{O}_{S,s}$ which is étale at a point s' over s such that the composition $\mathrm{Spec}\,\widehat{\mathcal{O}}_{S,s} \to S' \to \mathrm{Spec}\, B$ is equal to g.

affine scheme $S'' = \operatorname{Spec}(A_g[y_1, \ldots, y_n]/(f_1'', \ldots, f_m''))$ such that $S'' \times_{A_g} A_{\mathfrak{m}_s} \cong S'$ and such that $S'' \to \operatorname{Spec} A_g$ is étale. As F is limit preserving and $\Gamma(S'', \mathcal{O}_{S''}) = \operatorname{colim}_{h \notin \mathfrak{m}_s} A_{hg}[y_1, \ldots, y_n]/(f_1', \ldots, f_m')$, after replacing g with hg, we can find an element $\xi'' \in F(S'')$ restricting to ξ' and, in particular, agreeing with $\widehat{\xi}$ up to order N. $\qquad\square$

Exercise B.5.30 (Alternative formulations). Let $(A, \mathfrak{m})$ be a henselian local G-ring.

(1) Let $F = \operatorname{Hom}(-, X) \colon \operatorname{Sch}/S \to \operatorname{Sets}$ be the contravariant functor of an affine scheme $X = \operatorname{Spec} A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ of finite type over A. Note that for an A-algebra B,

$$F(B) = \{a = (a_1, \ldots, a_n) \in B^{\oplus n} \mid f_i(a) = 0 \text{ for all } i\}.$$

If $\widehat{a} = (\widehat{a}_1, \ldots, \widehat{a}_n) \in \widehat{A}_{\mathfrak{m}}$ is a solution to the equations $f_1(x) = \cdots = f_m(x) = 0$, show that Artin Approximation implies that for every $N \geq 0$, there is a solution $a = (a_1, \ldots, a_n) \in A^{\oplus n}$ to the equations $f_1(x) = \cdots = f_m(x) = 0$ such that $a \cong \widehat{a} \mod \mathfrak{m}^{N+1}$.

(2) Show that (1) implies Artin Approximation.

Hint: Use that F is limit preserving to find a finitely generated A-subalgebra $B \subset \widehat{\mathcal{O}}_{S,s}$ and an element $\xi \in F(B)$ restricting to $\widehat{\xi}$.

B.5.7   A first application of Artin Approximation

The next corollary states an important fact you may have taken for granted: if two schemes are formally isomorphic at two points, they are isomorphic in the étale topology.

Corollary B.5.31. *Let X and Y be schemes of finite type over a scheme S and let $s \in S$ be a point such that $\mathcal{O}_{S,s}$ is a G-ring. If $x \in X$ and $y \in Y$ are points over s such that $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{Y,y}$ are isomorphic as $\mathcal{O}_S$-algebras, then there exists étale morphisms*

$$\begin{array}{ccc} & (U, u) & \\ & \swarrow \qquad \searrow & \\ (X, x) & & (Y, y) \end{array} \qquad\qquad \text{(B.5.32)}$$

inducing isomorphisms $\kappa(x) \xrightarrow{\sim} \kappa(u)$ and $\kappa(y) \xrightarrow{\sim} \kappa(u)$ on residue fields.

Proof. The functor

$$F \colon \operatorname{Sch}/X \to \operatorname{Sets}, \quad (T \to X) \mapsto \operatorname{Mor}(T, Y)$$

is limit preserving as it can be identified with the representable functor $\operatorname{Mor}_X(-, Y \times X)$ corresponding to the finite type morphism $Y \times X \to X$. The isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{Y,y}$ provides an element of $F(\operatorname{Spec} \widehat{\mathcal{O}}_{X,x})$. By applying Artin Approximation with $N = 1$, we obtain a diagram as in (B.5.32) with $U \to X$ étale at u with $\kappa(x) \xrightarrow{\sim} \kappa(u)$ and such that $\mathcal{O}_{Y,y}/\mathfrak{m}_y^2 \to \mathcal{O}_{U,u}/\mathfrak{m}_u^2$ is an isomorphism. Since $\widehat{\mathcal{O}}_{U,u}$ is abstractly isomorphic to $\widehat{\mathcal{O}}_{Y,y}$, Complete Nakayama's Lemma (B.5.6(2)) implies that $\widehat{\mathcal{O}}_{Y,y} \to \widehat{\mathcal{O}}_{U,u}$ is an isomorphism and therefore that $(U, u) \to (Y, y)$ is étale. $\qquad\square$

Remark B.5.33. If $\phi\colon \widehat{\mathcal{O}}_{Y,y} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$ is the specified isomorphism, it is not always possible to arrange that the induced diagram

$$
\begin{array}{ccc}
 & \widehat{\mathcal{O}}_{U,u} & \\
 \nearrow & & \nwarrow \\
\widehat{\mathcal{O}}_{X,x} & \xleftarrow{\quad\phi\quad} & \widehat{\mathcal{O}}_{Y,y}
\end{array}
$$

is commutative, but the proof (using Artin Approximation with a given $N \geq 1$) shows that we can arrange that the diagram commutes modulo $\mathfrak{m}_y^{N+1}$. See also [SP, Tag 0CAV].

B.6 Birational geometry and positivity

We summarize some of the basic facts of birational geometry that are needed in our development of moduli theory. The birational geometry of surfaces in §B.6.1 is used crucially in Stable Reduction (Theorem 5.5.1) for $\overline{M}_g$, and the ampleness criteria in §B.6.3, the properties of nef vector bundles in §B.6.4, and the vanishing theorems in §B.6.5 are used to prove Kollár's Criterion (5.8.7), which we apply to prove that $\overline{M}_g$ is projective.

B.6.1 Birational geometry of surfaces

For an integral noetherian scheme X, a *resolution of singularities* is a proper birational morphism $X' \to X$ from an integral regular scheme X'.

Theorem B.6.1 (Existence of Resolutions)**.** *Every two dimensional integral noetherian scheme X has a resolution of singularities.*

Proof. This was first show by Zariski in characteristic 0 [Zar39], by Abhyankar in characteristic p [Abh56], and by Lipman in mixed characteristic [Lip78]. See also [Kol07, §2] and [SP, Tag 0BGP]. $\square$

Theorem B.6.2 (Existence of Minimal Resolutions)**.** *Let X be a two dimensional integral noetherian scheme. There exists a resolution of singularities $\pi\colon \widetilde{X} \to X$ such that every other resolution of singularities $Y \to X$ factors as $Y \to \widetilde{X} \to X$. Moreover, $K_{\widetilde{X}} \cdot E \geq 0$ for every π-exceptional curve E.*

Proof. See [Kol07, Thm. 2.16]. $\square$

Theorem B.6.3 (Existence of Embedded Resolutions)**.** *Let X be a regular scheme of dimension 2 and $Y \subset X$ be a subscheme of pure dimension one. Assume that for every irreducible component $Z \subset X_0$, the normalization $\widetilde{Z} \to Z$ is finite. Then there is a finite sequence of blowups*

$$X_n \to \cdots \to X_1 \to X$$

at reduced closed points such that the preimage Y_n of Y in X_n X_0 is an effective Cartier divisor supported on a normal crossings divisor, i.e. $(Y_n)_{\mathrm{red}}$ is nodal.

Proof. See [Har77, Thm V.3.9], [Kol07, Thm. 1.47], and [SP, Tag 0BIC]. $\square$

Theorem B.6.4 (Factorization of Birational Maps)**.** *Let X and Y be regular, integral, and noetherian schemes of dimension two. Every proper birational morphism $f\colon X \to Y$ is the composition of blowups at closed points.*

Proof. See [Har77, Thm V.5.5], [Kol07, Thm 2.13], and [SP, Tag 0C5R]. $\square$

Theorem B.6.5 (Hodge Index Theorem for Exceptional Curves)**.** *Let $f\colon X \to Y$ be a projective and generically finite morphism of noetherian schemes of dimension 2, where X is regular and Y is quasi-projective over a field or DVR. Let $E_1, \ldots, E_n$ be the exceptional curves. Then the intersection form matrix $(E_i \cdot E_j)$ is negative-definite. In particular, $E_i^2 < 0$ for each i.*

Proof. See [Kol07, Thm 2.12]. $\square$

Theorem B.6.6 (Castelnuovo's Contraction Theorem)**.** *Let X be a regular scheme of dimension 2 which is projective over either a field $\Bbbk$ or a DVR R with residue field $\Bbbk$, and let $E = \mathbb{P}^1_{\Bbbk} \subset X$ be a smooth rational curve with $E^2 < 0$. Then there is a projective morphism $X \to Y$ to a projective surface and a point $y \in Y$ such that $f^{-1}(y) = E$ and $X \setminus E \to Y \setminus \{y\}$ is an isomorphism. If $E^2 = -1$, then Y is smooth.*

Proof. See [Har77, Thm. V.5.7, Exer. V.5.2] and [Kol07, Thm. 2.14, Rmk. 2.15]. $\square$

One can show that the process of repeatedly contracting smooth rational -1 curves in a smooth projective surface terminates (see [Har77, Thm 5.8]). Thus by applying Castelnuovo's Contractibility Criterion a finite number of times, one obtains:

Corollary B.6.7 (Existence of Minimal Models)**.** *A smooth surface X admits a projective birational morphism $X \to X_{\min}$ to a smooth surface such that every projective birational morphism $X_{\min} \to Y$ to a smooth surface is an isomorphism. In particular, $X_{\min}$ has no smooth rational -1 curves.*

B.6.2 Positivity

We discuss positivity properties of line bundles, some of which are extended to algebraic spaces in §5.8.1. A good reference for this material is [Laz04a, Laz04b].

Ample line bundles. Let X be a proper scheme over a field $\Bbbk$. A line bundle L on X is *ample* if for some $m > 0$, $L^{\otimes m}$ is very ample, i.e. defines a closed embedding $|L^{\otimes m}|\colon X \hookrightarrow \mathbb{P}^N$ into projective space.

Proposition B.6.8 (Characterizations of Ampleness)**.** *Let X be a proper scheme over a field $\Bbbk$. For a line bundle L on X, the following are equivalent:*

(1) L is ample;

(2) for every $x \in X$, there exists a section $s \in \Gamma(X, L)$ such that $X_s = \{s \neq 0\}$ is affine and contains x;

(3) for every coherent sheaf F, the tensor product $F \otimes L^{\otimes m}$ is base point free for $m \gg 0$; and

(4) for every coherent sheaf F on X, $\mathrm{H}^i(X, F \otimes L^{\otimes m}) = 0$ for $m \gg 0$ and $i > 0$.

Proof. See [Har77, §II.7 and III.5.3], [EGA, II.4.5 and III.2.6], or [SP, Tags 01PR and 0B5U]. $\square$

Proposition B.6.9 (Properities of Ampleness)**.** *Let X be a proper scheme over a field $\Bbbk$ and L be a line bundle on X.*

(1) If $f\colon X' \to X$ is a finite surjective morphism, L is ample if and only if f^*L is.

(2) For a field extension $\Bbbk \to \Bbbk'$, L is ample on X if and only if $L_{\Bbbk'}$ is ample on $X_{\Bbbk'}$.

Proof. Both follow from the cohomological characteriziation of ampleness. For (1), see [Har77, Exer III.5.7], [EGA, III.2.6.2], and [SP, Tag 0B5V]. For prop:properties-of-ampleness-2, the ($\Rightarrow$) direction is clear. For the converse, it suffices to verify for every coherent sheaf F that $\mathrm{H}^i(X, F \otimes L^{\otimes m}) = 0$ for $m \gg 0$ and $i > 0$. By Flat Base Change (A.2.12), $\mathrm{H}^i(X, F \otimes L^{\otimes m}) \otimes_{\Bbbk} \Bbbk' = \mathrm{H}^i(X_{\Bbbk'}, F_{\Bbbk'} \otimes L_{\Bbbk'}^{\otimes m})$ vanishes for $m \gg 0$ and thus so does $\mathrm{H}^i(X, F \otimes L^{\otimes m})$. See also [SP, Tag 0D3C]. $\square$

Part (2) implies that a line bundle L on X is ample if and only if its restriction $L_{(X_i)_{\mathrm{red}}}$ to the reduced subscheme of each irreducible component X_i is ample.

Proposition B.6.10 (Openness of Ampleness)**.** *Let $f\colon X \to S$ be a proper, flat, and finitely presented morphism of schemes, and L be a line bundle on X. If for some $s \in S$, the restriction L_s of L to the fiber X_s is ample (resp. very ample and $\mathrm{H}^i(X_s, L_s) = 0$ for $i > 0$), then there exists an open neighborhood $U \subset S$ of s such that the restriction L_U on X_U is relatively ample (resp. relatively very ample) over U. In particular, for all $u \in U$, L_u is ample (resp. very ample) on X_u.*

Proof. If L_s is ample on X_s, then for $n \gg 0$, $L_s^{\otimes n}$ is very ample and $\mathrm{H}^i(X_s, L_s^{\otimes n}) = 0$ for $i > 0$. It therefore suffices to handle the very ample case. By Cohomology and Base Change (A.6.8), after replacing S with an open neighborhood of s, f_*L is a vector bundle and the comparison map $f_*L \otimes \kappa(t) \to \mathrm{H}^i(X_t, L_t)$ is an isomorphism for $t \in S$. By further replacing S with an affine open neighborhood, we can arrange that $\mathrm{H}^0(X, L)$ is freely generated by sections $t_0, \ldots, t_n$ that restrict to a basis in $\mathrm{H}^0(X_s, L_s)$. The vanishing locus $V := V(t_0, \ldots, t_n) \subset X$ is closed and disjoint from X_s. By replacing S with an affine open neighborhood of s contained in $S \setminus f(V)$, we may assume that the sections t_i generate L and that they define a morphism $g\colon X \to \mathbb{P}_S^n$ over S restricting to a closed immersion $g_s\colon X_s \hookrightarrow \mathbb{P}_{\kappa(s)}^n$. By upper semi-continuity of fiber dimension, there is a closed locus $Z \subset \mathbb{P}_S^n$ consisting of points z such that $\dim g^{-1}(z) > 0$. Since Z is disjoint from $\mathbb{P}_{\kappa(s)}^n$, we may shrink S further so that $g\colon X \to \mathbb{P}_S^n$ is quasi-finite, and hence finite (as g is proper). The cokernel of $\mathcal{O}_{\mathbb{P}_S^n} \to g_*\mathcal{O}_X$ is coherent and its support is a closed subscheme of $\mathbb{P}^n$ disjoint from $\mathbb{P}_{\kappa(s)}^n$. By shrinking S further, we may arrange that $g\colon X \to \mathbb{P}_S^n$ is a closed immersion and hence $L = g^*\mathcal{O}_{\mathbb{P}_S^n}(1)$ is very ample. See also [Laz04a, Thm. 1.2.17, Thm. 1.7.8], [EGA, III$_1$.4.7.1, IV$_3$.9.6.4], [KM98, Prop. 1.41] and [SP, Tag 0D3D]; the openness of ampleness holds without the flatness of $X \to S$. $\square$

Example B.6.11 (suggested by Brian Nugent)**.** It is not true that very ampleness is an open condition. If C is a non-hyperelliptic curve of genus 3, then K_C is very ample and we can write $K_C = \mathcal{O}_C(p_1 + \cdots + p_4)$. Considering the constant family $C \times C \to C$ with the constant sections p_1, p_2, and p_3 and the diagonal section Δ, then the fiber of $\mathcal{O}_{C \times C}(p_1 + p_2 + p_3 + \Delta)$ is very ample over $p_4 \in C$ but the fiber over a point $s \in C$ near to p_4 is not very ample.

Nef line bundles. A line bundle L on a proper scheme X over a field $\Bbbk$ is *nef* (or *numerically effective*) if

$$\int_C c_1(L) \geq 0$$

for every integral proper curve $C \subset X$. Here $\int_C c_1(L)$ is the same as $c_1(L) \cdot C$ (or simply $L \cdot C$) or $\deg L|_C$.

Theorem B.6.12 (Kleiman's Theorem). *If L is a line bundle on a proper scheme X over a field $\Bbbk$, then L is nef if and only if for every integral subscheme $Z \subset X$ of dimension k,*

$$\int_Z c_1(L)^k \geq 0.$$

Proof. See [Laz04a, Thm. 1.4.9], [Kol96, Thm. 2.17], or the original source [Kle66]. $\square$

It is often convenient to write line bundles in additive notation, so that $mL + H$ corresponds to $L^{\otimes m} \otimes H$.

Corollary B.6.13 (Characterization of Nefness). *Let X be a projective scheme over a field $\Bbbk$ and H be an ample line bundle. A line bundle L on X is nef if and only if $mL + H$ is ample for $m \gg 0$.*

Proof. See [Laz04a, Cor. 1.4.10]. $\square$

Proposition B.6.14 (Properties of Nefness). *Let X be a proper scheme over a field $\Bbbk$ and L be a line bundle on X.*

(1) *If $f \colon X' \to X$ is a surjective proper morphism, then L is nef if and only if f^*L is.*

(2) *For a field extension $\Bbbk \to \Bbbk'$, L is nef on X if and only if $L_{\Bbbk'}$ is nef on $X_{\Bbbk'}$.*

Proof. For (1), if $C' \subset X'$ is an integral curve and d is the degree of the induced map $C' \to f(C')$, then

$$f^*L \cdot C' = d(L \cdot f(C')) \tag{B.6.15}$$

by the projection formula. This gives the ($\Rightarrow$) implication. Conversely, if $C \subset X$ is an integral curve, we may choose an integral curve $C' \subset X'$ with $C = f(C')$, and thus (B.6.15) also implies the ($\Leftarrow$) implication. For (2), by Chow's Lemma and (1), we may assume that X is projective. In this case, the Characterization of Nefness (B.6.13) reduces us to the corresponding statement for ampleness (Proposition B.6.9(2)). $\square$

Proposition B.6.16 (Nefness is Stable under Generization). *Let X be a proper flat scheme over a DVR R and L be a line bundle on X. If the restriction L_0 of L to the central fiber X_0 is nef, then so is the restriction L_η to the generic fiber X_η.*

Proof. By Chow's Lemma and Item (1)(1), we may assume that X is projective with an ample line bundle H. By the Characterization of Nefness (B.6.13), $mL_0 + H_0$ is ample for $m \gg 0$. By Openness of Ampleness (B.6.10), $mL + H$ is ample, and thus so is $mL_\eta + H_\eta$. By applying again the Characterization of Nefness, we conclude that L_η is nef. $\square$

Remark B.6.17. For a surjective proper morphism $X \to S$ of varieties and a line bundle L on X whose fiber L_s over $s \in S$ is nef, there exists a countable union $B \subset S$ of proper subschemes not containing s such that L_t is nef for every $t \in S \setminus B$ [Laz04a, Prop 1.4.14]. It is not true that nefness is open in general; see [Lan13, Ex. 5.3] and [Les14, Thm. 1.2].

Remark B.6.18 (Ample and nef cones)**.** It's also worthwhile to keep in mind that ample and nef line bundles generate cones $\mathrm{Amp}(X), \mathrm{Nef}(X) \subset N^1(X)_{\mathbb{R}}$, called the *ample cone* and *nef cone*. For a projective variety, Kleiman's Theorem (B.6.12) implies that the nef cone is the closure of the ample cone and the ample cone is the interior of the nef cone; see [Laz04a, Thm. 1.4.23].

Effective, base point free, and semiample line bundles. We have the following notions for a line bundle L on a proper scheme X over a field $\Bbbk$:

- L is *effective* if $\Gamma(X, L) \neq 0$,
- L is *base point free* (or *globally generated*) if for every $x \in X$, there exists $s \in \Gamma(X, L)$ with $s(x) \neq 0$, or equivalent the complete linear series $|L|$ defines a morphism $X \to \mathbb{P}(\mathrm{H}^0(X, L))$, and
- L is *semiample* if for some $m > 0$, $L^{\otimes m}$ is base point free.

A semiample line bundle L is necessarily nef; indeed if for some $m > 0$, $L^{\otimes m}$ defines a morphism $f: X \to \mathbb{P}^N$ with $f^* \mathcal{O}(1) \cong L^{\otimes m}$, then the projection formula implies that $\int_C c_1(L^{\otimes m}) = \int_{f(C)} c_1(\mathcal{O}(1)) \geq 0$. We thus have the implications

$$\text{base point free} \implies \text{semiample} \implies \text{nef}.$$

Big line bundles. A line bundle L on a proper scheme X over a field $\Bbbk$ is *big* if there exists a constant $C > 0$ such that $\mathrm{h}^0(X, L^{\otimes m}) > C \cdot m^{\dim(X)}$ for $m \gg 0$.

Proposition B.6.19 (Kodaira's Lemma)**.** *Let X be a projective scheme over a field $\Bbbk$ and L be a big line bundle on X. If E is an effective line bundle, then $mL - E$ is effective for m sufficiently divisible.*

Proof. See [Laz04a, Prop. 2.2.6]. $\qquad\square$

Proposition B.6.20 (Characterizations of Bigness)**.** *For a projective scheme X over a field $\Bbbk$ and a line bundle L on X, the following are equivalent:*

(1) L is big;

(2) for every ample divisor A on X, there exists a positive integer $m > 0$ and an effective divisor N on X such that $mL = A + N$ (linear equivalence); and

(3) there exists an ample divisor A on X, a positive integer $m > 0$, and an effective divisor N on X such that $mL \equiv A + N$ (numerical equivalence).

If in addition X is normal, then the above are also equivalent to:

(4) For some $m > 0$, $|L^{\otimes m}|$ defines a rational map $X \to \mathbb{P}(\mathrm{H}^0(X, L^{\otimes m}))$ which is birational onto its image.

Proof. See [Laz04a, Cor. 2.2.7]. $\qquad\square$

As a consequence, we see that up to scaling (i.e. taking positive tensor powers), a big line bundle is the same as the sum of an ample and effective line bundle. This implies that the sum of a big and effective line bundle is also big. To summarize,

$$\text{big} \xLeftrightarrow{\text{up to scaling}} \text{ample} + \text{effective}$$

$$\text{big} + \text{effective} \implies \text{big}.$$

Proposition B.6.21 (Properties of Bigness)**.** *Let X be a proper scheme over a field $\Bbbk$ and L be a line bundle on X.*

(1) *If $f\colon X' \to X$ is a generically quasi-finite and proper morphism, then L is big if and only if f^*L is.*

(2) *For a field extension $\Bbbk \to \Bbbk'$, L is big on X if and only if $L_{\Bbbk'}$ is big on $X_{\Bbbk'}$.*

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Theorem B.6.22 (Asymptotic Riemann-Roch)**.** *Let X be a proper scheme over a field $\Bbbk$ of dimension n, and let L be a nef line bundle on X. Then the Euler characteristic $\chi(X, L^{\otimes m})$ is a polynomial of degree $\leq n$ in m*

$$\mathrm{h}^0(X, L^{\otimes m}) = \frac{(c_1(L)^n)}{n!}m^n + O(m^{n-1}).$$

Proof. See [Laz04a, Cor. 1.4.41] (projective case), [Kol96, Thm. VI.2.15], and [SP, Tag 0BJ8]. $\qquad\qquad$ $\square$

This gives a useful characterization of bigness for nef line bundles.

Corollary B.6.23 (Further Characterization of Bigness)**.** *On a proper scheme of dimension n, a nef line bundle L is big if and only if $c_1(L)^n > 0$.* $\qquad$ $\square$

Remark B.6.24 (Big and pseudo-effective cones)**.** Big and effective divisors generate the *big cone* $\mathrm{Big}(X)$ and *effective cone* $\mathrm{Eff}(X)$ in $N^1(X)_{\mathbb{R}}$, and the closure $\overline{\mathrm{Eff}}(X)$ is called the *pseudo-effective cone*. The big cone $\mathrm{Big}(X)$ is contained in the interior of $\overline{\mathrm{Eff}}(X)$, and $\overline{\mathrm{Eff}}(X) = \overline{\mathrm{Big}(X)}$ [Laz04a, Thm. 2.2.6].

B.6.3 Ampleness criteria

We review techniques to show that a line bundle L on a proper scheme X is ample. The first strategy to keep in mind is that if L is semiample and strictly nef, then L is ample.

Lemma B.6.25. *Let X be a proper scheme over an algebraically closed field $\Bbbk$. If L is a semiample line bundle on X and $\int_T c_1(L) = \deg L|_T > 0$ for all integral proper curves T over $\Bbbk$, then L is ample.*

Proof. For some $m > 0$, $L^{\otimes m}$ defines a morphism $f\colon X \to \mathbb{P}^N$ which does not contract any curves. It follows that $f\colon X \to \mathbb{P}^N$ is a proper and quasi-finite, thus finite. Therefore, $L^{\otimes m} = f^*\mathcal{O}(1)$ is ample. $\qquad\qquad$ $\square$

Remark B.6.26. The semiampleness condition can be very challenging to verify in practice. However, there are powerful base point free theorems in birational geometry stemming either from vanishing theorems or analytic methods that can reduce semiampleness to bigness and nefness. For instance, Kawamata's base point free theorem states that if (X, Δ) is a proper klt pair with Δ effective and D is a nef Cartier divisor such that $aD - K_X - \Delta$ is nef and big for some $a > 0$, then D is semiample [KM98, Thm. 3.3]. One can contrast this result with the Abundance Conjecture that states that if (X, Δ) is a proper log canonical pair with Δ effective, then the nefness of $K_X + \Delta$ implies semiampleness [KM98, Conj. 3.12].

Alternatively, it is a classical result of Zariski and Wilson that if X is a normal projective variety and D is a nef and big divisor, then D is semiample if and only if its graded section ring $\bigoplus_n \Gamma(X, \mathcal{O}_X(nD))$ is finitely generated; see [Laz04a, Thm. 2.3.15]. While [BCHM10] can sometimes be applied to verify the finite

generation, this result already presumes the projectivity of X. Nevertheless, it is sometimes useful.

In positive characteristic, Keel's theorem [Kee99] provides another technique: on a projective variety X over a field $\Bbbk$ of charaacteristic p, a nef line bundle L is semiample if and only if the restriction of L to the *exceptional locus* E is semiample, where the exceptional locus E is defined as the union of irreducible subvarieties the $Z \subset X$ satisfying $L^{\dim Z} \cdot Z = 0$.

Numerical criteria for ampleness. The Nakai–Moishezon Criterion[5] for ampleness provides a convenient method to establish projectivity. We will extend this criterion to algebraic spaces in Theorem 5.8.4, as it enters into the proof of Kollár's Criterion (5.8.7), which in turn is used for the projectivity of $\overline{M}_g$.

Theorem B.6.27 (Nakai–Moishezon Criterion). *If X is a proper scheme, a line bundle L is ample if and only if for all integral closed subschemes $Z \subset X$, $c_1(L)^{\dim Z} \cdot Z > 0$*

Remark B.6.28. By Corollary B.6.23, this translates to: L is ample $\iff$ L is nef and $L|_Z$ is big for all integral closed subschemes $Z \subset X$.

