Skew braces and solutions to the Yang–Baxter equation

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Contents

Lecture 1. 21/02/2024 3
§ 1.1. The Yang–Baxter equation 3
§ 1.2. The set-theoretic version 3
§ 1.3. A characterisation 3
§ 1.4. First examples 4
§ 1.5. Set-theoretic solutions to the Yang–Baxter equation and III Reidemeister move 4
§ 1.6. The derived solution 5
§ 1.7. Involutive solutions 5
§ 1.8. Skew braces 6
§ 1.9. Basic properties of skew braces 7
§ 1.10. Skew braces and solutions 8
§ 1.11. Subbraces and ideals. 9
§ 1.12. The isomorphism theorems 10
Lecture 2. 22/02/2024 11
§ 2.1. Useful definitions and results 11
§ 2.2. Selection of problems 11
§ 2.3. More exercises 12
Lecture 3. 23/02/2024 15
§ 3.1. From solutions to skew braces 15
§ 3.2. The permutation group of a solution 15
§ 3.3. Simple solutions 16
Appendix 18
§ 3.4. Radical rings 18
§ 3.5. An intriguing connection between group actions and solutions 20
§ 3.6. The retraction of a solution. 21
Some solutions 23
References 25
Index 26
Index 26

The notes correspond to the series of lectures on *Skew braces and solutions to the Yang–Baxter equation* taught as part of the conference Introduction to Modern Advances in Algebra.

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Lecture 1. 21/02/2024

§ 1.1. The Yang–Baxter equation. The Yang-Baxter equation (YBE) is one important equation in mathematical physics. It first appeared in two independent papers of Yang [**9**] and Baxter [**1**].

DEFINITION 1.1. A solution of the *Yang–Baxter equation* is a linear map $R : V \otimes V \to V \otimes V$, where V is a vector space such that

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where R_{ij} denotes the map $V \otimes V \otimes V \to V \otimes V \otimes V$ acting as R on the (i, j) factor and as the identity on the remaining factor.

Let $\tau : V \otimes V \to V \otimes V$ be the map $\tau(u \otimes v) = v \otimes u$ for $u, v \in V$. It's easy to check (try!) that $R : V \otimes V \to V \otimes V$ is a solution of the Yang–Baxter equation if and only if $\bar{R} := \tau R$ satisfies

$$\bar{R}_{12}\bar{R}_{23}\bar{R}_{12} = \bar{R}_{23}\bar{R}_{12}\bar{R}_{23}.$$

An interesting class of solutions of the Yang–Baxter equation arises when V has a R-invariant basis X. In this case, the solution is said to be set-theoretic.

§ 1.2. The set-theoretic version. Drinfeld in [**4**] observed it makes sense to consider the Yang–Baxter equation in the category of sets and stated that

it would be interesting to study set-theoretic solutions.

These lectures will focus on set-theoretic solutions to the Yang-Baxter equation and their connection with known and "new" algebraic structures.

DEFINITION 1.2. A *set-theoretic solution to the Yang–Baxter equation* is a pair (X, r) where X is a non-empty set and $r : X \times X \to X \times X$ is a map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r) \tag{1.1}$$

CONVENTION 1. If (X, r) is a set-theoretic solution to the Yang–Baxter equation, we write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

where $\lambda_x, \rho_x : X \to X$.

DEFINITION 1.3. Let (X, r) be a set-theoretic solution to the Yang–Baxter equation. We say that

- (X, r) is *bijective* if r is bijective.
- (X, r) is *finite* if X is finite.
- (X, r) is *non-degenerate* if λ_x, ρ_x are bijective for all $x \in X$.

§ 1.3. A characterisation.

PROPOSITION 1.4. *Let X be a non-empty set and $r : X \times X \to X \times X$ be a map, written ase $r(x, y) = (\lambda_x(y), \rho_y(x))$. Then r satisfies equation 1.1 if and only if*

1) $\lambda_x\lambda_y = \lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}$

2) $\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y) = \rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y)$

3) $\rho_z\rho_y = \rho_{\rho_z(y)}\rho_{\lambda_y(z)}$

for all $x, y, z \in X$.

In particular, (X, r) is a solution to the Yang–Baxter equation when r is bijective.

PROOF. Let us write $r_1 = r \times \mathrm{id}$ and $r_2 = \mathrm{id} \times r$. Then

$$\begin{aligned} r_1 r_2 r_1(x,y,z) &= r_1 r_2(\lambda_x(y), \rho_y(x), z) \\ &= r_1(\lambda_x(y), \lambda_{\rho_y(x)}(z), \rho_z\rho_y(x)) \\ &= (\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z), \rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y), \rho_z\rho_y(x)), \end{aligned}$$

and

$$\begin{aligned} r_2 r_1 r_2(x,y,z) =& r_2 r_1(x, \lambda_y(z), \rho_z(y)) \\ &= r_2(\lambda_x\lambda_y(z), \rho_{\lambda_y(z)}(x), \rho_z(y)) \\ &= (\lambda_x\lambda_y(z), \lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y), \rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x)). \end{aligned}$$

Therefore $r_1 r_2 r_1 = r_2 r_1 r_2$ if and only if 1), 2) and 3) hold. □

§ 1.4. First examples.

EXAMPLES 1.5. Let X be a non-empty set.

1) The pair $(X, \mathrm{id}_{X\times X})$ is a set-theoretic solution to the Yang–Baxter equation. Note that $(X, \mathrm{id}_{X\times X})$ is not non-degenerate, since $\lambda_x(y) = x$ and $\rho_y(x) = y$, for all $x, y \in X$.

2) Let $\tau : X \times X \to X \times X$ be the flip map, i.e. $\tau(x,y) = (y,x)$ for all $x, y \in X$. Then, the pair (X, τ) is a set-theoretic solution to the Yang–Baxter equation. Moreover, it is non-degenerate since $\lambda_x = \rho_x = \mathrm{id}_X$ for all $x \in X$.

3) Let λ, ρ be permutations of X. Then $r(x,y) = (\lambda(y), \rho(x))$ is a non-degenerate set-theoretic solution to the Yang–Baxter equation if and only if $\lambda\rho = \rho\lambda$. Moreover, (X,r) is involutive if and only if $\rho = \lambda^{-1}$. The solution (X,r) is called a *permutational solution* or a *Lyubashenko's solution*.

If we have a bit more structure on the set X, we can define more sophisticated solutions.

EXAMPLE 1.6. Let G be a group and let

$$\begin{aligned} r_1(x,y) &= (y, y^{-1}xy) \\ r_2(x,y) &= (x^2y, y^{-1}x^{-1}y). \end{aligned}$$

Then (X, r_1) and (X, r_2) are bijective non-degenerate set-theoretic solutions to the Yang–Baxter equation.

§ 1.5. Set-theoretic solutions to the Yang–Baxter equation and III Reidemeister move. Let us represent the map $r : X \times X \to X \times X$ as a crossing and the identity on X as a straight line; see Figure 1.

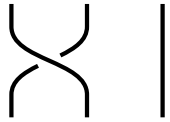


FIGURE 1. The map r is represented by a crossing and the identity as a straight line.

Then, the Yang–Baxter equation can be pictured as in Figure 2.
Moreover, we have the following lemma under the assumption of (X,r) being non-degenerate

LEMMA 1.7. *Let (X,r) be a solution to the Yang–Baxter equation.*

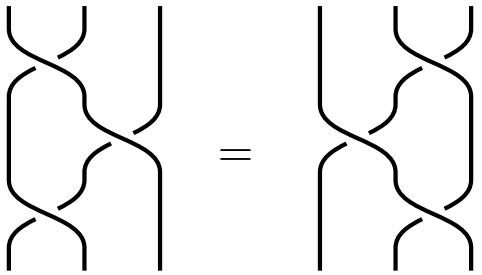


FIGURE 2. The Yang–Baxter equation.

1) *Given $x,u \in X$, there exist unique $y,v \in X$ such that $r(x,y) = (u,v)$.*
2) *Given $y,v \in X$, there exist unique $x,u \in X$ such that $r(x,y) = (u,v)$.*

PROOF. For the first claim take $y = \lambda_x^{-1}(u)$ and $v = \rho_y(x)$. For the second, $x = \rho_y^{-1}(v)$ and $u = \lambda_x(y)$. □

So, the bijectivity of r means that any row in Figure 3 determines the whole square. By Lemma 1.7 we have that non-degeneracy means that any column in Figure 3 also determines the entire square.

