1 Introduction

One time when I was binging YouTube, I stumbled upon a video by Grant Sanderson of 3Blue1Brown on Fourier series which featured many visualizations, prime of whose were drawings on a complex plain which were generated using Fourier series. Having done some reading on Fourier series as well as Fourier transform, I decided that it would be a great idea to actually understand those concepts. Since for me the ultimate way of learning is solving problems and since I feel best at expressing ideas in code, I decided to set myself a challenge of programming a program that will redraw my sketches with Fourier series.

2 What is Fourier series?

The purpose of Fourier series is to approximate any periodic function with a sum of sines and cosines. The formula is given by (1)

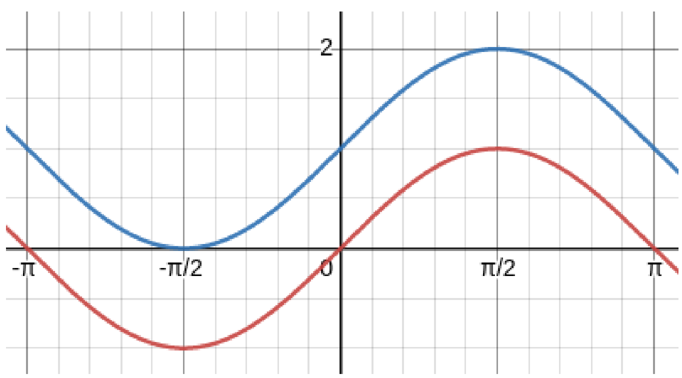
$$S_\infty(t) = a_0 + \sum_{n=1}^{\infty} a_n cos\frac{2\pi nt}{P} + b_n sin\frac{2\pi nt}{P} \tag{1}$$

From the formula we can observe two things, each sine and cosine in the sum has a weight assigned to it (a_n and b_n) and that n determines the frequency of the sinusoids. We can demistify the idea behind Fourier series by deriving formulas for all coefficients.

2.1 Deriving a_0

First coefficient we will find is a_0, which is the only coefficient that stands on its own. It is added before the actual series in order to compensate for the original function not oscillating around x axis on the plot. It basically is a vertical translation. We can see it if we add any constant to $sin(x)$ 1.

Figure 1: red - $sin(x)$, blue - $3 + sin(x)$



Now the problem to solve is what it the axis around which our sum of sines and cosines is to oscillate against. The solution to this problem is the average value of the original function over one period. The formula given for finding a_0 goes as follows (2)

$$a_0 = \frac{1}{P}\int_{\frac{P}{2}}^{\frac{-P}{2}} f(t)dt \tag{2}$$

Now, how does an integral grant us mean value of a function? Well, definite integral gives us the area under the graph of function $f(t)$ from $-\frac{1}{P}$ to $\frac{1}{P}$. If we have an area, we can express it with any figure that has this area, since we want an offset from the x axis, we can draw a rectangle whose base has length of P. Since we want to find out what the height of that rectangle is, we need to divide its area by its width, which is exactly what we are doing when dividing the integral by P.

2.2 Finding a_n and b_n

Weights a_n and b_n allow for the series to converge to desired function, which is quite obvious from the fact that the series is infinite, which means that the values on their own would shoot up to infinity. Now, how are we to find those weights to force the sinusoids to compliance? We can start with equating some function $f(t)$ to a Fourier series (3)

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cos\frac{2\pi nt}{P} + b_n sin\frac{2\pi nt}{P} \tag{3}$$

In order to find the coefficients, we can exploit the properties of definite integrals of sine and cosine. To do that, we will need to expand the whole expression by either $cos\frac{2\pi nt}{P}$ when looking for a_n, or by $sin\frac{2\pi nt}{P}$ if we are looking for b_n. Now we will go through the whole process by looking for a_n.

The expanded equation we will use to find a_n looks like this (4).

$$\begin{aligned}
\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)cos\frac{2\pi nt}{P}dt = \int_{-\frac{P}{2}}^{\frac{P}{2}} a_0 cos\frac{2\pi nt}{P}dt &+ \int_{-\frac{P}{2}}^{\frac{P}{2}} a_1 cos\frac{2\pi t}{P}cos\frac{2\pi nt}{P}dt + \int_{-\frac{P}{2}}^{\frac{P}{2}} b_1 sin\frac{2\pi t}{P}cos\frac{2\pi nt}{P}dt \\
&+ \int_{-\frac{P}{2}}^{\frac{P}{2}} a_2 cos\frac{4\pi t}{P}cos\frac{2\pi nt}{P}dt + \int_{-\frac{P}{2}}^{\frac{P}{2}} b_2 sin\frac{4\pi t}{P}cos\frac{2\pi nt}{P}dt \\
&\vdots \\
&+ \int_{-\frac{P}{2}}^{\frac{P}{2}} a_n cos^2(\frac{2\pi nt}{P})dt + \int_{-\frac{P}{2}}^{\frac{P}{2}} b_n sin\frac{2\pi nt}{P}cos\frac{2\pi nt}{P}dt
\end{aligned} \tag{4}$$

From this rather long expansion we can distill and solve for five cases appearing on the right side of the equation:

1. $$\int_{-\frac{P}{2}}^{\frac{P}{2}} a_0 cos\frac{2\pi nt}{P}dt = a_0 \cdot \left[\frac{P}{2\pi nt}sin\frac{2\pi nt}{P}\right]_{-\frac{P}{2}}^{\frac{P}{2}} = a_0 \cdot \frac{P}{2\pi nt}(sin(\pi n) - sin(-\pi n)) = 0$$