Proof. Let $n = \dim X$. First, if L is ample, then for some $m > 0$, $L^{\otimes m}$ is very ample and $m^n c_1(L)^{\dim Z} \cdot Z = c_1(L^{\otimes m})^{\dim Z} \cdot Z > 0$ as its the degree of Z under the projective embedding defined by $L^{\otimes m}$. For the converse, since L is nef, we may apply Lemma B.6.25 to reduce to showing that L is semiample. First, by Proposition B.6.9(1), we may assume that X is a normal, and we write $L = \mathcal{O}_X(D)$ for a divisor D is big on X, some positive multiple mD is effective. After replacing D by mD, there exists a nonzero section $s \in H^0(X, \mathcal{O}_X(D))$. This shows that $\mathcal{O}_X(D)$ has no base points on $X \setminus D$. We will show that for $m \gg 0$, $\mathcal{O}_X(mD)$ also has no base points on D. By induction on $n = \dim X$, we can assume that $\mathcal{O}_X(D)|_D$ is ample (where in the base case $n = 1$, a line bundle is ample if and only if it has positive degree). Consider the exact sequence

$$0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)|_D \to 0.$$

For $m \gg 0$, $\mathcal{O}_X(mD)|_D$ is base point free and $H^1(X, \mathcal{O}_X(mD)|_D) = 0$. It follows that $H^1(X, \mathcal{O}_X((m-1)D)) \twoheadrightarrow H^1(X, \mathcal{O}_X(mD))$ is surjective, but since each vector space is finite dimensional, we see that these surjections eventually become isomorphisms for $m \gg 0$. Thus, for $m \gg 0$, $H^0(X, \mathcal{O}_X(mD)) \to H^0(D, \mathcal{O}_X(mD)|_D)$ is surjective, and $\mathcal{O}_X(mD)$ has no base points on D. See also [Har77, Thm. V.1.10] (surface case), [Laz04a, Thm. 1.2.23] (projective case), [Kol96, Thm VI.2.18], [KM98, Thm. 1.37], and [Kle66, §III.1]. $\square$

The following criteria for ampleness will not be used in these notes but are included to offer a more complete treatment.

Theorem B.6.29 (Kleiman's Criterion). *If X is a projective scheme or a $\mathbb{Q}$-factorial (e.g. smooth) proper scheme, a line bundle L is ample if and only if for all $C \in \overline{\mathrm{Eff}(X)}$, $c_1(L) \cdot C > 0$.*

Remark B.6.30. See [Kle66, §III.1], [Kol96, Thm. VI.2.19], [KM98, Thm. 1.18], and [Laz04a, Thm. 1.4.23]. Note that it is not enough to check that $c_1(L) \cdot C > 0$

for only integral curves $C \subset X$; one must check it on curve classes in the closure $\overline{\mathrm{Eff}}(X)$ of the effective cone of curves. See [Har70, p.50-56] for a counterexample due to Mumford.

Theorem B.6.31 (Sesahdri's criterion)**.** *If X is a proper scheme, a line bundle L is ample if and only if there exists an $\epsilon > 0$ such that for every point $x \in X$ and every integral curve $C \subset X$, $c_1(L) \cdot C > \epsilon \, \mathrm{mult}_x(C)$, where $\mathrm{mult}_x(C)$ denotes the multiplicity of C at x.*

Remark B.6.32. See [Laz04a, Thm. 1.4.13] and [Kol96, Thm. 2.18].

B.6.4 Nef vector bundles

In Kollár's Criterion (5.8.7), nefness of vector bundles plays an essential role.

Definition B.6.33. A vector bundle E on a scheme X is called *nef* (or *semipositive*) if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E)$.

There is a related notion of an *ample vector bundle*, which we won't need, defined by requiring $\mathcal{O}_{\mathbb{P}(E)}(1)$ to be ample on $\mathbb{P}(E)$. This notion also has nice properties; see [Har66a] and [Laz04b, §6].

Proposition B.6.34 (Characterization of Nefness for Bundles)**.** *Let E be a vector bundle on a proper scheme X. Then the following are equivalent:*

(1) E is nef,

(2) for every map $f \colon C \to X$ from a smooth proper curve, every quotient line bundle of $f^*E \twoheadrightarrow L$ has nonnegative degree, and

(3) for every map $f \colon C \to X$ from a smooth proper curve, every quotient vector bundle $f^*E \twoheadrightarrow W$ has nonnegative degree.

Proof. See [Bar71, p.437], [Laz04b, Prop. 6.1.18], and [Kol90, Def.-Prop. 3.3] □

Proposition B.6.35 (Properties of Nefness for Bundles)**.** *Let X be a proper scheme over a field $\Bbbk$ and E be a vector bundle on X.*

(1) If $f \colon X' \to X$ is a surjective proper morphism, then E is nef if and only if f^*E is.

(2) For a field extension $\Bbbk \to \Bbbk'$, E is nef on X if and only if $E_{\Bbbk'}$ is nef on $X_{\Bbbk'}$.

(3) Quotients, extensions, and tensor products of nef vector bundles are nef. If E is nef, so is $\bigwedge^k E$, $\mathrm{Sym}^k E$, and $\mathrm{Sym}^k(E^\vee)^\vee$ for $k \geq 0$.

Proof. Parts (1) and (2) follow from the analogous properties of nef line bundles (Proposition B.6.14). Part (3) will require some work. By (2), we can assume $\Bbbk$ is algebraically closed. From the Characterization of Nefness for Bundles (B.6.34), it suffices to assume that X is a smooth curve of genus g. This characterization makes it clear that quotients of nef bundles are nef, and it is not hard to show that an extension of nef bundles is nef. Before we prove the remaining parts, we will show that if E is a nef vector bundle on X and L is a line bundle with $\deg L \geq 2g - 1$ (resp. $\deg L \geq 2g$), then $\mathrm{H}^1(X, E \otimes L) = 0$ (resp. $E \otimes L$ is globally generated). By Serre–Duality (5.1.2), $\mathrm{H}^1(X, E \otimes L) = \mathrm{Hom}_{\mathcal{O}_X}(E \otimes L, \omega_X)$. If $E \otimes L \to \omega_X$ were a nonzero map, the image $I \subset \omega_X$ would be a line bundle with $\deg I \leq \deg \omega_X = 2g-2$. If $\deg L \geq 2g - 1$, the induced quotient $E \twoheadrightarrow I \otimes L^\vee$ would have negative degree, contradicting the nefness of E, hence $\mathrm{H}^1(X, E \otimes L) = 0$. If $\deg L \geq 2g$, then for

every $p \in X$, $\mathrm{H}^1(X, E \otimes L(-p)) = 0$ and $\mathrm{H}^0(X, E \otimes L) \to \mathrm{H}^0(X, E \otimes L \otimes \kappa(p))$ is surjective.

Assume that E and F be nef bundles on X of rank e and f. Let L be a line bundle with $d := \deg L \geq 2g$. Since $E \otimes L$ is globally generated, we may choose e sections of $E \otimes L$ generating a basis of $E \otimes L \otimes \kappa(\xi)$, where $\xi \in X$ is the generic point. This gives a generically surjective map $(L^\vee)^{\oplus e} \to E$. Similarly, there is a generically surjective map $(L^\vee)^{\oplus f} \to F$. Taking the tensor product of these maps gives a generically surjective map

$$(L^\vee)^{\oplus ef} \to E \otimes F.$$

The image of $Q' \subset Q$ of $(L^\vee)^{\oplus ef}$ satisfies $\deg Q \geq \deg Q' \geq -d$. This shows that every quotient line bundle Q of the tensor product $E \otimes F$ of *any two* nef bundles E and F of rank e and f satisfies $\deg Q \geq -d$.

Assume that $\mathrm{char}(\Bbbk) = p$. Suppose that there is a line bundle quotient $E \otimes F \twoheadrightarrow Q$ with $\deg Q < 0$. Denote $\mathrm{Fr}^N \colon X \to X$ be the Nth power of the absolute Frobenius, the quotient $(\mathrm{Fr}^N)^* E \otimes (\mathrm{Fr}^N)^* F \twoheadrightarrow (\mathrm{Fr}^N)^* Q$ is a quotient line bundle of degree $Np \deg Q$. Taking N such that $Np \deg Q < -d$ gives a quotient line bundle of degree less than $-d$, contradicting the above fact.

Assume that $\mathrm{char}(\Bbbk) = 0$. Since X is of finite type over $\Bbbk$, its defining equations involve finitely many coefficients of $\Bbbk$. Thus there exists a finitely generated $\mathbb{Z}$-subalgebra $A \subset \Bbbk$, a scheme $\widetilde{X}$ of finite type over A, and a cartesian diagram

$$
\begin{array}{ccc}
X & \longrightarrow & \widetilde{X} \\
\downarrow & \square & \downarrow \\
\mathrm{Spec}\,\Bbbk & \longrightarrow & \mathrm{Spec}\,A.
\end{array}
$$

By possibly enlarging A, we can arrange that $X \to \mathrm{Spec}\,A$ is a family of smooth curves. Finally, by restricting along a map $\mathrm{Spec}\,R \to \mathrm{Spec}\,A$, we may assume that A is a DVR whose closed and generic points have characteristic p and 0, respectively.

See also [Kol90, Prop. 3.5], [Bar71, Thm. 3.3], and [Har66a, Thm. 5.2]. Remark: Kollár's generalization using semipositive reps, and even further ..... Δ' $\qquad\square$

Proposition B.6.36 (Nefness for Bundles is Stable under Generization)**.** *Let X be a proper and flat scheme over a DVR R and E be a vector bundle on X. If the restriction E_0 of E to the central fiber X_0 is nef, then so is the restriction E_η to the generic fiber X_η.*

Proof. This follows from Proposition B.6.16. $\qquad\square$

B.6.5 Vanishing theorems

Kollár's argument for the projectivity of $\overline{M}_g$ makes use of the following vanishing theorem in positive characteristic due to Ekedahl [Eke88]. The characteristic zero version is due to Bombieri [Bom73].

Theorem B.6.37 (Bombieri–Ekedahl Vanishing)**.** *Let S be a smooth projective surface over $\Bbbk$ which is minimal and of general type. If $\mathrm{char}(\Bbbk) \neq 2$, then $\mathrm{H}^1(S, K_S^{\otimes -n}) = 0$ for all $n \geq 1$. If $\mathrm{char}(\Bbbk) = 2$, then $\mathrm{h}^1(S, K_S^{\otimes -n}) \leq 1$ for all $n \geq 2$.*

Corollary B.6.38. *Let S be a smooth projective surface over an algebraically closed field $\Bbbk$ which is minimal and of general type. Let D be an effective divisor with $D^2 = 0$. If $\mathrm{char}(\Bbbk) \neq 2$, then $H^1(S, \omega_S^{\otimes n}(D)) = 0$ for all $n \geq 2$. If $\mathrm{char}(\Bbbk) = 2$, then $\mathrm{h}^1(S, \omega_S^{\otimes n}(D)) \leq 1$ for all $n \geq 2$.*

Proof. Bombieri–Ekedahl Vanishing implies that $H^1(S, K_S^{\otimes -n}) = 0$ for all $n \geq 1$. The Serre dual of this statement is that $H^1(S, K_S^{\otimes n}) = 0$ for all $n \geq 2$. The statement follows from using the short exact sequence

$$0 \to \omega_S^{\otimes n} \to \omega_S^{\otimes n}(D) \to \omega_S^{\otimes n}|_D \to 0$$

and adjunction (details to be added). □

Appendix C

Deformation theory

Deformation theory is the study of the local geometry of a moduli space $\mathcal{M}$ near an object $E_0 \in \mathcal{M}(\Bbbk)$. We focus primarily on the following three deformation problems:

(A) Embedded deformations of a closed subscheme Z_0 in a fixed projective scheme X over $\Bbbk$. Here the moduli problem is the Hilbert functor $\mathrm{Hilb}^P(X)$ and $E_0 = [Z_0 \subset X] \in \mathrm{Hilb}^P(X)(\Bbbk)$.

(B) Deformations of a scheme E_0 over $\Bbbk$. In this section, the main example for us is when E_0 is a smooth curve, in which case the moduli problem is $\mathcal{M}_g$ and $[E_0] \in \mathcal{M}_g(\Bbbk)$.

(C) Deformations of a coherent sheaf E_0 on a fixed projective scheme C over $\Bbbk$. The main example for us is when C is a smooth curve and E_0 is a vector bundle, in which case the moduli problem is $\underline{\mathrm{Bun}}_C$ and $[E_0] \in \underline{\mathrm{Bun}}_C(\Bbbk)$.

In this chapter, we sketch the local-to-global approach of deformation theory by zooming in around $E_0 \in \mathcal{M}(\Bbbk)$ and studying successively first-order neighborhoods of $\mathcal{M}$ at E_0, higher-order deformations of E_0, formal neighborhoods of E_0 and eventually étale or smooth neighborhoods of E_0.

(1) A *first-order deformation* of E_0 is an object $E \in \mathcal{M}(\Bbbk[\epsilon])$ over the dual numbers $\Bbbk[\epsilon] := \Bbbk[\epsilon]/(\epsilon^2)$ together with an isomorphism $\alpha\colon E_0 \to E|_{\mathrm{Spec}\,\Bbbk}$, or in other words a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}\,\Bbbk & \hookrightarrow & \mathrm{Spec}\,\Bbbk[\epsilon] \\
& {\scriptstyle [E_0]} \searrow & \downarrow {\scriptstyle [E]} \\
& & \mathcal{M}
\end{array}
$$

allowing us to view E as a tangent vector of $\mathcal{M}$ at E_0. We classify first-order deformations of Problems (A)–(C) in §C.1.

(2) Given a surjection $A' \twoheadrightarrow A$ of artinian local $\Bbbk$-algebras with residue field $\Bbbk$ and an object object $E \in \mathcal{M}(A)$ with an isomorphism $E_0 \to E|_{\mathrm{Spec}\,\Bbbk}$, a *deformation of E over A'* is an object $E' \in \mathcal{M}(A')$ with an isomorphism $\alpha\colon E \to E'|_{\mathrm{Spec}\,A}$. Pictorially, this corresponds to a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Spec}\,\Bbbk & \hookrightarrow & \mathrm{Spec}\,A & \hookrightarrow & \mathrm{Spec}\,A' \\
& {\scriptstyle [E_0]} \searrow & & {\scriptstyle [E]} \searrow & \downarrow {\scriptstyle [E']} \\
& & & & \mathcal{M}.
\end{array}
$$

For Problems (A)–(C), we determine when a deformation E' of E over A' exists and we classify them in §C.2.

(3) Given a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$, a *formal deformation of E_0 over R* is a compatible collection of deformations $E_n \in \mathcal{M}(R/\mathfrak{m}^{n+1})$ of E_0, and a formal deformation $\{E_n\}$ is *versal* if every other deformation factors through one of the E_n (see Definition C.3.5 for a precise definition). Rim–Schlessinger's Criteria (Theorem C.3.12) provides criteria for the existence of a versal deformation $\{E_n\}$ of E_0, and in §C.3 we verify the criteria for the Problems (A)–(C).

(4) A formal deformation $\{E_n\}$ over $(R, \mathfrak{m})$ is *effective* if there exists an object $\widehat{E} \in \mathcal{M}(R)$ extending the $\{E_n\}$, or in other words there exists a commutative diagram

$$\operatorname{Spec} R/\mathfrak{m} \hookrightarrow \operatorname{Spec} R/\mathfrak{m}^2 \hookrightarrow \operatorname{Spec} R/\mathfrak{m}^3 \hookrightarrow \cdots \hookrightarrow \operatorname{Spec} R$$

with morphisms $[E_0]$, $[E_1]$, $[E_2]$, $[\widehat{E}]$ into $\mathcal{M}$.

In §C.4, we show how Grothendieck's Existence Theorem (C.4.4) implies that formal deformations are effective for Problems (A)–(C).

(5) In §C.5, we take a detour from the local-to-global approach to provide a glimpse into the role of the cotangent complex in deformation theory.

(6) Given an effective versal formal deformation $\widehat{E}$ over R, Artin Algebraization (Theorem C.6.8) ensures the existence of a *finite type* $\Bbbk$-scheme U with a point $u \in U(\Bbbk)$ and an object $E \in \mathcal{M}(U)$ such that $R \cong \widehat{\mathcal{O}}_{U,u}$ and $\widehat{E}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}} \cong E|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}}$ for all n.

(7) Artin's Axioms for Algebraicity (Theorems C.7.1 and C.7.4) provides criteria to verify the algebraicity of a moduli problem $\mathcal{M}$. Namely, it provides conditions to ensure that the morphism $[E]\colon U \to \mathcal{M}$ constructed above is a smooth morphism in an open neighborhood of E_0.

In this chapter, $\Bbbk$ will denote an algebraically closed field. In §C.3, §C.6 and §C.7, we work over the category of $\Bbbk$-schemes for convenience but the results hold more generally.

C.1 First order deformations

Denote the *dual numbers* by $\Bbbk[\epsilon] := \Bbbk[\epsilon]/(\epsilon^2)$.

C.1.1 First order embedded deformations

Definition C.1.1. Let X be a projective scheme over a $\Bbbk$ and $Z_0 \subset X$ be a closed subscheme. A *first-order deformation of $Z_0 \subset X$* is a closed subscheme $Z \subset X_{\Bbbk[\epsilon]} := X \times_{\Bbbk} \Bbbk[\epsilon]$ flat over $\Bbbk[\epsilon]$ such that $Z_0 = Z \times_{\Bbbk[\epsilon]} \Bbbk$. Pictorially, a

first-order deformation is a filling of the diagram

with a scheme Z and dotted arrows making the diagram cartesian.

We say that $Z \subset X_{\Bbbk[\epsilon]}$ is *trivial* if $Z = Z_0 \times_{\Bbbk} \Bbbk[\epsilon]$.

Remark C.1.2. Since Z_0 and the central fiber $Z \times_{\Bbbk[\epsilon]} \Bbbk$ of Z are embedded in X, it makes sense to require that they are equal.

Remark C.1.3. The closed subscheme $Z_0 \subset X$ defines a $\Bbbk$-point $[Z_0 \subset X] \in \mathrm{Hilb}^P(X)$ of the Hilbert scheme where P is the Hilbert polynomial of Z_0 with respect to a fixed ample line bundle on X. A first-order deformation corresponds to a commutative diagram

or in other words a tangent vector $[Z \subset X_{\Bbbk[\epsilon]}] \in T_{\mathrm{Hilb}^P(X),[Z_0 \subset X]}$.

Proposition C.1.4. *Let X be a projective scheme over a $\Bbbk$ and $Z_0 \subset X$ be a closed subscheme defined by a sheaf of ideals $I_0 \subset \mathcal{O}_X$. There is a bijection*

$$\{\text{first-order deformations } Z \subset X_{\Bbbk[\epsilon]}\} \cong \mathrm{H}^0(Z_0, N_{Z_0/X})$$

where $N_{Z_0/X} = \mathcal{H}om_{\mathcal{O}_{Z_0}}(I_0/I_0^2, \mathcal{O}_{Z_0})$ is the normal sheaf. Under this correspondence, the trivial deformation corresponds to $0 \in \mathrm{H}^0(Z_0, N_{Z_0/X})$.

Remark C.1.5. In light of Remark C.1.3, this proposition gives a bijection $T_{\mathrm{Hilb}^P(X),[Z_0 \subset X]} \cong \mathrm{H}^0(Z_0, N_{Z_0/X})$.

Proof. We sketch the bijection and point the reader to [Har10, Prop. 2.3] and [Ser06, Prop. 3.2.1] for details. After reducing to the affine case $X = \mathrm{Spec}\, B$ and $Z_0 = \mathrm{Spec}\, B/I_0$, we need to show that the set of first-order deformations is bijective to

$$\mathrm{H}^0(Z_0, N_{Z_0/X}) \cong \mathrm{Hom}_{B/I_0}(I_0/I_0^2, B/I_0) \cong \mathrm{Hom}_B(I_0, B/I_0).$$

Given a first-order deformation $Z = \mathrm{Spec}\, B[\epsilon]/I$, the flatness of Z over $\Bbbk[\epsilon]$ ensures that tensoring the exact sequence $0 \to I \to B[\epsilon] \to B[\epsilon]/I \to 0$ of $\Bbbk[\epsilon]$-modules with $\Bbbk = \Bbbk[\epsilon]/(\epsilon)$ yields an exact sequence $0 \to I_0 \to B \to B/I_0 \to 0$. We define $\alpha \colon I_0 \to B/I_0$ as follows: for $x_0 \in I_0$, choose a preimage $x = a + b\epsilon \in I$ and set $\alpha(x_0) := \bar{b} \in B/I_0$. Conversely, given a B-module homomorphism $\alpha \colon I_0 \to B/I_0$, we define

$$I = \{a + b\epsilon \mid a \in I_0, b \in B \text{ such that } \bar{b} = \alpha(a) \in B/I_0\} \subset B[\epsilon].$$

Then $(B[\epsilon]/I) \otimes_{\Bbbk[\epsilon]} \Bbbk = B/I_0$. To see that $B[\epsilon]/I$ is flat over $\Bbbk[\epsilon]$, we need to check that the map $B/I_0 \xrightarrow{\epsilon} B[\epsilon]/I$ is injective (see Remark A.2.7): given $b \in B$ with $\epsilon b \in I$, then $b \in I_0$ by the definition of I. Thus $Z = \mathrm{Spec}\, B[\epsilon]/I$ defines a first-order deformation of Z_0. $\square$

C.1.2 Locally trivial first-order deformations of schemes

Definition C.1.6. Let X_0 be a scheme over a field $\Bbbk$. A *first-order deformation of* X_0 is a scheme X flat over $\Bbbk[\epsilon]$ together with an isomorphism $\alpha\colon X_0 \to X \times_{\Bbbk[\epsilon]} \Bbbk$, or in other words a cartesian diagram

$$
\begin{array}{ccc}
X_0 & \dashleftarrow\!\!\!-\!-\!-\!\!\dashrightarrow & X \\
\downarrow & \square & \downarrow \text{ flat} \\
\operatorname{Spec}\Bbbk & \longhookrightarrow & \operatorname{Spec}\Bbbk[\epsilon].
\end{array}
\tag{C.1.7}
$$

A *morphism of first-order deformations* (X,α) *and* (X',α') is a morphism $\beta\colon X \to X'$ of schemes over $\Bbbk[\epsilon]$ such that $(\beta \times_{\Bbbk[\epsilon]} \Bbbk) \circ \alpha = \alpha'$, or in other words considering X and X' in cartesian diagrams (C.1.7), we require the restriction of β to central fiber X_0 to be the identity.

We say that X is *trivial* if X is isomorphic to $X_0 \times_{\Bbbk} \Bbbk[\epsilon]$ as first-order deformations, and *locally trivial* if there exists a Zariski-cover $X = \bigcup_i U_i$ such that U_i is a trivial first-order deformation of $U_i \times_{\Bbbk[\epsilon]} \Bbbk \subset X_0$.

Every morphism of deformations is necessarily an isomorphism. This is a consequence of the following algebraic fact.

Lemma C.1.8. *Let A be a ring, $\mathfrak{m} \subset A$ be a nilpotent ideal (e.g. $(A,\mathfrak{m})$ is an artinian local ring), and $M \to N$ be a homomorphism of A-modules. Assume that N is flat over A. If $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is an isomorphism, so is $M \to N$.*

Proof. The right exact sequence $M \to N \to C \to 0$ becomes $M/\mathfrak{m}M \to N/\mathfrak{m}N \to C/\mathfrak{m}C \to 0$ after modding out by $\mathfrak{m}$, and we see that $C = \mathfrak{m}C$. As $\mathfrak{m}^n = 0$ for some n, we obtain that $C = \mathfrak{m}C = \mathfrak{m}^2C = \cdots = \mathfrak{m}^nC = 0$. Considering now the exact sequence $0 \to K \to M \to N \to 0$, the flatness of N implies that we get an exact sequence $0 \to K/\mathfrak{m}K \to M/\mathfrak{m}M \to N/\mathfrak{m}N \to 0$. Thus $K = \mathfrak{m}K = \cdots = \mathfrak{m}^nK = 0$, and we see that $M \to N$ is an isomorphism. $\square$

Proposition C.1.9. *Every first-order deformation of a smooth affine scheme X_0 over $\Bbbk$ is trivial. In other words, X_0 is rigid.*

Proof. Let X be a first-order deformation of X_0. Since $X_0 \to \operatorname{Spec}\Bbbk$ is smooth, we may apply the Infinitesimal Lifting Criterion for Smoothness (A.3.1) to construct a lift $X \to X_0$ making the diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\ \text{id}\ } & X_0 \\
\downarrow & \nearrow & \downarrow \text{smooth} \\
X & \longrightarrow & \operatorname{Spec}\Bbbk
\end{array}
$$

commute. This induces a morphism $X \to X_0 \times_{\Bbbk} \Bbbk[\epsilon]$ over $\Bbbk[\epsilon]$ which restricts to the identity on X_0, and thus is an isomorphism by Lemma C.1.8.

See also [Har77, Exer. II.8.7]. $\square$

Remark C.1.10. If X_0 is not smooth or affine, then first-order deformations are not necessarily trivial. For example, if $X_0 = \operatorname{Spec}\Bbbk[x,y]/(xy)$ is the nodal affine plane curve, then $X = \operatorname{Spec}\Bbbk[x,y,\epsilon]/(xy - \epsilon) \to \operatorname{Spec}\Bbbk[\epsilon]$ is a non-trivial first-order deformation.

On the other hand, consider an elliptic curve $E_0 = V(y^2 z - x(x-z)(x-2z)) \subset \mathbb{P}^2$ is a elliptic curve over $\Bbbk$ with $\mathrm{char}(\Bbbk) \neq 2, 3$. It is easy to write down global deformations by deforming the coefficients of the defining equations: $\mathcal{E} = V(y^2 z - (x - \lambda z)(x-z)(x-2z)) \subset \mathbb{P}^2 \times \mathbb{A}^1$ (where $\mathbb{A}^1$ has coordinate λ) defines a flat projective morphism $\mathcal{E} \to \mathbb{A}^1$ such the central fiber $\mathcal{E}_0$ is isomorphic to E_0. Restricting $\mathcal{E}$ to the family $E := \mathcal{E} \times_{\mathbb{A}^1} \mathrm{Spec}\, \Bbbk[\lambda]/\lambda^2$ over the dual numbers defines a *non-trivial* first-order deformation. For an affine open $U_0 \subset E_0$ and setting $U \subset E$ to be the open subscheme with the same topological space as U_0, then there is an isomorphism $U \xrightarrow{\sim} U_0 \times_{\Bbbk} \Bbbk[\epsilon]$. These local isomorphisms however do not glue to a global isomorphism $E \xrightarrow{\sim} E_0 \times_{\Bbbk} \Bbbk[\epsilon]$. Since every deformation of a smooth scheme is obtained by gluing together trivial deformations, we need to understand automorphisms of trivial deformations to classify global deformations.

