

Chapter 1

What is Operations Research?

Set 2.1a

Buy three roundtrip tickets for the first three weeks only—cost = $3 \times \$400 = \1200 . Though the cost is cheaper, it is not feasible because it covers only three out of the required five weeks.

Given a string of length L:

- (1) $h = .3L$, $w = .2L$, Area = $.06L^2$
- (2) $h = .1L$, $w = .4L$, Area = $.04L^2$

Solution (2) is better because the area is larger

$$L = 2(w + h)$$

$$w = L/2 - h$$

$$z = wh = h(L/2 - h) = Lh/2 - h^2$$

$$\delta z / \delta h = L/2 - 2h = 0$$

$$\text{Thus, } h = L/4 \text{ and } w = L/4.$$

Solution is optimal because z is a concave function

(a)

Let T = Total tie to move all four individuals to the other side of the river. the objective is to determine the transfer schedule that minimizes T.

(b)

Let t = crossing time from one side to the other. Use codes 1, 2, 5, and 10 to represent Amy, Jim, John, and Kelly.

4 cont.

East	Crossing	West
5, 10	$(1, 2) \rightarrow (t = 2)$	1, 2
1, 5, 10	$(t = 1) \leftarrow (1)$	2
1	$(5, 10) \rightarrow (t = 10)$	2, 5, 10
1, 2	$(t = 2) \leftarrow (2)$	5, 10
none	$(1, 2) \rightarrow (t = 2)$	2, 5, 10
Total = $2 + 1 + 10 + 2 + 2 = 17 \text{ minutes}$		

		Jim	
		Curve	Fast
Joe	Curve	.500	.200
	Fast	.100	.300

(a)

Alternatives:

Joe: Prepare for curve or fast ball.

Jim: Throw curve or fast ball.

(b)

Joe tries to improve his batting score and Jim tries to counter Joe's action by selecting a less favorable strategy. This means that neither player will be satisfied with a single (pure) strategy.

The problem is not an optimization situation in the familiar sense in which the objective is maximized or minimized. Instead, the conflicting situation requires a compromise solution in which neither layer is tempted to change strategy. Game theory (Chapter 14) provides such a solution.

continued...

Set 1.1a

Let L=ops. 1 and 2=20 sec, C=ops. 3 and 4=25 sec, U=op. 5=20 sec

Gant chart: L1=load horse 1, L2=load horse 2, etc.

one joist: 0---L1---20---C1---45---U1+L1---85---U2+L2---125---U1+L1---
165---U2+L2---205
20-L2-40 45---C2---70 85---C1---110 125---C2---140
165-C1-190

205---C2---230---U2---250

Total = 250

Loaders utilization=[250-(5+25)]/250=88%

Cutter utilization=[250-(20+15+15+15+15)]/250=68%

two joists: 0---2L1---40---2C1---90---2(U1+L1)---170---2C1---220---2U1---
260

40---2L2---80 90---2C2---140 170---2U2---210

Total = 260

Loaders utilization=[260-(10+10)]/260=92%

Cutter utilization=[260-(40+30+40)]/250=58%

three joists: 0---3L1---60---3C1---135---3C2---210---3U2---270
60---3L2---120 135---3U1---195

Total = 270

Loaders utilization=[270-(15+15)]/270=89%

Cutter utilization=[270-(60+60)]/270=56%

Recommendation: One joist at time gives the smallest time. The problem has other alternatives that combine 1, 2, and 3 joists. Cutter utilization indicates that cutter represents the bottleneck.

CHAPTER 2

Modeling with Linear Programming

Set 2.1a

- (a) $x_2 - x_1 \geq 1$ or $-x_1 + x_2 \geq 1$
- (b) $x_1 + 2x_2 \geq 3$ and $x_1 + 2x_2 \leq 6$
- (c) $x_2 \geq x_1$ or $x_1 - x_2 \leq 0$
- (d) $x_1 + x_2 \geq 3$
- (e) $\frac{x_2}{x_1 + x_2} \leq .5$ or $.5x_1 - .5x_2 \geq 0$

1

(a) $(x_1, x_2) = (1, 4)$

2

$$x_1, x_2 \geq 0$$

$$\begin{aligned} 6x_1 + 4x_2 &= 22 &< 24 \\ 1x_1 + 2x_2 &= 9 &\neq 6 \end{aligned}$$

infeasible

(b) $(x_1, x_2) = (2, 2)$

$$\begin{aligned} x_1, x_2 &\geq 0 \\ 6x_1 + 4x_2 &= 20 &< 24 \\ 1x_1 + 2x_2 &= 6 &= 6 \\ -1x_1 + 1x_2 &= 0 &< 1 \\ 1x_1 &= 2 &= 2 \end{aligned}$$

} feasible

$$Z = 5x_1 + 4x_2 = \$18$$

(c) $(x_1, x_2) = (3, 1.5)$

$$\begin{aligned} x_1, x_2 &\geq 0 \\ 6x_1 + 4x_2 &= 24 &= 24 \\ 1x_1 + 2x_2 &= 6 &= 6 \\ -1x_1 + 1x_2 &= -1.5 &< 1 \\ 1x_1 &= 1.5 &< 2 \end{aligned}$$

} feasible

$$Z = 5x_1 + 4x_2 = \$21$$

(d) $(x_1, x_2) = (2, 1)$

$$\begin{aligned} x_1, x_2 &\geq 0 \\ 6x_1 + 4x_2 &= 16 &< 24 \\ 1x_1 + 2x_2 &= 4 &< 6 \\ -1x_1 + 1x_2 &= -1 &< 1 \\ 1x_1 &= 1 &< 2 \end{aligned}$$

} feasible

$$Z = 5x_1 + 4x_2 = \$14$$

(e) $(x_1, x_2) = (2, -1)$

$x_1 \geq 0, x_2 < 0$, infeasible

Conclusion: (c) gives the best feasible solution

$(x_1, x_2) = (2, 2)$

Let S_1 and S_2 be the unused daily amounts of M1 and M2.

$$\text{For M1: } S_1 = 24 - (6x_1 + 4x_2) = 4 \text{ tons/day}$$

$$\begin{aligned} \text{For M2: } S_2 &= 6 - (x_1 + 2x_2) \\ &= 6 - (2 + 2x_2) = 0 \text{ tons/day} \end{aligned}$$

3

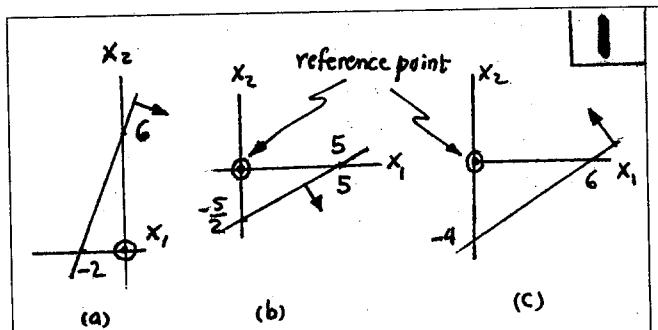
2-2

Quantity discount results in the **4** following nonlinear objective function:

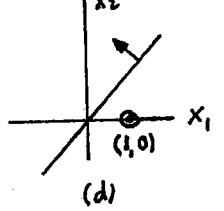
$$Z = \begin{cases} 5x_1 + 4x_2, & 0 \leq x_1 \leq 2 \\ 4.5x_1 + 4x_2, & x_1 > 2 \end{cases}$$

The situation cannot be treated as a linear program. Nonlinearity can be accounted for in this case using mixed integer programming (chapter 9).

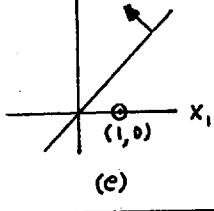
Set 2.2a



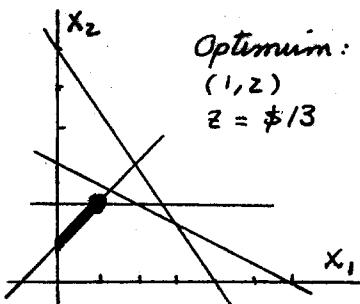
(a), (b), (c)



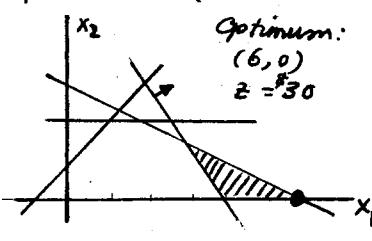
(c)



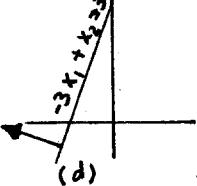
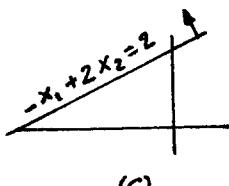
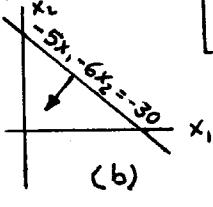
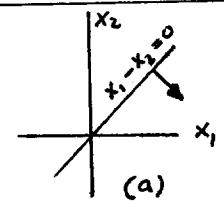
$$(c) -x_1 + x_2 = 1$$



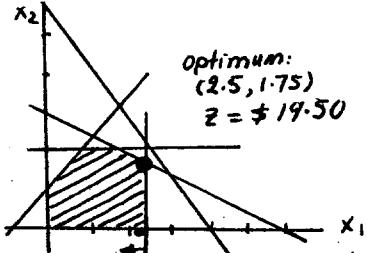
$$(d) 6x_1 + 4x_2 \geq 24$$



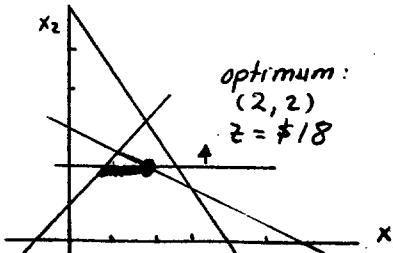
(c) No feasible space



$$(a) x_1 \leq 2.5$$



$$(b) x_2 \geq 2$$



2

x_1 = daily units of product 1
 x_2 = daily units of product 2

$$\text{Maximize } Z = 2x_1 + 3x_2$$

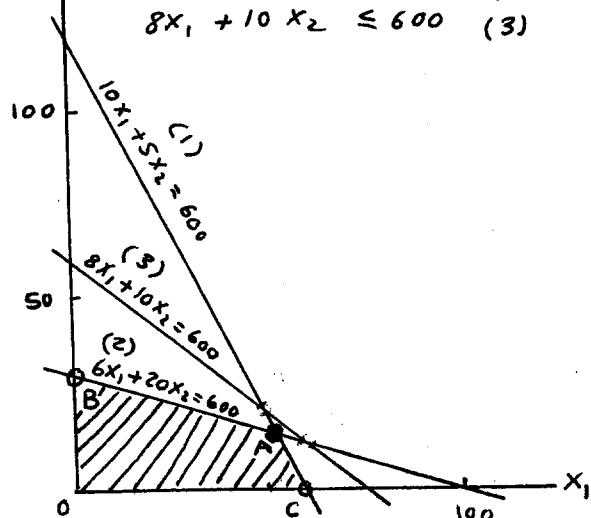
s.t.

$$10x_1 + 5x_2 \leq 600 \quad (1)$$

$$6x_1 + 20x_2 \leq 600 \quad (2)$$

$$8x_1 + 10x_2 \leq 600 \quad (3)$$

3



Optimum occurs at A:

$$x_1 = 52.94$$

$$x_2 = 14.12$$

$$Z = \$148.24$$

continued...

Set 2.2a

x_1 = number of units of A
 x_2 = number of units of B

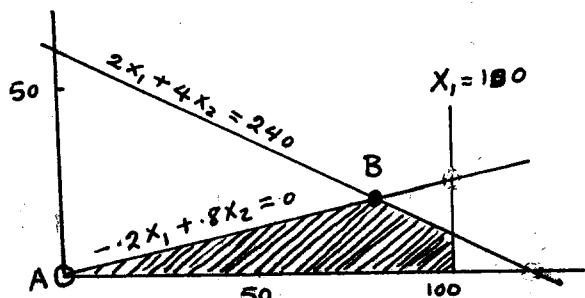
$$\text{Maximize } Z = 20x_1 + 50x_2$$

$$\frac{x_1}{x_1+x_2} \geq .8 \quad \text{or} \quad -2x_1 + 8x_2 \leq 0$$

$$x_1 \leq 100$$

$$2x_1 + 4x_2 \leq 240$$

$$x_1, x_2 \geq 0$$



Optimal occurs at B:

$$x_1 = 80 \text{ units}$$

$$x_2 = 20 \text{ units}$$

$$Z = \$2,600$$

5

x_1 = \$ invested in A

x_2 = \$ invested in B

$$\text{Maximize } Z = .05x_1 + .08x_2$$

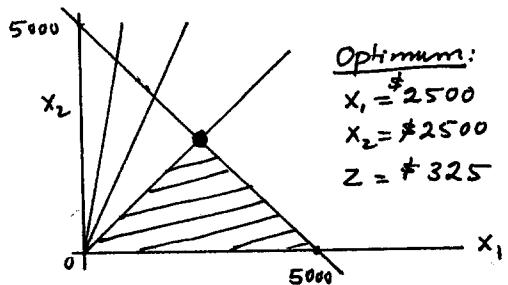
$$\text{s.t. } x_1 \geq .25(x_1 + x_2)$$

$$x_2 \leq .5(x_1 + x_2)$$

$$x_1 \geq .5x_2$$

$$x_1 + x_2 \leq 5000$$

$$x_1, x_2 \geq 0$$



7

x_1 = number of practical courses

x_2 = number of humanistic courses

$$\text{Maximize } Z = 1500x_1 + 1000x_2$$

s.t.

$$x_1 + x_2 \leq 30$$

$$x_1 \geq 10$$

$$x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

6

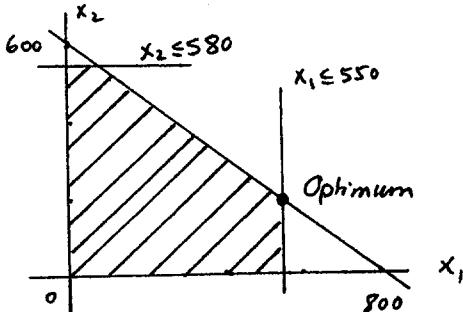
x_1 = number of sheets/day

x_2 = number of bars/day

$$\text{Maximize } Z = 40x_1 + 35x_2$$

$$\text{s.t. } \frac{x_1}{800} + \frac{x_2}{600} \leq 1$$

$$0 \leq x_1 \leq 550, \quad 0 \leq x_2 \leq 580$$



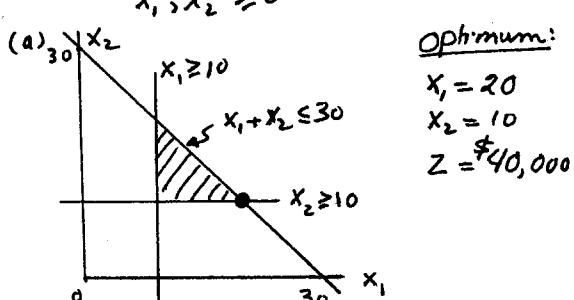
Optimum solution:

$$x_1 = 550 \text{ sheets}$$

$$x_2 = 187.13 \text{ bars}$$

$$Z = \$28,549.40$$

8



Optimum:

$$x_1 = 20$$

$$x_2 = 10$$

$$Z = \$40,000$$

(b) Change $x_1 + x_2 \leq 30$ to $x_1 + x_2 \leq 31$

$$\text{Optimum } Z = \$41,500$$

$$\Delta Z = \$41,500 - \$40,000 = \$1500$$

Conclusion: Any additional course will be of the practical type.

Set 2.2a

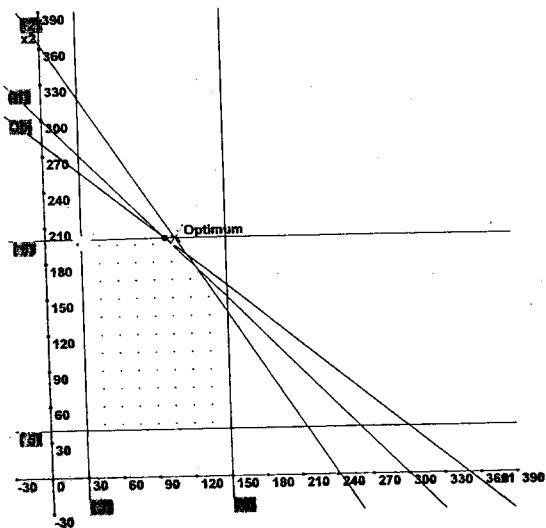
x_1 = units of solution A
 x_2 = units of solution B

$$\text{Maximize } Z = 8x_1 + 10x_2$$

subject to

$$\begin{aligned} .5x_1 + .5x_2 &\leq 150 \\ .6x_1 + .4x_2 &\leq 145 \\ x_1 &\geq 30 \\ x_1 &\leq 150 \\ x_2 &\geq 40 \\ x_2 &\leq 200 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Summary of Optimal Solution:
 Objective Value = 2800.00
 $x_1 = 100.00$
 $x_2 = 200.00$



x_1 = nbr. of grano boxes
 x_2 = nbr. of wheatie boxes

$$\text{Maximize } Z = x_1 + 1.35x_2$$

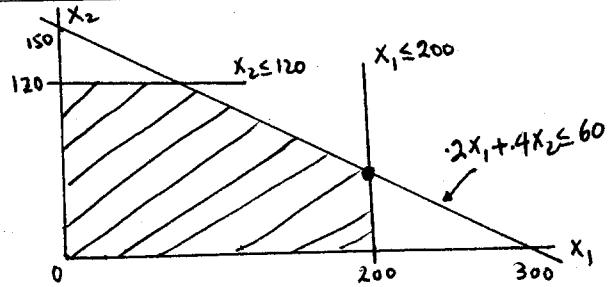
$$\text{s.t. } .2x_1 + .4x_2 \leq 60$$

$$x_1 \leq 200$$

$$x_2 \leq 120$$

$$x_1, x_2 \geq 0$$

9



Optimum: $x_1 = 200, x_2 = 50, Z = \267.50
Area allocation: 67% grano, 33% wheatie

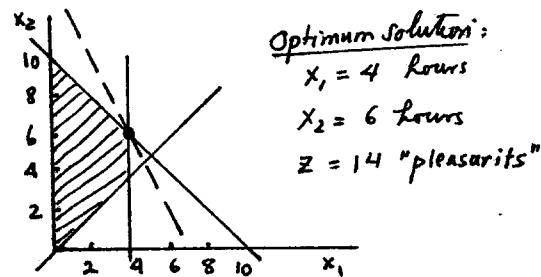
11

x_1 = play hours per day
 x_2 = work hours per day

$$\text{Maximize } Z = 2x_1 + x_2$$

s.t.

$$\begin{aligned} x_1 + x_2 &\leq 10 \\ x_1 - x_2 &\leq 0 \\ x_1 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Optimum solution:

$$x_1 = 4 \text{ hours}$$

$$x_2 = 6 \text{ hours}$$

$$Z = 14 \text{ "pleasurits"}$$

12

x_1 = Daily nbr. of type 1 hat
 x_2 = Daily nbr. of type 2 hat

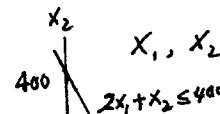
$$\text{Maximize } Z = 8x_1 + 5x_2$$

$$\text{s.t. } 2x_1 + x_2 \leq 400$$

$$x_1 \leq 150$$

$$x_2 \leq 200$$

$$x_1, x_2 \geq 0$$



Optimum:

$$x_1 = 100 \text{ type 1}$$

$$x_2 = 200 \text{ type 2}$$

$$Z = \$1800$$

10

continued...

continued...

Set 2.2a

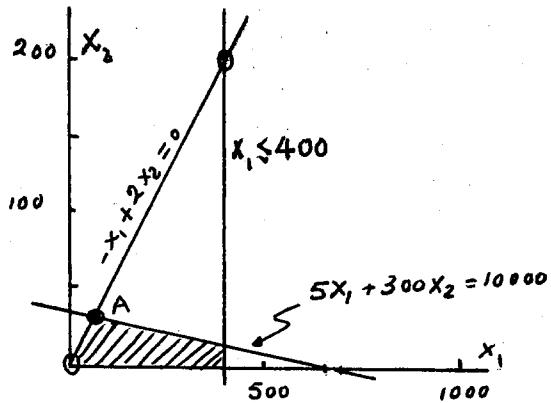
x_1 = radio minutes
 x_2 = TV minutes

$$\text{Maximize } Z = x_1 + 25x_2$$

$$\text{s.t. } 15x_1 + 300x_2 \leq 10,000$$

$$\frac{x_1}{x_2} \geq 2 \text{ or } -x_1 + 2x_2 \leq 0$$

$$x_1 \leq 400, x_1, x_2 \geq 0$$



Optimum occurs at A:

$$x_1 = 60.61 \text{ minutes}$$

$$x_2 = 30.3 \text{ minutes}$$

$$Z = 818.18$$

x_1 = tons of C_1 consumed per hour
 x_2 = tons of C_2 consumed per hour

$$\text{Maximize } Z = 12000x_1 + 9000x_2$$

s.t.

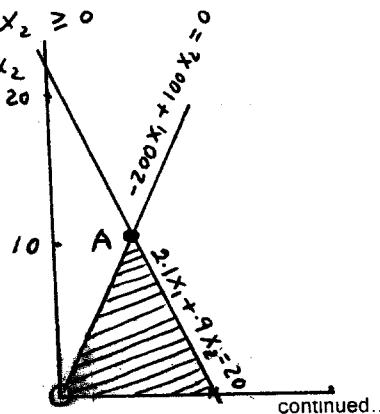
$$1800x_1 + 2100x_2 \leq 2000(x_1 + x_2)$$

or

$$-200x_1 + 100x_2 \leq 0$$

$$2.1x_1 + .9x_2 \leq 20$$

$$x_1, x_2 \geq 0$$



13

(a) Optimum occurs at A:

$$x_1 = 5.128 \text{ tons per hour}$$

$$x_2 = 10.256 \text{ tons per hour}$$

$$Z = 153,846 \text{ lb of Steam}$$

$$\text{Optimal ratio} = \frac{5.128}{10.256} = .5$$

$$(b) 2.1x_1 + .9x_2 \leq (20+1) = 21$$

$$\text{Optimum } Z = 161538 \text{ lb of Steam}$$

$$\Delta Z = 161538 - 153846 = 7692 \text{ lb}$$

15

x_1 = Nbr. of radio commercials beyond the first

x_2 = Nbr. of TV ads beyond the first

$$\text{Maximize } Z = 2000x_1 + 3000x_2 + 5000 + 2000$$

$$\text{s.t. } 300(x_1+1) + 2000(x_2+1) \leq 20,000$$

$$300(x_1+1) \leq .8 \times 20,000$$

$$2000(x_2+1) \leq .8 \times 20,000$$

$$x_1, x_2 \geq 0$$

or

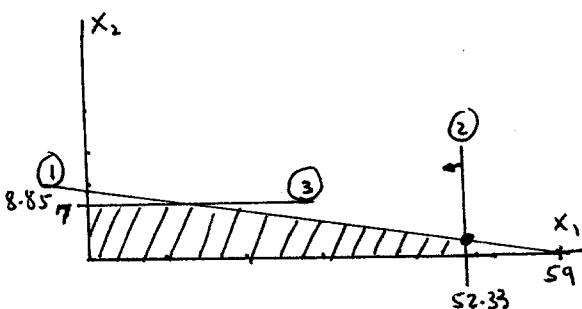
$$\text{Maximize } Z = 2000x_1 + 3000x_2 + 7000$$

$$\text{s.t. } 300x_1 + 2000x_2 \leq 17700 \quad (1)$$

$$300x_1 \leq 15700 \quad (2)$$

$$2000x_2 \leq 14000 \quad (3)$$

$$x_1, x_2 \geq 0$$



Optimum solution:

$$\text{Radio commercials} = 52.33 + 1 = 53.33$$

$$\text{TV ads} = 1 + 1 = 2$$

$$Z = 107666.67 + 7000 = 114666.67$$

Set 2.2a

x_1 = number of shirts per hour
 x_2 = number of blouses per hour

$$\text{Maximize } Z = 8x_1 + 12x_2$$

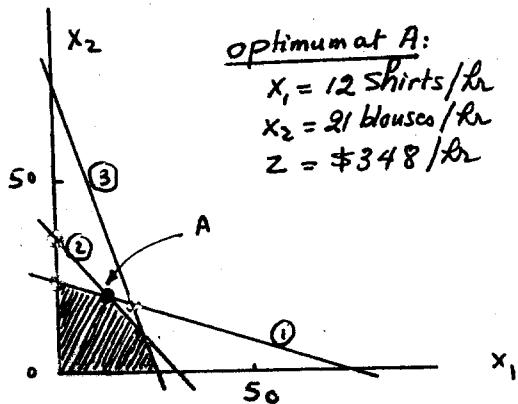
s.t.

$$20x_1 + 60x_2 \leq 25 \times 60 = 1500 \quad (1)$$

$$70x_1 + 60x_2 \leq 35 \times 60 = 2100 \quad (2)$$

$$12x_1 + 4x_2 \leq 5 \times 60 = 300 \quad (3)$$

$$x_1, x_2 \geq 0$$



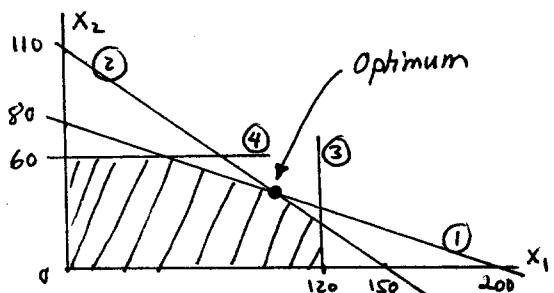
x_1 = Nbr. of desks per day
 x_2 = Nbr. of chairs per day

$$\text{Maximize } Z = 50x_1 + 100x_2$$

$$\frac{x_1}{200} + \frac{x_2}{80} \leq 1 \quad (1)$$

$$\frac{x_1}{150} + \frac{x_2}{110} \leq 1 \quad (2)$$

$$x_1 \leq 120, x_2 \leq 60 \quad (3,4)$$



Optimum:

$$x_1 = 90 \text{ desks}$$

$$x_2 = 44 \text{ chairs}$$

$$Z = \$8900$$

16

x_1 = number of HiFi 1 units
 x_2 = number of HiFi 2 units

18

Constraints:

$$6x_1 + 4x_2 \leq 480 \times 9 = 432$$

$$5x_1 + 5x_2 \leq 480 \times 8.6 = 412.8$$

$$4x_1 + 6x_2 \leq 480 \times 8.8 = 422.4$$

or

$$6x_1 + 4x_2 + s_1 = 432$$

$$5x_1 + 5x_2 + s_2 = 412.8$$

$$4x_1 + 6x_2 + s_3 = 422.4$$

Objective function:

$$\text{Minimize } S_1 + S_2 + S_3 = 1267.2 - 15x_1 - 15x_2$$

$$\text{Thus, } \min S_1 + S_2 + S_3 \equiv \max 15x_1 + 15x_2$$

$$\text{Maximize } Z = 15x_1 + 15x_2$$

s.t.

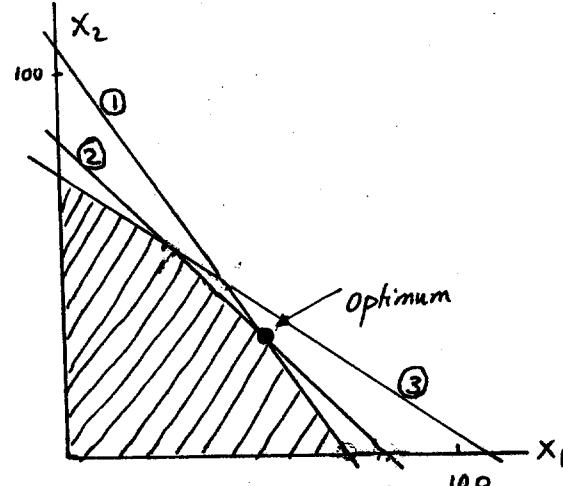
$$6x_1 + 4x_2 \leq 432 \quad (1)$$

$$5x_1 + 5x_2 \leq 412.8 \quad (2)$$

$$4x_1 + 6x_2 \leq 422.4 \quad (3)$$

$$x_1, x_2 \geq 0$$

17



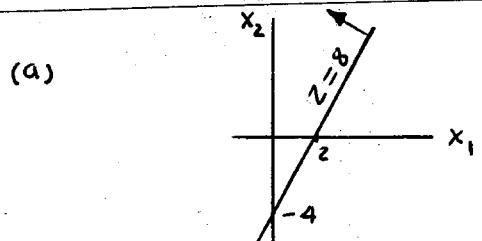
Optimum: (Problem has alternative optima)

$$x_1 = 50.88 \text{ units}$$

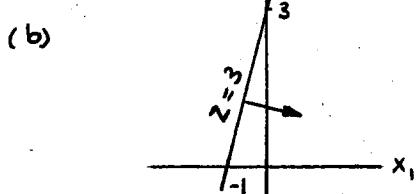
$$x_2 = 31.68 \text{ units}$$

$$Z = 1238.4 \text{ minutes}$$

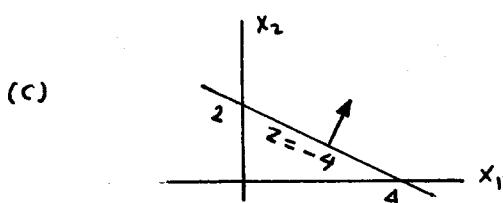
Set 2.2b



1

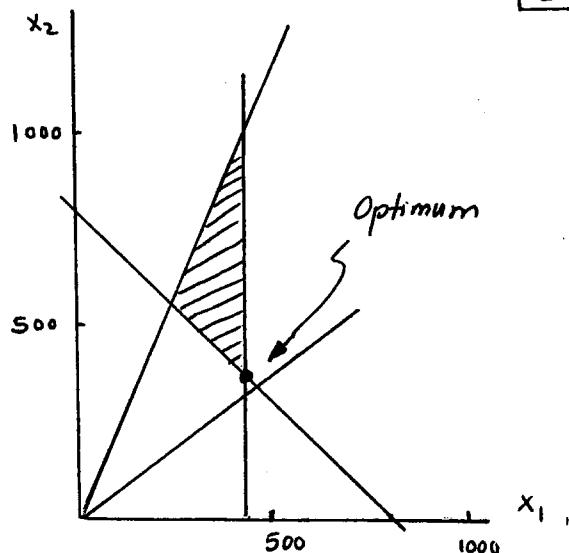


2



Additional constraint: $x_1 \leq 450$

2



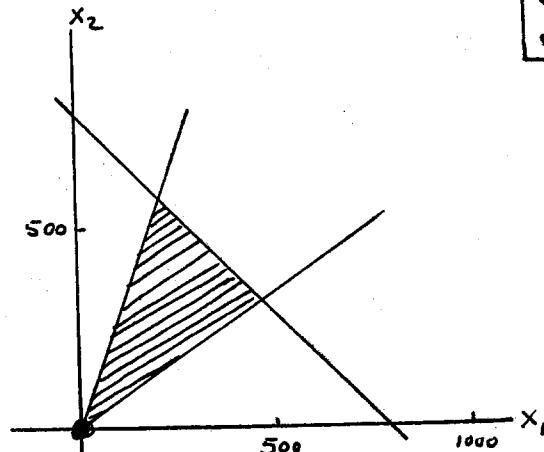
Optimum Solution:

$$x_1 = 450 \text{ lb}$$

$$x_2 = 350 \text{ lb}$$

$$Z = \$450$$

continued...



3

Optimum: $x_1 = 0, x_2 = 0, Z = 0$, which is nonsensical

4

x_1 = number of hours/week in store 1
 x_2 = number of hours/week in store 2

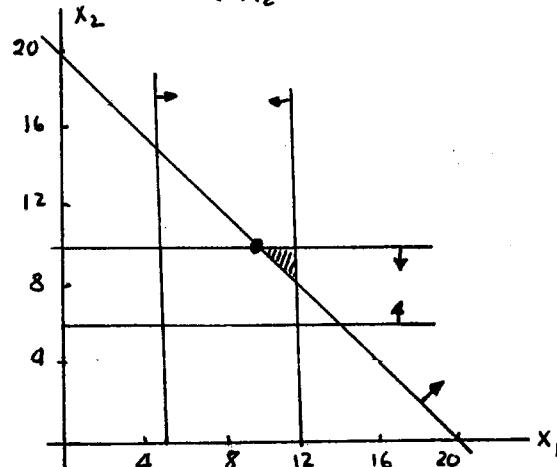
$$\text{Minimize } Z = 8x_1 + 6x_2$$

s.t.

$$x_1 + x_2 \geq 20$$

$$5 \leq x_1 \leq 12$$

$$6 \leq x_2 \leq 10$$



Optimum:

$$x_1 = 10 \text{ hours}$$

$$x_2 = 10 \text{ hours}$$

$$Z = 140 \text{ stress index}$$

continued...

Let

$$x_1 = 10^3 \text{ bbl/day from Iran}$$

$$x_2 = 10^3 \text{ bbl/day from Dubai}$$

$$\text{Refinery capacity} = x_1 + x_2 \quad 10^3 \text{ bbl/day}$$

$$\text{Minimize } Z = x_1 + x_2$$

Subject to

$$x_1 \geq .4(x_1 + x_2)$$

$$\text{or } -.6x_1 + .4x_2 \leq 0$$

$$.2x_1 + .1x_2 \geq 14$$

$$.25x_1 + .6x_2 \geq 30$$

$$.1x_1 + .15x_2 \geq 10$$

$$.15x_1 + .1x_2 \geq 8$$

$$x_1, x_2 \geq 0$$

Optimum solution from TORA:

5 Let

$$x_1 = 10^3 \text{ $ invested in blue chip stock}$$

$$x_2 = 10^3 \text{ $ invested in high-tech stocks}$$

$$\text{Minimize } Z = x_1 + x_2$$

Subject to

$$.1x_1 + .25x_2 \geq 10$$

$$.6x_1 - .4x_2 \geq 0$$

$$x_1, x_2 \geq 0$$

TORA optimum solution:

LINEAR PROGRAMMING – GRAPHICAL SOLUTION

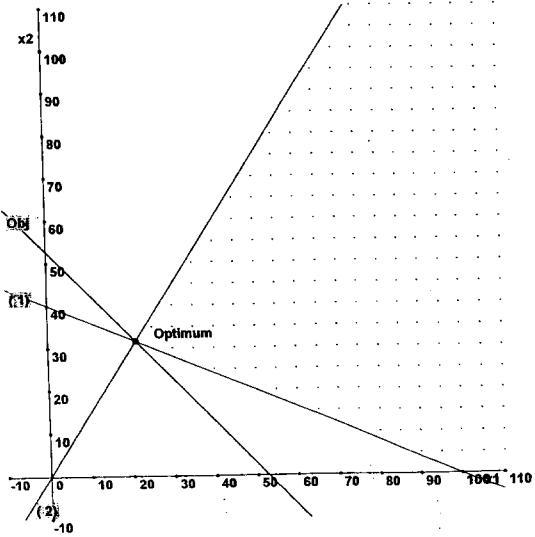
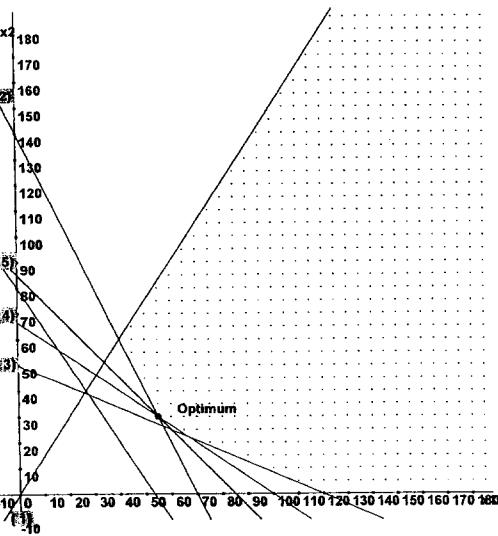
Title: diet problem

Summary of Optimal Solution:

Objective Value = 52.63

x1 = 21.05

x2 = 31.58



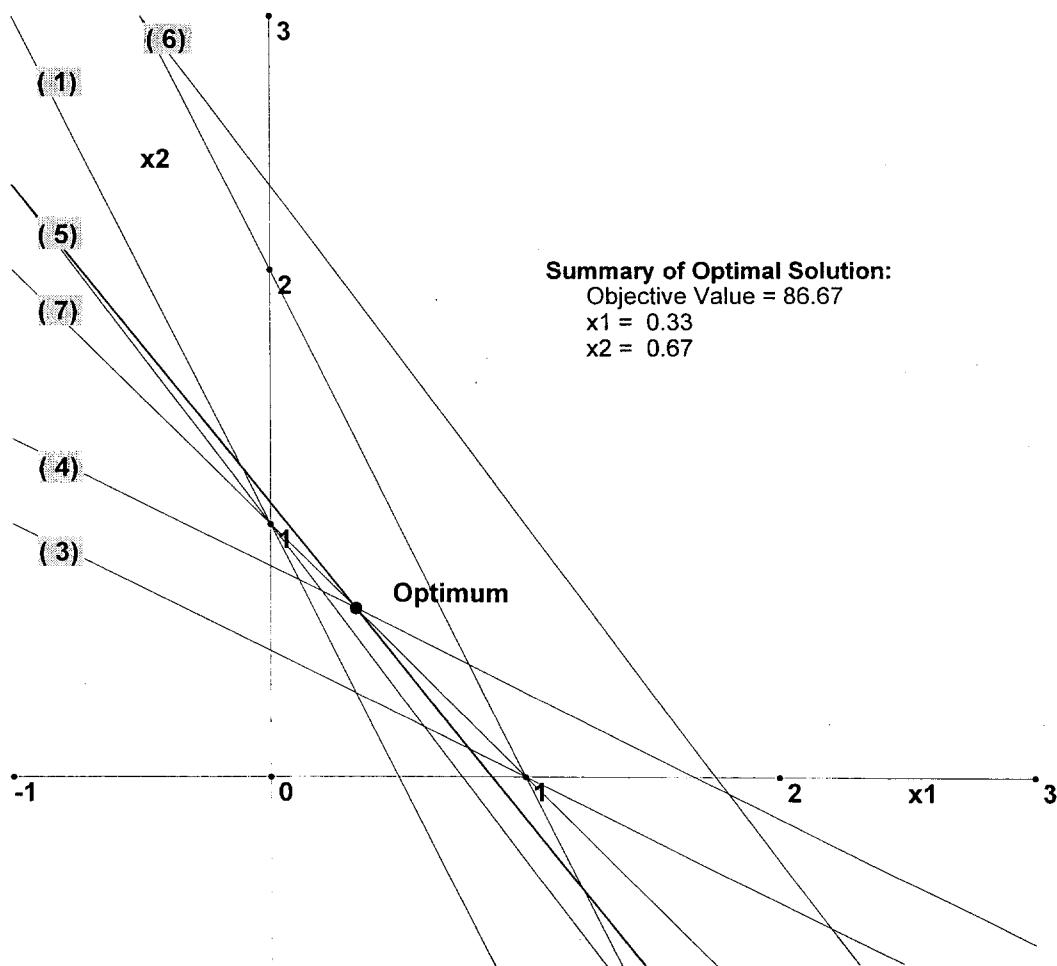
Set 2.2b

7

x_1 = Ratio of scrap A in alloy

x_2 = Ratio of scrap B in alloy

	x_1	x_2		
Minimize	100.00	80.00		
Subject to				
(1)	0.06	0.03	>=	0.03
(2)	0.06	0.03	<=	0.06
(3)	0.03	0.06	>=	0.03
(4)	0.03	0.06	<=	0.05
(5)	0.04	0.03	>=	0.03
(6)	0.04	0.03	<=	0.07
(7)	1.00	1.00	=	1.00



Set 2.3a

X_e = Nbr. of efficiency apartments

X_d = Nbr. of duplexes

X_s = Nbr. of single-family homes

X_r = Retail space in ft²

$$\text{Maximize } Z = 600X_e + 750X_d + 1200X_s + 100X_r$$

$$\text{s.t. } X_e \leq 500, X_d \leq 300, X_s \leq 250$$

$$X_r \geq 10X_e + 15X_d + 18X_s$$

$$X_r \leq 10000$$

$$X_d \geq \frac{X_e + X_s}{2}$$

$$X_e, X_d, X_s, X_r \geq 0$$

Optimal Solution:

$$Z = 1,595,714.29$$

$$X_e = 207.14, X_d = 228.57$$

$$X_s = 250, X_r = 10,000$$

LP does not guarantee integer solution.
Use rounded solution or apply integer LP algorithm (Chapter 9).

X_i = Acquired portion of property i

2

Each site is represented by a separate LP.
The site that yields the smaller objective value is selected.

Site 1 LP:

$$\text{Minimize } Z = 25 + X_1 + 2.1X_2 + 2.35X_3 + 1.85X_4 + 2.95X_5$$

$$\text{s.t. } X_4 \geq .75, \text{ all } X_i \geq 0, i=1,2,\dots,5$$

$$20X_1 + 50X_2 + 50X_3 + 30X_4 + 60X_5 \geq 200$$

$$\text{Optimum: } Z = 34.6625 \text{ million\$}$$

$$X_1 = .875, X_2 = X_3 = 1, X_4 = .75, X_5 = 1$$

Site 2 LP:

$$\text{Minimize } Z = 27 + 2.8X_1 + 1.9X_2 + 2.8X_3 + 2.5X_4$$

$$\text{s.t. } X_3 \geq .5, X_1, X_2, X_3, X_4 \geq 0$$

$$80X_1 + 60X_2 + 50X_3 + 70X_4 \geq 200$$

$$\text{Optimum: } Z = 34.35 \text{ million\$}$$

$$X_1 = X_2 = 1, X_3 = X_4 = .5$$

Select Site 2.

X_{ij} = portion of project i completed in year j

3

$$\begin{aligned} \text{Maximize } Z &= .05(4X_4 + 3X_{12} + 2X_{13}) + \\ &+ .07(3X_{22} + 2X_{23} + X_{24}) + \\ &+ .15(4X_{31} + 3X_{32} + 2X_{33} + X_{34}) + \\ &+ .02(2X_{43} + X_{44}) \end{aligned}$$

s.t.

$$\sum_{j=1}^3 X_{1j} = 1, \quad \sum_{j=3}^4 X_{4j} = 1$$

$$.25 \leq \sum_{j=2}^5 X_{2j} \leq 1, \quad .25 \leq \sum_{j=1}^5 X_{3j} \leq 1$$

$$5X_{11} + 15X_{31} \leq 3$$

$$5X_{12} + 8X_{22} + 15X_{32} \leq 6$$

$$5X_{13} + 8X_{23} + 15X_{33} + 1.2X_{43} \leq 7$$

$$8X_{24} + 15X_{34} + 1.2X_{44} \leq 7$$

$$8X_{25} + 15X_{35} \leq 7$$

Optimum:

$$Z = \$523,750$$

$$X_{11} = .6, X_{12} = .4$$

$$X_{24} = .225, X_{25} = .025$$

$$X_{32} = .267, X_{33} = .387, X_{34} = .346$$

$$X_{43} = 1$$

X_l = Nbr. of low income units

4

X_m = Nbr. of middle income units

X_u = Nbr. of upper income units

X_p = Nbr. of public housing units

X_s = Nbr. of school rooms

X_r = Nbr. of retail units

X_c = Nbr. of condemned homes

$$\text{Maximize } Z = 7X_l + 12X_m + 20X_u + 5X_p + 15X_r$$

$$-10X_s - 7X_c$$

$$100 \leq X_l \leq 200, \quad 125 \leq X_m \leq 190$$

$$75 \leq X_u \leq 260, \quad 300 \leq X_p \leq 600$$

$$0 \leq X_s \leq 2.045$$

$$.05X_l + .07X_m + .03X_u + .025X_p +$$

$$.045X_s + .1X_c \leq .85(50 + .25X_c)$$

$$X_r \geq .023X_l + .034X_m + .046X_u +$$

$$.023X_p + .034X_s$$

continued...

Set 2.3a

$$25x_S \geq 1.3x_L + 1.2x_m + .5x_u + 1.4x_p$$

Optimum: $Z = 8290.30$ thousand \$

$$x_L = 100, x_m = 125, x_u = 227.04$$

$$x_p = 300, x_S = 32.54, x_n = 25$$

$$x_c = 0$$

x_1 = Nbr. of single-family homes

5

x_2 = Nbr. of double-family homes

x_3 = Nbr. of triple-family homes

x_4 = Nbr. of recreation areas

$$\text{Maximize } Z = 10,000x_1 + 12,000x_2 + 15,000x_3$$

s.t.

$$2x_1 + 3x_2 + 4x_3 + x_4 \leq .85 \times 800$$

$$\frac{x_1}{x_1 + x_2 + x_3} \geq .5 \text{ or } .5x_1 - .5x_2 - .5x_3 \geq 0$$

$$x_4 \geq \frac{x_1 + 2x_2 + 3x_3}{200} \text{ or } 200x_4 - x_1 - 2x_2 - 3x_3 \geq 0$$

$$1000x_1 + 1200x_2 + 1400x_3 + 800x_4 \geq 100,000$$

$$400x_1 + 600x_2 + 840x_3 + 450x_4 \leq 200,000$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Optimum solution:

$$x_1 = 339.15 \text{ homes}$$

$$x_2 = 0$$

$$x_3 = 0$$

$$x_4 = 1.69 \text{ areas}$$

$$Z = \$339,521.20$$

New land use constraint:

6

$$2x_1 + 3x_2 + 4x_3 + x_4 \leq .85(800 + 100)$$

New Optimum Solution:

$$Z = \$381,5461.35$$

$$x_1 = 381.54 \text{ homes}$$

$$x_2 = x_3 = 0$$

$$x_4 = 1.91 \text{ areas}$$

$$\Delta Z = \$3815,461.35 - \$339,521.20$$

$$= \$423,940.35$$

$\Delta Z < \$450,000$, the purchasing cost of 100 acres. Hence, the purchase of the new acreage is not recommended.

