Third Collection of Python Problems

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This document contains the specifications for a third collection of Python problems created by Ilkka Kokkarinen starting in March 2023, to augment the original set of 109 Python Problems designed for the course **CCPS 109 Computer Science I** for Chang School of Continuing Education, Toronto Metropolitan University (formerly Ryerson University).

Same as in the original problem collection, these problems are listed roughly in order of increasing difficulty. The reader shouldn't get too hung up on that, since that estimation is basically a gut feeling anyway, and depends heavily on the propensities and interests of the reader. The complexity of the solutions for these additional problems ranges from straightforward loops up to convoluted branching recursive backtracking searches that require all sorts of clever optimizations to prune the branches of the search sufficiently to keep the total running time of the test suite within the twenty-second time limit.

The rules for solving and testing these problems are exactly the same as they were for the original 109 problems. You must implement all your functions in a single source code file that must be named `labs109.py`, the exact same way as you wrote your solutions to the original 109 problems. The automated test suite to check that your solutions are correct, along with all associated files, is available in the GitHub repository `ikokkari/PythonProblems`.

Python coders of all levels will hopefully find something interesting in this collection. The author believes there is room on Earth for all races to live, prosper and get strong in coding while having a good time solving these problems. Same as with the original 109 Python problems, if a problem doesn't spark joy to you, please don't get stuck with that problem, but skip it and move on to the next problem that interests you. Life is too short to be wasted on solving stupid problems.

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1. Multiply and sort

```
def multiply_and_sort(n, multiplier):
```

The following problem, unearthed from the post "Double (or triple) and sort" in Michael Lugo's blog "God Plays Dice" ("where randomness meets divine intervention"), explores an iterated numerical transformation with curious properties and general behaviour. Starting from a positive integer n, multiply the current number by the given `multiplier`, and sort the resulting digits in ascending order to produce the next number in the sequence. Leading zeros, being the shy digits that they are, quietly disappear from the front.

For example, with n = 513 and `multiplier` = 2, the product 1026 becomes 0126 after sorting, which simplifies to 126 (the zero, true to form, showing itself out). This process continues until some integer repeats itself in the sequence, at which point the sequence admits defeat and stops. Your task in writing this function is to return the largest integer encountered during this mathematical game of multiply-and-tidy-up.

n	multiplier	Expected result
2	5	4456
3	4	2667
5	5	4456
20	7	15556688
31	7	12345678888888888889

As one commenter of the original post pointed out, this process seems to always terminate up to multipliers up to 20, but then freezes for some multipliers from 21 onwards and behaves well for the others. Interested readers might want to further explore this behaviour and prove some necessary and sufficient conditions for termination. The automated tester will, of course, give your function only numbers for which this sequence will terminate in a reasonable time.

2. Approval voting

```
def approval_voting(ballots):
```

Approval voting is an alternative to the traditional first-past-the-post single winner election scheme. Instead of voting for just one favoured candidate, every voter marks every candidate in the ballot for either approval or disapproval. The candidate who gets the most approvals tallied over all the ballots wins the election.

For the purposes of this problem, assume that the n candidates have been numbered 0, …, $n - 1$, and each ballot is a text string of length n whose each character is either `'Y'` or `'N'` depending on whether that voter goes "yea" or "nay" for that candidate. This function should tally up the approvals for each candidate, and return the index of the winning candidate. If multiple candidates receive the same number of approvals, your function should resolve the tie for the candidate with the lowest position index.

ballots	Expected result
['NYN', 'NYN', 'NYN', 'YNN', 'YYN']	1
['NYY', 'NYY', 'YYN', 'NYY', 'NNY', 'NYY']	1
['NYNYN', 'NYNYY', 'YYYNY', 'YYYYN', 'YNNNY', 'YNNNY', 'NYYNN', 'YNNYN']	0

3. Vigenère cipher

```
def vigenere(text, key, direction):
```

The Vigenère cipher is a method of cryptography dating back almost 500 years. Simple enough to allow both encryption and decryption to be performed by hand even under battlefield conditions, this method was nearly impossible to crack with the methods of its era by someone who was not in possession of the agreed encryption key, at least within the time limit when the encrypted information would have made a difference in the battlefield.

Using the modern English alphabet, the Vigenère method encrypts the given text one character at a time, replacing each character with that character shifted as many positions to the right in the alphabet (wrapping around after z) as given by the corresponding character of the key. The character a denotes a shift of zero, the character b denotes a shift of one, and so on, up to the character z that denotes a shift of 25.. If the key is shorter than the text, the key is concatenated with itself as many times as needed to stretch it to sufficient length. To decrypt the text with the same key, simply shift each character that same amount in the opposite direction.

Given the text and key, in this problem both guaranteed to consist of the 26 lowercase letters of English only, this function should perform both encryption (if direction equals +1) and decryption (if direction equals –1) of the text with the given key.

text	key	direction	Expected result
'locoedturse'	'adsum'	+1	'lruiqdwmlee'
'locoedturse'	'adsum'	–1	'llkusdqcxge'
'marorcotoin'	'kafiz'	+1	'wawwqmoywhx'
'marorcotoin'	'kafiz'	–1	'camgssoogjd'

It is also easy to see that the Vigenère cipher contains both the Caesar cipher and rot-13 as trivial special cases, the key being a singleton letter in both cases.

4. Once you go down, you can't come up again

```
def is_organ_pipe(items):
```

We define a list of `items` to have an **organ-pipe shape** if its elements are first **strictly ascending** up to some <mark>pinnacle value</mark>, after which the elements are **strictly descending**. The two halves of the organ pipe are not required to have the same length, so the list `[3, 5, 8, 9,` <mark>10</mark>`, 4, 2, 1]` qualifies as having an organ-pipe shape. However, we do require both sides of the organ pipe to consist of at least one step, which means that a sorted list that is entirely strictly ascending or strictly descending does not qualify as an organ pipe in this problem. This function should determine whether the given list of `items` has an organ-pipe shape.

items	Expected result
`[5, 3]`	False
`[1, ` <mark>2</mark>`, 1]`	True
`[1, 2, 2, 1]`	False
`[1, 3, 4, ` <mark>6</mark>`, 3, 0]`	True
`[0, 2, 1, 4, 6, 4, 1]`	False
`[0, 2, 3, 4, 5, 8, 9, ` <mark>12</mark>`, 9, 4]`	True

The title of this problem contains the key insight to make this test simple and efficient.

5. Baum–Sweet sequence

```
def baum_sweet(n):
```

The <u>Baum–Sweet sequence</u> is defined with a simple function that, given a nonnegative integer parameter n, returns 1 if the binary representation of n does not contain any odd-length sequences of zero bits, and returns 0 if the binary representation contains at least one such sequences of zeros. Note that 1 is an odd number. This function should compute and return value in the Baum–Sweet sequence for the given integer parameter n.

You can iterate through the bits of n with a loop that repeatedly computes the remainder of n modulo 2 to extract the lowest bit, and then divide n by two to chop off that bit. Alternatively, you can use the Python core language function `bin` that returns the binary string representation of its integer parameter n, although prefixed with `'0b'` to indicate base 2 encoding.

n	Expected result
0	0
1	1
7	1
67	1
69	0

6. Damm those errors

```
def damm_checksum(digits):
```

<u>Check digits</u> are used for credit card numbers and other digit string identifiers to catch errors that humans tend to make when they are asked to enter such boring and meaningless digit sequences. At the time an identifier digit string is issued, the check digit algorithm is used to compute a single-digit checksum that becomes part of the identifier. The presence of the checksum digit in the identifier allows checking the validity of the preceding digits entered by the user on the spot, before proceeding to the actual authenticity checks of the identifier itself.

In this problem we shall implement the <u>Damm checksum algorithm</u>. This algorithm can be implemented as a simple **state machine** that processes each individual digit exactly once, but thanks to the clever design of this state machine, is powerful enough to detect any individual digit change or the transpose of any two consecutive digits. Damm's algorithm maintains an **internal state** that is an integer between 0 and 9, inclusive, initially set to 0. The digits of the given `digit` string are then processed one at the time from left to right.

The algorithm uses a two-dimensional state machine transition `table` that can be constructed in many different ways, but in this problem we shall use the one originally computed by Damm, as given on the Wikipedia page describing the algorithm. If the current internal state of the algorithm equals `s`, the internal state after processing the next digit `d` will be `table[s][d]`, that is, read from the transition table using the current state as the row index, and the current digit as the column index. Your function should perform this internal state update for each digit, and return the resulting internal state of the algorithm as the answer.

digits	Expected result
`'572'`	4
`'251'`	6
`'00759'`	9
`'2770753'`	0

7. Tanton necklace

```
def tanton_necklace(beads, m):
```

The Australian math educator <u>James Tanton</u> is blessed with the seemingly endless gift of creating deep problems simple enough to state in a single tweet in a few almost throwaway sentences, yet clear enough for any schoolboy to understand on the spot.

Inspired by <u>this recent tweet</u>, consider a cyclic **necklace** consisting of $2n$ `beads`, each either purple or orange, so that the necklace contains exactly n beads of each colour. This function should count how many contiguous **balanced sublists** of $2m$ beads consist of exactly m purple beads and m orange beads, and return that count. (In the table below, the starting positions of these sublists are marked with a green background.) Every necklace given to this function is nonempty and valid according to this specification.

To make this function efficient for thousands of pseudorandom test cases, you should devise a way to quickly determine whether the sublist of $2m$ starting from position i is balanced. Of course, you could iterate through all $2n$ possible starting positions, and for each such position, extract the $2m$ beads and count how many of each colour there are, incrementing the tally whenever the sublist contains exactly m of each colour of bead. However, this seems like a very "Shlemiel" way to solve this problem. Instead, you should note that if you have already counted how many beads of each colour there are in the subsequence of length $2m$ that starts from the position i, you can update these counts for the subsequence starting from the position $i + 1$ in a constant time.

beads	m	Expected result
'ooopp p'	2	2
'ppopoooop'	3	4
'ooooopppopoooppppppp'	3	6

As one commenter to Tanton's original tweet pointed out, at least one balanced subsequence with exactly m purple and exactly m orange beads is guaranteed to exist for any m, regardless of how the original purple and orange beads are distributed over the entire necklace. Discrete math enjoyers reading this can try their hand in proving this interesting topic in <u>Ramsey theory</u>.

8. Bug in a line

```
def bug_in_a_line(board):
```

The following problem is from Peter Winkler's "Mathematical Puzzles: A Connoisseur's Collection" where this simple system is shown to have interesting mathematical properties over all possible boards. Here we shall content ourselves into merely executing the system for the given starting `board`. A smol cartoon bug stands at position 0 of the `board`, which is a string whose characters are traffic lights, either Red, Yellow, or Green. The colour of the traffic light in the cell that the bug is at determines its next move:

- If the current light is **green**, the bug flips that light to be **yellow**, and moves one step **right**.
- If the current light is **yellow**, the bug flips that light to be **red**, and moves one step **right**.
- If the current light is **red**, the bug flips that light to be **green**, and moves one step **left**.

The bug therefore moves left roughly one third of the time, and right roughly two thirds of the time. This function should iterate through the steps taken by the bug on the `board` until it falls off either edge, and return the total number of steps it took for the bug to fall off. You will surely first convert the `board` to a `list` instead of an immutable string, so you can locally modify the traffic light that the bug flips in constant time, regardless of the `board` length.

board	Expected result
`'GGR'`	5
`'RRGY'`	1
`'GYRGYYRRRR'`	26
`'GGYRRYYYYRYRYG'`	34

The bug cannot wander forever on a finite `board`; it must eventually fall off from either end. If the bug were to traverse a finite board infinitely long, it would visit some cells infinitely often. Let k be the lowest position index of these infinitely often visited cells. Since that cell is red during one-third of these visits, the bug would also end up visiting the cell $k - 1$ infinitely often, a contradiction to the assumption that k was the lowest position index with that property.

9. Friendship paradox

```
def friendship_paradox(friends):
```

Each person in a group of n people numbered 0, ..., $n - 1$ has at least one friend in the group, with `friends[i]` giving the list of friends of person i. Friendship or its absence is always symmetric between any two people. (After all, unrequited friendship belongs in middle school diary entries of overly dramatic emo theatre kids, not in a serious and manly study of graph theory.)

This problem investigates a curious phenomenon known in the literature as the friendship paradox: on average, an individual's friends have more friends than that individual. While this may sound like a manifestation of social anxiety and everyone else seemingly having cooler lives than you, this phenomenon has a rigorous mathematical basis in that more popular people appear as someone's friends more frequently than less popular people, which skews the average number of friends that your friends have compared to your own friendship count.

Given the list of lists of `friends` for the people in the group, this function should compute and return two numerical quantities as a two-element tuple: the average number of friends per person, and the average number of friends among the friends per the average person. Both quantities should be computed and returned as exact `Fraction` objects, because real mathematicians don't round their social calculations.

friends	Expected result
[[1], [0, 2], [1]]	(Fraction(4, 3), Fraction(5, 3))
[[1, 3], [2, 0, 3], [1, 3], [1, 0, 2]]	(Fraction(5, 2), Fraction(8, 3))
[[1, 4, 3, 2], [3, 2, 4, 0], [1, 0, 3], [4, 1, 0, 2], [0, 3, 1]]	(Fraction(18, 5), Fraction(37, 10))

As a coding challenge, the enjoyers of Python **generator expressions** can try to write the part of the function that computes the sum of friends that are friends of a particular person as a one-liner.

10. Right turn at Albuquerque

```
def albuquerque_stretch(text):
```

This cute little problem, suitable for a coding exercise for a Python intro course, is "Albuquerque challenge" recently posted in Code Golf Stack Exchange but of course, here we shall aim for correctness and clarity instead of the extreme brevity of the source code. This function should first set the `result` to an empty string, then loop through the characters `c` of the `text` from left to right. At the first occurrence of that character `c`, simply append that character to the `result`. From the second occurrence onwards, append the substring of the original `text` spanning from the most recent occurrence of `c` to its current occurrence, both endpoints inclusive, to the `result`.

You can trace the steps of the first test case `'success'` either by yourself or follow its steps on the original web page linked above for a concrete explanation of these rules. It is probably easiest to use a Python dictionary to keep track of the characters you have already seen in the loop, each character associated with the position of its previous occurrence in the `text`.

text	Expected result
`'success'`	`'succcesuccesss'`
`'albuquerque'`	`'albuququerquerquerquerque'`
`'glarrota'`	`'glarrrotarrota'`
`'dontgangseedmmf'`	`'dontgantgangangseeedontgangseedmmf'`

11. Generalized Fibonacci sequence

```
def generalized_fibonacci(multipliers, n):
```

The Fibonacci sequence is a dusty cliche in teaching computer science, usually presented as the silly exponential time recursion, but better implemented as a loop that produces the elements one at a time based on the previous elements. Since we are going to write this function, we might as well implement a general version that can produce not just the terms of the plain old Fibonacci sequence, but any generalized <u>Lucas sequence</u> where each term F_i is computed as a sum of some m immediately preceding values F_{i-1}, F_{i-2}, ... , F_{i-m}, each preceding term multiplied with the given fixed multiplier a_j applied to the term F_{i-j}. For example, the classic Fibonacci sequence is defined with the recursive equation $F_i = F_{i-1} + F_{i-2}$ where the multipliers of the two preceding terms are $a_1 = a_2 = 1$. <u>Pell numbers</u> are defined with the equation $F_i = 2F_{i-1} + F_{i-2}$ with initial terms $a_1 = 2$ and $a_2 = 1$.

Given the `multipliers` as a tuple of m elements, compute the term in position n of the sequence. Position numbering starts from 0, and the first terms in positions 0, ..., $m - 1$ of the sequence all equal 1. The order of `multipliers` is such that the last multiplier in position −1 is applied to the term immediately preceding the current term being computed, the second last multiplier in position −2 is applied to the term before that term, and so on.

multipliers	n	Expected result
(1, 1)	5	8
(1, 2)	8	577
(4, 4, −3, −7, 5, 0, 7)	31	2579538296491128384859
(7, −7, 9, 9, −7, 1, −3, 9, 4)	35	416464779917747725480

12. Pollard's rho integer factorization

```
def pollard_rho(n):
```

Pollard's rho algorithm is an ingenious little algorithm to quickly factorize large integers. Based on the efficient cycle detection with the idea of **Floyd's tortoise and hare algorithm**, the tactically chosen function $g(x) = (x^2 + 1)$ mod n is iterated starting from $x = 2$ until some cycle is found, the final value of this cycle then magically revealing one of the prime factors of n.

In this problem, you should implement the algorithm from the pseudocode given on the linked Wikipedia page. Your function should return the list of prime factors of n in ascending sorted order. To keep this simple, the automated tester is guaranteed to give your function only values of n that are known **semiprimes**, that is, satisfy $n = pq$ for some two prime numbers p and q. Therefore, once your function has found one of these two prime factors of n, the other prime factor is given by a simple integer division.

n	Expected result
10	[2, 5]
79413038221179407	[26797907, 2963404501]
84997913392275761	[25537943, 3328299127]
65901324277161883	[22238383, 2963404501]
78644668293912529	[22682923, 3467131123]

13. Bays–Durham shuffle

```
def bays_durham_shuffle(item_stream, k):
```

The quality and the **period** of the sequence of pseudorandom numbers produced by any simple random number generator can be massively improved with the ingenious **Bays–Durham shuffle algorithm**, which is simple enough for us to implement here with a handful of lines of Python code.

This function receives a Python **iterator** named `item_stream` that produces a sequence of non-negative integers, and it should apply the Bays–Durham shuffle to this sequence as follows. Unlike in the other problems that you have seen in this and previous two Python problem collections, this iterator does not support **random access indexing** at arbitrary positions, but will produce the elements one at a time when you apply the Python `next` function to it, raising the `StopIteration` exception when it has no elements left.

Use the first `k` items from the stream to build a `table` of k elements, and initialize the `result` list to be empty. Then loop through the remaining $n - k$ items from the stream, and perform the following operation for each such item e: replace the element x currently in the position (e mod k) by the item e, and append the removed element x into the result list. For example, suppose that $k = 4$ and the current `table` contains the values `[17, 42, 99, 5]`, and the next item from the stream equals 55. Compute the table position with formula 55 % 4 = 3, replace the element 5 in that position by 55, and append that replaced element 5 into the result list, giving you the new `table` with values `[17, 42, 99, 55]`. This function should return the list of $n - k$ values computed from the stream in this manner, after the `table` of the first k elements of the stream.

(Inquiring minds would also want to know, of course, what would happen if the Bays–Durham shuffle were iteratively applied to the random sequence already shuffled in that manner.)

14. Double-ended pop

```
def double_ended_pop(items, k):
```

You are given a list of `items`, each one a nonnegative integer, and are allowed to do the following operation k times; remove an element from either the beginning or the end of the `items` list. This function should return the largest possible sum of items that can be built up with *k* such removal operations.

items	k	Expected result
[13, 0, 9, 11, 6, 8, 18, 13, 5, 11]	4	47
[7, 11, 17, 10, 5, 12, 10, 0, 1, 1, 14, 2, 12]	4	47
[17, 9, 0, 15, 19, 18, 7, 10, 20, 2, 0, 0, 20, 13, 3, 9, 14, 2, 2]	7	85
[21, 14, 24, 0, 21, 7, 21, 1, 4, 11, 18, 20, 0, 6, 0, 9, 21, 12, 16, 20]	5	95

15. First singleton

```
def first_singleton(text):
```

This function should find and return the first character in the `text` that occurs in that `text` exactly once. If the `text` does not contain any singleton characters, return `None`.

Of course, your function has to examine every character of the `text` before it can return the answer with absolute confidence. However, with the aid of Python sets and dictionaries to keep track of the characters that you have seen once or more than once, your function should be a single for-loop through the `text` that processes each character in constant time, and thus be blazingly fast even when tested with thousands of strings that contain thousands of characters each.

text	Expected result
`'gjexx'`	`'g'`
`'bababa'`	`None`
`'dddhmsddhdsmdrdhdddnddrs'`	`'n'`
`'fzmfomfpzmzfpomxfculzmfcqm'`	`'x'`

16. Candy sharing game

```
def candy_split(candies):
```

This problem from the 2003 Greater New York Region ACM Collegiate coding contest is similar to the candy-sharing problem in our original 109 Python Problems collection, with a different rule for distributing the candies. A number of students sit in a circle so that each student initially has some even number of `candies`. On teachers' marks, all students simultaneously give exactly half of their candies to the student sitting immediately to their right. Between rounds, the teacher gives each student with an odd number of candies one extra piece to ensure that every student always has an even number.

This system differs from the candy-sharing problem in the original 109 Python problems collection in that the total number of candies does not remain constant as the teacher keeps adding new candies to maintain even parity. However, it can still be proven that this system must eventually reach an equilibrium state in which each student has the same even number of candies. This function should return a two-tuple of the number of rounds needed to reach this equilibrium state, followed by the number of candies that each student ends up with in this equilibrium.

candies	Expected result
[2, 4, 0]	(3, 4)
[4, 4, 2]	(1, 4)
[8, 8, 2, 2, 2]	(7, 8)
[18, 6, 6, 8, 6, 14, 10, 22, 22]	(16, 22)
[50, 22, 56, 2, 24, 30, 14, 6, 24, 46, 40, 46, 36, 46, 24, 12, 14, 30, 32, 10, 16, 44]	(40, 44)

17. Haar wavelet transform: Encode

```
def haar_wavelet_encode(signal):
```

In **signal processing**, a series of observed `signals` gets transformed into a different space that allows certain operations on that signal to be computed more efficiently. We shall leave the theoretical underpinnings of these transformations for an advanced mathematics course. Instead, as an exercise of Python list processing with some simple recursion sprinkled in, we will implement the discrete **Haar wavelet transform**, especially since this transform does not require complex or floating-point numbers but can be implemented with straightforward integer arithmetic.

For simplicity, we assume that the `signal` is a list of integers whose length is some power of two. To perform the transform, split the `signal` into pairs of consecutive elements, and create the list of pairwise sums and differences of these element pairs. (The classical Haar transform also divides all sums and differences by two, but we shall skip that step to keep all values here as integers.)

For example, if the `signal` equals [2, 3, −1, 4, 7, 2, 4, 4], the resulting list of pairwise sums is [5, 3, 9, 8] and the list of pairwise differences is [−1, −5, 5, 0]. Compute the Haar wavelet transform of the list of pairwise sums recursively, and concatenate the transformed pairwise sum with the list of pairwise differences as it is without transform. The base case of the recursion is the singleton `signal` list that just transforms to itself.

signal	Expected result
[1, −5, −1, 0]	[−5, −3, 6, −1]
[4, −1, 2, 1]	[6, 0, 5, 1]
[−6, 4, −3, −8, 3, −2, 8, −5]	[−9, −17, 9, −2, −10, 5, 5, 13]
[2, 7, −3, 10, 4, 6, −7, 6, −10, 6, 7, 1, −3, 2, 5, −8]	[25, 25, 7, 8, 2, 11, −12, 2, −5, −13, −2, −13, −16, 6, −5, 13]

The Haar wavelet transform does not lose any information about the original `signal`, as you can see in the following problem asking you for the function to restore the `signal` from its Haar wavelet coefficients. Before that restoration, these coefficients can be used for compression (after smoothing and discarding small wavelet coefficients, the resulting coefficients compress better than the original noisy signal), feature extraction, edge detection, and data analysis on the signal.

18. Haar wavelet transform: Decode

```
def haar_wavelet_decode(coefficients):
```

This function is the inverse of the discrete Haar wavelet transform encoding function of the previous problem. Given the list of `coefficients` produced by the original Haar wavelet transform, restore the original `signal` that gave these `coefficients`. Again, for simplicity, you may assume that the length of the `coefficients` list is some power of two.

coefficients	Expected result
`[1, -5, -1, 0]`	`[-2, 0, 1, 2]`
`[-6, -4, 6, -2]`	`[0, -5, -2, 1]`
`[5, 2, 3, 0, 5, -1, 0, 6]`	`[4, -1, -1, 1, 0, 1, 3, -2]`
`[3, 8, -7, -5, 11, -4, 1, -12, -8, -9, -7, 0, -5, -9, -6, 11]`	`[-2, 7, -8, 2, -3, 4, 2, 3, -4, 2, -6, 4, -6, 1, 9, -2]`

19. Shell sort

```
def shell_sort(permutation, gaps):
```

Insertion sort is the first serious sorting algorithm that most computer science students analyze and implement as an exercise. Insertion sort is provably optimal among all comparison sort algorithms that are constrained to compare and swap adjacent elements, as it will perform exactly as many comparisons and swaps as are necessary and sufficient to sort the list with such swaps.

