A guide to tropicalizations

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Abstract

Tropicalizations form a bridge between algebraic and convex geometry. We generalize basic results from tropical geometry which are well-known for special ground fields to arbitrary non-archimedean valued fields. To achieve this, we develop a theory of toric schemes over valuation rings of rank 1. As a basic tool, we use techniques from non-archimedean analysis.

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1 Introduction

Let us consider a field K endowed with a non-archimedean absolute value $|\ |$. We fix coordinates $x_1, \ldots, x_n$ on the split multiplicative torus $\mathbb{G}_m^n$ over K. Using logarithmic coordinates $-\log|x_1|, \ldots, -\log|x_n|$, any closed subscheme X of $\mathbb{G}_m^n$ transforms into a finite union $\mathrm{Trop}(X)$ of polyhedra in $\mathbb{R}^n$. This process is called *tropicalization* and it can be used to transform a problem from algebraic geometry into a corresponding problem in convex geometry which is usually easier. If the toric coordinates are well suited to the problem, it is sometimes possible to use a solution of the convex problem to solve the original algebraic problem. Another strategy is to vary the ambient torus to compensate the loss of information due to the tropicalization process.

Tropicalization originates from a paper of Bergman [Berg] on logarithmic limit sets. The convex structure of the tropical variety $\mathrm{Trop}(X)$ was worked out by Bieri–Groves [BG] with applications to geometric group theory in mind. Sturmfels [Stu] pointed out that $\mathrm{Trop}(X)$ is a subcomplex of the Gröbner complex. In fact, the polyhedral complex $\mathrm{Trop}(X)$ has some natural weights satisfying a balancing condition which appears first in Speyer's thesis [Spe]. This relies on the description of the Chow cohomology of a toric variety given by Fulton and Sturmfels [FS]. An intrinsic approach to tropical geometry was proposed by Mikhalkin. The idea is to develop tropical geometry as some sort of algebraic geometry based on the min-plus algebra where every $\mathrm{Trop}(X)$ occurs as a natural object. This approach was used by Mikhalkin to prove celebrated results in enumerative geometry (see [Mik]). These results popularized tropical geometry generating a huge amount of interesting results and applications. For the relation of tropicalizations to idempotent mathematics and Maslov dequantization, we refer to [Lit].

In tropical literature, one usually considers tropicalizations under severe restrictions for the ground field suited to the situation at hand. These restrictions can be subdivided into four groups. First, many papers are written in case of the trivial valuation on K. Then $\mathrm{Trop}(X)$ is a finite union of rational cones. This is a setting which occurs very often in algebraic geometry. A second group of people is working under the assumption that K is the field of Puiseux series or a related field. This is often used by people interested in the combinatorial structure and effective computation of $\mathrm{Trop}(X)$. In this case, K has a natural grading, contains the residue

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field and the valuation has a canonical section which makes many arguments easier. A third group is assuming that the valuation is discrete. Most of the valuations occurring in applications to certain fields such as number theory are discrete. This makes it possible to use noetherian models over the valuation ring. Finally, a fourth group of people is working with algebraically closed ground fields endowed with a non-trivial complete absolute value. This is suitable for using arguments from the theory of rigid analytic spaces. As an excellent source for this case, we refer the reader to the recent paper of Baker–Payne–Rabinoff [BPR].

The goal of this paper is to survey basic results about tropicalizations and to generalize them to arbitrary non-archimedean absolute values on K. This will make these results accessible to all kind of applications. To handle the difficulties mentioned above, we will use methods from the theory of Berkovich analytic spaces which are very well-suited for this general framework. This is not surprising as even in the original Bieri–Groves paper, the analytification X^{an} of X with its Berkovich analytic topology was implicitly used before Berkovich introduced his new concept in rigid analytic geometry. Most parts of the paper can be read having just a topological understanding of X^{an} which is rather elementary. The paper is not meant as an introduction to the subject of tropical geometry. For this purpose, we refer the reader to the forthcoming book of Maclagan–Sturmfels [MS].

The paper is organized as follows: In Section 2, we introduce the analytification of an algebraic scheme X over K and we sketch how this fits into the theory of Berkovich analytic spaces. In Section 3, we define the tropicalization map and the tropicalization of a closed subscheme X of the split torus $\mathbb{G}_m^n$ over K. In Section 4, we study models over the valuation ring. For a potentially integral point of the generic fibre, we define its reduction to the special fibre. We compare this with the reduction map from the theory of strictly affinoid algebras. In Section 5, the initial degeneration of a closed subscheme of $\mathbb{G}_m^n$ at $\omega \in \mathbb{R}^n$ is studied leading to an alternative characterization of the tropicalization.

In Section 6, we investigate normal affine toric schemes over a valuation ring associated to polyhedra. In Section 7, we globalize these results assigning a normal toric scheme to every admissible fan in $\mathbb{R}^n \times \mathbb{R}_+$. This is rather new and it generalizes the theory from [KKMS] worked out in the special case of a discrete valuation. In Section 8, we introduce the tropical cone of X as a subset of $\mathbb{R}^n \times \mathbb{R}_+$. This new notion can be seen as a degeneration of the tropical variety with respect to the given valuation to the tropical variety with respect to the trivial valuation. It is very convenient to work with the tropical cone in the framework of tropical schemes over a valuation ring.

In Section 9, we study projectively embedded toric varieties over the valuation ring which are not necessarily normal. This generalizes work of Eric Katz. In Section 10, we show that the tropical variety is a subcomplex of the Gröbner complex. In Section 11, we study the closure of X in a toric scheme over the valuation ring and in Section 12, we generalize Tevelev's tropical compactifications to our setting. We introduce tropical multiplicities in Section 13 leading to the Sturmfels–Tevelev multiplicity formula for tropical cycles. In Section 14, we characterize proper compactifications of X in a toric scheme which intersect all orbits properly. In the appendix, we collect results from convex geometry which are needed in the paper.

Terminology

In $A \subset B$, A may be equal to B. The complement of A in B is denoted by $B \setminus A$ as we reserve $-$ for algebraic purposes. The zero is included in $\mathbb{N}$ and in $\mathbb{R}_+$.

All occurring rings and algebras are commutative with 1. If A is such a ring, then the group of multiplicative units is denoted by $A^{\times}$. A variety over a field is a separated reduced scheme of finite type. We denote by $\overline{F}$ an algebraic closure of

the field F.

For $\mathbf{m} \in \mathbb{Z}^n$, let $\mathbf{x^m} := x_1^{m_1} \cdots x_n^{m_n}$ and $|\mathbf{m}| := |m_1| + \cdots + |m_n|$. The standard scalar product of $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^n$ is denoted by $\mathbf{u} \cdot \mathbf{u}' := u_1 u_1' + \cdots + u_n u_n'$. The terminology from convex geometry is explained in the appendix.

In the whole paper, the base field is a *valued field* (K, v) which means that the field K is endowed with a non-archimedean absolute value $|\ |$ which might be trivial. The corresponding valuation is $v := -\log(|\cdot|)$ with value group $\Gamma := v(K^*)$. We get a valuation ring $K^\circ := \{\alpha \in K \mid v(\alpha) \geq 0\}$ with maximal ideal $K^{\circ\circ} := \{\alpha \in K \mid v(\alpha) > 0\}$ and residue field $\tilde{K} := K^\circ / K^{\circ\circ}$. Note that $K = K^\circ = \tilde{K}$ if the valuation is trivial. We call v a *discrete valuation* if $\Gamma \cong \mathbb{Z}$.

We fix a free abelian group M of rank n and $N := \mathrm{Hom}(M, \mathbb{Z})$ the dual abelian group. An element of M is usually denoted by u and an element of N is usually denoted by ω. We get the duality pairing $\langle u, \omega \rangle := \omega(u)$. We have the split torus $\mathbb{T} = \mathrm{Spec}(K^\circ[M])$ over K° with generic fibre T. Then M might be seen as the character group of this torus and the character corresponding to $u \in M$ is denoted by χ^u. If G is an abelian group, then $N_G := N \otimes_{\mathbb{Z}} G$ denotes the base change of N to G. Similarly, $\mathbb{T}_A$ denotes the base change of $\mathbb{T}$ to a K°-algebra A.

2 Analytification

In this section, we recall the construction of the Berkovich analytic space X^{an} associated to an algebraic scheme X over the field K with non-archimedean complete absolute value $|\ |$ and corresponding valuation $v := -\log(|\cdot|)$. Note that completeness is no restriction of generality as analytic constructions are always performed over complete fields. In general, this may be achieved by base change to the completion of K. The topological part 2.1–2.5 of the construction is elementary and essential for the understanding of the whole paper. The remaining analytic part is more technical and may be skipped in a first reading. Details and proofs for this section may be found in [Berk1] and [Tem].

2.1 We start with the construction for an affine scheme $X = \mathrm{Spec}(A)$ of finite type over K. Then the topological space underlying the *Berkovich analytic space associated to* X is the set of multiplicative seminorms on A extending the given absolute value on K. In other words, we consider maps $p : A \to \mathbb{R}_+$ characterized by the properties

(a) $p(fg) = p(f)p(g)$

(b) $p(1) = 1$

(c) $p(f + g) \leq p(f) + p(g)$

(d) $p(\alpha) = |\alpha|$

for all $f, g \in A$ and $\alpha \in K$. It is easy to see that the triangle inequality (c) is equivalent to the ultrametric triangle inequality

$$p(f + g) \leq \max(p(f), p(g)).$$

We denote this analytification of X by X^{an} and we endow it with the coarsest topology such that the maps $X^{\mathrm{an}} \to \mathbb{R}$, $p \mapsto p(f)$ are continuous for every $f \in A$.

We embed the set of closed points of X into X^{an} by mapping P to the seminorm p given by $p(f) = |f(P)|$.

Remark 2.2 Let $X = \operatorname{Spec}(A)$ be an affine scheme of finite type over K. For $p \in X^{\text{an}}$, the integral domain $A/\{a \in A \mid p(a) = 0\}$ is endowed with a canonical multiplicative norm induced by p. We conclude that its quotient field L is endowed with an absolute value $|\ |_w$ extending $|\ |$. The canonical homomorphism $A \to L$ gives an L-rational point P of $\operatorname{Spec}(A)$ and we may retrieve the seminorm by $p(a) = |a(P)|_w$ for any $a \in A$.

Conversely, any valued field (L, w) extending (K, v) and any L-rational point P of X give rise to an element $p \in X^{\text{an}}$ by $p(a) := |a(P)|_w$, Obviously, different L-valued points might induce the same seminorm on A.

Lemma 2.3 *Let* $X = \operatorname{Spec}(A)$ *be an affine scheme of finite type over* K *and let* (F, u) *be a complete valued field extending* (K, v). *Then the restriction map of seminorms gives a continuous surjective map from* $(X_F)^{\text{an}}$ *onto* X^{an}.

Proof: Continuity is obvious from the definitions. For $p \in X^{\text{an}}$, there is a valued field (L, w) and $P \in X(L)$ as in Remark 2.2. By Lemma 5.2 below, there is a valued field (F', u') extending both (L, w) and (F, u). We conclude that P is also an F'-rational point of X_F and hence it gives rise to a seminorm $p' \in (X_F)^{\text{an}}$. By construction, p' extends p proving surjectivity. $\qquad\square$

2.4 For any scheme X of finite type over K, we choose an open affine covering $\{U_i\}_{i \in I}$. Then we define the topological space underlying the *Berkovich analytic space* X^{an} *associated to* X by glueing the spaces U_i^{an} obtained in 2.1. We get a topological space which is locally compact. It is Hausdorff if and only if X is separated over K.

2.5 If $\varphi : X \to Y$ is a morphism of schemes of finite type over K, then we have a canonical map $\varphi^{\text{an}} : X^{\text{an}} \to Y^{\text{an}}$ between the associated Berkovich analytic spaces. It is easy to see that it is enough to define the map locally, i.e. we may assume that X and Y are affine. Then we set $\varphi^{\text{an}}(p) := p \circ \varphi^{\sharp}$ for any multiplicative seminorm p on $\mathscr{O}(X)$.

2.6 We will use analytifications also in situations where the valuation v is not complete. Then we define X^{an} over the completion K_v of K. Let $\overline{K}$ be the algebraic closure of K in an algebraic closure of K_v. The absolute value on $\overline{K}$ depends on the embedding of K into the algebraic closure of K_v if K is not complete. Then every $\overline{K}$-rational point of X induces a point in X^{an} as in Remark 2.2.

If the valuation is non-trivial, then the image of $X(\overline{K})$ is dense in X^{an}. To see this, we may assume that X is an irreducible affine variety of dimension d. By Noetherian normalization, there is a finite map φ from X onto the affine space $\mathbb{A}^d_K$. Let U be a non-empty open subset of X^{an}. By Lemma 3.2.7 in [Berk1], the map φ^{an} is open. Since the claim is obvious for $\mathbb{A}^d$, there is a $\overline{K}$-rational point in $\varphi^{\text{an}}(U)$. We choose a preimage in U which has to be $\overline{K}$-rational by finiteness of φ^{an}.

2.7 Now K is complete again. We will explain below how X^{an} is endowed with an analytic structure. Of course, the analytic structure will depend on the underlying scheme structure. First we handle the case of the affine space $\mathbb{A}^n := \operatorname{Spec}(K[x_1, \ldots, x_n])$. For $f(\mathbf{x}) = \sum_{\mathbf{m}} \alpha_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in K[x_1, \ldots, x_n]$, we have the *Gauss norm*

$$|f(\mathbf{x})| = \max_{\mathbf{m}} |\alpha_{\mathbf{m}}|. \tag{1}$$

The *Tate algebra* is defined as

$$K\langle x_1, \ldots, x_n \rangle := \{ \sum_{\mathbf{m}} \alpha_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in K[[x_1, \ldots, x_n]] \mid \lim_{|\mathbf{m}| \to \infty} |a_{\mathbf{m}}| = 0 \}$$

and it is the completion of $K[x_1, \ldots, x_n]$ with respect to the Gauss norm. The corresponding Banach norm $|\ |$ on $K\langle x_1, \ldots, x_n \rangle$ is also defined by (1). The closed ball $\mathbb{B}^n$ of radius 1 in $\mathbb{A}^n$ is defined as the set of multiplicative seminorms on $K\langle x_1, \ldots, x_n \rangle$ which are bounded by the Gauss norm, i.e. we have again properties (a)–(d) from 2.1 for all $a, b \in K\langle x_1, \ldots, x_n \rangle$ and the additional property $p(f) \leq |f|$. Note that a closed point of $\mathbb{A}^n$ is in $\mathbb{B}^n$ if and only if all its coordinates have absolute value at most 1. It is easy to see that the supremum norm on $\mathbb{B}^n$ is equal to the Gauss norm.

More generally, we may consider $\mathbf{r} = (r_1, \ldots, r_n)$ for strictly positive real numbers $r_1, \ldots, r_n$. Then the Banach algebra $K\langle r_1^{-1}x_1, \ldots, r_n^{-1}x_n \rangle$ is given by completion of $K[x_1, \ldots, x_n]$ with respect to the weighted Gauss norm

$$|f(\mathbf{x})|_{\mathbf{r}} := \max_{\mathbf{m}} |\alpha_{\mathbf{m}}| \mathbf{r}^{\mathbf{m}}.$$

If we repeat the above construction, we get the closed ball $\mathbb{B}^n_{\mathbf{r}}$ of radius $\mathbf{r}$ in $\mathbb{A}^n$. It is easy to see that $(\mathbb{A}^n)^{\mathrm{an}}$ may be covered by a union of such balls. They serve as compact charts for the analytic structure of $(\mathbb{A}^n)^{\mathrm{an}}$.

2.8 An *affinoid algebra* is a Banach algebra $(\mathscr{A}, \|\ \|)$ which is isomorphic to $K\langle r_1^{-1}x_1, \ldots, r_n^{-1}x_n \rangle / I$ for an ideal I and such that the norm $\|\ \|$ is equivalent to the quotient norm

$$\|f + I\|_{\mathrm{quot}} := \inf\{\|g\| \mid g \in f + I\}$$

on $\mathscr{A}/I$. It is called a *strictly affinoid algebra* if we may choose $r_i = 1$ for all $i = 1, \ldots, n$.

The Banach norm does not matter if the valuation is non-trivial (see [Tem], Fact 3.1.15). In this case, all such Banach norms on $\mathscr{A}$ are equivalent and every homomorphism between affinoid algebras is bounded. In classical rigid geometry as in [BGR], one considers only strictly affinoid algebras and they are called affinoid algebras there.

2.9 For an affinoid algebra as above, the *spectral radius* is defined by $\rho(a) := \inf\{\|a^n\|^{1/n} \mid n \geq 1\}$ for $a \in \mathscr{A}$. We set

$$\mathscr{A}^{\circ} := \{a \in \mathscr{A} \mid \rho(a) \leq 1\}, \quad \mathscr{A}^{\circ\circ} := \{a \in \mathscr{A} \mid \rho(a) < 1\}.$$

and the *residue algebra* is defined by $\tilde{\mathscr{A}} := \mathscr{A}^{\circ}/\mathscr{A}^{\circ\circ}$.

2.10 The *Berkovich spectrum* $\mathscr{M}(\mathscr{A})$ of a $\mathbb{K}$-affinoid algebra $\mathscr{A}$ is defined as the set of multiplicative bounded seminorms p on $\mathscr{A}$, i.e. for all $a, b \in \mathscr{A}$, we have

(a) $p(ab) = p(a)p(b)$

(b) $p(1) = 1$

(c) $p(a + b) \leq p(a) + p(b)$

(d) $p(a) \leq \rho(a)$.

It is endowed with the coarsest topology such that the maps $p \mapsto p(a)$ are continuous for all $a \in \mathscr{A}$. We get a compact space. The spectral radius $\rho(a)$ turns out to be equal to the supremum seminorm $\sup\{p(a) \mid p \in \mathscr{M}(\mathscr{A})\}$ of $a \in \mathscr{A}$.

Example 2.11 Let $\mathscr{A} = K\langle r_1^{-1}x_1, \ldots, r_n^{-1}x_n \rangle / I$. A *rational subdomain* of $X := \mathscr{M}(\mathscr{A})$ is defined by

$$X\left(\mathbf{s}^{-1}\frac{\mathbf{f}}{g}\right) := \{x \in X \mid |f_j(x)| \leq s_j|g(x)|, \, j = 1, \ldots, m\}$$

where $g, f_1, \ldots, f_m$ generate the unit ideal in $\mathscr{A}$ and $s_1, \ldots, s_m > 0$. The corresponding affinoid algebra is

$$\mathscr{A}\left\langle \mathbf{s}^{-1}\frac{\mathbf{f}}{g} \right\rangle := K\langle \mathbf{r}^{-1}\mathbf{x}, s_1^{-1}y_1, \ldots, s_m^{-1}y_m\rangle / \langle I, g(\mathbf{x})y_j - f_j(\mathbf{x}) \mid j = 1, \ldots, m\rangle$$

(see [Berk1], Remarks 2.2.2).

2.12 We will not give the precise definition of a *Berkovich analytic space* X (see [Berk2] for details). Roughly speaking it is a topological space endowed with an atlas such that each chart is homeomorphic to the Berkovich spectrum of an affinoid algebra and then there are some compatibility conditions. Analytic functions on such a chart $\mathscr{M}(\mathscr{A})$ are given by the elements of $\mathscr{A}$.

A morphism $\varphi : X_1 \to X_2$ between Berkovich spaces X_1 and X_2 is a continuous map such that for every chart U_1 of X_1 with $\varphi(U_1)$ contained in a chart U_2 of X_2 and every analytic function f_2 on U_2, the function $\varphi^\sharp(f_2) := f_2 \circ \varphi$ is an analytic function on U_1 and φ is induced by $\varphi^\sharp$.

2.13 If X is a scheme of finite type over K as at the beginning, then X^{an} is a Berkovich analytic space. As charts, we may choose $U^{\mathrm{an}} \cap \mathbb{B}_{\mathbf{r}}^n$, where U is an affine open subset of X realized as a closed subset of $\mathbb{A}^n$ and $\mathbb{B}_{\mathbf{r}}^n$ is a closed ball in $\mathbb{A}^n$. Serre's GAGA principle holds also in the non-archimedean framework. For details, we refer to [Berk1], §3.4 and §3.5.

Remark 2.14 In Example 2.11, we have defined rational subdomains of the Berkovich spectrum $X := \mathscr{M}(\mathscr{A})$. More generally, one can define *affinoid subdomains* of X by a certain universal property. They are Berkovich spectra contained in X which are used for localization arguments on Berkovich spaces. For details, we refer to [Berk1], Section 2.2. By the Gerritzen–Grauert theorem, every affinoid subdomain is a union of rational domains if the valuation v is non-trivial.

Roughly speaking, an *analytic subdomain* of a Berkovich analytic space X is a subset which behaves locally like an affinoid subdomain. For a precise definition and for properties, we refer to [Berk1], Section 3.1. In this paper, we need only analytic functions on affinoid subdomains of X^{an} where they are just elements of the corresponding affinoid algebra. However, it should be noted that analytic functions form a sheaf on open subsets giving X^{an} the structure of a locally ringed space (see [Berk1], or [BPS], §1.2, for a neat description).

3 Tropicalization

In this section, we consider a closed subscheme X of the multiplicative torus T over the valued field (K, v) and we will define the tropical variety $\mathrm{Trop}_v(X)$ associated to X. The tropicalization process is a transfer from algebraic geometry to convex geometry in $\mathbb{R}^n$. We will use the analytifications X^{an} and T^{an} from the previous section which are always performed over the completion of K.

3.1 Let M be a free abelian group of rank n and let $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual group. Then we consider the multiplicative torus $T := \mathrm{Spec}(K[M])$ with character group M. We have the *tropicalization map*

$$\mathrm{trop}_v : T^{\mathrm{an}} \to N_{\mathbb{R}}, \quad p \mapsto \mathrm{trop}_v(p),$$

where $\mathrm{trop}_v(p)$ is the element of $N_{\mathbb{R}} = \mathrm{Hom}(M, \mathbb{R})$ given by

$$\langle u, \mathrm{trop}_v(p)\rangle := -\log(p(\chi^u))$$

with χ^u the character of T corresponding to $u \in M$. Choosing coordinates $x_1, \ldots, x_n$ on $T = \mathbb{G}_m^n$, we may identify M and N with $\mathbb{Z}^n$ and we get an explicit description

$$\mathrm{trop}_v : T^{\mathrm{an}} \to \mathbb{R}^n, \quad p \mapsto (-\log(p(x_1)), \ldots, -\log(p(x_n))).$$

It is immediate from the definitions that the map trop_v is continuous. This is the big advantage of working with Berkovich analytic spaces in this framework as we may use their nice topological properties.

Definition 3.2 We define the *tropical variety associated to* X by $\mathrm{Trop}_v(X) := \mathrm{trop}_v(X^{\mathrm{an}})$. In Section 13, we will complete the definition of a tropical variety by assigning certain weights.

In the following result, we refer the reader to the appendix for the terminology borrowed from convex geometry.

Theorem 3.3 (Bieri–Groves) $Trop_v(X)$ *is a finite union of* Γ*-rational polyhedra in* $N_{\mathbb{R}}$. *If* X *is of pure dimension* d, *then we may choose all the polyhedra* d*-dimensional.*

Proof: The proof is given in [BG], Theorem A. Note that even the definition of X^{an} occurs implicitly in this paper. For a translation into tropical language, we refer to [EKL], Theorem 2.2.3. In Theorem 10.14, we will give a proof of the first statement using the Gröbner fan. A proof for dimensionality is given in [Gub3], Proposition 5.4, which generalizes to closed analytic subvarieties. $\quad\square$

Remark 3.4 If the absolute value on K is trivial, then a Γ-rational polyhedron is just a rational polyhedral cone. In this case, we conclude that $\mathrm{Trop}_v(X)$ is a finite union of such cones.

We illustrate the advantage of Berkovich spaces by giving the proof of the following well-known result (see [EKL], Theorem 2.2.7).

Proposition 3.5 If K is complete and X is connected, then $\mathrm{Trop}_v(X)$ is connected.

Proof: If X is connected, then X^{an} is connected ([Berk1], Theorem 3.4.8 and Theorem 3.5.3). This is a rather nontrivial fact. By continuity of the tropicalization map, we conclude that $\mathrm{Trop}_v(X)$ is connected. $\quad\square$

Remark 3.6 This is wrong if K is non-complete, but still true if K is algebraically closed. For more details, we refer to the paper [CP] of Cartwright and Payne, where they prove also that $\mathrm{Trop}_v(X)$ is connected in codimension 1 if X is irreducible over a complete or algebraically closed field K.

Proposition 3.7 Let (L, w) be a valued field extending (K, v). Then we have $\mathrm{Trop}_w(X_L) = \mathrm{Trop}_v(X)$.

Proof: Let $\varphi : (X_L)^{\mathrm{an}} \to X^{\mathrm{an}}$ be the restriction map of seminorms. We have seen in Lemma 2.3 that φ is surjective. Using $\mathrm{trop}_w = \mathrm{trop}_v \circ \varphi$, we get the claim. $\quad\square$

The following result shows that our definition of a tropical variety agrees with the usual one.

Proposition 3.8 Let (L, w) be an algebraically closed valued field extending (K, v) endowed with a non-trivial absolute value $| \ |_w$ and let $x_1, \ldots, x_n$ be torus coordinates on T. Then $\mathrm{Trop}_v(X)$ is equal to the closure of

$$\{(-\log |x_1|_w, \ldots, -\log |x_n|_w) \mid \mathbf{x} \in X(L)\}$$

in $\mathbb{R}^n$.

Proof: By base change and Proposition 3.7, we may assume that K is algebraically closed and that v is non-trivial. We have seen in 2.6, that $X(K)$ is dense in X^{an} and hence continuity of the tropicalization map yields the claim. $\quad\square$

4 Models over the valuation ring and reduction

In this section, (K, v) is a valued field with valuation ring K° and residue field $\tilde{K}$. We will study models of a scheme X of finite type over K. The models are flat schemes over K° but not necessarily of finite type. We will obtain a model of a closed subscheme of X by taking the closure. For integral points of a model, there is always a reduction modulo the maximal ideal $K^{\circ\circ}$ which is a point in the special fibre. We will compare it with the reduction from the theory of strictly affinoid algebras.

Definition 4.1 A K°-*model* of a scheme X over K is a flat scheme $\mathscr{X}$ over K° with generic fibre $\mathscr{X}_\eta := \mathscr{X}_K = X$. The special fibre $\mathscr{X}_{\tilde{K}}$ of $\mathscr{X}$ is denoted by $\mathscr{X}_s$. The model $\mathscr{X}$ is called *algebraic* if it is of finite type over K°.

Lemma 4.2 *A module Q over K° is flat if and only if Q is torsion-free.*

Proof: Any flat module is obviously torsion-free. If the base is a valuation ring, then the converse holds. It is enough to show that the map $I \otimes_{K^\circ} Q \to Q$ is injective for every finitely generated ideal I of K°. As such an ideal is generated by a single element α, injectivity follows immediately from Q torsion free. $\square$

4.3 Let $\mathscr{X} = \mathrm{Spec}(A)$ be a flat scheme over K° with generic fibre $X = \mathscr{X}_\eta$. Then we have $X = \mathrm{Spec}(A_K)$ for $A_K := A \otimes_{K^\circ} K$. Note that flatness implies $A \subset A_K$. A closed subscheme Y of X is given by an ideal I_Y in A_K. The *closure* $\overline{Y}$ *of* Y *in* $\mathscr{X}$ is defined as the closed subscheme of $\mathscr{X}$ given by the ideal $I_Y \cap A$.

Proposition 4.4 *The closure of Y is the unique closed subscheme of $\mathscr{X}$ with generic fibre Y which is flat over K°.*

Proof: It is clear that $A/(I_Y \cap A)$ is K°-torsion free and hence flat over K° by Lemma 4.2. For every $f \in A_K$, there is a non-zero $\lambda \in K^\circ$ with $\lambda f \in A$. We conclude that $I_Y \cap A$ generates I_Y as an ideal in A_K and hence the generic fibre of $\overline{Y}$ is Y.

Let $\mathscr{Y}$ be any closed subscheme of $\mathscr{X}$ with generic fibre Y which is flat over K°. Then $\mathscr{Y}$ is given by an ideal J in A such that J generates I_Y as an ideal in A_K. We conclude that $J \subset I_Y \cap A$. Hence we have a canonical homomorphism $h : A/J \to A/(I_Y \cap A)$. By flatness over K°, we have $A/J \subset A_K/I_Y$ and h factors through this inclusion. We deduce that h is one-to-one proving $J = I_Y \cap A$. $\square$

Corollary 4.5 *Let $\psi : \mathscr{X}' \to \mathscr{X}$ be a flat morphism of flat affine K°-schemes with generic fibre $\psi_\eta : X' \to X$. Then we have $\overline{(\psi_\eta)^{-1}(Y)} = \psi^{-1}(\overline{Y})$ for a closed subscheme Y of X, where the closures are taken in $\mathscr{X}'$ and $\mathscr{X}$.*

Proof: As $\psi^{-1}(\overline{Y})$ is a closed subscheme of $\mathscr{X}'$ with generic fibre $(\psi_\eta)^{-1}(Y)$ which is flat over K°, the claim follows from Proposition 4.4. $\square$

Remark 4.6 In particular, this shows that localization is compatible with taking the closure. Therefore the closure may be defined in any flat scheme $\mathscr{X}$ over K°. Indeed, let Y be a closed subscheme of $X := \mathscr{X}_\eta$. First, we define $\overline{Y}$ locally on affine charts as in 4.3 and then we glue the affine pieces to get a closed subscheme $\overline{Y}$ of $\mathscr{X}$ by compatibility of the affine construction with localization. The closure is still characterized by Proposition 4.4. Moreover, Corollary 4.5 immediately yields that the formation of the closure is compatible with flat pull-back. Note also that the underlying set of $\overline{Y}$ is the topological closure of Y in $\mathscr{X}$.

Corollary 4.7 *Let (L, w) be a valued field extension of (K, v) and let $\mathscr{X}$ be a flat scheme over K°. For a closed subscheme Y of $X = \mathscr{X}_\eta$, we have $(\overline{Y})_{L^\circ} = \overline{Y_L}$ with closures taken in $\mathscr{X}$ and $\mathscr{X}_{L^\circ}$.*

Proof: Note that the base change morphism $\mathscr{X}_{L^\circ} \to \mathscr{X}$ is flat. Taking the closure depends only on the model and not on the base and hence compatibility with flat pull-back (Corollary 4.5) yields the claim. □

4.8 For an L°-integral point P of $\mathscr{X}$, the reduction $\pi(P) \in \mathscr{X}_s$ is defined as the image of the closed point of $\mathrm{Spec}(L^\circ)$ with respect to the morphism $\mathrm{Spec}(L^\circ) \to \mathscr{X}$ defining P. If $\mathscr{X} = \mathrm{Spec}(A)$ is affine, then $\pi(P)$ is given by the prime ideal $\{a \in A \mid |a(P)|_w < 1\}$ in A.

4.9 Let X be a scheme of finite type over K with K°-model $\mathscr{X}$. Our goal is to introduce a reduction map π from X to the special fibre of $\mathscr{X}$. Such a map can be defined only at integral points and it turns out that it is better to work analytically.

We handle first the affine case, i.e. $\mathscr{X} = \mathrm{Spec}(A)$ for an algebra A of finite type over K. Then we define $X^\circ := \{p \in X^{\mathrm{an}} \mid p(f) \le 1 \; \forall f \in A\}$. Note that X° is the set of points in X^{an} which are induced by an L°-integral point of X for some valued field extension (L, w). Such points of X are called *potentially integral*. If $p \in X^\circ$, then the *reduction* $\pi(p) \in \mathscr{X}_s$ is given by the prime ideal $\{a \in A \mid p(a) < 1\}$ in A.

In general, we define X° as the union of all $U^\circ := \{p \in U^{\mathrm{an}} \mid p(f) \le 1 \; \forall f \in A\}$, where $\mathscr{U} = \mathrm{Spec}(A)$ ranges over the affine open subsets of $\mathscr{X}$ and $U := \mathscr{U}_\eta$. It is clear that the notions coincide in the affine case. The points of X° are induced by the potentially integral points of X as above. Proceeding locally, we get the *reduction map* $\pi : X^\circ \to \mathscr{X}_s$.

Note that if $\mathscr{X}$ is an algebraic K°-model, then X° is a compact analytic subdomain of X^{an}. Indeed, in the affine case we get an affinoid subdomain in X^{an} and in general, X° is a finite union of affinoids. If $\mathscr{X}$ is a proper scheme over K°, then rational and integral points are the same and hence $X^\circ = X^{\mathrm{an}}$.

If we assume that K is endowed with a non-trivial complete valuation v and if we assume that $\mathscr{X}$ is an algebraic K°-model, then we will see in 4.13 that X° is the generic fibre of the completion of $\mathscr{X}$ along the special fibre.

4.10 For a scheme X of finite type over K with algebraic K°-model $\mathscr{X}$, the reduction map π can be described algebraically in the following way: We consider an L°-integral point P of X for a valued field (L, w) extending (K, v). Integrality means here that there is an affine chart $\mathscr{U}$ of $\mathscr{X}$ with affine coordinates $x_1, \ldots, x_n$ such that $x_1(P), \ldots, x_n(P) \in L^\circ$. Then $\pi(P)$ is the point of the special fibre $\mathscr{U}_s$ given by using the coordinates modulo the maximal ideal $L^{\circ\circ}$ of L°. Note that this point is not closed if the residue field $\tilde{L}$ is an infinite extension of $\tilde{K}$.

In the theory of strictly affinoid algebras introduced in 2.8, there is a similar concept of reduction which we study next. For this, we assume that the valuation v on K is non-trivial and complete.

4.11 Let $\mathscr{A}$ be a strictly affinoid K-algebra with Berkovich spectrum $Y = \mathscr{M}(\mathscr{A})$. We define the *reduction* of Y by $\tilde{Y} := \mathrm{Spec}(\tilde{\mathscr{A}})$ and the *special fibre* of Y by $Y_s := \mathrm{Spec}(\mathscr{A}^\circ/(K^{\circ\circ}\mathscr{A}^\circ))$. The reduction is an algebraic variety over the residue field $\tilde{K}$ (see [BGR], Corollary 6.4.3/1). Since the maximal ideal $K^{\circ\circ}$ of K° generates an ideal in $\mathscr{A}^\circ$ contained in $\mathscr{A}^{\circ\circ}$, we get a canonical surjective homomorphism $\mathscr{A}^\circ/(K^{\circ\circ}\mathscr{A}^\circ) \to \tilde{\mathscr{A}}$. This induces a canonical morphism $\tilde{Y} \to Y_s$. Since the spectral radius is power-multiplicative, it is clear that this morphism is a bijection.

We have a map $Y \to \tilde{Y}$, given by mapping the seminorm p to the prime ideal $\{a \in \mathscr{A}^\circ \mid p(a) < 1\}/\mathscr{A}^{\circ\circ}$ of $\tilde{\mathscr{A}}$. It induces a *reduction map* $\pi : Y \to Y_s$.

As in algebraic geometry, a *Zariski closed* subset of Y is the zero set of a subset of $\mathscr{A}$ and this leads to the *Zariski topology* on Y. Note that a Zariski open subset is dense in the Berkovich topology if and only if it is dense in the Zariski topology.