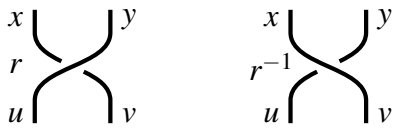


FIGURE 3. Any row or column determines the whole square.

§ 1.6. The derived solution.

PROPOSITION 1.8. *Let (X,r) be a non-degenerate set-theoretic solution to the Yang–Baxter equation. For any $x,y \in X$ consider*

$$\sigma_y(x) = \lambda_y \rho_{\lambda_x^{-1}(y)}(x).$$

Then $s : X \times X \to X \times X$ defined by $s(x,y) = (y, \sigma_y(x))$ is a solution to the Yang-Baxter equation. Moreover, r is bijective if and only if σ_y is bijective for every $y \in X$.

EXERCISE 1.9. Prove Proposition 1.8.

CONVENTION 2. From now on, a *solution* will always mean a non-degenerate bijective set-theoretic solution to the Yang–Baxter equation.

DEFINITION 1.10. Let (X,r) be a solution. The pair (X,s) is called the *derived solution* of the solution (X,r).

§ 1.7. Involutive solutions. In [**8**], Rump proved that radical rings provide examples of involutive solutions.

DEFINITION 1.11. A ring $(R,+,\cdot)$ is said to be *radical* if R with respect to the binary operation $\circ$ defined by $x \circ y = x + y + xy$ is a group. We denote by x' the inverse of x with respect to $\circ$.

DEFINITION 1.12. A solution (X,r) is said to be *involutive* if $r^2 = \mathrm{id}_{X\times X}$.

PROPOSITION 1.13. *Let R be a radical ring. Then (R,r), where*

$$r(x,y) = (-x+x\circ y, (-x+x\circ y)'\circ x\circ y) \tag{1.2}$$

is an involutive solution to the YBE.

EXERCISE 1.14. Let X be a non-empty set and $\lambda_x : X \to X$ bijective maps. Define $r : X\times X \to X\times X$ by $r(x,y) = (\lambda_x(y), \lambda^{-1}_{\lambda_x(y)}(x))$. Prove that (X,r) is an involutive map satisfying (1.1) if and only if

$$\lambda_x\lambda_{\lambda_x^{-1}(y)} = \lambda_y\lambda_{\lambda_y^{-1}(x)},$$

for all $x,y \in X$.

§ 1.8. Skew braces. Do we need radical rings to produce solutions of the form (1.2)?

DEFINITION 1.15. A *skew (left) brace* is a triple $(A,+,\circ)$, where $(A,+)$ and $(A,\circ)$ are (not necessarily abelian) groups and

$$a\circ(b+c) = (a\circ b) - a + (a\circ c) \tag{1.3}$$

holds for all $a,b,c \in A$.

The groups $(A,+)$ and $(A,\circ)$ are respectively the *additive* and *multiplicative* group of the skew brace A.

One says that a skew left brace A is of *abelian type* (it also is simply called a left brace) if $(A,+)$ is an abelian group. In general, the properties of the additive group determine the type of a skew brace[1].

REMARK 1.16. Even though we use the additive notation, the group $(A,+)$ is not necessarily abelian.

CONVENTION 3. The identity of $(A,+)$ will be denoted by 0 and the inverse of an element a will be denoted by $-a$.

As for radical rings, we write a' to denote the inverse of a with respect to the circle operation $\circ$.

Right skew braces are defined similarly.

DEFINITION 1.17. A *skew right brace* is a triple $(A,+,\circ)$, where $(A,+)$ and $(A,\circ)$ are groups and

$$(a+b)\circ c = a\circ c - c + b\circ c.$$

holds for all $a,b,c \in A$.

EXERCISE 1.18. Prove that there exists a bijective correspondence between skew left braces and skew right braces.

CONVENTION 4. From now on, with the term skew brace, we will always mean a skew left brace.

[1]Such terminology is borrowed from Hopf-Galois extension where the additive group determines the type of the extension.

EXAMPLE 1.19. Let $(A,+)$ be a group. Then A is a skew brace with $a\circ b = a+b$ for all $a,b\in A$. Such a skew brace is called a *trivial skew brace*.

EXAMPLE 1.20. Let $(A,+)$ be a group. The operation $a\circ b = b+a$ turns A into a skew brace. Such a skew brace is called an *almost trivial skew brace*.

§ 1.9. Basic properties of skew braces.

LEMMA 1.21. *Let A be a skew brace. The following statements hold.*

1) $1=0$*, where* 0 *denotes the identity of* $(A,+)$ *and by* 1 *the identity of* $(A,\circ)$.
2) $-(a\circ b) = -a+a\circ(-b)-a$*, for all* $a,b\in A$.

PROOF. By (1.3) we have

$$0 = 1\circ 0 = 1\circ(0+0) = 1\circ 0 - 1 + 1\circ 0 = -1.$$

Hence $0=1$. Now, let $a,b\in B$. From what we just proved and by (1.3) we have

$$a = a\circ 0 = a\circ(b-b) = a\circ b - a + a\circ(-b).$$

and 2) follows. □

PROPOSITION 1.22. *Let A be a skew brace. For each $a\in A$, the map*

$$\lambda_a : A\to A, b\mapsto -a+a\circ b$$

is an automorphism of $(A,+)$.
Moreover, the map $\lambda : (A,\circ)\to \mathrm{Aut}(A,+), a\mapsto\lambda_a$*, is a group homomorphism.*

PROOF. First, let us prove that λ_a is an endomorphism of $(A,+)$, for all $a\in A$. We have that

$$\lambda_a(b+c) = -a+a\circ(b+c) \overset{(1.3)}{=} -a+a\circ b - a + a\circ c,$$

for all $b,c\in A$. Now, for any $b\in A$,

$$\lambda_0(b) = -0+0\circ b \overset{1=0}{=} b,$$

hence $\lambda_0 = \mathrm{id}_A$. Moreover, for any $a,b,c\in A$,

$$\lambda_a\lambda_b(c) = -a+a\circ(-b+b\circ c) = -a+a\circ(-b)-a+a\circ b\circ c = -(a\circ b)+a\circ b\circ c = \lambda_{a\circ b}(c).$$

Hence, $\lambda_a\lambda_b = \lambda_{a\circ b}$, for all $a,b\in A$. It follows that for any $a\in A$, the map λ_a is bijective with inverse $\lambda_{a'}$. □

EXERCISE 1.23. Let A be a skew brace. Prove that

$$a\circ(a'+b) = \lambda_a(b),$$

for all $a,b\in A$. As a consequence, we have that $\rho_b(a) = (a'+b)'\circ b$, for all $a,b\in A$.

PROPOSITION 1.24. *Let A be a brace. For each $a\in A$, the map*

$$\rho_b : A\to A, \quad \mapsto (\lambda_a(b))'\circ a\circ b,$$

is bijective. Moreover, the map $\rho : (A,\circ)\to \mathrm{Sym}(A)$, $b\mapsto\rho_b$*, satisfies* $\rho_c\rho_b = \rho_{b\circ c}$*, for all* $b,c\in A$.

Proof. By Exercise 1.23, we get that $\rho_b(a) = (a'+b)'\circ b$, for all $a,b \in A$. Now, for all $a \in A$, we have that

$$\rho_0(a) = (a'+0)'\circ 0 = a,$$

i.e., $\rho_0 = \mathrm{id}_A$. Moreover, for all $a,b,c \in A$, we have

$$\begin{aligned}\rho_c\rho_b(a) &= ((\rho_b(a))'+c)'\circ c = (((a'+b)'\circ b)'+c)'\circ c\\ &= ((b'\circ(a'+b)+c)'\circ c = (b'\circ a' - b + c)'\circ c\\ &= (b'\circ(a'+b\circ c))'\circ c = (a'+b\circ c)'\circ b\circ c\\ &= \rho_{b\circ c}(a),\end{aligned}$$

i.e., $\rho_{b\circ c} = \rho_c\rho_b$. It also follows that ρ_b is bijective with inverse $\rho_{b'}$ for every $b \in A$. □

§ 1.10. Skew braces and solutions. Now we can state the theorem that gives a first connection of skew braces with solutions. Guarnieri and Vendramin have proved the following result in [**6**] extending an analogous result proved by Rump in [**8**] for involutive solutions.