2. let $m \in Z \wedge m \neq n$
$$\begin{aligned}
\int_{-\frac{P}{2}}^{\frac{P}{2}} a_m cos\frac{2\pi mt}{P}cos\frac{2\pi nt}{P}dt &= a_m \cdot \int_{-\frac{P}{2}}^{\frac{P}{2}} \frac{1}{2}\left(cos\frac{(m-n)2\pi t}{P} + cos\frac{(m+n)2\pi t}{P}\right)dt \\
&= a_m \cdot \left[\frac{P}{4\pi(m-n)}sin\frac{(m-n)2\pi t}{P} + \frac{P}{4\pi(m+n)}sin\frac{(m+n)2\pi t}{P}\right]_{-\frac{P}{2}}^{\frac{P}{2}} \\
&= a_m \cdot (\frac{P}{4\pi(m-n)}sin((m-n)\pi) + \frac{P}{4\pi(m+n)}sin(-(m+n)\pi) = 0
\end{aligned}$$

3. let $m \in Z$
$$\begin{aligned}
\int_{-\frac{P}{2}}^{\frac{P}{2}} b_m sin\frac{2\pi mt}{P}cos\frac{2\pi nt}{P}dt &= b_m \cdot \int_{-\frac{P}{2}}^{\frac{P}{2}} \frac{1}{2}\left(sin\frac{(m+n)2\pi t}{P} + sin\frac{(m+n)2\pi t}{P}\right)dt \\
&= b_m \cdot \left[\frac{-P}{4\pi(m+n)}cos\frac{(m+n)2\pi t}{P} + \frac{-P}{4\pi(m+n)}cos\frac{(m+n)2\pi t}{P}\right]_{-\frac{P}{2}}^{\frac{P}{2}} \\
&= b_m \cdot (\frac{-P}{4\pi(m-n)}cos((m-n)\pi) + \frac{-P}{4\pi(n+m)}cos(-(m+n)\pi) = 0
\end{aligned}$$

4. $$\begin{aligned}
\int_{-\frac{P}{2}}^{\frac{P}{2}} cos^2(\frac{2\pi nt}{P})dt = \frac{1}{2}\int_{-\frac{P}{2}}^{\frac{P}{2}} 1 + cos\frac{4\pi nt}{P}dt &= \frac{1}{2} \cdot \left[t + \frac{P}{4\pi nt}sin\frac{4\pi nt}{P}\right]_{-\frac{P}{2}}^{\frac{P}{2}} \\
&= \frac{1}{2} \cdot \left(\frac{P}{2} + \frac{1}{2\pi n}sin(2\pi n) + \frac{P}{2} + \frac{1}{2\pi n}sin(-2\pi n)\right) = \frac{2}{P}
\end{aligned}$$

We can see that the only term after expansion and integration that does not amount to zero is $\int_{-\frac{P}{2}}^{\frac{P}{2}} cos^2(\frac{2\pi nt}{P})dt$, which means that our equation will simplify to the following form (5)

$$\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)cos\frac{2\pi nt}{P}dt = a_n\frac{P}{2} \tag{5}$$

So in order to find a_n, we will get the following(6)

$$a_n = \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)cos\frac{2\pi nt}{P}dt \tag{6}$$

Formula for b_n is very similar (7) and stems from the same logic as one of a_n, see Appendix A for calculations.

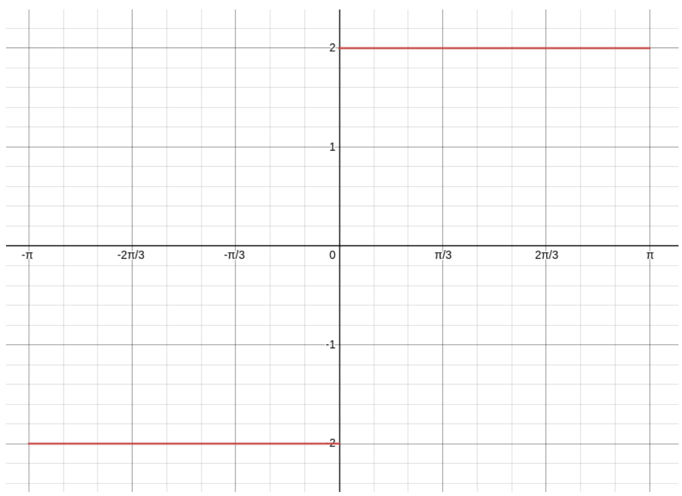
$$b_n = \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)sin\frac{2\pi nt}{P}dt \tag{7}$$

2.3 Approximating a step function

Before diving straight to other concepts and the drawings, let us take a second and manually approximate a step function defined in (8) and plotted in Figure 2

$$\begin{aligned} f(t) &= \begin{cases} -2 \text{ if } -\pi \leq t < 0 \\ 2 \text{ if } 0 \leq t \leq \pi \end{cases} \\ f(t) &= f(t+2\pi) \end{aligned} \tag{8}$$

Figure 2: Step function f(t)



If we look on the graph, we can already establish that the definite integral of the function over the whole period is zero, hence $a_0 = 0$.

