Sapienza University of Rome

Master in Artificial Intelligence and Robotics

Machine Learning

A.Y. 2025/2026

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Sapienza University of Rome, Italy - Machine Learning (2024/2025)

6. Probabilistic models for classification

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Overview

- Probabilistic generative models
- Probabilistic discriminative models
- Logistic regression

References
C. Bishop. Pattern Recognition and Machine Learning. Sect. 4.2, 4.3

Probabilistic Models for Classification

Given $f : \mathbf{X} \rightarrow C$ ($\mathbf{X} \subseteq \Re^d$), $D = \{(\mathbf{x}_n, c_n)_{n=1}^n\}$ and $\mathbf{x} \notin D$, estimate

$$P(C_i|\mathbf{x}, D)$$

Simplified notation without D in the formulas.

$P(C_i|\mathbf{x})$: posterior, $P(\mathbf{x}|C_i)$: class-conditional densities, $P(C_i)$: prior

Two families of models:

- Generative: estimate $P(\mathbf{x}|C_i)$ and then compute $P(C_i|\mathbf{x})$ with Bayes
- Discriminative: estimate $P(C_i|\mathbf{x})$ directly

Probabilistic Generative Models (2 classes)

Parametric model for Gaussian distributions

$$P(C_1) = \pi, \quad P(C_2) = 1 - \pi$$

$$P(\mathbf{x}|C_i) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$$

Parameters of the model: π, $\boldsymbol{\mu}_i$, $\boldsymbol{\Sigma}_i$

Probabilistic Generative Models (2 classes)

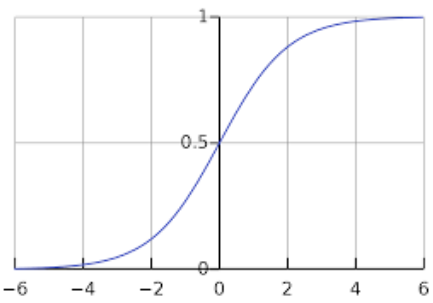
Generic formulation

$$P(C_1|\mathbf{x}) = \frac{P(\mathbf{x}|C_1)P(C_1)}{P(\mathbf{x})} = \frac{P(\mathbf{x}|C_1)P(C_1)}{P(\mathbf{x}|C_1)P(C_1) + P(\mathbf{x}|C_2)P(C_2)}$$
$$= \frac{1}{1 + \exp(-a)} = \sigma(a)$$

with:

$$a = \ln \frac{P(\mathbf{x}|C_1)P(C_1)}{P(\mathbf{x}|C_2)P(C_2)}$$

$\sigma(a) = \frac{1}{1+\exp(-a)}$ *sigmoid function*



Probabilistic Generative Models (2 classes)

For Gaussian model with same covariance $\mathbf{\Sigma}$

$$a = \ln \frac{P(\mathbf{x}|C_1)P(C_1)}{P(\mathbf{x}|C_2)P(C_2)} = \ln \frac{\mathcal{N}(\mathbf{x};\boldsymbol{\mu}_1,\mathbf{\Sigma})\,\pi}{\mathcal{N}(\mathbf{x};\boldsymbol{\mu}_2,\mathbf{\Sigma})\,(1-\pi)} = \ldots = \mathbf{w}^T\mathbf{x} + w_0$$

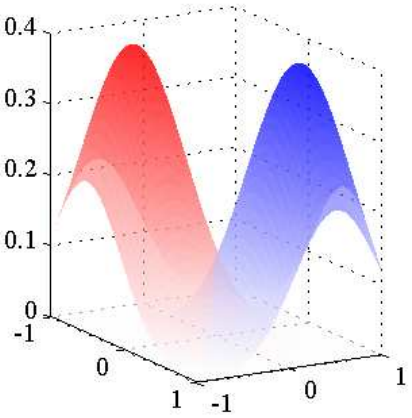
with:

$$\mathbf{w} = \mathbf{\Sigma}^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2),$$
$$w_0 = -\frac{1}{2}\boldsymbol{\mu}_1^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}_1 + \frac{1}{2}\boldsymbol{\mu}_2^T\mathbf{\Sigma}^{-1}\boldsymbol{\mu}_2 + \ln\frac{\pi}{1-\pi}.$$

