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Professor McDonnell | Assigned: March 16, 2017 | Due: March 28, 2017

Contents

5 Sequences, Mathematical Induction and Recursion

5.1 11, 21, 39, 53, 56

11) 0, $\frac{-1}{2}$, $\frac{2}{3}$, $\frac{-3}{4}$, $\frac{4}{5}$, $\frac{-5}{6}$, $\frac{6}{7}$, given this pattern, to get to the k^{th} term $\forall k \geq 0 \in \mathbb{Z}$

$$a_k = \frac{(-1)^k \ k}{k+1}$$

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21) $\displaystyle\sum_{k=-1}^{1}(k^2+3) =$

$$((-1)^2+3)+((0)^2+3)+((1)^3+3) = \boxed{11}$$

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39) $(1^3-1)-(2^3-1)+(3^3-1)-(4^3-1)+(5^3-1) \ =$

$$\boxed{\sum_{k=0}^{4}(-1)^k \cdot ((k+1)^3-1)}$$

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53) $\displaystyle\prod_{i=n}^{2n}\frac{n-i-1}{n+1} \to j = i-1$ and $j+1 = i \therefore n = i$. After subbing these in we get

$$\prod_{i=k}^{2n}\frac{-n+i-1}{1-n} \to \boxed{\prod_{j=n-1}^{2n-2}\frac{-n+j}{1-n}}$$

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56) $\displaystyle\prod_{k=1}^{n}\frac{k}{k+1} \cdot \prod_{k=1}^{n}\frac{k+1}{k+2} = \prod_{k=1}^{n}\frac{k}{k+1} \cdot \frac{k+1}{k+2} =$

$$\boxed{\prod_{k=1}^{n}\frac{k}{k+2}}$$

5.2 4, 9, 16, 27

4a) $\displaystyle\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$ that is $\forall n \geq 2 \in \mathbb{Z}$

Base Case: $n = 2 \Rightarrow \displaystyle\sum_{i=1}^{1} 1(2) = \frac{2 \cdot 1 \cdot 3}{3}$

4b) Inductive Step: Suppose $P(k) = \overbrace{1(1+1) + ... + k(k+1) = \frac{k(k-1)(k-2)}{3}}^{\textbf{Inductive Hypothesis}}$

4c) $P(k+1)$

$$\overbrace{1(1+1) + ... + k(k+1)}^{\text{What the I.H. is equal to}} + (k+1)((k+1)+1) = \frac{k(k+1)(k+2)}{3}$$

4d) What we have to do

- In the base step one must show that the lowest possible value is true
- In the Inductive Step, you assume that $P(k)$ is true such that k is any integer greater than or equal to 2. Then you show that $P(k+1)$ is true.

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9) Let $P(n)$ be the statement saying that $4^3 + 4^4 + 4^5 + ... + 4^n = \frac{4(4^n - 16)}{3}$ is true $\forall n \geq 3 \in \mathbb{Z}$.

Base Case: The Statement $P(3)$ would mean that we would use 3 for n and make sure that the original claim is true for the most basic value given the parameters of the claim (In this case its $\forall n \geq 3 \in \mathbb{Z}$).

Inductive Step: Assume that $P(k)$ is true $\forall k \geq 3 \in \mathbb{Z}$

This would make the claim that $\underbrace{4^3 + 4^4 + 4^5 + ... + 4^k = \frac{4(4^k - 16)}{3}}_{\textbf{Inductive Hypothesis}}$

Now we show that $P(k+1)$ is also true: $4^3 + 4^4 + 4^5 + ... + 4^k + 4^{k+1} = \frac{4(4^{k+1} - 16)}{3}$

$$\begin{aligned}
\frac{4(4^k - 16)}{3} + 4^{k+1} &= \frac{4(4^{k+1} - 16)}{3} \\
\frac{4(4^k - 16) + 3 \cdot 4^{k+1}}{3} &= \frac{4^{k+2}}{3} - \frac{16 \cdot 4}{3} \\
\frac{4^{k+1}}{3} - \frac{64}{3} + \frac{3 \cdot 4^{k+1}}{3} &= \frac{4^{k+2}}{3} - \frac{16 \cdot 4}{3} \\
\frac{4^{k+1}}{3} + \frac{3 \cdot 4^{k+1}}{3} - \frac{64}{3} &= \frac{4^{k+2}}{3} - \frac{64}{3} \\
4^{k+1} \cdot \left(\frac{1}{3} + \frac{3}{3}\right) - \frac{64}{3} &= \frac{4^{k+2}}{3} - \frac{64}{3} \\
4^{k+1} \cdot \left(\frac{4}{3}\right) - \frac{64}{3} &= \frac{4^{k+2}}{3} - \frac{64}{3} \\
\underbrace{4^{k+2} \cdot \left(\frac{1}{3}\right) - \frac{64}{3}}_{\text{This is what we wanted}} &= 4^{k+2} \cdot \left(\frac{1}{3}\right) - \frac{64}{3}
\end{aligned}$$

16) $P(n)$ states that $\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ is true $\forall(n\geq 2)\in\mathbb{Z}$

Base Case: The statement $P(2)$ would mean that we would use 2 for n and make sure that the original claim is true for the a value within the parameters of the claim (In this case its $\forall n\geq 2\in\mathbb{Z}$).