Faster sorting algorithms must necessarily compare and swap elements across longer distances over the list to quickly get them closer to their final resting places. Shell sort is a generalization of insertion sort that operates in multiple rounds, each time using a different gap value from the `gaps` sequence in decreasing order. In each round, loop through positions from the current gap value g up to $n - 1$, and repeatedly compare and swap the element in that position with the element located g steps before in in the list until either the gap g would lead past the beginning of the array, or the element g steps before the current position less than or equal to the current element. Since the last gap size is always one, the last round performs an ordinary insertion sort that is guaranteed to sort the list regardless of how the previous rounds have rearranged the elements.

This function should implement the shell sort algorithm as described in the pseudocode on the linked Wikipedia page. However, instead of returning the sorted array, as **proof of work** that you genuinely did implement the actual Shell sort algorithm without any underhanded activity, your function should instead return the two-tuple `(comparisons, swaps)` of how many element comparisons and element swaps were performed during the algorithm for the given `permutation` of integers $0, ..., n - 1$ using the given sequence of `gaps`.

permutation	gaps	Expected result
[5, 4, 1, 2, 0, 3]	[1, 3]	(12, 7)
[6, 0, 5, 1, 3, 4, 2]	[1, 3, 5]	(17, 8)
[9, 7, 3, 5, 0, 1, 12, 6, 4, 2, 10, 8, 11]	[1, 2, 3, 4, 6, 8, 9, 12]	(70, 15)
[13, 7, 11, 9, 10, 4, 8, 14, 3, 5, 0, 2, 12, 6, 1]	[1, 4, 9]	(59, 35)

20. Mixed type sorting

```
def mixed_sort(items):
```

Sorting a list of `items` implicitly requires that all elements are **pairwise comparable** so that these elements form a **total order** so that for any two items *x* and *y*, we can always tell which is smaller whenever they are not equal. However, the general sorting problem can be generalized to **partial orders** by loosening the expectation for the resulting list so that whenever the two original items x and y are not comparable with each other, their relative placement in the sorted list is irrelevant.

For an interesting coding challenge between these two approaches, your task is to return a sorted list of original `items` of both integers and text strings so that all integers end up being sorted in the same positions that originally contained an integer, and all text strings end up being sorted in the same positions that originally contained a text string. To determine the type of an element `x` in the list, use the Python built-in function `type` in something like `type(x) == int`.

items	Expected result
[12, -14, 'cashoo', 'konia', 4, 'goldi', 4, 14, 'yockel']	[-14, 4, 'cashoo', 'goldi', 4, 'konia', 12, 14, 'yockel']
['gratae', -11, 12, -9, 4, -1, 'lupus', 13, 10]	['gratae', -11, -9, -1, 4, 10, 'lupus', 12, 13]
['imbute', 2, -2, 'kroon', 17, 0, 'hagdin', 'rerent', 'alidad', 'cantic']	['alidad', -2, 0, 'cantic', 2, 17, 'hagdin', 'imbute', 'kroon', 'rerent']
[-10, 15, -23, 16, -1, 'gasify', 'osiers', -22, -6, 1, -13, -20, 'fusel', 12, 20, 'axine', 17]	[-23, -22, -20, -13, -10, 'axine', 'fusel', -6, -1, 1, 12, 15, 'gasify', 16, 17, 'osiers', 20]

21. The nine lives of Meaowsephus

```
def meaowsephus(lives, k):
```

The original *109 Python Problems* collection featured the classic Josephus problem where a group of men around the circle get unalived one at the time according to a cruel discipline that they all agreed to follow. However, this problem can be generalized into a more adorable form by, instead of sweaty men with swords and sandals, placing some cute kittens around the circle.

The positions of these kittens are numbered 0, ..., $n - 1$. Each kitten initially has a certain number of `lives` remaining. The elimination process starts at the beginning of the circle. Each elimination counts over k remaining kittens from the current position onwards, and takes away one of the lives of the kitten that the count landed at. Whenever a kitten loses its last life, it is removed from the circle to receive endless cuddles and hugs. The circle always tightens to compensate the eliminated kitten so that position is skipped entirely in the future rounds of the elimination count.

This function should return the elimination order of the kittens, each kitten numbered according to its original position. The main difficulty is making this function efficient by maintaining the structure of the kitten circle in a way that allows stepping from the current kitten to the next remaining kitten in small constant time, regardless of how many kittens have been eliminated between them.

lives	k	Expected result
[4, 1, 3]	2	[1, 0, 2]
[3, 6, 6, 1]	3	[3, 2, 0, 1]
[2, 1, 6, 7, 6, 1]	2	[1, 5, 2, 4, 3, 0]
[5, 2, 9, 5, 1, 8, 10, 10]	2	[1, 4, 3, 0, 5, 7, 6, 2]

22. Maximum repeated suffix

```
def maximum_repeated_suffix(items):
```

This function should find the length of the **maximum repeated suffix** of the given sequence of `items`, that is, the longest suffix that is immediately preceded in that sequence by that very same suffix. This function should not use list slicing to extract the suffix, but perform the comparison in the same list by looping through the position indices, which should make this function fast, even for the barrage of thousands of lists that each contain thousands of elements thrown at this function by our automated fuzz tester.

`items`	Expected result
`[1, 3, 2, 3]`	0
`[3, 0, 3, 0]`	2
`[2, 0, 1, 3, 1, 1, 3, 1]`	3
`[1, 0, 0, 3, 0, 1, 5, 0, 5, 5, 0, 3, 0, 1, 5, 0, 5, 5]`	

23. Count slices with given sum

```
def count_slices_with_sum(items, goal):
```

Given a list of `items`, each item guaranteed to be a positive integer, this function should count the number of separate **slices** in this list whose elements add up to the given `goal`. These slices are allowed to partially overlap if necessary, and can consist of a singleton element equal to the `goal`.

You could, of course, iterate through all possible starting positions of slices, and then loop from each starting position, add up elements until the total becomes greater than or equal to the `goal`. However, to make this algorithm way more efficient, you should note that if you have already tallied up the sum of elements of some particular slice, you can easily compute the sum of elements of that slice without its first element simply by subtracting that first element from the tally.

This approach allows you to compute the answer in a single linear-time loop over the entire list. Use two indices to mark the start and end positions of the current slice, both indices starting at the first element. If the sum of items in the current slice is less than `goal`, advance the end index, and otherwise advance the start index, first incrementing the tally whenever the sum of items equals `goal`. Since each iteration of the loop advances at least one index, the algorithm is guaranteed to finish in time that is linear in the number of elements in the list.

Such a **two-pointers** approach of using two indices to delimit the sublist in question is a common technique to solve many problems on a list in linear instead of quadratic time. In problems like this one, these two indices act as "point man" and "rear guard", never retreating but always advancing towards the enemy until victory is reached. (In some other problems, for example, the "two summers" problem from course examples, the indices start from opposite ends of the list and always advance towards each other until they meet somewhere in the middle.) As of this writing, the Leet-Code category "Two pointers" offers 219 problems amenable to two pointers technique.

items	goal	Expected result
[2, 1, 1, 1, 2, 1]	2	4
[1, 3, 2, 1, 2, 1, 3, 2, 1, 2]	3	7
[1, 4, 3, 1, 4, 3, 2, 1, 3, 1, 3, 1, 1, 2, 1]	8	8
[3, 4, 4, 1, 1, 3, 1, 4, 2, 3, 4, 1, 3, 5, 5, 4]	14	7

24. First sublist whose sum is a multiple of *k*

```
def sublist_sum_k(items, k):
```

Right on the heels of the previous problem, swerves in another one on the same theme: determine whether the given list of positive integer `items` contains at least one nonempty contiguous sublist whose sum of elements is an integer multiple of *k*. If at least one such sublist exists, return the start position of the earliest such sublist; otherwise, return `None`.

To avoid being a "Shlemiel" who solves this problem by looping through all possible sublists of `items`, we need to brush up a little bit on elementary number theory to get to a working solution that consists of a single for-loop over the `items` aided by some additional data structures. You should keep track of the running **prefix sum** of the elements that the loop has processed so far, and at each element, compute the remainder of the current prefix sum when divided by `k`. Keeping track of all the remainders that you have seen during this loop, along with the positions of these remainders, allows you to quickly recognize not only such a sublist but also its starting position.

items	k	Expected result
[1, 10, 15, 23, 35, 22]	25	1
[24, 14, 33, 8, 31, 4]	32	None
[79, 80, 40, 66, 37, 4, 27, 55, 51, 88]	133	6
[87, 187, 71, 125, 4, 175, 81, 110, 66, 124, 28, 66, 87, 193]	454	None

25. Ones mule, zeros pool

```
def one_zero(n):
```

The original 109 Python Problems collection featured a problem "Sevens rule, zeros drool" that asked the reader to find the smallest integer m whose positional base-10 representation is a solid nonempty sequence of sevens followed by a solid (possibly empty) sequence of zeros so that this m is divisible by the given n. An integer m of this restricted form satisfying this requirement was always guaranteed to exist. In the same vein, this problem now asks you to find the smallest integer m whose positional base-10 representation consists of a solid nonempty sequence of ones followed by a solid (possibly empty) sequence of zeros.

The clever proof to establish the existence of such m also immediately gives us a method to construct it. Your function through the **repdigit** integers 1, 11, 111, 1111, 11111, ..., noting the remainder left over when each repdigit integer is divided by n. If you come upon a repdigit integer that is divisible by n, well then, that's the one that you need. Alternatively, whenever you find two repdigit integers r_1 and r_2 with the same remainder, their difference $r_1 - r_2$ is also divisible by n, and is trivially of the required form! Use a dictionary to map the n possible remainders to the repdigit numbers that produced them, so that you can quickly recognize that you have already seen the current remainder earlier in the repdigit sequence and subtract those two repdigit numbers.

The famous **pigeonhole principle** guarantees that you must hit the answer after at most n steps. (In fact, the equally important **birthday paradox** ensures that you will usually hit the required answer much sooner, on average, roughly around the square root of n.)

n	Expected result
1	1
3	111
4	100
11	11
28	11111100
118	1111111111111111111111111111111111111111111111111111111111110

26. The backwards god

```
def count_dogs(grid):
```

You are given a two-dimensional `grid` whole each element is one of the three possible characters `'d'`, `'o'` or `'g'`. This grid is considered to be a **torus** with its opposite edges wrapped together, that is, moving off one edge makes you appear from the opposite edge. Your function should count how many times the word `'dog'` appears in this grid. The word can be positioned in any one of the eight directions, and can wrap around the edge.

This problem can be solved with a bunch of nested loops, but the point of this exercise is not repeating yourself for each of the eight possible directions and the logic of wrapping around the edges. In all test cases, both the width and height of the `grid` are at least three.

grid	Expected result
['ggd', 'ogg', 'ood', 'dgo', 'gdg']	4
['dood', 'ogod', 'odoo', 'ggdg']	6
['dgdggo', 'gdoood', 'ogdoog', 'gggdoo', 'odoooo', 'doggdg']	12

27. Klee's measure problem

```
def klee_measure(intervals):
```

A list of `intervals` consists of two-tuples of the form `(start, end)`, each such tuple denoting a closed **interval** on the one-dimensional real number line with the given `start` and `end` locations. The length of that individual interval is then `end – start`. This function should compute the total length of the **union** of these intervals so that any overlapping parts of multiple intervals are included only once in this total length.

This problem can be solved efficiently with the application of the **sweepline** technique, well known in **computational geometry** problems of all dimensions. Create a sorted list that contains all starting point and ending points of intervals as scalars, each such point paired with either +1 or –1 depending on whether it is a starting or ending point. Loop through the points in this list in their ascending sorted order, keeping a counter to measure the number of intervals that are currently active and overlapping during the journey. When this counter increases from zero to one, remember that point as the starting point of the current interval clump. Once the counter decreases back to zero, add the distance of the current point and the previously stored starting point to the total length.

intervals	Expected result
`[(28, 30), (-14, -9), (-4, -2), (-8, -7)]`	10
`[(12, 14), (31, 36), (6, 13), (-31, -29), (-21, -14), (27, 29)]`	24
`[(34, 39), (-20, -14), (-28, -23), (-16, -13), (27, 30), (-29, -24), (20, 26), (40, 47)]`	34

This Klee's measure problem is famous in that devising the provably optimal algorithm for higher dimensions *d* > 2 is still an open problem.

28. Poker test of randomness

```
def poker_test(sequence):
```

Statistical tests to evaluate the quality of **pseudorandom number generators** calculate various statistical quantities for the pseudorandom sequences that they generate, and compare those quantities to the quantities that a truly random sequence would be mathematically expected to have.

One such test looks at all the $n - 4$ overlapping sublists of five elements in the given `sequence` of n elements, and classifies each sublist according to which type of hand in **dice poker** that subsequence of five elements corresponds to. This function should return the list of counts of each type of hand in the exact order of the following seven hand types: five of a kind, four of a kind, full house, three of a kind, two pair, one pair, and high card.

sequence	Expected result
`[0, 3, 6, 6, 2, 6, 0, 3, 1, 1, 2, 1, 1, 4]`	`[0, 1, 0, 4, 0, 4, 1]`
`[6, 4, 6, 4, 4, 4, 4, 4, 4, 2, 4, 4, 2, 4, 2, 2, 2, 5, 1]`	`[2, 6, 5, 2, 0, 0, 0]`
`[4, 7, 1, 7, 7, 1, 3, 8, 3, 8, 8, 8, 8, 8, 8, 8, 8, 1, 1, 1, 2, 1]`	`[4, 4, 4, 2, 2, 2, 0]`
`[10, 4, 2, 0, 5, 2, 0, 6, 7, 11, 2, 3, 5, 3, 10, 6, 11, 10, 10, 7, 8, 3, 10, 2, 4, 10, 11]`	`[0, 0, 0, 1, 1, 12, 9]`

29. Three-way partition

```
def three_way_partition(items):
```

A list of `items` consists of three kinds of elements called "small", "medium" and "large". Your task is to tactically **swap** its elements with each other so that all small elements end up being together first in a consecutive block, followed by all medium elements together as another consecutive block, followed by all large elements together as the final consecutive block. The order of elements inside each block is irrelevant, as long as all original elements of the same kind end up together in one consecutive block.

For simplicity in this problem, the list of `items` is guaranteed to consist of elements 0, 1 and 2 only, denoting the original small, medium and large elements, respectively. Your function should compute and return the minimum number of swaps required to achieve the required partitioning. The famous Dutch national flag algorithm achieves this minimum number of swaps with a ravishingly beautiful technique that is often deservedly used as an educational example of proving the correctness of an algorithm using loop invariants.

The reader should implement this algorithm from the pseudocode given on the linked Wikipedia page with an additional swap counter whose final value the function then returns in the end. Note that the swap counter should not be incremented in the situation where the algorithm nominally performs a swap of two equal elements. After implementing this algorithm, the reader should read through it carefully to appreciate its elegance of doing only what is absolutely necessary to achieve the final result, but never spinning its wheels for no reason at any point during its execution.

items	Expected result
[0]	0
[1, 0, 2]	1
[0, 1, 0, 2, 1, 2]	2
[2, 0, 0, 2, 0, 1, 1]	5
[0, 0, 2, 2, 2, 0, 0, 1, 1, 0, 1]	6

Partitioning algorithms of this nature are used in the Quicksort sorting algorithm to rearrange the elements into separate blocks that can then be recursively sorted independently of each other, these days even in **parallel** for an even more dramatic speedup. The Dutch national flag partition algorithm has the advantage of producing a nice split even when some element is duplicated a large number of times in the list, whereas alternative partitioning methods might end up producing maximally unbalanced partitions of one element on one side and all the others on the other side.

30. Primality buildup

```
def is_prime(n):
```

Many algorithms exist to check if the given positive integer n is a **prime number**, that is, divisible only by one and by itself. Here we shall content ourselves with **trial division**, trying potential divisors up to the square root of n and concluding that n is prime after none of them worked. However, it is sufficient to try only trial divisors that are themselves prime numbers. We shall therefore speed up this process by maintaining a global list of known `primes` that we have found so far, initialized to something like `[2, 3, 5, 7, 11, 13, 17]` to, well, "prime the pump" to generate these trial divisors.

If n is less than or equal to the last element of the `primes` list, `is_prime` should check if n is a member of the `primes` list using the **binary search** function `bisect_left` in the standard library `bisect` module. Otherwise, the function should iterate through the trial divisors up to the integer square root of n, (use the `isqrt` function in the `math` module), returning `False` as soon as one working divisor is found, and returning `True` after all the potential divisors were tried and found wanting.

What if the square root n is large enough for the `primes` not to reach its square root? In addition to the `is_prime` function, write an auxiliary helper function `expand_primes(m)` that keeps appending new prime numbers to the end of the `primes` list until the last prime number is greater or equal to m. If the current last known prime number in the primes list equals *p*, just loop up through the numbers *p* + 2, *p* + 4, *p* + 6, ... until you hit a prime number. Use your function `is_prime` to quickly check whether the current candidate is a prime.

Implemented this way, your function should be swift even for numbers up to one trillion, requiring memory to cache the first prime numbers up to one million.

n	Expected result
67	True
8911	False
1234568837	True
1693101241	False

31. Goldbach verification

```
def goldbach(n):
```

Along with the twin prime conjecture, the equally famous Goldbach conjecture is another yet-to-be-proven hypothesis that states that every positive integer greater than two can be expressed as a sum of two primes in at least one way. For example, 42 = 5 + 37 and 112 = 3 + 109. (Note again that 1 is not a prime number, but an algebraic **unit**.) This conjecture has been verified for large values of n, but the definitive proof to establish this sweeping claim for every positive even integer is still waiting for someone to reach mathematical immortality by being the first man or machine to discover it.

For most even numbers, there actually exist multiple prime pairs that add up to it. This function should return the first prime number pair that adds up to `n` in the sense that its first number is as small as possible. Again, use the earlier utility function `is_prime` and the `primes` list that this function maintains to quickly iterate through the candidate prime numbers p needed for this search, returning the first pair $(p, n - p)$ where both p and $n - p$ are prime numbers.

n	Expected result
42	(5, 37)
724	(5, 719)
1682	(13, 1669)
5618	(37, 5581)

32. A square and twice a prime

```
def square_and_twice_prime(n):
```

The Goldbach conjecture that we met in the previous problem is the most famous conjecture about expressing arbitrary numbers using prime numbers, but other similar conjectures exist. A recent tweet by Mike Lawler presents another, still unsolved conjecture that says that for any integer $n > 1$, the integer $3 + 8n$ can be expressed in form $m^2 + 2p$, where m is some positive integer and p is a prime number. As we can see from the results below, this representation is not necessarily unique. This function should return the sorted list of all values of m that work for the given n in this sense.

n	Expected result
3	`[1]`
52	`[5, 11, 15, 19]`
79	`[1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 25]`
193	`[1, 3, 5, 9, 15, 19, 23, 25, 27, 29, 31, 33, 37, 39]`
9771391	(a list of 637 possible values of m)

33. Pseudo-Smarandache function

```
def pseudo_smarandache(n):
```

The curiously named **pseudo-Smarandache** function $Z(n)$ is defined for the given positive integer n as the smallest positive integer m for which the m-th **triangular number** $m(m + 1)/2$ is exactly divisible by n. For example, the reader can quickly verify that $Z(4) = 7$, since of the first seven triangular numbers 1, 3, 6, 10, 15, 21, and 28, only the seventh number 28 is divisible by 4. This function should return the value of m that gives that first triangular number.

This problem can, of course, be solved with a **brute force** search that goes through the positive integers from $m = 1$ upwards until it hits the first value of m that works for the given n. However, for some convenient values of n, $Z(n)$ can be computed directly with the following rules without having to resort to a brute-force search needed for other values of n.

- If n is any power of two greater than 1, then $Z(n) = 2n - 1$.
- If n is any power of some prime number greater than two, $Z(n) = n - 1$.
- If $n = 2p$ so that p is an odd prime number, then $Z(n) = p - 1$ whenever $p - 1$ is divisible by 4, and in the other cases of that divisibility, $Z(n) = p$.

n	Expected result
2	3
5	4
6	3
10	4
17	16
110	44

The pseudo-Smarandache function has mainly significance in recreational mathematics thanks to its chaotic behaviour and interesting properties, many of which are still open to conjecture. For example, it is known that $Z(n) \neq n$ for every n (as you can see in the above table, $Z(n)$ can be less than n), and that $Z(n) \neq Z(n + 1)$ for every n. The interested reader can search for articles that examine and document both resolved and still open properties of this function. Florentin Smarandache also invented many other integer functions worthy of examination.

34. Conway's subprime sequence

```
def conway_subprime(a, b):
```

We keep nibbling on the mysteries of the Fibonacci sequence with a simple variation invented by John Horton Conway, because, of course, once again, who else could it have been? (As the famous expression originally said about Isaac Newton puts it, you always recognize the lion by its claw.)

Conway's subprime Fibonacci sequence is defined the same way as the ordinary Fibonacci sequence, but whenever the next term would be a **composite** number (that is, not a prime number), it is divided by its smallest prime factor. For example, starting from the first two numbers 1, 1, the next three numbers are 2, 3, and 5 the same as in the original Fibonacci sequence, but since the following term 3 + 5 = 8 is a composite number whose smallest prime factor is 2, this term now becomes 8 / 2 = 4 instead. Use the function `is_prime` implemented in the earlier problem and the `primes` list that this function maintains to quickly iterate through the potential prime divisors of the next term.

Unlike the ordinary Fibonacci sequence that will shoot up exponentially towards infinity, the subprime sequence keeps bouncing up and down, and always seems to end up in a cycle by reaching some pair of consecutive terms that it has encountered earlier in the sequence. The conjecture that states that every subprime sequence that starts from any two positive integers will eventually reach a finite cycle is still open, similar to the more famous Collatz 3x + 1 conjecture. Of course, this conjecture has also been checked computationally for a large number of starting value pairs, but the general proof still remains out of reach. For our purposes in this problem set, this function should simply return the first repeating pair of terms for the Conway subprime sequence starting from the two initial terms `a` and `b`.

a	b	Expected result
1	4	(11, 9)
3	5	(48, 13)
1	8	(48, 13)
2	17	(24, 7)

35. Rowland's prime pump

```
def rowland_prime_pump(a_start, k):
```

Consider the integer sequence where after the first element in the first position 1, the element in each position *n* is given by the formula $a + \gcd(a, n)$ where *a* is the element in the previous position *n* – 1, and gcd is the **greatest common divisor** of its parameters, available as the function `gcd` in the `math` module of the Python standard library. The reader can verify that, after this sequence is started from 7, it will continue as 7, 8, 9, 10, 15, 18, 19, 20, 21, 22, 33, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, ...

The consecutive values usually increase by 1, but occasionally jump by much more. Weird but true: as discussed in the post "Pumping the primes" by Bit Player, assuming certain complex restrictions to the first element of the sequence (these restrictions are trivially satisfied by every prime number greater than six, but also by some sufficiently tricksy composite numbers), every time the difference between two consecutive elements is greater than one, this difference itself is guaranteed to be some prime number! This sequence can therefore be used to generate prime numbers in a seemingly random order: not very efficiently, but in an amusing fashion. (Why, you may ask. But why not?)

This function should return the list of the first *k* distinct prime numbers produced by the Rowland sequence starting from the given `a_start`, listed in the order in which these primes first appeared in the sequence. To make this function efficient, you need to implement the technique described in the linked Bit Player post to skip over the tedious sequences of consecutive ones in the sequence of differences. This technique requires the `is_prime` function from the earlier problem to determine whether $a - n$ is a prime number, necessary for this optimization to apply correctly.

a_start	k	Expected result
7	20	[5, 3, 11, 23, 47, 101, 7, 13, 233, 467, 941, 1889, 3779, 7559, 15131, 53, 30323, 60647, 121403, 242807]
1303	25	[1301, 3, 19, 2621, 7, 29, 5279, 10559, 5, 11, 21143, 42293, 251, 43, 17, 73, 31, 13, 23, 85061, 170123, 89, 340337, 680681, 1361363]
7759	43	[7757, 3, 5, 11, 7, 41, 15581, 31181, 47, 62417, 127, 17, 124991, 19, 53, 29, 31, 250169, 37, 617, 501029, 83, 1002143, 521, 2004809, 4009619, 109, 8019353, 16038707, 727, 32078177, 23, 13, 5431, 151, 179, 761, 64162961, 128325983, 193, 256652189, 73, 317]

36. Lehmer code: encoding

```
def lehmer_code(permutation):
```

In the theory of permutations, an **inversion** is a pair of positions $i < j$ in which the element in the position i is larger than the element in the position j. The count of how many inversions the permutation contains measures how "out of order" that permutation is compared to the perfectly sorted permutation whose elements are in ascending order.

