Skew braces and solutions to the Yang–Baxter equation

Ilaria Colazzo

Contents

The notes correspond to the series of lectures on *Skew braces and solutions to the Yang–Baxter equation* taught as part of the conference Introduction to Modern Advances in Algebra.

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Ilaria Colazzo
Exeter (UK)

University of Exeter – Exeter (UK)
E-mail address: ilariacolazzo@gmail.com.

Lecture 1. 21/02/2024

§ 1.1. The Yang–Baxter equation. The Yang-Baxter equation (YBE) is one important equation in mathematical physics. It first appeared in two independent papers of Yang [**4**] and Baxter [**1**].

DEFINITION 1.1. A solution of the *Yang–Baxter eqution* is a bijective linear map $R : V \otimes V \to V \otimes V$, where V is a vector space such that

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where R_{ij} denotes the map $V \otimes V \otimes V \to V \otimes V \otimes V$ acting as R on the (i, j) factor and as the identity on the remaining factor.

Let $\tau : V \otimes V \to V \otimes V$ be the map $\tau(u \otimes v) = v \otimes u$ for $u, v \in V$. It's easy to check (try!) that $R : V \otimes V \to V \otimes V$ is a solution of the Yang–Baxter equation if and only if $\bar{R} := \tau R$ satisfies

$$\bar{R}_{12}\bar{R}_{23}\bar{R}_{12} = \bar{R}_{23}\bar{R}_{12}\bar{R}_{23}.$$

An interesting class of solutions of the Yang–Baxter equation arises when V has a R-invariant basis X. In such a case the solution is said to be set-theoretic.

§ 1.2. The set-theoretic version. Drinfeld in [**2**] observed it makes sense to consider the Yang–Baxter equation in the category of sets and stated that

it would be interesting to study set-theoretic solutions.

These lectures will focus on set-theoretic solutions to the Yang-Baxter equation and their connection with known and "new" algebraic structures.

DEFINITION 1.2. A *set-theoretic solution to the Yang–Baxter equation* is a pair (X, r) where X is a non-empty set and $r : X \times X \to X \times X$ is a bijective map such that

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r) \tag{1.1}$$

CONVENTION 1. If (X, r) is a set-theoretic solution to the Yang–Baxter equation, we write

$$r(x, y) = (\lambda_x(y), \rho_y(x))$$

where $\lambda_x, \rho_x : X \to X$.

DEFINITION 1.3. Let (X, r) be a set-theoretic solution to the Yang–Baxter equation. We say that

- (X, r) is *finite* if X is finite.
- (X, r) is *non-degenerate* if λ_x, ρ_x are bijective for all $x \in X$.

§ 1.3. Set-theoretic solutions to the Yang–Baxter equation and III Reidemeister move. Let us represent the map $r : X \times X \to X \times X$ as a crossing and the identity on X as a straight line, see Figure 1.

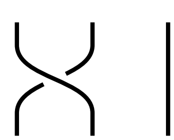

FIGURE 1. The map r represented as a crossing and the identity as a straight line.

Then the Yang–Baxter equation can be pictured as in Figure 2.

Moreover, under the assumption of (X, r) being non-degenerate we have the following lemma.

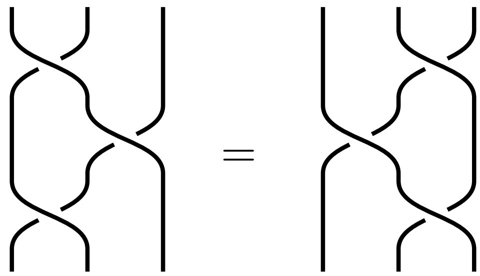

FIGURE 2. The Yang–Baxter equation.

LEMMA 1.4. *Let (X,r) be a solution to the Yang–Baxter equation.*

1) *Given $x,u \in X$, there exist unique $y,v \in X$ such that $r(x,y)=(u,v)$.*
2) *Given $y,v \in X$, there exist unique $x,u \in X$ such that $r(x,y)=(u,v)$.*

PROOF. For the first claim take $y=\lambda_x^{-1}(u)$ and $v=\rho_y(x)$. For the second, $x=\rho_y^{-1}(v)$ and $u=\lambda_x(y)$. $\square$

So, the bijectivity of r means that any row in Figure 3 determines the whole square. By Lemma 1.4 we have that non-degeneracy means that any column in Figure 3 also determines the whole square.

