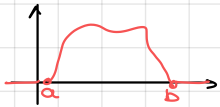
Tutoraggio distribuzioni - 9 aprile 2025

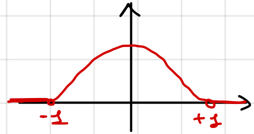
Spazio funzioni test $\mathcal{D}(\mathbb{R})$

$\varphi: \mathbb{R} \to \mathbb{R}$ (i) $\varphi \in C^\infty(\mathbb{R})$

(ii) φ supporto compatto

$\exists a,b : \varphi(x)=0 \quad \forall x \notin [a,b]$





Funzioni $f(x): \mathbb{R} \to \mathbb{R}$

Distribuzioni $T: \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ $\quad T = \int_0^1 \varphi'(x)\,dx$

$T \quad T(\varphi) \quad \langle T, \varphi \rangle$

(i) T lineare: $\langle T, a\varphi + b\psi \rangle = a\langle T,\varphi\rangle + b\langle T,\psi\rangle$

(ii) T continua

$$\langle T,\varphi\rangle = \int_0^1 \varphi'(x)\,dx$$

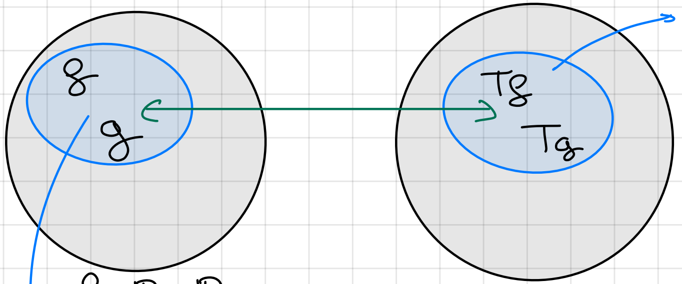
$$\langle T, a\varphi+b\psi\rangle = \int_0^1 (a\varphi+b\psi)'\,dx = \int_0^1 a\varphi'+b\psi'\,dx = \int_0^1 a\varphi'\,dx + \int_0^1 b\psi'\,dx = a\int_0^1 \varphi'\,dx + b\int_0^1 \psi'\,dx = a\langle T,\varphi\rangle + b\langle T,\psi\rangle$$

$$\langle T,\varphi\rangle = \varphi(x)^2$$

$$\langle T, a\varphi+b\psi\rangle = (a\varphi(x)+b\psi(x))^2 = a^2\varphi^2(x) + b^2\psi^2(x) + 2ab\,\varphi(x)\psi(x)$$

$$= a^2\langle T,\varphi\rangle + b^2\langle T,\psi\rangle + \dots$$

Distribuzioni regolari



distribuzioni regolari

$f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $T: \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}$

$f(x)$

$$\langle T_f, \varphi \rangle = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx$$

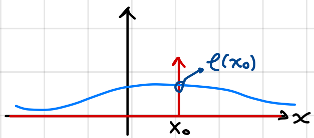
funzioni localmente sommabili

$$\int_a^b f(x)\,dx \,, \quad \int_a^b |f(x)|\,dx < \infty$$

$$\langle T, \varphi \rangle = \int_{-\infty}^{+\infty} \underset{\substack{\downarrow \\ f(x)}}{\sin(x)}\, \varphi(x)\,dx$$

Delta di Dirac

δ_{x_0} $\qquad$ $\langle \delta_{x_0}, \varphi(x) \rangle = \varphi(x_0)$



$$\langle \delta_2, \varphi(x) \rangle = \varphi(2)$$

Operazioni

TRASLAZIONE $\langle T(x-a),\varphi\rangle = \langle T,\varphi(x+a)\rangle$

RISCALAMENTO $\langle T(ax),\varphi\rangle = \langle T,\frac{1}{|a|}\varphi(\frac{x}{a})\rangle$

MOLTIPLICAZIONE PER $h\in C^\infty$ $\langle hT,\varphi\rangle = \langle T, h\varphi\rangle$
$h(x)$