The **right inversion count**, also called the Lehmer code of a permutation of n distinct elements, is itself a list of $n - 1$ elements. The element in position i of the Lehmer code counts the number of inversions where that position appears as the first position of the pair $i < j$. Alternatively, the element in the position i of the Lehmer code counts how many elements after that position are smaller than the element in position i. Since this count would always be zero for the last position of the permutation, this redundant zero is left out of the Lehmer code.

This function should return the Lehmer code for the given `permutation` of integers $0, ..., n - 1$.

perm	Expected result
[0]	[]
[1, 2, 0]	[1, 1]
[1, 0, 2]	[1, 0]
[0, 1, 2, 3]	[0, 0, 0]
[3, 2, 1, 0]	[3, 2, 1]
[5, 7, 0, 4, 8, 6, 3, 2, 3]	[5, 6, 0, 3, 3, 2, 1]

Since the result of the Lehmer code can always be further interpreted as a factoradic number (see the later problem "Factoradical dudes" in this same collection), this allows the permutations of integers $0, ..., n - 1$ to be encoded **bijectively** ("one-to-one") to the integers $0, ..., n! - 1$. The inverse versions of these two functions applied in reverse order will then decode these nonnegative integers back into the original permutations.

37. Sum of square roots

```
def square_root_sum(n1, n2):
```

One big thing that this instructor tries to do differently compared to most teachers of introductory Python courses is to train his students to instinctively avoid using floating point numbers, the same as how they would avoid following the voice coming from the sewer to tell them that they are all just "floating" down there. However, paraphrasing the life story of the reformed mob *caporegime* Michael Franzese with the popular YouTube channel, if you are going to be in the streets, then you have to be in the streets the right way. Instead of using the binary floating point type given by your computer's processor, you should use the `Decimal` type defined in the decimal module to perform your floating point calculations in base ten to the desired arbitrary precision.

Given two lists of positive integers, determine whether the sum of square roots of the numbers in the first list is strictly larger than the sum of square roots of the numbers in the second list. You should perform these computations using the `Decimal` type increasing the precision until the two sums of square roots are distinct enough.

n1	n2	Expected result
`[6, 9, 13]`	`[5, 10, 12]`	`True`
`[7, 13, 14, 18, 22]`	`[8, 12, 15, 17, 23]`	`False`
`[15, 20, 24, 28, 29]`	`[16, 19, 25, 27, 30]`	`False`
`[14, 17, 20, 23, 24, 28]`	`[15, 16, 21, 22, 25, 27]`	`False`

From the perspective of computational complexity, this problem is much harder than it might seem. The intricacies of integer mathematics and irrational numbers make it difficult to determine how many decimal places the addition needs to be performed to be absolutely sure about which sum is larger. The exact placement of this problem in a specific complexity class is still an open question in computability theory.

38. Night shift

```
def night_shift(data, rules, timeout=10000):
```

Based on the principle of repeated **string rewriting**, the esoteric programming language named Night shift is simple enough for us to implement here in its full glory. Yet this language is **Turing-complete**, meaning that any computation whatsoever that can be expressed in any programming language could also be expressed as an equivalent Night shift program! Such languages are a beautiful illustration of how little machinery is fundamentally needed to achieve all of computability by repeatedly doing small finite local decisions and modifications to linear (but unbounded) `data`.

A program written in Night shift is given as a list of `rules`, where each rule is a string of the form `"left-right"`, where both `left` and `right` are sequences made of digit characters zero and one. A minus sign inside the rule separates the `left` and `right` parts. The `right` part of the rule can also be empty, which is denoted by the special slash character. No spaces or any other characters than these previously mentioned exist inside the rule.

The program is given as input a piece of `data` that also consists of zeros and ones. The execution starts with prepending the bit pattern `'000'` to the beginning of the `data`. Each individual computation step first finds the rule whose `left` side occurs leftmost in the current `data`. If the `left` side of several rules occurs in that same leftmost position, use the rule placed the earliest in the `rules` list. If the `left` side of none of the rules matches any part of the `data`, the program terminates, and returns as its result the current `data`. Otherwise, replace the matching occurrence of the `left` side of the chosen rule in `data` by the `right` side of that very same rule, keeping the rest of the bits of `data` as they were. Repeat until done. To avoid falling into an **infinite loop**, we also terminate the program by force once it has taken `timeout` steps without terminating on its own.

data	rules (from the esolangs.org page)	Expected result
`'10'`	(remove first bit)	`'0'`
`'00001'`	(reverse bits)	`'10000'`
`'110110'`	(last bit)	`'0'`

39. Infinite self-square integer

```
def self_square(i):
```

In the domain of integers, the equation $x^2 = x$ has only the trivial solutions $x = 0$ and $x = 1$. However, as explained in the YouTube video "<u>Infinite Digits: A Number That Squares to Itself</u>", this equation has two additional solutions in **infinite integers** that consist of an infinite sequence of digits going **left**, unlike the familiar decimal part of a real number that is infinite going right! (If you accept the existence of the latter, why would you not also accept the former? Infinite to the left, infinite to the right, what's the difference? Infinite sequences of digits are not real, they can't hurt you...)

Since both addition and multiplication for infinite integers can be performed with the usual column-wise algorithms the same as for finite integers, the equation $x^2 = x$ has two such infinite solutions, one solution starting with the digit 5 in the units place, the other solution starting with the digit 6 in that place. This function should return the digit in the i:th position of the first of these two solutions, where i is the power of ten that corresponds to the i:th position in that number. The lowest 50 digits of that infinite self-square integer equal

```
...57423423230896109004106619977392256259918212890625
```

To find the digit in the position i of this Eldritch abomination using ordinary integer arithmetic, prepend each digit from 0 to 9 one at a time in front of the finite number that consists of the previously computed digits in positions $i - 1$ down to 0. Square the resulting finite integer, and compare the $(i + 1)$ lowest digits of this square to the original digits. Return the digit whose prepending makes both of those finite numbers equal. (We will simply accept that this works. Source is the classic "Trust me, bro.") For example, after producing the first two digits 25, only the digit 6 will work in the hundreds position, since only 625 × 625 = 390625 ends with that same 625.

For simplicity, the automated tester will call your function starting from $i = 0$ with this parameter increasing by exactly one in each subsequent call, so your function can cache the digit sequence produced so far in the previous calls to speed up the computation of its next digit.

i	Expected result
0	5
2	6
49	5
99	3

As explained in the video, the digits of the other infinite self-square with the digit 6 in the units place can be easily computed from this first solution simply by subtracting each digit from 9.

40. Strahler numbers

```
def strahler(tribute):
```

A river system consists of some number of river segments that occasionally come together in nodes numbered 0, …, *n* – 1, where 0 is the sea that the entire river system flows into. Each node is a tributary to some subset of lower-numbered nodes, so that `tribute[i]` gives the nodes that water flows to from node `i`. No node is ever a tributary to itself of any higher-numbered node. Note that the river system is not a tree, but a **directed acyclic graph** where two branches emanating from the same node may eventually meet again in some lower-numbered node.

Given this tributary information, this function should return the list of Strahler numbers for all *n* nodes. As explained on the linked Wikipedia page, the Strahler number of each node is computed recursively from the Strahler numbers of its tributaries, according to the following rules:

- A node with no tributaries has a Strahler number of 1.
- A node whose one tributary has a Strahler number equal to *k*, while all its other tributaries have a Strahler number strictly less than *k*, also has that same Strahler number *k*.
- A node with two or more tributaries with a Strahler number equal to *k*, where *k* is the maximum of the Strahler numbers of its tributaries, has a Strahler number equal to *k* + 1.

Since the nodes and their tributary flows in this problem have already been **topologically sorted**, you can simply compute the Strahler numbers of each node in reverse order from *n* – 1 down to 0.

tribute	Expected result
`[[], [0], [0]]`	`[2, 1, 1]`
`[[], [0], [1], [2]]`	`[1, 1, 1, 1]`
`[[], [0], [0], [0, 2], [0, 3, 2], [3, 4]]`	`[3, 1, 2, 2, 1, 1]`
`[[], [0], [0], [1, 0], [2, 3], [3], [5, 4, 3, 0], [0, 6, 4, 5, 1], [4, 6, 0, 2, 3, 5, 1]]`	`[4, 3, 2, 3, 2, 2, 2, 1, 1]`

41. Loopless walk

```
def loopless_walk(steps):
```

Another neat one from Stack Overflow Code Golf. A series of `steps` visits some letters in the given order. However, we want to shorten this journey by repeatedly performing the following operation as many times as possible: find the first letter *c* that occurs a second time in `steps`, and remove every letter in all positions following the first occurrence of *c*, up to and including the second occurrence. For example, this operation would turn `'baflac'` into `'bac'`. Once this operation cannot be performed because no letter occurs twice in `steps`, return the remaining `steps`.

This operation is pretty straightforward in theory, but making it efficient for long strings might require some thinking to avoid looping through the same prefixes of the string redundantly multiple times, or having to build up new strings by extracting some small part near the front.

steps	Expected result
`'zzz'`	`'z'`
`'aasaa'`	`'a'`
`'wczzzv'`	`'wczv'`
`'bhkzmrvfr'`	`'bnkzmr'`
`'llplllilllnell'`	`'l'`

42. Maximum number of blocks during shutdown

```
def max_blocks(permutation):
```

In a row of *n* lights numbered 0, ..., *n* – 1 where each light is either on or off, a **block** is a maximal sequence of consecutive lamps that are turned on. All lamps are initially turned on, forming exactly one block. These lamps are turned off one at a time, in the order given by the `permutation` given to the function received as a parameter. This function should keep track of the number of blocks formed by these lamps after turning off each lamp individually, and return the largest number of blocks encountered during the process.

To efficiently update the block count at each removal, your function should maintain a list of *n* truth values that indicate which lamps are still turned on. When the lamp in position *i* is turned off, this block count can be updated looking at the lamps in the neighbouring positions *i* – 1 and *i* + 1. Make sure that your logic works correctly at both ends of the row.

permutation	Expected result
[4, 2, 1, 0, 3]	2
[0, 4, 3, 2, 1]	1
[6, 2, 1, 3, 5, 0, 4, 7]	3
[9, 4, 7, 11, 6, 1, 0, 8, 5, 2, 14, 13, 3, 12, 10]	6

43. List of all factors

```
def all_factors(n):
```

This function should return the sorted list of all **factors** of the given positive integer *n*, that is, all positive integers *m* that exactly divide *n*, including *n* itself. For example, as the reader can readily verify, the factors of 120 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120. Instead of solving this important problem with **brute force** by looping through all integers from 1 to *n* / 2 and trying to divide *n* by each such candidate factor, we shall remember the **fundamental theorem of arithmetic** that says that every positive integer has exactly one factorization to its prime factors. Conversely, any two numbers that differ in at least one of their prime factors must themselves be different numbers. We will therefore first find all the prime factors of *n*,

Use your earlier `is_prime` function to produce the prime factors of *n*, each prime factor listed as many times as it divides *n*. For example, the prime factors of *n* = 120 are `[2, 2, 2, 3, 5]`. Simply loop through the list of candidate primes that your utility function generates, dividing *n* by *p* and appending p to your list of prime factors whenever *p* is a prime factor of *n*. Once *n* becomes 1 so that all its prime factors have been divided away, the list is complete.

Next, convert the list of prime factors into a list of lists whose each element list consists of the possible powers of the original prime factor, including the zeroth power equal to 1, for all possible ways that prime factor can be taken to build up a factor of the original *n*. For example, the prime factors `[2, 2, 2, 3, 5]` list would become `[[1, 2, 4, 8], [1, 3], [1, 5]]`.

To finish this off, loop through all possible ways to choose one prime power from each element list, adding the product of your result list of all factors. The `product` function in the standard library `itertools` module should achieve this with flourish. Sort this result list, and Bob's your uncle.

n	Expected result
4	[1, 2, 4]
20	[1, 2, 4, 5, 10, 20]
179	[1, 179]
180	[1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180]
1014	[1, 2, 3, 6, 13, 26, 39, 78, 169, 338, 507, 1014]

44. Fourth seat opening

```
def fourth_seat_opening(hand):
```

Sitting at the bridge table in yet another tournament instead of touching grass like a normal person, the hand sees three players pass in front of you, and you need to decide whether to open the bidding or pass out the hand altogether. A handy rule of thumb of <u>rule of fifteen</u>, also historically called the **Cansino count**, says to add up your raw **high card points** (4 four each ace, 3 for each king, 2 for each queen, and 1 for each jack) and the number of spade cards in your hand regardless of their ranks, and open the bidding if this count adds to 15 or more. The spade suit is special since it's the "boss suit", and you would like the opponents to afterwards lament "We missed our spade fit, partner, you should have known to open, herpa derpa derp" whenever you are short in that suit.)

Nothing is certain in life or bridge, but this rule has the highest expected value in this decision of uncertainty, over all possible distributions of cards for the other three players that is consistent with the observed evidence of all three passing out. According to bridge author Larry Cohen, it is also profitable to open in the fourth seat with a pre-emptive bid if you hold a major suit of at least six cards or a minor suit of at least seven cards, and have at least 10 high card points. With 14 high card points, you still open but at one level, since you don't want to miss your game contract when the partner has a good limit raise in your major. (Most serious partnerships don't use two-level pre-empts for minors, but give the 2♣ and 2♦ opening bids a more important artificial meaning.)

Your `hand` is once again encoded the same way as in the example program `cardproblems.py`. This function should determine whether it is profitable to open the given `hand` in the fourth seat.

hand	Expected result
[('ten', 'hearts'), ('three', 'diamonds'), ('king', 'hearts'), ('seven', 'hearts'), ('six', 'hearts'), ('four', 'diamonds'), ('four', 'hearts'), ('king', 'spades'), ('two', 'spades'), ('eight', 'hearts'), ('seven', 'diamonds'), ('two', 'diamonds'), ('ace', 'diamonds')]	True
[('ten', 'hearts'), ('five', 'hearts'), ('king', 'diamonds'), ('seven', 'hearts'), ('four', 'diamonds'), ('eight', 'spades'), ('ace', 'spades'), ('four', 'clubs'), ('eight', 'diamonds'), ('eight', 'hearts'), ('seven', 'diamonds'), ('four', 'spades'), ('ten', 'diamonds')]	False

45. Convert a fraction to a simple continued fraction

```
def to_simple_continued_fraction(a, b):
```

Consider a rational number a/b where $0 < a < b$ so that the value of that fraction is between 0 and 1. This fraction can be expressed equivalently as $1/(b/a)$. Since $b > a$, the denominator b/a can be written equivalently as $c + (b \bmod a)/a$, where c equals the integer part of the division b/a that will be at least equal to one. The leftover smaller fraction $(b \bmod a)/a$ can then be rewritten similarly. The remaining fraction will eventually be of the form $1/b$, giving us a <u>simple continued fractions</u>, an alternative way to express rational numbers as a list of positive integers. This function should return the simple continued fraction representation of the initial fraction a/b.

For example, consider the fraction 5/17. This can be written as $1/(17/5)$, which can be written as $1/(3 + 2/5)$. The inner fraction 2/5 can in turn be written as $1/(5/2) = 1/(2 + 1/2)$, giving the final complete representation of $5/17 = 1/(3 + 1/(2 + 1/2))$. Listing the integer terms of the in the order that they appear at the levels gives the result list `[3, 2, 2]`.

a	b	Expected result
3	13	`[4, 3]`
5	17	`[3, 2, 2]`
5	61	`[12, 5]`
11	97	`[8, 1, 4, 2]`
20086956	85050054	`[4, 4, 3, 1, 2, 8, 2, 20, 7, 4, 2]`

Simple continued fractions have all kinds of curious mathematical properties and practical applications. For example, finding the best rational approximation p/q for the given quantity x so that no fraction with a smaller denominator than q is closer to x than p/q. The simple continued fraction is always finite for any rational starting numbers and infinite for irrational numbers. This infinite sequence of terms eventually becomes periodic if and only if the original number is a **quadratic irrational**, that is, a solution of some polynomial equation of order two.

46. Convert a simple continued fraction to a fraction

```
def from_simple_continued_fraction(terms):
```

The function is the inverse of the previous problem of converting a rational number to a simple continued fraction in that this function should reverse the conversion and return the original rational number as a Python `Fraction` object. Your function should process the `terms` from end to beginning, adding each term to the accumulated fraction so far and taking the inverse of that result to be the accumulated fraction for the next round of this loop.

terms	Expected result (as `Fraction`)
[4, 3]	3/13
[4, 6]	6/25
[3, 2, 2]	5/17
[6, 1, 8, 6]	55/379
[13, 6, 11, 6, 2, 4, 1, 13, 7, 6, 15, 2]	90669457/1193592058
[5, 1, 4, 6, 6, 15, 6, 5, 9, 11, 5, 1, 1]	103597999/601519457

As a side note, in a situation like this one where you are writing two functions that are each other's inverses, it is trivially easy to write a powerful fuzz tester to convince yourself that the functions are correct. Just generate random inputs for one function, pass the result of the first function to the second function, and check that the result produced by the second function is identical to the original random input. Repeat this a billion times to root out all bugs from your two functions.

47. *Sturm und drangle*

```
def sturm_word(x, y):
```

On a grid paper where all the grid lines are drawn horizontally and vertically between integer points, a line segment is drawn from the origin point (0, 0) to the point (x, y) where x and y are non-negative integers so that $\gcd(x, y) = 1$. This line segment will intersect some of the horizontal and vertical grid lines along its route from origin to the point (x, y), but will not go through any corners of the grid until reaching the destination point (x, y). DUCY?

If we denote each crossing of some vertical grid line by `'0'` and each crossing of some horizontal grid line by `'1'`, writing these crossings into a single string creates a finite Sturmian word defined by the slope y/x of the line segment. This function should return the Sturmian word collected during the journey to the endpoint (x, y).

The simplest approach to this problem is probably to step in unit steps along one axis direction, using the slope of the line to increase the position in the direction of the other axis. As much as you may be tempted, please don't use any floating-point numbers in your function, but perform all calculations using exact `Fraction` values. God created the integers, the rest is a work of man.

x	y	Expected result
7	19	`'110111011101101110111011'`
19	7	`'001000100010010001000100'`
1	5	`'1111'`
6	29	`'11110111110111110111110111110111'`
6	31	`'11111011111011111011111011111011111'`

Sturmian words for the grid crossings of the infinitely long lines defined by their slope have interesting mathematical properties that the reader can check out on the linked Wikipedia page.

48. Rational roots of a polynomial

```
def rational_roots(coefficients):
```

The `coefficients` of the polynomial $a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ can be given as a tuple of $n + 1$ elements listed in increasing order of powers of x. For example, the polynomial $2x^4 - 3x^3 + x + 9$ would be given as (9, 1, 0, -3, 2), the missing second-order term simply having the coefficient zero. The famous <u>rational root theorem</u> of algebra says that if p/q is a rational root of a polynomial where p and q are reduced to lowest terms, p must be a factor of a_0, and q must be a factor of a_n. Since a known finite number of possible values of p and q fit this requirement (don't forget to consider both positive and negative values of p/q), this theorem can be used to effectively and efficiently find all rational roots of any polynomial, regardless of the order of that polynomial.

This function should return all rational roots of the polynomial with the given `coefficients` as `Fraction` objects, listed in ascending sorted order. You should return each root as a `Fraction` object, even if it is an integer. For simplicity, we will also ignore the multiplicity of these roots so that you don't need to implement the polynomial division algorithm. Both the first and the last term of the `coefficients` tuple given to your function are guaranteed to be nonzero, so the polynomial will not have a root at zero.

You probably want to implement the <u>Horner's method</u> as a subroutine to evaluate the polynomial with the given `coefficients` at each candidate root p/q. You could, of course, implement some weak sauce polynomial evaluation method instead, but why settle for the second best when the very best method is such a concise and beautiful piece of code, and only four lines of Python?

coefficients	Expected result
(-4, -5, -5, -1)	[Fraction(-4, 1)]
(1, 0, -4)	[Fraction(-1, 2), Fraction(1, 2)]
(-10, -1, 1, -4, -2, 8, 3)	[]
(728, 2658, -22225, -1335, 4262, 792)	[Fraction(-4, 1), Fraction(-7, 2), Fraction(-13, 99), Fraction(1, 4), Fraction(2, 1)]

49. Multiple winner election

```
def multiple_winner_election(votes, seats, method):
```

Anglosphere countries tend to use voting systems in which each district elects exactly one representative, typically using the first-past-the-post plurality method to determine the sole winner of each district. Many other countries use some form of <u>party-list proportional voting</u> system where each electoral district has multiple seats to fill. Each voter casts a vote for a particular party, either directly ("closed list") or indirectly ("open list"). The `seats` up for grabs in each district are allocated to the parties in proportion to their `votes` in that district, but so that each party gets a whole integer number of seats.

Given the number of `votes` received by parties numbered from 0, ..., $n - 1$, this function should compute the distribution of `seats` to these n parties according to these `votes`. Seats are handed out one at a time. Each party has a **priority quantity** that depends on v, the number of votes received by the party, which remains throughout this process, and s, the number of seats that have been given to that party so far, which increases throughout the process. The formula to compute the priority quantity depends on which one of the methods of D'Hondt, Webster and Imperiali is used. The priority quantity of the party that received v votes and has so far been given s seats is $v/(s + 1)$ in the D'Hondt method, $v/(2s + 1)$ in the Webster method, and $v/(1 + s/2)$ in the Imperiali method. Each seat always goes to the party with the highest priority quantity at the time the seat is given out, and as a result, the priority quantity of that party will be lower as the term s in the denominator of the priority quantity is increased.

You should perform all these calculations with exact integer arithmetic using the `Fraction` data type, of course. If two parties have the exact same priority quantity at the time a seat is given, you should break the tie by giving that seat to the party identified with the smaller number.

votes	seats	method	Expected result
[128, 115, 26, 23]	21	'dhondt'	[10, 8, 2, 1]
[128, 115, 26, 23]	21	'webster'	[9, 8, 2, 2]
[128, 115, 26, 23]	21	'imperiali'	[10, 9, 1, 1]
[40, 37, 35, 15, 12]	23	'webster'	[7, 6, 6, 2, 2]
[28, 20, 20, 17, 14]	17	'dhondt'	[5, 4, 3, 3, 2]

50. Spiral walk

```
def spiral_walk(dx, dy, n):
```

Starting at the origin point (0, 0) of the infinite two-dimensional integer plane, you will make a total of n moves. Each move consists of k steps according to the given **direction vector** (dx, dy), so that initially $k = 1$. After **every two moves**, the step count k is increased by one. Also, after **every move**, the current direction vector is rotated counterclockwise 90 degrees. This particular rotation can be performed easily without any trigonometry or linear algebra, since the direction vector (dx, dy) simply becomes ($-dy$, dx) in this rotation. For example, the direction vector (3, -2) rotates to (2, 3), which would then at the next step rotate to become (-3, 2).

With pen and grid paper, the reader can quickly verify, using the starting vector (1, 0) and some small number of steps, that these moves draw a spiral emanating from the origin. This function should return the final position that this spiral walk ends up after n moves. As illustrated in the last row of the table below, after a mellow start of small values suitable for debugging, the automated tester will give your function far larger parameter values to chew on. Analogous to the problem "Hippity hoppity, abolish loopity" in the original 109 Python Problems collection, your function should calculate the result in constant time using only integer arithmetic, and not use any loops whatsoever to explicitly iterate through the entire spiral path.

dx	dy	n	Expected result
1	0	2	(1, 1)
−1	1	3	(0, −2)
2	−3	6	(10, −2)
21770	4881	4516163594	(19068371743211, 30090068999249)

Using the initial direction vector (1, 0) and a for-loop nested inside a while-loop, this concept can be applied to write an infinite **generator** that iterates through all points (x, y) of the infinite two-dimensional integer grid spiralling away from the origin.