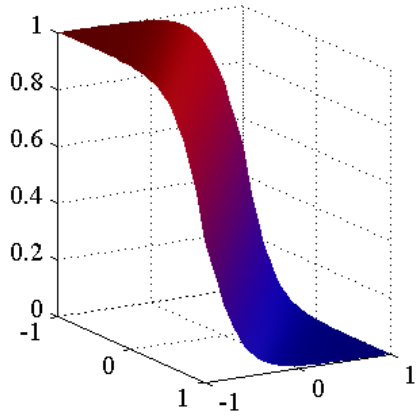
Thus

$$P(C_1|\mathbf{x}) = \sigma(\mathbf{w}^T\mathbf{x} + w_0),$$

Probabilistic Generative Models (2 classes)



$P(\mathbf{x}|C_1), P(\mathbf{x}|C_2)$



$P(C_1|\mathbf{x})$

Given a new sample $\mathbf{x}'$: *Decision rule*: $c = C_1 \iff P(c = C_1|\mathbf{x}') > 0.5$

Probabilistic Generative Models (2 classes)

Summary

Parametric model

$$P(C_1) = \pi \qquad P(C_2) = 1 - \pi$$

$$P(\mathbf{x}|C_1) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_1, \boldsymbol{\Sigma}) \qquad P(\mathbf{x}|C_2) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_2, \boldsymbol{\Sigma})$$

Unknown parameters (to be estimated from data)

$$\pi, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}$$

Posterior and classification of new samples

$$P(C_1|\mathbf{x}) = \sigma(a), \qquad a = \ln \frac{P(\mathbf{x}|C_1)P(C_1)}{P(\mathbf{x}|C_2)P(C_2)}$$

Maximum likelihood (2 classes)

Assuming Gaussian model $P(\mathbf{x}|C_i) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_i, \boldsymbol{\Sigma})$, $P(C_1) = \pi$

Given data set $D = \{(\mathbf{x}_n, t_n)_{n=1}^N\}$, $t_n = 1$ if $\mathbf{x}_n$ belongs to class C_1, $t_n = 0$ if $\mathbf{x}_n$ belongs to class C_2

Let N_1 be the number of samples in D belonging to C_1 and N_2 be the number of samples in C_2 ($N_1 + N_2 = N$)

Likelihood function

$$P(\mathbf{t}|\pi, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}, D) = \prod_{n=1}^{N} [\pi \mathcal{N}(\mathbf{x}_n; \boldsymbol{\mu}_1, \boldsymbol{\Sigma})]^{t_n} [(1-\pi)\mathcal{N}(\mathbf{x}_n; \boldsymbol{\mu}_2, \boldsymbol{\Sigma})]^{(1-t_n)}$$

ML solution:

$$\hat{\pi}, \hat{\boldsymbol{\mu}}_1, \hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\Sigma}} = \underset{\pi, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}}{\operatorname{argmax}} \ln P(\mathbf{t}|\pi, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}, D)$$

Maximum likelihood (2 classes)

Maximizing log likelihood function, we obtain

$$\hat{\pi} = \frac{N_1}{N}$$

$$\hat{\boldsymbol{\mu}}_1 = \frac{1}{N_1}\sum_{n=1}^{N} t_n \mathbf{x}_n \qquad \hat{\boldsymbol{\mu}}_2 = \frac{1}{N_2}\sum_{n=1}^{N}(1-t_n)\mathbf{x}_n$$

$$\hat{\boldsymbol{\Sigma}} = \frac{N_1}{N}S_1 + \frac{N_2}{N}S_2$$

with $S_i = \frac{1}{N_i}\sum_{n\in C_i}(\mathbf{x}_n - \hat{\boldsymbol{\mu}}_i)(\mathbf{x}_n - \hat{\boldsymbol{\mu}}_i)^T$, $i = 1, 2$

Note: details in C. Bishop. PRML. Section 4.2.2

Maximum likelihood (2 classes)

From the estimated parameters of the model $\hat{\pi}, \hat{\boldsymbol{\mu}}_1, \hat{\boldsymbol{\mu}}_2, \hat{\boldsymbol{\Sigma}}$, we obtain

$$\hat{\mathbf{w}} = \hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}}_1 - \hat{\boldsymbol{\mu}}_2),$$
$$\hat{w}_0 = -\frac{1}{2}\hat{\boldsymbol{\mu}}_1^T\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}_1 + \frac{1}{2}\hat{\boldsymbol{\mu}}_2^T\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}_2 + \ln\frac{\hat{\pi}}{1-\hat{\pi}}$$