Set 2.3b

The constraints remain unchanged, but the objective function is changed to

Maximize $Z = y - \text{commission}$

where

$$\begin{aligned} \text{Commission} &= .001(\text{all transactions in } \$) \\ &= .001[(X_{12} + X_{13} + X_{14} + X_{15}) + \\ &\quad \frac{1}{.769}(X_{21} + X_{23} + X_{24} + X_{25}) + \\ &\quad \frac{1}{.625}(X_{31} + X_{32} + X_{34} + X_{35}) + \\ &\quad \frac{1}{.105}(X_{41} + X_{42} + X_{43} + X_{45}) + \\ &\quad \frac{1}{.342}(X_{51} + X_{52} + X_{53} + X_{54})] \end{aligned}$$

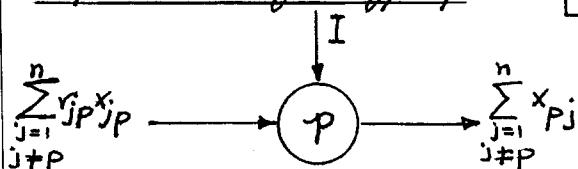
Optimum solution:

	Without	With
Z	5.09032	5.06211
y	5.09032	5.08986
Return	1.8064%	1.2421%

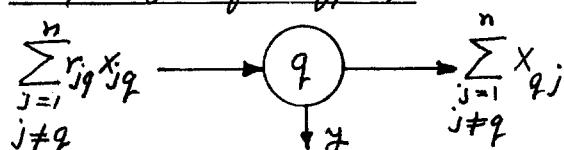
$$\begin{aligned} \text{Commission} &= 5.08986 - 5.06211 \\ &= \$27,750 \end{aligned}$$

or, .555% of the original investment of \$5 million

Input I in fund type p :



Output y in fund type q :

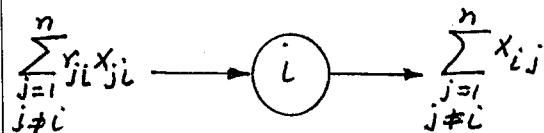


For specific p and q , the model below can be used to transform any fund to any other fund. In

continued...

the present problem, $p=1 (\$)$ and $q=2 (\text{€}), 3 (\text{£}), 4 (\text{¥}), \text{and } 5 (\text{KD})$.

General node i :



Maximize $Z = y$

s.t.

$$I + \sum_{\substack{j=1 \\ j \neq p}}^n r_{jp} x_{jp} = \sum_{\substack{j=1 \\ j \neq p}}^n x_{pj}$$

$$\sum_{\substack{j=1 \\ j \neq q}}^n r_{qj} x_{qj} = y + \sum_{\substack{j=1 \\ j \neq q}}^n x_{qj}$$

$$\sum_{\substack{j=1 \\ j \neq i}}^n r_{ij} x_{ij} = \sum_{\substack{j=1 \\ j \neq i}}^n x_{ij}, \quad i \neq p \text{ or } q$$

$$0 \leq x_{ij} \leq \text{Cap}_i, \quad \text{all } i \text{ and } j$$

Note: Solver or AMPL is ideal for solving this problem interactively. See files solver2.3b-2.xls and ampl2.3b-2.txt.

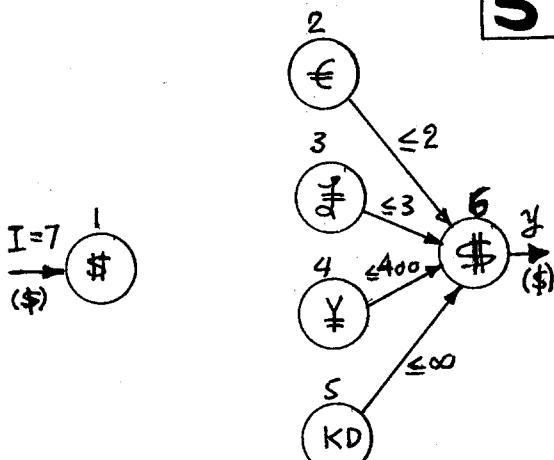
Results: (No commission)

p	q	Rate of return
\$	\$	1.8064%
\$	€	1.7966%
\$	£	1.8287%
\$	¥	2.8515%
\$	KD	1.0471%

Wide discrepancy in ¥ and KD currencies may be attributed to the fact that their exchange rates may not be consistent with the remaining rates. Nevertheless, the problem shows that there may be advantages in targeting accumulation in different currencies.

Set 2.3b

3



To formulate the objective function correctly, all output currencies are converted to a single currency (arbitrarily chosen to be #). Thus

$$Y = r_{21}x_{26} + r_{31}x_{36} + r_{41}x_{46} + r_{51}x_{56}$$

Maximize $Z = Y$

s.t. $x_{26} \leq 2, x_{36} \leq 3, x_{46} \leq 400; x_{5j} \leq 3.5$
 $x_{ij} \leq 5, x_{2j} \leq 3, x_{4j} \leq 100, x_{5j} \leq 2.8, \text{ all } j$

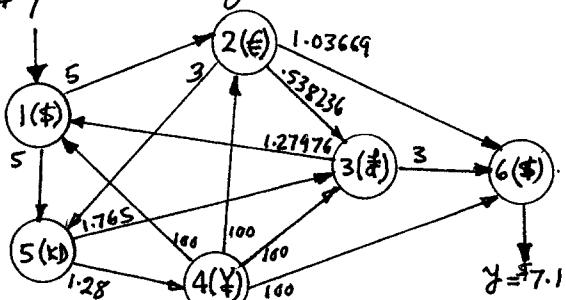
$$I + \sum_{i=2}^5 x_{ij} r_{i1} = \sum_{j=2}^5 x_{1j}$$

$$Y = r_{21}x_{26} + r_{31}x_{36} + r_{41}x_{46} + r_{51}x_{56}$$

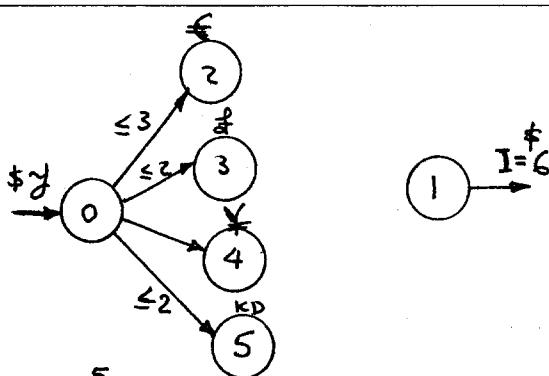
$$x_{i6} + \sum_{\substack{j=1 \\ j \neq i}}^5 x_{ij} = \sum_{\substack{j=1 \\ j \neq i}}^5 r_{ij} x_{jj}, i = 2, 3, 4, 5$$

all $x_{ij} \geq 0, i \neq j$

Solution: Total accumulation $Y = \$7.1$ million
Rate of return = 1.4354%



4



$$Y = \sum_{k=2}^5 r_{ki} x_{0k} = r_{21}x_{02} + r_{31}x_{03} + r_{41}x_{04} + r_{51}x_{05}$$

Minimize $Z = Y$

$$\text{s.t. } Y = r_{21}x_{02} + r_{31}x_{03} + r_{41}x_{04} + r_{51}x_{05}$$

$$x_{02} \leq 3, x_{03} \leq 2, x_{05} \leq 2$$

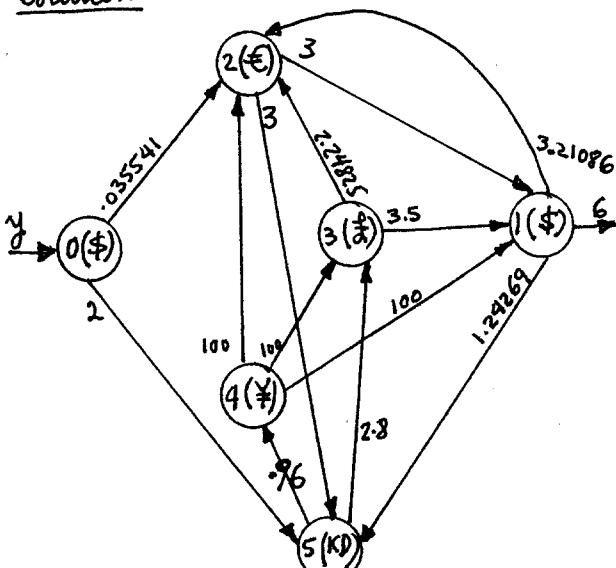
$$\text{all } j: x_{1j} \leq 5, x_{2j} \leq 3, x_{4j} \leq 3.5, x_{4j} \leq 100, x_{5j} \leq 2.8$$

$$x_{0j} + \sum_{\substack{i=1 \\ i \neq j}}^5 r_{ij} x_{ij} = \sum_{k=1}^5 x_{jk}, j = 2, 3, 4, 5$$

$$\sum_{i=2}^5 r_{i1} x_{i1} = \sum_{j=2}^5 x_{1j} + I$$

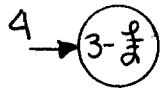
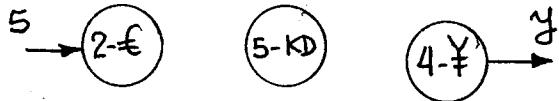
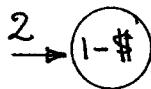
All $x_{ij} \geq 0$

Solution:



Rate of return = 1.7638%

5



$$\text{Maximize } z = y$$

s.t.

$$y = r_{14}x_{14} + r_{24}x_{24} + r_{34}x_{34} + r_{54}x_{54}$$

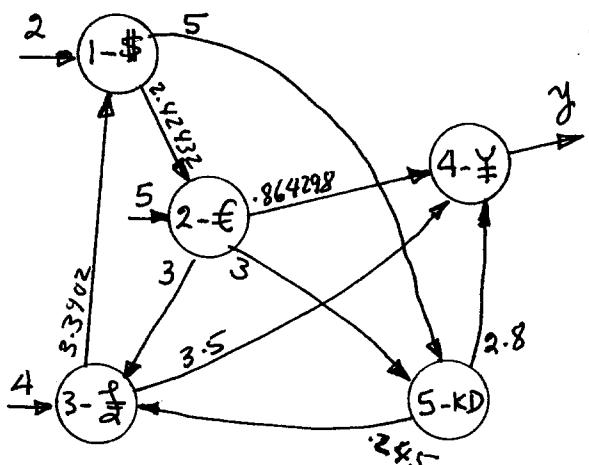
$$\sum_{\substack{i=1 \\ i \neq j}}^5 r_{ij}x_{ij} = \sum_{\substack{k=1 \\ k \neq j}}^5 x_{jk} - \begin{cases} 2, & j=1 \\ 5, & j=2 \\ 4, & j=3 \\ -y, & j=4 \\ 0, & j=5 \end{cases}$$

$$x_{ij} \leq C_i, \text{ for all } i \text{ and } j \quad \{i \neq j\}$$

$$x_{ij} \geq 0, \text{ for all } i \text{ and } j$$

Solution: $y = 1584.91 \text{ million } \text{¥}$

Rate of return = .8853%



Set 2.3c

(a) x_i = undertaken portion of project i

Maximize

$$Z = 32.4x_1 + 35.8x_2 + 17.75x_3 + 14.8x_4 + 18.2x_5 + 12.35x_6$$

Subject to

$$\begin{aligned} 10.5x_1 + 8.3x_2 + 10.2x_3 + 7.2x_4 + 12.3x_5 + 9.2x_6 &\leq 60 \\ 14.4x_1 + 12.6x_2 + 14.2x_3 + 10.5x_4 + 10.1x_5 + 7.8x_6 &\leq 70 \\ 2.2x_1 + 9.5x_2 + 5.6x_3 + 7.5x_4 + 8.3x_5 + 6.9x_6 &\leq 35 \\ 2.4x_1 + 3.1x_2 + 4.2x_3 + 5.0x_4 + 6.3x_5 + 5.1x_6 &\leq 20 \\ 0 \leq x_j &\leq 1, j=1,2,\dots,6 \end{aligned}$$

TOA optimum solution:

$$x_1 = x_2 = x_3 = x_4 = 1, x_5 = 0.84, x_6 = 0, Z = 116.06$$

(b) Add the constraint $x_2 \leq x_6$

TOA optimum solution:

$$x_1 = x_2 = x_3 = x_4 = x_5 = 1, x_6 = 0.03, Z = 113.68$$

(c) Let s_i be the unused funds at the end of year i and change the right-hand sides of constraints 2, 3, and 4 to $70+s_1$, $35+s_2$, and $20+s_3$, respectively.

TOA optimum solution:

$$x_1 = x_2 = x_3 = x_4 = x_5 = 1, x_6 = 0.71$$

$$Z = \$127.72 \text{ (thousand)}$$

The solution is interpreted as follows:

i	s_i	$s_i - s_{i-1}$	Decision
1	4.96	-	-
2	7.62	+2.66	Don't borrow from yr 1
3	4.62	-3.00	Borrow \$3 from year 2
4	0	-4.62	Borrow \$4.62 from yr 2

The effect of availing excess money for use in later years is that the first five projected are completed and 71% of project 6 is undertaken.

The total revenue increases from \$116,060 to 127,720.

(d) The slack s_i in period i is treated as an unrestricted variable.

TOA optimum solution: $Z = \$131.30$

$$s_1 = 2.3, s_2 = 4, s_3 = -5, s_4 = -6.1$$

This means that additional funds are needed in years 3 and 4.

$$\text{Increase in return} = 131.30 - 116.06$$

$$= \$15.24$$

Ignoring the time value of money, the amount borrowed $5+6.1-(2.3+4)$ $= \$8.4$. Thus,

$$\text{rate of return} = \frac{15.24 - 8.4}{8.4} \approx 81\%$$

2

x_i = dollar investment in project i , $i=1,2,3,4$

y_j = dollar investment in bank in year j , $j=1,2,3,4,5$

Maximize $Z = y_5$

Subject to

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + y_1 &\leq 10,000 \\ .5x_1 + .6x_2 - x_3 + .4x_4 + 1.065y_1 - y_2 &= 0 \\ .3x_1 + .2x_2 + .8x_3 + .6x_4 + 1.065y_2 - y_3 &= 0 \\ 1.8x_1 + 1.5x_2 + 1.9x_3 + 1.8x_4 + 1.065y_3 - y_4 &= 0 \\ 1.2x_1 + 1.3x_2 + .8x_3 + .95x_4 + 1.065y_4 - y_5 &= 0 \end{aligned}$$

All variables ≥ 0

TOA optimal solution:

$$x_1 = 0, x_2 = \$10,000, x_3 = \$6000, x_4 = 0$$

$$y_1 = 0, y_2 = 0, y_3 = \$6800, y_4 = \$33,642$$

$$Z = \$53,628.73 \text{ at the start of year 5}$$

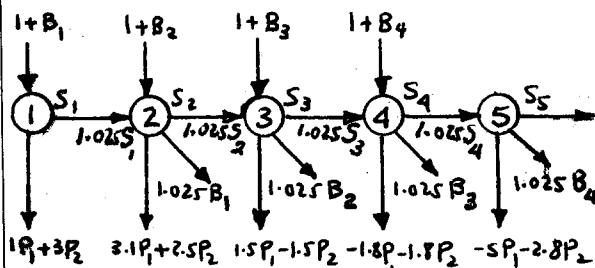
continued...

P_i = fraction undertaken of project

$i, i = 1, 2$

B_j = million dollars borrowed in quarter j , $j = 1, 2, 3, 4$

S_j = surplus million dollars at the start of quarter j , $j = 1, 2, 3, 4, 5$



$$(a) \text{Maximize } Z = S_5$$

subject to

$$\begin{aligned} P_1 + 3P_2 + S_1 - B_1 &= 1 \\ 3.1P_1 + 2.5P_2 - 1.02S_1 + S_2 + 1.025B_1 - B_2 &= 1 \\ 1.5P_1 - 1.5P_2 - 1.02S_2 + S_3 + 1.025B_2 - B_3 &= 1 \\ -1.8P_1 - 1.8P_2 - 1.02S_3 + S_4 + 1.025B_3 - B_4 &= 1 \\ -5P_1 - 2.8P_2 - 1.02S_4 + S_5 + 1.025B_4 &= 1 \\ 0 \leq P_1 \leq 1, 0 \leq P_2 \leq 1 & \\ 0 \leq B_j \leq 1, j = 1, 2, 3, 4 & \end{aligned}$$

Optimum Solution:

$$P_1 = .7113 \quad P_2 = 0$$

$Z = 5.8366$ million dollars

$B_1 = 0, B_2 = .9104$ million dollars

$B_3 = 1$ million dollars, $B_4 = 0$

(b) $B_1 = 0, S_1 = -.2887$ million \$

$B_2 = .9104, S_2 = 0$

$B_3 = 1, S_3 = 0$

$B_4 = 0, S_4 = 1.2553$

The solution shows that $B_i \cdot S_i = 0$, meaning that you can't borrow and also end up with surplus in any quarter. The result makes sense because the cost of borrowing (2.5%) is higher than the return on surplus funds (2%).

Assume that the investment program ends at the start of year 11. Thus, the 6-year bond option can be exercised in years 1, 2, 3, 4, and 5 only. Similarly, the 9-year bond can be used in years 1 and 2 only. Hence, from year 6 on, the only option available is insured savings at 7.5%.

Let

I_i = insured savings investments in year i , $i = 1, 2, \dots, 10$

G_i = 6-year bond investment in year i , $i = 1, 2, \dots, 5$

M_i = 9-year bond investment in year i , $i = 1, 2$

The objective is to maximize total accumulation at the end of year 10; that is,

$$\text{maximize } Z = 1.075 I_{10} + 1.079 G_5 + 1.085 M_2$$

The constraints represent the balance equation for each year's cash flow.

$$I_1 + .98G_1 + 1.02M_1 = 2$$

$$I_2 + .98G_2 + 1.02M_2 = 2 + 1.075I_1 + .079G_1 + .085M_1$$

$$\begin{aligned} I_3 + .98G_3 &= 2.5 + 1.075I_2 + .079(G_1 + G_2) + \\ &\quad .085(M_1 + M_2) \end{aligned}$$

$$\begin{aligned} I_4 + .98G_4 &= 2.5 + 1.075I_3 + \\ &\quad .079(G_1 + G_2 + G_3) + \\ &\quad .085(M_1 + M_2) \end{aligned}$$

$$\begin{aligned} I_5 + .98G_5 &= 3 + 1.075I_4 + \\ &\quad .079(G_1 + G_2 + G_3 + G_4) + \\ &\quad .085(M_1 + M_2) \end{aligned}$$

$$\begin{aligned} I_6 &= 3.5 + 1.075I_5 + \\ &\quad .079(G_1 + G_2 + G_3 + G_4 + G_5) + \\ &\quad .085(M_1 + M_2) \end{aligned}$$

continued...

Set 2.3c

$$\begin{aligned}
 I_7 &= 3.5 + 1.075 I_6 + 1.079 G_1 \\
 &\quad + .079 (G_2 + G_3 + G_4 + G_5) \\
 &\quad + .085 (M_1 + M_2) \\
 I_8 &= 4 + 1.075 I_7 + 1.079 G_2 \\
 &\quad + .079 (G_3 + G_4 + G_5) \\
 &\quad + .085 (M_1 + M_2) \\
 I_9 &= 4 + 1.075 I_8 + 1.079 G_3 \\
 &\quad + .079 (G_4 + G_5) \\
 &\quad + .085 (M_1 + M_2) \\
 I_{10} &= 5 + 1.075 I_9 + 1.079 G_4 \\
 &\quad + .079 G_5 + 1.085 M_1 + .085 M_2 \\
 \text{all variables} &\geq 0
 \end{aligned}$$

*** OPTIMUM SOLUTION SUMMARY ***				
Title: Problem 26a-14 Final iteration No: 14 Objective value (max) = 46.8500				
Variable	Value	Obj Coeff	Obj Val	Contrib
x1 11	0.0000	0.0000	0.0000	
x2 12	0.0000	0.0000	0.0000	
x3 13	0.0000	0.0000	0.0000	
x4 14	0.0000	0.0000	0.0000	
x5 15	0.0000	0.0000	0.0000	
x6 16	4.6331	0.0000	0.0000	
x7 17	9.6137	0.0000	0.0000	
x8 18	15.4678	0.0000	0.0000	
x9 19	24.6663	0.0000	0.0000	
x10 110	37.5201	1.0750	40.3341	
x11 61	0.0000	0.0000	0.0000	
x12 62	0.0000	0.0000	0.0000	
x13 63	2.9053	0.0000	0.0000	
x14 64	3.1395	0.0000	0.0000	
x15 65	3.9028	1.0790	4.2111	
x16 M1	1.9608	0.0000	0.0000	
x17 M2	2.1242	1.0850	2.3047	
Constraint	RHS	Slack(-)/Surplus(+)		
1 (=)	2.0000	0.0000		
2 (=)	2.0000	0.0000		
3 (=)	2.5000	0.0000		
4 (=)	2.5000	0.0000		
5 (=)	3.0000	0.0000		
6 (=)	3.5000	0.0000		
7 (=)	3.5000	0.0000		
8 (=)	4.0000	0.0000		
9 (=)	4.0000	0.0000		
10 (=)	5.0000	0.0000		

Year	Recommendation
1	Invest all in 9-yr bond
2	Invest all in 9-yr bond
3	Invest all in 6-yr bond
4	Invest all in 6-yr bond
5	Invest all in 6-yr bond
7	Invest all in insured savings
8	Invest all in insured savings
9	Invest all in insured savings
10	Invest all in insured savings

x_{iA} = amount invested in year i , plan A (\$1000)
5

x_{iB} = amount invested in year i , plan B (\$1000\$)

$$\text{Maximize } Z = 3x_{2B} + 1.7x_{3A}$$

subject to

$$\begin{aligned}
 x_{1A} + x_{1B} &\leq 100 \\
 -1.7x_{1A} + x_{2A} + x_{2B} &= 0 \\
 -3x_{1B} - 1.7x_{2A} + x_{3A} &= 0
 \end{aligned}$$

$$x_{iA}, x_{iB} \geq 0 \text{ for } i = 1, 2, 3$$

*** OPTIMUM SOLUTION SUMMARY ***

Title: Problem 2.6a-15
Final iteration No: 4
Objective value (max) = 510.0000
=> ALTERNATIVE solution detected at x2

Variable	Value	Obj Coeff	Obj Val	Contrib
x1 x1A	100.0000	0.0000	0.0000	
x2 x1B	0.0000	0.0000	0.0000	
x3 x2A	0.0000	0.0000	0.0000	
x4 x2B	170.0000	3.0000	510.0000	
x5 x3A	0.0000	1.7000	0.0000	

Constraint	RHS	Slack(-)/Surplus(+)
1 (=)	100.0000	0.0000-
2 (=)	0.0000	0.0000-
3 (=)	0.0000	0.0000-

Optimum solution: Invest \$100,000 in A in yr 1 and \$170,000 in B in yr 2.

Alternative optimum: Invest \$100,000 in B in yr 1 and \$300,000 in A in yr 3.

x_i = dollars allocated to choice i ,
 $i = 1, 2, 3, 4$

6

y = minimum return

$$\begin{aligned}
 \text{Maximize } Z &= \min \left\{ -3x_1 + 4x_2 - 7x_3 + 15x_4 \right. \\
 &\quad \left. 5x_1 - 3x_2 + 9x_3 + 4x_4 \right. \\
 \text{subject to} &\quad \left. 3x_1 - 9x_2 + 10x_3 - 8x_4 \right.
 \end{aligned}$$

$$x_1 + x_2 + x_3 + x_4 \leq 500$$

$$x_1, x_2, x_3, x_4 \geq 0$$

The problem can be converted to a linear program as

continued...

Set 2.3c

Maximize $Z = y$

subject to

$$-3x_1 + 4x_2 - 7x_3 + 15x_4 \geq y$$

$$5x_1 - 3x_2 + 9x_3 + 4x_4 \geq y$$

$$3x_1 - 9x_2 + 10x_3 - 8x_4 \geq y$$

$$x_1 + x_2 + x_3 + x_4 \leq 500$$

$$x_1, x_2, x_3, x_4 \geq 0$$

if unrestricted

*** OPTIMUM SOLUTION SUMMARY ***

Title:

Final iteration No: 5

Objective value (max) = 1175.0000

Variable	Value	Obj Coeff	Obj Val Contrib
x1	0.0000	0.0000	0.0000
x2	0.0000	0.0000	0.0000
x3	287.5000	0.0000	0.0000
x4	212.5000	0.0000	0.0000
x5 y	1175.0000	1.0000	1175.0000

Constraint	RHS	Slack(-)/Surplus(+)
1 (>)	0.0000	0.0000+
2 (>)	0.0000	2262.5000+
3 (>)	0.0000	0.0000+
4 (<)	500.0000	0.0000-

Allocate \$287.50 to choice 3
and \$212.50 to choice 4. Return =
\$1175.00

$$i = \begin{cases} 1, \text{ regular savings} \\ 2, \text{ 3-month CD} \\ 3, \text{ 6-month CD} \end{cases}$$

7

x_{it} = Deposit in plan i at start of month t

$$t = \begin{cases} 1, 2, \dots, 12 & \text{if } i=1 \\ 1, 2, \dots, 10 & \text{if } i=2 \\ 1, 2, \dots, 7 & \text{if } i=3 \end{cases}$$

y_i = initial amount on hand to
ensure a feasible solution

r_i = interest rate for plan $i = 1, 2, 3$

$$\bar{J}_i = \begin{cases} 12, & i=1 \\ 10, & i=2 \\ 7, & i=3 \end{cases}$$

continued...

$$P_i = \begin{cases} 1, & i=1 \\ 3, & i=2 \\ 6, & i=3 \end{cases} \quad d_t = \$\text{demand for period } t$$

$$\text{Maximize } Z = \sum_{t=1}^{12} \sum_{i=1}^3 r_i x_{i,t} - y_i$$

$$t - P_i > 0$$

s.t.

$$y_i - x_{1i} - x_{2i} - x_{3i} \geq d_{i,3}$$

$$1000 + \sum_{i=1}^3 (1+r_i) x_{i,t} - \sum_{i=1}^3 x_{it} \geq d_t, t=2, \dots, 12$$

$$t - P_i > 0 \quad t \leq J_i$$

$$x_{it}, y_i \geq 0$$

Solution: (see file ampl2.3c-7.txt)

$$y_i = \$1200, Z = -1136.29$$

$$\text{Interest amount} = 1200 - 1136.29 = \$63.71$$

Deposits:

t	x_{1t}	x_{2t}	x_{3t}
1	0	0	0
2	0	200	0
3	286.48	313.53	0
4	0	587.43	0
5	314.37	289.30	0
6	0	734.69	0
7	0	98.20	0
8	0	294.60	0
9	0	848.16	0
10	0	0	0
11	0	0	0
12	0	0	0

Set 2.3d

X_{W1} = # wrenches/wk using regular time
 X_{W2} = # wrenches/wk using overtime
 X_{W3} = # wrenches/wk using subcontracting
 X_{C1} = # chisels/wk using regular time
 X_{C2} = # chisels/wk using overtime
 X_{C3} = # chisels/wk using subcontracting

$$\text{Minimize } Z = 2X_{W1} + 2.8X_{W2} + 3X_{W3} + 2.1X_{C1}$$

$$\text{Subject to } + 3.2X_{C2} + 4.2X_{C3}$$

$$X_{W1} \leq 550, X_{W2} \leq 250$$

$$X_{C1} \leq 620, X_{C2} \leq 280$$

$$X_{C1} + X_{C2} + X_{C3} \geq 2$$

$$X_{W1} + X_{W2} + X_{W3}$$

or

$$2X_{W1} + 2X_{W2} + 2X_{W3} - X_{C1} - X_{C2} - X_{C3} \leq 0$$

$$X_{W1} + X_{W2} + X_{W3} \geq 1500$$

$$X_{C1} + X_{C2} + X_{C3} \geq 1200$$

$$\text{all variables } \geq 0$$

(a) Optimum from TORA:

$$X_{W1} = 550, X_{W2} = 250, X_{W3} = 700$$

$$X_{C1} = 620, X_{C2} = 280, X_{C3} = 2100$$

$$Z = \$14,918$$

(b) Increasing marginal cost ensures that regular time capacity is used before that of overtime, and that overtime capacity is used before that of subcontracting. If the marginal cost function is not monotonically increasing, additional constraints are needed to ensure that the capacity restriction is satisfied.

continued...

1
 x_j = number of units produced of product j , $j = 1, 2, 3, 4$

2

Profit per unit:

$$\text{Product 1} = 75 - 2 \times 10 - 3 \times 5 - 7 \times 4 = \$12$$

$$\text{Product 2} = 70 - 3 \times 10 - 2 \times 5 - 3 \times 4 = \$18$$

$$\text{Product 3} = 55 - 4 \times 10 - 1 \times 5 - 2 \times 4 = \$2$$

$$\text{Product 4} = 45 - 2 \times 10 - 2 \times 5 - 1 \times 4 = \$11$$

$$\text{Maximize } Z = 12x_1 + 18x_2 + 2x_3 + 11x_4$$

s.t.

$$2x_1 + 3x_2 + 4x_3 + 2x_4 \leq 500$$

$$3x_1 + 2x_2 + x_3 + 2x_4 \leq 380$$

$$7x_1 + 3x_2 + 2x_3 + x_4 \leq 450$$

TORA Solution:

$$x_1 = 0, x_2 = 133.33, x_3 = 0, x_4 = 50$$

$$Z = \$2950$$

3

x_j = number of units of model j

$$\text{Maximize } Z = 30x_1 + 20x_2 + 50x_3$$

Subject to

$$(1) \quad 2x_1 + 3x_2 + 5x_3 \leq 4000$$

$$(2) \quad 4x_1 + 2x_2 + 7x_3 \leq 6000$$

$$(3) \quad x_1 + .5x_2 + \frac{1}{3}x_3 \leq 1500$$

$$(4) \quad \frac{x_1}{3} = \frac{x_2}{2}, \text{ or } 2x_1 - 3x_2 = 0$$

$$(5) \quad \frac{x_2}{2} = \frac{x_3}{5}, \text{ or } 5x_2 - 2x_3 = 0$$

$$x_1 \geq 200, x_2 \geq 200, x_3 \geq 150$$

*** OPTIMUM SOLUTION SUMMARY ***

Title: Problem 2.6a-12
 Final iteration No: 4
 Objective value (max) = 41081.0820

Variable	Value	Obj Coeff	Obj Val Contrib
x1	324.3243	30.0000	9729.7305
x2	216.2162	20.0000	4324.3242
x3	540.5405	50.0000	27027.0273
Constraint	RHS	Slack(-)/Surplus(+)	
1 (<)	4000.0000	0.0000-	
2 (<)	6000.0000	486.4865-	
3 (<)	1500.0000	887.3875-	
4 (=)	0.0000	0.0000	
5 (=)	0.0000	0.0000	
LB-x1	200.0000	124.3243+	
LB-x2	200.0000	16.2162+	
LB-x3	150.0000	390.5405+	

Set 2.3d

x_{ij} = Nbr. Cartons in month i from supplier j

I_i = End inventory in period i , $I_0 = 0$

C_{ij} = Price per unit of x_{ij}

h = Holding cost/unit/month

C = Supplier capacity/month

d_i = Demand for month i

$i = 1, 2, 3, j = 1, 2$

$$\text{Minimize } Z = \sum_{i=1}^3 \sum_{j=1}^2 C_{ij} x_{ij} + \frac{h}{2} \left(\sum_{i=1}^3 \left(\sum_{j=1}^2 x_{ij} + I_{i-1} + I_i \right) \right)$$

s.t. $x_{ij} \leq C$, all i and j

$$\sum_{j=1}^2 x_{ij} + I_{i-1} - I_i = d_i, \text{ all } i$$

Optimum solution:

i	x_{i1}	x_{i2}	I
1	400	100	0
2	400	400	200
3	200	0	0

Total cost = \$167,450.

X_i = Production amount in quarter i

I_i = End inventory for quarter i

$$\text{Minimize } Z = 20X_1 + 22X_2 + 24X_3 + 26X_4 + 3.5(I_1 + I_2 + I_3)$$

s.t.

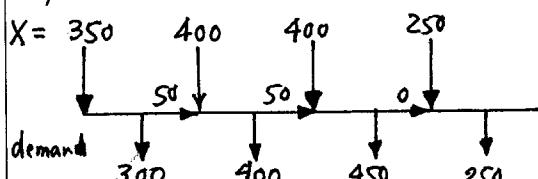
$$X_1 = 300 + I_1, \quad X_i \leq 400, i=1,2,3$$

$$I_1 + X_2 = 400 + I_2, \quad I_i \leq 100, i=1,2,3$$

$$I_2 + X_3 = 450 + I_3, \quad I_0 = I_4 = 0$$

$$I_3 + X_4 = 250$$

Optimum solution:



Total cost = \$32,250

4

x_{ij} = Qty of product i in month j ,
 $i=1,2, j=1,2,3$

I_{ij} = End inventory of product i in month j

$$\text{Minimize } Z = 30(x_{11} + x_{12} + x_{13}) + 28(x_{21} + x_{22} + x_{23}) + .9(I_{11} + I_{12} + I_{13}) + .75(I_{21} + I_{22} + I_{23})$$

s.t.

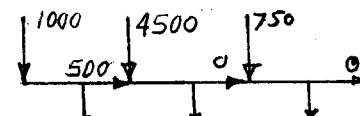
$$(X_{1j}/1.25) + x_{2j} \leq \begin{cases} 3000, & j=1 \\ 3500, & j=2 \\ 3000, & j=3 \end{cases}$$

$$I_{1,j-1} + X_{1j} - I_{1j} = \begin{cases} 500, & j=1 \\ 5000, & j=2 \\ 750, & j=3 \end{cases}$$

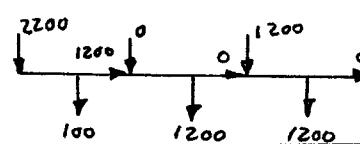
$$I_{2,j-1} + X_{2j} - I_{2j} = \begin{cases} 1000, & j=1 \\ 1200, & j=2 \\ 1200, & j=3 \end{cases}$$

Optimum solution: Cost = \$284,050

Product 1:



Product 2:



5

x_{ij} = Qty by operation i in month j ,
 $i=1,2, j=1,2,3$

$$\text{Minimize } Z = .2 \sum_{j=1}^3 I_{1j} + .4 \sum_{j=1}^3 I_{2j} + 10x_{11} + 12x_{12} + 11x_{21} + 15x_{22} + 18x_{23}$$

$$+ 6x_{11} \leq 800, .6x_{12} \leq 700, .6x_{13} \leq 550$$

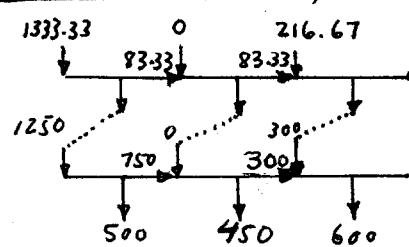
$$.8x_{21} \leq 1000, .8x_{22} \leq 850, .8x_{23} \leq 700$$

$$x_{1j} + I_{1,j-1} = x_{2j} + I_{1j} \quad \} j=1,2,3$$

$$x_{2j} + I_{2,j-1} = I_{2j} + d_j \quad \} j=1,2,3$$

$$I_{i0} = 0, i=1,2$$

Solution: Cost = \$39,720



6

7

I_{ij} = Ending inv. of op. i in month j

Set 2.3d

x_j = Units of product j , $j=1, 2$

8

y_i^- = Unused hours of machine i } $i=1, 2$

y_i^+ = Overtime hours of machine i }

$$\text{Maximize } Z = 110x_1 + 118x_2 - 100(y_1^+ + y_2^+)$$

s.t.

$$\frac{x_1}{5} + \frac{x_2}{5} + y_1^- - y_1^+ = 8$$

$$\frac{x_1}{8} + \frac{x_2}{4} + y_2^- - y_2^+ = 8$$

$$y_1^+ \leq 4, \quad y_2^+ \leq 4$$

$$x_1, x_2, y_1^-, y_1^+, y_2^-, y_2^+ \geq 0$$

Solution:

$$\text{Revenue} = \$6,232$$

$$x_1 = 56, \quad y_1^+ = 4 \text{ hrs}$$

$$x_2 = 4, \quad y_2^+ = 0$$

$$y_1^-, y_2^- = 0$$

Set 2.3e

x_s = tons of strawberry / day

x_g = tons of grapes / day

x_a = tons of apples / day

x_A = cans of drink A / day
 x_B = cans of drink B / day } Each can holds one lb
 x_C = cans of drink C / day }

x_{SA} = 1b of strawberry used in drink A / day

x_{SB} = 1b of strawberry used in drink B / day

x_{gA} = 1b of grapes used in drink A / day

x_{gB} = 1b of grapes used in drink B / day

x_{gC} = 1b of grapes used in drink C / day

x_{aB} = 1b of apples used in drink B / day

x_{aC} = 1b of apples used in drink C / day

$$\text{Maximize } Z = 1.1x_A + 1.25x_B + 1.2x_C - 200x_s$$

$$\text{s.t. } -100x_g - 90x_a$$

$$x_s \leq 200, x_g \leq 100, x_a \leq 150$$

$$x_{SA} + x_{SB} = 1500x_s$$

$$x_{gA} + x_{gB} + x_{gC} = 1200x_g$$

$$x_{aB} + x_{aC} = 1000x_a$$

$$x_A = x_{SA} + x_{gA}$$

$$x_B = x_{SB} + x_{gB} + x_{aB}$$

$$x_C = x_{gC} + x_{aC}$$

$$x_{SA} = x_{gA}$$

$$x_{SB} = x_{gB}, x_{gB} = .5x_{aB}$$

$$3x_{gC} = 2x_{aC}$$

$$\text{all variables} \geq 0$$

Optimum solution:

$$x_A = 90,000 \text{ cans}, x_B = 300,000 \text{ cans}, x_C = 0$$

<u>x_{ij}</u>	<u>j</u>		
<u>i</u>	<u>A</u>	<u>B</u>	<u>C</u>
S	45,000	75,000	0
g	45,000	75,000	0
a	0	150,000	0
	90,000	300,000	0

$$x_s = 80 \text{ tons}, x_g = 100 \text{ tons}, x_a = 150 \text{ tons}$$

$$Z = \$439,000/\text{day}$$

x_s = 1b of screws per package

x_b = 1b of bolts per package

x_n = 1b of nuts per package

x_w = 1b of washers per package

$$\text{Minimize } Z = 1.1x_s + 1.5x_b + \frac{70}{80}x_n + \frac{20}{30}x_w$$

$$\text{s.t. } Y = x_s + x_b + x_n + x_w$$

$$x_s \geq .1Y$$

$$x_b \geq .25Y, \frac{x_b}{50} \leq x_w, \frac{x_b}{10} \leq x_n$$

$$x_n \leq .15Y$$

$$x_w \leq .1Y$$

$$Y \geq 1$$

All variables are nonnegative

Optimum solution:

$$Y = 1, x_s = .5, x_b = .25, x_n = .15, x_w = .1$$

$$\text{Cost} = \$1.12$$

x_o = 1b of oats in cereals A, B, C

x_r (A, C) = 1b of raisins in cereals A, C

x_c (B, C) = 1b of coconuts in cereals B, C

x_a (A, B, C) = 1b of almond in cereals A, B, C

$$Y_o = x_{oA} + x_{oB} + x_{oC}$$

$$Y_r = x_{rA} + x_{rC}$$

$$Y_c = x_{cB} + x_{cC}$$

$$Y_a = x_{aA} + x_{aB} + x_{aC}$$

$$W_A = x_{oA} + x_{rA} + x_{aA}$$

$$W_B = x_{oB} + x_{cB} + x_{aB}$$

$$W_C = x_{oC} + x_{rC} + x_{cC} + x_{aC}$$

$$\text{Maximize } Z = \frac{1}{5} (2W_A + 2.5W_B + 3W_C)$$

$$- \frac{1}{2000} (100Y_o + 120Y_r + 110Y_c + 200Y_a)$$

$$\text{s.t. } W_A \leq 500 \times 5 = 2500$$

$$W_B \leq 600 \times 5 = 3000$$

$$W_C \leq 500 \times 5 = 4000$$

continued...

2

Set 2.3e

$$Y_0 \leq 5X_{2000} = 10,000$$

$$Y_r \leq 2X_{2000} = 4,000$$

$$Y_c \leq 1X_{2000} = 2,000$$

$$Y_a \leq 1X_{2000} = 2,000$$

$$X_{OA} = \frac{50}{5} X_{rA}, X_{OA} = \frac{50}{2} X_{aA}$$

$$X_{OB} = \frac{60}{2} X_{cB}, X_{OB} = \frac{60}{3} X_{aB}$$

$$X_{OC} = \frac{60}{3} X_{rC}, X_{OC} = \frac{60}{4} X_{cC}, X_{OC} = \frac{60}{2} X_{aC}$$

all variables are nonnegative.

$$\text{Optimum solution: } Z = \$5384.84/\text{day}$$

$$W_A = 2500 \text{ lb or } 500 \text{ boxes/day}$$

$$W_B = 3000 \text{ lb or } 600 \text{ boxes}$$

$$W_C = 5793.45 \text{ lb or } \approx 1158 \text{ boxes}$$

$$X_0 = 10,000 \text{ lb or } 5 \text{ tons/day}$$

$$X_r = 471.19 \text{ lb or } .236 \text{ ton}$$

$$X_c = 428.16 \text{ lb or } .214 \text{ ton}$$

$$X_a = 394.11 \text{ lb or } .197 \text{ ton}$$

$$\begin{aligned} X_{Ai} &= \text{bbl of gasoline A in fuel i} \\ X_{Bi} &= \text{bbl of gasoline B in fuel i} \\ X_{Ci} &= \text{bbl of gasoline C in fuel i} \\ X_{Di} &= \text{bbl of gasoline D in fuel i} \end{aligned} \quad i = 1, 2$$

$$Y_A = X_{A1} + X_{A2}$$

$$Y_B = X_{B1} + X_{B2}$$

$$Y_C = X_{C1} + X_{C2}$$

$$Y_D = X_{D1} + X_{D2}$$

$$F_1 = X_{A1} + X_{B1} + X_{C1} + X_{D1}$$

$$F_2 = X_{A2} + X_{B2} + X_{C2} + X_{D2}$$

$$\text{Maximize } Z = 200F_1 + 250F_2$$

$$- (120Y_A + 90Y_B + 100Y_C + 150Y_D)$$

s.t.

$$X_{A1} = X_{B1}, X_{A1} = .5X_{C1}, X_{A1} = .25X_{D1}$$

$$X_{A2} = X_{B2}, X_{A2} = 2X_{C2}, X_{A2} = \frac{2}{3}X_{D2}$$

$$Y_A \leq 1000, Y_B \leq 1200, Y_C \leq 900, Y_D \leq 1500$$

$$F_1 \geq 200, F_2 \geq 400$$

$$\text{Optimum solution: } Z = \$495,416.67$$

$$Y_A = 958.33 \text{ bbl/day}$$

$$Y_B = 958.33 \text{ bbl/day}$$

$$Y_C = 516.67 \text{ bbl/day}$$

$$Y_D = 1500 \text{ bbl/day}$$

$$F_1 = 200 \text{ bbl/day}$$

$$F_2 = 3733.33 \text{ bbl/day}$$

5

$$A = \text{bbl of crude A/day}$$

$$B = \text{bbl of crude B/day}$$

$$R = \text{bbl of regular gasoline/day}$$

$$P = \text{bbl of premium gasoline/day}$$

$$J = \text{bbl of jet gasoline/day}$$

$$\begin{aligned} \text{Maximize } Z &= 50(R - R^+) + 70(P - P^+) \\ &+ 120(J - J^+) - (10R^- + 15P^- + 20J^-) \\ &- (2R^+ + 3P^+ + 4J^+) - (30A + 40B) \end{aligned}$$

s.t.

$$A \leq 2500, B \leq 3000$$

$$R = .2A + .25B, R + R^- - R^+ = 500$$

$$P = .1A + .3B, P + P^- - P^+ = 700$$

$$J = .25A + .1B, J + J^- - J^+ = 400$$

All variables ≥ 0

Optimum solution:

$$Z = \$21,852.94$$

$$A = 1176.47 \text{ bbl/day}$$

$$B = 1058.82 \text{ bbl/day}$$

$$R = 500 \text{ bbl/day}$$

$$P = 435.29 \text{ bbl/day}$$

$$J = 400 \text{ bbl/day}$$

continued...

Set 2.3e

$$NR = 661/\text{day of naphtha used in regular}$$

$$NP = 661/\text{day of naphtha used in premium}$$

$$NJ = 661/\text{day of naphtha used in jet}$$

$$LR = 661/\text{day of light used in regular}$$

$$LP = 661/\text{day of light used in premium}$$

$$LJ = 661/\text{day of light used in jet}$$

Using the other notation in Problem 5,

$$\text{Maximize } Z = 50(R - R^+) + 70(P - P^+) + 12(J - J^+)$$

$$-(10R^+ + 15P^+ + 20J^+) - (2R^+ + 3P^+ + 4J^+)$$

$$-(30A + 40B)$$

s.t.

$$A \leq 2500, B \leq 3000$$

$$R + R^- - R^+ = 500$$

$$P + P^- - P^+ = 700$$

$$J + J^- - J^+ = 400$$

$$.35A + .45B = NR + NP + NJ$$

$$.6A + .5B = LR + LP + LJ$$

$$R = NR + LR$$

$$P = NP + LP$$

$$J = NJ + LJ$$

all variables are nonnegative

Optimum solution: $Z = \$71,473.68$

$$A = 1684.21, B = 0$$

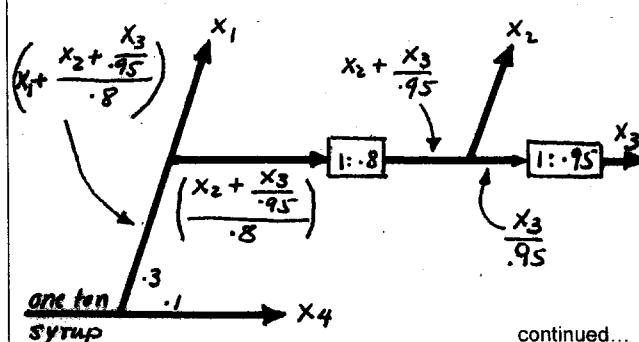
$$R = 500, P = 700, J = 400$$

$$x_1 = \text{tons of brown sugar per week}$$

$$x_2 = \text{tons of white sugar per week}$$

$$x_3 = \text{tons of powdered sugar per week}$$

$$x_4 = \text{tons of molasses per week}$$



6

$$\text{Maximize } Z = 150x_1 + 200x_2 + 280x_3 + 35x_4$$

s.t.

$$x_4 \leq 4000 \times .1$$

$$\text{or } x_4 \leq 400$$

$$x_1 + \frac{(x_2 + \frac{x_3}{.95})}{.8} \leq .3 \times 4000$$

$$\text{or } .76x_1 + .95x_2 + x_3 \leq 912$$

$$x_1 \geq 25, x_2 \geq 25$$

$$x_3 \geq 25, x_4 \geq 0$$

Optimum solution from TORA:

$$x_1 = 25 \text{ tons per week}$$

$$x_2 = 25 \text{ tons per week}$$

$$x_3 = 869.25 \text{ tons per week}$$

$$x_4 = 400 \text{ tons per week}$$

$$Z = \$222,677.50$$

8

$$A = 661/\text{hr of stock A}$$

$$B = 661/\text{hr of stock B}$$

$$Y_{Ai} = 661/\text{hr of A used in gasoline i} \quad i=1,2$$

$$Y_{Bi} = 661/\text{hr of B used in gasoline i} \quad i=1,2$$

$$\text{Maximize } Z = 7(Y_{A1} + Y_{B1}) + 10(Y_{A2} + Y_{B2})$$

s.t.

$$A = Y_{A1} + Y_{A2}, A \leq 450$$

$$B = Y_{B1} + Y_{B2}, B \leq 700$$

$$98Y_{A1} + 89Y_{B1} \geq 91(Y_{A1} + Y_{B1})$$

$$98Y_{A2} + 89Y_{B2} \geq 93(Y_{A2} + Y_{B2})$$

$$10Y_{A1} + 8Y_{B1} \leq 12(Y_{A1} + Y_{B1})$$

$$10Y_{A2} + 8Y_{B2} \leq 12(Y_{A2} + Y_{B2})$$

all variables are nonnegative

Optimum solution:

$$Z = \$10,675$$

$$A = 450 \text{ bbl/hr}$$

$$B = 700 \text{ bbl/hr}$$

$$\begin{aligned} \text{Gasoline 1 production} &= Y_{A1} + Y_{B1} \\ &= 61.11 + 213.89 = 275 \text{ bbl/hr} \end{aligned}$$

$$\begin{aligned} \text{Gasoline 2 production} &= Y_{A2} + Y_{B2} \\ &= 388.89 + 486.11 = 875 \text{ bbl/hr} \end{aligned}$$

Set 2.3e

S = tons of steel scrap / day
 A = tons of alum. scrap / day
 C = tons of Cast iron scrap / day
 A_b = tons of alum. briquettes / day
 S_b = tons silicon briquettes / day
 a = tons of alum. / day
 g = tons of graphite / day
 s = tons of silicon / day

aI = tons of alum. in ingot I / day
 aII = tons of alum. in ingot II / day
 gI = tons of graphite in ingot I / day
 gII = tons of graphite in ingot II / day
 SI = tons of silicon in ingot I / day
 SII = tons of silicon in ingot II / day
 I_1 = tons of ingot I / day
 I_2 = tons of ingot II / day.

