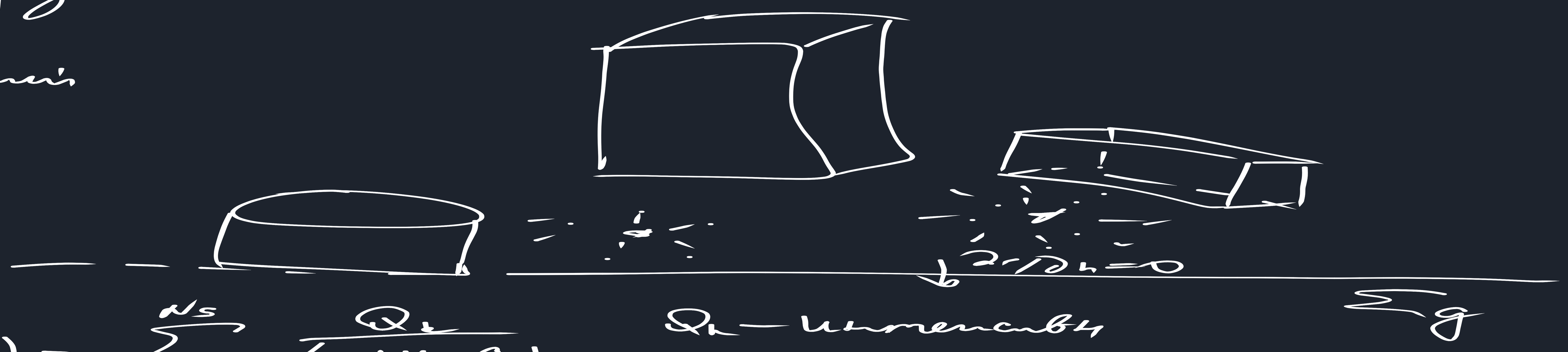
Задача 1. Диффузия.

$\Omega = \mathbb{R}^3$, Σ − гр. здания

$C(x)$ − концентр. примеси

$$\begin{cases} \Delta C(x) = 0 & \text{в } \Omega \\ \dfrac{\partial c}{\partial n} = 0 & \text{на } \Sigma \end{cases}$$



$$C = C_0(x) + \underset{\text{вклад}}{\tilde{C}(x)}, \quad C_0(x) = \sum_{k=1}^{N_s} \frac{Q_k}{4\pi|x-q_k|}$$

Q_k − интенсивн.

q_k − координата

$$\begin{cases} \Delta \tilde{C}(x) = 0 & \text{в } \Omega \\ \dfrac{\partial \tilde{C}}{\partial n} = -\dfrac{\partial C_0}{\partial n} & \text{на } \Sigma \end{cases} \Rightarrow \tilde{C} = V[\Sigma, g] = \int_\Sigma g(y) F(x-y)\, dy, \quad F = \frac{1}{4\pi|x-y|}$$

$$\frac{\partial c^+}{\partial n} = -\frac{1}{2} g(x) + \int_\Sigma g(y) \frac{\partial F(x-y)}{\partial n_x} dy = -\sum_{k=1}^{N_s} \frac{Q_k (n_x, x-q_k)}{4\pi |x-q_k|^3}$$

Кусочно-пост. аппроксим. и коллокация

$\Sigma \approx \{\sigma_i\}_{i=1,N}$, $g(x \in \sigma_i) = g_i$, x_i − центр σ_i

