Skew braces and solutions to the Yang–Baxter equation

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Contents

The notes correspond to the series of lectures on *Skew braces and solutions to the Yang–Baxter equation* taught as part of the conference Introduction to Modern Advances in Algebra.

This version was compiled on Wednesday 14$^{\text{th}}$ February, 2024 at 16:34.

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Lecture 1. 21/02/2024

§ 1.1. The Yang–Baxter equation. The Yang-Baxter equation is one important equation in mathematical physics. It first appeared in two independent papers of Yang [**?**] and Baxter [**?**].

DEFINITION 1.1. A solution of the *Yang–Baxter eqution* is a linear map $R : V \otimes V \to V \otimes V$, where V is a vector space such that

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where R_{ij} denotes the map $V \otimes V \otimes V \to V \otimes V \otimes V$ acting as R on the (i, j) factor and as the identity on the remaining factor.

Let $\tau : V \otimes V \to V \otimes V$ be the map $\tau(u \otimes v) = v \otimes u$ for $u, v \in V$. It's easy to check (try!) that $R : V \otimes V \to V \otimes V$ is a solution of the Yang–Baxter equation if and only if $\bar{R} := \tau R$ satisfies

$$\bar{R}_{12}\bar{R}_{23}\bar{R}_{12} = \bar{R}_{23}\bar{R}_{12}\bar{R}_{23}.$$

An interesting class of solutions of the Yang–Baxter equation arises when V has a R-invariant basis X. In such a case the solution is said to be set-theoretic.

§ 1.2. The set-theoretic version. Drinfeld in [**?**] observed it makes sense to consider the Yang–Baxter equation in the category of sets and stated that

it would be interesting to study set-theoretic solutions.

These lectures will focus on set-theoretic solutions to the Yang-Baxter equation and their connection with known and "new" algebraic structures.

DEFINITION 1.2. A *set-theoretic solution (to the Yang–Baxter equation)* is a pair (X, r) where X is a non-empty set and $r : X \times X \to X \times X$ is a map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$$

CONVENTION 1. If (X, r) is a set-theoretic solution to the YBE, we write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

where $\lambda_x, \rho_x : X \to X$.

itemize
(X, r) is *bijective* if r is bijective.
(X, r) is *finite* if X is finite.
(X, r) is *non-degenerate* if λ_x, ρ_x are bijective for all $x \in X$.