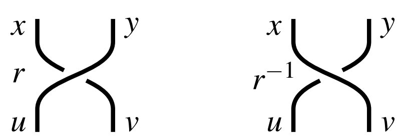

FIGURE 3. Any row or column determines the whole square.

§ 1.4. First examples.

EXAMPLES 1.5. Let X be a non-empty set.

1) The pair $(X,\mathrm{id}_{X\times X})$ is a set-theoretic solution to the Yang–Baxter equation. Note that $(X,\mathrm{id}_{X\times X})$ is not non-degenerate, since $\lambda_x(y)=x$ and $\rho_y(x)=y$, for all $x,y\in X$.
2) Let $\tau : X\times X \to X\times X$ be the flip map, i.e. $\tau(x,y)=(y,x)$ for all $x,y\in X$. Then, the pair (X,τ) is a set-theoretic solution to the Yang–Baxter equation. Moreover, it is non-degenerate since $\lambda_x=\rho_x=\mathrm{id}_X$ for all $x\in X$.
3) Let λ,ρ be permutaions of X. Then $r(x,y)=(\lambda(y),\rho(x))$ is a non-degenerate set-theoretic solution to the Yang–Baxter equation if and only if $\lambda\rho=\rho\lambda$. Morever, (X,r) is involutive if and only if $\rho=\lambda^{-1}$. The solution (X,r) is called a *permutational solution* or a *Lyubashenko's solution*.

If on the set X we have a bit more structure we can define some more sophisticated solutions.

EXAMPLE 1.6. Let G be a group and let

$$r_1(x,y)=(y,y^{-1}xy)$$
$$r_2(x,y)=(x^2y,y^{-1}x^{-1}y).$$

Then (X,r_1) and (X,r_2) are bijective non-degenerate set-theoretic solutions to the Yang–Baxter equation.

§ 1.5. A characterisation.

Proposition 1.7. *Let X be a non-empty set and $r: X \times X \to X \times X$ be a map, written ase $r(x,y) = (\lambda_x(y), \rho_y(x))$. Then r satisfies equation 1.1 if and only if*

1) $\lambda_x\lambda_y = \lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}$
2) $\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y) = \rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y)$
3) $\rho_z\rho_y = \rho_{\rho_z(y)}\rho_{\lambda_y(z)}$

for all $x,y,z \in X$.

In particular, (X,r) is a solution to the Yang–Baxter equation when r is bijective.

Proof. Let us write $r_1 = r \times \mathrm{id}$ and $r_2 = \mathrm{id} \times r$. Then

$$\begin{aligned} r_1r_2r_1(x,y,z) &= r_1r_2(\lambda_x(y), \rho_y(x), z) \\ &= r_1(\lambda_x(y), \lambda_{\rho_y(x)}(z), \rho_z\rho_y(x)) \\ &= (\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z), \rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y), \rho_z\rho_y(x)), \end{aligned}$$

and

$$\begin{aligned} r_2r_1r_2(x,y,z) =& r_2r_1(x, \lambda_y(z), \rho_z(y)) \\ &= r_2(\lambda_x\lambda_y(z), \rho_{\lambda_y(z)}(x), \rho_z(y)) \\ &= (\lambda_x\lambda_y(z), \lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y), \rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x)). \end{aligned}$$

Therefore $r_1r_2r_1 = r_2r_1r_2$ if and only if $1), 2)$ and $3)$ hold. □

Exercise 1.8. Let (X,r) be a set-theoretic solution to the Yang–Baxter equation. Define for all $x,y \in X$

$$\bar{r}(x,y) = \tau r \tau(x,y) = (\rho_x(y), \lambda_y(x)).$$

Then $(X,\bar{r})$ is a set-theoretic solution to the Yang–Baxter equation.

§ 1.6. Shelfs and racks.

