CS 22 Problem 9.5c

Uthsav Chitra

February 25, 2016

Problem (Homework 9, Problem 5c)
In the airplane problem (see here, although there it's framed as people picking lunches from the fridge), what's the expected number of mismatched people?

Solution. Label the people $1, \ldots, n$ in order of when they pick items from the fridge. Let E_n be the desired quantity, the expected number of people who pick a lunch different from their own. We will find a recurrence relation between E_n and E_{n-1}, and use that to solve for E_n. Since

$$E_n = \sum_i E[E_n | \text{person 1 picks lunch i}], \tag{1}$$

fi we will find each term.

Fix some $i > 1$ and suppose person 1 picks person i's lunch, which happens with probability $\frac{1}{n}$. Then persons 2 through $i-1$ will pick their own lunch. After this, the people left are $i, i+1, \ldots, n$, while the lunches left are $1, i+1, \ldots, n$.

Relabeling person i as person 1, person $i+1$ as person 2, and so on, we have seemingly reduced our problem to the $n-i-1$ person case, so one would think that the contribution to E_n is $\frac{1}{n}(E_{n-i+1})$; that is, $E[E_n | \text{person 1 picks lunch i}] = \frac{1}{n}(E_{n-i+1})$. However, there are two adjustments we have to make:

- First, note that in this reduced case, person i (relabeled as person 1) will always be mismatched, as the original person 1 took her lunch. Thus, there is an extra $+1$ factor we must add to this term, leaving us at $\frac{1}{n}(E_{n-i+1} + 1)$. (Why do we add $+1$ inside the parentheses?)
- Second, since we relabel person i as person 1, the E_{n-i+1} term will think that, if person i is matched with the original lunch 1, that is not a mismatch. However, we would count this as a mismatch. Since person i picks lunch 1 (and thus mismatches) with probability $\frac{1}{n-i+1}$, we must add $\frac{1}{n-1+1}$ as well. Doing this gives us $E[E_n | \text{person 1 picks lunch i}] = \frac{1}{n}\left(E_{n-i+1} + 1 + \frac{1}{n-i+1}\right)$.

With these two corrections, we are ready to derive a recurrence relation. Noting that, $E_1 = 0$, using 1 we get the following:

$$\begin{aligned} E_n &= \frac{1}{n}\left(E_{n-1} + 1 + \frac{1}{n-1}\right) + \frac{1}{n}\left(E_{n-2} + 1 + \frac{1}{n-2}\right) + \ldots + \frac{1}{n}\left(E_1 + 1 + \frac{1}{1}\right) \\ &= \frac{1}{n}\left(S_{n-1} + n - 1 + H_{n-1}\right), \end{aligned} \tag{2}$$

where $S_n = E_1 + \ldots + E_n$ and $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$. Rearranging the above equation yields that $S_{n-1} = nE_n - n + 1 - H_{n-1}$.

If we replace n with $n + 1$ in 2, we get the following:

$$\begin{aligned}
E_{n+1} &= \frac{1}{n+1}(S_n + n + H_n) \\
&= \frac{1}{n+1}(E_n + S_{n-1} + n + H_n) \\
&= \frac{1}{n+1}(E_n + nE_n - n + 1 - H_{n-1} + n + H_n) \\
&= \frac{1}{n+1}\left((n+1)E_n + 1 + \frac{1}{n}\right), \text{ since } H_n - H_{n-1} = \frac{1}{n} \\
&= E_n + \frac{1}{n}
\end{aligned}$$

Indexing $n + 1 \to n$ yields that $E_n = \frac{1}{n-1} + E_{n-1}$. Telescoping this recurrence, we get that

$$E_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} = H_{n-1}.$$

So E_n is the $(n-1)$-th Harmonic number. Neat, huh? $\square$