Sapienza University of Rome

Master in Artificial Intelligence and Robotics

# Machine Learning

A.Y. 2025/2026

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# 5. Bayesian Learning

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# Outline

- Bayes Theorem
- MAP, ML hypotheses
- MAP learners
- Bayes optimal classifier
- Naive Bayes learner
- Example: Learning over text data

*References*
T. Mitchell. Machine Learning. Chapter 6





# Two Roles for Bayesian Methods

Provides practical learning algorithms:

- Naive Bayes learning (examples affect prob. that a hypothesis is correct)
- Combine prior knowledge (prior probabilities) with observed data
- Make probabilistic predictions (new instances classified by weighted combination of multiple hypotheses)
- Requires prior probabilities (often estimated from available data)

Provides useful conceptual framework

- Provides "gold standard" for evaluating other learning algorithms

# Basic Formulas for Probabilities

- *Product Rule*: probability of conjunction of A and B:

$$P(A \wedge B) = P(A|B)P(B) = P(B|A)P(A)$$

- *Sum Rule*: probability of disjunction of A and B:

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

- *Theorem of total probability*: if events $A_1, \ldots, A_n$ are mutually exclusive with $\sum_{i=1}^{n} P(A_i) = 1$, then

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

- *Bayes theorem*:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$





# Classification as Probabilistic estimation

Given target function $f : X \to V$, dataset $D$ and a new instance $x'$, best prediction $\hat{f}(x') = v^*$

$$v^* = \operatorname*{argmax}_{v \in V} P(v|x', D)$$

More general formulation: given $D$ and $x'$, compute the probability distribution over $V$

$$P(V|x', D)$$

# Learning as Probabilistic estimation

Given dataset $D$ and hypothesis space $H$, compute a probability distribution over $H$ given $D$.

$$P(H|D)$$

Bayes rule

$$P(h|D) = \frac{P(D|h)P(h)}{P(D)}$$

- $P(h)$ = prior probability of hypothesis $h$ (prior)
- $P(D)$ = prior probability of training data $D$ (normalization factor)
- $P(h|D)$ = probability of $h$ given $D$ (posterior)
- $P(D|h)$ = probability of $D$ given $h$ (likelihood)





# MAP Hypotheses

$$P(h|D) = \frac{P(D|h)P(h)}{P(D)}$$

Generally we want the most probable hypothesis $h$ given $D$

*Maximum a posteriori* hypothesis $h_{MAP}$:

$$\begin{aligned} h_{MAP} &\equiv \operatorname*{argmax}_{h \in H} P(h|D) = \operatorname*{argmax}_{h \in H} \frac{P(D|h)P(h)}{P(D)} \\ &= \operatorname*{argmax}_{h \in H} P(D|h)P(h) \end{aligned}$$

# ML Hypotheses

$$P(h|D) = \frac{P(D|h)P(h)}{P(D)}$$

If assume $P(h_i) = P(h_j)$, we can further simplify, and choose the *Maximum likelihood* (ML) hypothesis

$$h_{ML} = \operatorname*{argmax}_{h \in H} P(D|h)$$





# Brute Force MAP Hypothesis Learner

1. For each hypothesis $h$ in $H$, calculate the posterior probability

$$P(h|D) = \frac{P(D|h)P(h)}{P(D)}$$

2. Output the hypothesis $h_{MAP}$ with the highest posterior probability

$$h_{MAP} = \operatorname*{argmax}_{h \in H} P(h|D)$$

# Most Probable Classification of New Instances

$h_{MAP}$: most probable *hypothesis* given data $D$.

Given a new instance $x'$, what is its most probable *classification* of $x'$?

$h_{MAP}(x')$ may not be the most probable classification !!!





# Most Probable Classification of New Instances

Consider:

- Three possible hypotheses $h_1$, $h_2$, $h_3$:
  $$P(h_1|D) = 0.4,\ P(h_2|D) = 0.3,\ P(h_3|D) = 0.3$$
- Given a new instance $x$,
  $$h_1(x) = \oplus,\ h_2(x) = \ominus,\ h_3(x) = \ominus$$
- What is the most probable classification of $x$ ?