Theorem 1.25. *Let A be a skew brace. Then (A, r_A), where*

$$r_A(x,y) = (-x+x\circ y, (-x+x\circ y)'\circ x\circ y)$$

is a bijective solution to the YBE. Moreover, (A,r_A) is involutive if and only if A is of abelian type.

Proof. As before, let us set

$$\begin{aligned}\lambda_x(y) &= -x+x\circ y\\ \rho_y(x) &= (\lambda_x(y))'\circ x\circ y.\end{aligned}$$

By Proposition 1.22 and Proposition 1.24, we have that $\lambda : A \to \mathrm{Aut}(A,+)$ is a left action of A on itself and $\rho : A \to \mathrm{Sym}(A)$ is a right action of A on itself. Moreover, by definition

$$\lambda_x(y)\circ\rho_y(x) = x\circ y,$$

i.e. condition (3.1) in Theorem 3.21 is satisfied. Hence, by Theorem 3.21, (A,r_A) is a solution.

Now let us compute r_A^2,

$$r_A^2(x,y) = (-\lambda_x(y)+\lambda_x(y)\circ\rho_y(x), (-\lambda_x(y)+\lambda_x(y)\circ\rho_y(x))'\circ\lambda_x(y)\circ\rho_y(x)).$$

First if we assume $(A,+)$ abelian, we have

$$\begin{aligned}-\lambda_x(y)+\lambda_x(y)\circ\rho_y(x) &= -(-x+x\circ y)+x\circ y = -(x\circ y)+x+x\circ y\\ &\overset{\text{Lemma 1.21}}{=} -x+x\circ(-y)+x\circ y = x\circ(-y)-x+x\circ y\\ &\overset{(1.3)}{=} x\circ(-y+y) = x\end{aligned}$$

and

$$(-\lambda_x(y)+\lambda_x(y)\circ\rho_y(x))'\circ\lambda_x(y)\circ\rho_y(x) = x'\circ x\circ y = y.$$

Hence, (A,r_A) is involutive.

Now let us assume (A,r_A) involutive. In particular, for all $x,y \in A$

$$x = -\lambda_x(y)+\lambda_x(y)\circ\rho_y(x) = -(x\circ y)+x+x\circ y.$$

For the arbitrary of y and since $(A,\circ)$ is a group, it follows $x = -y+x+y$, for all $x,y \in A$, i.e. $(A,+)$ is abelian. □

Exercise 1.26. Let A be a skew brace. Prove that

$$a+b=a\circ\lambda_a^{-1}(b)$$

and

$$a\circ b=a+\lambda_a(b)$$

§ 1.11. Subbraces and ideals.

Definition 1.27. Let A be a skew brace.

A *subbrace* f A is a subset B of A such that $(B,+)$ is a subgroup of $(A,+)$ and $(B,\circ)$ is a subgroup of $(A,\circ)$.

A *left ideal* of A is a subgroup $(I,+)$ of $(A,+)$ such that $\lambda_b(I)\subseteq I$ for all $b\in B$, i.e. $\lambda_b(x)\in I$ for all $b\in A$ and $x\in I$.

A *strong left ideal* of A is a left ideal I of A such that $(I,+)$ is a normal subgroup of $(A,+)$.

Lemma 1.28. *A left ideal I of a skew brace A is a subbrace of B.*

Proof. We need to prove that $(I,\circ)$ is a subgroup of $(A,\circ)$. Clearly, I is non-empty, as it is an additive subgroup of A. If $x,y\in I$, then

$$x\circ y=x-x+x\circ y=x+\lambda_x(y)\in I+I=I$$

and

$$x'=-\lambda_{x'}(x)\in I.$$

□

Exercise 1.29. Let A be a skew brace. Then

$$\mathrm{Fix}(A)=\{b\in B\colon\ \lambda_x(b)=b,\ \forall b\in A\}$$

is a left ideal of A.

Definition 1.30. An *ideal* of a skew brace A is a strong left ideal I of A such that $(I,\circ)$ is a normal subgroup of $(A,\circ)$.

In general, left ideals, strong left ideals and ideals are different notions.

Definition 1.31. Let A be a skew brace. The subset $\mathrm{Soc}(A)=\ker\lambda\cap Z(A,+)$ is the *socle* of A.

Proof. First, $\mathrm{Soc}(A)\neq\emptyset$, since $0\in\mathrm{Soc}(A)$. Moreover, if $x\in\mathrm{Soc}(A)$, then $x'=\lambda_x(x')=-x$. It follows that if $x,y\in\mathrm{Soc}(A)$ then

$$\lambda_{x-y}=\lambda_{x\circ y'}=\lambda_x\lambda_{y'}=\lambda_x\lambda_y^{-1}=\mathrm{id}_A,$$

and, clearly $x-y\in Z(A,+)$. Hence, $\mathrm{Soc}(A)$ is an additive subgroup of A and since $\mathrm{Soc}(A)$ is a subgroup of $Z(A,+)$ it is also a normal additive subgroup of A. Moreover, for all $x\in\mathrm{Soc}(A)$ and $a\in A$:

$$\lambda_a(x)=a\circ x-a \tag{1.4}$$

$$\lambda_a(x)=a\circ x\circ a'. \tag{1.5}$$

For the first equality we have that applying Exercise 1.26

$$\lambda_a(x) = a \circ (a' + x) = a \circ (x + a') \overset{(1.3)}{=} a \circ x - a,$$

for the second equality

$$\lambda_a(x) = a \circ (a' + x) = a \circ (x \circ \lambda_x(a')) = a \circ x \circ a'.$$

It follows that, for all $x \in \mathrm{Soc}(A)$ and $a, b \in A$, we have

$$\lambda_{\lambda_a(x)} \overset{(1.4)}{=} \lambda_{a \circ x \circ a'} = \lambda_a \lambda_x \lambda_a^{-1} = \lambda_a \lambda_a^{-1} = \mathrm{id}_A$$

and, by Exercise 1.26,

$$\begin{aligned} b + \lambda_a(x) = b \circ \lambda_b^{-1} \lambda_a(x) = b \circ \lambda_{b' \circ a}(x) &\overset{(1.5)}{=} a \circ x \circ a \circ b \\ &\overset{(1.5)}{=} \lambda_a(x) \circ b = \lambda_a(x) + \lambda_{\lambda_a(x)}(b) = \lambda_a(x) + b, \end{aligned}$$

i.e., $\lambda_a(x) \in Z(A, +)$. Finally, it also follows that for any $a \in A$ and $x \in \mathrm{Soc}(A)$, $a \circ x \circ a' \in \mathrm{Soc}(A)$. Therefore, $\mathrm{Soc}(A)$ is an ideal of A. □

Exercise 1.32. Let A be a skew brace. Prove that $\mathrm{Soc}(A) = \ker \lambda \cap \ker \rho$.

Definition 1.33. Let A be a skew brace. The subset $\mathrm{Ann}(A) = \mathrm{Soc}(A) \cap Z(B, \circ)$ is the *annihilator* of B.

Proposition 1.34. *The annihilator of a skew brace A is an ideal of A.*

Proof. First, if $x, y \in \mathrm{Ann}(A)$, then $x - y \in \mathrm{Soc}(A)$ and for any $a \in A$

$$(x - y) \circ a = x \circ y' \circ a = x \circ a \circ y' a \circ x \circ y' = a \circ (x - y),$$

i.e. $x - y \in \mathrm{Ann}(A)$. Now, since $\mathrm{Ann}(A) \subseteq Z(A, +) \cap Z(A, \circ)$, we only need to prove $\lambda_a(x) \in \mathrm{Ann}(A)$, for all $x \in \mathrm{Ann}(A)$ and $a \in A$. By (1.4) we have that $\lambda_a(x) = a \circ x \circ a' = x \circ a \circ a' = x \in \mathrm{Ann}(A)$. □

§ 1.12. The isomorphism theorems. If A is a skew brace and I is an ideal of A, then $a + I = a \circ I$ for all $a \in A$.

This allows us to prove that there exists a unique skew brace structure over A/I such that the map

$$A \mapsto A/I, \quad a \mapsto a + I = a \circ I,$$

is a homomorphism of skew braces.