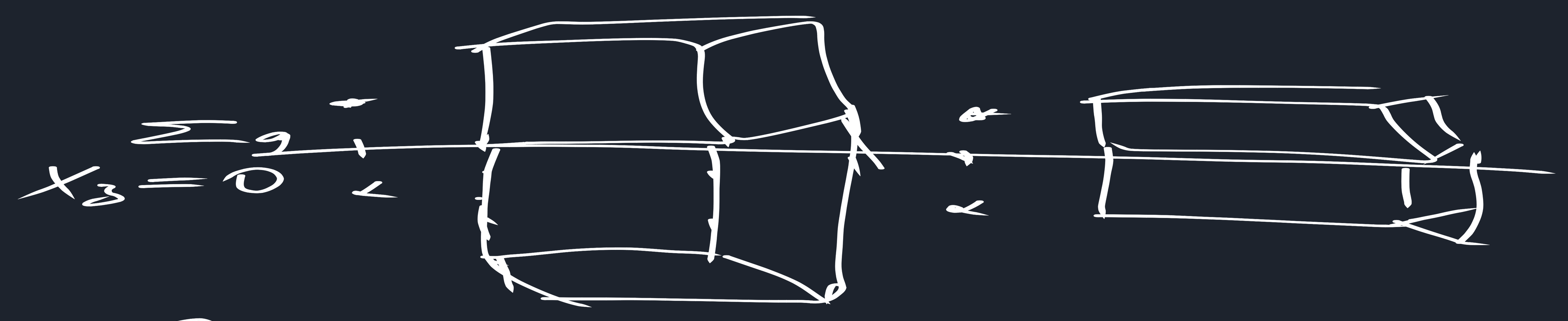
$$+\frac{1}{2} g(x_i) + \sum_{j=1}^{N} g_j \int_{\sigma_j} \frac{(n_i, x_i - y)}{4\pi|x_i - y|^3} dy = +\sum_{k=1}^{N_s} \frac{Q_k (n_i, x_i - q_k)}{4\pi |x_i - q_k|^3}, \quad i = 1, N$$

$$A\underline{g} = b, \quad a_{ij} = \frac{\delta_{ij}}{2} + \int_{\sigma_j} \frac{(n_i, x_i - y)}{4\pi|x_i - y|^3} dy, \quad b_i = \sum_{k=1}^{N_s} \frac{Q_k (n_i, x_i - q_k)}{4\pi|x_i - q_k|^3}$$

$$\tilde{C}(x \in \Omega) = V[\Sigma, g] = \int_\Sigma g(y) F(x-y)\, dy = \sum_{j=1}^{N} g_j \int_{\sigma_j} \frac{dy}{4\pi|x-y|}$$