# Bayes Optimal Classifier

Consider target function $f : X \mapsto V$, $V = \{v_1, ..., v_k\}$, data set $D$ and a new instance $x \notin D$:

$$P(v_j|x, D) = \sum_{h_i \in H} P(v_j|x, h_i, D)P(h_i|x, D) = \sum_{h_i \in H} P(v_j|x, h_i)P(h_i|D)$$

total probability over $H$

probability that $h_i(x) = v_j$ is independent from $D$ given $h_i$
$\Rightarrow P(v_j|x, h_i, D) = P(v_j|x, h_i)$
$h_i$ does not depend on $x \notin D \Rightarrow P(h_i|x, D) = P(h_i|D)$





# Bayes Optimal Classifier

**Bayes Optimal Classifier**
Class of a new instance $x$:

$$v_{OB} = \operatorname*{argmax}_{v_j \in V} \sum_{h_i \in H} P(v_j|x, h_i)P(h_i|D)$$

# Bayes Optimal Classifier

Example:

$$
\begin{array}{lll}
P(h_1|D) = 0.4, & P(\ominus|x,h_1) = 0, & P(\oplus|x,h_1) = 1 \\
P(h_2|D) = 0.3, & P(\ominus|x,h_2) = 1, & P(\oplus|x,h_2) = 0 \\
P(h_3|D) = 0.3, & P(\ominus|x,h_3) = 1, & P(\oplus|x,h_3) = 0
\end{array}
$$

therefore

$$
\begin{array}{rcl}
\sum_{h_i \in H} P(\oplus|x,h_i)P(h_i|D) & = & 0.4 \\
\sum_{h_i \in H} P(\ominus|x,h_i)P(h_i|D) & = & 0.6
\end{array}
$$

and

$$
v_{OB} = \arg\max_{v_j \in V} \sum_{h_i \in H} P(v_j|x,h_i)P(h_i|D) \quad = \quad \ominus
$$





# Bayes Optimal Classifier

*Optimal learner*: no other classification method using the same hypothesis space and same prior knowledge can outperform this method on average.

It maximizes the probability that the new instance $x$ is classified correctly, i.e., $\mathrm{argmax}_{v_j \in V} P(v_j|x,D)$.

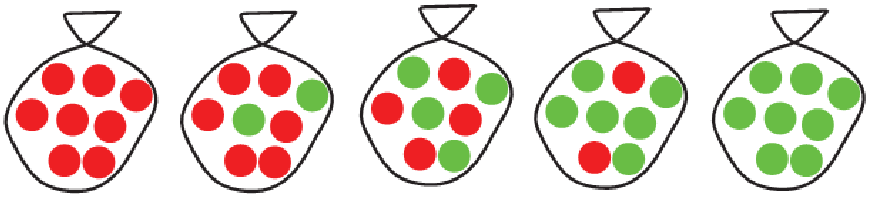
Very powerful: labelling new instances $x$ with $\mathrm{argmax}_{v_j \in V} P(v_j|x,D)$ can correspond to none of the hypotheses in $H$.

Impractical: requires iteration on $H$

# Bayesian Learning Example

Five kinds of bags of candiers:

1. 10% are $h_1$: 100% cherry
2. 20% are $h_2$: 75% cherry, 25% lime
3. 40% are $h_3$: 50% cherry, 50% lime
4. 20% are $h_4$: 25% cherry, 75% lime
5. 10% are $h_5$: 100% lime







# Bayesian Learning Example

We choose a random bag (not knowing which type it is) and extract some candies from it.

What kind of bag is it? What is the probability of extracting a candy of a specific flavor next?

## Bayesian Learning Example

Prior probability distribution:

$$\mathrm{P}(H) =< 0.1, 0.2, 0.4, 0.2, 0.1 >$$

Likelihood for lime candy:

$$\mathrm{P}(l|H) =< 0, 0.25, 0.5, 0.75, 1 >$$

Probability of extracting a lime candy (without data set):

$$\sum_{h_i} P(l|h_i)P(h_i) = 0 \cdot 0.1 + 0.25 \cdot 0.2 + 0.5 \cdot 0.4 + 0.75 \cdot 0.2 + 1 \cdot 0.1 = 0.5$$



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## Bayesian Learning Example