Minimize $Z = 100S + 150A + 75C + 900A_b + 380S_b$
 s.t. $S \leq 1000$, $A \leq 500$, $C \leq 2500$

$$\begin{aligned} a &= .1S + .95A + A_b \\ g &= .05S + .01A + .15C \\ S &= .04S + .02A + .08C + S_b \end{aligned}$$

$$\begin{aligned} I_1 &= aI + gI + SI \\ I_2 &= aII + gII + SII \\ aI + gII &\leq 18, SI + SII \leq 8, gI + gII \leq 9 \\ .08I_1 &\leq aI \leq .108I_1 \\ .015I_1 &\leq gI \leq .03I_1 \\ .025I_1 &\leq SI < \infty \\ .062I_2 &\leq aII \leq .089I_2 \\ .041I_2 &\leq gII \leq \infty \\ .028I_2 &\leq SII \leq .041I_2 \\ I_1 &\geq 130, I_2 \geq 250 \end{aligned}$$

Optimum solution:

$$Z = \$117,435.65$$

$$S = 0, A = 38.2, C = 1489.41$$

$$A_b = S_b = 0$$

$$I_1 = 130, I_2 = 250$$

$$a = 36.29, g = 223.79, s = 119.92$$

9

10

x_{ij} = tons of ore i allocated to alloy k
 w_k = tons of alloy k produced

$$\begin{aligned} \text{Maximize } Z &= 200w_A + 300w_B \\ &- 30(x_{1A} + x_{1B}) \\ &- 40(x_{2A} + x_{2B}) \\ &- 50(x_{3A} + x_{3B}) \end{aligned}$$

Subject to

Specification constraints:

$$\begin{aligned} .2x_{1A} + .1x_{2A} + .05x_{3A} &\leq .8w_A \quad (1) \\ .1x_{1A} + .2x_{2A} + .05x_{3A} &\leq .3w_A \quad (2) \\ .3x_{1A} + .3x_{2A} + .2x_{3A} &\geq .5w_A \quad (3) \\ .1x_{1B} + .2x_{2B} + .05x_{3B} &\geq .4w_B \quad (4) \\ .1x_{1B} + .2x_{2B} + .05x_{3B} &\leq .6w_B \quad (5) \\ .3x_{1B} + .3x_{2B} + .7x_{3B} &\geq .3w_B \quad (6) \\ .3x_{1B} + .3x_{2B} + .2x_{3B} &\leq .7w_B \quad (7) \end{aligned}$$

Other constraints:

$$\begin{aligned} x_{1A} + x_{1B} &\leq 1000 \\ x_{2A} + x_{2B} &\leq 2000 \\ x_{3A} + x_{3B} &\leq 3000 \end{aligned}$$

*** OPTIMUM SOLUTION SUMMARY ***

Title: Problem 26a-17
 Final iteration No: 12
 Objective value (max) = 400000.0000

Variable	Value	Obj Coeff	Obj Val Contrib
x1 wA	1799.9999	200.0000	359999.9688
x2 wB	1000.0001	300.0000	300000.0312
x3 x1A	1000.0000	-30.0000	-30000.0000
x4 x1B	0.0000	-30.0000	-0.0000
x5 x2A	0.0000	-40.0000	-0.0000
x6 x2B	2000.0001	-40.0000	-80000.0078
x7 x3A	3000.0000	-50.0000	-150000.0000
x8 x3B	0.0000	-50.0000	0.0000

Constraint	RHS	Slack(<)/Surplus(+)
1 (<)	0.0000	1090.0000-
2 (<)	0.0000	290.0000-
3 (>)	0.0000	0.0000+
4 (>)	0.0000	0.0000+
5 (<)	0.0000	200.0000-
6 (>)	0.0000	300.0002+
7 (<)	0.0000	100.0000-
8 (<)	1000.0000	0.0000-
9 (<)	2000.0000	0.0000-
10 (<)	3000.0000	0.0000-

Solution:

Produce 1800 tons of alloy A
 and 1000 tons of alloy B.

Set 2.3f

6

Let x_i = Nbr. starting on day i and lasting for 7 days

y_{ij} = Nbr. starting shift on day i and *starting* their 2 days off on day j, $i \neq j$

Thus, of the x_1 workers who start on Monday, y_{12} will take T and W off, y_{13} will take W and Th off, and so on, as the following table shows.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
1	start on Mon	y_{12}	$y_{12}+y_{13}$	$y_{13}+y_{14}$	$y_{14}+y_{15}$	$y_{15}+y_{16}$	y_{16}
2	y_{27}	Tue	y_{23}	$y_{23}+y_{24}$	$y_{24}+y_{25}$	$y_{25}+y_{26}$	$y_{26}+y_{27}$
3	$y_{31}+y_{37}$	y_{31}	Wed	y_{34}	$y_{34}+y_{35}$	$y_{35}+y_{36}$	$y_{36}+y_{37}$
4	$y_{41}+y_{47}$	$y_{41}+y_{42}$	y_{42}	Th	y_{45}	$y_{45}+y_{46}$	$y_{46}+y_{47}$
5	$y_{51}+y_{57}$	$y_{51}+y_{52}$	$y_{52}+y_{53}$	y_{53}	Fri	y_{56}	$y_{56}+y_{57}$
6	$y_{61}+y_{67}$	$y_{61}+y_{62}$	$y_{62}+y_{63}$	$y_{63}+y_{64}$	y_{64}	Sat	y_{67}
7	y_{71}	$y_{71}+y_{72}$	$y_{72}+y_{73}$	$y_{73}+y_{74}$	$y_{74}+y_{75}$	y_{75}	Su

Minimize $Z = x_1+x_2+x_3+x_4+x_5+x_6+x_7$

Each employee has 2 days off: $x_i = \sum\{j \text{ in } 1..7, j \neq i\} y_{ij}$

Mon (1) constraint: $s - (y_{27} + y_{31} + y_{37} + y_{41} + y_{47} + y_{51} + y_{57} + y_{61} + y_{67} + y_{71}) \geq 12$

Tue (2) constraint: $s - (y_{12} + y_{31} + y_{41} + y_{42} + y_{51} + y_{52} + y_{61} + y_{62} + y_{71} + y_{72}) \geq 18$

Wed (3) constraint: $s - (y_{12} + y_{13} + y_{23} + y_{42} + y_{52} + y_{53} + y_{62} + y_{63} + y_{72} + y_{73}) \geq 20$

Th (4) constraint: $s - (y_{13} + y_{14} + y_{23} + y_{24} + y_{24} + y_{53} + y_{63} + y_{64} + y_{73} + y_{74}) \geq 28$

Fri (5) constraint: $s - (y_{14} + y_{15} + y_{24} + y_{25} + y_{34} + y_{35} + y_{45} + y_{64} + y_{74} + y_{75}) \geq 32$

Sat(6) constraint: $s - (y_{15} + y_{16} + y_{25} + y_{26} + y_{35} + y_{36} + y_{45} + y_{46} + y_{56} + y_{75}) \geq 40$

Sun(7) constraint: $s - (y_{16} + y_{26} + y_{27} + y_{36} + y_{37} + y_{46} + y_{47} + y_{56} + y_{57} + y_{67}) \geq 40$

continued

Set 2.3f

Solution: 42 employees

Starting		Nbr off						
On	Nbr	M	Tu	Wed	Th	Fri	Sat	Sun
M	16		16	16				
Tu	8				8	8		
Wed	8	8	8					
Th	0							
Fri	6			6	6			
Sat	2	2					2	
Sun	2				2	2		
Nbr off		10	24	22	14	10	2	2
Nbr at work		32	18	20	28	32	40	40
Surplus above minimum		20	0	0	0	0	0	0

Set 2.3g

Setting		Number produced	Surplus rolls	1
1	3			
5'	0	200	50	
7'	1	200	0	
9'	1	300	0	
Loss/ft ²	4	1		
No. rolls	200	100		

Trim loss area =

$$L(200 \times 4 + 100 \times 1 + 50 \times 5) = 1150 \text{ ft}^2$$

(b) 15' standard roll:

Setting				
	1	2	3	4
5'	3	1	1	0
7'	0	1	0	2
9'	0	0	1	0
trim loss				
perf ft	0	3	1	1

$$(c) x_1 + x_2 + 2x_5 \geq 120$$

New solution calls for decreasing the number of standard 20'-rolls by 30

$$(d) x_1 + x_3 + 2x_6 \geq 240$$

New solution calls for increasing the number of standard 20'-rolls by 50

x_i = Space (in^2) allocated to cereal i

$$\text{Maximize } Z = 1.1x_1 + 1.3x_2 + 1.08x_3 + 1.25x_4 + 1.2x_5$$

s.t.

$$16x_1 + 24x_2 + 18x_3 + 22x_4 + 20x_5 \leq 5000$$

$$x_1 \leq 100, x_2 \leq 85, x_3 \leq 140, x_4 \leq 80, x_5 \leq 90$$

$$x_i \geq 0 \text{ for all } i = 1, 2, \dots, 5$$

Solution:

$$Z = \$314/\text{day}$$

$$x_1 = 100, x_3 = 140, x_5 = 44$$

$$x_2 = x_4 = 0$$

2

x_i = Nbr. of ads for issue i , $i = 1, 2, 3, 4$

3

$$\text{Minimize } Z = S_1^- + S_2^- + S_3^- + S_4^-$$

s.t.

$$(-30,000 + 60,000 + 30,000)x_1 + S_1^- - S_1^+ = .51 \times 400,000$$

$$(80,000 + 30,000 - 45,000)x_2 + S_2^- - S_2^+ = .51 \times 400,000$$

$$(40,000 + 10,000)x_3 + S_3^- - S_3^+ = .51 \times 400,000$$

$$(90,000 - 25,000)x_4 + S_4^- - S_4^+ = .51 \times 400,000$$

$$1500(x_1 + x_2 + x_3 + x_4) \leq 100,000$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution:

$$x_1 = 3.4, x_2 = 3.14, x_3 = 4.08, x_4 = 3.14$$

4

x_{ij} = Units of part j produced by department i , $i = 1, 2, 3, j = 1, 2$

$$\text{Maximize } Z = \min \{x_{11} + x_{21}, x_{12} + x_{22}, x_{13} + x_{23}\}$$

or

$$\text{Maximize } Z = y$$

s.t.

$$y \leq x_{11} + x_{21}$$

$$y \leq x_{12} + x_{22}$$

$$y \leq x_{13} + x_{23}$$

$$\frac{x_{11}}{8} + \frac{x_{12}}{5} + \frac{x_{13}}{10} \leq 100$$

$$\frac{x_{21}}{6} + \frac{x_{22}}{12} + \frac{x_{23}}{4} \leq 80$$

$$\text{all } x_{ij} \geq 0$$

Solution:

$$\text{Nbr. of assembly units} = y = 556.2 \approx 557$$

$$x_{11} = 354.78, x_{21} = 0$$

$$x_{12} = 556.52, x_{22} = 201.74$$

$$x_{13} = 556.52, x_{23} = 0$$

5

x_i = tons of coal i , $i = 1, 2, 3$

$$\text{Minimize } Z = 30x_1 + 35x_2 + 33x_3$$

s.t.

$$2500x_1 + 1500x_2 + 1600x_3 \leq 2000(x_1 + x_2 + x_3)$$

$$x_1 \leq 30, x_2 \leq 30, x_3 \leq 30$$

$$x_1 + x_2 + x_3 \geq 50$$

Solution: $Z = \$1361.11$

$$x_1 = 22.22 \text{ tons}, x_2 = 0, x_3 = 27.78 \text{ tons.}$$

Set 2.3g

t_i = Green time in secs for highway i ,
 $i = 1, 2, 3$

$$\text{Maximize } Z = 3\left(\frac{500}{3600}\right)t_1 + 4\left(\frac{600}{3600}\right)t_2 + 5\left(\frac{400}{3600}\right)t_3$$

s.t.

$$\left(\frac{500}{3600}\right)t_1 + \left(\frac{600}{3600}\right)t_2 + \left(\frac{400}{3600}\right)t_3 \leq \frac{510}{3600} (2.2 \times 60 - 3 \times 10)$$

$$t_1 + t_2 + t_3 + 3 \times 10 \leq 2.2 \times 60, t_1 \geq 25, t_2 \geq 25, t_3 \geq 25$$

Solution: $Z = \$58.04/\text{hr}$

$$t_1 = 25, t_2 = 43.6, t_3 = 33.4 \text{ sec}$$

6

Cost (\$) per cubic yd:

	(5) A2	(6) A4
(1) A1	$.2 + 2x \cdot .15 = .50$	$.20 + 7x \cdot .15 = 1.25$
(2) A3	$.20 + 2x \cdot .15 = .50$	$.20 + 3x \cdot .15 = .65$
(3) P1	$1.70 + 3x \cdot .15 = 2.15$	$1.70 + 8x \cdot .15 = 2.90$
(4) P3	$2.10 + 7x \cdot .15 = 3.15$	$2.10 + 2x \cdot .15 = 2.40$

Using the code $A1 \equiv 1, A3 \equiv 2, P1 \equiv 3, P3 \equiv 4, A2 \equiv 5, A4 \equiv 6$, let

$x_{ij} = 10^3 \text{ Yd}^3$ from source i to destination j
 $i = 1, 2, 3, 4, j = 5, 6$

$$\text{Minimize } Z = 1000(.5x_{15} + 1.25x_{16} + .5x_{25} + .65x_{26} + 2.15x_{35} + 2.9x_{36} + 3.15x_{45} + 2.4x_{46})$$

s.t.

$$x_{15} + x_{16} \leq 1760 \quad x_{35} + x_{36} \leq 20,000$$

$$x_{25} + x_{26} \leq 1760 \quad x_{45} + x_{46} \leq 15,000$$

$$x_{15} + x_{25} + x_{35} + x_{45} \geq 3520$$

$$x_{16} + x_{26} + x_{36} + x_{46} \geq 3520$$

Solution:

$$A1 \rightarrow A2: x_{15} = 1760 \text{ (1000 Cu Yd)}$$

$$A1 \rightarrow A4: x_{16} = 0$$

$$A3 \rightarrow A2: x_{25} = 0$$

$$A3 \rightarrow A4: x_{26} = 1760$$

$$P1 \rightarrow A2: x_{35} = 1760$$

$$P1 \rightarrow A4: x_{36} = 0$$

$$P2 \rightarrow A2: x_{45} = 0$$

$$P2 \rightarrow A4: x_{46} = 1760$$

$$\text{Cost} = \$10,032,000$$

8



$$A1 = 2 \times 1760 \times 10 \times 50 = 1760 \text{ (thousand) Yd}^3$$

$$A2 = 3520, A3 = 1760, A4 = 3520$$

Distances (center to center) in miles:

	A2	A4
A1	2	7
A3	2	3
P1	3	8
P2	7	2

continued...

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x_{ij} = Blue regulars on front i in defense line $j, i =$

y_{ij} = Blue reserves on front i in defense line j .

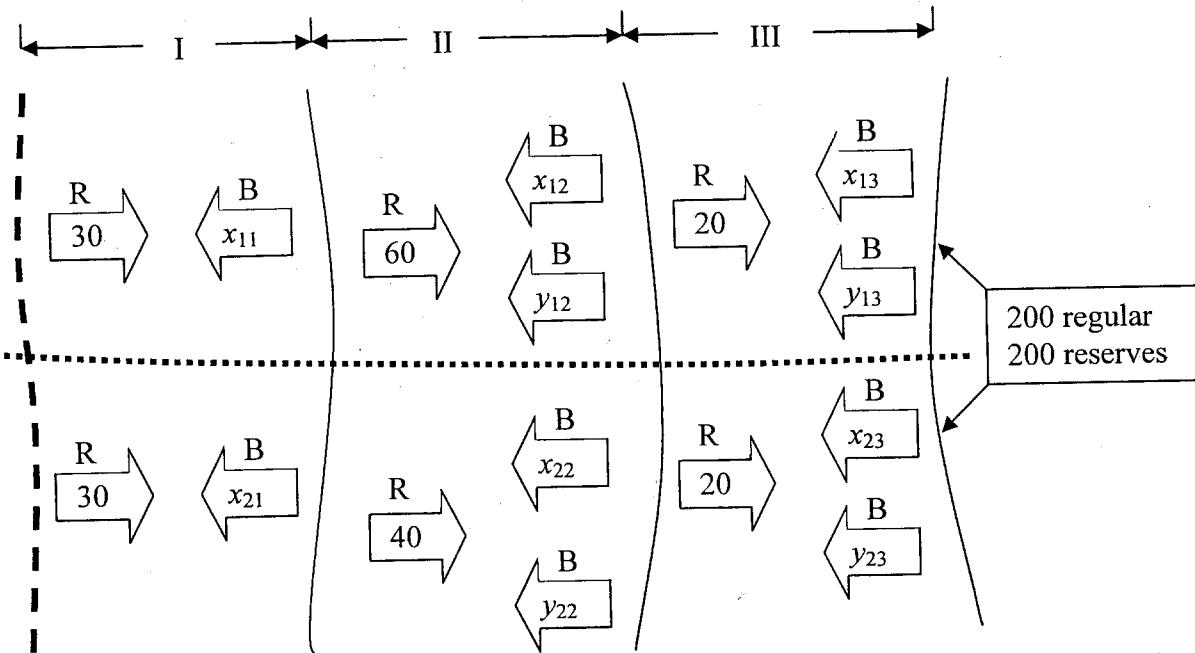
t_{ij} = Delay days on front i in defense line j :

$$\text{Maximize } Z = \min \{t_{11} + t_{12} + t_{13}, t_{21} + t_{22} + t_{23}\}$$

or

continued...

Set 2.3g



Maximize $Z = T$

s.t.

$$T \leq t_{11} + t_{12} + t_{13}$$

$$T \leq t_{21} + t_{22} + t_{23}$$

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} \leq 200$$

$$y_{12} + y_{13} + y_{22} + y_{23} \leq 200$$

$$t_{11} = .5 + 8.8 \frac{x_{11}}{30}$$

$$t_{12} = .75 + 7.9 \frac{x_{12} + y_{12}}{60}$$

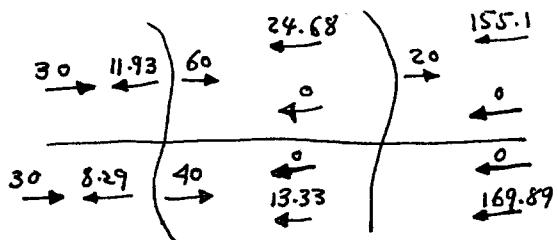
$$t_{13} = .55 + 10.2 \frac{x_{13} + y_{13}}{20}$$

$$t_{21} = 1.1 + 10.5 \frac{x_{21}}{30}$$

$$t_{22} = 1.3 + 8.1 \frac{x_{22} + y_{22}}{40}$$

$$t_{23} = 1.5 + 9.2 \frac{x_{23} + y_{23}}{20}$$

Solution: Battle duration = 87.65 days



continued...

x_i = Efficiency of plant i

10

$$\text{Minimize } Z = .2(500)x_1 + .25(3000)x_2 + .15(6000)x_3 + .18(1000)x_4$$

s.t.

$$500(1-x_1) \leq .00085 \times 215,000$$

$$.94(500)(1-x_1) + 3000(1-x_2) \leq .0009 \times 220,000$$

$$.94^2(500)(1-x_1) + .94(3000)(1-x_2) + 6000(1-x_3) \leq .0008 \times 200,000$$

$$.94^3(500)(1-x_1) + .94^2(3000)(1-x_2) + .94(6000)(1-x_3) + 1000(1-x_4) \leq .0008 \times 210,000$$

$$0 \leq x_1 \leq .99$$

$$0 \leq x_2 \leq .99$$

$$0 \leq x_3 \leq .99$$

$$0 \leq x_4 \leq .99$$

Solution

Cost per hour = \$1891.41

Plant 1 efficiency = .99

Plant 2 efficiency = .9661

Plant 3 efficiency = .99

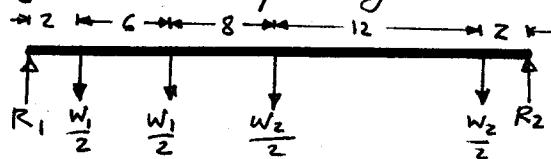
Plant 4 efficiency = .9824

Set 2.3g

w_i = Capacity of yoke i (Kips)

R_1 = Reaction in Kips at left end

R_2 = Reaction in Kips at right end



$$\text{Maximize } Z = w_1 + w_2$$

s.t.

$$R_1 + R_2 = w_1 + w_2$$

$$2\left(\frac{w_1}{2}\right) + 8\left(\frac{w_1}{2}\right) + 16\left(\frac{w_2}{2}\right) + 28\left(\frac{w_2}{2}\right) \\ = 30R_2$$

$$R_1 \leq 25, \quad R_2 \leq 25$$

$$\frac{w_1}{2} \leq 20, \quad \frac{w_2}{2} \leq 20$$

Solution:

$$w_1 = 20.59 \text{ Kips}$$

$$w_2 = 29.41 \text{ Kips}$$

x_{ij} = Nbr. of aircraft of type i

allocated to route j

($i = 1, 2, 3, 4$, $j = 1, 2, 3, 4$)

s_j = Nbr. of passengers not served on route j , $j = 1, 2, 3, 4$

$$\begin{aligned} \text{Minimize } Z &= 1000(3x_{11}) + 1100(2x_{12}) \\ &+ 1200(2x_{13}) + 1500(x_{14}) \\ &+ 800(4x_{21}) + 900(3x_{22}) \\ &+ 1000(3x_{23}) + 1000(2x_{24}) \\ &+ 600(5x_{31}) + 800(5x_{32}) \\ &+ 800(4x_{33}) + 900(2x_{34}) \\ &+ 40S_1 + 50S_2 + 45S_3 + 70S_4 \end{aligned}$$

Subject to

$$\sum_{j=1}^4 x_{1j} \leq 5, \quad \sum_{j=1}^4 x_{2j} \leq 8, \quad \sum_{j=1}^4 x_{3j} \leq 10$$

$$50(3x_{11}) + 30(4x_{21}) + 20(5x_{31}) + S_1 = 1000$$

$$50(2x_{12}) + 30(3x_{22}) + 20(5x_{32}) + S_2 = 2000$$

$$50(2x_{13}) + 30(3x_{23}) + 20(4x_{33}) + S_3 = 900$$

$$50(x_{14}) + 30(2x_{24}) + 20(2x_{34}) + S_4 = 1200$$

All x_{ij} and $s_j \geq 0$

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*** OPTIMUM SOLUTION SUMMARY ***

Title: Problem 26a-16
Final iteration No: 16
Objective value (min) = 221900.0000
=> ALTERNATIVE solution detected at x13

Variable	Value	Obj Coeff	Obj Val Contrib
x1 x11	5.0000	3000.0000	14999.9990
x2 x12	0.0000	2200.0000	0.0000
x3 x13	0.0000	2400.0000	0.0000
x4 x14	0.0000	1500.0000	0.0000
x5 x21	0.0000	3200.0000	0.0000
x6 x22	0.0000	2700.0000	0.0000
x7 x23	0.0000	3000.0000	0.0000
x8 x24	8.0000	2000.0000	15999.9990
x9 x31	2.5000	3000.0000	7500.0015
x10 x32	7.5000	4000.0000	29999.9980
x11 x33	0.0000	3200.0000	0.0000
x12 x34	0.0000	1800.0000	0.0000
x13 s1	0.0000	40.0000	0.0000
x14 s2	1250.0000	50.0000	62500.0000
x15 s3	899.9998	45.0000	40499.9922
x16 s4	720.0001	70.0000	50400.0078

Constraint	RHS	Slack(-)/Surplus(+)
1 (<)	5.0000	0.0000-
2 (<)	8.0000	0.0000-
3 (=)	10.0000	0.0000-
4 (=)	1000.0000	0.0000
5 (=)	2000.0000	0.0000
6 (=)	900.0000	0.0000
7 (=)	1200.0000	0.0000

Solution:

Aircraft Type	Route	Nbr. Aircraft
1	1	5
2	4	8
3	1	2.5
3	2	7.5

Fractional solution must be rounded.

Cost = \$ 221,900

continued...

CHAPTER 3

The Simplex Method and Sensitivity Analysis

Set 3.1a

$$(x_1, x_2) = (3, 1)$$

$$M1: S_1 = 24 - (6x_3 + 4x_1) = 2 \text{ tons/day}$$

$$M2: S_2 = 6 - (1x_3 + 2x_1) = 1 \text{ ton/day}$$

$$S_1 = x_1 + x_2 - 800$$

$$= 500 + 600 - 800 = 300/16$$

$$10x_1 - 3x_2 \geq -5 \equiv -10x_1 + 3x_2 \leq 5$$

$$\text{Thus, } -10x_1 + 3x_2 + S_1 = 5 \quad ①$$

$$\text{Also, } 10x_1 - 3x_2 \geq -5 \equiv 10x_1 - 3x_2 - S_2 = -5$$

$$\text{Thus, } -10x_1 + 3x_2 + S_2 = 5 \quad ②$$

① and ② are the same

x_{ij} = number of units of product **4**
 i manufactured on machine j

LP model

$$\text{Maximize } Z = 10(x_{11} + x_{12}) + 15(x_{21} + x_{22})$$

Subject to

$$|(x_{11} + x_{21}) - (x_{12} + x_{22})| \leq 5$$

$$x_{11} + x_{21} \leq 200$$

$$x_{12} + x_{22} \leq 250$$

$$x_{ij} \geq 0 \text{ for all } i \neq j$$

Equation form:

$$|(x_{11} + x_{21}) - (x_{12} + x_{22})| \leq 5$$

to

$$x_{11} + x_{21} - x_{12} - x_{22} \leq 5$$

$$x_{11} + x_{21} - x_{12} - x_{22} \geq -5$$

$$\text{Maximize } Z = 10x_{11} + 10x_{12} + 15x_{21} + 15x_{22}$$

Subject to

$$x_{11} + x_{21} - x_{12} - x_{22} + S_1 = 5$$

$$-x_{11} - x_{21} + x_{12} + x_{22} + S_2 = 5$$

$$x_{11} + x_{21} + S_3 = 200$$

$$x_{12} + x_{22} + S_4 = 250$$

$$x_{ij} \geq 0 \text{ for all } i \text{ and } j$$

$$S_i \geq 0 \text{ for all } i$$

continued...

$$y = \max \{|x_1 - x_2 + 3x_3|, |-x_1 + 3x_2 - x_3|\}$$

Hence

$$|x_1 - x_2 + 3x_3| \leq y$$

$$|-x_1 + 3x_2 - x_3| \leq y$$

LP model:

$$\text{minimize } Z = y$$

Subject to

$$x_1 - x_2 + 3x_3 \leq y$$

$$x_1 - x_2 + 3x_3 \geq -y$$

$$-x_1 + 3x_2 - x_3 \leq y$$

$$-x_1 + 3x_2 - x_3 \geq -y$$

$$x_1, x_2, x_3, y \geq 0$$

Equation form:

$$\text{minimize } Z = y$$

Subject to

$$-y + x_1 - x_2 + 3x_3 + S_1 = 0$$

$$-y - x_1 + x_2 - 3x_3 + S_2 = 0$$

$$-y - x_1 + 3x_2 - x_3 + S_3 = 0$$

$$-y + x_1 - 3x_2 + x_3 + S_4 = 0$$

$$x_1, x_2, x_3, y, S_1, S_2, S_3, S_4 \geq 0$$

$$\sum_{j=1}^n a_{ij} x_j = b_i \Leftrightarrow \begin{cases} \sum_{j=1}^n a_{ij} x_j \leq b_i & ① \\ \sum_{j=1}^n a_{ij} x_j \geq b_i & ② \end{cases} \quad 6$$

From ②, for $i = 1, 2, \dots, m$, we have

$$\sum_{j=1}^n a_{ij} x_j \geq b_i \Leftrightarrow \sum_{i=1}^m (\sum_{j=1}^n a_{ij} x_j) \geq \sum_{i=1}^m b_i$$

$$\Leftrightarrow \sum_{j=1}^n (\sum_{i=1}^m a_{ij}) x_j \geq \sum_{i=1}^m b_i$$

Thus, ① and ② are equivalent to

$$\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots, m$$

$$\sum_{j=1}^n (\sum_{i=1}^m a_{ij}) x_j \geq \sum_{i=1}^m b_i$$

Set 3.1b

$$X_1 = \text{Nbr. } \frac{1}{4} - \text{lb / day}$$

$$X_2 = \text{Nbr. cheeseburgers/day}$$

$$\text{Maximize } Z = .2X_1 + .15X_2 - .25X_3^+$$

s.t.

$$.25X_1 + .2X_2 + X_3^- - X_3^+ = 200$$

$$X_1 + X_2 \leq 900$$

$$\underline{\text{Solution: }} Z = \$173.35$$

$$X_1 = 900, X_2 = 0, X_3^+ = 25 \text{ lb}$$

$$(a) X_j = \# \text{ units of product } j \text{ per day}, j=1,2$$

$$X_3^+ = \text{unused minutes of machine time / day}$$

$$X_3^- = \text{machine overtime per day in minutes}$$

$$\text{Maximize } Z = 6X_1 + 7.5X_2 - .5X_3^-$$

Subject to

$$10X_1 + 12X_2 + X_3^+ - X_3^- = 2500$$

$$150 \leq X_1 \leq 200$$

$$X_2 \leq 45$$

$$X_1, X_2 \geq 0$$

$$X_3^+, X_3^- \geq 0$$

TORA optimum solution:

$$X_1 = 200 \text{ units/day}$$

$$X_2 = 45 \text{ units/day}$$

$$X_3^- = \text{overtime minutes} \\ = 40 \text{ minutes/day}$$

$$Z = \$1517.50$$

(b) Overtime at \$1.50/min yields $X_3^- = 0$, which means no overtime is needed

$$X_j = \# \text{ of units of products } 1, 2, \text{ and } 3$$

$$\text{Maximize } Z = 2X_1 + 5X_2 + 3X_3 - 15X_4^+ - 10X_5^+$$

Subject to

$$2X_1 + X_2 + 2X_3 + X_4^+ - X_4^- = 80$$

$$X_1 + X_2 + 2X_3 + X_5^+ - X_5^- = 65$$

$$\text{all variables } \geq 0$$

$$\underline{\text{Solution: }} Z = \$325$$

$$X_2 = 65 \text{ units}, X_4^- = 15$$

$$\text{All other variables} = 0$$

continued...

Formulation 1:

$$\text{Maximize } Z = -2X_1 + 3X_2^+ - 3X_2^- - 2X_3^+ + 2X_3^-$$

Subject to

$$4X_1 - X_2^+ + X_2^- - 5X_3^+ + 5X_3^- = 10$$

$$2X_1 + 3X_2^+ - 3X_2^- + 2X_3^+ - 2X_3^- = 12$$

all variables ≥ 0

Optimum solution:

$$X_1 = 0$$

$$X_2^+ = 6.15 \quad \} \Rightarrow X_2 = 6.15$$

$$X_2^- = 0$$

$$X_3^+ = 0 \quad \} \Rightarrow X_3 = -3.23$$

$$X_3^- = 3.23$$

$$Z = 24.92$$

Formulation 2:

$$\text{Maximize } Z = -2X_1 + 3X_2^+ - 2X_3^+ - w$$

Subject to

$$4X_1 - X_2^+ - 5X_3^+ + 6w = 10$$

$$2X_1 + 3X_2^+ + 2X_3^+ - 5w = 12$$

all variables ≥ 0

Optimum solution:

$$X_1 = 0$$

$$X_2^+ = 9.38 \quad \} \Rightarrow X_2 = 9.38 - 3.23 = 6.15$$

$$w = 3.23$$

$$X_3^+ = 0 \quad \} \Rightarrow X_3 = 0 - 3.23 = -3.23$$

$$w = 3.23$$

$$Z = 24.92$$

continued...

Set 3.2a

(a)

Equation form:

$$\text{Maximize } Z = 2x_1 + 3x_2$$

Subject to

$$x_1 + 3x_2 + x_3 = 6$$

$$3x_1 + 2x_2 + x_4 = 6$$

$$x_1, x_2, x_3, x_4 \geq 0$$

(b) Basic (x_1, x_2) (Point B):

$$x_1 + 3x_2 = 6$$

$$3x_1 + 2x_2 = 6$$

$$\text{Solution: } (x_1, x_2) = \left(\frac{6}{7}, \frac{12}{7}\right), Z = 6 \frac{6}{7}$$

Basic (x_1, x_3) (Point E):

$$x_1 + x_3 = 6$$

$$3x_1 = 6$$

$$\text{Solution: } (x_1, x_3) = (2, 4), Z = 4$$

Basic (x_1, x_4) (Point C):

$$x_1 = 6$$

$$3x_1 + x_4 = 6$$

$$\text{Solution: } (x_1, x_4) = (6, -12)$$

unique but infeasible

Basic (x_2, x_3) (Point A):

$$3x_2 + x_3 = 6$$

$$2x_2 = 6$$

$$\text{Solution: } (x_2, x_3) = (3, -3)$$

unique but infeasible

Basic (x_2, x_4) (Point D):

$$3x_2 = 6$$

$$2x_2 + x_4 = 6$$

$$\text{Solution: } (x_2, x_4) = (2, 2), Z = 6$$

Basic (x_3, x_4) (Point F):

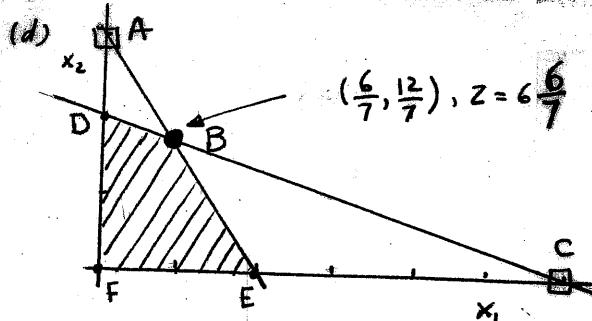
$$x_3 = 6$$

$$x_4 = 6$$

$$\text{Solution: } (x_3, x_4) = (6, 6), Z = 0$$

(c) Optimum solution occurs at B:

$$(x_1, x_2) = \left(\frac{6}{7}, \frac{12}{7}\right) \text{ with } Z = 6 \frac{6}{7}$$



(e) From the graph in (d), we have

$$A: x_2 = 3, x_3 = -3$$

$$C: x_1 = 6, x_4 = -12$$

2

(a) Maximize $Z = 2x_1 - 4x_2 + 5x_3 - 6x_4$

Subject to

$$x_1 + 4x_2 - 2x_3 + 8x_4 + x_5 = 2$$

$$-x_1 + 2x_2 + 3x_3 + 4x_4 + x_6 = 1$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Combination	Solution	Status	Z
x_1, x_2	0, 1/2	Feasible	-2
x_1, x_3	8, 3	Feasible	31
x_1, x_4	0, 1/4	Feasible	-3/2
x_1, x_5	-1, 3	Infeasible	-
x_1, x_6	2, 3	Feasible	4
x_2, x_3	1/2, 0	Feasible	-2
x_2, x_4	1/2, 0	Feasible	-2
x_2, x_5	1/2, 0	Feasible	-2
x_2, x_6	1/2, 0	Feasible	-2
x_3, x_4	0, 1/4	Feasible	-3/2
x_3, x_5	1/3, 8/3	Feasible	5/3
x_3, x_6	-1, 4	Infeasible	-
x_4, x_5	1/4, 0	Feasible	-3/2
x_4, x_6	1/4, 0	Feasible	-3/2
x_5, x_6	2, 1	Feasible	0

Optimum Solution:

$$x_1 = 8, x_2 = 0, x_3 = 3, x_4 = 0$$

$$Z = 31$$

continued...

Set 3.2a

(b) Minimize $Z = x_1 + 2x_2 - 3x_3 - 2x_4$
Subject to

$$x_1 + 2x_2 - 3x_3 + x_4 = 4$$

$$x_1 + 2x_2 + x_3 + 2x_4 = 9$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Combination Solution Status Z

x_1, x_2	infinity of solutions	-
x_1, x_3	4, 0	Feasible 4
x_1, x_4	4, 0	Feasible 4
x_2, x_3	2, 0	Feasible 4
x_2, x_4	2, 0	Feasible 4
x_3, x_4	$-\frac{4}{7}, \frac{16}{7}$	Infeasible -

Alternative optima:

x_1	x_2	x_3	x_4	Z
4	0	0	0	4
0	2	0	0	4

maximize $Z = x_1 + x_2$
Subject to

$$x_1 + 2x_2 + x_3 = 6$$

$$2x_1 + x_2 - x_4 = 16$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Combination Solution Status

x_1, x_2	$2\frac{6}{3}, -4\frac{1}{3}$	Infeasible
x_1, x_3	8, -2	Infeasible
x_1, x_4	6, -4	Infeasible
x_2, x_3	16, -26	Infeasible
x_2, x_4	3, -13	Infeasible
x_3, x_4	6, -16	Infeasible

3

Maximize $Z = 2x_1 + 3x_2 - 3x_2^+ - 5x_3^+$

4

Subject to

$$-6x_1 + 7x_2^- - 7x_2^+ - 9x_3^- - x_4 = 4$$

$$x_1 + x_2^- - x_2^+ + 4x_3^- = 10$$

$$x_1, x_2^-, x_2^+, x_3^-, x_4 \geq 0$$

(x_2^-, x_2^+) :

$$7x_2^- - 7x_2^+ = 4$$

$$x_2^- - x_2^+ = 10$$

Since $(7x_2^- - 7x_2^+)$ and $(x_2^- - x_2^+)$ are dependent, it is impossible for x_2^- and x_2^+ to be basic simultaneously. This means that at least x_2^- and x_2^+ must be nonbasic at zero level; thus making it impossible for x_2^- and x_2^+ to assume positive values simultaneously in any basic solution.

maximize $Z = x_1 + 3x_2$

5

Subject to

$$x_1 + x_2 + x_3 = 2$$

$$-x_1 + x_2 + x_4 = 4$$

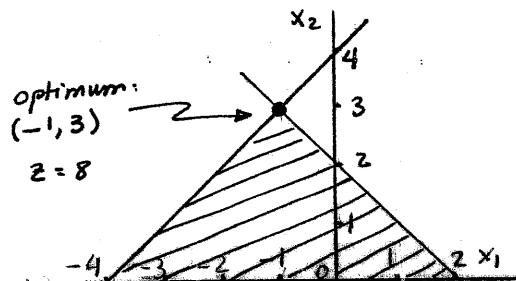
x_i unrestricted

$$x_2, x_3 \geq 0$$

Combination	Solution	Status	Z
x_1, x_2	-1, 3	Feasible	8
x_1, x_3	-4, 6	Feasible	-4
x_1, x_4	2, 6	Feasible	2
x_2, x_3	4, -2	Infeasible	-
x_2, x_4	2, 2	Feasible	6
x_3, x_4	2, 4	Feasible	0

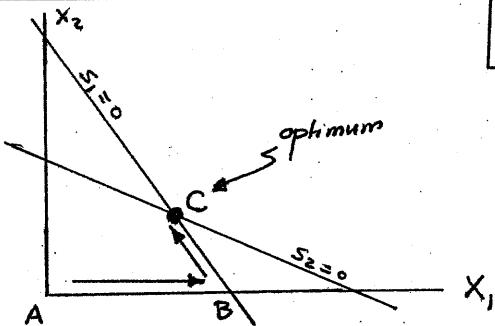
Optimum: $x_1 = -1, x_2 = 3, Z = 8$

(c)

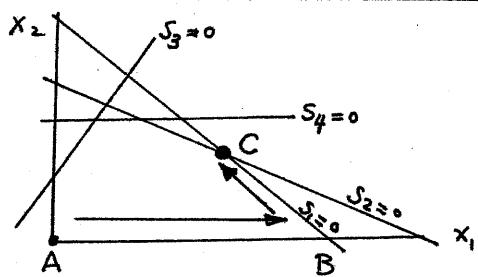


continued...

Set 3.3a



Extreme point	Basic	Nonbasic
A	S_1, S_2	X_1, X_2
B	X_1, S_2	X_2, S_1
C	X_1, X_2	S_1, S_2



Extreme point	Basic	Nonbasic
A	S_1, S_2, S_3, S_4	X_1, X_2
B	X_1, S_2, S_3, S_4	S_1, X_2
C	X_1, X_2, S_3, S_4	S_1, S_2

(a) (A, B) adjacent, hence can be on a simplex path. Remaining pairs cannot be on a simplex path because they are not adjacent.

- (b) (i) Yes, because connects adjacent extreme points
(ii) No, because C and I are not adjacent.
(iii) No, because the path returns to a previous extreme point.

1

4

Extreme Point	Basic	Nonbasic
A	S_1, S_2, S_3, S_4	X_1, X_2, X_3
B	S_1, X_1, S_3, S_4	S_2, X_2, S_3
C	X_1, S_2, S_3, S_4	S_1, X_1, X_3
D	S_1, S_2, X_3, S_4	X_1, X_2, S_3
E	X_1, X_2, S_3, S_4	S_1, S_2, X_3
F	X_1, S_2, X_3, S_4	X_1, S_1, S_3
G	S_1, X_1, X_3, S_4	S_2, X_2, S_3
H	S_1, X_1, X_2, X_3	S_2, S_3, S_4
I	X_1, X_2, X_3, S_3	S_1, S_2, S_4
J	X_1, S_2, X_2, X_3	S_1, S_3, S_4

2

5

- (a) x_3 enters at value 1
 $Z = 0 + 3 \times 1 = 3$
- (b) x_1 enters at value 1
 $Z = 0 + 5 \times 1 = 5$
- (c) x_2 enters at value 1
 $Z = 0 + 7 \times 1 = 7$
- (d) Tie broken arbitrarily between x_1, x_2 , and x_3 . Entering value = 1
 $Z = 0 + 1 \times 1 = 1$

3

Set 3.3b

								1
Basic	Z	X ₁	X ₂	S ₁	S ₂	S ₃	S ₄	Sol
Z	1	-5	-4	0	0	0	0	0
S ₁	0	6	4	1	0	0	0	24
S ₂	0	1	2	0	1	0	0	6
S ₃	0	-1	1	0	0	1	0	1
S ₄	0	0	1	0	0	0	1	2
Z	1	0	6	0	5	0	0	30
S ₁	0	0	-8	1	-6	0	0	-12
X ₁	0	1	2	0	1	0	0	6
S ₃	0	0	3	0	1	1	0	7
S ₄	0	0	1	0	0	0	1	2

								2	
(a)	Basic	x ₁	x ₂	x ₃	x ₄	sx ₅	sx ₆	sx ₇	Solution
Z	-2.00	-1.00	3.00	-5.00	0.00	0.00	0.00	0.00	0.00
1)sx ₅	1.00	2.00	2.00	4.00	1.00	0.00	0.00	40.00	
2)sx ₆	2.00	-1.00	1.00	2.00	0.00	1.00	0.00	8.00	
3)sx ₇	4.00	-2.00	1.00	-1.00	0.00	0.00	1.00	10.00	
Z	3.00	-3.50	5.50	0.00	0.00	2.50	0.00	20.00	
1)sx ₅	-3.00	4.00	0.00	0.00	1.00	-2.00	0.00	24.00	
2)sx ₄	1.00	-0.50	0.50	1.00	0.00	0.50	0.00	4.00	
3)sx ₇	5.00	-2.50	1.50	0.00	0.00	0.50	1.00	14.00	
Z	0.38	0.00	5.50	0.00	0.88	0.75	0.00	41.00	
1)sx ₂	-0.75	1.00	0.00	0.00	0.25	-0.50	0.00	6.00	
2)sx ₄	0.62	0.00	0.50	1.00	0.12	0.25	0.00	7.00	
3)sx ₇	3.12	0.00	1.50	0.00	0.62	-0.75	1.00	29.00	

								3
Basic	Ratios							
Basic	X ₁	X ₂	X ₃	X ₄				
X ₅	4/1	4/2	--					4/5
X ₆	8/5	--	--					8/6
X ₇	3/2	3/3	--					3/3
X ₈	--	--	0/1	--				
Value	1.5	1	0	0.8				
Leaving var	X ₇	X ₇	X ₈	X ₅				

(a) Nonbasic x_1 will improve solution.	4
Basic x_i -ratios	
x_2 $\frac{4}{5} \Rightarrow x_2$ leaves, $x_i = \frac{4}{5}$	
x_3 $\frac{8}{16}$	
x_4 $\frac{3}{3}$	
$X_1 = \frac{4}{5} = .8, X_3 = 8 - 6 \times .8 = 3.6, X_4 = 3 - 3 \times .8 = .6$	
$X_2 = 0, Z = .8 \times 1 = .8$	
(b) x_1 remains nonbasic at zero. Current solution, $X_2 = 4, X_3 = 8, X_4 = 3, Z = 0$ is optimum	
Basic solutions consist of one variable each. Thus,	5
$X_1 = 90/1 = 90, Z = 5 \times 90 = 450$	
$X_2 = 90/3 = 30, Z = -6 \times 30 = -180$	
$X_3 = 90/5 = 18, Z = 3 \times 18 = 54$	
$X_4 = 90/6 = 15, Z = -5 \times 15 = -75$	
$X_5 = 90/3 = 30, Z = 12 \times 30 = 360$	
Optimum solution:	
$X_1 = 90, X_2 = X_3 = X_4 = X_5 = 0, Z = 450$	

(a) Basic: $(X_8, X_3, X_1) = (12, 6, 0), Z = 620$	6
Nonbasic: $(X_2, X_4, X_5, X_6, X_7) = (0, 0, 0, 0, 0)$	
(b) X_2, X_5, X_6 will improve solution.	
X_2 enters: $X_2 = \min(\frac{12}{3}, \frac{6}{1}, -) = 4$. Thus,	
X_8 leaves, $\Delta Z = 4 \times 5 = 20$	
continued...	