$$C(x) = C_0(x) + \tilde{C}(x)$$

Отражение от земли



$$\tilde{C}(x) = V[\Sigma, g] + V[\Sigma^*, g]$$

$q_k = (x_1, x_2, x_3)$

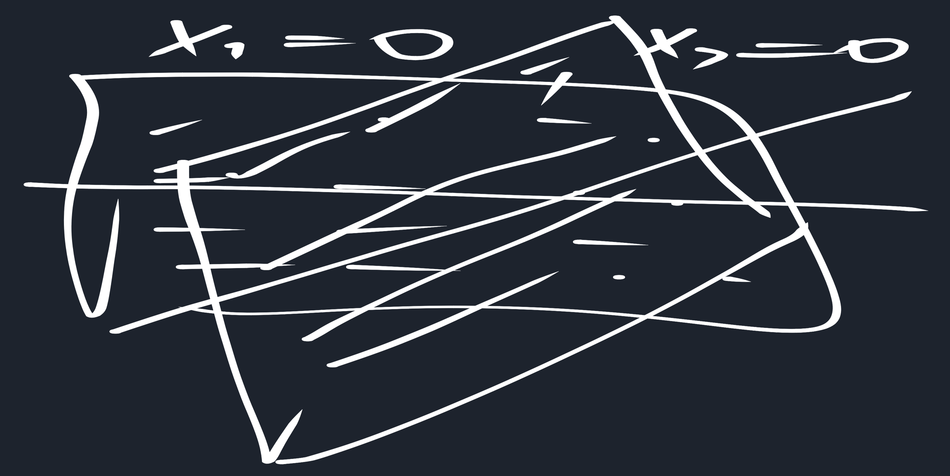
$q_k^* = (x_1, x_2, -x_3)$

$$\Sigma = \{\sigma_i\} \cup \{\sigma_i^*\}$$

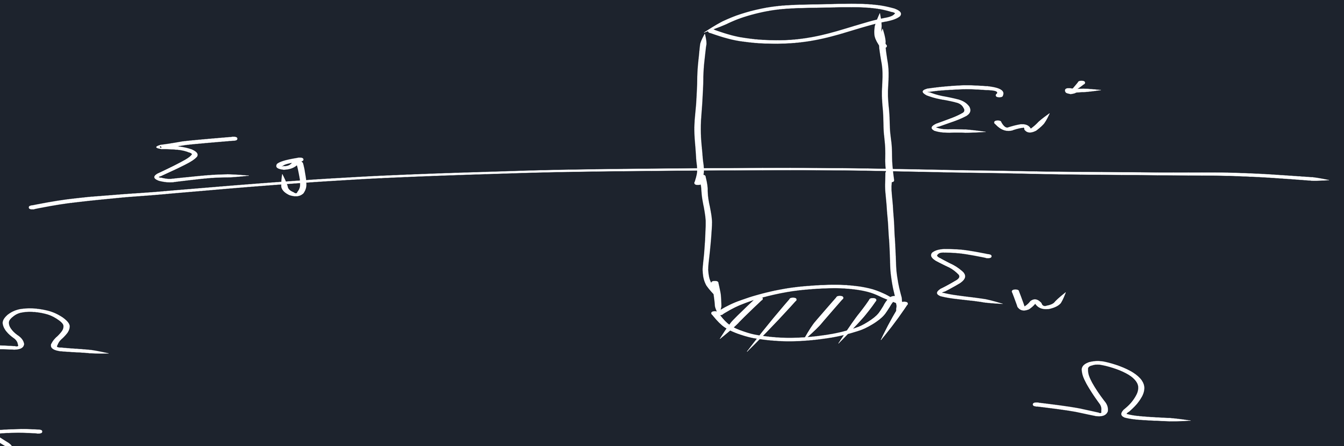
$$\frac{1}{2} g_i + \sum_{j=1}^{N} g_j \int_{\sigma_j} \frac{\partial F(x_i - y)}{\partial n_i} dy + \sum_{j=1}^{N} g_j \int_{\sigma_j^*} \frac{\partial F(x_i - y)}{\partial n_i} dy = \underbrace{\sum_{k=1}^{N_s} \frac{Q_k}{4\pi} \left(\frac{(n_i, x_i - q_k)}{|x_i - q_k|^3} + \frac{(n_i, x_i - q_k^*)}{|x_i - q_k^*|^3} \right)}_{= b_i}$$

$$a_{ij} = \frac{\delta_{ij}}{2} + \int_{\sigma_j} \frac{\partial F}{\partial n_i} + \int_{\sigma_j^*} \frac{\partial F}{\partial n_i},$$

$$C(x) = C_0(x) + \tilde{C}(x) = \sum_{k=1}^{N_s \cdot 2} \ldots + V[\Sigma, g] + V[\Sigma^*, g]$$



Задача 2. Фильтрация



$$\begin{cases} u = -k\nabla p \\ \nabla\cdot u = 0 \end{cases} \Rightarrow \begin{cases} \Delta p = 0 \ |\ \Omega \\ p = p_w \ |\ \Sigma_w \\ \frac{\partial p}{\partial n} = 0 \ |\ \Sigma_g \longrightarrow p = p_w \ |\ \Sigma_w^* \end{cases}$$

$$W[\Sigma, g] = \int_\Sigma g(y) \frac{\partial F(x-y)}{\partial n_y} dy$$

$$p_w(x^*) = p(x) \qquad x^* = (x_1, x_2, -x_3), \quad x = (x_1, x_2, x_3)$$

$$p(x) = W[\Sigma_w \cup \Sigma_w^*, g]$$

$$\exists\, x \in \Sigma_w \quad p^+ = \frac{1}{2} g(x) + \int_{\Sigma_w} g(y) \frac{\partial F}{\partial n_y} dy + \int_{\Sigma_w^*} g(y) \frac{\partial F}{\partial n_y} dy = p_w(x)$$

мет. коллок. $\quad g(x \in \sigma_i) = g_i,$

$$\frac{1}{2} g_i + \sum_{j=1}^{N} g_j \left(\int_{\sigma_j} \frac{\partial F(x_i - y)}{\partial n_y} dy + \int_{\sigma_j^*} \frac{\partial F(x_i - y)}{\partial n_y} dy \right) = \underset{b_i}{\underset{\parallel}{p_w(x_i)}}$$

$$Ag = b \qquad a_{ij} = \frac{\delta_{ij}}{2} + \int_{\sigma_j} \frac{(n_y, x_i - y)}{4\pi |x_i - y|^3} dy + \int_{\sigma_j^*} \frac{(n_y, x_i - y)}{4\pi |x_i - y|^3} dy$$

$$\int_{ABC} \frac{\partial F(x-y)}{\partial n_y} dy = -2 \operatorname{arctg} \Theta(x, ABC)$$