Lemma C.1.11. *If $X_0 = \mathrm{Spec}\, A$ is an affine scheme over $\Bbbk$ and $X = \mathrm{Spec}\, A[\epsilon]$ is the trivial first-order deformation, then there are identifications*

$$\{automorphisms\ X \to X\ of\ first\text{-}order\ defs\} \cong \mathrm{Der}_{\Bbbk}(A, A) \cong \mathrm{Hom}_A(\Omega_{A/\Bbbk}, A).$$

Proof. The second equivalence is given by the universal property of the module of differentials. An automorphism of the trivial first-order deformation corresponds to a $\Bbbk[\epsilon]$-algebra isomorphism $\phi \colon A \oplus A\epsilon \to A \oplus A\epsilon$ which is the identity modulo ϵ. Therefore, ϕ is determined by the images $\phi(a) = a + d(a)\epsilon$ of elements $a \in A$ where $d \colon A \to A$ is $\Bbbk$-linear map. Since ϕ is a ring homomorphism, for elements $a, a' \in A$, we must have that $aa' + d(aa')\epsilon = (a + d(a)\epsilon)(a' + d(a')\epsilon) = aa' + (ad(a') + a'd(a))\epsilon$ and we see that $d \colon A \to A$ is a $\Bbbk$-derivation. $\square$

For a scheme X_0 over $\Bbbk$, let $\mathrm{Def}(X_0)$ and $\mathrm{Def}^{\mathrm{lt}}(X_0)$ denote the sets of isomorphism classes of first-order and locally trivial first-order deformations.

Proposition C.1.12. *For a scheme X_0 of finite type over $\Bbbk$ with affine diagonal, there is a bijection*

$$\mathrm{Def}^{\mathrm{lt}}(X_0) \cong \mathrm{H}^1(X_0, T_{X_0}),$$

where $T_{X_0} = \mathscr{H}om_{\mathcal{O}_{X_0}}(\Omega_{X_0/\Bbbk}, \mathcal{O}_{X_0})$. The trivial deformation corresponds to $0 \in \mathrm{H}^1(X_0, T_{X_0})$.

In particular, if in addition X_0 is smooth over $\Bbbk$, then there is a bijection

$$\mathrm{Def}(X_0) \cong \mathrm{H}^1(X_0, T_{X_0}).$$

Proof. Let $X \to \mathrm{Spec}\, \Bbbk[\epsilon]$ be a locally trivial first-order deformation of X_0. Choose an affine cover $\{U_i\}$ of X_0 and isomorphisms $\phi_{ij} := \phi_i \colon U_i \times_{\Bbbk} \Bbbk[\epsilon] \xrightarrow{\sim} X|_{U_i}$, where $X|_{U_i} \subset X$ denotes the open subscheme with the same topological space as U_i. Letting $U_{ij} = U_i \cap U_j$, we have automorphisms $\phi_j^{-1}|_{U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]} \circ \phi_i|_{U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]}$ of the trivial deformation $U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]$ which by Lemma C.1.11 corresponds to elements $\phi_{ij} \in \mathrm{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega_{U_{ij}/\Bbbk}, \mathcal{O}_{U_{ij}})$. Since $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $U_{ijk} := U_i \cap U_j \cap U_k$, we have that $\phi_{ij} + \phi_{jk} = \phi_{ik} \in T_{X_0}(U_{ijk})$. Recall that $\mathrm{H}^1(X_0, T_{X_0})$ can be computed using the Céch complex

$$0 \longrightarrow \bigoplus_i T_{X_0}(U_i) \xrightarrow{d_0} \bigoplus_{i,j} T_{X_0}(U_{ij}) \xrightarrow{\quad d_1 \quad} \bigoplus_{i,j,k} T_{X_0}(U_{ijk})$$

$$(s_{ij}) \longmapsto (s_{ij}|_{U_{ijk}} - s_{ik}|_{U_{ijk}} + s_{jk}|_{U_{ijk}})_{ijk}$$

As $\{\phi_{ij}\} \in \bigoplus_{i,j} T_{X_0}(U_{ij})$ is in the kernel of d_1, it defines an element of $\mathrm{H}^1(X_0, T_{X_0}) = \ker(d_1)/\mathrm{im}(d_0)$. Conversely, given an element of $\mathrm{H}^1(X_0, T_{X_0})$ and a choice of

representative $\{\phi_{ij}\} \in \ker(d_1)$, then viewing each ϕ_{ij} as an automorphism ϕ_{ij} of the trivial deformation of U_{ij}, we may glue together the trivial deformations $U_i \times_{\Bbbk} \Bbbk[\epsilon]$ along $U_{ij} \times_{\Bbbk} \Bbbk[\epsilon]$ via ϕ_{ij} to construct a global first-order deformation X of X_0.

For the final statement, observe that since X_0 is smooth over $\Bbbk$, every first-order deformation is locally trivial by Proposition C.1.9. See also [Har77, Exer. III.4.10 and Ex. III.9.13.2]. $\qquad\square$

Example C.1.13. If C is a smooth projective curve of genus $g \geq 2$, then we've computed that

$$T_{\mathcal{M}_g, [C]} = \mathrm{H}^1(C, T_C) \overset{\mathrm{SD}}{=} \mathrm{H}^0(C, \Omega_{C/\Bbbk}^{\otimes 2})$$

which by Riemann–Roch is a $3g - 3$ dimensional vector space.

Exercise C.1.14 (easy)**.** Use the Euler exact sequence to show that $\mathrm{H}^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0$ and conclude that every first-order deformation of $\mathbb{P}^n$ is trivial, i.e. $\mathbb{P}^n$ is rigid.

Exercise C.1.15 (moderate)**.** Let C_0 be a smooth and proper curve over a field $\Bbbk$ with marked points $p_1, \ldots, p_n$. Let $\mathrm{Def}(C_0, p_i)$ denote the set of first-order deformations of (C_0, p_i), i.e. flat morphisms $C \to \mathrm{Spec}\,\Bbbk[\epsilon]$ with n sections σ_i extending (C_0, p_i). Show that

$$\mathrm{Def}(C_0, p_i) \cong \mathrm{H}^1\big(C_0, T_{C_0}(-\sum_i p_i)\big).$$

C.1.3 First order deformations of vector bundles and coherent sheaves

Definition C.1.16. Let X be a projective scheme over $\Bbbk$ and E_0 be a coherent sheaf. A *first-order deformation of E_0* is a pair (E, α) where E is a coherent sheaf on $X \times_{\Bbbk} \Bbbk[\epsilon]$ flat over $\Bbbk[\epsilon]$ and $\alpha \colon E_0 \overset{\sim}{\to} E|_X$ is an isomorphism. Pictorially, we have

$$
\begin{array}{ccc}
E_0 & & E \\
| & & \Big| \,{\scriptstyle \mathrm{flat}/\Bbbk[\epsilon]} \\
X & \hookrightarrow & X_{\Bbbk[\epsilon]}.
\end{array}
$$

A *morphism $(E, \alpha) \to (E', \alpha')$ of first-order deformations* is a morphism $\beta \colon E \to E'$ (equivalently an isomorphism by Lemma C.1.8) of coherent sheaves on X' such that $\alpha' = \beta|_X \circ \alpha$.

We say that (E, α) is *trivial* if it isomorphic as first-order deformations to (p^*E_0, id) where $p \colon X_{\Bbbk[\epsilon]} \to X$.

Proposition C.1.17. *Let X be a scheme over $\Bbbk$ and E_0 be a coherent sheaf. There is a bijection*

$$\{\textit{first-order deformations } (E, \alpha) \textit{ of } E_0\}/\!\sim \ \cong \mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$$

Under this correspondence, the trivial deformation corresponds to $0 \in \mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$.

If in addition E_0 is a vector bundle (resp. line bundle), then the set of isomorphism classes of first-order deformations of E_0 is bijective to $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E_0))$ (resp. $\mathrm{H}^1(X, \mathcal{O}_X)$).

Proof. If (E, α) is a first-order deformation then since E is flat over $\Bbbk[\epsilon]$, we may tensor the exact sequence $0 \to \Bbbk \xrightarrow{\epsilon} \Bbbk[\epsilon] \to \Bbbk \to 0$ of $\Bbbk[\epsilon]$-modules with E to obtain an exact sequence $0 \to E_0 \xrightarrow{\epsilon} E \to E_0 \to 0$ (after identifying $E \otimes_{\Bbbk[\epsilon]} \Bbbk$ with E_0 via α). Since $\mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$ parameterizes isomorphism classes of extensions [Har77, Exer. III.6.1], we have constructed an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$. Conversely, given an exact sequence $0 \to E_0 \xrightarrow{\alpha} E \to E_0 \to 0$, then E is a coherent sheaf on $X_{\Bbbk[\epsilon]}$ and is flat over $\Bbbk[\epsilon]$ by the flatness criterion over the dual numbers (see Remark A.2.7). The restriction $E|_X$ is isomorphic to E_0 via α.

See also [Har10, Thm. 2.7]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark C.1.18. The classifications of Propositions C.2.2, C.2.7 and C.2.17 give a vector space structure to the set of isomorphism classes of first-order deformations. This vector space structure can also be witnessed as a consequence of Rim–Schlessinger's homogeneity condition; see Lemma C.3.15.

C.2 Higher-order deformations and obstructions

Let $\mathcal{M}$ be a moduli problem and $E \in \mathcal{M}(A)$ be an object defined over a ring A. If $A' \twoheadrightarrow A$ is a surjection of rings with square-zero kernel, this section addresses the following two questions:

(1) Does E deform to an object $E' \in \mathcal{M}(A')$?

(2) If so, can we classify all such deformations?

Pictorially, we have:

$$
\begin{array}{ccc}
E & & E' \\
\big| & & \vdots \\
\big| & & \vdots \\
\mathrm{Spec}\, A & \hookrightarrow & \mathrm{Spec}\, A'.
\end{array}
$$

where $\mathrm{Spec}\, A \hookrightarrow \mathrm{Spec}\, A'$ is a closed immersion of schemes with the same topological space. Note that since $J = \ker(A' \to A)$ is square-zero, $J = J/J^2$ is naturally a module over $A = A'/J$. In other words, Question (1) asks whether there is some "obstruction" to the existence of a deformation E' while (2) seeks to classify all higher-order deformations given that there is no obstruction.

An interesting case is when A and A' are local artinian algebras with residue field $\Bbbk$ and the kernel $J = \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$ (which implies that $J^2 = 0$). In this case, $J = J/\mathfrak{m}_{A'} J$ is naturally a vector space over $\Bbbk = A'/\mathfrak{m}_{A'}$. Setting $E_0 := E|_\Bbbk \in \mathcal{M}(\Bbbk)$, we can view E as a deformation over E_0 over A, and we are attempting to classify the higher-order deformations over A'. If there are no obstructions to deforming, then the Infinitesimal Lifting Criterion for Smoothness (3.7.1) implies that $\mathcal{M}$ is smooth at $[E_0]$.

The previous section studied the specific case when $A = \Bbbk$ and $A' = \Bbbk[\epsilon]$ in which case deformations of an object $E_0 \in \mathcal{M}(\Bbbk)$ over A' correspond to first-order deformations. In this case, the obstruction vanishes as there is always the trivial deformation (i.e. the pullback of E_0 along $\mathrm{Spec}\,\Bbbk[\epsilon] \to \mathrm{Spec}\,\Bbbk$). Other examples of $A' \to A$ to keep in mind are $\Bbbk[x]/x^{n+1} \twoheadrightarrow \Bbbk[x]/x^n$ and $\mathbb{Z}/p^{n+1} \twoheadrightarrow \mathbb{Z}/p^n$ where we inductively attempt to deform E_0 over the nilpotent thickenings $\mathrm{Spec}\,\Bbbk[x]/x^{n+1} \hookrightarrow \mathbb{A}^1$ and $\mathrm{Spec}\,\mathbb{Z}/p^{n+1} \hookrightarrow \mathrm{Spec}\,\mathbb{Z}$.

C.2.1 Higher order embedded deformations

Definition C.2.1. Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel. Let X' be a scheme over A' and set $X := X' \times_{A'} A$. Let $Z \subset X$ be a closed subscheme flat over A. A *deformation of $Z \subset X$ over A'* is a closed subscheme $Z' \subset X'$ flat over A' such that $Z' \times_{A'} A = Z$ as closed subschemes of X. Pictorially, a deformation is a filling of the cartesian diagram

The formulation of the next proposition uses the following notion: a *torsor of a group G* is a transitive and free action of G on a set.

Proposition C.2.2. *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. Let X' be a scheme over A' with affine diagonal (e.g. separated) and $Z \subset X := X' \times_{\Bbbk} A$ be a closed subscheme flat over A defined by a sheaf of ideals $I \subset \mathcal{O}_X$. Then*

> *(1) If there exists a deformation $Z' \subset X'$ of $Z \subset X$ over A', then the set of such deformations is a torsor under $\mathrm{H}^0(Z, N_{Z/X} \otimes_A J) = \mathrm{Ext}^0_{\mathcal{O}_Z}(I/I^2, J)$.*

> *(2) There exists an element $\mathrm{ob}_Z \in \mathrm{Ext}^1_{\mathcal{O}_Z}(I/I^2, J)$ (depending on Z and $A' \twoheadrightarrow A$) such that there exists a deformation of $Z \subset X$ over A' if and only if $\mathrm{ob}_Z = 0$.*

Remark C.2.3. An interesting example is when $X = X_0 \times_{\Bbbk} A$ and $X' = X_0 \times_{\Bbbk} A'$ are the base changes of a $\Bbbk$-scheme X_0. If the closed subscheme $Z \subset X$ has constant Hilbert polynomial P (i.e. for each $s \in \mathrm{Spec}\, A$, the Hilbert polynomial of $Z_s \subset X_0 \times_{\Bbbk} \kappa(s)$, with respect to a fixed ample line bundle on X_0, is independent of s), then we have an object $[Z \subset X] \in \mathrm{Hilb}^P(X_0)(A)$ of the Hilbert scheme. In this case, a deformation of $Z \subset X$ over A' is an object $[Z' \subset X'] \in \mathrm{Hilb}^P(X_0)(A')$ which restricts to $[Z \subset X]$. Note that when $A' \twoheadrightarrow A$ is a surjection of local artinian $\Bbbk$-algebras with $\mathfrak{m}_{A'} J = 0$, then there is an identification $\mathrm{H}^0(Z, N_{Z/X} \otimes_A J) = \mathrm{H}^0(Z_0, N_{Z_0/X_0} \otimes_{\Bbbk} J)$ where $Z_0 = Z \times_A \Bbbk$.

Remark C.2.4. In the case that deformations of $Z \subset X$ over A' exist Zariski-locally on X, then there is an obstruction element $\mathrm{ob}_Z \in H^1(Z, N_{Z/X} \otimes_A J)$.

Proof. Suppose first that $X' = \mathrm{Spec}\, B'$, $X = \mathrm{Spec}\, B$ where $B = B' \otimes_{A'} A$ and $Z = \mathrm{Spec}\, B/I$. If there exists a deformation $Z' = \mathrm{Spec}\, B'/I'$, then there is an exact

diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I \otimes_A J & \longrightarrow & I' & \longrightarrow & I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B \otimes_A J & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (B/I) \otimes_A J & \longrightarrow & B'/I' & \longrightarrow & B/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & \\
\end{array}
$$

The exactness of the bottom row (resp. middle row) is equivalent to the flatness of B'/I' (resp. B') over A' by the Local Criterion of Flatness (Corollary A.2.6), while the exactness of the left column follows from the flatness of B/I over A. Conversely, an exact diagram defines a deformation $Z' = \operatorname{Spec} B'/I'$.

We will define an action $\operatorname{Hom}_B(I, (B/I) \otimes_A J)$ on the set of deformations as follows: given $\phi \in \operatorname{Hom}_B(I, (B/I) \otimes_A J)$ and a deformation $Z' = \operatorname{Spec} B'/I'$, define $I'' \subset B'$ as the set of elements $x'' \in B'$ such that its image $\overline{x}'' \in B$ lies in I and such that a lifting $x' \in I'$ of $\overline{x}'' \in I$ satisfies $x'' - x' = \phi(\overline{x}'') \in (B/I) \otimes_A J$ (noting that this condition is independent of the choice of lifting x'). One checks that $\operatorname{Spec} B'/I''$ is another deformation.

On the other hand, given two deformations defined by ideals I' and I'', we define $\phi \colon I \to (B/I) \otimes_A J$ by $\phi(x) = \overline{x' - x''}$ where $x' \in I'$ and $x'' \in I''$ are lifts of x (which forces $x' - x'' \in B \otimes_A J$). One checks that this is a B-module homomorphism providing an inverse to the above construction. We have natural identifications

$$
\operatorname{Hom}_B(I, (B/I) \otimes_A J) = \operatorname{Hom}_{B/I}(I/I^2, B/I \otimes_A J) = \mathrm{H}^0(Z, N_{Z/X} \otimes_A J).
$$

The above constructions globalize to X and establish (1).

For (2), we will assume that there is an open cover $\{U_i\}$ of X such that there exists deformations $Z'_i \subset X' \cap U_i$ of $Z \cap U_i \subset X \cap U_i$ (noting that X and X' are homeomorphic). On $U_{ij} = U_i \cap U_j$, the two deformations $Z'_i|_{U_{ij}}$ and $Z'_j|_{U_{ij}}$ defines an element $\phi_{ij} \in \mathrm{H}^0(U_{ij}, N_{Z/X} \otimes_A J)$ which in turn defines a Čech 1-cocycle $(\phi_{ij}) \in \mathrm{H}^1(X, N_{Z/X} \otimes_A J)$. We leave the reader to check that the vanishing of (ϕ_{ij}) characterizes whether there is a deformation of $Z \subset X$ over A'.

See also [Har10, Thm. 6.2]. $\qquad\square$

C.2.2 Higher-order deformations of schemes

In this chapter, we discuss higher-order deformations and obstructions for schemes that are smooth or more generally local complete intersections.

Definition C.2.5. Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel and $X \to \operatorname{Spec} A$ be a flat morphism of schemes. A *deformation of* $X \to \operatorname{Spec} A$ *over* A' is a flat morphism $X' \to \operatorname{Spec} A'$ together with an isomorphism $\alpha \colon X \xrightarrow{\sim} X' \times_{A'} A$ over A, or in other words a cartesian diagram

$$
\begin{array}{ccc}
X & \dashrightarrow & X' \\
\downarrow{\scriptstyle\text{flat}} & & \downarrow{\scriptstyle\text{flat}} \\
\operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A'.
\end{array}
\tag{C.2.6}
$$

A *morphism of deformations over A'* is a morphism of schemes over A' restricting to the identity on X. By Lemma C.1.8, every morphism of deformations is an isomorphism.

Proposition C.2.7 (Higher-order deformations of Smooth Schemes). *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. If $X \to \operatorname{Spec} A$ is a smooth morphism of schemes where X has affine diagonal (e.g. separated), then*

(1) *The group of automorphisms of a deformation $X' \to \operatorname{Spec} A'$ of $X \to \operatorname{Spec} A$ over A' is bijective to $\mathrm{H}^0(X, T_{X/A} \otimes_A J)$.*

(2) *If there exists a deformation of $X \to \operatorname{Spec} A$ over A', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, T_{X/A} \otimes_A J)$.*

(3) *There is an element $\mathrm{ob}_X \in \mathrm{H}^2(X, T_{X/A} \otimes_A J)$ with the property that there exists a deformation of $X \to \operatorname{Spec} A$ over A' if and only if $\mathrm{ob}_X = 0$.*

Remark C.2.8. If A and A' are local artinian rings with residue field $\Bbbk$ such that $\mathfrak{m}_{A'} J = 0$ and we set $X_0 := X \times_A \Bbbk$, then automorphisms, deformations and obstructions are classified by $\mathrm{H}^i(X_0, T_{X_0} \otimes_{\Bbbk} J)$ for $i = 0, 1, 2$.

Proof. When $X = \operatorname{Spec} A$ is an affine scheme, the same argument of Lemma C.1.11 shows that group of automorphisms of X' is identified with $\operatorname{Hom}_A(\Omega_{A/\Bbbk}, J) = \mathrm{H}^0(X, T_{X/A} \otimes J)$. Since $T_{X/A} \otimes J$ and the assignment of an open subscheme U to the group of automorphisms of the first-order deformation $X'|_U$ are sheaves, Part (1) follows.

Let $\{U_i\}$ be an affine cover of X. Part (2) follows from a similar argument to Proposition C.1.12: Fix a deformation $X' \to \operatorname{Spec} A$ of X. For every other deformation $X'' \to \operatorname{Spec} A$, we know that over each affine U_i, there is an isomorphism $\phi_i \colon X'|_{U_i} \to X''|_{U_i}$ and we let $\phi_{ij} = \phi_j^{-1}|_{X'|_{U_{ij}}} \circ \phi_i|_{X'|_{U_{ij}}}$ viewed as an element of $\mathrm{H}^0(U_i, T_{X/A} \otimes J)$. The Céch 1-cycle (ϕ_{ij}) defines an element in $\mathrm{H}^1(X, T_{X/A} \otimes J)$.

For (3), again using that U_i is affine, we can choose a deformation $U_i' \to \operatorname{Spec} A'$ of U_i. We can also choose isomorphisms $\phi_{ij} \colon U_i'|_{U_{ij}} \to U_j'|_{U_{ij}}$. This defines gluing data for a deformation X' if $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on the triple intersections U_{ijk}. The automorphism $\Psi_{ijk} = \phi_{ik}^{-1} \circ \phi_{jk} \circ \phi_{ij}$ restricts to the identity on U_{ijk} and thus defines an element of $\mathrm{H}^0(U_{ijk}, T_{X/A} \otimes J)$. Consider the Céch complex with $F = T_{X/A} \otimes J$

$$\bigoplus_{i,j} F(U_{ij}) \xrightarrow{d_1} \bigoplus_{i,j,k} F(U_{ijk}) \xrightarrow{d_2} \bigoplus_{i,j,k,l} F(U_{ijkl})$$

$$(s_{ij}) \longmapsto (s_{ij} - s_{ik} + s_{jk})_{ijk}$$

$$(s_{ijk}) \longmapsto (s_{ijk} - s_{ijl} + s_{ikl} - s_{jkl})_{ijkl}.$$

One checks that $d_2(\Psi_{ijk}) = 0$ and that if ϕ_{ij}' is a different choice of isomorphisms then the corresponding element (Ψ_{ijk}') differs from (Ψ_{ijk}) by an element in the image of d_1. Thus (Ψ_{ijk}) is a well-defined element of $\mathrm{H}^2(X, T_{X/A} \otimes J)$.

See also [Har10, Cor. 10.3]. $\qquad\square$

Exercise C.2.9 (Interpretation of deformations and obstruction using gerbes). With the hypotheses of Proposition C.2.7, consider the category $\mathcal{G}$ over Sch/X whose objects over $S \to X$ are cartesian diagrams

$$\begin{array}{ccc} S & \hookrightarrow & S' \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} A & \hookrightarrow & \operatorname{Spec} A' \end{array}$$

where $S \to \operatorname{Spec} A$ is the composition $S \to X \to \operatorname{Spec} A$. A morphism $(S \to X, S \hookrightarrow S' \to \operatorname{Spec} A') \to (T \to X, T \hookrightarrow T' \to \operatorname{Spec} A')$ is the data of a morphism $\phi \colon S' \to T'$ over A' such that ϕ restricts to a morphism $S \to T$ over X.

(a) Show that $\mathcal{G}$ is a gerbe banded by the sheaf of groups $T_{X/A} \otimes_A J$ on X. (*Hint: Use Lemma C.1.8 to show it is a prestack. See Definition 6.2.22 for the definition of a banded gerbe.*)

(b) Give an alternate proof of Proposition C.2.7. (*Hint:* For part (3), use Exercise 6.2.40.)

Exercise C.2.10 (Deformations of principal G-bundles)**.** Let G be a smooth affine algebraic group over a field $\Bbbk$ with Lie algebra $\mathfrak{g}$. Let $X \hookrightarrow X'$ be a closed immersion of finite type $\Bbbk$-schemes defined by a square-zero sheaf of ideals J and assume that X has affine diagonal. If $P \to X$ is a principal G-bundle, one can define deformations over X' and automorphisms of deformations analogous to the case of smooth morphisms. Show that

(1) The group of automorphisms of a deformation $P' \to X'$ of $P \to X$ is bijective to $\mathrm{H}^0(X, \mathfrak{g} \otimes J)$.

(2) If there exists a deformation over X', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, \mathfrak{g} \otimes J)$.

(3) There is an element $\mathrm{ob}_X \in \mathrm{H}^2(X, \mathfrak{g} \otimes J)$ with the property that there exists a deformation over X' if and only if $\mathrm{ob}_X = 0$.

Example C.2.11 (Abelian varieties)**.** If X_0 is an abelian variety over $\mathbb{C}$ of dimension n, then it turns out that deforming X_0 as an abstract scheme is equivalent to deforming it as an abelian variety, and that obstructions to deforming X_0 as an abelian variety also live in $\mathrm{H}^2(X_0, T_{X_0})$. Using that $\Omega_{X_0} = \mathcal{O}_{X_0}^n$ is trivial and the Hodge symmetries, we see that $\mathrm{H}^2(X_0, T_{X_0}) = \mathrm{H}^2(X_0, \mathcal{O}_{X_0})^{\oplus n} = \mathrm{H}^0(X_0, \bigwedge^2 \mathcal{O}_{X_0}^n)^{\oplus n}$ is nonzero. Nevertheless, Grothendieck and Mumford showed that given every deformation problem as in (C.2.6), the obstruction $\mathrm{ob}_X \in \mathrm{H}^2(X, T_{X/A} \otimes_A J)$ vanishes! This shows that abelian varieties are unobstructed, and their moduli space is formally smooth. See [Oor71].

Proposition C.2.12 (Higher-order deformations of Complete Intersections)**.** *Let X_0 be a scheme of finite type over a field $\Bbbk$ such that X_0 is generically smooth and a local complete intersection. Let $A' \twoheadrightarrow A$ be a surjection of noetherian local rings with residue field $\Bbbk$. Assume that the kernel $J := \ker(A' \to A)$ satisfies $\mathfrak{m}_{A'} J = 0$. If $X \to \operatorname{Spec} A$ is a flat morphism of schemes with central fiber $X \times_A \Bbbk \cong X_0$, then*

(1) *The group of automorphisms of a deformation $X \to \operatorname{Spec} A$ over A' is bijective to $\mathrm{Ext}^0_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$.*

(2) *If there exists a deformation of $X \to \operatorname{Spec} A$ over A', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$.*

(3) *There is an element $\mathrm{ob}_X \in \mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}, J)$ with the property that there exists a deformation of $X \to \operatorname{Spec} A$ over A' if and only if $\mathrm{ob}_X = 0$.*

Proof. See [Vis97, Thm. 4.4] for an explicit argument. Alternatively, since X_0 is generically smooth and a local complete intersection, the cotangent complex L_{X_0} is quasi-isomorphic to Ω_{X_0} (see Example C.5.12) and thus the result follows from the fact that the cotangent complex controls deformations (Theorem C.5.11). $\square$

Exercise C.2.13 (hard)**.**

(a) If in addition $\dim X_0 = 1$ and $\sigma_1, \ldots, \sigma_n \colon \operatorname{Spec} A \to X$ are sections such that $p_i = \sigma_i(0) \in X_0$ are smooth points, show that automorphisms, isomorphism classes, and obstructions of a deformation $(X' \to \operatorname{Spec} A', \sigma'_i)$ of $(X \to \operatorname{Spec} A, \sigma_i)$ are classified by $\operatorname{Ext}^2_{\mathcal{O}_{X_0}}(\Omega_{X_0}(\sum_i p_i), J)$.