Set 3.3b

x_5 enters: $x_5 = \min(-, \frac{6}{1}, \frac{0}{6}) = 0$. Thus,
 $\Delta Z = 1 \times 0 = 0$ (x_5 leaves)

x_6 enters: $x_6 = \min(-, -, -)$. Thus, no
leaving variable and x_6 can
be increased to ∞ . $\Delta Z = +\infty$

(C) x_4 can improve solution.

x_4 enters: $x_4 = \min(-, \frac{6}{3}, -) = 2$. Thus,
 x_3 leaves. $\Delta Z = -4 \times 2 = -8$

(d) As shown in (b), x_5 cannot
change Z because it enters the
solution at level zero. x_7 cannot change
 Z either because its objective equation
coefficient = 0. $\Delta Z = 0 \times \min(\frac{12}{5}, \frac{6}{3}, -) = 0$

(a) Maximize $Z = 3x_1 + 6x_2$: 7

x_2 is the first entering variable.
Resulting path is $A \rightarrow G \rightarrow F \rightarrow E$.

(b) Maximize $Z = 4x_1 + x_2$:

Entering variable $x_1 = \min$ (intercept with)
 x_1 -axis)

$$x_1 = \min(2, 3, 5) = 2 \text{ at } B$$

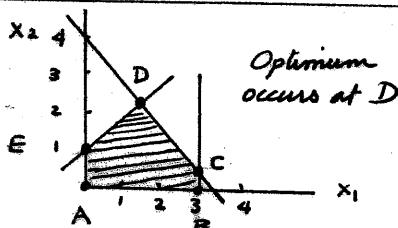
$$\Delta Z = 4 \times 2 = 8$$

(c) Maximize $Z = x_1 + 4x_2$:

Entering variable $x_2 = \min$ (intercept)
with x_2 -axis)

$$x_2 = \min(1, 2, 4) = 1$$

$$\Delta Z = 4 \times 1 = 4$$



8

(a) x_1 will enter first and the iterations
will follow the path $A \rightarrow B \rightarrow C \rightarrow D$

(b) x_2 enters first and the iterations will
follow the path $A \rightarrow E \rightarrow D$

(c) The most-negative criterion requires
more iterations (4 vs. 3). This criterion
is only a heuristic, and although it does
not guarantee the smallest number of

iterations, computational experience
demonstrates that, on the average,
the most-negative criterion is
more efficient.

(d) Iterations are identical, with
the exception of the objective row, which
should appear with an opposite sign

Optimum tableau:

Basic	x_1	x_2	s_1	s_2	s_3	s_4	
Z	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

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If s_1 enters, its value = $\min\{\frac{3}{1/4}, -\frac{5/2}{3/4}, \frac{1/2}{1/8}\} = 4$

$$\text{New } Z = 21 - \frac{5}{2} \times 4 = 18$$

If s_2 enters, its value = $\min\{-\frac{3/2}{3/4}, -\frac{1/2}{1/8}\} = 2$

$$\text{New } Z = 21 - \frac{1}{2} \times 2 = 20. \text{ The second best } Z \text{ is associated with } s_2 \text{ entering the basic solution}$$

Not easily extendable because the third best solution may not be an adjacent corner point of the current optimum point.

10

x_1 = number of purses per day

x_2 = number of bags per day

x_3 = number of backpacks per day

$$\text{Maximize } Z = 24x_1 + 22x_2 + 45x_3$$

Subject to

$$2x_1 + x_2 + 3x_3 \leq 42$$

$$2x_1 + x_2 + 2x_3 \leq 40$$

$$x_1 + 5x_2 + x_3 \leq 45$$

$$x_1, x_2, x_3 \geq 0$$

TORA's optimum solution:

$$x_1 = 0, x_2 = 36, x_3 = 2, Z = \$882$$

Status of resources:

Resource	slack	Status
Leather	0	scarce
Sewing	0	scarce
Finishing	25	abundant

continued...

From TORA Iterations module, **12**

Click **All Iterations**, then go to the optimal iteration and click any of the associated nonbasic variables (X_4, Sx_6, Sx_7, Sx_8). Now, click **Next Iteration** to produce the new iteration in which the selected variable becomes basic. The associated value of Z will deteriorate.

To determine the next-best solution, follow the procedure in Problem 1. First, let X_4 enter the basic solution and record the associated value of Z . Next, click **View/Modify Input Data** and re-solve the problem to produce the same optimum tableau that was used before X_4 was entered into the basic solution. Now, enter Sx_6 into the basic solution and record the associated value of Z . Repeat the procedure for Sx_7 and Sx_8 . You will get the following results:

Entering Variable	Z
X_4	2.63
Sx_6	1.00
Sx_7	6.40
Sx_8	1.90

The next-best solution is associated with entering Sx_7 into the basic solution. Associated values of the variables are

$$X_1 = 1.6$$

$$X_2 = 0$$

$$X_3 = 1.6$$

$$X_4 = 0$$

$$Z = 6.40$$

Set 3.4a

Iteration	Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
0 (starting)	z	$-4 + 7M$	$-1 + 4M$	$-M$	0	0	0	$9M$
	R_1	3	1	0	1	0	0	3
	R_2	4	3	-1	0	1	0	6
	x_4	1	2	0	0	0	1	4
1	z	0	$\frac{1+5M}{3}$	$-M$	$\frac{4-7M}{3}$	0	0	$4+2M$
	x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
	R_2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2
	x_4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3
2	z	0	0	$\frac{1}{5}$	$\frac{8}{5} - M$	$-1/5 - M$	0	$18/5$
	x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
	x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
	x_4	0	0	1	1	-1	1	1
3	z	0	0	0	$\frac{7}{5} - M$	$-M$	$-\frac{1}{5}$	$17/5$
	x_1	1	0	0	$\frac{2}{5}$	0	$-\frac{1}{5}$	$\frac{2}{5}$
	x_2	0	1	0	$-\frac{1}{5}$	0	$\frac{3}{5}$	$\frac{9}{5}$
	x_3	0	0	1	1	-1	1	1
(optimum)								

M = 1:

Optimum solution: $x_1 = 0, x_2 = 2, x_3 R4 = 1$
 $Z = 3$

Solution is infeasible because $xR4$ is positive. The reason $M=1$ produces an infeasible solution is that it does not play the role of a penalty relative to the objective coefficients of the real variables, x_1 and x_2 . Using $M=1$ makes $xR4$ more attractive than x_1 from the standpoint of minimization.

M = 10:

Optimum solution: $x_1 = -4, x_2 = 1.8, Z = 3.4$

The solution is feasible because it does not include artificials at positive level. $M=10$ is relatively much larger than the objective coefficients of x_1 and x_2 , and hence properly plays the role of a penalty.

M = 1000:

It produces the optimum solution as with $M=10$. The conclusion is that it suffices to select M reasonably larger than the objective coefficients of the real variables. Actually, $M=1000$ is an "over kill" in this case, and selecting such huge values could result in adverse round-off error.

(a) Minimize $Z = 4x_1 + x_2 + M(R_1 + R_2 + R_3)$
 subject to

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 - S_2 + R_2 &= 6 \\ x_1 + 2x_2 - S_3 + R_3 &= 4 \\ x_1, x_2, S_2, S_3, R_1, R_2, R_3 &\geq 0 \end{aligned}$$

3

Basic	x_1	x_2	S_2	S_3	R_1	R_2	R_3	
Z	-4	-1			(-M)	(-M)	(-M)	0
R_1	3	1			1			3
R_2	4	3	-1			1		6
R_3	1	2		-1			1	4
Z	$-4+8M$	$-1+6M$	$-M$	$-M$	0	0	0	$10M$
R_1	3	1			1			3
R_2	4	3	-1			1		6
R_3	1	2		-1			1	4

(b) Minimize $Z = 4x_1 + x_2 + M R_1$
 subject to

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 + S_2 &= 6 \\ x_1 + 2x_2 + S_3 &= 4 \end{aligned}$$

Basic	x_1	x_2	R_1	S_2	S_3	
Z	-4	-1	(-M)			0
R_1	3	1	1			3
S_2	4	3		1		6
R_3	1	2			1	4
Z	$-4+3M$	$-1+M$		0	0	$3M$
R_1	3	1	1			3
S_2	4	3		1		6
R_3	1	2			1	4

(c) Minimize $Z = 4x_1 + x_2 + M(R_1 + R_2)$

subject to

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 + R_2 &= 6 \\ x_1 + 2x_2 + S_3 &= 4 \end{aligned}$$

Basic	x_1	x_2	R_1	R_2	S_3	
Z	-4	-1	(-M)	(-M)	0	0
R_1	3	1	1			3
R_2	4	3		1		6
S_3	1	2			1	4
Z	$-4+7M$	$-1+4M$	0	0	0	$9M$
R_1	3	1	1			3
R_2	4	3		1		6
S_3	1	2			1	4

continued...

Set 3.4a

5

(d) Maximize $Z = 4x_1 + x_2 - M(R_1 + R_2)$
subject to

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 - S_2 + R_2 &= 6 \\ x_1 + 2x_2 + S_3 &= 4 \end{aligned}$$

Basic	x_1	x_2	S_2	R_1	R_2	S_3	
Z	-4	-1	0	(M)	(M)	0	0
R_1	3	1		1			3
R_2	4	3	-1		1		6
S_3	1	2				1	4
Z	-4-7M	-1-4M	M	0	0	0	-9M
R_1	3	1		1			3
R_2	4	3	-1		1		6
S_3	1	2				1	4

(a) Maximize $Z = 5x_1 + 6x_2 - M(R_1)$

Subject to

$$\begin{aligned} -2x_1 + 3x_2 + (R_1) &= 3 \quad (1) \\ x_1 + 2x_2 + S_3 &= 5 \quad (3) \\ 6x_1 + 7x_2 + S_4 &= 3 \quad (4) \end{aligned}$$

$$Z - (5-2M)x_1 - (6+3M)x_2 = -3M$$

(b) Maximize $Z = 2x_1 - 7x_2 - M(R_1 + R_2 + R_5)$

Subject to

$$\begin{aligned} -2x_1 + 3x_2 + R_1 &= 3 \quad (1) \\ 4x_1 + 5x_2 - S_2 + R_2 &= 10 \quad (2) \\ 6x_1 + 7x_2 + S_4 &= 3 \quad (4) \\ 4x_1 + 8x_2 - S_5 + R_5 &= 5 \quad (5) \end{aligned}$$

$$Z - (2+6M)x_1 - (-7+16M)x_2 + MS_2 + MS_5 = -18M$$

(c) Minimize $Z = 3x_1 + 6x_2 + MR_S$

Subject to

$$\begin{aligned} x_1 + 2x_2 + S_1 &= 5 \quad (3) \\ 6x_1 + 7x_2 + S_2 &= 3 \quad (4) \\ 4x_1 + 8x_2 - S_5 + R_S &= 5 \quad (5) \end{aligned}$$

$$Z - (3-4M)x_1 - (6-8M)x_2 - MS_5 = 5M$$

(d) Minimize $Z = 4x_1 + 6x_2 + M(R_1 + R_2 + R_5)$

Subject to

$$\begin{aligned} -2x_1 + 3x_2 + R_1 &= 3 \quad (1) \\ 4x_1 + 5x_2 - S_2 + R_2 &= 10 \quad (2) \\ 4x_1 + 8x_2 - S_5 + R_5 &= 5 \quad (5) \end{aligned}$$

$$Z - (4-6M) - (6-16M)x_2 - MS_2 - MS_5 = 18M$$

(e) Minimize $Z = 3x_1 + 2x_2 + M(R_1 + R_5)$

Subject to

$$\begin{aligned} -2x_1 + 3x_2 + R_1 &= 3 \quad (1) \\ 4x_1 + 8x_2 - S_5 + R_5 &= 5 \quad (5) \end{aligned}$$

$$Z - (3-2M)x_1 - (2-11M)x_2 - MS_5 = 8M$$

(a)

Basic	x_1	x_2	x_3	S_2	R_1	R_2	
$\bar{3}$	-2	-3	5	-M	0	0	-17M
0	R_1	1	1	1	0	1	0
R_2	[2]	-5	1	-1	0	1	10
$\bar{3}$	0	-8	6	-1	1	1	16
I	R_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	2
x_1	1	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	5
$\bar{3}$	0	0	$50/7$	$1/7$	$16/7$	$-1/7$	$\frac{102}{7}$
II	x_2	0	1	$1/7$	$1/7$	$3/7$	$-1/7$
x_1	1	0	$6/7$	$-1/7$	$5/7$	$1/7$	$\frac{45}{7}$

(b)

Basic	x_1	x_2	x_3	S_2	R_1	R_2	S_0/M
$\bar{3}$	-2	-3	5	-M	0	0	-17M
0	R_1	1	1	1	0	1	7
R_2	2	-5	1	-1	0	1	10
$\bar{3}$	0	-8	6	-1	1	1	15
I	R_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	2
x_1	1	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	5
$\bar{3}$	0	0	$50/7$	$1/7$	$16/7$	$-1/7$	$\frac{102}{7}$
II	x_2	0	1	$1/7$	$1/7$	$3/7$	$-1/7$
x_1	1	0	$6/7$	$-1/7$	$5/7$	$1/7$	$\frac{45}{7}$
$\bar{3}$	0	-50	0	-7	-12	7	-14
III	x_3	0	7	1	1	2	-1
x_1	1	-6	0	-1	-1	1	3

continued...

continued...

Set 3.4a

(c)

	Basic	x_1	x_2	x_3	s_1	R_1	R_2	Solution
0	\bar{Z}	-1	-2	-1	0	m	m	-
0	R_1	1	1	1	0	1	0	7
0	R_2	2	-5	1	-1	0	1	10
I	\bar{Z}	-1	-2	-1	m	0	0	$-17m$
I	R_1	1	1	1	0	1	0	7
I	R_2	2	-5	1	-1	0	1	10
II	\bar{Z}	0	$-\frac{9}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{2}$	$-\frac{2}{2}m$
II	R_1	0	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	2
II	x_1	1	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	5
III	\bar{Z}	0	0	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}m$	$-\frac{1}{7}m$	$\frac{56}{7}$
III	x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{4}{7}$
III	x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{5}{7}$	$\frac{45}{7}$	$\frac{45}{7}$

(d)

	Basic	x_1	x_2	x_3	s_1	s_2	R_1	R_2	Solution
0	\bar{Z}	-4	8	-3	0	$-m$	$-m$	0	-
0	R_1	1	1	1	0	1	0	7	
0	R_2	2	-5	1	-1	0	1	10	
I	\bar{Z}	-4	8	-3	$-m$	0	0	$17m$	
I	R_1	1	1	1	0	1	0	7	
I	R_2	2	-5	1	-1	0	1	10	
II	\bar{Z}	0	-2	-1	-2	2	$\frac{20}{7}$		
II	R_1	0	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{2}{7}$	
II	x_1	1	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	5	
III	\bar{Z}	0	0	$-\frac{5}{7}$	$-\frac{12}{7}$	$\frac{4}{7}m$	$-\frac{1}{7}m$	$\frac{45}{7}$	
III	x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}$	$-\frac{1}{7}$	$\frac{41}{7}$	
III	x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$-\frac{5}{7}$	$\frac{1}{7}$	$\frac{45}{7}$	

In the first iteration, we must substitute out the starting solution variables, x_3 and x_4 , in the Z-equation, exactly as we do with the artificial variables.

6

Basic	x_1	x_2	x_3	x_4	Solution
Z	-2	-4	<u>(-4)</u>	<u>(3)</u>	-
x_3	1	1	<u>(1)</u>	0	4
x_4	1	4	0	<u>(1)</u>	8
Z	-1	-12	0	0	-8
x_3	1	1	0	0	4
x_4	1	<u>(4)</u>	0	1	8
Z	2	0	0	3	16
x_3	$\frac{3}{4}$	0	1	$-\frac{1}{4}$	2
x_2	$\frac{1}{4}$	1	0	$\frac{1}{4}$	2

After adding surplus s_1 and s_2 , substitute out x_3 in the Z-equation.

7

Basic	x_1	x_2	s_1	s_2	x_3	x_4	Solution
Z	-3	-2	0	0	<u>(-3)</u>	0	-
x_3	1	4	-1	0	<u>(1)</u>	0	7
x_4	2	1	0	-1	0	1	10
Z	0	10	-3	0	0	0	21
x_3	1	4	-1	0	1	0	7
x_4	2	1	0	-1	0	1	10
Z	$-\frac{5}{2}$	0	$-\frac{1}{2}$	0	$-\frac{5}{2}$	0	$\frac{7}{2}$
x_2	$\frac{1}{4}$	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{7}{4}$
x_4	$\frac{7}{4}$	0	$\frac{1}{4}$	-1	$-\frac{1}{4}$	1	$\frac{33}{4}$

Both x_3 and R (the starting solution variables) must be substituted out in the Z-equation.

8

Basic	x_1	x_2	x_3	R	Solution
\bar{Z}	-1	-5	<u>(-3)</u>	<u>(M)</u>	-
x_3	1	2	<u>(1)</u>	0	3
R	2	-1	0	<u>(1)</u>	4
\bar{Z}	$2-2M$	$1+M$	0	0	$9-4M$
x_3	1	2	1	0	3
R	<u>(2)</u>	-1	0	1	4
\bar{Z}	0	2	0	$-1+M$	5
x_3	0	$\frac{5}{2}$	1	$-\frac{1}{2}$	1
x_1	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	2

Maximize $Z = 2x_1 + 5x_2 - MR_1$
 subject to

$$\begin{aligned} 3x_1 + 2x_2 - S_1 + R_1 &= 6 \\ 2x_1 + x_2 &+ S_2 = 2 \\ x_1, x_2, S_1, R_1, S_2 &\geq 0 \end{aligned}$$

9

Basic	x_1	x_2	S_1	R_1	S_2	
Z	-2	-5	0	M	0	-
R_1	3	2	-1	1	0	6
S_2	2	1	0	0	1	2
Z	$-2-3M$	$-5-2M$	M	0	0	$-6M$
R_1	3	2	-1	1	0	6
S_2	2	1	0	0	1	2
Z	0	$-4-M/2$	M	0	$1+3M/2$	$\frac{-2}{+3M}$
R_1	0	$1/2$	-1	1	$-3/2$	3
x_1	1	$1/2$	0	0	$1/2$	1
Z	$8+M$	0	M	0	$5+2M$	$\frac{10}{-2M}$
R_1	-1	0	-1	1	-2	2
x_2	2	1	0	0	1	2

The Z-row shows that the solution is optimal (all nonbasic coefficients in the Z-row are ≥ 0). However, the solution is infeasible because the artificial variable R_1 assumes a positive value. Having a positive value for the artificial variable R_1 is the same as regarding the constraint $3x_1 + 2x_2 \geq 6$ as $3x_1 + 2x_2 \leq 6$, which violates the constraints of the original model.

Set 3.4b

In Phase I, we always minimize the sum of the artificial variables because the sum represents a measure of infeasibility in the problem

- (a) Minimize $r = R_1$
 (b) Minimize $r = R_1 + R_2 + R_5$
 (c) Minimize $r = R_5$
 (d) Minimize $r = R_1 + R_2 + R_5$
 (e) Minimize $r = R_1 + R_5$

(a) Phase I:

Basic	x_1	x_2	x_3	s_2	R_1	R_2	Sol ^{1a}
r	0	0	0	0	-1	-1	0
R_1	1	1	1	0	1	0	7
R_2	2	-5	1	-1	0	1	10
r	3	-4	2	-1	0	0	17
R_1	1	1	1	0	1	0	7
R_2	2	-5	1	-1	0	1	10
r	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{3}{2}$	2
R_1	0	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	2
x_1	1	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	5
r	0	0	0	0	-1	-1	0
x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{4}{7}$
x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	$\frac{45}{7}$

2

(c) Phase I is the same as in (a)

Phase II:

Basic	x_1	x_2	x_3	s_2	s_1	Sol ^{1a}
r	-1	-2	-1	0	0	0
x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{4}{7}$
x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{45}{7}$	$\frac{45}{7}$
r	0	0	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{53}{7}$	$\frac{53}{7}$
x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{4}{7}$
x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{48}{7}$	$\frac{48}{7}$

(d) Phase I is the same as in (a)

Phase II:

Basic	x_1	x_2	x_3	x_4	s_1	Sol ^{1a}
Z	-4	8	-3	0	0	0
x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{4}{7}$
x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{45}{7}$	$\frac{45}{7}$
Z	0	0	$-\frac{5}{7}$	$-\frac{12}{7}$	$\frac{21}{7}$	$\frac{21}{7}$
x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{4}{7}$
x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{45}{7}$	$\frac{45}{7}$

Minimize $r = R_1$

Subject to

$$3x_1 + 2x_2 - s_1 + R_1 = 6 \\ 2x_1 + x_2 + s_2 = 2 \\ x_1, x_2, s_1, R_1, s_2 \geq 0$$

Solution of Phase I by TORA yields $r = 2$, which indicates that the problem has no feasible space

Basic	x_1	x_2	x_3	s	Sol ^{1a}
r	-2	-3	5	0	0
x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}$
x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{45}{7}$
r	0	0	$\frac{50}{7}$	$\frac{1}{7}$	$\frac{102}{7}$
x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}$
x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{45}{7}$

(b) Phase I is the same as in (a)

Basic	x_1	x_2	x_3	s_2	Sol ^{1a}
r	-2	-3	5	0	0
x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}$
x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{45}{7}$
r	0	0	$\frac{50}{7}$	$\frac{1}{7}$	$\frac{102}{7}$
x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}$
x_1	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{45}{7}$
r	0	-50	0	-7	-14
x_3	0	7	1	1	4
x_4	1	-6	0	-1	3

continued...

Minimize $Z = R_2$

Subject to

$$2x_1 + x_2 + x_3 + s_1 = 2 \\ 3x_1 + 4x_2 + 2x_3 - s_2 + R_2 = 8 \\ x_1, x_2, x_3, s_1, s_2, R_2 \geq 0$$

Phase I Optimal solution:

Basic	x_1	x_2	x_3	s_1	s_2	R_2	Sol ^{1a}
r	-5	0	-2	-1	-4	0	0
x_2	2	1	1	0	1	0	2
R_2	-5	0	-2	-1	-4	1	0

$R_2 = 0$ is basic in the Phase I Solution
continued...

Set 3.4b

6

(b)

Phase I (continued): R2 leaves, x₁ enters (also x₃, s₂, and s₁ are candidates for the entering variable).

	x ₁	x ₂	x ₃	s ₂	s ₁	R ₂	Sol.
r	-5	0	-2	-1	-4	0	0
x ₂	2	1	1	0	1	0	2
R ₂	-5	0	-2	-1	-4	1	0
r	0	0	0	0	0	-1	
x ₂	0	1	1/5	-2/5	-3/5	2/5	2
x ₁	1	0	2/5	1/5	4/5	-1/5	0

Drop R₂-column.

Phase II:

	x ₁	x ₂	x ₃	s ₂	s ₁	Sol.
z	-2	-2	-4	0	0	0
x ₂	0	1	1/5	-2/5	-3/5	2
x ₁	1	0	2/5	1/5	4/5	0
z	0	0	-14/5	-2/5	2/5	4
x ₂	0	1	1/5	-2/5	-3/5	2
x ₁	1	0	2/5	1/5	4/5	0
z	7	0	0	1	6	4
x ₂	-1/2	1	0	-1/2	-1	2
x ₃	5/2	0	1	1/2	2	0

Optimum solution:

$$x_1 = 0, x_2 = 2, x_3 = 0, z = 4$$

Phase I:

	x ₁	x ₂	x ₃	R ₁	R ₂	R ₃	Sol.
r	-10	0	-4	-8	0	0	0
x ₂	2	1	1	1	0	0	2
R ₂	-5	0	-2	-3	1	0	0
R ₃	-5	0	-2	-4	0	1	0
r	0	0	1	-2	-2	0	0
x ₂	0	1	1/5	-1/5	2/5	0	2
x ₁	1	0	2/5	3/5	-1/5	0	0
R ₃	0	0	0	-1	-1	1	0

Remove R₁- and R₂ columns, which gives

	x ₁	x ₂	x ₃	R ₃	Sol.
r	0	0	1	0	0
x ₂	0	1	1/5	0	2
x ₁	1	0	2/5	0	0
R ₃	0	0	0	1	0

The R₃-row is R₃ = 0, which is redundant. Hence the R₃-row and R₃-column can be dropped from the tableau with no consequences.

Phase II:

	x ₁	x ₂	x ₃	Sol.
z	-3	-2	-3	0
x ₂	0	1	1/5	2
x ₁	1	0	2/5	0
z	0	0	-7/5	4
x ₂	0	1	1/5	2
x ₁	1	0	2/5	0
z	7/2	0	0	4
x ₂	-1/2	1	0	2
x ₁	5/2	0	1	0

Optimum solution:

$$x_1 = 0, x_2 = 2, x_3 = 0, z = 4$$

Set 3.4b

If x_1, x_3, x_4 , or x_5 assume a positive value, the value of the objective function at the end of Phase I must necessarily become positive. This follows because these variables have nonzero Z-row coefficients in the optimal Phase I tableau. A positive objective value at the end of Phase I means that Phase I solution is infeasible. Since Phase II uses the same constraints as in Phase I, it follows that Phase II must have $x_1 = x_3 = x_4 = x_5 = 0$ as well.

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Phase II:

Basic	x_2	R	S _{0/Z}
Z	(-2)	0	0
x_2	1	0	2
R	0	1	0
Z	0	0	4
x_2	1	0	2
R	0	1	0

Optimum solution:

$$x_1 = 0 \quad x_2 = 2 \quad x_3 = x_4 = x_5 = 0 \\ Z = 4$$

8

$$\begin{array}{lcl} -5x_1 + 6x_2 - 2x_3 + x_4 & = -5 \\ x_1 - 3x_2 - 5x_3 + x_5 & = -8 \\ 2x_1 + 5x_2 - 4x_3 + x_6 & = 9 \end{array}$$

x_1	x_2	x_3	x_4	x_5	x_6	R
0	0	0	0	0	0	-1
-5	6	-2	1	0	0	-1
1	-3	-5	0	1	0	-1
2	5	-4	0	0	1	9
-1	3	5	0	-1	0	8
-6	9	3	1	-1	0	3
-1	3	5	0	-1	0	1
2	5	-4	0	0	1	9

Phase I problem:

$$\text{minimize } r = R$$

subject to

$$\begin{array}{lcl} -6x_1 + 9x_2 + 3x_3 + x_4 - x_5 & = 3 \\ -x_1 + 3x_2 + 5x_3 - x_5 + R & = 8 \\ 2x_1 + 5x_2 - 4x_3 + x_6 & = 9 \end{array}$$

all variables ≥ 0

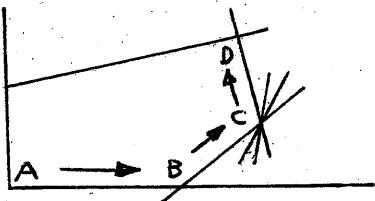
The logic of the procedure is as follows:

In the R-column, enter -1 for any constraint with negative RHS and 0 for all other constraints.

Next, use the R-column as a pivot column and select the pivot element as the one corresponding to the most negative RHS. This procedure will always require one artificial variable regardless of the number of constraints.

Set 3.5a

(a)

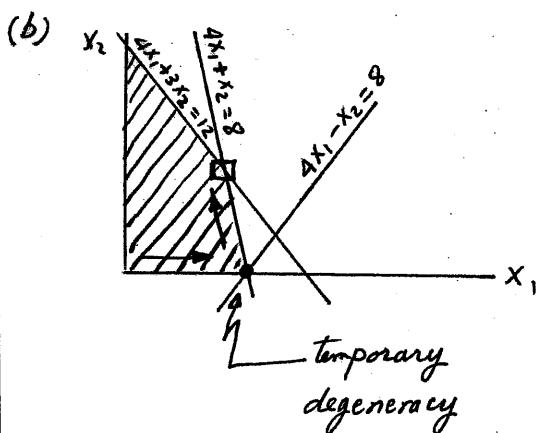


1

(b) $A: 1, B: 1, C: \binom{3}{2} = 3, D: 1$

(a) From TORA, iterations 2 and 3 are degenerate. Degeneracy is removed in iteration 4.

2



(a) Four iterations

3

(b) Three iterations: In iteration 2, degeneracy is removed because basic $SX_5 = 0$ corresponds to a negative constraint coefficient in the entering variable column (x_2).

(c) In part (a), solution encounters 2 degenerate basic solutions at the same corner point. In part (b), only one basic solution was encountered.

Set 3.5b

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Solution
\bar{Z}	-1	-2	-3	0	0	0	0
s_1	1	2	$\boxed{3}$	1	0	0	10
s_2	1	1	0	0	1	0	5
s_3	1	0	0	0	0	1	1
\bar{Z}	0	$\boxed{0}$	0	1	0	0	10
x_3	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$	0	0	$\frac{10}{3}$
s_2	1	$\boxed{1}$	0	0	1	0	5
s_3	1	0	0	0	0	1	1
\bar{Z}	$\boxed{0}$	0	0	1	0	0	10
x_3	$-\frac{1}{3}$	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	0
x_2	1	1	0	0	1	0	5
s_3	$\boxed{1}$	0	0	0	0	1	1
\bar{Z}	0	0	0	1	$\boxed{0}$	0	10
x_3	0	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
x_2	0	1	0	0	$\boxed{1}$	-1	4
x_1	1	0	0	0	0	1	1
\bar{Z}	0	0	0	1	0	0	10
x_3	0	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	3
s_2	0	1	0	0	1	-1	4
x_1	1	0	0	0	0	1	1

Three alternative basic optima:

$$(x_1, x_2, x_3) = \begin{cases} (0, 0, 10/3) \\ (0, 5, 0) \\ (1, 4, 1/3) \end{cases}$$

The associated nonbasic alternative optima are

$$\hat{x}_1 = \lambda_3$$

$$\hat{x}_2 = 5\lambda_2 + 4\lambda_3$$

$$\hat{x}_3 = 10/3\lambda_1 + 1/3\lambda_3$$

where

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$0 \leq \lambda_i \leq 1, i=1,2,3$$

Basic	x_1	x_2	x_3	s_1	s_2	s_3	2
Z	-2	1	3	0	0	0	0
s_1	1	-1	$\boxed{5}$	1	0	10	
s_2	2	-1	3	0	1	40	
Z	$-7/5$	$2/5$	0	$3/5$	0	6	
x_3	$\boxed{1/5}$	$-1/5$	1	$1/5$	0	2	
s_2	$7/5$	$-2/5$	0	$-3/5$	1	34	
Z	0	-1	7	2	0	20	
x_1	1	-1	5	1	0	10	
s_2	0	$\boxed{1}$	-7	-2	1	20	
Z	0	0	$\boxed{0}$	$\boxed{0}$	1	40	
x_1	1	0	-2	-1	0	30	
x_2	0	1	-7	-2	1	20	

x_3 and s_1 can yield alternative optima. However, because all their constraint coefficients are negative (in general, ≤ 0), none can yield an alternative (corner point) basic solution.

Basic	x_1	x_2	x_3	s_1	s_2	s_3	3
\bar{Z}	-3	-1	0	0	0	0	0
s_1	1	2	0	1	0	0	5
s_2	$\boxed{1}$	1	-1	0	1	0	2
s_3	7	3	-5	0	0	1	20
\bar{Z}	0	2	-3	0	3	0	6
s_1	0	1	$\boxed{1}$	1	-1	0	3
x_1	1	1	-1	0	1	0	2
s_3	0	-4	2	0	-7	1	6
\bar{Z}	0	5	0	3	$\boxed{0}$	0	15
x_3	0	1	1	1	-1	0	3
x_1	1	2	0	1	0	0	5
s_3	0	-6	0	-2	-5	1	0

The optimum solution is degenerate because s_3 is basic and equal to zero. Also, it has alternative nonbasic solutions because s_2 has a zero coefficient in the Z -row and all its constraint coefficients are ≤ 0 .

Basic	x_1	x_2	s_1	s_2	
Z	-2	-1	0	0	0
s_1	1	-1	1	0	10
s_2	2	0	0	1	40
Z	0	-3	2	0	20
x_1	1	-1	1	0	10
s_2	0	2	-2	1	20
Z	0	0	-1	$\frac{3}{2}$	50
x_1	1	0	0	$\frac{1}{2}$	20
x_2	0	1	-1	$\frac{1}{2}$	10

unbounded \rightarrow

(a)

x_2
-10
43
0
71

2

\Rightarrow Solution space unbounded
in the direction of x_2

(b) Objective value is unbounded
because each unit increase
in x_2 increases Z by 10

If, at any iteration, all the
constraint coefficients of a
variable are ≤ 0 , then the
solution space is unbounded in
the direction of that variable.

3

A more "foolproof" way of
accomplishing this task is to solve
a sequence of LPs in which the
objective function is

Maximize $Z = x_j$, $j=1, 2, \dots, n$
Subject to the constraints of the
problem. For the unbounded
variables, $Z = \infty$.

Set 3.5d

x_1 = number of units of T1
 x_2 = number of units of T2
 x_3 = number of units of T3

Constraints:

$$3x_1 + 5x_2 + 6x_3 \leq 1000$$

$$5x_1 + 3x_2 + 4x_3 \leq 1200$$

$$x_1 + x_2 + x_3 \geq 500$$

$$x_1, x_2, x_3 \geq 0$$

We can use Phase I to see whether the problem has a feasible solution; that is,

minimize $r = R_3$

subject to

$$3x_1 + 5x_2 + 6x_3 + S_1 = 1000$$

$$5x_1 + 3x_2 + 4x_3 + S_2 = 1200$$

$$x_1 + x_2 + x_3 - S_3 + R_3 = 500$$

$$x_1, x_2, x_3, S_1, S_2, S_3, R_3 \geq 0$$

Optimum solution from TORA:

$$R_3 = r = 225 \text{ units}$$

This is interpreted as a deficiency of 225 units. The most that can be produced is $500 - 225 = 275$ units

Basic	x_1	x_2	x_3	S_1	S_2	R_3	Sol^n
Z	-3	-2	-3				
	-3M	-4M	-2M	M	0	0	-8M
S_1	2	1	0	1	0	0	2
R_1	3	4	2	-1	0	1	8
Z	-1	-1	2				
	+5M	0	+2M	M	+4M	0	4
x_2	2	1	0	1	0	0	2
R_1	-5	0	-2	-1	-4	1	0

Because $R_1 = 0$ in the optimal tableau, the problem has a feasible solution. The optimum solution is

$$x_1 = 0, x_2 = 2, Z = 4$$

Note that in the first iteration, R_1 could have been used as the leaving variable, in which case it would not be basic in the optimum iteration.

Set 3.6a

X_1 = Nbr. units of product A
 X_2 = Nbr. units of product B

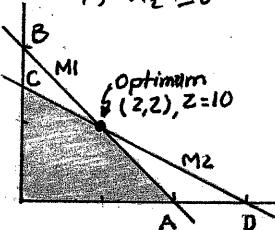
$$\text{Maximize } Z = 2X_1 + 3X_2$$

S.t.

$$2X_1 + 2X_2 \leq 8 \quad (\text{M1})$$

$$3X_1 + 6X_2 \leq 18 \quad (\text{M2})$$

$$X_1, X_2 \geq 0$$



	M1	M2	Z
A = (4, 0)	12	8	
B = (0, 4)	24	12	
C = (0, 3)	6	9	
D = (2, 2)	12	12	

$$(a) M1 \text{ at } C = 2(0) + 2(3) = 6$$

$$M1 \text{ at } D = 2(6) + 2(0) = 12$$

$$Z \text{ at } C = 2(0) + 3(3) = 9$$

$$Z \text{ at } D = 2(6) + 3(0) = 12$$

$$\text{Dual price} = \frac{12-9}{12-6} = .50/\text{unit}$$

$$\text{Allowable range} = (6 \leq M1 \leq 12)$$

$$M2 \text{ at } A = 3(4) + 6(0) = 12$$

$$M2 \text{ at } B = 3(0) + 6(4) = 24$$

$$Z \text{ at } A = 2(4) + 3(0) = 8$$

$$Z \text{ at } B = 2(0) + 3(4) = 12$$

$$\text{Dual price} = \frac{12-8}{24-12} = .33/\text{unit}$$

$$\text{Range: } 12 \leq M2 \leq 24$$

(b) Dual price = \$.50/unit valid in the range $6 \leq M1 \leq 12$

$$\text{Increase in revenue} = .5 \times 4 = \$2.00$$

$$\text{Increase in cost} = .3 \times 4 = \$1.20$$

Cost < Revenue - purchase recommended

(c) Dual price = \$.33/unit valid in

$$\text{the range } 12 \leq M2 \leq 24$$

$$\text{Purchase price/unit} < .33$$

(d) Dual price = \$.33/unit valid in

$$\text{the range } 12 \leq M2 \leq 24. M2 \text{ is}$$

increased from 18 to 23 units

Increase in revenue

$$= 5 \times .33 = \$1.65$$

$$\text{New optimum revenue} = 10 + 1.65 = \$11.65$$

Set 3.6a

2

X_1 = daily number of type 1 rats

X_2 = daily number of type 2 rats

$$\text{Maximize } Z = 8X_1 + 5X_2$$

$$2X_1 + X_2 \leq 400$$

$$X_1 \leq 150$$

$$X_2 \leq 200$$

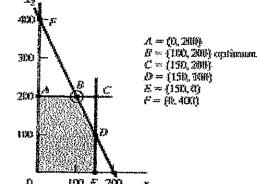
$$X_1, X_2 \geq 0$$

(a) Optimum occurs at B:

$$X_1 = 100 \text{ type 1 rats}$$

$$X_2 = 200 \text{ type 2 rats}$$

$$Z = \$1800$$



$$(b) A = (0,200), C = (150,200)$$

capacity

$$A \quad 2 \times 0 + 1 \times 200 = 200 \quad 8 \times 0 + 5 \times 200 = 1000$$

$$C \quad 2 \times 150 + 1 \times 200 = 500 \quad 8 \times 150 + 5 \times 200 = 2200$$

$$\text{worth/capacity unit} = \frac{2200 - 1000}{500 - 200}$$

$$= \$4 \text{ per type 2 rat}$$

$$\text{Range: } (200, 500)$$

$$(c) \text{ Dual price} = 0 \text{ in the range } (100, \infty)$$

Thus, change from $X_1 \leq 150$ to $X_1 \leq 120$

has no effect on optimum Z.

(d) Let d = demand limit for type 2 rat

d Z

$$D(150,100) \quad 100 \quad 8(150) + 5(100) = \$1700$$

$$F(0,400) \quad 400 \quad 8(0) + 5(400) = \$2000$$

$$\text{Dual price} = \frac{2000 - 1700}{400 - 100} = \$1.00$$

$$\text{Range } (100, 400)$$

Maximum increase in demand limit for type 2 rat = $400 - 200 = 200$ rats

Set 3.6b

(a) $\frac{3}{6} \leq \frac{C_A}{C_B} \leq \frac{2}{2}$, or
 $0.5 \leq \frac{C_A}{C_B} \leq 1$ or $1 \leq \frac{C_B}{C_A} \leq 2$

(b) Maximize $Z = 2x_A + 3x_B$

$C_B = 3$: $3 \times 0.5 \leq C_A \leq 3 \times 1$
 $1.5 \leq C_A \leq 3$

$C_A = 2$: $2 \times 0.5 \leq C_B \leq 2 \times 2$
 $1 \leq C_B \leq 4$

(c) $\frac{C_A}{C_B} = \frac{5}{4} = 1.25$, which falls outside the range $0.5 \leq \frac{C_A}{C_B} \leq 1$. Optimum solution changes and must be computed anew.
New solution: $x_A = 4$, $x_B = 0$, $Z = \$20$.

(d) Case 1: $Z = 5x_A + 3x_B$

$C_A = 5$ falls outside the range $(1.5, 3)$, hence the optimum changes. New Optimum is $x_A = 4$, $x_B = 0$, $Z = \$20$.

Case 2: $Z = 2x_A + 4x_B$

$C_B = 4$ falls in the range $(1, 4)$, hence optimum is unchanged at $x_A = x_B = 2$,
 $Z = 2(2) + 4(2) = \$12$

(a) $\frac{1}{2} \leq \frac{C_1}{C_2} \leq \frac{6}{4}$, or

$0.5 \leq \frac{C_1}{C_2} \leq 1.5$ or $\frac{2}{3} \leq \frac{C_2}{C_1} \leq 2$

(b) Given $C_1 = 5$, then

$5\left(\frac{2}{3}\right) \leq C_2 \leq 5(2)$, or $\frac{10}{3} \leq C_2 \leq 10$

(c) $\frac{C_1}{C_2} = \frac{5}{3} = 1.67$, which falls

outside the range $0.5 \leq \frac{C_1}{C_2} \leq 1.5$.

Hence the solution changes

1

(a) $0 \leq \frac{C_1}{C_2} \leq \frac{2}{1}$, or

$0 \leq \frac{C_1}{C_2} \leq 2$

(b) $\frac{C_1}{C_2} = 1$, which falls in the range $0 \leq \frac{C_1}{C_2} \leq 2$. Hence, the solution is unchanged.

3

2

Set 3.6c

Feasibility conditions:

$$X_2 = 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2$$

$$X_3 = 230 + \frac{1}{2}D_2$$

$$X_6 = 20 - 2D_1 + D_2 + D_3$$

$$(a) D_1 = 438 - 430 = 8 \text{ min}$$

$$D_2 = 500 - 460 = 40$$

$$D_3 = 410 - 420 = -10$$

$$X_2 = 100 + \frac{1}{2}(8) - \frac{1}{4}(40) = 94 > 0$$

$$X_3 = 230 + \frac{1}{2}(40) = 250 > 0$$

$$X_6 = 20 - 2(8) + 40 - 10 = 34 > 0$$

Dual prices:

$$\text{Resource 1} = \$1/\text{min}, -200 \leq D_1 \leq 10$$

$$2 = \$2/\text{min}, -20 \leq D_2 \leq 400$$

$$3 = \$0/\text{min}, -20 \leq D_3 < \infty$$

$$\text{New profit} = 1350 + D_1 + 2D_2 + 0D_3 \\ = 1350 + 8 + 2 \times 40 = \$1438$$

$$(b) D_1 = 460 - 430 = 30 \text{ min}$$

$$D_2 = 440 - 460 = -20$$

$$D_3 = 380 - 420 = -40$$

$$X_2 = 100 + \frac{1}{2}(30) - \frac{1}{4}(-20) = 120 > 0$$

$$X_3 = 230 + \frac{1}{2}(-20) = 220 > 0$$

$$X_6 = 20 - 2(30) - 20 - 40 = -100 < 0$$

$$(a) Overtime cost \frac{50}{60} = \$0.83/\text{min}$$

Revenue (dual price) for operation 1 is \\$1/min.

Cost < Revenue \Rightarrow advantageous

$$(b) \text{Dual price for operation 2} = \$2/\text{min} \text{ valid in the range } -20 \leq D_2 \leq 400$$

$$D_2 = 120 \text{ minutes}$$

$$\text{Revenue increase} = 120 \times 2 = \$240$$

$$\text{Cost increase} = 2 (\$55) = \$110$$

Revenue > Cost \Rightarrow accept.

(c) No, resource 3 is already abundant. This is the reason its dual price = 0

(d) Dual price for operation 1 is \\$1/min, valid in the range $-200 \leq D_1 \leq 10$

continued...

$$D_1 = 440 - 430 = 10 \text{ min}$$

$$\text{Cost} = \frac{10}{60} \times 40 = \$6.67$$

$$\text{New revenue} = 1350 + 1 \times 10 = \$1360$$

$$\text{Net revenue} = 1360 - \$6.67 = \$1353.33$$

$$(e) \text{Dual price} = \$2/\text{min}, -20 \leq D_2 \leq 400$$

$$D_2 = - \text{ min}$$

$$\text{Decrease in cost} = \frac{15}{60} \times 30 = \$7.50$$

$$\text{Lost revenue} = 15 \times \$2.00 = \$30.00$$

Lost revenue > Decrease in cost

Not recommended.

$$x_j = \text{units of product } i = 1, 2, 3$$

$$\text{Maximize } Z = 20x_1 + 50x_2 + 35x_3$$

s.t.

$$-.5x_1 + .5x_2 + .5x_3 \leq 0$$

$$x_1 \leq 75$$

$$2x_1 + 4x_2 + 3x_3 \leq 240$$

$$x_1, x_2, x_3 \geq 0$$

$$(a) \text{Solution: } Z = \$2800$$

$$x_1 = x_2 = 40, x_3 = 0$$

	x_1	x_2	x_3	S_1	S_2	S_3	
Z	0	0	$10/3$	$20/3$	0	$35/3$	2800
x_2	0	0	$5/6$	$2/3$	0	$1/6$	40
S_2	1	0	$1/6$	$4/3$	1	$-1/6$	35
x_1	0	1	$-1/6$	$-4/3$	0	$1/6$	40

$$(b) Z + 10/3x_3 + 20/3S_1 + 0S_2 + 35/3S_3 = 2800$$

$$\text{Dual price for raw material} = \$35/3 / 16$$

$$\left. \begin{aligned} x_2 &= 40 + \frac{D_3}{6} \\ S_2 &= 35 - \frac{D_3}{6} \end{aligned} \right\} \Rightarrow -240 \leq D_3 \leq 210$$

$D_3 = 120$ falls in the range (-240, 210)

New solution:

$$x_1 = 40 + \frac{120}{6} = 60 \text{ units}$$

$$x_2 = 40 + \frac{120}{6} = 60 \text{ units}$$

$$x_3 = 0$$

$$\text{New revenue} = 2800 + (35/3)(120) \\ = \$4200$$

3

Set 3.6c

(e) Dual price = 0, $-35 \leq D_2 < \infty$
 $\pm 10\% \text{ of } 75 = \pm 7.5$ or
 Change has no effect on the solution

X_j = units of product j , $j = 1, 2, 3$,

Maximize $Z = 4.5X_1 + 5X_2 + 4X_3$
 s.t.

$$\begin{aligned} 10X_1 + 5X_2 + 6X_3 &\leq 600 \\ 6X_1 + 8X_2 + 9X_3 &\leq 600 \\ 8X_1 + 10X_2 + 12X_3 &\leq 600 \\ X_1, X_2, X_3 &\geq 0 \end{aligned}$$

(a) Solution: $Z = \$325$

$$X_1 = 50, X_2 = 20, X_3 = 0$$

(b) Optimum tableau

X_1	X_2	X_3	S_1	S_2	S_3	
Z	0	0	2	.083	0	.458 325
X_1	1	0	0	.167	0	-.083 50
S_2	0	0	-.6	.067	1	-.833 140
X_2	0	1	1.2	-.133	0	.167 20

$$Z + 2X_3 + .083S_1 + 0S_2 + .458S_3 = 325$$

Dual prices:

Process 1: \$.083/min

2: \$0/min

3: \$.458/min

Process 3 > Process 1

(c) Process 1: $60X_1 \cdot .083 = \$4.98$

2: 0

3: $60X_1 \cdot .458 = \$27.48$

X_1 = Nbr. of practical courses

X_2 = Nbr. of humanistic courses

Maximize $Z = 1500X_1 + 1000X_2$

$$X_1 + X_2 + S_1 = 30 \quad (1)$$

$$X_1 - S_2 = 10 \quad (2)$$

$$X_2 - S_3 = 10 \quad (3)$$

$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

(a) Solution:

$$Z = \$40,000$$

$$X_1 = 20 \text{ courses}$$

$$X_2 = 10 \text{ courses}$$

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(b) From TORA,
 $Z + 1500S_1 + 0S_2 + 500S_3 = 40,000$
 S_1 is a slack, S_2 and S_3 are surplus

Dual prices:

constraint 1: \$1500/course
 constraint 2: \$0/min limit course
 constraint 3: -\$500/min limit course

Dual price for constraint 1 equals the revenue per practical course. Hence, an additional course must necessarily be of the practical type.