$$\Theta(x, ABC) = \frac{(r_a, r_b, r_c)}{(r_a, r_b) + (r_b, r_c) + (r_c, r_a) + 1} \qquad r_a = \frac{a - x}{|a - x|}, \ r_b = \frac{b - x}{|b - x|}, \ r_c = \frac{c - x}{|c - x|}$$

$$A\underline{\underline{g}} = b \longrightarrow p(x),\ u(x)$$

$$p(x) = W[\Sigma_w \cup \Sigma_w^*, g] = \sum g_j \left(\int_{\sigma_j} \frac{\partial F}{\partial n_y} dy + \int_{\sigma_j^*} \frac{\partial F}{\partial n_y} dy \right)$$

$$u(x) = -k\nabla W[\Sigma_w \cup \Sigma_w^*, g] = -k \sum g_j \left(\int_{\sigma_j} \nabla_x \frac{\partial F}{\partial n_y} dy + \int_{\sigma_j^*} \nabla_x \frac{\partial F}{\partial n_y} dy \right)$$

$$\sigma_i = ABC \longrightarrow \partial\sigma_i = AB + BC + CA$$

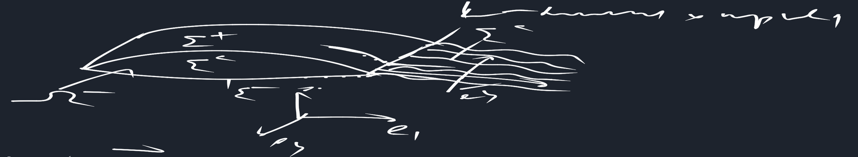
$$\int_{\sigma_j} \nabla_x \frac{\partial F(x-y)}{\partial n_y} dy = f(x, a, b) + f(x, b, c) + f(x, c, a)$$

$$f(x, a, b) = \frac{[b - a, x - a]}{|[b - a, x - a]|^2} \left(\frac{(x - a, b - a)}{|x - a|} - \frac{(x - b, b - a)}{|x - b|} \right)$$

Закон Био-Савара.

Задача 3. Аэродинамика

$\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega^-} \setminus \Sigma^c$



$\vec{W}(x)$ – потенциально в Ω^+

на Σ^c $w(x)$ разрыв[illegible]

$\vec{W}_{tot}(x) = \vec{W}_\infty(x) + \vec{W}(x)$

α – угол атаки

β – угол скольжения

$w_\infty(x) = W_\infty(\cos\alpha\cos\beta, \sin\alpha\cos\beta, \sin\beta)$

$\nabla \cdot w = 0$, $\nabla \times w = 0$ $\quad | \text{ в } \Omega^+$

$n \cdot w_{tot} = 0 \quad | \Sigma = \Sigma^+ \cup \Sigma^- \cup \Sigma^s$

$w_1^+ = w_1^-$, $w_2^+ = w_1^- \quad | \Sigma^c$

w_{tot} – [illegible] в Ω^+ $\quad \vec{w}(x) = \nabla u(x) \to \Delta u(x) = 0$ в Ω

$\frac{\partial u}{\partial n} = -(w_\infty, n) \quad | \Sigma$

$\lim\limits_{\substack{x \in \Sigma^+ \\ x \to L}} u^+ = \lim\limits_{\substack{x \in \Sigma^c \\ x \to L}} u^+$

$\lim\limits_{\substack{x \in \Sigma^- \\ x \to L}} u^+ = \lim\limits_{\substack{x \in \Sigma^- \\ x \to L}} u^-$

$\frac{\partial u^+ - \partial u^-}{\partial x_1} = \frac{\partial u^+ - \partial u^-}{\partial x_2} = 0 \quad | \Sigma^c$

$\frac{\partial u^+}{\partial n} = \frac{\partial u^-}{\partial n} \quad | \Sigma_c$