Lemma 4.12 *For a Zariski open and dense subset S of Y, we have $\pi(S) = Y_s$.*

Proof: We first note that the reduction map π is surjective (see [Berk1], Proposition 2.4.4). If z is a closed point of Y_s, then $\pi^{-1}(z)$ is an open non-empty subset of Y ([Berk1], Lemma 2.4.1). By density, there is $y \in S$ with $\pi(y) = z$. In general, there is a complete valued field (L, w) extending (K, v) and an L-rational point P of Y such that $\pi(P) = z$ (see Remark 2.2). Then the reduction $\pi_L(P)$ of P in the special fibre of $Y_L = \mathscr{M}(\mathscr{A} \widehat{\otimes}_K L)$ is $\tilde{L}$-rational. The preimage S_L of S is Zariski open and dense in Y_L. Using the above, there is $y_L \in S_L$ with $\pi_L(y_L) = \pi_L(P)$. Then we have $\pi(y) = z$ for the image y of y_L in Y. $\qquad\square$

4.13 We compare the two concepts for a reduction map in the following situation: Let (K, v) be an arbitrary valued field and let (L, w) be a complete valued field extending (K, v) with w non-trivial. We consider a flat affine scheme $\mathscr{X} = \mathrm{Spec}(A)$ of finite type over K° with generic fibre $X = \mathrm{Spec}(A_K)$. For convenience, we choose coordinates $x_1, \ldots, x_n$ on $\mathscr{X}$, i.e. $A = K^\circ[x_1, \ldots, x_n]/I$ for an ideal I in $K^\circ[x_1, \ldots, x_n]$. Then we complete the base change $\mathscr{X}_{L^\circ}$ along the special fibre (more precisely, we take the ν-adic completion for some non-zero $\nu \in K^{\circ\circ}$) to get a flat formal scheme $\mathcal{Y} = \mathrm{Spf}(L^\circ\langle x_1, \ldots, x_n\rangle/\langle I\rangle)$ over L° (see [Ull]). The generic fibre Y of $\mathcal{Y}$ is the Berkovich spectrum of the strictly affinoid algebra $\mathscr{A}$ defined by

$$\mathscr{A} := (L^\circ\langle x_1, \ldots, x_n\rangle/\langle I\rangle) \otimes_{L^\circ} L = L\langle x_1, \ldots, x_n\rangle/\langle I\rangle.$$

By construction, Y is the affinoid subdomain $(X_L)^\circ = \{p \in (X_L)^{\mathrm{an}} \mid p(x_1) \leq 1, \ldots, p(x_n) \leq 1\}$ in $(X_L)^{\mathrm{an}}$. It is easy to see that we have a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \pi\ } & Y_s \\
\downarrow & & \downarrow \\
X^\circ & \xrightarrow{\ \pi\ } & \mathscr{X}_s
\end{array}
\tag{2}
$$

where the vertical maps are induced by base change and the horizontal maps are the reduction maps. Applying Theorem 6.3.4/2 of [BGR] to the surjective homomorphism $L\langle x_1, \ldots, x_n\rangle \to \mathscr{A}$, it follows that the canonical morphism $\tilde{Y} \to \mathcal{Y}_s$ is a finite map. Both spaces have dimension equal to $\dim(X)$ and an easy localization argument shows that this finite map is surjective. As the base change morphism $\mathcal{Y}_s = (\mathscr{X}_L)_s \to \mathscr{X}_s$ is also surjective and since the canonical morphism $\tilde{Y} \to Y_s$ is a bijection, we deduce that the morphism $Y_s \to \mathscr{X}_s$ is surjective. By Lemma 4.12, we conclude that the reduction map $\pi : X^\circ \to \mathscr{X}_s$ is surjective.

Proposition 4.14 *Let $\mathscr{X}$ be a flat scheme of finite type over K° with generic fibre X and let U be an open dense subset of X. Then we have $\pi(U^{\mathrm{an}} \cap X^\circ) = \mathscr{X}_s$. If K is algebraically closed and v is non-trivial, then every $\tilde{K}$-rational point of $\mathscr{X}_s$ is the reduction of a K°-integral point contained in U.*

Proof: We may assume that $\mathscr{X}$ is affine and hence we are in the situation of 4.13. We choose a valued field (L, w) extending (K, v) with w complete and non-trivial. Since the base change U_L is open and dense in X_L, we conclude that $S := (U_L)^{\mathrm{an}} \cap Y$ is Zariski open and dense in the analytic space Y. Using surjectivity of the map $Y_s \to \mathscr{X}_s$ and Lemma 4.12, we deduce $\pi(U^{\mathrm{an}} \cap X^\circ) = \mathscr{X}_s$ from the commutative diagram (2).

If K is algebraically closed and v is non-trivial, then we note first that $\tilde{K}$ is algebraically closed (see [BGR], 3.4.1). For any closed point z of the special fibre, the above and anticontinuity of the reduction map show that $\pi^{-1}(z) \cap U^{\mathrm{an}}$ is a non-empty open subset of X°. If we embed U into affine space, then we see that $\pi^{-1}(z) \cap U^{\mathrm{an}}$ is the intersection of U^{an} with an open ball and hence $\pi^{-1}(z) \cap U^{\mathrm{an}}$ is even an open subset of X^{an}. Density of the K-rational points yields the claim (see 2.6). $\qquad\square$

Example 4.15 We assume that the absolute value is trivial on K. Let $X = \mathbb{P}^1_K$ with projective coordinates x_0, x_1. For $i = 0, 1$, we consider the affine charts $U_i := \{x \in X \mid x_i \neq 0\}$ isomorphic to $\mathbb{A}^1_K$. For any $r > 0$, we get an element $p_r \in U_0^{\mathrm{an}}$ given as the seminorm $p_r(f) := \max_i |a_i| r^i$ for $f(y) = \sum_i a_i y^i \in K[y]$ with $y := \frac{x_1}{x_0}$. If $r \leq 1$, then $p_r(f) = r^j$ for j minimal with $a_j \neq 0$. Then we have $p_r \leq 1$ and hence the reduction of p_r is defined by $\pi(p_r) = \{f \in K[y] \mid p_r(f) < 1\} \in \mathrm{Spec}(K[y]) \subset \mathbb{P}^1_K$. If $r < 1$, then $\pi(p_r) = (1 : 0)$. If $r = 1$, then $\pi(p_r)$ is the generic point of $\mathbb{P}^1_K$. If $r > 1$, then we use the other chart U_1 with affine coordinate $z := \frac{x_0}{x_1}$. For $g(z) = \sum_i a_i z^i \in K[z]$, we have $p_r(g) = \max_i |a_i| r^{-i}$ and hence $\pi(p_r) = (0 : 1)$.

5   Initial degeneration

In this section, we study the initial degeneration $\mathrm{in}_\omega(X)$ of a closed subscheme X of the multiplicative torus $T = \mathbb{G}^n_m$ over the valued field (K, v) at $\omega \in N_{\mathbb{R}}$. We follow here the original definition of the initial degeneration using a translation to the origin of the torus. Then $\mathrm{in}_\omega(X)$ is a closed subscheme of the torus $\mathbb{T}_{\tilde{K}}$ which is only well-defined up to translations. This approach fits very well to Hilbert schemes as we will see in Section 10. For an intrinsic approach, we refer to [OP].

Definition 5.1 Let (L, w) be a valued field extending (K, v) and let $t \in T(L)$. Then the *initial degeneration of X at t* is defined as the special fibre of the closure of $t^{-1} X_L$ in the split multiplicative torus $\mathbb{T}_{L^{\circ}}$ over the valuation ring L°. It is a closed subscheme of the split torus $\mathbb{T}_{\tilde{L}}$ over the residue field $\tilde{L}$ which we denote by $\mathrm{in}_t(X)$.

Lemma 5.2 *Let (L, w) and (L', w') be valued fields extending (K, v). Then there is a valued field (L'', w'') extending (L, w) and (L', w').*

Proof: This is proved in [Duc], §0.3.2, using Berkovich's theory. $\qquad\square$

Proposition 5.3 *Let (L, w) and (L', w') be valued fields extending (K, v). Suppose that there is $\omega \in N_{\mathbb{R}}$ such that $\omega = \mathrm{trop}_w(t) = \mathrm{trop}_{w'}(t')$ for $t \in T(L)$ and $t' \in T(L')$. For any field (L'', w'') as in Lemma 5.2, there is $g \in \mathbb{T}(\tilde{L}'')$ with*

$$\mathrm{in}_{t'}(X)_{\tilde{L}''} = g \cdot \mathrm{in}_t(X)_{\tilde{L}''}. \tag{3}$$

Proof: Since t, t' have the same tropicalizations, the point $t/t' \in T(L'')$ is in fact an $(L'')^{\circ}$-integral point of $\mathbb{T}$ and hence it has a well-defined reduction $g \in \mathbb{T}(\tilde{L}'')$. The relation

$$(t')^{-1} X_{L''} = (t/t') \cdot t^{-1} X_{L''}$$

and Corollary 4.7 give immediately the claim. $\qquad\square$

5.4 The proposition shows that the initial degeneration depends essentially only on ω. For any $\omega \in N_{\mathbb{R}}$, there is a valued field (L, w) extending (K, v) and $t \in T(L)$ with $\mathrm{trop}_w(t) = \omega$. We define the *initial degeneration $\mathrm{in}_\omega(X)$ of X at ω* as $\mathrm{in}_t(X)$ which is well-defined as an equivalence class for the equivalence relation (3). We call the residue field $\tilde{L}$ or any extension of it a *field of definition* for $\mathrm{in}_\omega(X)$.

Proposition 5.5 *Let (L, w) be a valued field extending (K, v) and let $\omega \in N_{\mathbb{R}}$. Then we have $\mathrm{in}_\omega(X_L) = \mathrm{in}_\omega(X)$ up to the equivalence relation given by (3).*

Proof: By Corollary 4.7, the formation of the closure is compatible with base change and this yields easily the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The next result is called the *fundamental theorem* of tropical algebraic geometry. It is due to Kapranov in the hypersurface case (unpublished manuscript, later incorporated in [EKL]) and to Speyer–Sturmfels [SS], Draisma [Dra], Payne [Pay] in general.

Theorem 5.6 *For a closed subscheme X of T, the tropical variety $\mathrm{Trop}_v(X)$ may be characterized in the following two equivalent ways:*

(a) $\mathrm{Trop}_v(X) = \mathrm{trop}_v(X^{\mathrm{an}})$

(b) The set $\{\omega \in N_{\mathbb{R}} \mid \mathrm{in}_\omega(X) \neq \emptyset\}$ in $N_{\mathbb{R}}$ is equal to $\mathrm{Trop}_v(X)$.

Proof: We have to prove that $\omega \in N_{\mathbb{R}}$ is in $\mathrm{trop}_v(X^{\mathrm{an}})$ if and only if $\mathrm{in}_\omega(X) \neq \emptyset$. By base change, we may assume that (K, v) is a non-trivially valued complete algebraically closed field such that $\omega = \mathrm{trop}_v(t)$ for some $t \in T(K)$ (see Propositions 3.7 and 5.5). Passing to $t^{-1}X$, we may assume that $t = e$ and $\omega = 0$. Let $\mathscr{X}$ be the closure of X in $\mathbb{T}$. It is an algebraic K°-model of X. We recall that the reduction map π to the special fibre $\mathscr{X}_s$ is defined on the affinoid subdomain X° of X^{an} from 4.9. Using that the regular function on $\mathscr{X}$ are generated by the characters χ^u with u ranging over a basis of M, we deduce easily that $X^\circ = \mathrm{trop}_v^{-1}(0) \cap X^{\mathrm{an}}$. We have seen in Proposition 4.14 that $\pi(X^\circ) = \mathscr{X}_s$ and hence $\mathrm{in}_0(X) = \mathscr{X}_s$ is empty if and only if $\mathrm{trop}_v^{-1}(0) \cap X^{\mathrm{an}} = \emptyset$. The latter is equivalent to $0 \notin \mathrm{trop}_v(X^{\mathrm{an}})$ proving the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.7 Initial degenerations may be studied using methods from the theory of Gröbner bases. Let (L, w) be a valued field extending (K, v) and let $t \in T(L)$ with $\omega = \mathrm{trop}_w(t)$. For a Laurent polynomial $f = \sum_{u \in M} \alpha_u \chi^u \in K[M]$, we define *the initial form* $\mathrm{in}_t(f) \in \tilde{L}[M]$ in the following way. If $f = 0$, then $\mathrm{in}_t(f)$ is the zero polynomial in $\tilde{L}[M]$. If $f \neq 0$, then we choose $\lambda \in L$ with $v(\lambda) = \min_{u \in M} v(\alpha_u) + \langle u, \omega \rangle$ and we set $\mathrm{in}_t(f) := \sum_{u \in M} \pi(\lambda^{-1}\alpha_u \chi^u(t))\chi^u$, where $\pi : L^\circ \to \tilde{L}$ is the reduction map. Note that the initial form is only well-defined up to multiplication by $\tilde{L}^\times$.

For a closed subscheme X of T given by the ideal I_X in $K[M]$, we define the *initial ideal* of X at t as the ideal $\mathrm{in}_t(I_X)$ in $\tilde{L}[M]$ generated by $\{\mathrm{in}_t(f) \mid f \in I_X\}$. By construction, the initial degeneration $\mathrm{in}_t(X)$ from 5.1 is given by the initial ideal $\mathrm{in}_t(I_X)$ in $\tilde{L}[M]$. If there is a canonical homomorphism $\tau : v(L^\times) \to L^\times$ with $v(\tau(r)) = r$ for all $r \in \mathbb{R}$, then the initial form $\mathrm{in}_\omega(f)$ of f can be defined canonically at $\omega \in N_{\mathbb{R}}$ using $t := \tau(\omega)$ and $\lambda := \tau(\min_{u \in M} v(\alpha_u) + \langle u, \omega \rangle)$. For more details, we refer to [MS], §2.4. For an intrinsic approach to initial forms $\mathrm{in}_\omega(f)$ and initial degenerations $\mathrm{in}_\omega(X)$ without using translations to the origin, we refer to [OP].

Example 5.8 Suppose that X is a hypersurface in T. Then I_X is generated by some $f \in K[M] \setminus \{0\}$. For $t \in T(K)$, the initial ideal $\mathrm{in}_t(X)$ is generated by the initial form $\mathrm{in}_t(f)$. This is clear from $\mathrm{in}_t(fg) = \mathrm{in}_t(f)\mathrm{in}_t(g)$ (up to multiplication by $K^\times$) for every $g \in K[M]$.

6   Affine toric schemes over a valuation ring

First, we recall some facts from the theory of normal toric varieties which will be very important in the sequel. We refer to [CLS], [Ful2], [KKMS] or [Oda] for details.

They are independent of any valuations on the field K. Then we assume that K is endowed with a non-archimedean absolute value $|\ |$ with valuation $v := -\log|\ |$ and value group $\Gamma := v(K^\times)$. We consider the split torus $\mathbb{T} = \mathrm{Spec}(K^\circ[M])$ over the valuation ring K° with generic fibre T. The main focus will be laid on the theory of affine $\mathbb{T}$-toric schemes over K° associated to a pointed Γ-rational polyhedron. While the generic fibre of such a scheme is a T-toric variety over K, the geometry of the special fibre is more complicated and is closely related to the combinatorics of the polyhedron. This section can be seen as a generalization of §4.3 in [KKMS], where the case of a discrete valuation is handled. Further references: [Rab], [BPR].

Definition 6.1 Let K be a field and let T be a split torus over K. A *T-toric variety* is a variety Y over K containing T as an open dense subset such that the translation action of T on itself extends to an algebraic T-action on Y.

6.2 There are bijective correspondences between

(a) rational polyhedral cones σ in $N_\mathbb{R}$ which do not contain a line;

(b) finitely generated saturated semigroups S in M which generate M as a group;

(c) affine normal T-toric varieties Y over K (up to equivariant isomorphisms restricting to the identity on T).

The correspondences are given by $S = \check{\sigma} \cap M$ and $Y = \mathrm{Spec}(K[S])$. We refer the reader to the appendix for the terminology from convex geometry.

6.3 In general, there is a bijective correspondence between normal T-toric varieties Y over K (up to equivariant isomorphisms restricting to the identity on T) and pointed rational fans in $N_\mathbb{R}$. We denote the toric variety associated to the fan Σ by Y_Σ. Every cone σ of Σ induces an open affine toric subset U_σ of Y_Σ by the affine case considered above and Y_Σ is covered by such affine charts.

We extend the above definition to the case of valuation rings:

Definition 6.4 A $\mathbb{T}$-*toric scheme* over the valuation ring K° is an integral separated flat scheme $\mathscr{Y}$ over K° such that the generic fibre $\mathscr{Y}_\eta$ contains T as an open subset and such that the translation action of T on T extends to an algebraic action of $\mathbb{T}$ on $\mathscr{Y}$ over K°. We call it a $\mathbb{T}$-*toric variety* if $\mathscr{Y}$ is of finite type over K°.

Definition 6.5 For a Γ-rational polyhedron Δ in $N_\mathbb{R}$, we set

$$K[M]^\Delta := \{ \sum_{u \in M} \alpha_u \chi^u \in K[M] \mid v(\alpha_u) + \langle u, \omega \rangle \geq 0 \ \forall \omega \in \Delta \}.$$

Proposition 6.6 *We get a K°-subalgebra $K[M]^\Delta$ of $K[M]$ which is flat over K°. Moreover, $K[M]^\Delta$ is an integral domain with $K[M]^\Delta \otimes_{K^\circ} K = K[\check{\sigma} \cap M]$ and quotient field $K(\check{\sigma} \cap M) = K(\rho^\perp \cap M)$ where $\rho = \sigma \cap (-\sigma)$ is the largest linear subspace contained in the recession cone σ of Δ.*

Proof: It is easy to show that $K[M]^\Delta$ is a K°-subalgebra of $K[M]$ and hence it is an integral domain. In particular, $K[M]^\Delta$ has no K°-torsion and hence it is flat over K° (see Lemma 4.2). For $\sum_{u \in M} \alpha_u \chi^u \in K[M]^\Delta$, it follows from the Minkowski-Weil theorem (see A.7) that $\alpha_u = 0$ for every $u \notin M \cap \check{\sigma}$ and that $K[M]^\Delta \otimes_{K^\circ} K = K[\check{\sigma} \cap M]$. The last claim is now obvious. $\qquad\square$

The algebra $K[M]^\Delta$ was studied by [KKMS] in case of a discrete valuation and by [BPR] in case of an algebraically closed ground field endowed with a non-trivial complete absolute value. We will see in the following that most of their results hold in our more general setting. If the valuation v is trivial, then Δ is a rational cone σ and the above shows that $K[M]^\Delta = K[M \cap \check{\sigma}]$ leading to the classical case in 6.2.

Proposition 6.7 *If the value group Γ is either discrete or divisible in $\mathbb{R}$, then the algebra $K[M]^\Delta$ is of finite presentation over K°.*

Proof: It is enough to prove that $K[M]^\Delta$ is a finitely generated K°-algebra. This follows from the fact that every finitely generated flat algebra over an integral domain is of finite presentation ([RG], Corollaire 3.4.7). If Γ is discrete, then either v is a discrete valuation or v is trivial. The latter is covered by the divisible case.

If v is a discrete valuation, then we may assume $\Gamma = \mathbb{Z}$. We consider the closure σ of the cone in $N_\mathbb{R} \times \mathbb{R}_+$ generated by $\Delta \times \{1\}$. It is a rational polyhedral cone (see 7.1 and refadmissible cones for an argument). If π is a uniformizing parameter for K°, then $K[M]^\Delta$ is generated by $\pi^k \chi^u$ with (u, k) ranging over the semigroup $S_\sigma := \check{\sigma} \cap (M \times \mathbb{Z})$. This semigroup is finitely generated (see 6.2) and hence we get the claim in the case of a discrete valuation.

If the value group is divisible in $\mathbb{R}$, we argue as follows: We reduce to the case of a pointed Γ-rational polyhedron by the procedure described in 6.11 below. Then the same proof as for Proposition 4.11 in [BPR] works. Indeed, the crucial point in this proof is that the vertices $\omega_1, \ldots, \omega_r$ of Δ are in N_Γ which is always the case for Γ divisible in $\mathbb{R}$. Then it is shown that $K[M]^\Delta$ is generated by the functions $\alpha_{ij} \chi^{u_{ij}}$, where $(u_{ij})_j$ is a finite set of generators for $\check{\sigma}_i \cap M$ with σ_i equal to the local cone $\mathrm{LC}_{\omega_i}(\Delta)$ and where $\alpha_{ij} \in K$ with $v(\alpha_{ij}) + \langle u_{ij}, \omega_i \rangle = 0$. $\qquad\square$

6.8 For $\omega \in N_\mathbb{R}$, we will use the ω-*weight*

$$v_\omega\left(\sum_u \alpha_u \chi^u\right) := \min_u v(\alpha_u) + \langle u, \omega \rangle$$

on $K[M]$ which extends obviously to a valuation on the field $K(T)$. We may view it as a weighted Gauss-valuation similarly as in 2.7.

For a Γ-rational polyhedron Δ in $N_\mathbb{R}$, we have $K[M]^\Delta \otimes_{K^\circ} K = K[\check{\sigma} \cap M]$ (see Proposition 6.6). It is clear that $v_\Delta := \min_{\omega \in \Delta} v_\omega$ is not necessarily a valuation on $K[\check{\sigma} \cap M]$. However, $\|\ \|_\Delta := \exp(-v_\Delta)$ is a K-*algebra norm* on $K[\check{\sigma} \cap M]$, i.e. we have $\|f \cdot g\|_\Delta \leq \|f\|_\Delta \cdot \|g\|_\Delta$ and $\|\lambda \cdot f\|_\Delta = |\lambda| \cdot \|f\|_\Delta$ for $\lambda \in K$ and $f, g \in K[\check{\sigma} \cap M]$.

If Δ is pointed, then any affine form on Δ which is bounded below takes its minimum in a vertex and so we have

$$v_\Delta(f) = \inf_{\omega \in \Delta} v_\omega(f) = \min\{v_\omega(f) \mid \omega \text{ vertex of } \Delta\} \tag{4}$$

for every $f \in K[\check{\sigma} \cap M]$.

If the value group Γ is neither discrete nor divisible in $\mathbb{R}$, then the K°-algebra $K[M]^\Delta$ is not necessarily of finite presentation over K° as one can verify in the example $K[x]^\omega$ for any $\omega \in \mathbb{R} \setminus \Gamma$ with a non-zero $n \in \mathbb{N}$ such that $n\omega \in \Gamma$. The referee has pointed out (and proved) that the following precise equivalence is valid.

Proposition 6.9 *Let us assume that the value group Γ is not discrete in $\mathbb{R}$ and let Δ be a pointed Γ-rational polyhedron in $N_\mathbb{R}$. Then the following are equivalent:*

 (a) the K°-algebra $K[M]^\Delta$ is of finite presentation;

 (b) the K°-algebra $K[M]^\Delta$ is finitely generated;

 (c) the vertices of Δ are in N_Γ;

 (d) $v_\Delta(K[M]^\Delta \setminus \{0\}) \subset \Gamma$.

Proof: The equivalence of (a) and (b) follows from [RG], Corollaire 3.4.7. Since Δ is a pointed Γ-rational polyhedron, it is clear that $M \cap \check{\sigma}$ generates M as a group. This and (4) prove immediately the equivalence of (c) and (d). The same arguments as in the proof of Proposition 6.7 show that (c) implies (b).

Finally, we prove that (b) yields (d). Let A be the completion of $K[M]^\Delta$ with respect to the algebra norm $\| \; \|_\Delta$ from 6.8. Since A has no $(K_v)^\circ$-torsion for the completion K_v of K, we have $A \subset \mathscr{A} := A \otimes_{(K_v)^\circ} K_v$ by using Lemma 4.2. There is a canonical algebra norm on $\mathscr{A}$ which extends the norm of the completion A. We denote this non-archimedean norm again by $\| \; \|_\Delta$. By construction, A is the closed unit ball in $\mathscr{A}$ with respect to $\| \; \|_\Delta$.

For a nonzero ρ in the maximal ideal $K^{\circ\circ}$, it is easy to see that the ρ-adic completion of $K[M]^\Delta$ is equal to A. Since A is of finite presentation over K°, we see that A is a topologically finitely generated $(K_v)^\circ$-algebra and hence $\mathscr{A}$ is an affinoid K-algebra (see 4.13). By construction, $\| \; \|_\Delta$ is a complete power-multiplicative algebra norm on $\mathscr{A}$. By [BGR], Theorem 6.2.4/1, we easily deduce that $\| \; \|_\Delta$ is the spectral radius of $\mathscr{A}$ (i.e. the supremum norm) and hence $A = \mathscr{A}^\circ$ in the notation of 2.9. Since A is topologically finitely generated and using also that v is not discrete, we deduce from Corollary 6.4.3/6 of [BGR] that the affinoid algebra $\mathscr{A}$ is distinguished. This means that $\mathscr{A}$ is a quotient of a Tate algebra such that the spectral radius on $\mathscr{A}$ agrees with the quotient norm. In particular, we get $\|\mathscr{A}\|_\Delta = |K|$ and hence (d) holds. $\qquad\square$

Proposition 6.10 *For a Γ-rational polyhedron Δ in $N_\mathbb{R}$, the algebra $K[M]^\Delta$ is integrally closed.*

Proof: Using the procedure described in 6.11 below, we may assume that Δ is pointed. Since every affine form on Δ which is bounded below takes its minimum in a vertex, we deduce that $K[M]^\Delta = \bigcap_\omega K[M]^\omega$ with ω ranging over the vertices of Δ. Hence it is enough to show that $K[M]^\omega$ is integrally closed in $K[M]$. Since the ω-weight v_ω from 6.8 is a valuation and $K[M]^\omega = \{f \in K[M] \mid v_\omega(f) \geq 0\}$, the same argument as in the case of valuation rings proves the claim. Indeed, let $f^m + a_{m-1} f^{m-1} + \cdots + a_0 = 0$ with $f \in K[M]$ and all $a_i \in K[M]^\omega$. Then the ultrametric triangle inequality applied to $f^m = -a_{m-1} f^{m-1} - \cdots - a_0$ and $v_\omega(a_i) \geq 0$ yield $v_\omega(f) \geq 0$. $\qquad\square$

6.11 Let Δ be a Γ-rational polyhedron in $N_\mathbb{R}$ with recession cone σ. We call $\mathscr{U}_\Delta := \mathrm{Spec}(K[M]^\Delta)$ the *polyhedral scheme over K° associated to Δ*. By Proposition 6.6 and Proposition 6.10, $\mathscr{U}_\Delta$ is a normal scheme which is flat over K°. If the value group is discrete or divisible then Proposition 6.7 shows that $\mathscr{U}_\Delta$ is of finite type over K°.

The K°-algebra $K[M]^\Delta$ is M-graded and hence $\mathbb{T}$ acts on $\mathscr{U}_\Delta$. It follows from Proposition 6.6 that $\mathscr{U}_\Delta$ is a $\mathbb{T}$-toric scheme over K° if and only if Δ is a pointed polyhedron in the sense of A.8. In this case, the generic fibre is the affine T-toric variety U_σ associated to σ (see Proposition 6.6).

In general, we consider the smallest linear subspace $\rho = \sigma \cap (-\sigma)$ contained in σ. Then $\mathscr{U}_\Delta$ is a toric scheme over K° with respect to the split torus $\mathrm{Spec}(K^\circ[M(\rho)])$. Here, we have used the lattice $M(\rho) := M \cap \rho^\perp$ with dual lattice $N(\rho) = N/N_\rho$ where N_ρ is the group $N \cap \rho$. The image of Δ in $N(\rho)_\mathbb{R}$ is a pointed polyhedron whose associated polyhedral scheme is $\mathscr{U}_\Delta$. This procedure is often used to reduce to the case of pointed polyhedra.

Proposition 6.12 *Let Δ' be a closed face of the Γ-rational polyhedron Δ in $N_\mathbb{R}$. Then the canonical $\mathbb{T}$-equivariant morphism $\mathscr{U}_{\Delta'} \to \mathscr{U}_\Delta$ is a distinguished open immersion.*

Proof: There is a halfspace $\{\omega \in N_{\mathbb{R}} \mid \langle u, \omega \rangle + v(\alpha) \geq 0\}$ containing Δ such that the face Δ' is cut out from Δ by the hyperplane $\{\omega \in N_{\mathbb{R}} \mid \langle u, \omega \rangle + v(\alpha) = 0\}$ for suitable $u \in M$ and $\alpha \in K$. We claim that $\mathscr{U}_{\Delta'}$ is the complement of the closed subscheme of $\mathscr{U}_{\Delta}$ given by the equation $\alpha \chi^u = 0$. To see this, we will show that $K[M]^{\Delta'}$ is the localization $(K[M]^{\Delta})_f$ for $f := \alpha \chi^u$. Using 6.11, we may assume that Δ is pointed. By construction, f is in $K[M]^{\Delta} \subset K[M]^{\Delta'}$ and f is invertible in $K[M]^{\Delta'}$. This yields $(K[M]^{\Delta})_f \subset K[M]^{\Delta'}$. To prove the reverse inclusion, it is enough to show that any homogeneous element $f' = \alpha' \chi^{u'} \in K[M]^{\Delta'}$ is contained in $(K[M]^{\Delta})_f$. From our assumptions, we deduce that there is a sufficiently large $m \in \mathbb{N}$ such that

$$m\left(v(\alpha) + \langle u, \omega \rangle\right) + v(\alpha') + \langle u', \omega \rangle \geq 0$$

for all vertices ω of Δ. By (4), we get $f' f^m \in K[M]^{\Delta}$ proving the claim. $\qquad \square$

Let Δ be any Γ-rational polyhedron in $N_{\mathbb{R}}$. It follows from 6.11 that the split torus $\mathbb{T}_{\tilde{K}}$ acts on the special fibre of $\mathscr{U}_{\Delta}$. Our goal is the description of the orbits of this action and hence only the induced reduced structure $((\mathscr{U}_{\Delta})_s)_{\mathrm{red}}$ is relevant.

Lemma 6.13 *The reduced induced structure on the special fibre is given by*

$$((\mathscr{U}_{\Delta})_s)_{\mathrm{red}} = \mathrm{Spec}(K[M]^{\Delta}/\{f \in K[M]^{\Delta} \mid v_{\Delta}(f) > 0\}).$$

The special fibre $(\mathscr{U}_{\Delta})_s$ is always of finite type over $\tilde{K}$. If the valuation v is not discrete or if Δ is pointed with all vertices contained in N_{Γ}, then $(\mathscr{U}_{\Delta})_s$ is reduced.

Proof: If v is the trivial valuation, then the special fibre is equal to the generic fibre and the claims are obvious. So we may assume that v is non-trivial. The special fibre of $\mathscr{U}_{\Delta}$ is a closed subscheme of $\mathscr{U}_{\Delta}$ given by the ideal $I = K^{\circ\circ}K[M]^{\Delta}$ in $K[M]^{\Delta}$. Since v_{Δ} is power-multiplicative, it is clear that the radical ideal $\sqrt{I}$ of I is contained in the ideal $J = \{f \in K[M]^{\Delta} \mid v_{\Delta}(f) > 0\}$. On the other hand, J is an M-homogeneous ideal in $K[M]^{\Delta}$ and so it is enough to show that every $f = \alpha \chi^u \in J$ is contained in $\sqrt{I}$. Since the valuation v on K is non-trivial, there is β in the maximal ideal $K^{\circ\circ}$ of K and $v(\beta) \leq v_{\Delta}(f^m)$ for $m \in \mathbb{N}$ sufficiently large. We conclude that $f^m \in I$ proving $I \subset \sqrt{J}$ and the first claim.

We handle now the remaining claims first in the case of a discrete valuation. Then Proposition 6.7 yields that the special fibre is of finite type. If Δ is pointed with all vertices contained in N_{Γ}, then (4) shows that we may choose β as the uniformizing parameter and $m = 1$ in the above argument. This proves $I = J$ and hence $(\mathscr{U}_{\Delta})_s$ is reduced.

It remains to handle the case of a value group Γ which is not discrete in $\mathbb{R}$. Then Γ is dense in $\mathbb{R}$ and hence we may choose $m = 1$ in the above argument proving again that $I = J$. We conclude that $(\mathscr{U}_{\Delta})_s$ is reduced. To prove that the special fibre is of finite type over $\tilde{K}$, we may assume that Δ is pointed using the procedure described in 6.11, . Since Δ is Γ-rational, we have a non-zero $m \in \mathbb{N}$ such that $m\omega \in N_{\Gamma}$ for every vertex ω of Δ. We conclude that $M_{\omega} := \{u \in M \mid \langle u, \omega \rangle \in \Gamma\}$ is a subgroup of M of finite index. Let σ_{ω} be the local cone of Δ at ω and let $\Delta_{\omega} := \omega + \sigma_{\omega}$. By Proposition 6.9, the K°-algebra $K[M_{\omega}]^{\Delta_{\omega}}$ is generated by a finite set S_{ω}.

To see that $(\mathscr{U}_{\Delta})_s$ is of finite type, it is enough to show that $K[M]^{\Delta}/K^{\circ\circ}K[M]^{\Delta}$ is generated by the reductions of $S := \bigcup_{\omega} S_{\omega}$, where ω ranges over the vertices of Δ. Using that Γ is not discrete in $\mathbb{R}$, we have seen above that $K[M]^{\Delta}/K^{\circ\circ}K[M]^{\Delta} = K[M]^{\Delta}/\{f \in K[M]^{\Delta} \mid v_{\Delta}(f) > 0\}$. Hence it is enough to show that any $f = \alpha \chi^u$ with $\alpha \in K$, $u \in M$ and $v_{\Delta}(f) = 0$ is in the algebraic span of S over K° modulo the ideal $J = \{f \in K[M]^{\Delta} \mid v_{\Delta}(f) > 0\}$. By (4), there is a vertex ω of Δ such that $v(\alpha) + \langle u, \omega \rangle = v_{\omega}(f) = v_{\Delta}(f) = 0$. This yields $u \in M_{\omega}$ and $f \in K[M_{\omega}]^{\Delta_{\omega}}$. We conclude that f is in the algebraic span of S_{ω} over K° proving the claim. $\qquad \square$

Proposition 6.14 *Let Δ be a pointed Γ-rational polyhedron in $N_{\mathbb{R}}$. Then there is a bijection between the vertices of Δ and the irreducible components of $(\mathscr{U}_{\Delta})_s$. The irreducible component corresponding to the vertex ω is the closed subscheme Y_{ω} of $\mathscr{U}_{\Delta}$ given by the prime ideal $\{f \in K[M]^{\Delta} \mid v_{\omega}(f) > 0\}$ of $K[M]^{\Delta}$.*

Proof: Since v_{ω} is a valuation on $K[M]^{\Delta}$ for any $\omega \in \Delta$, it is clear that $I_{\omega} := \{f \in K[M]^{\Delta} \mid v_{\omega}(f) > 0\}$ is a prime ideal in $K[M]^{\Delta}$. Since Δ is a pointed polyhedron, the restriction of any affine form $v_{\omega}(\alpha \chi^u)$ to Δ with $\alpha \chi^u \in K[M]^{\Delta}$ takes its minimum in a vertex ω and for every vertex, there is such an affine form which has its minimum precisely in this vertex. This means that the set of prime ideals I_{ω}, with ω ranging over the vertices of Δ, is a minimal primary decomposition of the ideal $\{f \in K[M]^{\Delta} \mid v_{\Delta}(f) > 0\}$. We have seen in Lemma 6.13 that the latter is the ideal of the reduced scheme underlying the special fibre $(\mathscr{U}_{\Delta})_s$ in $K[M]^{\Delta}$ and hence the I_{ω} are the ideals of the irreducible components of $(\mathscr{U}_{\Delta})_s$. $\qquad\square$

Corollary 6.15 *The irreducible component Y_{ω} of $(\mathscr{U}_{\Delta})_s$ is naturally $\mathbb{T}_{\tilde{K}}$-equivariantly isomorphic to $((\mathscr{U}_{\Delta(\omega)})_s)_{\mathrm{red}}$ where $\Delta(\omega) = \omega + \mathrm{LC}_{\omega}(\Delta)$. Moreover, $M_{\omega} := \{u \in M \mid \langle u, \omega \rangle \in \Gamma\}$ is a sublattice of finite index in M and Y_{ω} is equivariantly (but non-canonically) isomorphic to the $\mathrm{Spec}(\tilde{K}[M_{\omega}])$-toric variety over $\tilde{K}$ associated to the local cone $\mathrm{LC}_{\omega}(\Delta)$.*

Proof: Since $\Delta \subset \Delta(\omega)$, we have a canonical injective homomorphism

$$\varphi : K[M]^{\Delta(\omega)}/\{f \in K[M]^{\Delta(\omega)} \mid v_{\omega}(f) > 0\} \to K[M]^{\Delta}/\{f \in K[M]^{\Delta} \mid v_{\omega}(f) > 0\}.$$

To show surjectivity, it is enough to show that the residue class of $f = \alpha \chi^u \in K[M]^{\Delta}$ is in the image of φ. We may assume that $v_{\omega}(f) = 0$ otherwise this is trivial. Then the affine form $\Delta \to \mathbb{R}$, $\nu \mapsto v_{\nu}(f)$ takes its minimum in the vertex ω. This even holds if we extend the affine form to $\Delta(\omega)$ by definition of the local cone $\mathrm{LC}_{\omega}(\Delta)$. We conclude that $f \in K[M]^{\Delta(\omega)}$ proving that φ is an isomorphism. By Lemma 6.13 and Proposition 6.14, we deduce $Y_{\omega} \cong ((\mathscr{U}_{\Delta(\omega)})_s)_{\mathrm{red}}$. Equivariance of this isomorphism follows from the fact that φ is an M-graded isomorphism.