1. First candy is lime: $D_1 = \{l\}$

$P(h_i|\{d_1\}) = \alpha P(\{d_1\}|h_i)P(h_i)$ (Bayes rule)

$$\begin{aligned}
\mathrm{P}(H|D_1) &= \alpha < 0, 0.25, 0.5, 0.75, 1 > \cdot < 0.1, 0.2, 0.4, 0.2, 0.1 > \\
&= \alpha < 0, 0.05, 0.2, 0.15, 0.1 > \\
&= < 0, 0.1, 0.4, 0.3, 0.2 >
\end{aligned}$$

Note: in these slides the operator $\cdot$ represents element-wise product (not the dot product)

# Bayesian Learning Example

2. Second candy is lime: $D_2 = \{l, l\}$

$P(h_i|\{d_1, d_2\}) = \alpha P(\{d_1, d_2\}|h_i) P(h_i)$ (Bayes rule)
$= \alpha P(\{d_2\}|h_i)\, P(\{d_1\}|h_i) P(h_i)$ (independent data samples)

$$
\begin{aligned}
\mathrm{P}(H|D_2) &= \alpha < 0, 0.25, 0.5, 0.75, 1 > \cdot < 0, 0.1, 0.4, 0.3, 0.2 > \\
&= \alpha < 0, 0.025, 0.2, 0.225, 0.2 > \\
&= < 0, 0.038, 0.308, 0.346, 0.308 >
\end{aligned}
$$



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# Bayesian Learning Example

3. Third candy is lime: $D_3 = \{l, l, l\}$

$P(h_i|\{d_1, d_2, d_3\}) = \alpha P(\{d_1, d_2, d_3\}|h_i) P(h_i)$ (Bayes rule)
$= \alpha P(\{d_3\}|h_i)\, P(\{d_2\}|h_i)\, P(\{d_1\}|h_i) P(h_i)$ (independent data samples)

$$
\begin{aligned}
\mathrm{P}(H|D_3) &= \alpha < 0, 0.25, 0.5, 0.75, 1 > \cdot < 0, 0.038, 0.308, 0.346, 0.308 > \\
&= \alpha < 0, 0.01, 0.154, 0.260, 0.308 > \\
&= < 0, 0.013, 0.211, 0.355, 0.421 >
\end{aligned}
$$

# Bayesian Learning Example

What is probability of having another lime candy after $D_3 = \{l, l, l\}$ ?

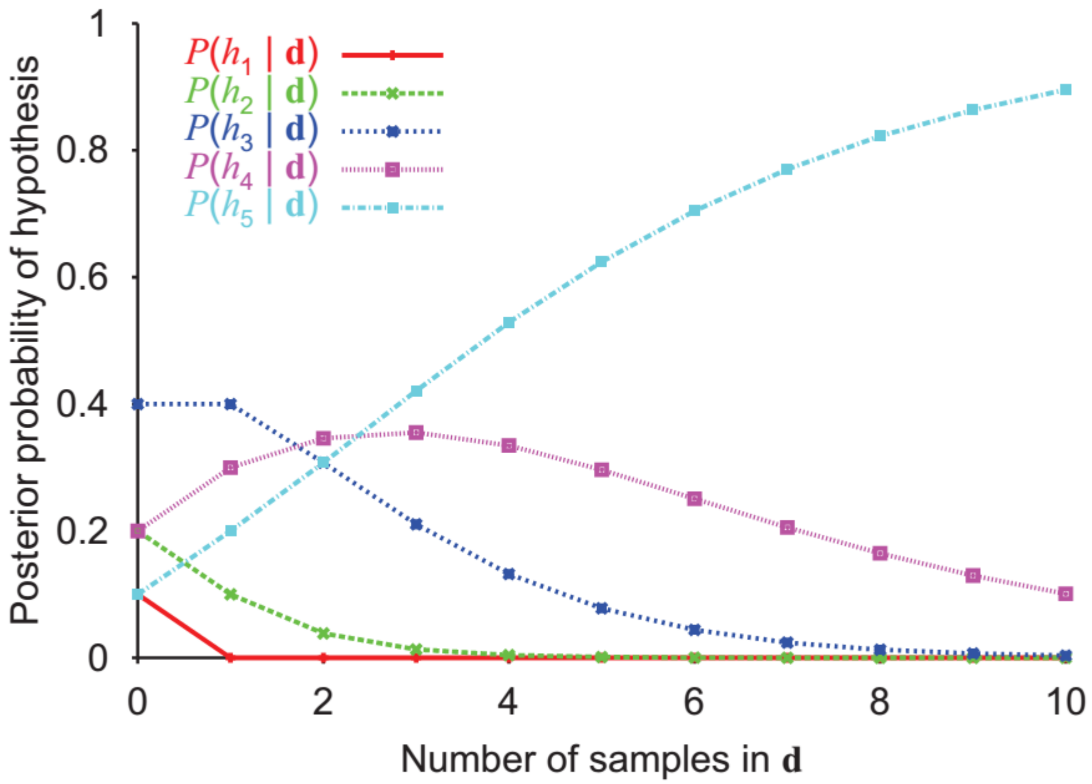
$$
\begin{aligned}
P(l|D_3) &= \sum_{h_i} P(l|h_i)P(hi|D_3) \\
&= 0 \cdot 0 + 0.25 \cdot 0.013 + 0.5 \cdot 0.211 + 0.75 \cdot 0.355 + 1 \cdot 0.421 \\
&= 0.8
\end{aligned}
$$