(c) From TORA,

$$\begin{cases} S_2 = 10 + D_1 \geq 0 \\ X_1 = 20 + D_1 \geq 0 \\ X_2 = 10 \end{cases} \quad -10 \leq D_1 < \infty$$

Thus, the dual price of \$1500 for constraint 1 is valid for any number of courses $\geq 30 - 10 = 20$.

(d) Dual price = -\$500. To determine the range where it applies, we have from TORA

$$\begin{cases} S_3 = 10 - D_3 \geq 0 \\ X_1 = 20 - D_3 \geq 0 \\ X_2 = 10 + D_3 \geq 0 \end{cases} \quad -10 \leq D_3 \leq 10$$

A unit increase in lower limit on humanistic course offering (i.e. from 10 to 11) decreases revenue by \$500

X_1 = Radio minutes

X_2 = TV minutes

X_3 = Newspaper ads

Maximize $Z = X_1 + 50X_2 + 5X_3$

s.t.

$$15X_1 + 300X_2 + 50X_3 \leq 10000 \quad (1)$$

$$X_3 \geq 5 \quad (2)$$

$$X_1 \leq 400 \quad (3)$$

$$-X_1 + 2X_2 \leq 0 \quad (4)$$

$$X_1, X_2, X_3 \geq 0$$

Solution: $Z = 1561.36$

$$X_1 = 59.09 \text{ min}, X_2 = 29.55 \text{ min}, X_3 = 5 \text{ ads}$$

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continued...

continued...

Set 2.3c

(b) S_1, S_3, S_4 = slacks associated with constraints 1, 3, and 4
 S_2 = surplus associated with constraint 2

From TORA's optimum tableau:

$$\begin{aligned} Z + 2.879S_2 + .158S_1 + 0S_2 + 1.364S_3 &= 1561.36 \\ 59.091 + .006D_1 - .303D_2 &\quad -.909D_4 \geq 0 \\ 5 &+ D_2 \\ 340.909 - .006D_1 + .303D_2 + D_3 &+.909D_4 \geq 0 \\ 29.545 + .003D_1 - .152D_2 &+.045D_4 \geq 0 \end{aligned}$$

Constraint	Dual Price	RHS Range
1.	.158	(250, 66250)
2	-2.879*	(0, 2000)
3	0	(59.09, ∞)
4	1.3636	(-375, 65)

* Negative because S_2 is a surplus variable
 + These results are taken from TORA output. They differ from those computed from the given D_i -conditions because of roundoff error

Conclusions:

- Increasing the lower limit on the number of newspaper ads is not advantageous because the associated dual price is negative ($= -2.879$)
- Increasing the upper limit on radio minutes is not warranted because its dual price is zero (the current limit is already abundant).
- Dual price $= .158$ /budget \$ valid in the range $250 \leq \$ \leq 66250$.

50% budget increase = \$5000, or budget will be increased to 15,000.

Increase in $Z = .158 \times 5000 = 790$

(a) X_1 = Nbr. shirts / week
 X_2 = Nbr. blouses / week

Maximize $Z = 8X_1 + 12X_2$

s.t.
 $20X_1 + 60X_2 \leq 25 \times 60 \times 40 = 60,000$
 $70X_1 + 60X_2 \leq 35 \times 60 \times 40 = 84,000$
 $12X_1 + 4X_2 \leq 5 \times 60 \times 40 = 12,000$
 $X_1, X_2 \geq 0$

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Solution: $Z = \$13920/\text{week}$

$X_1 = 480$ shirts, $X_2 = 840$ blouses

(b) Let S_1, S_2 , and S_3 be the slack variables associated with the cutting, sewing, and packaging constraints. From the optimum TORA tableau, we have

$$Z + .12S_1 + .08S_2 + 0S_3 = 13920$$

Dept. Worth/hr (Dual price)

Cutting	\$.12 /	= \$7.20/hr
Sewing	\$.08/min	= \$4.80/hr
Packaging	\$ 0/hr	

(c) Break-even wages are \$7.20/hr for cutting and \$4.80 for sewing

(a) X_1 = units of solution A

X_2 = units of solution B

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$$\text{Maximize } Z = 8X_1 + 10X_2$$

$$\text{s.t. } .5X_1 + .5X_2 \leq 150 \quad (1)$$

$$.6X_1 + .4X_2 \leq 145 \quad (2)$$

$$30 \leq X_1 \leq 150 \quad (3)$$

$$40 \leq X_2 \leq 200 \quad (4)$$

Solution: $Z = \$2800$

$X_1 = 100$ units, $X_2 = 200$ units

(b) Define

S_1, S_2, S_3, S_4 = slacks in constraints 1, 2, 3, 4

S_5, S_6 = surplus variables associated with the lower bounds of constraints 3 and 4.

From TORA's optimum tableau:

$$Z + 16S_1 + 0S_2 + 0S_3 + 2S_4 + 0S_5 + 0S_6 = 2800$$

Conditions:

$$S_1 = 70 + 2D_1 - D_2 - D_5 \geq 0$$

$$S_2 = 5 - 1.2D_1 + D_2 + 2D_4 \geq 0$$

$$S_3 = 50 - 2D_1 + D_3 + D_4 \geq 0$$

$$X_1 = 100 + 2D_1 - D_4 \geq 0$$

$$X_2 = 200 + D_4 \geq 0$$

$$S_4 = 160 + D_4 - D_6 \geq 0$$

continued...

continued...

Set 3.6c

Constraint	Dual price	RHS-range
1	16	(115, 154.17)
2	0	(140, ∞)
3 (upper)	0	(100, ∞)
3 (lower)	0	(- ∞ , 100)
4 (upper)	2	(175, 270)
4 (lower)	0	(- ∞ , 200)

Increase in raw material 1 and in the upper bound on solution B is advantageous because their dual prices (16 and 2) are positive.

(c) Increase in revenue/unit = \$16

Increase in cost/unit = \$20

Not recommended!

(d) Dual price for raw material 2 is zero because it is abundant. No increase is warranted.

$$X_1 = \text{Nbr. } D_1 G_i - 1$$

$$X_2 = \text{Nbr. } D_2 G_i - 2$$

S_i = Idle minutes for station i , $i=1, 2, 3$

The objective is to minimize $S_1 + S_2 + S_3$.

To express the objective function in terms of X_1 and X_2 , consider

$$6X_1 + 4X_2 + S_1 = .9 \times 480 = 432$$

$$5X_1 + 4X_2 + S_2 = .86 \times 480 = 412.8$$

$$4X_1 + 6X_2 + S_3 = .88 \times 480 = 422.4$$

Thus, $S_1 + S_2 + S_3 = 1267.2 - 15X_1 - 14X_2$

(a)

Maximize $Z = 15X_1 + 14X_2$

s.t.

$$6X_1 + 4X_2 + S_1 = 432$$

$$5X_1 + 4X_2 + S_2 = 412.8$$

$$4X_1 + 6X_2 + S_3 = 422.4$$

$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

Z represents the total used time in the three stations in minutes.

Solution: $Z = 1241.28$ minutes
 $X_1 = 45.12$ units, $X_2 = 40.32$ units

Utilization = $\frac{1241.28}{1267.20} \times 100 = 97.95\%$

(b) From TORA,

$$Z + 1.7S_1 + 0S_2 + 1.2S_3 = 1241.28$$

Conditions:

$$X_1 = .3D_1 - .2D_2 + 45.12 \geq 0$$

$$S_2 = -.7D_1 + D_2 - .2D_3 + 25.92 \geq 0$$

$$X_2 = -.2D_1 + .3D_2 + 40.32 \geq 0$$

Station Dual Price RHS Range

Station	Dual Price	RHS Range
1	1.7	281.6, 469.03
2	0	386.88, ∞
3	1.2	288, 552

1% decrease in maintenance time is equivalent to $D_1 = D_2 = D_3 = 4.8$ minutes. This is equivalent to having Daily minutes

Station	Daily minutes
1	436.8
2	417.6
3	427.2

All three daily minutes fall within the allowable ranges. Thus

Station	Increase in utilized time/day
1	$4.8 \times 1.7 = 8.16$ minutes
2	$4.8 \times 0 = 0$
3	$4.8 \times 1.2 = 5.76$

(c) $D_1 = .9(600 - 480) = 108$ min

$$D_2 = .86(600 - 480) = 103.2$$

$$D_3 = .88(600 - 480) = 105.6$$

From the conditions in (b)

$$X_1 = .3 \times 108 - .2 \times 105.6 + 45.12 = 56.4$$

$$S_2 = -.7 \times 108 + 103.2 - .2 \times 105.6 + 25.92 = 32.4$$

$$X_2 = -.2 \times 108 + .3 \times 105.6 + 40.32 = 50.4$$

Solution is feasible. Hence dual prices remain applicable and the net utilization is increased by $.1.7 \times 108 + 0 \times 103.2 + 1.2 \times 105.6 = 310.32$ minutes. Because station 2 has zero dual price, its capacity need not be increased. The associated cost thus equals $1.5(600 - 480) + 0 + 1.5(600 - 480) = \360 .

The proposal can be improved by recommending that station 2 time remain unchanged.

Set 3.6c

$$\begin{aligned} X_1 &= \text{Nbr. purses/day} \\ X_2 &= \text{Nbr. bags/day} \\ X_3 &= \text{Nbr. backpacks/day} \end{aligned}$$

$$\begin{aligned} \text{Maximize } Z &= 24X_1 + 22X_2 + 45X_3 \\ \text{s.t.} \end{aligned}$$

$$2X_1 + X_2 + 3X_3 \leq 42$$

$$2X_1 + X_2 + 2X_3 \leq 40$$

$$X_1 + .5X_2 + X_3 \leq 45$$

$$X_1, X_2, X_3 \geq 0$$

Solution: $Z = \$882, X_1 = 0, X_2 = 2, X_3 = 36$

Letting S_1, S_2, S_3 be the slacks in constraints 1, 2, and 3, we get

$$Z + 20X_1 + S_1 + 21S_2 + 0S_3 = 882$$

Conditions:

$$X_3 = 2 + D_1 - D_2 \geq 0$$

$$X_2 = 36 - 2D_1 + 3D_2 \geq 0$$

$$S_3 = 25 - .5D_2 + D_3 \geq 0$$

Resource	Dual price	RHS Range
Leather	1	(40, 60)
Sewing	21	(28, 42)
Finishing	0	(20, ∞)

(a) Available leather = 45 ft² falls in the RHS range. Solution remains feasible.

$$D_1 = 45 - 42 = 3. \text{ New solution:}$$

$$X_1 = 0$$

$$X_2 = 36 - 2X_3 = 30$$

$$X_3 = 2 + 3 = 5$$

$$Z = 882 + 1 \times D_1 = 882 + 1 \times 3 = \$885$$

(b) Available leather = 41 ft² falls in the RHS range and the solution remains feasible. $D_1 = 41 - 42 = -1$

$$X_2 = 36 - (2X_3 - 1) = 38$$

$$X_3 = 2 - 1 = 1$$

$$Z = 882 + (1 \times -1) = \$881$$

(c) Sewing hours = 38 falls within the RHS range. $D_2 = 38 - 40 = -2$. Dual price = 21

$$X_2 = 36 + 3X_3 - 2 = 30$$

$$X_3 = 2 - (-2) = 4$$

$$Z = 882 + (21 \times -2) = \$840$$

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(d) Sewing hours = 46 hours falls outside the RHS range. Thus, the current optimum basic solution is infeasible. To obtain the new solution, either solve the problem anew or use the algorithms in chapter 4.

(e) Finishing hours = 15, which falls outside the RHS range. Hence, resolve the problem.

(f) Sewing hours = 50, which falls in the RHS range. $D_3 = 50 - 45 = 5$. Solution remains unchanged because dual price is zero and D_3 does not appear in the expression for X_2 or X_3 .

(g) Dual price = \$21/hr, which is higher than the cost of an additional worker per hour. Hiring is recommended.

$$X_1 = \text{Nbr. model 1 units}$$

$$X_2 = \text{Nbr. model 2 units}$$

$$\text{Maximize } Z = 3X_1 + 4X_2$$

s.t.

$$2X_1 + 3X_2 \leq 1200$$

$$2X_1 + X_2 \leq 1000$$

$$4X_2 \leq 800$$

$$X_1, X_2 \geq 0$$

Solution: $Z = \$1750$

$$X_1 = 450, X_2 = 100$$

(a) $S_1 = 0 \Rightarrow$ Resistors scarce

$S_2 = 0 \Rightarrow$ Capacitors scarce

$S_3 = 400 \Rightarrow$ chips abundant

$$(b) Z + \frac{5}{4}S_1 + \frac{1}{4}S_2 = 1750$$

Resource	Dual price
Resistors	\$1.25/resistor
Capacitors	\$0.25/capacitor
Chips	\$0/chip

(c) Conditions:

$$X_1 = 450 - \frac{1}{4}D_1 + \frac{3}{4}D_2 \geq 0$$

$$S_3 = 400 - 2D_1 + 2D_2 + D_3 \geq 0$$

$$X_2 = 100 + \frac{1}{2}D_1 - \frac{1}{2}D_2 \geq 0$$

Feasibility ranges:

$$\begin{cases} 450 - .25D_1 \geq 0 \\ 400 - 2D_1 \geq 0 \\ 100 + .5D_1 \geq 0 \end{cases} \Rightarrow -200 \leq D_1 \leq 200$$

continued...

continued...

Set 3.6c

$$\begin{aligned} 450 + .75D_2 &\geq 0 \\ 400 + 2D_2 &\geq 0 \\ 100 - .5D_2 &\geq 0 \end{aligned} \Rightarrow -200 \leq D_2 \leq 200$$

$$400 + D_3 \geq 0 \Rightarrow -400 \leq D_3 < \infty$$

(d) $D_1 = 1300 - 1200 = 100$ in the allowable range $-200 \leq D_1 \leq 200$.

$$\Delta Z = 100 \times 1.25 = \$125$$

$$X_1 = 450 - .25 \times 100 = 425$$

$$X_2 = 100 + .5 \times 100 = 150$$

$$\text{New } Z = 1750 + \Delta Z = \$1875$$

(e) $D_3 = 350 - 800 = -450$, which falls outside allowable range $-400 \leq D_3$.

Thus, basic solution and dual price change and the problem must be solved anew.

(f) $-200 \leq D_2 \leq 200$, dual price = .25.

Thus,

$$-200 \times .25 \leq \Delta Z \leq 200 \times .25$$

$$-50 \leq \Delta Z \leq 50$$

$$\$1700 \leq Z \leq \$1800$$

$$450 - .75 \times 200 \leq X_1 \leq 450 + .75 \times 200$$

$$100 - \frac{1}{2}(-200) \leq X_2 \leq 100 - \frac{1}{2}(+200)$$

(g) Cost of purchasing 500 additional resistors = $500 \times .40 = \$200$

$$D_1 = 500 \text{ resistors}$$

Dual price of \$1.25 is valid in $-200 \leq D_1 \leq 200$. Thus, for the first 200 resistors alone, H/D/C will get an additional revenue of $200 \times 1.25 = \$250$, which is more than the cost of all 500 resistors. Accept.

From Example 3.6-2, we have for the TOYCO model

$$\begin{aligned} -200 \leq D_1 &\leq 10 \\ -20 \leq D_2 &\leq 400 \\ -20 \leq D_3 &< \infty \end{aligned}$$

(g) $D_1 = 8, D_2 = 40, D_3 = -10$

All $D_i, i=1, 2, 3$ fall within the feasibility ranges. Thus

continued...

$$r_1 = \frac{8}{10}, r_2 = \frac{40}{400}, r_3 = \frac{-10}{-20}$$

$$r_1 + r_2 + r_3 = -8 + .1 + .5 = 1.4 > 1$$

Hence, no conclusion can be made about the feasibility of the new RHS (438, 500, 410). Problem 1(a) shows that these new values do produce a feasible solution.

(b) $D_1 = 30, D_2 = -20, D_3 = -40$. Because D_1 and D_3 fall outside the given feasibility ranges, the 100% rule cannot be applied in this case.

(a) From TORA,

$$x_1 = 2 + \frac{2}{3}D_1 + \frac{1}{3}D_2 \geq 0$$

$$x_2 = 2 - \frac{1}{3}D_1 + \frac{2}{3}D_2 \geq 0$$

Feasibility ranges:

$$-3 \leq D_1 \leq 6$$

$$-3 \leq D_2 \leq 6$$

(b) $D_1 = D_2 = \Delta > 0$. Thus

$$x_1 = 2 + \Delta/3 > 0 \quad \text{for all } \Delta > 0$$

$$x_2 = 2 + \Delta/3 > 0$$

100% rule for $0 < \Delta \leq 3$:

$$r_1 = r_2 = \frac{\Delta}{6} \leq \frac{3}{6} \Rightarrow r_1 + r_2 \leq 1, \text{ which confirms feasibility for } 0 < \Delta \leq 3$$

100% rule for $3 < \Delta \leq 6$:

$$r_1 = r_2 = \frac{\Delta}{6} \Rightarrow \frac{3}{6} \leq r_1, r_2 \leq \frac{6}{6}$$

$r_1 + r_2 \geq 1 \Rightarrow$ cannot confirm feasibility.

100% rule for $\Delta > 6$:

Δ is outside $-3 \leq D_1, D_2 \leq 6$. Thus, the rule is not applicable.

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Set 3.6d

From Section 3.6.3, we have the following optimality conditions for the TOYCO model:

$$x_1: 4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 \geq 0$$

$$x_4: 1 + \frac{1}{2}d_2 \geq 0$$

$$x_5: 2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 \geq 0$$

$$(i) Z = 2x_1 + x_2 + 4x_3$$

$$d_1 = 2-3 = -1, d_2 = 1-2 = -1, d_3 = 4-5 = -1$$

$$x_1: 4 - \frac{1}{4}(-1) + \frac{3}{2}(-1) - (-1) = 3.75 > 0$$

$$x_4: 1 + \frac{1}{2}(-1) = -0.5 > 0$$

$$x_5: 2 - \frac{1}{4}(-1) + \frac{1}{2}(-1) = 1.75 > 0$$

Conclusion: Solution is unchanged

$$(ii) Z = 3x_1 + 6x_2 + x_3$$

$$d_1 = 3-3 = 0, d_2 = 6-2 = 4, d_3 = 1-5 = -4$$

$$x_1: 4 - \frac{1}{4}(4) + \frac{3}{2}(4) - (0) = -3 < 0$$

Conclusion: Solution changes

$$(iii) Z = 8x_1 + 3x_2 + 9x_3$$

$$d_1 = 8-3 = 5, d_2 = 3-2 = 1, d_3 = 9-5 = 4$$

$$x_1: 4 - \frac{1}{4}(1) + \frac{3}{2}(4) - (5) = 4.75 > 0$$

$$x_4: 1 + \frac{1}{2}(1) = 1.5 > 0$$

$$x_5: 2 - \frac{1}{4}(1) + \frac{1}{2}(4) = 3.75 > 0$$

Conclusion: Solution is unchanged

x_1 = Nbr. cans of A1

x_2 = Nbr. cans of A2

x_3 = Nbr. cans of BK

$$\text{Maximize } Z = 80x_1 + 70x_2 + 60x_3$$

$$\text{s.t. } x_1 + x_2 + x_3 \leq 500 \quad \leftarrow S_1$$

$$x_1 \geq 100 \quad \leftarrow S_2$$

$$4x_1 - 2x_2 - 2x_3 \leq 0 \quad \leftarrow S_3$$

$$x_1, x_2, x_3 \geq 0$$

TORA optimum tableau:

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Solution
Z	0	0	10	73.33	0	1.67	3666.67
x_2	0	1	1	-67	0	-17	333.33
x_1	1	0	0	.33	0	.17	166.67
S_2	0	0	0	.33	1	.17	66.67

continued...

(a) $Z = \$366.67$

$$x_1 = 166.67, x_2 = 333.33, x_3 = 0$$

(b) Reduced cost for $x_3 = 10$ cents. Price should be increased by more than 10 cents/can

(c) $d_1 = d_2 = d_3 = -5$ cents

From the optimum tableau, reduced costs:

$$x_3: 10 + d_2 - d_3 = 10 - 5 - (-5) = 10 > 0$$

$$S_1: 73.33 + .67d_2 + .33d_3$$

$$= 73.33 + .67(-5) + .33(-5) = 68.33 > 0$$

$$S_3: 1.67 - .17d_2 + .17d_3 = 1.67 - .17(-5) + .17(-5) = 1.67 > 0$$

Conclusion: Solution is unchanged.

(d) Available carpenter hours in a 10-day period = $4 \times 10 \times 8 = 320$

x_1 = Nbr. chairs assembled in 10 days

x_2 = Nbr. tables assembled in 10 days

$$\text{Maximize } Z = 50x_1 + 135x_2$$

s.t.

$$5x_1 + 2x_2 \leq 320$$

$$4 \leq \frac{x_1}{x_2} \leq 6 \Rightarrow \begin{cases} x_1 - 4x_2 \geq 0 \\ x_1 - 6x_2 \leq 0 \end{cases}$$

$$x_1, x_2 \geq 0$$

$$\text{Solution: } Z = \$27,840, x_1 = 384, x_2 = 64$$

(e) Optimum tableau:

	x_1	x_2	S_1	S_2	S_3	Solution
Z	0	0	87	0	6.5	27840
x_2	0	1	-2	0	-1	64
x_1	1	0	1.2	0	.4	384
S_2	0	0	.4	1	.8	128

Optimality conditions:

$$S_1: 87 + 1.2d_1 + 2d_2 \geq 0$$

$$S_3: 6.5 + 4d_1 - 1d_2 \geq 0$$

For $d_1 = -5, d_2 = -13.5$:

$$S_1: 87 + 1.2(-5) + 2(-13.5) = 78.3 > 0$$

$$S_3: 6.5 + 4(-5) - 1(-13.5) = 5.85 > 0$$

Solution remains the same

$$(f) d_1 = 25 - 50 = -25, d_2 = 120 - 135 = -15$$

$$S_1: 87 + 1.2(-25) + 2(-15) = 58.5 > 0$$

$$S_3: 6.5 + 4(-25) - 1(-15) = -2 < 0$$

Solution changes

Set 3.6d

(a) x_1 = Amt. of personal loan (\$) x_2 = Amt. of car loan (\$)	4
Maximize $Z = .14(x_1 - .03x_1) + .12(x_2 - .02x_2)$ $= .1058x_1 + .0976x_2$	
S.t.	
$x_1 + x_2 \leq 200,000$	
$\frac{x_2}{x_1} \geq 2$ or $2x_1 - x_2 \leq 0$	
$x_1, x_2 \geq 0$	
Solution: $Z = \$20,067$ $x_1 = \$66,667, x_2 = \$133,333$ Rate of return = $\frac{20,067}{200,000} \times 100 = 10.03\%$	

(b) Optimum tableau:

x_1	x_2	s_1	s_2	Solution
0	0	.1003	.0027	200666.67
0	1	.6667	-.3333	133333.33
1	0	.3333	.3333	66666.67

Optimality conditions:

$$S_1: .1003 + .333d_1 + .6667d_2 \geq 0$$

$$S_2: .0027 + .3333d_1 - .3333d_2 \geq 0$$

$$\text{New } x_1 \text{-objective coefficient} = .14(1 - .04) - .04 \\ = .0944$$

$$\text{New } x_2 \text{-objective coefficient} = .12(1 - .03) - .03 \\ = .0864$$

$$d_1 = .0944 - .1058 = -.0114$$

$$d_2 = .0864 - .0976 = -.0112$$

$$S_1: .1003 + .3333(-.0114) + .6667(-.0112) \\ = .08907 > 0$$

$$S_2: .0027 + .3333(-.0114) - .3333(-.0112) \\ = .00267 > 0$$

Solution does not change

(a) x_i = Nbr of units of motor $i, i=1, 2, 3, 4$	5
Maximize $Z = 60x_1 + 40x_2 + 25x_3 + 30x_4$	
S.t.	
$8x_1 + 5x_2 + 4x_3 + 6x_4 \leq 8000$	
$x_1 \leq 500, x_2 \leq 500, x_3 \leq 800, x_4 \leq 750$	
$x_1, x_2, x_3, x_4 \geq 0$	

$$\text{Solution: } Z = \$59,375, x_1 = 500, x_2 = 500, x_3 = 375, x_4 = 0$$

(b) Optimality conditions (from TORA):

$$x_4: 7.5 + 1.5d_3 - d_4 \geq 0$$

$$S_1: 6.25 + .25d_3 \geq 0$$

$$S_2: 10 - 2d_3 + d_1 \geq 0$$

$$S_3: 8.75 - 1.25d_3 + d_2 \geq 0$$

$$\text{From } S_3, 8.75 + d_2 \geq 0 \Rightarrow -8.75 \leq d_2 < \infty$$

Thus, price of type 2 motor can be reduced by at most \$8.75 without causing a solution change.

$$(c) d_1 = -15, d_2 = -10, d_3 = -6.25, d_4 = -7.5$$

Solution remains the same because

$$x_4: 7.5 + 1.5(-6.25) - (-7.5) = 5.625 > 0$$

$$S_1: 6.25 + .25(-6.25) = 4.6875 > 0$$

$$S_2: 10 - 2(-6.25) + (-15) = 7.5 > 0$$

$$S_3: 8.75 - 1.25(-6.25) + (-10) = 6.5625 > 0$$

(d) Reduced cost for $x_4 = 7.5$. Increase price of type 4 motor by more than \$7.50.

$$(a) x_1 = Cases of juice/day$$

$$x_2 = Cases of sauce/day$$

$$x_3 = Cases of pasta/day$$

$$\text{Maximize } Z = 21x_1 + 9x_2 + 12x_3$$

S.t.

$$(1 \times 24)x_1 + (\frac{1}{2} \times 24)x_2 + (\frac{3}{4} \times 24)x_3 \leq 60,000$$

$$x_1 \leq 2000, x_2 \leq 5000, x_3 \leq 6000$$

$$x_1, x_2, x_3 \geq 0$$

$$\text{Solution: } Z = \$51,000$$

$$x_1 = 2000, x_2 = 1000, x_3 = 0$$

(b) From TORA, optimality conditions given d_2 :

$$x_3: 1.5 + 1.5d_2 \geq 0 \Rightarrow d_2 \geq -1$$

$$S_1: .75 + .083d_2 \geq 0 \Rightarrow d_2 \geq -9$$

$$S_2: 3 - 2d_2 \geq 0 \Rightarrow d_2 \leq 1.5$$

Thus, $-1 \leq d_2 \leq 1.5$, or

$$9-1 \leq \text{price/case of sauce} \leq 9+1.5$$

Solution mix remains the same if the price per case of sauce remains between \$8 and \$10.50.

continued...

Set 3.6e

See file solver 3.6e-1.xls in ch3Files

Dual prices for years 1, 2, 3, and 4 are 0, 0, 0, 2.89. Thus, for year 4, one thousand additional dollars increases Z by \$2.89 thousand. It is worthwhile to increase the funding for year 4.

See file tora 3.6e-2.txt

Constraint	Dual Price	Range
1	5.36	(0, ∞)
2	-3.73	(-∞, 6000)
3	-1.13	(-∞, 6800)
4	-1.07	(-∞, 33642)
5	-1.00	(-∞, 53628.73)

2

(a) Constraint 1: $x_1 + x_2 + x_3 + y_1 \leq 10,000$

Dual price = \$5.36 / invested \$

Rate of return = 536%

(b) Constraint 2: \$1000 spent on pleasure

$$.5x_1 + .6x_2 - x_3 + .4x_4 + 1.065y_2 - y_1 = 1000$$

Dual price = -3.73 / pleasure \$

Range = (-∞, 6000)

Spending \$1000 at end of year 1 reduces total return by \$3730.

See file tora 3.6e-3.txt in ch3Files

Quarter	Dual price	Range
1	1.2488	.6647, 2.5806
2	1.2443	.6580, 2.6122
3	1.1945	-.2646, 1.1245
4	1.0200	-.2553, 00
5	1.0000	-4.8366, 00

3

(a) An additional \$ available at the start of quarter 1 is worth \$1.2488 at the end of 4 quarters. Similarly, an additional dollar at the start of periods 2, 3, and 4 is worth \$1.2443, \$1.1945, and \$1.02, respectively. The dual price for quarter 4 (= \$1.02) shows that all we can do with the money then is to invest it at 2% for the quarter.

We can use the dual price to determine

continued...

the rate of return for each quarter — namely,

quarter 1:

$$1.2488 = 1.2243(1+i_1) \Rightarrow i_1 = .02$$

quarter 2:

$$1.2243 = 1.1945(1+i_2) \Rightarrow i_2 = .025$$

quarter 3:

$$1.1945 = 1.02(1+i_3) \Rightarrow i_3 = .171$$

quarter 4:

$$1.02 = 1.0(1+i_4) \Rightarrow i_4 = .02$$

(b) The dual price associated with the upper bound on B_3 (UB-X10) is \$.149. It represents the networth per dollar borrowed in period 3. Also, an extra dollar in period 3 is worth \$.11945 at the end of the horizon. However, if that dollar is borrowed, it must be repaid as \$1.025 in the next quarter. The repayment is equivalent to forgoing making 2% in interest. Thus, the networth of borrowing in period 3 is

$$1.1945 - 1.025 \times 1.02 = .149$$

This result is consistent with the dual price for the upper bound on B_3 .

Constraint	Current RHS	Min RHS	Max RHS	Dual Price
1 (=)	2.0000	0.0000	infinity	2.1756
2 (=)	2.0000	-0.1667	infinity	2.0173
3 (=)	2.5000	-0.3472	infinity	1.8647
4 (=)	2.5000	-0.5767	infinity	1.7296
5 (=)	3.0000	-0.8248	infinity	1.6044
6 (=)	3.5000	-1.1331	infinity	1.4356
7 (=)	3.5000	-6.1137	infinity	1.3355
8 (=)	4.0000	-11.4678	infinity	1.2423
9 (=)	6.0000	-20.6663	infinity	1.1556
10 (=)	9.0000	-32.5201	infinity	1.0759

4

The dual price provides the worth per additional \$ at the end of year 10.

Annual rate of return:

$$\text{Period 1: } 2.1756 = 2.0173(1+i_1) \Rightarrow i_1 = .0785$$

$$\text{Period 2: } 2.0173 = 1.8647(1+i_2) \Rightarrow i_2 = .0818$$

$$\text{Period 3: } 1.8647 = 1.7296(1+i_3) \Rightarrow i_3 = .0781$$

$$\text{Period 4: } 1.7296 = 1.6044(1+i_4) \Rightarrow i_4 = .0780$$

etc...

Set 3.6e

See file `tora3.6e-5.txt` in ch3files

The dual price for constraint 1

$$x_{1A} + x_{1B} \leq 100,000$$

is \$5.10. Thus, each invested \$ is worth \$5.10 at the end of the investment horizon. Range (0, ∞)

5

See file `tora3.6e-9.txt` in ch3files

(a) Constraint $2x_1 + 3x_2 + 5x_3 \leq 4000$

corresponds to raw material A. Its dual price is \$10.27/lb. For a purchase price of \$12/lb, acquisition of additional raw material A is not recommended.

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(b) Constraint $4x_1 + 2x_2 + 7x_3 \leq 6000$

is associated with raw material B. Its dual price is \$0/lb. Resource B is already abundant. Thus, no additional purchase is recommended.

Dual price for the constraint

$$x_1 + x_2 + x_3 + x_4 \leq 500$$

is \$2.35 per \$ invested, range (0, ∞)

The gambler should bet the largest amount possible.

6

See file `tora3.6e-7.txt` in ch3files

7

For, $x_{W1} + x_{W2} + x_{W3} \geq 1500$, the dual price is \$11.4, range (800, ∞)

One extra wrench automatically requires the production of two chisels, thus leading to the following changes:

Cost of one wrench using subcont. = \$3.00
Cost of 2 chisels using subcont. = $2 \times \$4.20$
total = \$11.40

$x_{W1} \leq 550$, dual price = -\$1, range (- ∞ , 1250). If regular time capacity for wrenches is increased by 1 unit, one less wrench will be produced by subcontractor, which saves $\$3 - \$2 = \$1$.

Similar interpretations can be given for the remaining dual prices

(a) See file `tora3.6e-10.txt`

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Constraint	Dual price
1	0
2	0
3	-400
4	-750
5	0
6	0
7	0

Constraints 3 and 4 have negative dual price. These correspond respectively to the third specification for alloy A and the first specification for alloy B. Changes in these specifications affects profit adversely.

(b) For the ore constraints, the dual prices are \$90, \$110, and \$30 per additional ton of ores 1, 2, and 3, respectively. These are the maximum prices the company should pay.

See file `tora3.6e-8.txt` in ch3files

8

Machine Capacity Dual price Range

1	500	2	(253.33, 570)
2	380	12	(333.33, 750)

The company should pay less than \$2/hr for machine 1 and less than \$12/hr for machine 2.

CHAPTER 4

Duality and Post-Optimal Analysis

Set 4.1a

Primal:

$$\begin{aligned} \text{Minimize } Z &= 5x_1 + 12x_2 + 4x_3 \\ \text{subject to} \\ x_1 + 2x_2 + x_3 + s_1 &= 10 \\ 2x_1 - x_2 + 3x_3 &= 8 \\ x_1, x_2, x_3, s_1 &\geq 0 \end{aligned}$$

Dual:

$$\begin{aligned} \text{Maximize } w &= 10y_1 + 8y_2 \\ \text{subject to} \\ y_1 + 2y_2 &\leq 5 \\ 2y_1 - y_2 &\leq 12 \\ y_1 + 3y_2 &\leq 4 \\ y_1 &\leq 0 \\ y_2 &\text{ unrestricted} \end{aligned}$$

Primal:

$$\begin{aligned} \text{Minimize } Z &= 15x_1 + 12x_2 \\ \text{subject to} \\ x_1 + 2x_2 - x_3 &= 3 \\ 2x_1 - 4x_2 + x_4 &= 5 \\ 3x_1 + x_2 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Dual:

$$\begin{aligned} \text{Maximize } Z &= 3y_1 + 5y_2 + 4y_3 \\ \text{subject to} \\ y_1 + 2y_2 + 3y_3 &\leq 15 \\ 2y_1 - 4y_2 + y_3 &\leq 12 \\ -y_1 - y_2 &\leq 0 \Rightarrow y_1 \geq 0 \\ y_3 &\text{ unrestricted} \end{aligned}$$

Primal:

$$\begin{aligned} \text{Minimize } Z &= 5x_1^+ - 5x_1^- + 6x_2 \\ \text{subject to} \\ x_1^+ - x_1^- + 2x_2 &= 5 \\ -x_1^+ + x_1^- + 5x_2 - x_3 &= 3 \\ 4x_1^+ - 4x_1^- + 7x_2 + x_4 &= 8 \\ x_1^+, x_1^-, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Dual:

$$\begin{aligned} \text{Maximize } Z &= 5y_1 + 3y_2 + 8y_3 \\ \text{subject to} \\ y_1 - y_2 + 4y_3 &\leq 5 \\ -y_1 + y_2 - 4y_3 &\leq -5 \Rightarrow y_1 - y_2 + 4y_3 = 5 \\ 2y_1 + 5y_2 + 7y_3 &\leq 6 \\ -y_2 - y_3 &\leq 0 \Rightarrow y_2 \geq 0 \\ y_3 &\leq 0 \\ y_1 &\text{ unrestricted} \end{aligned}$$

(a) Primal:

$$\begin{aligned} \text{Maximize } Z &= -5x_1 + 2x_2 \\ \text{s.t.} \\ x_1 - x_2 - x_3 &= 2 \\ 2x_1 + 3x_2 + x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Dual:

$$\begin{aligned} \text{Minimize } w &= 2y_1 + 5y_2 \\ \text{subject to} \\ y_1 + 2y_2 &\geq -5 \\ -y_1 + 3y_2 &\geq 2 \\ -y_1 &\geq 0 \Rightarrow y_1 \leq 0 \\ y_2 &\geq 0 \end{aligned}$$

(b) Primal:

$$\begin{aligned} \text{Minimize } Z &= 6x_1 + 3x_2 \\ \text{subject to} \\ 6x_1 - 3x_2 + x_3 - x_4 &= 2 \\ 3x_1 + 4x_2 + x_3 - x_5 &= 5 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Dual:

$$\begin{aligned} \text{Maximize } w &= 2y_1 + 5y_2 \\ \text{subject to} \\ 6y_1 + 3y_2 &\leq 6 \\ -3y_1 + 4y_2 &\leq 3 \\ y_1 + y_2 &\leq 0 \\ -y_1 - y_2 &\leq 0 \} \Rightarrow y_1, y_2 \geq 0 \end{aligned}$$

(c) Primal:

$$\begin{aligned} \text{Maximize } Z &= x_1 + x_2 \\ \text{subject to} \\ 2x_1 + x_2 &= 5 \\ 3x_1 - x_2 &= 6 \\ x_1, x_2 &\text{ unrestricted} \end{aligned}$$

Dual:

$$\begin{aligned} \text{Minimize } w &= 5y_1 + 6y_2 \\ \text{subject to} \\ 2y_1 + 3y_2 &= 1 \\ y_1 - y_2 &= 1 \\ y_1, y_2 &\text{ unrestricted} \end{aligned}$$

Primal:

$$\text{Maximize } Z = 5x_1 + 12x_2 + 4x_3 - MR_2$$

$$x_1 + 2x_2 + x_3 + S_1 = 10$$

$$2x_1 - x_2 + 3x_3 + R_2 = 8$$

$$x_1, x_2, x_3, S_1, R_2 \geq 0$$

Dual

$$\text{Minimize } w = 10y_1 + 8y_2$$

Subject to

$$y_1 + 2y_2 \geq 5$$

$$2y_1 - y_2 \geq 12$$

$$y_1 + 3y_2 \geq 4$$

$$y_1 \geq 0$$

$$y_2 \geq -M$$

y_2 unrestricted } same

All parts, (a) through (e),
are true

5

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(1) max + (\geq constraints):

$$\sum a_{ij}x_j - S_i = b_i \Rightarrow -y_i \geq 0 \Rightarrow y_i \leq 0$$

(2) min + (\geq constraints):

$$\sum a_{ij}x_j - S_i = b_i \Rightarrow -y_i \leq 0 \Rightarrow y_i \geq 0$$

(3) max + (\leq constraints):

$$\sum a_{ij}x_j + S_i = b_i \Rightarrow y_i \geq 0$$

(4) min + (\leq constraints):

$$\sum a_{ij}x_j + S_i = b_i \Rightarrow y_i \leq 0$$

(5) max or min + (= constraint)

$$\sum a_{ij}x_j = b_i \Rightarrow y_i \text{ unrestricted}$$

(6) max + ($x_j \geq 0$):

$$\boxed{\frac{c_j \cdot x_j}{a_{ij}x_j}} \Rightarrow \sum_{i=1}^m a_{ij}y_i \geq c_j$$

(7) max + ($x_j \leq 0$):

$$\text{Let } x_j = -x_j' \text{, } x_j' \geq 0$$

$$\boxed{-c_j \cdot x_j' - a_{ij}x_j'} \Rightarrow -\sum_{i=1}^m a_{ij}y_i \geq -c_j \\ \Rightarrow \sum_{i=1}^m a_{ij}y_i \leq c_j$$

(8) min + ($x_j \geq 0$):

$$\boxed{\frac{c_j \cdot x_j}{a_{ij}x_j}} \Rightarrow \sum_{i=1}^m a_{ij}y_i \leq c_j$$

(9) min + ($x_j \leq 0$):

$$\text{Let } x_j = -x_j' \text{, } x_j' \geq 0$$

$$\boxed{-c_j \cdot x_j' - a_{ij}x_j'} \Rightarrow -\sum_{i=1}^m a_{ij}y_i \leq -c_j \\ \Rightarrow \sum_{i=1}^m a_{ij}y_i \geq c_j$$

(10) max or min + (x_j unrestricted)

$$\boxed{\frac{c_j \cdot x_j}{a_{ij}x_j}} \Rightarrow \sum_{i=1}^m a_{ij}y_i = c_j$$

Set 4.2a

$$(a) A_{3 \times 2} V_1^T_{1 \times 2} \text{ undefined}$$

$$(b) AP_1 = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \\ 15 \end{pmatrix}_{3 \times 1}$$

$$(c) A P_2^T_{3 \times 2} \text{ undefined}$$

$$(d) V_1^T A_{1 \times 2} \text{ undefined}$$

$$(e) V_2^T A = (-1, -2, -3) \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \\ = (-14, -32)_{1 \times 2}$$

$$(f) P_1 P_2^T_{2 \times 1} \text{ undefined}$$

$$(g) V_1^T P_1^T_{1 \times 2} = (11, 22) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ = 55_{1 \times 1}$$

Set 4.2b

(a)

$$\text{inverse} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 24 \\ 6 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3/2 \\ 5/2 \\ 1/2 \end{pmatrix}$$

(a)

$$\text{inverse} = \begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

2

Set 4.2c

Dual: Maximize $w = 50y$

s.t.

$$5y_1 \leq 10, -7y_1 \leq 4, 3y_1 \leq 5, y \geq 0$$

The constraints simplify to

$$0 \leq y_1 \leq 5/3$$

$$\text{Thus, } \max w = 50 \times \frac{5}{3} = \frac{250}{3} = \min z$$

Dual:

$$\text{Maximize } w = 50y_1 + 20y_2 + 30y_3 + 35y_4 + 10y_5 + 90y_6 + 20y_7$$

s.t.

$$5y_1 + y_2 + 7y_3 + 5y_4 + 2y_5 + 12y_6 \leq 5$$

$$5y_1 + y_2 + 6y_3 + 5y_4 + 4y_5 + 10y_6 + y_7 \leq 6$$

$$3y_1 - y_2 - 9y_3 + 5y_4 - 15y_5 - 10y_7 \leq 3$$

$$-y_j \leq 0 \Rightarrow y_j \geq 0, j = 1, 2, \dots, 7$$

From TORA, optimal objective equation is

$$Z + 50y_1 + 0y_2 + 90y_3 + 65y_4 + 70y_5 + 110y_6 + 0y_7 + 0S_1 + 20S_2 + 0S_3 = 120$$

(S_1, S_2, S_3) are slack variables.

$$\text{Thus, } x_1 = 0, x_2 = 20, x_3 = 0$$

Obtaining the solution from the dual is advantageous computationally because the dual has a smaller number of constraints.

Dual: Minimize $w = 30y_1 + 40y_2$

s.t.

$$y_1 + y_2 \geq 5$$

$$5y_1 - 5y_2 \geq 2$$

$$2y_1 - 6y_2 \geq 3$$

$$y_2 \geq 0, y \text{ unrestricted}$$

$$\text{Method 1: } Z + 0x_1 + 23x_2 + 7x_3 + 105x_4 + 0x_5 = 150$$

$$\text{Coefficient of } x_4 = 105 \Rightarrow y_1 = 105 + (-100) = 5$$

$$\text{Coefficient of } x_5 = 0 \Rightarrow y_2 = 0$$

Method 2:

$$(y_1, y_2) = (5, 0) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= (5, 0)$$

$$w = 30 \times 5 + 40 \times 0 = 150$$

Dual: Maximize $w = 3y_1 + 6y_2 + 4y_3$

s.t.

$$3y_1 + 4y_2 + y_3 \leq 4$$

$$y_1 + 3y_2 + 2y_3 \leq 1$$

$$-y_2 \leq 0 \Rightarrow y_2 \geq 0$$

$$y_3 \leq 0$$

y_j unrestricted

$$\text{Method 1: } Z - 98.6x_4 - 100x_5 - 1.2x_6 = 3.4$$

$$\text{Coefficient of } x_4 = -98.6 \Rightarrow y_1 = -98.6 + 100 = 1.4$$

$$\text{Coefficient of } x_5 = -100 \Rightarrow y_2 = -100 + 100 = 0$$

$$\text{Coefficient of } x_6 = -1.2 \Rightarrow y_3 = -1.2$$

Method 2:

$$(y_1, y_2, y_3) = (1.4, 0, 0) \begin{pmatrix} .4 & 0 & -1.2 \\ -2 & 0 & .6 \\ 1 & -1 & 1 \end{pmatrix} = (1.4, 0, -1.2)$$

$$w = 3 \times 1.4 + 6 \times 0 + 4 \times -1.2 = 3.4$$

Dual: Minimize $w = 4y_1 + 8y_2$

s.t.

$$y_1 + y_2 \geq 2$$

$$y_1 + 4y_2 \geq 4$$

$$y_1 \geq 4$$

$$y_2 \geq -3$$

$$\text{Method 1: } Z + 2x_1 + 0x_2 + 0x_3 + 3x_4 = 16$$

$$\text{Coefficient of } x_3 = 0 \Rightarrow y_1 = 0 + 4 = 4$$

$$\text{Coefficient of } x_4 = 3 \Rightarrow y_2 = 3 + (-3) = 0$$

Method 2:

$$(y_1, y_2) = (4, 0) \begin{pmatrix} 1 & -25 \\ 0 & .25 \end{pmatrix} = (4, 0)$$

$$w = 4 \times 4 + 8 \times 0 = 16$$

Dual: Minimize $w = 3y_1 + 4y_2$

s.t.

$$y_1 + 2y_2 \geq 1$$

$$2y_1 - y_2 \geq 5$$

$$y_1 \geq 3, y_2 \text{ unrestricted}$$

$$\text{Method 1: } Z + 2x_2 + 0x_3 + 99x_4 = 5$$

$$\text{Coefficient of } x_3 = 0 \Rightarrow y_1 = 0 + 3 = 3$$

$$\text{Coefficient of } x_4 = 99 \Rightarrow y_2 = 99 + (-100) = -1$$

Method 2:

$$(y_1, y_2) = (3, -1) \begin{pmatrix} 1 & -5 \\ 0 & .5 \end{pmatrix} = (3, -1)$$

$$w = 3 \times 3 + 4 \times (-1) = 5$$

Set 4.2c

Set 4.2d

From TORA using $M = 100$:

	x_1	x_2	x_3	x_4	x_5	
Z	-20.5	88	-30.4	0	0	-800
x_4	1	2	1	1	0	10
x_5	2	-1	3	0	1	8
Z	-7/3	-40/3	0	0	304/3	32/3
x_4	1/3	7/3	0	1	-1/3	22/3
x_3	2/3	-1/3	1	0	1/3	8/3

Primal

Dual

Maximize $Z = 5x_1 + 12x_2 + 4x_3$	Minimize $w = 10y_1 + 8y_2$
s.t.	s.t.
$x_1 + 2x_2 + x_3 \leq 10$	$y_1 + 2y_2 \geq 5$
$2x_1 - x_2 + 3x_3 = 8$	$2y_1 - y_2 \geq 12$
$x_1, x_2, x_3 \geq 0$	$y_1 + 3y_2 \geq 4$
	$y_1 \geq 0$
	y_2 unrestricted

Iteration 1: x_5 artificial, $M = 100$

$$\text{Inverse} = \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix}, C_B = (0, 4)$$

Constraints:

$$\begin{aligned} LHS &= \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/3 & 7/3 & 0 & 1 & -1/3 \end{pmatrix} \\ RHS &= \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 10 \\ 8 \end{pmatrix} = \begin{pmatrix} 22/3 \\ 8/3 \end{pmatrix} \end{aligned}$$

Objective row:

$$\text{Dual values } (y_1, y_2) = (0, 4) \begin{pmatrix} 1 & -1/3 \\ 0 & 1/3 \end{pmatrix} = (0, 4/3)$$

Variable Objective coefficient

$$\begin{aligned} x_1 & y_1 + 2y_2 - 5 = 0 + 2(4/3) - 5 = -7/3 \\ x_2 & 2y_1 - y_2 - 12 = 2(0) - (4/3) - 12 = -40/3 \\ x_3 & y_1 + 3y_2 - 4 = 0 + 3(4/3) - 4 = 0 \\ x_4 & y_1 - 0 = 0 - 0 = 0 \\ x_5 & y_2 - (-M) = 4/3 - (-100) = 304/3 \end{aligned}$$

Dual:

$$\text{Minimize } w = 21y_1 + 21y_2$$

Subject to

$$2y_1 + 7y_2 \geq 4$$

$$7y_1 + 2y_2 \geq 14$$

$$y_1, y_2 \geq 0$$

$$(a) \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/7 & 0 \\ -2/7 & 1 \end{pmatrix} \begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \end{pmatrix} \Rightarrow \text{feasible}$$

$$(y_1, y_2) = (14, 0) \begin{pmatrix} 1/7 & 0 \\ -2/7 & 1 \end{pmatrix} = (2, 0)$$

$$\text{Obj coeff of } x_1 = 2y_1 + 7y_2 - 4$$

$$= 2 \times 2 + 7 \times 0 - 4 = 0$$

$$\text{Obj coeff of } x_3 = y_1 - 0 = 2 - 0 = 2 \Rightarrow \text{optimal}$$

(b) Feasibility:

continued...

