Lagrangian Coherent Set Detection with Topological Advection

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We present a topological framework for detecting Lagrangian coherent sets (LC-sets) directly from sparse two-dimensional particle trajectory data. Unlike conventional Lagrangian coherent structure (LCS) diagnostics that require full knowledge of the velocity field, our method relies solely on discrete trajectories and exploits topological invariants of material curves. Closed material loops are encoded via intersection coordinates relative to a dynamically maintained Delaunay triangulation. Their evolution is computed through local edge-flip operations, enabling efficient topological advection without geometric reconstruction. Coherent sets are identified as regions enclosed by loops exhibiting minimal finite-time stretching. We demonstrate this method using the Bickley jet model as a benchmark flow and recover known coherent vortex cores under time-aperiodic forcing. This framework provides a scalable and velocity-field-free approach to coherent structure detection in experimental, geophysical, and data-driven flow systems.

INTRODUCTION

Fluid mixing and scalar transport are central to many problems in oceanography, meteorology, global climate modeling, and engineering. Lagrangian Coherent Structures (LCSs) are important geometric features of flow fields and constitute the main organizing principle for these transport and mixing phenomena. Traditionally, standard approaches to LCS detection have relied on computing the full velocity field and have involved variational principles, finite-time Lyapunov exponent (FTLE) fields, or Cauchy-Green strain tensor analysis.

However, in many experimental and observational contexts, including particle tracking velocimetry, ocean drifter data, and planetary atmospheres, only discrete Lagrangian trajectory data are available. Reconstructing a velocity field from sparse trajectories can introduce substantial errors and computational overhead. In order to resolve this problem, we use an alternative concept of Lagrangian coherent sets (LC-sets), defined as collections of trajectories that can be enclosed by a material curve that does not appreciably stretch over a given time interval. Such curves act as finite-time transport barriers and serve as practical stand-ins for elliptic LCSs in discrete advected 2D particle trajectories.

Leveraging this, we introduce a new computational framework extending the Ensemble-based Topological Entropy Calculation (E-tec) algorithm — originally developed to compute topological entropy in planar flows — into the DualEtecPeriodicBC algorithm. This extension accommodates doubly periodic boundary conditions, allowing trajectory data defined on the torus where coordinates are defined in the ranges $[0, D_x]$ and $[0, D_y]$ in x and y respectively.

ALGORITHM OVERVIEW

To demonstrate proof-of-concept that our algorithm detects coherent structures, the following high-level steps are followed:

- Input: 2D Lagrangian particle trajectories (Bickley Jet used as benchmark)
- Topological Advection of particle trajectories
- Output: Coherent Set Detection

We represent the evolving particle configuration through a time-dependent Delaunay triangulation. The triangulation provides a discrete geometric skeleton of the flow efficiently updated through local operations (edge flips). Closed material loops are encoded via intersection coordinates relative to this triangulation, which serve as a combinatorial representation of loop boundaries (see Fig. 1).

Edge-Flip Operations and Coordinate Updates

As particles advect, the Delaunay triangulation changes locally via edge flips (Fig. 2). An edge flip occurs when a particle enters the circumcircle of an adjacent triangle. When flipping edge E in a quadrilateral bounded by points (A, B, C, D), the intersection coordinate updates according to

$$E' = \max(A + C, B + D) - E. \qquad (1)$$

This rule ensures that the encoded loop representation remains consistent under topological deformation.

COHERENT SET DETECTION

The boundary of a coherent region experiences minimal stretching under advection. By tracking the relative change in loop intersection coordinates, we quantify the loop's stretching without geometric reconstruction. Seed loops initially encircle adjacent particle pairs in the triangulation and are evaluated based on the finite-time

