# Tutoraggio Laplace - 04 giugno 2025

# Trasformata di Laplace

$f(t) \xrightarrow{\mathcal{L}} F(s)$

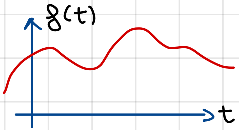
dominio $t \in \mathbb{R}$ — dominio $s \in \mathbb{C}$

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} f(t)\, e^{-st}\, dt$$

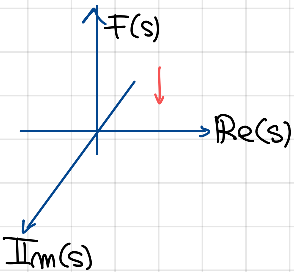
$\mathcal{L}$-trasformabile?

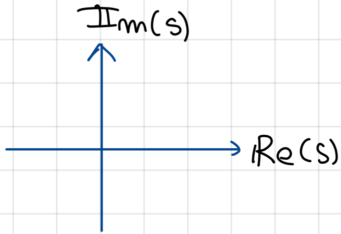
- funzione $f$
  - ↓ $f(t)$ localmente sommabile
  - $\int_a^b f(t)\,dt < +\infty$
  - $\int_a^b |f(t)|\,dt < +\infty$
- distribuzione $T$
  1. $T$ a supporto compatto $\longrightarrow$ trasf.
  2. $T_f$ : $f$ é trasformabile
     ↓
     $T_f$ é trasformabile

## Dominio

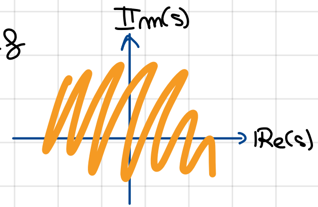


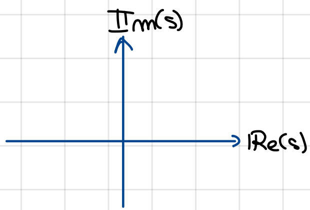

$\xrightarrow{\mathcal{L}}$



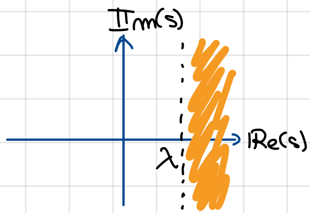




$\Omega_f$

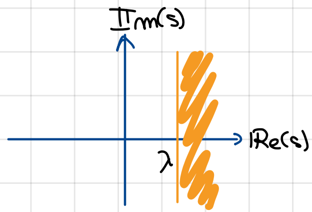



$\Omega_f : \mathrm{Re}(s) > \lambda$




$\Omega_f : \mathrm{Re}(s) \geq \lambda$



$f(t) = \sin(\omega t) H(t)$

$F(s) = \mathcal{L}\{f(t)\} = \dfrac{\omega}{s^2+\omega^2} \qquad \mathrm{Re}(s) > 0$

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Trova l'insieme $\Omega_f$ e la trasformata di Laplace di:

$$f(t) = t \cdot sin(\omega t) \cdot H(t)$$

Quale risposta è corretta?

1. Non posso trasformare la funzione
2. $\Omega_f = \{s \in \mathbb{C} : \text{Re(s)} > 0\}, \mathscr{L}[f(t)](s) = \frac{2\omega s}{(s^2+\omega^2)^2}$
3. $\Omega_f = \{s \in \mathbb{C} : \text{Re(s)} > 0\}, \mathscr{L}[f(t)](s) = \frac{2\omega}{(s^2+\omega^2)^2}$
4. $\Omega_f = \{s \in \mathbb{C} : \text{Re(s)} > 1\}, \mathscr{L}[f(t)](s) = \frac{2\omega}{(s^2+\omega^2)^2}$

$f(t) = t \cdot \sin(\omega t) \cdot H(t)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{t \cdot \sin(\omega t) H(t)\} = -\mathcal{L}\{\sin(\omega t) H(t)\}' = -\left(\dfrac{\omega}{s^2+\omega^2}\right)' = +\dfrac{2s\omega}{(s^2+\omega^2)^2}$

(d) $[\mathcal{L}(T(t))]'(s) = -\mathcal{L}(tT(t))(s),$ $\longleftarrow$

Trova l'insieme $\Omega_f$ e calcola la trasformata di Laplace di:

$$f(t) = (t-3) \cdot H(t-2) \cdot e^{t+1}$$

Quale risposta è corretta?