Definition 1.35. The skew brace A/I is the *quotient skew brace* of A modulo I.

It is possible to prove the isomorphism theorems for skew braces. (See Exercises 2.9–2.12).

Lecture 2. 22/02/2024

§ 2.1. Useful definitions and results.

DEFINITION 2.1. A *set-theoretic solution to the Yang–Baxter equation* is a pair (X,r) where X is a non-empty set and $r : X \times X \to X \times X$ is a map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r) \tag{2.1}$$

DEFINITION 2.2. Let (X,r) be a set-theoretic solution to the Yang–Baxter equation. We say that

- (X,r) is *bijective* if r is bijective.
- (X,r) is *finite* if X is finite.
- (X,r) is *non-degenerate* if λ_x, ρ_x are bijective for all $x \in X$.

CONVENTION 5. From now on, a *solution* will always mean a bijective non-degenerate set-theoretic solution to the Yang–Baxter equation.

DEFINITION 2.3. A *skew (left) brace* is a triple $(A,+,\circ)$, where $(A,+)$ and $(A,\circ)$ are (not necessarily abelian) groups and

$$a \circ (b+c) = (a \circ b) - a + (a \circ c) \tag{2.2}$$

holds for all $a,b,c \in A$.

The groups $(A,+)$ and $(A,\circ)$ are respectively the *additive* and *multiplicative* group of the skew brace A.

FACT 2.4. *Let A be a skew brace. The following statements hold.*

1) $1 = 0$*, where* 0 *denotes the identity of* $(A,+)$ *and by* 1 *the identity of* $(A,\circ)$.
2) $-(a \circ b) = -a + a \circ (-b) - a$*, for all* $a,b \in A$.

FACT 2.5. *Let A be a skew brace. For each $a \in A$, the map*

$$\lambda_a : A \to A, b \mapsto -a + a \circ b$$

is an automorphism of $(A,+)$.

Moreover, the map $\lambda : (A,\circ) \to \mathrm{Aut}(A,+), a \mapsto \lambda_a$*, is a group homomorphism.*

FACT 2.6. *Let A be a brace. For each $a \in A$, the map*

$$\rho_b : A \to A, \quad \mapsto (\lambda_a(b))' \circ a \circ b,$$

is bijective. Moreover, the map $\rho : (A,\circ) \to \mathrm{Sym}(A)$*, $b \mapsto \rho_b$, satisfies* $\rho_c \rho_b = \rho_{b \circ c}$*, for all* $b,c \in A$.

DEFINITION 2.7. An *ideal* of a skew brace A is a strong left ideal I of A such that $(I,\circ)$ is a normal subgroup of $(A,\circ)$.

§ 2.2. Selection of problems.

It is possible to prove the isomorphism theorems for skew braces. (See Exercises 2.9–2.12).

EXERCISE 2.8. A map $f : A \to B$ between two skew braces A and B is a *homomorphism* of skew braces if $f(a+b) = f(a) + f(b)$ and $f(a \circ b) = f(a) \circ f(b)$, for all $a,b \in A$.The *kernel* of f is

$$\ker f = \{a \in A : f(a) = 0\}.$$

Let $f : A \to B$ be a homomorphism of two skew braces A and B. Prove that $\ker f$ is an ideal of A.

Exercise 2.9. Let $f : A \to B$ be a homomorphism of skew braces. Prove that $A/\ker f \cong f(A)$.

Exercise 2.10. Let A be a skew brace and let B be a subbrace of A. Prove that if I is an ideal of A, then $B \circ I$ is a subbrace of A, $B \cap I$ is an ideal of B and $(B \circ I)/I \cong B/(B \cap I)$.

Exercise 2.11. Let A be a skew brace and I and J be ideals of A. Prove that if $I \subseteq J$, then $A/J \cong (A/I)/(J/I)$.

Exercise 2.12. Let A be a skew brace and let I be an ideal of A. Prove that there is a bijective correspondence between (left) ideals of A containing I and (left) ideals of A/I.

Exercise 2.13. Let A be a skew brace and I be a characteristic subgroup of the additive. Prove that I is a left ideal of A.

Let us get some concrete examples of skew braces.

Exercise 2.14. Let p be a prime number and let $A = \mathbb{Z}/(p^2)$ the ring of integers modulo p^2. Prove that A with respect to the usual sum and the operation given by $x \circ y = x + y + pxy$ is a skew brace.

Exercise 2.15 (The semidirect product). Let A, B be skew braces. Let $\alpha : (B, \circ) \to \mathrm{Aut}(A, +, \circ)$ be a homomorphism of groups. Define two operations on $A \times B$ by

$$(a,x) + (b,y) = (a+b, x+y)$$
$$(a,x) \circ (b,y) = (a \circ \alpha_x(b), x \circ y),$$

for all $a, b \in A$ and $x, y \in B$. Prove that $(A \times B, +, \circ)$ is a skew brace.
This skew brace is the *semidirect product* of the skew brace A by B via α, and it is denoted by $A \rtimes_\alpha B$.

Exercise 2.16. Let $(A, +)$ be a group with an *exact factorisation* through the subgroups B and C (i.e. B and C are subgroups of A such that $B \cap C = \{0\}$ and $A = B + C$). This means that each $x \in A$ can be written in a unique way as $x = x_B + x_C$, for some $x_B \in B$ and $x_C \in C$. Set

$$x \circ y = x_B + y + x_C.$$

Prove that

1) $(A, \circ)$ is a group isomorphic to $B \times C$, the direct product of B and C.
2) $(A, +, \circ)$ is a skew brace.

§ 2.3. More exercises.

Exercise 2.17. Let (X,r) be a set-theoretic solution to the Yang–Baxter equation. Define for all $x,y \in X$

$$\bar{r}(x,y) = \tau r \tau(x,y) = (\rho_x(y), \lambda_y(x)).$$

Then $(X,\bar{r})$ is a set-theoretic solution to the Yang–Baxter equation.

A *(right) shelf* is a pair $(X,\triangleleft)$ where X is a non-empty set and $\triangleleft$ is a binary operation such that

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z).$$

If, in addition, the maps $\rho_y : X \to X, x \mapsto x \triangleleft y$ are bijective for all $y \in X$, then $(X,\triangleleft)$ is called a *(right) rack*.

Exercise 2.18. Let X be a non-empty set. Let $\triangleleft : X \times X \to X$ be a binary operation and define $r : X \times X \to X \times X$ such that $r(x,y) = (y, x \triangleleft y)$. Then r satisfies equation 1.1 if and only if $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ holds for all $x,y,z \in X$. Moreover, r is bijective if and only if the maps $\rho_y : X \to X, x \mapsto x \triangleleft y$ are bijective.

Exercise 2.19. Let G be a group. Prove that G with respect to the binary operation $\triangleleft$ defined by $x \triangleleft y = y^{-1}xy$ is a rack.

Exercise 2.20. Let (X,r) be a solution. Define

$$x \triangleleft y = \lambda_y \rho_{\lambda_x^{-1}(xy)}(x).$$

Prove that $(X,\triangleleft)$ is a shelf.

Exercise 2.21. Let A be a skew brace. Prove that

$$\rho_b(a) = \lambda^{-1}_{\lambda_a(b)}(-(a \circ b) + a + a \circ b)$$

Exercise 2.22. Let $(A,+)$ be a (not necessarily abelian) group.

1) Prove that a structure of skew brace over A is equivalent to an operation $A \times A \to A$ $(a,b) \mapsto a * b$, such that

$$a * (b+c) = a * b + b + a * c - b$$

holds for all $a,b,c \in A$ and the operation $a \circ b = a + a * b + c$ turns A into a group.

2) Deduce that radical rings are examples of skew braces.

Exercise 2.23. Let A be a skew brace and $a * b = \lambda_a(b) - b = -a + a \circ b - b$. Prove the following identities:

1) $a * (b+c) = a * b + b + a * c - b$.

2) $(a \circ b) * c = (a * (b * c)) + b * c + a * c$.

Exercise 2.24. Let $(A,+,\circ)$ be a triple, where $(A,+)$ and $(A,\circ)$ are groups, and $\lambda : A \to \mathrm{Sym}(A)$, $a \mapsto \lambda_a$ with $\lambda_a(b) = -a + a \circ b$. Prove that the following statements are equivalent:

1) $(A,+,\circ)$ is a skew brace.
2) $\lambda_a\lambda_b(c) = \lambda_{a\circ b}(c)$, for all $a,b,c \in A$.
3) $\lambda_a(b+c) = \lambda_a(b) + \lambda_a(c)$, for all $a,b,c \in A$.

