| AMAT 491 | Assignment 4 | F2017 |
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## Direct and Iterative Methods

Due: Monday, December 11th

**01.** A matrix $\boldsymbol{A}$ is called tridiagonal, if

$$\boldsymbol{A} = \begin{bmatrix} a_1 & c_1 & 0 & & & \cdots & \cdots & \cdots & & 0 \\ b_2 & a_2 & c_2 & 0 & & & & & & \\ 0 & b_3 & a_3 & c_3 & 0 & & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & & 0 & b_{n-2} & a_{n-2} & c_{n-2} & 0 \\ & & & & & & 0 & b_{n-1} & a_{n-1} & c_{n-1} \\ 0 & & \cdots & \cdots & \cdots & & & 0 & b_n & a_n \end{bmatrix}$$

Find an $\boldsymbol{L}\boldsymbol{U}$ factorization of $\boldsymbol{A}$ with

$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & & & \cdots & \cdots & 0 \\ l_2 & 1 & 0 & & & & \\ 0 & l_3 & 1 & 0 & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & 0 & l_{n-1} & 1 & 0 \\ 0 & & & & 0 & l_n & 1 \end{bmatrix}$$

and

$$\boldsymbol{U} = \begin{bmatrix} u_1 & c_1 & 0 & & & \cdots & \cdots & 0 \\ 0 & u_2 & c_2 & 0 & & & & \\ & 0 & u_3 & c_3 & 0 & & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & & & 0 & u_{n-2} & c_{n-2} & 0 \\ & & & & & 0 & u_{n-1} & c_{n-1} \\ 0 & & & & & & 0 & u_n \end{bmatrix}$$

**a.** Write down the algorithm that computes $\boldsymbol{L}$ and $\boldsymbol{U}$.

**b.** Write a MATLAB function TriLU( ) that implements your algorithm from the previous question. The function should take as arguments the diagonal $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^t$, the subdiagonal $\boldsymbol{b} = \begin{bmatrix} b_2 & b_3 & \cdots & b_n \end{bmatrix}^t$, the superdiagonal $\boldsymbol{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} \end{bmatrix}^t$ of $\boldsymbol{A}$, and the number of columns/rows $n$ of the matrix $\boldsymbol{A}$. It should return the subdiagonal $\boldsymbol{\ell} = \begin{bmatrix} \ell_2 & \ell_3 & \cdots & \ell_n \end{bmatrix}^t$ of $\boldsymbol{L}$ and the diagonal $\boldsymbol{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}^t$ of $\boldsymbol{U}$.
Test your function on the matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 4 & 3 & 3 & 0 & 0 \\ 0 & -3 & 12 & 4 & 0 \\ 0 & 0 & -6 & -10 & 1 \\ 0 & 0 & 0 & -10 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

**02.** Write two MATLAB functions myJacobi( ) and myGaussSeidel( ) that implement the Jacobi and the Gauss-Seidel methods. The functions should take as arguments the matrix $\boldsymbol{A}$, the right hand side $\boldsymbol{b}$, the initial guess $\boldsymbol{x}^{(0)}$, the maximum number of iterations nmax and a tolerance tol to stop the iterations whenever the relative error

$$\frac{\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\|}{\|\boldsymbol{x}^{(k)}\|}$$

is smaller than tol. The functions should return the solution $\boldsymbol{x}$ and the number of iterations it took the method to converge. You may use the MATLAB norm to compute the norm of a vector. Test your functions on the system

$$\begin{cases} 10\,x_1 - x_2 + 2\,x_3 = 6 \\ -x_1 + 11\,x_2 - x_3 + 3\,x_4 = 25 \\ 2\,x_1 - x_2 + 10\,x_3 - x_4 = -11 \\ 3\,x_2 - x_3 + 8\,x_4 = 15 \end{cases} \qquad \text{with solution} \qquad \boldsymbol{x} = \begin{bmatrix} 1 & 2 & -1 & 1 \end{bmatrix}^t$$

Start with $\boldsymbol{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^t$, and iterate until $\dfrac{\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\|_\infty}{\|\boldsymbol{x}^{(k)}\|_\infty} < 10^{-3}$

**03.** Show that if $\boldsymbol{A}$, is an $n \times n$ matrix, then

$$\|\boldsymbol{A}\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

**04.** Recall that Jacobi and Gauss-Seidel Algorithms applied to the system $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$ reads

$$\left\| \begin{array}{l} \boldsymbol{x}^{(0)} = \boldsymbol{0} \\ \boldsymbol{x}^{(k)} = \boldsymbol{B}_{\boldsymbol{J}}\,\boldsymbol{x}^{(k-1)} + \boldsymbol{c} \end{array} \right. \qquad \text{and} \qquad \left\| \begin{array}{l} \boldsymbol{x}^{(0)} = \boldsymbol{0} \\ \boldsymbol{x}^{(k)} = \boldsymbol{B}_{\boldsymbol{GS}}\,\boldsymbol{x}^{(k-1)} + \boldsymbol{d} \end{array} \right.$$

**Part 1** Suppose $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -1 \\ 4 \\ -5 \end{bmatrix}$

**(i)** Find the matrices $\boldsymbol{B}_{\boldsymbol{J}}$ and $\boldsymbol{B}_{\boldsymbol{GS}}$

**(ii)** Show that $\rho\Big(\boldsymbol{B}_{\boldsymbol{J}}\Big) = \dfrac{\sqrt{5}}{2}$ and $\rho\Big(\boldsymbol{B}_{\boldsymbol{GS}}\Big) = \dfrac{1}{2}$

**(iii)** Apply Jacobi and Gauss-Seidel to the system with $\boldsymbol{x}^{(0)} = \boldsymbol{0}$ and 25 iterations. Comment on the results.

**Part 2** Suppose $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$

**(iv)** Find the matrices $\boldsymbol{B}_{\boldsymbol{J}}$ and $\boldsymbol{B}_{\boldsymbol{GS}}$

**(vi)** Show that $\rho\Big(\boldsymbol{B}_{\boldsymbol{J}}\Big) = 0$ and $\rho\Big(\boldsymbol{B}_{\boldsymbol{GS}}\Big) = 2$

**(vii)** Apply Jacobi and Gauss-Seidel to the system with $\boldsymbol{x}^{(0)} = \boldsymbol{0}$ and 25 iterations. Comment on the results.

Notice that both systems have solution $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

**05.** Given $\boldsymbol{A}$, $\widetilde{\boldsymbol{A}}$, be two invertible $n \times n$ matrices, and $\boldsymbol{b}$, $\widetilde{\boldsymbol{b}}$, two vectors in $\mathbb{R}^n$, let $\boldsymbol{x}$, $\widetilde{\boldsymbol{x}}$, be the unique solutions of $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$, and $\widetilde{\boldsymbol{A}}\,\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{b}}$, respectively. Show

$$\frac{\|\boldsymbol{x} - \widetilde{\boldsymbol{x}}\|}{\|\boldsymbol{x}\|} \leq \frac{\operatorname{cond}\big(\boldsymbol{A}\big)}{1 - \left|\left|\left|\boldsymbol{A}^{-1}\Big(\boldsymbol{A} - \widetilde{\boldsymbol{A}}\Big)\right|\right|\right|} \left( \frac{\|\boldsymbol{b} - \widetilde{\boldsymbol{b}}\|}{\|\boldsymbol{b}\|} + \frac{\left|\left|\left|\boldsymbol{A} - \widetilde{\boldsymbol{A}}\right|\right|\right|}{|||\boldsymbol{A}|||} \right)$$