1. $\lambda_f = 1, \mathscr{L}[f(t)](s) = e^{3-2s} \cdot \frac{2-s}{(s-1)^2}$
2. $\lambda_f = 0, \mathscr{L}[f(t)](s) = e^{3-2s} \cdot \frac{2-s}{(s-1)^2}$
3. $\lambda_f = 1, \mathscr{L}[f(t)](s) = e^{3-3s} \cdot \frac{2-s}{(s-2)^2}$
4. $\lambda_f = 0, \mathscr{L}[f(t)](s) = e^{3-3s} \cdot \frac{2-s}{(s-2)^2}$

$$g(t) = (t-3)H(t-2)e^{t+1} = e\,(t-3)H(t-2)\,e^{t} = e(t-3)H(t-3+1)e^{t-3+3}$$

$$\mathscr{L}(g(t-3))(s) = e^{-3s}\mathscr{L}(e^4 t\, H(t+1)\, e^t)(s) = e^{4-3s}\mathscr{L}((t+1-1)H(t+1)e^{t+1-1})(s) = e^{4-3s}e^{s}\mathscr{L}((t-1)H(t)e^{t-1})(s)$$

$$e^{3-2s}\mathscr{L}(tH(t)e^t)(s) - e^{3-2s}\mathscr{L}(H(t)e^t)(s) = e^{3-2s}(-1)\mathscr{L}(H(t)e^t)'(s) - e^{3-2s}\mathscr{L}(H(t)e^t)(s)$$

$$-e^{3-2s}\left[\frac{1}{s-1}\right]' - e^{3-2s}\left[\frac{1}{s-1}\right] = e^{3-2s}\frac{1}{(s-1)^2} - e^{3-2s}\frac{1}{s-1} = e^{3-2s}\left[\frac{1-(s-1)}{(s-1)^2}\right] = e^{3-2s}\frac{2-s}{(s-1)^2}$$

$\mathrm{Re}\, s > 1$

Trova l'insieme $\Omega_f$ e calcola la trasformata di Laplace di:

$$f(t) = H(t) \cdot e^{t-1} + H(t-2) \cdot cos(t)$$

Quale risposta è corretta?

1. $\lambda_f = 1, \mathscr{L}[f(t)](s) = \frac{1}{e} \cdot \frac{1}{s-1} + cos(2) \cdot e^{-2s} \cdot \frac{s}{s^2+1} - sin(2) \cdot e^{-2s} \cdot \frac{1}{(s-2)^2+1}$
2. $\lambda_f = 0, \mathscr{L}[f(t)](s) = \frac{1}{e} \cdot \frac{1}{s-1} + cos(2) \cdot e^{-2s} \cdot \frac{s}{s^2+1} - sin(2) \cdot e^{-2s} \cdot \frac{1}{(s-2)^2+1}$
3. $\lambda_f = 0, \mathscr{L}[f(t)](s) = \frac{1}{e} \cdot \frac{1}{s-1} + cos(2) \cdot e^{-2s} \cdot \frac{s}{s^2+1} - sin(2) \cdot e^{-2s} \cdot \frac{1}{s^2+1}$
4. $\lambda_f = 1, \mathscr{L}[f(t)](s) = \frac{1}{e} \cdot \frac{1}{s-1} + cos(2) \cdot e^{-2s} \cdot \frac{s}{s^2+1} - sin(2) \cdot e^{-2s} \cdot \frac{1}{s^2+1}$

$$f(t) = H(t)e^{t-1} + H(t-2)\cos(t-2+2)$$
$$= H(t)e^t e^{-1} + H(t-2)[\cos(t-2)\cos(2) - \sin(t-2)\sin(2)]$$
$$= \frac{1}{e}H(t)e^t + \cos(2)H(t-2)\cos(t-2) - \sin(2)H(t-2)\sin(t-2)$$

$$\mathscr{L}\{f(t)\} = \frac{1}{e}\frac{1}{s-1} + \cos(2)\underbrace{e^{-2s}}_{\text{traslaz.}}\mathscr{L}\{H(t)\cos(t)\} - \sin(2)\underbrace{e^{-2s}}_{\text{trasl.}}\mathscr{L}\{H(t)\sin(t)\}$$
$$= \frac{1}{e}\frac{1}{s-1} + \cos(2)\,e^{-2s}\frac{s}{s^2+1} - \sin(2)e^{-2s}\frac{1}{s^2+1}$$

Calcola la trasformata di Laplace di:

$$T = t^4 \cdot \delta_3$$

Quale risposta è corretta?

