Skew braces and solutions to the Yang–Baxter equation

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Contents

The notes correspond to the series of lectures on *Skew braces and solutions to the Yang–Baxter equation* taught as part of the conference Introduction to Modern Advances in Algebra.

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§ 1.1. The Yang–Baxter equation. The Yang-Baxter equation (YBE) is one important equation in mathematical physics. It first appeared in two independent papers of Yang [**3**] and Baxter [**1**].

Definition 1.1. A solution of the *Yang–Baxter eqution* is a bijective linear map $R : V \otimes V \to V \otimes V$, where V is a vector space such that

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where R_{ij} denotes the map $V \otimes V \otimes V \to V \otimes V \otimes V$ acting as R on the (i, j) factor and as the identity on the remaining factor.

Let $\tau : V \otimes V \to V \otimes V$ be the map $\tau(u \otimes v) = v \otimes u$ for $u, v \in V$. It's easy to check (try!) that $R : V \otimes V \to V \otimes V$ is a solution of the Yang–Baxter equation if and only if $\bar{R} := \tau R$ satisfies

$$\bar{R}_{12}\bar{R}_{23}\bar{R}_{12} = \bar{R}_{23}\bar{R}_{12}\bar{R}_{23}.$$

An interesting class of solutions of the Yang–Baxter equation arises when V has a R-invariant basis X. In such a case the solution is said to be set-theoretic.

§ 1.2. The set-theoretic version. Drinfeld in [**2**] observed it makes sense to consider the Yang–Baxter equation in the category of sets and stated that

it would be interesting to study set-theoretic solutions.

These lectures will focus on set-theoretic solutions to the Yang-Baxter equation and their connection with known and "new" algebraic structures.

Definition 1.2. A *set-theoretic solution to the Yang–Baxter equation* is a pair (X, r) where X is a non-empty set and $r : X \times X \to X \times X$ is a bijective map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r) \tag{1.1}$$

Convention 1. If (X, r) is a set-theoretic solution to the Yang–Baxter equation, we write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

where $\lambda_x, \rho_x : X \to X$.

Definition 1.3. Let (X, r) be a set-theoretic solution to the Yang–Baxter equation. We say that

- (X, r) is *finite* if X is finite.
- (X, r) is *non-degenerate* if λ_x, ρ_x are bijective for all $x \in X$.

§ 1.3. Set-theoretic solutions to the Yang–Baxter equation and III Reidemeister move. Let us represent the map $r : X \times X \to X \times X$ as a crossing and the identity on X as a straight line, see Figure 1.

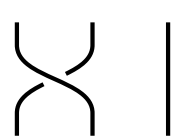

Figure 1. The map r represented as a crossing and the identity as a straight line.

Then the Yang–Baxter equation can be pictured as in Figure 2.

Moreover, under the assumption of (X, r) being non-degenerate we have the following lemma.

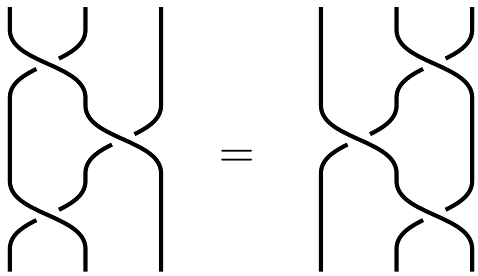

FIGURE 2. The Yang–Baxter equation.

LEMMA 1.4. *Let (X,r) be a solution to the Yang–Baxter equation.*

1) *Given $x,u\in X$, there exist unique $y,v\in X$ such that $r(x,y)=(u,v)$.*
2) *Given $y,v\in X$, there exist unique $x,u\in X$ such that $r(x,y)=(u,v)$.*

PROOF. For the first claim take $y=\lambda_x^{-1}(u)$ and $v=\rho_y(x)$. For the second, $x=\rho_y^{-1}(v)$ and $u=\lambda_x(y)$. □

So, the bijectivity of r means that any row in Figure 3 determines the whole square. By Lemma 1.4 we have that non-degeneracy means that any column in Figure 3 also determines the whole square.

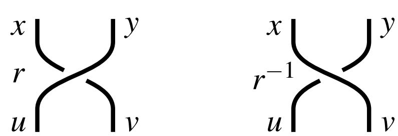

FIGURE 3. Any row or column determines the whole square.

§ 1.4. First examples.

EXAMPLES 1.5. Let X be a non-empty set.

1) The pair $(X,\mathrm{id}_{X\times X})$ is a set-theoretic solution to the Yang–Baxter equation. Note that $(X,\mathrm{id}_{X\times X})$ is not non-degenerate, since $\lambda_x(y)=x$ and $\rho_y(x)=y$, for all $x,y\in X$.
2) Let $\tau : X\times X\to X\times X$ be the flip map, i.e. $\tau(x,y)=(y,x)$ for all $x,y\in X$. Then, the pair (X,τ) is a set-theoretic solution to the Yang–Baxter equation. Moreover, it is non-degenerate since $\lambda_x=\rho_x=\mathrm{id}_X$ for all $x\in X$.
3) Let λ,ρ be permutaions of X. Then $r(x,y)=(\lambda(y),\rho(x))$ is a non-degenerate set-theoretic solution to the Yang–Baxter equation if and only if $\lambda\rho=\rho\lambda$. Morever, (X,r) is involutive if and only if $\rho=\lambda^{-1}$. The solution (X,r) is called a *permutational solution* or a *Lyubashenko's solution*.