Note: if we use $h_{MAP} = h_5$ to answer this question, the result will be 1, which is not correct!

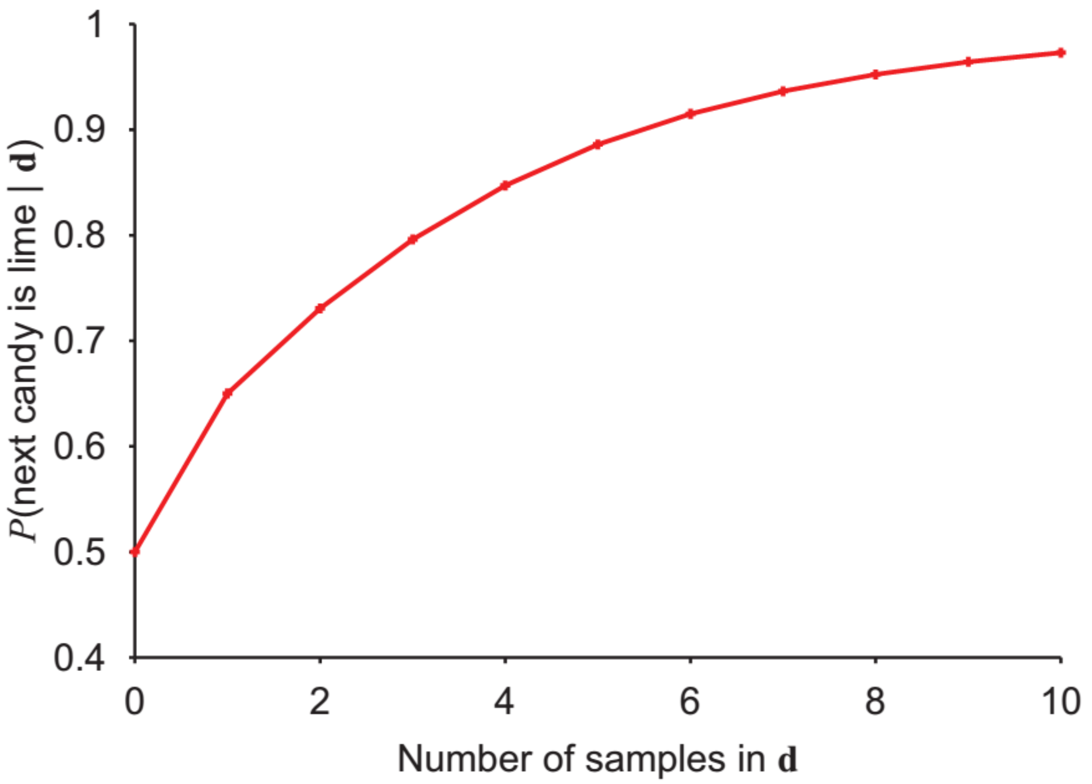




# Bayesian Learning Example



# Bayesian Learning Example








# Bayesian Learning Example 2

Consider a new manufacturer producing bags with an arbitrary choice of cherry/lime candies. $\theta \equiv \frac{nr.\ of\ cherry\ candies}{N} \in [0, 1]$.
Continuous space for hypotheses: $h_\theta$

Data set: $D = \{c \text{ cherries}, l \text{ lime}\}$, $N = c + l$

$P(c|h_\theta) = \theta$
$P(l|h_\theta) = 1 - \theta$

- What is the ML hypothesis?

