1 Introduction

One time when I was binging YouTube, I stumbled upon a video by Grant Sanderson of 3Blue1Brown on Fourier series which featured many visualizations, prime of whose were drawings on a complex plain which were generated using Fourier series. Having done some reading on Fourier series as well as Fourier transform, I decided that it would be a great idea to actually understand those concepts. Since for me the ultimate way of learning is solving problems and since I feel best at expressing ideas in code, I decided to set myself a challenge of programming a program that will redraw my sketches with Fourier series.

2 Overview of the program

The program has three main stages:

1. taking input in the form of drawing from the user
2. processing the drawing using discrete Fourier transform (DFT)
3. printing the drawing using discrete Fourier Series

The output of the program is the following: (ADD IMAGES)

3 What is Fourier series?

The purpose of Fourier series is to approximate any periodic function with a sum of sines and cosines. The formula is given by (1)

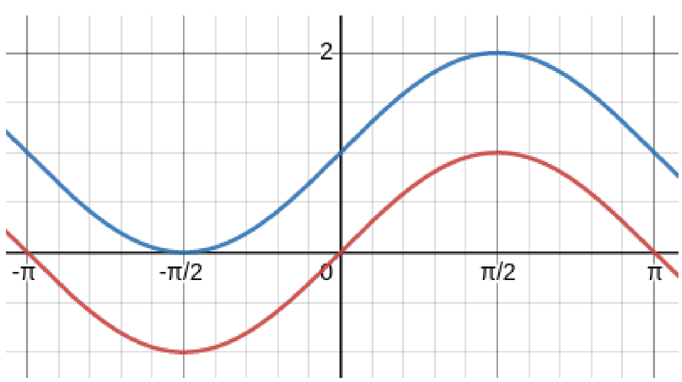
$$S_\infty(t) = a_0 + \sum_{n=1}^{\infty} a_n cos\frac{2\pi nt}{P} + b_n sin\frac{2\pi nt}{P} \tag{1}$$

From the formula we can observe two things, each sine and cosine in the sum has a weight assigned to it (a_n and b_n) and that n determines the frequency of the sinusoids. We can demistify the idea behind Fourier series by deriving formulas for all coefficients.

3.1 Deriving a_0

First coefficient we will find is a_0, which is the only coefficient that stands on its own. It is added before the actual series in order to compensate for the original function not oscillating around x axis on the plot. It basically is a vertical translation. We can see it if we add any constant to $sin(x)$ 1.

Figure 1: red - $sin(x)$, blue - $3 + sin(x)$



Now the problem to solve is what it the axis around which our sum of sines and cosines is to oscillate against. The solution to this problem is the average value of the original function over one period. The formula given for finding a_0 goes as follows (2)

$$a_0 = \frac{1}{P}\int_{\frac{P}{2}}^{\frac{-P}{2}} f(t)dt \tag{2}$$

Now, how does an integral grant us mean value of a function? Well, definite integral gives us the area under the graph of function $f(t)$ from $-\frac{1}{P}$ to $\frac{1}{P}$. If we have an area, we can express it with any figure that has this area, since we want an offset from the x axis, we can draw a rectangle whose base has length of P. Sine we want to find out what the height of that rectangle is, we need to divide its area by its width, which is exactly what we are doing when dividing the integral by P.

3.2 Finding a_n and b_n

Weights a_n and b_n allow for the series to converge to desired function, which is quite obvious from the fact that the series is infinite, which means that the values on their own would shoot up to infinity. Now, how are we to find those weights to force the sinusoids to compliance? We can start with equating some function $f(t)$ to a Fourier series (3)

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n cos\frac{2\pi nt}{P} + b_n sin\frac{2\pi nt}{P} \tag{3}$$

In order to find the coefficients, we can exploit the properties of definite integrals of sine and cosine. To do that, we will need to expand the whole expression by either $cos\frac{2\pi nt}{P}$ when looking for a_n, or by $sin\frac{2\pi nt}{P}$ if we are looking for b_n. Now we will go through the whole process by looking for a_n.

The expanded equation we will use to find a_n looks like this (4).

$$\begin{aligned}
\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)cos\frac{2\pi nt}{P}dt = \int_{-\frac{P}{2}}^{\frac{P}{2}} a_0 cos\frac{2\pi nt}{P}dt + \int_{-\frac{P}{2}}^{\frac{P}{2}} a_1 cos\frac{2\pi t}{P}cos\frac{2\pi nt}{P}dt + \int_{-\frac{P}{2}}^{\frac{P}{2}} b_1 sin\frac{2\pi t}{P}cos\frac{2\pi nt}{P}dt \\
+ \int_{-\frac{P}{2}}^{\frac{P}{2}} a_2 cos\frac{4\pi t}{P}cos\frac{2\pi nt}{P}dt + \int_{-\frac{P}{2}}^{\frac{P}{2}} b_2 sin\frac{4\pi t}{P}cos\frac{2\pi nt}{P}dt \\
\vdots \\
+ \int_{-\frac{P}{2}}^{\frac{P}{2}} a_n cos^2(\frac{2\pi nt}{P})dt + \int_{-\frac{P}{2}}^{\frac{P}{2}} b_n sin\frac{2\pi nt}{P}cos\frac{2\pi nt}{P}dt
\end{aligned} \tag{4}$$

