Amat 491	**Assignment 1**	**F2017**

Nonlinear Scalar Equations

DUE: OCTOBER 9^{TH}

01. (15 marks) Write a MATLAB function `newton_bisect()` that combines the Bisection and Newton's methods in a way that uses the best features of each.
Starting with a bracketing interval $[a_1, b_1]$, select the first iterate $x_1 = a_1$, then compute the next iterate x_2 using a Newton step.
If $x_2 \in [a_1, b_1]$, proceed with either $[a_2, b_2] = [a_1, x_2]$ or $[a_2, b_2] = [x_2, b_1]$ whichever brackets the root.
If $x_2 \notin [a_1, b_1]$, take a Bisection step and select $x_2 = \dfrac{a_1 + b_1}{2}$. Set $[a_2, b_2] = [a_1, x_2]$ or $[a_2, b_2] = [x_2, b_1]$ whichever brackets the root.

The function should take the following arguments

- a and b, the end points of the interval where the root of $f(x) = 0$ is being sought
- fname is the handle of the function that evaluates both f(x) and f'(x)
- tolx, a positive termination criteria
- nmax, maximum number of function evaluations

tolx, and nmax should be used to guarantee termination, i.e., the iterations should stop if the current interval has length smaller than tolx, or the number of times $f(x)$ got evaluated exceeds nmax.

newton_bisect() should return

- x, the approximate root of the equation $f(x) = 0$
- fx, the value of f at x
- nf, the number of times $f(x)$ and $f'(x)$ have been evaluated
- af and bf, the end points of the final bracketing interval

Test your function on the equation

$$\frac{1}{4} + \frac{x^2}{4} - x\,\sin(x) - \frac{1}{2}\cos(2\,x) = 0$$

with $a = 0.5$ and $b = 3$. The root is $2.380\,061\,273\,139\,339\,\cdots$

02. (10 marks) Consider the equation $x^2 - a = 0 \ \cdots \ (*)$, where $a > 0$ is given.

a. Write down the Secant algorithm for $(*)$. Make sure to simplify your answer completely.

b. Use the algorithm with $a = 3$, $x_1 = 1$, and $x_2 = 2$ to compute the first 10 iterates. Using $\sqrt{3} = 1.732\,050\,807\,568\,877\,293\,\cdots$ as the exact value, comment on the rate of convergence of the sequence.

03. (10 marks) Consider the equation $f(x) = 0$, where $f(x) = \mathrm{e}^x - \mathrm{e}\,x$.

a. Show that $f(x)$ has a unique root r, and find its multiplicity p

b. Let $\left\{x_n\right\}_{n\geq 1}$ be the Newton sequence with initial guess $x_1 = 0$. Compute 20 iterates and verify the linear convergence of the sequence.

c. Show that the convergence of $\left\{x_n\right\}_{n\geq 1}$ is indeed linear.

d. Let $\left\{z_n\right\}_{n\geq 1}$ be the sequence generated by the modified Newton's algorithm

$$z_1 = 0, \quad z_{n+1} = z_n - p\,\frac{f(z_n)}{f'(z_n)}, \quad n = 1, 2, \cdots$$

where p is the multiplicity of the root r. Verify that the convergence of $\left\{z_n\right\}_{n\geq 1}$ is of order p.

04. (5 marks) Show that if $g(x)$ is continuous in $[a, b]$ and if $a \leq g(x) \leq b$, for any x in $[a, b]$, then $g(x)$ has at least one fixed point in $[a, b]$. Is the fixed point unique? Justify your answer.