Amat 491	**Assignment 2**	**F2017**

Polynomial Interpolation

Due: Monday, October 30th

01. Let $P_n(x)$ be the degree n polynomial that interpolates the function $f(x)$ at the pairwise distinct nodes $\left\{ x_i \right\}_{1 \le i \le n+1}$, written in Newton's form

$$P_n(x) = a_1 + a_2(x - x_1) + a_3(x - x_1)(x - x_2) + \cdots + a_{n+1}(x - x_1)(x - x_2)\cdots(x - x_n)$$

where the coefficients are given by the divided differences

$$a_1 = f[x_1],\ a_2 = f[x_1, x_2], a_3 = f[x_1, x_2, x_3], \cdots, a_{n+1} = f[x_1, x_2, \cdots, x_{n+1}]$$

a. Write a Matlab function newton_dd() that takes as arguments the nodes $X = [x_1 ; \cdots ; x_{n+1}]$, the function values $F = [f(x_1) ; \cdots ; f(x_{n+1})]$, the degree n of the polynomial $P_n(x)$, and returns the coefficients $a = [a_1 ; a_2 ; \cdots ; a_{n+1}]$.

b. Write a Matlab function plot_newton_poly() that sketches the graph of $P_n(x)$ and the interpolation points $\left\{ (x_i, f(x_i)) \right\}_{1 \le i \le n+1}$, and displays both on the same plot. The function should take as arguments the number of points, the nodes $X = [x_1 ; \cdots ; x_{n+1}]$, and the function values $F = [f(x_1) ; \cdots ; f(x_{n+1})]$. The plot should be based on at least 100 evaluations of $P_n(x)$. You may use the Matlab function Horner_Newton(), posted on D2L, to perform the 100 evaluations of $P_n(x)$.

To test your functions, use the data points

$$(-1, f(-1)),\ (-3/4, f(-3/4)),\ (-1/4, f(-1/4)),\ (0, f(0)),\ (1/2, f(1/2),\ (1, f(1))$$

where $f(x) = \dfrac{4}{1 + 4x^2}$.

02. The degree n Lagrange polynomials associated with the points $x_0 < x_1 < \cdots < x_n$, are defined by

$$L_{n,j}(x) = \prod_{\substack{i=0 \\ i \ne j}}^{n} \frac{x - x_i}{x_j - x_i}, \quad j = 0, 1, \cdots, n$$

Assuming the points to be equidistant, with $x_0 = 0$ and $x_n = 1$, use Matlab to sketch in the same plot the graph of $L_{3,0}(x)$, $L_{3,1}(x)$, $L_{3,2}(x)$, and $L_{3,3}(x)$

03. Let n be a natural number, and define

$$T_n(x) = \cos\big(n \cos^{-1}(x)\big), \qquad -1 \le x \le 1$$

Recall that $\cos^{-1}(x)$ is the inverse function of $\cos(x)$. Its domain is $[-1\,,\,1]$, and its range is $[0\,,\,\pi]$

a. Display in the same plot the graphs of $T_i(x)$, $i = 1,\, 2,\, 3,\, 4,\, 5$

b. Show that $T_n(x)$ is a polynomial of degree n. $T_n(x)$ is known as the degree n Chebyshev polynomial.
<u>**Hint:**</u> Make use of the identity $\cos\big((n+1)\,\theta\big) + \cos\big((n-1)\,\theta\big) = 2\,\cos(\theta)\,\cos(n\,\theta)$ to show that $T_{n+1}(x) = 2\,x\,T_n(x) - T_{n-1}(x)$

c. Find the roots $\Big\{\ \xi_i\ \Big\}_{1\le i\le n}$ of $T_n(x)$

04. Recall that if $P_n(x)$ is the degree n polynomial that interpolates $f(x)$ at the set of points $\big\{x_0, x_1, \cdots, x_n\big\}$, and if $f(x)$ has $(n+1)$ continuous derivatives in an interval $[a,b]$ that contains the interpolating points, then

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}\,(x-x_0)\,(x-x_1)\,\cdots\,(x-x_n) \qquad (*)$$

Assuming $a \le x_0 < x_1 < \cdots < x_n \le b$, and taking absolute values, $(*)$ leads to the error estimation

$$|f(x) - P_n(x)| \le \frac{\max\limits_{x_0\le x\le x_n}\left|f^{(n+1)}(x)\right|}{(n+1)!}\ \max_{x_0\le x\le x_n}\ |(x-x_0)\,(x-x_1)\,\cdots\,(x-x_n)|$$

Let $f(x) = \dfrac{1}{1+x^2}$.

a. Assuming $x_i = -1 + 2\,\dfrac{i}{n}$, $i = 0, 1, \cdots, n$, sketch the graphs of $f(x), P_5(x),, P_{10}(x)$, $P_{20}(x)$, and display them in the same plot. Comment on the closeness of the interpolating polynomials to the function $f(x)$.

b. <u>**Bonus**</u>
Assuming that x_i, $i = 0, 1, \cdots, n$, are the roots of the degree $(n+1)$- Chebyshev polynomial $T_{n+1}(x)$, sketch the graphs of $f(x), P_5(x),, P_{10}(x)$, $P_{20}(x)$, and display them in the same plot. Comment on the closeness of the interpolating polynomials to the function $f(x)$.

05. Let $P_n(t)$ be the degre n polynomial that interpolates $f(t)$ at the distinct points $\big\{x_0, x_1, \cdots, x_n\big\}$. Show that if $t \ne x_i$, for $i = 0, 1, \cdots, n$, then

$$f(t) = P_n(t) + f\big[x_0, x_1, \cdots, x_n, t\big]\,\big(t-x_0\big)\big(t-x_1\big)\cdots\big(t-x_n\big)$$

<u>**Hint:**</u> do you recognize what the right side is?