2

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 1 & -7/2 \end{pmatrix} \begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 10.5 \\ -105 \end{pmatrix} \Rightarrow \text{infeasible}$$

Optimality:

$$(y_1, y_2) = (14, 0) \begin{pmatrix} 0 & 1/2 \\ 1 & -7/2 \end{pmatrix} = (0, 7)$$

$$\text{Obj coeff of } x_1: 2y_1 + 7y_2 - 4 = 2 \times 0 + 7 \times 7 - 4 = 45 > 0$$

$$\text{Obj coeff of } x_4: y_1 - 0 = 7 - 0 > 0$$

Solution is optimal but infeasible

(c) Feasibility:

$$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7/45 & -2/45 \\ -2/45 & 7/45 \end{pmatrix} \begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 7/3 \\ 7/3 \end{pmatrix} \Rightarrow \text{feasible}$$

Optimality:

$$(y_1, y_2) = (14, 4) \begin{pmatrix} 7/45 & -2/45 \\ -2/45 & 7/45 \end{pmatrix} = (2, 0)$$

$$\text{Obj coeff of } x_3: y_1 - 0 = 2 - 0 > 0 \} \text{ optimal}$$

$$\text{Obj coeff of } x_4: y_2 - 0 = 0 - 0 = 0 \} \text{ optimal}$$

Solution is optimal and feasible

(d) Feasibility:

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -7/2 & 1 \end{pmatrix} \begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 21/2 \\ -105/2 \end{pmatrix} \Rightarrow \text{infeasible}$$

Optimality:

$$(y_1, y_2) = (4, 0) \begin{pmatrix} 1/2 & 0 \\ -7/2 & 1 \end{pmatrix} = (2, 0)$$

$$\text{Obj coeff of } x_2: 7y_1 + 2y_2 - 14 = 0 \} \text{ optimal}$$

$$\text{Obj coeff of } x_3: y_1 - 0 = 2 - 0 = 2 \} \text{ optimal}$$

Solution optimal but infeasible

Dual:

$$\begin{aligned} \text{minimize } w &= 30y_1 + 60y_2 + 20y_3 \\ \text{subject to} \end{aligned}$$

$$y_1 + 3y_2 + y_3 \geq 3$$

$$2y_1 + 4y_3 \geq 2$$

$$y_1 + 2y_2 \geq 5$$

$$y_1, y_2, y_3 \geq 0$$

(a) Feasibility:

$$\begin{pmatrix} x_4 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 30 \\ 60 \\ 20 \end{pmatrix} = \begin{pmatrix} 0 \\ 50 \\ 20 \end{pmatrix} \Rightarrow \text{feasible}$$

Optimality:

$$(y_1, y_2, y_3) = (0, 5/2, 0) \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (0, 5/2, 0)$$

$$\text{Obj coeff of } x_1: y_1 + 3y_2 + y_3 - 3 = 0 + 3(\frac{5}{2}) + 0 - 3 = 9/2$$

$$\text{Obj coeff of } x_2: 2y_1 + 4y_3 - 2 = 2 \times 0 + 4 \times 0 - 2 = -2 < 0$$

Solution feasible but not optimal

continued...

3

4-8

Set 4.2d

b) Feasibility:

$$\begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 30 \\ 60 \\ 20 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 15 \\ 10 \end{pmatrix} \Rightarrow \text{feasible}$$

Optimality:

$$(y_1, y_2, y_3) = (2, 5, 3) \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix} = (5, 0, -2)$$

$$\text{obj. coeff of } x_4 : y_1 - 0 = 5$$

$$\text{obj. coeff of } x_5 : y_2 - 0 = 0$$

$$\text{obj. coeff of } x_6 : y_3 - 0 = -2 \Rightarrow \text{not optimal}$$

c) Feasibility:

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 30 \\ 60 \\ 20 \end{pmatrix} = \begin{pmatrix} 0 \\ 30 \\ 20 \end{pmatrix} \Rightarrow \text{feasible}$$

Optimality:

$$(y_1, y_2, y_3) = (2, 5, 0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} = (1, 2, 0)$$

$$\text{obj. coeff of } x_1 : y_1 + 3y_2 + y_3 - 3 = 1 + 6 + 0 - 3 = 4$$

$$\text{obj. coeff of } x_4 : y_1 - 0 = 1 - 0 = 1 \quad \left. \begin{array}{l} \text{optimal} \\ \text{obj. coeff of } x_5 : y_2 - 0 = 2 - 0 = 2 \end{array} \right\}$$

Constraints:

$$\begin{aligned} LHS &= \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 4 & 3 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/5 & -3/5 & 1/5 & 0 \\ 0/5 & 4/5 & -3/5 & 0 \\ 0/5 & -1 & 1 & 1 \end{pmatrix} \end{aligned}$$

$$RHS = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{Objective coefficients: } & (y_1, y_2, y_3) = (2, 1, 0) \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \\ &= (2/5, 1/5, 0) \end{aligned}$$

$$\text{obj. coeff of } x_3 = -y_1 - 0 = -2/5$$

$$\text{obj. coeff of } x_4 = -y_2 - 0 = -1/5$$

$$Z = 2 \times 2/5 + 1 \times 1/5 = 12/5$$

	x_1	x_2	x_3	x_4	x_5	
Z	0	0	-2/5	-1/5	0	12/5
x_1	1	0	-3/5	1/5	0	3/5
x_2	0	1	4/5	-3/5	0	6/5
x_5	0	0	-1	1	1	0

continued...

$$\begin{aligned} (a) \begin{pmatrix} x_4 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1 & -1/5 \\ 0 & 1/5 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 28/5 \\ 2/5 \end{pmatrix} \\ Z &= 4 \times 2/5 = 8/5 \end{aligned}$$

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$$\begin{aligned} (ii) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 18/5 \\ 14/5 \end{pmatrix} \\ Z &= 5 \times \frac{14}{5} + 12 \times \frac{18}{5} = 57.2 \end{aligned}$$

$$\begin{aligned} (iii) \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 3/7 & -1/7 \\ 1/7 & 2/7 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\ Z &= 12 \times 4 + 4 \times 2 = 56 \end{aligned}$$

Solution in (b) is the best

$$(b) \quad y_1, y_2 = (12, 5) \begin{pmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{pmatrix} = \left(\frac{29}{5}, -\frac{2}{5} \right)$$

$$\text{obj. coeff of } x_3 : y_1 + 3y_2 - 4 = \frac{29}{5} + 3(-\frac{2}{5}) - 4 = \frac{3}{5}$$

$$\text{obj. coeff of } x_4 : y_1 - 0 = \frac{29}{5} - 0 = \frac{29}{5}$$

Solution is optimal.

$$\text{Inverse} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

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$$(a) \begin{pmatrix} x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 10 \end{pmatrix}$$

$$\text{Thus, } b_1 = 30, b_2 = 40$$

(b) Optimal dual solution:

$$(y_1, y_2) = (5, 0) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (5, 0)$$

$$(c) (d, e) = (y_1, y_2) = (5, 0)$$

$$a = 5y_1 - 5y_2 - 2 = 5 \times 5 - 5 \times 0 - 2 = 23$$

$$\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$$

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Objective value:

$$\text{initial} = b_1 y_1 + b_2 y_2 + b_3 y_3$$

$$\text{final primal} = C_1 x_1 + C_2 x_2$$

$$\begin{pmatrix} x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 8 \end{pmatrix}$$

$$\text{Thus, } b_1 = 4, b_2 = 6, b_3 = 8$$

continued...

Set 4.2d

$$(y_1, y_2, y_3) = (0, c_2, c_1) \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \\ = (0, c_2 - c_1, c_1)$$

$$\left. \begin{array}{l} \text{Obj coeff of } x_3 = 0 = y_1 - 0 \\ \text{Obj coeff of } x_4 = 3 = y_2 - 0 \\ \text{Obj coeff of } x_5 = 2 = y_3 - 0 \end{array} \right\} \begin{array}{l} y_1 = 0, y_2 = 3, y_3 = 2 \\ \end{array}$$

Thus, $c_2 - c_1 = 3$ and $c_1 = 2 \Rightarrow c_1 = 2, c_2 = 5$

Now we can determine the objective value as follows:

$$\begin{aligned} \text{Dual} &= b_1 y_1 + b_2 y_2 + b_3 y_3 \\ &= 4x_0 + 6x_3 + 8x_2 = 34 \end{aligned}$$

$$\begin{aligned} \text{Primal} &= C_1 x_1 + C_2 x_2 \\ &= 2x_2 + 5x_6 = 34 \end{aligned}$$

Dual:

$$\text{Minimize } w = 4y_1 + 8y_2$$

Subject to

$$\begin{aligned} y_1 + y_2 &\geq 2 \\ y_1 + 4y_2 &\geq 4 \\ y_1 &\geq 4 \\ y_2 &\geq -3 \end{aligned}$$

For basic (x_1, x_2) , we have

$$\left. \begin{array}{l} y_1 + y_2 - 2 = 0 \\ y_1 + 4y_2 - 4 = 0 \end{array} \right\} \Rightarrow y_1 = \frac{4}{3}, y_2 = \frac{2}{3}$$

$$\text{Obj coeff of } x_3 = y_1 - 4 = \frac{4}{3} - 4 = -\frac{8}{3} < 0$$

The result shows that the solution is not optimal.

For a slack starting basic variable, the dual constraint is of the form

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$$y \geq 0$$

(assuming primal maximization).

Thus,

$$\text{Optimal obj coeff. of basic variable} = y - 0$$

For artificial starting basic variable, the dual constraint is $y \geq -M$ if the primal is max maximization, and $y \leq M$ if the primal is minimization.

Thus,

$$\text{Optimal obj coeff} = \begin{cases} y + M, \text{ for maximization} \\ y - M, \text{ for minimization} \end{cases}$$

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Set 4.3a

From TORA output:

	y_1	y_2	y_3	y_4
Range:	.75	.5	0	0

(20,36) (4,6.7) (-1.5,0) (1.5,0)

$$(a) \# 750 \times (22-24) = - \$1500$$

$$(b) \Delta Z = \$500(4.5-6) = - \$750$$

$$(c) \Delta Z = \$0 (10-2) = \$0$$

x_1, x_2, x_3, x_4 = daily units of cables
320, 325, 340, and 370

(a) Maximize $Z = 9.4x_1 + 10.8x_2 + 8.75x_3 + 7.8x_4$
subject to

$$10.5x_1 + 9.3x_2 + 11.6x_3 + 8.2x_4 \leq 4800$$

$$20.4x_1 + 24.6x_2 + 17.7x_3 + 26.5x_4 \leq 9600$$

$$3.2x_1 + 2.5x_2 + 3.6x_3 + 5.5x_4 \leq 4700$$

$$5x_1 + 5x_2 + 5x_3 + 5x_4 \leq 4500$$

$$x_1 \geq 100, x_2 \geq 100, x_3 \geq 100, x_4 \geq 100$$

2

(b) Only soldering capacity can be increased because its dual price is positive.

(c) The fact that the dual prices of the lower bounds on x_1 , x_2 , and x_4 are negative shows that the lower bounds have adverse effect on profitability. Specifically, one unit decrease in the production of cables SC320, SC325, and SC370 will respectively increase the profit by \$.68, \$1.36, and \$5.30 per cable. These values are valid considering the cables one at a time.

(d) Dual price for soldering is \$.49 per minute, valid in the range (8920, 10201.7) minutes. Hence, the \$.49 additional profit per minute is guaranteed only for up to $\frac{10201.7 - 8920}{.49} = 6.26\%$ capacity increase.

*** OPTIMUM SOLUTION SUMMARY ***

Title: Final iteration No: 3 Objective value (max) = 4011.1582				
Variable	Value	Obj Coeff	Obj Val	Contrib
x1	100.0000	9.4000	939.9999	
x2	100.0000	10.8000	1080.0000	
x3	138.4181	8.7500	1211.1582	
x4	100.0000	7.8000	780.0000	

Constraint	RHS	Slack(-)/Surplus(+)
1 (<)	4800.0000	304.3503-
2 (<)	9600.0000	0.0000-
3 (<)	4700.0000	3081.6968-
4 (<)	4500.0000	2307.9097-
LB-x1	100.0000	0.0000+
LB-x2	100.0000	0.0000+
LB-x3	100.0000	38.4181+
LB-x4	100.0000	0.0000+

*** SENSITIVITY ANALYSIS ***

Objective coefficients -- Single Changes:				
Variable	Current Coeff	Min Coeff	Max Coeff	Reduced Cost
x1	9.4000	-infinity	10.0847	0.6847
x2	10.8000	-infinity	12.1610	1.3610
x3	8.7500	8.1559	infinity	0.0000
x4	7.8000	-infinity	13.1003	5.3003

Right-hand Side -- Single Changes:

Constraint	Current RHS	Min RHS	Max RHS	Dual Price
1 (<)	4800.0000	4405.6497	infinity	0.0000
2 (<)	9600.0000	8919.9999	10201.7242	0.4944
3 (<)	4700.0000	1618.3052	infinity	0.0000
4 (<)	4500.0000	2192.0903	infinity	0.0000
LB-x1	100.0000	0.0000	133.3333	-0.6847
LB-x2	100.0000	42.1946	127.6423	-1.3610
LB-x3	100.0000	-infinity	138.4181	0.0000
LB-x4	100.0000	56.9826	125.6604	5.3003

continued...

(b) Only soldering capacity can be increased because its dual price is positive.

(c) The fact that the dual prices of the lower bounds on x_1 , x_2 , and x_4 are negative shows that the lower bounds have adverse effect on profitability. Specifically, one unit decrease in the production of cables SC320, SC325, and SC370 will respectively increase the profit by \$.68, \$1.36, and \$5.30 per cable. These values are valid considering the cables one at a time.

(d) Dual price for soldering is \$.49 per minute, valid in the range (8920, 10201.7) minutes. Hence, the \$.49 additional profit per minute is guaranteed only for up to $\frac{10201.7 - 8920}{.49} = 6.26\%$ capacity increase.

x_1 = number of jackets per week

x_2 = number of handbags per week

Maximize $Z = 350x_1 + 120x_2$

Subject to

$$8x_1 + 2x_2 \leq 1200$$

$$12x_1 + 5x_2 \leq 1850$$

$$x_1, x_2 \geq 0$$

TORA optimum solution:

$$x_1 = 144, x_2 = 25, Z = \$53,312.50$$

Resource	Dual price	Range
Leather	\$19.38/m ²	(740, 1233.33)
Labor	\$16.25/hr	(1800, 3000)

BagCo should not pay more than \$19.38/m² of leather and \$16.25/hr of labor time.

Set 4.3b

Dual prices: $y_1 = 1$, $y_2 = 2$, $y_3 = 0$
all in \$/min

$$(1 - r_1) y_1 + 1.25 y_2 + y_3 \geq 3$$

$$\text{Reduced cost of } x_2 = (1 - r_1)x_1 + 1.25x_2 + x_0 - 3 \\ = .5 - r_1$$

For x_2 to be just profitable, its reduced cost must be (at least) zero; that is, $.5 - r_1 \leq 0$ or $r_1 \geq .5$. This means a reduction of at least 50%

Dual constraint for fire trucks:

$$y_2 + 3y_3 \geq 4$$

$$\text{Reduced cost} = y_2 + 3y_3 - 4 \\ = 1x_2 + 3x_0 - 4 = -2 < 0$$

New toy is recommended.

x_j = number of units of PP_j , $j=1,2,3,4$

$$\text{Maximize } Z = 3x_1 + 6x_2 + 5x_3 + 4x_4$$

Subject to

$$\begin{aligned} 2x_1 + 5x_2 + 3x_3 + 4x_4 &\leq 5300 \\ 3x_1 + 4x_2 + 6x_3 + 4x_4 &\leq 5300 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

*** OPTIMUM SOLUTION SUMMARY ***

Title: Problem 4.4b-3
Final iteration No: 4
Objective value (max) = 6814.2856

Variable	Value	Obj Coeff	Obj Val	Contrib
x_1	757.1429	3.0000	2271.4287	
x_2	757.1428	6.0000	4542.8569	
x_3	0.0000	5.0000	0.0000	
x_4	0.0000	4.0000	0.0000	

Constraint	RHS	Slack(-)/Surplus(+)
1 (<)	5300.0000	0.0000-
2 (<)	5300.0000	0.0000-

*** SENSITIVITY ANALYSIS ***

Objective coefficients -- Single Changes:

Variable	Current Coeff	Min Coeff	Max Coeff	Reduced Cost
x_1	3.0000	2.9444	4.5000	0.0000
x_2	6.0000	4.0000	6.3333	0.0000
x_3	5.0000	-infinity	5.1429	0.1429
x_4	4.0000	-infinity	5.1429	1.1429

Right-hand Side -- Single Changes:

Constraint	Current RHS	Min RHS	Max RHS	Dual Price
1 (<)	5300.0000	3533.3334	6625.0000	0.8571
2 (<)	5300.0000	4240.0000	7949.9998	0.4286

continued...

From TORA solution:

Variable	Reduced cost
x_3	.1429
x_4	1.1429

Thus,

$$(\text{Rate of deterioration in } z) = \$.14 \text{ per unit of } x_3$$

$$(\text{Rate of deterioration in } z) = \$ 1.14 \text{ per unit of } x_4$$

Resource	Dual price	Range
Lathe	\$.8571	(5333.33, 6625)
Drill	\$.4286	(4240, 7950)

Reduced cost for x_3

$$\begin{aligned} &= .8(3y_1 + 6y_2) - 5 \\ &= .8(3x \cdot .8571 + 6x \cdot .4286) - 5 \\ &= -.8857 < 0 \end{aligned}$$

Reduced cost for x_4

$$\begin{aligned} &= .8(4y_1 + 4y_2) - 4 \\ &= .8(4x \cdot .8571 + 4x \cdot .4286) - 4 \\ &= .1142 > 0 \end{aligned}$$

Only PP_3 will be profitable.

PP_4 needs more than

$$1 - \frac{4}{4x \cdot .8571 + 4x \cdot .4286} = 22.2\%$$

improvement to be profitable

Set 4.4a

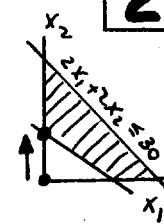
- (a) No, because A is feasible.
 (b) No, because E is feasible. Dual simplex iterations remain infeasible until the last iteration is reached.
 (c) $L \rightarrow I \rightarrow F$.

(a)

Minimize $Z = 2x_1 + 3x_2$
 subject to

$$\begin{aligned} 2x_1 + 2x_2 &\leq 30 \\ -x_1 - 2x_2 &\leq -10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

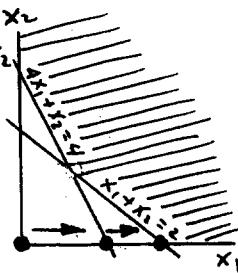
2



Basic	x_1	x_2	x_3	x_4	Sol ⁿ
Z	-2	-3	0	0	0
x_3	2	2	1	0	30
x_4	-1	-2	0	1	-10
Z	-1/2	0	0	-3/2	15
x_3	1	0	1	1	20
x_2	1/2	1	0	-1/2	5

(b)

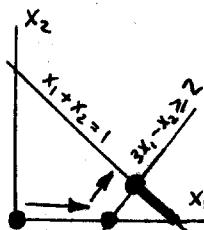
Minimize $Z = 5x_1 + 6x_2$
 subject to
 $\begin{aligned} -x_1 - x_2 &\leq -2 \\ -4x_1 - x_2 &\leq -4 \\ x_1, x_2 &\geq 0 \end{aligned}$



Basic	x_1	x_2	x_3	x_4	Sol ⁿ
Z	-5	-6	0	0	0
x_3	-1	-1	1	0	-2
x_4	-4	-1	0	1	-4
Z	0	-19/4	0	-5/4	5
x_3	0	-3/4	1	-1/4	-1
x_1	1	1/4	0	-1/4	1
Z	0	-1	-5	0	10
x_4	0	3	-4	1	4
x_1	1	1	-1	0	2

(c)

Minimize $Z = 4x_1 + 2x_2$
 Subject to
 $\begin{aligned} x_1 + x_2 &\leq 1 \\ x_1 + x_2 &\geq 1 \\ 3x_1 - x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$

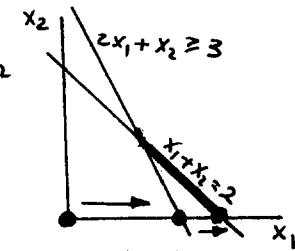


(Convert the equation into two inequalities to fit the dual simplex format.)

Basic	x_1	x_2	x_3	x_4	x_5	Sol ⁿ
Z	-4	-2	0	0	0	0
x_3	1	1	1	0	0	1
x_4	-1	-1	0	1	0	-1
x_5	-3	1	0	0	1	-2
Z	0	-4/3	0	0	-4/3	8/3
x_3	0	4/3	1	0	1/3	1/3
x_4	0	-4/3	0	1	-1/3	-1/3
x_1	1	-1/3	0	0	-1/3	2/3
Z	0	0	0	-5/2	-1/2	7/2
x_3	0	0	1	1	0	0
x_2	0	1	0	-3/4	1/4	1/4
x_1	1	0	0	-1/4	-1/4	3/4

(d)

Minimize $Z = 2x_1 + 3x_2$
 Subject to
 $\begin{aligned} 2x_1 + x_2 &\geq 3 \\ x_1 + x_2 &\leq 2 \\ x_1 + x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$



Basic	x_1	x_2	x_3	x_4	x_5	Sol ⁿ
Z	-2	-3	0	0	0	0
x_3	-2	-1	1	0	0	-3
x_4	1	1	0	1	0	2
x_5	-1	-1	0	0	1	-2
Z	0	-2	-1	0	0	3
x_1	1	1/2	-1/2	0	0	3/2
x_4	0	1/2	1/2	1	0	1/2
x_5	0	-1/2	-1/2	0	1	-1/2
Z	0	-1	0	0	-2	4
x_1	1	1	0	0	-1	2
x_4	0	0	0	1	1	0
x_3	0	1	1	0	-2	1

continued...

Set 4.4a

Add the constraint $x_1 + x_3 \leq M$

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Basic	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
Z	-2	1	-1	0	0	0	0	0
S_1	-2	-3	5	1	0	0	0	-4
S_2	1	-9	1	0	1	0	0	-3
S_3	4	6	3	0	0	1	0	8
S_4	1	0	1	0	0	0	1	M
Z	0	0	1	0	0	0	2	$2M$
S_1	0	-3	7	1	0	0	2	$-4+2M$
S_2	0	-9	0	0	1	0	-1	$-3-M$
S_3	0	6	-1	0	0	1	-4	$8-4M$
x_1	1	0	1	0	0	0	1	M

The second tableau is now optimal but infeasible. We can thus apply the dual simplex to the second tableau.

Optimal solution is:

$$x_1 = 1.286, x_2 = .476, x_3 = 0$$

$$Z = 2.095$$

(a) Add the constraint $x_3 \leq M$

4

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
Z	0	0	-2	0	0	0	0	0
x_4	1	-2	2	1	0	0	0	-8
x_5	-1	1	1	0	1	0	0	4
x_6	2	-1	4	0	0	1	0	10
x_7	0	0	1	0	0	0	1	M
Z	0	0	0	0	0	0	2	$2M$
x_4	1	-2	0	1	0	0	-2	$-8-2M$
x_5	-1	1	0	0	1	0	-1	$4-M$
x_6	2	-1	0	0	0	1	-4	$10-4M$
x_7	0	0	1	0	0	0	1	M

Last tableau is optimal but infeasible. Application of the dual simplex method yields the solution:

$$x_1 = 56/9, x_2 = 26/3, x_3 = 14/9$$

$$Z = 28/9$$

(b) Add the constraint $x_1 \leq M$

	x_1	x_2	s_1	s_2	s_3	s_4	
\bar{Z}	-1	3	0	0	0	0	0
S_1	1	-1	1	0	0	0	2
S_2	-1	-1	0	1	0	0	-4
S_3	-2	2	0	0	1	0	-3
S_4	1	0	0	0	0	1	M
\bar{Z}	0	3	0	0	0	1	M
S_1	0	-1	1	0	0	-1	$2-M$
S_2	0	-1	0	1	0	1	$-4+M$
S_3	0	2	0	0	1	2	$-3+2M$
x_1	1	0	0	0	0	1	M

Optimum: $x_1 = 3, x_2 = 1, z = 0$

(c) Add the constraint $x_1 \leq M$

	x_1	x_2	s_1	s_2	s_3	s_4	
\bar{Z}	1	-1	0	0	0	0	0
S_1	-1	4	1	0	0	0	-5
S_2	1	-3	0	1	0	0	1
S_3	-2	5	0	0	1	0	-1
S_4	1	0	0	0	0	1	M
\bar{Z}	0	-1	0	0	0	-1	$-M$
S_1	0	4	1	0	0	1	$-5+M$
S_2	0	-3	0	1	0	-1	$1-M$
S_3	0	5	0	0	1	2	$-1+2M$
x_1	1	0	0	0	0	1	M

Problem has no feasible solution

(d) Add the constraint $x_3 \leq M$

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	
\bar{Z}	0	0	-2	0	0	0	0	0
S_1	1	-3	7	1	0	0	0	-5
S_2	-1	1	-1	0	1	0	0	1
S_3	3	1	-10	0	0	1	0	8
S_4	0	0	1	0	0	0	1	M
\bar{Z}	0	0	0	0	0	0	2	$2M$
S_1	1	-3	0	1	0	0	-7	$-5-7M$
S_2	-1	1	0	0	1	0	1	$1+M$
S_3	3	1	0	0	0	1	10	$8+10M$
S_4	0	0	1	0	0	0	1	M

Solution is unbounded

continued...

Method 1: M-technique (or two-phase method)

5

Starting tableau:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	s_3	R_1	R_2	R_3	Sol^n
Z	-6	-7	-3	-5	0	0	0	-1	-M	-M	-
R_1	5	6	-3	4	-1	0	0	1	0	0	12
R_2	0	1	-5	-6	0	-1	0	0	1	0	10
R_3	2	5	1	1	0	0	-1	0	0	1	8

Method 2: Solve the dual problem

Starting tableau:

Basic	y_1	y_2	y_3	s_1	s_2	s_3	Sy
w^*	-12	-10	-8	0	0	0	0
s_1	5	0	2	1	0	0	6
s_2	6	1	5	0	1	0	7
s_3	-3	-5	1	0	0	1	0
s_4	4	-6	1	0	0	0	5

Method 3: Dual simplex

Starting tableau:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Sol^n
Z	-6	-7	-3	-5	0	0	0	0
s_1	-5	-6	3	-4	1	0	0	-12
s_2	0	-1	5	6	0	1	0	-10
s_3	-2	-5	-1	-1	0	0	1	-8

Optimal solution: $x_1 = 0, x_2 = 10, x_3 = x_4 = 0$

$$Z = 70$$

Method	Number of iterations
1	5
2	3
3	

The dual simplex is the best. It follows because it requires the smallest number of iterations and has the smallest number of constraints.

Set 4.4b

Basic	x_1	x_2	x_3	x_4	x_5	
Z	1	-1	0	0	0	0
x_3	-1	4	1	0	0	-5
x_4	1	-3	0	1	0	1
x_5	-2	5	0	0	1	-1
Z						
x_1	1	-4	-1	0	0	5
x_4	0	1	1	1	0	-4
x_5	0	-3	-2	0	1	9

In the second iteration, row 2 has all nonnegative coefficients on the left-hand side. This means that the infeasibility of x_4 cannot be removed, and the problem has no feasible solution.

2

	x_1	x_2	x_3	x_4	x_5	x_6	
Z	0	0	-2	0	0	0	0
x_4	1	-3	7	1	0	0	-5
x_5	-1	1	-1	0	1	0	1
x_6	3	1	-10	0	0	1	8
Z	0	0	-2	0	0	0	0
x_2	-1/3	1	-7/3	-1/3	0	0	5/3
x_5	-2/3	0	4/3	1/3	1	0	-2/3
x_6	10/3	0	-23/3	1/3	0	1	19/3
Z			-2			0	
x_1			-4/3			2	
x_1			-2			1	
x_6			-1			3	

Iteration 3 is feasible but nonoptimal. However, x_3 shows that the solution is unbounded.

Set 4.5a

$$\text{new RHS} = \begin{pmatrix} 430 \\ 480 \\ 400 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 430 \\ 480 \\ 400 \end{pmatrix} = \begin{pmatrix} 95 \\ 240 \\ 20 \end{pmatrix}$$

The new solution is feasible with $x_1 = 0, x_2 = 95, x_3 = 240$. $Z = 3x_0 + 2x_95 + 5x_240 = \1390 , which is better than the current value of Z .

$$(a) \begin{pmatrix} x_0 \\ x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 460 \\ 500 \\ 400 \end{pmatrix} = \begin{pmatrix} 105 \\ 250 \\ -20 \end{pmatrix}$$

Solution is unfeasible

	x_1	x_2	x_3	x_4	x_5	x_6	
Z	4	0	0	1	2	0	1460
x_2	-1/4	1	0	1/2	-1/4	0	105
x_3	3/2	0	1	0	1/2	0	250
x_6	z	0	0	-2	1	1	-20
Z	5	0	0	0	5/2	1/2	1450
x_2	1/4	1	0	0	0	1/4	100
x_3	3/2	0	1	0	1/2	0	250
x_4	-1	0	0	1	-1/2	-1/2	10

$$(b) \begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 500 \\ 400 \\ 600 \end{pmatrix} = \begin{pmatrix} 150 \\ 200 \\ 0 \end{pmatrix}$$

New solution is feasible. $Z = \$1300$

$$(c) \begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 300 \\ 800 \\ 200 \end{pmatrix} = \begin{pmatrix} -50 \\ 400 \\ 400 \end{pmatrix}$$

	x_1	x_2	x_3	x_4	x_5	x_6	
Z	4	0	0	1	2	0	1900
x_2	-1/4	1	0	1/2	-1/4	0	-50
x_3	3/2	0	1	0	1/2	0	400
x_6	2	0	0	-2	1	1	400
Z	2	8	0	5	0	0	1500
x_5	1	-4	0	-2	1	0	200
x_3	1	2	1	1	0	0	300
x_6	1	4	0	0	0	1	200

continued...

$$(d) \begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 450 \\ 700 \\ 350 \end{pmatrix} = \begin{pmatrix} 50 \\ 350 \\ 150 \end{pmatrix}$$

Solution is feasible. $Z = \$1850$

$$\begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \end{pmatrix} \begin{pmatrix} 28 \\ 8 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5/2 \\ 3/2 \\ -1 \end{pmatrix}$$

3

	x_1	x_2	s_1	s_2	s_3	s_4	
Z	0	0	3/4	1/2	0	0	25
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	5/2
s_3	0	0	3/8	-5/4	1	0	3/2
s_4	0	0	1/8	-3/4	0	1	-1
Z	0	0	5/6	0	0	2/3	24 $\frac{2}{3}$
x_1	1	0	1/6	0	0	-2/3	10/3
x_2	0	1	0	0	0	1	2
s_3	0	0	1/6	0	1	-5/3	7/3
s_2	0	0	-1/6	1	0	-4/3	2/3

$x_1 = 1$ lb limestone in weekly mix

$x_2 = 1$ lb corn in weekly mix

$x_3 = 1$ lb soybean meal in weekly mix

Minimize $Z = .12x_1 + .45x_2 + 1.6x_3$

s.t.

$$\begin{aligned} x_1 + x_2 + x_3 &\geq Q \\ .38x_1 + .001x_2 + .002x_3 &\geq .008(x_1 + x_2 + x_3) \\ .38x_1 + .001x_2 + .002x_3 &\leq .012(x_1 + x_2 + x_3) \\ .09x_2 + .5x_3 &\geq .22(x_1 + x_2 + x_3) \\ .02x_2 + .08x_3 &\leq .05(x_1 + x_2 + x_3) \end{aligned}$$

$x_1, x_2, x_3 \geq 0$

$Q = \text{weekly mix}$

The constraints simplify to

$$\begin{aligned} x_1 + x_2 + x_3 &\geq Q \\ .372x_1 - .007x_2 - .006x_3 &\geq 0 \\ .368x_1 - .001x_2 - .01x_3 &\leq 0 \\ -.22x_1 - .13x_2 + .28x_3 &\geq 0 \\ -.05x_1 - .03x_2 + .03x_3 &\leq 0 \end{aligned}$$

Week	1	2	3	4	5	6	7	8
$Q(1b)$	5200	9600	15600	20000	26000	32000	38000	42000

continued...

Set 4.5a

5

First, we solve the problem using $Q = 5200 \text{ lb}_s$, feed requirements for week 1. Then we use sensitivity analysis for the remaining weeks.
Week 1 Solution (using TORA)

$$(\text{Basic vector}) = \begin{pmatrix} x_2 \\ x_1 \\ Sx_5 \\ x_3 \\ Sx_{11} \end{pmatrix}, Z = \$4224.74$$

$$\text{inverse} = \begin{pmatrix} .649 & 0 & -3.216 & -2.431 & 0 \\ .028 & 0 & 2.637 & -.006 & 0 \\ .004 & -1 & 1.000 & -.000 & 0 \\ .323 & 0 & .579 & 2.438 & 0 \\ .011 & 0 & .018 & .146 & 1 \end{pmatrix}$$

Solution given Q :

$$\begin{pmatrix} x_2 \\ x_1 \\ Sx_5 \\ x_3 \\ Sx_{11} \end{pmatrix} = (\text{inverse}) \begin{pmatrix} Q \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} .649Q \\ .028Q \\ .004Q \\ .323Q \\ .011Q \end{pmatrix}$$

General solution:

$$x_1 = .028Q$$

$$x_2 = .649Q$$

$$x_3 = .323Q$$

$$Z = (.12x_1 + .45x_2 + 1.6x_3)Q$$

$$= .81221Q$$

B^{-1} inverse

$D_i = \text{change in RHS of constraint } i, i=1, 2, \dots, m$

Simultaneous feasibility conditions:

$$B^{-1} \begin{pmatrix} b_1 + D_1 \\ \vdots \\ b_m + D_m \end{pmatrix} \geq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (1)$$

Let $P_i \leq D_i \leq q_i$ be the feasibility range computed from the single-change conditions:

$$B^{-1} \begin{pmatrix} b_1 \\ b_1 + D_i \\ \vdots \\ b_m \end{pmatrix} \geq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (2)$$

Define

$$\Delta_i = \begin{cases} P_i, & \text{if } D_i < 0 \\ q_i, & \text{if } D_i > 0 \end{cases}$$

Condition (2) holds true for $D_i = \Delta_i$ also.

Now, define $r_i \geq 0, i=0, 1, 2, \dots, m$ such that $r_0 + r_1 + \dots + r_m = 1$. Then

$$B^{-1} \left[r_0 \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} + r_1 \begin{pmatrix} b_1 + \Delta_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} + \dots + r_m \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m + \Delta_m \end{pmatrix} \right]$$

must also be feasible. The last expression reduces to

$$B^{-1} \left[\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} + \begin{pmatrix} r_1 \Delta_1 \\ \vdots \\ r_m \Delta_m \end{pmatrix} \right] \geq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (3)$$

Next, select $r_i = \frac{D_i}{\Delta_i}, i=1, 2, \dots, m$. Then (3) is the same as condition (1). However, because $r_0 + r_1 + \dots + r_m = 1$, it must be true that $r_1 + r_2 + \dots + r_m \leq 1$. The condition

$$r_1 + r_2 + \dots + r_m \leq 1$$

thus implies that (3), and hence (1), is feasible. The condition is not sufficient because (3) can be satisfied for arbitrary values of $r_0, r_1, \dots, \text{and } r_m$.

Set 4.5a

(a)

$$\bar{B}^{-1} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

$$Y = (1, 4, 0, 0) \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} = (-1/4, 5/2, 0, 0)$$

$$X_B = \bar{B}^{-1} b$$

$$= \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 28 \\ 8 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5/2 \\ 3/2 \\ -1/2 \end{pmatrix}$$

The simplex tableau is

	x_1	x_2	x_3	x_4	x_5	x_6	Solution
Z	0	0	-1/4	5/2	0	0	13
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	5/2
x_5	0	0	3/8	-5/4	1	0	3/2
x_6	0	0	1/8	-3/4	0	1	-1/2

The tableau is both nonoptimal and infeasible.

(b) Apply the primal simplex to the tableau above, disregarding the x_6 row in the ratio test. Thus, x_3 enters the basic solution and x_5 leaves. The resulting tableau is

6

	x_1	x_2	x_3	x_4	x_5	x_6	
Z	0	0	0	5/3	2/3	0	14
x_1	1	0	0	1/3	-2/3	0	2
x_2	0	1	0	1/3	1/3	0	3
x_3	0	0	1	-10/3	8/3	0	4
x_6	0	0	0	-1/3	-1/3	1	-1

The tableau is now optimal but infeasible. Application of the dual simplex method should then lead to feasibility while maintaining the tableau optimal.

continued...

continued...

Set 4.5b

Current optimum is

$$x_1 = 0, x_2 = 100, x_3 = 230$$

$$(a) 4x_1 + x_2 + 2x_3 \leq 570:$$

Since $4x_0 + 1x100 + 2x230 = 560 < 570$, the additional constraint is redundant and the solution remains unchanged.

$$(b) 4x_1 + x_2 + 2x_3 \leq 548:$$

The current solution violates the new constraints. We use the dual simplex method to determine the new solution.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
Z	4	0	0	1	2	0	0	1350
x_2	-1/4	1	0	1/2	-1/4	0	0	100
x_3	3/2	0	1	0	1/2	0	0	230
x_6	2	0	0	-2	1	1	0	20
x_7	4	1	2	0	0	0	1	548
Z	4	0	0	1	2	0	0	1350
x_2	-1/4	1	0	1/2	-1/4	0	0	100
x_3	3/2	0	1	0	1/2	0	0	230
x_6	2	0	0	-2	1	1	0	20
x_7	5/4	0	0	-1/2	-3/4	0	1	-12
Z	13/2	0	0	0	1/2	0	2	1326
x_2	-1/4	1	0	0	-1	0	1	88
x_3	3/2	0	1	0	1/2	0	0	230
x_6	-3	0	0	0	1	1	-4	68
x_4	-5/2	0	0	1	3/2	0	-2	24

Optimum solution:

$$x_1 = 0, x_2 = 88, x_3 = 230$$

$$Z = \$1326$$

1

$$\text{Maximize } Z = 5x_1 + 6x_2 + 3x_3$$

Subject to

2

$$5x_1 + 5x_2 + 3x_3 \leq 50 \quad (1)$$

$$x_1 + x_2 - x_3 \leq 20 \quad (2)$$

$$7x_1 + 6x_2 - 9x_3 \leq 30 \quad (3)$$

$$5x_1 + 5x_2 + 5x_3 \leq 35 \quad (4)$$

$$12x_1 + 6x_2 \leq 90 \quad (5)$$

$$x_2 - 9x_3 \leq 20 \quad (6)$$

$$x_1, x_2, x_3 \geq 0$$

Start with constraints (1), (3), and (4). The associated solution is

$$x_1 = 0, x_2 = 6.2, x_3 = -8$$

This solution automatically satisfies the remaining constraints (2), (5), and (6). Hence these constraints are discarded as redundant and the optimum solution for the problem is as given above.

Set 4.5c

$$\begin{pmatrix} \text{Basic} \\ \text{vector} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_6 \end{pmatrix} \quad \text{Inverse} = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

Nonbasic variables: x_1, x_4, x_5

$$(a) Z = 2x_1 + x_2 + 4x_3$$

$$(y_1, y_2, y_3) = (1, 4, 0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} = (1/2, 7/4, 0)$$

Reduced costs:

$$x_1: (1/2, 7/4, 0) \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - 2 = 15/4$$

$$x_4: (1/2, 7/4, 0) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 0 = 1/2$$

$$x_5: (1/2, 7/4, 0) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 0 = 7/4$$

current solution remains optimal

$$(b) Z = 3x_1 + 6x_2 + x_3$$

$$(y_1, y_2, y_3) = (6, 1, 0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} = (3, -1, 0)$$

Reduced costs:

$$x_1: 1x_3 + 3x_2 - 1 + 1x_0 - 3 = -3 < 0$$

$$x_4: 1x_3 + 0x_2 - 1 + 0x_0 - 0 = 3$$

$$x_5: 0x_3 + 1x_2 - 1 + 0x_0 - 0 = -1 < 0$$

Solution is not optimal.

Z	x_1	x_2	x_3	x_4	x_5	x_6	
Z	-3	0	0	3	-1	0	830
x_2	-1/4	1	0	1/2	-1/4	0	100
x_3	3/2	0	1	0	1/2	0	230
x_6	2	0	0	-2	1	1	20
Z	0	0	0	0	1/2	3/2	860
x_2	0	1	1/4	1/4	-1/4	1/8	102 1/2
x_3	0	0	0	0	1/2	0	215
x_1	1	0	-1	-1	1/2	1/2	10

Optimum solution: $x_1 = 10, x_2 = 10, x_3 = 215$
Problem has alternative optima. $Z = \$860$

$$(c) Z = 8x_1 + 3x_2 + 9x_3$$

$$(y_1, y_2, y_3) = (3, 9, 0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} = (\frac{3}{2}, \frac{15}{4}, 0)$$

Reduced costs:

$$x_1: 1x_{\frac{3}{2}} + 3x_{\frac{15}{4}} + 1x_0 - 8 = 19/4$$

$$x_4: 1x_{\frac{3}{2}} + 3x_0 + 1x_0 - 0 = 3/2$$

$$x_5: 0x_{\frac{3}{2}} + 1x_{\frac{15}{4}} + 0x_0 - 0 = 15/4$$

Solution remains optimal

$$\begin{pmatrix} \text{Basic} \\ \text{vector} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix}, \text{inverse} = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix}$$

2

Dual problem:

$$\text{Minimize } w = 24y_1 + 6y_2 + y_3 + 2y_4$$

Subject to

$$6y_1 + y_2 - y_3 \geq 5$$

$$4y_1 + 2y_2 + y_3 + y_4 \geq 4$$

$$y_1, y_2, y_3, y_4 \geq 0$$

$$(a) Z = 3x_1 + 2x_2$$

$$(y_1, y_2, y_3, y_4) = (3, 2, 0, 0) \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} = (1/2, 0, 0, 0)$$

Reduced costs:

$$x_3: y_1 - 0 = 3/4 - 0 = 3/4$$

$$x_4: y_2 - 0 = 7/2 - 0 = 7/2$$

Solution remains optimal

$$(b) Z = 8x_1 + 10x_2$$

$$(y_1, y_2, y_3, y_4) = (8, 10, 0, 0) \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} = (3/4, 7/2, 0, 0)$$

Reduced costs:

$$x_3: y_1 - 0 = 3/4 - 0 = 3/4$$

$$x_4: y_2 - 0 = 7/2 - 0 = 7/2$$

Solution remains optimal

$$(c) Z = 2x_1 + 5x_2$$

$$(y_1, y_2, y_3, y_4) = (2, 5, 0, 0) \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} = (-1/8, 11/4, 0, 0)$$

Reduced costs:

$$x_3: y_1 - 0 = -1/8 - 0 = -1/8 < 0$$

$$x_4: y_2 - 0 = 11/4 - 0 = 11/4$$

current solution is not optimal.

continued...

continued...

Set 4.5c

	x_1	x_2	x_3	x_4	x_5	x_6	
Z	0	0	-1/8	11/4	0	0	27/2
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	3/2
x_5	0	0	3/8	-5/4	1	0	5/2
x_6	0	0	1/8	-3/4	0	1	1/2
Z	0	0	0	2	0	1	14
x_1	1	0	0	1	0	-2	2
x_2	0	1	0	0	0	1	2
x_5	0	0	0	1	1	-3	1
x_3	0	0	1	-6	0	8	4

Optimum solution:

$$x_1 = 2, x_2 = 2, x_3 = 4, Z = 14$$

Let d_j = change in the objective coefficient c_j , $j = 1, 2, \dots, n$ 3

The simultaneous changes yield the same optimum if (for maximization)

$$(Z_j - c_j - d_j) \geq 0, j = 1, 2, \dots, n \quad (1)$$

where Z_j = left-hand of constraint dual $j = \sum_{i=1}^m a_{ij} y_i$

Let $u_j \leq d_j \leq v_j$ be the optimality range computed from the single-change condition

$$Z_j - c_j - d_j \geq 0 \quad (2)$$

and define

$$\delta_j = \begin{cases} u_j, & \text{if } d_j < 0 \\ v_j, & \text{if } d_j > 0 \end{cases}$$

Condition (2) holds true also for $d_j = \delta_j$.