Since Δ is Γ-rational, there is a non-zero $m \in \mathbb{N}$ with $m\omega \in N_{\Gamma}$ and hence M_{ω} is a sublattice of finite index in M. It is trivial to show that the canonical homomorphism from $K[M_{\omega}]^{\Delta(\omega)}/\{f \in K[M_{\omega}]^{\Delta(\omega)} \mid v_{\omega}(f) > 0\}$ to $K[M]^{\Delta(\omega)}/\{f \in K[M]^{\Delta(\omega)} \mid v_{\omega}(f) > 0\}$ is an isomorphism. We conclude that we may replace M by M_{ω} and so we may assume $M = M_{\omega}$. Then there is $t \in T(K)$ with $\mathrm{trop}_v(t) = \omega$. We may replace Δ by $\Delta - \omega$ which means geometrically that we use translation by t^{-1} on T. Then $\omega = 0$ is the given vertex of Δ. By the first claim, the irreducible component Y_{ω} is equivariantly isomorphic to

$$\mathrm{Spec}(K[M]^{\Delta(\omega)}/\{f \in K[M]^{\Delta(\omega)} \mid v_{\omega}(f) > 0\}) = \mathrm{Spec}(\tilde{K}[M]^{\mathrm{LC}_{\omega}(\Delta)})$$

which is the $\mathbb{T}_{\tilde{K}}$-toric variety associated to $\mathrm{LC}_{\omega}(\Delta)$. $\qquad\square$

6.16 Next, we describe the reduction map with respect to the $\mathbb{T}$-toric scheme $\mathscr{U}_{\Delta}$ over K° associated to the pointed Γ-rational polyhedron Δ in $N_{\mathbb{R}}$. Recall from 6.11 that the T-toric variety U_{σ} is the generic fibre of $\mathscr{U}_{\Delta}$ where σ is the recession cone of Δ. We have seen in 4.9 that the reduction is a map to the special fibre $(\mathscr{U}_{\Delta})_s$ which is defined on the set $U_{\sigma}^{\circ} := \{p \in U_{\sigma}^{\mathrm{an}} \mid p(f) \leq 1 \; \forall f \in K[M]^{\Delta}\}$. The points of U_{σ}° are induced by the potentially integral points of $\mathscr{U}_{\Delta}$.

We will describe the analytic structure of U_{σ}° using the following result of Joe Rabinoff.

Proposition 6.17 *We assume that the valuation v on K is complete. Let Δ be a pointed Γ-rational polyhedron in $N_\mathbb{R}$ with recession cone σ and let $\| \ \|$ be any norm on $M_\mathbb{R}$. Then the set of Laurent series*

$$\mathscr{A}_\Delta := \left\{ \sum_{u \in \check{\sigma} \cap M} a_u \chi^u \mid \lim_{\|u\| \to \infty} v(a_u) + \langle u, \omega \rangle = \infty \ \forall \omega \in \Delta \right\}$$

is a strictly affinoid algebra with spectral radius

$$\rho\left(\sum_{u \in \check{\sigma} \cap M} a_u \chi^u \right) = \sup_{\omega \in \Delta, \, u \in \check{\sigma} \cap M} |a_u| e^{-\langle u, \omega \rangle} = \max_{\omega \text{ vertex}, \, u \in \check{\sigma} \cap M} |a_u| e^{-\langle u, \omega \rangle}. \quad (5)$$

Proof: In the case of a non-trivial valuation, we use [Rab], Proposition 6.9. If v is trivial, then the sums in the definition of $\mathscr{A}_\Delta$ are finite and hence $\mathscr{A}_\Delta = K[M]^\Delta$ which immediately yields the claims. $\qquad\square$

Remark 6.18 The special case of polytopal domains was studied in [Gub3]. Using Hochster's theorem for toric varieties, Rabinoff has shown that $\mathscr{A}_\Delta$ is Cohen-Macauley for any Γ-rational polyhedron Δ (see [Rab], Proposition 6.9). If the valuation is discrete or K algebraically closed, then Wilke [Wil] has shown that $\mathscr{A}_\Delta$ is a factorial ring for Γ-rational polytopes Δ.

Proposition 6.19 *Using the notation from Proposition 6.17, the Berkovich spectrum $\mathscr{M}(\mathscr{A}_\Delta)$ is an affinoid subdomain of U_σ^{an} which is equal to U_σ°. Moreover, the special fibres of $\mathscr{M}(\mathscr{A}_\Delta)$ and $\mathscr{U}_\Delta$ agree which means $\mathrm{Spec}(\mathscr{A}_\Delta^\circ / (K^{\circ\circ} \mathscr{A}_\Delta^\circ)) = (\mathscr{U}_\Delta)_s$.*

Proof: If v is trivial, then Δ is a rational pointed cone and we have seen $\mathscr{A}_\Delta^\circ = \mathscr{A}_\Delta = K[M]^\Delta$ which makes the claims obvious. So we may assume that v is non-trivial. By Proposition 6.17, $\mathscr{O}(U_\sigma) = K[\check{\sigma} \cap M]$ is dense in $\mathscr{A}_\Delta$ and hence $\mathscr{M}(\mathscr{A}_\Delta)$ may be seen as a subset of U_σ^{an}. In fact, it is shown in [Rab], Proposition 6.17, that $\mathscr{M}(\mathscr{A}_\Delta)$ is an affinoid subdomain of U_σ^{an}. Moreover, we deduce from Rabinoff's result that $K[M]^\Delta$ is a subset of $\mathscr{A}_\Delta^\circ$ and hence $\mathscr{M}(\mathscr{A}_\Delta) \subset U_\sigma^\circ$. We will prove the reverse inclusion (which would follow immediately from 4.13 if $K[M]^\Delta$ is finitely generated) and so we choose $p \in U_\sigma^\circ$.

We claim first that $p(\chi^u) \leq \rho(\chi^u)$ for any $u \in \check{\sigma} \cap M$ where $\rho(\chi^u)$ is the spectral radius in $\mathscr{A}_\Delta$. There is a vertex ω_0 of Δ such that the halfspace $H := \{\omega \in N_\mathbb{R} \mid \langle u, \omega \rangle \geq 0\} + \omega_0$ contains Δ. By Γ-rationality of Δ, there is a non-zero $m \in \mathbb{N}$ such that $m\omega_0 \in N_\Gamma$. We conclude that there is a non-zero $\alpha \in K$ such that $v(\alpha) + \langle mu, \omega_0 \rangle = 0$. Using $H = \{\omega \in N_\mathbb{R} \mid v(\alpha) + \langle mu, \omega \rangle \geq 0\}$, we get $\alpha\chi^{mu} \in K[M]^\Delta$ and $\rho(\alpha\chi^{mu}) = |\alpha| e^{-\langle mu, \omega \rangle} = 1$ follows from Proposition 6.17. Using power multiplicativity of both $p \in U_\sigma^\circ$ and ρ, we get

$$|\alpha| p(\chi^u)^m = p(\alpha\chi^{mu}) \leq 1 = \rho(\alpha\chi^{mu}) = |\alpha| \rho(\chi^u)^m.$$

This proves $p(\chi^u) \leq \rho(\chi^u)$ for any $u \in \check{\sigma} \cap M$.

Next, we will prove $p(f) \leq \rho(f)$ for any $f \in \mathscr{O}(U_\sigma)$. Note that $f = \sum_u \alpha_u \chi^u$ where u ranges over a finite subset of $\check{\sigma} \cap M$. Using the above and Proposition 6.17, we get

$$p(f) \leq \max_u |\alpha_u| p(\chi^u) \leq \max_u |\alpha_u| \rho(\chi^u) = \max_u \rho(\alpha_u \chi^u) = \rho(f)$$

as desired. Now $p \leq \rho$ yields that p extends uniquely to a multiplicative seminorm of $\mathscr{M}(\mathscr{A}_\Delta)$. This proves $\mathscr{M}(\mathscr{A}_\Delta) = U_\sigma^\circ$. The claim about the special fibres follows immediately from Proposition 6.17. $\qquad\square$

In the following, we do not necessarily assume that the valuation v on K is complete as the analytifications are defined on the completion of K anyway.

Corollary 6.20 *Let Δ be any pointed Γ-rational polyhedron in $N_\mathbb{R}$ with recession cone σ. Then the reduction map from 4.9 maps $U_\sigma^\circ \cap T^{\mathrm{an}}$ surjectively onto $(\mathscr{U}_\Delta)_s$.*

Proof: Using the procedure described in 6.11, we may assume that Δ is a pointed polyhedron. Passing to the completion does not change the special fibre and so we may assume K complete. By Proposition 6.19, the special fibre of $\mathscr{U}_\Delta$ agrees with the special fibre of $U_\sigma^\circ = \mathscr{M}(\mathscr{A}_\Delta)$. Since T is the dense orbit in the generic fibre U_σ, it is clear that $T^{\mathrm{an}} \cap U_\sigma^\circ$ is Zariski open and dense in the affinoid subdomain U_σ°. Now the claim follows from Lemma 4.12. $\qquad\square$

Lemma 6.21 *We have $U_\sigma^\circ \cap T^{\mathrm{an}} = \mathrm{trop}_v^{-1}(\Delta)$ for a pointed Γ-rational polyhedron Δ.*

Proof: By definition, $U_\sigma^\circ \cap T^{\mathrm{an}}$ is the set of multiplicative seminorms p on $K[M]$ extending $|\ |$ with the additional condition that $p(f) \leq 1$ for every $f \in K[M]^\Delta$. The latter is equivalent to the condition that if there are $u \in M$, $\alpha \in K$ with $v(\alpha) + \langle u, \omega \rangle \geq 0$ for every $\omega \in \Delta$, then $p(\alpha \chi^u) \leq 1$. Since Δ is defined by such inequalities and since $-\log p(\chi^u) = \langle u, \mathrm{trop}_v(p) \rangle$, this is also equivalent to $\mathrm{trop}_v(p) \in \Delta$. This proves the claim.

If v is non-trivial, then we could use the above Proposition 6.19 and Proposition 6.9 from [Rab] to get an alternative proof. $\qquad\square$

As we have defined the tropicalization map only on T^{an}, we restrict the reduction map in the following proposition to $U_\sigma^\circ \cap T^{\mathrm{an}}$. By abuse of notation, we denote this restriction $U_\sigma^\circ \cap T^{\mathrm{an}} \to (\mathscr{U}_\Delta)_s$ also by π. In the following result, we use the partial order on the set of orbits (resp. open faces) given by inclusion of closures.

Proposition 6.22 *Let Δ be a pointed Γ-rational polyhedron in $N_\mathbb{R}$ and let $\mathscr{U}_\Delta$ be the associated $\mathbb{T}$-toric scheme over K°. Then there is a bijective order reversing correspondence between $\mathbb{T}$-orbits Z of $(\mathscr{U}_\Delta)_s$ and open faces τ of Δ given by*

$$Z = \pi(\mathrm{trop}_v^{-1}(\tau)), \quad \tau = \mathrm{trop}_v(\pi^{-1}(Z)).$$

Moreover, we have $\dim(Z) + \dim(\tau) = n$.

Proof: Let τ be an open face of Δ. The affinoid torus $\{p \in T^{\mathrm{an}} \mid p(\chi^u) = 1 \ \forall u \in M\}$ operates on $\mathrm{trop}_v^{-1}(\tau)$. By passing to the reductions, we see that $Z := \pi(\mathrm{trop}_v^{-1}(\tau))$ is $\mathbb{T}_{\tilde{K}}$-invariant. Note that Z is well-defined by Lemma 6.21. It is clear that distinguished open faces give rise to distinguished $\mathbb{T}_{\tilde{K}}$-invariant subsets of $(\mathscr{U}_\Delta)_s$. It remains to show that Z is an orbit. We are allowed to pass to a $\mathbb{T}$-invariant open subset and hence we may assume that $\tau = \mathrm{relint}(\Delta)$ by using Proposition 6.12. Since Δ is pointed, it has a vertex ω. We have seen in Corollary 6.15 that the irreducible component Y_ω is non-canonically isomorphic to the $\mathrm{Spec}(\tilde{K}[M_\omega])$-toric variety over $\tilde{K}$ associated to the local cone $\mathrm{LC}_\omega(\Delta)$. We claim that Z is the unique closed orbit Z' of Y_ω. Since Z is invariant and Z' is an orbit, it is enough to show that $\pi(p) \in Z'$ for every $p \in \mathrm{trop}_v^{-1}(\tau)$.

We will first prove that $\pi(p) \in Y_\omega$. By Proposition 6.14, the latter is given by the M-graded prime ideal $\{f \in K[M]^\Delta \mid v_\omega(f) > 0\}$. So let us choose $f = \alpha \chi^u \in K[M]^\Delta$ with $v_\omega(f) > 0$. Then $v_\nu(f) > 0$ for all $\nu \in \tau = \mathrm{relint}(\Delta)$. In particular, this holds for $\nu = \mathrm{trop}_v(p)$ and hence $-\log p(f) = v_\nu(f) > 0$. We conclude $p(f) < 1$ which means that f is contained in the prime ideal of $\pi(p)$ in $K[M]^\Delta$. This proves $\pi(p) \in Y_\omega$.

The well-known orbit-cone correspondence for toric varities over a field shows that the closed orbit Z' of Y_ω is given by the ideal in $\tilde{K}[Y_\omega]$ generated by $\{\chi^u \mid u \in M_\omega, \langle u, \omega' \rangle > 0 \ \forall \omega' \in \tau'\}$ where $\tau' := \mathrm{LC}_\omega(\tau)$. Taking into account how Y_ω is defined as a $\mathrm{Spec}(\tilde{K}[M_\omega])$-toric variety, we conclude that Z' is given as a closed

subscheme of $\mathscr{U}_\Delta$ by the ideal generated by $\{f = \beta\chi^u \mid \beta \in K, u \in M_\omega, v_{\omega'}(f) > 0 \ \forall \omega' \in \omega + \tau'\}$. For such an f, we conclude $-\log p(f) = v_\nu(f) > 0$ as above and hence $p(f) < 1$. Again this means $\pi(p) \in Z'$ proving that $Z = Z'$. We conclude that $Z = \pi(\mathrm{trop}_v^{-1}(\tau))$ is a $\mathbb{T}$-orbit.

Since π maps $U_\sigma^\circ \cap T^{\mathrm{an}} = \mathrm{trop}_v^{-1}(\Delta)$ onto $(\mathscr{U}_\Delta)_s$ by Corollary 6.20, we get a bijective correspondence between open faces of Δ and torus orbits of $(\mathscr{U}_\Delta)_s$. Since Z is the torus orbit of Y_ω corresponding to the open cone τ', we get $\dim(\tau) + \dim(Z) = \dim(\tau') + \dim(Z) = n$ from the theory of toric varieties over a field. Moreover, we conclude from the special case Y_ω that the correspondence is order reversing.

Conversely, let Z be any torus orbit of $(\mathscr{U}_\Delta)_s$. By the above, we have $Z = \pi(\mathrm{trop}_v^{-1}(\tau))$ for a unique open face τ of Δ. This yields $\tau \subset \mathrm{trop}_v(\pi^{-1}(Z))$. Equality follows from the fact that the torus orbits (resp. open faces) form a partition of $(\mathscr{U}_\Delta)_s$ (resp. Δ). $\qquad\square$

Remark 6.23 The bijective correspondence between open faces and orbits holds more generally for the polyhedral scheme $\mathscr{U}_\Delta$ associated to any Γ-rational polyhedron Δ in $N_\mathbb{R}$. This follows from the reduction to the case of pointed polyhedra described in 6.11.

If Δ' is any Γ-rational polyhedron contained in Δ, then the canonical equivariant morphism $\mathscr{U}_{\Delta'} \to \mathscr{U}_\Delta$ is an open immersion if and only if Δ' is a closed face of Δ. We have seen one direction in Proposition 6.12 and the converse follows easily from the orbit-face correspondence.

If v is trivial, then Δ is a pointed rational cone and the arguments in the proof of Proposition 6.22 show that we get the classical orbit-cone correspondence for toric varieties over a field from [Ful2], §3.1, or from [CLS], §3.2.

We consider a field extension L/K and an arbitrary valuation w on L (not necessarily of rank 1) extending v with valuation ring L° and value group Γ_L. For $P \in T(L)$, we define $\mathrm{trop}_w(P) \in N_{\Gamma_L} = \mathrm{Hom}(M, \Gamma_L)$ by $u \mapsto w(\chi^u(P))$ similarly as in the case of a valuation of rank 1. We recall from A.14 that a Γ-rational polyhedron Δ in $N_\mathbb{R}$ induces a canonical polyhedron $\Delta(\Gamma_L)$ in N_{Γ_L}. Then we have the following analogue of Lemma 6.21 for L°-integral points:

Proposition 6.24 *Under the assumptions above, let Δ be a pointed Γ-rational polyhedron in $N_\mathbb{R}$ and let $P \in T(L)$. Then P is an L°-integral point of $\mathscr{U}_\Delta$ if and only if $\mathrm{trop}_w(P) \in \Delta(\Gamma_L)$.*

Proof: The point P is L°-integral if and only if $v(\alpha) + w(\chi^u(P)) \geq 0$ for all $\alpha\chi^u \in K[M]^\Delta$. We note that $w(\chi^u(P)) = \langle u, \mathrm{trop}_w(P)\rangle$. Since Δ is a Γ-rational polyhedron, Δ is the intersection of the half-spaces $H_{u,\alpha}^+ := \{\omega \in N_\mathbb{R} \mid v(\alpha) + \langle u, \omega\rangle \geq 0\}$ with $\alpha\chi^u$ ranging over $K[M]^\Delta$. Moreover, Δ is the intersection of finitely many such half-spaces. It follows from A.13 that $\Delta(\Gamma_L)$ is the intersection of the sets $H_{u,\alpha}^+(\Gamma_L) = \{\omega \in N_{\Gamma_L} \mid v(\alpha) + \langle u, \omega\rangle \geq 0\}$ and hence we get the claim. $\square$

7 Toric schemes over a valuation ring

In this section, K is a field endowed with a non-archimedean valuation v and Γ is the valuation group of v. We extend the theory of toric schemes over a discrete valuation ring from [KKMS] to this more general situation. More precisely, we will use the affine toric schemes associated to pointed polyhedra from the previous section to define toric schemes. For the glueing process, it is necessary to work with fans in $N_\mathbb{R} \times \mathbb{R}_+$ rather than polyhedral complexes in $N_\mathbb{R}$. Recall that $\mathbb{T} = \mathrm{Spec}(\mathbb{G}_m^n)$ is the split multiplicative torus over K° with generic fibre T. The character group

of T is M and N is the dual lattice. Further references for the special case of a discrete valuation are [BPS] (with a lot of arithmetic applications) and [Smi] (from the projective point of view).

7.1 As a building block, we will use the affine $\mathbb{T}$-toric scheme $\mathscr{U}_\Delta$ over K° from 6.11 for any pointed Γ-rational polyhedron Δ in $N_\mathbb{R}$. For glueing, it is better to replace Δ by the closed cone $\sigma = c(\Delta)$ in $N_\mathbb{R} \times \mathbb{R}_+$ generated by $\Delta \times \{1\}$. For $s \in \mathbb{R}_+$, let $\sigma_s := \{\omega \in N_\mathbb{R} \mid (\omega, s) \in \sigma\}$. For $s > 0$, we have $\sigma_s = s\Delta$ and σ_0 is the recession cone of Δ. This follows easily from $c(\Delta) = \{(\omega, s) \in N_\mathbb{R} \times \mathbb{R}_+ \mid \langle u_i, \omega \rangle + sc_i \geq 0 \, \forall i\}$ using that Δ is the intersection of finitely many halfspaces $\{\omega \in W \mid \langle u_i, \omega \rangle + c_i \geq 0)\}$ with $u_i \in M$ and $c_i \in \Gamma$.

7.2 A cone σ in $N_\mathbb{R} \times \mathbb{R}_+$ is called Γ-*admissible* if it may be written as

$$\sigma = \bigcap_{i=1}^{N} \{(\omega, s) \mid \langle u_i, \omega \rangle + sc_i \geq 0\}$$

for $u_1, \ldots, u_N \in M$ and $c_1, \ldots, c_N \in \Gamma$ and if σ does not contain a line. For $s \in \mathbb{R}_+$, we define σ_s as above.

Note that $\Delta \mapsto c(\Delta)$ gives a bijection between the set of non-empty pointed Γ-rational polyhedra in $N_\mathbb{R}$ and the set of Γ-admissible cones in $N_\mathbb{R} \times \mathbb{R}_+$ which are not contained in $N_\mathbb{R} \times \{0\}$. The inverse map is $\sigma \mapsto \sigma_1$.

Definition 7.3 For a Γ-admissible cone σ in $N_\mathbb{R} \times \mathbb{R}_+$, we define

$$K[M]^\sigma := \{\sum_{u \in M} \alpha_u \chi^u \in K[M] \mid cv(\alpha_u) + \langle u, \omega \rangle \geq 0 \, \forall (\omega, c) \in \sigma\}$$

and $\mathscr{V}_\sigma := \mathrm{Spec}(K[M]^\sigma)$.

Proposition 7.4 *$\mathscr{V}_\sigma$ is an affine normal $\mathbb{T}$-toric scheme over K° with generic fibre equal to the affine toric variety U_{σ_0} associated to σ_0. If the value group Γ of K is discrete or divisible in $\mathbb{R}$, then $\mathscr{V}_\sigma$ is an affine normal $\mathbb{T}$-toric variety over K°.*

Proof: If σ is contained in the hyperplane $N_\mathbb{R} \times \{0\}$, then $\mathscr{V}_\sigma$ is the normal toric variety U_{σ_0} over K associated to σ_0. Since K is of finite type over the valuation ring K°, it is also a normal toric variety over K°.

If σ is not contained in $N_\mathbb{R} \times \{0\}$, then σ_1 is a non-empty Γ-rational polyhedron Δ in $N_\mathbb{R}$ with $\mathscr{V}_\sigma = \mathscr{U}_\Delta$ and the claim follows from 6.11. $\square$

Definition 7.5 A Γ-*admissible fan* Σ in $N_\mathbb{R} \times \mathbb{R}_+$ is a fan of Γ-admissible cones. For $s \geq 0$, let Σ_s be the polyhedral complex $\{\sigma_s \mid \sigma \in \Sigma\}$ in $N_\mathbb{R}$.

Remark 7.6 It was noticed by Burgos and Sombra [BS] that if $\mathscr{C}$ is a Γ-rational polyhedral complex in $N_\mathbb{R}$, then $c(\mathscr{C}) := \{c(\Delta) \mid \Delta \in \mathscr{C}\}$ is not necessarily a fan in $N_\mathbb{R} \times \mathbb{R}_+$. However if the support of $\mathscr{C}$ is convex, then they prove that $c(\mathscr{C})$ is a fan. This gives a bijective correspondence between complete Γ-rational pointed polyhedral complexes of $N_\mathbb{R}$ and complete Γ-admissible fans of $N_\mathbb{R} \times \mathbb{R}_+$.

7.7 Let Σ be a Γ-admissible fan in $N_\mathbb{R} \times \mathbb{R}_+$. Then the affine $\mathbb{T}$-toric schemes $\mathscr{V}_\sigma$, $\sigma \in \Sigma$, may be glued along the open subschemes $\mathscr{V}_\rho$ from common subfaces ρ to get a normal $\mathbb{T}$-toric scheme $\mathscr{Y}_\Sigma$ over K°. The generic fibre of $\mathscr{Y}_\Sigma$ is the normal T-toric variety Y_{Σ_0} over K associated to the fan Σ_0 in $N_\mathbb{R}$. This follows all from the affine case except separatedness which we shall prove next:

Lemma 7.8 *The scheme $\mathscr{Y}_\Sigma$ is separated over K°.*

Proof: Let $\sigma := \sigma' \cap \sigma''$ for $\sigma', \sigma'' \in \Sigma$. We have to show that the canonical morphism $\mathscr{V}_\sigma \to \mathscr{V}_{\sigma'} \times_{K^\circ} \mathscr{V}_{\sigma''}$ is a closed embedding. To prove that we may assume that σ', σ'' are not contained in $N_\mathbb{R} \times \{0\}$ (as the claim is well-known for toric varieties over a field). Then we have pointed Γ-rational polyhedra $\Delta' := \sigma_1'$, $\Delta'' := \sigma_1''$ and $\Delta := \Delta' \cap \Delta'' = \sigma_1$ in $N_\mathbb{R}$ with $\mathscr{U}_{\Delta'} = \mathscr{V}_{\sigma'}$, $\mathscr{U}_{\Delta''} = \mathscr{V}_{\sigma''}$ and $\mathscr{U}_\Delta = \mathscr{V}_\sigma$. We have to show that $K[M]^\Delta$ is generated by $K[M]^{\Delta'}$ and $K[M]^{\Delta''}$ as a K°-algebra.

Let $G := \{\gamma \in \mathbb{R} \mid \exists k \in \mathbb{N} \setminus \{0\} \;\; k\gamma \in \Gamma\}$; then the affine subspace of $N_\mathbb{R}$ generated by the face of a Γ-rational polyhedron is an N_G-translate of a rational linear subspace. We conclude that there is $u_0 \in M$ and $\omega_0 \in N_G$ such that $\Delta = \Delta' \cap (\omega_0 + \{u_0\}^\perp)$, $\Delta' \subset \omega_0 + \{\omega \in N_\mathbb{R} \mid \langle u_0, \omega \rangle \geq 0\}$ and $\Delta'' \subset \omega_0 + \{\omega \in N_\mathbb{R} \mid \langle u_0, \omega \rangle \leq 0\}$. There is $\alpha_0 \in K$ and a non-zero $k \in \mathbb{N}$ with $v(\alpha_0) = k\langle u_0, \omega_0 \rangle$. We consider a vertex ω' of Δ'. By construction,

$$v_{\omega'}((\chi^{ku_0}/\alpha_0)^m f) = km\langle u_0, \omega' - \omega_0 \rangle + v_{\omega'}(f)$$

is non-negative for $m \gg 0$. We conclude that $v_{\Delta'}(g) \geq 0$ for $g := (\chi^{ku_0}/\alpha_0)^m f$.

Since

$$v_{\omega''}(\alpha_0 \chi^{-ku_0}) = k\langle u_0, \omega_0 - \omega'' \rangle \geq 0$$

for every $\omega'' \in \Delta''$, we conclude that $\alpha_0 \chi^{-ku_0} \in K[M]^{\Delta''}$. Using $f = (\alpha_0 \chi^{-ku_0})^m g$, we get the claim proving that $\mathscr{Y}_\Sigma$ is separated. $\hfill\square$

7.9 We have a bijective correspondence between torus orbits of $\mathscr{Y}_\Sigma$ and open faces of Σ. The torus orbits in the generic fibre correspond to the open faces contained in $N_\mathbb{R} \times \{0\}$ using the theory of toric varieties over a field. The torus orbits in the special fibre correspond to the remaining open faces of Σ using the fact that the latter are the open faces of the polyhedral complex Σ_1 in $N_\mathbb{R}$. Indeed, the orbits are contained in an affine $\mathbb{T}$-toric scheme $\mathscr{V}_\sigma$ for some $\sigma \in \Sigma$ and so we may use Proposition 6.22. We will later describe the orbit correspondence for $\mathscr{Y}_\Sigma$ in a neat way (see Proposition 8.8).

Note that $\mathscr{Y}_\Sigma$ is a noetherian topological space which follows from the fact that both the generic and the special fibre are noetherian. If $\mathscr{X}$ is a closed irreducible subset of $\mathscr{Y}_\Sigma$ with non-empty generic fibre $\mathscr{X}_\eta$, then it follows from Proposition 11.3 below that $\mathscr{X}_s$ is of pure dimension equal to $\dim(X_\eta)$. This means that the dimension and the codimension of $\mathscr{X}$ can be computed using any maximal chain of closed subset and hence we get $\dim(\mathscr{X}) = \dim(\mathscr{X}_\eta) + 1$. Moreover, if Z is a closed subsets of $\mathscr{X}_s$, then we get $\mathrm{codim}(Z, \mathscr{X}) = \mathrm{codim}(Z, \mathscr{X}_s) + 1$ (see [OP], Lemma 4.2.3 for similar arguments in the case of any irreducible flat scheme of finite type over K°).

If Z_τ is the torus orbit corresponding to the open face τ of Σ, then the above and Proposition 6.22 show that

$$\dim(\tau) = \mathrm{codim}(Z_\tau, \mathscr{Y}_\Sigma).$$

In particular, the $\mathbb{T}$-invariant prime divisors on $\mathscr{Y}_\Sigma$ correspond to the halflines in Σ. The irreducible components of the special fibre of $\mathscr{Y}_\Sigma$ correspond to the halflines of Σ not contained in $N_\mathbb{R} \times \{0\}$ or in other words to the vertices of Σ_1.

Lemma 7.10 *Suppose that the valuation on K has value group $\Gamma = \mathbb{Z}$ and suppose $v(\pi) = 1$ for $\pi \in K$. Let Y_σ be an irreducible component of the special fibre of $\mathscr{Y}_\Sigma$ corresponding to the halfline σ of Σ. Then the multiplicity of the divisor Y_σ in $\mathscr{Y}_\Sigma$ is equal to k, where (ω, k) is the primitive generator of the monoid $\sigma \cap (N \times \mathbb{Z})$.*

Proof: See [KKMS], §4.3. $\hfill\square$

Proposition 7.11 *If the valuation on K is discrete, then the following conditions are equivalent for a Γ-admissible fan Σ in $N_\mathbb{R} \times \mathbb{R}_+$:*

(a) *The vertices of Σ_1 are contained in N_Γ.*

(b) *The special fibre $(\mathscr{Y}_\Sigma)_s$ is reduced.*

(c) *$(\mathscr{Y}_\Sigma)_s$ is geometrically reduced.*

(d) *For all valued fields (L, w) extending (K, v), the formation of $\mathscr{Y}_\Sigma$ is compatible with base change to L°.*

(e) *For all $\Delta \in \Sigma_1$, the canonical map $K[M]^\Delta \otimes_{K^\circ} \tilde{K} \to \mathscr{A}_\Delta^\circ / \mathscr{A}_\Delta^{\circ\circ}$ is an isomorphism, where we refer to Proposition 6.17 for the definition of $\mathscr{A}_\Delta$.*

Proof: The equivalence of (a) and (b) follows from Lemma 7.10. Clearly, (c) implies (b).

Now let $\sigma \in \Sigma$ and $\Delta := \sigma_1$. Suppose that the vertices ω_j of Δ are contained in N_Γ. In this case, we may use also the last part of the proof of Proposition 6.7 to get a set of generators of $K[M]^\Delta$ which depends only on the combinatorics of Δ and hence it generates also $L[M]^\Delta$. This proves

$$L[M]^\Delta = K[M]^\Delta \otimes_{K^\circ} L^\circ$$

and hence (a) implies (d).

Now suppose that (d) holds. There is a finite extension L/K such that (a) holds for the value group of L. By the equivalence of (a) and (b), we conclude that the special fibre of $(\mathscr{Y}_\Sigma)_L$ is reduced and hence the special fibre of $\mathscr{Y}_\Sigma$ is also reduced. We may repeat this for any finite extension of K and hence (d) yields (c). Since the residue algebra $\mathscr{A}_\Delta^\circ / \mathscr{A}_\Delta^{\circ\circ}$ of a strictly affinoid algebra is always reduced, we see that (e) implies (b).

Finally we show that (a) implies (e). Since the vertices are in N_Γ, it is easy to see that the kernel of the quotient homomorphism $K[M]^\Delta \to K[M]^\Delta / \langle K^{\circ\circ} \rangle = K[M]^\Delta \otimes_{K^\circ} \tilde{K}$ is equal to $\{\sum_{u \in M} a_u \chi^u \in K[M] \mid v(a_u) + \langle u, \omega \rangle > 0 \ \forall \omega \in \Delta\}$. By density of $K[M]^\Delta$ in $\mathscr{A}_\Delta^\circ$, we deduce (e). $\square$

Proposition 7.12 *If v is not a discrete valuation, then (b), (c) and (e) hold always. Moreover, properties (a) and (d) are equivalent, but are not always true. In particular, if Γ is a divisible group in $\mathbb{R}$, then (a)–(e) hold.*

Proof: We first prove that (c) always holds. Let L be an algebraic closure of K. We choose a valuation w on L extending K. Then the residue field $\tilde{L}$ is also algebraically closed (see [BGR], Lemma 3.4.1/4). We have to show that $(\mathscr{U}_\Delta)_s \otimes_{\tilde{K}} \tilde{L}$ is reduced for any $\Delta \in \Sigma_1$. We consider the L°-subalgebra $B := K[M]^\Delta \otimes_{K^\circ} L^\circ$ of $L[M]^\Delta$ and we have to prove that $I := L^{\circ\circ} B = L^{\circ\circ} K[M]^\Delta$ is a radical ideal in B. Similarly as in the proof of Lemma 6.13, I is contained in the radical ideal $J := \{f \in B \mid w_\Delta(f) > 0\}$ of B. To prove the reverse inclusion, it is enough to show that f is contained in I for any $f = \gamma \alpha \chi^u \in J$ with $\gamma \in L^\circ$ and $\alpha \chi^u \in K[M]^\Delta$. If $\gamma \in L^{\circ\circ}$, then the claim is trivial and so we may assume that $\gamma = 1$. Then $f \in K[M]^\Delta$ and the claim follows from Lemma 6.13. This proves (c).