Inductive Step: Assume that $P(k)$ is true $\forall k\geq 2\in\mathbb{Z}$

This would make the claim that $\underbrace{\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}}_{\text{This is the Inductive Hypothesis}}$

Now we show that $P(k+1)$ is also true: $\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\cdot\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)}$

$$\begin{aligned}
\frac{k+1}{2k}\cdot\left(1-\frac{1}{(k+1)^2}\right) &= \frac{(k+1)+1}{2(k+1)}\\
\frac{(k+1)\cdot((k+1)^2-1)}{2k\cdot(k+1)\cdot(k+1)} &= \frac{k+2}{2k+2}\\
\frac{((k+1)^2-1)}{2k\cdot(k+1)} &= \frac{k+2}{2k+2}\\
\frac{k^2+2k+1-1}{2k^2+2k} &= \frac{k+2}{2k+2}\\
\frac{k^2+2k}{2k^2+2k} &= \frac{k+2}{2k+2}\\
\frac{k\cdot(k+2)}{k\cdot(2k+2)} &= \frac{k+2}{2k+2}\\
\underbrace{\frac{k+2}{2k+2}}_{\text{This is what we wanted}} &= \frac{k+2}{2k+2}
\end{aligned}$$

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27) $5^3+5^4+5^5+...+5^n=\underbrace{\frac{r^{n+1}-1}{r-1}}_{S_n}$ This is dependent on the value of k so we can only get a sum in terms of k

$$S_k=\frac{5^{k+1}-1}{5-1}=\frac{5^{k+1}-1}{4}$$

This is the end of 5.2, 5.3 is on the next page

5.3 15, 22, 27

15) $P(n)$ states that $n \cdot (n^2 + 5)$ is divisible by 6 $\forall n \geq 0 \in \mathbb{Z}$

Base Case:
$n = 0$, $0(0^2 + 5) = 0$ and $6|0$ is true.

Inductive Step:
Suppose $P(k)$, that would make the claim that $\forall k \geq 0 \in \mathbb{Z}$, $\underbrace{6|k(k^2+5) \text{ or } 6|(k^3+5k)}_{\text{This is the Inductive Hypothesis}}$ are both true.

Now we need to show $P(k + 1)$ is true.

$$\begin{aligned}
&6|(k+1)((k+1)^2+5) \\
&6|(k^3+2k^2+k+5k+k^2+2k+1+5) \\
&6|(k^3+3k^2+3k+5k+6) \\
&6|\underbrace{(k^3+5k)}_{\text{I.H.}}+(3k^2+3k+6)
\end{aligned}$$

We need to prove that $6|(3k^2 + 3k + 6)$ is true:

We have $(3k^2 + 3k + 6)$ which can be rewritten as $3k(k + 1) + 6$ then $3(k(k + 1) + 2)$.

From this we can say that $k(k + 1)$ is an even number because if k is odd, $k + 1$ is even and an even time an odd is even. If k is even, $k(k + 1)$ is even because if the multiples of a number consists an even number, the product is even.

If $k(k + 1)$ is even then $k(k + 1) + 2$ is also even as 2 is even and an even plus an even is also even. 3 times $(k(k + 1) + 2)$ has to be divisible by 6 $\forall k \geq 0 \in \mathbb{Z}$ as 3 times an even number is divisible by 6. This makes $6|(3 \cdot (k(k + 1) + 2))$ true $\forall k \geq 0 \in \mathbb{Z}$.
This in the makes makes out final claim of $6|(k^3 + 3k^2 + 3k + 5k + 6)$ true as the sum of two numbers $((k^3 + 5k)$ and $3k^2 + 3k + 6)$ that are each divisible by another number (6) is also divisible by that number (6).

