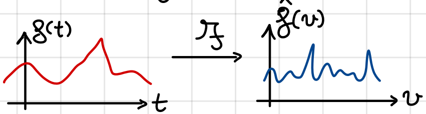
# Tutoraggio Fourier - 14 maggio 2025

# Trasformata di Fourier

- funzione $f(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \hat{f}(\nu)$ $\qquad \mathcal{F}\{f(t)\} = \hat{f}(\nu) = \int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i \nu t}\, dt$




- distribuzione $T(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \hat{T}(\nu)$ $\qquad \mathcal{F}\{T\} = \langle \mathcal{F}\{T\}, \varphi \rangle = \langle T, \mathcal{F}\{\varphi\} \rangle$

$\mathcal{F}$-trasformabile?

**funzione $f(t)$**

funzione sommabile?

$$\int_{-\infty}^{+\infty} f(t)\,dt < \infty \ \wedge \int_{-\infty}^{+\infty} |f(t)|\,dt < \infty$$

- SÌ → ✓
- NO → $T_f$ → distribuzione $T$

**distribuzione $T$**

temperata?

(1) $T_f$ : $f$ sommabile

(2) $T_f$ : $f$ a crescita lenta

$\exists c \ \exists p \quad |f(t)| \leq c\,|1+|x|^p|$

(2.1) $f$ é un polinomio

(2.2) $f$ é limitata

(3) $T$ a supporto compatto

- SÌ → ✓
- NO → ✗

$$\mathcal{F}\{\mathcal{F}\{g(t)\}\} = g(-\nu)$$

ex

$$g(t) = \frac{1}{8+2\pi i t} = \mathcal{F}\{H(t)e^{-8t}\}$$

$$\mathcal{F}\{g(t)\} = \mathcal{F}\{\mathcal{F}\{H(t)e^{-8t}\}\} = H(-\nu)e^{8\nu}$$

ex

$$g(t) = e^{-2t^2+4t} = e^{-2(t^2-2t)} = e^{-2(t^2-2t+1-1)}$$

$$= e^{-2(t^2-2t+1)+2} = e^{-2(t-1)^2+2} = e^2 e^{-2(t-1)^2}$$

$$\mathcal{F}\{g(t)\}(\nu) = e^2\, \mathcal{F}\{e^{-2(t-1)^2}\}(\nu)$$

(b) $\mathcal{F}(T(t-t_0))(\nu) = e^{-2\pi i t_0 \nu}\,\mathcal{F}(T(t))(\nu)$ $\quad (t_0 \in \mathbb{R})$

$$\underset{t_0=1}{=} e^{-2\pi i \nu} e^2\, \mathcal{F}\{e^{-2t^2}\}(\nu) = e^{-2\pi i \nu} e^2 \sqrt{\frac{\pi}{2}}\, e^{-\pi^2\nu^2/2}$$

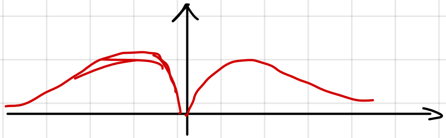
$$= \sqrt{\frac{\pi}{2}}\, e^{-2\pi i \nu + 2 - \pi^2\nu^2/2}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-\frac{\pi^2}{2}\left(\frac{4i\nu}{\pi} - \frac{4}{\pi^2} + \nu^2\right)}$$

quadrato di binomio

$$= \sqrt{\frac{\pi}{2}}\, e^{-\frac{\pi^2}{2}\left(\nu + \frac{2i}{\pi}\right)^2}$$

ex $g(t)=|t|e^{-|t|}$



$$g(t)=\begin{cases} te^{-t} & t\geq 0 \quad (\cdot H(t)) \\ -te^{t} & t<0 \quad (\cdot H(-t)) \end{cases}$$

$$g(t)=\underline{H(t)\cdot te^{-t}}\ \underline{-H(-t)\cdot te^{t}}$$

$\#$ $$\mathcal{F}\{t\cdot H(t)e^{-t}\}(\nu) \underset{\uparrow}{=} \left(-\frac{1}{2\pi i}\right)\mathcal{F}\{H(t)e^{-t}\}'(\nu)$$

(d) $\mathcal{F}(t^kT(t))(\nu) = \left(-\frac{1}{2\pi i}\right)^k (\mathcal{F}(T))^{(k)}(\nu)$ $\quad K=1$

$$=\left(-\frac{1}{2\pi i}\right)\frac{d}{d\nu}\left(\frac{1}{1+2\pi i\nu}\right)$$

$$=\left(-\frac{1}{\cancel{2\pi i}}\right)\frac{-\cancel{2\pi i}}{(1+2\pi i\nu)^2}=\frac{1}{(1+2\pi i\nu)^2}$$

$$\mathcal{F}\{-H(-t)\cdot te^{t}\} \underset{\substack{\uparrow \\ a=-1}}{=} \mathcal{F}\{H(t)te^{-t}\}(-\nu) = \frac{1}{(1-2\pi i\nu)^2}$$