51. Equal sum partition

```
def equal_sum_partition(items):
```

A **partition** of a list is a splitting into nonempty sublists, possibly of different lengths. For example, one possible way to partition the list `[10, 4, 2, 12, 14]` is to split it into `[10, 4]`, `[2, 12]` and `[14]`. In an **equal-sum partition** such as this one, the elements of each piece add up to the same value. Every list can always be trivially partitioned into an equal sum partition by treating the list itself as a single piece, but some lists allow more efficient equal sum partitions composed of smaller pieces.

This function should determine the best way to create an equal-sum partition from the given list of positive integer `items`, by minimizing the sum of elements in each piece, and return this sum as the final answer. Given a list of *n* integer `items`, your function should iterate through all possible lengths 1, ..., *n* – 1 of the first piece, and for each such potential first piece, iterate over the remaining elements to determine if they can be partitioned into pieces with that exact same element sum.

Since all `items` are guaranteed to be positive integers, this test does not require any branching choices, but can be performed with a single linear loop over the elements after the chosen first piece. This loop should maintain a running total of the elements covered so far, initialized to zero. Whenever this running total equals exactly *s*, the sum of the elements chosen in the first piece, restore the running total back to zero; and whenever the running total exceeds *s*, conclude that this attempt for equal sum partition fell flat, and move on to the next higher first piece length. Make sure to handle the final piece correctly in this process.

items	Expected result
[7, 2, 8, 1, 5, 4]	9
[2, 11, 10, 3, 7, 2, 4]	13
[1, 4, 3, 4, 4, 4, 2, 2, 1, 7, 2, 6, 1, 6, 1]	8
[5, 2, 10, 1, 2, 13, 1, 16, 1, 4, 6, 7, 16, 1, 17, 9, 5, 2, 1]	17

52. Maximum interval clique

```
def maximum_interval_clique(intervals):
```

A closed **interval** on the real number line is given as a two-tuple (*s*, *e*) of its **start** and **end** points, and contains every point *x* for which $s \leqslant x \leqslant e$. Two intervals **overlap** if they have at least one common point, that is, neither ends before the other one starts.

Consider the situation of having *n* such `intervals` whose start and end points have been chosen from integers 0, ..., 2*n* – 1 so that each one of these 2*n* integers appears exactly once as a start or end point of some interval. Your task is to find the **maximum clique** of these `intervals`, that is, the largest possible subset of intervals so that every pair of intervals in this subset overlaps. Since the clique is not necessarily unique, this function should only return its size.

Unlike in general **graphs** whose maximal cliques can be enumerated with the **Bron–Kerbosch algorithm** seen later in this same problem collection, the highly constrained overlap structure of real line intervals allows for a far more efficient **sweepline algorithm** to determine the size of the maximum clique in linear time. We start by noticing that not only does every pair of intervals in the clique contain some common point by definition, but this common point can be chosen to be the exact same point for all pairs of intervals in that clique! Furthermore, since the start and end points of all intervals are known to be integers from 0, ..., 2*n* – 1, the shared common point can be one of these integer points. Therefore, you should loop through these candidate integer points from left to right, maintaining a running count of the number of intervals that contain the current point. Use a preprocessing loop through the `intervals` to set up an auxiliary list to tell you what happens at each point, so you can update the running count in constant time at each point during the actual count.

intervals	Expected result
`[(4, 5), (1, 2), (0, 3)]`	2
`[(1, 7), (6, 8), (3, 5), (0, 4), (2, 9)]`	4
`[(8, 10), (12, 14), (2, 4), (1, 13), (5, 6), (0, 7), (9, 15), (3, 11)]`	4

53. Maximum overlap intervals

```
def maximum_overlap_intervals(intervals):
```

Same as in the previous problem of finding the maximum interval clique, a closed **interval** on the real number line is given as a two-tuple (s, e) of its **start** and **end** points, and contains every point x for which $s \le x \le e$. Two intervals **overlap** if they have at least one point in common.

Consider again the situation of n such `intervals` whose start and end points have been chosen from integers $0, ..., 2n – 1$ so that each one of these $2n$ integers appears exactly once as a start or end point of some interval. This time, your task is to find the intervals that overlap the largest number of other intervals, but unlike in the interval clique problem, those other intervals do not need to overlap each other. Since there can be several such intervals with the same overlap count, your function should return the list of all these intervals, listed in ascending sorted order of their starting points.

Instead of looping through every interval to compare it to every other interval in an inner loop, this problem can also be solved far more efficiently with the classic **sweepline algorithm** approach. Loop through the integers $0, ..., 2n – 1$ while maintaining an **active set** of intervals that contain the current point. At the same time, maintain a **counter dictionary** to keep track of the number of intervals that each interval overlaps. When this loop reaches the point s that is the start point of some interval (s, e), set the count for that interval to the current size of the active set. Next, increment the count by one for each member of the active set, and finally add (s, e) to the active set. When this loop comes to a point e that is the end point of some interval (s, e), simply remove that interval from the active set. A linear preprocessing loop through the `intervals` again allows you to set up an auxiliary list to tell you exactly what to do at each point that the main loop iterates over.

intervals	Expected result
`[(0, 5), (1, 6), (2, 3), (4, 7)]`	`[(0, 5), (1, 6)]`
`[(0, 5), (1, 6), (2, 4), (3, 9), (7, 8)]`	`[(3, 9)]`
`[(0, 1), (2, 13), (3, 8), (4, 10), (5, 14), (6, 7), (9, 12), (11, 15)]`	`[(2, 13), (5, 14)]`

Peter Winkler's "<u>Mathematical Puzzles: A Connoisseur's Collection</u>" presents the following extremely surprising result; for n intervals constructed randomly from the endpoints $0, ..., 2n – 1$, the probability of one interval overlapping all other $n – 1$ intervals **is the exact same 2/3, for every $n > 1$**. Yes, that highly counterintuitive probability is exactly 2/3, even when n is up in the gazillions!

54. Split at None

```
def split_at_none(items):
```

The original Fizzbuzz problem, used to quickly screen out the utterly hopeless candidates for programming jobs, will soon be almost two decades old and too well known, so our current situation requires more sophisticated tests for this purpose. Modern problems require modern solutions, as the famous expression goes.

A recent tweet by Hasen Judi offers another interview screening problem that is more fitting the power of modern scripting languages, yet requires no special techniques past the level of basic imperative programming taught in any introductory programming course. Given a list of items, some of which equal None, return a list containing, as its elements, the sublists of consecutive items split around the None elements that serve as the separator marks. Each consecutive block of elements is a separate list in the result. Note the expected correct behaviour whenever these None separators are located at the beginning or at the end of the list, and the expected correct behaviour whenever multiple None values are consecutive items.

items	Expected result
[-4, 8, None, 5]	[[-4, 8], [5]]
[None, None, 12, None, 3]	[[], [], [12], [3]]
[None, None, None, None, 5, None]	[[], [], [], [], [5], []]
[None, -36, 25, 28, None, -27, -24, 34, -28, 24, 44, 6, 33, 20, None]	[[], [-36, 25, 28], [-27, -24, 34, -28, 24, 44, 6, 33, 20], []]
[53, 20, -49, 34, None, 13, -43, -14, 53, -6, None, None, None, -26, None, 14, None, None, -11]	[[53, 20, -49, 34], [13, -43, -14, 53, -6], [], [], [-26], [14], [], [-11]]

(Cue the haughty reply tweets of "I have no idea how to solve this totes stupid and meaningless problem that has nothing to do with real work! It's just that those stupid employers don't see what a smart and productive programmer I am in practice, but I just always panic and freeze when I am given simple tests" in three, two, one...)

55. Chunk sorting

```
def chunk_sorting(permutation):
```

Sorting the given random-access sequence of elements in ascending order has a solved problem during the entire lifetime of most students reading this, and the best algorithms have long been implemented in the standard libraries of all serious programming languages.

The sorting problem remains fruitful for teaching algorithm analysis and related design techniques. Consider the following interesting variation of sorting, given as the problem "Max Chunks To Make Sorted" at LeetCode. Your task is to split the given `permutation` of integers 0, ..., $n - 1$ into non-empty contiguous chunks so that when each chunk is sorted separately in place (using any one of the standard sorting algorithms, it doesn't matter which one) independently of the other chunks, the entire list becomes sorted. Of course one chunk spanning the entire `permutation` would always work, but this function should determine the largest number of chunks that can work.

Our automated tester will run your function on 20,000 test cases, most of which contain thousands of elements, so your algorithm needs to operate **in place** on the given `permutation` list and not extract any slices, nor use other inefficient list operations or keep looping back and forth through the list like some Shlemiel. To develop the swift and sure **greedy algorithm** for this problem that will operate in a single for-loop through the entire list, never straying or turning back, you should start by noticing that, of course, the smallest element 0 must be part of the first chunk, but where will that first chunk end? Nope, not necessarily at that element 0, as the third and fifth test cases in the table below illustrate. After that first chunk, where does the second chunk end?

permutation	Expected result
[1, 0, 2]	2
[0, 1, 4, 5, 3, 2]	3
[3, 0, 2, 4, 1, 6, 5, 7]	2
[0, 4, 3, 6, 5, 1, 7, 8, 2]	2
[2, 0, 1, 6, 8, 7, 4, 3, 5, 9]	3
[1, 2, 0, 3, 8, 5, 10, 4, 7, 6, 11, 9]	3

56. Merge the two biggest

```
def merge_biggest(items):
```

Your task is to repeatedly perform the following operation for the given list of `items`; remove its two largest elements *a* and *b*, and if their absolute difference *c* = abs(*a* – *b*) is nonzero, insert *c* into the `items`. Rinse and repeat until only one element remains, and return that element as the answer. Should the list become empty after removing the last two elements that were equal, return 0.

For example, if the `items` list initially contains the elements [7, 1, 5], removing the two largest elements 7 and 5 and inserting their absolute difference 2 makes the list to be [2, 1]. Repeating the same operation makes the list to be [1] whereupon the function terminates, returning 1.

The automated tester will run your function on ten thousand test cases, most of them involving thousands of elements. To allow your function to perform this operation efficiently to pass the automated tester within the time limit, this would be a good time to turn the `items` list into a **binary heap priority queue** using the functions of the <u>heapq</u> module in the Python standard library. The powerful operations `heappop` and `heappush` will then perform the required operations efficiently even for humongous lists... but how to finagle these operations to extract the largest element from the queue instead of the smallest? You need to think up some way to, uh, *negate* the direction of the ordering that is used to organize the elements inside the priority queue.

items	Expected result
[2, 10, 3]	5
[7, 1, 5]	1
[7, 11, 11, 1, 15, 11]	2
[57, 18, 47, 45, 52, 59, 66, 1, 49, 13, 66, 51, 39, 50, 13, 49, 9, 43, 37, 39, 19, 59]	1

57. Lehmer code: decoding

```
def lehmer_decode(lehmer):
```

This problem is the inverse of the earlier problem of calculating the Lehmer code encoding for the given permutation. This function should reconstruct and return the original permutation of the nonnegative integers $0, ..., n - 1$ that produces the given Lehmer code.

This problem is not quite as straightforward as the mechanistic Lehmer encoding seen in an earlier problem, but requires a bit of combinatorial thinking to get started. Note how the first element of the reconstructed permutation must necessarily equal the first element of the Lehmer code. But how would you then construct the remaining required permutation?

lehmer	Expected result
[1]	[1, 0]
[1, 0]	[1, 0, 2]
[1, 1]	[1, 2, 0]
[2, 0, 1, 0, 1]	[2, 0, 3, 1, 5, 4]
[2, 3, 0, 2, 2, 0]	[2, 4, 0, 5, 6, 1, 3]
[7, 1, 1, 4, 1, 0, 1]	[7, 1, 2, 6, 3, 0, 5, 4]

58. Bayesian updating of dice

```
def bayes_dice_update(dice, rolls):
```

You have a box of `dice`, each with a different number of sides used in the game of Dungeons and Dragons. Your friend randomly takes out one die from the box and makes a series of `rolls` with that same die out of your view, and then tells you the results of these rolls. Your task is to calculate, for each die in the box, the exact probability that your friend used that particular die.

Initially, each of the n dice has the same **prior** probability $1/n$ of being the chosen die. However, the **posterior** probabilities of these dice will usually be very different after seeing the resulting rolls. According to the famous Bayes' theorem, the conditional probability $P(d \mid \text{rolls})$ can be written equivalently in the form $P(\text{rolls} \mid d) \, P(d) / P(\text{rolls})$. The term $P(d)$ is known to equal $1 / n$, and since the individual rolls are **independent** of each other, the term $P(\text{rolls} \mid d)$ can be computed simply as the product of the probabilities of the individual rolls.

To get rid of the problematic term $P(\text{rolls})$, we only need to note that this term is the same unknown constant factor for each die, and can therefore be completely ignored! Therefore, all we need to do is to calculate $P(\text{rolls} \mid d) \, P(d)$ for each die, and then **normalize** the resulting probabilities so that they add up to one, seeing that precisely one of these dice must be the one that your friend drew from the box. Of course, you should perform all calculations using exact `Fraction` objects.

dice	rolls	Expected result
[5, 6]	[2]	[Fraction(6, 11), Fraction(5, 11)]
[3, 4, 5]	[2, 2]	[Fraction(400, 769), Fraction(225, 769), Fraction(144, 769)]
[2, 7, 10, 11]	[2, 7, 3]	[Fraction(1331000, 2130533)], Fraction(456533, 2130533), Fraction(343000, 2130533)

59. Haircut

```
def haircut(speed, n):
```

You need a haircut, so go to a popular barbershop that employs barbers numbered 0, ..., *M*, each individual barber renting a chair. In this shop, deeply committed to the finest Taylorist principles of standardization and efficiency, the barber number `i` will always take the same number of minutes `speed[i]` to complete a haircut, regardless of the customer. You arrive at the barbershop in the morning before it opens, and take position `n` in the queue of customers already waiting ahead of you. Each customer always walks into the first available chair, regardless of the barber's speed at that chair. Which barber will eventually service you?

speed	n	Expected result
[36, 36, 10]	7	0
[28, 16, 51, 36]	8	3
[15, 41, 20, 31, 24]	11	2
[10, 11, 19, 46, 47, 60, 60]	22	2

This is the problem "Haircut" from Google Code Jam 2015.

60. Kadane in triplicate

```
def max_three_disjoint_sublists(items):
```

The maximum subarray problem is a classic exercise in algorithmic thinking that asks you to find the contiguous sublist of the given list of integers that has the largest sum. (Some of the original list elements are negative so that the answer isn't the trivial "duh, just take that entire list, bro".) A bit surprisingly, the maximum subarray can be found with one simple linear loop through the original list with Kadane's algorithm, instead of iterating through a quadratic number of possible pairs of start and end positions of all possible sublists.

Initialize a variable m to zero, and loop through all the elements e in the list. For each list element e, simply update $m = \max(m + e, e)$. This way, the variable m always contains the largest possible sum of any sublist that ends with that element e in that position of the entire list. Note how we don't care about which preceding elements added up to that maximum sum m, only about the value of that maximum sum itself. On the side, keep track of the largest sum that you have seen so far along in the entire list, with the position of that largest sum. Once the loop has terminated, you can reconstruct the sublist using another linear loop going through that sublist in reverse from its stored position.

Now that you know all that, you are hopefully ready to tackle the following problem: find the largest sum that can be formed by adding up the elements of **three** contiguous sublists of your choice from the given list of `items`. All three sublists must contain at least one element, and their positions must be fully **disjoint** so that no position is included in any two of your chosen lists. (Your chosen sublists can be consecutive; there is no need to leave a nonempty gap between them.) Your function should be fast enough to handle thousands of pseudorandom lists with thousands of elements each within the time limit of the automated fuzz tester.

items	Expected result
`[-2, -2, -2]`	−6
`[2, 3, 4, -5, -5, -5]`	9
`[-1, -2, 7, -7, 5, -1, -3, -5]`	11
`[-17, -15, 21, 44, -20, 40, -21, -11, -15, 31, -17, -21, -11, 25, -4, -4, 47, -11, 44, -17, -9, -8, -19, -22]`	213

61. Yellowstone sequence

```
def yellowstone_sequence(n):
```

This pleasantly chaotic **infinite permutation** of all positive integers comes from the Numberphile video "The Yellowstone Permutation". This sequence is based on a common structure of giving the first couple values of the sequence, after which the next element is always the smallest so far **unseen** positive integer that satisfies the simple local constraint given for the sequence. This local requirement for the Yellowstone permutation is that if b is the most recent previous element and a is the second most recent one, the next element c of the sequence **must** be relatively prime to b (that is, it must be that $\gcd(b, c) = 1$), and it **may not** be relatively prime to c (that is, $\gcd(a, c) > 1$).

Setting the first three elements of the sequence to be simply 1, 2, and 3, the patient reader can verify that this sequence starts off as

1, 2, 3, 4, 9, 8, 15, 14, 5, 6, 25, 12, 35, 16, 7, 10, 21, 20, 27, 22, 39, 11, 13, 33, 26, 45, 28, 51, ...

Using 0-based position indexing since we pretend to be proper computer scientists, this function should return the element in position n of the Yellowstone sequence. The automated tester will give your function the consecutive values of n from 0 up to 9999 in this order, so that you can store the numbers seen so far in a global set that maintains its contents between the function calls.

n	Expected result
0	1
4	9
9	6
99	165
999	920
9999	11649

As with many other sequences of this nature, it is not at all self-evident that this infinite sequence truly is an infinite permutation of positive integers, that is, it contains every positive integer exactly once without leaving any gaps in its wake. The Numberphile video linked above contains a lovely proof of this fact, as presented by Neil Sloane himself.

62. Largest rectangle in a histogram

```
def largest_rectangle(histogram):
```

This problem is "Largest Rectangle in a Histogram" in Peking University Online Judge. You are given a `histogram` that specifies the heights of the bars for positions 0, ..., $n - 1$. This function should find the largest axis-aligned rectangle that fits inside this histogram, and return its area.

The difficulty in this seemingly simple problem is devising an efficient algorithm that works even for thousands of pseudorandom problem instances with thousands of elements. There are many ways to achieve this efficiently. The simplest one might be to use a Python list as a stack to the possible starting *x*-coordinates of rectangles, removing them from the stack when you encounter a bar that is low enough to end that rectangle.

histogram	Expected result
[3, 9, 8, 12]	24
[3, 1, 4, 2, 1]	5
[14, 13, 13, 2, 8, 3]	39
[21, 21, 18, 2, 1, 4, 5, 4, 5, 7, 7, 4]	54

63. Frobenius nuggets

```
def frobenius(coins):
```

The famous <u>Frobenius coin problem</u>, sometimes popularized as the **McNugget problem**, asks for the largest positive integer that cannot be broken down into a sum of available `coins`, when you are allowed to use each coin denomination as many times as necessary. For example, 43 is the smallest integer that cannot be expressed as sum of [6, 9, 20], whereas all integers less than 43 can be expressed in this manner. This function should determine the Frobenius number for the given list of `coins`. For simplicity, you may assume that the `coins` are co-prime (which guarantees a finite solution), and the `coins` are given in ascending sorted order. So that the answer isn't trivially zero, all coins are greater than 1.

This problem can be solved best with **dynamic programming**. Loop through the integer values from $k = 2$ upwards, filling in a list of truth values telling whether each k can be expressed as a sum of the available `coins`. This is possible precisely when at least one of the integers $k - c$ can be expressed this way, when c goes over the available `coins`, easily checked in a simple nested loop. Once you have found a run of m consecutive `True` elements, where m is the smallest coin denomination available, the integer that precedes that run is the desired Frobenius number.

As you can see in the fourth example case in the table below, the Frobenius number can be smaller than the largest available coin.

coins	Expected result
[6, 9, 20]	43
[29, 37]	1007
[13, 67, 97]	412
[3, 23, 47, 53]	43
[11, 29, 41, 67]	112

As an aside, 1 is not considered a prime number. While it's often explained through the lens of the usefulness of unique prime factorization, the deeper reason is that 1 is what mathematicians call a **unit**, a number that "divides" everything. This concept generalizes beyond integers to more abstract algebraic structures that the reader will encounter in later studies.

64. Postage stamp problem

```
def postage_stamp(stamps, k):
```

The Postage stamp problem is very similar to the previous Frobenius problem, and can be solved with a similar general approach. You are given an unlimited supply of `stamps` of given face values. All stamps are physically the same size, and there is room on the postcard to fit at most k stamps. What is the smallest total postage that cannot be placed on the postcard under these restrictions? (The smallest stamp is always worth one cent, so any amount of postage is theoretically possible, if we ignore the limit k.)

The simplest way to solve this problem is to use a dynamic programming approach similar to the one we used in the Frobenius problem. Fill in a list of values $M[v]$ of the number of stamps required to achieve total of v. Assuming that you have filled this list up to the value $v - 1$, the value $M[v]$ equals one plus the minimum of the values $M[v - s]$, where s loops through the given `stamps`. Once you reach a position v where $M[v] = k + 1$, the current position v is the final answer.

stamps	k	Expected result
[1, 3, 7, 8]	6	49
[1, 3, 10, 13]	6	61
[1, 8, 12, 15, 19, 21]	12	245
[1, 11, 16, 21, 25, 27, 30]	13	383
[1, 2, 14, 27, 34, 39, 49, 50, 53, 68]	5	11

65. Knaves of the round table

```
def knaves_of_round_table(answers):
```

Some <u>knights and knaves</u> are gathered cyclically around King Arthur's round table. These men are numbered 0, ..., $n - 1$ according to their positions. As we have learned from the works of <u>Raymond Smullyan</u>, knights always tell the truth and knaves always lie whenever they open their mouths. Unfortunately, we cannot tell which men are knights and which are knaves. (If asked directly which kind of man he is, each man will just claim to be a knight; not very helpful to us.)

To start resolving this mess, we ask each man how many of the two men sitting immediately to his left and right are knights, and collect the `answers`. These `answers` do not always uniquely pinpoint the positions of the knights and knaves, but at least we can identify all possible assignments consistent with these `answers`. For example, there are two possible solutions consistent with the three men's answers `[2, 2, 2]`; either all three men are knights telling the truth, or all three men are lying knaves with their pants on fire.

This function should return the list of possible counts of how many knights are seated at the round table. Note that the function does not return the possible positions of these knights, but the possible total number of knights. The automated tester will only give your function `answers` for which there exists at least one assignment of knights and knaves consistent with those `answers`.

This function can recursively fill out all possible assignments of these men to knights and knaves from left to right, backtracking from the partial assignment built so far as soon as it hits a local contradiction. The vast majority of possible branches in this search will be immediately pruned away with some local contradiction, making this recursive search fast even if hundreds of men were seated around King Arthur's table.

answers	Expected result
`[1, 2, 2]`	`[0]`
`[0, 2, 0]`	`[1]`
`[2, 1, 1, 1]`	`[0, 3]`
`[2, 2, 2, 2, 2, 2]`	`[0, 6]`
`[1, 2, 2, 1, 2, 1, 2]`	`[0, 3, 4]`
`[0, 1, 2, 2, 2, 2, 2, 2, 1]`	`[8]`

66. Six of this, half a dozen of the other

```
def assign_sides(costs):
```

You are given 2*n* items, and you need to assign exactly *n* of these items of your choice on this side, and the remaining *n* items on the other side. Each item is represented as a two-tuple (this, other) that gives the cost of assigning that item on this and on the other side, respectively. This function should compute the optimal assignment of these 2*n* items to this side and the other side to minimize the total overall cost of the assignment.

This problem may initially seem like a backtracking recursion where each item can be assigned either way, terminating at the base case where neither side has *n* items assigned. However, this would be serious overpaying for overkill; some clever preprocessing of these costs actually allows the optimal assignment of these items to be computed in a single linear pass over the costs. The key insight starts from the realization that you must assign every item somewhere, and that one dollar saved equals that exact same one dollar, no matter where it comes from.

costs	Expected result
[(4, 6), (1, 3), (3, 2), (3, 5), (4, 2), (3, 1)]	13
[(1, 1), (8, 6), (7, 9), (4, 6), (2, 9), (3, 2), (8, 3), (6, 4), (4, 3), (3, 5)]	35
[(5, 9), (5, 1), (4, 7), (7, 11), (10, 7), (11, 11), (11, 8), (10, 9), (5, 9), (5, 9), (2, 11), (8, 8), (11, 3), (7, 1)]	76

67. Factoradical dudes

```
def factoradic_base(n):
```

Several problems in our original two Python problem collections have showcased problems of representing integers in positional number systems in all sorts of weird bases. Instead of the familiar positive integer bases ten and two, these bases could be negative, rational or even complex numbers whose powers multiplied by the coefficient of that position add up to that integer.