# Bayesian Learning Example 2

$$h_{ML} = \underset{h_\theta}{\operatorname{argmax}} P(D|h_\theta) = \underset{h_\theta}{\operatorname{argmax}} L(D|h_\theta)$$

with $L(D|h_\theta) = \log P(D|h_\theta)$

$$P(D|h_\theta) = \prod_{j=1\ldots N} P(d_j|h_\theta) = \theta^c \cdot (1-\theta)^l$$

$$L(D|h_\theta) = c \log \theta + l \log(1-\theta)$$

$$\frac{dL(D|h_\theta)}{d\theta} = \frac{c}{\theta} - \frac{l}{1-\theta} = 0 \Rightarrow \theta_{ML} = \frac{c}{c+l} = \frac{c}{N}$$





# General approach

Given dataset $D = \{d_i\}$ with $d_i \in \{0,1\}$,
assuming a probability distribution $P(d_i; \Theta)$

Maximum likelihood estimation (MLE)

$$\Theta_{ML} = \underset{\Theta}{\operatorname{argmax}} \log P(D|\Theta)$$

For some distributions $P$, this problem can be solved analytically.

Example: for Bernoulli distribution $P(X = k; \theta) = \theta^k (1-\theta)^{1-k}$

$$\theta_{ML} = \frac{|\{d_i = 1\}|}{|D|}$$

# Bernoulli distribution

Probability distribution of a binary random variable $X \in \{0, 1\}$

$$P(X = 1) = \theta \quad P(X = 0) = 1 - \theta$$

(e.g., observing head after flipping a coin, extracting a lime candy, ...).

$$P(X = k; \theta) = \theta^k \, (1 - \theta)^{1-k}$$





# Multi-variate Bernoulli distribution

Joint probability distribution of a set of binary random variables $X_1, \ldots X_n$, each random variable following Bernoulli distribution

$$P(X_1 = k_1, \ldots, X_n = k_n \, ; \, \theta_1, \ldots, \theta_n)$$

$k_i \in \{0, 1\}$

(e.g., observing head after flipping a coin **and** extracting a lime candy, ...).

Under the assumption that random variables $X_i$ are mutually independent, Multi-variate Bernoulli distribution is the product of $n$ Bernoulli distributions

$$P(X_1 = k_1, \ldots \, ; \, \theta_1, \ldots, \theta_n) = \prod_{i=1}^{n} P(X_i = k_i; \theta_i) = \prod_{i=1}^{n} \theta_i^{k_i} (1 - \theta_i)^{1-k_i}$$

## Binomial distribution

Probability distribution of $k$ outcomes from $n$ Bernoulli trials

$$P(X = k; n, \theta) = \binom{n}{k} \theta^k (1 - \theta)^{n-k}$$

(e.g., flipping a coin $n$ times and observing $k$ heads, extracting $k$ lime candies after $n$ extractions, ...).





## Multinomial distribution

Generalization of binomial distribution for discrete valued random variables with $d$ possible outcomes.
Probability distribution of $k_1$ outcomes for $X_1$, ..., $k_d$ outcomes for $X_d$, after $n$ trials (with $\sum_{i=1\ldots d} k_i = n$)

$$P(X_1 = k_1, \ldots, X_d = k_d; n, \theta_1, \ldots, \theta_d) = \frac{n!}{k_1! \ldots k_d!} \theta_1^{k_1} \cdot \ldots \cdot \theta_d^{k_d}$$

(e.g., rolling a $d$-sided die $n$ times and observing $k$ times a particular value, extracting $k$ lime candies after $n$ extractions form a bag containing $d$ different flavors, ...).

# Remarks

Probabilistic classification

$$\underset{v_j \in V}{\operatorname{argmax}} P(v_j | x, D)$$

Discrete model

- Bayes Optimal Classifier
  provides best result, not practical when hypothesis space is large

Continuous model

- Maximum likelihood estimation
  efficiently solved when analytical solutions are available

What are more practical and general solutions?





# Naive Bayes Classifier

Naive Bayes Classifier uses conditional independence to approximate the solution.

$X$ is *conditionally independent* of $Y$ given $Z$

$$P(X, Y|Z) = P(X|Y, Z)P(Y|Z) = P(X|Z)P(Y|Z)$$

# Naive Bayes Classifier

Assume target function $f : X \to V$, where each instance $x$ is described by attributes $\langle a_1, a_2, \ldots, a_n \rangle$.