Other thing we want to observe is that the function $f(t)$ is odd, meaning that $f(t) = -f(-t)$ and the function that is odd in our series is the sine, we know that all coefficients a_n will be zero because otherwise they would distort our approximation.

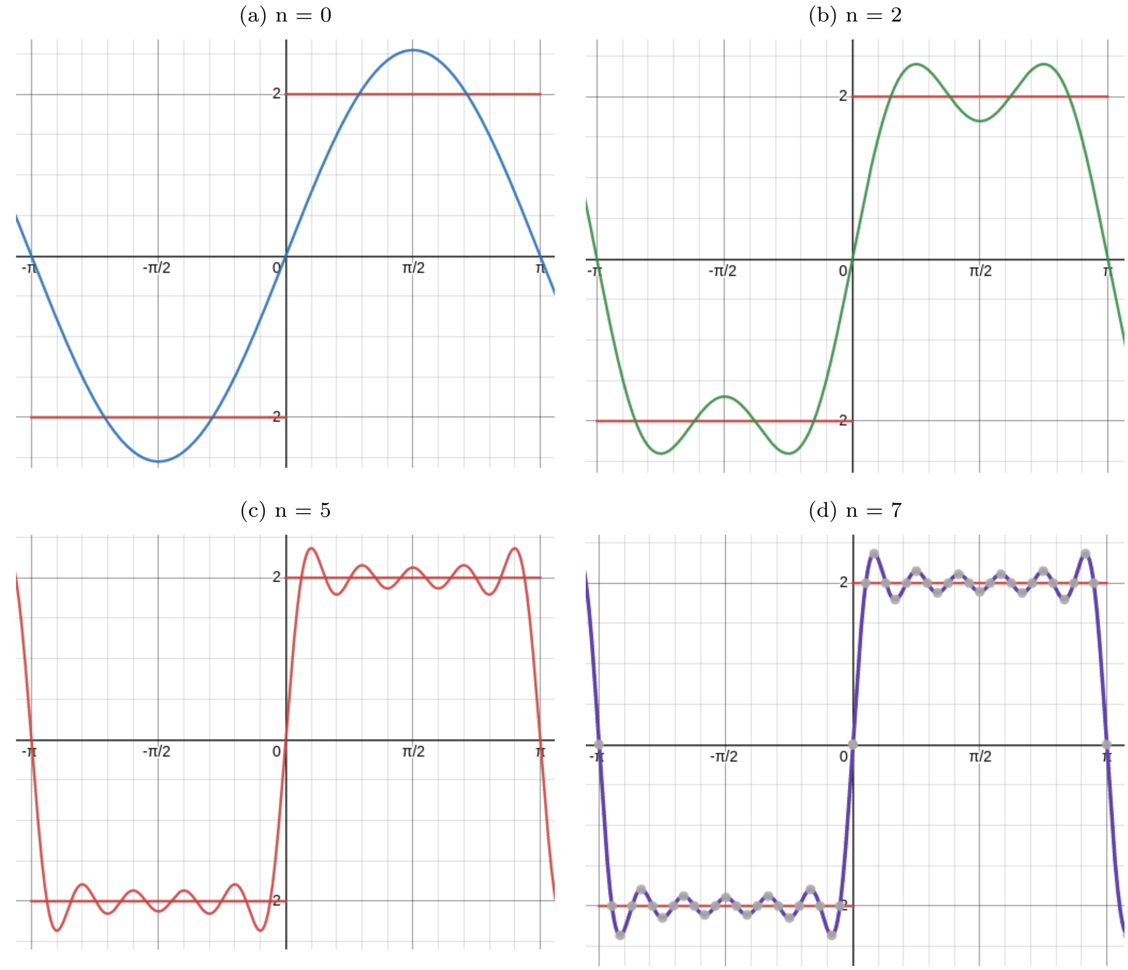
$$\begin{aligned} b_n &= \frac{2}{\pi}\int_{-\pi}^{\pi} f(t)sin(nt)dt = \frac{2}{\pi}\cdot\left[f(t)\cdot t\cdot\frac{1}{-nt}cos(nt)\right]_{-\pi}^{\pi} \\ &= \frac{2}{\pi}\cdot\left[\frac{2}{n}\left(cos(n\pi)+cos(n\pi)-(-cos(-(n\pi)+cos(-n\pi)\right)\right] \\ &= \frac{4}{n\pi}\left(2cos(n\pi)+2cos(-n\pi)\right) = \begin{cases} \frac{8}{n\pi} \text{ if n is odd} \\ 0 \text{ if n is even} \end{cases} \end{aligned} \tag{9}$$

The series for our step function looks like this (10)

$$S_\infty(t) = \sum_{n=1}^{\infty}\frac{8}{(2n-1)\pi}sin((2n-1)t) \tag{10}$$

Now we can make a few plots of the series with increasing number of terms and see it converge to $f(t)$, Figure 11

Figure 3



We now know what the pure form of Fourier series is, but now we need to use it to draw images, so from now on, we will keep on walking through all the concepts, but in the context of the program itself.

3 Overview of the program

The program has three main stages:

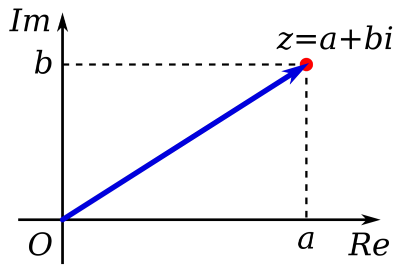
1. taking input from the user and mapping it onto complex plane
2. processing the input using discrete Fourier transform (DFT)
3. redrawing the sketch using complex form of Fourier series

In this short list we have two completely new concepts and one slightly mutated, so let us now go through each component in their applied form.