From this rather long expansion we can distill and solve for five cases appearing on the right side of the equation:

1. $$\int_{-\frac{P}{2}}^{\frac{P}{2}} a_0 cos\frac{2\pi nt}{P}dt = a_0 \cdot \left[\frac{P}{2\pi nt}sin\frac{2\pi nt}{P}\right]_{-\frac{P}{2}}^{\frac{P}{2}} = a_0 \cdot \frac{P}{2\pi nt}(sin(\pi n) - sin(-\pi n)) = 0$$

2. let $m \in Z \land m \neq n$

$$\begin{aligned}
\int_{-\frac{P}{2}}^{\frac{P}{2}} a_m cos\frac{2\pi mt}{P}cos\frac{2\pi nt}{P}dt &= a_m \cdot \int_{-\frac{P}{2}}^{\frac{P}{2}} \frac{1}{2}\left(cos\frac{(m-n)2\pi t}{P} + cos\frac{(m+n)2\pi t}{P}\right)dt \\
&= a_m \cdot \left[\frac{P}{4\pi(m-n)}sin\frac{(m-n)2\pi t}{P} + \frac{P}{4\pi(m+n)}sin\frac{(m+n)2\pi t}{P}\right]_{-\frac{P}{2}}^{\frac{P}{2}} \\
&= a_m \cdot (\frac{P}{4\pi(m-n)}sin((m-n)\pi) + \frac{P}{4\pi(m+n)}sin(-(m+n)\pi) = 0
\end{aligned}$$

3. let $m \in Z$

$$\begin{aligned}
\int_{-\frac{P}{2}}^{\frac{P}{2}} b_m sin\frac{2\pi mt}{P}cos\frac{2\pi nt}{P}dt &= b_m \cdot \int_{-\frac{P}{2}}^{\frac{P}{2}} \frac{1}{2}\left(sin\frac{(m+n)2\pi t}{P} + sin\frac{(m+n)2\pi t}{P}\right)dt \\
&= b_m \cdot \left[\frac{-P}{4\pi(m+n)}cos\frac{(m+n)2\pi t}{P} + \frac{-P}{4\pi(m+n)}cos\frac{(m+n)2\pi t}{P}\right]_{-\frac{P}{2}}^{\frac{P}{2}} \\
&= b_m \cdot (\frac{-P}{4\pi(m-n)}cos((m-n)\pi) + \frac{-P}{4\pi(n+m)}cos(-(m+n)\pi) = 0
\end{aligned}$$

4.

$$\int_{-\frac{P}{2}}^{\frac{P}{2}} cos^2(\frac{2\pi nt}{P})dt = \frac{1}{2}\int_{-\frac{P}{2}}^{\frac{P}{2}} 1 + cos\frac{4\pi nt}{P}dt = \frac{1}{2}\cdot\left[t + \frac{P}{4\pi nt}sin\frac{4\pi nt}{P}\right]_{-\frac{P}{2}}^{\frac{P}{2}}$$
$$= \frac{1}{2}\cdot\left(\frac{P}{2} + \frac{1}{2\pi n}sin(2\pi n) + \frac{P}{2} + \frac{1}{2\pi n}sin(-2\pi n)\right) = \frac{2}{P}$$

We can see that the only term after expansion and integration that does not amount to zero is $\int_{-\frac{P}{2}}^{\frac{P}{2}} cos^2(\frac{2\pi nt}{P})dt$, which means that our equation will simplify to the following form (5)

$$\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)cos\frac{2\pi nt}{P}dt = a_n\frac{P}{2} \tag{5}$$

So in order to find a_n, we will get the following(6)

$$a_n = \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)cos\frac{2\pi nt}{P}dt \tag{6}$$

Formula for b_n is very similar (7) and stems from the same logic as one of a_n, see Appendix A for calculations.

$$b_n = \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)sin\frac{2\pi nt}{P}dt \tag{7}$$

3.3 Approximating a step function

Before diving straight to other concepts and finally the drawings, let us take a second and manually approximate a step function (8).

$$\begin{aligned} f(t) &= \begin{cases} -5 \text{ if } \frac{-P}{2} \leq t < 0 \\ 5 \text{ if } 0 \leq t \leq \frac{P}{2} \end{cases} \\ f(t) &= f(t+P) \end{aligned} \tag{8}$$