Maximum likelihood (2 classes)

Prediction of new sample $\mathbf{x}' \notin D$

$$P(C_1|\mathbf{x}') = \sigma(\hat{\mathbf{w}}^T\mathbf{x}' + \hat{w}_0) \begin{cases} > 0.5 \Rightarrow C_1 \\ \leq 0.5 \Rightarrow C_2 \end{cases}$$

Generative model

$\hat{P}(\mathbf{x}|C_i) = \mathcal{N}(\mathbf{x}; \hat{\boldsymbol{\mu}}_i, \hat{\boldsymbol{\Sigma}})$ can be used to generate new samples for C_i

Probabilistic Generative Models (K classes)

Given

$f : \mathbf{X} \rightarrow \{C_1, \ldots, C_K\}$, $D = \{(\mathbf{x}_n, t_n)_{n=1}^N\}$

$t_n = (0, \ldots, 1, \ldots, 0)$: 1-of-K (one-hot) encoding ($t_{nk} = 1$ iff $f(\mathbf{x}_n) = C_k$)

estimate

$$P(C_k|\mathbf{x}) = \frac{P(\mathbf{x}|C_k)P(C_k)}{\sum_j P(\mathbf{x}|C_j)P(C_j)} = \frac{exp(a_k)}{\sum_j exp(a_j)}$$

(normalized exponential or softmax function)

with $a_k = \ln P(\mathbf{x}|C_k)P(C_k)$

Maximum likelihood (K classes)

Gaussian model (with same covariance $\mathbf{\Sigma}$)

$$P(C_k) = \pi_k \quad P(\mathbf{x}|C_k) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_k, \mathbf{\Sigma})$$

Parameters of the model: $\pi_k, \boldsymbol{\mu}_k, \mathbf{\Sigma}$

ML solution:

$$\hat{\pi_1}, \ldots, \hat{\pi_K}, \hat{\boldsymbol{\mu}}_1, \ldots, \hat{\boldsymbol{\mu}}_K, \hat{\mathbf{\Sigma}} = \underset{\pi_k, \boldsymbol{\mu}_k, \mathbf{\Sigma}}{\operatorname{argmax}} \ln P(\mathbf{t}|\pi_1, \ldots, \pi_K, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_K, \mathbf{\Sigma}, D)$$

Maximum likelihood (K classes)

Training: Estimate from D

$$\hat{\pi}_k = \frac{N_k}{N}$$

$$\hat{\boldsymbol{\mu}}_k = \frac{1}{N_k} \sum_{n=1}^{N} t_{nk} \mathbf{x}_n$$

$$\hat{\mathbf{\Sigma}} = \sum_{k=1}^{K} \frac{N_k}{N} S_k \ , \qquad S_k = \frac{1}{N_k} \sum_{n=1}^{N} t_{nk} (\mathbf{x}_n - \boldsymbol{\mu}_k)(\mathbf{x}_n - \boldsymbol{\mu}_k)^T$$

Maximum likelihood (K classes)

Prediction of new sample $\mathbf{x}' \notin D$

$$\underset{C_k \in C}{\operatorname{argmax}} \hat{P}(C_k|\mathbf{x}') = \underset{k}{\operatorname{argmax}} \frac{exp(a_k)}{\sum_j exp(a_j)}$$

with $a_k = \ln \hat{P}(\mathbf{x}'|C_k)\hat{P}(C_k) = \ln \hat{\pi}_k \mathcal{N}(\mathbf{x}'; \hat{\boldsymbol{\mu}}_k, \hat{\boldsymbol{\Sigma}})$

Probabilistic Models

Represent posterior distributions with parametric models.

For two classes

$$P(C_1|\mathbf{x}) = \sigma(a)$$

For $k \geq 2$ classes

$$P(C_i|\mathbf{x}) = \frac{exp(a_k)}{\sum_j exp(a_j)}$$

$$a_k = \mathbf{w}^T \mathbf{x} + w_0$$

This is valid for all the class-conditional distributions in the exponential family (including Gaussians). [Bishop, Sect. 4.2.4]

Gaussian Naive Bayes

$\mathbf{X} \subseteq \Re^m, \qquad D = \{((a_1, \ldots, a_m)_n, t_n)_{n=1}^N\}$

Naive assumption: conditional independence of features given the class.