Exercise 2.25. Let $(A,+)$ and $(M,+)$ be groups and let $\alpha\colon A \to \mathrm{Aut}(M)$ be a group homomorphism. Prove that $M \times A$ with

$$(x,a) + (y,b) = (x+y, a+b),$$
$$(x,a) \circ (y,b) = (x + \alpha_a(y), a+b)$$

is a skew brace. Similarly, prove that $M \times A$ with

$$(x,a) + (y,b) = (x + \alpha_a(y), a+b),$$
$$(x,a) \circ (y,b) = (x+y, b+a)$$

is a skew brace.

Exercise 2.26. Consider the semidirect product $A = \mathbb{Z}/(3) \rtimes \mathbb{Z}/(2)$ of the trivial skew braces $\mathbb{Z}/(3)$ and $\mathbb{Z}/(2)$ via the non-trivial action of $\mathbb{Z}/(2)$ over $\mathbb{Z}/(3)$.
Prove that $\mathrm{Fix}(A) = \{b \in B\colon \lambda_x(b) = b,\ \forall b \in A\}$ is not an ideal of A.

Lecture 3. 23/02/2024

§ 3.1. From solutions to skew braces.

DEFINITION 3.1. Let (X,r) be a solution. The *structure group* of (X,r) is the group

$$G(X,r) = \langle X : xy = \lambda_x(y)\rho_y(x) \text{ for all } x,y \in X \rangle.$$

The *derived structure group* of (X,r) is the group

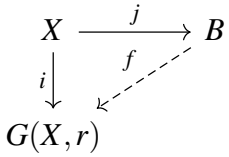
$$A(X,r) = \langle X : x\lambda_x(y) = \lambda_x(y)\lambda_{\lambda_x(y)}\rho_y(x) \text{ for all } x,y \in X \rangle.$$

THEOREM 3.2. *[*[**5**] *or* [**7**]*] Let (X,r) be a solution. Then, there exists a unique structure of skew brace with multiplicative group the structure group isomorphic to $G(X,r)$, additive structure isomorphic to $A(X,r)$ and such that $\lambda_{i(x)}(i(y)) = i(\lambda_x(y))$ for all $x,y \in X$, where $i : X \to G(X,r), x \mapsto x$ is the canonical map.*

PROOF. Omitted. □

The structure skew brace defined in the previous theorem satisfies the following universal property.

PROPOSITION 3.3. *Let $(B,+,\circ)$ be a skew brace, $j\colon X \to B$ be a map such that $\lambda_{j(x)}(j(y)) = j(\lambda_x(y))$ and $j(x)\circ j(y) = j(\lambda_x(y))\circ j(\rho_y(x))$, for all $x,y \in X$. Then there exists a unique homomorphism of skew braces $f\colon G(X,r) \to B$ such that $fi = j$, i.e.*



PROOF. Omitted. □

§ 3.2. The permutation group of a solution. Let (X,r) be a solution. Consider the structure group $G(X,r)$ of the solution (X,r). Let $i\colon X \to G(X,r)$ be the natural map.

The *permutation group* of (X,r) is the subgroup

$$\mathscr{G}(X,r) = \langle (\lambda_x, \sigma_x^{-1}) : x \in X \rangle \subseteq \mathrm{Sym}_X \times \mathrm{Sym}_X .$$

Since

$$\lambda_x\lambda_y = \lambda_{\lambda_x(y)}\lambda_{\rho_y(x)} \quad \text{and} \quad \rho_x^{-1}\rho_y^{-1} = \rho_{\lambda_x(y)}^{-1}\rho_{\rho_y(x)}^{-1}$$

for all $x,y \in X$, there exists a unique group homomorphism $h\colon G(X,r) \to \mathscr{G}(x,r)$ such that $hi(x) = (\lambda_x, \rho_x^{-1})$ for all $x \in X$. We write

$$h(a) = (\lambda_a, \rho_a^{-1})$$

for all $a \in G(X,r)$. By Theorem 3.2, $G(X,r)$ has a unique structure of skew brace with multiplicative group the structure group $G(X,r)$ and $\lambda_{i(x)}(i(y)) = i(\lambda_x(y))$ for all $x,y \in X$. One can prove that $\ker h$ is an ideal of the skew brace $G(X,r)$. This allows to give a skew brace structure to the permutation group.

§ 3.3. Simple solutions. Describing all solutions is a very difficult task. A strategy to tackle such a problem is to focus on "building blocks" such as indecomposable and simple solutions.

Definition 3.4. A solution (X,r) is said to be *decomposable* if there exist $\emptyset \neq Y,Z \subseteq X$ such that $Y \cup Z = X$, $Y \cap Z = \emptyset$ and $r(Y \times Y) \subseteq Y \times Y$, $r(Z \times Z) \subseteq Z \times Z$. Otherwise, (X,r) is said *indecomposable*.

It is not difficult to prove that a solution (X,r) is indecomposable if the group $\langle \lambda_x, \rho_y \colon x,y \in \rangle$ acts transitively on X.

Definition 3.5. A solution (X,r) is said to be *simple* if for any epimorphism of solutions $\varphi : (X,r) \to (Y,t)$, we have either φ an isomorphism or Y a singleton.

One can prove that a simple solution is, in particular, indecomposable.

Cedó and Okniński in [**2**] proved that some simple skew braces with additive structure abelian provide examples of (involutive) simple solutions.

Joyce, in 1982, studied simple racks and provided an algebraic characterisation of such racks. By Exercise 2.18, a rack provides a solution, and it is not difficult to see that simple racks are, in particular, examples of simple solutions.

In joint work with Jespers, Kubat, and Van Antwerpen [**3**], we obtain a brace theoretic classification of simple solutions.

Theorem 3.6. *A simple solution (X,r) is one of the following type:*

1) *(X,r) is a simple permutational solution,*
2) *(X,r) is a simple square-free[2] derived solution (i.e. it is a simple quandle[3]),*
3) *(X,r) is a simple solution embedded in a solution associated with a finite skew brace.*

3.3.1. *Simple permutational solution.* Recall that a (X,r) is a permutational solution if there exist λ, ρ commuting permutations such that $r(x,y) = (\lambda(y), \rho(x))$.

We can give a combinatorial characterisation of simple permutational solutions.

Proposition 3.7. *A permutational solution (X,r) is simple if and only if the cardinality of X is a prime number p, the group $H = \langle \lambda, \rho \rangle$ is cyclic of order p.*

In particular, (X,r) solution is isomorphic to $(\mathbb{Z}_p,t)$ such that $t(x,y) = (y+a, x+b)$ with $,a,b \in \mathbb{Z}$ and $(a,b) \neq (0,0)$.

3.3.2. *Simple quandle.* The description of simple square-free derived solutions coincides with the description obtained by Joyce of simple quandles.

Proposition 3.8. *A square-free derived solution (X,r) is simple if and only if (X,r) embeds in a solution (A,r) where A is a finite group and $r(y,y^{-1}xy)$ such that*

- *X is a conjucacy class generating A,*
- *the derived subgroup $[A,A]$ is the smallest non-zero normal subgroup of A,*
- *the quotient group $A/[A,A]$ is a cyclic group,*
- *the center $Z(A)$ is trivial.*

[2] A solution (X,r) is square-free if $r(x,x) = (x,x)$ for every $x \in X$.

[3] A rack $(X, \triangleleft)$ is a quandle if $x \triangleleft x = x$, for every $x \in X$.

3.3.3. *Simple solutions embedded in solutions associated with finite skew braces.* Let $(A,+,\circ)$ be a skew brace. In analogy with radical rings, we can associate to A another binary operation $*: A\times A\to A$ such that $a*b=-a+a\circ b+b$, for every $a,b\in A$.

EXERCISE 3.9. Let $(A,+,\circ)$ be a skew brace. Define A^2 the additive subgroup generated by $\{a*b=-a+a\circ b-b\colon a,b\in A\}$. Prove that A^2 is an ideal of A.

The ideal A^2 plays for skew braces, a similar role as that the derived subgroup plays for groups. With this observation in mind, the following proposition might not be surprising.