$$\boxed{\text{QED}}$$

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22) $P(n)$ states that $1 + nx \leq (1 + x)^n$ is true $\forall n \geq 2 \in \mathbb{Z}$ and $\forall x > -1 \in \mathbb{R}$
Base Case: $n = 2$

$$\begin{aligned}
1 + nx &\leq (1+x)^n \\
1 + 2x &\leq (1+x)^2 \\
1 + 2x &\leq 1 + 2x + x^2 \\
0 &\leq x^2
\end{aligned}$$

This is true $\forall x > -1 \in \mathbb{R}$

Inductive Step: $\forall k \geq 2 \in \mathbb{Z}$ and $\forall x > -1 \in \mathbb{R}$, $\underbrace{P(k) \text{ states that } 1 + kx \leq (1+x)^k}_{\text{Inductive Hypothesis}}$

Now we need to show true for $P(k + 1)$

On the Next Page

Now we need to show true for $P(k+1)$

$$1+(k+1)x \leq (1+x)^{k+1} \tag{1}$$
$$1+kx+x \leq (1+x)^k \cdot (1+x)^1 \tag{2}$$

If we take a look at the I.H. and manipulate that by adding x to both sides...

$$1+kx+x \leq (1+x)^k \tag{3}$$

We need to show that

$$(1+x)^k + x \leq (1+x)^{k+1} \tag{4}$$
$$(1+x)^k + x \leq (1+x)^k \cdot (1+x) \tag{5}$$
$$(1+x)^k + x \leq (1+x)^k \cdot 1 + (1+x)^k \cdot x \tag{6}$$
$$x \leq (1+x)^k \cdot x \tag{7}$$
$$1 \leq (1+x)^k \tag{8}$$

This is true for $\forall x > -1 \in \mathbb{R}$ and $\forall k \geq 2 \in \mathbb{Z}$

$$\boxed{\text{This means that } 1+kx+x \leq (1+x)^k + x \leq (1+x)^{k+1} \text{ is true, which is what we wanted}}$$

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27) Given $d_1, d_2, d_3, ...$ and $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$, $\forall k \geq 2 \in \mathbb{Z}$

Prove the following, $P(n)$ states that $\forall n \geq 1 \in \mathbb{Z}$, $d_n = \frac{2}{n!}$ is true

Base Case: Let $n = 2$ so show $P(2)$; $d_2 = \frac{2}{2} = 1$ and $d_2 = \frac{2}{2!} = 1$; so the base case is true for the thing that we are trying to prove.

Inductive Step: Let $P(c)$ state that $\underbrace{d_c = \frac{2}{c!}}_{I.H.}$ is true $\forall c \geq 1 \in \mathbb{Z}$. This would make it so that $d_c = \frac{d_{c-1}}{c}$

Now we need to show $P(c+1)$

$d_{c+1} = \frac{2}{(c+1)!} = \frac{d_{c+1-1}}{c+1} = \frac{d_c}{c+1} = \frac{2/(c!)}{c+1} = \frac{2}{c!(c+1)} = \boxed{\frac{2}{(c+1)!}}$ Which is what we wanted.

This is the end of 5.3, 5.4 is on the next page

5.4 2, 7

2) Given that $b_1 = 4$, $b_2 = 12$ and $b_k = b_{k-2} + b_{k-1}$ $\forall k \geq 3 \in \mathbb{Z}$
Prove that $(4|b_k)$ is true.

Base Case: $(4|b_1) = (4|4)$, This is true. $(4|b_2) = (4|12)$, This is also true, so the base case holds true.

Inductive Step: Let the statement $P(c)$ be that $(4|b_c)$ whilst $b_c = b_{c-2} + b_{c-1}$ $\forall c \geq 1 \in \mathbb{Z}$. Suppose its true. This is the **Inductive Hypothesis**.

Now we need to show $P(c+1)$ and that its true. $P(c+1)$ would say that $b_{c+1} = b_{c-1} + b_c$ and that $(4|b_{c+1})$.
In b_{c-1}, we can concur that $c-1$ has to be greater than or equal to 1 so in this case, $c \geq 2$, which means that the lowest possible value for c is 2, which makes the lowest possible for $b_c = 12$ which is divisible by 4 as seen in the base case. so for any value greater than 2, b_c has b_2 in it and shows multiple iterations of it. So this MUST mean that b_c is divisible by 4 $\forall c \geq 1 \in \mathbb{Z}$ because a multiple of 4 is divisible by 4. $\boxed{\text{QED}}$

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7) Given the series $g_1, g_2, g_3, ...$ the initial conditions $g_1 = 3$, $g_2 = 5$ and the general equation $g_k = 3g_{k-1} - 2g_{k-2}$ $\forall k \geq 3 \in \mathbb{Z}$
Prove that $P(n)$, $g_n = 2^n + 1$, is true $\forall n \geq 1 \in \mathbb{Z}$