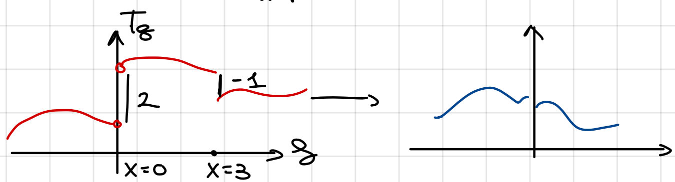
$$e^x\delta_0 = e^0\delta_0 = \delta_0$$

$$(\sin(x)+x^3+e^x)\delta_0 = \delta_0$$

Derivata

$f \to f'$ $\quad T \to T'$ $\quad$ DISTR. $\langle T',\varphi\rangle = -\langle T,\varphi'\rangle$

DISTR. REGOLARI $\langle T_f',\varphi\rangle = T_{f'} + \sum_{k=1}^{m}[f(x_k^+)-f(x_k^-)]\delta_{x_k}$

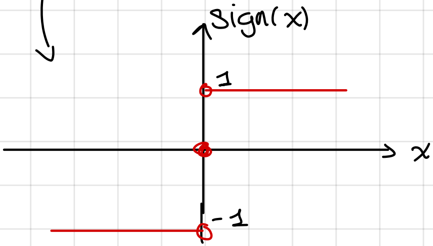


$$+2\delta_0 - 1\delta_3$$

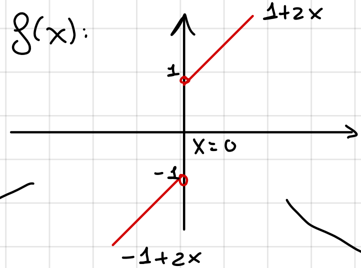
ex

$f(x) = \text{sign}(x) + 2x$

$\langle T_f, \varphi \rangle$ $\quad \langle T'_f, \varphi \rangle = T_{f'} + \sum_{k=1}^{m} [f(x_k^+) - f(x_k^-)]\delta_{x_k}$



$f(x)$:

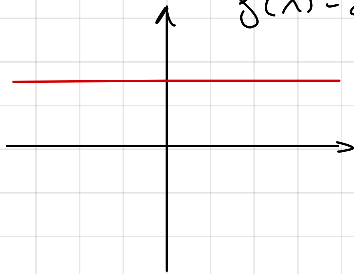


$x > 0 \quad 1 + 2x$

$x < 0 \quad -1 + 2x$

disc. in $x = 0$

$f'(x) = 2$



$[f(0^+) - f(0^-)] = (1 - (-1))\delta_0$

$= 2\delta_0$

$T_{f'} = T_2$

$\langle T'_f, \varphi \rangle = T_2 + 2\delta_0$

ex

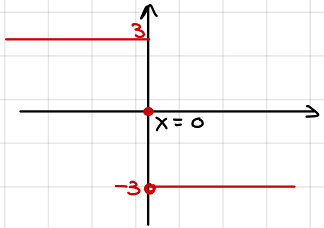
$T = e^{x^2}\delta_{-1} + T_{-3\,\mathrm{sign}(x)}$

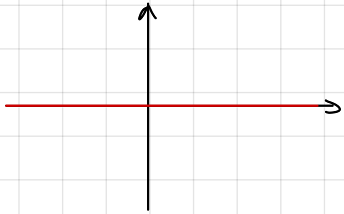
$T' = e\delta'_{-1} + \cancel{T_0}^{\emptyset} - 6\delta_0$

$(e^{x^2}\delta_{-1})' = (e^{(-1)^2}\delta_{-1})' = (e\delta_{-1})' = e\delta'_{-1}$