(b) Show that the conclusion of Proposition C.2.12 holds if $X_0 = \operatorname{Spec} A$ is the spectrum of a complete noetherian local $\Bbbk$-algebra A which is a global complete intersection with an isolated singularity.

Exercise C.2.14 (good practice)**.**

(1) Show that under the bijection $\operatorname{Def}(\Bbbk[\![x,y]\!]/(xy)) \cong \Bbbk$ an element $t \in \Bbbk$ corresponds to the first-order deformation $\operatorname{Spec} \Bbbk[\![x,y,\epsilon]\!]/(xy - t\epsilon)$.

(2) Classify first-order deformations of the A_k-singularity $\Bbbk[\![x,y]\!]/(y^2 - x^{k+1})$.

Exercise C.2.15 (Higher-order deformations of Morphisms)**.** Let $\Bbbk$ be a field and $A' \twoheadrightarrow A$ be a surjection of local artinian rings with residue field $\Bbbk$. Let $f \colon X \to Y$ be a morphism of schemes over A. A *deformation of $f \colon X \to Y$ over A'* is a morphism $f' \colon X' \to Y'$ of schemes over $\operatorname{Spec} A'$ together with isomorphisms $\alpha' \colon X \xrightarrow{\sim} X' \times_{A'} A$ and $\beta' \colon Y \xrightarrow{\sim} Y' \times_{A'} A$ such that both X' and Y' are flat over A' and such that the base change of f' to A is equal to f under the isomorphisms α and β. In other words, a deformation is a cartesian diagram

$$
\begin{array}{ccc}
X & \hookrightarrow & X' \\
\downarrow f & \square & \downarrow f' \\
Y & \hookrightarrow & Y' \\
\downarrow & \square & \downarrow \\
\operatorname{Spec} A & \hookrightarrow & \operatorname{Spec} A'.
\end{array}
$$

A *morphism* of deformations $(X' \to Y', \alpha', \beta') \to (X'' \to Y'', \alpha'', \beta'')$ consists of morphisms $X' \to X''$ and $Y' \to Y''$ over A' compatible with the given isomorphisms.

Assume that X and Y are proper A, and that $f_* \mathcal{O}_X = \mathcal{O}_Y$ and $\mathrm{R}^1 f_* \mathcal{O}_X = 0$. Show that the functor taking a deformation $f' \colon X' \to Y'$ of $f \colon X \to Y$ over A' to the deformation X' over X over A' induces an isomorphism of categories.

Hint: Given a deformation X' over X, define Y' as the ringed space $(Y, f_* \mathcal{O}_{X'})$ (using that X and X' have the same topological space). Use the conditions of f and the flatness of X' over A' to show that Y' is a scheme flat over A'. See also [Vak06, §5.3], [Ran89, Thm. 3.3], and [SP, Tag 0E3X]. (For additional properties of deformations of morphisms, see [Ser06, §3.4].)

C.2.3 Higher-order deformations of vector bundles

Definition C.2.16. Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. Let $X' \to \operatorname{Spec} A'$ be finite type morphism of schemes and set $X := X' \times_{A'} A$. Given a coherent sheaf E on X flat over A, a *deformation of E over $A' \to A$* is a pair (E', α) where E' is a coherent sheaf on X' flat over A' and $\alpha \colon E \to E'|_X$ is an isomorphism. Pictorially, we have

$$
\begin{array}{ccc}
E & & E' \\
| & & | \\
|\,\text{flat}/A & & |\,\text{flat}/A' \\
| & & | \\
X & \hookrightarrow & X'.
\end{array}
$$

A *morphism* $(E, \alpha) \to (E', \alpha')$ *of deformations* is a morphism $\beta \colon E \to E'$ of coherent sheaves on $X_{A'}$ such that $\alpha' = \beta|_X \circ \alpha$. By Lemma C.1.8, every morphism of deformations is an isomorphism.

Proposition C.2.17. *Let $A' \twoheadrightarrow A$ be a surjection of noetherian rings with square-zero kernel J. Let $X' \to \operatorname{Spec} A'$ be a flat and finite type morphism of schemes such that X' has affine diagonal (e.g. separated) and set $X := X' \times_{A'} A$. Let E be a vector bundle on X over A.*

(1) *The group of automorphisms of a deformation E' of E over A' is bijective to $\mathrm{H}^0(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$.*

(2) *If there exists a deformation of E over A', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$.*

(3) *There is an element $\mathrm{ob}_E \in \mathrm{H}^2(X, \mathscr{E}nd_{\mathcal{O}_X}(E) \otimes_A J)$ with the property that there exists a deformation of E over A' if and only if $\mathrm{ob}_E = 0$.*

Remark C.2.18. If X and X' are base changes of a finite type $\Bbbk$-scheme X_0 with affine diagonal, and A and A' are local artinian rings with residue field $\Bbbk$ such that $\mathfrak{m}_{A'} J = 0$, then automorphisms, deformations and obstructions are classified by $\mathrm{H}^2(X_0, \mathscr{E}nd_{\mathcal{O}_{X_0}}(E_0) \otimes_{\Bbbk} J)$ for $i = 0, 1, 2$ where $E_0 = E|_{X_0}$.

Proof. See [Har10, Thm. 7.1]. $\qquad\square$

Exercise C.2.19. Give an alternative proof of Proposition C.2.17 using the technique outlined in Exercise C.2.9.

C.3 Versal formal deformations and Rim–Schlessinger's Criteria

C.3.1 Functors of artin rings

For an algebraically closed field $\Bbbk$, let $\mathrm{Art}_{\Bbbk}$ denote the category of artinian local $\Bbbk$-algebras with residue field $\Bbbk$. The opposite category $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ is equivalent to the category of local artinian $\Bbbk$-schemes (S, s) with $\kappa(s) = \Bbbk$.

Definition C.3.1. We say that a covariant functor $F \colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ is *prorepresentable* if there exists a noetherian complete local $\Bbbk$-algebra R such that for all $A \in \mathrm{Art}_{\Bbbk}$, there is a isomorphism $F \xrightarrow{\sim} h_R$ where $h_R := \operatorname{Hom}_{\Bbbk-\mathrm{alg}}(R, -)$.

Remark C.3.2. If $F \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ is a contravariant functor and $x_0 \in F(\Bbbk)$, then we can consider the induced functor of artin rings

$$F_{x_0} \colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}, \quad A \mapsto \{x \in F(A) \mid x|_{\Bbbk} = x_0 \in F(\Bbbk)\}$$

where $x|_{\Bbbk}$ denotes the image of x under $F(A) \to F(A/\mathfrak{m}_A)$. If F is representable by a scheme X and $x \in X$ is the $\Bbbk$-point corresponding to x_0, then F_{x_0} is prorepresentable by $\widehat{\mathcal{O}}_{X,x}$.

Exercise C.3.3. Provide an example of a non-representable contravariant functor $F \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ and an object $x_0 \in F(\Bbbk)$ such that F_{x_0} is prorepresentable.

Many functors of artin rings are not prorepresentable. For example, if C_0 is a smooth, connected, and projective curve with a non-trivial automorphism group, then the covariant functor $F_{C_0} \colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ where $F_{C_0}(A)$ consists of isomorphism

classes of smooth proper families of curves $\mathcal{C} \to \operatorname{Spec} A$ such that $\mathcal{C} \times_A A/\mathfrak{m}_A$ is isomorphic to C_0, is not prorepresentable. Nevertheless, many moduli functors admit *versal* deformations.

Remark C.3.4. To work over a more general base (e.g. of mixed characteristic), one can consider instead the following setup: let Λ be a noetherian complete local ring with residual field $\Bbbk$ (not necessarily algebraically closed) and $\operatorname{Art}_\Lambda$ be the category of artinian local Λ-algebras $(A, \mathfrak{m})$ with an identification $\Bbbk \xrightarrow{\sim} A/\mathfrak{m}$. Rim–Schlessinger's Criteria (Theorem C.3.12) holds after replacing $\operatorname{Art}_\Bbbk$ with $\operatorname{Art}_\Lambda$. More generally, one can consider the setup where $A \to \Bbbk$ is a finite morphism to a field, not assumed to be the residue field.

Setting $\Lambda = \Bbbk$ recovers our setup, but in many applications, it is often useful to take Λ to be a ring of Witt vectors, e.g. $\Lambda = \mathbb{Z}_p$. In this way, one can consider deforming an object E_0 over $\mathbb{F}_p$ inductively along extensions $\mathbb{Z}/p^{n+1} \twoheadrightarrow \mathbb{Z}/p^n$ with the hope of applying Rim–Schlessinger's Criteria (Theorem C.3.12) and Grothendieck's Existence Theorem (C.4.4) to deform E_0 to an object $\widehat{E}$ over the *characteristic zero ring* $\mathbb{Z}_p$; see Section C.4.3

C.3.2   Versal deformations

As it's important to keep track of automorphisms, we will present Rim–Schlessinger's Criteria, a generalization of Schlessinger's Criterion from functors to prestacks. Therefore we will formulate the definition of versality for prestacks $\mathcal{X}$ over $\operatorname{Art}_\Bbbk^{\operatorname{op}}$. We will assume that $\mathcal{X}(\Bbbk)$ is equivalent to a set consisting of a single object, i.e. there is a unique morphism between any two objects in $\mathcal{X}(\Bbbk)$.

Definition C.3.5. Let $\mathcal{X}$ be a prestack over $\operatorname{Art}_\Bbbk^{\operatorname{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$.

(1) A *formal deformation* $(R, \{x_n\})$ *of* x_0 is the data of a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m}_R)$ together with objects $x_n \in \mathcal{X}(R/\mathfrak{m}_R^{n+1})$ and morphisms $x_{n-1} \to x_n$ over $\operatorname{Spec} R/\mathfrak{m}_R^n \to \operatorname{Spec} R/\mathfrak{m}_R^{n+1}$, or in other words an element of $\varprojlim \mathcal{X}(R/\mathfrak{m}^n)$. When $\mathcal{X} = F$ is a covariant functor $\operatorname{Art}_\Bbbk \to \operatorname{Sets}$, a formal deformation is a compatible sequence of elements $x_n \in F(R/\mathfrak{m}_R^{n+1})$.

(2) A formal deformation $(R, \{x_n\})$ is *versal* if for every surjection $A \twoheadrightarrow A_0$ in $\operatorname{Art}_\Bbbk$ with $\mathfrak{m}_A^{n+1} = 0$, object $\eta \in \mathcal{X}(A)$ and $\Bbbk$-algebra homomorphism $\phi_0 \colon R/\mathfrak{m}_R^{n+1} \to A_0$ with an isomorphism $\alpha_0 \colon x_n|_{A_0} \xrightarrow{\sim} \eta|_{A_0}$ in $\mathcal{X}(A_0)$, there exists a $\Bbbk$-algebra homomorphism $\phi \colon R/\mathfrak{m}_R^{n+1} \to A$ and an isomorphism $\alpha \colon x_n|_A \xrightarrow{\sim} \eta$ in $\mathcal{X}(A)$ such that ϕ_0 is the composition $R/\mathfrak{m}_R^{n+1} \xrightarrow{\phi} A \twoheadrightarrow A_0$ $\alpha|_{A_0} = \alpha_0$.

(3) A versal formal deformation $(R, \{x_n\})$ is *miniversal* (or *a prorepresentable hull*) if the induced map $\operatorname{Hom}_{\Bbbk-\mathrm{alg}}(R, \Bbbk[\epsilon]) \to \mathcal{X}(\Bbbk[\epsilon])/\sim$ on isomorphism classes, defined by $(R \to R/\mathfrak{m}_R^2 \xrightarrow{\phi} \Bbbk[\epsilon]) \mapsto \phi(x_1)$, is bijective.

Remark C.3.6. The deformation $x_n \in \mathcal{X}(R/\mathfrak{m}_R^{n+1})$ can be viewed via Yoneda's 2-Lemma as a morphism $\operatorname{Spec} R/\mathfrak{m}_R^{n+1} \to \mathcal{X}$ or more precisely as $h_{R/\mathfrak{m}_R^{n+1}} \to \mathcal{X}$. Likewise, we can view a formal deformation as a morphism $\{x_n\} \colon h_R \to \mathcal{X}$ where $h_R = \operatorname{Hom}_{\Bbbk-\mathrm{alg}}(R, -)$ (see Exercise C.3.9). With this terminology, $\{x_n\}$ is versal if

there exists a lift for every commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A_0 & \longrightarrow & h_R \\
\cap \downarrow & \nearrow & \downarrow {\{x_n\}} \\
\operatorname{Spec} A & \xrightarrow{\ \eta\ } & \mathcal{X}
\end{array}
\qquad\qquad (\text{C.3.7})
$$

of solid arrows where $A \twoheadrightarrow A_0$ is a surjection in $\operatorname{Art}_{\Bbbk}$. In this way, we see that a versal formal deformation corresponds to the Infinitesimal Lifting Criterion for Smoothness (see Theorem A.3.1(2) and Theorem 3.7.1) with respect to artinian local $\Bbbk$-algebras. Meanwhile, a miniversal deformation is a versal formal deformation inducing an isomorphism on tangent spaces $h_R(\Bbbk[\epsilon]) \to \mathcal{X}(\Bbbk[\epsilon])/\sim$.

Remark C.3.8. The condition of versality can be checked on surjections $A \twoheadrightarrow A_0$ with $\ker(A \to A_0) \cong \Bbbk$. Indeed, the kernel of every surjection $A \twoheadrightarrow A_0$ in $\operatorname{Art}_{\Bbbk}$ is a finite dimensional $\Bbbk$-vector space, and $A \to A_0$ can be factored into surjections where each kernel is one-dimensional.

Exercise C.3.9. Let R be a noetherian complete local $\Bbbk$-algebra and let $h_R = \operatorname{Hom}_{\Bbbk-\mathrm{alg}}(R, -)$ be the covariant functor $\operatorname{Art}_{\Bbbk} \to \operatorname{Sets}$ which we can also view as a prestack over $\operatorname{Art}_{\Bbbk}^{\mathrm{op}}$. If $\mathcal{X}$ is a prestack over $\operatorname{Art}_{\Bbbk}^{\mathrm{op}}$, show that giving a formal deformation $(R, \{x_n\})$ is equivalent to giving a morphism $h_R \to \mathcal{X}$ of prestacks.

Remark C.3.10. If F is prorepresentable by R, then letting $x_n \in F(R/\mathfrak{m}_R^{n+1})$ correspond to the surjection $(R \twoheadrightarrow R/\mathfrak{m}_R^{n+1}) \in h_R(R/\mathfrak{m}_R^{n+1})$, it is easy to see that $\{x_n\}$ is a versal formal deformation. In this case, there is a unique lift in (C.3.7)

Remark C.3.11 (Global prestacks to local deformation prestacks). If $\mathcal{X}$ is a prestack over $\operatorname{Sch}/\Bbbk$ and $x_0 \in \mathcal{X}(\Bbbk)$, we can consider the *local deformation prestack* $\mathcal{X}_{x_0}$ *at* x_0 as the prestack of morphisms $x_0 \to x$ over $\operatorname{Art}_{\Bbbk}^{\mathrm{op}}$ where a morphism $(x_0 \xrightarrow{\alpha} x) \to (x_0 \xrightarrow{\alpha'} x')$ is a morphism $\beta \colon x \to x'$ such that $\alpha' = \alpha \circ \beta$. In other words, an object of $\mathcal{X}_{x_0}$ is a pair (x, α) where $x \in \mathcal{X}(A)$ and $\alpha \colon x_0 \to x|_{\Bbbk}$ is an isomorphism. Note that the fiber category $\mathcal{X}_{x_0}(\Bbbk)$ is equivalent to the set $\{x_0 \xrightarrow{\mathrm{id}} x_0\}$.

If $\mathcal{X}$ is algebraic with a smooth presentation $U \to \mathcal{X}$ from a scheme and $u \in U(\Bbbk)$ is a point mapping to x_0, then we may set $x_n \in \mathcal{X}(\mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1})$ to be the composition $\operatorname{Spec} \mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1} \hookrightarrow U \to \mathcal{X}$. Then $\{x_n\}$ is a versal formal deformation.

On the other hand, if $\mathcal{X}$ is not yet known to be algebraic, one can sometimes verify the existence of versal formal deformation via Rim–Schlessinger's Criteria (Theorem C.3.12) as a first step to verifying the algebraicity of $\mathcal{X}$ via Artin's Axioms for Algebraicity (Theorem C.7.1).

C.3.3 Rim–Schlessinger's Criteria

Rim–Schlessinger's Criteria provides necessary and sufficient conditions for a prestack $\mathcal{X}$ over $\operatorname{Art}_{\Bbbk}^{\mathrm{op}}$ or covariant functor $F \colon \operatorname{Art}_{\Bbbk} \to \operatorname{Sets}$ to admit a versal formal deformation.

Theorem C.3.12 (Rim–Schlessinger's Criteria). *Let $\mathcal{X}$ be a prestack over $\operatorname{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$. For morphisms $B_0 \to A_0$ and $A \to A_0$ in $\operatorname{Art}_{\Bbbk}$, consider the natural functor*

$$
\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A) \qquad\qquad (\text{C.3.13})
$$

Then $\mathcal{X}$ admits a miniversal formal deformation if and only if

(RS$_1$) *the functor* (C.3.13) *is essentially surjective whenever* $A \twoheadrightarrow A_0$ *is surjection with kernel* $\Bbbk$;

(RS$_2$) *the map* (C.3.13) *is essentially surjective when* $A_0 = \Bbbk$ *and* $A = \Bbbk[\epsilon]$, *and given two commutative diagrams*

there exists an isomorphism $\beta \colon y \to y'$ *in* $\mathcal{X}(\Bbbk[\epsilon] \times_\Bbbk B_0)$ *such that* $\alpha_2 = \beta \circ \alpha_1$.

(RS$_3$) $\dim_\Bbbk T_\mathcal{X} < \infty$ *where* $T_\mathcal{X} := \mathcal{X}(\Bbbk[\epsilon])/\sim$.

Moreover, $\mathcal{X}$ *is prorepresentable if and only if* $\mathcal{X}$ *is equivalent to a functor and*

(RS$_4$) *the map* (C.3.13) *is an equivalence whenever* $A \twoheadrightarrow A_0$ *is a surjection with kernel* $\Bbbk$.

Conditions (RS$_2$)–(RS$_3$) (sometimes referred to as *semi-homogeneity*) may be difficult to parse[1] but in practice, it is almost always just as easy to verify the stronger condition (RS$_4$) (called *homogeneity*), and in fact the even stronger condition (RS$_4^*$) (called *strong homogeneity*) introduced in §C.3.4.

Remark C.3.14 (Schlessinger's Criteria). When $\mathcal{X}$ is a covariant functor $F \colon \mathrm{Art}_\Bbbk \to \mathrm{Sets}$ with $F(\Bbbk) = \{x_0\}$, then (RS$_1$)–(RS$_4$) translate into Schlessinger's conditions as introduced in [Sch68]:

(H$_1$) the map (C.3.13) is surjective whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\Bbbk$;

(H$_2$) the map (C.3.13) is bijective when $A_0 = \Bbbk$ and $A = \Bbbk[\epsilon]$;

(H$_3$) $\dim_\Bbbk F(\Bbbk[\epsilon]) < \infty$; and

(H$_4$) the map (C.3.13) is bijective whenever $A \twoheadrightarrow A_0$ is a surjection with kernel $\Bbbk$.

The functor F admits a miniversal formal deformation if (H$_1$)–(H$_3$) hold and is prorepresentable if (H$_3$)–(H$_4$) hold.

If $\mathcal{X}$ satisfies (RS$_1$)–(RS$_3$), then the functor $F_\mathcal{X} \colon \mathrm{Sch}/\Bbbk \to \mathrm{Sets}$ parameterizing isomorphism classes of objects satisfies (H$_1$)–(H$_3$) but the converse does not always hold. On the other hand, the essential surjectivity of $\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$ implies the surjectivity of $F_\mathcal{X}(B_0 \times_{A_0} A) \to F_\mathcal{X}(B_0) \times_{F_\mathcal{X}(A_0)} F_\mathcal{X}(A)$ and the fully faithfulness for $\mathcal{X}$ implies the injectivity of $F_\mathcal{X}$ as long as $\mathrm{Aut}_{\mathcal{X}(B_0)}(y_0) \to \mathrm{Aut}_{\mathcal{X}(A_0)}(y_0|_{A_0})$ is surjective for an object $y_0 \in \mathcal{X}(B_0)$. This latter condition holds in the case when $F_\mathcal{X}(A_0)$ is a set, e.g. when $A_0 = \Bbbk$. If $\mathcal{X}$ is the local deformation prestack arising from an object $x_0 \in \widetilde{\mathcal{X}}(\Bbbk)$ of an algebraic stack $\widetilde{\mathcal{X}}$ over $\mathrm{Sch}/\Bbbk$ as in Remark C.3.11, then the surjectivity condition on automorphisms translates to the inertia stack $I_\mathcal{X} \to \mathcal{X}$ being smooth at $e(x_0)$, where $e \colon \mathcal{X} \to I_\mathcal{X}$ is the identity section, by the Infinitesimal Lifting Criterion for Smoothness (Theorem 3.7.1).

While the existence of a miniversal formal deformation of $F_\mathcal{X}$ suffices for many applications, for moduli problems with automorphisms it is more natural to ask for the existence of a miniversal formal deformation of $\mathcal{X}$ and this generality is needed for some applications, e.g. Artin's Algebraization (Theorem C.6.8) and Artin's Axioms for Algebraicity (Theorem C.7.4).

---

[1]The second part of (RS$_2$) is slightly stronger than requiring that two objects in $\mathcal{X}(\Bbbk[\epsilon] \times_\Bbbk B_0)$ are isomorphic if and only if their images are. Note that (RS$_2$) does not require the isomorphism $\beta \colon y \to y'$ to be compatible with the given morphisms $x \to y$ and $x \to y'$.

Before proceeding to the proof, we first show that (RS_1)–(RS_2) yield natural structures on sets of deformations. In particular, they induce a vector space structure on the tangent space $T_{\mathcal{X}} = \mathcal{X}(\Bbbk[\epsilon])/\sim$, which allows us to make sense of condition (RS_3).

Lemma C.3.15. *Let $\mathcal{X}$ be a prestack over $\mathrm{Art}_{\Bbbk}^{\mathrm{op}}$ such that the groupoid $\mathcal{X}(\Bbbk)$ is equivalent to the set $\{x_0\}$, and let $F_{\mathcal{X}}\colon \mathrm{Art}_{\Bbbk} \to \mathrm{Sets}$ be the covariant functor assigning $A \in \mathrm{Art}_{\Bbbk}$ to the set of isomorphism classes $\mathcal{X}(A)/\sim$. Assume that Condition (RS_2) holds for $\mathcal{X}$.*

(1) The tangent space $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ has a natural structure of a $\Bbbk$-vector space. More generally, for every finite dimensional $\Bbbk$-vector space V, denoting $\Bbbk[V]$ as the $\Bbbk$-algebra $\Bbbk \oplus V$ defined by $V^2 = 0$, the set $F_{\mathcal{X}}(\Bbbk[V])$ has a natural structure of a $\Bbbk$-vector space and there is a functorial bijection $F_{\mathcal{X}}(\Bbbk[V]) = T_{\mathcal{X}} \otimes_{\Bbbk} V$.

(2) Consider a surjection $B \twoheadrightarrow A$ in $\mathrm{Art}_{\Bbbk}$ with square-zero kernel I and an element $x \in \mathcal{X}(A)$, and let $\mathrm{Lift}_x(B)$ be the set of morphisms $x \to y$ over $\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ where $x \xrightarrow{\alpha} y$ is declared equivalent to $x \xrightarrow{\alpha'} y'$ if there is an isomorphism $\beta\colon y \to y'$ such that $\alpha' = \beta \circ \alpha$. There is an action of $T_{\mathcal{X}} \otimes I$ on $\mathrm{Lift}_x(B)$ which is functorial in $\mathcal{X}$. Assuming $\mathrm{Lift}_x(B)$ is non-empty, this action is transitive if Condition (RS_1) holds for $\mathcal{X}$ and free and transitive (i.e. $\mathrm{Lift}_x(B)$ is a torsor under $T_{\mathcal{X}} \otimes I$) if Condition (RS_4) holds for $\mathcal{X}$.

Proof. We first note if V is a finite dimensional vector space, then $\Bbbk[V] = \Bbbk[\epsilon] \times_{\Bbbk} \cdots \times_{\Bbbk} \Bbbk[\epsilon]$ and by applying (RS_2) inductively, we see that the statement of Condition (RS_2) also holds for $A_0 = \Bbbk$ and $A = \Bbbk[V]$. For $B_0 \in \mathrm{Art}_{\Bbbk}$, the first part of (RS_2) implies that $F_{\mathcal{X}}(B_0 \times_{\Bbbk} \Bbbk[V]) \xrightarrow{\sim} F_{\mathcal{X}}(B_0) \times F_{\mathcal{X}}(\Bbbk[V])$ is a bijection. In particular, $F_{\mathcal{X}}(\Bbbk[V] \times_{\Bbbk} \Bbbk[W]) \xrightarrow{\sim} F_{\mathcal{X}}(\Bbbk[V]) \times F_{\mathcal{X}}(\Bbbk[W])$ is bijective for every pair of finite dimensional vector spaces, or in other words the functor $V \mapsto F_{\mathcal{X}}(\Bbbk[V])$ commutes with finite products.

The vector space structure of $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ follows from the bijectivity of

$$F_{\mathcal{X}}(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon']) \xrightarrow{\sim} F_{\mathcal{X}}(\Bbbk[\epsilon]) \times F_{\mathcal{X}}(\Bbbk[\epsilon']). \tag{C.3.16}$$

Indeed, addition of $\tau_1, \tau_2 \in F_{\mathcal{X}}(\Bbbk[\epsilon])$ is defined by using the above identification to view $(\tau_1, \tau_2) \in F_{\mathcal{X}}(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon'])$ and then taking its image under $F(\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon']) \to F(\Bbbk[\epsilon])$ induced by the ring map $\Bbbk[\epsilon] \times_{\Bbbk} \Bbbk[\epsilon'] \to \Bbbk[\epsilon]$ taking $(\epsilon, 0)$ and $(0, \epsilon')$ to ϵ. Scalar multiplication of $c \in \Bbbk$ on $\tau \in F_{\mathcal{X}}(\Bbbk[\epsilon])$ is defined by taking the image of τ under $F_{\mathcal{X}}(\Bbbk[\epsilon]) \to F_{\mathcal{X}}(\Bbbk[\epsilon])$ induced by the map $\Bbbk[\epsilon] \to \Bbbk[\epsilon]$ taking ϵ to $c\epsilon$.