Exercise 1.9. Let X be a non-empty set. Let $\triangleleft : X \times X \to X$ be a binary operation and define $r: X \times X \to X \times X$ such that $r(x,y) = (y, x \triangleleft y)$. Then r satisfies equation 1.1 if and only if $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ holds for all $x,y,z \in X$. Moreover, r is bijective if and only if the maps $\rho_y : X \to X, x \mapsto x \triangleleft y$ are bijective.

Definition 1.10. A *(right) shelf* is a pair $(X, \triangleleft)$ where X is a non-empty set and $\triangleleft$ is a binary operation such that

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z).$$

If, in addition, the maps $\rho_y : X \to X, x \mapsto x \triangleleft y$ are bijective for all $y \in X$, then $(X, \triangleleft)$ is called a *(right) rack*.

Proposition 1.11. *Let X be a non-empty set with a binary operation $\triangleleft : X \times X \to X$. Then $r(x,y) = (y, x \triangleleft y)$ is a set-theoretic solution to the Yang–Baxter equation if and only if $(X, \triangleleft)$ is a rack.*

Proof. Follows from exercise 1.9. □

EXERCISE 1.12. Let G be a group. Then (G,r) where $r(x,y) = (y, y^{-1}xy)$ is a non-degenerate set-theoretic solution to the Yang–Baxter equation.

CONVENTION 2. From now on, a *solution* will always mean a non-degenerate set-theoretic solution to the Yang–Baxter equation.

§ 1.7. An intriguing connection between group actions and solutions. The following theorem is the core result of the paper [**3**] by Lu, Yan Zhu.

THEOREM 1.13. *Let G be a group, let $\lambda : G \times G \to G, (x,y) \mapsto \lambda_x(y)$ a left group action of G on itself as a set and $\rho : G \times G \to G, (x,y) \mapsto \rho_y(x)$ a right group action of G on itself as a set. If the "compatibility" condition*

$$uv = \lambda_u(v)\rho_v(u) \tag{1.2}$$

holds, then (G,r), where

$$r : G \times G \to G \times G, \qquad (x,y) \mapsto (\lambda_x(y), \rho_y(x))$$

is a solution.

EXERCISE 1.14. Prove Theorem 1.13

Some solutions

1.9. For every $x,y\in X$ let us write $\lambda_x=\mathrm{id}_X$ and $\rho_y(x)=x\triangleleft y$. We want to apply Proposition 1.7. First note that clearly $\lambda_x\lambda_y=\mathrm{id}_X=\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}$, i.e. 1) is satisfied. Moreover,$\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y)=\rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y)$ reduce to the trivial identity $\rho_z(y)=\rho_z(y)$. Finally, $\rho_z\rho_y(x)=\rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x)$ is equivalent to $(x\triangleleft y)\triangleleft z=(x\triangleleft z)\triangleleft(y\triangleleft z)$.

Now assume that r is bijective. If $x_1,x_2\in X$ such that $\rho_y x_1=\rho_y(x_2)$, then $r(x_1,y)=r(x_2,y)$ and so $x_1=x_2$, i.e. ρ_y is injective. Now, let $z\in X$ and let $x\in X$ such that $r(x,y)=(y,z)$. It follows that $\rho_y(x)=z$ and ρ_y is bijective. Similarly one obtains the converse.

1.14. Let us write $r_1=r\times\mathrm{id}$ and $r_2=\mathrm{id}\times r$,

$$\begin{aligned} r_1r_2r_1(x,y,z)&=(\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z),\rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y),\rho_z\rho_y(x))\\ &=(u_1,v_1,w_1),\end{aligned}$$

and

$$\begin{aligned} r_2r_1r_2(x,y,z)&=(\lambda_x\lambda_y(z),\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y),\rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x))\\ &=(u_2,v_2,w_2).\end{aligned}$$