$g(x) = -3\,\mathrm{sign}(x)$

$g'(x) = \emptyset$





DISC. $x = 0$: $(g(0^+) - g(0^-))\delta_0 = (-3-3)\delta_0 = -6\delta_0$

ex $T = \langle \sin(x)\delta_0'', \varphi(x) \rangle$ $\qquad \langle T', \varphi \rangle = -\langle T, \varphi' \rangle$

$$T = \langle \delta_0'', \sin(x)\varphi(x) \rangle = -\langle \delta_0', (\sin(x)\varphi(x))' \rangle = -\langle \delta_0', \sin'(x)\varphi(x) + \sin(x)\varphi'(x) \rangle$$

$$= \langle \delta_0, (\sin'(x)\varphi(x) + \sin(x)\varphi'(x))' \rangle$$

$$= \langle \delta_0, \sin''(x)\varphi(x) + \sin'(x)\varphi'(x) + \sin'(x)\varphi'(x) + \sin(x)\varphi''(x) \rangle$$

$$= \langle \delta_0, -\sin(x)\varphi(x) + \cos(x)\varphi'(x) + \cos(x)\varphi'(x) + \sin(x)\varphi''(x) \rangle$$

$\varphi'(0) = -2$

$$= 0\varphi(0) + \varphi'(0) + \varphi'(0) + 0\varphi''(0) = 2\varphi'(0) = 2(-2) = -4$$

Convergenza di distribuzioni

T_n , $n \in \mathbb{N}$ Si dice che $T_n \to T$
se $\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \ \forall \varphi$

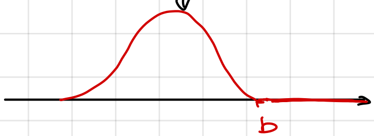
(i) $\varphi(x)$ supp. compatto

(ii) $\langle \delta_{x_0}, \varphi(x) \rangle = \varphi(x_0)$

ex $\langle T_n, \varphi \rangle = \langle n^n \delta_n, \varphi \rangle = n^n \langle \delta_n, \varphi \rangle = n^n \varphi(n)$

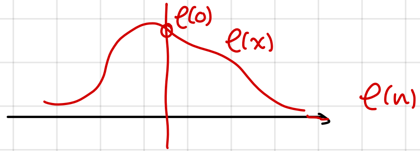
$$\lim_{n\to+\infty} \langle T_n, \varphi \rangle = \lim_{n\to+\infty} n^n \overset{\emptyset}{\varphi(n)} = n^n \cdot \emptyset = \emptyset$$

$$\boxed{\lim_{n\to+\infty} \varphi(n) = \emptyset}$$



ex $T_n = n(\delta_{1/n}+\delta_0) = \underline{n\delta_{1/n}} + \underline{n\delta_0}$ non converge

* $\lim\limits_{n\to+\infty} n\delta_0 = \lim\limits_{n\to+\infty} n\langle\delta_0,\varphi\rangle = n\varphi(0) = +\infty$



* $\lim\limits_{n\to+\infty} n\delta_{1/n} = \lim\limits_{n\to+\infty} n\langle\delta_{1/n},\varphi\rangle = n\varphi(1/n) = n\varphi(0) = +\infty$

ex

$$g_n = n \cdot P_{[-2/n,+2/n]}(x)$$

$$T_{g_n} \xrightarrow{?} T$$

$$\int_a^b g(x)\,dx = g(c)\cdot(b-a)$$
$$c \in [a,b]$$

$$T_{g_n} = \int_{-\infty}^{+\infty} g_n(x)\varphi(x)\,dx = \int_{-\infty}^{+\infty} n \cdot P_{[-2/n,+2/n]}(x)\varphi(x)\,dx = \int_{-2/n}^{+2/n} n\varphi(x)\,dx = n\int_{-2/n}^{+2/n} \varphi(x)\,dx$$

$$= n\underbrace{\varphi(c)}_{c\,\in[-2/n,+2/n]}\cdot\left(\frac{2}{n}-\left(-\frac{2}{n}\right)\right) = \cancel{n}\varphi(c)\cdot\frac{4}{\cancel{n}} = 4\underbrace{\varphi(c)}_{c\,\in[-2/n,+2/n]} \xrightarrow{+\infty} 4\varphi(0) = 4\delta_0 \qquad \text{CONVERGE}$$

$n \to +\infty \quad \in [0,0] = \emptyset$