The same argument gives $F_{\mathcal{X}}(\Bbbk[V])$ the structure of a vector space such that the assignment $V \mapsto F_{\mathcal{X}}(\Bbbk[V])$ is a $\Bbbk$-linear functor $\mathrm{Vect}_{\Bbbk}^{\mathrm{fd}} \to \mathrm{Vect}_{\Bbbk}$ defined on finitely dimensional $\Bbbk$-vector spaces. The natural map

$$F_{\mathcal{X}}(\Bbbk[\epsilon]) \times \mathrm{Hom}_{\Bbbk}(\Bbbk[\epsilon], \Bbbk[V]) \to F_{\mathcal{X}}(\Bbbk[V]), \qquad (\tau, \phi) \mapsto \phi^* \tau$$

is $\Bbbk$-bilinear and under the equivalences $T_{\mathcal{X}} = F_{\mathcal{X}}(\Bbbk[\epsilon])$ and $V = \mathrm{Hom}_{\Bbbk}(\Bbbk[\epsilon], \Bbbk[V])$ corresponds to a linear map $T_{\mathcal{X}} \otimes V \to F_{\mathcal{X}}(\Bbbk[V])$, which is an isomorphism. This finishes the proof of (1).

For (2), we observe that the natural map

$$B \times_A B \to \Bbbk[I] \times_{\Bbbk} B, \qquad (b_1, b_2) \mapsto (\overline{b_1} + b_2 - b_1, b_1)$$

is an isomorphism. We therefore have a diagram

$$\mathcal{X}(\Bbbk[I]) \times \mathcal{X}(B) \leftarrow \mathcal{X}(\Bbbk[I] \times_{\Bbbk} B) \cong \mathcal{X}(B \times_A B) \xrightarrow{p_1^*} \mathcal{X}(B) \tag{C.3.17}$$

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where the left functor is essentially surjective by the first part of (RS$_2$). Given $\tau \in T_{\mathcal{X}} \otimes I = F_{\mathcal{X}}(\Bbbk[I])$ (with a choice of representative in $\mathcal{X}(\Bbbk[I])$) and $(x \xrightarrow{\alpha} y) \in \mathrm{Lift}_x(B)$, we would like to define $\tau \cdot (x \xrightarrow{\alpha} y)$ as the image under p_1^* of a choice of preimage of (τ, y). To see that this is well-defined, consider two elements $z, z' \in \mathcal{X}(\Bbbk[I] \times_{\Bbbk} B)$ whose images in $\mathcal{X}(\Bbbk[I]) \times \mathcal{X}(B)$ are isomorphic to (τ, y). This yields a diagram

$$
\begin{array}{ccc}
x_0 \longrightarrow y & & \mathrm{Spec}\,\Bbbk \lhook\joinrel\longrightarrow \mathrm{Spec}\,B \\
\downarrow \;\; {}_{\alpha_1}\downarrow \;\; {}^{\alpha_2}\searrow & \text{over} & \downarrow \qquad\qquad \downarrow \\
\tau \longrightarrow z \dashrightarrow^{\beta} z' & & \mathrm{Spec}\,\Bbbk[I] \lhook\joinrel\longrightarrow \mathrm{Spec}(\Bbbk[I] \times_{\Bbbk} B)
\end{array}
$$

and by the second part of (RS$_2$), there exists a dotted arrow β such that $\alpha_2 = \beta \circ \alpha_1$. Therefore choices of pullbacks $p_1^* z$ and $p_1^* z'$ in $\mathcal{X}(B)$ defines the same element in $\mathrm{Lift}_x(B)$. If (RS$_1$) holds, then the statement of Condition (RS$_1$) also holds for every surjection $A \mapsto A_0$ (since we may factor it as a composition of surjections whose kernels are $\Bbbk$). Therefore if (RS$_1$) holds (resp. (RS$_4$) holds), then $\mathcal{X}(B \times_A B) \to \mathcal{X}(B) \times_{\mathcal{X}(A)} \mathcal{X}(B)$ is essentially surjective (resp. an equivalence), and we see that the action is transitive (resp. free and transitive). $\qquad\square$

Proof Theorem C.3.12. The details of the necessity of these conditions are left to the reader. We will establish the sufficiency. The tangent space $T_{\mathcal{X}} := \mathcal{X}(\Bbbk[\epsilon])/\sim$ has the structure of a vector space by Lemma C.3.15(1) and is finite dimensional by (RS$_2$). Let $N = \dim_{\Bbbk} T_{\mathcal{X}}$ with basis $x_1, \ldots, x_N$ and define $S = \Bbbk[\![x_1, \ldots, x_N]\!]$. We will construct inductively a decreasing sequence of ideals $J_0 \supset J_1 \supset \cdots$ and objects $\eta_n \in \mathcal{X}(S/J_n)$ together with morphisms $\eta_n \to \eta_{n+1}$ over $\mathrm{Spec}\,S/J_n \hookrightarrow \mathrm{Spec}\,S/J_{n+1}$. We set $J_0 = \mathfrak{m}_S$ and $\eta_0 = x_0 \in \mathcal{X}(\Bbbk)$. We also set $J_1 = \mathfrak{m}_S^2$ so that $S/J_1 \cong \Bbbk[T_{\mathcal{X}}]$. Using the bijection $F_{\mathcal{X}}(\Bbbk[T_{\mathcal{X}}]) \cong T_{\mathcal{X}} \otimes_{\Bbbk} T_{\mathcal{X}}$ of Lemma C.3.15(1), the element $\sum_i x_i \otimes x_i$ defines an isomorphism class of an object $\eta_1 \in \mathcal{X}(S/J_1)$ such that the induced map $\mathrm{Spec}\,S/J_1 \to \mathcal{X}$ induces a bijection on tangent spaces. By construction, we have a morphism $\eta_0 \to \eta_1$ over $\mathrm{Spec}\,\Bbbk \hookrightarrow \mathrm{Spec}\,S/J_1$.

Suppose we've constructed J_n and $\eta_{n-1} \to \eta_n$. We claim that the set of ideals

$$
\Sigma = \big\{ J \subset S \mid \mathfrak{m}_S J_n \subset J \subset J_n \text{ and there exists } \eta_n \to \eta \\
\text{over } \mathrm{Spec}\,S/J_n \hookrightarrow \mathrm{Spec}\,S/J \big\} \tag{C.3.18}
$$

has a minimal element. Indeed, it is non-empty since $J_n \in \Sigma$ and given $J, K \in \Sigma$, we must check that $J \cap K \in \Sigma$. To achieve this, choose an ideal $J' \subset S$ satisfying $J \subset J' \subset I$ with $J \cap K = J' \cap K$ and $J' + K = I$. Then $A/(J' \cap K) \cong A/J' \times_{A/I} A/K$. Letting $\eta_J \in \mathcal{X}(S/J)$ and $\eta_K \in \mathcal{X}(S/K)$ be the objects corresponding to J and K, the data of $(\eta_J|_{S/J'} \leftarrow \eta_n \to \eta_K)$ defines an object of $\mathcal{X}(S/J') \times_{\mathcal{X}(S/I)} \mathcal{X}(S/K)$. The functor $\mathcal{X}(A/(J \cap K)) \to \mathcal{X}(S/J') \times_{\mathcal{X}(S/I)} \mathcal{X}(S/K)$ is essentially surjective by (RS$_1$) and the existence of preimage of $(\eta_J|_{S/J'} \leftarrow \eta_n \to \eta_K)$ shows that $J \cap K \in \Sigma$.

Setting $J = \bigcap_n J_n$, then $R = S/J$ is a noetherian complete local $\Bbbk$-algebra with ideals $I_n := J_n/J$. Since $\mathfrak{m}_S J_n \subset J_{n+1}$, we have that $\mathfrak{m}_R^{n+1} \subset I_n$ and thus $\xi_n := \eta_n|_{R/\mathfrak{m}_R^{n+1}}$ defines a formal deformation of x_0 over R.

We must check that $\xi := \{\xi_n\}$ is versal. Suppose $B \twoheadrightarrow A$ is a surjection in $\mathrm{Art}_{\Bbbk}$ with kernel $\Bbbk$ and that we have a diagram

$$
\begin{array}{ccc}
x \longrightarrow \xi & & \mathrm{Spec}\,A \xrightarrow{\;g\;} h_R \\
\downarrow & \text{over} & \big\uparrow \qquad {}^{\widetilde{g}}\nearrow \\
y & & \mathrm{Spec}\,B.
\end{array}
$$

We must construct a morphism $y \to \xi$ extending $x \to \xi$. We claim that it suffices to construct a morphism $\widetilde{g}\colon \operatorname{Spec} B \to h_R$ (i.e. a ring map $R \to B$) extending g. Since $h_R(\Bbbk[\epsilon]) \to T_{\mathcal{X}}$ is bijective, Lemma C.3.15(2) implies that there are actions of $T_{\mathcal{X}}$ on the sets $\operatorname{Lift}_x(B)$ and $\operatorname{Lift}_g(B)$ of isomorphism classes of lifts of x and g to objects in $\mathcal{X}(B)$ and $h_R(B)$ which are compatible with the map $\operatorname{Lift}_g(B) \to \operatorname{Lift}_x(B)$ where $\widetilde{g} \mapsto \widetilde{g}^*\xi$. Thus, we can find $\tau \in T_{\mathcal{X}}$ such that $y = \tau \cdot (\widetilde{g}^*\xi) = (\tau \cdot \widetilde{g})^*\xi$. This gives an arrow $y \to \xi$ over $\tau \cdot \widetilde{g}\colon \operatorname{Spec} B \to h_R$.

To construct $\widetilde{g}$, choose n such that $R \to A$ factors as $R \to R/I_n = S/J_n \to A$. It suffices to show that $\operatorname{Spec} A \to \operatorname{Spec} S/J_n$ extends to a map $\operatorname{Spec} B \to \operatorname{Spec} S/J_{n+1}$ and for this, it suffices to show the existence of a dotted arrow making the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A & \longrightarrow & \operatorname{Spec} S/J_n \\
\Big\uparrow & & \Big\uparrow \\
\operatorname{Spec} B & \longrightarrow & \operatorname{Spec} B \times_A (S/J_n) \\
& & \searrow \\
& & \operatorname{Spec} S/J_{n+1}
\end{array}
$$

commutative. As $S = \Bbbk[\![x_1, \ldots, x_n]\!]$, we may choose an extension $S \to B$ of $S \to S/J_n \to A$. Then $B \times_A (S/J_n) = S/K$ where K is the kernel of the induced map $S \to B \times_A (S/J_n)$. The kernel K lies in the set of ideals defined in (C.3.18): the inclusion $K \subset J_n$ is clear, the inclusion $\mathfrak{m}_S J_n \subset K$ is implied by the equality $\ker(B \to A) = \Bbbk$, and the existence of $\eta_n \to \eta$ over $\operatorname{Spec} S/J_n \hookrightarrow \operatorname{Spec} S/K$ follows from applying (RS$_1$) to the above square. Thus $J_{n+1} \subset K$ and we have a ring map $S/J_{n+1} \to S/K = B \times_A (S/J_n)$ inducing the desired dotted arrow.

Finally, we must show that if $\mathcal{X}$ is equivalent to a functor F and (RS$_4$) holds, then F is prorepresentable by $\xi = \{\xi_n\}$. Given a surjection $B \to A$ with kernel $\Bbbk$ and $x \in F(A)$, it suffices to show the existence of a *unique* lift in every diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A & \xrightarrow{\;g\;} & h_R \\
\Big\uparrow & \nearrow & \Big\downarrow \\
\operatorname{Spec} B & \longrightarrow & F.
\end{array}
$$

This holds because the map $\operatorname{Lift}_g(B) \to \operatorname{Lift}_x(B)$ is bijective by Lemma C.3.15(2) as both are torsors under $T_{\mathcal{X}}$.

See also [Sch68, Thm. 2.11], [SGA7-I, Thm. VI.1.11] and [SP, Tag 06IX], where the result is established more generally for prestacks over the category $\operatorname{Art}_\Lambda$ introduced in Remark C.3.4. $\qquad\square$

C.3.4 Verifying Rim–Schlessinger's Conditions

Consider the following *strong homogeneity* condition:

(RS$_4^*$) $\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$ is an equivalence for every map $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$ of rings with square-zero kernel (where the rings are not necessarily local artinian);

If $\mathcal{X}$ is a prestack over $(\operatorname{Sch}/\Bbbk)$ satisfying (RS$_4^*$), then the local deformation prestack $\mathcal{X}_{x_0}$ at x_0 (see Remark C.3.11) is easily checked to satisfy (RS$_4$). On the other hand, it turns out that every algebraic stack satisfies (RS$_4^*$); see [SP, Tag 07WN].

In other words, the Ferrand pushout $\operatorname{Spec}(B_0 \times_{A_0} A)$ is a pushout in the category of algebraic stacks. Condition (RS_4^*) will appear in our second version of Artin's Axioms for Algebraicity (Theorem C.7.4) as it will be useful to verify openness of versality (in addition to implying (RS_2)–(RS_3) ensuring the existence of versal formal deformations).

For a moduli problem $\mathcal{M}$, it is often possible to verify (RS_4^*) (and thus (RS_4) as well as (RS_1)–(RS_2)) as a consequence of Proposition B.4.10: for a ring map $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$, the functor $\operatorname{Mod}(B_0 \times_{A_0} A) \to \operatorname{Mod}(B_0) \times_{\operatorname{Mod}(A_0)} \operatorname{Mod}(A)$ restricts to an equivalence on flat modules. When B_0, A_0, and A are artinian, there is an elementary argument for this fact since flatness translates to freeness for modules over an artinian ring (Proposition A.2.3).

We say that a prestack $\mathcal{X}$ over $\mathrm{Sch}/\Bbbk$ *admits miniversal formal deformations* if for every $\Bbbk$-point x_0, the local deformation prestack $\mathcal{X}_{x_0}$ (Remark C.3.11) admits a miniversal formal deformation.

Proposition C.3.19. *Each of the moduli problems* $\operatorname{Hilb}^P(X)$, $\mathcal{M}_g$ *(with* $g \geq 2$*) and* $\underline{\operatorname{Bun}}_C$ *over* $\Bbbk$ *satisfy* (RS_3) *and* (RS_4^*)*, and therefore admit miniversal formal deformations.*

Proof. To check (RS_3) for objects $[Z_0 \subset X]$, C_0 and E_0 of $\mathcal{X} = \operatorname{Hilb}^P(X)$, $\mathcal{M}_g$ and $\underline{\operatorname{Bun}}_C$ defined over $\Bbbk$, we have identifications of the tangent spaces $T_\mathcal{X}$ with the *finite dimensional* $\Bbbk$-vector spaces $\mathrm{H}^0(Z_0, N_{Z_0/X})$, $H^1(C_0, T_{C_0})$ and $\mathrm{H}^1(X, \mathscr{E}nd_{\mathcal{O}_X}(E_0))$ by Propositions C.1.4, C.1.12 and C.1.17.

For (RS_4^*), let $B_0 \to A_0$ be a ring map and $A \twoheadrightarrow A_0$ be a surjection with square-zero kernel. Set $B = B_0 \times_{A_0} A$. For $\operatorname{Hilb}^P(X)$, Proposition B.4.10(1)–(2) implies that the diagram

$$
\begin{array}{ccc}
X_{A_0} & \hookrightarrow & X_A \\
\downarrow & & \downarrow \\
X_{B_0} & \hookrightarrow & X_B
\end{array}
$$

is a pushout, and that the functor

$$\operatorname{QCoh}(X_B) \to \operatorname{QCoh}(X_{B_0}) \times_{\operatorname{QCoh}(X_{A_0})} \operatorname{QCoh}(X_A) \tag{C.3.20}$$

restricts to an equivalence between the full subcategory of finitely presented $\mathcal{O}_{X_B}$-modules flat over B and the fiber product of the full subcategories of finitely presented $\mathcal{O}$-modules flat over B_0 and A. This implies the desired equivalence $\operatorname{Hilb}^P(X)(B) \to \operatorname{Hilb}^P(X)(B_0) \times_{\operatorname{Hilb}^P(X)(A_0)} \operatorname{Hilb}^P(X)(A)$ between closed subschemes flat over the base.

For $\mathcal{M}_g$, the essential surjectivity of $\mathcal{M}_g(B) \to \mathcal{M}_g(B_0) \times_{\mathcal{M}_g(A_0)} \mathcal{M}_g(A)$ translates into the existence of an extension

of smooth families of curves. The existence of $\mathcal{D}$ as a pushout of the top face follows from Theorem B.4.1. The fact that $\mathcal{D}$ is smooth over B follows from

Proposition B.4.10(4). The properness of $\mathcal{D} \to \operatorname{Spec} B$ follows from the properness of $\mathcal{D}_0 \to \operatorname{Spec} B_0$. The fully faithfulness translates to the bijectivity of

$$\operatorname{Aut}(\mathcal{D}/B) \to \operatorname{Aut}(\mathcal{D}_0/B_0) \times_{\operatorname{Aut}(\mathcal{C}_0/A_0)} \operatorname{Aut}(\mathcal{C}/A)$$

and follows directly from the fact that $\mathcal{D}$ is a pushout of the top face. Alternatively, one can replicate the above argument for $\operatorname{Hilb}^P(X)$ using the tricanonical embedding.

For $\underline{\operatorname{Bun}}_C$, Proposition B.4.10(1) implies that the functor (C.3.20) restricts to an equivalence on finitely presented $\mathcal{O}$-modules flat over the base and therefore also on vector bundles. $\square$

Exercise C.3.21. Let $X = \operatorname{Spec} A$ be an affine scheme with isolated singularities over a field $\Bbbk$, and let $F_X \colon \operatorname{Art}_\Bbbk \to \operatorname{Sets}$ be the functor, where $F_X(R)$ is the set of isomorphism classes of pairs (B, β) where B is a flat R-algebra and $\beta \colon A \to B \otimes_R R/\mathfrak{m}_R$ is an isomorphism. Show that F_X admits a miniversal formal deformations. See also [Art76, Thm. 8.1].

C.4 Effective formal deformations and Grothendieck's Existence Theorem

Grothendieck's Existence Theorem—sometimes referred to as Formal GAGA—is a powerful technique to show that formal deformations are effective. It is analogous to Serre's GAGA Theorem [Ser56] that for a proper scheme X over $\mathbb{C}$ with analytification X^{an}, the natural functor $\operatorname{Coh}(X) \to \operatorname{Coh}(X^{\mathrm{an}})$, taking a coherent sheaf F to its analytification F^{an}, is an equivalence. The proofs follow very similar strategies.

C.4.1 Effective formal deformations

Definition C.4.1. Let $\mathcal{X}$ be a prestack (or functor) over $(\operatorname{Sch}/\Bbbk)$. Let $x_0 \in \mathcal{X}(\Bbbk)$ and consider a formal deformation $(R, \{x_n\})$ of x_0 (or more precisely a formal deformation of the deformation stack $\mathcal{X}_{x_0}$ at x_0 as defined in Remark C.3.11). We say that $\{x_n\}$ is *effective* if there exists an object $\widehat{x} \in \mathcal{X}(R)$ and compatible isomorphisms $x_n \xrightarrow{\sim} \widehat{x}|_{\operatorname{Spec} R/\mathfrak{m}^{n+1}}$.

Remark C.4.2. A formal deformation $(R, \{x_n\})$ is effective if it is in the essential image of the natural functor $\mathcal{X}(R) \to \varprojlim \mathcal{X}(R/\mathfrak{m}^n)$ or in other words if there exists a dotted arrow making the diagram

commutative.

Example C.4.3. If $F \colon \operatorname{Sch}/\Bbbk \to \operatorname{Sets}$ is a contravariant functor representable by a scheme X over $\Bbbk$, then every formal deformation $(R, \{x_n\})$ is effective. Indeed, x_n corresponds to a morphism $\operatorname{Spec} R/\mathfrak{m}^{n+1} \to X$ with image $x \in X(\Bbbk)$ and thus to a $\Bbbk$-algebra homomorphism $\phi_n \colon \widehat{\mathcal{O}}_{X,x} \to R/\mathfrak{m}^{n+1}$. By taking the inverse image of ϕ_n, we have a local homomorphism $\widehat{\mathcal{O}}_{X,x} \to R$ which in turn defines a morphism $\widehat{x} \colon \operatorname{Spec} R \to X$ extending $\{x_n\}$.

C.4.2 Grothendieck's Existence Theorem

The following is often applied when $(R, \mathfrak{m})$ is a complete *local* ring.

Theorem C.4.4 (Grothendieck's Existence Theorem). *Let X be a scheme proper over a noetherian ring A which is complete with respect to an ideal $\mathfrak{m} \subset A$. Let $X_n := X \times_R R/\mathfrak{m}^{n+1}$. The functor*

$$\mathrm{Coh}(X) \to \varprojlim \mathrm{Coh}(X_n), \qquad E \mapsto (E/\mathfrak{m}^{n+1}E) \qquad (\mathrm{C.4.5})$$

is an equivalence of categories.

The essential surjectivity of (C.4.5) translates to an extension of the diagram

Using the language of formal schemes and setting $\widehat{X} = X \times_{\mathrm{Spec}\,R} \mathrm{Spf}\,R$ to be the $\mathfrak{m}$-adic completion of X, then Grothendieck's Existence Theorem asserts that the functor $\mathrm{Coh}(X) \to \mathrm{Coh}(\widehat{X})$, defined by $E \mapsto \widehat{E}$, is an equivalence.

Remark C.4.6 (Limits of categories). An object of $\varprojlim \mathrm{Coh}(X_n)$ is a sequence (E_n) of coherent $\mathcal{O}_{X_n}$-modules E_n together with isomorphisms $E_{n+1}/\mathfrak{m}^{n+1}E_{n+1} \xrightarrow{\sim} E_n$. A morphism $\phi = (\phi_n) \colon (E_n) \to (F_n)$ is the data of compatible $\mathcal{O}_{X_n}$-module homomorphisms $\phi_n \colon E_n \to F_n$. If $X \cong \mathrm{Spec}\,R$, then $\varprojlim \mathrm{Coh}(X_n)$ is equivalent to the category of finite A-modules. This fact can be used to show that $\varprojlim \mathrm{Coh}(X_n)$ is an abelian category and that (C.4.5) is an exact functor. While the cokernel of a map $(\phi_n) \colon (E_n) \to (F_n)$ in $\varprojlim \mathrm{Coh}(X_n)$ is simply $(\mathrm{coker}\,\phi_n)$, kernels are more subtle. The Artin–Rees Lemma (B.5.4) implies that for each m, the image of $\ker(\alpha_l) \to E_m$ stabilizes for $l \gg m$ to a subsheaf $E'_m \subset E_m$, and that for each n, the quotients $E'_m/\mathfrak{m}^{n+1}E'_m$ stabilizes for $m \gg n$ to a coherent sheaf K_n on X_n. The kernel of (ϕ_n) is (K_n) (which is usually different from $(\ker \phi_n)$). See [SP, Tag 0EHN].

Proof. Fully faithfulness of (C.4.5) translates into the bijection $\mathrm{Ext}^0_{\mathcal{O}_X}(E, F) \xrightarrow{\sim} \varprojlim_n \mathrm{Ext}^0_{\mathcal{O}_{X_n}}(E|_{X_n}, F|_{X_n})$, which is a consequence of Formal Functions (A.5.5). It remains to show essential surjectivity.

Projective case: The key claim is that for every $(E_n) \in \mathrm{Coh}(X_n)$, there exists integers m and r together with compatible surjections $\mathcal{O}_{X_n}(-m)^{\oplus r} \twoheadrightarrow E_n$. Consider the finite type $\mathcal{O}_{X_0}$-algebra $\mathcal{A} := \bigoplus_{i \geq 0} \mathfrak{m}^i\mathcal{O}_{X_0}/\mathfrak{m}^{i+1}\mathcal{O}_{X_0}$ and the finitely generated $\mathcal{O}_{\mathcal{A}}$-module $\mathcal{G} := \bigoplus_{i \geq 0} \mathfrak{m}^i E_i$ (noting that when $(E_n) = (E/\mathfrak{m}^{n+1})$ for a coherent sheaf E on X, then $\mathcal{G}$ is the associated graded $\bigoplus_{i \geq 0} \mathfrak{m}^i E/\mathfrak{m}^{i+1}E$). Viewing $\mathcal{G}$ as a coherent sheaf on $\mathrm{Spec}_{X_0} \mathcal{A}$ and applying Serre's Vanishing Theorem [Har77, Thm. II.5.2] to the projective morphism $\mathrm{Spec}_{X_0} \mathcal{A} \to \mathrm{Spec} \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ gives an integer m_0 such that for all $m \geq m_0$ $\mathrm{H}^1(X_0, \mathcal{G}(m)) = 0$. This yields that $\mathrm{H}^1(X_0, \mathfrak{m}^{n+1}E_n(m)) = 0$ for all n, which in turn implies that

$$\mathrm{H}^0(X_0, E_{n+1}(m)) \twoheadrightarrow \mathrm{H}^0(X_0, E_n(m)) \qquad (\mathrm{C.4.7})$$

is surjective. After possibly increasing m_0, we can assure that $E_0(m)$ is globally generated by sections $s_{0,1}, \ldots, s_{0,r}$. By the surjectivity of (C.4.7), we can find compatible lifts $s_{n,i} \in \mathrm{H}^0(X_0, E_n(m))$ of the sections $s_{0,i}$. This gives compatible maps $\mathcal{O}_{X_n}^{\oplus r} \to E_n(m)$, each which is surjective by Nakayama's Lemma.

With the claim established, given $(E_n) \in \mathrm{Coh}(X_n)$, we choose compatible surjections $\phi_n \colon \mathcal{O}_{X_n}(-m)^{\oplus r} \twoheadrightarrow E_n$. Let (K_n) be the kernel of (ϕ_n) in the category $\varprojlim \mathrm{Coh}(X_n)$ as described in Remark C.4.6. Applying the claim again to (K_n) induces compatible right exact sequences

$$\mathcal{O}_{X_n}(-m')^{\oplus r'} \xrightarrow{\alpha_n} \mathcal{O}_{X_n}(-m)^{\oplus r} \to E_n \to 0.$$

By fully faithfulness, the morphism (α_n) is induced by a morphism $\widetilde{\alpha} \colon \mathcal{O}_X(-m')^{\oplus r'} \to \mathcal{O}_X(-m)^{\oplus r}$, and it follows that $E := \mathrm{coker}(\widetilde{\alpha})$ is a coherent sheaf on X such that $(E/\mathfrak{m}^{n+1}E) \cong (E_n)$ in $\varprojlim \mathrm{Coh}(X_n)$.