$u(x) = W[\Sigma, g] + W[\Sigma^c, h]$

$g(x)$ на Σ

$h(x)$ на Σ^c

$u^+ - u^- = g \quad | \Sigma$

$u^+ - u^- = h \quad | \Sigma^c \qquad \frac{\partial h}{\partial x_1} = 0$

$\Delta u = 0$ в Ω^-

$\frac{\partial u}{\partial n} = 0$ в Ω^-

$u = 0$ в Ω^-

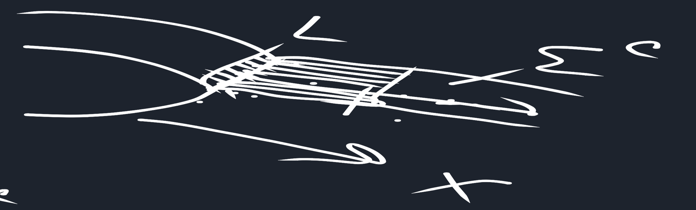
$u^- = 0$ на Σ

(1) $\int\limits_{\Sigma} g(y) \frac{\partial^2 F(x-y)}{\partial n_x \partial n_y} dy + \int\limits_{\Sigma^c} h(y) \frac{\partial^2 F(x-y)}{\partial n_x \partial n_y} dy = -(w_\infty, n_x)$

$(u^+ - u^-) = h \quad | \Sigma^c$

$u^+ = u^+_{\Sigma^+}$, $u^- = u^+_{\Sigma^-}$

$\underline{h(x \in \Sigma^c) = g(x \in \Sigma^+) - g(x \in \Sigma^-)}, x \in L$ (2)



$h(\Sigma^c) \leftarrow h(L)$

Метод коллокаций

$A = \begin{pmatrix} \int \frac{\partial^2 F(x_i - y)}{\partial n_i \partial n_j} dy \\ \int \text{——} \end{pmatrix}$

(1) $\sum\limits_{j=1}^{N} g_j \underbrace{\int\limits_{\sigma_j} \frac{\partial^2 F(x_i - y)}{\partial n_i \partial n_j} dy}_{a_{ij}} + \sum\limits_{j=1}^{N_c} h_j \int\limits_{\sigma_j} \frac{\partial^2 F(x_i - y)}{\partial n_i \partial n_j} dy = \underbrace{-(w_\infty, n_i)}$

(2) $h_i = g_{idx(i,+)} - g_{idx(i,-)}$ $\qquad idx(i, \pm)$ – номер $\sigma_n \in \Sigma^\pm$

(3) $\int\limits_{\Sigma} g(x) \underline{dx} = \text{const} \Rightarrow \sum\limits_{j=1}^{N} g_j \cdot s_j = \text{const}$

$\left(\begin{array}{c} g_1 A_{11} \end{array} \right)$

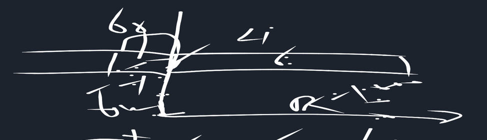
$(N + N_c, 1)$

$(1, N + N_c)$

$A \cdot x$

$\underline{Ax = b}$

$$\begin{array}{r} \\ N \\ N_c \\ 1 \end{array} \begin{array}{|c|c|c|} \multicolumn{1}{c}{N} & \multicolumn{1}{c}{N_c} & \multicolumn{1}{c}{1} \\ \hline a_{ij} & a_{ij} & 1 \, \vdots \, 1 \\ \hline idx(i,j) & -I & 0 \\ \hline s_1 \ldots s_N & 0 & 0 \\ \hline \end{array} \quad \begin{array}{|c|} \hline -(w_\infty, n_i) \\ \vdots \\ \hline 0 \\ \hline \text{const} \\ \hline \end{array}$$



$idx(i,j) = \begin{cases} 1, & \sigma_j \in \Sigma^+, \sigma_j \cap \sigma_i = L_i \\ -1, & \sigma_j \in \Sigma^-, \sigma_j \cap \sigma_i = L_i \\ 0, & \text{else} \end{cases}$

$x = \begin{array}{|c|} g_1 \\ \vdots \\ g_N \\ h_1 \\ \vdots \\ h_{N_c} \end{array} \begin{array}{l} N \\ \\ N_c \\ 2 \end{array}$

$ABCD_i$, $ABCD_j$

$BC_i = AD_j$ или AB_j –

$B_i = C_j$ X

!!(!!))!!

const

g, h

Закон Бернулли

$p(x) + \frac{\rho W(x)^2}{2} = p_\infty + \frac{\rho W_\infty^2}{2}$

$x \to \Sigma \quad p^+(x) + \frac{\rho (w^+_{tot}(x))^2}{2} = p_\infty + \frac{\rho w_\infty^2}{2}$

коэфф. давл.