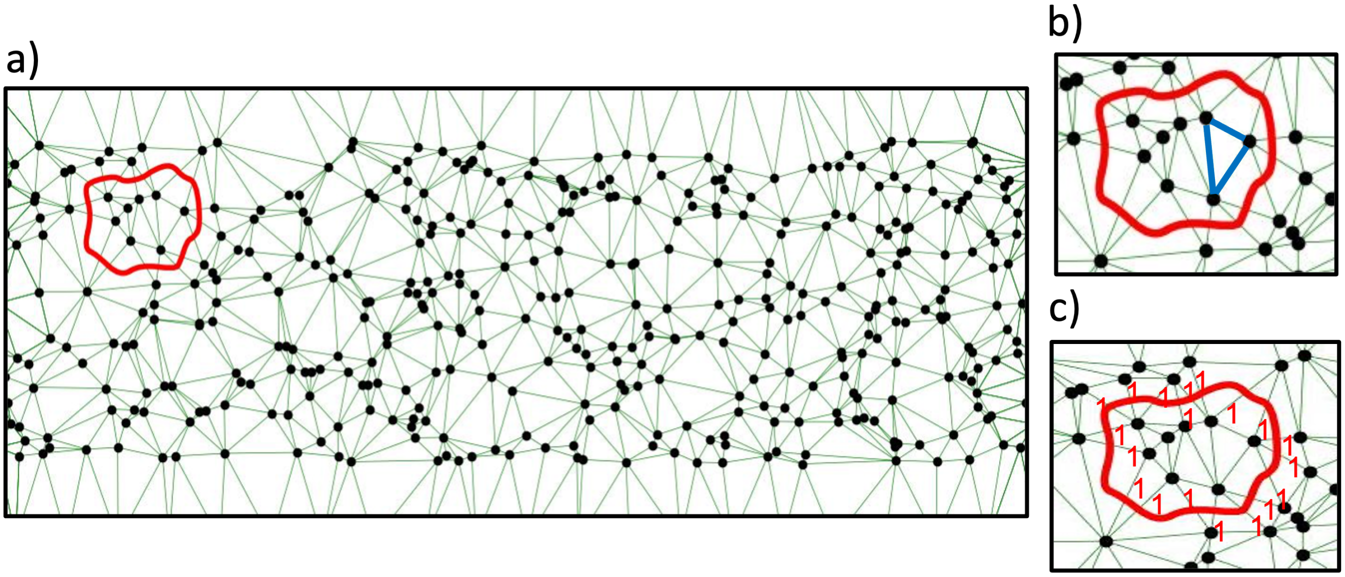

FIG. 1. **Topological Advection and Loop Encoding.** (a) Advected particle ensemble connected by Delaunay triangulation; a red contour denotes a loop enclosing a region of trajectories. (b) Zoom view showing an example loop (red) intersecting edges of the triangulation (blue triangle). (c) Loops are represented by integer-valued intersection coordinates (red numbers) that encode how many times each triangulation edge is crossed. This discrete encoding allows efficient non-geometric advection and quantitative tracking of loop deformation.

growth of their encoded length. Loops exhibiting minimal stretching serve as candidates for coherent boundaries (see Fig. 3).

Through iterative merging of adjacent seed loops with shared trajectories, larger coherent sets are formed hierarchically. Growth halts when no further merge preserves the low-stretch criterion. The final coherent set boundaries correspond to regions of dynamically persistent particle groupings.

BENCHMARK FLOW: BICKLEY JET

We apply our algorithm to the Bickley Jet — a canonical model for barotropic zonal jets with embedded vortices under time-aperiodic forcing. The detected coherent sets correspond to slowly deforming vortex cores consistent with analytical and LCS-based results (Fig. 4).

COMPUTATIONAL COMPLEXITY

Let N denote the number of particles and F the number of edge flips. Maintaining the Delaunay triangulation scales as $\mathcal{O}(N \log N + F)$, while intersection coordinate updates scale as $\mathcal{O}(SF)$, where S is the number of candidate loops post-pruning. Since $S \ll 2^N$, the algorithm remains computationally efficient even for large trajectory datasets.

DISCUSSION AND CONCLUSION

This topological framework enables coherent structure detection directly from discrete trajectories—free from velocity-field reconstruction. The approach is:

- Velocity-field free
- Robust to sparse sampling
- Naturally suited to experimental trajectory datasets
- Computationally scalable

By encoding material boundaries topologically and evolving them through combinatorial operations, coherent sets are recovered as minimally stretching loop regions. This method bridges geometric, topological, and data-driven perspectives, providing a robust new tool for Lagrangian analysis.