If on the set X we have a bit more structure we can define some more sophisticated solutions.

EXAMPLE 1.6. Let G be a group and let

$$r_1(x,y)=(y,y^{-1}xy)$$
$$r_2(x,y)=(x^2y,y^{-1}x^{-1}y).$$

Then (X,r_1) and (X,r_2) are bijective non-degenerate set-theoretic solutions to the Yang–Baxter equation.

§ 1.5. A characterisation.

PROPOSITION 1.7. *Let X be a non-empty set and $r : X \times X \to X \times X$ be a map, written ase $r(x,y) = (\lambda_x(y), \rho_y(x))$. Then r satisfies equation 1.1 if and only if*

1) $\lambda_x\lambda_y = \lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}$
2) $\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y) = \rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y)$
3) $\rho_z\rho_y = \rho_{\rho_z(y)}\rho_{\lambda_y(z)}$

for all $x,y,z \in X$.

In particular, (X,r) is a solution to the Yang–Baxter equation when r is bijective.

PROOF. Let us write $r_1 = r \times \mathrm{id}$ and $r_2 = \mathrm{id} \times r$. Then

$$\begin{aligned} r_1r_2r_1(x,y,z) &= r_1r_2(\lambda_x(y), \rho_y(x), z) \\ &= r_1(\lambda_x(y), \lambda_{\rho_y(x)}(z), \rho_z\rho_y(x)) \\ &= (\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z), \rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y), \rho_z\rho_y(x)), \end{aligned}$$

and

$$\begin{aligned} r_2r_1r_2(x,y,z) =& r_2r_1(x, \lambda_y(z), \rho_z(y)) \\ &= r_2(\lambda_x\lambda_y(z), \rho_{\lambda_y(z)}(x), \rho_z(y)) \\ &= (\lambda_x\lambda_y(z), \lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y), \rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x)). \end{aligned}$$

Therefore $r_1r_2r_1 = r_2r_1r_2$ if and only if $1), 2)$ and $3)$ hold. □

EXERCISE 1.8. Let (X,r) be a set-theoretic solution to the Yang–Baxter equation. Define for all $x,y \in X$

$$\bar{r}(x,y) = \tau r \tau(x,y) = (\rho_x(y), \lambda_y(x)).$$

Then $(X,\bar{r})$ is a set-theoretic solution to the Yang–Baxter equation.

§ 1.6. Shelfs and racks.

EXERCISE 1.9. Let X be a non-empty set. Let $\triangleleft : X \times X \to X$ be a binary operation and define $r : X \times X \to X \times X$ such that $r(x,y) = (y, x \triangleleft y)$. Then r satisfies equation 1.1 if and only if $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ holds for all $x,y,z \in X$. Moreover, r is bijective if and only if the maps $\rho_y : X \to X, x \mapsto x \triangleleft y$ are bijective.

DEFINITION 1.10. A *(right) shelf* is a pair $(X,\triangleleft)$ where X is a non-empty set and $\triangleleft$ is a binary operation such that

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z).$$

If, in addition, the maps $\rho_y : X \to X, x \mapsto x \triangleleft y$ are bijective for all $y \in X$, then $(X,\triangleleft)$ is called a *(right) rack*.

PROPOSITION 1.11. *Let X be a non-empty set with a binary operation $\triangleleft : X \times X \to X$. Then $r(x,y) = (y, x \triangleleft y)$ is a set-theoretic solution to the Yang–Baxter equation if and only if $(X,\triangleleft)$ is a rack.*

PROOF. Follows from exercise 1.9. □

Some solutions

1.9. For every $x,y\in X$ let us write $\lambda_x=\mathrm{id}_X$ and $\rho_y(x)=x\triangleleft y$. We want to apply Proposition 1.7. First note that clearly $\lambda_x\lambda_y=\mathrm{id}_X=\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}$, i.e. 1) is satisfied. Moreover,$\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y)=\rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y)$ reduce to the trivial identity $\rho_z(y)=\rho_z(y)$. Finally, $\rho_z\rho_y(x)=\rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x)$ is equivalent to $(x\triangleleft y)\triangleleft z=(x\triangleleft z)\triangleleft(y\triangleleft z)$.

Now assume that r is bijective. If $x_1,x_2\in X$ such that $\rho_y x_1=\rho_y(x_2)$, then $r(x_1,y)=r(x_2,y)$ and so $x_1=x_2$, i.e. ρ_y is injective. Now, let $z\in X$ and let $x\in X$ such that $r(x,y)=(y,z)$. It follows that $\rho_y(x)=z$ and ρ_y is bijective. Similarly one obtains the converse.

References

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[3] C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.*, 19:1312–1315, 1967.

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