PROPOSITION 3.10. *A square-free solution (X,r), which is not a derived solution nor a permutational solution, is simple if and only if (X,r) is embedded in a solution associated with a non-trivial skew brace (A,r_A) such that*

- *X generates additively A,*
- *the ideal A^2 is the smallest non-zero ideal of A,*
- *the quotient skew brace A/A^2 is a trivial skew brace with additive (and multiplicative) group cyclic,*
- *the action of A^2 on X is transitive.*

Let us explain what the last item means. Let A be a skew brace, $X\subset A$ and I an ideal of A. We say that I *acts* on X if X is invariant under the action of the group $(I,+)\rtimes(I,\circ)$ (where the *lambda*-map gives the semidirect product).

Appendix

§ 3.4. Radical rings.

DEFINITION 3.11. A non-empty set R with two binary operations the addition $+$ (addition) and the multiplication $\cdot$ is a *ring* if

- $(R,+)$ is an abelian group,
- $(R,\cdot)$ is a semigroup (i.e. $\cdot$ is associative),
- The multiplication is distributive with respect to the addition, i.e.

$$a\cdot(b+c)=(a\cdot b)+(a\cdot c) \qquad \text{(left distributivity)}$$
$$(b+c)\cdot a=(b\cdot a)+(c\cdot a) \qquad \text{(right distributivity)}$$

for all $a,b,c\in R$.

A ring $(R,+,\cdot)$ is *unitary* if there is an element 1 in R such that $a\cdot 1=1\cdot a=a$ for all $a\in R$ (i.e., 1 is the *multiplicative identity*).

Let R be a non-unitary ring. Consider $R_1=\mathbb{Z}\times R$ with the addition defined component-wise and multiplication

$$(k,a)(l,b)=(kl,kb+la+ab)$$

for all $k,l\in\mathbb{Z}$ and $a,b\in R$.

Then R_1 is a ring and $(1,0)$ is its multiplicative identity.

Note that $\{0\}\times R$ is isomorphic to R as non-unitary rings.

EXERCISE 3.12. Let R be a non-unitary ring. Consider $R_1=\mathbb{Z}\times R$ as before. If $(k,x)\in R_1$ is invertible, then $k\in\{1,-1\}$.

DEFINITION 3.13. Let R be a unitary ring. The *(Jacobson) radical* $J(R)$ of R is defined as the intersection of all maximal left ideals[4] of R.

EXERCISE 3.14. Let R be a unitary ring.

1) Prove that $J(R)$ in an ideal of R.

2) Prove that $x\in J(R)$ if and only if $1+rx$ is invertible for all $r\in R$.

DEFINITION 3.15. A non-unitary ring R is a *(Jacobson) radical ring* if it is isomorphic to the Jacobson radical of a unitary ring.

PROPOSITION 3.16. *Let R be a non-unitary ring. The following statements are equivalent.*

1) *R is a radical ring.*

2) *For all $a\in R$ there exists a unique $b\in R$ such that $a+b+ab=a+b+ba=0$.*

3) *R is isomorphic to $J(R_1)$.*

PROOF. Let us first prove that 1) implies 2). Let M be a unitary ring such that R is isomorphic to its Jacobson radical $J(M)$ and let $\psi:R\to M$ be a homomorphism such that $\psi(R)$ is isomorphic to $J(M)$. Now, if $a\in R$, then $\psi(a)\in J(M)$. By Exercise 3.14, $1+\psi(a)$ is invertible, i.e. there exists $c\in M$ such that

$$(1+\psi(a))(1+c)=1=(1+c)(1+\psi(a)).$$

[4] A *left ideal* of R is an additive subgroup I of R such that $ax\in I$ for all $a\in R$ and $x\in I$.

It follows that $c \in J(M)$, i.e. $c = \psi(b)$ for some $b \in R$. Moreover, since ψ is a homomorphism

$$\begin{aligned} 1 &= (1+\psi(a))(1+c) = (1+\psi(a))(1+\psi(b)) \\ &= 1+\psi(a)+\psi(b)+\psi(a)\psi(b) = 1+\psi(a+b+ab) \end{aligned}$$

and

$$\begin{aligned} 1 &= (1+c)(1+\psi(a)) = (1+\psi(b))(1+\psi(a)) \\ &= 1+\psi(b)+\psi(a)+\psi(b)\psi(a) = 1+\psi(a+b+ba). \end{aligned}$$

Hence, 2) holds.

Now let us prove 2) implies 3). Let $a \in R$, we aim to prove that $(1,a) \in R_1$ is invertible. By 2) there exists $b \in R$ such that

$$\begin{aligned} (1,a)(1,b) &= (1,a+b+ab) = (1,0) \\ (1,b)(1,a) &= (1,b+a+ba) = (1,0). \end{aligned}$$

Now, consider $(k,a) \in J(R_1)$. We want to prove that $k=0$, i.e. $J(R_1) \subseteq \{0\} \times R$. Since $(k,a) \in J(R_1)$ follows that $(1,0)+(3,0)(k,a) =)(1+3k,3a)$ is invertible by Exercise 3.14, and so $k=0$. Therefore $J(R_1) \subseteq \{0\} \times R$. Moreover, let $(0,R) \in \{0\} \times R$. then

$$(1,0)+(k,a)(0,x) = (1,0)+(0,kx+ka) = (1,kx+ka)$$

which is invertible. So $(0,x) \in J(R_1)$. Finally the implication 3) implies 1) is trivially true. □

Definition 3.17. Let R be any ring. Define on R the binary operation $\circ$ called the *adjoint multiplication* of R

$$a \circ b = a+b+ab,$$

for all $a,b \in R$.

Lemma 3.18. *Then $(R,\circ)$ is a monoid with neutral element* 0.

Exercise 3.19. Prove Lemma 3.18.

Convention 6. If $a \in R$ is invertible in the monoid $(R,\circ)$, we will denote by a' its inverse.

Examples 3.20.

1) Let p be a prime and let $A = \mathbb{Z}/(p^2) = \mathbb{Z}/p^2\mathbb{Z}$ be the ring of integers modulo p^2. Then $(A,+)$ with a new multiplication $*$ defined by $a*b = pab$ is a radical ring. In this case, $a \circ b = a+b+pab$, and $a' = -a+pa^2$.

2) Let n be an integer such that $n > 1$. Let

$$A = \left\{ \frac{nx}{ny+1} : x,y \in \mathbb{Z} \right\} \subseteq \mathbb{Q}.$$

A is a (non-unitary) subring of $\mathbb{Q}$. In fact, A is a radical ring. A straightforward computation shows

$$\left(\frac{nx}{ny+1}\right)' = \frac{-nx}{n(x+y)+1}.$$

§ 3.5. An intriguing connection between group actions and solutions. The following theorem is the core result of the paper [7] by Lu, Yan Zhu.

THEOREM 3.21. *Let G be a group, let $\lambda : G \times G \to G, (x,y) \mapsto \lambda_x(y)$ a left group action of G on itself as a set and $\rho : G \times G \to G, (x,y) \mapsto \rho_y(x)$ a right group action of G on itself as a set. If the "compatibility" condition*

$$uv = \lambda_u(v)\rho_v(u) \tag{3.1}$$

holds, then (G,r), where

$$r : G \times G \to G \times G, \qquad (x,y) \mapsto (\lambda_x(y), \rho_y(x))$$

is a solution.