Define $r_j \geq 0$, $j = 0, 1, 2, \dots, n$, such that $r_0 + r_1 + \dots + r_n = 1$. Then

$$r_0(Z_1 - c_1, \dots, Z_n - c_n) + r_1(Z_1 - c_1 - \delta_1, \dots, Z_n - c_n) + \dots + r_n(Z_1 - c_1, \dots, Z_n - c_n - \delta_n)$$

continued...

must be nonnegative. However, the last expression reduces to

$$(Z_1 - c_1, \dots, Z_n - c_n) - (r_1 \delta_1, \dots, r_n \delta_n) \geq 0$$

$$\text{or } Z_j - c_j - r_j \delta_j \geq 0, j = 1, 2, \dots, n \quad (3)$$

Now, set $r_j = \frac{d_j}{\delta_j}$, then (3) is identical to (1), the desired condition. However, since $r_0 + r_1 + \dots + r_n = 1$ and $r_0 \geq 0$, then for optimality we must have

$$r_1 + r_2 + \dots + r_n \leq 1$$

3

Set 4.5d

Dual constraint for toy trains is

$$y_1 + 3y_2 + y_3 \geq 3$$

where $y_1 = 1$, $y_2 = 2$, and $y_3 = 0$

new reduced cost for x_1 is

$$\frac{P}{100} (y_1 + 3y_2 + y_3) - 3.$$

For toy trains to be just profitable, we must have

$$\frac{P}{100} (1 + 3x_2 + x_0) - 3 \geq 0$$

$$\text{or } P \geq 42.86\%$$

$$x_1 - \text{reduced cost} = .5y_1 + y_2 + .5y_3 - 3$$

$$= .5x_1 + x_2 + .5x_0 - 3 = -.5$$

$$x_1 - \text{columns} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} .5 \\ 1 \\ .5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$$

x_1	x_2	x_3	x_4	x_5	x_6	Z
$-\frac{1}{2}$	0	0	1	2	0	1350
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	100
x_3	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	230
x_6	$\frac{1}{2}$	0	0	-2	1	20
Z	0	0	-1	3	1	1370
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	100
x_3	0	0	1	$\boxed{2}$	$-\frac{1}{2}$	210
x_1	1	0	0	-4	2	40
Z	0	0	$\frac{1}{2}$	0	$\frac{11}{4}$	1475
x_2	0	1	$-\frac{1}{4}$	0	$-\frac{1}{8}$	$47\frac{1}{2}$
x_4	0	0	$\frac{1}{2}$	1	$-\frac{1}{4}$	105
x_1	1	0	2	0	1	460

(a) New dual constraint for fire engines is

$$3y_1 + 2y_2 + 4y_3 \geq 5, y_1 = 1, y_2 = 2, y_3 = 0$$

$$\text{Reduced cost} = 3x_1 + 2x_2 + 4x_0 - 5$$

$$= 2 > 0$$

Fire engines are not profitable

$$(b) \text{ Reduced cost} = 3x_1 + 2x_2 + 4x_0 - 10 = -3$$

$$\text{(Tableau Column)} = \begin{pmatrix} y_2 & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
Z	4	0	0	-3	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	1	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	1	0	$\frac{1}{2}$	0	230
x_7	2	0	0	0	-2	1	1	20
Z	$\frac{13}{4}$	3	0	0	$\frac{5}{2}$	$\frac{5}{4}$	0	1650
x_4	$-\frac{1}{4}$	1	0	1	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{7}{4}$	-1	1	0	$-\frac{1}{2}$	$\frac{3}{4}$	0	130
x_7	2	0	0	-2	1	1	1	20

x_3 = daily tons of new exterior paint

$$\text{Maximize } Z = 5x_1 + 4x_2 + 3.5x_3$$

subject to

$$6x_1 + 4x_2 + \frac{3}{4}x_3 \leq 24$$

$$x_1 + 2x_2 + \frac{3}{4}x_3 \leq 6$$

$$-x_1 + x_2 + x_3 \leq 1$$

$$x_2 \leq 2$$

$$x_1, x_2, x_3 \geq 0$$

$$\text{New dual constraint: } \frac{3}{4}y_1 + \frac{3}{4}y_2 + y_3 \geq 3.5$$

$$\text{Dual solution: } y_1 = \frac{3}{4}, y_2 = \frac{1}{2}, y_3 = 0$$

$$\text{Reduced cost} = \frac{3}{4}(\frac{3}{4} + \frac{1}{2}) + 0 - 3.5 = -\frac{41}{16}$$

$$\text{(Constraint Column)} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{8} & \frac{3}{4} & 0 & 0 \\ \frac{3}{8} & -\frac{5}{4} & 1 & 0 \\ \frac{1}{8} & -\frac{3}{4} & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{4} \\ \frac{3}{4} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{16} \\ \frac{15}{32} \\ \frac{13}{16} \\ -\frac{15}{32} \end{pmatrix}$$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	0	$-\frac{41}{16}$	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	1	0	$-\frac{3}{16}$	$\frac{1}{4}$	$-\frac{1}{2}$	0	3
x_2	0	1	$\frac{15}{32}$	$-\frac{1}{8}$	$\frac{3}{4}$	0	$\frac{3}{2}$
x_6	0	0	$\frac{13}{16}$	$\frac{3}{8}$	$-\frac{5}{4}$	1	$\frac{5}{2}$
x_7	0	0	$-\frac{15}{32}$	$\frac{1}{8}$	$-\frac{3}{4}$	0	$\frac{1}{2}$
Z	0	5.47	0	.07	4.6	0	29.2
x_1	1	.4	0	-2	-2	0	3.6
x_3	0	2.13	1	-27	1.6	0	3.2
x_6	0	-73	0	.47	-1.8	1	1.4
x_7	0	1	0	0	0	1	2.0

Optimum solution:

$$x_1 = 3.6 \text{ tons}, x_2 = 0, x_3 = 3.2 \text{ tons}$$

$$Z = \$29,200$$

continued...

CHAPTER 5

Transportation Model and its Variants

Set 5.1a

- (a) False
 (b) True
 (c) True

(a) $\sum a_i = 25, \sum b_j = 31$

Add a dummy source whose supply amount is $31 - 25 = 6$ units

(b) $\sum a_i = 74, \sum b_j = 65$

Add a dummy destination whose demand amount is $74 - 65 = 9$ units

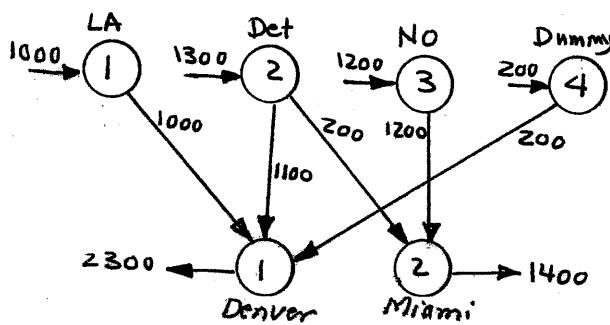
Denver will be 150 cars short.

Similarly, Miami will be 50 cars short of satisfying its demand

Assign a very high cost M to the route from Detroit to Dummy

	Den	Miami	5
	1	2	
LA 1	80	M	
	1000		1000
	100	108	use $M = 1000$ in TORA
Det 2	1100	200	1300
	102	68	
NO 3		1200	1200
	200	300	
Dummy 4	200		200
	2300	1400	

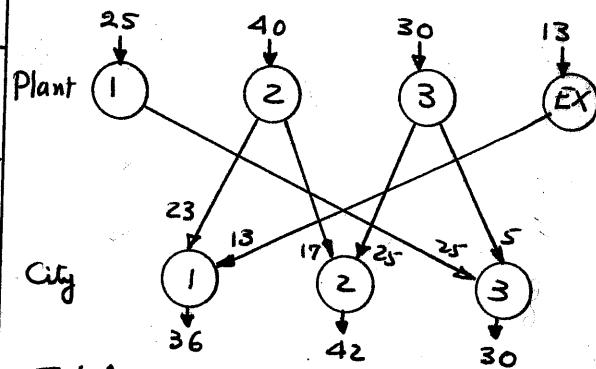
Optimum solution from TORA



Denver is 200 cars short, Cost = \$33,200

	City			6
	1	2	3	
Plant 1	600	700	400	25
Plant 2	320	300	350	40
Plant 3	500	480	450	30
Excess plant 4	1000	1600	M	13
	36	42	30	

(b) $M = \$10,000$ in TORA

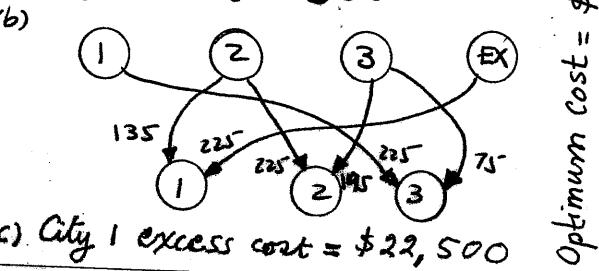


Total cost = \$ 49,710

(c) City 1 excess cost = $13 \times 1000 = \$13,000$

Assume units in 100,000 kWh

	City 1	2	3	7
	60	70	40	
Plant 1			225	225
2	32	30	35	360
3	50	48	45	270
EX	225	195	75	225
	360	420	300	



(c) City 1 excess cost = \$22,500

Optimum cost = \$55,300*

Set 5.1a

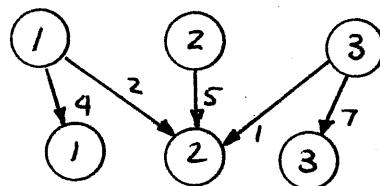
Unit transportation cost in thousand \$ per million gallons

$$\text{gallons} = \left(\frac{10\text{¢}}{1000} \times 10^6 \times \text{mileage} \right) \times \frac{1}{100} \times \frac{1}{1000}$$

$$= \frac{\text{mileage}}{10}$$

Distribution area

	1	2	3	M	6
Ref. 1	4	2	18		6
	30	5	10	8	5
	20	1	25	12	8
	4	8	7		



Total cost = \$243,000

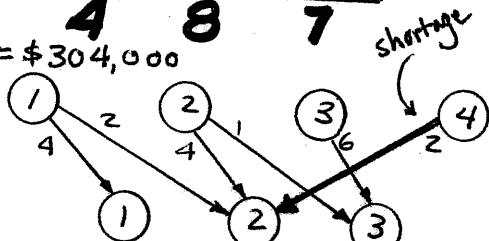
Unit cost in thousand \$ from Dummy source to distribution areas 2 or 3

$$= \frac{5\text{¢}}{100} \times \frac{10^6}{10^3} = 50 \text{ thousand } \$/\text{million gal}$$

Distribution area

	1	2	3	M	6
Ref. 1	4	2	18		6
	30	4	10	8	5
	20	1	25	12	6
	M	2	50	50	2
	4	8	7		

Cost = \$304,000



Unit costs in thousand \$ per million gallons:

from refinery 1 to Dummy
 $= \frac{1.50}{100} \times \frac{10^6}{10^3} = 15$

from refinery 2 to Dummy
 $= \frac{22.20}{100} \times \frac{10^6}{10^3} = 22$

	1	2	3	Dummy	15
Ref. 1	4	2	18	M	15
	30	5	10	8	22
	20	1	25	12	30
	4	8	4	3	

Refinery 3 diverts 3 million gallons for use within.

Total cost = \$207,000

(a) Total supply = $150 + 200 + 250 = 600$ crates
 Total demand = $150 + 150 + 400 + 100 = 800$ crates
 (Potential overtime supply)
 (by each of orchards 1 & 2) = $800 - 600 = 200$ crates

Orchard 1: $150 + 200 = 350$

Orchard 2: $150 + 200 = 350$

Orchard 3: $150 + 200 = 350$

Orchard 4: $100 + 100 = 200$

Orchard 5: $100 + 100 = 200$

Orchard 6: $100 + 100 = 200$

Orchard 7: $100 + 100 = 200$

Orchard 8: $100 + 100 = 200$

Orchard 9: $100 + 100 = 200$

Orchard 10: $100 + 100 = 200$

Orchard 11: $100 + 100 = 200$

Orchard 12: $100 + 100 = 200$

Orchard 13: $100 + 100 = 200$

Orchard 14: $100 + 100 = 200$

Orchard 15: $100 + 100 = 200$

Orchard 16: $100 + 100 = 200$

Orchard 17: $100 + 100 = 200$

Orchard 18: $100 + 100 = 200$

Orchard 19: $100 + 100 = 200$

Orchard 20: $100 + 100 = 200$

Orchard 21: $100 + 100 = 200$

Orchard 22: $100 + 100 = 200$

Orchard 23: $100 + 100 = 200$

Orchard 24: $100 + 100 = 200$

Orchard 25: $100 + 100 = 200$

Orchard 26: $100 + 100 = 200$

Orchard 27: $100 + 100 = 200$

Orchard 28: $100 + 100 = 200$

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Orchard 130: $100 + 100 = 200$

Orchard 131: $100 + 100 = 200$

Orchard 132: $100 + 100 = 200$

Orchard 133: $100 + 100$

Set 5.1a

Supply/demand quantities are expressed in truck loads, determined by dividing the number of cars by 18 and rounding the result up, if necessary. For example, supply amount at center 1 is $\frac{400}{18} = 22.22$ or 23 truck loads.

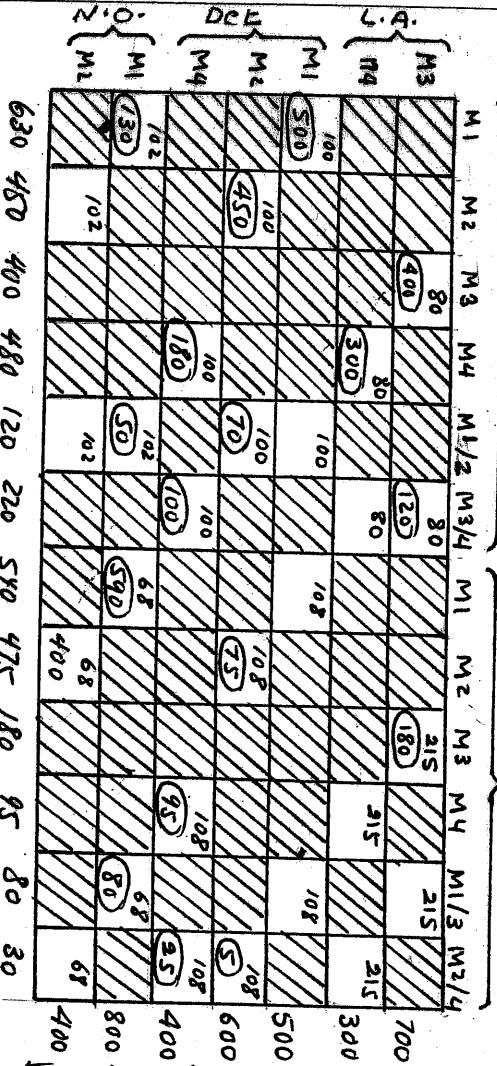
¹⁸ Expressing unit transportation costs in \$1000 per truck load, we get

	1	2	3	4	5	
1	2.5 (6)	3.75	5	3.5 (9)	875 (8)	23
2	1.25	1.75 (3)	1.5 (9)	1.625	2	12
3	1	2.25 (9)	2.5	3.75	3.25	9
	6	12	9	9	8	

(b) Alternative solution exists
Cost = \$ 92,500

12

13



Optimum Solutions:

LA - Denver M4	=	300	CARS
Det. - Denver M1	=	500	CARS
Det. - Denver M2	=	450	CARS
Det. - Denver M1/M2	=	70	CARS
Det. - Miami M2	=	75	CARS
Det. - Miami M2/4	=	5	CARS
Det. - Denver M4	=	180	CARS
Det. - Denver M3/4	=	100	CARS
Det. - Miami M4	=	95	CARS
Det. - Miami M2/4	=	25	CARS
N.O. - Denver M1	=	130	CARS
N.O. - Denver M1/2	=	50	CARS
N.O. - Miami M1	=	540	CARS
N.O. - Miami M1/3	=	80	CARS
N.O. - Miami M2	=	400	CARS
Total Cost =	\$	343,620	

Set 5.2a

	1	2	3	4		1	Sharpening Service					
1	40 50	40.4 42 50	40.7 40.3 42 70	41.4 41 30 270		50	Day	New	Overnite	2-day	3-day	Disposal
2					180	180	Mon	24	0	6	18	0
3					280	280	Tue	12	12	0	0	0
4					270	270	Wed	2	14	0	0	0
	100	200	180	300			Thu	0	0	20	0	0
							Fri	0	14	0	0	4
							Sat	0	2	0	0	12
							Sun	0	0	0	0	22

	1	2	3	4		1	3
	50	180	280	270		50	12
	100	200	180	300		180	12
	50	50	130	70		12	12
	200	200	180	180		8	8
	300	300	300	270		12	12
						92	92

Cost = \$31,461

	Mon	Tue	Wed	Thu	Fri	Sat	Sun	Disposal	
New	12 24	12 12	12 2	12 12	12 12	12 12	12 86	0	124
Mon	6	6	3 6	1 18	1	1	1	0	24
Tue	6	6	3	1	1	1	0		12
Wed	6	6	3	1	1	0			14
Thu	6	6	3	0					20
Fri	6	6	3	0					18
Sat							6	0	14
Sun							14	0	22
	24	12	14	20	18	14	22	124	

**Least-cost starting solution.
(Problem has alternative optima.)**

2

	Mon	Tue	Wed	Thu	Fri	Sat	Sun	Disposal	
New	12 24	12 12	12 2	12 12	12 12	12 12	12 86	0	124
Mon	6	6	3 6	1 18	1	1	1	0	24
Tue	6	6	3	1	1	1	0		12
Wed	6	6	3	1	1	0			14
Thu	6	6	3	0					20
Fri	6	6	3	0					18
Sat							6	0	14
Sun							14	0	22
	24	12	14	20	18	14	22	124	

The given optimum solution is interpreted as summarized below.

Total cost = \$804

continued...

	Day	New	Overnite	2-day	Disposal
Mon	24	12	12	0	
Tue	0	6	6	0	
Wed	8	8	6	0	
Thu	0	12	8	0	
Fri	0	8	0	10	
Sat	0	14	0	0	
Sun	0	0	0	22	

Sharpening Service

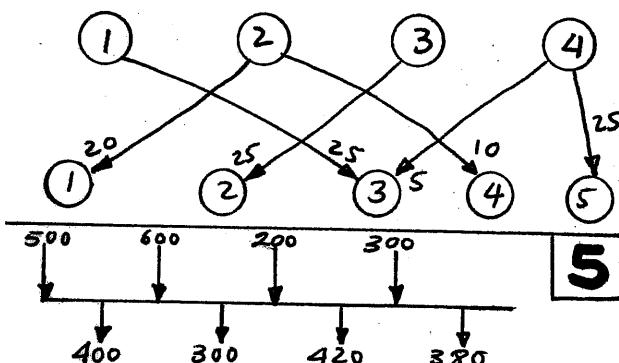
Total cost = \$840

alternative solution exists

Set 5.2a

	Task					
	1	2	3	4	5	4
Machine	10	2	3	15	9	25
1			(25)			
2	5	10	15	(10)	4	30
3	15	5	14	7	15	20
4	20	15	(5)	M	(25)	30
	20	20	30	10	25	

Total cost = \$560



C: \$100 \$140 \$120 \$150
h: \$3 \$3 \$3 \$3

	1	2	3	4	Surplus
1	100	103	106	M	0
2	M	140	143	146	0
3	M	M	120	123	0
4	M	M	M	150	0
	400	300	420	380	100

Cost = \$190,040, Alternative solution exists

Period	Capacity	Actual Prod.	Delivery
1	500	500	400 for 1 100 for 2
2	600	600	200 for 2 220 for 3 180 for 4
3	200	200	200 for 3
4	300	200	200 for 4

	1	2	3	4	5	Surplus	
R ₁	100	180					180
O ₁	150	20	154	158	162	166	90
R ₂		96	150	80	104	108	230
O ₂		144	148	152	156	115	115
R ₃			116	220	120	124	430
O ₃			174	178	182	186	215
R ₄				102	106	0	300
O ₄				250	50	150	150
R ₅					106	0	300
O ₅					300	159	150
	200	150	300	250	400	860	

Cost = \$137,720

Alternative solution exists.

Period	Production schedule
1	Regular - 180 engines Overtime - 20 engines
2	Regular: 230 engines
3	Regular 270 engines
4	Regular 300 engines
5	Regular 300 engines

Set 5.2a

New	1	2	3	4	5	6	Disposal
	200	210	220.5	231.53	243.1	255.26	0
1	200	180	140				878
	120	121.5	35	36.5	38	0	1398
2		120	121.5	35	36.5	0	
		148	32				
3		120	121.5	35	0		
		10	290				
4		120	121.5	0			
		198					
5		120	0				
		230					
6			0				
	200	180	300	198	230	290	1398

Cost = \$ 170,698

Alternative solution exists

Month	New	1-day	3-day	Disposal
1	200	12	188	0
2	180	148	32	0
3	140	10	290	0
4	0	198	0	0
5	0	0	0	230
6	0	0	0	290

7

(a) Use negative cost values

		Bidder			
Loc		1	2	3	4
1		-520	M	-650	-180
2		-210	-510	M	-430
3		-570	-495	-240	-710
Dummy		30	10	20	0
		30	30	30	30

8

(b) Bidder 1 = 0 acre

Bidder 2 = 20 acres (location 1)

Bidder 3 = 10 acres (location 2)

Bidder 4 = 30 acres (location 3)

Set 5.3a

(a)

Northwest:

Cost = \$42

0	2	1	6
5	1	3	5
2	4	7	7
5	5	10	

Least-cost:

Cost = \$37

0	2	1	6
5	5	2	5
2	4	7	7

Vogel:

Cost = \$37

0	2	1	6	Penalty	Penalty
5	1	3	5	1	1
2	5	2	5	1	4
2	4	7	7	1	1

Penalty

Penalty

(b)

Northwest:

Cost = \$94

1	2	6	7
7	9	2	12
3	1	10	11

Least-Cost:

Cost = \$61

1	2	6	7
10	4	2	12
3	10	1	11

continued...

VAM:

Cost = \$40

1	2	6	Penalties
7	0	4	2 4 -
2	10	5	2 2 2
1	10	-	3

Penalties

(c)

Northwest:

Cost = \$104

5	3	1	8	12
2	4	0	7	14
3	6	7	4	4

Least-Cost

Cost = \$38

5	10	1	8	12
3	2	4	0	14
4	3	6	7	4

VAM:

Cost = \$38

5	10	1	8	Penalties
2	10	0	6	4 4
3	2	4	11	2 2
4	3	6	7	3 3

Penalties

{ 1
1
3
3
- }

Set 5.3b

(i)

<u>u</u>	<u>v</u>	0	2	6	
0		(5) 0	1 2	5	6
-1		2	1 -	5	9
-3		-3	(4) +	(5) -	
-3		2	4	(5) 3	5
		5	5	10	

<u>u</u>	<u>v</u>	0	-3	1	
0		(5) 0	-5	1	6
4		2	1	5	9
7		0	-5	(5) 3	5
		5	5	10	

<u>u</u>	<u>v</u>	0	-1	1	
0		(1) 0	2	(5) 1	6
2		(4) 2	(5) 1	5	9
2		0	-3	(5) 3	5
		5	5	10	

Cost = \$33

Alternative solution exists

(ii)

<u>u</u>	<u>v</u>	0	4	2	
0		(7) 0	(1) 4	0	8
-1		2	(5) 3	4	5
-2		1	2	(6) 0	6
		7	6	6	

Problem has alternative optima. Cost = \$19

Note: If x_{23} were selected as the zero in place of x_{32} , solution would require one more iteration.

(i)

<u>u</u>	<u>v</u>	M	M-3	M-5	
0		(4) M	3	5	4
7-M		1	M-6	M-10	7
11-M		0	8	(19) 6	19
		5	6	19	

<u>u</u>	<u>v</u>	6	3	1	
0		6-M	(4) 3	-4	4
1		(5) 7	(2) 4	9	7
5		1	8	(19) 6	19
		10	6	19	

<u>u</u>	<u>v</u>	6	3	11	
0		6-M	(4) 3	5	4
1		(5) 7	(2) 4	9	7
-5		1	8	(19) 6	19
		10	6	19	

<u>u</u>	<u>v</u>	0	-3	5	
0		-M	-6	(4) 5	4
1		(1) -	(6) 3	9	7
1		(4) +	-10	(15) -	19
		5	6	19	

<u>u</u>	<u>v</u>	0	0	5	
-M		-3	(4) 5	4	
-3		7	(6) 4	(1) 9	7
(5)		1	8	(14) 6	19
		5	6	19	

Cost = \$142

continued...

Set 5.3b

(c)

Method

	(i)	Nbr. of iterations	(ii)	(iii)
NW	3	4	5	
Least-cost	2	2	2	
Vogel	2	1	1	

Least-cost starting solution:

u v 2 1 2

5	10	1	7
-3	10	-5	
70	6	4	10
5	10	2	5

75 20 50

u v 2 1 3

5	10	1	7
-2	10	-4	
60	10	4	10
15	3	2	5
-1	5	3	2

75 20 50

Destination 3 will be 40 units short. Optimum cost = \$595

Least-cost starting solution:

u v 2 1 1

5	10	1	7
-3	10	-6	
30	6	4	50
5	10	2	5

75 20 50

u v 3 1 3	10	80
0 -2 10 -4	20	10 50
3 6 4 9	15	5
0 0 2 -2	40	M

75 20 50

Total cost = \$515. Dest. 1 is 40 units short.

Vogel method:

1	2	1	5
3	4	5	M
2	3	3	20

1	2	20
3	4	5
2	3	3

3	4	5
2	3	0

20	20	4
10	3	

Set 5.3b

$u \backslash v$	0	1	1	1	
0	-1	-1	20	1	20
3	20	20	4	5	M
2	10	2	3	3	20
	30	20	20	20	30

Cost = \$240 - Alternative solution exists

$u \backslash v$	2	5	10	
-2	(15)	c_{12}	c_{13}	15
3	(5)	(25)	c_{23}	30
5	c_{31}	(5)	(80)	c_{33}
	20	30	80	

(a) $C_{ij} = u_i + v_j$ for basic x_{ij}

Thus,

$$C_{11} = 2 - 2 = 0$$

$$C_{21} = 3 + 2 = 5$$

$$C_{22} = 3 + 5 = 8$$

$$C_{32} = 5 + 5 = 10$$

$$C_{33} = 5 + 10 = 15$$

$$\text{Cost} = 15 \times 0 + 5 \times 5 + 25 \times 8 + 5 \times 10 \\ + 80 \times 15 = \$1475$$

(b) $u_i + v_j - C_{ij} \leq 0$ for nonbasic x_{ij}

$$-2 + 5 - C_{12} \leq 0 \Rightarrow C_{12} \geq 3$$

$$-2 + 10 - C_{13} \leq 0 \Rightarrow C_{13} \geq 8$$

$$3 + 10 - C_{23} \leq 0 \Rightarrow C_{23} \geq 13$$

$$5 + 2 - C_{31} \leq 0 \Rightarrow C_{31} \geq 7$$

5

Problems 6 and 7 on
next page

continued...

continued...

Set 5.3b

(a) For basic x_{ij} , $c_{ij} = u_i + v_j$.

u	2	2	5	
1	$c_{11}=3$ (10)	$1+2\theta$	$1+3\theta$	10
-1	$2+\theta$	$c_{21}=1$ (20)	$c_{23}=4$ (20)	40

6

$$\text{Cost} = 3 \times 10 + 1 \times 20 + 4 \times 20 = \$130$$

(b) For nonbasic x_{ij} : $u_i + v_j - c_{ij} \leq 0$ to satisfy optimality. Hence

$$2+1-(1+2\theta) \leq 0 \Rightarrow \theta \geq 1$$

$$5+1-(1+3\theta) \leq 0 \Rightarrow \theta \geq 5/3$$

$$2-1-(2+\theta) \leq 0 \Rightarrow \theta \geq -1$$

Take $\theta = \frac{5}{3}$ to yield $x_{13} = 0$ as the zero basic variable.

7

$$\begin{array}{ccccccc} & x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\ \text{Min } Z = & 1 & 1 & 2 & 6 & 5 & 1 \end{array}$$

s.t.

$$\begin{array}{ccccc} 1 & 1 & 1 & & \geq 5 \\ & 1 & 1 & 1 & \geq 6 \\ 1 & & 1 & & \geq 2 \\ 1 & & & 1 & \geq 7 \\ & 1 & & 1 & \geq 1 \end{array}$$

$x_{ij} \geq 0$ for all i and j

Optimum LP solution using TORA:

$$Z = 15, x_{11} = 2, x_{12} = 3, x_{23} = 6$$

If we replace the first two constraints with equations, we get the optimum solution:

$$Z = 27, x_{11} = 2, x_{12} = 3,$$

$$x_{22} = 4, x_{23} = 2$$

The new solution is worse!

Set 5.3c

U_1	U_2	U_3	V_1	V_2	V_3	V_4	
Max 15	25	10	5	15	15	15	

s.t.

$$\begin{array}{l} | \\ | \quad \quad \quad \leq 10 \\ | \quad \quad \quad \leq 2 \\ | \quad \quad \quad \leq 20 \\ | \quad \quad \quad \leq 11 \\ | \quad \quad \quad \leq 12 \\ | \quad \quad \quad \leq 7 \\ | \quad \quad \quad \leq 9 \\ | \quad \quad \quad \leq 20 \\ | \quad \quad \quad \leq 4 \\ | \quad \quad \quad \leq 14 \\ | \quad \quad \quad \leq 16 \\ | \quad \quad \quad \leq 18 \end{array}$$

From Table 5-25:

$$\begin{aligned} U_1 &= 0, U_2 = 5, U_3 = 7 \\ V_1 &= -3, V_2 = 2, V_3 = 4, V_4 = 11 \end{aligned}$$

$$\begin{aligned} \text{Optimum } W &= 15x_0 + 25x_5 + 10x_7 \\ &\quad + 5x_{-3} + 15x_2 + \\ &\quad 15x_4 + 15x_{11} \\ &= \$435 \end{aligned}$$

Minimize $Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$

subject to

$$\sum_{j=1}^n x_{ij} = a_i, \quad i=1, 2, \dots, m$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad j=1, 2, \dots, n$$

Next, consider

$$\begin{aligned} Z' &= \sum_{i=1}^m \sum_{j=1}^n (c_{ij} + K) x_{ij} \\ &= \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} + K \sum_{i=1}^m \left(\sum_{j=1}^n x_{ij} \right) \\ &= \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} + K \sum_{i=1}^m a_i \end{aligned}$$

continued...

$$\begin{aligned} &= \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} + K, \quad K \text{ is a constant} \\ &= Z + K \end{aligned}$$

This result shows that optimization using Z and Z' yield the same optimum values of x_{ij} .

To show why the dual values associated with a given primal basic solution are not unique, note that, for any constant K ,

$$(\text{Dual Values}) = \begin{pmatrix} \text{Original basic} \\ \text{obj. coefficients} \\ + \\ K \end{pmatrix} \times \text{Inverse}$$

This means that even though the optimal primal solution is unique for all K , there are infinity of dual values, each corresponding to a given value of K .

The conclusion is that an arbitrary value assigned to one of the dual variables (e.g., $U_1 = 0$) implies a specific value for the constant K .

2

Set 5.4a

(a-i)

3	8	2	10	3	2
8	7	2	9	7	2
6	4	2	7	5	2
8	4	2	3	5	2
9	10	6	9	10	6

Row min

1

0	7	0	0	5
4	0	4	4	5
5	1	4	6	0
0	4	3	0	0
6	4	0	1	4

Optimum:

1-1

2-2

3-5

4-4

5-3

Cost = \$11

1	6	0	8	1
6	5	0	7	5
4	2	0	5	3
6	2	0	1	3
3	4	0	3	4

Col min → 1 2 0 1 1

Assignment:

0	4	2	7	0
3	1	0	4	2
3	0	2	4	2
5	0	2	0	2
0	0	0	0	1

Cost = \$21

(a-ii)

3	9	2	2	7	2
6	1	5	6	6	1
9	4	7	10	3	3
2	5	4	2	1	1
9	6	2	4	6	2

1	7	0	1	5
5	0	4	5	5
6	1	4	7	0
1	4	3	1	0
7	4	0	2	4

Col min 1 0 0 1 0

continued...

5	5	M	2
7	4	2	3
9	3	5	M
7	2	6	7

3	3	M-2	0
5	2	0	1
6	0	2	M-3
5	0	4	5

(All entries are divided by 10 for convenience)

0	3	M-2	0
2	2	0	1
3	0	2	M-3
2	0	4	5

0	5	M-2	0
2	4	0	1
1	0	0	M-5
0	0	4	5

Optimum: 1-4, 2-3, 3-2, 4-1
Cost = \$140

1	50	50	M	20	0
2	70	40	20	30	0
3	90	30	50	M	0
4	70	20	60	70	0
5	60	45	30	80	0

Job 5 is dummy

1	2	3	4	5
0	30	M-20	0	0
20	20	0	10	0
40	10	30	M-20	0
20	0	40	50	0

1	2	3	4	5
0	30	M-20	0	10
20	20	0	10	10
30	0	20	M-30	0
20	0	40	50	10

Optimum:

1-4

2-3

3-5

4-2

5-1

Worker 3 is assigned to dummy job 5.
Thus, worker 5 must replace worker 3.

Set 5.4a

Add a "dummy" operator with zero assignment cost to each job (including the fifth). The optimal solution will show the replacement by indicating which of the current jobs (1 thru 4) is assigned to the dummy operator. If the dummy operator is assigned to the new job, then the new job must assume lower priority to the current four jobs. (All assignment cost are divided by 10 for convenience.)

	Job				
1	2	3	4	5	
Operator	5	5	M	2	2
2	7	4	2	3	1
3	9	3	5	M	2
4	7	2	6	7	8
5	0	0	0	0	0

← Dummy

3	3	M-3	0	0
6	3	1	2	0
7	1	3	M-2	0
5	0	4	5	0
0	0	0	0	0

Optimum:

2	2	M-4	0	0
5	2	0	2	0
6	0	2	M-2	0
5	0	4	6	7
0	0	0	1	0

1 - 4
2 - 3
3 - 5
4 - 2
(S-1)

Since dummy operator is assigned to job 1, new job 5 has higher priority over job 1.

Define the following two sets:

Set 1: (DA,3), (DA,10), (DA,17), (DA,25)

continued

Set 2: (AT,7), (AT,12), (AT,21), (AT,28).

The idea is to match one element from set 1 with another element from set 2. The matching automatically decides the date and location for the purchase of each ticket. For example, consider the following assignment:

- (DA,3) - (AT,21)
- (DA,10) - (AT,7)
- (DA,17) - (AT,28)
- (DA,25) - (AT,12)

This assignment can be interpreted as follows:

Ticket 1: June 3 DA → AT
June 21 AT → DA

Ticket 2: June 7 AT → DA
June 10 DA → AT

Ticket 3: June 17 DA → AT
June 28 AT → DA

Ticket 4: June 12 AT → DA
June 25 DA → AT

The complete assignment model is given below

	A,7	A,12	A,21	A,28
D,3	400	300	300	280
D,10	300	400	300	300
D,17	300	300	400	300
D,25	300	300	300	400

Optimum:

- (D,3) - (A,28) (A,21) - (D,25)
- (A,7) - (D,10) (A,12) - (D,17)

Problem has alternative optima.

Set 5.4a

Distance matrix in meters:

	candidate areas				
	a	b	c	d	
existing centers	1	50	50	95	45
	2	30	30	55	65
	3	70	50	25	55
	4	100	60	55	25

A measure of the optimal assignment of new centers to candidate locations must reflect both distance and frequency of trips; that is

	existing				candidate				
	1	2	3	4	a	b	c	d	
new	I	10	7	0	11	50	50	95	45
	II	2	1	8	4	30	30	55	65
	III	4	9	6	0	70	50	25	55
	IV	3	5	2	7	100	60	55	25

	a	b	c	d	
New	I	1810	1370	1940	(1180)
	II	1090	770	(665)	695
	III	(890)	770	1025	1095
	IV	1140	(820)	995	745

TORA optimum assignment:

- I - d
- II - c
- III - a
- IV - b

6

The ranking of the projects by the different teams can use the following numeric score

7

1: Highest preference

10: Lowest preference

A tie in preference between two or more projects is indicated by assigning the projects the same score. For example, the scores

Project	1	2	3	4	5	6	7	8	9	10
Score	9	9	8	7	3	5	4	1	2	6

indicate that project 8 is the most preferred and projects 1 and 2 tie for the least preferred status.

For the development of the model, we use the following numeric designations for the projects

Project nbr.	Project name
1	Boeing-F15
2	Boeing-F18
3	Boeing Simulation
4	Cargil
5	Cobb-Vantress
6	ConAgra
7	Cooper
8	DaySpring(layout)
9	DaySpring(Materials)
10	JB Hunt
11	Raytheon
12	Tyson South
13	Tyson East
14	WAL-MART
15	Yellow

continued...

Set 5.4a

The following is a typical summary of preference scores submitted by the 11 teams:

	1*	2	3	Team	4	5	6*	7*	8*	9	10	11
1*	-	①	2	2	1	-	-	-	-	2	15	
2*	-	1	3	①	2	-	-	-	-	10	12	
3	1	2	5	3	2	13	5	1	4	15	①	
4	②	3	6	4	10	5	14	2	1	4	14	
5	3	5	4	5	9	4	12	3	3	13	13	
6	3	4	2	5	9	8	12	①	2	1	13	
7	4	6	①	12	8	9	10	2	5	2	5	
8	5	6	7	14	7	9	10	4	6	3	15	
9	7	8	9	14	7	1	①	15	1	15	1	
10	7	9	12	15	6	3	9	5	4	7	5	
11*	-	9	13	6	5	-	-	7	-	6	7	
12	13	10	14	7	4	②	8	9	15	4	9	
13	14	11	1	8	3	13	7	8	①	8	9	
14	15	12	5	9	①	14	7	6	2	9	10	
15	15	13	7	10	2	15	6	1	3	①	11	

* Team does not meet citizenship requirements

② Project requiring US citizenship

The problem is modeled as an assignment model. Entries — are replaced by M, a very large value. The model is unbalanced. Thus, 4 artificial teams must be added to balance the model. In the end four projects will not be assigned.

TOA Solution:

Project	Team	Score
1	2	1
2	4	1
3	11	1

continued...

Project	Team	Score
4	1	1
5	None	—
6	8	1
7	3	1
8	None	—
9	7	1
10	None	—
11	None	—
12	6	2
13	10	1
14	5	1
15	10	1

Total score 13

$$\text{Average score} = \frac{13}{11} = 1.18$$

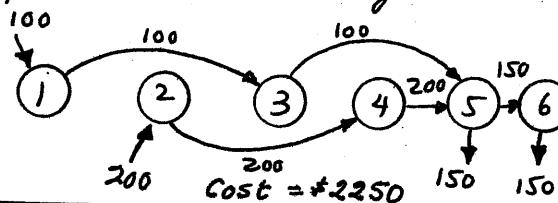
The average score is close to 1, meaning that all preferences are well met.

Set 5.5a

	1	4	M	M	6
1	(100)				100
2	3	2	M	M	
3	(200)	1	(100)	6	M
4	3	0	5	8	
5	M	M	(150)	(150)	1
	B	B	150+B	150	

Let $B = 300$ units

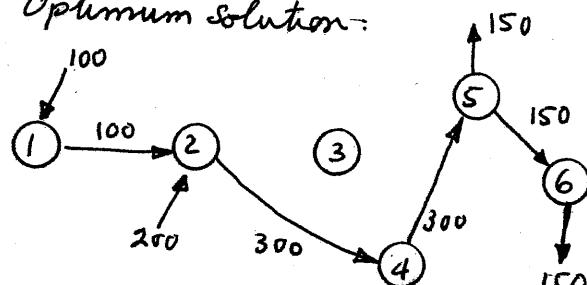
Optimum solution using TORA:



$B = 300$ units

	1	2	3	4	5	6	7
1	(100)						100
2	0	3	(300)	2	M	M	
3	M	0	1	6	M		
4	M	3	0	(300)	5	8	
5	M	M	M	(150)	(150)	1	
	B	B	B	150+B	150		

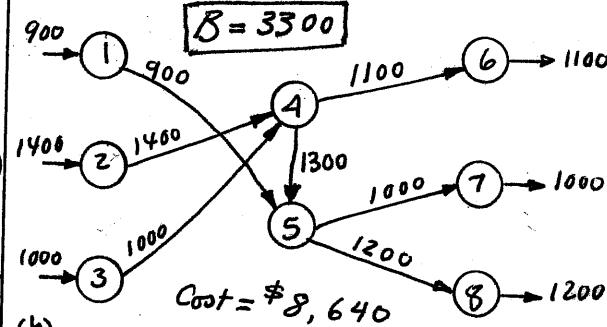
Optimum solution:



Cost = \$2,350

	4	5	6	7	8	9
1		(900)	M	M	M	900
2	8	4.3	M	M	M	1400
3	2	4.6	M	M	M	1000
4	0	1.5	.2	4.5	6	B
5	900	1300	1100	2.1	1.9	B
	M	0	3	1000	1200	
	1100	1000	1200			

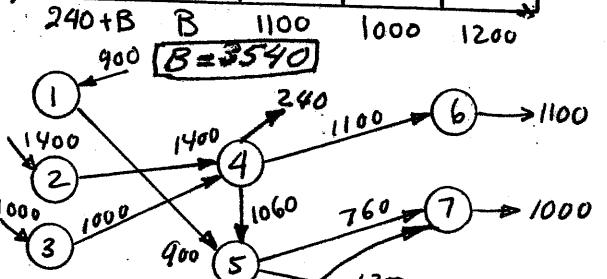
$B = 3300$



(b) 2

	4	5	6	7	8	9
1		(900)	M	M	M	900
2	8	4.3	M	M	M	1400
3	2	4.6	M	M	M	1000
4	0	1.5	.2	4.5	6	B
5	1300	1060	1100	2.1	1.9	B
	M	0	3	760	1200	
	1580	0	0	240	0	940

(b) 2



(b) 2

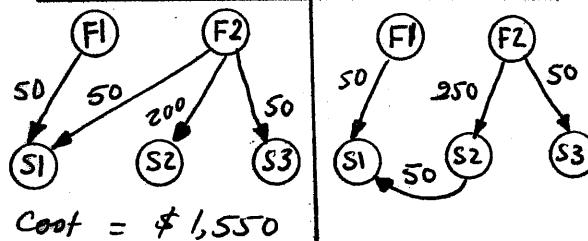
Cost = \$8,016

Set 5.5a

	F1	F2	S1	S2	S3	Dum.
F1	0 (500)	6 (50)	7 (50)	8 (50)	9 (50)	0 (150)
F2	6 (500)	0 (50)	5 (200)	4 (50)	3 (50)	0 (300)
S1	7 (500)	2 (500)	0 (500)	5 (500)	1 (500)	0 (500)
S2	1 (500)	5 (500)	1 (500)	0 (500)	4 (500)	0 (500)
S3	8 (500)	9 (500)	7 (500)	6 (500)	0 (500)	0 (500)
	B	B	$\frac{100}{B}$	$\frac{200}{B}$	$\frac{50}{B}$	150

$$B = 500$$

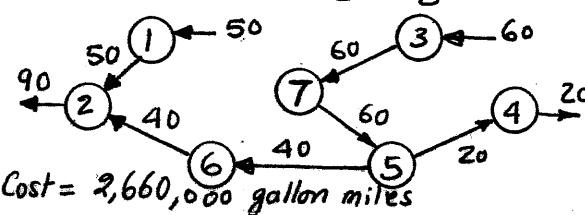
Alternative optima using TORA:
#1 #2



$$\text{Cost} = \$1,550$$

Assume that units of supply and demand are in thousand gallons. $B = 110$

	2	4	5	6	7	
1	20 (50)	m	m	m	3 (50)	50
3	m	30	m	m	9 (60)	60
5	m	2	0	4	10 (40)	B
6	8 (40)	m	4	0 (70)	m	B
7	40	m	10 (60)	m	0 (50)	B
	90	20	B	B	B	



$$\text{Cost} = 2,660,000 \text{ gallon miles}$$

	2	3	4	5	6	7	
1	5 (1)	3 (1)	m	m	m	m	1
2	0 (1)	4	1	7	m	m	0+1
3	6	0	5	1	2 (1)	m	0+1
4	m	m	0 (1)	9	m	4	0+1
5	m	m	2	0	5	8	0+1
6	m	3	m	7	0	3 (1)	0+1
	0+1	0+1	0+1	0+1	0+1	1	

Optimum route using TORA:

$$1 \rightarrow 3 \rightarrow 6 \rightarrow 7$$

$$\text{Distance} = 3 + 2 + 3 = 8$$

minimize Z.

$$x_{13} x_{14} x_{23} x_{24} x_{34} x_{35} x_{36} x_{46} x_{47} x_{56} x_{67}$$

$$z = 3 \quad 4 \quad 2 \quad 5 \quad 7 \quad 8 \quad 6 \quad 4 \quad 9 \quad 5 \quad 3$$

①	1	1									= 1000
②			1	1							= 1200
③	-1	-1	1	1	1						= 0
④	-1	-1	-1	-1		1	1				= 0
⑤					-1			1			= -800
⑥					-1	-1	-1	1			= -900
⑦					-1	-1	-1	-1	-1	-1	= -500

Each node yields a constraint. The special characteristics of the model show that each column has exactly +1 and -1, with the remainder of the elements equal to zero.

Set 5.5a

x_{ij} = number of laborers hired at the start of period i
and terminated at the start of period j .

8

Define nodes 1, 2, 3, 4, and 5 to correspond to the five months of the horizon. Node 6 is added to allow defining the variables x_{i6} that terminate at the end of the five-month planning horizon. The associated LP is defined below.

	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{23}	x_{24}	x_{25}	x_{26}	x_{34}	x_{35}	x_{36}	x_{45}	x_{46}	x_{56}	
	100	130	180	220	250	100	130	180	220	100	130	180	100	130	100	min
(1)	1	1	1	1	1											≥ 100
(2)		1	1	1	1	1	1	1								≥ 120
(3)			1	1	1		1	1	1	1	1					≥ 80
(4)				1	1			1	1	1	1	1	1			≥ 170
(5)						1			1		1		1	1	1	≥ 50

Let s_1, s_2, s_3, s_4 , and s_5 be the surplus variables associated with constraints 1, 2, 3, 4, and 5, respectively. The LP after adding the surplus variables thus appears as

	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{23}	x_{24}	x_{25}	x_{26}	x_{34}	x_{35}	x_{36}	x_{45}	x_{46}	x_{56}	s_1	s_2	s_3	s_4	s_5		
	100	130	180	220	250	100	120	180	220	100	130	180	100	130	100		min					
1	1	1	1	1	1											-1					100	
	1	1	1	1	1	1	1	1									-1				120	
		1	1	1		1	1	1	1	1	1							-1			80	
			1	1			1	1	1	1	1	1	1					-1			170	
						1			1		1		1	1					-1			50

Next, perform the following transformation:

1. Leave equation (1) unchanged.
2. Replace equation (2) with $(2) - (1)$.
3. Replace equation (3) with $(3) - (2)$.
4. Replace equation (4) with $(4) - (3)$.
5. Replace equation (5) with $(5) - (4)$.
6. Add a new equation that equals $-(5)$.