The above shows that $\{\sum_{u \in M} a_u \chi^u \in K[M] \mid v(a_u) + \langle u, \omega \rangle > 0 \ \forall \omega \in \Delta\}$ is equal to the kernel of $K[M]^\Delta \to K[M]^\Delta / \langle K^{\circ\circ} \rangle$. As in the proof of Proposition 7.11, we conclude that (e) holds. Moreover, this proof shows that (a) yields (d).

It remains to prove that (d) implies (a). We choose (L, w) as above and we note that the value group of w is divisible. Using Proposition 6.7 and (d), we see that $(\mathscr{Y}_\Sigma)_{L^\circ}$ is of finite type over L°. By [EGA IV], Proposition 2.7.1, the toric scheme $\mathscr{Y}_\Sigma$ is of finite type over K° and (a) follows from Proposition 6.9.

If Γ is a divisible group in $\mathbb{R}$, then the vertices of Σ_1 are always in N_Γ proving the last claim. It is also clear that (a) has not always to be true if Γ is not divisible. $\square$

7.13 For a Γ-rational polyhedron Δ in $N_{\mathbb{R}}$, we introduce the following notation: The affine space generated by Δ is a translate of $(\mathbb{L}_{\Delta})_{\mathbb{R}}$ for a rational linear subspace $\mathbb{L}_{\Delta}$ of $N_{\mathbb{Q}}$. Then $N_{\Delta} := N \cap \mathbb{L}_{\Delta}$ and $N(\Delta) := N/N_{\Delta}$ are free abelian groups of finite rank with quotient homomorphism $\pi_{\Delta} : N \to N(\Delta)$. Dually, we have $M(\Delta) := \mathbb{L}_{\Delta}^{\perp} \cap M = \operatorname{Hom}(N(\Delta), \mathbb{Z})$.

We return to an arbitrary valued field (K, v). Let Σ be a Γ-admissible cone in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let Z be an orbit of $\mathscr{Y}_{\Sigma}$ contained in the generic fibre. By 7.9, Z corresponds to the relative interior of a rational cone $\sigma \in \Sigma_0$

Proposition 7.14 *Under the hypothesis above, the closure $\overline{Z}$ of Z in $\mathscr{Y}_{\Sigma}$ is isomorphic to the $\operatorname{Spec}(K^{\circ}[M(\sigma)])$-toric scheme over K° associated to the Γ-admissible fan $\Sigma_{\sigma} := \{(\pi_{\sigma} \times \operatorname{id}_{\mathbb{R}_+})(\nu) \mid \nu \in \Sigma, \nu \supset \sigma\}$ in $N(\sigma)_{\mathbb{R}} \times \mathbb{R}_+$.*

Proof: Let $\nu \in \Sigma$ with $\nu \supset \sigma$ and let $\nu_{\sigma} := (\pi_{\sigma} \times \operatorname{id}_{\mathbb{R}_+})(\nu)$. Then there is a canonical surjective K°-algebra homomorphism

$$K[M]^{\nu} \to K[M(\sigma)]^{\nu_{\sigma}}, \quad \alpha \chi^u \mapsto \begin{cases} \alpha \chi^u, & \text{if } u \in M(\sigma), \\ 0, & \text{if } u \in M \setminus M(\sigma). \end{cases}$$

We conclude that the $\operatorname{Spec}(K^{\circ}[M(\sigma)])$-toric scheme over K° associated to the Γ-admissible fan Σ_{σ} in $N(\sigma)_{\mathbb{R}} \times \mathbb{R}_+$ is a closed subscheme of $\mathscr{Y}_{\Sigma}$. By [Ful2], §3.1, its generic fibre is the closure of Z in the generic fibre of $\mathscr{Y}_{\Sigma}$. By Proposition 4.4, we get the claim. $\square$

Now we assume that the orbit Z of $\mathscr{Y}_{\Sigma}$ is contained in the special fibre. By 7.9, Z corresponds to the relative interior τ of $\Delta \in \Sigma_1$. Similarly as in Proposition 6.15, Γ-rationality of Δ yields that $M(\Delta)_{\tau} := \{u \in M(\Delta) \mid \langle u, \omega \rangle \in \Gamma \; \forall \omega \in \tau\}$ is a lattice of finite index in $M(\Delta)$. For $\nu \in \Sigma_1$ with face Δ, we define $\operatorname{LC}_{\tau}(\nu) := \operatorname{LC}_{\omega}(\nu)$ which is independent of the choice of $\omega \in \tau$ and where we use the local cones from A.6.

Proposition 7.15 *Under the hypothesis above, the closure $\overline{Z}$ of Z in $\mathscr{Y}_{\Sigma}$ is equivariantly isomorphic to the $\operatorname{Spec}(\tilde{K}[M(\Delta)_{\tau}])$-toric variety over $\tilde{K}$ associated to the rational fan $\{\pi_{\Delta}(\operatorname{LC}_{\tau}(\nu)) \mid \nu \in \Sigma_1, \nu \supset \Delta\}$ in $N(\Delta)_{\mathbb{R}}$.*

Proof: If ω is a vertex of Σ_1, this follows immediately from Corollary 6.15. The general case follows from the corresponding generalization of Corollary 6.15 which can be proved completely analogous. We leave the details to the reader. $\square$

8 Tropical cone of a variety

In this section, K denotes a field with a non-trivial non-archimedean absolute value $|\;|_v$, corresponding valuation $v := -\log|\;|_v$ and valuation ring K°. We consider $W := \{\varepsilon v \mid \varepsilon \geq 0\}$ which is induced by all valuations equivalent to v together with the trivial absolute which we denote by 0. We may identify W with $\mathbb{R}_+$ using $\varepsilon v \leftrightarrow \varepsilon$. The value group of $w \in W$ is denoted by Γ_w and the residue field by $k(w)$. Obviously, we have $k(w) = k(v)$ for $w \neq 0$ and $k(0) = K$. At the end of this section, we show how to adjust the notation so that everything works also for the trivial valuation.

We have seen the advantage of using fans Σ in $N_{\mathbb{R}} \times \mathbb{R}_+$ rather than polyhedral complexes in $N_{\mathbb{R}}$ to define an associated toric scheme $\mathscr{Y}_{\Sigma}$ over K°. It is not surprising that the consideration of the closed cone in $N_{\mathbb{R}} \times \mathbb{R}_+$ generated by $\operatorname{Trop}_v(X) \times \{1\}$ is useful to describe information about the closure of the closed subscheme X of T in $\mathscr{Y}_{\Sigma}$ in a uniform way. Moreover, we will see that the tropical variety of X with respect to the trivial valuation is just the intersection of this tropical cone with $N_{\mathbb{R}} \times \{0\}$.

8.1 For an algebraic scheme X over K, we have defined in Section 2 the analytification with respect to the valuation w which we denote here by X_w^{an}. In fact, we can define the *analytification X_W^{an} of X with respect to W* by the same process allowing all multiplicative seminorms p with restriction $w_p := p|_K \in W$. This gives again a locally compact Hausdorff space which is as a set equal to the disjoint union of all X_w^{an} with w ranging over W. The proof follows from Tychonoff's theorem similarly as in the case of a single valuation.

8.2 For $w \in W$, let $\mathrm{trop}_w : T_w^{\mathrm{an}} \to N_{\mathbb{R}}$ be the tropicalization map. Proceeding fibrewise, we get the *W-tropicalization map*

$$\mathrm{trop}_W : T_W^{\mathrm{an}} \to N_{\mathbb{R}} \times W, \quad t \mapsto (\mathrm{trop}_{w_t}(t), w_t).$$

It is clear that trop_W is continuous.

Definition 8.3 Let X be a closed subscheme of T. Then we define the *tropical cone* associated to X as $\mathrm{trop}_W(X_W^{\mathrm{an}})$ and we denote it by $\mathrm{Trop}_W(X)$.
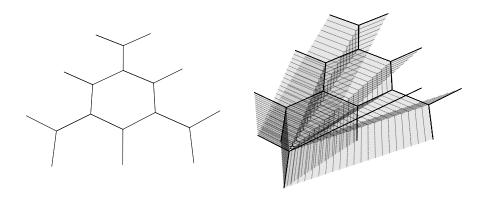

Figure 1: Tropical curve and its tropical cone

Remark 8.4 We will see in Corollary 11.13 that $\mathrm{Trop}_W(X)$ is equal to the closure C of the cone in $N_{\mathbb{R}} \times \mathbb{R}_+$ generated by $\mathrm{Trop}_v(X) \times \{1\}$. It follows from the Bieri–Groves theorem that $\mathrm{Trop}_W(X)$ is a finite union of Γ-admissible cones in $N_{\mathbb{R}} \times \mathbb{R}_+$. If X is of pure dimension d, then we may choose these cones $d + 1$-dimensional.

We will not use these facts until we have proved them in Corollary 11.13. At the moment, it is only clear from the definitions that C agrees with $\mathrm{Trop}_W(X)$ on $N_{\mathbb{R}} \times \mathbb{R}_+$.

Proposition 8.5 *The tropical cone $\mathrm{Trop}_W(X)$ of X is closed in $N_{\mathbb{R}} \times \mathbb{R}_+$.*

Proof: We have seen that trop_W is a continuous map. Moreover, it is easy to see that trop_W is a proper map, i.e. the preimage of a compact subset is a compact subset. This shows immediately that $\mathrm{Trop}_W(X) = \mathrm{trop}_W(X_W^{\mathrm{an}})$ is closed in $N_{\mathbb{R}} \times \mathbb{R}_+$. $\qquad\square$

In the remaining part of this section, let Σ be a Γ-admissible fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let $\mathscr{Y}_\Sigma^v$ be the associated toric scheme over K°. We may identify T with the dense open orbit in $\mathscr{Y}_\Sigma^v$ and this orbit is contained in the generic fibre Y_{Σ_0}.

In the following proposition, we consider a field extension L/K and a valuation u on L (not necessarily of rank 1) extending v with valuation ring L° and value group Γ_L. Since Γ_L is a totally ordered abelian group, any Γ-rational polyhedron Δ in $N_\mathbb{R}$ induces a polyhedron $\Delta(\Gamma_L)$ in N_{Γ_L}. We denote by $|\Sigma_1(\Gamma_L)|$ the union of all $\Delta(\Gamma_L)$ with Δ ranging over the Γ-rational polyhedral complex Σ_1 defined in 7.2.

Proposition 8.6 *Under the assumptions above, $P \in T(L)$ is an L°-integral point of $\mathscr{Y}_\Sigma^v$ if and only if $\mathrm{trop}_u(P)$ is contained in $|\Sigma_1(\Gamma_L)|$.*

Proof: This follows immediately from Proposition 6.24. $\qquad\square$

8.7 We conclude from Proposition 8.6 that we have a well-defined reduction map $\pi_W : \mathrm{trop}_W^{-1}(|\Sigma|) \to \mathscr{Y}_\Sigma^v$. Indeed, we have $\mathscr{Y}_{\varepsilon\Sigma}^{\varepsilon v} = \mathscr{Y}_\Sigma^v$ for all $\varepsilon > 0$ and so we may use the reduction map $\pi_w : \mathrm{trop}_w^{-1}(|\Sigma_\varepsilon|) \to (\mathscr{Y}_{\varepsilon\Sigma}^w)_s = (\mathscr{Y}_\Sigma^v)_s$ in the fibre over $w = \varepsilon v$. For $w = 0$, the special fibre agrees with the generic fibre Y_{Σ_0} and we use the reduction $\pi_0 : \mathrm{trop}_0^{-1}(|\Sigma_0|) \to Y_{\Sigma_0}$. Note that we may use Proposition 8.6 also for the trivial valuation $v = 0$.

Then we can describe the orbit-face correspondence in the following uniform way.

Proposition 8.8 *There is a bijective order reversing correspondence between $\mathbb{T}$-orbits Z of $\mathscr{Y}_\Sigma^v$ and open faces τ of Σ given by*

$$Z = \pi_W(\mathrm{trop}_W^{-1}(\tau)), \quad \tau = \mathrm{trop}_W(\pi_W^{-1}(Z)).$$

Proof: We easily reduce to the case of an invariant open subset $\mathscr{V}_\sigma$ of $\mathscr{Y}_\Sigma^v$ for $\sigma \in \Sigma$. Then the claim follows from Proposition 6.22 applied to every $w \in W$. To prove that the correspondence is order reversing we use also Proposition 7.14 to handle an orbit in the generic fibre whose closure contains orbits in the special fibre. $\quad\square$

Remark 8.9 If v is the trivial valuation, then we have to adjust the notation of this section by using the set $\mathbb{R}_+$ rather than $W = \{0\}$. We define $X_W^{\mathrm{an}} := X_0^{\mathrm{an}} \times \mathbb{R}_+$ which is a locally compact Hausdorff space. Then everything works as above.

9 Projectively embedded toric varieties

In this section, K denotes a field with a non-archimedean absolute value $|\ |$, corresponding valuation $v := -\log |\ |$ and value group $\Gamma := v(K^\times)$. We have defined toric varieties in Definition 6.1. Here, we consider projective toric varieties over K° with an equivariant embedding into projective space. These toric varieties are not necessarily normal. This section is inspired by the introductory article of E. Katz ([Kat], section 4) and we will generalize his results. Further references: [CLS], §2.1, §3.A; [GKZ], Chapter 5.

Recall that $\mathbb{T} = \mathrm{Spec}(\mathbb{G}_m^n)$ is a split multiplicative torus over K° with generic fibre T. The character group of T is M and the character corresponding to $u \in M$ is denoted by χ^u. For convenience, we always choose coordinates on the projective space $\mathbb{P}_{K^\circ}^N$ defined over the valuation ring K°.

9.1 We first recall the following well-known way to construct a not necessarily normal toric subvariety Y from $A = (u_0, \dots, u_N) \in M^{N+1}$ and $\mathbf{y} = (y_0 : \cdots : y_N) \in \mathbb{P}^N(K)$ (see [GKZ], Chapter 5). The torus T acts on $\mathbb{P}_K^N$ by

$$t \cdot \mathbf{x} := (\chi^{u_0}(t)x_0 : \cdots : \chi^{u_N}(t)x_N)$$

and we define Y as the closure of the orbit $T\mathbf{y}$. Then there is a bijective correspondence between T-orbits of Y and faces of the weight polytope $\mathrm{Wt}(\mathbf{y})$ which is defined as the convex hull of $A(\mathbf{y}) := \{u_j \mid y_j \neq 0\}$. If Q is a face of $\mathrm{Wt}(\mathbf{y})$, then the corresponding orbit is given by

$$Z := \{\mathbf{z} \in Y \mid z_j \neq 0 \iff u_j \in Q\}.$$

Duality gives also a bijective correspondence to the normal fan Σ of $\mathrm{Wt}(\mathbf{y})$. The cone σ corresponding to the face Q is the set of $\omega \in N_\mathbb{R}$ such that the linear functional $\langle \cdot, \omega \rangle$ achieves its minimum on $\mathrm{Wt}(\mathbf{y})$ precisely in the face Q. The torus corresponding to the orbit Z has character group $\{\sum m_j u_j \mid \sum m_j = 0\}$, where j ranges over $0, \ldots, N$ with $z_j \neq 0$ for any $\mathbf{z} \in Z$. This character group is of finite index in $\mathbb{Z}(\sigma^\perp \cap M)$ and hence $\dim(Z) = \dim(Q) = n - \dim(\sigma)$ (see [CLS], Section 3.A).

9.2 The goal of this section is to perform a similar construction over the valuation ring K°. Let $A = (u_0, \ldots, u_N) \in M^{N+1}$ and let $\mathbf{y} = (y_0 : \cdots : y_N) \in \mathbb{P}^N(K)$. We define the *height function* of $\mathbf{y}$ by

$$a : \{0, \ldots, N\} \to \Gamma \cup \{\infty\}, \quad j \mapsto v(y_j).$$

The torus $\mathbb{T}$ operates on $\mathbb{P}^N_{K^\circ}$ by

$$\mathbb{T} \times_{K^\circ} \mathbb{P}^N_{K^\circ} \to \mathbb{P}^N_{K^\circ}, \quad (t, \mathbf{x}) \mapsto (\chi^{u_0}(t)x_0 : \cdots : \chi^{u_N}(t)x_N).$$

The closure of the orbit $T\mathbf{y}$ in $\mathbb{P}^N_{K^\circ}$ is a projective toric variety with respect to the split torus over K° with generic fibre $T/\mathrm{Stab}(\mathbf{y})$. We denote this projective toric variety by $\mathscr{Y}_{A,a}$ and its generic fibre by $Y_{A,a}$. Using the base point $\mathbf{y}$, the torus $T/\mathrm{Stab}(\mathbf{y})$ may be seen as an open dense subset of $\mathscr{Y}_{A,a}$. If we apply the following result to the special case M equal to the lattice in M' generated by A and F equal to a translation, then we see that $\mathscr{Y}_{A,a}$ depends only on the affine geometry of (A, a).

Proposition 9.3 *Suppose that $\mathbb{T}'$ is another split multiplicative torus over K° with character lattice M' and that there is an injective affine transformation $F : M \to M'$ of lattices. Let $A = (u_0, \ldots, u_N) \in M^{N+1}, A' = (u'_0, \ldots, u'_N) \in (M')^{N+1}$ and let $\mathbf{y}, \mathbf{y}' \in \mathbb{P}^N(K)$ with height functions a (resp. a'). Let $\mathscr{Y}_{A,a}$ (resp. $\mathscr{Y}_{A',a'}$) be the projective toric variety with respect to $A, \mathbf{y}$ (resp. $A', \mathbf{y}'$). We assume that $F(u_j) = u'_j$ for every j with $y_j \neq 0$. If there is $\lambda \in \Gamma$ such that $a' = a + \lambda$, then $\mathscr{Y}_{A,a}$ is canonically isomorphic to $\mathscr{Y}_{A',a'}$.*

Proof: The injective linear map corresponding to F induces a surjective homomorphism $\mathbb{T}' \to \mathbb{T}$ of multiplicative tori. If $\mathbf{y} = \mathbf{y}'$, then we deduce that $\mathscr{Y}_{A,a} = \mathscr{Y}_{A',a}$. In general, we have $\mathbf{y}' = g\mathbf{y}$ for some $g = (g_0, \ldots, g_N) \in K^{N+1}$ with $|g_0| = \cdots = |g_N| \neq 0$. Then g induces a linear automorphism of $\mathbb{P}^N_{K^\circ}$ mapping $\mathscr{Y}_{A,a}$ onto $\mathscr{Y}_{A',a'}$. If $y_j \neq 0$, then g_j is uniquely determined and hence we have constructed a canonical isomorphism. $\qquad\square$

Corollary 9.4 *The open dense orbit $T/\mathrm{Stab}(\mathbf{y})$ of $\mathscr{Y}_{A,a}$ is a torus with character lattice $\{\sum m_j u_j \mid \sum m_j = 0\}$, where j ranges over $0, \ldots, N$ with $y_j \neq 0$. This orbit has dimension equal to $\dim(\mathbb{A})$, where $\mathbb{A}$ is the smallest affine subspace of $M_\mathbb{R}$ containing $A(\mathbf{y}) := \{u_j \mid y_j \neq 0\}$.*

Proof: Using Proposition 9.3, we may assume that $u_0 = 0$ and that $u_1, \ldots, u_N$ form a basis of M. Then it is easy to see that the stabilizer of $\mathbf{y}$ is trivial. The corollary is also a special case of the result mentioned at the end of 9.1. $\qquad\square$

The following result is well-known for fields or discrete valuation rings. We need it for arbitrary valuation rings of rank 1 which are not noetherian in general and hence we may not use algebraic intersection theory. However, there is an intersection theory with Cartier divisors in this situation (see [Gub1]) which together with the result for the generic fibre will easily imply the claim.

Proposition 9.5 *The restriction map gives an isomorphism* $\mathrm{Pic}(\mathbb{P}_{K^\circ}^N) \to \mathrm{Pic}(\mathbb{P}_K^N)$ *and pull-back with respect to the second projection gives an isomorphism* $\mathrm{Pic}(\mathbb{P}_{K^\circ}^N) \to \mathrm{Pic}(\mathbb{T} \times_{K^\circ} \mathbb{P}_{K^\circ}^N)$

Proof: To prove the first claim, we have to show that every line bundle $\mathscr{L}$ on $\mathbb{P}_{K^\circ}^N$ is isomorphic to $O_{\mathbb{P}^N}(m)$ over K° for some $m \in \mathbb{Z}$. We consider a Cartier divisor $\mathscr{D} = \{U_i, \gamma_i\}_{i \in I}$ which is trivial on the generic fibre $\mathbb{P}_K^N$ and we have to prove that $\mathscr{D}$ is trivial. We may assume that $\mathscr{U}_i = \mathrm{Spec}(K^\circ[x_1, \ldots, x_N]_{h_i})$ for a polynomial $h_i \in K^\circ[x_1, \ldots, x_N]$. Obviously, we may skip all charts with empty special fibre. This means that the prime factors $p_1, \ldots, p_r$ of h_i in $K^\circ[x_1, \ldots, x_N]$ are non-constant. Using unique factorization, we get $\mathscr{O}(U_i)^\times = K^\times p_1^{\mathbb{Z}} \cdots p_r^{\mathbb{Z}}$ for the generic fibre U_i of $\mathscr{U}_i$. By triviality of $\mathscr{D}$ on the generic fibre $\mathbb{P}_K^N$, we get $\gamma_i = \lambda_i h_i'$ for some $\lambda_i \in K^\times$ and $h_i' \in p_1^{\mathbb{Z}} \cdots p_r^{\mathbb{Z}}$. We want to show that these factorizations fit on an overlapping $\mathscr{U}_i \cap \mathscr{U}_j$. As $\mathscr{U}_i \cap \mathscr{U}_j$ intersects the special fibre, there is a valued field (L, w) extending (K, v) and an L°-integral point P of $\mathscr{U}_i \cap \mathscr{U}_j$. Using $h_i \in \mathscr{O}(\mathscr{U}_i)^\times$, we get $|h_i'(P)|_w = 1$. The multiplicity $m(\mathscr{D}, \mathbb{P}_{\tilde{K}}^N)$ of $\mathscr{D}$ along the special fibre $\mathbb{P}_{\tilde{K}}^N$ was defined in [Gub1], Section 3. Since the special fibre is irreducible and smooth, it is shown in Proposition 7.6 of [Gub2] that

$$m(\mathscr{D}, \mathbb{P}_{\tilde{K}}^n) = -\log|\gamma_i(P)| = v(\lambda_i).$$

We conclude that $v(\lambda_i) = v(\lambda_j)$. Dividing the equations of $\mathscr{D}$ by a fixed λ_i, we deduce that $\mathscr{D}$ is trivial on $\mathbb{P}_{K^\circ}^N$ proving the first claim.

Similarly, we prove the second claim. The claim holds on the generic fibre and hence it is enough to show that a Cartier divisor $\mathscr{D}$ on $\mathbb{T} \times_{K^\circ} \mathbb{P}_{K^\circ}^N$ which is trivial on the generic fibre $T \times_K \mathbb{P}_K^N$ is trivial on $\mathbb{T} \times_{K^\circ} \mathbb{P}_{K^\circ}^N$. This is done as above replacing K by the unique factorization domain $\mathscr{O}(T) = K[M]$ and using $K[M]^\times = \{\lambda \chi^u \mid u \in M, \lambda \in K^\times\}$. $\qquad\square$

Remark 9.6 It was pointed out to the author by Qing Liu and C. Pépin that the second claim holds more generally for any integral normal scheme $\mathscr{X}$ instead of $\mathbb{P}_{K^\circ}^N$. Their argument is as follows: Injectivity follows from the existence of a section for p_2. It remains to prove surjectivity. Let ξ be the generic point of $\mathscr{X}$ with residue field $\kappa(\xi)$. Then the fibre $(\mathbb{T} \times_{K^\circ} \mathscr{X})_\xi$ is isomorphic to the split torus $T_{\kappa(\xi)}$ and hence the restriction of any line bundle on $\mathbb{T} \times_{K^\circ} \mathscr{X}$ to this fibre is trivial. We conclude that it is enough to show that pull-back with respect to the second projection p_2 gives an isomorphism from the group of Cartier divisors of $\mathscr{X}$ onto the group of those Cartier divisors of $\mathbb{T} \times_{K^\circ} \mathscr{X}$ whose restriction to $(\mathbb{T} \times_{K^\circ} \mathscr{X})_\xi$ is zero. Injectivity follows again from the existence of a section for p_2. Moreover, this shows that surjectivity is a local question on $\mathscr{X}$ and so we may assume $\mathscr{X}$ affine. By the descent argument in [EGA IV], Proposition 8.9.1, we may assume that $\mathscr{X}$ is noetherian. Then surjectivity follows from Proposition 21.4.11 in the list of Errata and Addendum in [EGA IV].

The referee gave the following alternative argument for the first claim in Proposition 9.5. It is obvious that the restriction $\mathrm{Pic}(\mathbb{P}_{K^\circ}^N) \to \mathrm{Pic}(\mathbb{P}_K^N)$ is surjective. He noted that injectivity of $\mathrm{Pic}(\mathscr{X}) \to \mathrm{Pic}(\mathscr{X}_K)$ holds more generally for any projective and flat scheme $\mathscr{X}$ over K° with irreducible and reduced geometric fibres if the structure morphism $f : \mathscr{X} \to S := \mathrm{Spec}(K^\circ)$ admits a section and if $f_*(\mathscr{O}_\mathscr{X}) = \mathscr{O}_S$. Note that these assumptions hold for $\mathbb{P}_{K^\circ}^N$. His argument uses

that $\mathrm{Pic}_{\mathscr{X}/S}(S) = \mathrm{Pic}(\mathscr{X})$ and $\mathrm{Pic}_{\mathscr{X}/S}(\mathrm{Spec}(K)) = \mathrm{Pic}(\mathscr{X}_K)$ for the relative Picard functor $\mathrm{Pic}_{\mathscr{X}/S}$ (see [BLR], Proposition 8.1.4). By a result of Grothendieck, $\mathrm{Pic}_{\mathscr{X}/S}$ is represented by a separated S-scheme. Note that Grothendieck's original result ([Gro], no. 232, Theorem 3.1) was written for locally noetherian schemes, but using the technique of noetherian approximation introduced in [EGA IV], this holds in general as stated in [BLR], Theorem 8.2.1. Now the valuative criterion of separatedness ([EGA II], Proposition 7.2.3) shows that the map $\mathrm{Pic}(\mathscr{X}) \to \mathrm{Pic}(\mathscr{X}_K)$ is injective.

Lemma 9.7 *Suppose that the torus $\mathbb{T}$ acts linearly on $\mathbb{P}^N_{K^\circ}$. Then this action lifts to a linear representation of $\mathbb{T}$ on $\mathbb{A}^{N+1}_{K^\circ}$.*

Proof: The action $\sigma : \mathbb{T} \times_{K^\circ} \mathbb{P}^N_{K^\circ} \to \mathbb{P}^N_{K^\circ}$ is linear which means that it is given by a homomorphism $\mathbb{T} \to PGL(N+1)$ defined over K°. We are looking for a lift to a homomorphism $\mathbb{T} \to GL(N+1)$. This is equivalent to the existence of a $\mathbb{T}$-linearization on the line bundle $L := O(1)$ of $\mathbb{P}^N_{K^\circ}$, i.e. an action of $\mathbb{T}$ on L which is compatible with the given group action σ. Here, we use the language of [Mum], §1.3, which is written for schemes over a base field. However, the argument for the existence of a $\mathbb{T}$-linearization in [Mum], Proposition 1.5, extends to the case of a valuation ring. Indeed, the essential point is the existence of an isomorphism $\sigma^*(L) \cong p_2^*(L)$ which follows from Proposition 9.5 and then we may conclude as at the end of the proof of [Mum], Proposition 1.5, to prove that $L = O(1)$ has a $\mathbb{T}$-linearization. $\qquad\square$

Proposition 9.8 *Let $\mathscr{Y}$ be a closed irreducible subvariety of $\mathbb{P}^N_{K^\circ}$. Suppose that the torus $\mathbb{T}$ operates linearly on $\mathbb{P}^N_{K^\circ}$ and leaves $\mathscr{Y}$ invariant. We assume that $\mathscr{Y}$ has an open dense orbit containing a K-rational point $\mathbf{y}$. Then after a suitable change of coordinates on $\mathbb{P}^N_{K^\circ}$, there is $A \in \check{T}^{N+1}$ such that $\mathscr{Y} = \mathscr{Y}_{A,a}$ for the height function $a : \{0, \dots, N\} \to \Gamma$ of $\mathbf{y}$.*

Proof: By Lemma 9.7, the projective representation of $\mathbb{T}$ on $\mathbb{P}^N_{K^\circ}$ lifts to a representation S of $\mathbb{T}$ on $\mathbb{A}^{N+1}_{K^\circ}$. Since the multiplicative torus T is split over K, it follows that the vector space $V := K^{N+1}$ has a simultaneous eigenbasis $v_0, \dots, v_N$ for the T-action ([Bor], Proposition III.8.2). For $j = 0, \dots, N$, we have $S_t(v_j) = \chi^{u_j}(t)v_j$ for all $t \in T(K)$ and some $u_j \in M$.

We endow V with the norm $\|\mathbf{x}\| := \max_{j=0,\dots,N} |x_j|$ for $\mathbf{x} \in K^{N+1}$. By definition, this gives a K-cartesian space and hence every subspace U of V is also K-cartesian meaning that there is a basis $u_1, \dots, u_r$ of U such that $\|\sum_{j=1}^r \alpha_j u_j\| = \max_{j=1,\dots,r} |\alpha_j|$ for all $\alpha_1, \dots, \alpha_r \in K$ (see [BGR], Proposition 2.4.1/5). In this non-archimedean situation, such a basis is called orthonormal. We apply this to the simultaneous eigenspaces V_{u_j} for the T-action and so we may choose the simultaneous eigenbasis $v_0, \dots, v_N$ above in such a way that a suitable subset is an orthonormal basis of V_{u_j} for every $j = 0, \dots, N$.

We consider the subgroup $U := \mathbb{T}(K^\circ) = \{t \in T(K) \mid v(t_1) = \dots = v(t_n) = 0\}$ of $T(K)$. For $t \in U$, we have $S_t \in GL(N+1, K^\circ)$ and hence the eigenvalues $\chi^{u_j}(t)$ have absolute value 1. If we use reduction modulo $K^{\circ\circ}$, then the U-action becomes a $(\mathbb{G}^n_m)_{\tilde{K}}$-operation on $\tilde{K}^{N+1}$. We note that the reduction of an orthonormal basis in a subspace of V is linearly independent in $\tilde{K}^{N+1}$. Using that eigenvectors for distinguished eigenvalues are linearly independent, we conclude that the reduction of $v_0, \dots, v_N$ is a a simultaneous eigenbasis for the $(\mathbb{G}^n_m)_{\tilde{K}}$-action. By Nakayama's Lemma, it follows that $v_0, \dots, v_N$ is a K°-basis for $(K^\circ)^{N+1}$. We choose the coordinates of $\mathbb{P}^N_{K^\circ}$ according to this basis and let a be the corresponding height function of $\mathbf{y}$. For $A = (u_0, \dots, u_N)$, we get $\mathscr{Y} = \mathscr{Y}_{A,a}$. $\qquad\square$

Remark 9.9 Every projective normal toric variety over a field can be equivariantly embedded into some projective space endowed with a linear torus action (see [Mum], §1.3). There are projective non-normal toric varieties over a field for which this is not true (see [GKZ], Remark 5.1.6).

9.10 In the following, we consider the toric variety $\mathscr{Y}_{A,a}$ for some $A \in M^{N+1}$ and $\mathbf{y} \in \mathbb{P}^{N+1}_{K^\circ}$ with height function a.

The *weight polytope* $\mathrm{Wt}(\mathbf{y})$ is the convex hull of $A(\mathbf{y}) := \{u_j \mid a(j) < \infty\}$ in $M_\mathbb{R}$. The *induced subdivision* of $\mathrm{Wt}(\mathbf{y})$ is given by projection of the faces of the convex hull of $\{(u_j, \lambda) \in M_\mathbb{R} \times \mathbb{R} \mid j = 0, \ldots, N, \lambda \geq a(j)\}$. The weight subdivision is a polytopal complex denoted by $\mathrm{Wt}(\mathbf{y}, a)$. The vertices of $\mathrm{Wt}(\mathbf{y}, a)$ are contained in $A(\mathbf{y})$.

9.11 In the following, we need some additional notions from convex geometry which we have introduced in the appendix. By construction, there is a unique proper polyhedral function f on $M_\mathbb{R}$ such that the epigraph of f is equal to the convex hull of $\{(u_j, \lambda) \in M_\mathbb{R} \times \mathbb{R} \mid j = 0, \ldots, N, \lambda \geq a(j)\}$. The domain of f is equal to $\mathrm{Wt}(\mathbf{y})$ and $f(u_j) = a(j)$ for all vertices u_j of the weight subdivision $\mathrm{Wt}(\mathbf{y}, a)$.

We define the *dual complex* $\mathscr{C}(A, a)$ of $\mathrm{Wt}(\mathbf{y}, a)$ as the complete polyhedral complex in $N_\mathbb{R}$ characterized by the fact that the n-dimensional polyhedra in $\mathscr{C}$ are the domains of linearity of the affine function

$$g(\omega) := \min_{j=0,\ldots,N} a(j) + \langle u_j, \omega \rangle.$$

Obviously, all polyhedra in $\mathscr{C}(A, a)$ are Γ-rational. There is a bijective order reversing correspondence between the faces of $\mathrm{Wt}(\mathbf{y}, a)$ and polyhedra in $\mathscr{C}(A, a)$. The polyhedron $\widehat{Q} \in \mathscr{C}(A, a)$ corresponding to the face Q of $\mathrm{Wt}(\mathbf{y}, a)$ is given by

$$\widehat{Q} = \{\omega \in N_\mathbb{R} \mid g(\omega) = \langle u, \omega \rangle + f(u) \; \forall u \in Q\}$$
$$= \{\omega \in N_\mathbb{R} \mid g(\omega) = \langle u_j, \omega \rangle + a(j) \; \forall u_j \in A(\mathbf{y}) \cap Q\}.$$

Conversely, the face $\widehat{\sigma}$ of $\mathrm{Wt}(\mathbf{y}, a)$ corresponding to $\sigma \in \mathscr{C}(A, a)$ is given by

$$\widehat{\sigma} = \{u \in M_\mathbb{R} \mid g(\omega) = \langle u, \omega \rangle + f(u) \; \forall \omega \in \sigma\}$$

and it is also the convex hull of $\{u_j \in A \mid g(\omega) = \langle u_j, \omega \rangle + a(j) \; \forall \omega \in \sigma\}$. All this can be seen using the dual complex $\mathrm{Wt}(\mathbf{y}, a)^f$ from A.11 and the conjugate polyhedral function f^* of f from A.10. Indeed, we have $f^*(\omega) = -g(-\omega)$ and hence $\mathrm{Wt}(\mathbf{y}, a)^f = -\mathscr{C}(A, a)$.

In the next results, we will also use the tropicalization map $\mathrm{trop}_v : T^{\mathrm{an}} \to N_\mathbb{R}$ and the reduction map $\pi : Y^{\mathrm{an}}_{A,a} \to (\mathscr{Y}_{A,a})_s$.