3.1 Complex numbers

A drawing in case of the program is a set of x and y coordinates, however, for all calculations we need something called complex numbers. Akin to points that are desrcibed in terms of coordinates, complex numbers are described in two dimensions. One of them everyone knows really well, because this is the the real component, which, as the name suggests, consists of real numbers. The component that allows us to move vertically does not belong on the real number line because this number is $\sqrt{-1}$, usually denoted as i. So any complex number can be denoted as following: $a + bi$ and can be depicted like in figure 4 [1]

Figure 4



The arithmetics with complex numbers look exactly the same like in case of any other numbers.

It is already apparent why we would use a set complex numbers when trying to draw instead, for example a function like we did in previous section. It gives us two dimensions of freedom which is excactly what we need since our sketches will be two-dimensional.

3.2 Discrete Fourier transform (DFT)

Next step in the program is to find the coefficients for the Fourier series to work with. However, the process will be a bit different since we are working with a set of complex numbers rather than a real-valued function. This is where DFT, or discrete Fouirer transform comes into play. Remember how the coefficients a_n and b_n described the amplitude of sinusoids with frequency of n? What Fourier transform does is finding the amplitude of signals with given frequency. The formula for DFT is as follows (11).

$$X_k = \sum_{n=0}^{N-1} x_n \cdot \left[cos\left(\frac{2\pi}{N}kn\right) - isin\left(\frac{2\pi}{N}kn\right)\right] \tag{11}$$

Now let us walk through what each variable means:

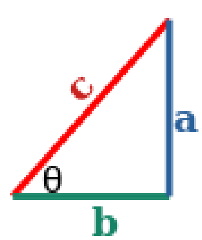
- X_k - output complex number of DFT
- N - size of input set
- k - frequency
- x_n - n^{th} complex number in the input set

The first thing to go through is why we are multiplying our input complex number by a complex sum?

3.2.1 Expressing a point on a unit circle using complex numbers

Let us take a right triangle with hypotenuse of length 1 5 and angle θ between that hypotenuse (c) and the base b of triangle.

Figure 5

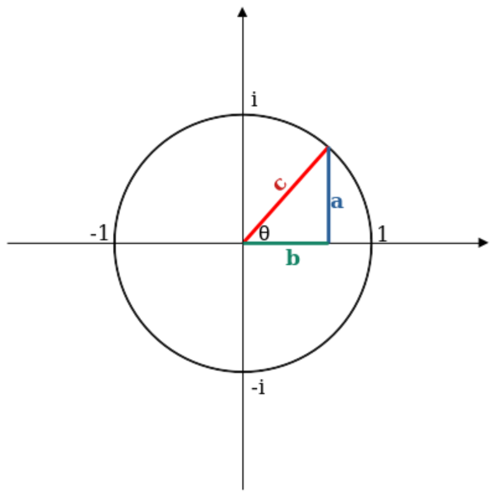


[1]Graphic taken from `https://en.wikipedia.org/wiki/Complex_number`

If we wanted to find the length of the base (b), it would be equal to the $cos\theta$ since the hypotenuse has the length 1. Similarly, if we wanted to find the height of the triangle (a), it would be the $sin\theta$.

Now we can inscribe such right triangle into a unit circle on a complex plane where the hypotenuse of the triangle would simultaneusly be the radius of the triangle 6 with some angle θ between radius and the horizontal axis.

Figure 6



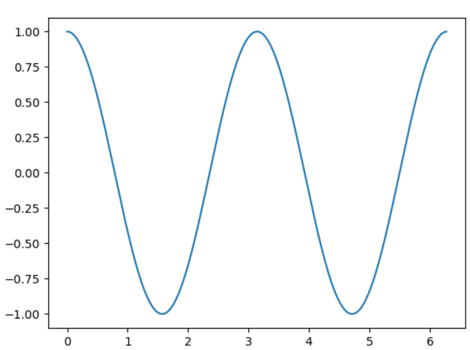
We can find the coordinate of the point where the radius touches the circumference using the inscribed triangle, so $cos\theta$ would be the horizontal coordinate and $i \cdot sin\theta$ would be the vertical component since we are on complex plane. So the whole position can be described as $cos\theta + isin\theta$.

3.2.2 The crux of DFT matter

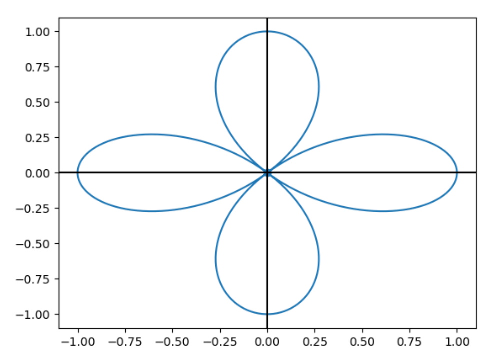
Now, having the basis, we can go through the DFT inductively. Let us find Fourier transform for $cos(2t)$. We already know that the expression $cos\left(\frac{2\pi}{N}kn\right) - isin\left(\frac{2\pi}{N}kn\right)$ describes unit circle on a complex plane. What we are doing with DFT mapping our input set around this unit circle - Figure 7. In plot (b) we can see the heart of Fouerier transform. Since we are summing all numbers in the set, and, as we can see, the plot is symmetrical, so all the values will cancel out, leaving us with 0.