In this problem, we look at an interesting positional number system that uses a **mixed base** where the base changes with each position. The idea of a variable base might initially seem weird in general. However, you already use such a system every day in expressing time with seconds and minutes both given in base 60, whereas hours are expressed in either base 12 or 24, depending on which side of the pond you live in. Imperial measurements for weights and lengths also use mixed bases, with each unit expressed as a multiple of the next lower unit.

The factoradic number system uses the factorials 1, 2, 6, 24, 120, ... as variable bases. If the positions are numbered starting from 1, the base in position k equals $k!$, the product of the first k positive integers. The possible coefficients for the position k are then 0, ..., k. If the coefficient of the position k were equal to $k + 1$, the resulting represented term $(k + 1)k!$ would equal $(k + 1)!$ and could therefore be carried over to the next position, analogous to any other positional number system.

This function should return the factoradic representation of the given positive integer n as the list of its factoradic coefficients so that the coefficient in the position $k - 1$ of the result list gives the coefficient of the factorial $k!$ in the representation. The automated tester will give your function some pretty humongous numbers to convert, but since factorials grow exponentially fast, your function should return the list of coefficients rapidly, even for numbers that have hundreds of digits.

n	Expected result
1	[1]
11	[1, 2, 1]
96	[0, 0, 0, 4]
1781	[1, 2, 0, 4, 2, 2]
4146434844171	[1, 1, 0, 2, 3, 1, 6, 7, 2, 10, 4, 11, 7, 2, 3]

Factoradic numbers can be used to encode arbitrary permutations of distinct elements uniquely into positive integers with the **Lehmer coding** seen in earlier problems.

68. Tarjan's strongly connected components

```
def tarjans_scc(edges):
```

The basic graph search algorithms, breadth-first search and depth-first search, can be augmented to compute other graph properties. Consider the problem of finding the strongly connected components of the given directed graph, that is, maximal subsets of nodes so that you can get from any node in that component to any other node in that same component, but once you leave that component, there is no coming back. (If there were such a way back, the node that you exited to would, by definition, be part of that same strongly connected component.) The strongly connected components of a graph form a **partition** of its nodes, which each node belonging to exactly one strongly connected component. If nothing else, each node by itself is a singleton component.

Tarjan's strongly connected components algorithm is an ingenious application of depth-first search that, as a byproduct of a single depth-first search, enumerates all strongly connected components of that graph. Your task here is to translate the pseudocode from the Wikipedia page (or any other source where fine algorithms are sold) into Python. The basic list data type of Python will work nicely as the `index`, `lowlink` and `on_stack` bookkeeper, but also as the **stack** from which to pop the nodes of the strongly connected component represented by the current node whose `index` and `lowlink` ended up being equal.

Your function is given a directed graph whose nodes are integers 0, ..., $n - 1$, and the list of `edges` emanating from each node. The function should return the list of strongly connected components so that the nodes of each component are listed in ascending sorted order, and the components themselves are listed in ascending order of their lowest-numbered nodes.

edges	Expected result
[[1, 4, 3, 2], [3, 4, 0], [4, 1, 0, 3], [2], [0, 3, 2, 1]]	[[0, 1, 2, 3, 4]]
[[4, 3], [2, 5, 4], [4, 1, 0, 3, 5], [4, 0], [], [1, 2]]	[[0, 3], [1, 2, 5], [4]]
[[2, 11, 10, 3, 12], [0, 3, 12, 2, 11, 4], [1, 0, 12], [4, 1, 6], [7], [4, 8], [8, 4], [4, 5], [6], [8, 10, 6, 12], [9, 12, 11, 7], [10], [1, 10, 0, 11]]	[[0, 1, 2, 3, 9, 10, 11, 12], [4, 5, 6, 7, 8]]

As a nostalgic trip down memory lane, your author took two days to understand and implement this algorithm in the C programming language needed for his work in his Master's thesis back in 1995, whereas now that same deal took about fifteen minutes.

69. Copeland harder

```
def copeland_method(ballots):
```

Consider an election in which multiple candidates compete to be the single winner in that district. The first-past-the-post plurality voting used in most Anglosphere elections at least has simplicity and tradition on its side, but this method lacks many desirable properties of fair election systems.

In Copeland's method, instead of voting just for the one preferred candidate, each voter ranks all candidates in descending preference order on the ballot. To determine the winner, each candidate has a one-on-one match against all other candidates. In this one-on-one match, all other candidates are ignored, and the match is won by the candidate that most voters prefer against their opponent, looking at only those two candidates on each ballot. The candidate who wins the one-on-one match gets one matchpoint for this win. Should the candidates tie, both receive half a matchpoint.

Note that the magnitude of winning a single match does not affect the matchpoint gain; beating the opponent by one vote gives that candidate the exact same one matchpoint as winning by a million votes. It is also irrelevant how much each voter prefers one candidate over the other one, as only the ordinal preference matters, regardless of the cardinal magnitude of this preference.

The winner of the overall election is the candidate with the most matchpoints. However, unless the voter preferences determine a unique Condorcet winner, the mechanism can result in a tie. This function should return the sorted list of winners in the election among candidates 0, ..., $n - 1$, where each ballot is a permutation of these candidates indicating that voter's preferences.

ballots	Expected result
[[1, 0, 2], [2, 0, 1], [2, 1, 0], [0, 2, 1], [1, 2, 0]]	[2]
[[1, 2, 0, 3], [3, 2, 0, 1], [2, 3, 0, 1], [1, 3, 0, 2], [3, 1, 0, 2], [2, 1, 0, 3]]	[1, 2, 3]
[[1, 0, 2, 3], [3, 0, 2, 1], [0, 3, 2, 1], [3, 0, 2, 1], [0, 3, 2, 1], [1, 3, 2, 0]]	[0, 3]

70. Instant-runoff voting

```
def instant_runoff(ballots):
```

Following the previous Copeland method problem, consider again an election where multiple candidates compete to be the single winner of that district. Same as with the Copeland method, <u>instant runoff voting</u> makes the voters rank all candidates in that voter's preference order in the ballot. The difference is that the winner is not computed with pairwise matches, but by repeatedly eliminating the least favoured candidate in a plurality vote until a strict majority winner emerges.

Initially, each ballot goes for the candidate who was that voter's first preference. If none of the candidates gets a majority of votes, the candidate with the fewest ballots is eliminated from the race, and his ballots are distributed to the remaining candidates, each ballot to the candidate who was that voter's second preference. Elimination of candidates in this manner continues until one remaining candidate has the strict majority of votes (that is, *more* than half) and is declared the winner.

This function should determine the winner of candidates numbered 0, ..., $n - 1$ in the instant runoff election, when each ballot is a permutation of these candidates listed in the preference order of the voter casting that ballot. To make the results unambiguous for the automated fuzz tester, if several remaining candidates have the same smallest number of ballots in the current round, eliminate the highest-numbered such candidate.

To implement this function correctly, note that a candidate who has been eliminated in any previous round cannot come back to life by receiving ballots from candidates eliminated in the current round, but each one of their ballots is passed on to that voter's next preferred candidate who is still alive and kicking in the election.

ballots	Expected result
[[1, 0, 2], [2, 0, 1], [2, 1, 0], [0, 2, 1], [1, 2, 0]]	2
[[1, 0, 2], [0, 1, 2], [1, 0, 2], [2, 0, 1], [0, 2, 1]]	1
[[1, 4, 2, 3, 0], [3, 4, 2, 1, 0], [2, 4, 3, 1, 0], [3, 4, 2, 1, 0], [2, 4, 3, 1, 0], [4, 2, 3, 1, 0], [3, 2, 4, 1, 0], [4, 2, 3, 1, 0]]	4
[[1, 0, 2, 3, 4, 5], [5, 0, 2, 3, 4, 1], [0, 5, 2, 3, 4, 1], [1, 2, 5, 0, 4, 3], [0, 2, 5, 1, 4, 3], [3, 5, 4, 0, 2, 1], [4, 5, 3, 0, 2, 1], [5, 2, 0, 3, 4, 1]]	5

71. Cousin explainer

```
def cousin_explainer(a, b):
```

Women in a matrilineal family tree are numbered so that Eve at the root has the number 1, and the woman whose number is *i* has exactly two daughters, numbered 2*i* and 2*i* + 1. For example, the two daughters of woman 7 would be the two women numbered 14 and 15. In this scheme, the mother of woman number *i* is given by the formula *i* / 2, rounded down to an integer for odd *i*. As you can see, this formula gives the mother's number 7 for both of her daughters, 14 and 15. (This is also exactly how a binary tree structure is imposed on a list when treating it as a **binary heap**, as implemented in the `heapq` standard library module.)

Given two women numbered `a` and `b`, this function should return in English what woman `b` is to woman `a`. For simplicity, we assume that all the men whom these women procreated were genetically sufficiently distinct from each other so that the familial relationship between any two women in this problem is solely determined by this matrilineal tree.

To determine the relationship between women `a` and `b`, trace the parent lineages from both women to their **nearest common ancestor**, keeping track of the number of steps from each woman to that ancestor. These two counts determine the relationship, as illustrated in the tweet "The cousin explainer". For example, if one of these step counts is zero, that woman is a direct ancestor of the other one. If one of the step counts is one, that woman is the (some number of `great`) aunt of her (some number of `great`) niece. Otherwise, these two women are cousins to the degree determined by the smaller of the two counts of steps to their common ancestor. The count of **removals** is determined by the difference between these two step counts. For this count of removals, use `'once'`, `'twice'` and `'thrice'` for the first three, after which use `'four times'`, and so on. Do not use hyphens; say "great grandniece" instead of "great-grandniece".

a	b	Expected result
1	3	`'daughter'`
3	5	`'niece'`
5	18	`'grandmother'`
9	7	`'first cousin once removed'`
3	18	`'great great grandniece'`
344	28	`'third cousin four times removed'`
612	5	`'great great great great great grandaunt'`
1133	3668	`'ninth cousin once removed'`

72. One-dimensional random walk

```
def random_walk(moves, n, positions):
```

Starting from the origin point zero of the one-dimensional integer line, you take a **random walk** of exactly n steps so that independently of the previous steps, each step either moves one step to the left, stays put in the current position, or moves one step to the right according to the probabilities given in the list `moves=[left, stay, right]`, these three transition probabilities always adding up to one. This function should compute and return, as a batch job, the exact probability (as a `Fraction` object) of this random walk finishing in each of the given `positions`.

Combinatorics mavens could probably come up with a closed-form formula for these probabilities, but here we can solve them with a recursive formula in terms of the number of steps n. The base case with $n = 0$ is trivial. After that, the probability $P(n, k)$ of ending in the position k after taking n steps can be computed from the probabilities $P(n - 1, k - 1)$, $P(n - 1, k)$ and $P(n - 1, k + 1)$ for ending up after $n - 1$ steps in a position where you can move to position k in a single step, and adding up these probabilities weighted by their corresponding transition probabilities to position k.

You can either implement this recursion as a nested recursive function using the `@lru_cache` to memoize the overlapping subproblem results to prevent exponential blowup of running time, or implement it with **dynamic programming** by computing the table of probabilities directly. You also only need to maintain the current and the previous row of this table, instead of lugging around the entire two-dimensional structure.

moves	n	pos	Expected result
[Fraction(1, 2), Fraction(1, 10), Fraction(2, 5)]	2	[0]	[Fraction(41, 100)]
[Fraction(2, 3), Fraction(1, 5), Fraction(2, 15)]	4	[-4, -2]	[Fraction(16, 81), Fraction(536, 2025)]
[Fraction(1, 2), Fraction(1, 18), Fraction(4, 9)]	7	[1, 4]	[Fraction(14159887, 76527504), Fraction(97888, 4782969)]

73. Knight watch

```
def knight_watch(n, x, y, k, pillars):
```

One of the blind chess knights from the problem "Sneaking" in the earlier Additional Python problems collection is the lone guardian of an *n*-by-*n* chessboard outside of which nothing exists except impenetrable stone walls that continue solid to infinity in every possible direction. Some stone `pillars` have also been placed on some locations of the board. This long-forgotten knight can observe the square that he is currently at, and all the squares that are one knight's move away from his current position. The knight cannot move inside a pillar (but neither can the king), but otherwise, these pillars don't hinder his movement or sensing. Even in his old days, this knight still has his powers of movement so that he can move and see over a pillar, just not into it.

Waking up at the coordinates `(x, y)`, the weary old knight can only make k moves before going to sleep again. A loyal servant of his lost empire long since ground to dust, the knight still wants to be maximally effective to guard his board. Even though he can't necessarily check the entire board, he wants to choose his sequence of k moves to allow him to observe the largest number of squares along the way, to catch any wayward enemy kings that might still be sneaking around after millennia of abandonment. Since this optimal sequence of moves is not necessarily unique, this function should return the largest number of squares that this knight can observe with k moves before he goes back to sleep, only to repeat this same thing again the next day.

n	x	y	k	pillars	Expected result
8	4	4	0	[]	9
8	4	4	1	[]	16
4	0	3	2	[(0, 0), (3, 1), (3, 2), (3, 0), (1, 0), (1, 1)]	5
4	2	0	2	[(3, 2), (1, 1), (0, 0)]	8
6	0	3	3	[(5, 3), (2, 3), (4, 2), (5, 2), (5, 5), (1, 4)]	19

Enjoyers of chessboard combinatorics are surely familiar with the knight's tour problem, where the chess knight has to visit each square exactly once and return to his starting place. An interesting variation of this problem is to ask for the shortest path for our blind knight to fully observe the entire 8-by-8 chessboard.

74. Longest Fibonacci-like subsequence

```
def longest_fib_subsequence(items):
```

A sequence of positive integers is said to be **Fibonacci-like** if, after the first two elements, each element equals the sum of the two immediately preceding elements. (By this definition, every sequence of two elements is trivially Fibonacci-like.) This function should find the longest Fibonacci-like subsequence of the given list of positive integer `items`. Since this subsequence is not necessarily unique, the function should return only the length of that subsequence. Also, as you can see in the second and third example test cases in the table below, the second element of the subsequence does not need to be strictly larger than the first element.

The main difficulty of this problem is in making this function fast enough to pass the automated fuzz tester within the time limit. The trivial O(n^3) algorithm of iterating through all possible positions of the first two elements and finding the longest Fibonacci-like sequence emanating from each such element pair in a single linear pass through the remaining elements is too slow to pass the fuzz test suite within the time limit, so you might want to preprocess the sequence to optimize the subsequence generation, or think up some other algorithmic optimization that is applicable here.

items	Expected result
[7, 4, 10, 6, 4]	2
[6, 4, 9, 3, 7, 13]	3
[6, 3, 6, 12, 15, 12, 18, 30, 27]	5
[6, 11, 17, 23, 28, 45, 73, 118, 191, 309, 500, 326]	10

75. Maximize exam confusion

```
def maximal_confusion(answer_key, k):
```

This deceptively simple problem is problem 2024 in LeetCode, "Maximize the Confusion of an Exam". Too dumb to create new original exams himself, this one lazy teacher has inherited a multiple choice exam whose `answer_key` consists of characters `'T'` and `'F'` to indicate whether the correct answer to each particular question is `True` or `False`. Being generally unfit for his teaching position and an insecure impostor who knows that his students who paid hard earned money to take the course will soon notice his incompetence unless their minds are distracted elsewhere, he decides to play sadistic mind games with his students and change at most *k* questions to opposite answers to maximize the longest run of consecutive questions with the same answer, either all `True` or all `False`, to gaslight the exam takers to question their sanity.

Of course, even this *mamaluke* will soon realize that all *k* changes should be to the same direction (either all change some `True` answer to `False`, or all change some `False` answer to `True`), and soon after that, realize that there is no point leaving gaps between these changes, but all changes should take place in questions that are consecutive in the subsequence of questions with the same answer. After these flashes of insight, perhaps you can hold your nose and help him to achieve this maximal confusion.

Your function should return the length of the longest run achievable by at most *k* changes. Despite the despicable premise, it is possible to polish this turd enough to reveal the diamond underneath. Try to solve this problem in a single for-loop with O(1) extra memory, for a solution so ravishingly beautiful in its simplicity that any student working through it and proving it correct will instantly level up in his or her algorithmic and coding problem skills.

answer_key	k	Expected result
'FTFFFF'	3	6
'TFFTTFFTT'	2	6
'TTFFFTFFTFT'	3	9
'TFFFFTTTTTFFFF'	3	8

76. Burrows–Wheeler transform

```
def burrows_wheeler_encode(text, positions):
```

Burrows–Wheeler transform is a famous algorithm that converts any `text` string into a string of equal length, with the original characters permuted so that various compression algorithms can compress them much better compared to how they would compress the original string. Of course, after decompressing to get back the transformed string, this transform can be effectively and efficiently reversed to restore the original `text`, as a topic for a follow-up question left hanging here.

This function should compute the Burrows–Wheeler transform of the given `text` string. However, instead of returning the entire transformed string, you should only return the characters in the given `positions` of the result as **proof of work** of having computed this transformation, while keeping the result small for our automated fuzz tester and the `expected_answers` file.

The educational point of this exercise is to take the inefficient Python implementation that explicitly constructs all cyclic rotations as the starting point, and rewrite it to operate **in place** so that only one copy of the `text` is kept in memory at all times. The fuzz tester will give your function long strings composed of random words from the `words_sorted.txt` file, so efficient operation of your function here is a must. Each `text` is also guaranteed to end with the terminator `'$'`.

Your function should create the list of all position indices in the text and sort this list with a **custom comparator** function that, given two positions `i` and `j`, compares the two cyclic rotations of `text` starting from those positions in lexicographical order, traversing the characters in lockstep from both positions and comparing them pairwise until a mismatch is found. Use the `cmp_to_key` function of the Python functools standard library module to pass your custom comparator function to the `sort` method, and then just extract the last characters of the cyclic substrings starting from the indices in the `positions` given.

text	positions	Expected result
`'a$'`	`range(2)`	`'a$'`
`'abc$'`	`range(4)`	`'c$ab'`
`'abab$'`	`range(5)`	`'bb$aa'`
`'baab$'`	`range(5)`	`'bbaa$'`
`'pilloriesafterpartlepidopterous$'`	`[10, 12, 17, 23, 27]`	`'rtree'`

77. Bron–Kerbosch maximal clique enumeration

```
def maximal_cliques(edges):
```

The pairwise friendship relation of a group of *n* people is expressed as a graph, where `edges[u]` is the list of friends of the person *u*. Friendship is assumed to be symmetric in that *v* is a friend of *u* whenever *u* is a friend of *v*, so this graph is effectively undirected. Some people are loners with no friends, so their friend list is empty.

In graph theory, a **clique** is a set of nodes that are all pairwise friends with each other. A clique is said to be **maximal** if no node can be added to it without breaking the defining clique property of the universal pairwise friendship. This function should return the list of all maximal cliques of three or more nodes in the graph with the given `edges`. The nodes of each maximal clique should be listed in ascending order, and the cliques themselves should be listed in descending order of size, with cliques of equal size listed in ascending lexicographic order. Note that, unlike in the earlier Tarjan's strongly connected components problem, maximal cliques are not an equivalence partitioning of the nodes in that graph, but it is perfectly possible for some social person to simultaneously belong to multiple separate maximal cliques.

You should implement the Bron–Kerbosch maximal clique enumeration algorithm as a nested function inside this function. Maintain a list of maximal cliques outside the recursion, and append the maximal cliques to this list as your recursion finds them. The pseudocode of the Bron–Kerbosch algorithm should translate neatly into Python and its built-in operations on sets. To ensure that your function will pass our pseudorandom fuzz test suite in a reasonable time, you should implement the version of the Bron–Kerbosch algorithm that uses pivot nodes, making a good choice of a pivot node each time to ruthlessly prune the redundant branches of the search.

edges	Expected result
[[1, 3, 2, 5], [0, 2, 4, 3], [1, 0], [0, 1, 5], [1], [0, 3]]	[[0, 1, 2], [0, 1, 3], [0, 3, 5]]
[[1, 4], [0, 2, 5, 4, 7, 3], [1, 7, 3], [2, 1, 5], [0, 1, 7], [1, 7, 3], [], [2, 5, 1, 4]]	[[0, 1, 4], [1, 2, 3], [1, 2, 7], [1, 3, 5], [1, 4, 7], [1, 5, 7]]
[[1, 3, 6, 2, 5], [0, 2, 5, 3], [1, 3, 0, 5], [0, 2, 6, 1, 5], [], [1, 0, 2, 3], [0, 3]]	[[0, 1, 2, 3, 5], [0, 3, 6]]

78. Queue jockeys

```
def queue_jockeys(permutation, moves):
```

A simulated system consists of two queues. The first queue initially contains the integers 0, ..., $n - 1$ as listed from front to back in the given `permutation`, whereas the second queue is initially empty. A series of `moves`, each move naming one of the stored elements in this system, is executed in order so that each move removes that named element from whichever queue it is currently in, and places that element in the front of the other queue. This function should return a two-tuple giving the contents of the first and second queues as lists after all these `moves` have been executed.

Of course, at this point in your studies, this function should not be hard to write using two Python lists to store the contents of these queues, and using simple Python list slicing and dicing operations to perform each individual move. Alas, after the smooth start of small inputs suitable for debugging, the automated tester will give your function inputs where n is in the thousands, and the number of `moves` is in tens of thousands, making sure that any solutions based on list slicing and dicing will not pass the automated tests within the set time limit. You need to design a more efficient representation of the system state that allows you to perform each move in constant time, regardless of the number of elements in both queues.

Note also that even though in the examples below `moves` is always a Python list, for the later inputs the automated tester will make `moves` to be a lazy **generator** object that allows you to iterate through the `moves` with a for-loop, but does not allow random access to jump back and forth or rewind the moves. This should not really affect any reasonable implementation of this function, though. (Any generator object turns into a `list` with a single function call anyway, if needed.)

permutation	moves	Expected result
[0, 1]	[0, 1, 1, 0]	([0, 1], [])
[3, 1, 0, 2]	[0, 0, 0, 1, 3, 0, 1, 2]	([1, 0], [2, 3])
[4, 3, 0, 1, 2]	[4, 2, 0, 1, 3, 3, 4, 3]	([4], [3, 1, 0, 2])

79. Distance from land

```
def distance_from_land(map):
```

Each square in a two-dimensional `map` grid is either `'L'` for land or `'S'` for sea. This function should return a two-dimensional list of lists of the same size as the `map` whose each element gives the integer distance from that square to its nearest land square. The distances between two squares are computed as **Manhattan distances**, that is, the sum of their absolute horizontal distance and their absolute vertical distance. This function should be efficient even for large `map` instances to pass the automated fuzz tester.

map	Expected result
['LSS', 'SSS', 'SSS']	[[0, 1, 2], [1, 2, 3], [2, 3, 4]]
['SSSLS', 'SSSSS', 'SSSSS']	[[3, 2, 1, 0, 1], [4, 3, 2, 1, 2], [5, 4, 3, 2, 3]]
['SSSSLSLLS', 'LSLLLSSSS', 'SSLLLLSSL', 'SSLSSLSSS', 'LSSLSLSSS', 'SLLLSLSSS', 'SSSSSLLSS', 'SSSSLLSSL']	[[1, 2, 1, 1, 0, 1, 0, 0, 1], [0, 1, 0, 0, 0, 1, 1, 1, 1], [1, 1, 0, 0, 0, 0, 1, 1, 0], [1, 1, 0, 1, 1, 0, 1, 2, 1], [0, 1, 1, 0, 1, 0, 1, 2, 2], [1, 0, 0, 0, 1, 0, 1, 2, 2], [2, 1, 1, 1, 1, 0, 0, 1, 1], [3, 2, 2, 1, 0, 0, 1, 1, 0]]

This problem is adapted from LeetCode problem 1162, "As far from land as possible".