PROOF. Let us write $r_1 = r \times \mathrm{id}$ and $r_2 = \mathrm{id} \times r$,

$$\begin{aligned} r_1 r_2 r_1(x,y,z) &= (\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z), \rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y), \rho_z\rho_y(x)) \\ &= (u_1, v_1, w_1), \end{aligned}$$

and

$$\begin{aligned} r_2 r_1 r_2(x,y,z) &= (\lambda_x\lambda_y(z), \lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y), \rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x)) \\ &= (u_2, v_2, w_2). \end{aligned}$$

Then we obtain

$$\begin{aligned} u_1 v_1 w_1 &= \lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z)\rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y)\rho_z\rho_y(x) \\ &\overset{(3.1)}{=} \lambda_x(y)\lambda_{\rho_y(x)}(z)\rho_z\rho_y(x) \\ &\overset{(3.1)}{=} \lambda_x(y)\rho_y(x)z \\ &\overset{(3.1)}{=} xyz \end{aligned}$$

and, similarly

$$\begin{aligned} u_2 v_2 w_2 &= \lambda_x\lambda_y(z)\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y)\rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x) \\ &\overset{(3.1)}{=} \lambda_x\lambda_y(z)\rho_{\lambda_y(z)}(x)\rho_z(y) \\ &\overset{(3.1)}{=} x\lambda_y(z)\rho_z(y) \\ &\overset{(3.1)}{=} xyz. \end{aligned}$$

Hence

$$u_1 v_1 w_1 = xyz = u_2 v_2 w_2. \tag{3.2}$$

Moreover, since λ is a left action of G on itself, we get

$$u_1 = \lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z) = \lambda_{\lambda_x(y)\rho_y(x)}(z) \overset{(3.1)}{=} \lambda_{xy}(z) = \lambda_x\lambda_y(z) = u_2.$$

Similarly, since ρ is a right action

$$w_2 = \rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x) = \rho_{\lambda_y(z)\rho_z(y)}(x) \overset{(3.1)}{=} \rho_{yz}(x) = \rho_z\rho_y(x) = w_1.$$

From (3.2) and G being a group it follows that also $v_1 = v_2$. Moreover, λ_x and ρ_x are bijective maps by assumption. It is left to prove that r is bijective. First let us write $r(u,v) = (x,y)$, hence $\lambda_u(v) = x$, $\rho_v(u) = y$, and $uv = xy$. Now, since λ is an action and in particular $\lambda_v^{-1} = \lambda_{v^{-1}}$, we get

$$\lambda_y(v^{-1})u = \lambda_y(v^{-1})\rho_v^{-1}(y) = \lambda_y(v^{-1})\rho_{v^{-1}}(y) \overset{(3.1)}{=} yv^{-1} = x^{-1}u = (\lambda_u(v))^{-1}u,$$

and so

$$(\lambda_u(v))^{-1} = \lambda_{\rho_v(u)}(v^{-1}). \tag{3.3}$$

Similarly, expanding $v\rho_x(u^{-1})$ one proves

$$(\rho_v(u))^{-1} = \rho_{\lambda_u(v)}(u^{-1}). \tag{3.4}$$

Define

$$r'(x,y) = ((\rho_{x^{-1}}(y^{-1}))^{-1}, (\lambda_{y^{-1}}(x^{-1}))^{-1}).$$

Then

$$\begin{aligned} rr'(x,y) &= (\lambda_{(\rho_{x^{-1}}(y^{-1}))^{-1}((\lambda_{y^{-1}}(x^{-1}))^{-1})}, \rho_{(\lambda_{y^{-1}}(x^{-1}))^{-1}((\rho_{x^{-1}}(y^{-1}))^{-1})}) \\ &\overset{(3.3)\&(3.4)}{=} (\lambda^{-1}_{\rho_{x^{-1}}(y^{-1})}\lambda_{\rho_{x^{-1}}(y^{-1})}(x), \rho^{-1}_{\lambda_{y^{-1}}(x^{-1})}\rho_{\lambda_{y^{-1}}(x^{-1})}(y)) \\ &= (x,y). \end{aligned}$$

And

$$\begin{aligned} r'r(x,y) &= ((\rho_{(\lambda_x(y))^{-1}((\rho_y(x))^{-1})})^{-1}, (\lambda_{(\rho_y(x))^{-1}((\lambda_x(y))^{-1})})^{-1}) \\ &\overset{(3.3)\&(3.4)}{=} ((\rho^{-1}_{\lambda_x(y)}\rho_{\lambda_x(y)}(x^{-1}))^{-1}, (\lambda^{-1}_{\rho_y(x)}\lambda_{\rho_y(x)}(y^{-1}))^{-1}) \\ &= ((x^{-1})^{-1}, (y^{-1})^{-1}) = (x,y). \end{aligned}$$

□

§ 3.6. The retraction of a solution. Let (X,r) be a solution and define on X the following relation

$$x \sim y \quad \Longleftrightarrow \quad \lambda_x = \lambda_y \text{ and } \rho_x = \rho_y.$$

Let $\bar{X} = X/\sim$ denote the set of equivalence classes and let $[x]$ denote the class of x.

LEMMA 3.22. *Let (X,r) be a solution and write $r^{-1}(x,y) = (\hat{\lambda}_x(y), \hat{\rho}_y(x))$. Then*

$$\hat{\lambda}_y^{-1}(x) = \rho_{\lambda_x^{-1}(y)}(x), \tag{3.5}$$

$$\lambda_x^{-1}(y) = \hat{\rho}_{\hat{\lambda}_y^{-1}(x)}(y), \tag{3.6}$$

$$\hat{\rho}_x^{-1}(y) = \lambda_{\rho_y^{-1}(y)}(y), \tag{3.7}$$

$$\rho_y^{-1}(x) = \hat{\lambda}_{\hat{\rho}_x^{-1}(y)}(x). \tag{3.8}$$

EXERCISE 3.23. Prove Lemma 3.22.

THEOREM 3.24. *Let (X,r) be a solution. Then r induce a solution $\bar{r}$ on $\bar{X}$ by*

$$\bar{r}([x],[y]) = ([\lambda_x(y)], [\rho_y(x)]),$$

for all $x,y \in X$.

PROOF. Omitted. □

DEFINITION 3.25. Let (X,r) be a solution. The solution $\mathrm{Ret}(X,r) = (\bar{X},\bar{r})$ induced by the equivalence relation $\sim$ is the *retraction* of (X,r).

We define inductively $\mathrm{Ret}^0(X,r) = (X,r)$, $\mathrm{Ret}^1(X,r) = \mathrm{Ret}(X,r)$ and

$$\mathrm{Ret}^{n+1}(X,r) = \mathrm{Ret}(\mathrm{Ret}^n(X,r)), \quad n \geq 1.$$

DEFINITION 3.26. A solution (X,r) is said to be a *multipermutation solution of level n*, if n is the smallest non-negative integer such that $|\mathrm{Ret}^n(X,r)| = 1$.

DEFINITION 3.27. The solution (X,r) is said to be *irretractable* if $\mathrm{Ret}(X,r) = (X,r)$.

EXAMPLE 3.28. The trivial solution (X,τ) over a set with one element is a multipermutation solution of level zero.

EXAMPLE 3.29. Permutation solutions are multipermutation solutions of level 1.

EXAMPLE 3.30. et $X = \{1,2,3,4\}$ and let $r : X \times X \to X \times X$ defined by $r(x,y) = (\varphi_x(y), \varphi_y(x))$ where

$$\varphi_1 = \varphi_2 = \mathrm{id}, \quad \varphi_3 = (3\ 4), \quad \varphi_4 = (1\ 2)(3\ 4).$$

Then (X,r) is an involutive multipermutation solution of level 3.

PROPOSITION 3.31. *Let B be a skew brace and let (B,r_B) be the associated solution. Then, the retraction $\mathrm{Ret}(B,r_B)$ is the solution associated with the quotient skew brace $B/\operatorname{Soc}(B)$.*

THEOREM 3.32. *Let (X,r) be a finite multipermutation solution to the YBE. If $|X| > 1$, then r has even order.*

PROOF. Since $(X,r) \to \mathrm{Ret}(X,r)$, $x \mapsto [x]$ is a homomorphism of solutions, it follows that the order of the solution $\bar{r}$ divides the order of r. Assume that (X,r) has multipermutation level n. There exists a homomorphism of solutions $(X,r) \to \mathrm{Ret}^{n-1}(X,r)$, thus it is enough to prove the theorem when $r(x,y) = (\lambda(y), \rho(x))$ for commuting permutations λ and ρ, i.e. multipermutation solutions of level 1. If r has order $2k+1$, then

$$(x,y) = r^{2k+1}(x,y) = (\lambda^{k+1}\rho^k(y), \lambda^k\rho^{k+1}(x)).$$

This implies that $\lambda^{k+1}\rho^k(y) = x$ for all $x,y \in X$. This equality in particular implies that $x = y$ because $\lambda^{k+1}\rho^k$ is a permutation, a contradiction. □