Compute

$$\operatorname*{argmax}_{v_j \in V} P(v_j | x, D) = \operatorname*{argmax}_{v_j \in V} P(v_j | a_1, a_2 \ldots a_n, D)$$

without explicit representation of hypotheses.





# Naive Bayes Classifier

Given a data set $D$ and a new instance $x = \langle a_1, a_2 \ldots a_n \rangle$, most probable value of $f(x)$ is:

$$\begin{aligned} v_{MAP} &= \operatorname*{argmax}_{v_j \in V} P(v_j | a_1, a_2 \ldots a_n, D) \\ &= \operatorname*{argmax}_{v_j \in V} \frac{P(a_1, a_2 \ldots a_n | v_j, D) P(v_j | D)}{P(a_1, a_2 \ldots a_n | D)} \\ &= \operatorname*{argmax}_{v_j \in V} P(a_1, a_2 \ldots a_n | v_j, D) P(v_j | D) \end{aligned}$$

(Bayes rule)

# Naive Bayes Classifier

Naive Bayes assumption:

$$P(a_1, a_2, \ldots, a_n | v_j, D) = \prod_i P(a_i | v_j, D)$$

**Naive Bayes classifier**
Class of new instance $x$:

$$v_{NB} = \underset{v_j \in V}{\operatorname{argmax}} P(v_j | D) \prod_i P(a_i | v_j, D)$$





# Naive Bayes Algorithm

Target function $f : X \mapsto V$, $X = A_1 \times \ldots \times A_n$, $V = \{v_1, ..., v_k\}$,
data set $D$, new instance $x = \langle a_1, a_2 \ldots a_n \rangle$.

Naive_Bayes_Learn($A, V, D$)

for each target value $v_j \in V$
  $\hat{P}(v_j | D) \leftarrow$ estimate $P(v_j | D)$
  for each attribute $A_k$
    for each attribute value $a_i \in A_k$
      $\hat{P}(a_i | v_j, D) \leftarrow$ estimate $P(a_i | v_j, D)$

Classify_New_Instance($x$)

$$v_{NB} = \underset{v_j \in V}{\operatorname{argmax}} \hat{P}(v_j | D) \prod_{a_i \in x} \hat{P}(a_i | v_j, D)$$

# Naive Bayes estimation

$$\hat{P}(v_j|D) = \frac{|\{< \ldots, v_j >\}|}{|D|}$$

$$\hat{P}(a_i|v_j, D) = \frac{|\{< \ldots, a_i, \ldots, v_j >\}|}{|\{< \ldots, v_j >\}|}$$

Note: if none of the training instances with target value $v_j$ have attribute value $a_i$, then $\hat{P}(a_i|v_j, D) = 0$ and thus $\hat{P}(v_j|D) \prod_i \hat{P}(a_i|v_j, D) = 0$





# Naive Bayes estimation

Typical solution is Bayesian estimate with prior estimates

$$\hat{P}(a_i|v_j, D) = \frac{|\{< \ldots, a_i, \ldots, v_j >\}| + mp}{|\{< \ldots, v_j >\}| + m}$$

where

- $p$ is a prior estimate for $P(a_i|v_j, D)$
- $m$ is a weight given to prior (i.e. number of "virtual" examples)

# Example *PlayTennis*: Training data

| Day | Outlook | Temperature | Humidity | Wind | PlayTennis |
|---|---|---|---|---|---|
| D1 | Sunny | Hot | High | Weak | No |
| D2 | Sunny | Hot | High | Strong | No |
| D3 | Overcast | Hot | High | Weak | Yes |
| D4 | Rain | Mild | High | Weak | Yes |
| D5 | Rain | Cool | Normal | Weak | Yes |
| D6 | Rain | Cool | Normal | Strong | No |
| D7 | Overcast | Cool | Normal | Strong | Yes |
| D8 | Sunny | Mild | High | Weak | No |
| D9 | Sunny | Cool | Normal | Weak | Yes |
| D10 | Rain | Mild | Normal | Weak | Yes |
| D11 | Sunny | Mild | Normal | Strong | Yes |
| D12 | Overcast | Mild | High | Strong | Yes |
| D13 | Overcast | Hot | Normal | Weak | Yes |
| D14 | Rain | Mild | High | Strong | No |





# Naive Bayes: Example

Consider *PlayTennis* again, and new instance

$$\langle Outlook = sun, Temp = cool, Humid = high, Wind = strong \rangle$$

We want to compute:

$$v_{NB} = \underset{v_j \in V}{\mathrm{argmax}}\, P(v_j|D) \prod_i P(a_i|v_j, D)$$

without making any hypothesis space explicit.