$c_p = \frac{p - p_\infty}{q}, \quad q = \frac{\rho W_\infty^2}{2}$

$c_p = 1 - \frac{(w^+_{tot}(x))^2}{W_\infty^2} = \boxed{1 - 4 \frac{w_{tot}^2}{w_\infty^2}}$

$\overline{W}_{tot} = \overline{W} + \overline{W}_\infty$

$w^+_{tot} = 2 w_{tot}$

$w_{tot} = \frac{1}{2}(w^+_{tot} + w^-_{tot})$

$w_{tot} = w_\infty + \nabla W[\Sigma, g] + \nabla W[\Sigma^c, h]$

$\int\limits_{\Sigma} \vec{n} \cdot \text{const} \, dx = 0$

$q = \frac{\rho w_\infty^2}{2}$

1) $c_p(x_i) = 1 - 4 \frac{w_{tot}(x_i)^2}{w_\infty^2}$

2) $F = -\int\limits_{\Sigma} n \cdot p^+(x) dx = -\int n \cdot (c_p \cdot q + p_\infty) dx = -\int\limits_{\Sigma} \vec{n} (c_p \cdot q) dx$

$F = -\sum\limits_{j=1}^{N} (\vec{n}_j \cdot c_p(x_j) q \cdot s_j)$

3) $M = (x - c_0) \times F$

c_0

Задача 4. Акустика.

$\rho = \rho_0 + \rho'$, $p = p_0 + p'$, $w = w'$

$$\begin{cases} \frac{\partial \rho'}{\partial t} + \rho_0 (\nabla \cdot w) = 0 \\ \frac{\partial w}{\partial t} + \frac{\nabla p'}{\rho_0} = 0 \\ p' = c^2 \rho' \end{cases} \qquad \begin{cases} \frac{\partial \rho'}{\partial t} + \rho_0 (\nabla \cdot w) = 0 \\ \frac{\partial w}{\partial t} + \frac{c^2}{\rho_0} \nabla \rho' = 0 \end{cases}$$

$c = \sqrt{\frac{\partial p}{\partial \rho}}\Big|_{\rho_0}$ – скор. звука

$\vec{w} = \nabla u(x) \cdot e^{-i\omega t}$

$\frac{\partial w}{\partial t} = -i\omega e^{-i\omega t} \nabla u$

$-i\omega e^{-i\omega t} \nabla u = \frac{c^2}{\rho_0} \nabla \rho'$

$\nabla \cdot w = \Delta u \, e^{-i\omega t}$

$\rho' = \frac{i\omega\rho_0}{c^2} e^{-i\omega t} u + \text{const}$

$$\frac{\omega^2}{c^2} \cancel{\rho_0} \cancel{e^{-i\omega t}} u + \cancel{\rho_0} \Delta u \cancel{e^{-i\omega t}} = 0$$

$$\underline{\Delta u + k^2 u = 0} \qquad k = \frac{\omega}{c}$$

$u_{tot} = u_0 + u$ $\qquad u_0 = e^{-ik(\bar{v}_0, \bar{x})}$ $\qquad |\bar{v}_0| = 1$

$(w_{tot}, n) = 0 \Rightarrow \frac{\partial u_{tot}}{\partial n} = 0 \Rightarrow \frac{\partial u}{\partial n} = -\frac{\partial u_0}{\partial n}$ $\qquad \vec{v_0}$

Жесткое тело

$$\begin{cases} \Delta u + k^2 u = 0 & | \, \Omega^+ \\ \frac{\partial u^+}{\partial n} = -\frac{\partial u_0}{\partial n} & | \, \Sigma \end{cases}$$

Мягкое тело

$$\begin{cases} \Delta u + k^2 u = 0 & | \, \Omega^+ \\ u = -u_0 & | \, \Sigma \end{cases}$$

$$u = V[\Sigma, g] = \int_\Sigma g(y) F(x-y) \, dy \qquad F = \frac{e^{ikr}}{4\pi r} \qquad \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

$$\frac{\partial u^+}{\partial n} = -\frac{1}{2} g(x) + \int_\Sigma g(y) \frac{\partial F}{\partial n_x} dy = -\frac{\partial u_0(x)}{\partial n_x}$$