Proposition 9.12 *There are bijective order reversing correspondences between*

  (a) *faces Q of the weight subdivision $\mathrm{Wt}(\mathbf{y}, a)$;*

  (b) *polyhedra σ of the dual complex $\mathscr{C}(A, a)$;*

  (c) *$\mathbb{T}$-orbits Z of the special fibre of $\mathscr{Y}_{A,a}$.*

The correspondences are given as follows: The face $Q = \widehat{\sigma}$ is the face of $\mathrm{Wt}(\mathbf{y}, a)$ spanned by those u_j with $x_j \neq 0$ for $x \in Z$. The polyhederon σ is given by $\sigma = \widehat{Q}$ and $\mathrm{relint}(\sigma) = \mathrm{trop}_v(\{t \in T^{\mathrm{an}} \mid \pi(t\mathbf{y}) \in Z\})$. The orbit Z is equal to

$$\{\mathbf{x} \in (\mathscr{Y}_{A,a})_s \mid x_j \neq 0 \iff u_j \in A(\mathbf{y}) \cap Q\} = \{\pi(t\mathbf{y}) \mid t \in T^{\mathrm{an}} \cap \mathrm{trop}_v^{-1}(\mathrm{relint}(\sigma))\}.$$

The correspondence $Q \leftrightarrow Z$ is preserving the natural orders and the other correspondences are order reversing. Moreover, we have $\dim(Q) = \dim(Z) = n - \dim(\sigma)$.

Proof: We have discussed the correspondence $Q \leftrightarrow \sigma$ in 9.11. Next, we note that every point z of $(\mathscr{Y}_{A,a})_s$ is the reduction of a point in $T^{\mathrm{an}}\mathbf{y}$. Since $T\mathbf{y}$ is an open dense subset of the generic fibre of $\mathscr{Y}_{A,a}$, this follows from Lemma 4.12.

Now let σ be a polyhedron from $\mathscr{C}(A,a)$. We will show next that $Z := \{\pi(t\mathbf{y}) \mid t \in T^{\mathrm{an}} \cap \mathrm{trop}_v^{-1}(\mathrm{relint}(\sigma))\}$ is a $\mathbb{T}$-invariant subset of $(\mathscr{Y}_{A,a})_s$. Let us consider the formal affinoid torus T° which is the affinoid subdomain of T^{an} given by the equations $|x_1| = \cdots = |x_n| = 1$. The reduction map induces a surjective group homomorphism $T^\circ \to \mathbb{T}_s$ and $\pi : Y_{A,a}^{\mathrm{an}} \to (\mathscr{Y}_{A,a})_s$ is equivariant with respect to this homomorphism. Since T° leaves $\mathrm{trop}_v^{-1}(\mathrm{relint}(\sigma))$ invariant, we conclude that Z is invariant under the $\mathbb{T}_s$-action.

For $z \in (\mathscr{Y}_{A,a})_s$, we have seen above that there is $t \in T^{\mathrm{an}}$ with $z = \pi(t\mathbf{y})$. It follows from 9.11 that $\omega \in \mathrm{relint}(\sigma)$ if and only if

$$A(\mathbf{y}) \cap Q = \{u_j \in A \mid g(\omega) = a(j) + \langle u_j, \omega \rangle\},$$

i.e. precisely the functions $a(j) + \langle u_j, \omega \rangle$ with $u_j \in A(\mathbf{y}) \cap Q$ are minimal in ω. If we apply this with $\omega := \mathrm{trop}_v(t)$, then we have $a(j) + \langle u_j, \omega \rangle = v(\chi^{u_j}(t) \cdot y_j)$ and we deduce

$$Z = \{\mathbf{x} \in (\mathscr{Y}_{A,a})_s \mid x_j \neq 0 \iff u_j \in A(\mathbf{y}) \cap Q\}. \tag{6}$$

Next, we prove that Z is a $\mathbb{T}_s$-orbit. We have already seen that Z is $\mathbb{T}_s$-invariant. It remains to show that the action is transitive and so we consider $z_1, z_2 \in Z$. There is a complete valued field (F, u) extending (K, v) such that z_1, z_2 are $\tilde{F}$-rational. Let $L = F((\mathbb{R}))$ be the the Mal'cev-Neumann ring. Note that L is a complete field consisting of certain power series in the variable x and with real exponents (see [Poo] for details). The advantage is that we have a canonical homomorphism $\rho : \mathbb{R} \to L^*$ with $v \circ \rho = \mathrm{id}$. Using suitable coordinates, we get a homomorphism $N_{\mathbb{R}} \to T(L)$ which is a section of trop_v and which we also denote by ρ.

For $i = 1, 2$, there is $t_i \in T^{\mathrm{an}}$ with $z_i = \pi(t_i \mathbf{y})$ and $\mathrm{trop}_v(t_i) \in \mathrm{relint}(\sigma)$. Choosing F sufficiently large, we may assume that t_i is induced by an F-rational point in T which we also denote by t_i. For $t \in T(L)$, we set $t^\circ := t \cdot \rho(-\mathrm{trop}_v(t))$. This is an element of the formal affinoid torus $T^\circ(L)$ and hence reduces to an element $\tilde{t^\circ} \in \mathbb{T}(\tilde{L})$. The map $t \mapsto t^\circ$ is a homomorphism as well as the reduction. We will use this construction for t_1, t_2 and $t := t_2/t_1$. We claim that $\tilde{t^\circ} z_1 = z_2$. To see this, we note for $j \in A(\mathbf{y}) \cap Q$ that

$$(t^\circ t_1 \mathbf{y})_j = \chi^{u_j}(t^\circ t_1) y_j = \chi^{u_j}(\rho \circ \mathrm{trop}_v(t_1/t_2) t_2) y_j = \lambda_j (t_2 \mathbf{y})_j$$

with factor

$$\lambda_j := \chi^{u_j}(\rho \circ \mathrm{trop}_v(t_1/t_2)) = \rho(\langle u_j, \mathrm{trop}_v(t_1) - \mathrm{trop}_v(t_2)\rangle).$$

From the above considerations and using that $\mathrm{trop}_v(t_i) \in \mathrm{relint}(\sigma)$, we conclude that $\langle u_j, \mathrm{trop}_v(t_i)\rangle + a(j)$ does not depend on the choice of $u_j \in A(\mathbf{y}) \cap Q$ and hence the factor λ_j does not depend on $j \in A(\mathbf{y}) \cap Q$ as well. For $i = 1, 2$, let $x_i \in \mathbb{P}^N$ be the point with coordinates $(t_i \mathbf{y})_j$ for $j \in A(\mathbf{y}) \cap Q$ and with all other coordinates 0. Then the above shows $t^\circ x_1 = x_2$ and hence $\tilde{t^\circ} z_1 = z_2$ by the equivariance of the reduction maps. This proves transitivity.

Conversely, if the orbit Z is given, then we may recover $A(\mathbf{y}) \cap Q$ by (6) and this set generates the face Q of $\mathrm{Wt}(\mathbf{y}, a)$. Let $\sigma = \widehat{Q}$ be the corresponding polyhedron in the dual complex $\mathscr{C}(A, a)$; then

$$\mathrm{trop}_v(\{t \in T^{\mathrm{an}} \mid \pi(t\mathbf{y}) \in Z\}) \subset \mathrm{relint}(\sigma)$$

is also clear from what we have proven in (6). Then we get immediately equality as the left hand side forms a partition of $N_{\mathbb{R}}$ for varying Z.

The character group of the torus corresponding to the orbit Z has finite index in $\mathbb{Z}(\sigma^\perp \cap M)$. This is clear as we may choose a base point $\mathbf{y}'$ in Z and then apply 9.1 with $A, \mathbf{y}'$ and with $\tilde{K}$ replacing K. This and A.11 prove immediately the identities relating the dimensions. Finally, the claims about the orders are evident. $\qquad\square$

Remark 9.13 If v is the trivial valuation, then the dual complex $\mathscr{C}(A, a)$ is just the normal fan of the weight polytope $\mathrm{Wt}(\mathbf{y})$.

Corollary 9.14 *There are bijective order correspondences between*

  (a) *faces Q of the weight polytope $\mathrm{Wt}(\mathbf{y})$;*

  (b) *polyhedra σ of the normal fan of $\mathrm{Wt}(\mathbf{y})$;*

  (c) *T-orbits Z of the generic fibre of $\mathscr{Y}_{A,a}$.*

Proof: The generic fibre $Y_{A,a}$ is the closure of the T-orbit of $\mathbf{y}$ in $\mathbb{P}_K^N$ with respect to the T-action induced by the character set A and so we have seen the claim already in 9.1. We note here that the equivalence is also a special case of Proposition 9.12. To see this, we replace v by the trivial valuation and then the special fibre is equal to the generic fibre. $\qquad\square$

9.15 Let Z be an orbit of $\mathscr{Y}_{A,a}$ corresponding to a face Q of the weight subdivision $\mathrm{Wt}(\mathbf{y}, a)$ (resp. the weight polytope $\mathrm{Wt}(\mathbf{y})$). We choose a base point $\mathbf{z} \in Z(K)$. Then the closure of Z in $\mathbb{P}^N$ is the projective toric variety $\mathscr{Y}_{A,a}(\mathbf{z})$ in $\mathbb{P}_{\tilde{K}}^N$ (resp. in $\mathbb{P}_K^N$) constructed from $\mathbf{z}$ and $A(\mathbf{y}) \cap Q$ as in 9.1. We conclude that Q is its weight polytope.

Remark 9.16 The polyhedra of $\mathscr{C}(A, a)$ are pointed if and only if $\mathrm{Wt}(\mathbf{y})$ has dimension n. In other words, this means that the smallest affine space containing $A(\mathbf{y})$ is $N_\mathbb{R}$ and this is equivalent to $\dim(\mathrm{Stab}(\mathbf{y})) = 0$ (see Corollary 9.4).

By passing to a sublattice of M, we may always achieve this situation and we may even assume that $M = \mathbb{Z}A(\mathbf{y})$ (see Proposition 9.3). Since $\mathscr{C}(A, a)$ is a complete complex, it follows from 7.6 that $\mathscr{C}(A, a) = \Sigma_1$ for a complete Γ-admissible fan Σ in $N_\mathbb{R} \times \mathbb{R}_+$.

10   The Gröbner complex

In this section, K denotes a field with a non-archimedean absolute value $|\ |$, corresponding valuation $v := -\log|\ |$, valuation ring K°, residue field $\tilde{K}$ and value group $\Gamma = v(K^\times)$. Then we consider a closed subscheme X of the split multiplicative torus T over K. We will introduce its Gröbner complex on $N_\mathbb{R}$ which is related to the natural orbit of X in the Hilbert scheme of a projective compactification. This is a certain complete Γ-rational complex which has a subcomplex with support equal to $\mathrm{Trop}_v(X)$. At the end, we relate the Gröbner complex to the initial degenerations of X. This section is inspired by [Kat], Section 5, which in turn was influenced by Tevelev. We work here with more general base fields, but the ideas are the same. For an elementary approach using Gröbner bases and for examples, we refer to [MS], Section 2.4.

10.1 First, we recall the following property of the *Hilbert scheme* $\mathrm{Hilb}_p(\mathbb{P}_S^m)$ for the projective space $\mathbb{P}_S^m$ over a locally noetherian scheme S and for a Hilbert polynomial $p(x) \in \mathbb{Q}[x]$. This property characterizes the Hilbert scheme up to unique isomorphism:

There is a projective scheme $\mathrm{Hilb}_p(\mathbb{P}_S^m)$ over the base scheme S and a closed subscheme $\mathrm{Univ}_p(\mathbb{P}_S^m)$ of $\mathbb{P}_S^m \times_S \mathrm{Hilb}_p(\mathbb{P}_S^m)$ which is flat over $\mathrm{Hilb}_p(\mathbb{P}_S^m)$ and which

has Hilbert polynomial p such that for every scheme Z over S, the map from the set of morphisms $Z \to \mathrm{Hilb}_p(\mathbb{P}_S^m)$ to the set of closed subschemes of $\mathbb{P}_Z^m$ with Hilbert polynomial p and flat over Z, given by mapping f to the inverse image scheme

$$(\mathrm{id} \times f)^{-1}(\mathrm{Univ}_p(\mathbb{P}_S^m)) = \mathrm{Univ}_p(\mathbb{P}_S^m) \times_{\mathrm{Hilb}_p(\mathbb{P}_S^m)} Z,$$

is a bijection. In other words, there is a bijective correspondence $Y \mapsto [Y]$ between the set of closed subschemes of $\mathbb{P}_Z^m$ which are flat over Z and which have Hilbert polynomial p and the set of Z-valued points of $\mathrm{Hilb}_p(\mathbb{P}_S^m)$. For a proof, we refer to [Kol], Section 1.1. Note that the Hilbert polynomial of a closed subscheme Y of $\mathbb{P}_Z^m$ is defined for every fibre over a point z of Z. If Y is flat over Z and if Z is connected, then the Hilbert polynomial does not depend on the choice of z.

The valuation ring K° has not to be noetherian and so we cannot directly apply the above. However, the Hilbert scheme exists also for an arbitrary base scheme S if we require that it represents the functor mapping Z to the set of closed subschemes of $\mathbb{P}_Z^m$ with Hilbert polynomial p and which are flat and *finitely presented* over Z (see [AK], Corollary 2.8). Note that if $S = \mathrm{Spec}(K^\circ)$, then every closed subscheme of $\mathbb{P}_S^m$ which is flat over S is of finite presentation (use [RG], Corollaire 3.4.7), hence it is defined over a noetherian subring of K° and so we can construct the Hilbert scheme $\mathrm{Hilb}_p(\mathbb{P}_S^m)$ from the noetherian case by base change.

10.2 We briefly sketch the construction of the Hilbert scheme as far as we need it later. For simplicity, we restrict to the case $S = \mathrm{Spec}(F)$ for a field F. The general case follows similary using graded ideal sheaves instead of graded ideals. For details, we refer to [Kol], Section 1.1.

Let $I_Y(k)$ be the k-th graded piece of the graded ideal I_Y in $F[x_0, \ldots, x_m]$ of a closed subscheme Y of $\mathbb{P}_F^m$ with Hilbert polynomial p. For sufficiently large k depending only on p, we have $\dim(I_Y(k)) = q(k) - p(k)$ and the map $Y \mapsto I_Y(k)$ is an injective map from the set of closed subschemes of $\mathbb{P}_F^m$ to the Grassmannian $G(q(k) - p(k), q(k))$, where q is the Hilbert polynomial of $\mathbb{P}^m$. The image is $\mathrm{Hilb}(\mathbb{P}_F^m)$ which we may endow with a suitable structure as a closed subscheme of the Grassmannian and with a family $\mathrm{Univ}_p(\mathbb{P}_F^m)$ which satisfies the required universal property. Using the Grassmann coordinates $L \mapsto \bigwedge^{q(k)-p(k)}(L)$, we get $G(q(k) - p(k), q(k))$ as a closed subscheme of $\mathbb{P}_F^N$ for $N := \binom{q(k)}{p(k)} - 1$ and hence $\mathrm{Hilb}(\mathbb{P}_F^m)$ may be seen as a closed subscheme of $\mathbb{P}_F^N$ as well.

10.3 We consider a linear action of the torus $\mathbb{T}$ on $\mathbb{P}_{K^\circ}^m$. It follows easily from the universal property of the Hilbert scheme that $\mathbb{T}$ operates also on $\mathrm{Hilb}_p(\mathbb{P}_{K^\circ}^m)$ such that for any scheme Z over K° and any closed subscheme Y of $\mathbb{P}_Z^m$ with Hilbert polynomial p which is finitely presented and flat over Z, we have $g \cdot [Y] = [g^{-1}Y]$. It makes the following formulas more natural if the action is by pull-back with respect to multiplication by g rather than push-forward. If we use the closed embedding of $\mathrm{Hilb}_p(\mathbb{P}_{K^\circ}^m)$ into $\mathbb{P}_{K^\circ}^N$ similarly as in 10.2, then the $\mathbb{T}$-action on $\mathrm{Hilb}_p(\mathbb{P}_{K^\circ}^m)$ extends to a linear action of $\mathbb{T}$ on $\mathbb{P}_{K^\circ}^N$. Indeed, if A_t is the $(m+1) \times (m+1)$-matrix representing the action of t on $\mathbb{P}_{K^\circ}^m$ similarly as in the proof of Lemma 9.7, then $(A_t \mathbf{x})^{\mathbf{m}}$ is a linear combination of monomials of degree $|\mathbf{m}|$ and this shows easily the claim using the Grassmann coordinates.

Proposition 10.4 *Let Y be a closed subscheme of $\mathbb{P}_K^m$ with Hilbert polynomial p. Then the closure of the $\mathbb{T}$-orbit of $[Y]$ in $\mathrm{Hilb}_p(\mathbb{P}_{K^\circ}^m)$ is equal to $\mathscr{Y}_{A,a}$ for suitable $A \in M^{N+1}$ and height function $a : \{0, \ldots, N\} \to \Gamma \cup \{\infty\}$ using suitable coordinates on $\mathbb{P}_{K^\circ}^N$.*

Proof: This follows from Proposition 9.8. $\square$

Definition 10.5 The dual complex $\mathscr{C}(A, a)$ from 9.11 is called the *Gröbner complex of Y*.

Definition 10.6 Let (L, w) be a valued field extension of (K, v). For $t \in T(L)$, the special fibre of the closure of $t^{-1}Y_L$ in $\mathbb{P}^m_{L^\circ}$ is called the *initial degeneration of Y at t*. This is a closed subscheme of $\mathbb{P}^m_{\tilde{L}}$ defined over the residue field $\tilde{L}$ which we denote by $\mathrm{in}_t(Y)$.

For $\omega = \mathrm{trop}_w(t)$, we set $\mathrm{in}_\omega(Y) = \mathrm{in}_t(Y)$. Similarly as in Proposition 5.3, this is independent of the choice of t up to multiplication by an element from $\mathbb{T}$ defined over a suitable field extension of $\tilde{K}$. Since trop_v is surjective, $\mathrm{in}_\omega(Y)$ is defined for every $\omega \in N_\mathbb{R}$.

10.7 In the situation above, $[t^{-1}Y_L] = t \cdot [Y_L]$ is an L-rational point of $\mathrm{Hilb}(\mathbb{P}^m)$. By projectivity of the Hilbert scheme, we conclude that $[t^{-1}Y_L]$ extends uniquely to an L°-valued point h_t of $\mathrm{Hilb}(\mathbb{P}^m)$ and hence corresponds to a closed subscheme of $\mathbb{P}^m_{L^\circ}$ which is flat over L° and has generic fibre $t^{-1}Y_L$. By Proposition 4.4 and Remark 4.6, this has to be the closure of $t^{-1}Y_L$ and hence the special fibre is $\mathrm{in}_t(Y)$. In other words, $[\mathrm{in}_t(Y)]$ is equal to the reduction of h_t in $\mathrm{Hilb}(\mathbb{P}^m)(\tilde{L})$.

Proposition 10.8 *Suppose that $\mathbb{T}$ acts linearly on $\mathbb{P}^m_{K^\circ}$. Let Y be a closed subscheme of $\mathbb{P}^m_K$ and let (L, w) be a valued field extending (K, v). For $t_1, t_2 \in T(L)$, the following conditions are equivalent:*

  (a) There is a polyhedron σ of the Gröbner complex $\mathscr{C}(A, a)$ of Y such that for $i = 1, 2$, we have $\mathrm{trop}_w(t_i) \in \mathrm{relint}(\sigma)$.

  (b) There is $g \in \mathbb{T}(\tilde{L})$ with $\mathrm{in}_{t_2}(Y) = g \cdot \mathrm{in}_{t_1}(Y)$.

Proof: This follows from Proposition 9.12, Proposition 10.4 and 10.7. $\qquad\square$

Proposition 10.9 *Let $\omega_1 = \omega_0 + \Delta\omega$ in $N_\mathbb{R}$ and suppose that there is a polyhedron σ from the Gröbner complex $\mathscr{C}(A, a)$ with $\omega_0 \in \sigma$ and $\omega_1 \in \mathrm{relint}(\sigma)$. Then we have*

$$\mathrm{in}_{\omega_1}(Y) = \mathrm{in}_{\Delta\omega}(\mathrm{in}_{\omega_0}(Y)), \tag{7}$$

where we consider $\mathrm{in}_{\omega_0}(Y)$ as a closed subscheme of $\mathbb{P}^m$ over a trivially valued extension of the residue field $\tilde{K}$ and then we take its initial degeneration with respect to $\Delta\omega$. In particular, we have (7) for all $\omega_1 \in N_\mathbb{R}$ in a sufficiently small neighbourhood of ω_0.

Proof: It follows from Proposition 10.8 that $z_1 := [\mathrm{in}_{\omega_1}(Y)]$ is in the orbit Z_σ of the special fibre of $\mathscr{Y}_{A,a}$ corresponding to σ. If ρ is the closed face of σ with $\omega_0 \in \mathrm{relint}(\rho)$, then $z_0 := [\mathrm{in}_{\omega_0}(Y)]$ is in the orbit $Z := Z_\rho$ corresponding to ρ.

Now we repeat the procedure taking the closure of the orbit Z with respect to the base point z_0 in $\mathrm{Hilb}(\mathbb{P}^m_{\tilde{K}})$. We have seen in 9.15 that the dual polytope $\hat{\rho}$ is the weight polytope of the projective toric variety $\overline{Z}$. Since we use the trivial valuation on $\tilde{K}$, the dual complex of $\hat{\rho}$ is the complete fan formed by the local cones $\mathrm{LC}_{\omega_0}(\nu)$ with ν ranging over all polyhedra from $\mathscr{C}(A, a)$ containing ρ. Then $z := [\mathrm{in}_{\Delta\omega}(\mathrm{in}_{\omega_0}(Y))]$ is in the orbit of $\overline{Z}$ corresponding to the fan $\mathrm{LC}_{\omega_0}(\nu)$ containing $\Delta\omega$ in its relative interior. Obviously, this holds for $\nu = \sigma$.

Recall that $\mathrm{Hilb}(\mathbb{P}^m_{\tilde{K}})$ is the special fibre of $\mathrm{Hilb}(\mathbb{P}^m_{K^\circ})$ and we have $Z = Z_\rho$. Moreover, $\overline{Z}$ is contained in the special fibre of $\mathscr{Y}_{A,a}$. We note that every orbit of $\overline{Z}$ is an orbit of $(\mathscr{Y}_{A,a})_s$ and the corresponding fan $\mathrm{LC}_{\omega_0}(\nu)$ transforms to ν taking into account that the base point has changed from $[Y]$ to z_0. We conclude that z and z_1 are in the same orbit. This proves (7). Finally, the last claim is obvious from the fact that the above local fan in ω_1 is complete. $\qquad\square$

10.10 In the remaining part of this section, we consider the following important special case, where we can compare the definitions in 10.6 and in 5.1: We consider a projective toric variety $\mathscr{Y}_{B,0}$ over K° given by $B \in M^{m+1}$ and height function identically zero, i.e. the base point $\mathbf{z} \in \mathbb{P}^m(K)$ in the open dense orbit satisfies $v(z_j) = 0$ for $j = 0, \ldots, m$. Recall that $\mathscr{Y}_{B,0}$ is a closed subvariety of $\mathbb{P}^m_{K^\circ}$ and the torus action extends to a linear action on $\mathbb{P}^m_{K^\circ}$ (see 9.2). We assume further that the stabilizer of $\mathbf{z}$ is trivial and so we may identify T with the open dense orbit $T\mathbf{z}$. By Corollary 9.4, the affine span of B is $M_\mathbb{R}$. For example, the standard embedding of $\mathbb{T}$ in $\mathbb{P}^n_{K^\circ}$ fulfills all these requirements.

The triviality of the height function implies that the weight polytope is equal to the weight subdivision and the dual complex is just the normal fan of $\mathrm{Wt}(\mathbf{z})$. Moreover, we may identify $\mathbb{T}$ with the $\mathbb{T}$-invariant open subset of $\mathscr{Y}_{B,0}$ whose generic fibre is the open dense orbit and whose special fibre is the orbit corresponding to the vertex 0 of the cones.

10.11 We consider a closed subscheme X of T and we denote by Y its closure in $\mathbb{P}^m_K$. For a valued field (L, w) extending (K, v) and $t \in T(L)$, it follows immediately from comparing Definitions 5.1 and 10.6 that

$$\mathrm{in}_t(X_L) = \mathrm{in}_t(Y_L) \cap \mathbb{T}_{\tilde{L}}.$$

Corollary 10.12 *For $\omega_0 \in N_\mathbb{R}$, there is a neighbourhood Ω of ω_0 in $N_\mathbb{R}$ such that*

$$\mathrm{in}_{\omega_1}(X) = \mathrm{in}_{\Delta\omega}(\mathrm{in}_{\omega_0}(X)),$$

for every $\omega_1 \in \Omega$ and $\Delta\omega := \omega_1 - \omega_0$. On the right hand side, the initial degeneration of $\mathrm{in}_{\omega_0}(X)$ at $\Delta\omega$ is with respect to a trivially valued field of definition for $\mathrm{in}_{\omega_0}(X)$.

Proof: This follows from Proposition 10.9 and 10.11. $\qquad\qquad\square$

10.13 We apply the above for $Y = \overline{X}$ leading to a polyhedral complex $\mathscr{C}(A, a)$ in $N_\mathbb{R}$ which we call the *Gröbner complex for X*. It depends on the choices from 10.10.

Theorem 10.14 *The Gröbner complex $\mathscr{C}(A, a)$ of X is a complete Γ-rational complex in $N_\mathbb{R}$ and $\{\sigma \in \mathscr{C}(A, a) \mid \sigma \subset \mathrm{Trop}_v(X)\}$ is a subcomplex $\mathscr{C}_X$ of $\mathscr{C}(A, a)$ with support equal to $\mathrm{Trop}_v(X)$.*

Proof: All statements are evident by construction except the claim about the support. Let $\omega \in \mathrm{Trop}_v(X)$. By completeness of the Gröbner complex, there is $\sigma \in \mathscr{C}(A, a)$ with $\omega \in \mathrm{relint}(\sigma)$. We have to prove that $\sigma \subset \mathrm{Trop}_v(X)$. Since $\mathrm{Trop}_v(X)$ is closed in $N_\mathbb{R}$, it is enough to show that every $\omega' \in \mathrm{relint}(\sigma)$ is contained in $\mathrm{Trop}_v(X)$. There is a valued field (L, w) extending (K, v) and $t, t' \in T(L)$ with $\mathrm{trop}_w(t) = \omega$ and $\mathrm{trop}_w(t') = \omega'$. By Proposition 10.8, there is $g \in \mathbb{T}(\tilde{L})$ with $\mathrm{in}_{t'}(Y) = g \cdot \mathrm{in}_t(Y)$. By 10.11 , we conclude that $\mathrm{in}_{\omega'}(X) = \mathrm{in}_\omega(X)$. Using $\omega \in \mathrm{Trop}_v(X)$, Theorem 5.6 implies that $\mathrm{in}_\omega(X)$ is non-empty and hence the same is true for $\mathrm{in}_{\omega'}(X)$. Using this equivalence the other way round, we deduce that $\omega' \in \mathrm{Trop}_v(X)$ proving the claim. $\qquad\square$

The following result is very useful for reducing local statements about the tropical variety to the case of trivial valuations. We will see in Proposition 13.7 that this is also compatible with tropical multiplicities.

Proposition 10.15 *Let X be a closed subscheme of T and let $\omega \in N_\mathbb{R}$. Using the local cone at ω from Appendix A.6, we have*

$$\mathrm{Trop}_0(\mathrm{in}_\omega(X)) = \mathrm{LC}_\omega(\mathrm{Trop}_v(X)).$$

Proof: The fundamental theorem of tropical algebraic geometry (Theorem 5.6) says that $\Delta\omega \in N_\mathbb{R}$ is in $\mathrm{Trop}_0(\mathrm{in}_\omega(X))$ if and only if $\mathrm{in}_{\Delta\omega}(\mathrm{in}_\omega(X))$ is non-empty. If we choose $\Delta\omega$ sufficiently small, then we deduce from Corollary 10.12 that these conditions are also equivalent to $\mathrm{in}_{\omega+\Delta\omega}(X) \neq \emptyset$. Theorem 5.6 again shows that this is equivalent to $\omega + \Delta\omega \in \mathrm{Trop}_v(X)$. As we are working in a sufficiently small neighbourhood of ω, this is equivalent to $\Delta\omega \in \mathrm{LC}_\omega(\mathrm{Trop}_v(X))$ proving the claim. $\square$

10.16 For a polyhedron Δ in $N_\mathbb{R}$, let us recall that $c(\Delta)$ denotes the closed cone in $N_\mathbb{R} \times \mathbb{R}_+$ generated by $\Delta \times \{1\}$. We call $\Sigma(A,a) := c(\mathscr{C}(A,a))$ the *Gröbner fan of X in $N_\mathbb{R} \times \mathbb{R}_+$*.

The following result uses that the tropical cone $\mathrm{Trop}_W(X)$ is the closure of the cone in $N_\mathbb{R} \times \mathbb{R}_+$ generated by $\mathrm{Trop}_v(X) \times \{1\}$ which is proved in the next section (see Corollary 11.13). We will not use Corollary 10.17 and the following consequence in 10.18 in the next section.

Corollary 10.17 *The Gröbner fan $\Sigma(A,a)$ of X in $N_\mathbb{R} \times \mathbb{R}_+$ is a complete Γ-rational fan and $\Sigma_X := \{\sigma \in \Sigma(A,a) \mid \sigma \subset \mathrm{Trop}_W(X)\}$ is a subfan of $\Sigma(A,a)$ with support equal to the tropical cone $\mathrm{Trop}_W(X)$ from 8.3.*

Proof: Since $\mathscr{C}(A,a)$ is a complete Γ-rational polyhedral complex, it follows from Remark 7.6 that $\Sigma(A,a)$ is a complete Γ-rational fan in $N_\mathbb{R} \times \mathbb{R}_+$. Then the claim follows from Theorem 10.14 and Corollary 11.13 below. $\square$

10.18 By 9.16, $\mathscr{C}(A,a)$ is a pointed polyhedral complex if and only if $\mathrm{Stab}(\mathbf{y})$ is zero-dimensional. By definition of the torus action on the Hilbert scheme, we have $\mathrm{Stab}(\mathbf{y}) = \mathrm{Stab}(Y) = \mathrm{Stab}(X)$, where Y is the closure of X in $\mathbb{P}_K^m$. In general, it is clear that $\mathscr{C}(A,a)$ is isomorphic to the product of an affine space and the Gröbner complex of $X/\mathrm{Stab}(X)$. By the above, the latter is pointed and so it is obvious that $\mathscr{C}(A,a)$ has always a Γ-rational subdivision $\mathscr{C}$ consisting of pointed polyhedra. By Corollary 10.17, $c(\mathscr{C})$ is a Γ-admissible fan in $N_\mathbb{R} \times \mathbb{R}_+$ with support $\mathrm{Trop}_W(X)$.

11 Compactifications in toric schemes

Let K be a field with a non-archimedean absolute value $|\ |$, corresponding valuation $v := -\log|\ |$, valuation ring K°, residue field $\tilde{K}$ and value group $\Gamma = v(K^\times)$. Let $\mathbb{T}$ be the split torus over K° with generic fibre T associated to the character lattice M of rank n and dual lattice N. We keep the usual notation. In this section, we consider a closed subscheme X of T and we study its closure $\mathscr{X}$ in the toric scheme $\mathscr{Y}_\Sigma$ associated to a Γ-admissible fan Σ in $N_\mathbb{R} \times \mathbb{R}_+$ (see 7.5). First, we prove surjectivity of the reduction map which is called the tropical lifting lemma. Then, we show Tevelev's lemma which is a tropical characterization of the orbits intersecting $\mathscr{X}$. Finally, we give several equivalences for properness of the occurring schemes.

We start with a lemma due to Draisma.

Lemma 11.1 *Let (L,w) be a valued field extending (K,v) and let $r,s \in \mathbb{N}$. For $a_{ij}, b_i \in K$ and $\lambda_i \in \mathbb{R}$, we consider the following system of equalities*

$$a_{i1}x_1 + \cdots + a_{it}x_t = b_i \quad (1 \leq i \leq r)$$

and inequalities

$$w(a_{i1}x_1 + \cdots + a_{it}x_t) \geq \lambda_i \quad (r+1 \leq i \leq r+s).$$

If this system has a solution $y \in L^t$, then it has also a solution $z \in K^t$.

Proof: This follows from the same arguments as Lemma 4.3 in [Dra]. □

Lemma 11.2 *Let (L, w) be a valued field extending (K, v) and let $\mathscr{X}'$ be the closure of X_L in the toric scheme over L° associated to Σ. Then the canonical morphism $\phi : (\mathscr{X}')_s \to \mathscr{X}_s$ is surjective.*

Proof: We will first prove the claim if the value group Γ is a divisible subgroup of $\mathbb{R}$ and then we will reduce the claim to this special case in several steps.

Step 1: If the value group Γ is a divisible subgroup of $\mathbb{R}$, then ϕ is surjective.

In this case, we have seen in Proposition 7.12 that the toric scheme over L° associated to Σ is the base change of $\mathscr{Y}_\Sigma$ to L°. By Corollary 4.7, we have $\mathscr{X}' = \mathscr{X}_{L^\circ}$ and hence $(\mathscr{X}')_s$ is the base change of $\mathscr{X}_s$ to the residue field $\tilde{L}$. This yields surjectivity of ϕ.

In particular, this proves the claim for v trivial. We may assume that v is non-trivial and that $\mathscr{Y}_\Sigma = \mathscr{U}_\Delta$ for a pointed Γ-rational polyhedron Δ in $N_\mathbb{R}$. Let σ be the recession cone of Δ. Then X is given by an ideal I_X in $K[M]^\sigma$ and its closure $\mathscr{X}$ is given by the ideal $I_X \cap K[M]^\Delta$ in $K[M]^\Delta$. Similarly, $\mathscr{X}'$ is the closed subscheme given by the ideal $(I_X L[M]^\sigma) \cap L[M]^\Delta$ in $L[M]^\Delta$.

Step 2: The morphism ϕ is dominant.

Let $f \in K[M]^\Delta$ such that the residue class of f in $L[M]^\Delta / ((I_X L[M]^\sigma) \cap L[M]^\Delta)) \otimes_{\tilde{K}} \tilde{L}$ is zero. We have to prove that there is $m \in \mathbb{N}$ such that $f^m \in (I_X \cap K[M]^\Delta) + K^{\circ\circ} K[M]^\Delta$. By assumption, we have

$$f = g_1 h_1 + \cdots + g_r h_r + \lambda f_1 + \cdots + \lambda f_s \qquad (8)$$

with $g_i \in I_X$, $h_i \in L[M]^\sigma$, $\lambda \in L^{\circ\circ}$ and $f_j \in L[M]^\Delta$. We may assume that $h_i = \beta_i \chi^{u_i}$ for some $\beta_i \in L$ and $u_i \in \check{\sigma} \cap M$. Similarly, we may assume that $f_j = \gamma_j \chi^{v_j}$ for some $\gamma_j \in L$ and $v_j \in M$. Since the valuation v is non-trivial, there is $m \in \mathbb{N}$ such that λ^m is divisible by an element of K°. Replacing f by f^m, we may assume that $\lambda \in K^{\circ\circ}$. If we compare the coefficients on both sides of equation (8), then we get a finite system of linear equations with coefficients in K and unknowns $\beta_1, \ldots, \beta_r$ and $\gamma_1, \ldots, \gamma_s$. The conditions $f_j \in L[M]^\Delta$ are equivalent to the finitely many inequalities $v(\gamma_j) + \langle v_j, \omega \rangle \geq 0$, where ω ranges over the vertices of Δ. By assumption, this system of equalities and inequalities has a solution in L^{r+s}. By Lemma 11.1, there is a solution with $\beta_1, \ldots, \beta_r, \gamma_1, \ldots, \gamma_s \in K$ which means that we find a representation in (8) with all $h_i \in K[M]^\sigma$ and all $f_j \in K[M]^\Delta$. We conclude that $f \in (I_X \cap K[M]^\Delta) + K^{\circ\circ} K[M]^\Delta$ proving Step 2.