Proper case: By Chow's Lemma [Har77, Exer. II.4.10], there exists a projective morphism $g \colon X' \to X$ which is an isomorphism over a dense open subset $U \subset X$ such that X' is projective over the $\mathrm{Spec}\, R$. Given $(E_n) \in \mathrm{Coh}(X_n)$, consider the pull back $(g^*E_n) \in \mathrm{Coh}(X'_n)$, where $X'_n := X' \times_R R/\mathfrak{m}^{n+1}$. Since X' is projective over R, there exists a coherent sheaf E' on X' an an isomorphism

$$(\beta_n) \colon (g^*E_n) \xrightarrow{\sim} (E'/\mathfrak{m}^{n+1}E')$$

in $\varprojlim \mathrm{Coh}(X'_n)$. By Finiteness of Cohomology (A.5.3), g_*E' is coherent. Conceptually, the argument is now very straightforward: adjunction $E_n \to g_*g^*E_n$ induces an exact sequence

$$0 \to (K_n) \to (E_n) \xrightarrow{(\alpha_n)} \big((g_*E')/\mathfrak{m}^{n+1}(g_*E')\big) \xrightarrow{(q_n)} (Q_n) \to 0 \qquad \text{(C.4.8)}$$

in $\varprojlim \mathrm{Coh}(X_n)$ such that the $\ker(\alpha) = (K_n)$ and $\mathrm{coker}(\alpha) = (Q_n)$ are supported on $X \setminus U$, i.e. each K_n and Q_n are supported on $X \setminus U$. By noetherian induction, we can assume that there are coherent sheaves K and Q on X and isomorphisms $(K_n) \cong (K/\mathfrak{m}^{n+1}K)$ and $(Q_n) \cong (Q/\mathfrak{m}^{n+1}Q)$. By full faithfulness and exactness of (C.4.5), the map (q_n) is induced by a surjection $g_*E' \to Q$, and we define the kernel $F = \ker(g_*E' \to Q)$. Since $\mathrm{Ext}^i_{\mathcal{O}_X}(K, F) \xrightarrow{\sim} \varprojlim_n \mathrm{Ext}^i_{\mathcal{O}_{X_n}}(K_n, F/\mathfrak{m}^{n+1}F)$, there is an extension $0 \to K \to E \to F \to 0$ giving a coherent sheaf E on X such that $(E_n) \cong (E/\mathfrak{m}^{n+1}E)$.

The existence of (α_n) in (C.4.8), however, takes some work. The formalism of formal schemes can be useful here as one can consider the map of formal schemes $\widehat{g} \colon \widehat{X}' \to \widehat{X}$ over $\mathrm{Spf}\, R$, the coherent sheaf $\widehat{E} = \varprojlim E_n$ on $\widehat{X}$, and the adjunction morphism $\widehat{E} \to \widehat{g}_*\widehat{g}^*\widehat{E}$ in $\mathrm{Coh}(\widehat{X})$, and apply a version of formal functions [EGA, III$_1$.4.1.5]—sometimes called the 'comparison theorem'—giving identifications $\mathrm{R}^i\widehat{g}_*(\widehat{g}^*\widehat{E}) \cong \varprojlim \mathrm{R}^n g_*(g^*E_n) \cong (\mathrm{R}^n g_*E')\widehat{}$.

We argue more directly. We first show that there are unique maps α_n filling in the diagram

$$
\begin{array}{ccc}
E_n & \xrightarrow{\quad c_n \quad} & g_*g^*E_n \\
\Big\downarrow{\scriptstyle \alpha_n} & & \Big\downarrow{\scriptstyle g_*\beta_n} \\
(g_*E')/\mathfrak{m}^{n+1}(g_*E') & \xrightarrow{\quad d_n \quad} & g_*(E'/\mathfrak{m}^{n+1}E'),
\end{array}
$$

where c_n and d_n are the natural maps. By the uniqueness, the existence is local on X, so we may assume that $X = \mathrm{Spec}\, B$. Since $B \to \widehat{B} := \varprojlim B/\mathfrak{m}^{n+1}B$ is flat and

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all the coherent sheaves in the diagram are annihilated by a power of $\mathfrak{m}$, Flat Base Change (A.2.12) further reduces us to the case that B is complete with respect to $\mathfrak{m}B$. In this case $E := \varprojlim E_n$ corresponds to a coherent sheaf on X mapping to (E_n) in $\varprojlim \mathrm{Coh}(X_n)$. Applying Formal Functions (A.5.4) to $X' \to \mathrm{Spec}\, B$ and E' yields that $g_* E' \cong \varprojlim g_*(E'/\mathfrak{m}^{n+1}E')$, which shows that $\varprojlim d_n$ is an isomorphism. Therefore the map $E \to g_* E'$ defined by the inverse limit of the maps $g_* \beta_n \circ c_n$ shows that there are unique maps α_n as desired. We now show that the kernel and cokernel of (C.4.8) are supported on $X \setminus U$. Since $g^* E \cong \varprojlim g^* E_n$, the isomorphism $(\beta_n) \colon (g^* E_n) \to (E'/\mathfrak{m}^{n+1}E')$ comes from an isomorphism $g^* E \xrightarrow{\sim} E'$ such that $(\alpha_n) \colon (E_n) \to \big((g_* E')/\mathfrak{m}^{n+1}(g_* E')\big)$ comes from $E \to g_* g^* E \xrightarrow{\sim} g_* E'$. Since g is an isomorphism over U, the adjunction map $E \to g_* g^* E$ is an isomorphism over U, and it follows that $\ker(\alpha_n)$ and $\mathrm{coker}(\alpha_n)$ are supported on $X \setminus U$. See also [EGA, III$_1$.5.1.4], [FGI$^+$05, Thm. 8.4.2] and [SP, Tag 088E]. $\qquad\square$

Corollary C.4.9. *Let $(R, \mathfrak{m})$ be a noetherian complete local ring and $X_n \to \mathrm{Spec}\, R/\mathfrak{m}^{n+1}$ be a sequence of proper morphisms such that $X_n \times_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \cong X_{n-1}$. If L_n is a compatible sequence of line bundles on X_n such that L_0 is ample, then there exists a projective morphism $X \to \mathrm{Spec}\, R$ and an ample line bundle L on X and compatible isomorphisms $X_n \cong X \times_R R/\mathfrak{m}^{n+1}$ and $L_n \xrightarrow{\sim} L|_{X_n}$.*

Remark C.4.10. It follows that there is an extension in the cartesian diagram

such that X is projective over R. We say that the formal deformation $\{X_n \to \mathrm{Spec}\, R/\mathfrak{m}^{n+1}\}$ of X_0 is *effective* (which is sometimes referred to as *algebraizable.*

Proof. Consider the finitely generated graded $\Bbbk$-algebra $B = \bigoplus \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and the quasi-coherent graded $\mathcal{O}_{X_0}$-algebra $\mathcal{A} = B \otimes_{\Bbbk} \mathcal{O}_{X_0}$. By applying Serre's Vanishing Theorem to $\mathrm{Spec}_{X_0} \mathcal{A}$ and the ample line bundle $L_0 \otimes_{\mathcal{O}_{X_0}} \mathcal{O}_{X_0'}$, we see that $\mathrm{H}^1(X_0, \mathcal{A} \otimes L_0^{\otimes d}) = 0$ for $d \gg 0$. We have a closed immersion $X_0 \hookrightarrow \mathbb{P}^N$ defined by a basis $s_{0,0}, \ldots, s_{0,N}$ of $\mathrm{H}^0(X_0, L_0^{\otimes d})$. Noting that $\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}$ is identified with $\ker(\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n})$, we may tensor the corresponding short exact sequence by $L_{n+1}^{\otimes d}$ to obtain a short exact sequence

$$0 \to (\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}) \otimes L_0^{\otimes d} \to L_{n+1}^{\otimes d} \to L_n^{\otimes d} \to 0,$$

where we've used that that $(\mathfrak{m}^n \mathcal{O}_{X_{n+1}}/\mathfrak{m}^{n+1}\mathcal{O}_{X_{n+1}}) \otimes L_{n+1}^{\otimes d}$ is supported on X_0 along with the identifications $L_{n+1} \otimes \mathcal{O}_{X_m} \cong L_m$ for $m \leq n$. The vanishing of $\mathrm{H}^1(X_0, \mathcal{A} \otimes L_0^{\otimes d})$ implies that we may lift the sections $s_{0,0}, \ldots, s_{0,N}$ inductively to compatible sections $s_{n,0}, \ldots, s_{n,N}$ of $\mathrm{H}^0(X_n, L_n^{\otimes d})$. By Nakayama's Lemma, the induced morphisms $X_n \hookrightarrow \mathbb{P}^N_{R/\mathfrak{m}^{n+1}}$ are closed immersions giving a commutative diagram

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Grothendieck's Existence Theorem (C.4.4) gives an equivalence $\mathrm{Coh}(\mathbb{P}^N_R) \to \varprojlim \mathrm{Coh}(\mathbb{P}^N_{R/\mathfrak{m}^{n+1}})$. Essential surjectivity gives a coherent sheaf E on $\mathbb{P}^N_R$ extending $\{\mathcal{O}_{X_n}\}$ and full faithfulness gives a surjection $\mathcal{O}_{\mathbb{P}^N_R} \to E$ extending $\mathcal{O}_{\mathbb{P}^N_{R/\mathfrak{m}^{n+1}}} \to \mathcal{O}_{X_n}$. We take $X \subset \mathbb{P}^N_R$ to be the closed subscheme defined by $\ker(\mathcal{O}_{\mathbb{P}^N_R} \to E)$.

See also [EGA, III.5.4.5], [FGI$^+$05, Thm. 8.4.10] and [SP, Tag 089A]. $\qquad\square$

Remark C.4.11. Suppose that X is flat over R and that we are only given an ample line bundle L_0 on X_0 (but not the line bundles L_n). Then the obstruction to deforming L_{n-1} to L_n is an element $\mathrm{ob}_{L_{n-1}} \in \mathrm{H}^2(X, \mathcal{O}_X \otimes_{\Bbbk} \mathfrak{m}^n)$ by Proposition C.2.17. If these cohomology groups vanish (e.g. if X is of dimension 1), then there exists compatible extensions L_n, and thus the formal deformation $\{X_n \to \mathrm{Spec}\, R/\mathfrak{m}^{n+1}\}$ are effective.

Without the existence of deformations L_n of L_0, it is not necessarily true that formal deformations are effective. For instance, there is a projective K3 surface (X_0, L_0) and a first-order deformation $X_1 \to \mathrm{Spec}\,\Bbbk[\epsilon]$ which is not projective (so L_0 does not deform to X_1), and a formal deformation which is not effective; see [Har10, Ex. 21.2.1]. Similarly, formal deformations of abelian varieties may not be effective. Note that for both K3 surfaces and abelian varieties, Rim–Schlessinger's Criteria applies to construct versal formal deformations.

Corollary C.4.12. *For each of the moduli problems* $\mathrm{Hilb}^P(X)$, $\mathcal{M}_g$ *(with $g \geq 2$) and* $\underline{\mathrm{Bun}}_C$ *over $\Bbbk$, every formal deformation is effective. In particular, there exists effective versal formal deformations.*

Proof. For $\mathrm{Hilb}^P(X)$, we show the effectivity of a formal deformation $\{Z_n \subset X_{R/\mathfrak{m}^{n+1}}\}$ by following the argument at the end of the proof of Corollary C.4.9 (with $X_n \subset \mathbb{P}^N_{R/\mathfrak{m}^{n+1}}$ replaced with $Z_n \subset X_{R/\mathfrak{m}^{n+1}}$): Grothendieck's Existence Theorem (C.4.4) implies the existence of a coherent sheaf E on X_R extending $\{\mathcal{O}_{Z_n}\}$ and a surjection $\mathcal{O}_{X_R} \to E$ extending $\{\mathcal{O}_{X_n} \to \mathcal{O}_{Z_n}\}$. We take $Z \subset X_R$ defined by $\ker(\mathcal{O}_{X_R} \to E)$.

For $\mathcal{M}_g$, the effectivity of a formal deformation $\{C_n \to \mathrm{Spec}\, R/\mathfrak{m}^{n+1}\}$ follows from Corollary C.4.9 by taking L_n be the ample bundle $\Omega_{C_n/(R/\mathfrak{m}^{n+1})}$, or by taking L_0 to be any ample bundle on C_0 and using Proposition C.2.17 and the vanishing of $\mathrm{H}^2(C_0, \mathcal{O}_{C_0})$ to inductively deform L_0 to a compatible sequence of line bundle L_n on C_n.

For $\underline{\mathrm{Bun}}_C$, the effectivity of a formal deformation of vector bundles E_n on $C_{R/\mathfrak{m}^{n+1}}$ follows directly from Grothendieck's Existence Theorem (C.4.4) noting that the coherent extension is necessarily a vector bundle.

The last statement follows from the existence of versal formal deformations of these moduli problems (Proposition C.3.19). $\qquad\square$

Exercise C.4.13. Let X and Y be proper schemes over the spectrum $S = \mathrm{Spec}\, R$ of a complete noetherian local ring R. Denote by X_n and Y_n the restrictions of X and Y to $S_n = \mathrm{Spec}\, R/\mathfrak{m}^{n+1}_R$. Show that a compatible sequence of morphisms $f_n \colon X_n \to Y_n$ over S_n extends to a unique morphism $f \colon X \to Y$.

Exercise C.4.14. If $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$ and $(R, \mathfrak{m})$ is a noetherian complete local ring with residue field $\Bbbk$, show that the functor

$$\mathcal{X}(R) \to \varprojlim \mathcal{X}(R/\mathfrak{m}^{n+1})$$

is an equivalence of categories. In particular, every formal deformation is effective.

Remark C.4.15. Formal deformations of abstract K3 surfaces may fail to be effective, and thefore the moduli stack parameterizing K3 surfaces is not algebraic. See [Sta06, Claim 3.5].

C.4.3 Lifting to characteristic 0

One striking application of deformation theory is to "lift" a smooth variety X_0 over a field $\Bbbk$ of $\operatorname{char}(\Bbbk) = p$ to characteristic 0. We say that X_0 is *liftable to characteristic* 0 if there exists a noetherian complete local ring $(R, \mathfrak{m})$ of characteristic 0 such that $R/\mathfrak{m} = \Bbbk$ and a smooth scheme $X \to \operatorname{Spec} R$ such that $X_0 \cong X \times_R \Bbbk$.[2] One can hope to then use characteristic 0 techniques (e.g. Hodge theory) on X and deduce properties of X_0. The strategy to lift a variety X_0 is to inductively deform X_0 to smooth schemes X_n over $R/\mathfrak{m}^{n+1}$ and then apply Grothendieck's Existence Theorem to effective the formal deformation. Note however that to achieve this, we must work over a mixed characteristic base as in Remark C.3.4 rather than over a fixed field $\Bbbk$.

Smooth curves are liftable as obstructions to deforming both the curve and the ample line bundle both vanish. Serre produced an example of a non-liftable projective threefold (see [Har10, Thm. 22.4]), which Mumford extended to a non-liftable projective surface (see [FGI$^+$05, Cor. 8.6.7]). On the other hand, Mumford showed that principally polarized abelian varieties are liftable [Mum69] while Deligne showed that K3 surfaces are liftable [Del81]. These examples are quite interesting as, in both cases, formal deformations are not necessarily effective (see Remark C.4.11) and additional techniques are needed.

C.5 Cotangent complex

In this chapter, we summarize properties of the cotangent complex of a morphism of schemes as introduced in [Ill71] globalizing work of André [And67] and Quillen [Qui68, Qui70] on the cotangent complex of a ring homomorphism. One advantage of the cotangent complex is that it allows us to describe the deformations and obstruction of singular schemes; see Theorem C.5.11.

C.5.1 Properties of the cotangent complex

Theorem C.5.1. *For every morphism $f\colon X \to Y$ of schemes (resp. finite type morphism of noetherian schemes), there exists a complex*

$$L_{X/Y}\colon \cdots \to L_{X/Y}^{-1} \to L_{X/Y}^0 \to 0$$

of flat $\mathcal{O}_X$-modules with quasi-coherent (resp. coherent) cohomology, whose image in $D_{\mathrm{QCoh}}^-(\mathcal{O}_X)$ (resp. $D_{\mathrm{Coh}}^-(\mathcal{O}_X)$) is also denoted by $L_{X/Y}$. It satisfies the following properties:

(1) $\mathrm{H}^0(X, L_{X/Y}) \cong \Omega_{X/Y}$;

(2) f is smooth if and only if f is locally of finite presentation and $L_{X/Y}$ is a perfect complex supported in degree 0. In this case $L_{X/Y}$ is quasi-isomorphic to the complex where the vector bundle $\Omega_{X/Y}$ sits in degree 0;

---

[2]There are some variants to this definition, e.g. when R is already given as a complete DVR with residue field $\Bbbk$.

(3) If f is flat and finitely presented, then f is syntomic if and only if $L_{X/Y}$ is a perfect complex supported in degrees $[-1, 0]$. Explicitly, if f factors as a local complete intersection $X \hookrightarrow \widetilde{Y}$ defined by a sheaf of ideals I and a smooth morphism $\widetilde{Y} \to Y$, then $L_{X/Y}$ is quasi-isomorphic to $0 \to I/I^2 \xrightarrow{d} \Omega_{X/Y} \to 0$ (with $\Omega_{X/Y}$ in degree 0);

(4) If

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
$$

is a cartesian diagram with either f or g flat (or more generally f and g are tor-independent), then there is a quasi-isomorphism $g'^* L_{X/Y} \to L_{X'/Y'}$. (Note that without any flatness condition $g'^* \Omega_{X/Y} \cong \Omega_{X'/Y'}$.)

(5) If $X \xrightarrow{f} Y \to Z$ is a composition of morphisms of schemes, then there is an exact triangle in $D^-_{\mathrm{QCoh}}(\mathcal{O}_X)$

$$
f^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to f^* L_{Y/Z}[1].
$$

This induces a long exact sequence on cohomology

$$
\begin{array}{c}
\cdots \longrightarrow \mathrm{H}^{-2}(L_{X/Y}) \longrightarrow \\
\mathrm{H}^{-1}(f^* L_{X/Z}) \longrightarrow \mathrm{H}^{-1}(L_{X/Z}) \longrightarrow \mathrm{H}^{-1}(L_{X/Y}) \longrightarrow \\
f^* \Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0
\end{array}
$$

extending the usual right exact sequence on differentials [Har77, II.8.12]. (Note that if f is smooth, then $H^{-1}(L_{X/Y}) = 0$ and $f^* \Omega_{Y/Z} \to \Omega_{X/Z}$ is injective.)

Proof. See [Ill71, II.1.2.3], [SP, Tag 08T2] for the definition of the cotangent complex of a morphism of schemes (and more generally for morphisms of ringed topoi). For (1)–(5), see [Ill71, II.1.2.4.2, II.3.1.2, II.3.2.6, II.2.2.3 and II.2.1.2] and [SP, Tags 08UV, 0D0N, 0FK3, 08QQ, and 08T4] (noting that [SP, Tag 08RB] relates the naive cotangent complex $NL_{X/Y}$ to $L_{X/Y}$). $\square$

C.5.2 Truncations of the cotangent complex

The definition of the cotangent complex relies on simplicial techniques and we won't attempt an exposition here. We will however give an explicit description of its truncation, which often suffice for applications.

First, if $X \to Y$ factors as a closed immersion $X \hookrightarrow P$ defined by a sheaf of ideals I and a smooth morphism $P \to Y$, then the truncation $\tau_{\geq -1}(L_{X/Y})$ of $L_{X/Y}$ in degrees $[-1, 0]$ is quasi-isomorphic to $0 \to I/I^2 \xrightarrow{d} \Omega_{X/Y} \to 0$ (with $\Omega_{X/Y}$ in degree 0). In the case that $X \to Y$ is smooth or syntomic, then $X \hookrightarrow \widetilde{Y}$ is a regular immersion, I/I^2 is a vector bundle and $L_{X/Y}$ is quasi-isomorphic to $\tau_{\geq -1}(L_{X/Y})$ (Theorem C.5.1(3)).

For a morphism $X = \operatorname{Spec} A \to \operatorname{Spec} B = Y$ of affine schemes, Lichtenbaum–Schlessinger [LS67] offer an explicit description of $\tau_{\geq -2}(L_{X/Y})$. Choose a polynomial ring $P = B[x_i]$ (with possibly infinitely many generators) and a surjection $P \twoheadrightarrow A$

as B-algebras with kernel I. Choose a free P-module $F = \oplus_{\lambda \in \Lambda} P$ and a surjection $p\colon F \twoheadrightarrow I$ of P-modules with kernel $K = \ker(p)$. Let $K' \subset K$ be the submodule generated by $p(x)y - p(y)x$ for $x, y \in F$. Then the truncation $\tau_{\geq -2}(L_{X/Y})$ (or rather $\tau_{\geq -2}(L_{B/A})$) is quasi-isomorphic to the complex of A-modules

$$K/K' \to F \otimes_P A \to \Omega_{P/B} \otimes_P A \qquad (\text{C.5.2})$$

with the last term in degree 0; see [SP, Tag 09CG].

One defines the T^i *functors* on the category of A-modules by

$$T^i(A/B, -) := \mathrm{H}^i(\mathrm{Hom}_A(L_{A/B}, -)),$$

which can be used for instance to describe deformations of schemes (see Example C.5.12). See also [LS67, §2.3] and [Har10, §1.3].

C.5.3   Extensions of algebras and schemes

Definition C.5.3. An *extension* of a ring homomorphism $R \to A$ by an A-module J is an exact sequence of R-modules

$$0 \to J \to A' \to A \to 0$$

where $A' \to A$ is an R-algebra homomorphism and $J \subset A'$ is an ideal with $J^2 = 0$. (Note that since $J^2 = 0$, $J = J/J^2$ is a module over $A = A'/J$.) The *trivial extension* is $A[J] := A \oplus J$ where multiplication is defined by $J^2 = 0$.

A morphism of extensions is a morphism of short exact sequences which is the identity on J and A. By the five lemma, a morphism of extensions is necessarily an isomorphism. We let $\underline{\mathrm{Exal}}_R(A, J)$ be the groupoid of extensions of $R \to A$ by J, and $\mathrm{Exal}_R(A, J)$ the set of isomorphism classes.

Remark C.5.4. Geometrically, an extension is a commutative diagram of schemes

$$\begin{array}{ccc}
\mathrm{Spec}\, A & \hookrightarrow & \mathrm{Spec}\, A' \\
\downarrow & \swarrow & \\
\mathrm{Spec}\, R &
\end{array}$$

such that $J \cong \ker(A' \to A)$ and $J^2 = 0$.

The set of extensions $\mathrm{Exal}_R(A, J)$ is functorial with respect to A and J:

(a) Given a map $B \to A$ of R-algebras, there is a map $\mathrm{Exal}_R(A, J) \to \mathrm{Exal}_R(B, J)$ given by mapping a complex $0 \to J \to A' \to A \to 0$ to $0 \to J \to A' \times_A B \to B \to 0$.

(b) Given an A-module map $\alpha\colon J \to J$, there is a map $\alpha_*\colon \mathrm{Exal}_R(A, J) \to \mathrm{Exal}_R(A, J)$ given by mapping a complex $0 \to J \to A' \to A \to 0$ to $0 \to J \to (A' \oplus J)/\{(-x, \alpha(x)), x \in J\} \to A \to 0$.

(c) Given modules J and K, the natural map $(p_{1,*}, p_{2,*})\colon \mathrm{Exal}_R(A, J \oplus K) \to \mathrm{Exal}_R(A, J) \oplus \mathrm{Exal}_R(A, K)$, induced from (b) by the projections $p_1\colon J \oplus K \to J$ and $p_2\colon J \oplus K \to K$, is a bijection.

Moreover, $\mathrm{Exal}_R(A, J)$ naturally has the structure of an A-module: scalar multiplication by $x \in A$ is defined using (b) with $x\colon J \to J$ and addition is defined by $\mathrm{Exal}_R(A, J) \times \mathrm{Exal}_R(A, J) \cong \mathrm{Exal}_R(A, J \oplus J) \xrightarrow{\Sigma_*} \mathrm{Exal}_R(A, J)$ using the bijection in (c) and the map Σ_* of (b) where $\Sigma\colon J \oplus J \to J$ is addition. The maps (a)–(c) are in fact maps of A-modules. See [Ill71, §III.1.1] for details.

Proposition C.5.5. *Let R be a ring.*

(1) Given a R-algebra A and an exact sequence $0 \to J' \to J \to J'' \to 0$ of A-modules, there is an exact sequence

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Der}_R(A, J') & \longrightarrow & \mathrm{Der}_R(A, J) & \longrightarrow & \mathrm{Der}_R(A, J'') \\
& & \mathrm{Exal}_R(A, J) & \longrightarrow & \mathrm{Exal}_R(A, J) & \longrightarrow & \mathrm{Exal}_R(A, J'')
\end{array}
$$

of A-modules.

(2) Given a homomorphism $B \to A$ of R-algebras, there is an exact sequence

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Der}_B(A, J) & \longrightarrow & \mathrm{Der}_R(A, J) & \longrightarrow & \mathrm{Der}_R(B, J) \\
& & \mathrm{Exal}_B(A, J) & \longrightarrow & \mathrm{Exal}_R(A, J) & \longrightarrow & \mathrm{Exal}_R(B, J)
\end{array}
$$

of A-modules.

Proof. See [EGA, 0.20.2.3] and [Ill71, III.1.2.4.3, III.1.2.5.4]. $\qquad\square$

Remark C.5.6. The top row of (C.5.5) can be realized using the right exact sequence $\Omega_{B/R} \otimes_B A \to \Omega_{A/R} \to \Omega_{A/B} \to 0$. Namely, apply $\mathrm{Hom}_A(-, J)$ and use the identities $\mathrm{Hom}_A(\Omega_{A/B}, J) = \mathrm{Der}_B(A, J)$, $\mathrm{Hom}_A(\Omega_{A/R}, J) = \mathrm{Der}_R(A, J)$ and $\mathrm{Hom}_A(\Omega_{B/R} \otimes_B A, J) = \mathrm{Hom}_B(\Omega_{B/R}, J) = \mathrm{Der}_R(B, J)$.

The cotangent complex can be applied to extend these sequences to long exact sequences; see Remark C.5.9.

The definition of Exal extends naturally to schemes (and more generally to ringed topoi).

Definition C.5.7. An *extension* of a morphism $X \to S$ of schemes by a quasi-coherent $\mathcal{O}_X$-module J is a short exact sequence

$$ 0 \to J \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0 $$

where $X \hookrightarrow X'$ is a closed immersion of schemes defined by the sheaf of ideals $J \subset \mathcal{O}_{X'}$ with $J^2 = 0$. (Note that the condition $J^2 = 0$ implies that the $J \subset \mathcal{O}_{X'}$ is naturally a $\mathcal{O}_X$-module.) The *trivial extension* is $X[J] := (X, \mathcal{O}_X \oplus J)$ where the ring structure is defined by $J^2 = 0$.

A *morphism of extensions* is a morphism of short exact sequences which is the identity on J and $\mathcal{O}_X$. We let $\underline{\mathrm{Exal}}_S(X, J)$ be the category of extensions of $X \to S$ by J, and $\mathrm{Exal}_S(X, J)$ be the set of isomorphism classes.

The set $\mathrm{Exal}_S(X, J)$ is naturally an $\mathcal{O}_X$-module and is functorial in X and J. In fact, the groupoid $\underline{\mathrm{Exal}}_S(X, J)$ is a *Picard category*, and the prestack over Sch/S whose fiber category over $f : T \to S$ is $\underline{\mathrm{Exal}}_T(X_T, f^*J)$ is a *Picard stack*; see [Ill71, III.1.1.5] and [SGA4, XVIII.1.4].