Base Case:
$P(1)$, $g_1 = 3$ (Given)
$P(1)$, $g_1 = 2^1 + 1 = 3$ - True for $n = 1$
$P(2)$, $g_2 = 5$ (Given)
$P(2)$, $g_2 = 2^2 + 1 = 5$ - True for $n = 2$

Inductive Step:
We need to suppose $P(c)$ which would say that $\underbrace{g_c = 2^c + 1 \quad \forall c \geq 1 \in \mathbb{Z}}_{\text{Inductive Hypothesis}}$

Now we need to show that $P(c+1)$

$$
\begin{aligned}
g_{c+1} &= 2^{c+1} + 1 \\
g_{c+1} &= 3 \cdot g_c - 2 \cdot g_{c-1} \\
g_{c-1} &= 2^{c-1} + 1 \\
g_{c+1} &= 3(2^c + 1) - 2(2^{c-1} + 1) \\
g_{c+1} &= 3 \cdot 2^c + 3 - 2^c - 2 \\
g_{c+1} &= 2 \cdot 2^c + 1 \\
g_{c+1} &= 2^{c+1} + 1
\end{aligned}
$$

$g_{c+1} = 2^{c+1} + 1$ is what we wanted, this is the Inductive Hypothesis but for c+1 instead of c

$\boxed{\text{Q.E.D.}}$

End of 5.4, 5.5 begins on the next page.

5.5 4, 12, 28, 42

4) Given $k(d_{k-1})^2$ is true $\forall k \geq 1 \in \mathbb{Z}$ and $d_0 = 3$. Find the first four terms.

$$\begin{aligned}
d_0 &= 3 \\
d_1 &= 1(d_0)^2 = 1(9) = 9 \\
d_2 &= 2(d_1)^2 = 2(81) = 162 \\
d_3 &= 3(d_2)^2 = 3(26244) = 78732
\end{aligned}$$

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12) Given the sequence $s_0, s_1, s_2...$ and let that be defined by $s_n = \dfrac{(-1)^n}{n!}$ $\forall n \geq 0 \in \mathbb{Z}$
Show that it satisfies $s_k = \dfrac{-s_{k-1}}{k}$.

<u>Base Case:</u> $P(1) = 1 = \dfrac{-s_0}{1}$ = true. Base case holds true for the claim $\forall k \geq 1 \in \mathbb{Z}$.

<u>Inductive Step:</u>
Let $P(c)$ state that $s_c = \dfrac{-s_{c-1}}{c}$ and suppose this is true $\forall c \geq 1 \in \mathbb{Z}$. We also need to consider that $s_c = \dfrac{(-1)^c}{c!}$ within the same bounds. If we check for s_{c+1} in the given equation, we get $\dfrac{(-1)^{c+1}}{(c+1)!}$
We need to show that $P(k+1)$ is true

$$\begin{aligned}
s_{c+1} &= \frac{-s_c}{c+1} \\
s_{c+1} &= \frac{-1 \cdot \dfrac{(-1)^c}{c!}}{c+1} \\
s_{c+1} &= \frac{(-1)^{c+1}}{c! \cdot (c+1)} \\
s_{c+1} &= \frac{(-1)^{c+1}}{(c+1)!}
\end{aligned}$$

$$\boxed{\text{This is what we wanted, Q.E.D.}}$$

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28) The question we are dealing with is a Fibonacci series so we can safely suppose that $F_{k+1} = F_k + F_{k-1}$
Prove the first statement $\forall k \geq 1 \in \mathbb{Z}$

$$\begin{aligned}
(F_{k+1})^2 - (F_k)^2 - (F_{k-1})^2 &= 2F_kF_{k-1} \\
n = 1; F_2^2 - F_1^2 - F_0^2 &= 2(F_1)(F_0) \\
2^2 - 1^2 - 1^2 &= 2(1)(1) \\
2 &= 2
\end{aligned}$$

This is true for the Base case, n=1

$$\begin{aligned}
(F_{k+1})^2 - (F_k)^2 - (F_{k-1})^2 &= 2F_kF_{k-1} \\
(F_k + F_{k-1})^2 - (F_k)^2 - (F_{k-1})^2 &= 2F_kF_{k-1} \\
(F_k)^2 + 2F_kF_{k-1} + (F_{k-1})^2 - (F_k)^2 - (F_{k-1})^2 &= 2F_kF_{k-1} \\
2F_kF_{k-1} &= 2F_kF_{k-1} \text{ - } \boxed{\text{This is what we wanted}}
\end{aligned}$$

42) Prove inductively $\prod_{i=1}^{n}(ca_i) = c^n \prod_{i=1}^{n}(a_i)$

Base Case: $i = 1$, $ca_1 = c(a_1)$, works for the base case.