Then we obtain

$$\begin{aligned} u_1v_1w_1&=\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z)\rho_{\lambda_{\rho_y(x)}(z)}\lambda_x(y)\rho_z\rho_y(x)\\ &\overset{(1.2)}{=}\lambda_x(y)\lambda_{\rho_y(x)}(z)\rho_z\rho_y(x)\\ &\overset{(1.2)}{=}\lambda_x(y)\rho_y(x)z\\ &\overset{(1.2)}{=}xyz\end{aligned}$$

and, similarly

$$\begin{aligned} u_2v_2w_2&=\lambda_x\lambda_y(z)\lambda_{\rho_{\lambda_y(z)}(x)}\rho_z(y)\rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x)\\ &\overset{(1.2)}{=}\lambda_x\lambda_y(z)\rho_{\lambda_y(z)}(x)\rho_z(y)\\ &\overset{(1.2)}{=}x\lambda_y(z)\rho_z(y)\\ &\overset{(1.2)}{=}xyz.\end{aligned}$$

Hence

$$u_1v_1w_1=xyz=u_2v_2w_2. \tag{1.3}$$

Moreover, since λ is a left action of G on itself, we get

$$u_1=\lambda_{\lambda_x(y)}\lambda_{\rho_y(x)}(z)=\lambda_{\lambda_x(y)\rho_y(x)}(z)\overset{(1.2)}{=}\lambda_{xy}(z)=\lambda_x\lambda_y(z)=u_2.$$

Similarly, since ρ is a right action

$$w_2=\rho_{\rho_z(y)}\rho_{\lambda_y(z)}(x)=\rho_{\lambda_y(z)\rho_z(y)}(x)\overset{(1.2)}{=}\rho_{yz}(x)=\rho_z\rho_y(x)=w_1.$$

From (1.3) and G being a group it follows that also $v_1=v_2$. Moreover, λ_x and ρ_x are bijective maps by assumption. It is left to prove that r is bijective. First let us write $r(u,v)=(x,y)$, hence

$\lambda_u(v) = x$, $\rho_v(u) = y$, and $uv = xy$. Now, since λ is an action and in particular $\lambda_v^{-1} = \lambda_{v^{-1}}$, we get

$$\lambda_y(v^{-1})u = \lambda_y(v^{-1})\rho_v^{-1}(y) = \lambda_y(v^{-1})\rho_{v^{-1}}(y) \overset{(1.2)}{=} yv^{-1} = x^{-1}u = (\lambda_u(v))^{-1}u,$$

and so

$$(\lambda_u(v))^{-1} = \lambda_{\rho_v(u)}(v^{-1}). \tag{1.4}$$

Similarly, expanding $v\rho_x(u^{-1})$ one proves

$$(\rho_v(u))^{-1} = \rho_{\lambda_u(v)}(u^{-1}). \tag{1.5}$$

Define

$$r'(x,y) = ((\rho_{x^{-1}}(y^{-1}))^{-1}, (\lambda_{y^{-1}}(x^{-1}))^{-1}).$$

Then

$$\begin{aligned} rr'(x,y) &= (\lambda_{(\rho_{x^{-1}}(y^{-1}))^{-1}}((\lambda_{y^{-1}}(x^{-1}))^{-1}), \rho_{(\lambda_{y^{-1}}(x^{-1}))^{-1}}((\rho_{x^{-1}}(y^{-1}))^{-1})) \\ &\overset{(1.4)\&(1.5)}{=} (\lambda^{-1}_{\rho_{x^{-1}}(y^{-1})}\lambda_{\rho_{x^{-1}}(y^{-1})}(x), \rho^{-1}_{\lambda_{y^{-1}}(x^{-1})}\rho_{\lambda_{y^{-1}}(x^{-1})}(y)) \\ &= (x,y). \end{aligned}$$

And

$$\begin{aligned} r'r(x,y) &= ((\rho_{(\lambda_x(y))^{-1}}((\rho_y(x))^{-1}))^{-1}, (\lambda_{(\rho_y(x))^{-1}}((\lambda_x(y))^{-1}))^{-1}) \\ &\overset{(1.4)\&(1.5)}{=} ((\rho^{-1}_{\lambda_x(y)}\rho_{\lambda_x(y)}(x^{-1}))^{-1}, (\lambda^{-1}_{\rho_y(x)}\lambda_{\rho_y(x)}(y^{-1}))^{-1}) \\ &= ((x^{-1})^{-1}, (y^{-1})^{-1}) = (x,y). \end{aligned}$$

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