80. Smallie Bigs

```
def partition_point(items):
```

The famous quicksort algorithm, to, well, *quickly sort* the elements of the given list, starts by **partitioning** the elements of the list into two lists of "small" and "big" elements so that every element in the list of small elements is less than or equal to every element in the list of big elements. Various clever ways to perform this partition to make both sides roughly equal in size have been invented, thereby balancing the "divide and conquer" recursion of the quicksort algorithm, as all things should be.

However, in this problem, we want to determine whether a good partition point already exists for the given list of items. This function should determine whether the list of items can already be split at some position into two sublists so that every element in the left sublist is less than or equal to every element in the right sublist. This function should return the number of elements in the left sublist of the resulting partition. If multiple solutions exist, you should return the one that makes the number of elements in the left sublist as close to half of the length of the original items list.

The automated fuzz tester will give this function only lists where such a partition exists so that both sides end up having at least one element. Furthermore, the tester will give your function 10,000 test cases that contain up to thousands of elements, so your function needs to operate in linear time with respect to the length of the list, instead of sawing back and forth in an inner loop for each individual element like some kind of Shlemiel.

items	Expected result
[0, 9, 15, 13, 14]	2
[5, 11, 3, 13, 18, 22]	3
[53, 23, 31, 23, 3, 47, 42, 22, 79, 65]	8
[108, 148, 107, 6, 44, 39, 130, 281, 185, 274, 175, 240, 168, 188]	7

81. Gauss circle

```
def gauss_circle(r):
```

The <u>Gauss circle problem</u> of **combinatorial geometry** asks how many points (x, y) where both co-ordinates x and y are integers lie within the disk of radius r centered at the origin. (Points at the edge of the disk are counted as being within the disk.) This famous problem is still open in that no-body has discovered a closed-form solution to this problem as a function of the radius r. Of course, in this course, we can still use this classic problem as an exercise in writing loops to solve the prob-lem efficiently, at least for reasonably small r.

One simple way to count up the points is to add them up one row of points at a time. Initialize the total count to 0 and start traversing the boundary of the disk at its topmost point $(0, r)$, surely part of the disk. Whenever you are standing at the point (x, y), add the $2x + 1$ points on the row y to the total count, and do the same for the points in the row $–y$ whenever $y > 0$. Decrement the y-coordi-nate, and increment the x-coordinate as long as the point (x, y) remains inside the disk. Once you have added up the points in all rows in this manner, return the final total.

r	Expected result
1	5
2	13
7	149
58	10557
1809	10280737
1000000	3141592649625

The related <u>Euclid's orchard</u> problem is otherwise the same, but counts only the points (x, y) that are directly visible from the origin without any other points standing directly in between the origin and that point, which happens precisely when x and y are **relatively prime** with no factors higher than one in common.

82. Zeckendorf decoding

```
def zeckendorf_decode(fits):
```

Fibonacci numbers are often used to teach recursion in computer science. However, they can also be used as building blocks of all positive integers due to Zeckendorf's theorem, that says that any positive integer can be expressed as a sum of distinct non-consecutive Fibonacci numbers in exactly one way, and that representation can be easily computed with the greedy algorithm that always uses the largest Fibonacci number that still fits in the remaining number. For example, the unique representation of 72 as such a sum is 1 + 3 + 13 + 55.

As an alternative to expressing a positive integer in binary "bits" as a sum of distinct powers of two, **Zeckendorf encoding** turns this sum of distinct Fibonacci numbers in binary "fits" where the ones indicate the Fibonacci numbers included and the zeros indicate the numbers excluded in the sum. Unlike in ordinary **positional number** systems, where the digits are written in order of decreasing powers of the base, Zeckendorf encoding writes the fits in order of increasing Fibonacci numbers. For example, the Zeckendorf encoding of 72 is 101001001. This order helps greatly when encoding an arbitrary long list of integers, each of an arbitrarily large magnitude, where the Zeckendorf encoding shines since the impossible fit pattern 11 can be used to denote the "comma" that separates two consecutive numbers in the list, the first of these two fits serving a double duty as the highest order fit of the integer being encoded! For example, the list [72, 6, 81, 2, 5] would be encoded as a string of fits 10100100110011000100101101100011.

This function should decode the given sequence of `fits` back to the original list of integers. Your function can assume that the last two `fits` are always 11.

fits	Expected result
'01001110001101011'	[10, 9, 7]
'011'	[2]
'1001000001110001001011100100110101011111'	[95, 132, 27, 20, 1]
'00101011'	[32]
'001001010100011010000000001011011'	[749, 523, 2]

When integers are expressed in binary positional notation, encoding the separator between integers of arbitrary magnitude can get tricky. Elias gamma coding is one clever way to achieve even tighter compression than Zeckendorf encoding.

83. Sultan's daughter

```
def sultans_daughter(expression):
```

Similar in spirit and style to the "McCulloch's Second Machine" problem in the original collection of *109 Python Problems*, **Sultan's daughter** is an esoteric programming language based on the [string rewriting paradigm](), devised by none other than [Raymond Smullyan]() himself.

Expressions written in Sultan's daughter are strings that consist of digit characters 0 to 9, with no other characters allowed. To evaluate an expression of the form dX, where d is some leading digit character and X is some string of digit characters (possibly empty), the first digit d determines which operation is applied to the recursively computed value of the rest of the expression X.

Evaluation rules for possible values of d are listed on [the Wiki page for Sultan's daughter]() at the website [esolangs.org]() for the reader to consult. For example, since the expression 19872 evaluates to 987, the expression 319872 evaluates to 987987, and therefore the expression 4319872 evaluates to 789789. This function should iterate the given `expression` of digit characters 0 to 9 until the evaluation either terminates when none of the rewriting rules apply, or is nonterminating due to reaching some expression that has already been encountered during that evaluation. This function should return the number of iteration steps required to reach termination. In case of a nonterminating `expression`, return −1 instead.

expression	Expected result
'663653'	1
'1171112222'	4
'131156166331311152222222'	8
'11313116113147222222222'	10
'47431474312'	−1

That last expression 47431474312 is actually a [quine]() in Sultan's daughter, evaluating to itself.

84. Limited swap sorting

```
def limited_swaps(perm, swaps):
```

All **comparison sort** algorithms operate by swapping two elements according to some discipline that guarantees that after these comparisons and conditional swaps, the resulting list will be in sorted order. This problem considers an interesting restriction of sorting problem: the function is given the list of `swaps` as tuples `(i, j)` that the algorithm is allowed to perform, where `i` and `j` are position indices.

Given the list `perm` that is guaranteed to be some **permutation** of integers 0, ..., $n - 1$, this function should determine how close to sorted order that permutation can be brought when limited to using only the given `swaps`. This function should return the **lexicographically smallest** list that is reachable from the given `perm` using only the allowed `swaps`. Note that the same individual swap may be used multiple times during the sorting process to marshal the elements towards their final resting positions.

perm	swaps	Expected result
[2, 0, 1]	[(0, 1), (0, 2)]	[0, 1, 2]
[2, 3, 1, 0]	[(1, 2), (2, 3)]	[2, 0, 1, 3]
[0, 3, 2, 4, 1]	[(0, 1), (0, 2), (1, 2)]	[0, 2, 3, 4, 1]
[3, 4, 0, 6, 1, 2, 5]	[(1, 4), (3, 4), (5, 6)]	[3, 1, 0, 4, 6, 2, 5]

This problem is "Mosey's Birthday" in the DMOJ online problem collection. (Why that problem is called that over there, well, your guess is basically as good as mine.)

85. Average run length test of randomness

```
def average_run_length(sequence):
```

As in the earlier "Poker test" problem, statistical tests to evaluate the quality of **pseudorandom number generators** first produce a long sequence of pseudorandom numbers from that generator, and then calculate various statistical quantities for that pseudorandom sequence and compare those quantities to the quantities that a truly random sequence would be expected to have.

Another such test computes the average length of the non-ascending and non-descending runs of the given sequence. For example, the sequence `[2, 2, 0, 1, 2, 6]` starts with a non-ascending run `[2, 2, 0]` of length 3, followed by a non-descending run `[0, 1, 2, 6]` of length 4. Note how the turning point element, where the direction of the run turns, is included in both runs. The average run length of this example sequence is therefore (3 + 4) / 2 = 7/2.

This function should compute the average run length of the given `sequence` of integers, and return the answer as an exact `Fraction` object. Be careful to ensure that your function also correctly handles the sequences that start with a run of equal values, forcing you to wait and see whether that first run is non-ascending or non-descending.

sequence	Expected result (as `Fraction`)
[3, 1]	2
[1, 1, 1, 3, 4, 2, 5]	3
[1, 1, 1, 5, 5, 2]	7/2
[1, 3, 8, 8, 11, 9, 10, 11]	10/3
[4, 4, 8, 14, 15, 11, 11, 7, 0, 3, 1, 1, 9]	17/5
[42, 42, 42, 42]	4

Students interested in this kind of stuff can also ponder what the expected average run length would be for a truly random sequence of integers from 1, ..., *n*.

86. Count increasing paths

```
def count_increasing_paths(matrix):
```

You are given an *n*–by–*n* element square `matrix` that contains each one of the integers 0, ..., $n^2 - 1$ in some order. The two-dimensional square `matrix` is given as the list of its individual rows, which allows you to index the element in row `i` and column `j` with the expression `matrix[i][j]`, just like you did in your linear algebra courses.

Your task is to count how many separate increasing paths there exist inside the entire `matrix` when you are allowed to start the path from any cell and keep moving to any of the four axis-aligned compass directions, as long as you always take each step into an adjacent cell that contains a higher value than the cell that you are moving away from. The edges of the `matrix` do not wrap around to the other side of the matrix in a toroidal fashion, but are again solid walls reaching to infinity, with nothing existing outside these numbers in this matrix.

The increasing paths in the matrix have a nice self-similar structure that allows you to solve this problem recursively. The number of increasing paths of length *k* that terminate in the cell (*i*, *j*) can be computed recursively by adding up the number of paths of length *k* – 1 that lead to those neighbouring cells that contain a lower value than the cell (*i*, *j*). The base case of this recursion is the count of singleton paths starting and ending at any one cell being equal to 1. You can either write a recursive function to solve this and use the `@lru_cache` decorator to **memoize** it, or you can use **dynamic programming** to fill in the table of path counts for the length *k* from the previous table that contains the path counts for the length *k* – 1. If you choose to take the latter route, note that you only need to maintain the **sliding window** of two such tables, assuming that you add up path counts in ascending order, since each count of paths of the length *k* depends only on the counts of paths of the length *k* – 1.

matrix	Expected result
`[[1, 0], [2, 3]]`	11
`[[3, 0], [1, 2]]`	8
`[[1, 2, 8], [6, 3, 4], [0, 5, 7]]`	39
`[[10, 15, 1, 12], [14, 9, 7, 0], [11, 6, 2, 13], [8, 5, 3, 4]]`	89
`[[36, 31, 39, 5, 17, 1, 23], [13, 35, 48, 4, 38, 18, 43], [21, 28, 10, 12, 25, 34, 6], [16, 2, 8, 19, 3, 11, 44], [41, 32, 47, 7, 37, 29, 0], [46, 14, 22, 27, 45, 26, 15], [24, 33, 40, 9, 42, 30, 20]]`	258

87. Cocke–Younger–Kasami parsing

```
def cyk_parser(text, rules):
```

A context-free grammar consists of **nonterminal symbols** (here, uppercase letters) of which `'A'` is conventionally the **start symbol**, and **terminal symbols** (here, lowercase letters) that the `text` string is made of. The grammar itself is defined by its rewrite `rules` of the form $S \to \alpha$ that determine the sequences of terminal and nonterminal characters that the left-hand side S can be replaced with. The `rules` are given as a Python dictionary whose keys are the left-hand side nonterminals and the values are the lists of possible right-hand sides for those nonterminals. The grammar **produces** the strings of terminal characters that can be somehow produced by application of rewrite rules in some order from the start symbol, each rule applied one at a time to one occurrence of its left-hand side in the current string of symbols.

The context-free grammar is in Chomsky normal form if the right-hand side of each rewrite rule is either a single terminal symbol, or exactly two nonterminal symbols. Whether the given terminal string α can be produced from a grammar whose `rules` are in Chomsky normal form can always be efficiently determined with the Cocke–Younger–Kasami algorithm. This algorithm is a simple recursion on the current nonterminal symbol S and terminal string α, equaling the start symbol and the entire `text` in the top-level call.

The base case of recursion is when α is one terminal symbol, easily answered by checking if there is a rewrite rule $S \to \alpha$. Otherwise, check whether α can be split into two nonempty pieces $\alpha = \beta\gamma$, so that the grammar has some rewrite rule $S \to BC$ for which B produces the left piece β, and C produces the right piece γ. You can write this parsing algorithm as straightforward recursion using the `@lru_cache` decorator to tame the exponentially overlapping subproblems, or as a bottom-up **dynamic programming** approach, systematically tabulating subproblem results by translating the algorithm pseudocode from the linked Wikipedia page to Python.

text	rules	Expected result
`'ccb'`	`{'A': ['a', 'b', 'BA'], 'B': ['c', 'BC', 'b', 'CA'], 'C': ['CA', 'a', 'c', 'BC']}`	True
`'bccb'`	`{'A': ['BC', 'DB', 'b'], 'B': ['d', 'BA', 'c', 'b'], 'C': ['AC', 'DC', 'AB', 'a'], 'D': ['DB']}`	False

For some reason, it's practically a law that whenever you are talking about grammar and theory of automata and parsing like you were some fifties mathematician with a crew cut and horn-rimmed glasses, it's compulsory to use Greek letters to denote various strings.

88. Avoidant subset of points on a circular line

```
def farthest_points(points, p, k):
```

A number of `points` have been placed along a line segment of length `p` that has been wrapped around **cyclically** so that its endpoints are connected to each other, making that line segment a weird little thing a bit like a circle, but not quite so. The `points` are given in sorted order in the list, and each point lies in the range $[0, p)$ falling on the line.

Your task is to choose a subset of exactly `k` of these `points` to maximize the minimum pairwise distance among your chosen subset of points. The distances of the other pairs of points in your chosen subset do not matter, but the two points that lie closest to each other determine the solution cost. Since this distance-maximizing subset of points is not necessarily unique, your function should return the maximum distance of the two closest points in your chosen subset.

points	p	k	Expected result
[3, 5, 8, 11, 13, 20]	21	3	6
[3, 5, 7, 12, 13, 20, 21]	22	4	4
[0, 1, 3, 6, 11, 15, 16, 22]	24	4	5
[0, 1, 7, 10, 11, 13, 14, 19, 20, 27]	29	4	6
[3, 4, 9, 10, 12, 14, 18, 20, 21, 23, 24, 25, 28, 31, 34, 40, 42]	43	4	10

89. Ehrenfeucht–Mycielski sequence

```
def ehrenfeucht_mycielski(prefix, k):
```

The blog post "Guess again? The Ehrenfeucht–Mycielski sequence" by Jim Propp gives us a lively discussion of another famous self-referential sequence of bits where the iteration of a simple rule generates highly chaotic and unpredictable behaviour. This function should generate and return the sequence extended by *k* bits from the given initial `prefix`.

The bits of the prefix are extended individually according to the following rule. Start by finding the **longest proper suffix** of that prefix that occurs somewhere else inside that same prefix. Overlapping occurrences are accepted, so the longest suffix of the sequence `'010101'` that occurs somewhere else would be `'0101'`, not `'01'`. In the edge case where the current prefix is either `'0'`, `'1'`, `'01'` or `'10'`, use the trivial empty suffix that occurs at the beginning of the prefix.

Once you have found this longest proper suffix that works, find its rightmost occurrence strictly earlier in the prefix. Extend the prefix with the negation of the bit that follows this rightmost earlier occurrence. For example, in the previous example `'010101'`, the bit following the rightmost earlier occurrence of its longest suffix `'0101'` is 0, and the new extended prefix therefore `'0101011'`.

The test cases given to your function are short enough so that a good optimized search will pass the tests within the time limit, and there is no need to implement any kind of suffix tree or some such data structure.

prefix	k	Expected result
`'0'`	10	`'01001101011'`
`'01'`	14	`'0100110101110001'`
`'101'`	21	`'101100101000111011110000'`
`'01000'`	42	`'01000110101110010011110110000101000001110100101'`

The interested reader can compare this sequence to **Kimberling's repetition-resistant sequence** included in the Additional Python Problems collection, and also to the Linus sequence included in this author's collection of Java coding problems for the course CCPS 209. Both of these sequences produce self-avoidant chaos with a similar but subtly different approaches.

90. Diamond sequence

```
def diamond_sequence(k):
```

One example in the book "Mathematical Diamonds" by the late great Ross Honsberger features a beautiful, infinitely intricate sequence defined by a simple self-referential rule. As a mathematician rather than a computer scientist, the numbering of sequence positions starts at 1, at which position the first element of the sequence equals one. After that, the element in each position k equals the smallest previously unseen positive integer that makes the sum of all elements up to and including that position divisible by k. The reader can verify that the sequence starts as

1, 3, 2, 6, 8, 4, 11, 5, 14, 16, 7, 19, 21, 9, 24, 10, 27, 29, 12, 32, 13, 35, 37, 15, 40, ...

This function should return the element in position k of this sequence. For simplicity, the automated tester will call this function for values of k from 2 to 999999 in this order, so your function does not need to remember the entire sequence so far, but can store the previous element and the set of values seen so far in `global` variables that persist between the function calls.

Furthermore, since this set of values seen so far consists of positive integers and is **monotonic** (that is, an element once seen will never be unseen), you should also maintain in a global variable the lowest unseen positive integer that has not appeared in the sequence so far. You can save plenty of memory and some running time by removing the elements from the set of seen elements that are lower than this lowest unseen element. (Modify the set inclusion tests accordingly in the code.)

k	Expected result
2	3
7	11
11	7
1000	1618
1618	1000

This sequence has an amazing property illustrated in the above table, If the element in position k equals m, then the element in position m equals k. In other words, this sequence is a **permutation** of all positive integers that, after the position 1 that gets mapped to itself, consists of nothing but two-cycles. This fact is all the more astonishing in that generating this sequence seemingly has nothing to do with this property, and yet it still chugs out the elements in seemingly random order made of these neat two-cycles as certainly as a train travelling on its tracks, not even once slipping or missing a beat on its journey to infinity.

91. Minimize largest triangle

```
def min_max_triangle(points):
```

You are given a list of $3n$ points on a two-dimensional plane. Your task is to divide these points into n groups of three points each so that when each group of points is considered a triangle, the area of the largest triangle is as small as possible. This function should return the area of the largest triangle multiplied by two, to ensure that the result is always an integer. For example, if the area of the largest triangle is 5 1/2, your function should return 11.

This problem can be solved recursively by realizing that the first point has to be part of some triangle anyway, so loop through the possibilities of choosing the other two points for that triangle. For each such triplet, compute the best grouping for the remaining points. Once the area of some triangle created in the recursion is not smaller than that of the best solution found so far, you can terminate that branch of the search immediately and return to the previous level.

The example program `geometry.py` in the `ikokkari/PythonExamples` repository contains a utility function that computes the **cross product** of three points that gives twice the signed area of that triangle. Note that depending on the orientation of the three points, the signed area can be negative, so remember to take the absolute value of this quantity to use as the area of the triangle.

```
def cross(p1, p2, p3):
    (x1, y1), (x2, y2), (x3, y3) = p1, p2, p3
    return (x2 - x1) * (y3 - y1) - (y2 - y1) * (x3 - x1)
```

points	Expected result
[(2, 4), (1, 2), (4, 3), (0, 3), (3, 0), (2, 2)]	4
[(10, 15), (13, 14), (11, 0), (14, 1), (4, 12), (12, 9), (0, 12), (10, 10), (15, 3), (1, 3), (3, 5), (0, 11)]	6
[(6, 12), (25, 16), (12, 17), (5, 14), (20, 17), (23, 19), (11, 22), (8, 9), (3, 3), (8, 22), (22, 0), (16, 18), (21, 18), (4, 17), (5, 2)]	13

92. Cutting sticks

```
def cutting_sticks(sticks):
```

An individual stick by itself can be easily broken, but together in a bunch, they are capable of mighty things. Consider a bunch of integer-length `sticks` whose sum of lengths equals $n(n + 1) / 2$ for some positive integer n. Furthermore, assume that the length of each individual stick is guaranteed to be at least n. Under these two assumptions, it is always possible to cut these `sticks` into n separate pieces whose lengths are precisely the first n positive integers $1, ..., n$, with no glue needed.

The cuts to achieve this are not necessarily unique. For example, if given the sticks `[5, 10]`, we can keep the stick 5 as it is, and split the stick 10 into 1 + 2 + 3 + 4. However, we could also split the stick 5 into 1 + 4, and split the stick 10 into 2 + 3 + 5. The number of cuts is always the same in every solution (DUCY?), but we wish to find the best way to cut the sticks by distributing these cuts evenly across the sticks with a cost function where the cost of cutting a stick into k separate pieces equals $(k - 1)^2$ dollars. (For example, cutting a stick into three pieces would cost $(3 - 1)^2 = 4$ dollars.) According to this cost function, the second way to cut the previous example sticks is better with the cost of 1 + 4 = 5, as opposed to the cost 0 + 9 = 9 of the first, more unbalanced solution.

This function should return the cost of the optimal way to cut the `sticks` to produce the required n pieces. You can solve this with **backtracking** that examines the possible ways to extract the piece of length k from the sticks, and for each such cut, recursively explores the ways to extract the pieces from k – 1 down for the remaining sticks. At the top-level call we have $k = n$, and the base case is when $k = 0$ with all required pieces having been cut. As usual, you are free to tune up this backtracking algorithm every which way but loose, the only rule being that your code has to be standard Python that produces the correct expected results for every automated test case.

pieces	Expected result
[5, 10]	5
[12, 14, 10]	9
[18, 11, 15, 11]	10
[58, 72, 19, 22]	50

Dropping the assumption of the initial length of each stick being at least n gives up the guarantee of a legal solution existing. In fact, determining whether a legal solution exists for the given bunch of `sticks` is known to be an NP-complete decision problem.

93. Optimal text filling

```
def optimal_ab_filling(text, ab, ba):
```

You are given a `text` string guaranteed to consist only of characters `'a'` and `'b'`, and the period character that acts as a placeholder for an empty location. Your task is to fill in these empty locations with characters `'a'` and `'b'` of your choice to minimize the cost over the entire string, and return the minimal cost that you achieved. Two adjacent characters that are equal to each other incur no cost, whereas any two adjacent characters `'ab'` incur the cost `ab`, and any two adjacent characters `'ba'` incur the cost `ba`.

This problem can be solved recursively with a function that takes two parameters: the position `i` to the `text` string, and the character that was placed at the previous position `i-1`. This function should try the possible characters at position `i` (whenever the `text` already contains a non-placeholder character in that position, the choice is forced without branching) and choose the character that minimizes the total overall cost continuing from that position. Since this recursion visits the same subproblems exponentially many times, use the `lru_cache` magic to prune this into a linear number of branches.

text	ab	ba	Expected result
`'b.a'`	3	4	3
`'b.b.a'`	1	2	2
`'....ab'`	4	1	4
`'b.aaa..a'`	3	2	2
`'..a.baa.'`	5	5	10

This problem is adapted from the Google Code Jam 2021 problem "Moons and Umbrellas".

94. S-expression evaluator

```
def s_eval(expression):
```

<u>Lisp</u>, the great granddaddy of all functional programming languages, is known for its **homoiconic** syntax that makes no distinction between code and data, since both code and data are presented in the exact same syntactic form of **s-expressions** of fully parenthesized expressions written in **prefix notation**. An s-expression is either an **atom** (such as 42) or a structure `(f a1 ... an)` where `f` is a function symbol, and each `ai` is an s-expression whose value becomes that argument to the function `f`. S-expressions are evaluated in order of **inside out** so that, for example, the expression `(* (+ 1 1) (- 7 3))` first simplifies to `(* 2 4)`, which then simplifies to 8.

This problem asks you to write a function that evaluates the given `s-expression` limited to the situation where the only functions are the three basic arithmetic operators `'+'`, `'-'` and `'*'` that always receive exactly two arguments each, and the atoms are nonnegative integers. Your function does not need to be able to deal with syntactically incorrect or invalid s-expressions.