1. $L[T](s) = 81 \cdot e^{3s}$
2. $L[T](s) = -81 \cdot e^{3s}$
3. $L[T](s) = 81 \cdot e^{-3s}$
4. $L[T](s) = -81 \cdot e^{-3s}$

$T = t^4 \delta_3 = 3^4 \delta_3$

$\mathcal{L}\{T\} = 3^4 e^{-3s} = 81 e^{-3s}$

Calcola la anti trasformata di Laplace di:

$$F(s) = \frac{s^2 + s + 4}{s^3 + 3s^2 + s + 3}$$

Quale risposta è corretta?

1. $[e^{3t} + cos(t)] \cdot H(t)$
2. $[e^{3t} + cos(t)] \cdot H(t-3)$
3. $[e^{-3t} + cos(t)] \cdot H(t-3)$
4. $[e^{-3t} + sin(t)] \cdot H(t)$

$$F(s) = \frac{s^2+s+4}{s^3+3s^2+s+3} = \frac{s^2+s+4}{s^2(s+3)+(s+3)} = \frac{s^2+s+4}{(s^2+1)(s+3)}$$

$$= \frac{s^2+s+4}{(s+j)(s-j)(s+3)}$$

$$F(s) = \frac{s^2+s+4}{(s+j)(s-j)(s+3)} = \frac{A}{s+j} + \frac{B}{s-j} + \frac{C}{s+3}$$

$$A = F(s)(s+j)\Big|_{-j} = \frac{s^2+s+4}{\cancel{(s+j)}(s-j)(s+3)}\cancel{(s+j)}\Big|_{-j} = \frac{\cancel{-1-j+4}}{(-2j)\cancel{(3-j)}} = \frac{1\cdot j}{-2j\cdot j} = j/2$$

$$B = F(s)(s-j)\Big|_{j} = \frac{s^2+s+4}{(s+j)(s+3)}\Big|_{j} = \frac{\cancel{-1+j+4}}{(2j)\cancel{(3+j)}} = -j/2$$

$$C = F(s)(s+3)\Big|_{-3} = \frac{s^2+s+4}{(s+j)(s-j)}\Big|_{-3} = \frac{9-3+4}{(-3+j)(-3-j)} = \frac{10}{10} = 1$$

$$F(s) = j/2\cdot\frac{1}{s+j} - j/2\cdot\frac{1}{s-j} + \frac{1}{s+3}$$

$$e^{s_0 t}H(t) \longleftarrow \Big|\ \frac{1}{s-s_0}$$

$$f(t) = \left(j/2\, e^{-jt} - j/2\, e^{jt} + e^{-3t}\right)H(t)$$

$$= \left(e^{-3t} - \frac{j}{2}\underbrace{\left(e^{jt} - e^{-jt}\right)}_{2j\sin(t)}\right)H(t) = \left(e^{-3t} + \sin(t)\right)H(t)$$

Calcola la anti trasformata di Laplace di:

$$F(s) = \frac{s^2 - 3s + 4}{(s+1)(s-2)^2}$$

Quale risposta è corretta?