Step 3: If L is an algebraic closure of K, then ϕ induces a finite surjective map $(\mathscr{X}')_s \to \mathscr{X}_s \otimes_{\tilde{K}} \tilde{L}$.

We use first that the value group Γ_L of w is equal to $\{\lambda \in \mathbb{R} \mid \exists m \in \mathbb{N} \setminus \{0\}, \, m\lambda \in \Gamma\}$. It follows that the vertices of Δ are in N_{Γ_L} and there is a non-zero $m \in \mathbb{N}$ such that $m\omega \in N_\Gamma$ for every vertex ω of Δ. For every $u \in \check{\sigma} \cap M$, there is $\beta_u \in L$ with $v_\Delta(\beta_u \chi^u) = 0$. For each vertex ω of Δ, we choose a finite generating set of the semigroup $\check{\sigma}_\omega \cap M$, where σ_ω is the local cone of Δ at ω. We have seen in the proof of Proposition 6.7 that $L[M]^\Delta$ is generated as an L°-algebra by $\beta_u \chi^u$, where u ranges over the union S of all these generating sets.

We claim that the finite set $H := \{\prod_{u \in S} (\beta_u \chi^u)^{k_u} \mid 0 \leq k_u < m\}$ generates $L[M]^\Delta$ as a $K[M]^\Delta \otimes_{K^\circ} L^\circ$-module. Indeed, every $f \in L[M]^\Delta$ has the form $f = \sum_{h,k} \lambda_{hk} h \prod_{u \in S} (\beta_u \chi^u)^{mk_u}$ where h ranges over H, k over $\mathbb{N}^S$ and only finitely many coefficients $\lambda_{hk} \in L^\circ$ are non-zero. The construction of m yields that $\langle mu, \omega \rangle \in \Gamma$ for every vertex ω of Δ and hence there is $\alpha_u \in K$ with $v_\Delta(\alpha_u \chi^{mu}) = 0$. We

conclude that $\beta_u^m = \alpha_u \gamma_u$ for some $\gamma_u \in L^\circ$. Since $\alpha_u \chi^{mu} \in K[M]^\Delta$, this implies that $\prod_{u \in S}(\beta_u \chi^u)^{mk_u} \in L^\circ K[M]^\Delta$ proving that H generates the module $L[M]^\Delta$.

Since $(\mathscr{X}')_s$ is a closed subscheme of $\mathrm{Spec}(L[M]^\Delta)$ and since $\mathscr{X}_s \otimes_{\tilde{K}} \tilde{L}$ is a closed subscheme of $\mathrm{Spec}(K[M]^\Delta \otimes_{K^\circ} L^\circ)$, we conclude that $(\mathscr{X}')_s \to \mathscr{X}_s \otimes_{\tilde{K}} \tilde{L}$ is a finite map. It follows from Step 2 that this map is dominant and hence it is surjective proving Step 3.

We will now deduce the claim from Step 3. We endow an algebraic closure E of L with a valuation u extending w. Let F be the algebraic closure of K in E endowed with the restriction of u. Let $\mathscr{X}''$ (resp. $\mathscr{X}'''$) be the closure of X_F (resp. X_E) in the toric scheme over F° (resp. E°) associated to Δ. Then we have a commutative diagram

$$
\begin{array}{ccc}
(\mathscr{X}''')_s & \longrightarrow & \mathscr{X}_s \otimes_{\tilde{K}} \tilde{F} \\
\downarrow & & \downarrow \\
(\mathscr{X}')_s & \overset{\phi}{\longrightarrow} & \mathscr{X}_s
\end{array}
$$

of canonical morphisms. The first row has the factorization $(\mathscr{X}''')_s \to (\mathscr{X}'')_s \to \mathscr{X}_s \otimes_{\tilde{K}} \tilde{F}$ and hence it is surjective by Steps 1 and 3. Since the second column is surjective as well, we deduce that ϕ is surjective. □

Proposition 11.3 *The closure $\mathscr{X}$ of X in $\mathscr{Y}_\Sigma$ is a separated flat scheme over K°. The special fibre $\mathscr{X}_s$ is of finite type over $\tilde{K}$. If X is of pure dimension d and $\mathscr{X}_s$ is non-empty, then $\mathscr{X}_s$ is of pure dimension d. If the value group Γ is divisible, then $\mathscr{X}$ is of finite presentation over K°.*

Proof: It follows from Lemma 7.8 that $\mathscr{X}$ is separated over K°. Flatness is a consequence of 6.11 and 4.6. By Lemma 6.13, $\mathscr{X}_s$ is of finite type over $\tilde{K}$. The dimensionality claim is clear in case of a divisible value group Γ as in this case $\mathscr{X}$ is a flat scheme of finite type over K° (see Proposition 6.7). In this case, it follows from [RG], Corollaire 3.4.7, that $\mathscr{X}$ is of finite presentation over K°. In general, we will reduce to the divisible case: We may assume that v is non-trivial and that $\mathscr{Y}_\Sigma = \mathscr{U}_\Delta$ for a Γ-rational polyhedron Δ in $N_\mathbb{R}$. Then the dimensionality claim follows from Step 3 in the proof of Proposition 11.2. □

11.4 We recall from §4 that the reduction map is defined on an analytic subdomain $(Y_{\Sigma_0})^\circ$ of the generic fibre Y_{Σ_0} and maps to the special fibre of the K°-model $\mathscr{Y}_\Sigma$. The points of $(Y_{\Sigma_0})^\circ$ are induced by potentially integral points and Proposition 8.6 shows that $(Y_{\Sigma_0})^\circ \cap T^{\mathrm{an}} = \mathrm{trop}_v^{-1}(|\Sigma_1|)$. We conclude that the potentially integral points of X with respect to $\mathscr{X}$ induce an analytic subdomain $X^\circ = \mathrm{trop}_v^{-1}(|\Sigma|) \cap X^{\mathrm{an}}$ of X^{an} where we have a well-defined reduction map $\pi : X^\circ \to \mathscr{X}_s$.

We have here the following generalization of Jan Draisma's tropical lifting lemma (see [Dra], Lemma 4.4).

Proposition 11.5 *Using the above notation, we have $\pi(U^{\mathrm{an}} \cap X^\circ) = \mathscr{X}_s$ for every open dense subset U of X. Moreover, if K is algebraically closed and v is non-trivial, then every closed point of $\mathscr{X}_s$ is the reduction of a closed point of U.*

Proof: The additional difficulty here in contrast to Draisma's paper is that $\mathscr{X}$ and the ambient toric scheme $\mathscr{Y}_\Sigma$ might be not of finite type (see Proposition 6.9). Let L be an algebraic closure of K and let us choose a valuation u on L extending v. Let $\mathscr{X}'$ be the closure of X_L in the toric scheme over L° associated to the fan Σ. Then $\mathscr{X}'$ is a flat scheme of finite type over L° by Proposition 11.3. By Proposition

4.14, the reduction map $\pi_L : (U_L)^{\mathrm{an}} \cap (X_L)^\circ \to (\mathscr{X}')_s$ is surjective. We have a canonical commutative diagram

$$
\begin{array}{ccc}
(U_L)^{\mathrm{an}} \cap (X_L)^\circ & \xrightarrow{\ \pi_L\ } & \mathscr{X}'_s \\
\downarrow & & \downarrow \\
U^{\mathrm{an}} \cap X^\circ & \xrightarrow{\ \pi\ } & \mathscr{X}_s
\end{array}
$$

where the second column is surjective by Lemma 11.2. This proves surjectivity of π. The last claim follows directly from Proposition 4.14. $\hfill\square$

The following result is called *Tevelev's Lemma*. We will use the tropical cone $\mathrm{Trop}_W(X)$ and the notation of the previous section. The bijective correspondence between open faces and orbits from Proposition 8.8 will be important for the understanding of the following.

Lemma 11.6 *Let $\sigma \in \Sigma$. Then the orbit Z_σ corresponding to $\mathrm{relint}(\sigma)$ intersects $\mathscr{X}$ if and only if $\mathrm{Trop}_W(X) \cap \mathrm{relint}(\sigma)$ is non-empty.*

Proof: If $\omega \in \mathrm{Trop}_W(X) \cap \mathrm{relint}(\sigma)$, then there is $x \in X_W^{\mathrm{an}}$ with $\mathrm{trop}_W(x) = \omega$. Let $\pi_W : \mathrm{trop}_W^{-1}(|\Sigma|) \to \mathscr{Y}_\Sigma$ be the reduction map. We deduce from Proposition 8.8 that $\pi_W(x) \in Z_\sigma$. Since $\pi_W(x)$ is also contained in $\mathscr{X}$, we see that $\mathscr{X} \cap Z_\sigma$ is non-empty.

Suppose that $z \in \mathscr{X} \cap Z_\sigma$. By Proposition 11.5, π_W induces a surjective map $X_W^{\mathrm{an}} \cap \mathrm{trop}_W^{-1}(|\Sigma|) \to \mathscr{X}$ and hence there is $x \in X_W^{\mathrm{an}}$ with $z = \pi_W(x)$. Again by Proposition 8.8, we see that $\mathrm{trop}_W(x) \in \mathrm{relint}(\sigma)$. $\hfill\square$

Remark 11.7 We will give a procedure which can be often used to reduce questions about $\mathscr{X}$ to the case of the trivial valuation:

Let ω be a vertex of Σ_1. By 7.9 or Proposition 6.14, we have a corresponding irreducible component Y_ω of the toric scheme $\mathscr{Y}_\Sigma$. It is the closure of the orbit Z_ω corresponding to the vertex ω. By Proposition 6.15, Y_ω may be viewed as a normal toric variety over $\tilde{K}$ associated to the fan $\mathrm{LC}_\omega(\Sigma_1) := \{\mathrm{LC}_\omega(\Delta) \mid \Delta \in \Sigma_1\}$. Note that the acting torus is $T_\omega := \mathrm{Spec}(\tilde{K}[M_\omega])$, where $M_\omega := \{u \in M \mid \langle u, \omega \rangle \in \Gamma\}$ is a sublattice of M of finite index. To identify it with the dense open orbit Z_ω of Y_ω involves the choice of a basepoint in $Z_\omega(\tilde{K})$ which does not influence tropical varieties of closed subschemes of $Z_\omega(\tilde{K})$ as we deal with the trivial valuation on $\tilde{K}$.

We assume now that the vertex ω is also contained in $\mathrm{Trop}_v(X)$. In local problems involving ω, the relevant closed subscheme of T_ω is $X_\omega := \mathscr{X} \cap Z_\omega$. By Tevelev's Lemma 11.6, X_ω is non-empty and its closure $\mathscr{X}_\omega$ is contained in $\mathscr{X} \cap Y_\omega$. We claim that $\mathrm{Trop}_0(X_\omega)$ is the local cone of $\mathrm{Trop}_v(X)$ at ω.

To prove the claim, we note first that the induced reduced structure of the special fibre is compatible with base change by Lemma 6.13. As the tropical variety is also invariant under base change (Proposition 3.7), we may assume that $\omega \in N_\Gamma$. Then there is $t \in T(K)$ with $\omega = \mathrm{trop}_v(t)$ and we may choose the basepoint of Z_ω equal to $\pi(t)$. Using translation by t^{-1}, we conclude easily that X_ω is isomorphic to $\mathrm{in}_\omega(X)$ and hence the claim follows from Proposition 10.15.

Proposition 11.8 *For a Γ-admissible fan Σ in $N_\mathbb{R} \times \mathbb{R}_+$, the following conditions are equivalent:*

  (a) $|\Sigma| = N_\mathbb{R} \times \mathbb{R}_+$;

  (b) $|\Sigma_1| = N_\mathbb{R}$;

  (c) the special fibre of $\mathscr{Y}_\Sigma$ is non-empty and proper over $\tilde{K}$;

(d) $\mathscr{Y}_\Sigma$ is universally closed over K°.

If the equivalent conditions (a)–(d) hold, then the generic fibre of $\mathscr{Y}_\Sigma$ is also proper over K. If the value group Γ is divisible or discrete in $\mathbb{R}$, then (a)–(d) are equivalent to $\mathscr{Y}_\Sigma$ proper over K°.

Proof: Clearly, (a) and (b) are equivalent. Suppose that (a) holds. Then Σ_0 is a complete fan and hence the generic fibre Y_{Σ_0} of $\mathscr{Y}_\Sigma$ is complete (see [Ful2], §2.4). The special fibre of $\mathscr{Y}_\Sigma$ is the union of its finitely many irreducible components corresponding to the vertices ω_j of Σ_1 (see 7.9). Such an irreducible component is a toric variety with fan $\mathrm{LC}_\omega(\Sigma_1)$ generated by the local cones in ω_j (see Proposition 7.15). Since Σ_1 satisfies (b), all these fans are also complete and hence the irreducible components are proper over $\tilde{K}$. Proposition 11.3 shows that the special fibre is separated and of finite type over $\tilde{K}$. We conclude that $(\mathscr{Y}_\Sigma)_s$ is proper over $\tilde{K}$ by [EGA IV], Corollaire 5.4.5. This proves (a) $\Rightarrow$ (c).

Next, we show that (c) implies (b). If the special fibre $(\mathscr{Y}_\Sigma)_s$ is proper over $\tilde{K}$, then every irreducible component of $(\mathscr{Y}_\Sigma)_s$ is complete. As we have seen above, such an irreducible component Y is associated to a vertex ω of Σ_1 and Y is a toric variety over $\tilde{K}$ associated to the fan $\mathrm{LC}_\omega(\Sigma_1)$. By completeness of Y, the fan $\mathrm{LC}_\omega(\Sigma_1)$ is also complete (see [Ful2], §2.4) . As this holds for every vertex of Σ_1, we conclude that Σ_1 is complete. This proves (c) $\Rightarrow$ (b).

We have seen now that (a)–(c) are equivalent and that the generic fibre of $\mathscr{Y}_\Sigma$ is proper over K in this case. Now we will prove that (a) yields (d). So we assume that the equivalent conditions (a)–(c) hold. We note that $\mathscr{Y}_\Sigma$ is a quasi-compact separated scheme over K° and so we may apply the valuative criterion of universal closedness ([EGA II], Théorème 7.3.8) which holds also in the non-noetherian situation. Let L° be a valuation ring with fraction field L and suppose that we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(L) & \xrightarrow{\ \psi\ } & \mathscr{Y}_\Sigma \\
\downarrow & & \downarrow \\
\mathrm{Spec}(L^\circ) & \longrightarrow & \mathrm{Spec}(K^\circ)
\end{array}
$$

of morphisms. To prove universal closedness of $\mathscr{Y}_\Sigma$, the criterion says that it is enough to show that there is a morphism $g : \mathrm{Spec}(L^\circ) \to \mathscr{Y}_\Sigma$ over K° which factors through ψ. First, we assume that the homomorphism $K^\circ \to L^\circ$ is not injective. Then the kernel is the maximal ideal $K^{\circ\circ}$ of K° and ψ factors through the special fibre of $\mathscr{Y}_\Sigma$. It follows from (c) that the special fibre is proper over $\tilde{K}$ and hence the valuative criterion of properness gives the existence of g. So we may assume that $K^\circ \subset L^\circ$. Then the intersection of the maximal ideal $L^{\circ\circ}$ of L° with K° is either $\{0\}$ or $K^{\circ\circ}$. In the first case, we may replace the second column in the diagram by its generic fibre. We have seen at the beginning of the proof that the generic fibre of $\mathscr{Y}_\Sigma$ is proper over K. By the valuative criterion of properness again, we get the existence of g.

Finally, we have to consider the case $L^{\circ\circ} \cap K^\circ = K^{\circ\circ}$. In this case, L° is the valuation ring of a valuation w on L extending v. Since the value group Γ_L of w is a totally ordered abelian group, the polyhedra $\Delta \in \Sigma_1$ induce polyhedra $\Delta(\Gamma_L)$ in N_{Γ_L} and (b) yields that they are covering the whole space (see A.14). We note that ψ corresponds to an L-rational point P of the generic fibre $(\mathscr{Y}_\Sigma)_\eta = Y_{\Sigma_0}$ and we have to show that P is an L°-point of $\mathscr{Y}_\Sigma$. If P is contained in the dense orbit T, then $\mathrm{trop}_w(P) \in \Delta(\Gamma_L)$ for some $\Delta \in \Sigma_1$ and it follows from Proposition 8.6 that P extends to an L°-integral point of $\mathscr{Y}_\Sigma$. If P is contained in another orbit Z of Y_{Σ_0}, then this extension property holds as well by using Proposition 7.14 to reduce

to the previous case. This proves that $\mathscr{Y}_\Sigma$ is universally closed over K° and hence we get (a)–(c) $\Rightarrow$ (d).

Conversely, if (d) holds, then the special fibre is non-empty and universally closed over $\tilde{K}$. By Proposition 11.3, the special fibre is also separated and of finite type over $\tilde{K}$. We conclude that $(\mathscr{Y}_\Sigma)_s$ is proper over $\tilde{K}$. This proves (d) $\Rightarrow$ (c) and hence all four properties are equivalent. If the value group Γ is divisible or discrete in $\mathbb{R}$, Proposition 11.3 shows that $\mathscr{Y}_\Sigma$ is separated and of finite type over K°. If (d) holds, then it follows that $\mathscr{Y}_\Sigma$ is proper over K°. $\qquad\square$

11.9 We consider now another free abelian group M' of finite rank with dual N' and split multiplicative torus $T' = \mathrm{Spec}(K[M'])$ over K. Then a Γ-admissible fan Σ' in $N'_\mathbb{R} \times \mathbb{R}_+$ induces a toric scheme $\mathscr{Y}_{\Sigma'}$ over K° with dense orbit T'. We assume that $f : N' \to N$ is a homomorphism such that $f_\mathbb{R} \times \mathrm{id}_{\mathbb{R}_+}$ maps each cone σ' of Σ' into a suitable cone σ of Σ. Then the dual homomorphism of f induces a canonical equivariant morphism $\mathscr{V}_{\sigma'} \to \mathscr{V}_\sigma$. We can patch these homomorphisms together to get an equivariant morphism $\varphi : \mathscr{Y}_{\Sigma'} \to \mathscr{Y}_\Sigma$ of toric schemes over K° which is canonically determined by f through the fact that φ restricts to the homomorphism $T' \to T$ of tori induced by f.

Proposition 11.10 *Under the hypothesis above, the following properties are equivalent:*

(a) *the morphism φ is closed with generic fibre φ_η and special fibre φ_s both proper;*

(b) $(f_\mathbb{R} \times \mathrm{id}_{\mathbb{R}_+})^{-1}(|\Sigma|) = |\Sigma'|$;

(c) *the morphism φ is universally closed.*

If the value group Γ is divisible or discrete in $\mathbb{R}$, then (a)–(c) are also equivalent to φ proper over K°.

Proof: We assume that (a) holds. By the criterion of properness for homomorphisms of toric varieties over a field ([Ful2], §2.4), we have $f^{-1}(|\Sigma_0|) = |\Sigma'_0|$. To prove (b), it remains to see that $f^{-1}(|\Sigma_1|) = |\Sigma'_1|$. Let $\omega' \in N_\mathbb{R}$ with $f(\omega') \in |\Sigma_1|$. There is $t' \in (T')^{\mathrm{an}}$ with $\mathrm{trop}_v(t') = \omega'$ and hence $\mathrm{trop}_v(t) = f(\omega') \in |\Sigma_1|$ for $t := \varphi^{\mathrm{an}}(t')$. By Proposition 8.6, we have $t \in Y^\circ_{\Sigma_0}$ and hence we have a well-defined reduction $\pi(t)$ in the special fibre of the closure of $\varphi(T')$ in $\mathscr{Y}_\Sigma$. Since φ is closed, we have $\pi(t) = \varphi_s(z')$ for some $z' \in (\mathscr{Y}_{\Sigma'})_s$. By Proposition 11.5, there is $t'_0 \in (T')^{\mathrm{an}} \cap Y^\circ_{\Sigma'_0}$ with reduction $\pi(t'_0) = z'$. Again Proposition 8.6 shows that $\omega'_0 := \mathrm{trop}_v(t'_0) \in \Sigma'_1$. We have $\pi(\varphi^{\mathrm{an}}(t'_0)) = \varphi_s(\pi(t'_0)) = \pi(t)$. The orbit correspondence in Proposition 8.8 yields that $f(\omega'_0) = \mathrm{trop}_v(\varphi^{\mathrm{an}}(t'_0))$ is in the same open face τ of Σ_1 as $f(\omega') = \mathrm{trop}_v(t)$.

Arguing by contradiction, we assume that $\omega' \notin |\Sigma'_1|$. We consider now the closed segment $[\omega'_0, \omega']$ in $N'_\mathbb{R}$. Let ω'_1 be the point of $[\omega'_0, \omega'] \cap |\Sigma'_1|$ which is closest to ω'. Then ω'_1 is contained in an open face τ' of Σ'_1. Let ω'_2 be a vertex of $\overline{\tau'}$. Using $[\omega'_1, \omega'] \cap |\Sigma'_1| = \{\omega'_1\}$ and moving ω' sufficiently close to ω'_1, we may assume also that

$$[\omega'_2, \omega'] \cap |\Sigma'_1| = \{\omega'_2\}. \tag{9}$$

Now we use the notation from Proposition 7.15 and we apply this result two times. The irreducible component $Y_{\omega'_2}$ corresponding to the vertex ω'_2 is the toric variety over $\tilde{K}$ associated to the fan $\mathrm{LC}_{\omega'_2}(\Sigma'_1)$. The closure $\overline{Z}$ of the orbit Z of $(\mathscr{Y}_\Sigma)_s$ corresponding to τ is the toric variety over $\tilde{K}$ associated to the fan in $N(\overline{\tau})_\mathbb{R}$ which is given by the projections of $\mathrm{LC}_\tau(\Sigma_1) = \{\mathrm{LC}_\tau(\nu) \mid \nu \in \Sigma_1, \nu \supset \tau\}$ to $N(\overline{\tau})_\mathbb{R}$. We have an equivariant morphism $\varphi_{\omega'_2} : Y_{\omega'_2} \to \overline{Z}$ induced by φ_s. We deduce from (9) that $|\mathrm{LC}_{\omega'_2}(\Sigma'_1)|$ is a proper subset of $f_\mathbb{R}^{-1}(|\mathrm{LC}_\tau(\Sigma_1)|)$. By the criterion of properness

for homomorphisms of toric varieties over a field (see [Ful2], §2.4), $\varphi_{\omega'_2}$ is not proper. This contradicts properness of φ_s. We conclude that (a) implies (b).

To prove the converse, we assume that (b) holds. We get $f^{-1}(|\Sigma_0|) = |\Sigma'_0|$ and hence φ_η is proper again by the criterion in [Ful2], §2.4. By 7.9, the irreducible components of $(\mathscr{Y}_{\Sigma'})_s$ correspond to the vertices ω' of Σ_1. Moreover, let Z be the orbit of $(\mathscr{Y}_\Sigma)_s$ corresponding to the open face τ of Σ_1 containing $f(\omega')$. As above, we get an equivariant morphism $\varphi_{\omega'} : Y_{\omega'} \to \overline{Z}$ of toric varieties over $\overline{K}$. It follows from (b) that $f_{\mathbb{R}}^{-1}(|\mathrm{LC}_\tau(\Sigma_1)|) = |\mathrm{LC}_{\omega'}(\Sigma'_1)|$. The criterion in [Ful2], §2.4, shows that φ_ω is proper. As this holds for any irreducible component $Y_{\omega'}$, we get properness of φ_s. Indeed, it follows from Proposition 11.3 that φ_s is separated and of finite type and so we may use [EGA II], Corollaire 5.4.5, to deduce properness of φ_s.

It remains to see that $\varphi(\mathscr{X}')$ is closed for any closed subset $\mathscr{X}'$ of $\mathscr{Y}_{\Sigma'}$. We may assume that $\mathscr{X}'$ is irreducible. Since φ_s is proper, we may also assume that $\mathscr{X}'$ is the closure of a closed subvariety X' of $(\mathscr{Y}_{\Sigma'})_\eta$. Using Proposition 7.14, we may reduce the claim to the case $X' \cap T' \neq \emptyset$. Since φ_η is proper, the generic fibre of $\varphi(\mathscr{X}')$ is a closed subvariety X of $\mathscr{Y}_\Sigma$. It remains to show that any point z in the special fibre of the closure $\mathscr{X}$ of $\varphi(\mathscr{X}')$ is contained in $\varphi(\mathscr{X}')$. By Proposition 11.5, the reduction map $\pi : X^\circ \cap T^{\mathrm{an}} \to \mathscr{X}_s$ is surjective. We conclude that $z = \pi(x)$ for some $x \in X^\circ \cap T^{\mathrm{an}}$ and hence $\mathrm{trop}_v(x) \in |\Sigma_1|$ by the orbit correspondence in Proposition 8.8. There is $x' \in (X')^{\mathrm{an}}$ with $\varphi^{\mathrm{an}}(x') = x$. By Chevalley's theorem, $\varphi_\eta(X' \cap T')$ is a constructible dense subset of X and hence contains an open dense subset of X. We conclude that we may choose x and x' in the above argument such that $x' \in (T')^{\mathrm{an}}$ additionally. Using $f(\mathrm{trop}_v(x')) = \mathrm{trop}_v(\varphi^{\mathrm{an}}(x')) = \mathrm{trop}_v(x) \in |\Sigma_1|$, our assumption (b) on the fans leads to $\mathrm{trop}_v(x') \in |\Sigma'_1|$. By Proposition 8.6, we have $x' \in (X')^\circ$ and hence its reduction $z' := \pi(x')$ is well-defined in $(\mathscr{X}')_s$. We get $\varphi(z') = \varphi_s \circ \pi(x') = \pi \circ \varphi^{\mathrm{an}}(x') = z$. This proves $z \in \varphi(\mathscr{X}')$ and therefore the morphism φ is closed. We conclude that (b) implies (a).

Next we prove that (a) $\Rightarrow$ (c). We will use similar arguments as for (a) $\Rightarrow$ (c) in the proof of Proposition 11.8. By the above, we may assume that the equivalent properties (a) and (b) hold. We note that φ is a quasi-compact separated morphism (see Proposition 11.3) and so we may apply the valuative criterion of universal closedness ([EGA II], Théorème 7.3.8). Let L° be a valuation ring with fraction field L and suppose that we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(L) & \xrightarrow{\ \psi\ } & \mathscr{Y}_{\Sigma'} \\
\downarrow & & \downarrow{\varphi} \\
\mathrm{Spec}(L^\circ) & \xrightarrow{\ h\ } & \mathscr{Y}_\Sigma
\end{array}
$$

of morphisms. The criterion says that the morphism φ is universally closed, if there is a morphism $g : \mathrm{Spec}(L^\circ) \to \mathscr{Y}_{\Sigma'}$ over h which factors through ψ. If h maps the generic point of $\mathrm{Spec}(L^\circ)$ into the special fibre of $\mathscr{Y}_\Sigma$, then we may replace the second column in the diagram by the special fibre φ_s and the existence of g follows from the valuative criterion of properness for φ_s. If h maps the closed point of $\mathrm{Spec}(L^\circ)$ into the generic fibre of $\mathscr{Y}_{\Sigma'}$, then we may replace the second column in the diagram by the generic fibre φ_η and the existence of g follows from the valuative criterion of properness for φ_η. So we may assume that h maps the generic point of $\mathrm{Spec}(L^\circ)$ into the generic fibre and the special point of $\mathrm{Spec}(L^\circ)$ into the special fibre. Then $L^{\circ\circ} \cap K^\circ = K^{\circ\circ}$ and it follows that L° is the valuation ring of a valuation w on L extending v. The morphism ψ corresponds to an L-rational point P of the generic fibre $(\mathscr{Y}_{\Sigma'})_\eta = Y_{\Sigma'_0}$ and we have to show that P is an L°-point of $\mathscr{Y}_{\Sigma'}$. It follows from Proposition 7.14 again that we may assume $P \in T(L)$. Using that $\varphi(P)$ is an L°-integral point of $\mathscr{Y}_\Sigma$, Proposition 8.6 implies that $f_{\Gamma_L}(\mathrm{trop}_w(P)) = \mathrm{trop}_w(\varphi(P)) \in \Delta(\Gamma_L)$ for some $\Delta \in \Sigma_1$. By (b), we have

$f^{-1}(|\Sigma_1|) = |\Sigma'_1|$. It follows from A.15 that $\mathrm{trop}_w(P) \in \Delta'$ for some $\Delta' \in \Sigma_1$. Again Proposition 8.6 shows that P is an L°-integral point of $\mathscr{Y}_{\Sigma'}$. This proves (a) $\Rightarrow$ (c).

Conversely, we prove (c) $\Rightarrow$ (a). If (c) holds, then the generic and the special fibre of φ are universally closed. As both of them are separated and of finite type (use Proposition 11.3), they are proper and hence we get (a). If the value group Γ is divisible or discrete in $\mathbb{R}$, Proposition 11.3 implies that φ is separated and of finite type over K°. If (c) holds, then φ is proper over K°. $\qquad\square$

We have the following generalization of the above proposition to the case of a closed subscheme:

Proposition 11.11 *Using the assumptions and notations from 11.10, let X' be a closed subscheme of T' with closure $\mathscr{X}'$ in $\mathscr{Y}_{\Sigma'}$ and let ϕ be the restriction of the equivariant morphism $\varphi : \mathscr{Y}_{\Sigma'} \to \mathscr{Y}_\Sigma$ to $\mathscr{X}'$. Then the following conditions are equivalent:*

(a) $\mathrm{Trop}_W(X') \cap (f_\mathbb{R} \times \mathrm{id}_{\mathbb{R}_+})^{-1}(|\Sigma|) = \mathrm{Trop}_W(X') \cap |\Sigma'|$;

(b) $\phi : \mathscr{X}' \to \mathscr{Y}_\Sigma$ is universally closed.

If the equivalent conditions (a) and (b) hold, then the generic fibre of ϕ is proper. If Γ is divisible or discrete in $\mathbb{R}$, then (a) and (b) are equivalent to ϕ proper.

Proof: We assume (a) and we will show (b): Similarly as in [BS], Proposition 3.15, one can construct a Γ-admissible fan Σ'' which subdivides $(f \times \mathrm{id}_{\mathbb{R}_+})^{-1}(\Sigma)$ and such that Σ'' has a subfan Σ''' which subdivides Σ'. We get a commutative diagram

$$
\begin{array}{ccc}
\mathscr{Y}_{\Sigma'''} & \longrightarrow & \mathscr{Y}_{\Sigma''} \\
\downarrow{\scriptstyle \varphi'''} & & \downarrow{\scriptstyle \varphi''} \\
\mathscr{Y}_{\Sigma'} & \xrightarrow{\;\varphi\;} & \mathscr{Y}_\Sigma
\end{array}
$$

of canonical equivariant morphisms. Since Σ''' is a subfan of Σ'', it is obvious that $\mathscr{Y}_{\Sigma'''}$ is an open subset of $\mathscr{Y}_{\Sigma''}$. By Proposition 11.10, the morphisms φ'' and φ''' are universally closed. By Tevelev's Lemma 11.6, the assumption (a) yields that the closure $\mathscr{X}''$ of X' in $\mathscr{Y}_{\Sigma''}$ is contained in the open subset $\mathscr{Y}_{\Sigma'''}$. It follows that $\mathscr{X}''$ is the closure of X' in $\mathscr{Y}_{\Sigma'''}$. Using that φ''' is closed and restricts to the identity on the dense orbit T', we get $\varphi'''(\mathscr{X}'') = \mathscr{X}'$. Let $\phi'' : \mathscr{X}'' \to \mathscr{Y}_\Sigma$ (resp. $\phi''' : \mathscr{X}'' \to \mathscr{X}'$) be the restriction of φ'' (resp. φ''') to the closed subscheme $\mathscr{X}''$. Since φ'' is universally closed, the same is true for ϕ''. We have the factorization $\phi'' = \phi \circ \phi'''$. Using that the surjective morphism φ''' remains surjective after base change ([EGA I], Proposition 3.5.2(ii)) and that ϕ'' is universally closed, we deduce universal closedness of ϕ. This proves (a) $\Rightarrow$ (b).

We consider now the property

(a') $\mathrm{Trop}_v(X') \cap f_\mathbb{R}^{-1}(|\Sigma_1|) = \mathrm{Trop}_v(X') \cap |\Sigma'_1|$.

If we would have assumed (a') instead of (a) in the above argument, then we could still show that the special fibre of $\mathscr{X}''$ is contained in $\mathscr{Y}_{\Sigma'''}$ and the same proof would show that the special fibre of ϕ is universally closed. This will be important in the proof of Proposition 11.12.

We assume now that (b) holds and we will prove first (a'). It is clear that $\mathrm{Trop}_v(X') \cap f_\mathbb{R}^{-1}(|\Sigma_1|) \supset \mathrm{Trop}_v(X') \cap |\Sigma'_1|$ and we have to prove the reverse inclusion. Let $\omega' \in \mathrm{Trop}_v(X) \cap f_\mathbb{R}^{-1}(|\Sigma_1|)$. By Remark 2.2, there is a valued field (L, w) extending (K, v) and an L-rational point P of X' such that $\mathrm{trop}_w(P) = \omega'$. We have $\mathrm{trop}_w(\phi(P)) = f(\mathrm{trop}_w(P)) = f(\omega') \in |\Sigma_1|$ and hence $\phi(P)$ is an L°-integral point of $\mathscr{Y}_\Sigma$ (see Proposition 8.6). We note that ϕ is a quasi-compact separated

morphism (see Proposition 11.3) and hence we may apply the valuative criterion of universal closedness ([EGA II], Théorème 7.3.8). It follows that P is an L°-integral point of $\mathscr{X}'$. By Proposition 8.6, this means $\omega' = \mathrm{trop}_w(P) \in |\Sigma_1'|$ proving (a').

Now we show (b) $\Rightarrow$ (a). We have seen that (b) implies (a'). If we apply this with the trivial valuation instead of v, we get

$$\mathrm{Trop}_0(X') \cap f_{\mathbb{R}}^{-1}(|\Sigma_0|) = \mathrm{Trop}_0(X') \cap |\Sigma_0'|.$$

This together with (a') implies that (a) holds.

Finally, we assume that Γ is divisible or discrete in $\mathbb{R}$. It follows from Proposition 11.3 and [EGA I], Proposition 5.5.1, that ϕ is a separated morphism of finite type. Then (b) yields that ϕ is proper. $\qquad\square$

Proposition 11.12 *Let X be a closed subscheme of T. For the closure $\mathscr{X}$ of X in $\mathscr{Y}_\Sigma$, the following are equivalent:*

(a) $\mathrm{Trop}_W(X) \subset |\Sigma|$;

(b) $\mathrm{Trop}_v(X) \subset |\Sigma_1|$;

(c) $\mathscr{X}$ *is universally closed over* K°.