In a more advanced course, this problem could be solved with a classic **recursive descent parser**, but it can actually be solved even more simply with one loop aided by an initially empty `list` used as a **stack**. Loop through the characters of the `expression` and skip all whitespace and left parenthesis characters, but `append` each arithmetic operator and integer atom that you encounter to the stack. Whenever you encounter a closing parenthesis, `pop` the preceding two arguments and the operator from the stack, perform that corresponding arithmetic operation on those two arguments, and `append` the result back to the stack. Once you reach the end of a legal S-expression, the stack will contain exactly one integer atom to return as the result. Also make sure to correctly handle the special case where the `expression` is a single integer atom.

expression	Expected result
42	42
(+ 2 2)	4
(* (* 0 4) (+ 2 3))	0
(* (* 58 (- 47 49)) (* 40 (- 40 49)))	41760

95. Expand recursively run-length encoded string

```
def expand_string(text):
```

Text strings that contain repeated patterns can sometimes be efficiently compressed with **run-length encoding**, where each repeated pattern is stored only once with its repetition count. If the pattern itself contains repeated subpatterns, these subpatterns can themselves be run-length encoded, and this recursive structure can potentially be nested arbitrarily far and deep.

Let us use the syntax k`[pattern]` to indicate that the `pattern` is to be repeated k times. For example, `2[abc]` would expand to `abcabc`. If the pattern itself contains such recursive repetitions, they are expanded inside out before the outer expansion is performed. For example, `2[a3[bc]]` would first expand to `2[abcbcbc]`, which then expands to `abcbcbcabcbcbc`.

This function should expand the given `text` in this manner. Your function may assume that the `text` will follow the syntax rules of this encoding, so it does not need to be able to deal with any malformed input strings. Same as in the previous problem of evaluating recursively nested S-expressions, of course, you could, but you don't need to write a full-on **recursive descent parser**, since a stack-based approach that executes in a single linear for-loop through the `text` without ever having to backtrack is actually able to handle this job. Ain't properly parenthesized expressions just dandy?

text	Expected result
`'4[x]'`	`'xxxx'`
`'4[2[u]]xqk4[r]2[i]'`	`'uuuuuuuuxqkrrrrii'`
`'t5[4[n]]'`	`'tnnnnnnnnnnnnnnnnnnnn'`
`'l3[io]bs2[d]iiyg2[2[kw]i]'`	`'lioioiobsddiiygkwkwikwkwi'`

This cute little problem is the Leetcode problem 394, "Decode string".

96. Longest common strictly increasing subsequence

```
def lcsis(items1, items2):
```

The <u>longest common subsequence problem</u> is a well-known first example used to teach dynamic programming.. (In fact, you can recognize the algorithm courses and instructors whose learning was based on the <u>White Book</u> from the fact that they use this particular problem as the first example of dynamic programming!) This problem asks you to find the longest possible sequence of items that is simultaneously a **subsequence** of the two lists `items1` and `items2`. For example, the longest common subsequence of the sequences `[3, 9, 1, 0, 17, 2]` and `[9, 0, 5, 6, 2]` would be `[9, 0, 2]`.

For two sequences consisting of n and m elements, this problem can be solved with a recursive function $L(n, m)$ whose base cases are $L(n, 0) = L(0, m) = 0$. For the general recursive case $L(n, m)$ where n and m are the lengths of the two respective strings, the result is the maximum of the three numbers $L(n - 1, m)$, $L(n, m - 1)$ and $1 + L(n - 1, m - 1)$ where the third case applies only if the last elements of both sequences are equal. As usual, this recursion can be made efficient either through **memoization** or **dynamic programming** to tabulate the results bottom up.

However, this problem now adds an extra twist to this classic recursion example in that your recursive function needs to find the longest common subsequence whose elements are in **strictly increasing order**. Your recursive function needs to have three parameters, where the new third parameter is the upper limit of the value of the next element that can be taken in the subsequence. Since this subsequence is not necessarily unique, return only its length.

items1	items2	Expected result
[7, 13, 1]	[2, 15, 3, 8, 2]	0
[8, 17, 13, 17]	[2, 23, 18, 5, 8, 13, 2, 17]	3
[47, 39, 78, 4, 35, 63, 6, 81, 3, 90, 112]	[4, 124, 102, 35, 81, 90, 111, 112]	5

97. Maximum palindromic integer

```
def maximum_palindrome(digits):
```

Here is a cute little puzzle from a technical interview problem collection for which writing the actual code is not too difficult, but the difficulty is in coming up with the algorithm to solve the problem under the time limit of solving the question under pressure of getting the job and securing your future on the good side of the barbed wire fence in the diminishing race to the bottom that the entire computing industry has suddenly turned into since the golden years of the youth of this author. Yes, children, gather around to hear how there once was a peaceful time when you could get an $80K entry-level job (purchasing power equivalent to about $200K total compensation in 2025 money) for merely knowing how to write loops in Visual Basic.

Given a string of `digits` that you are allowed to use, your task is to construct the largest possible integer that is a palindrome, that is, reads the same from left to right as from right to left. The resulting number may not contain any leading zero digits, but may contain zeros inside the number itself. Each digit can be used as many times as it appears inside the `digits` string.

digits	Expected result
'0'	0
'011'	101
'8698'	898
'123123123'	3213123
'225588770099'	987520025789

98. Optimal palindromic split

```
def palindrome_split(text):
```

Any `text` string can always be split into disjoint contiguous pieces so that each individual piece is a palindrome, that is, reads the same from left to right and right to left. Of course, making each piece a singleton character trivially satisfies the definition of a palindrome, but some strings allow for a more elegant partition that consists of significantly longer pieces.

Following the general principle of how you get what you reward, let us define that each palindromic piece whose length is k characters is worth k^2 points. For example, a piece with five characters would give you 25 points, much better than using those same five characters as five separate pieces worth a measly one point each. This function should determine the optimal way to split the given `text` string into palindrome pieces to maximize the overall score, and return that maximal score.

This problem cannot be solved with a **greedy** approach where you make the first palindromic piece as long as possible, and then try to maximize the score over the remaining `text` in the same greedy vein. The greedy choice for the first palindrome might consume some characters that were necessary for the next piece to be maximally long, such as in `'SOSAIPPUAKAUPPIAS'`. You should instead use **dynamic programming** to compute the maximum score M(i) as the maximum over the immediate scores of each possible palindrome that starts from position i, plus the maximum score achievable from the position after that initial palindrome. You can solve this as a recursive function using the `@lru_cache` decorator to **memoize** it to avoid iterating through the same subproblems an exponential number of times, or you can go hardcore and tabulate the maximum scores backwards from the end of the `text` to the beginning, and then return `table[0]` as your final answer.

text	Expected result
`'AXA'`	8
`'ABAB'`	10
`'())('`	16
`'CABCCBA'`	37
`'CACAACCAABBAACCAA'`	205
`'AACBAACBCAACCA'`	60

Also, do not fail to appreciate afterwards how your algorithm is not actually dependent on the fact that the acceptability of a piece is being a palindrome and that the scoring function is the square of piece length, but the dynamic programming algorithm itself is exactly the same for literally any acceptability criterion and scoring function that can be freely plugged in.

99. Semiconnected guys

```
def is_semiconnected(edges):
```

A **directed graph** consists of nodes numbered 0, ..., $n - 1$. The `edges` of this graph are given as a list where each element `edges[u]` is the list of the edges emanating from the node u to its immediate neighbours. This function should determine whether this graph is **semiconnected**, that is, for each pair of nodes u and v, there exists a directed path from u to v, or from v to u. Note that this question is not the same as testing whether the graph is fully connected when each edge is treated as undirected, since the edges in each directed path still have to be consistently pointing in the same direction. Each individual node is considered to be automatically connected to itself with an empty path.

The simplest way to solve this problem would be to implement the classic Floyd–Warshall all-pairs reachability algorithm to determine the reachability between all pairs of nodes simultaneously, and then verify from the resulting reachability matrix that the nodes of each pair of nodes in the graph are reachable from each other in at least one direction. Since the paths connecting nodes inevitably have many subpaths, the Floyd–Warshall algorithm is superior to performing a breadth-first search separately from each node and collating the final results of these searches.

edges	Expected result
`[[1, 3], [], [0], [1, 0]]`	True
`[[3, 2], [4], [], [1], [0]]`	False
`[[5], [6, 2], [5, 4, 7, 0, 6], [6], [3], [4, 7, 6, 3], [5, 0], [0, 3, 2, 5]]`	True
`[[1, 7, 4, 3], [4, 6], [5], [1, 7, 6, 5], [7], [1, 6], [2, 3], [2, 5]]`	True
`[[1], [5], [1], [7], [6, 5], [1], [2, 4], [1, 0, 8], [4, 6]]`	False

100. Falling squares

```
def falling_squares(squares):
```

Starting with an initially flat surface with a uniform height of zero, some `squares` fall straight down from a large height, one square at a time, in the style of Tetris. Each square is represented as a two-tuple `(x, w)`, where `x` is the *x*-coordinate of the left edge of the square, and `w` is the width (and thus also the height) of that square. Each square falls straight down until some non-zero part of it lands on a flat surface, either the floor or some previous square. The surfaces between squares are assumed to be infinitely sticky, so that the square will stick to where it lands, again in Tetris style, without tipping over, regardless of any dramatic overhang of the column squares set up so far.

Given the list of `squares` in the order in which they fall one by one, this function should return a list of the same length whose each element equals the largest height that the entire resulting structure reaches after dropping that square. Your function should somehow keep the surface's current shape, and efficiently update that surface after each individual square.

This problem is a hardened version of problem 699 of LeetCode, "Falling squares". Their version keeps the coordinates and square sizes small enough for you to simply maintain a **heightfield**, a list of current height values for the range of x-coordinates that the squares fall on, and update it with a trivial for-loop for each new square. In our version, after a smooth start with test cases small enough to help your debugging, the automated tester will start shooting these numbers up exponentially. Instead of using a heightfield, you therefore need to set up some kind of higher-level representation for the outline of the shape created by the squares so far that remains efficient in both time and space, even for square positions and widths up in the gazillions.

squares	Expected result
`[(4, 3), (6, 1), (5, 1)]`	`[3, 4, 4]`
`[(8, 1), (2, 1), (11, 1), (3, 2)]`	`[1, 1, 1, 2]`
`[(4, 3), (6, 3), (2, 1), (2, 2), (13, 1)]`	`[3, 6, 6, 6, 6]`
`[(7, 2), (3, 2), (7, 1), (9, 3), (5, 2), (9, 3), (7, 4), (10, 1)]`	`[2, 2, 3, 3, 3, 6, 10, 11]`

101. Tchuka Ruma

```
def tchuka_ruma(board):
```

Tchuka Ruma is an interesting solitaire variation of Mancala, the family of **sowing games**. The `board` consists of a **store** in position 0 and **holes** in positions 1, ..., $n - 1$. The store is initially empty, and each hole initially contains some number of pebbles. Your goal is to get as many pebbles as you can in the store before you inevitably paint yourself in a corner with no more possible moves. This function should return the maximal number of pebbles that end up in the store.

An individual move consists of picking up all pebbles of some nonempty non-store hole, and sowing these pebbles into holes starting from the position immediately preceding the chosen hole and continuing leftward one pebble at a time. If this sowing process extends past the left edge of the board after placing a pebble in the store, it continues cyclically in the holes at the right end of the `board`. This sowing move can then end in three possible ways:

- If the last pebble ends up in the store, that move is successfully completed. The player gets to freely choose his next move from the available nonempty holes.
- If the last pebble ends up in an empty non-store hole, the game is over with the final score equal to the number of pebbles that ended up in the store.
- Otherwise, the move is not over, but continues by the player picking up all the pebbles in that last hole and sowing them leftward in the exact same fashion, with the same end conditions.

board	Expected result
[0, 3, 0]	1
[0, 3, 1]	3
[0, 3, 2, 4]	7
[0, 1, 4, 4, 5]	13
[0, 5, 4, 7, 7, 0, 0, 8, 1]	2
[0, 5, 9, 3, 0, 1, 9, 2, 3]	32

Since the sowing process can only move these pebbles leftward and cannot skip over the store from where no pebbles ever get picked up from, we see that even though a single move can consist of several stages of picking up and sowing pebbles, each move must eventually terminate after a finite number of steps, since during each round trip around the board, the number of pebbles available for seeding will decrease by one.

102. Slater–Vélez sequence

```
def slater_velez(k):
```

This problem greatly resembles the earlier "Diamond sequence" problem and can actually be solved with the same technique, even though the sequence of positive integers that this self-referential process chugs out is even more chaotically whimsical than in the diamond sequence. The numbering of sequence positions again starts at 1 instead of 0, and the first element in that first position also equals 1. After that, each element is always the smallest positive integer n that has not appeared so far in the sequence, under the constraint that the absolute difference of n and its immediate predecessor in the previous position of the sequence also has not previously appeared as the absolute difference of any two consecutive elements earlier in the sequence. With some quick arithmetic, the reader can verify that this sequence starts its journey as

1, 2, 4, 7, 3, 8, 14, 5, 12, 20, 6, 16, 27, 9, 21, 34, 10, 25, ...

since the sequence of absolute differences between the consecutive elements of the previous sequence begins as

1, 2, 3, 4, 5, 6, 9, 7, 8, 14, 10, 11, 18, 12, 13, 24, 15, ...

This function should return the value in position k of this sequence. The automated tester will again give your function consecutive values of k starting from 2 upwards, so your function can store the previous element and the sets of previously seen values and previously seen absolute differences as global variables. Since both sets involved in this problem are monotonic, you can also apply the same technique as in the diamond sequence problem of keeping track of the lowest unseen value in both sets, which eliminates the need to explicitly store the elements lower than that value and massively alleviates the memory use of this function.

103. Odds and evens

```
def odds_and_evens(first, second):
```

This function is given two strings `first` and `second`, both guaranteed to consist only of binary digit characters `'0'` and `'1'`. These strings are to be converted to lists of positive integers so that each `'0'` becomes an even integer and each `'1'` becomes an odd integer, under the two constraints that (a) both resulting lists must be in ascending sorted order and (b) each integer may appear at most once in these two lists. Your task is to determine, over all possible ways to convert these strings into lists of integers under these constraints, the minimum possible value of the largest integer to appear in either list.

first	second	Expected result
`'1111'`	`'1'`	9
`'000111'`	`'000111'`	12
`'0001'`	`'0001'`	9
`'10111011'`	`'10111011'`	21

This cute little problem is the problem "Zrinka" in the DMOJ online problem collection. It is actually a bit harder than it initially seems in that this author misunderstood his own problem and made an error in the original version that one tyro student pointed out. However, when this specification was given to Claude Sonnet 4, it made the same error as the original author!

104. The prodigal sequence

```
def front_back_sort(perm):
```

When you are allowed to repeatedly swap any two elements in the list, many efficient **comparison sort** algorithms are known to sort any list of items in the sorted order. However, suppose that your moves are restricted to picking any one element from the list, and moving that element to either to the front or to the back of the list. Given the `permutation` of integers 0, ..., $n - 1$, how many such moves would be needed to rearrange the elements of that permutation in sorted order?

Of course, $n - 1$ moves always trivially suffice by repeatedly moving each element to the back, starting from one and proceeding in sorted order. But how do you recognize if some smaller number of moves would also work?

perm	Expected result
[0, 1]	0
[1, 0, 2]	1
[0, 2, 3, 1]	2
[2, 1, 3, 4, 0]	2
[2, 1, 4, 0, 3]	4
[0, 1, 5, 2, 4, 3]	2

The solution to this problem can also benefit a lot from thinking instead of just brute forcing through all possible move permutations.

105. Descending suffix game

```
def descending_suffix_game(board, n):
```

This cute little problem was apparently used as a job interview question for a quant position at Jane Street, the trading firm known for its love of puzzles (solutions). You have a shuffled deck made of *n* cards marked with integers 1, ..., *n*, and are invited to play a blackjack-type game where you draw cards randomly from the deck, one card at a time, to build a board where, unlike in blackjack, the order of the cards matters. Each round, you get to choose whether to **stand** with the board that you have built so far, or if the deck still has cards remaining, **hit** to draw the next random card and append it at the end of your current board. Once you choose to stand, your final score equals the sum of the cards in the longest descending suffix of the board. For example, if you choose to stand with the board [9, 3, 8, 4, 2], your final score would be 8 + 4 + 2 = 14.

This function should determine the expected value of the given `board`, assuming that the player chooses his moves optimally, analogous to a blackjack player mechanistically following the basic strategy card to maximize his expected result. Under the basic assumptions of sufficiently small stakes in a repeated game so that your utility curve of points is linear, the expected value and thus the expected utility of the given `board` equals max(*s*, *h*), where *s* is the value of standing with the descending suffix that you have so far (easy enough to compute), and *h* is the expected value of hitting, averaged over all the possible cards that can come out next under the premise that you will continue making optimal decisions of standing or hitting regardless of whatever card the gods of randomness happen to give you. You should do all calculations with exact `Fraction` objects.

board	n	Expected result
[4, 1, 2]	5	13/2
[1, 4, 3, 7]	7	77/6
[11, 1, 9, 5, 12]	12	100339/5040
[12, 10, 7, 5, 3]	12	37

The original question asked to calculate the expected value of the game for *n* = 9 from the empty board. This turns out to be 2749727/181440, approximately 15.155, which agrees with the commenters in the Math Overflow thread linked above.

106. The Borgesian dictionary

```
def find_all_words(letters, words):
```

Wordscapes, a popular phone and tablet game, gives players a bunch of `letters` to be used to create as many words as possible. Since this mindless task of patience and memory of iterating through all possibilities seems better suited for a machine than a man, it's time to automate this task and relieve humans from at least this one burden imposed on us by these modern times. This function should return the alphabetical list of all legal `words` that can be created from these `letters`. You can use each letter in the word as many times as it appears in the `letters`. Since the wordlist that we use in these problems is pretty huge, this function should return only the words that have at least seven letters. For simplicity, your function may assume that both `letters` and `words` are given in sorted order.

This function should be a relatively simple exercise on backtracking search, building the result word one character at a time by looping over all the remaining letters, and recursively trying to extend the new word with each such letter until a dead end is reached. Make sure also not to scrub through redundant search paths of duplicated letters in the same level of recursion, and use **binary search** from the Python standard library `bisect` module to quickly determine whether the word that you have built up so far it itself a legal word, and whether it could in principle be extended into a longer word. The efficiency of all backtracking algorithms depends on their ability to turn back as soon as possible once they have painted themselves into a dead-end corner.

`letters`	Expected result (using the wordlist `words_sorted.txt`)
`'aghimoopss'`	`['agomphosis', 'amishgo', 'gaposis', 'gomphosis', 'goosish', 'mishaps', 'samshoo', 'shamois', 'shampoo', 'shampoos', 'sophism']`
`'aeekrstuy'`	`['aesture', 'austere', 'estuary', 'euaster', 'euskera', 'keyseat', 'keyster', 'restake', 'retakes', 'sakeret', 'sautree', 'streaky', 'turkeys', 'tuyeres']`
`'aabcgkostt'`	`['attacks', 'catboat', 'catboats', 'costata', 'gasboat', 'sackbag', 'tabasco']`

Alternatively, you can preprocess the wordlist into a global data structure that maps each word into its Gödel number (the product of prime numbers that correspond to letters in that word), to then determine with integer arithmetic whether a given word can be constructed from `letters`.

107. The Dictorian borgesitory

```
def maximum_word_select(words, letters):
```

Kind of like an upside-down version of the previous "Borgesian dictionary" problem, this function is given a list of `words` and a multiset of `letters` that you are allowed to use. Your task is to choose a subset of `words` so that all these chosen words can be made from the given `letters`, when each individual letter can be used at most as many times as it appears in `letters`. The score for the subset of words is the sum of the lengths of those words, so longer words are better. This function should return the highest total score that can be achieved with the given `words` and `letters`.

This problem is a pretty basic backtracking recursion (you either take in each word or leave it), but this problem is also an opportunity for you to try out the excellent `Counter` data structure in the Python standard library `collections` module to keep track of the remaining letters and quickly determine if the current word can still be constructed from your remaining letters.

words	letters	Expected result
['hymnody', 'lycodes', 'phrasal', 'chams']	'aaabccddehhjllmnoopqrsssyy'	19
['dicht', 'dangler', 'rosal', 'ostend']	'aabbddddeeghillnoorstyy'	12
['nonfluids', 'ululate', 'tentability', 'shook', 'ticktacker', 'spectacularism', 'bourre', 'inexposure', 'sassan']	'aaaaaabbccccddeeeeeeeffiiiii ikkkllllmmnnnnooooppqrrrrrrrss ssssstttttttuuuuuuwxxxy'	60

108. The art of strategic sacrifice

```
def pick_it(items):
```

You are given a list of `items`, each item an integer. The list contains a total of *n* items, and you need to perform the following step exactly *n* – 2 times: choose any one of the `items` that is neither the first nor the last remaining item, and remove that item from the list. At each such removal, your score increases by the square of the current predecessor of that item plus the item itself, and decreases by the square of the current successor of that item. This function should compute the highest score achievable by performing these *n* – 2 item removals in the optimal order.

This problem is adapted from the problem "Pick it" in the DMOJ online problem collection by adding squaring and subtracting the neighbouring elements to make the whole thing more nonlinear.

Similar to the previous problem, "The Borgesian Dictionary", this problem is a pretty basic exercise in backtracking, so that each level of this recursion loops through the remaining elements, and for each such element, recursively tries to build up the rest of the score from the remaining elements. To allow the code to find the current predecessor and successor of each element in constant time, you should use the **dancing links technique** with two lists `prev` and `succ`, both of length *n*. The elements `prev[i]` and `succ[i]` give the current predecessor and successor of the element in the position *i*, respectively. To remove the element in position *i*, update the predecessor and successor information of its neighbours, and after returning from the backtracking recursion, downdate the same information of its neighbours back to how they were before the removal.

items	Expected result
[1, 4, 3, 5]	−25
[1, 5, 7, 1, 6, 2, 2]	142
[3, 7, 1, 2, 2, 5, 6, 7, 7]	117
[10, 8, 1, 6, 7, 6, 6, 7, 9, 9]	481

109. Colonel Blotto and the spymaster

```
def colonel_blotto(friend, enemy, prize):
```

"Colonel Blotto game" is an umbrella term for a family of games in which players have to blindly commit their resources in separate battlefields before knowing how the enemy will allocate their resources. Once both sides have made this commitment, the `prize` for each separate battlefield goes to the player who allocated more resources in that battlefield in a winner-takes-all fashion.

In our version of such a game, the `friend` list gives the strengths of n units to be distributed over n battlefields, listed in descending order. However, instead of us having to allocate these units under fog of war, our valiant spymaster has bravely run through the eye of the needle to acquire the complete allocation of `enemy` forces over these battlefields! This function should compute the maximum total prize available by allocating our units wisely with this knowledge. To ensure that there are no ties on any battlefield, our heroic forces all have an even-valued strength, whereas those dastardly invaders all have an odd-valued strength. You must allocate exactly one unit in each individual battlefield, and are not allowed to combine multiple units into one for a larger strength. All three lists, `friend`, `enemy` and `prize` are guaranteed to have the same number of elements.

A common principle in all Colonel Blotto games (for example, electoral **gerrymandering**) is that if you are going to lose some battlefield, you really want to lose it by as much as possible, since this doesn't make the enemy's gain there any larger while saving our precious resources for better use, while the enemy allocates its power to fight air. In the same vein, if you are planning to win a battlefield, you should try to eke out as small a victory as possible. This idea gives you a simple two-way choice for each battlefield: either give it up and assign your weakest remaining unit there, or choose to win and assign your weakest remaining unit that can beat the enemy force there.

friend	enemy	prize	Expected result
[14, 10, 6]	[9, 13, 21]	[7, 1, 5]	8
[18, 12, 4, 2]	[9, 3, 21, 17]	[7, 3, 1, 12]	22
[18, 16, 16, 14, 12, 8, 6, 4]	[11, 9, 13, 5, 17, 19, 17, 21]	[1, 3, 15, 11, 7, 2, 4, 7]	37

Bonus Problem 110: The way of a man with array

```
def pebblery(predecessors):
```

Jack, an outwardly respectable Victorian gentleman who is secretly a malignant narcissist, knows n ladies numbered 0, ..., $n - 1$. The social hierarchy of these ladies forms a **directed acyclic graph** so that each lady i has immediate `predecessors[i]` who are all numbered higher than her. Jack's ultimate goal is to claim the lady number 0 in his harem for his ultimate validation.