1. $\frac{8}{9} \cdot e^{-t} \cdot H(t) + \frac{1}{9} \cdot e^{2t} \cdot H(t) + \frac{2}{3} \cdot e^{2t} \cdot t$
2. $\frac{1}{9} \cdot e^{-t} \cdot H(t) + \frac{8}{9} \cdot e^{2t} \cdot H(t) + \frac{2}{3} \cdot e^{2t} \cdot t \cdot H(t)$
3. $\frac{8}{9} \cdot e^{2t} \cdot H(t) + \frac{1}{9} \cdot e^{-t} \cdot H(t) + \frac{2}{3} \cdot e^{-t} \cdot t \cdot H(t)$
4. $\frac{8}{9} \cdot e^{-t} \cdot H(t) + \frac{1}{9} \cdot e^{2t} \cdot H(t) + \frac{2}{3} \cdot e^{2t} \cdot t \cdot H(t)$

$$F(s) = \frac{A}{s+1} + \underbrace{\frac{B_1}{(s-2)} + \frac{B_2}{(s-2)^2}}_{\text{polo multiplo}}$$

$$A = F(s)(s+1)\Big|_{s=-1} = \frac{s^2-3s+4}{(s-2)^2}\Big|_{s=-1} = 8/9$$

$$B_1 = \frac{1}{(2-1)!}\left(\frac{d}{ds}\left[(s-2)^2 F(s)\right]\right)\Big|_{s=2} = \frac{d}{ds}\,\frac{s^2-3s+4}{s+1}\Big|_{s=2} = 1/9$$

$$B_2 = \frac{1}{(2-2)!}(s-2)^2 F(s)\Big|_{s=2} = F(s)(s-2)^2\Big|_{s=2} = 2/3$$

$$F(s) = 8/9\,\frac{1}{s+1} + 1/9\,\frac{1}{s-2} + 2/3\,\frac{1}{(s-2)^2}$$

$$f(t) = \left(8/9\,e^{-t} + 1/9\,e^{2t} + 2/3\,t e^{2t}\right)H(t)$$

Calcola la anti trasformata di Laplace di:

$$F(s) = \frac{s^3 + s^2 + 2s + 3}{s^3 - 4s^2 - 15s + 18}$$

Quale risposta è corretta?

1. $\delta_0 - \frac{7}{20} \cdot e^{t} \cdot H(t) - \frac{7}{12} \cdot e^{-3t} \cdot H(t) + \frac{89}{15} \cdot e^{-6t} \cdot H(t)$
2. ~~2.~~ $-\frac{7}{20} \cdot e^{t} \cdot H(t) - \frac{7}{12} \cdot e^{-3t} \cdot H(t) + \frac{89}{15} \cdot e^{6t} \cdot H(t)$
3. $\delta_0 - \frac{7}{20} \cdot e^{t} \cdot H(t) - \frac{7}{12} \cdot e^{-3t} \cdot H(t) + \frac{89}{15} \cdot e^{6t} \cdot H(t)$
4. $-\frac{7}{20} \cdot e^{t} \cdot H(t) - \frac{7}{12} \cdot e^{-3t} \cdot H(t) + \frac{89}{15} \cdot e^{-6t} \cdot H(t)$

$S^3 - 4S^2 - 15S + 18$

Ruffini

$$\begin{array}{r|rrr|r} & 1 & -4 & -15 & 18 \\ +1 & \downarrow & 1 & -3 & -18 \\ \hline & 1 & -3 & -18 & 0 \end{array}$$

$(S-1)(S^2 - 3S - 18)$

$\downarrow$

$(S-1)(S+3)(S-6)$

$$F(S) = \frac{S^3 + S^2 + 2S + 3}{S^3 - 4S^2 - 15S + 18} = \frac{S^3 + S^2 + 2S + 3}{(S-1)(S+3)(S-6)} = \frac{A}{S-1} + \frac{B}{S+3} + \frac{C}{S-6}$$

$$A = F(S)(S-1)\big|_{S=1} = \frac{S^3 + S^2 + 2S + 3}{(S+3)(S-6)} = -7/20$$

$$B = F(S)(S+3)\big|_{S=-3} = \frac{S^3 + S^2 + 2S + 3}{(S-1)(S-6)} = -7/12$$

$$C = F(S)(S-6)\big|_{S=6} = \frac{S^3 + S^2 + 2S + 3}{(S+3)(S-1)} = 89/15$$

$$F(S) = -7/20 \, \frac{1}{S-1} - 7/12 \, \frac{1}{S+3} + 89/15 \, \frac{1}{S-6} \Rightarrow f(t) = \left(-7/20 \, e^{t} - 7/12 \, e^{-3t} + 89/15 \, e^{6t}\right) H(t)$$