Figure 7

(a) $cos(2t)$

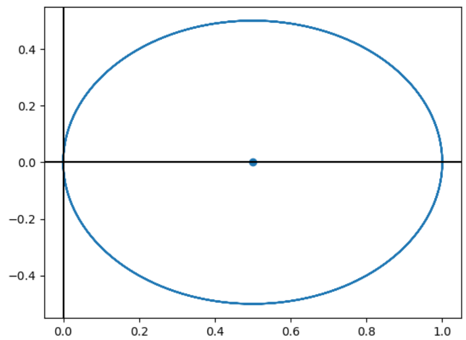


(b) $cos(2t)$ - wrapped



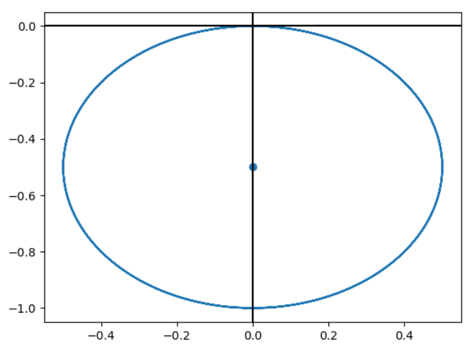
However, if we map the values to a circle with matching frequency k, we observe something interesting - Figure 8. All values have moved to one side, so now, if we sum all the values, we get a complex number. In the case of cosine, the result is real, which means that we the function consists of a sinewave with 0 phase change and amplitude of 1 (the raw results from DFT have to be divided by N to get representation below and by $2N$ to get the real amplitude).

Figure 8



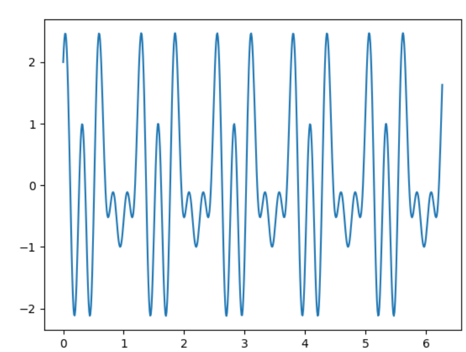
If we wrap $sin(2t)$ the same way, we will see that we will have the phase change represented by the complex component - Figure 9. This should make intuitive sense, because we saw that cosine function starts at completely horizontal position and the sine starts from completely vertical position.

Figure 9



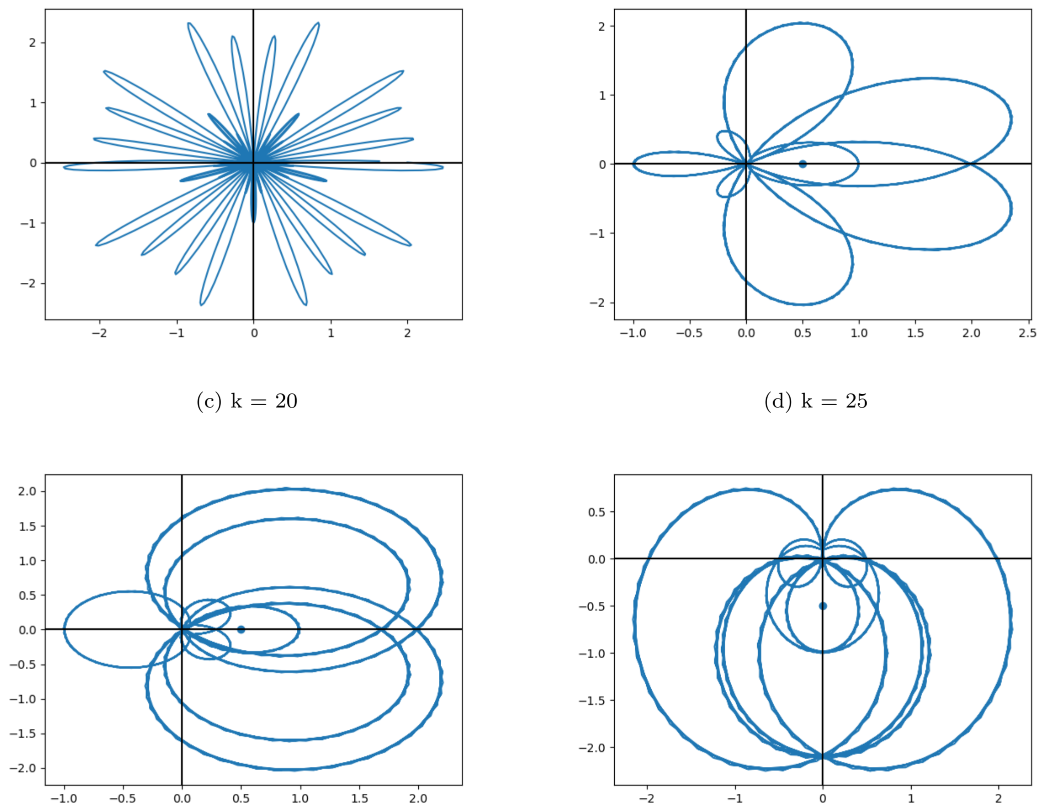
We can already see that we will use DFT to set radii and initial positions of our "circles" that will spin at k frequencies. Before that let us have one last bit of fun with DFT and decompose a function with more than one component sine wave, say $f(t) = cos(10t) + cos(20t) + sin(25t)$, which looks like in Figure 10. Now, all we have to do is apply Fourier transform.