Метод коллокаций $\quad \exists \, \Sigma \simeq \{\sigma_i\}$, $\quad g(x \in \sigma_i) = g(x_i) = g_i$, $\quad i = \overline{1, N}$

x_i – центр масс $\quad n_i$ – нормаль к σ_i, $|n_i| = 1$

$$-\frac{1}{2} g_i + \sum_{j=1}^N g_j \int_{\sigma_j} \frac{(n_i, \overline{x_i - y})}{4\pi r_i^3} e^{ikr_i} (ikr_i - 1) \, dy = -ik(\bar{v}_0, \bar{n}_i) e^{-ik(\bar{v}_0, \bar{x}_i)}$$

$r_i = |x_i - y|$

$i = \overline{1, N}$

$Ag = b$ $\qquad a_{ij} = -\frac{1}{2}\delta_{ij} + \int_{\sigma_j} \frac{(n_i, x_i - y)}{4\pi r_i^3} e^{ikr_i} (ikr_i - 1) \, dy$ $\qquad b_i = -ik(v_0, n_i) e^{-ik(v_0, x_i)}$

$u(x) = \int g(y) \frac{e^{ik|x-y|}}{4\pi|x-y|} dy$, $x \in \Omega^+$

$u(x) \simeq \sum_{j=1}^N g_j \int_{\sigma_j} \frac{e^{ik|x-y|}}{4\pi|x-y|} dy$

ЭПР: $\quad |\tau| = 1$

$$\sigma(\tau) = \lim_{R \to \infty} 4\pi \frac{|u(R\tau)|^2 \cdot R^2}{|u_0|^2} = 4\pi \lim_{R \to \infty} \left| \int_\Sigma g(y) \frac{e^{ikr}}{4\pi r} \cdot R \, dy \right|^2$$

$|u_0|^2 \to 1$

$r = |x - y| = |R\tau - y|$ $\qquad \frac{R}{r} \to 1$

$e^{ikr} = e^{ik(r-R)} \cdot e^{ikR} \to e^{-ik(\tau, y)} \cdot e^{ikR}$

$(r - R) \to -(\tau, y)$

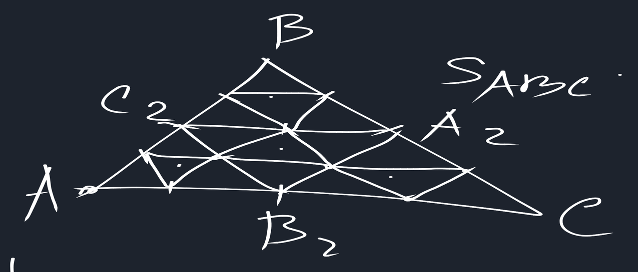
$$\frac{(|R\tau - y| - R)(|R\tau - y| + R)}{|R\tau - y| + R} = \frac{\cancel{R^2\tau^2} - 2\cancel{R}(\tau, y) + (y)^2 - \cancel{R^2}}{\cancel{R}|\tau - \frac{y}{R}| + \cancel{R}}$$

$$\sigma(\tau) = \frac{1}{4\pi} \cancel{\lim_{R \to \infty} |e^{ikR}|^2} \left| \int_\Sigma g(y) e^{-ik(\tau, y)} dy \right|^2 = \frac{1}{4\pi} \left| \int_\Sigma g(y) e^{-ik(\tau, y)} dy \right|^2 \simeq \frac{1}{4\pi} \left| \sum_{j=1}^N g_j \int_{\sigma_j} e^{-ik(\tau, y)} dy \right|^2$$

$\tau = (\cos\varphi, \sin\varphi, 0)^T$, $\varphi \in [0, 2\pi)$

$$= \frac{1}{4\pi} \left| \sum_{j=1}^N g_j e^{-ik(\tau, y_j)} \cdot s_j \right|^2$$

$$\int_{ABC} f(y, \vec{\sigma}) \, dy = \sum f_i \Delta_i = \left(f(M_1) + f(M_2) + f(M_3) + f(M_4) \right) \cdot S_{ABC} \cdot \frac{1}{4}$$



$S_{ABC} \cdot \frac{1}{4}$

$C_2 = \frac{A+B}{2}$, ...

