Skew braces and solutions to the Yang–Baxter equation

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Contents

The notes correspond to the series of lectures on *Skew braces and solutions to the Yang–Baxter equation* taught as part of the conference Introduction to Modern Advances in Algebra.

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Lecture 1. 21/02/2024

§ 1.1. The Yang–Baxter equation. The Yang-Baxter equation (YBE) is one important equation in mathematical physics. It first appeared in two independent papers of Yang [**3**] and Baxter [**1**].

DEFINITION 1.1. A solution of the *Yang–Baxter eqution* is a bijective linear map $R : V \otimes V \to V \otimes V$, where V is a vector space such that

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where R_{ij} denotes the map $V \otimes V \otimes V \to V \otimes V \otimes V$ acting as R on the (i, j) factor and as the identity on the remaining factor.

Let $\tau : V \otimes V \to V \otimes V$ be the map $\tau(u \otimes v) = v \otimes u$ for $u, v \in V$. It's easy to check (try!) that $R : V \otimes V \to V \otimes V$ is a solution of the Yang–Baxter equation if and only if $\bar{R} := \tau R$ satisfies

$$\bar{R}_{12}\bar{R}_{23}\bar{R}_{12} = \bar{R}_{23}\bar{R}_{12}\bar{R}_{23}.$$

An interesting class of solutions of the Yang–Baxter equation arises when V has a R-invariant basis X. In such a case the solution is said to be set-theoretic.

§ 1.2. The set-theoretic version. Drinfeld in [**2**] observed it makes sense to consider the Yang–Baxter equation in the category of sets and stated that

it would be interesting to study set-theoretic solutions.

These lectures will focus on set-theoretic solutions to the Yang-Baxter equation and their connection with known and "new" algebraic structures.

DEFINITION 1.2. A *set-theoretic solution to the Yang–Baxter equation* is a pair (X, r) where X is a non-empty set and $r : X \times X \to X \times X$ is a bijective map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$$

CONVENTION 1. If (X, r) is a set-theoretic solution to the Yang–Baxter equation, we write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

where $\lambda_x, \rho_x : X \to X$.

DEFINITION 1.3. Let (X, r) be a set-theoretic solution to the Yang–Baxter equation. We say that

- (X, r) is *finite* if X is finite.
- (X, r) is *non-degenerate* if λ_x, ρ_x are bijective for all $x \in X$.

§ 1.3. Set-theoretic solutions to the Yang–Baxter equation and III Reidemeister move.

§ 1.4. First examples.

EXAMPLES 1.4. Let X be a non-empty set.

1) The pair $(X, \mathrm{id}_{X \times X})$ is a set-theoretic solution to the Yang–Baxter equation. Note that $(X, \mathrm{id}_{X \times X})$ is not non-degenerate, since $\lambda_x(y) = x$ and $\rho_y(x) = y$, for all $x, y \in X$.

2) Let $\tau : X \times X \to X \times X$ be the flip map, i.e. $\tau(x, y) = (y, x)$ for all $x, y \in X$. Then, the pair (X, τ) is a set-theoretic solution to the Yang–Baxter equation. Moreover, it is non-degenerate since $\lambda_x = \rho_x = \mathrm{id}_X$ for all $x \in X$.

3) Let λ,ρ be permutaions of X. Then $r(x,y)=(\lambda(y),\rho(x))$ is a non-degenerate set-theoretic solution to the Yang–Baxter equation if and only if $\lambda\rho=\rho\lambda$. Morever, (X,r) is involutive if and only if $\rho=\lambda^{-1}$. The solution (X,r) is called a *permutational solution* or a *Lyubashenko's solution*.

If on the set X we have a bit more structure we can define some more sophisticated solutions.

Example 1.5. Let G be a group and let

$$r_1(x,y)=(y,y^{-1}xy)$$
$$r_2(x,y)=(x^2y,y^{-1}x^{-1}y).$$

Then (X,r_1) and (X,r_2) are bijective non-degenerate set-theoretic solutions to the Yang–Baxter equation.

References

[1] R. J. Baxter. Eight-vertex model in lattice statistics. *Phys. Rev. Lett.*, 26:832–833, 1971.

[2] V. G. Drinfeld. On some unsolved problems in quantum group theory. In *Quantum Groups (Leningrad 1990)*, volume 1510 of *Lecture Notes in Math.*, pages 1–8. Springer, Berlin, 1992.

[3] C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.*, 19:1312–1315, 1967.