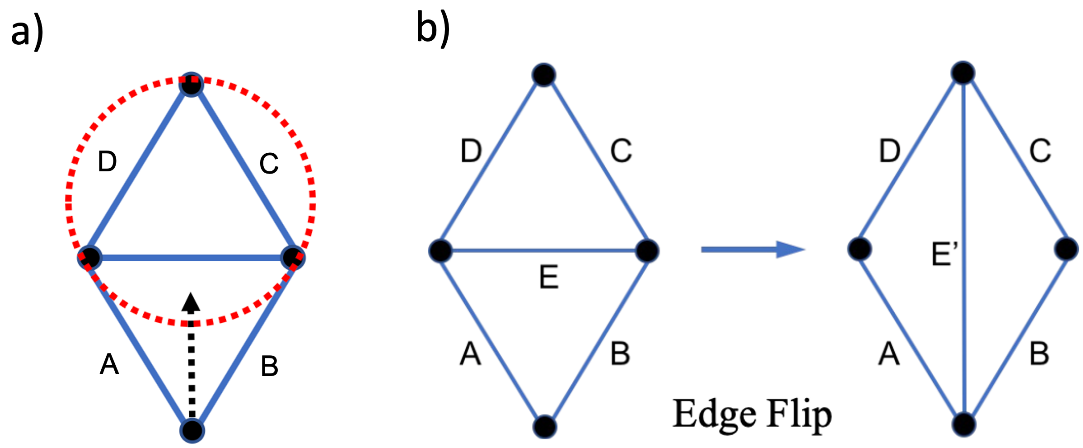

FIG. 2. **Edge Flip Operation.** (a) Local Delaunay violation occurs when a vertex enters the circumcircle (red dashed line) of a triangle, prompting a flip of the diagonal edge. (b) The flip replaces diagonal E with E' between the opposite vertices, restoring the Delaunay property. This local update triggers a corresponding algebraic update to intersection coordinates, maintaining topological consistency.

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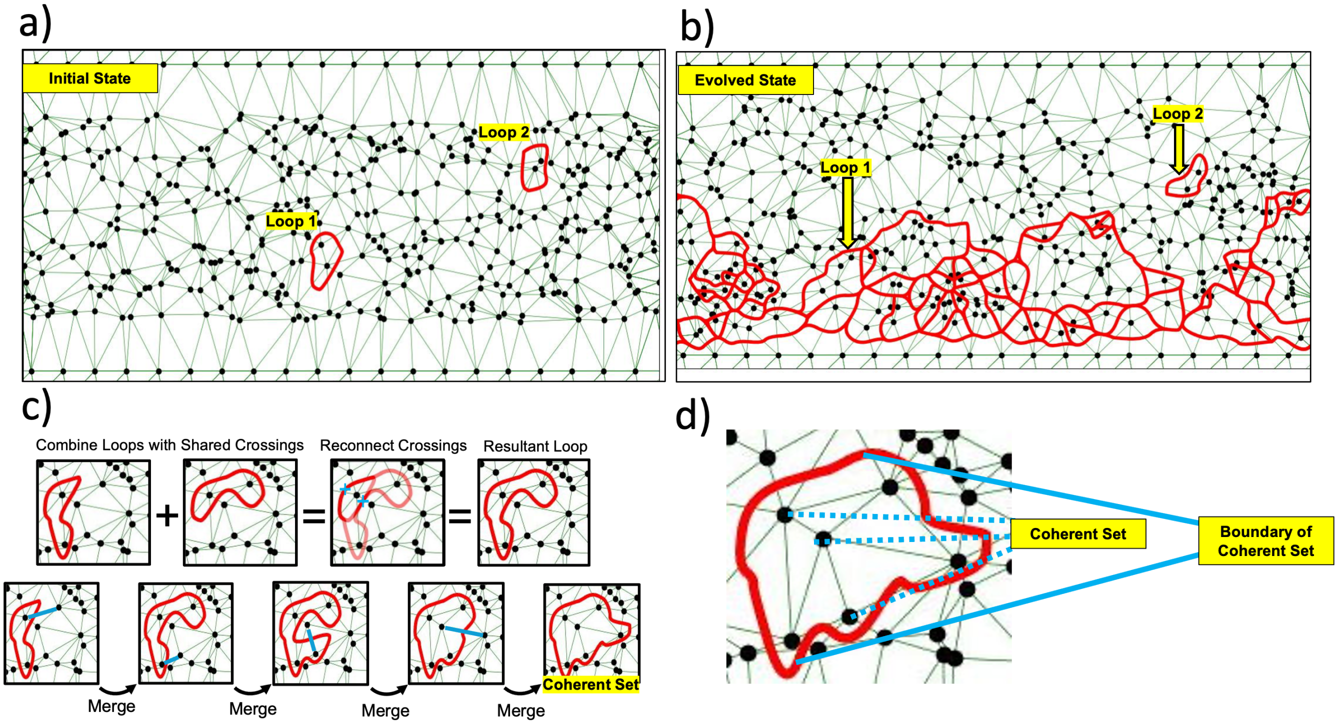