If X is geometrically connected, then (a)–(c) are equivalent to:

(d) the special fibre of $\mathscr{X}$ is non-empty and proper over $\tilde{K}$.

If the value group Γ is divisible or discrete in $\mathbb{R}$, then (a)–(c) are equivalent to:

(e) $\mathscr{X}$ *is proper over* K°.

If (a) holds, then the generic fibre of $\mathscr{X}$ is proper over K.

Proof: It follows from Proposition 11.11 that (a) and (c) are equivalent and that they imply properness of the generic fibre $\mathscr{X}_\eta$ over K. If the value group Γ is divisible or discrete in $\mathbb{R}$, the same result shows that (a) and (c) are also equivalent to (e). If we apply this with the trivial valuation, then we get the following result:

Step 1: Suppose that the valuation v is trivial. Then the closure $\mathscr{X}$ of X in the toric variety Y_{Σ_0} associated to the rational fan Σ_0 in $N_{\mathbb{R}}$ is proper if and only if $\mathrm{Trop}_0(X) \subset |\Sigma_0|$.

Obviously, (a) yields (b). We assume now (b) and we will show (d). We have seen in the proof of Proposition 11.11 that (b) (which is a special case of (a') there) implies (c). In particular, the special fibre $\mathscr{X}_s$ is non-empty and universally closed over $\tilde{K}$. Since $\mathscr{X}_s$ is always separated and of finite type over $\tilde{K}$ (see Proposition 11.3), we conclude that $\mathscr{X}_s$ is proper over $\tilde{K}$. This proves (b) $\Rightarrow$ (d) without assuming irreducibility of X or completeness of K.

For the converse, we assume X geometrically connected. By the first step, the converse holds for the trivial valuation and so we may assume that v is non-trivial complete valuation. Arguing by contradiction, we assume that (d) holds and that $\mathrm{Trop}_v(X)$ is not a subset of $|\Sigma_1|$. Since the special fibre of $\mathscr{X}$ is non-empty, Tevelev's Lemma 11.6 yields that $\mathrm{Trop}_v(X)$ intersects $|\Sigma_1|$. Since $\mathrm{Trop}_v(X)$ is a connected finite union of Γ-rational polyhedra (see Theorem 3.3 and Proposition 3.5), there is $\omega \in \mathrm{Trop}_v(X) \cap |\Sigma_1|$ such that $\Omega \cap \mathrm{Trop}_v(X)$ is not contained in $|\Sigma_1|$ for every neighbourhood Ω of ω. Moreover, we may assume $m \cdot \omega \in N_\Gamma$ for some non-zero $m \in \mathbb{N}$. Then there is a Γ-admissible subdivision Σ' of Σ such that ω is a vertex of Σ_1'. Let $\mathscr{X}'$ be the closure of X in $\mathscr{Y}_{\Sigma'}$. By Proposition 11.10, the canonical $\mathbb{T}$-equivariant morphism $\varphi : \mathscr{Y}_{\Sigma'} \to \mathscr{Y}_\Sigma$ is closed and the special fibre φ_s is proper.

This shows $\varphi(\mathscr{X}') = \mathscr{X}$ and that the special fibre of $\mathscr{X}'$ is also non-empty and proper over $\tilde{K}$. To simplify the notation, we may assume that $\mathscr{X} = \mathscr{X}'$.

By Remark 11.7, $X_\omega := \mathscr{X} \cap Z_\omega$ is a closed subscheme of the dense orbit Z_ω of the toric variety Y_ω over $\tilde{K}$ with $\mathrm{Trop}_0(X_\omega)$ equal to the local cone of $\mathrm{Trop}_v(X)$ at ω. This means that $\mathrm{Trop}_0(X_\omega)$ is not contained in the fan $\mathrm{LC}_\omega(\Sigma_1)$ of the toric variety Y_ω. By Step 1, we conclude that the closure $\mathscr{X}_\omega$ of X_ω in Y_ω is not proper over $\tilde{K}$. On the other hand, $\mathscr{X}_\omega$ is a closed subscheme of $\mathscr{X} \cap Y_\omega$. Since the special fibre of $\mathscr{X}$ is assumed to be proper over $\tilde{K}$, this has to be true also for its closed subscheme $\mathscr{X}_\omega$. This is a contradiction and hence (d) implies (b).

Next, we assume X geometrically connected and that (d) holds. We will show that the generic fibre $\mathscr{X}_\eta$ is proper over K. By base extension and Proposition 3.7, we may assume that K is an algebraically closed complete field and hence the value group Γ is divisible in $\mathbb{R}$. Then $\mathscr{X}$ is flat, separated and of finite presentation over K° (see Proposition 11.3). Using that $\mathscr{X}_s$ is non-empty, we deduce that $\mathscr{X}$ is faithfully flat over K. We have seen in 4.9 that the reduction map $\pi : \mathscr{X}_\eta^\circ \to \mathscr{X}_s$ is defined on the compact analytic subdomain $\mathscr{X}_\eta^\circ$ of $\mathscr{X}_\eta^{\mathrm{an}}$. We have already seen that (d) yields (b) and hence $\mathscr{X}_\eta^\circ \cap T^{\mathrm{an}} = X^\circ = \mathrm{trop}_v^{-1}(|\Sigma_1|) \cap X^{\mathrm{an}} = X^{\mathrm{an}}$ implies that $\mathscr{X}_\eta^\circ = \mathscr{X}_\eta^{\mathrm{an}}$. Since $\mathscr{X}_\eta$ is connected and K is complete, $\mathscr{X}_\eta^{\mathrm{an}}$ is connected (see [Berk1], Theorem 3.4.8 and Theorem 3.5.3). By anticontinuity of the reduction map π, we deduce that $\mathscr{X}_s$ is connected. Since $\tilde{K}$ is algebraically closed as well ([BGR], Lemma 3.4.1/4), we conclude that the special fibre $\mathscr{X}_s$ is geometrically connected. The same holds obviously for the generic fibre $\mathscr{X}_\eta$. We conclude that all assumptions of [EGA IV], Corollaire 15.7.11, are satisfied and this result shows that $\mathscr{X}_\eta$ is proper over K.

Finally, we prove that (b) yields (a). By base change and Proposition 3.7, we may assume that K is algebraically closed and that X is irreducible. We have seen that (b) yields (d) and the above shows that $\mathscr{X}_\eta$ is proper over K. By Step 1, we get $\mathrm{Trop}_0(X) \subset |\Sigma_0|$. Moreover, (b) yields $\mathrm{Trop}_{\varepsilon v}(X) \subset |\Sigma_\varepsilon|$ for every $\varepsilon > 0$ and so we conclude $\mathrm{Trop}_W(X) \subset |\Sigma|$. $\qquad\square$

Corollary 11.13 *Let X be a closed subscheme of T. Then $\mathrm{Trop}_W(X)$ is the closure of the cone in $N_\mathbb{R} \times \mathbb{R}_+$ which is generated by $\mathrm{Trop}_v(X) \times \{1\}$.*

Proof: By base change and Proposition 3.7, we may assume K algebraically closed and X irreducible. Let C be the closure of the cone in $N_\mathbb{R} \times \mathbb{R}_+$ generated by $\mathrm{Trop}_v(X) \times \{1\}$. It follows easily from the definitions that C agrees with $\mathrm{Trop}_W(X)$ on $N_\mathbb{R} \times (0, \infty)$ and hence $C \subset \mathrm{Trop}_W(X)$ using that $\mathrm{Trop}_W(X)$ is closed in $N_\mathbb{R} \times \mathbb{R}_+$ (see Proposition 8.5). The Gröbner complex $\mathscr{C}(A, a)$ from 10.13 is a complete Γ-rational polyhedral complex in $N_\mathbb{R}$. We have seen in Theorem 10.14 that $\mathscr{C}(A, a)$ has a subcomplex $\mathscr{C}$ with support $\mathrm{Trop}_v(X)$. It follows from 10.18 that $\mathscr{C}$ has a pointed Γ-rational subdivision $\mathscr{C}'$. By Remark 7.6, $\Sigma := c(\mathscr{C}')$ is a Γ-admissible fan in $N_\mathbb{R} \times \mathbb{R}_+$ with support C. Since $\Sigma_1 = \mathrm{Trop}_v(X)$, Proposition 11.12 yields that $\mathrm{Trop}_W(X) \subset |\Sigma| = C$ and we get the claim. $\qquad\square$

12 Tropical compactifications

We keep the notation from Section 11, where we have studied the closure $\mathscr{X}$ of a closed subscheme X of T in the toric scheme $\mathscr{Y}_\Sigma$ over K°. In this section, we study tropical compactifications $\mathscr{X}$ related to certain fans Σ supported on the tropical cone $\mathrm{Trop}_W(X)$ introduced in Section 8. This generalizes results of Tevelev who handled the case of an integral X over an algebraically closed field with trivial valuation (see [Tev]) and of Qu who obtained some results in the case of a discrete valuation (see [Qu]). Their definition of a tropical fan seems simpler, but our

definition is better suited to handle the case of a non-reduced X and the definitions agree in the case of reduced closed subschemes.

Let (K, v) be an arbitrary valued field and let X be any closed subscheme of T.

Definition 12.1 A Γ-*admissible tropical fan* for X is a Γ-admissible fan Σ in $N_\mathbb{R} \times \mathbb{R}_+$ such that $\mathrm{Trop}_v(X) \subset |\Sigma_1|$ and such that there is a closed subscheme $\mathscr{F}$ of $\mathbb{T} \times_{K^\circ} \mathscr{Y}_\Sigma$ with the following properties:

(a) The second projection induces a faithfully flat map $f : \mathscr{F} \to \mathscr{Y}_\Sigma$.

(b) The map $\Phi : \mathbb{T} \times_{K^\circ} \mathscr{Y}_\Sigma \to \mathbb{T} \times_{K^\circ} \mathscr{Y}_\Sigma, (t, y) \mapsto (t^{-1}, t \cdot y)$ maps $T \times_K X$ isomorphically onto $f^{-1}(T)$.

In this case, we call the closure $\mathscr{X}$ of X in $\mathscr{Y}_\Sigma$ a *tropical compactification* of X.

Remark 12.2 Let us consider just multiplication $m : T \times_K X \to T$. Then this is isomorphic to the trivial fibre bundle $X \times_K T$ over T. The isomorphism is given by $(t, x) \mapsto (x, t \cdot x)$. In particular, we see that m is faithfully flat. A tropical fan asks for extension of faithful flatness for p_2 from $\Phi(T \times_K X)$ to a closed subscheme $\mathscr{F}$ of $\mathbb{T} \times_{K^\circ} \mathscr{Y}_\Sigma$.

If Σ is a Γ-admissible tropical fan, then it follows from flatness that the open subset $f^{-1}(T)$ of $\mathscr{F}$ is dense. Using the isomorphism Φ, we get $(\Phi^{-1}(\mathscr{F}))_{\mathrm{red}} = \mathbb{T} \times_{K^\circ} \mathscr{X}_{\mathrm{red}}$ since the right hand side is reduced by [EGA IV], Proposition 17.5.7. We conclude that the multiplication map $m : \mathbb{T} \times_{K^\circ} \mathscr{X}_{\mathrm{red}} \to \mathscr{Y}_\Sigma$ is surjective and has the same topological properties as f.

If X is reduced, then the closure $\mathscr{X}$ is reduced. If we assume additionally that Γ is divisible or discrete in $\mathbb{R}$, then Proposition 11.12 implies that Σ is a Γ-admissible tropical fan if and only if $\mathscr{X}$ is proper and the multiplication map induces a faithfully flat map $\mathbb{T} \times_{K^\circ} \mathscr{X} \to \mathscr{Y}_\Sigma$. Hence our definition is the same as Tevelev's definition of a tropical fan for varieties over a trivially valued algebraically closed field.

We can now generalize Tevelev's result to our framework:

Theorem 12.3 *Let $\Sigma(A, a)$ be the Gröbner fan for X in $N_\mathbb{R} \times \mathbb{R}_+$ and let Σ_X be the subcomplex with support $\mathrm{Trop}_W(X)$ as in Corollary 10.17. Then every Γ-admissible fan Σ which subdivides Σ_X is a Γ-admissible tropical fan for X. In particular, Γ-admissible tropical fans exist for every closed subscheme X of T.*

Proof: We keep the notation introduced in Section 10 about Gröbner complexes. Let $\mathscr{Y}_{A,a}$ be the orbit closure of $\mathbf{y} := [\overline{X}]$ in $\mathrm{Hilb}(\mathbb{P}^m_{K^\circ})$. Since the Γ-admissible fan Σ subdivides the subcomplex Σ_X of $\Sigma(A, a)$, the canonical morphism $T \to T\mathbf{y}$ between the dense orbits extends to a $\mathbb{T}$-equivariant morphism $\varphi : \mathscr{Y}_\Sigma \to \mathscr{Y}_{A,a}$ (see 11.9). Indeed, φ is given on T by $t \mapsto (\chi^u(t)y_u)_{u \in A} \in \mathbb{P}^N_K$ (see 9.2) and it is easy to see that this extends to the desired morphism.

We consider the closed subscheme $\mathscr{G} := (\mathrm{id} \times \varphi)^{-1}(\mathrm{Univ}(\mathbb{P}^m_{K^\circ}))$ of $\mathbb{P}^m_{K^\circ} \times_{K^\circ} \mathscr{Y}_\Sigma$ which is flat over $\mathscr{Y}_\Sigma$. The fibre $\mathscr{G}_y$ over $y \in T$ is equal to $y^{-1}\overline{X} \subset \mathbb{P}^m_K$. This makes it easy to check that

$$h : \mathscr{G}|_T \to \overline{X} \times_K T, \quad (z, y) \mapsto (y \cdot z, y) \tag{10}$$

is an isomorphism over T. Let $\mathscr{F}$ be the restriction of $\mathscr{G}$ to $\mathbb{T} \times_{K^\circ} \mathscr{Y}_\Sigma$. Then the second projection restricts to a flat morphism $f : \mathscr{F} \to \mathscr{Y}_\Sigma$. Moreover, axiom (b) from the definition of a tropical fan follows from (10). By Corollary 10.17, $\mathrm{Trop}_W(X)$ is the support of Σ_X and hence also from its subdivision Σ. By Tevelev's Lemma 11.6, we conclude that every orbit intersects $\mathscr{X}$ and hence the multiplication map $m : \mathbb{T} \times_{K^\circ} \mathscr{X}_{\mathrm{red}} \to \mathscr{Y}_\Sigma$ is surjective. By the same argument as in Remark 12.2,

we conclude that f is surjective and hence faithfully flat. This means that Σ is a tropical fan for X. Finally, we have seen in 10.18 that a Γ-admissible fan exists which subdivides Σ_X. $\qquad\square$

Proposition 12.4 *Let Σ be a Γ-admissible tropical fan for X and let Σ' be a Γ-rational fan which subdivides Σ. Then Σ' is a Γ-admissible tropical fan for X.*

Proof: Since Σ' is a subdivision of Σ, we have $\mathrm{Trop}_v(X) \subset |\Sigma_1| = |\Sigma_1'|$. By 11.9, we get a canonical $\mathbb{T}$-equivariant morphism $\varphi : \mathscr{Y}_{\Sigma'} \to \mathscr{Y}_\Sigma$ which is the identity on the dense open orbits T. Let us define the closed subscheme $\mathscr{G}$ of $\mathbb{T} \times_{K^\circ} \mathscr{Y}_{\Sigma'}$ by the following Cartesian diagram:

$$
\begin{array}{ccc}
\mathscr{G} & \xrightarrow{\ f'\ } & \mathscr{Y}_{\Sigma'} \\
\downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} \\
\mathscr{F} & \xrightarrow{\ f\ } & \mathscr{Y}_\Sigma
\end{array}
$$

Since f' is obtained from f by base change, we conclude that f' is faithfully flat. Since we have $(\varphi')^{-1}(f^{-1}(T)) = (f')^{-1}(T)$ and φ is the identity on T, we deduce easily axiom (b) from the definition of a tropical fan. This proves the claim. $\qquad\square$

Proposition 12.5 *Every Γ-admissible tropical fan for X in $N_\mathbb{R} \times \mathbb{R}_+$ has support equal to $\mathrm{Trop}_W(X)$.*

Proof: It follows from Proposition 11.12 that the support of a Γ-admissible tropical fan Σ contains $\mathrm{Trop}_W(X)$. We have to show that every $\sigma \in \Sigma$ is contained in $\mathrm{Trop}_W(X)$. We argue by contradiction and so we assume that σ is not contained in Σ. Passing to a subdivision and using Proposition 12.4, we may assume that σ is disjoint from $\mathrm{Trop}_W(X)$. It follows from Lemma 11.6 that the tropical compactification $\mathscr{X}$ is disjoint from the orbit corresponding to $\mathrm{relint}(\sigma)$. We conclude that the multiplication map $m : \mathbb{T} \times_{K^\circ} \mathscr{X}_{\mathrm{red}} \to \mathscr{Y}_\Sigma$ is not surjective. This contradicts Remark 12.2. $\qquad\square$

Proposition 12.6 *Let X be a pure dimensional closed subscheme of T and let Σ be a Γ-admissible tropical fan for X with tropical compactification $\mathscr{X}$ of X. If Z is any torus orbit in the generic (resp. special) fibre of $\mathscr{Y}_\Sigma$, then $Z \cap \mathscr{X}$ is a non-empty pure dimensional scheme over K (resp. $\tilde{K}$) with*

$$
\dim(Z \cap \mathscr{X}) = \dim(X) + \dim(Z) - n.
$$

In particular, Z intersects the generic (resp. special) fibre of $\mathscr{X}$ properly.

Proof: By Proposition 11.3, the special fibre of $\mathscr{X}$ is also pure dimensional of the same dimension as X. By flatness of f and Remark 12.2, the multiplication map $m : \mathbb{T} \times_{K^\circ} \mathscr{X} \to \mathscr{Y}_\Sigma$ has pure dimensional fibres of constant fibre dimension. By Remark 12.2, this fibre dimension is equal to $\dim(X)$. For a closed point $z \in Z$, the fibre $m^{-1}(z) = \{(t,x) \in \mathbb{T} \times_{K^\circ} \mathscr{X} \mid t \cdot x = z\}$ projects onto $\mathscr{X} \cap Z$. The fibres of this projection are isomorphic to $\mathrm{Stab}(z)$ and hence they have dimension $n - \dim(Z)$. By the fibre dimension theorem, we get

$$
\dim(X) = \dim(m^{-1}(z)) = \dim(Z \cap \mathscr{X}) + n - \dim(Z)
$$

proving the claim. The fibre dimension theorem yields also that $Z \cap \mathscr{X}$ is pure dimensional. $\qquad\square$

For a tropical fan and $z \in (\mathscr{Y}_\Sigma)_s$, Remark 12.2 shows $(m^{-1}(z))_{\mathrm{red}} \cong f^{-1}(z)_{\mathrm{red}}$; this is closely related to a certain initial degeneration, as we will see in the next remark.

Remark 12.7 Let Σ be a Γ-admissible tropical fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let $z \in (\mathscr{Y}_\Sigma)_s$. By the tropical lifting lemma (Proposition 11.5), there is $y \in T^{\mathrm{an}} \cap Y_{\Sigma_0}^{\circ}$ with reduction $\pi(y) = z$. There is a valued field (L, w) extending (K, v) such that y is an L-rational point of T in the sense of Remark 2.2. Let $f : \mathscr{F} \to \mathscr{Y}_\Sigma$ be the faithfully flat family from Definition 12.1. We claim that $f^{-1}(z)_{\tilde{L}}$ is isomorphic to the special fibre of the closure of $y^{-1}X$ in $\mathbb{T}_{L^{\circ}}$. This means that $\mathrm{in}_{\mathrm{trop}_w(y)}(X)$ is represented by the embedding $f^{-1}(z) \to \mathbb{T}_{\tilde{L}}$ given by the first projection.

To prove the claim, we may assume $K = L$. By flatness of f, the closure of $f^{-1}(y)$ is equal to $f^{-1}(\overline{y})$ (see Corollary 4.5 and Remark 4.6). We restrict the flat family f to the closure $\overline{y}$ of y in $\mathscr{Y}_\Sigma$. The generic fibre of this restriction is $f^{-1}(y)$ which is isomorphic to $y^{-1}X$ using the first projection of $T \times_K Y_{\Sigma_0}$ and axiom (b) in 12.1. Note that the first projection also gives a closed embedding of $f^{-1}(\overline{y})$ into $\mathbb{T}$ and hence the special fibre $f^{-1}(z)$ is isomorphic to the special fibre of the closure of $y^{-1}X$ in $\mathbb{T}$ as claimed.

Proposition 12.8 *Let Σ be a Γ-admissible tropical fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ and let z be an F-rational point of $\mathscr{Y}_\Sigma$ for a field F. If we use the first projection to identify the following fibres of f with closed subschemes of $\mathbb{T}_F$ as in Remark 12.7, then we have $f^{-1}(sz) = s^{-1} \cdot f^{-1}(z)$ for all $s \in \mathbb{T}(F)$.*

Proof: Note that F is either an extension of K or of the residue field $\tilde{K}$. It is enough to consider the case $F/\tilde{K}$ as we may deduce the case F/K from the previous one by using the trivial valuation on K. There is a valued field (L, w) extending (K, v) such that the residue field $\tilde{L}$ contains F. Since all the objects are defined over F, it is enough to show $f^{-1}(sz) = s^{-1} \cdot f^{-1}(z)$ over $\tilde{L}$. Let $t \in \mathbb{T}(L^{\circ})$ be a lift of s, i.e. $\pi(t) = s$. By enlarging L, we may assume that $z = \pi(y)$ for some $y \in T(L)$ (see Proposition 11.5). Using Remark 12.7, we see that the special fibre of the closure of $(ty)^{-1}(X)$ is equal to $f^{-1}(sz)$. On the other hand, multiplication with t^{-1} induces an automorphism of $\mathbb{T}$ and hence is compatible with taking closures. This automorphism is given on the special fibre by multiplication with s^{-1} and hence we get the claim. $\qquad\square$

Corollary 12.9 *Let Σ be a Γ-admissible tropical fan in $N_{\mathbb{R}} \times \mathbb{R}_+$. Suppose that $\omega, \nu \in \mathrm{relint}(\sigma_1)$ for some $\sigma \in \Sigma$. Then we have $\mathrm{in}_\omega(X) = \mathrm{in}_{\omega'}(X)$.*

Proof: This follows immediately from the orbit correspondence (Proposition 8.8), Remark 12.7 and Proposition 12.8. $\qquad\square$

Proposition 12.10 *Let σ be a cone of the Γ-admissible tropical fan Σ in $N_{\mathbb{R}} \times \mathbb{R}_+$. For every $\omega \in \sigma_1$ and every $\omega' = \omega + \Delta\omega \in \mathrm{relint}(\sigma_1)$, we have*

$$\mathrm{in}_{\omega'}(X) = \mathrm{in}_{\Delta\omega}(\mathrm{in}_\omega(X)),$$

where the initial degeneration at $\Delta\omega$ is with respect to the trivial valuation.

Proof: We have seen in Corollary 10.12 that the identity holds in a neighbourhood of ω. Now the claim follows from Corollary 12.9. $\qquad\square$

13 Tropical multiplicities

In this section, X is a closed subscheme of T. We will define a tropical multiplicity function on the tropical variety $\mathrm{Trop}_v(X)$. It will be used to define $\mathrm{Trop}_v(X)$ as a tropical cycle, i.e. a weighted polyhedral complex satisfying the balancing condition. This appeared first in Speyer's thesis [Spe]. The balancing condition relies on the description of the Chow cohomology of a toric variety given by Fulton–Sturmfels

[FS]. This is very implicit in the presentation here as we reduce the claim to the case of the trivial valuation where the balancing condition of $\mathrm{Trop}_v(X)$ is a result of Sturmfels and Tevelev based on [FS]. Further references: [AR], [BPR], [ST].

Definition 13.1 A point ω of $\mathrm{Trop}_v(X)$ is called *regular* if there is a polytope $\sigma \subset \mathrm{Trop}_v(X)$ such that $\mathrm{relint}(\sigma)$ is a neighbourhood of ω in $\mathrm{Trop}_v(X)$.

Proposition 13.2 *A point ω of $\mathrm{Trop}_v(X)$ is regular if and only if 0 is regular in $\mathrm{Trop}_0(\mathrm{in}_\omega(X))$.*

Proof: This follows immediately from Proposition 10.15. $\qquad\square$

13.3 For $\omega \in N_\mathbb{R}$, we have seen that $\mathrm{in}_\omega(X)$ is a closed subscheme of the special fibre of $\mathbb{T}$ defined over a field extension of the residue field and it is well-defined up to multiplication with elements $g \in \mathbb{T}$ which are rational over a possibly larger field extension. Let F be an algebraically closed field extension over which $\mathrm{in}_\omega(X)$ is defined. Then the irreducible components of $\mathrm{in}_\omega(X)$ over F are also irreducible components over every field extension of F and hence the following definition makes sense.

Definition 13.4 The *tropical multiplicity* $m(\omega, X)$ of $\omega \in N_\mathbb{R}$ is defined as the sum of the multiplicities of the irreducible components of $\mathrm{in}_\omega(X)$ in $\mathrm{in}_\omega(X)$ over the algebraically closed field F. For a cycle $Z = \sum m_Y Y$ of T with prime components Y, we define the *tropical multiplicity of Z in ω* by $m(\omega, Z) := \sum_Y m_Y m(\omega, Y)$.

We have defined the initial degeneration as an equivalence class of closed subschemes up to multiplication by torus elements over an extension of the residue field (see 5.4). In the next result, we form the cycle of an initial degeneration. This means that we consider cycles up to the obvious linear extension of the above equivalence relation from prime components to all cycles. Note that the following result is a special case of [OP], Theorem 4.4.5. Here, we give a different proof using intersection theory with Cartier divisors. We have to deal with the fact that the models are usually non-noetherian and hence we cannot use algebraic intersection theory, but there is an analytic replacement introduced in [Gub1].

Lemma 13.5 *Let $\mathrm{cyc}(X) = \sum_Y m_Y Y$ be the representation of the cycle associated to X as a sum of its irreducible components Y counted with multiplicities. Then we have*
$$\mathrm{cyc}(\mathrm{in}_\omega(X)) = \sum_Y m_Y \mathrm{cyc}(\mathrm{in}_\omega(Y)).$$

Proof: By base change, we may assume that v is non-trivial and that K is an algebraically closed complete field such that all occurring initial degenerations are defined over $\tilde{K}$. Moreover, we may suppose that $\omega = \mathrm{trop}_v(t)$ for some $t \in T(K)$. Replacing X by $t^{-1}X$, we may assume $t = e$ and $\omega = 0$. Then $\mathrm{in}_\omega(X)$ is the special fibre $\mathscr{X}_s$ of the closure $\mathscr{X}$ of X in $\mathbb{T}$.

By Proposition 6.7, we have $\mathscr{X} = \mathrm{Spec}(A)$ for a flat K°-algebra of finite type. Let us choose a non-zero $\nu \in K^{\circ\circ}$. We have seen in 4.13 that the ν-adic completion $\hat{A}$ of A is a flat K°-algebra which is topologically of finite type, i.e. $\hat{\mathscr{X}} := \mathrm{Spf}(\hat{A})$ is an admissible formal affine scheme over K° in the theory of Raynaud, Bosch and Lütkebohmert (see [BL], §1). Its generic fibre is defined as the Berkovich spectrum $\mathscr{M}(\mathscr{A})$ of the strictly affinoid algebra $\mathscr{A} := \hat{A} \otimes_{K^\circ} K$ and it is equal to the affinoid subdomain X° of X^{an} from 4.9. If T° is the formal affinoid torus constructed in the same way from T, then we have $X^\circ = X^{\mathrm{an}} \cap T^\circ$.

Using that $\mathscr{A}$ is a noetherian algebra, we have a theory of cycles and Cartier divisors on X° (see [Gub1], §2, for details). Hence we have a cycle decomposition

$\mathrm{cyc}(X^\circ) = \sum_{W \in S} m_W W$ for a finite set S of prime cycles of X°. If Y is an irreducible component of X, then the GAGA principle shows that Y° is a closed reduced analytic subvariety of X°, but Y° is not necessarily irreducible. Hence we have $\mathrm{cyc}(Y^\circ) = \sum_{W \in S_Y} W$ for a subset S_Y of S. It is clear that S is the disjoint union of the sets S_Y. By [Gub1], Proposition 6.3, we have $m_Y = m_W$ for all $W \in S_Y$. Moreover, it is obvious that $S_Y \neq \emptyset$ if and only if Y meets the affinoid torus T° and the latter is equivalent to $\mathrm{in}_0(Y) \neq \emptyset$. By [Gub2], Lemma 4.5, we have

$$\mathrm{cyc}(\mathscr{X}_s) = \sum_{W \in S} m_W \mathrm{cyc}(\overline{W}_s) \tag{11}$$

where $\overline{W}_s$ is the special fibre of the closure $\overline{W}$ of W in $\hat{\mathscr{X}}$. Similarly, we get

$$\mathrm{cyc}(\overline{Y}_s) = \sum_{W \in S_Y} \mathrm{cyc}(\overline{W}_s) \tag{12}$$

where $\overline{Y}$ is the closure of Y in $\mathscr{X}$. Using (11), (12) and the above facts, we get

$$\mathrm{cyc}(\mathrm{in}_0(X)) = \sum_Y \sum_{W \in S_Y} m_Y \mathrm{cyc}(\overline{W}_s) = \sum_Y m_Y \mathrm{cyc}(\overline{Y}_s) = \sum_Y m_Y \mathrm{cyc}(\mathrm{in}_0(Y))$$

proving the claim. $\qquad\square$

Proposition 13.6 *Tropical multiplicities have the following properties:*

(a) *They are invariant under base change of X or Z to valued field (L, w) extending (K, v).*

(b) *The tropical multiplicity $m(\omega, Z)$ is linear in the cycle Z.*

(c) *For the cycle $\mathrm{cyc}(X)$ associated to X, we have $m(\omega, X) = m(\omega, \mathrm{cyc}(X))$.*

Proof: Property (a) follows from Proposition 5.5 and (b) is obvious. Finally, (c) follows from Lemma 13.5. $\qquad\square$

The following result shows that we may compute tropical multiplicities locally over the trivially valued residue field.

Proposition 13.7 *For $\omega_0 \in N_\mathbb{R}$, there is a neighbourhood Ω of ω_0 in $N_\mathbb{R}$ such that $m(\omega, X) = m(\omega - \omega_0, \mathrm{in}_{\omega_0}(X))$ for all $\omega \in \Omega$.*

Proof: This follows from Corollary 10.12. $\qquad\square$

We have now the setup to generalize the following result of Sturmfels–Tevelev, which was given in the case of trivial valuations.

Theorem 13.8 *The restriction of the tropical multiplicity function $m(\cdot, X)$ to the open subset of regular points in $\mathrm{Trop}_v(X)$ is locally constant.*

Proof: By Proposition 13.7, we reduce to the case of a trivially valued base field. By Proposition 13.6, we may assume that base field is algebraically closed and that X is an irreducible subvariety. Then the claim follows from [ST], Corollaries 3.8 and 3.15. $\qquad\square$

13.9 A Γ-rational polyhedral complex $\mathscr{C}$ in $N_\mathbb{R}$ is called *of pure dimension d* if every maximal $\sigma \in \mathscr{C}$ has dimension d. Such a complex is called *weighted* if it is endowed with a *multiplicity function m* which maps every d-dimensional $\sigma \in \mathscr{C}$ to a number $m_\sigma \in \mathbb{Z}$.

A polyhedron $\sigma \in \mathscr{C}$ generates an affine space in $N_\mathbb{R}$ which is a translate of a linear space $\mathbb{L}_\sigma$. By Γ-rationality of σ, the vector space $\mathbb{L}_\sigma$ is defined over $\mathbb{Q}$ and $N_\sigma := \mathbb{L}_\sigma \cap N$ is a lattice in $\mathbb{L}_\sigma$.

We say that a weighted Γ-rational polyhedral complex $\mathscr{C}$ in $N_\mathbb{R}$ of pure dimension d satisfies the *balancing condition* if for every $d - 1$-dimensional polyhedron ν, we have

$$\sum_{\sigma \supset \nu} m_\sigma n_{\sigma,\nu} \in N_\nu,$$

where σ ranges over all d-dimensional polyhedra of $\mathscr{C}$ containing ν, and $n_{\sigma,\nu}$ is any representative of the generator of the 1–dimensional lattice N_σ/N_ν pointing in the direction of σ.

A weighted Γ-rational polyhedral complex $\mathscr{C}$ in $N_\mathbb{R}$ of pure dimension d is called a *tropical cycle* if it satisfies the balancing condition. We identify tropical cycles if there is a common Γ-rational subdivision of both complexes for which the multiplicities coincide. This allows us to add tropical cycles. In general, we define a Γ-rational tropical cycle $\mathscr{C}$ in $N_\mathbb{R}$ as a formal sum $\mathscr{C} = \sum_{j=0}^{n} \mathscr{C}_j$, where $\mathscr{C}_j$ is a tropical cycle in $N_\mathbb{R}$ of pure dimension j. For details about tropical cycles, we refer to [AR].

13.10 We suppose that X is a pure-dimensional closed subscheme of T and we set $d := \dim(X)$. Let $\mathscr{C}$ be any Γ-rational polyhedral complex with support equal to $\mathrm{Trop}_v(X)$. By Theorem 10.14, we know that such complexes exist. Theorem 3.3 shows that $\mathscr{C}$ is of pure dimension d. Note that the relative interior of a d-dimensional polyhedron $\sigma \in \mathscr{C}$ is contained in the regular part of $\mathrm{Trop}_v(X)$. By Theorem 13.8, the multiplicity function $m(\cdot, X)$ is constant on $\mathrm{relint}(\sigma)$ and this constant is denoted by m_σ. We call m_σ the *tropical multiplicity* of σ.

Theorem 13.11 *Under the hypothesis of 13.10, the complex $\mathscr{C}$ endowed with the tropical multiplicities is a Γ-rational tropical cycle of pure dimension d.*

Proof: The balancing condition is a local condition in any $\omega \in N_\mathbb{R}$. By Propositions 10.15 and 13.7, it is enough to check the balancing condition for $\mathrm{in}_\omega(X)$ in a neighbourhood of 0. Hence we have reduced the claim to the case of trivial valuation. Again, we may assume that the base field is algebraically closed and that X is an irreducible subvariety. This case is proved in [ST], Corollary 3.8. $\square$

Remark 13.12 It follows from Theorem 13.8 that the tropical cycle from Theorem 13.11 does not depend on the choice of the complex $\mathscr{C}$ from 13.10. We conclude that $\mathrm{Trop}_v(X)$ is canonically a tropical cycle which we denote also by $\mathrm{Trop}_v(X)$.

If X is any closed subscheme of T, then we define $\mathrm{Trop}_v(X)$ by linearity in its irreducible components, i.e. we set $\mathrm{Trop}_v(X) := \sum_Y m_Y \mathrm{Trop}_v(Y)$ as a tropical cycle, where m_Y is the multiplicity of X in the irreducible component Y. This is a tropical cycle in $N_\mathbb{R}$ with support equal to the set-theoretic tropical variety of X. By Proposition 13.6, this agrees with the above construction in the pure dimensional case.

If $Z = \sum_Y m_Y Y$ is any cycle on X with prime components Y, then we define the *tropical cycle associated to* Z by $\mathrm{Trop}_v(Z) := \sum_Y m_Y \mathrm{Trop}_v(Y)$, where we use the induced reduced structure on every Y.