Figure 10



Here are plots of our function mapped to the unit circle with following frequencies: $1, 10, 20, 25$, where the dot signifies the sum mapped with $\frac{1}{N}$

Figure 11



(a) k = 1

(b) k = 10

(c) k = 20

(d) k = 25

See Appendix B for source code of script used to generate all plots above.

3.2.3 Why naive implementation of DFT is inefficient

Now we can take a look at how DFT is implemented in C++.

```
C_set *DFT(C_set input)
{
    C_set *out = new C_set;
    C temp, X_k;
    unsigned N = input.size();
    double a = (2 * M_PI) / N, x;

    for (unsigned k = 0; k < N; ++k)
    {
        for (unsigned n = 0; n < N; ++n)
        {
            x = a * n * k;
            temp.re = cos(x);
            temp.im = -sin(x);

            X_k += (temp * input[n]);
        }

        out->push_back(X_k);
```

```
        X_k = C(0,0);
    }

    return out;
}
```

The function takes set of complex numbers which is our drawing and calculates coefficients for frequencies from 0 to $N-1$. Of course this number could be arbitrarly high, but N seems to do the trick. The algorithm is a direct implementation of the formula and the source of inefficiency is in the two nested for loops that go from 0 to N. What this means is that the algorithm has the time complexity of $O(N^2)$, which means that supplied set of N elements, the computer will always perform N^2 operations because the function does not contain any break statements which could add best and worst case scenario to the considerations. In real world, this algorithm is rarely used, as it has its much more efficient counterpart - Fast Fourier Transform (FFT), which has the complexity of $O(Nlog(N))$ which is significantly faster than $O(N^2)$, but since the program is made as a bit of fun, we can live with poor efficiency.

3.3 Fourier series and making drawings

At this point we have a set of complex numbers describing our sketch in terms of frequency and we have to come back to time domain. To do that we use inverse discrete Fourier transform (IDFT), which might as well be called discrete Fourier series. Akin to approximating step function in section 2, we have a bunch of weights on our hands and all we have to do is apply them to what we will now call our rotating circles - epicycles. DFT gives us two pieces of information about each frequency - how big is the epicycle and where is starts. The formula for IDFT is as follows (12)

$$x_n = \frac{1}{N}\sum_{n=0}^{N-1} X_k \cdot \left[cos\left(\frac{2\pi}{N}kn\right) + isin\left(\frac{2\pi}{N}kn\right)\right] \tag{12}$$

Notice that what we are doing is applying Fourier transform to Fourier transform. The $\frac{1}{N}$ weight is the same weight we applied when depicting DFT. Now, having all tools needed, we can take a look at the code. Our goal is to control the number of terms. Provided that number, we can calculate the set making up our drawing by adding n for every angle from 0 to 2π. Below is function called *draw IDFT*, which does just that.

```
void draw_IDFT(C_set &x_k, VertexArray &drawing, unsigned n)
{
    C temp, x_n;
    unsigned N = x_k.size();
    double step = (2 * M_PI) / N, theta;

    for (unsigned t = 0; t < N; ++t)
    {
        for (unsigned k = 0; k < n; ++k)
        {
            theta = step * t * k;
            temp = C(cos(theta), sin(theta));

            x_n += (temp * x_k[k]) / N;
        }

        drawing[t] = Vector2f(x_n.re + 400, -x_n.im + 400);
        x_n = C(0,0);
    }
}
```