C.5.4 The cotangent complex and deformation theory

Theorem C.5.8. *If $X \to Y$ is a morphism of schemes and J is a quasi-coherent $\mathcal{O}_Y$-module, there is a natural isomorphism*

$$ \mathrm{Exal}_Y(X, J) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/Y}, J). $$

Proof. See [Ill71, III.1.2.3]. □

Remark C.5.9. This identification allows us to use the cotangent complex to extend the 6-term left exact sequences of Proposition C.5.5 to long exact sequences. Namely, applying $\mathrm{Hom}_{\mathcal{O}_X}(L_{X/Y}, -)$ to the exact sequence $0 \to J' \to J \to J''$ extends C.5.5(1) and applying $\mathrm{Hom}_{\mathcal{O}_X}(-, J)$ to the exact triangle $f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y}$ extends C.5.5(2).

When $X = \mathrm{Spec}\, A \to \mathrm{Spec}\, B = Y$ is a morphism of affine schemes, using the T^i functors of §C.5.2, the above equivalence translates to $\mathrm{Exal}_B(A, J) = T^1(A/B, J)$. This can be established using the explicit description of the Lichtenbaum–Schlessinger truncated cotangent complex (C.5.2); see [LS67, 4.2.2] and [Har10, Thm. 5.1]. The T^i functors can also be used to extend the 6-term sequences of Proposition C.5.5 to 9-term sequences; see [LS67, 2.3.5-6] and [Har10, Thms. 3.4-5].

Remark C.5.10. More generally, there is an equivalence between the groupoid $\underline{\mathrm{Exal}}_Y(X, J)$ and the groupoid obtained from the 2-term complex

$$[C^{-1} \xrightarrow{d} C^0] := \tau_{\leq 0}(\mathrm{RHom}_{\mathcal{O}_X}(\tau_{\geq -1}L_{X/Y}, J)[1])$$

where objects are elements of C^0 and $\mathrm{Mor}(c, c') = d^{-1}(c - c')$; see [Ill71, III.1.2.2].

Theorem C.5.11. *Consider the following deformation problem*

where $f \colon X \to Y$ is a morphism of schemes and $i \colon Y \hookrightarrow Y'$ is a closed immersion of schemes defined by an ideal sheaf $J \subset \mathcal{O}_{Y'}$ with $J^2 = 0$. A deformation is a morphism $f' \colon X' \to Y'$ making the above diagram cartesian, and a morphism of deformations is a morphism over Y' restricting to the identity on X.

(1) *The group of automorphisms of a deformation $f' \colon X' \to Y'$ is isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X}(L_{X/Y}, f^*J)$.*

(2) *If there exists a deformation, then the set of deformations is a torsor under $\mathrm{Ext}^1_{\mathcal{O}_X}(L_{X/Y}, f^*J)$.*

(3) *There exists an element $\mathrm{ob}_X \in \mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/Y}, f^*J)$ with the property that there exists a deformation if and only if $\mathrm{ob}_X = 0$.*

Proof. See [Ill71, III.2.1.7] and [SP, Tag 08UZ]. See also [LS67, 4.2.5] and [Har10, Thm. 10.1] for descriptions in the affine case using the truncated cotangent complex. □

Example C.5.12. As a reality check, let's first consider a smooth morphism $f \colon X \to \mathrm{Spec}\, A$ is smooth and a surjection $A' \twoheadrightarrow A$ of noetherian rings with square-zero kernel J. By the identification

$$\mathrm{Ext}^i_{\mathcal{O}_X}(L_{X/A}, f^*J) = \mathrm{H}^i(X, T_{X/A} \otimes_A J),$$

we recover Proposition C.2.7.

Second, let's consider a scheme X_0 locally of finite type over a field $\Bbbk$ which is generically smooth and a local complete intersection. In this case, every point of X_0

has an open neighborhood U such that $U = V(I) \subset \mathbb{A}^n$ where I is generated by a regular sequence. We always have a right exact sequence

$$I/I^2 \xrightarrow{d} \Omega_{\mathbb{A}^n}|_U \to \Omega_U \to 0, \tag{C.5.13}$$

and by properties of the cotangent complex (C.5.1(3)), we have that $L_U = [I/I^2 \to \Omega_{\mathbb{A}^n}|_U]$ and supported in degrees $[-1, 0]$. On the other hand, since U is generally smooth, $I/I^2 \to \Omega_{\mathbb{A}^n}|_U$ is generically injective. But as I/I^2 is a vector bundle (as I is generated by a regular sequence), it can have no torsion subsheaves. It follows that the sequence C.5.13 is also left exact and that Ω_U is quasi-isomorphic to L_U. Thus, Ω_{X_0} is also quasi-isomorphic to L_{X_0}, and automorphisms, deformations, and obstructions are classified by $\operatorname{Ext}^i_{\mathcal{O}_X}(\Omega_{X_0}, J)$ recovering Proposition C.2.12.

One major advantage of the cotangent complex is that for $\operatorname{Ext}^i_{\mathcal{O}_X}(L_{X/A}, f^*J)$ for $i = 0, 1, 2$ classifies automorphisms, deformations, and obstructions for an arbitrary morphism. Moreover, the truncated cotangent complex $\tau_{\geq -2} L_{X/A}$ suffices to compute automorphisms, deformations and obstructions; for instance when $X = \operatorname{Spec} B$ is affine, we get equivalent descriptions using the T^i functors $T^i(L_{B/A}, f^*J)$ as defined in §C.5.2.

Remark C.5.14. There are analogous results for other deformation problems. For instance, for the deformation problem

where the horizontal morphisms are closed immersions defined by square-zero ideal sheaves J_X, J_Y and J_Z, then automorphisms, deformations, and obstructions are classified by $\operatorname{Ext}^i_{\mathcal{O}_X}(f^*L_{Y/Z}, J_X)$ for $i = -1, 0, 1$ [Ill71, III.2.2.4]. An important special case is when $Y = Y'$ and $Z = Z'$.

C.6 Artin Algebraization

Artin Algebraization is a procedure to "algebraize" or extend an effective versal formal deformation $\xi \in \mathcal{M}(R)$ to an object $\eta \in \mathcal{M}(U)$ over a *finite type* $\Bbbk$-scheme U. In this section, we show how Artin Algebraization follows from Artin Approximation following the ideas of Conrad and de Jong [CdJ02].

C.6.1 Limit preserving prestacks

Extending the definition of a limit preserving functor §B.5.6, we say that a prestack $\mathcal{X}$ over Sch/$\Bbbk$ is *limit preserving* (or *locally of finite presentation*) if for every system B_λ of $\Bbbk$-algebras, the natural functor

$$\operatorname{colim} \mathcal{X}(B_\lambda) \to \mathcal{X}(\operatorname{colim} B_\lambda)$$

is an equivalence of categories. When $\mathcal{X}$ is an algebraic stack over $\Bbbk$, then this equivalent to the morphism $\mathcal{X} \to \operatorname{Spec} \Bbbk$ being locally of finite presentation; see Exercise 3.3.31).

Lemma C.6.1. *Each of the prestacks* $\mathrm{Hilb}^P(X)$, $\mathcal{M}_g$ *(with $g \geq 2$) and $\underline{\mathrm{Bun}}_C$ over* $(Sch/\Bbbk)$ *are limit preserving.*

Proof. To add. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

C.6.2   Conrad–de Jong Approximation

In Artin Approximation (Theorem B.5.26), the initial data is an object over a noetherian complete local $\Bbbk$-algebra $\widehat{\mathcal{O}}_{S,s}$ which is assumed to be the completion of a finitely generated $\Bbbk$-algebra at a maximal ideal. We will now see that a similar approximation result still holds if this latter hypothesis is dropped and one approximates *both* the complete local ring and the object.

Recall also that if $(A, \mathfrak{m})$ is a local ring and M is an A-module, then the *associated graded module of M* is defined as $\mathrm{Gr}_{\mathfrak{m}}(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M / \mathfrak{m}^{n+1} M$; it is a graded module over the graded ring $\mathrm{Gr}_{\mathfrak{m}}(A)$.

Theorem C.6.2 (Conrad–de Jong Approximation). *Let $\mathcal{X}$ be a limit preserving prestack over $\mathrm{Sch}/\Bbbk$. Let $(R, \mathfrak{m}_R)$ be a noetherian complete local $\Bbbk$-algebra and let $\xi \in \mathcal{X}(R)$. Then for every integer $N \geq 0$, there exist*

(1) an affine scheme $\mathrm{Spec}\,A$ of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in \mathrm{Spec}\,A$,

(2) an object $\eta \in \mathcal{X}(A)$,

(3) an isomorphism $\phi_{N+1} \colon R/\mathfrak{m}_R^{N+1} \cong A/\mathfrak{m}_u^{N+1}$,

(4) an isomorphism of $\xi|_{R/\mathfrak{m}_R^{N+1}}$ and $\eta|_{A/\mathfrak{m}_u^{N+1}}$ via ϕ_N, and

(5) an isomorphism $\mathrm{Gr}_{\mathfrak{m}_R}(R) \cong \mathrm{Gr}_{\mathfrak{m}_u}(A)$ of graded $\Bbbk$-algebras.

The proof of this theorem will proceed by simultaneously approximating equations and relations defining R and the object ξ. The statements (1)–(4) will be easily obtained as a consequence of Artin Approximation. A nice insight of Conrad and de Jong is that condition (5) can be ensured by Artin Approximation, and moreover that this condition suffices to imply the isomorphism of complete local $\Bbbk$-algebras in Artin Algebraization. As such, condition (5) takes the most work to establish.

We will need some preparatory results controlling the constant appearing in the Artin–Rees Lemma (B.5.4).

Definition C.6.3 (Artin–Rees Condition). Let $(A, \mathfrak{m})$ be a noetherian local ring. Let $\varphi \colon M \to N$ be a morphism of finite A-modules. Let $c \geq 0$ be an integer. We say that $(\mathsf{AR})_c$ *holds for φ* if

$$\varphi(M) \cap \mathfrak{m}^n N \subset \varphi(\mathfrak{m}^{n-c} M), \quad \forall n \geq c.$$

The Artin–Rees Lemma (B.5.4) implies that $(\mathsf{AR})_c$ holds for φ if c is sufficiently large; see [AM69, Prop. 10.9] or [Eis95, Lem. 5.1].

Lemma C.6.4. *Let $(A, \mathfrak{m})$ be a noetherian local ring. Let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \qquad and \qquad L' \xrightarrow{\alpha'} M \xrightarrow{\beta'} N$$

be two complexes of finite A-modules. Let c be a positive integer. Assume that

(a) the first sequence is exact,

(b) the complexes are isomorphic modulo $\mathfrak{m}^{c+1}$, and

(c) $(\mathsf{AR})_c$ holds for α and β.

Then there exists an isomorphism $\mathrm{Gr}_{\mathfrak{m}}(\mathrm{coker}\,\beta) \to \mathrm{Gr}_{\mathfrak{m}}(\mathrm{coker}\,\beta')$ of graded $\mathrm{Gr}_{\mathfrak{m}}(A)$-modules.

Proof. The proof, while technical, is rather straightforward. First, by taking free presentations of L and L', we can assume that $L = L'$. One shows that $(\mathsf{AR})_c$ holds for β' and that the second sequence is exact. Then one establishes the equality

$$\mathfrak{m}^{n+1}N + \beta(M) \cap \mathfrak{m}^n N = \mathfrak{m}^{n+1}N + \beta'(M) \cap \mathfrak{m}^n N$$

by using that $(\mathsf{AR})_c$ holds for β to show the containment "$\subset$" and then using $(\mathsf{AR})_c$ holds for β' to get the other containment. The statement then follows from the description $\mathrm{Gr}_{\mathfrak{m}}(\mathrm{coker}\,\beta)_n = \mathfrak{m}^n N/(\mathfrak{m}^{n+1}N + \beta(M) \cap \mathfrak{m}^n N)$ and the similar description of $\mathrm{Gr}_{\mathfrak{m}}(\mathrm{coker}\,\beta')_n$. For details, see [CdJ02, §3] and [SP, Tag 07VF]. $\square$

Proof of Conrad–de Jong Approximation (Theorem C.6.2). Since $\mathcal{X}$ is limit preserving and R is the colimit of its finitely generated $\Bbbk$-subalgebras, there is an affine scheme $V = \mathrm{Spec}\,B$ of finite type over $\Bbbk$ and an object γ of $\mathcal{X}$ over V together with a 2-commutative diagram

$$\mathrm{Spec}\,R \xrightarrow{\quad} V \xrightarrow{\gamma} \mathcal{X}.$$

with ξ over the arrow $\mathrm{Spec}\,R \to V$.

Let $v \in V$ be the image of the maximal ideal $\mathfrak{m} \subset R$. After adding generators to the ring B if necessary, we can assume that the composition $\widehat{\mathcal{O}}_{V,v} \to R \to R/\mathfrak{m}^2$ is surjective. This implies that $\widehat{\mathcal{O}}_{V,v} \to R$ is surjective by Complete Nakayama's Lemma (B.5.6(2)). The goal now is to simultaneously approximate over V the equations and relations defining the closed immersion $\mathrm{Spec}\,R \hookrightarrow \mathrm{Spec}\,\widehat{\mathcal{O}}_{V,v}$ and the object ξ. To accomplish this goal, we choose a resolution

$$\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}} \widehat{\mathcal{O}}_{V,v} \to R \to 0 \tag{C.6.5}$$

as $\widehat{\mathcal{O}}_{V,v}$-modules and consider the functor

$$F\colon (\mathrm{Sch}\,/V) \to \mathrm{Sets}$$
$$(T \to V) \mapsto \big\{\text{complexes } \mathcal{O}_T^{\oplus r} \xrightarrow{\alpha} \mathcal{O}_T^{\oplus s} \xrightarrow{\beta} \mathcal{O}_T\big\}.$$

It is not hard to check that this functor is limit preserving. The resolution in (C.6.5) yields an element of $F(\widehat{\mathcal{O}}_{V,v})$. Applying Artin Approximation (Theorem B.5.26) yields an étale morphism $(V' = \mathrm{Spec}\,B', v') \to (V, v)$ and an element

$$(B'^{\oplus r} \xrightarrow{\alpha'} B'^{\oplus s} \xrightarrow{\beta'} B') \in F(V') \tag{C.6.6}$$

such that α', β' are equal to $\widehat{\alpha}, \widehat{\beta}$ modulo $\mathfrak{m}^{N+1}$.

Let $U = \mathrm{Spec}\,A \hookrightarrow \mathrm{Spec}\,B' = V'$ be the closed subscheme defined by $\mathrm{im}\,\beta'$ and let $u = v' \in U$. Consider the composition

$$\eta\colon U \hookrightarrow V' \to V \xrightarrow{\gamma} \mathcal{X}$$

As $R = \mathrm{coker}\,\widehat{\beta}$ and $A = \mathrm{coker}\,\beta'$, we have an isomorphism $R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$ together with an isomorphism of $\xi|_{R/\mathfrak{m}^{N+1}}$ and $\eta|_{A/\mathfrak{m}_u^{N+1}}$. This gives statements (1)–(4).

To establish (5), we need to show that there are isomorphisms $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \mathfrak{m}_u^n/\mathfrak{m}_u^{n+1}$. For $n \leq N$, this is guaranteed by the isomorphism $R/\mathfrak{m}^{N+1} \cong A/\mathfrak{m}_u^{N+1}$. On the other hand, for $n \gg 0$, this can be seen to be a consequence of the Artin–Rees Lemma (B.5.4). To handle the middle range of n, we need to control the constant appearing in the Artin–Rees Lemma. First, note that before we applied Artin Approximation, we could have increased N to ensure that $(\mathsf{AR})_N$ holds for $\widehat{\alpha}$ and $\widehat{\beta}$. We are thus free to assume this. Now statement (5) follows directly if we apply Lemma C.6.4 to the exact complex $\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}} \widehat{\mathcal{O}}_{V,v}$ of (C.6.5) and the complex $\widehat{\mathcal{O}}_{V,v}^{\oplus r} \xrightarrow{\widehat{\alpha}'} \widehat{\mathcal{O}}_{V,v}^{\oplus s} \xrightarrow{\widehat{\beta}'} \widehat{\mathcal{O}}_{V,v}$ obtained by restricting (C.6.6) to $F(\widehat{\mathcal{O}}_{V,v})$.

See also [CdJ02] and [SP, Tag 07XB]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Exercise C.6.7. Show that Conrad–de Jong Approximation implies Artin Approximation.

C.6.3 Artin Algebraization

Artin Algebraization has a stronger conclusion than Artin Approximation or Conrad–de Jong Approximation in that no approximation is necessary. It guarantees the existence of an object η over a pointed affine scheme $(\operatorname{Spec} A, u)$ of finite type over $\Bbbk$, which agrees with the given effective formal deformation ξ to all orders. To ensure this, we need to impose that ξ is *versal at u*, i.e. that the restrictions $\xi_n = \xi|_{A/\mathfrak{m}_u^{n+1}}$ define a versal formal deformation $\{\xi_n\}$ over A (Definition C.3.5).

Theorem C.6.8 (Artin Algebraization). *Let $\mathcal{X}$ be a limit preserving prestack over $\mathrm{Sch}/\Bbbk$. Let $(R, \mathfrak{m})$ be a noetherian complete local $\Bbbk$-algebra and $\xi \in \mathcal{X}(R)$ be an effective versal formal deformation. There exist*

- *(1) an affine scheme $\operatorname{Spec} A$ of finite type over $\Bbbk$ and a $\Bbbk$-point $u \in \operatorname{Spec} A$;*
- *(2) an object $\eta \in \mathcal{X}(A)$;*
- *(3) an isomorphism $\alpha \colon R \xrightarrow{\sim} \widehat{A}_{\mathfrak{m}_u}$ of $\Bbbk$-algebras; and*
- *(4) a compatible family of isomorphisms $\xi|_{R/\mathfrak{m}^{n+1}} \cong \eta|_{A/\mathfrak{m}_u^{n+1}}$ (under the identification $R/\mathfrak{m}^{n+1} \cong A/\mathfrak{m}_u^{n+1}$) for $n \geq 0$.*

Remark C.6.9. If $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$, then there exists an isomorphism $\xi \cong \eta|_{\widehat{A}_{\mathfrak{m}_u}}$.

Remark C.6.10. If R is known to be the completion of a finitely generated $\Bbbk$-algebra, this theorem can be viewed as an easy consequence of Artin Approximation. Indeed, one applies Artin Approximation with $N = 1$ and then uses versality to obtain compatible maps $R \to A/\mathfrak{m}_u^{n+1}$ and therefore a map $R \to \widehat{A}_{\mathfrak{m}_u}$ which is an isomorphism modulo $\mathfrak{m}^2$. As R and $\widehat{A}_{\mathfrak{m}_u}$ are abstractly isomorphic, the homomorphism $R \to \widehat{A}_{\mathfrak{m}_u}$ is an isomorphism by Complete Nakayama's Lemma (B.5.6(2)) and the statement follows. The argument in the general case is analogous, except we use Conrad–de Jong Approximation instead of Artin Approximation.

Proof of Artin Algebraization (Theorem C.6.8). Applying Conrad–de Jong Approximation (Theorem C.6.2) with $N = 1$, we obtain an affine scheme $\operatorname{Spec} A$ of finite type over $\Bbbk$ with a $\Bbbk$-point $u \in \operatorname{Spec} A$, an object $\eta \in \mathcal{X}(A)$, an isomorphism $\phi_2 \colon \operatorname{Spec} A/\mathfrak{m}_u^2 \to \operatorname{Spec} R/\mathfrak{m}^2$, an isomorphism $\alpha_2 \colon \xi|_{R/\mathfrak{m}^2} \to \eta|_{A/\mathfrak{m}_u^2}$, and an isomorphism $\operatorname{Gr}_{\mathfrak{m}}(R) \cong \operatorname{Gr}_{\mathfrak{m}_u}(A)$ of graded $\Bbbk$-algebras. We claim that ϕ_2 and α_2 can be extended inductively to a compatible family of morphisms $\phi_n \colon \operatorname{Spec} A/\mathfrak{m}_u^{n+1} \to \operatorname{Spec} R$

and isomorphisms $\alpha_n \colon \xi|_{A/\mathfrak{m}_u^{n+1}} \to \eta|_{A/\mathfrak{m}_u^{n+1}}$. Given ϕ_n and α_n, versality of ξ implies that there is a lift ϕ_{n+1} filling in the commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A/\mathfrak{m}_u^n & \xrightarrow{\;\phi_n\;} & \operatorname{Spec} R \\
\downarrow & \nearrow{\scriptstyle \phi_{n+1}} & \downarrow{\scriptstyle \xi} \\
\operatorname{Spec} A/\mathfrak{m}_u^{n+1} & \xrightarrow[\eta|_{A/\mathfrak{m}_u^{n+1}}]{} & \mathcal{X},
\end{array}
$$

which establishes the claim. By taking the limit, we have a homomorphism $\widehat{\phi} \colon R \to \widehat{A}_{\mathfrak{m}_u}$ which is surjective by Complete Nakayama's Lemma (B.5.6(2)). On the other hand, for each n the $\Bbbk$-vector spaces $\mathfrak{m}^N/\mathfrak{m}^{N+1}$ and $\mathfrak{m}_u^N/\mathfrak{m}_u^{N+1}$ have the same dimension. This implies that $\widehat{\phi}$ is an isomorphism.

See also [Art69b, Thm. 1.6] and [CdJ02, §4], where the statement is established more generally when $\mathcal{X}$ is defined over a scheme S whose local rings are G-rings where it is required that $\operatorname{Spec} R/\mathfrak{m} \xrightarrow{\xi_0} \mathcal{X} \to S$ be of finite type. $\qquad\square$

C.7 Artin's Axioms for Algebraicity

A spectacular application of Artin Algebraization is a criterion—often verifiable in practice—ensuring that a given stack is algebraic. This is called Artin's Axioms for Algebraicity and we provide two versions below Theorems C.7.1 and C.7.4. This foundational result was proved by Artin in the very same paper [Art74] where he introduced algebraic stacks.

The first version can be proved easily using Artin Algebraization.

Theorem C.7.1. *(Artin's Axioms for Algebraicity—first version) Let $\mathcal{X}$ be a stack over $\Bbbk$. Then $\mathcal{X}$ is an algebraic stack locally of finite type over $\Bbbk$ if and only if the following conditions hold:*

(1) (Limit preserving) The stack $\mathcal{X}$ is limit preserving over $\mathrm{Sch}/\Bbbk$, i.e. for every system B_λ of $\Bbbk$-algebras, the functor

$$\operatorname{colim} \mathcal{X}(B_\lambda) \to \mathcal{X}(\operatorname{colim} B_\lambda)$$

is an equivalence of categories.

(2) (Representability of the diagonal) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.

(3) (Existence of versal formal deformations) Every $x_0 \in \mathcal{X}(\Bbbk)$ has a versal formal deformation $\{x_n\}$ over a noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$.

(4) (Effectivity) For every noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$ with residue field $\Bbbk$, the natural functor

$$\mathcal{X}(\operatorname{Spec} R) \to \varprojlim \mathcal{X}(\operatorname{Spec} R/\mathfrak{m}^n)$$

is an equivalence of categories.

(5) (Openness of versality) For every morphism $U \to \mathcal{X}$ from a finite type $\Bbbk$-scheme which is versal at $u \in U(\Bbbk)$ (i.e. the formal deformation $\{\operatorname{Spec} \widehat{\mathcal{O}}_{U,u}/\mathfrak{m}_u^{n+1} \to \mathcal{X}\}$ is versal), there exists an open neighborhood V of u such that $U \to \mathcal{X}$ is versal at every $\Bbbk$-point of V.

Proof. We first note that for a representable and locally of finite type morphism $U \to \mathcal{X}$ from a finite type $\Bbbk$-scheme U, the Infinitesimal Lifting Criterion for Smoothness (Theorem A.3.1, Theorem 3.7.1) implies that $U \to \mathcal{X}$ is smooth if and only if it is versal at all $\Bbbk$-points $u \in U$. Indeed, this is clear when $U \to \mathcal{X}$ is representable by schemes, and the general case follows as one can see that both properties are étale-local on U.

For $(\Rightarrow)$, (1) holds by Exercise 3.3.31, (2) holds by Theorem 3.2.1 and (4) holds by Exercise C.4.14. If $U \to \mathcal{X}$ is a morphism from a finite type $\Bbbk$-scheme, then it is necessarily representable and locally of finite type. By using the above equivalence between versality and smoothness, (3) holds by choosing a smooth presentation $U \to \mathcal{X}$ and a preimage $u \in U(\Bbbk)$ of x_0 and taking the formal deformation $\{\operatorname{Spec} \mathcal{O}_{U,u}/\mathfrak{m}_u^{n+1} \to \mathcal{X}\}$, and (5) holds by openness of smoothness.

For the converse, we first note that representability of the diagonal, i.e. condition (2), implies that every morphism $U \to \mathcal{X}$ from a scheme U is representable and the limit preserving property (1) implies that $U \to \mathcal{X}$ is locally of finite type. For every object $x_0 \in \mathcal{X}(\Bbbk)$, we will construct a smooth morphism $U \to \mathcal{X}$ from a scheme and a preimage $u \in U(\Bbbk)$ of x_0. Conditions (3)–(4) guarantee that there exists an effective versal formal deformation $\widehat{x} \colon \operatorname{Spec} R \to \mathcal{X}$ of x_0 where $(R, \mathfrak{m})$ is a noetherian complete local $\Bbbk$-algebra with residue field $\Bbbk$. By Artin Algebraization (Theorem C.6.8), there exists a finite type $\Bbbk$-scheme U, a point $u \in U(\Bbbk)$, a morphism $p \colon U \to \mathcal{X}$, an isomorphism $R \cong \widehat{\mathcal{O}}_{U,u}$ and compatible isomorphisms $p|_{R/\mathfrak{m}^{n+1}} \xrightarrow{\sim} \widehat{x}|_{R/\mathfrak{m}^{n+1}}$. By (5), we can replace U with an open neighborhood of u so that $U \to \mathcal{X}$ is versal at every $\Bbbk$-point of U. By the equivalence in the first paragraph, we have obtained a smooth morphism $(U, u) \to (\mathcal{X}, x_0)$.

See also [Art74], [LMB00, Cor. 10.11] and [SP, Tag 07Y4] where the result is established more generally. $\square$

Remark C.7.2. In practice, condition (1)–(4) are often easy to verify directly with (3) a consequence of Rim–Schlessinger's Criteria (Theorem C.3.12) and (4) a consequence of Grothendieck's Existence Theorem (C.4.4). Also note that (2) can sometimes be established by applying the theorem to the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$, i.e. to the Isom sheaves $\operatorname{Isom}_T(x, y)$ of objects $x, y \in \mathcal{X}(T)$ over a scheme T. In some cases, Condition (5) can be checked directly, while for more general moduli problems, it is often a consequence of a well-behaved deformation and obstruction theory, as will be explained in the next section.