Proposition 13.13 *Let X be any closed subscheme of T and let $\omega \in N_\mathbb{R}$. Replacing the polyhedra in the tropical cycle $\mathrm{Trop}_v(X)$ by its local cones in ω and using the same tropical multiplicities, we get a tropical cycle in $N_\mathbb{R}$ which is equal to the tropical cycle $\mathrm{Trop}_0(\mathrm{in}_\omega(X))$ with respect to the trivial absolute value 0.*

Proof: This follows from Proposition 13.7 and Proposition 10.15. $\square$

13.14 Let T' be another split torus over K with lattice N' of one–parameter–subgroups. Let $\varphi : T \to T'$ be a homomorphism of split tori over K. This induces a homomorphism $M' \to M$ of character lattices and hence we get a linear map $N \to N'$. The base change of this map to $\mathbb{R}$ is easily seen to be the unique map $\mathrm{Trop}_v(\varphi) : N_\mathbb{R} \to N'_\mathbb{R}$ such that $\mathrm{Trop}_v(\varphi) \circ \mathrm{trop}_v = \mathrm{trop}_v \circ \varphi$.

13.15 The push-forward of a cycle Z on X with respect to the homomorphism φ is a cycle $\varphi_*(Z)$ on the closure X' of $\varphi(X)$ defined in the following way: If Z is a prime cycle and Z' is the closure of $\varphi(Z)$, then

$$\varphi_*(Z) := \begin{cases} [K(Z) : K(Z')]Z', & \text{if } [K(Z) : K(Z')] < \infty, \\ 0, & \text{if } [K(Z) : K(Z')] = \infty. \end{cases}$$

In general, $\varphi_*(Z)$ is defined by linearity in its prime components.

Usually, the push-forward of cycles is defined with respect to proper morphisms. This could be easily obtained by using tropical compactifications as in Section 12, but as we are not interested in compatibility with rational equivalence of cycles, this plays no role here.

13.16 We will explain how the linear map $f := \mathrm{Trop}_v(\varphi) : N_\mathbb{R} \to N'_\mathbb{R}$ induces a push–forward map of tropical cycles. For details, we refer to [AR], §7. Let $\mathscr{C}$ be a tropical cycle in $N_\mathbb{R}$ of pure dimension d. After a subdivision of $\mathscr{C}$, we may assume that

$$f_*(\mathscr{C}) := \{f(\sigma) \mid \sigma \text{ is a face of } \nu \in \Sigma \text{ with } \dim(f(\nu)) = d\}$$

is a (d-dimensional Γ-rational) polyhedral complex in $N'_\mathbb{R}$. We define the multiplicity of a d-dimensional $f(\sigma) \in f_*(\mathscr{C})$ by

$$m_{f(\sigma)} := \sum_{\nu \subset f^{-1}(f(\sigma))} [N_\nu : N_{f(\sigma)}]m_\nu,$$

where ν ranges over all d-dimensional $\nu \in \mathscr{C}$ contained in $f^{-1}(f(\sigma))$. Endowed with these multiplicities, we get a weighted polyhedral complex which is a tropical cycle in $N'_\mathbb{R}$. It might happen that $f_*(\mathscr{C})$ is empty, then we get the tropical zero-cycle.

The following result is the Sturmfels–Tevelev multiplicity formula (see [ST]). It was generalized to the case of non-trivial valuations in [BPR], Corollary 8.4 and Appendix A.

Theorem 13.17 *Let $\varphi : T \to T'$ be a homomorphism of split tori over K and let Z be cycle on T. Then we have*

$$\mathrm{Trop}_v(\varphi)_*(\mathrm{Trop}_v(Z)) = \mathrm{Trop}_v(\varphi_*(Z))$$

as an identity of tropical cycles.

Proof: By base change, we may assume that K is an algebraically closed field with a complete non-trivial valuation. Using linearity of the identity in the prime components of Z, we may assume that $X = Z$ is an integral closed subscheme of T. If $\dim(X') < \dim(X)$ for $X' := \varphi(X)$, then $\varphi_*(X) = 0$ by definition. Since $\mathrm{Trop}_v(X)$ is a polyhedral complex of pure dimension $d := \dim(X)$ and since $\varphi_*(\mathrm{Trop}_v(X))$ is a tropical cycle supported in $\mathrm{Trop}_v(X')$ which is of lower dimension, we conclude $\varphi_*(\mathrm{Trop}_v(X)) = 0$ as well. So we may assume that φ induces a generically finite map $X \to X'$ and then we may deduce the claim from [BPR], Corollary 8.4. $\square$

14   Proper intersection with orbits

As usual, (K, v) is a valued field which serves as a ground field. Let Σ be a Γ-admissible fan in $N_{\mathbb{R}} \times \mathbb{R}_+$ with associated toric scheme $\mathscr{Y}_{\Sigma}$ over K°. Let X be a closed subscheme of the dense torus T with closure $\mathscr{X}$ in $\mathscr{Y}_{\Sigma}$. We have seen in Proposition 12.6 that $\mathscr{X}$ intersects the orbits of $\mathscr{Y}_{\Sigma}$ properly in case of a tropical fan. In this section, we will generalize this result and we prove that this property is a purely combinatorial property of the fan Σ. I am very grateful to Sam Payne for explaining to me some of the arguments for this nice result.

14.1 Let $\sigma \in \Sigma_1$ and $\tau = \mathrm{relint}(\sigma)$. Then σ generates an affine space in $N_{\mathbb{R}}$ which is a translate of a linear space $\mathbb{L}_{\sigma}$. By Γ-rationality of σ, the vector space $\mathbb{L}_{\sigma}$ is defined over $\mathbb{Q}$. Then $N_{\sigma} := N \cap \mathbb{L}_{\sigma}$ and $N(\sigma) := N/N_{\sigma}$ are free abelian groups of finite rank with quotient homomorphism $\pi_{\sigma} : N \to N(\sigma)$. Dually, we have $M(\sigma) := \mathbb{L}_{\sigma}^{\perp} \cap M = \mathrm{Hom}(N(\sigma), \mathbb{Z})$.

For $S \subset N_{\mathbb{R}}$, we define the *local cone of S at τ* by

$$\mathrm{LC}_{\tau}(S) := \bigcup_{\omega \in \tau} \mathrm{LC}_{\omega}(S)$$

using the local cones at points from A.6. If S is a polyhedron containing τ, then we have $\mathrm{LC}_{\tau}(S) = \mathrm{LC}_{\omega}(S)$ for any $\omega \in \tau$.

14.2 We recall from 7.9 that τ corresponds to an orbit $Z = Z_{\tau}$ of the special fibre of $\mathscr{Y}_{\Sigma}$. By choosing a base point $z_0 \in Z(\tilde{K})$, Proposition 7.15 shows that Z may be identified with the torus $\mathrm{Spec}(\tilde{K}[M(\sigma)_{\tau}])$ for the sublattice $M(\sigma)_{\tau} := \{u \in M(\sigma) \mid \langle u, \omega \rangle \in \Gamma \; \forall \omega \in \tau\}$ of finite index in $M(\sigma)$. We get tropical varieties of closed subschemes of Z with respect to the trivial valuation which do not depend on the choice of the base point z_0. This is used in the following result which generalizes Remark 11.7.

Proposition 14.3 *Using the notions from above, we have*

$$\mathrm{Trop}_0(\mathscr{X} \cap Z_{\tau}) = \pi_{\sigma}(\mathrm{LC}_{\tau}(\mathrm{Trop}_v(X))).$$

Proof: By base change and Lemma 11.2, we may assume that Γ is divisible and hence we have $M(\sigma) = M(\sigma)_{\tau}$. We assume first that Σ has a tropical subfan for X (i.e. a subfan Σ' which is a tropical fan for X, see Definition 12.1 and A.5). If $\tau \cap \mathrm{Trop}_v(X)$ is empty, then Tevelev's Lemma 11.6 shows that both sides of the claim are empty. By Proposition 12.5, we have $\mathrm{Trop}_v(X) = |\Sigma_1'|$ and so we may assume that $\tau \subset \mathrm{Trop}_v(X)$. We choose $\omega \in \tau \cap N_{\Gamma}$. By translation, we may assume that $\omega = 0$ and therefore the affine toric scheme $\mathscr{U}_{\omega}$ from 6.11 is just the split torus $\mathbb{T}$ over K°. We conclude that $\mathrm{in}_{\omega}(X)$ is the special fibre of the closure of X in $\mathscr{U}_{\omega}$. To identify Z_{τ} with $T(\sigma) := \mathrm{Spec}(\tilde{K}[M(\sigma)])$, we choose the base point z_0 of Z_{τ} as the reduction of the unit element in $T(K)$. Then the canonical quotient homomorphism $q : \mathbb{T}_{\tilde{K}} \to T(\sigma)$ of tori over $\tilde{K}$ maps $\mathrm{in}_{\omega}(X)$ into $\mathscr{X} \cap Z_{\tau}$. Since τ is an open face of a tropical subfan of Σ, the proof of Proposition 12.6 and Remark 12.7 show that $\mathrm{in}_{\omega}(X) = q^{-1}(\mathscr{X} \cap Z_{\tau})$ holds set theoretically. This yields

$$\pi_{\sigma}(\mathrm{Trop}_0(\mathrm{in}_{\omega}(X))) = \mathrm{Trop}_0(\mathscr{X} \cap Z_{\tau}). \tag{13}$$

By Proposition 10.15, we have $\mathrm{Trop}_0(\mathrm{in}_{\omega}(X)) = \mathrm{LC}_{\omega}(\mathrm{Trop}_v(X))$. Since $\mathrm{Trop}_v(X)$ is a finite union of polyhedra which either contain τ or are disjoint from τ, we get $\mathrm{LC}_{\omega}(\mathrm{Trop}_v(X)) = \mathrm{LC}_{\tau}(\mathrm{Trop}_v(X))$. Inserting these facts into (13), we get the proposition.

Now we prove the proposition in the case of an arbitrary Γ-admissible fan Σ. The affine toric scheme $\mathscr{U}_{\sigma}$ associated to the closure σ of τ is an open subset of

$\mathscr{Y}_\Sigma$ containing Z_τ. The claim in the proposition depends only on $\mathscr{U}_\sigma$ and hence we may change Σ outside of σ. So we may assume that Σ_1 is a complete Γ-rational polyhedral complex containing σ. By Theorem 12.3 and Proposition 12.4, there is a Γ-admissible fan Σ' which is a subdivision of Σ and which has a tropical subfan. Then we have a canonical $\mathbb{T}$-equivariant morphism $\varphi : \mathscr{Y}_{\Sigma'} \to \mathscr{Y}_\Sigma$ of $\mathbb{T}$-toric schemes over K°. It follows from Proposition 11.10 that φ is closed and surjective. For the closure $\mathscr{X}'$ of X in $\mathscr{Y}_{\Sigma'}$, we conclude that $\varphi(\mathscr{X}') = \mathscr{X}$. Relevant for our purposes is that τ has a subdivision into open faces $\tau_1, \ldots, \tau_r$ of Σ'. The orbit correspondence in Proposition 8.8 leads to the partition of $\varphi^{-1}(Z_\tau)$ into the orbits $Z_{\tau_1}, \ldots, Z_{\tau_r}$. We conclude that $\mathscr{X} \cap Z_\tau$ is the union of the sets $\varphi(\mathscr{X}' \cap Z_{\tau_i})$. Let σ_i be the closure of τ_i and let $\pi_i : N(\sigma_i) \to N(\sigma)$ be the canonical homomorphism. Then we get

$$\mathrm{Trop}_0(\mathscr{X} \cap Z_\tau) = \bigcup_{i=1}^r \mathrm{Trop}_0(\varphi(\mathscr{X}' \cap Z_{\tau_i})) = \bigcup_{i=1}^r \pi_i(\mathrm{Trop}_0(\mathscr{X}' \cap Z_{\tau_i})). \quad (14)$$

Using the special case above, we have

$$\mathrm{Trop}_0(\mathscr{X}' \cap Z_{\tau_i}) = \pi_{\sigma_i}(\mathrm{LC}_{\tau_i}(\mathrm{Trop}_v(X))).$$

Inserting this in (14) and using $\pi_i \circ \pi_{\sigma_i} = \pi_\sigma$, we get the claim. $\qquad\square$

For simplicity, we assume now that the closed subscheme X of T is of pure dimension d. By the Bieri–Groves Theorem 3.3, there is a finite set S of d-dimensional Γ-rational polyhedra in $N_\mathbb{R}$ such that $\mathrm{Trop}_v(X) = \bigcup_{\Delta \in S} \Delta$.

Corollary 14.4 *Under the hypothesis above, we have*

$$\dim(\mathscr{X} \cap Z_\tau) = d - \inf\{\dim(\Delta \cap \tau) \mid \Delta \in S, \, \Delta \cap \tau \neq \emptyset\}.$$

Proof: By Tevelev's Lemma 11.6, $\mathscr{X} \cap Z_\tau$ is empty if and only if no $\Delta \in S$ intersects τ. We see that the claim holds in this special case as the dimension of the empty set is defined as $-\infty$ and the infimum over an empty set is ∞. Proposition 14.3 shows that we have

$$\mathrm{Trop}_0(\mathscr{X} \cap Z_\tau) = \bigcup_{\Delta \in S} \pi_\sigma(\mathrm{LC}_\tau(\Delta)). \quad (15)$$

For $\Delta \in S$ with $\Delta \cap \tau \neq \emptyset$, we have $\dim(\pi_\sigma(\mathrm{LC}_\tau(\Delta))) = d - \dim(\Delta \cap \tau)$. Using this in (15), we get the claim. $\qquad\square$

Remark 14.5 We recall from 7.9 that the open faces τ of Σ_0 correspond to the orbits Z_τ contained in the generic fibre Y_{Σ_0} of $\mathscr{Y}_\Sigma$. If we use a decomposition $\mathrm{Trop}_0(X) = \bigcup_{\Delta \in S} \Delta$ into d-dimensional rational cones Δ in $N_\mathbb{R}$ (see Remark 3.4), then Corollary 14.4 holds also for these orbits. This follows immediately from Corollary 14.4 replacing v by the trivial valuation. Then the generic fibre is equal to the special fibre.

14.6 Let X be a closed subscheme of T of pure dimension d with closure $\mathscr{X}$ in $\mathscr{Y}_\Sigma$ and let Z_τ be the orbit of $\mathscr{Y}_\Sigma$ induced by the open face τ of Σ_1 (resp. Σ_0). We say that $\mathscr{X}$ *intersects* Z_τ *properly* if $\dim(\mathscr{X} \cap Z_\tau) = d - \dim(\tau)$. We emphasize that in this case, $\mathscr{X} \cap Z_\tau$ is not empty. Note that $\mathscr{Y}_\Sigma$ is a noetherian topological space and one can easily show that $\mathscr{X}$ intersects Z_τ properly if and only if every irreducible component of $\mathscr{X} \cap Z_\tau$ has codimension in $\mathscr{X}$ equal to $\mathrm{codim}(Z_\tau, \mathscr{Y}_\Sigma)$.

The following result was shown to me by Sam Payne.

Proposition 14.7 *If $\mathscr{X}$ intersects Z_τ properly, then $\mathscr{X} \cap Z_\tau$ is pure dimensional.*

Proof: We may assume that $\tau \in \Sigma_1$ and hence Z_τ is contained in the special fibre of $\mathscr{Y}_\Sigma$. Indeed, the case $\tau \in \Sigma_0$ follows as usual from this replacing v by the trivial valuation. We choose a vertex ω of $\overline{\tau}$. By Proposition 7.15, the irreducible component Y_ω of $\mathscr{X}_s$ corresponding to ω is a toric variety over $\tilde{K}$ associated to the fan $\mathrm{LC}_\omega(\Sigma_1) := \{\mathrm{LC}_\omega(\Delta) \mid \Delta \in \Sigma_1\}$. We note that Z_τ is also an orbit of Y_ω and we have $\mathrm{codim}(Z_\tau, Y_\omega) = \dim(\tau)$. For every $z \in Z_\tau$, there is a neighbourhood U of z in Y_ω such that $Z_\tau \cap U$ is set theoretically the intersection of $\mathrm{codim}(Z_\tau, Y_\omega)$ effective Cartier divisors. This follows from Hochter's theorem which says that a toric variety is Cohen-Macaulay (see [CLS], Theorem 9.2.9). We conclude that every irreducible component of $\mathscr{X} \cap Z_\tau$ has dimension at least $d - \dim(\tau)$. $\qquad\square$

Proposition 14.8 *Let τ be an open face of Σ_1. If $\mathscr{X}$ intersects Z_τ properly, then $\tau \subset \mathrm{Trop}_v(X)$.*

Proof: Assuming that $\mathscr{X}$ intersects Z_τ properly, we deduce from Corollary 14.4 the following fact which is crucial for the proof: If Δ is any d-dimensional polyhedron contained in $\mathrm{Trop}_v(X)$ and if $\Delta \cap \tau \neq \emptyset$, then we have

$$\dim(\Delta \cap \tau) = \dim(\tau). \tag{16}$$

By assumption, $\mathscr{X} \cap Z_\tau$ is non-empty and hence $\tau \cap \mathrm{Trop}_v(X) \neq \emptyset$ by Tevelev's Lemma 11.6. By the Bieri–Groves theorem and Theorem 10.14, there is a complete Γ-rational polyhedral complex $\mathscr{D}$ in $N_\mathbb{R}$ of pure dimension d with a subcomplex $\mathscr{C}$ such that $\mathrm{Trop}_v(X) = |\mathscr{C}|$.

Using these facts, the proposition will follow from elementary arguments in convex geometry. Especially important is the collection $\mathscr{E}$ of all $\sigma \in \mathscr{D}$ with $\dim(\mathrm{relint}(\sigma) \cap \tau) = \dim(\tau)$. We note that $(\sigma \cap \tau)_{\sigma \in \mathscr{E}}$ is a covering of the open face τ which is like a tiling of τ. We have seen above that $\tau \cap \mathrm{Trop}_v(X) \neq \emptyset$ and hence there is a d-dimensional polyhedron $\Delta \in \mathscr{C}$ with $\Delta \cap \tau \neq \emptyset$. Using an appropriate closed face of Δ, we get the existence of a polyhedron $\sigma \in \mathscr{C} \cap \mathscr{E}$.

Let $\sigma' \in \mathscr{E}$ such that $\sigma' \cap \tau$ is a direct neighbor of $\sigma \cap \tau$ which means that $\nu := \sigma \cap \sigma' \cap \tau$ has dimension equal to $\dim(\tau) - 1$. Using the above tiling of τ, the open face τ may be covered by using successively such neighboring $\sigma' \cap \tau$. We conclude that it is enough to show that $\sigma' \cap \tau \subset \mathrm{Trop}_v(X)$.

Remembering that $\mathscr{C}$ is a polyhedral complex of pure dimension d, there is a d-dimensional polyhedron $\Delta \in \mathscr{C}$ with closed face σ. Note that ν is obtained by intersecting τ with the proper closed face $\sigma \cap \sigma'$ of σ. We conclude that there is a closed face ρ of Δ with $\dim(\rho) = d - 1$ which contains ν but not σ. We have $\dim(\mathrm{relint}(\sigma) \cap \tau) = \dim(\tau)$ and hence $\Delta \cap \tau = \sigma \cap \tau$ contains $\rho \cap \tau$ as a proper subset. Since ν is of codimension 1 in $\sigma \cap \tau$, we get $\nu = \rho \cap \tau$.

We choose a hyperplane H in $N_\mathbb{R}$ which contains ρ but not Δ. By the balancing condition in Theorem 13.11, there is a d-dimensional polyhedron $\Delta' \in \mathscr{C}$ with closed face ρ on the other side of H than Δ. Since ν is the border of $\sigma \cap \tau$ and $\sigma' \cap \tau$ in the above tiling of τ, we conclude that $\Delta' \cap \tau \subset \sigma' \cap \tau$. Using (16) for Δ', we get $\dim(\Delta' \cap \tau) = \dim(\tau)$ and hence $\Delta' \cap \tau = \sigma' \cap \tau$. This proves $\sigma' \cap \tau \subset \mathrm{Trop}_v(X)$. $\square$

To deal with orbits in the special fibre and in the generic fibre simultaneously, one has to use the Γ-admissible fan Σ in $N_\mathbb{R} \times \mathbb{R}_+$ and the tropical cone $\mathrm{Trop}_W(X)$ of X in $N_\mathbb{R} \times \mathbb{R}_+$ (see Definition 8.3).

Theorem 14.9 *Let Σ be a Γ-admissible fan in $N_\mathbb{R} \times \mathbb{R}_+$ and let X be a closed subscheme of T of pure dimension d. Then the following properties are equivalent for the closure $\mathscr{X}$ of X in the toric scheme $\mathscr{Y}_\Sigma$:*

  (a) The special fibre $\mathscr{X}_s$ is non-empty, proper over $\tilde{K}$, and $\mathscr{X}$ intersects all the orbits of $\mathscr{Y}_\Sigma$ properly.

(b) The support of Σ is equal to the tropical cone $\mathrm{Trop}_W(X)$.

If the value group Γ is divisible or discrete in $\mathbb{R}$, then (a) and (b) are also equivalent to the condition that $\mathscr{X}$ is a proper scheme over K° which intersects all the orbits properly.

Proof: We assume that (a) holds. By Proposition 14.8, the assumption that $\mathscr{X}$ intersects all orbits properly yields that $|\Sigma_1|$ is contained in $\mathrm{Trop}_v(X)$. If we replace v by the trivial valuation, then the same argument shows that $|\Sigma_0| \subset \mathrm{Trop}_0(X)$. Since $\mathrm{Trop}_W(X)$ is the closed cone in $N_\mathbb{R} \times \mathbb{R}_+$ generated by $\mathrm{Trop}_v(X) \times \{1\}$ (see Proposition 8.4), we conclude that $|\Sigma| \subset \mathrm{Trop}_W(X)$. On the other hand, $\mathscr{X}_s$ is a non-empty proper scheme over $\tilde{K}$ and hence $\mathrm{Trop}_v(X)$ is contained in $|\Sigma_1|$ by Proposition 11.12. We conclude that (a) yields (b).

Now we suppose that (b) holds. Then Σ_1 is a Γ-rational complex with support equal to $\mathrm{Trop}_v(X)$. We choose an open face τ of Σ_1 with corresponding orbit Z_τ in the special fibre of $\mathscr{Y}_\Sigma$. For any d-dimensional polyhedron $\Delta \in \Sigma_1$, either $\Delta \cap \tau$ is empty or τ. From Corollary 14.4, we deduce that $\mathscr{X}$ intersects Z_τ properly. Using $\mathrm{Trop}_0(X) = |\Sigma_0|$, Remark 14.5 shows that $\mathscr{X}$ intersects the orbits in the generic fibre of $\mathscr{Y}_\Sigma$ properly. It follows from Proposition 11.12 that $\mathscr{X}_s$ is a non-empty proper scheme over $\tilde{K}$.

If Γ is divisible or discrete in $\mathbb{R}$, then the last claim follows immediately from Proposition 11.12. $\qquad\qquad\square$

Remark 14.10 We have seen in Proposition 12.5 that every tropical fan satisfies the equivalent properties (a) and (b) of Theorem 14.9. However, the converse does not hold as it was shown by Sturmfels–Tevelev in Example 3.10 of [ST] and by Cartwright in Section 1 of [Car].

A   Convex geometry

In this appendix, we collect the notation used from convex geometry. We denote by Γ a subgroup of $\mathbb{R}$. We consider a free abelian group M of rank n with dual $N := \mathrm{Hom}(M, \mathbb{Z})$ and the corresponding real vector spaces $V := M \otimes_\mathbb{Z} \mathbb{R}$ and $W := \mathrm{Hom}(V, \mathbb{R}) = N \otimes_\mathbb{Z} \mathbb{R}$. The natural duality between $u \in V$ and $\omega \in W$ is denoted by $\langle u, \omega \rangle$. References: [Roc], [McM].

A.1 A *polyhedron* Δ in W is an intersection of finitely many closed half-spaces $\{\omega \in W \mid \langle u_i, \omega \rangle \geq c_i\}$. We say that Δ is Γ-*rational* if we may choose all $u_i \in M$ and all $c_i \in \Gamma$. If $\Gamma = \mathbb{Q}$, then we say that Δ is *rational*. A *closed face* of Δ is either Δ itself or has the form $H \cap \Delta$ where H is the boundary of a closed half-space containing Δ. An *open face* of Δ is a closed face without all its properly contained closed faces. We denote by $\mathrm{relint}(\Delta)$ the unique open face of Δ which is dense in Δ.

A.2 A bounded polyhedron is called a *polytope*. This is equivalent to being the convex hull of finitely many points. Let $G := \{\lambda \in \mathbb{R} \mid \exists m \in \mathbb{N} \setminus \{0\}, \, m\lambda \in \Gamma\}$ be the divisible hull of Γ in $\mathbb{R}$. Simple linear algebra shows that a polytope is Γ-rational if and only if all vertices are G-rational and the edges have rational slopes. Similarly, a polyhedron is Γ-rational if and only if every closed face spans an affine subspace which is a translate of a rational linear subspace by a G-rational vector.

A.3 A *polyhedral complex* $\mathscr{C}$ in W is a finite set of polyhedra such that

(a) $\Delta \in \mathscr{C} \Rightarrow$ all closed faces of Δ are in $\mathscr{C}$;

(b) $\Delta, \sigma \in \mathscr{C} \Rightarrow \Delta \cap \sigma$ is either empty or a closed face of Δ and σ.

The polyhedral complex is called Γ-*rational* if every $\Delta \in \mathscr{C}$ is Γ-rational. The *support* of $\mathscr{C}$ is defined as

$$|\mathscr{C}| := \bigcup_{\Delta \in \mathscr{C}} \Delta.$$

The polyhedral complex $\mathscr{C}$ is called *complete* if $|\mathscr{C}| = W$. A *subcomplex* of a polyhedral complex $\mathscr{C}$ is a polyhedral complex $\mathscr{D} \subset \mathscr{C}$.

A.4 A polyhedral complex $\mathscr{D}$ *subdivides* the polyhedral complex $\mathscr{C}$ if they have the same support and if every polyhedron Δ of $\mathscr{D}$ is contained in a polyhedron of $\mathscr{C}$. In this case, we say that $\mathscr{D}$ is a *subdivision* of $\mathscr{C}$.

A.5 A *cone* σ in W is centered at 0, i.e. it is characterized by $\mathbb{R}_+\sigma = \sigma$. Its *dual* is defined by

$$\check{\sigma} := \{u \in V \mid \langle u, \omega \rangle \geq 0 \ \forall \omega \in \sigma\}.$$

A *fan* is a polyhedral complex consisting of polyhedral cones. A *subfan* is a subcomplex of a fan.

A.6 The *local cone* $\mathrm{LC}_\omega(S)$ of $S \subset W$ at ω is defined by

$$\mathrm{LC}_\omega(S) := \{\omega' \in W \mid \omega + [0, \varepsilon)\omega' \subset S \text{ for some } \varepsilon > 0\}.$$

A.7 The *recession cone* of a polyhedron Δ is defined by

$$\mathrm{rec}(\Delta) := \{\omega \in W \mid \omega + \Delta \subset \Delta\}.$$

By the Minkowski–Weil theorem, the recession cone is the unique convex polyhedral cone σ such that $\Delta = \sigma + \rho$ for a polytope ρ of W. If Δ is Γ-rational, then $\mathrm{rec}(\Delta)$ is a rational convex polyhedral cone.

A.8 A polyhedron Δ is called *pointed* if it does not contain an affine line. Note that Δ is a pointed polyhedron if and only if $\mathrm{rec}(\Delta)$ has the origin 0 as a vertex. This explains the terminology. A *pointed polyhedral complex* is a polyhedral complex consisting of pointed polyhedra.

A.9 We say that $f : W \to \mathbb{R} \cup \{\infty\}$ is a *proper polyhedral function* if the *epigraph* $\mathrm{epi}(f) := \{(\omega, s) \in W \times \mathbb{R} \mid f(\omega) \leq s\}$ is a non-empty polyhedron. Then the faces of the polyhedron $\mathrm{epi}(f)$ contained in the graph of f form a polyhedral complex in $W \times \mathbb{R}$ called the *graph complex*. The projection of the graph complex onto W gives a polyhedral complex in W. Such a complex is called a *coherent polyhedral complex* in W.

Note that f is a proper polyhedral function if and only if there is a non-empty polyhedron Σ in W and a function $f_\Sigma : \Sigma \to \mathbb{R}$ with the following properties:

(a) f_Σ is continuous and piecewise affine;

(b) f_Σ is a convex function in the usual sense, i.e.

$$f_\Sigma(r\omega + s\omega') \leq r f_\Sigma(\omega) + s f_\Sigma(\omega') \tag{17}$$

for $\omega, \omega' \in \Sigma$ and $r, s \in [0, 1]$ with $r + s = 1$.

(c) f agrees with f_Σ on Σ and $f = \infty$ outside of Σ.

We call Σ the *domain of* f. The *domains of linearity* are the maximal subsets of W where f_Σ is affine. They are just the maximal dimensional polyhedra Δ from the coherent polyhedral complex corresponding to f. On such a Δ, we have

$$f(\omega) = c_\Delta + \langle u_\Delta, \omega \rangle$$

for some $u_\Delta \in V$ and $c_\Delta \in \mathbb{R}$. We call u_Δ the *peg* of Δ.

A.10 Let f be a proper polyhedral function with associated coherent polyhedral complex $\mathscr{C}$. Then the *conjugate* of f is the proper polyhedral function $f^* : V \to \mathbb{R} \cup \{\infty\}$ given by

$$f^*(u) := \sup\{\langle u, \omega \rangle - f(\omega) \mid \omega \in W\}.$$

We have $f^{**} = f$.

A.11 Let f be a proper polyhedral function on W with associated coherent polyhedral complex $\mathscr{C}$. The coherent polyhedral complex in V associated to f^* is called the *dual complex* $\mathscr{C}^f$ of $\mathscr{C}$. The duality is a bijective order reversing correspondence $\sigma \mapsto \sigma^f$ between polyhedra of $\mathscr{C}$ and polyhedra of $\mathscr{C}^f$ given by

$$\sigma^f = \{u \in V \mid f^*(u) = \langle u, \omega \rangle - f(\omega) \ \forall \omega \in \sigma\}$$

and we have

$$\dim(\sigma) + \dim(\sigma^f) = n.$$

This follows from [McM], Theorem 7.1 and its proof. Note also that $\mathscr{C}^f$ is complete if and only if the support of $\mathscr{C}$ is bounded (which is then a polytope).

A.12 Let G be a totally ordered abelian group which is not necessarily a subgroup of $\mathbb{R}$. Then G-rational polyhedra can be defined as subsets of $N_G := N \otimes_{\mathbb{Z}} G$ in the same way as in A.1. For us, compatibilities are of interest in the situation when G contains Γ as an ordered subgroup where Γ is our given subgroup of $\mathbb{R}$. Let $H^+ := \{\omega \in W \mid \langle u, \omega \rangle \geq c\}$ be a Γ-rational half-space in W with $u \in M$ and $c \in \Gamma$. By base change, the homomorphism $N \to \mathbb{Z}, \omega \mapsto \langle u, \omega \rangle$ extends canonically to a homomorphism $N_G \to G$ of abelian groups and hence we get a pairing $M \times N_G \to G$ which we also denote by $\langle \ , \ \rangle$. Then we set $H^+(G) := \{\omega \in N_G \mid \langle u, \omega \rangle \geq c\}$.

A.13 *Let H_i^+ be a finite family of Γ-rational half-spaces in W defining the Γ-rational polyhedron $\Delta := \bigcap_i H_i^+$. Using the assumptions and the notation from A.12, the intersection $\bigcap_i H_i^+(G)$ depends only on Δ and not on the choice of the half-spaces.*

Proof: The given half-spaces have the form $H_i^+ := \{\omega \in W \mid \langle u_i, \omega \rangle \geq c_i\}$ with $u_i \in M$ and $c_i \in \Gamma$. It is enough to show that $H^+(G) \supset \bigcap_i H_i^+(G)$ for any Γ-rational half-space $H^+ := \{\omega \in W \mid \langle u, \omega \rangle \geq c\} \supset \Delta$. We may assume that H^+ is a supporting half-space of Δ, i.e. there is $\omega_0 \in W$ such that $\langle u, \omega_0 \rangle = c$. Moreover, we may assume that all H_i^+ are supporting hyperplanes in ω_0, i.e. $\langle u_i, \omega_0 \rangle = c_i$. We note that the recession cone $\rho = \{\omega \in W \mid \langle u, \omega \rangle \geq 0\}$ of H^+ contains the recession cone $\sigma = \{\omega \in W \mid \langle u_i, \omega \rangle \geq 0 \ \forall i\}$ of Δ. By the bijective correspondence between rational polyhedral cones in $W = N_{\mathbb{R}}$ and finitely generated saturated semigroups of M (see 6.2), we see that $\check{\rho} \cap M \subset \check{\sigma} \cap M = \{\sum_i m_i u_i \in M \mid 0 \leq m_i \in \mathbb{Q} \ \forall i\}$. We conclude that $u = \sum_i m_i u_i$ with $0 \leq m_i \in \mathbb{Q}$ for all i. Replacing u by a positive multiple, we may assume that all $m_i \in \mathbb{N}$. For $\omega \in \bigcap_i H_i^+(G)$, we get

$$\langle u, \omega \rangle = \sum_i m_i \langle u_i, \omega \rangle \geq \sum_i m_i c_i = \sum_i m_i \langle u_i, \omega_0 \rangle = \langle u, \omega_0 \rangle = c$$

and hence $\omega \in H^+(G)$. This proves the claim. $\qquad\square$

A.14 Let Δ be a Γ-rational polyhedron in W as above. Then A.13 shows that $\Delta(G) := \bigcap_i H_i^+(G)$ is well-defined. For another Γ-rational polyhedron Δ' in W, we get $\Delta(G) \cap \Delta'(G) = (\Delta \cap \Delta')(G)$. In particular, if $\Delta' \subset \Delta$, then $\Delta'(G) \subset \Delta(G)$.

More generally, if finitely many Γ-rational polyhedra Δ_i cover Δ, then $\Delta(G) \subset \bigcup_i \Delta_i(G)$. To see this, let $(H_j^+)_{j=1,\ldots,r}$ be the half-spaces occurring in the definitions of the polyhedra. We denote by H_j^- the half-space on the other side of the boundary

of H_j^+. Since G is a totally ordered group, the sets $\Delta(G) \cap H_1^\pm(G) \cap \cdots \cap H_r^\pm(G)$ cover $\Delta(G)$. Every polyhedron $\Delta \cap H_1^\pm \cap \cdots \cap H_r^\pm$ is contained in a Δ_i and we conclude from A.13 and the above that the sets $\Delta_i(G)$ cover $\Delta(G)$.

A.15 Let us still consider a totally ordered abelian group G containing Γ as an ordered subgroup and a Γ-rational polyhedron Δ in $W = N_\mathbb{R}$. If $\varphi : N' \to N$ is a homomorphism of free abelian groups of finite rank and if $\psi : N'_\Gamma \to N_\Gamma$ is given by $\psi = \varphi_\Gamma + \omega_0$ for a fixed $\omega_0 \in N_\Gamma$, then we have

$$(\psi^{-1}(\Delta))(G) = \psi_G^{-1}(\Delta(G)),$$

where φ_Γ and ψ_G denote base changes of φ and ψ. This is easily seen for a Γ-rational half-space and then the claim follows from A.13.

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