$$P(\mathbf{x}|C_k) = \prod_{j=1}^{m} P(a_j|C_k)$$

Model

$$P(C_k) = \pi_k \quad P(a_j|C_k) = \mathcal{N}(a_j; \mu_{j,k}, \sigma_{j,k})$$

Parameters of the model:
$\pi_k, \mu_{j,k}, \sigma_{j,k}$, for $j = 1, \ldots, m, k = 1, \ldots, K$

Gaussian Naive Bayes

Maximum likelihood solution

$$\hat{\pi}_k = \frac{N_k}{N}$$

$$\hat{\mu}_{j,k} = \frac{1}{N_k} \sum_{n=1}^{N} t_{nk}\, a_{j,n}$$

$$\hat{\sigma_{j,k}} = \sum_{n=1}^{N} t_{nk}(a_{j,n} - \mu_{j,k})^2$$

Exercise

For a classification problem $f : \Re^8 \rightarrow \{C_1, \ldots, C_4\}$ compute the size of the two models: 1) a probabilistic generative model with Gaussian distributions and different covariance matrices: 2) Gaussian Naive Bayes.

Parameters of the models
1) $\pi_k, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k$
2) $\pi_k, \mu_{j,k}, \sigma_{j,k}$

Size of the models (number of independent parameters)
1) for π_k, $k-1$ (i.e., 3), for $\boldsymbol{\mu}_k$, $k \cdot m$ (i.e., 32), for $\boldsymbol{\Sigma}_k$, $k \cdot m \cdot (m+1)/2$ (i.e., $4 \cdot 36 = 144$), total $= 179$
2) for π_k, $k-1$ (i.e., 3), for $\mu_{j,k}, \sigma_{j,k}$ $m \cdot k$ (i.e., $2 \cdot 32 = 64$), total $= 67$

Simplified Notation

Target function: $f : \Re^d \mapsto C$

Dataset $D = \{(\mathbf{x}_n, t_n)_{n=1}^N\} = \langle \mathbf{X}, \mathbf{t} \rangle$

$\mathbf{X}$: matrix $(N \times d)$ of input values

$\mathbf{t}$: vector (N) of output values

Compact notation

Model

$$\mathbf{w}^T\mathbf{x} + w_0 = (w_0\ \mathbf{w}) \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}$$

$$\tilde{\mathbf{w}} = \begin{pmatrix} w_0 \\ \mathbf{w} \end{pmatrix}, \tilde{\mathbf{x}} = \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}$$

$$a_k = \mathbf{w}^T\mathbf{x} + w_0 = \tilde{\mathbf{w}}^T\tilde{\mathbf{x}}$$

Maximum Likelihood for parametric models

Likelihood for a parametric model $\mathcal{M}_\Theta$ of dataset $D = \langle \mathbf{X}, \mathbf{t} \rangle$: $P(\mathbf{t}|\Theta, \mathbf{X})$,

Maximum likelihood solution:

$$\Theta^* = \underset{\Theta}{\operatorname{argmax}} \ln P(\mathbf{t}|\Theta, \mathbf{X})$$

When $\mathcal{M}_\Theta$ belongs to the exponential family, likelihood $P(\mathbf{t}|\Theta, \mathbf{X})$ can be expressed in the form $P(\mathbf{t}|\tilde{\mathbf{w}}, \mathbf{X})$, with maximum likelihood

$$\tilde{\mathbf{w}}^* = \underset{\tilde{\mathbf{w}}}{\operatorname{argmax}} \ln P(\mathbf{t}|\tilde{\mathbf{w}}, \mathbf{X})$$

Probabilistic Discriminative Models

Estimate directly

$$P(C_k|\tilde{\mathbf{x}}, D) = \frac{exp(a_k)}{\sum_j exp(a_j)} \qquad a_k = \tilde{\mathbf{w}}^T \tilde{\mathbf{x}}$$

with maximum likelihood

$$\tilde{\mathbf{w}}^* = \underset{\tilde{\mathbf{w}}}{\operatorname{argmax}} \ln P(\mathbf{t}|\tilde{\mathbf{w}}, \mathbf{X})$$

without estimating the model parameters.

Simplified notation (dataset omitted): $P(\mathbf{t}|\tilde{\mathbf{w}})$

Logistic regression

Probabilistic discriminative model based on maximum likelihood.