Inductive Step: $\underbrace{(ca_1) \cdot (ca_2) \cdot (ca_3) \cdots (ca_n) = c^n(a_1 \cdot a_2 \cdot a_3 \cdots a_n)}_{\text{Inductive Hypothesis}}$

Show true for $n+1$, $c^{n+1} \prod_{i=1}^{n+1} a_i$

$$\begin{aligned}
(ca_1) \cdot (ca_2) \cdot (ca_3) \cdots (ca_n) \cdot (ca_{n+1}) &= c^{n+1}(a_1 \cdot a_2 \cdot a_3 \cdots a_n \cdot a_{n+1}) \\
c^n(a_1 \cdot a_2 \cdot a_3 \cdots a_n) \cdot (ca_{n+1}) &= c^{n+1}(a_1 \cdot a_2 \cdot a_3 \cdots a_n \cdot a_{n+1}) \\
c^n \cdot c \cdot (a_1 \cdot a_2 \cdot a_3 \cdots a_n \cdot a_{n+1}) &= c^{n+1}(a_1 \cdot a_2 \cdot a_3 \cdots a_n \cdot a_{n+1}) \\
c^{n+1} \prod_{i=1}^{n+1} a_i &= c^{n+1} \prod_{i=1}^{n+1} a_i
\end{aligned}$$

$$\boxed{\text{This is what we wanted}}$$

5.6 8, 13, 38

8) $f_k = f_{k-1} + 2^k$ and $f_1 = 1$

$$\begin{aligned}
f_1 &= 1 \\
f_2 &= 1 + 2^2 = 5 \\
f_3 &= 5 + 2^3 = 13 \\
f_4 &= 13 + 2^4 = 29 \\
f_5 &= 29 + 2^5 = 61
\end{aligned}$$

Claim: $f_k = 2^{k+1} - 3$

Base Case: $k = 1$ so $f_1 = 1 = 2^2 - 1$, this works so its valid for the base case.

Inductive Step: Suppose $f_k = 2^{k+1} - 3$, we want to show $f_{k+1} = 2^{k+2} - 3$

$$\begin{aligned}
f_k &= 2^{k+1} - 3 \\
f_{k+1} &= f_k + 2^{k+1} \\
f_{k+1} &= 2^{k+1} - 3 + 2^{k+1} \\
f_{k+1} &= 2 \cdot 2^{k+1} - 3 \\
f_{k+1} &= 2^{k+2} - 3
\end{aligned}$$

$$\boxed{\text{This is what we wanted, Q.E.D}}$$

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13 and 38) $t_k = t_{k-1} + 3k + 1$ and $t_0 = 0$

$$\begin{aligned} t_0 &= 0 \\ t_1 &= 0 + 3(1) + 1 = 4 \\ t_2 &= 4 + 3(2) + 1 = 11 \\ t_3 &= 11 + 3(3) + 1 = 21 \\ t_4 &= 21 + 3(4) + 1 = 34 \\ t_5 &= 34 + 3(5) + 1 = 50 \end{aligned}$$

Claim: $t_k = 3 \cdot \left(\frac{k \cdot (k+1)}{2} \right) + k$

Base Case: $k = 1$ so $t_1 = 3 \cdot \left(\frac{1 \cdot (1+1)}{2} \right) + 1 = 4$, this works so its valid for the base case.

38) Inductive Step: Suppose $t_k = 3 \cdot \left(\frac{k \cdot (k+1)}{2} \right) + k$, we want to show $t_k = 3 \cdot \left(\frac{(k+1) \cdot (k+2)}{2} \right) + (k+1)$

Going back to the sequence that we were given

$$\begin{aligned} t_{k+1} &= t_k + 3k + 3 + 1 \\ t_{k+1} &= 3\left(\frac{k(k+1)}{2} \right) + k + 3k + 3 + 1 \\ t_{k+1} &= \frac{3k^2 + 11k + 8}{2} \end{aligned}$$

Going back to the Inductive Hypothesis and the Inductive Step

$$\begin{aligned} t_k &= 3 \cdot \left(\frac{(k+1) \cdot (k+2)}{2} \right) + (k+1) \\ t_{k+1} &= \frac{3 \cdot (k^2 + 3k + 2)}{2} + \frac{2 \cdot (k+1)}{2} \\ t_{k+1} &= \frac{3k^2 + 11k + 8}{2} \end{aligned}$$

This is what we wanted $\boxed{\text{Q.E.D.}}$