$\frac{1}{3}(A + B_2 + C_2) = M_1$

$I_1 = f(M_1^1) \cdot S_{ABC}$

$I_2 = [f(M_1^2) + \ldots + f(M_4^2)] \cdot \frac{S_{ABC}}{4}$ $\qquad |I_2 - I_1|$

$I_3 = [f(M_1^3) + \ldots + f(M_{16}^3)] \cdot \frac{S_{ABC}}{16}$ $\qquad |I_3 - I_2| < \varepsilon \cdot |I_2|$

Задача 5. Электродинамика

$$\nabla\cdot E = 0$$
$$\nabla\cdot H = 0$$
$$\nabla\times E = -\mu\frac{\partial H}{\partial t}$$
$$\nabla\times H = \varepsilon\frac{\partial E}{\partial t}$$

$$E(x,t) = E(x)\cdot e^{-i\omega t}$$
$$H(x,t) = \sqrt{\frac{\varepsilon}{\mu}}\, H(x)\cdot e^{-i\omega t}$$

$$\begin{cases} \nabla\times E(x) = ikH \\ \nabla\times H(x) = -ikE \\ \nabla\cdot E = \nabla\cdot H = 0 \end{cases} \qquad k = \omega\sqrt{\varepsilon\mu}$$

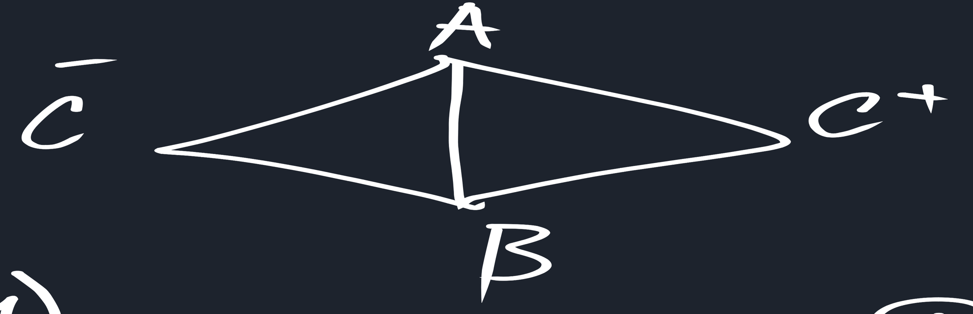
$$\Delta E_{tot} + k^2 E_{tot} = 0, \quad E_{tot} = E_0 + E, \quad \overline{E}_0 = \overline{p}\cdot e^{ik(v_0,x)}$$
$$n\times E_{tot} = 0 \;\big|_\Sigma$$

$\overline{p}$ – поляризация
v_0 – направление

$$E(x) = \mathcal{K}[\Sigma, \overline{g}] = \operatorname{rot}\operatorname{rot}\int_\Sigma g(y)F(x-y)\,dy = (\operatorname{grad}\operatorname{div} + k^2)\int_\Sigma gF\,dy$$

$$n\times K[\Sigma, g](x) = -n\times E_0$$

RWG $\quad g(x) = \sum_{i=1}^{N} g_i\,\overline{e_i(x)}$



$$e_i(x) = \begin{cases} \frac{C^+ - x}{S^+} & ABC^+ \\ \frac{x - C^-}{S^-} & ABC^- \\ 0 & \text{else} \end{cases}$$

$$\mathcal{D}_i = \operatorname{div} e_i = \begin{cases} -2/S_i^+ \\ 2/S_i^- \\ 0 \end{cases}$$

$$a_{ij} = (e_i, \mathcal{K}[\operatorname{supp}_j, e_j]) = \int_{\operatorname{supp}_i}\int_{\operatorname{supp}_j} [k^2 \underbrace{(e_i, e_j)}_{e_i(x)\cdot e_j(y)} - \mathcal{D}_i\mathcal{D}_j]\,F(x-y)\,dy\,dx$$

$$\frac{e^{ikr}}{r} = \frac{1}{r} + \frac{e^{ikr} - 1}{r} \xrightarrow[r\to 0]{} ik, \qquad e^{ikr} - 1 \to ikr + \frac{(ikr)^2}{2}$$

$$b_i = -(e_i, E_0) = -\int_{\operatorname{supp}_i} (e_i, p)\, e^{ik(v_0,x)}\,dx$$

$$Ag = b$$

$$\sigma(\tau) = 4\pi\lim_{R\to\infty}\frac{|E(x)|^2}{|E_0|^2}R^2 = \frac{1}{4\pi}\left|\int_\Sigma k^2 e^{-ik(\tau,y)}\big(g(y) - \tau(\tau, g(y))\big)\,dy\right|^2$$