C.7.1 Refinements of Artin's Axioms

We state a refinement of Artin's Axioms for Algebraicity that is often easier to verify in practice. To formulate the statements, we will need a bit of notation. Let $\xi \in \mathcal{X}(A)$ be an object over a finitely generated $\Bbbk$-algebra A. Let M be a finite A-module and denote by $A[M]$ the ring $A \oplus M$ defined by $M^2 = 0$. Let $\operatorname{Def}_\xi(M)$ the set of isomorphism classes of diagrams

$$
\begin{array}{ccc}
\operatorname{Spec} A & \xrightarrow{\ \xi\ } & \mathcal{X} \\
\cap\ \downarrow & \nearrow^{\eta} & \\
\operatorname{Spec} A[M], & &
\end{array}
$$

where an isomorphism of two extensions $\eta, \eta' \colon \operatorname{Spec} A[M] \to \mathcal{X}$ is by definition an isomorphism $\eta \xrightarrow{\sim} \eta'$ in $\mathcal{X}(A[M])$ restricting to the identity on ξ. Let $\operatorname{Aut}_\xi(M)$ be the

group of automorphisms of the trivial deformation $\xi'\colon \operatorname{Spec} A[M] \to \operatorname{Spec} A \to \mathcal{X}$. Note that when $\xi \in \mathcal{X}(\Bbbk)$, then $\operatorname{Def}_\xi(\Bbbk)$ is precisely the tangent space of $\mathcal{X}$ at ξ and is identified with $T\mathcal{X}_\xi = \mathcal{X}_\xi(\Bbbk[\epsilon])/\sim$ of the local deformation prestack at ξ while $\operatorname{Aut}_\xi(\Bbbk)$ is the group of infinitesimal automorphism of ξ and is identified with the kernel $\operatorname{Aut}_{\mathcal{X}(\Bbbk[\epsilon])}(\xi') \to \operatorname{Aut}_{\mathcal{X}(\Bbbk)}(\xi)$.

Lemma C.7.3. *Suppose that $\mathcal{X}$ is a prestack over $\operatorname{Sch}/\Bbbk$ satisfying the strong homogeneity condition (RS_4^*). Let $\xi \in \mathcal{X}(A)$ be an object over a finitely generated $\Bbbk$-algebra A.*

(1) For every A-module M, $\operatorname{Def}_\xi(M)$ and $\operatorname{Aut}_\xi(M)$ are naturally A-modules, and the functors

$$\operatorname{Aut}_\xi(-)\colon \operatorname{Mod}(A) \to \operatorname{Mod}(A)$$
$$\operatorname{Def}_\xi(-)\colon \operatorname{Mod}(A) \to \operatorname{Mod}(A)$$

are A-linear.

(2) Consider a surjection $B \twoheadrightarrow A$ in $\operatorname{Art}_\Bbbk$ with square-zero kernel I, and let $\operatorname{Lift}_\xi(B)$ be the set of morphisms $\xi \to \eta$ over $\operatorname{Spec} A \to \operatorname{Spec} B$ where $\xi \xrightarrow{\alpha} \eta$ is declared equivalent to $\xi \xrightarrow{\alpha'} \eta'$ if there is an isomorphism $\beta\colon \eta \to \eta'$ such that $\alpha' = \beta \circ \alpha$. There is an action of $\operatorname{Def}_\xi(I)$ on $\operatorname{Lift}_\xi(B)$ which is functorial in B and I. Assuming $\operatorname{Lift}_\xi(B)$ is non-empty, this action is free and transitive.

Proof. This can be established by arguing as in Lemma C.3.15. For instance, scalar multiplication by $x \in A$ is defined by pulling back along the morphism $\operatorname{Spec} A[M] \to \operatorname{Spec} A[M]$ induced by the A-algebra homomorphism $A[M] \to A[M], a+m \mapsto a+xm$. Condition (RS_4^*) implies that the functor $\mathcal{X}(A[M \oplus M]) \to \mathcal{X}(A[M]) \times_{\mathcal{X}(A)} \mathcal{X}(A[M])$ is an equivalence. Addition $M \oplus M \to M$ induces an A-algebra homomorphism $A[M \oplus M] \to A[M]$ and thus a functor

$$\mathcal{X}(A[M]) \times_{\mathcal{X}(A)} \mathcal{X}(A[M]) \cong \mathcal{X}(A[M \oplus M]) \to \mathcal{X}(A[M])$$

which defines addition on $\operatorname{Def}_\xi(M)$ and $\operatorname{Aut}_\xi(M)$. $\qquad\square$

Theorem C.7.4 (Artin's Axioms for Algebraicity—second version). *A stack $\mathcal{X}$ over $(\operatorname{Sch}/\Bbbk)_{\text{ét}}$ is an algebraic stack locally of finite type over $\Bbbk$ if the following conditions hold:*

(AA$_1$) *(Limit preserving) The stack $\mathcal{X}$ is limit preserving;*

(AA$_2$) *(Representability of the diagonal) The diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable;*

(AA$_3$) *(Finiteness of tangent spaces) For every object $\xi\colon \operatorname{Spec}\Bbbk \to \mathcal{X}$, $\operatorname{Def}_\xi(\Bbbk)$ is a finite dimensional $\Bbbk$-vector space;*

(AA$_4$) *(Strong homogeneity) For every $\Bbbk$-algebra homomorphism $B_0 \to A_0$ and surjection $A \twoheadrightarrow A_0$ of $\Bbbk$-algebras with square-zero kernel, the functor*

$$\mathcal{X}(B_0 \times_{A_0} A) \to \mathcal{X}(B_0) \times_{\mathcal{X}(A_0)} \mathcal{X}(A)$$

is an equivalence, i.e. Condition (RS_4^*) holds;

(AA$_5$) *(Effectivity) For every noetherian complete local $\Bbbk$-algebra $(R, \mathfrak{m})$, the natural functor*

$$\mathcal{X}(\operatorname{Spec} R) \to \varprojlim \mathcal{X}(\operatorname{Spec} R/\mathfrak{m}^n)$$

is an equivalence of categories;

(AA$_6$) *(Existence of an obstruction theory) For every object $\xi \in \mathcal{X}(A)$ over a finitely generated $\Bbbk$-algebra A, there exists the following data*

(a) *there is an A-linear functor*

$$\mathrm{Ob}_\xi(-)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A),$$

and for every surjection $B \to A$ with square-zero kernel I, there is an element $\mathrm{ob}_\xi(B) \in \mathrm{Ob}_\xi(I)$ such that there is an extension

if and only if $\mathrm{ob}_\xi(B) = 0$, and

(b) *for every composition $B \to B' \to A$ of $\Bbbk$-algebras such that $B \twoheadrightarrow A$ and $B' \twoheadrightarrow A$ are surjective with square-zero kernels I and I', the image of $\mathrm{ob}_\xi(B)$ under $\mathrm{Ob}_\xi(I) \to \mathrm{Ob}_\xi(I')$ is $\mathrm{ob}_\xi(B')$; and*

(AA$_7$) *(Coherent deformation theory) For every object $\xi \in \mathcal{X}(A)$ over a $\Bbbk$-algebra A, the functors $\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commute with products.*

Moreover (AA$_2$) *can be removed if we replace* (AA$_3$) *and* (AA$_7$) *with:*

(AA$_{3'}$) *For every object $\xi\colon \mathrm{Spec}\,\Bbbk \to \mathcal{X}$, $\mathrm{Aut}_\xi(\Bbbk)$ and $\mathrm{Def}_\xi(\Bbbk)$ are finite dimensional $\Bbbk$-vector spaces; and*

(AA$_{7'}$) *For every object $\xi \in \mathcal{X}(A)$ over a $\Bbbk$-algebra A, the functors $\mathrm{Aut}_\xi(-)$, $\mathrm{Def}_\xi(-)$ and $\mathrm{Ob}_\xi(-)$ commute with products.*

Proof. Conditions (AA$_3$)–(AA$_4$) above allow us to apply Rim–Schlessinger's Criteria (Theorem C.3.12) to deduce the existence of versal formal deformations, i.e. Condition C.7.1(3) holds. It remains to check openness of versality, i.e. Condition C.7.1(5), in order to apply the first version (Theorem C.7.1) to establish this version.

Let $\xi_0\colon U_0 \to \mathcal{X}$ be a morphism from an affine scheme $U_0 = \mathrm{Spec}\,B_0$ of finite type over $\Bbbk$ which is versal at a point $u_0 \in U_0(\Bbbk)$. By (AA$_1$)–(AA$_2$), the morphism $\xi_0\colon U_0 \to \mathcal{X}$ is representable and locally of finite type. Let $\Sigma = \{u \in U_0(\Bbbk) \mid \xi_0\colon U_0 \to \mathcal{X}$ is not versal at $u\}$. If openness of versality does not hold, then $u_0 \in \overline{\Sigma}$ and there exists a countably infinite subset $\Sigma' = \{u_1, u_2, \ldots\} \subset \Sigma$ of distinct points with $u_0 \in \overline{\Sigma'}$.

Step 1. We claim that there exists a commutative diagram

where each closed immersion $U_{n-1} \hookrightarrow U_n$ is defined by a short exact sequence

$$0 \to \kappa(u_n) \to \mathcal{O}_{U_n} \to \mathcal{O}_{U_{n-1}} \to 0,$$

and for each n and open neighborhood $W \subset U_n$ of u_n, the restriction $\xi_n|_W$ is not the trivial deformation of $\xi_0|_{W \cap U_0}$, i.e. there is no morphism $r\colon \xi_n|_W \to \xi_0|_{W \cap U_0}$ such that $\xi_n|_W \xrightarrow{r} \xi_0|_{W \cap U_0} \to \xi_n|_W$ is the identity. Note that for each $m \geq n$, $U_n \hookrightarrow U_m$ is a closed immersion which is square-zero (i.e. $\ker(\mathcal{O}_{U_m} \to \mathcal{O}_{U_n})$ is

square-zero). We will inductively construct $U_n = \operatorname{Spec} B_n$ and $\xi_n \in \mathcal{X}(U_n)$, Since $\xi_0 \colon U_0 \to \mathcal{X}$ and $\xi_{n-1} \colon U_{n-1} \to \mathcal{X}$ are isomorphic in an open neighborhood of u_n, the morphism $\xi_{n-1} \colon U_{n-1} \to \mathcal{X}$ is also not versal at u_n. By definition of versality (using Remark C.3.8) there exists a surjection $A \to A_0$ in $\operatorname{Art}_\Bbbk$ with $\ker(A \to A_0) = \Bbbk$ and a commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A_0 & \longrightarrow & U_{n-1} \\
\cap \Big\downarrow & \nearrow\!\!\!\!\!\!\!\!\!\overset{\exists}{\diagup} & \Big\downarrow {\scriptstyle \xi_{n-1}} \\
\operatorname{Spec} A & \longrightarrow & \mathcal{X},
\end{array}
\tag{C.7.5}
$$

such that u_n is the image of $\operatorname{Spec} A_0 \to U_{n-1}$, which does not admit a lift $\operatorname{Spec} A \to U_{n-1}$. Using strong homogeneity (AA_4), there exists an extension of the commutative diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A_0 & \longrightarrow & U_{n-1} = \operatorname{Spec} B_{n-1} \\
\cap \Big\downarrow & & \cap \Big\downarrow \qquad \searrow {\scriptstyle \xi_{n-1}} \\
\operatorname{Spec} A & \longrightarrow & U_n = \operatorname{Spec}(A \times_{A_0} B_{n-1}) \\
& \searrow & \qquad \dashrightarrow {\scriptstyle \xi_n} \\
& & \qquad\qquad \mathcal{X}
\end{array}
$$

yielding an object ξ_n over $U_n = \operatorname{Spec} B_n$ with $B_n := A \times_{A_0} B_{n-1}$. If ξ_n were the trivial deformation of ξ_0 in an open neighborhood of u_n, then $\operatorname{Spec} A \to \mathcal{X}$ would be the trivial deformation of $\operatorname{Spec} A_0$ contradicting the obstruction to a lift of (C.7.5). Finally note that $\ker(B_n \to B_{n-1}) = \Bbbk$ since $\ker(A \to A_0) = \Bbbk$. This establishes the claim.

Step 2. Letting $\widehat{B} = \varprojlim B_n$ and $\widehat{U} = \operatorname{Spec} \widehat{B}$, we claim that there exists an object $\widehat{\xi} \in \mathcal{X}(\widehat{U})$ extending each $\xi_n \in \mathcal{X}(U_n)$. Let $M_n = \ker(B_n \to B_0)$ (noting that $M_0 = 0$). Since $M_n^2 = 0$, we can view M_n as a B_0-module. The $\Bbbk$-algebra

$$
\widetilde{B} := \{(b_0, b_1, \ldots) \in \prod_{n \geq 0} B_n \mid \text{the image of each } b_n \text{ under } B_n \to B_0 \text{ is } b_0\}
$$

has the following properties:

- The surjective $\Bbbk$-algebra homomorphism $\widetilde{B} \to B_0$ defined by $(b_i) \mapsto b_0$ has kernel $M := \prod_{n \geq 0} M_n$;
- The map $\widetilde{B} \to B_0[M]$ defined by $(b_0, b_1, b_2, \ldots) \mapsto (b_0, b_1 - b_0, b_2 - b_1, b_3 - b_2, \ldots)$ is a surjective $\Bbbk$-algebra homomorphism with square-zero kernel;
- The composition $\widehat{B} \to \widetilde{B} \to B_0[M]$ induces a short exact sequence

$$
\begin{array}{ccccccccc}
0 & \to & \ker(\widehat{B} \to B_0) & \to & \ker(\widetilde{B} \to B_0) & \longrightarrow & \ker(B_0[M] \to B_0) & \longrightarrow & 0 \\
& & \| & & \| & & \| & & \\
0 & \longrightarrow & \varprojlim_{n \geq 0} M_n & \longrightarrow & \prod_{n \geq 0} M_n & \longrightarrow & \prod_{n \geq 0} M_n & \longrightarrow & 0
\end{array}
$$

$$
(b_0, b_1, \ldots) \longmapsto (b_1 - b_0, b_2 - b_1, \ldots)
$$

- There is an identification $\widehat{B} = \widetilde{B} \times_{B_0[M]} B_0$.

Since the lift $\xi_n \in \mathcal{X}(B_n)$ of ξ_0 exists for each n, $\mathrm{ob}_\xi(B_n) = 0 \in \mathrm{Ob}_\xi(M_n)$. By $(\mathrm{AA}_6)(\mathrm{b})$, the element $\mathrm{ob}_\xi(\widetilde{B})$ maps to $\mathrm{ob}_\xi(B_n)$ under $\mathrm{Ob}_\xi(M) \to \mathrm{Ob}_\xi(M_n)$. By (AA_7), the map $\mathrm{Ob}_\xi(M) \hookrightarrow \prod_n \mathrm{Ob}_\xi(M_n)$ is injective[3] and thus $\mathrm{ob}_\xi(\widetilde{B}) = 0 \in \mathrm{Ob}_\xi(M)$ which shows that there exists a lift $\widetilde{\xi} \in \mathcal{X}(\widetilde{B})$ of ξ_0.

The restrictions $\widetilde{\xi}|_{B_n}$ are not necessarily isomorphic to ξ_n. However, we may use the free and transitive action $\mathrm{Def}_\xi(M_n) = \mathrm{Lift}_\xi(B_0[M_n])$ on the non-empty set of liftings $\mathrm{Lift}_\xi(\widetilde{B_n})$ to find elements $t_n \in \mathrm{Def}_\xi(M_n)$ such that $\xi_n = t_n \cdot \widetilde{\xi}|_{B_n}$ (Lemma C.7.3). Since $\mathrm{Def}_\xi(M) \xrightarrow{\sim} \prod_n \mathrm{Def}_\xi(M_n)$ by (AA_7), there exists $\widetilde{t} \in \mathrm{Def}_\xi(M)$ mapping to (t_n). After replacing $\widetilde{\xi}$ with $\widetilde{t} \cdot \widetilde{\xi}$, we can arrange that $\widetilde{\xi}|_{B_n}$ and ξ_n are isomorphic for each n.

We now show that each restriction $\widetilde{\xi}|_{B_0[M_n]} \in \mathrm{Def}_\xi(M_n)$ under the composition $\widetilde{B} \to B_0[M] \to B_0[M_n]$ is the trivial deformation. Indeed, the map $M = \ker(\widetilde{B} \to B_0) \to \ker(B_0[M_n] \to B_0) = M_n$ induces a map $\mathrm{Def}_\xi(M) \to \mathrm{Def}_\xi(M_n)$ on deformation modules which under the identification $\mathrm{Def}_\xi(M) \xrightarrow{\sim} \prod_n \mathrm{Def}_\xi(M_n)$ of (AA_7) sends an element $(\eta_0, \eta_1, \ldots)$ to $(\eta_{n+1}|_{B_n} - \eta_n)$. The ring map $\widetilde{B} \to B_0[M_n]$ also induces a map $\mathrm{Lift}_\xi(\widetilde{B}) \to \mathrm{Lift}_\xi(B_0[M_n])$ which is equivariant with respect to $\mathrm{Def}_\xi(M) \to \mathrm{Def}_\xi(M_n)$. It follows that the image of $\widetilde{\xi}$ in $\mathrm{Lift}_\xi(B_0[M_n]) = \mathrm{Def}_\xi(M_n)$ is $\xi_{n+1}|_{B_n} - \xi_n = 0$.

The existence of $\widehat{\xi} \in \mathcal{X}(\widehat{B})$ extending $(\xi_n) \in \varprojlim \mathcal{X}(B_n)$ now follows from applying the identity $\widehat{B} = \widetilde{B} \times_{B_0[M]} B_0$ and strong homogeneity (AA_4) to the diagram

Step 3. We now use the versality of $\xi_0 \colon U_0 \to \mathcal{X}$ at u_0 to arrive at a contradiction. Since $\mathcal{X}$ is limit preserving (AA_1), there exists a finitely generated $\Bbbk$-subalgebra $B' \subset \widehat{B}$ and an object $\xi' \in \mathcal{X}(B')$ together with an isomorphism $\widehat{\xi} \xrightarrow{\sim} \xi|_{\widehat{B}}$. After possibly enlarging B', we may assume that the composition $B' \hookrightarrow \widehat{B} \to B_0$ is surjective. There is thus a closed immersion $U_0 \hookrightarrow U' := \mathrm{Spec}\, B'$, and we can consider the commutative diagram

where the fiber product $U_0 \times_\mathcal{X} U'$ is an algebraic space locally of finite type over $\Bbbk$. Since $\xi_0 \colon U_0 \to \mathcal{X}$ is versal at u_0, it follows from the artinian version of the

---

[3]The hypotheses of (AA_7) can be weakened to only require the injectivity of $\mathrm{Ob}_\xi(M) \hookrightarrow \prod_n \mathrm{Ob}_\xi(M_n)$ although in practice one usually verifies that this map is bijective.

Infinitesimal Lifting Criterion for Smoothness (Theorem A.3.1) that $U_0 \times_{\mathcal{X}} U' \to U'$ is smooth at $i(u_0)$. After replacing U_0 with an open affine neighborhood of u_0, U' with the corresponding open and $\{u_1, u_2, \ldots\}$ with an infinite subsequence contained in this open, we can arrange that $U_0 \times_{\mathcal{X}} U' \to U'$ is smooth. The non-artinian version of the Infinitesimal Lifting Criterion for Smoothness implies the section of $U_0 \times_{\mathcal{X}} U' \to U'$ over U_0 extends to a global section $U' \to U_0 \times_{\mathcal{X}} U'$. This implies that ξ' is the trivial deformation of ξ_0, contradicting our choice of $\xi' \colon U' \to \mathcal{X}$.

Our exposition follows [SP, Tag 0CYF] and [Hal17, Thm. A]. See also [Art74, Thm. 5.3] and [HR19a, Main Thm.]. $\qquad\square$

Remark C.7.6. The converse of the theorem also holds. For the necessity of the conditions, we only need to check (AA_3), (AA_4), (AA_6) and (AA_7). Condition (AA_3) (finiteness of the tangent spaces) holds as $\mathcal{X}$ is of finite type over $\Bbbk$. The strong homogeneity condition (AA_4) holds by [SP, Tag 07WN]. Condition (AA_6) (existence of an obstruction theory) follows from the existence of a cotangent complex $L_{\mathcal{X}/\Bbbk}$ for $\mathcal{X}$ satisfying properties analogous to Theorem C.5.1; see [Ols06]. If $\xi \colon \operatorname{Spec} A \to \mathcal{X}$ is a morphism from a finitely generated $\Bbbk$-algebra A and I is an A-module, then we set $\operatorname{Ob}_\xi(I) := \operatorname{Ext}^1_A(\xi^* L_{\mathcal{X}/\Bbbk}, I)$. Property $(AA_6)(b)$ holds as a consequence of [Ols06, Thm. 1.5], a generalization of [Ill71, III.2.2.4] (which was discussed in Remark C.5.14) from morphisms of schemes to representable morphisms of algebraic stacks. Finally, Condition (AA_7) ($\operatorname{Def}_\xi(-)$ and $\operatorname{Ob}_\xi(-)$ commutes with products) follows from cohomology and base change.

C.7.2 Verifying Artin's Axioms

Theorem C.7.7. *Each of the stacks* $\operatorname{Hilb}^P(X)$, $\mathcal{M}_g$ *(with $g \geq 2$) and $\underline{\operatorname{Bun}}_C$ over* $(Sch/\Bbbk)_{\text{ét}}$ *are algebraic stacks locally of finite type over $\Bbbk$.*

Proof. We check the conditions of Theorem C.7.4. Condition (AA_1) (limit preserving) was verified in Lemma C.6.1. For $(AA_{3'})$, the finite dimensionality of the vector spaces $\operatorname{Def}_\xi(\Bbbk)$ and $\operatorname{Aut}_\xi(\Bbbk)$ for an object $\xi \in \mathcal{X}(\Bbbk)$ can be identified with:

- $\mathrm{H}^0(Z, N_{Z/X})$ and $\{0\}$ for $\xi = [Z \subset X] \in \operatorname{Hilb}^P(X)(\Bbbk)$ (Proposition C.1.4),

- $\mathrm{H}^1(C, T_C)$ and $\mathrm{H}^0(C, T_C)$ for $\xi = [C] \in \mathcal{M}_g(\Bbbk)$ (Lemma C.1.11 and Proposition C.1.12) and

- $\operatorname{Ext}^1_{\mathcal{O}_C}(E, E)$ and $\operatorname{Ext}^0_{\mathcal{O}_C}(E, E)$ for $\xi = [E] \in \underline{\operatorname{Bun}}_C(\Bbbk)$ (Proposition C.1.17).

Condition (AA_4) (the strong homogeneity condition of (RS_4^*)) was checked in Proposition C.3.19. Condition (AA_5) (effectivity) was checked in Corollary C.4.12 as a consequence of Grothendieck's Existence Theorem. For Condition (AA_6), we define obstruction theories as follows: for a finitely generated $\Bbbk$-algebra A and an A-module M, we set

- $\operatorname{Ob}_\xi(M) := \operatorname{Ext}^1_{\mathcal{O}_Z}(I_Z/I_Z^2, M)$ for $\xi = [Z \subset X_A] \in \operatorname{Hilb}^P(X)(A)$ where $I_Z \subset \mathcal{O}_{X_A}$ is the sheaf of ideal defining Z.

- $\operatorname{Ob}_\xi(M) := \mathrm{H}^2(\mathcal{C}, T_{\mathcal{C}/A} \otimes_A M) = 0$ for $\xi = [\mathcal{C} \to \operatorname{Spec} A] \in \mathcal{M}_g(A)$, and

- $\operatorname{Ob}_\xi(M) := \mathrm{H}^2(C_A, \mathscr{E}nd_{\mathcal{O}_{C_A}}(E) \otimes_A M) = 0$ for $\xi = [E] \in \underline{\operatorname{Bun}}_C(A)$.

Property $(AA_6)(a)$ holds for these obstruction theories as a consequence of Propositions C.2.2, C.2.7 and C.2.17; these results also show that $\operatorname{Aut}_\xi(M)$ and $\operatorname{Def}_\xi(M)$ are identified with the analogous cohomology groups. Condition $(AA_{7'})$ ($\operatorname{Aut}_\xi(-)$, $\operatorname{Def}_\xi(-)$ and $\operatorname{Ob}_\xi(-)$ commutes with products) follows from Lemma C.7.8. $\qquad\square$

Lemma C.7.8. *Let $X \to \operatorname{Spec} A$ be a proper morphism of noetherian schemes. Let E and F be coherent sheaves on X with F flat over A. Then the functors*

$$\operatorname{H}^i(X, F \otimes_A -) \colon \operatorname{Mod}(A) \to \operatorname{Mod}(A) \quad \text{and}$$
$$\operatorname{Ext}^i_{\mathcal{O}_X}(E, F \otimes_A -) \colon \operatorname{Mod}(A) \to \operatorname{Mod}(A)$$

commute with products.

Proof. Since F is flat over A, there is a perfect complex $K^\bullet$ of A-modules such that $\operatorname{H}^i(X, F \otimes_A -) \cong \operatorname{H}^i(K^\bullet \otimes_A -)$ (Theorem A.6.2). Write $K^d = A^{\oplus r_d}$. For every set of A-modules $\{M_\alpha\}$ we have an identification of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_\alpha M_\alpha^{\oplus r_0} & \longrightarrow & \prod_\alpha M_\alpha^{\oplus r_1} & \longrightarrow \cdots \longrightarrow & \prod_\alpha M_\alpha^{\oplus r_n} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & (\prod_\alpha M_\alpha)^{\oplus r_0} & \longrightarrow & (\prod_\alpha M_\alpha)^{\oplus r_1} & \longrightarrow \cdots \longrightarrow & (\prod_\alpha M_\alpha)^{\oplus r_n} & \longrightarrow & 0.
\end{array}
$$

The top row is the product of the complexes $K^\bullet \otimes_A M_\alpha$ and its cohomology is identified with $\prod_\alpha \operatorname{H}^i(X, F \otimes_A M_\alpha)$ while the bottom row is $K^\bullet \otimes_A (\prod_\alpha M_\alpha)$ with cohomology groups $\operatorname{H}^i(X, F \otimes_A (\prod_\alpha M_\alpha))$. For the second statement, one needs to apply alternative versions of cohomology and base change (see [EGA, III.7.7.5], [SP, Tag 08JR] and [Hal14, Thm. E]). $\qquad\square$

Exercise C.7.9. If $X \to S$ is a proper flat morphism of noetherian schemes, show that the stack $\underline{\operatorname{Coh}}(X/S)$, whose objects over an S-scheme T are finitely presented, quasi-coherent sheaves on X_T flat over T, is an algebraic stack locally of finite type over S.

Exercise C.7.10. Using Artin's Axioms to show the algebraicity of the stack of all curves $\mathcal{M}_g^{\mathrm{all}}$ as defined in Definition 5.4.4. This is shown in Theorem 5.4.6 by explicitly exhibiting a smooth presentation. See also [SP, Tag 0D5A].

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