# Naive Bayes: Example

Note: easy notation with conditioning on $D$ omitted.

$$P(PlayTennis = yes) = P(y) = 9/14 = 0.64$$
$$P(PlayTennis = no) = P(n) = 5/14 = 0.36$$

$$\begin{aligned} P(Wind = strong|y) &= 3/9 = 0.33 \\ P(Wind = strong|n) &= 3/5 = 0.60 \end{aligned}$$

$$\ldots$$

$$P(y)\ P(sun|y)\ P(cool|y)\ P(high|y)\ P(strong|y) = .005$$
$$P(n)\ P(sun|n)\ P(cool|n)\ P(high|n)\ P(strong|n) = .021$$

$$\rightarrow v_{NB} = n$$





# Naive Bayes Remarks

Conditional independence assumption is often violated

$$P(a_1, \ldots, a_n|v_j, D) \approx \prod_i P(a_i|v_j, D)$$

...but it works surprisingly well anyway.

Note: don't need estimated posteriors $\hat{P}(v_j|x, D)$ to be correct;
need only that

$$\operatorname*{argmax}_{v_j \in V} \hat{P}(v_j|D) \prod_i \hat{P}(a_i|v_j, D) = \operatorname*{argmax}_{v_j \in V} P(v_j|D) P(a_1, \ldots, a_n|v_j, D)$$

Issue: Naive Bayes posteriors often unrealistically close to 1 or 0

# Learning to classify text

Input:
set of documents (sequences of words) $MyDocs \subset Docs$ ,
each classified as $c_1, \ldots, c_k$

Learn target function $f : Docs \mapsto \{c1, \ldots, c_k\}$

Examples:

- spam classification (e-mail, SMS, ...)
- sentiment analysis (facebook/twitter posts, web reviews, ...)
- ...





# Learning to Classify Text: Naive Bayes approach

Classification of documents $Docs$ in classes $C = \{c_1, \ldots, c_k\}$

Target function $f : Docs \mapsto C$

Data set $D = \{< doc, c >_i\}$

Given a new document $doc \notin D$, compute

$$c_{NB} = \underset{c_j \in C}{\operatorname{argmax}} P(c_j|D)P(doc|c_j, D)$$

# Learning to Classify Text: Naive Bayes approach

$doc = w_1 w_2 \cdots w_m \ (m = length(doc))$

Naive Bayes conditional independence assumption

$$P(doc|c_j, D) = \prod_{i=1}^{length(doc)} P(p_i = w_i|c_j, D)$$

- $P(p_i = w_i|c_j)$: probability that, in document *doc* of class $c_j$, word in position $i$ is $w_i$

One more assumption: $\forall i, k, P(p_i = w_i|c_j, D) = P(p_k = w_i|c_j, D)$,
thus consider only:

- $P(w_i|c_j, D)$: probability that $w_i$ occurs in document *doc* of class $c_j$





# Document representations

Vocabulary $V = \{w_1, \ldots, w_n\}$
set of all words appearing in any document $doc \in D$

$n = |V|$: size of vocabulary

Representation of *doc*: $n$-dimensional feature vector $d$, each feature $d_i$ represents the encoding of word $w_i$ in *doc*

Issues:

This representation looses context information (order of words important!)

Vectors are very large and sparse.

# Document representations

Fix an (arbitrary) order on words in $V$: $w_1, w_2, \ldots, w_n$

Representing *doc* $\in D$ as a fixed-length feature vector $d =< d_1, \ldots, d_n >$:

1. boolean feature vector: $d_i = 1$ if $w_i$ appears in *doc*, 0 otherwise (Multivariate Bernoulli distribution) $\rightarrow$ binary encoding
2. ordinal feature vector: $d_i = k$ if word $w_i$ occurs $k$ times in *doc* (Multinomial distribution) $\rightarrow$ Bag of Words
3. real-valued feature vecor (tf-idf): $d_i = tf(w_i, doc)\, idf(w_i, D)$ $\rightarrow$ term frequency-inverse document frequency (tf-idf)

Note that the meaning of $P(w_k|c_j, D)$ can be generalized to capture the *importance* of $w_k$ to represent class $c_j$.