(c) $\mathcal{F}(T(at))(\nu) = \frac{1}{|a|}\mathcal{F}(T(t))\left(\frac{\nu}{a}\right)$

$$\mathcal{F}\{g(t)\} = \frac{1}{(1+2\pi i\nu)^2} + \frac{1}{(1-2\pi i\nu)^2}$$

**ex**

$$g(t) = \frac{t}{(9+4t^2)^2} = -\frac{1}{8}\frac{-8t}{(9+4t^2)^2} = -\frac{1}{8}\frac{d}{dt}\frac{1}{9+4t^2}$$

$$= -\frac{1}{8}\frac{d}{dt}\frac{1}{4(9/4+t^2)} = -\frac{1}{32}\frac{d}{dt}\frac{1}{9/4+t^2}$$

$$\mathcal{F}\{g(t)\} = -\frac{1}{32}\mathcal{F}\left\{\frac{d}{dt}\frac{1}{9/4+t^2}\right\} = -\frac{1}{32}\mathcal{F}\left\{\left(\frac{1}{9/4+t^2}\right)^{(1)}\right\}$$

(e) $\mathcal{F}(T^{(k)})(\nu) = (2\pi i\nu)^k\,\mathcal{F}(T)(\nu)$

$$= -\frac{1}{32}(2\pi i\nu)\,\mathcal{F}\left\{\frac{1}{9/4+t^2}\right\}(\nu)$$

$$= -\frac{1}{32}(2\pi i \nu)\cdot\frac{2\pi}{3}e^{-3\pi|\nu|} = -\frac{\pi^2 i\nu}{24}e^{-3\pi|\nu|}$$

$a = 3/2$

**ex** $g(t) = \cos(2t+1)$

$$\boxed{\cos(t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}}$$

$$= \frac{1}{2}e^{i(2t+1)} + \frac{1}{2}e^{-i(2t+1)}$$

$$= \frac{1}{2}e^{2it}\cdot e^{i} + \frac{1}{2}e^{-2it}\cdot e^{-i}$$

$$\mathcal{F}\{g(t)\} = \frac{1}{2}e^{i}\mathcal{F}\{e^{2it}\}(\nu) + \frac{1}{2}e^{-i}\mathcal{F}\{e^{-2it}\}(\nu)$$

| $e^{2\pi i x_0 t}$ | $\delta_{x_0}$ |
|---|---|

$$= \frac{1}{2}e^{i}\mathcal{F}\{e^{2it\frac{\pi}{\pi}}\}(\nu) + \frac{1}{2}e^{-i}\mathcal{F}\{e^{-2it\frac{\pi}{\pi}}\}(\nu)$$

$x_0 = 1/\pi$ $\qquad$ $x_0 = -1/\pi$

$$= \frac{1}{2}e^{i}\delta_{1/\pi} + \frac{1}{2}e^{-i}\delta_{-1/\pi} = \frac{1}{2}(e^{i}\delta_{1/\pi} + e^{-i}\delta_{-1/\pi})$$

ex

$$g(t) = \frac{K+t^2}{1+t^2} = \frac{K-1+1+t^2}{1+t^2} = \frac{(K-1)}{1+t^2} + 1$$

$$\mathcal{F}\{g(t)\}(\nu) = (K-1)\mathcal{F}\left\{\frac{1}{1+t^2}\right\} + \mathcal{F}\{1\}(\nu)$$

$$= (K-1)\pi e^{-2\pi|\nu|} + \mathcal{F}\{1\}(\nu)$$

$$= (K-1)\pi e^{-2\pi|\nu|} + \mathcal{F}\{e^{2\pi i\cdot\emptyset\cdot t}\}(\nu)$$

$$= (K-1)\pi e^{-2\pi|\nu|} + \delta_0$$

ex

$$T = t^4\delta_2 + \delta_0'' = t^4\delta_2 + \delta_0^{(2)}$$

$$= 2^4\delta_2 + \delta_0^{(2)}$$

$$\mathcal{F}\{T\}(\nu) = 16\mathcal{F}\{\delta_2\}(\nu) + \mathcal{F}\{\delta_0^{(2)}\}$$

$$= 16\, e^{-2\pi i \cdot 2 \cdot \nu} + \mathcal{F}\{\delta_0^{(2)}\}(\nu)$$

(e) $\mathcal{F}(T^{(k)})(\nu) = (2\pi i\nu)^k\, \mathcal{F}(T)(\nu)$

$$= 16\, e^{-4\pi i \nu} + (2\pi i \nu)^2\, \mathcal{F}\{\underbrace{\delta_0}_{1}\}(\nu)$$

$$= 16\, e^{-4\pi i \nu} - 4\pi^2 \nu^2$$

ex $\quad g(t) = t \cdot H(t)$

(d) $\mathcal{F}(t^k T(t))(\nu) = \left(-\frac{1}{2\pi i}\right)^k (\mathcal{F}(T))^{(k)}(\nu)$

$$\mathcal{F}\{g(t)\} = -\frac{1}{2\pi i}\, \mathcal{F}\{H(t)\}'(\nu)$$

$\frac{1}{2\pi i}\,\text{v.p.}\,\frac{1}{\nu} + \frac{\delta_0}{2}$

$$= -\frac{1}{2\pi i}\, \frac{d}{d\nu}\left(\frac{1}{2\pi i}\, v.p.\, \frac{1}{\nu} + \frac{\delta_0}{2}\right)$$

$$= \frac{1}{4\pi^2}\, v.p.'\, \frac{1}{\nu} + \frac{i}{4\pi}\, \delta_0'$$