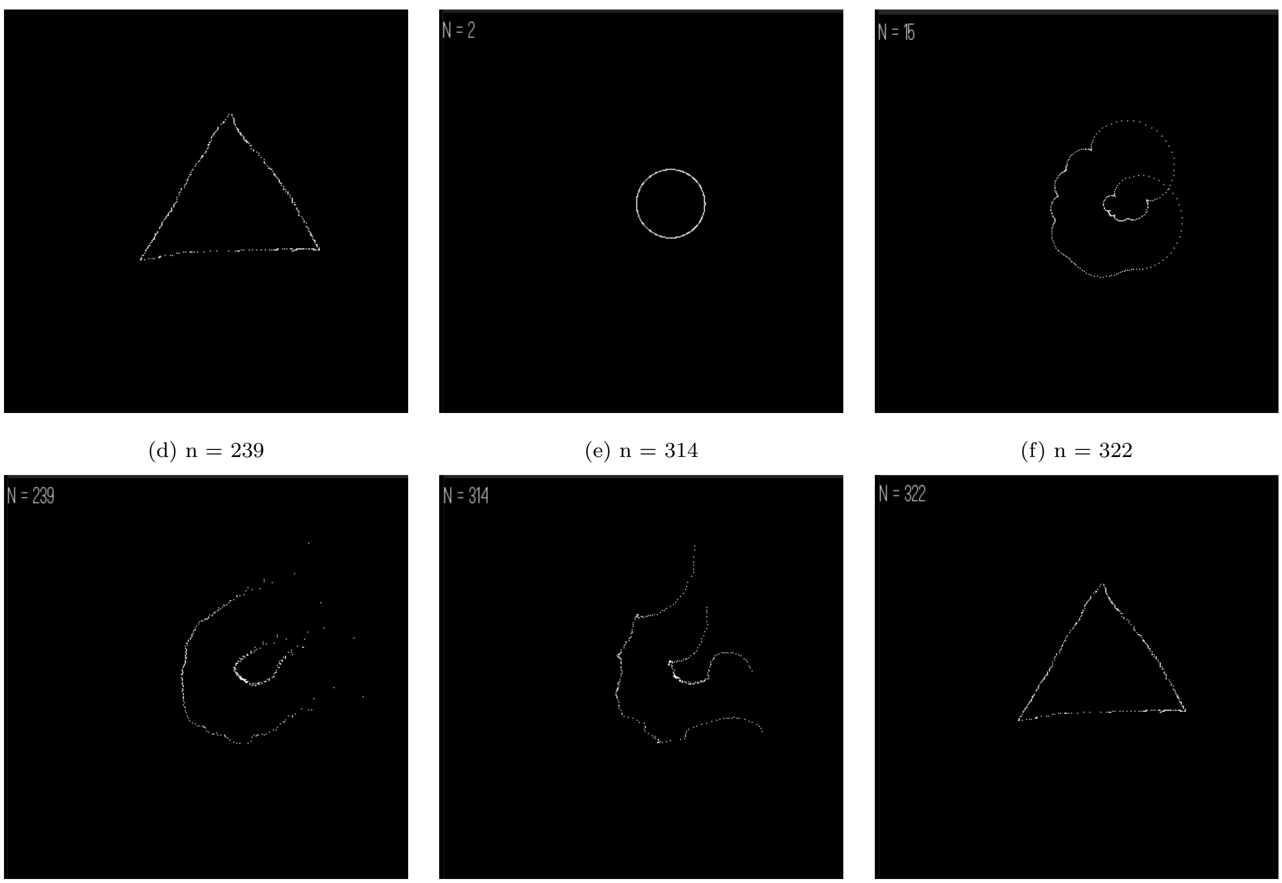
If we go look at the code, it looks very similar to DFT implementation complete with that doubly nested loops. Starting from the top for loop, we have variable t, which is there to iterate over the full range, so in other terms - make one full revolution at frequency 1. The next loop is what allows us to observe the convergence to our input. Variable k, consistent with previous variable- naming is our frequency. Inside the loop, we are calculating the angle at which epicycle revolving at frequency k is, after which we express it as a complex number in variable *temp* to finally calculate its position using IDFT. We are adding the terms to x_n, since we want the outline rather than path of each epicycle. Having calculated x_n, we add it to the buffer holding points making up our drawing. Rince and repeat.

4 Drawings

This moment might be somewhat anticlimactic, since the program works best as a kind of animation, I will leave link to code repository complete with instructions to compile the program. In this paper, however I will show off some simpler sketches that converge fast so that we can see it well without perusing a 100-page document containing every frame.