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# Multi-variate Bernoulli Naive Bayes distribution

For boolean feature vector $d =< d_1, \ldots, d_n >$ of generic $doc \in Docs$:

$$P(d|c_j, D) = \prod_{i=1}^{n} P(w_i|c_j, D)^{d_i} \cdot (1 - P(w_i|c_j, D))^{1-d_i}$$

Maximum-likelihood solution:

$$\hat{P}(w_i|c_j, D) = \frac{t_{i,j} + 1}{t_j + 2}$$

$t_{i,j}$: number of documents in $D$ of class $c_j$ containing word $w_i$
$t_j$: number of documents in $D$ of class $c_j$
1, 2: parameters for Laplace smoothing

# Multinomial Naive Bayes distribution

For ordinal feature vector $d =< d_1, \dots, d_n >$ of generic $doc \in Docs$:

$$P(d|c_j, D) = \frac{n!}{d_1! \cdots d_n!} \prod_{i=1}^{n} P(w_i|c_j, D)^{d_i}$$

Maximum-likelihood solution:

$$\hat{P}(w_i|c_j, D) = \frac{\sum_{doc \in D} tf_{i,j} + \alpha}{\sum_{doc \in D} tf_j + \alpha \cdot |V|}$$

$tf_{i,j}$: term frequency (# occurrences) of $w_i$ in document $doc$ of class $c_j$
$tf_j$: all-term frequency of document $doc$ of class $c_j$
$\alpha$: smoothing parameter ($\alpha = 1$ for Laplace smoothing)





# Naive Bayes Text Classification algorithm

Estimate $\hat{P}(c_j)$ and $\hat{P}(w_i|c_j)$ using *Bernoulli distribution*.

LEARN_NAIVE_BAYES_TEXT_BE($D$, $C$)
- $V \leftarrow$ all distinct words in $D$
- for each target value $c_j \in C$ do
  - $docs_j \leftarrow$ subset of $D$ for which the target value is $c_j$
  - $t_j \leftarrow |docs_j|$: total number of documents in $c_j$
  - $\hat{P}(c_j) \leftarrow \frac{t_j}{|D|}$
  - for each word $w_i$ in $V$ do
    - $t_{i,j} \leftarrow$ number of documents in $c_j$ containing word $w_i$
    - $\hat{P}(w_i|c_j) \leftarrow \frac{t_{i,j}+1}{t_j+2}$

# Naive Bayes Text Classification algorithm

Estimate $\hat{P}(c_j)$ and $\hat{P}(w_i|c_j)$ using *Multinomial distribution*.

$\textsc{Learn\_Naive\_Bayes\_text\_Mu}(D, C)$

$V \leftarrow$ all distinct words in $D$
for each target value $c_j \in C$ do
  $docs_j \leftarrow$ subset of $D$ for which the target value is $c_j$
  $t_j \leftarrow |docs_j|$: total number of documents in $c_j$
  $\hat{P}(c_j) \leftarrow \frac{t_j}{|D|}$
  $TF_j \leftarrow$ total number of words in $docs_j$ (counting duplicates)
  for each word $w_i$ in $V$ do
    $TF_{i,j} \leftarrow$ total number of times word $w_i$ occurs in $docs_j$
    $\hat{P}(w_i|c_j) \leftarrow \frac{TF_{i,j}+1}{TF_j+|V|}$





# Naive Bayes Text Classification algorithm

Use estimated $\hat{P}(c_j)$ and $\hat{P}(w_i|c_j)$ to classify a new document.

$\textsc{Classify\_Naive\_Bayes\_text}(doc)$

remove from $doc$ all words not included in vocabulary $V$
return

$$v_{NB} = \operatorname*{argmax}_{c_j \in C} \hat{P}(c_j) \prod_{i=1}^{length(doc)} \hat{P}(w_i|c_j)$$

# Text Classification improvements

- Stop words: remove from all the documents common words ("the", "a", etc.)
- Stemming: replace words with basic forms ("likes" $\rightarrow$ "like", "liking" $\rightarrow$ "like", etc.)
- Bi-gram, n-gram: token is a sequence of words
- ...