Some solutions

1.9. First, let us prove note that $s(x,y) = (y, \sigma_y(x))$ satisfies the Yang–Baxter equation if and only if

$$\sigma_z\sigma_y = \sigma_{\sigma_z(y)}\sigma_z.$$

Note that 2) in Proposition 1.4, implies that

$$\lambda_x\sigma_y = \sigma_{\lambda_x(y)}\lambda_x. \tag{3.9}$$

Indeed, for any $x,y,z \in X$ it holds

$$\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y) = \lambda_{\rho_{\lambda_y(z)}(x)}\lambda^{-1}_{\lambda_y(z)}\sigma_{\lambda_y(z)}(y) \overset{1)}{=} \lambda^{-1}_{\lambda_x\lambda_y(z)}\lambda_x\sigma_{\lambda_y(z)}(y)$$

and

$$\rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y) = \lambda^{-1}_{\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z)}\sigma_{\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z)}\lambda_x(y) \overset{1)}{=} \lambda^{-1}_{\lambda_x\lambda_y(z)}\sigma_{\lambda_x\lambda_y(z)}\lambda_x(y)$$

Moreover, 3) in Proposition 1.4, implies that

$$\sigma_z\sigma_y = \sigma_{\sigma_z(y)}\sigma_z.$$

Indeed

$$\begin{aligned}\rho_z\rho_y(x) &= \lambda^{-1}_{\lambda_{\rho_y(x)}(z)}\sigma_{\lambda_{\rho_y(x)}(z)}\lambda^{-1}_{\lambda_x(y)}\sigma_{\lambda_x(y)}(x) \overset{(3.9)}{=} \lambda^{-1}_{\lambda_{\rho_y(x)}(z)}\lambda^{-1}_{\lambda_x(y)}\sigma_{\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z)}\sigma_{\lambda_x(y)}(x)\\ &\overset{1)}{=} \lambda^{-1}_{\lambda_{\rho_y(x)}(z)}\lambda^{-1}_{\lambda_x(y)}\sigma_{\lambda_x\lambda_y(z)}\sigma_{\lambda_x(y)}(x).\end{aligned}$$

and

$$\begin{aligned}\rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x) &= \lambda^{-1}_{\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y)}\lambda^{-1}_{\lambda_x\lambda_y(z)}\sigma_{\lambda_x\lambda_{\lambda_y(z)}\rho_z(y)}\sigma_{\lambda_x\lambda_y(z)}(x)\\ &\overset{1)\&2)}{=} \lambda^{-1}_{\rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y)}\lambda^{-1}_{\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z)}\sigma_{\lambda_x\sigma_{\lambda_y(z)}(y)}\sigma_{\lambda_x\lambda_y(z)}(x)\\ &\overset{1)}{=} \lambda^{-1}_{\lambda_{\rho_y(x)}(z)}\lambda^{-1}_{\lambda_x(y)}\sigma_{\sigma_{\lambda_x\lambda_y(z)}\lambda_x(y)}\sigma_{\lambda_x\lambda_y(z)}(x).\end{aligned}$$

Hence, for all $x,y,z \in X$

$$\sigma_{\lambda_x\lambda_y(z)}\sigma_{\lambda_x(y)}(x) = \sigma_{\sigma_{\lambda_x\lambda_y(y)}\lambda_x(z)}\sigma_{\lambda_x\lambda_y(z)}(x)$$

and the wanted equality follows.

This prove that s satisfies the Yang–Baxter equation.

Now to prove that r is bijective if and only if s is non-degenerate (i.e. all σ_y bijective). Let us first notice that $\varphi r\varphi^{-1} = s$ where $\varphi(x,y) = (x, \lambda_x(y))$. Indeed

$$\varphi r\varphi^{-1}(x,y) = \varphi r(x, \lambda^{-1}x(y)) = \varphi(\lambda_x\lambda^{-1}x(y), \rho_{\lambda^{-1}x(y)}(x)) = (y, \lambda_y\rho_{\lambda^{-1}x(y)}(x)) = (y, \sigma_y(x)).$$

It follows that r is bijective if and only if s is bijective. Finally clearly s is bijective if and only if σ_y is bijective for every $y \in X$.

2.18. For every $x,y \in X$ let us write $\lambda_x = \mathrm{id}_X$ and $\rho_y(x) = x \triangleleft y$. We want to apply Proposition 1.4. First note that clearly $\lambda_x\lambda_y = \mathrm{id}_X = \lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}$, i.e. 1) is satisfied. Moreover,$\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y) = \rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y)$ reduce to the trivial identity $\rho_z(y) = \rho_z(y)$. Finally, $\rho_z\rho_y(x) = \rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x)$ is equivalent to $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.

Now assume that r is bijective. If $x_1, x_2 \in X$ such that $\rho_y x_1 = \rho_y(x_2)$, then $r(x_1, y) = r(x_2, y)$ and so $x_1 = x_2$, i.e. ρ_y is injective. Now, let $z \in X$ and let $x \in X$ such that $r(x, y) = (y, z)$. It follows that $\rho_y(x) = z$ and ρ_y is bijective. Similarly one obtains the converse.

2.16. Consider the map $\varphi : A \to B \times C, (x) \mapsto (x_B, -x_C)$. Clearly, φ is bijective. Moreover, for $x, y \in A$ we have

$$\varphi(x \circ y) = \varphi(x_B + y + x_C) = \varphi(x_B + y_B + y_C + x_C) = (x_B + y_B, -(y_C + x_C)) = (x_B + y_B, -x_C - y_C),$$

and

$$\varphi(x) + \varphi(y) = (x_B, -x_C) + (y_B, -y_C) = (x_B + y_B, -x_C - y_C).$$

Hence φ is an isomorphism from $(A, \circ$ to the direct product $B \times C$.

Now, let $x, y, z \in A$. Then

$$\begin{aligned} x \circ y - x + x \circ z &= x_B + y + x_C - (x_B + x_C) + x_B + z + x_C \\ &= x_B + y + z + x_C = x \circ (y + z). \end{aligned}$$

Hence $(A, +, \circ)$ is a skew brace.

1.23. Let $a, b, c \in A$. We have that

$$a \circ (a' + b) \overset{(1.3)}{=} a \circ a' - a + a \circ b \overset{0=1}{=} 0 - a + a \circ b = \lambda_a(b).$$

Hence, $\lambda_a(b) = a \circ (a' + b)$. Moreover,

$$\rho_b(a) = (\lambda_a(b))' \circ a \circ b = (a \circ (a' + b))' \circ a \circ b = (a' + b)' \circ b.$$

2.26. Note that

$$\begin{aligned} \lambda_{(a,x)}(b, y) &= -(a, x) + (a, x) \circ (b, y) \\ &= -(a, x) + (a + (-1)^x b, x + y) \\ &= ((-1)^x b, y). \end{aligned}$$

and hence $\mathrm{Fix}(A) = \{(0,0), (0,1)\}$ is not a normal subgroup of $(A, \circ)$. In particular, $\mathrm{Fix}(A)$ is not an ideal of A.

3.23. Let us compute

$$(x, y) = r^{-1} r(x, y) = (\hat{\lambda}_{\lambda_x(y)} \rho_y(x), \hat{\rho}_{\rho_y(x)} \lambda_x(y)).$$

It follows that

$$(x, \lambda_x^{-1} y) = (\hat{\lambda}_y \rho_{\lambda_x^{-1}(y)}(x), \hat{\rho}_{\rho_{\lambda_x^{-1}(y)}(x)}(y)),$$

hence $\hat{\lambda}_y^{-1}(x) = \rho_{\lambda_x^{-1}(y)}(x)$ and $\lambda_x^{-1}(y) = \hat{\rho}_{\hat{\lambda}_y^{-1}(x)}(y)$. Similarly

$$(\rho_y^{-1}(x), y) = (\hat{\lambda}_{\lambda_{\rho_y^{-1}(x)}(y)}(x), \hat{\rho}_x \lambda_{\rho_y^{-1}(x)}(y)),$$

hence $\hat{\rho}_x^{-1}(y) = \lambda_{\rho_y^{-1}(y)}(y)$ and $\rho_y^{-1}(x) = \hat{\lambda}_{\hat{\rho}_x^{-1}(y)}(x)$.

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Index

I

III Reidemeister move 4

R

Rack 13
Ring 18

S

Set-theoretic solution 3
  Finite 3, 11
  Non-degenerate 3, 11
Shelf 13

Y

Yang–Baxter equation 3