4.1 Triangle

Figure 12

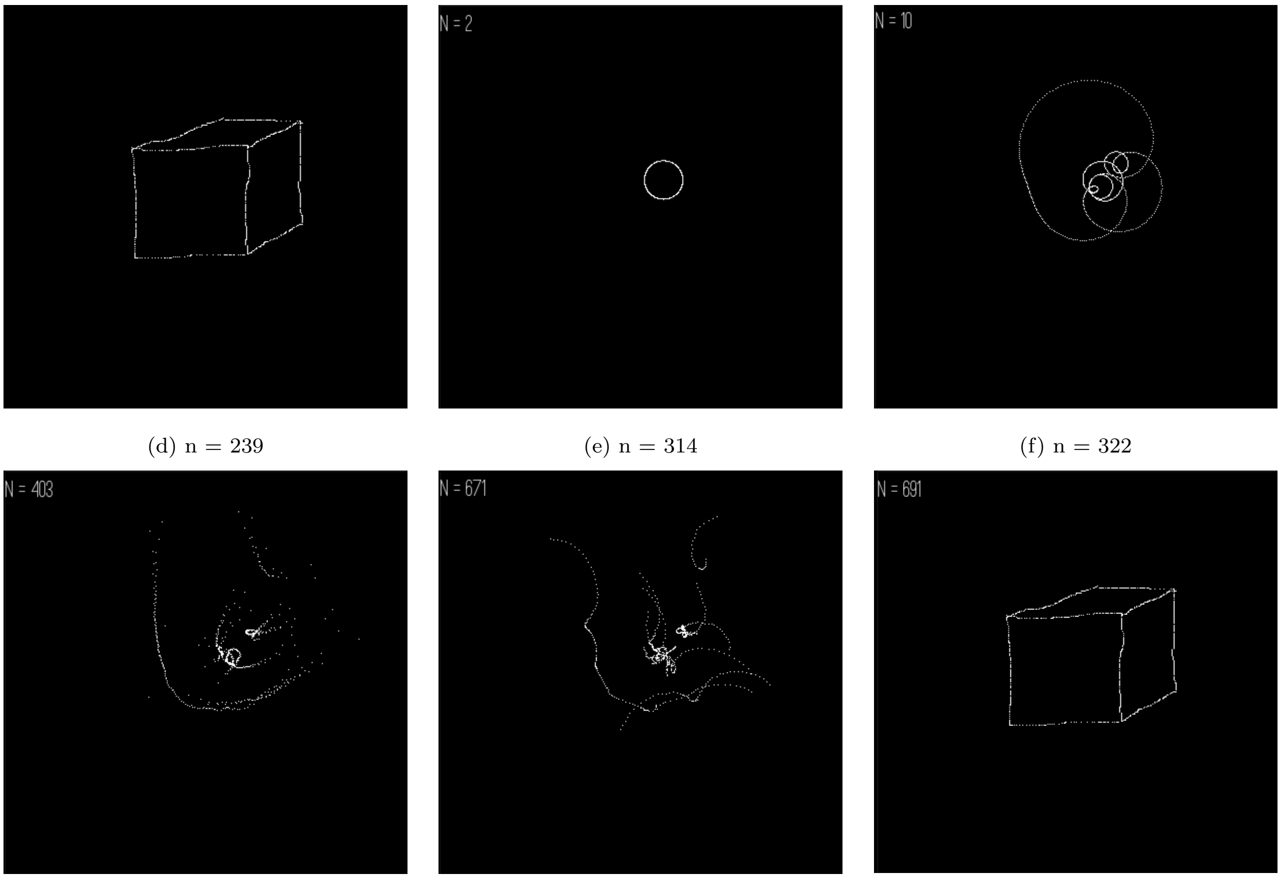


(a) original input (b) n = 2 (c) n = 15

(d) n = 239 (e) n = 314 (f) n = 322

4.2 Cube

Figure 13

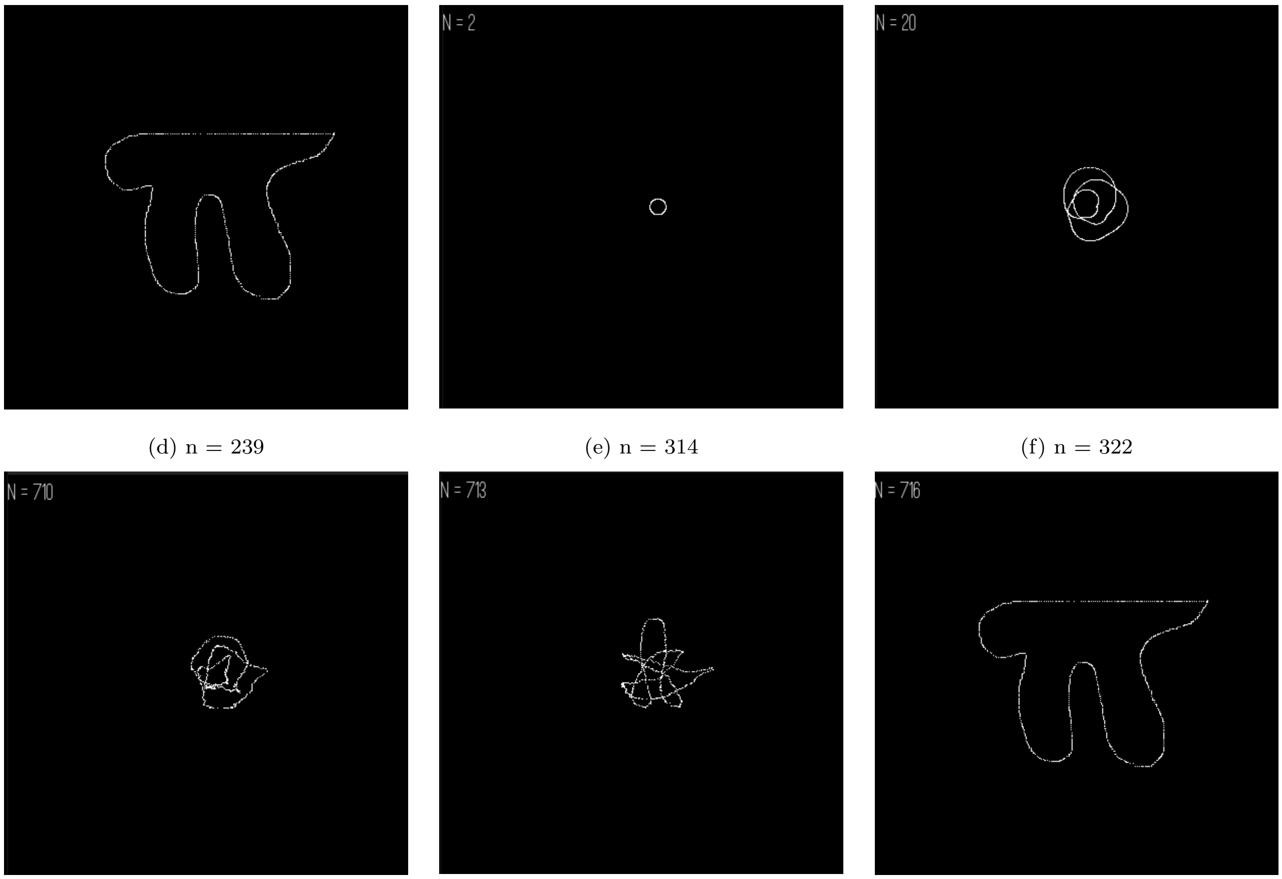


(a) original input (b) n = 2 (c) n = 15

(d) n = 239 (e) n = 314 (f) n = 322

4.3 Pi

Figure 14



(a) original input (b) n = 2 (c) n = 15

(d) n = 239 (e) n = 314 (f) n = 322