AMAT 491 Assignment 4

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Dec 11 2017

1

a.

Let a be an nxn tridiagonal matrix. Let a be the diagonal, b be the subdiagonal, and c be the superdiagonal. Then to find l, u, and d such that l is the subdiagonal of a lower matrix; and u and d are the superdiagonal and diagonal of the upper matrix such that $L * U = A$ we follow the following algorithm.

1. First, prepend a 0 to the b vector and append a 0 to the c vector so that all vectors are of length n.
2. Set u = c.
3. Set the first element of d equal to the first element of a.
4. For each k from 2 to n,
   4.1. Set $l(k) = b(k)/d(k-1)$
   4.2. Set $d(k) = a(k) - l(k) \cdot c(k-1)$
5. Remove the first element of l and the last element of u.

b.

See `TriLU.m` and `queston1.m`.

2

See `myJacobi.m`, `myGaussSeidel.m`, `relative_error.m`, and `question2.m`.

3

Let's start from the definition of a matrix norm

$$|||A||| = \max_{\substack{x \epsilon \mathbb{R} \\ x \neq 0}} \frac{||Ax||}{||x||}$$

When we specify that we are looking for the infinity-norm, this leaves us with

$$\begin{aligned}
|||A|||_\infty &= \max_{\substack{x \epsilon \mathbb{R} \\ x \neq 0}} \frac{||Ax||_\infty}{||x||_\infty} \\
&= \frac{\max\limits_{1 \leq i \leq n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|}{\max\limits_{1 \leq i \leq n} |x_j|} \\
&= \frac{\max\limits_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \, |x_j|}{\max\limits_{1 \leq i \leq n} |x_j|} \\
&= \frac{\max\limits_{1 \leq i \leq n} |x_j| \max\limits_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|}{\max\limits_{1 \leq i \leq n} |x_j|} \\
&= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|
\end{aligned}$$

4

4.1

4.1.1

For the Jacobi Method we have

$$B_j = (I - D^{-1}A) = \begin{bmatrix} 0 & 1/2 & -1/2 \\ -1 & 0 & -1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

and for the Gauss-Seidel Method we have

$$B_{GS} = (I - L^{-1}A) = \begin{bmatrix} 0 & 1/2 & -1/2 \\ 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 \end{bmatrix}$$

4.1.2

For the Jacobi matrix we have

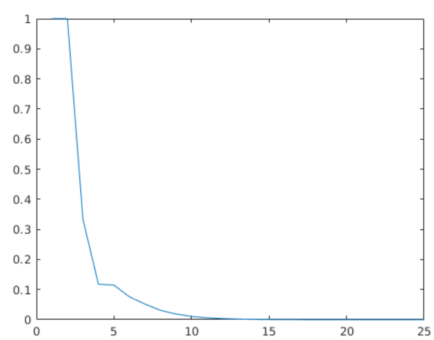
$$\rho(B_J) = max\left(abs\left(\left\{0, \frac{\sqrt{5}}{2}i, \frac{-\sqrt{5}}{2}i\right\}\right)\right) = \frac{\sqrt{5}}{2}$$

and for the Gauss-Seidel matrix we have

$$\rho(B_{GS}) = max\left(abs\left(\left\{0, \frac{-1}{2}, \frac{-1}{2}\right\}\right)\right) = \frac{1}{2}$$

4.1.3

After 25 iterations, the Jacobi method finds the solution of $[-20.8279, 2, -22.8279]'$. Even after many more iterations, it fails to converge to the correct solution.
On the other had, the Gauss-Seidel method converges quickly, and the relative error rapidly decreases.



These results are expected because of the size of the spectral radius.

4.2

4.2.1

For the Jacobi Method we have

$$B_j = (I - D^{-1}A) = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$$

and for the Gauss-Seidel Method we have

$$B_{GS} = (I - L^{-1}A) = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

4.2.2

For the Jacobi matrix we have

$$\rho(B_J) = max\left(abs\left(\{0,0,0\}\right)\right) = 0$$

and for the Gauss-Seidel matrix we have

$$\rho(B_{GS}) = max\left(abs\left(\{0,2,2\}\right)\right) = 2$$

4.2.3

After only four iterations, the Jacobi method converges to the correct answer.
For the Gauss-Seidel method, it results in a very incorrect answer, and with increasing iterations the result gets worse.
Again these results are expected because of the size of the spectral radius.

5

First we begin with the definitions $Ax = b$ and $\tilde{A}\tilde{x} = \tilde{b}$. From that we can see

$$\begin{aligned} A\tilde{x} &= \tilde{b} - (\tilde{A} - A)\tilde{x} \\ A\tilde{x} - Ax &= \tilde{b} - b - (\tilde{A} - A)\tilde{x} \\ A^{-1}A(\tilde{x} - x) &= A^{-1}(\tilde{b} - b(\tilde{A} - A)\tilde{x}) \end{aligned}$$

And taking the norm of both sides we find

$$\begin{aligned} ||\tilde{x} - x|| &= ||A^{-1}(\tilde{b} - b - (\tilde{A} - A)\tilde{x})|| \\ &\leq |||A^{-1}||| \cdot ||(\tilde{b} - b - (\tilde{A} - A)\tilde{x}||) \\ &\leq |||A^{-1}||| \cdot (||\tilde{b} - b|| + || - 1 * (\tilde{A} - A)|| \cdot ||\tilde{x}||) \\ &\leq |||A^{-1}||| \cdot |||\tilde{A} - A||| \cdot ||\tilde{x} - x|| + |||A^{-1}||| \cdot (||\tilde{b} - b|| + |||\tilde{A} - A||| \cdot ||x||) \end{aligned}$$

And by rearranging, we see

$$\begin{aligned} (1 - |||A^{-1}||| \cdot |||\tilde{A} - A|||)\frac{||\tilde{x} - x||}{||x||} &\leq |||A^{-1}||| \left(\frac{||\tilde{b} - b||}{||x||} + |||\tilde{A} - A|||\right) \\ \left(1 - |||A^{-1}||| \cdot |||A|||\frac{|||\tilde{A} - A|||}{|||A|||}\right) &\leq |||A^{-1}||| \cdot |||A||| \left(\frac{||\tilde{b} - b||}{||b||} + \frac{|||\tilde{A} - A|||}{|||A|||}\right) \end{aligned}$$

And finally we find

$$\frac{||\tilde{x} - x||}{||x||} \leq \frac{cond(A)}{1 - |||A^{-1}||| \cdot |||\tilde{A} - A|||}\left(\frac{||\tilde{b} - b||}{||b||} + \frac{|||A - \tilde{A}|||}{|||A|||}\right)$$