FIG. 3. **Coherent Set Extraction through Loop Growth.** (a–b) Two sample loops in the initial (a) and evolved (b) states. Loop 1 elongates significantly, while Loop 2 retains its shape, marking potential coherence. (c) Adjacent low-stretch loops sharing crossings are merged progressively through reconnection operations. (d) The process yields a final minimally stretching closed loop defining the boundary of a coherent set (yellow labels).

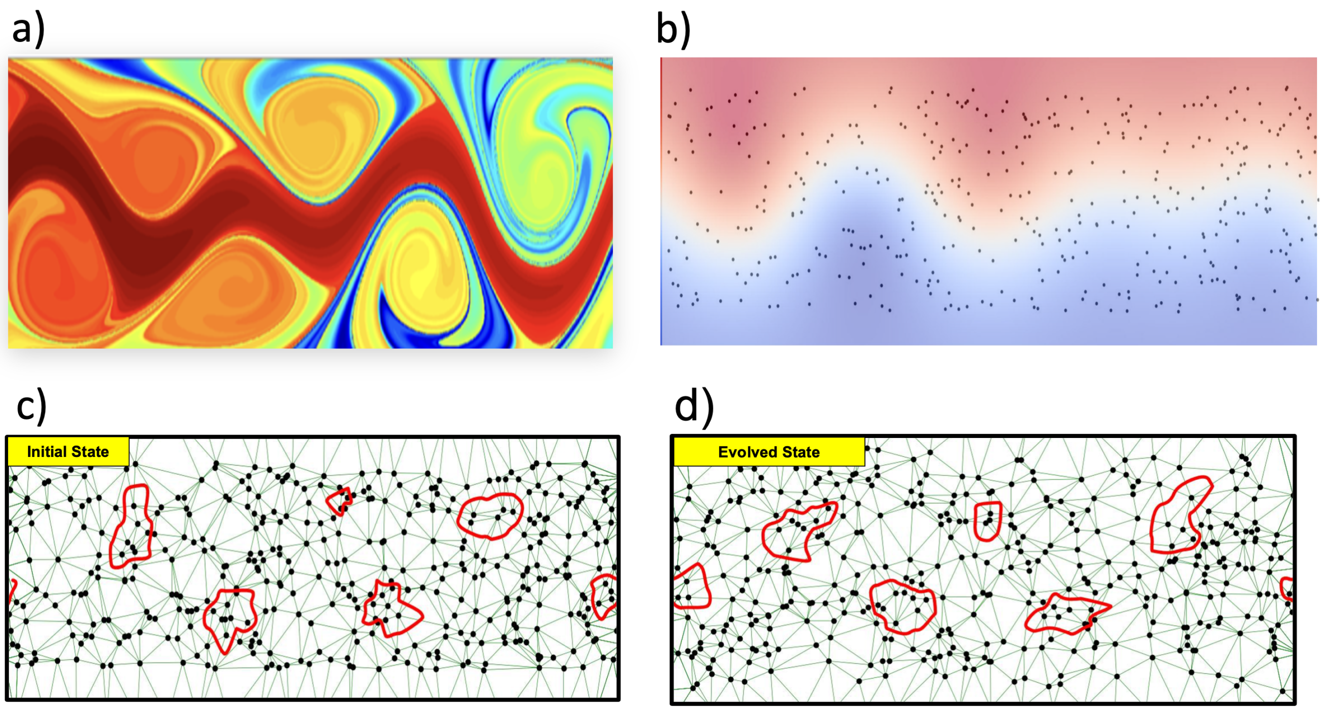

FIG. 4. **Application to Bickley Jet Flow.** (a) Stream function visualization of the time-aperiodic Bickley Jet highlighting alternating vortex and jet regions (adapted from Hadjighasem et al., 2017). (b) Initial particle distribution colored by latitude, showing sampling of the jet and surrounding shear zones. (c) Detected coherent sets (red loops) at initial time t_1 identified via the topological advection algorithm. (d) The same coherent sets advected to later time t_2, retaining compactness and minimal deformation—demonstrating successful coherent set identification.