Two classes

Given data set $D = \{(\tilde{\mathbf{x}}_n, t_n)_{n=1}^N\}$, with $t_n \in \{0, 1\}$

Likelihood function:

$$p(\mathbf{t}|\tilde{\mathbf{w}}) = \prod_{n=1}^{N} y_n^{t_n}(1 - y_n)^{1-t_n}$$

with $y_n = p(C_1|\tilde{\mathbf{x}}_n) = \sigma(\tilde{\mathbf{w}}^T \tilde{\mathbf{x}}_n)$

Note: t_n: value in the data set corresponding to $\mathbf{x}_n$,
y_n: posterior prediction of the current model $\tilde{\mathbf{w}}$ for $\mathbf{x}_n$.

Logistic regression

Cross-entropy error function (negative log likelihood)

$$E(\tilde{\mathbf{w}}) \equiv -\ln p(\mathbf{t}|\tilde{\mathbf{w}}) = -\sum_{n=1}^{N}[t_n \ln y_n + (1-t_n)\ln(1-y_n)]$$

Solution concept: solve the optimization problem

$$\tilde{\mathbf{w}}^* = \underset{\tilde{\mathbf{w}}}{\operatorname{argmin}}\, E(\tilde{\mathbf{w}})$$

Many solvers available.

Iterative reweighted least squares

Apply *Newton-Raphson* iterative optimization for minimizing $E(\tilde{\mathbf{w}})$.

Gradient of the error with respect to $\tilde{\mathbf{w}}$

$$\nabla E(\tilde{\mathbf{w}}) = \sum_{n=1}^{N}(y_n - t_n)\tilde{\mathbf{x}}_n$$

Gradient descent step

$$\tilde{\mathbf{w}} \leftarrow \tilde{\mathbf{w}} - \boldsymbol{H}(\tilde{\mathbf{w}})^{-1}\nabla E(\tilde{\mathbf{w}})$$

$\boldsymbol{H}(\tilde{\mathbf{w}}) = \nabla\nabla E(\tilde{\mathbf{w}})$ is the Hessian matrix of $E(\tilde{\mathbf{w}})$ (second derivatives with respect to $\tilde{\mathbf{w}}$).

Iterative reweighted least squares

Given

$$\tilde{\mathbf{X}} = \begin{pmatrix} \tilde{\mathbf{x}}_1^T \\ \dots \\ \tilde{\mathbf{x}}_N^T \end{pmatrix} \qquad \mathbf{t} = \begin{pmatrix} t_1 \\ \dots \\ t_N \end{pmatrix},$$

$\mathbf{y}(\tilde{\mathbf{w}}) = (y_1, \dots, y_n)^T$ posterior predictions of model $\tilde{\mathbf{w}}$

$\boldsymbol{R}(\tilde{\mathbf{w}})$: diagonal matrix with $R_{nn} = y_n(1 - y_n)$

we have

$$\nabla E(\tilde{\mathbf{w}}) = \tilde{\mathbf{X}}^T(\mathbf{y}(\tilde{\mathbf{w}}) - \mathbf{t})$$

$$\boldsymbol{H}(\tilde{\mathbf{w}}) = \nabla\nabla E(\tilde{\mathbf{w}}) = \sum_{n=1}^{N} y_n(1 - y_n)\tilde{\mathbf{x}}_n\tilde{\mathbf{x}}_n^T = \tilde{\mathbf{X}}^T\boldsymbol{R}(\tilde{\mathbf{w}})\tilde{\mathbf{X}}$$

Iterative reweighted least squares

Iterative method:

1. Initialize $\tilde{\mathbf{w}}$
2. Repeat until termination condition

$$\tilde{\mathbf{w}} \leftarrow \tilde{\mathbf{w}} - (\tilde{\mathbf{X}}^T\,\boldsymbol{R}(\tilde{\mathbf{w}})\,\tilde{\mathbf{X}})^{-1}\tilde{\mathbf{X}}^T(\mathbf{y}(\tilde{\mathbf{w}}) - \mathbf{t})$$

Multiclass logistic regression

K classes

$$P(C_k|\tilde{\mathbf{x}}) = \frac{exp(a_k)}{\sum_j exp(a_j)} \qquad a_k = \tilde{\mathbf{w}}_k^T \tilde{\mathbf{x}} \qquad k = 1, \ldots, K$$

$$\tilde{\mathbf{X}} = \begin{pmatrix} \tilde{\mathbf{x}}_1^T \\ \ldots \\ \tilde{\mathbf{x}}_N^T \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} t_1^T \\ \ldots \\ t_N^T \end{pmatrix} \text{ 1-of-}K \text{ encoding of labels}$$