Jack's demented game of courtship proceeds as follows. Each round, Jack can execute one of his two possible moves. He can successfully add to his harem any lady whose every immediate predecessor is currently in his harem. (Thus, if some lady has no predecessors, Jack can claim her in his harem at any time.) Alternatively, he can coldly discard any one of the ladies from his harem. Thanks to Jack's well-honed skills of psychological manipulation, any lady so discarded can later be courted to rejoin the harem under the same precondition of all her immediate predecessors being in Jack's harem at that moment, making her powerless to resist Jack's incessant hoovering and love bombing with promises of how it will all be totally different this time.

However, being paranoid of the women of his harem secretly scheming together against him, Jack wants to minimize the size of his harem at any one time on the path of acquiring the favour of the lady number 0, not including the lady zero herself in this count since Jack doesn't give a jack about anything after that final conquest and will be dumping the rest of his harem anyway. The total number of moves required to reach the ultimate lady is irrelevant to him; the mastery of delicate placement of favours along the way is reward enough anyway. Given the social ranking `predecessors` structure of these ladies, compute Jack's optimal courtship strategy that minimizes the maximum number of ladies in his harem at any given time, and return that minimum. The strategic removal and replacement of ladies along the way is the key to success in this work, but the complexity of this search increases dramatically when multiple paths lead to the intermediate goal. As any skilled practitioner surely knows, a direct approach is rarely optimal.

predecessors	Expected result
`[[1, 3], [2], [3], []]`	2
`[[1, 2], [2], [3, 4, 5], [4, 5], [], []]`	4
`[[1, 2, 4, 5], [4, 5, 6], [3, 4, 6], [], [5], [6], []]`	5
`[[1, 2, 6], [3, 5, 6, 8, 9], [], [4, 10], [8], [6, 7, 8], [9], [11], [11], [], [], []]`	6

Bonus Problem 111: Levine's sequence

```
def levine_sequence(items, k):
```

This excellent problem comes from the Numberphile video "The Levine Sequence" the reader might want to watch before embarking on this coding problem itself. Given a starting list of `items`, each item a positive integer, you will perform the following operation `k` times. Counting from the end of the current `items` list, these individual items represent the number of consecutive occurrences of the corresponding positive integer in the new `items` list. This function should return the sum of elements in the resulting `items` list after this transformation has been iterated `k` times.

For example, starting with the list `[3]`, the first iteration yields `[1,  1,  1]`, with the element 1 occurring three times. The second iteration turns this into the list `[1,  2,  3]`, each of the three elements 1, 2 and 3 occurring exactly once. The third iteration will then turn this into the longer list `[1,  1,  1,  2,  2,  3]`. As the reader can verify, one more iteration step will then turn this into the list `[1,  1,  1,  2,  2,  3,  3,  4,  5,  6]`.

Since the new `items` list contains exactly `sum(items)` elements after each iteration step, the list grows exponentially. The difficulty in this problem is therefore coming up with some more compact representation for the intermediate lists, most likely using some variation of **run-length encoding**. Knowing that the `items` are guaranteed to be consecutive positive integers after the first iteration step, this encoding can and should be made even more compact. However, despite the tantalizing simplicity of this construction, as explained in the post "Levine's sequence" in Futility Closet, nobody has yet managed to compute the last term of Levine's sequence after a measly eighteen steps started from the initial list `[1,  1]`.

items	k	Expected result
[3]	4	28
[2, 1]	2	9
[3, 3, 2]	4	7868
[4, 4, 1, 4, 1]	6	58361987124448

Bonus Problem 112: Post correspondence problem

```
def post_correspondence_problem(first, second, lo, hi):
```

<u>Post Correspondence Problem</u> is a famous problem on string matching, first discovered way back in the year 1946, after all the research silently done to explore the theoretical limits of computability during the totalitarian world war. Given two lists `first` and `second` of equal length of strings made of letters a and b, such as `['a', 'ab', 'bba']` and `['baa', 'aa', 'bb']`, the problem is to find some nonempty sequence of position indices so that concatenating the elements of both of these lists separately as listed in this sequence produces the exact same result for both lists. For example, as the reader can quickly verify, the position sequence `(2, 1, 2, 0)` produces the same result string `'bbaabbbaa'` and `'bbaabbbaa'` for both of these lists, here colour-coded for your convenience.

This function should determine whether there exists a position sequence that produces the same result for the `first` and `second` lists whose length is at least `lo` characters and at most `hi` characters. This problem should be a pretty straightforward exercise on backtracking, but you should again look for ways to prune the search and alternatives at each level of recursion.

first	second	lo	hi	Expected result
`['a', 'ab', 'bba']`	`['baa', 'aa', 'bb']`	5	10	True
`['ab', 'bbab', 'bbb']`	`['aba', 'aabb', 'abba']`	5	5	False
`['baa', 'bb', 'bba', 'bbbb', 'a']`	`['b', 'baba', 'baaa', 'aaaa', 'aa']`	6	12	True

In the absence of `lo` and `hi` parameters, this famous problem is **algorithmically undecidable** in that there simply does not exist any finite algorithm that is guaranteed to determine the existence or nonexistence of a nonempty solution for the two given parameter lists in finite time in all possible cases. The problem is **semidecidable** in that if a solution exists, systematic generation of all solution paths in the breadth-first manner will eventually find one, but there is no general way to determine in finite time whether the current partial solution cannot be extended to a solution.

Bonus Problem 113: Optimal bridge placement

```
def optimal_bridges(perm, k):
```

A straight river, uniformly one unit wide, flows through the peaceful land from east to west. On the north side of that river are *n* houses, placed at uniform distances at positions 0, ..., *n* – 1, with an idyllic linear street connecting all these houses along the riverbank. On the south side of the river, *n* workplaces have been similarly positioned at 0, ..., *n* – 1 along another linear riverbank street. The homeowner `i` works in the workplace `perm[i]` and is the only homeowner working in that place, so `perm` is guaranteed to be a permutation of integers 0, ..., *n* – 1.

Your task is to build exactly *k* bridges across the river. Each bridge must be placed at an exact integer position, and cross the river directly to the same position on the other side. Your task is to place these bridges optimally to minimize the sum of distances from the homeowners to their workplace, each homeowner using the bridge that gives him the shortest path to where he is going. The same bridge can be shared by any number of homeowners on their way to work. The distance travelled by each individual homeowner `i` to his workplace `perm[i]` is therefore given by the formula `abs(i−b) + 1 + abs(perm[i]−b)`, using the best bridge position b for that homeowner. This function should return the minimum total distance across all homeowners.

The automated fuzz tester will give your function enough test cases that are large enough so that brute forcing through the `combinations(range(n), k)` possible bridge positions will not pass these tests within the set time limit. Instead, you need to loop through all possible groupings of which block of homeowners will be using which bridge, after which you determine where to place that particular bridge to give the shortest total trip within only that group of homeowners.

perm	k	Expected result
[0, 1]	2	2
[4, 2, 1, 0, 3]	2	15
[1, 4, 2, 3, 0, 6, 5]	2	21
[5, 7, 10, 4, 0, 6, 11, 2, 13, 8, 9, 1, 12, 14, 3]	3	87

This problem is "Palembang bridges" in the DMOJ problem collection and online judge.

Bonus Problem 114: No dogs, no masters

```
def no_dogs_allowed(grid):
```

The earlier problem "Backwards god" had your function count how many times the word `'dog'` appears in the given `grid` of letters. In that problem, the occurrences of that word were allowed to appear in any of the eight compass directions and wrap around the edge. However, if we disallow the wrapping around the edges and only look at the four cardinal directions, it turns out that you can always remove all occurrences of `'dog'` by swapping some letters pairwise with each other, regardless of how the letters are placed in the original grid. These letters can be swapped with each other freely across the entire grid, instead of just between adjacent letters or letters in the same column or row.

This function should return the smallest number of pairwise swaps necessary to erase all occurrences of the words `'dog'` and `'god'` from the `grid`. Same as in the previous problem of optimal filling of a box of dominoes, the symmetries in this problem can heavily constrain the search for the optimal solution. For example, consider how swapping all occurrences of `'d'` and `'g'` with each other in the `grid` creates a problem equivalent to the original instance.

grid	Expected result
['odg', 'ggo', 'dog']	1
['dggodg', 'dooodg', 'ddoggg', 'oododd', 'dddgdo', 'gogogg']	2

Bonus Problem 115: Car match game

```
def car_match(rows, trucks):
```

The actual steak of a mobile game is seldom as tasty as the sizzle promised in the advertisement video reel, but at least the ad for the game "Car match" gives us a fun combinatorial optimization problem to solve. Each individual car has a colour, expressed as a nonnegative integer. A number of cars in the lot have been placed in `rows` so that only the car at the end of each row can move out to exit that row, since it is blocking the cars waiting in that row behind it.

A series of `trucks`, coloured the same way as the cars, will then drive in the lot in the order given. Each truck can carry exactly three cars, all the same colour as that truck. Once exactly three cars of that same colour have been loaded onto the truck, that truck will drive away, and the next truck in line can move in to load another three cars of its colour. A truck will not drive out of the lot to make room for the next truck until it has exactly three cars. Once it is not possible to place three cars of the correct colour on the current truck, the entire lot comes to a screeching halt to end this game.

Your task is to load as many cars as possible onto the `trucks` by tactically choosing each row to `pop` the next car to load onto the truck currently taking cars. This function should return the maximum number of cars that can be removed from the lot under these rules. Again, remember that the last truck will not drive out carrying only one or two cars, so the result will always be divisible by three. The difficulty in this problem is avoiding your search from exploring equivalent moves multiple times in the recursion, which can lead to exponential blowup.

rows	trucks	Expected result
[[0, 0, 2, 0], [1, 1, 1, 0], [1, 0], [1, 0, 0, 0, 0]]	[0, 0, 0, 1, 1]	6
[[3, 3, 4, 0, 1], [2, 3, 2, 4, 3], [0, 0, 1], [2, 3, 4, 4, 1]]	[4, 4, 4, 3, 3, 3]	0
[[2, 1, 3, 3, 3, 3], [2, 3, 3], [2, 2, 2], [2, 1], [1, 3, 3], [0, 1, 1, 1, 1, 1, 3]]	[3, 3, 3, 1, 1, 1, 2, 2]	15

Bonus Problem 116: Liquid sort

```
def liquid_sort(tubes):
```

Liquid sort is a popular mobile game in which some `tubes` are filled with liquids of different colours so that do not mix. Each tube has room for exactly four portions of liquid. Each move pours all consecutive portions of liquid of the same colour from the top of one test tube into another tube that is either empty or also contains liquid of that same colour at the top. The goal of the puzzle is to complete each tube so that it contains four portions of liquid of the same colour. Note also that in our version of this game, each move must pour all liquid of the same colour into the other tube, and is not allowed to leave in any liquid of the colour being poured in the first tube.

Liquid colours are represented as nonnegative integers. Given a list of `tubes`, each individual tube itself a list whose last element is on the top in the position `-1` (so that you can easily `pop` the portion of liquid from one tube to `append` it to the other), this function should return the maximum number of tubes completed with some optimal sequence of moves. Some of the `tubes` are initially empty to give you some room to maneuver, represented as empty lists.

The main difficulty in this problem is again efficiently pruning the backtracking search through the search tree of all possible move sequences, to make the function pass the automated tester within the time limit. Also make sure not to swish and swosh the same liquid back and forth between one tube filled with just that one colour and another tube that is initially empty. All empty tubes can also be treated as the same empty tube at the same level of recursion.

tubes	Expected result
`[[1, 0, 1, 0], [1, 2, 1, 2], [2, 0, 2, 0], [], []]`	3
`[[0, 1, 1, 3], [2, 2, 2, 1], [3, 3, 0, 0], [3, 1, 0, 2], []]`	1
`[[9, 8, 5, 8], [3, 3, 9, 8], [1, 7, 3, 2], [4, 0, 5, 4], [2, 6, 5, 9], [1, 0, 2, 7], [4, 9, 8, 6], [1, 6, 7, 0], [7, 5, 1, 6], [4, 2, 0, 3], []]`	3

Bonus Problem 117: Count bowling paths

`def bowling_paths(score):`

In the popular game of **ten-pin bowling**, the same final score can be reached by many different sequences of throws. There is, of course, only one way to bowl a dead 0 or a perfect 300, but many different paths lead to the same scores in between. This function should compute the number of different sequences of throws that lead to the same final `score`. For simplicity, we assume that fouls don't exist, but gutter balls that knock over zero pins are possible on any throw.

To solve this problem, you need to define a recursive function whose parameters are the current frame number, the score to achieve, and the raw pin results of the next two throws that follow this frame. This function should go through all possible ways to make the two throws in this frame, and recursively count the number of ways to achieve the remaining score in the previous frames. (Note that the frame number **decreases** at each level of recursion, and equals ten at the top-level call.) You also need to handle the last frame separately due to the third bonus throw whenever there is a strike or a spare. Since the subproblems of this recursion will massively overlap, you should use the `@lru_cache` decorator to memoize the results for fast future access in other recursion branches.

score	Expected result
1	20
74	161209444961060944
147	46583823560371
200	1399365362
289	10
290	11
291	1

Bonus Problem 118: Expiring cookies

```
def expiring_cookies(board):
```

The following wacky variation of the game of <u>nim</u> was examined in an article by <u>Jim Propp</u>. The state of the game is a `board` made of positive integers. Two players alternate turns to remove any one integer of their choice from the board. Unlike in nim, the player always takes away that entire number, not just some part of it. After each move, all remaining numbers are decremented by one, and all zeros disappear. For example, removing one occurrence of 4 from `[1, 2, 4, 4, 6, 7]` first removes the chosen 4 to get `[1, 2, 4, 6, 7]`. Decrementing these remaining values gives the list `[0, 1, 3, 5, 6]`, from which removing all zeros gives the new board `[1, 3, 5, 6]`.

The player who makes the board empty by removing the last remaining number wins the game. This function should determine whether the player whose turn it is to play on the given `board` has a winning strategy. This can be determined with the recursive **minimax search algorithm** that examines all possible moves in the current board, and for each such move, computes recursively whether the opponent has a winning strategy in the successor board. A position is winning for the current player if there exists at least one move that leads to a losing position for the opponent.

Since the same board position can be reached in potentially exponentially many different ways, you should maintain the sets of the known winning and losing boards encountered during the search. Whenever those boards are encountered later, the answer can be returned immediately. As you can see from the first two test cases in the table below, unlike in ordinary nim, a pair of equal numbers is not *miai* that would cancel each other out. However, also unlike in ordinary nim, any integer greater than the current length of the board can be considered to be equal to that length. For example, the board `[1, 5, 6, 7]` plays exactly the same as the board `[1, 4, 4, 4]`.

board	Expected result
[1, 2, 5, 6]	True
[1, 2, 4, 4, 5, 6]	False
[1, 1, 2, 4, 6, 8, 10]	True
[1, 4, 4, 4, 7, 10, 11, 14, 16, 17]	False

Bonus Problem 119: Sonata pattern matching

```
def sonata_match(text, pattern):
```

The pattern matching subroutine of the **Turing-complete** esoteric programming language <u>Sonata</u> makes for an interesting exercise of Python string processing along with some associated bookkeeping. This function should determine whether the `pattern` matches the **entire `text`** when all its lowercase characters match themselves, variables of the form `^X` match any single character, and variables of the form `_X` match an arbitrary substring of zero or more characters. In both variable forms, `X` can be any single uppercase letter. If the same variable occurs multiple times inside the `pattern`, that variable must match the exact same substring of `text` at each occurrence.

This function should return the list of tuples of form `(variable, match)` of the bindings for all variables of the given `pattern` that give a successful match of the entire `text`. All caret variables must be listed first in alphabetical order, followed by all underscore variables in alphabetical order. If there is no way to match the `pattern` to the entire given `text`, this function should return `None`.

If the `pattern` can match the entire `text` in different ways, the returned binding should be the one that minimizes the length of the first variable in the `pattern` when read from left to right, followed by the binding of the second variable that minimizes the length of that variable, and so on.

text	pattern	Expected result
`'tltgt'`	`'^A^B^Ag^A'`	`[('^A', 't'), ('^B', 'l')]`
`'pxplayssplaysf'`	`'px_As_Af'`	`[('_A', 'plays')]`
`'iluikelkkelkkelk'`	`'^Al^Ci_B_B_B'`	`[('^A', 'i'), ('^C', 'u'), ('_B', 'kelk')]`
`'hmaidaqffaaidaeyi'`	`'h_A^Eff_Aeyi'`	None

You can write your own recursive matcher function based on a **backtracking** search. Maintain an auxiliary dictionary to keep track of the variable bindings made and unmade during the recursive dive down the `text`). The readers familiar with them can alternatively harness <u>Python's regular expression standard library</u> to do this hard work. (**Backreferences** allow you to ensure that the same part of the regular expression is matched identically in each occurrence of the same variable.)

Bonus Problem 120: Fox and hounds

```
def fox_and_hounds(fox, hounds):
```

Fox and hounds is an abstract strategy game whose complexity and state space are small enough for us to calculate the optimal moves for both players in any possible game situation. As explained on the linked Wikipedia page, the game is played on black squares of an 8-by-8 checkerboard, with the four hounds starting from row zero and aiming to capture one fox who starts from row seven. In its turn, a hound can only move forward towards the last row, whereas the fox can move one step to any one of the four diagonal directions. There are no captures, promotions or jumps, but the fox wins if it reaches the zero row, and the hounds win if they block the fox and make it unable to move.

From the starting position, this game is known to be a win for the hounds, but this function should compute whether the `fox` has a winning strategy against the four `hounds` in the given position on his turn to move. The **minimax search algorithm** that can solve not just this but any **complete information two player game of alternating moves** consists of two **mutually recursive** functions `fox_move` and `hounds_move` that each loop through the possible moves for that player in the given situation, and for each move, call the other function to determine if the opponent has a winning strategy in the state resulting from that move. Both functions return `True` if there exists a move for which the opponent does not have a winning response, and `False` otherwise.

Despite the small state space of this game, the space of possible move sequences is still exponentially large, especially since many possible sequences of moves lead to the same game states. This requires the use of **transposition tables** to store the states encountered during the recursive search. Should the search reach the same game state again, just look up the result of that state from the table. The easiest might be to use four sets `fox_win`, `fox_lose`, `hounds_win` and `hounds_lose` to store each state after the first evaluation. The reader should still think up additional ways to speed up this algorithm in various places.

fox	hounds	Expected result
(7, 0)	[(0, 1), (0, 3), (0, 5), (0, 7)]	False
(6, 1)	[(1, 2), (3, 2), (4, 3), (4, 5)]	False
(6, 5)	[(0, 5), (2, 1), (3, 0), (5, 6)]	True
(5, 0)	[(1, 2), (2, 7), (3, 0), (3, 4)]	False
(7, 0)	[(1, 0), (3, 6), (4, 7), (5, 4)]	True

Bonus Problem 121: Toads and frogs

`def toads_and_frogs(board):`

Continuing on the theme of complete information abstract strategy games with alternating moves made by cute animals, the game of <u>toads and frogs</u> produces surprisingly complex situations in combinatorial game theory despite its bare and simple rules and setup. Some toads (`'T'`) and frogs (`'F'`), at least one of each type of animal, have been placed on a one-dimensional `board` of squares. On the toads turn to move, one toad can either move one step to the right to an adjacent empty square (`'.'`), or hop over a frog immediately to its right to an empty square immediately behind that frog. (Unlike in checkers, hopping over a frog does not capture it.) Frogs move the exact same way, except to the left. The blocked side, unable to make any move on its turn, loses the game. The board does not wrap around cyclically, but the animals that reach the end of the board either way are blocked.

This function should determine whether there exists a winning strategy for the toads on the given `board` when it is the toads turn to move. The two mutually recursive functions for finding the best moves for toads and frogs (or just one suitably parameterized function for this symmetric game) in the given situation are similar as in the previous game of fox and hounds. However, the use of **transposition tables** to memoize game states that have been previously encountered during the recursive search is even more essential in this game, since this game is basically nothing but transposed move sequences leading to the same game states. In addition, you still need to think up ways to speed up your code to pass the automated fuzz tester within the set time limit.

board	Expected result
`'T..F'`	False
`'T...T.F..F'`	False
`'TTT...FF.F..'`	True
`'..T.TT.TTFFFF.F.'`	True
`'T.TFTTT.FT..F.FF.F'`	False

Finding the winning strategy for toads and frogs is a PSPACE-complete problem, so every algorithm will occasionally run in exponential time, but most instances can be solved in polynomial time.

Bonus Problem 122: Unamerican gladiators

```
def gladiators(our, their):
```

Here is another banger from Peter Winkler's "Mathematical Puzzles: A Connoisseur's Collection". Each gladiator has a strength that is a positive integer. When two gladiators with strengths x and y duel, they each win with probabilities of $x / (x + y)$ and $y / (x + y)$ proportional to their respective strengths, so that with lots of luck, even David ($x = 1$) can slay Goliath ($y = 1000$). The winner absorbs his fallen opponent's strength, *Highlander* style, so that his new strength becomes $x + y$.

Two teams of gladiators, `our` valiant heroes versus `their` dastardly villains, take turns to choose any one of their gladiators and any one of their opponent's gladiators to a duel, and continue in this manner until one team has been utterly vanquished. If `our` boys go first, and both teams always choose the gladiators to maximize their chances of winning, what is the probability that our team of heroes emerges victorious in the end? You should, of course, use exact `Fraction` objects for all calculations and the final result. Only the mission is important, and the survival chances of any individual gladiator don't matter, so in the best samurai spirit, each gladiator will gladly face even hopeless odds if it helps maximize his team's chances of triumph.

This problem can be solved with **expectimax** search, a variation of minimax search where at each level of recursion, the consequences of the duels between each possible choice of gladiators for this round, win and lose, is evaluated recursively from the opposing team's point of view, and the possible results are added together weighted by the win/lose probabilities. Then just return the expected value of the best possible choice for the duelling pair at that level. However, as it always seems to happen with expectimax, the search tree massively branches sideways and soon becomes prohibitive. But should the reader add some debugging outputs for the expected values of each move, they should soon realize something very interesting that ought to help speed up this search by several orders of magnitude.

our	their	Expected result
`[4, 1]`	`[3, 3]`	5/11
`[9, 6, 1]`	`[4, 2, 8]`	8/15
`[5, 11, 7, 5]`	`[10, 10, 1, 7]`	1/2
`[15, 14, 15, 15, 24]`	`[23, 19, 7, 14, 8]`	83/154

Bonus Problem 123: Life is like a box of dominoes

```
def domino_box(tiles):
```

A set of *n* domino `tiles` is to be placed inside a 2-by-*n* box. It is guaranteed that none of the domino tiles is a double (that is, both halves of that domino have an equal number of pips), but the set can contain copies of identical domino tiles. Placing the halves of some two dominoes adjacent to each other so that both halves have an equal number of pips gives you as many points as the number of those pips in each half. The same domino can participate in more than one such pairing with its neighbouring dominoes, but each pairing is counted only once.

Your task is to place these dominoes inside the box to maximize the total score. Each tile can be placed in its position in either orientation, and tiles can be placed inside the box horizontally or vertically as desired. Since the optimal placement is not necessarily unique, this function should just return this score.

Since there are $Fib(n) \times 2^n \times n!$ different ways to place these dominoes inside the box (DUCY?), far too many to enumerate exhaustively for any nontrivial value of *n*, your search needs to utilize the many explicit and implicit symmetries available in the possible placements of dominoes to allow your code to pass the automated tests within reasonable time. From this Thanksgiving turkey that is a product of three exponential quantities, you can keep carving off polynomial slices forever to push the limit of how large instances of this problem you can solve.

tiles	Expected result
`[(2, 3), (3, 1), (4, 3), (1, 5)]`	7
`[(2, 1), (2, 3), (1, 2), (2, 3)]`	12
`[(3, 2), (5, 4), (3, 5), (1, 5), (1, 5), (2, 5), (6, 3)]`	29

From the author's first working solution with backtracking search, using new symmetries always sped up the search by a factor of three to five, so that the fuzz tester had to be changed to produce test cases both more numerous and longer.