$\mathbf{y}_n^T = (y_{n1} \ldots y_{nK})^T$ posterior prediction of $\tilde{\mathbf{x}}_n$ for model $\tilde{\mathbf{w}}_1, \ldots, \tilde{\mathbf{w}}_K$

$$\mathbf{Y}(\tilde{\mathbf{w}}_1, \ldots, \tilde{\mathbf{w}}_K) = \begin{pmatrix} \mathbf{y}_1^T \\ \ldots \\ \mathbf{y}_N^T \end{pmatrix} \text{ posterior predictions of model } \tilde{\mathbf{w}}_1, \ldots, \tilde{\mathbf{w}}_K$$

Multiclass logistic regression

Discriminative model

$$P(\mathbf{T}|\tilde{\mathbf{w}}_1, \ldots, \tilde{\mathbf{w}}_K) = \prod_{n=1}^{N} \prod_{k=1}^{K} P(C_k|\tilde{\mathbf{x}}_n)^{t_{nk}} = \prod_{n=1}^{N} \prod_{k=1}^{K} y_{nk}^{t_{nk}}$$

with $y_{nk} = \mathbf{Y}[n, k]$ and $t_{nk} = \mathbf{T}[n, k]$.

Multiclass logistic regression

Cross-entropy error function

$$E(\tilde{\mathbf{w}}_1, \dots \tilde{\mathbf{w}}_K) = -\ln P(\mathbf{T}|\tilde{\mathbf{w}}_1, \dots \tilde{\mathbf{w}}_K) = -\sum_{n=1}^{N}\sum_{k=1}^{K} t_{nk} \ln y_{nk}$$

Iterative algorithm

gradient $\nabla_{\tilde{\mathbf{w}}_j} E(\tilde{\mathbf{w}}_1, \dots \tilde{\mathbf{w}}_K) = \dots$

Hessian matrix $\nabla_{\tilde{\mathbf{w}}_k} \nabla_{\tilde{\mathbf{w}}_j} E(\tilde{\mathbf{w}}_1, \dots \tilde{\mathbf{w}}_K) = \dots$

Summary

Given a target function $f : \mathbf{X} \to C$, and data set D

assume a parametric model for the posterior probability $P(C_k|\tilde{\mathbf{x}}, \tilde{\mathbf{w}})$
sigmoid $\sigma(\tilde{\mathbf{w}}^T\tilde{\mathbf{x}})$ (2 classes) or softmax $\frac{exp(\tilde{\mathbf{w}}_k^T\tilde{\mathbf{x}})}{\sum_{j=1}^{K} exp(\tilde{\mathbf{w}}_j^T\tilde{\mathbf{x}})}$ (K classes)

Define an error function $E(\tilde{\mathbf{w}})$ (negative log likelihood)

Solve the optimization problem

$$\tilde{\mathbf{w}}^* = \underset{\tilde{\mathbf{w}}}{\operatorname{argmin}}\, E(\tilde{\mathbf{w}})$$

Classify new sample $\tilde{\mathbf{x}}'$ as C_{k^*} where $k^* = \operatorname{argmax}_{k=1,\dots,K} P(C_k|\tilde{\mathbf{x}}', \tilde{\mathbf{w}}^*)$

Generalization

Given a target function $f : \mathbf{X} \to C$, and data set D

assume a prediction parametric model $y(\mathbf{x}; \theta)$, $\quad y(\mathbf{x}; \theta) \approx f(\mathbf{x})$

Define an error function $E(\theta)$

Solve the optimization problem

$$\theta^* = \underset{\theta}{\operatorname{argmin}}\, E(\theta)$$

Classify new sample $\mathbf{x}'$ as $y(\mathbf{x}'; \theta^*)$

Learning in feature space

All methods described above can be applied in a transformed space of the input (*feature space*).

Given a function $\phi : \tilde{\mathbf{X}} \mapsto \mathbf{\Phi}$ ($\mathbf{\Phi}$ is the *feature space*)
each sample $\tilde{\mathbf{x}}_n$ can be mapped to a feature vector $\boldsymbol{\phi}_n = \phi(\tilde{\mathbf{x}}_n)$

Replacing $\tilde{\mathbf{x}}_n$ with $\boldsymbol{\phi}_n$ in all the equations above, makes the learning system to work in the feature space instead of the input space.

We will see in the next lectures why this trick is useful.