These transformations lead to the following LP

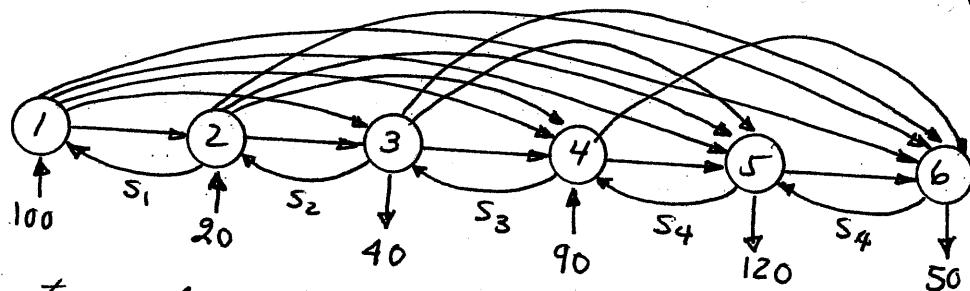
	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}	x_{23}	x_{24}	x_{25}	x_{26}	x_{34}	x_{35}	x_{36}	x_{45}	x_{46}	x_{56}	s_1	s_2	s_3	s_4	s_5		
	100	130	180	220	250	100	130	180	220	100	130	180	100	130	100		min					
1	1	1	1	1	1											-1					100	
	-1					1	1	1	1							1	-1				20	
		-1				-1				1	1	1					1	-1				-40
			-1				-1		-1	1	1	1					1	-1				90
				-1				-1	-1	-1	-1	-1	1	1			1	-1				-120
					-1				-1	-1	-1	-1	-1	-1	-1		1	-1				-50

continued...

Set 5.5a

The last LP has the structure of a transhipment model (See Problem 7). Let
 $S_1 = x_{21}$ $S_3 = x_{43}$ $S_5 = x_{65}$
 $S_2 = x_{32}$ $S_4 = x_{54}$

Then the LP above can be translated as a network as follows:

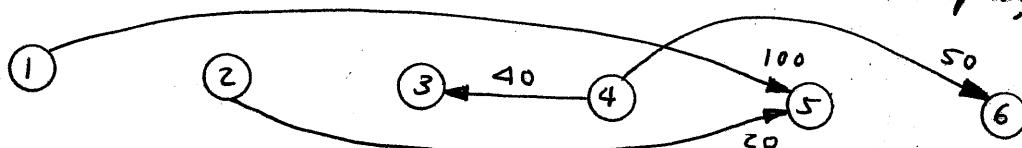


The transhipment model thus appears as

	1	2	3	4	5	6	
1	0	100	130	180	220	250	
2	0	0	100	130	180	220	
3	M	0	0	100	130	180	
4	M	M	0	0	100	130	
5	M	M	M	0	0	100	
6	M	M	M	M	0	0	
	B	B	40+B	B	120+B	50+B	

$$B = 550$$

The optimum solution from TORA is (Problem has alternative optima)



This solution can be interpreted as follows

1. Hire 100 laborers at the start of period 1 and terminate them at the start of period 5.
2. Hire 20 workers at the start of period 2 and terminate them at the start of period 5.
3. Hire 50 workers at the start of period 4 and terminate them at the start of period 6.

The solution satisfies the labor requirements exactly, except for period 3 where there is a surplus of 40 workers ($x_{43} = 40$).

CHAPTER 6

Network Models

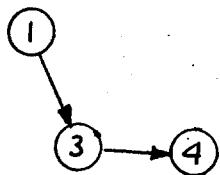
Set 6.1a

(i)

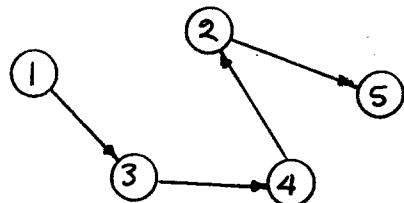
(a) Path: 1-3-4-2

(b) Cycle: 1-3-4-5-1

(c) Tree



(d) Spanning tree:

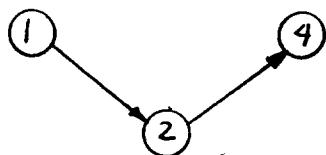


(ii)

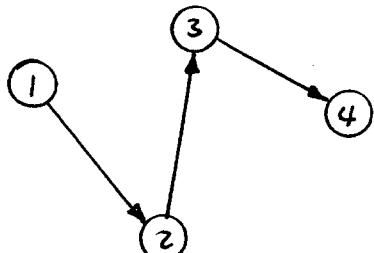
(a) Path: 1-2-3

(b) Cycle: 1-2-3-1

(c) Tree



(d) Spanning Tree:



1

(i) $N = \{1, 2, 3, 4, 5\}$

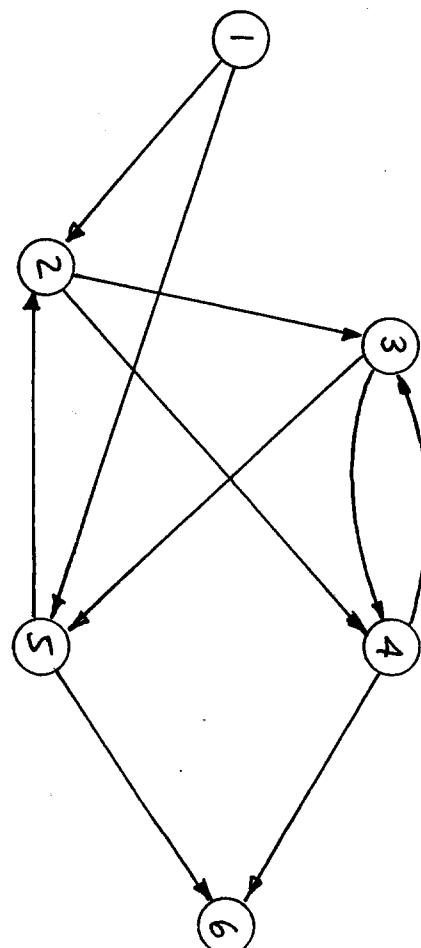
$A = \{1-2, 1-3, 2-5, 3-4, 3-5, 4-2, 4-5, 5-1\}$

(ii) $N = \{1, 2, 3, 4\}$

$A = \{1-2, 1-3, 2-3, 2-4, 3-4\}$

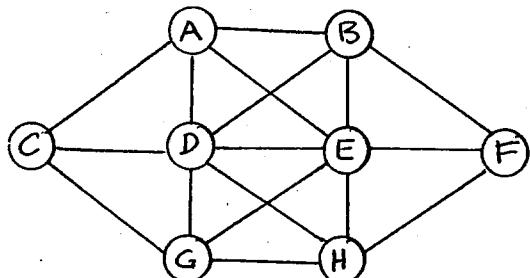
2

3

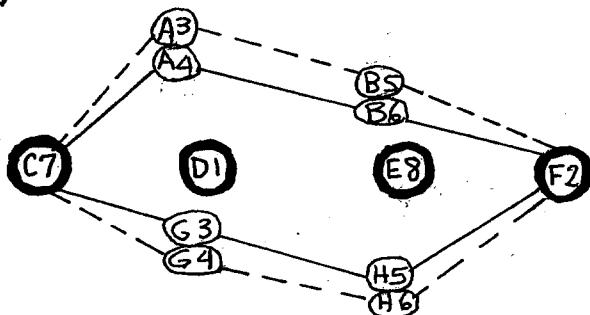


4

A	B		
C	D	E	F
G	H		



The network shows that nodes connected by an arc cannot hold consecutive numbers. Nodes D and E each has 6 emanating arcs, whereas all the remaining nodes have at most 4 emanating arcs. Because 1 and 8 each can have 6 nonconsecutive neighbors (namely, 1-3, 1-4, 1-5, 1-6, 1-7, 1-8 or 8-6, 8-5, 8-4, 8-3, 8-2, 8-1) and no other number has this property, 1 and 8 must be assigned to D and E. Letting D=1 and E=8, we must assign C=7 and F=2 because 2 and 7 can't be assigned anywhere else without violating the sequence condition. Next, we have the following possibilities:



Two possible solutions indicated by the solid and dashed arcs:

4	6		
7	1	8	2
3	5		

3	5		
7	1	8	2
4	6		

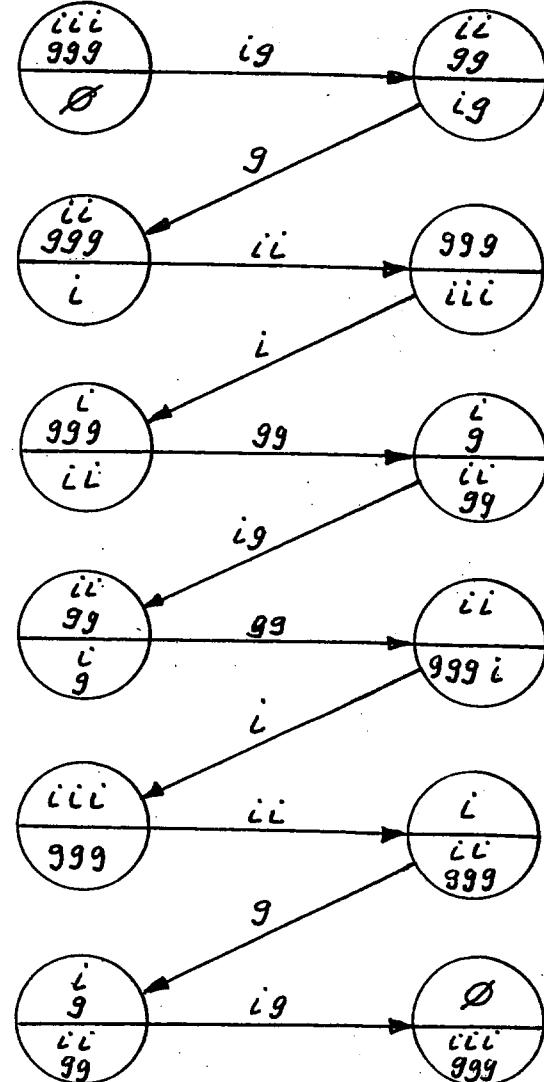
Switch D=1 and E=8 to two mirror arrangements.

Let

$i \equiv \text{innate}$
 $g \equiv \text{guard}$

For each node, top half represents the number of i 's and g 's on the mainland side. The bottom half is that of Alcatraz.

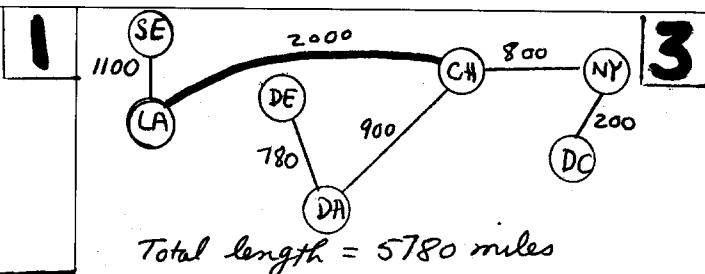
5



Set 6.2a

Spanning tree length = 16

0. Start at node N5
1. Connect N2 to N5: Length = 3.
2. Connect N1 to N2: Length = 1.
3. Connect N4 to N2: Length = 4.
4. Connect N6 to N4: Length = 3.
5. Connect N3 to N4: Length = 5.



(a) Spanning tree length = 14

0. Start at node N1
1. Connect N2 to N1: Length = 1.
2. Connect N5 to N2: Length = 3.
3. Connect N6 to N5: Length = 2.
4. Connect N4 to N6: Length = 3.
5. Connect N3 to N4: Length = 5.

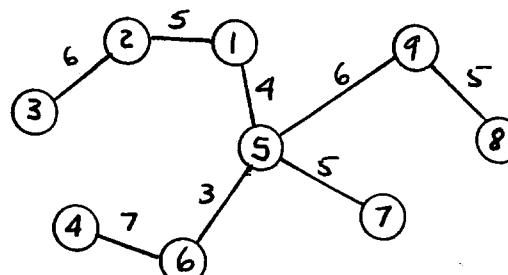
2

Order of selection : 1-5, 5-6,
1-2, 5-7, 2-3, 5-9, 9-8, 6-4

4

(b) Spanning tree length = 21

0. Start at node N1
1. Connect N2 to N1: Length = 1.
2. Connect N4 to N2: Length = 4.
3. Connect N6 to N4: Length = 3.
4. Connect N3 to N4: Length = 5.
5. Connect N5 to N4: Length = 8.



Total length = 41

(c) Spanning tree length = 16

0. Start at node N1
1. Connect N2 to N1: Length = 1.
2. Connect N5 to N2: Length = 3.
3. Connect N6 to N2: Length = 4.
4. Connect N4 to N6: Length = 3.
5. Connect N3 to N4: Length = 5.

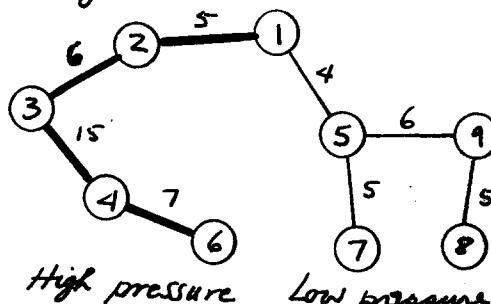
Set length of arcs 3-5, 5-3, 4-5, 5-4, 4-7,
7-4, 5-6, and 6-5 to ∞

Total length = 53

5

(d) Spanning tree length = 20

0. Start at node N1
1. Connect N3 to N1: Length = 5.
2. Connect N4 to N3: Length = 5.
3. Connect N6 to N4: Length = 3.
4. Connect N2 to N4: Length = 4.
5. Connect N5 to N2: Length = 3.



(e) Spanning tree length = 13

0. Start at node N1
1. Connect N2 to N1: Length = 1.
2. Connect N5 to N2: Length = 3.
3. Connect N3 to N5: Length = 2.
4. Connect N4 to N2: Length = 4.
5. Connect N6 to N4: Length = 3.

(a) $d_{ij} = 1 - \frac{n_{ij}}{n_{ij} + m_{ij}}$

6

$i-j$	n_{ij}	m_{ij}	d_{ij}
1-2	0	10	1
1-3	0	6	1
1-4	0	8	1
1-5	0	7	1
1-6	1	5	.83
1-7	0	8	1
1-8	0	5	1
1-9	0	4	1
1-10	0	7	1

continued...

(f) Spanning tree length = 21

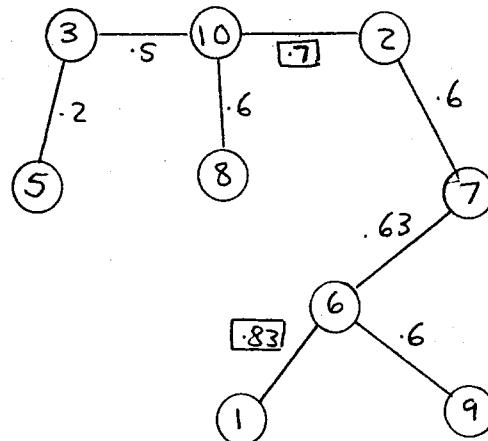
0. Start at node N1
1. Connect N2 to N1: Length = 1.
2. Connect N4 to N2: Length = 4.
3. Connect N6 to N4: Length = 3.
4. Connect N3 to N4: Length = 5.
5. Connect N5 to N4: Length = 8.

Set 6.2a

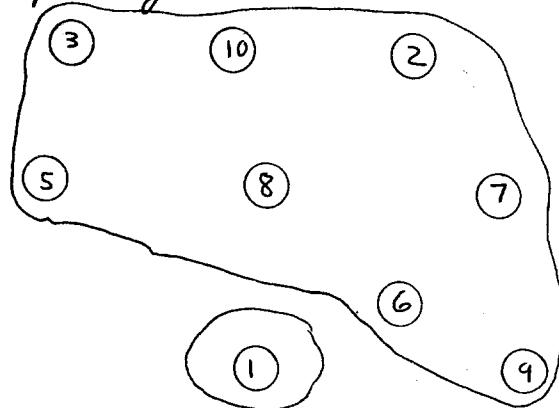
$i-j$	n_{ij}	m_{ij}	d_{ij}
2-3	1	10	.91
2-4	5	4	.44
2-5	1	11	.92
2-6	1	11	.92
2-7	4	6	.6
2-8	2	7	.78
2-9	0	10	1
2-10	3	7	.7
3-4	0	10	1
3-5	4	1	.2
3-6	2	5	.71
3-7	2	6	.75
3-8	1	5	.83
3-9	1	4	.8
3-10	3	3	.5
4-5	1	9	.9
4-6	0	11	1
4-7	3	6	.67
4-8	0	9	1
4-9	0	8	1
4-10	1	9	.9
5-6	2	6	.75
5-7	2	7	.78
5-8	1	6	.86
5-9	1	5	.83
5-10	3	4	.57
6-7	3	5	.63
6-8	1	6	.86
6-9	2	3	.60
6-10	1	8	.89
7-8	0	9	1
7-9	1	6	.86
7-10	1	9	.9
8-9	1	3	.75
8-10	2	4	.67
9-10	1	5	.83

continued...

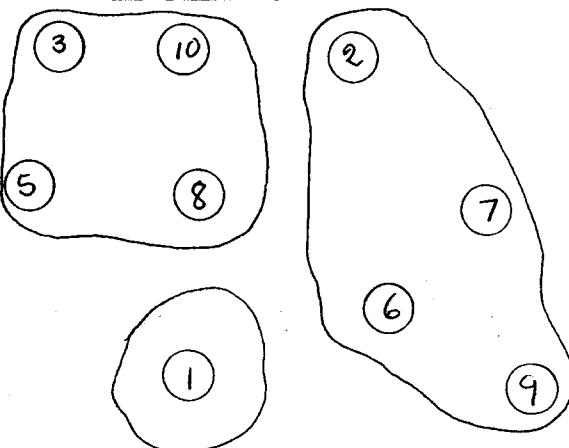
(b) Spanning Tree



(c) A 2-cell solution is formed by removing the highest link in the minimal spanning tree.

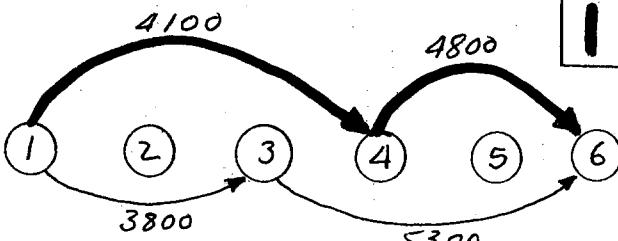


3-cell solution:



continued...

Set 6.3a



Optimal route: 1-4-6

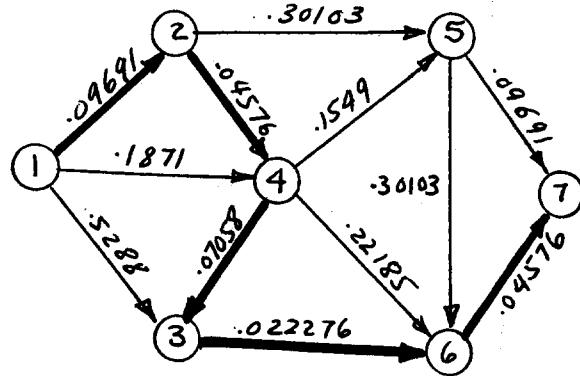
Decision: Buy new cars in 2001 & 2004.

$$\max(P_1 P_2 \cdots P_n)$$

2

$$\equiv \max(\log P_1 + \log P_2 + \cdots + \log P_n)$$

$$\equiv \min(-\log P_1 - \log P_2 - \cdots - \log P_n)$$



Optimum solution by TORA:

1-2-4-3-6-7

$$\sum_{i=1}^7 \log P_i = -281286. \text{ Thus,}$$

$$\sum_{i=1}^7 \log P_i = -281286.$$

Hence,

$$P = 10^{-281286} = .52326$$

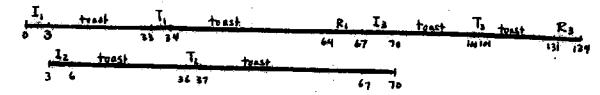
Define

(i, j, k) = number of sides toasted of slices 1, 2, and 3

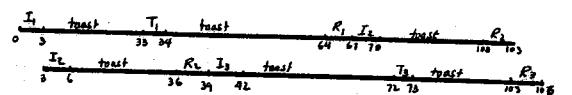
5

The two time charts below provides a summary of the times between the successive nodes.

Problem 4 on p. 6-7

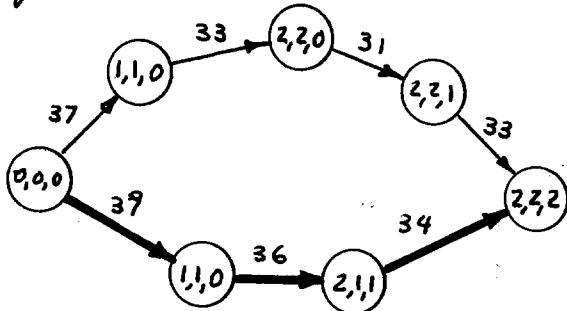


(0,0,0) (1,1,0) (2,2,0) (2,2,1) (2,2,2)



(0,0,0) (1,1,0) (2,2,1) (2,2,2)

The associated network is shown given as



The optimal sequence is $(0,0,0) \rightarrow (1,1,0) \rightarrow (2,1,1) \rightarrow (2,2,2)$. It is interpreted as follows:

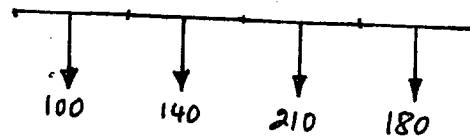
- Toast both sides of slice 1 successively (without interruption) on Side A.
- Toast side 1 of slice 2 in Side B, then remove slice 2.
- Toast both sides of slice 3 in Side B
- Toast side 2 of slice 2 in Side A after slice 1 is toasted.

Total time = 106 seconds.

Summary of the problem data

3

h:	\$1.20	\$1.20	\$1.20
C:	\$15	\$12	\$10

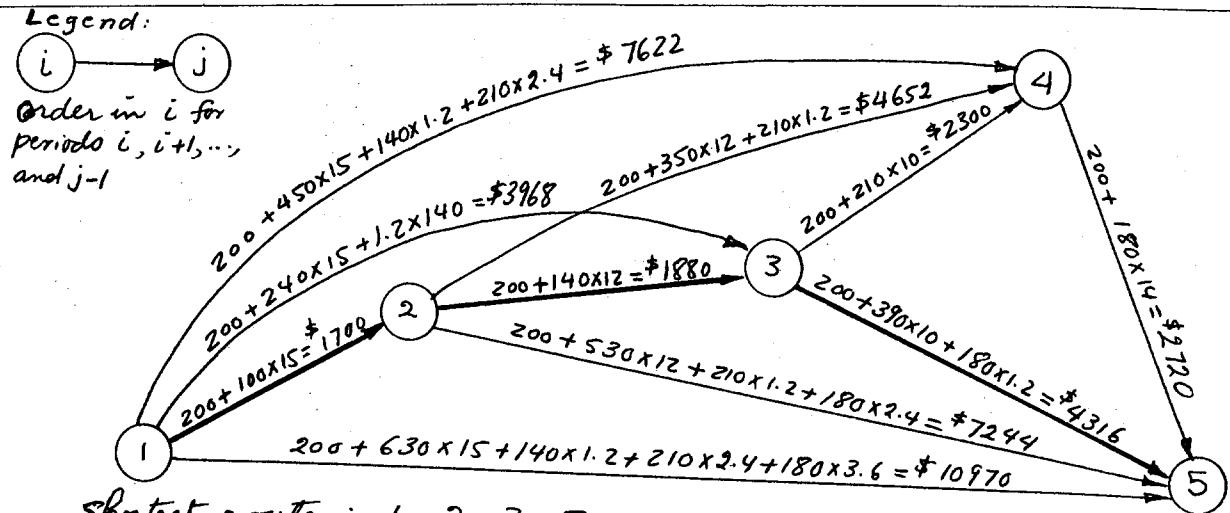


Setup cost = \$200

continued...

continued...

Set 6.3a



Shortest route : 1 - 2 - 3 - 5

Interpretation of the solution: order 100 units in Period 1, 140 units in Period 2, and 390 units in Period 3. Total cost = \$ 7896

Define node (i, v) , where i is the item number and v is the volume remaining before item i is selected. Each arc represents a feasible value of the number of units of item i .

Item i	1	2	3
Volume/unit	2	3	4
Value/unit	30	50	70
Total available volume = 5 ft^3			

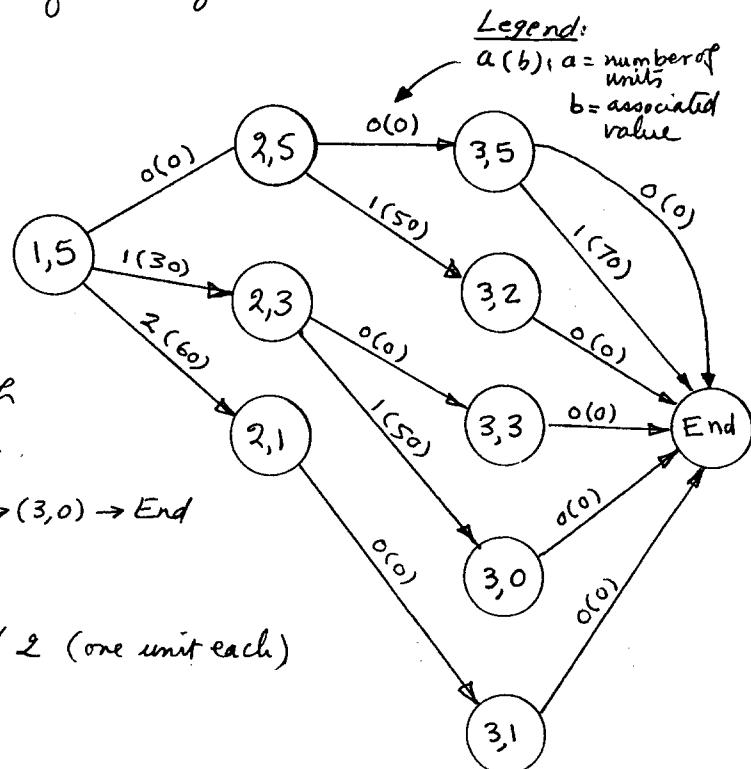
The objective is to determine the longest path between $(1, 5)$ and (End) .

Longest path : $(1, 5) \rightarrow (2, 3) \rightarrow (3, 0) \rightarrow \text{End}$

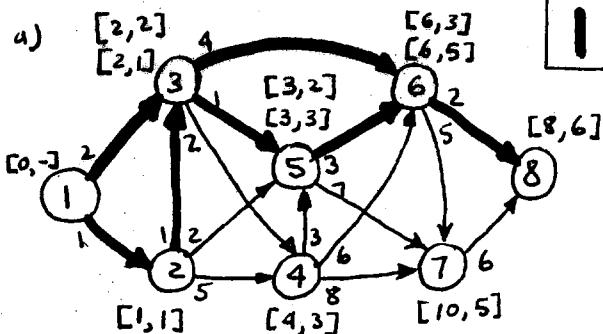
Interpretation of the solution:

Select items 1 and 2 (one unit each)

Total value = 80



Set 6.3b

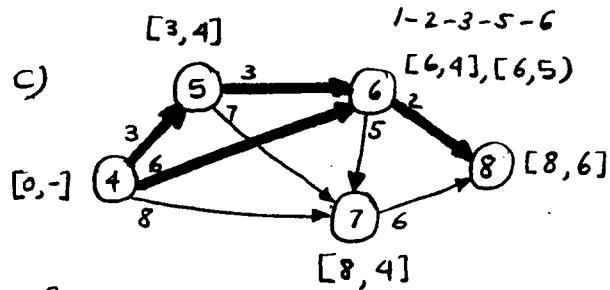


Shortest distance = 8:

alternative routes: 1-3-6-8
1-2-3-6-8
1-3-5-6-8
1-2-3-5-6-8

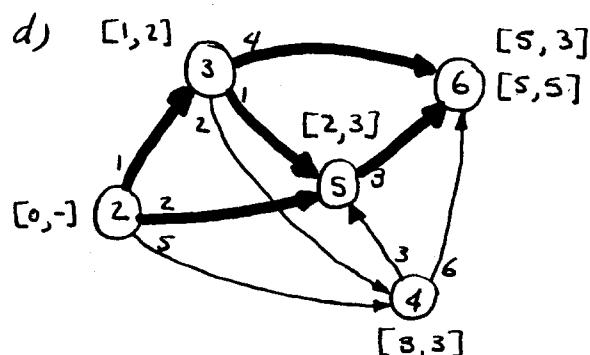
b) From part (a), shortest distance between ① and ⑥ is 6.

alternative routes: 1-3-6
1-3-5-6
1-2-3-6
1-2-3-5-6



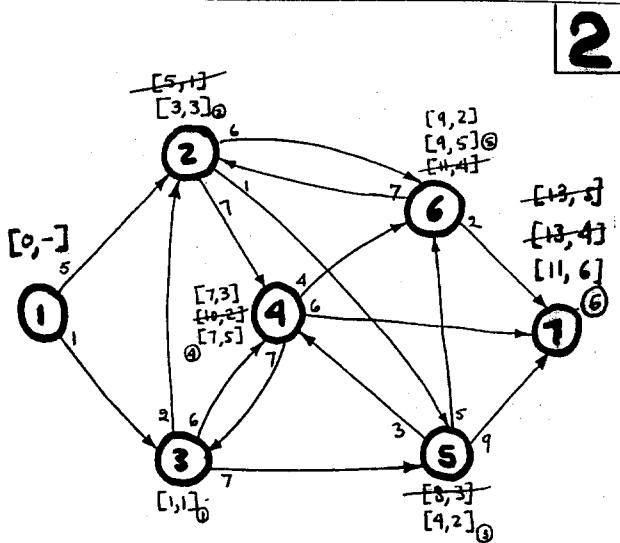
Shortest distance = 8

alternative routes: 4-5-6-8
4-6-8



Shortest distance = 5

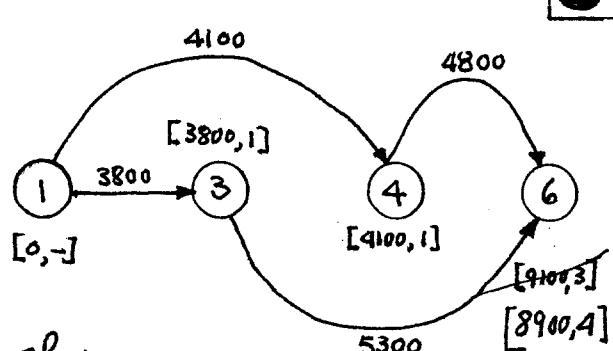
Alternative routes = {
2-3-6
2-3-5-6
2-5-6}



Shortest routes:

	Length
1-2 : 1-3-2	3
1-3 : 1-3	1
1-4 : { 1-3-4 } 1-3-2-5-4	7
1-5 : 1-3-2-5	4
1-6 : { 1-3-2-5-6 } 1-3-2-6	9
1-7 : { 1-3-2-5-6-7 } 1-3-2-6-7	11

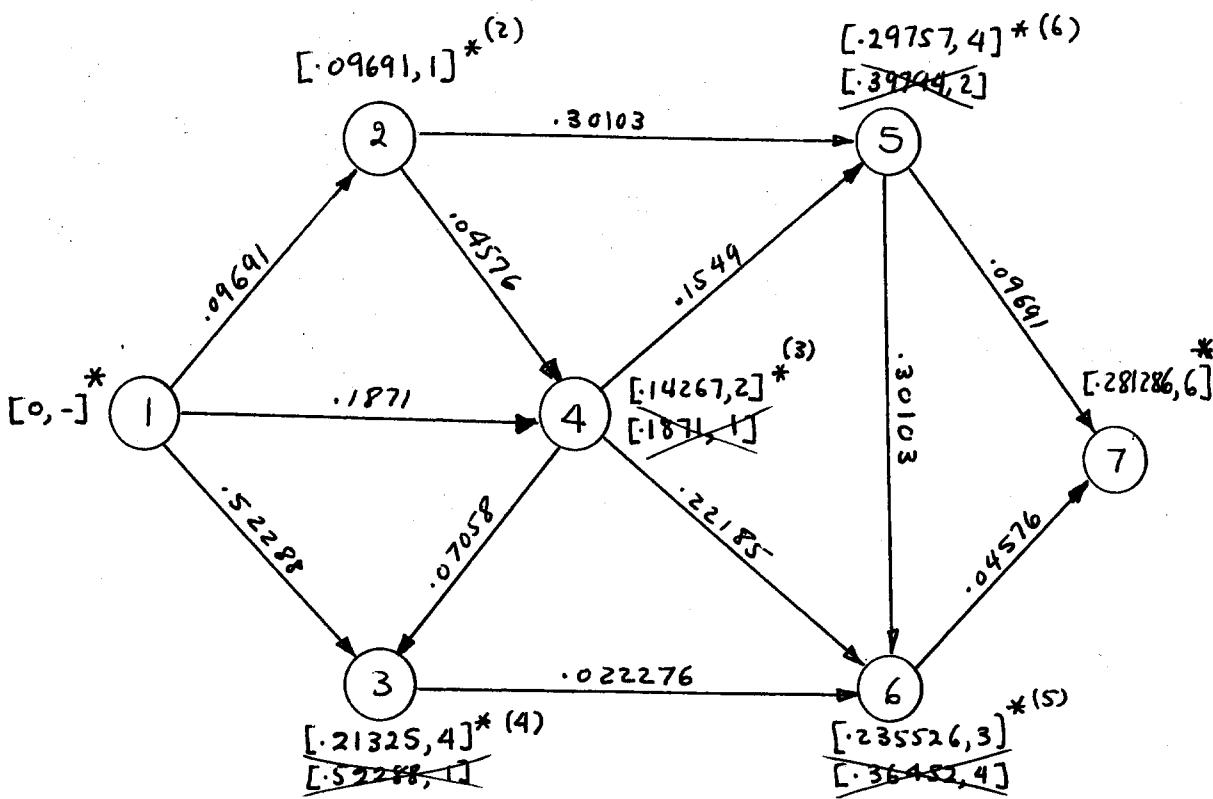
(a)



Shortest route: 1-4-6. Cost = \$ 8900
Buy in 2001 & 2004

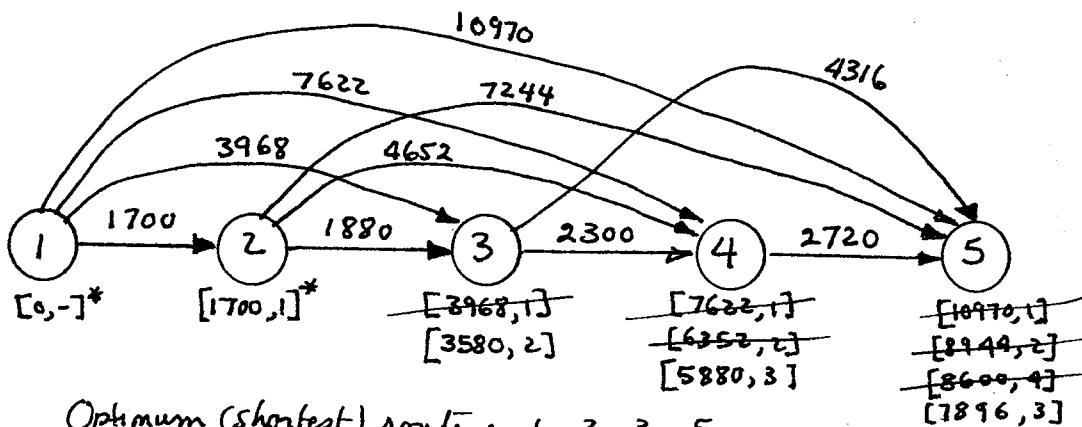
Set 6.3b

3(b)



Solution: 1-2-4-3-5-6, Route value = .281286
 Probability = $10^{-.281286} = .52326$

3(c)



Optimum (shortest) route: 1-2-3-5

Solution: Order in 1 for 1

Order in 2 for 2

Order in 3 for 3 and 4

Set 6.3c

3

Iteration 5							Iteration 0						
Array D5							Array D0						
N1:	N2:	N3:	N4:	N5:	N6:	N7:	N1:	N2:	N3:	N4:	N5:	N6:	N7:
N1:	4.00	3.00	9.00	6.00	7.00	12.00	N1:	infinity	700.00	200.00	infinity	infinity	infinity
N2:	4.00	1.00	5.00	2.00	3.00	8.00	N2:	1	300.00	200.00	infinity	400.00	infinity
N3:	3.00	1.00	6.00	3.00	4.00	9.00	N3:	200.00	300.00	700.00	600.00	infinity	100.00
N4:	9.00	5.00	6.00	3.00	4.00	3.00	N4:	infinity	200.00	700.00	300.00	100.00	500.00
N5:	6.00	2.00	3.00	3.00	1.00	6.00	N5:	infinity	infinity	600.00	300.00	100.00	500.00
N6:	7.00	3.00	4.00	1.00	1.00	4.00	N6:	infinity	400.00	infinity	100.00	500.00	
N7:	12.00	8.00	9.00	3.00	6.00	4.00							
Array S5							Array S0						
N1:	N2:	N3:	N4:	N5:	N6:	N7:	N1:	N2:	N3:	N4:	N5:	N6:	N7:
N1:	3	3	3	3	5	4	N1:	2	3	4	5	6	
N2:	3	3	4	5	5	4	N2:	1	3	4	5	6	
N3:	1	2	2	2	5	4	N3:	1	2	3	4	5	
N4:	3	2	2	5	5	7	N4:	1	2	3	4	5	
N5:	3	2	2	4	6	4	N5:	1	2	3	4	5	
N6:	5	5	5	4	5	4	N6:	1	2	3	4	5	
N7:	4	4	4	4	4	6							
Iteration 6							Iteration 1						
Array D6							Array D1						
N1:	N2:	N3:	N4:	N5:	N6:	N7:	N1:	N2:	N3:	N4:	N5:	N6:	N7:
N1:	4.00	3.00	8.00	6.00	7.00	11.00	N1:	infinity	700.00	200.00	infinity	infinity	infinity
N2:	4.00	1.00	4.00	2.00	3.00	7.00	N2:	200.00	300.00	200.00	infinity	400.00	infinity
N3:	3.00	1.00	5.00	3.00	4.00	8.00	N3:	infinity	200.00	700.00	600.00	300.00	100.00
N4:	9.00	5.00	6.00	3.00	4.00	3.00	N4:	infinity	200.00	700.00	300.00	100.00	500.00
N5:	6.00	2.00	3.00	2.00	1.00	5.00	N5:	infinity	infinity	600.00	300.00	100.00	500.00
N6:	7.00	3.00	4.00	1.00	1.00	4.00	N6:	infinity	400.00	infinity	100.00	500.00	
Array S6							Array S1						
N1:	N2:	N3:	N4:	N5:	N6:	N7:	N1:	N2:	N3:	N4:	N5:	N6:	N7:
N1:	3	3	6	3	5	6	N1:	2	3	4	5	6	
N2:	3	3	6	5	5	6	N2:	1	3	4	5	6	
N3:	1	2	6	2	5	6	N3:	1	2	3	4	5	
N4:	3	2	2	5	5	7	N4:	1	2	3	4	5	
N5:	3	2	2	6	6	6	N5:	1	2	3	4	5	
N6:	5	5	5	4	5	4	N6:	1	2	3	4	5	
N7:	6	6	6	4	6	6							
(a) 1-7 distance = 11							Iteration 2						
$1-6-7 \Rightarrow 1-5-6-7 \Rightarrow 1-3-5-6-7 \Rightarrow$							Array D2						
$1-3-2-5-6-7 \Rightarrow 1-3-2-5-6-4-7$							N1:	N2:	N3:	N4:	N5:	N6:	N7:
(b) 7-1 distance = 11							N1:	infinity	700.00	200.00	900.00	infinity	1100.00
7-6-1							N2:	200.00	300.00	200.00	500.00	600.00	700.00
7-6-5-1							N3:	infinity	200.00	500.00	300.00	100.00	500.00
7-6-5-3-1							N4:	infinity	infinity	600.00	300.00	100.00	500.00
7-6-5-2-3-1							N5:	infinity	400.00	700.00	100.00	500.00	
(c) 6-7 distance = 4							Array S2						
6-4-7							N1:	N2:	N3:	N4:	N5:	N6:	N7:
							N1:	2	3	2	5	2	
							N2:	1	3	4	5	6	
							N3:	1	2	2	5	2	
							N4:	1	2	2	3	5	
							N5:	1	2	3	4	6	
							N6:	1	2	2	4	5	
(d) 5-6 distance = 4							Iteration 3						
5-4-6							Array D3						
							N1:	N2:	N3:	N4:	N5:	N6:	N7:
							N1:	500.00	200.00	700.00	800.00	900.00	
							N2:	500.00	300.00	200.00	900.00	400.00	
							N3:	200.00	300.00	500.00	600.00	700.00	
							N4:	700.00	200.00	500.00	300.00	100.00	
							N5:	800.00	900.00	600.00	300.00	100.00	
							N6:	900.00	400.00	700.00	100.00	500.00	
							Array S3						
							N1:	N2:	N3:	N4:	N5:	N6:	N7:
							N1:	3	3	3	3	3	
							N2:	3	3	4	3	6	
							N3:	1	2	2	5	2	
							N4:	3	2	2	5	6	
							N5:	3	3	4	4	6	
							N6:	3	2	2	4	5	

continued...

Set 6.3c

Iteration 4						Iteration 4						
Array D4						Array S4						
N1:		N2:	N3:	N4:	N5:	N6:	5	4	300.00	5- 4		
N1:	500.00		200.00	700.00	800.00	800.00	5	6	400.00	5- 4- 6		
N2:	500.00		300.00	200.00	500.00	300.00	6	1	800.00	6- 4- 2- 3- 1		
N3:	200.00		300.00		500.00	600.00	6	2	300.00	6- 4- 2		
N4:	700.00		200.00	500.00		300.00	6	3	600.00	6- 4- 2- 3		
N5:	800.00		500.00	600.00	300.00		6	4	100.00	6- 4		
N6:	800.00		300.00	600.00	100.00	400.00	6	5	400.00	6- 4- 5		
Array S4												
N1:		N2:	N3:	N4:	N5:	N6:	Iteration 0					
N1:		3	3	3	3	4	4	infinity	infinity	infinity	infinity	infinity
N2:	3		3	4	4	4	1	infinity	infinity	infinity	infinity	infinity
N3:	1	2		2	5	4	2	1.00	1.00	1.00	infinity	
N4:	3	2	2		5	6	3	infinity	infinity	1.00	infinity	infinity
N5:	3	4	3	4		4	4	infinity	infinity	infinity	infinity	1.00
N6:	4	4	4	4	4		5	1.00	1.00	infinity	infinity	infinity
Iteration 5						Iteration 0						
Array D5						Array D0						
N1:		N2:	N3:	N4:	N5:	N6:	N1:joe	1.00	infinity	infinity	infinity	1.00
N1:	500.00		200.00	700.00	800.00	800.00	N2:bob	infinity	1.00	infinity	infinity	infinity
N2:	500.00		300.00	200.00	500.00	300.00	N3:kay	infinity	1.00	1.00	1.00	infinity
N3:	200.00		300.00		500.00	600.00	N4:jim	infinity	infinity	1.00	infinity	infinity
N4:	700.00		200.00	500.00		300.00	N5:rae	infinity	infinity	infinity	infinity	1.00
N5:	800.00		500.00	600.00	300.00		N6:kim	1.00	1.00	infinity	infinity	infinity
N6:	800.00		300.00	600.00	100.00	400.00						
Array S5						Iteration 0						
N1:		N2:	N3:	N4:	N5:	N6:	N1:joe	2	3	4	5	6
N1:		3	3	3	3	4	N2:bob	1	3	4	5	6
N2:	3		3	4	4	4	N3:kay	1	2	4	5	6
N3:	1	2		2	5	4	N4:jim	1	2	3		6
N4:	3	2	2		5	6	N5:rae	1	2	3	4	6
N5:	3	4	3	4		4	N6:kim	1	2	3	4	5
N6:	4	4	4	4	4							
<i>Shortest routes :</i>												
From	To	Distance	Route	Iteration 0								
1	2	500.00	1- 3- 2	Array S0								
1	3	200.00	1- 3	N1:joe N2:bob N3:kay N4:jim N5:rae N6:kim								
1	4	700.00	1- 3- 2- 4	Iteration 1								
1	5	800.00	1- 3- 5	Array D1								
1	6	800.00	1- 3- 2- 4- 6	N1:joe N2:bob N3:kay N4:jim N5:rae N6:kim								
2	1	500.00	2- 3- 1	Iteration 1								
2	3	300.00	2- 3	Array D1								
2	4	200.00	2- 4	N1:joe N2:bob N3:kay N4:jim N5:rae N6:kim								
2	5	500.00	2- 4- 5	Iteration 1								
2	6	300.00	2- 4- 6	Array D1								
3	1	200.00	3- 1	N1:joe N2:bob N3:kay N4:jim N5:rae N6:kim								
3	2	300.00	3- 2	Iteration 1								
3	4	500.00	3- 2- 4	Array D1								
3	5	600.00	3- 5	N1:joe N2:bob N3:kay N4:jim N5:rae N6:kim								
3	6	600.00	3- 2- 4- 6	Iteration 1								
4	1	700.00	4- 2- 3- 1	Array S1								
4	2	200.00	4- 2	N1:joe N2:bob N3:kay N4:jim N5:rae N6:kim								
4	3	500.00	4- 2- 3	Iteration 1								
4	5	300.00	4- 5	Array D1								
4	6	100.00	4- 6	N1:joe N2:bob N3:kay N4:jim N5:rae N6:kim								
5	1	800.00	5- 3- 1	Iteration 1								
5	2	500.00	5- 4- 2	Array D1								
5	3	600.00	5- 3	N1:joe N2:bob N3:kay N4:jim N5:rae N6:kim								

continued...

continued...

Set 6.3e

2

(a)

Solver Shortest Route Model (Example 6.3-6)						
	A	B	C	D	E	F
1	distance	N2	N3	N4	N5	
2	N1	100	30			1
3	N2		20			
4				10	60	
5	N3					
6	N4	15			50	
7						
8	solution	N2	N3	N4	N5	1
9	N1	0	1	0	0	1.1E-11
10	N2	0	2E-13	0	0	2.2E-13
11	N3	0	0	0	1	1.1E-12
12	N4	0	0	0	0	0
13	N5					0
14		0	1	0	4.6E-12	totalDist= 90
15	Solver Parameters					
16	Set Target Cell:	TotalDist	Solve			
17	Equal To:	<input type="radio"/> Max <input checked="" type="radio"/> Min <input type="radio"/> Value of: 0				
18	By Changing Cells:		Close			
19	Subject to the Constraints:					
20	netFlow = 0					
21	solution >= 0					
22						
23						
24						
25						
26						
27						
28						
29						
30						

(b)

Solver Shortest Route Model (Example 6.3-6)						
	A	B	C	D	E	F
1	distance	N2	N3	N4	N5	
2	N1	100	30			1
3	N2		20			
4				10	60	
5	N3					
6	N4	15			50	1
7						
8	solution	N2	N3	N4	N5	cutFlow
9	N1	0	-1E-13	0	0	-1.1E-13
10	N2	0	1	0	0	1
11	N3	0	0	0	0	-5E-12
12	N4	1	0	0	1.1E-11	4.6E-12
13	N5					0
14		1	-6E-12	0	1.1E-11	totalDist= 35
15	Solver Parameters					
16	Set Target Cell:	TotalDist	Solve			
17	Equal To:	<input type="radio"/> Max <input checked="" type="radio"/> Min <input type="radio"/> Value of: 0				
18	By Changing Cells:		Close			
19	Subject to the Constraints:					
20	netFlow = 0					
21	solution >= 0					
22						
23						
24						
25						
26						
27						
28						
29						
30						

```

param n;
param start;
param end;
param p{1..n,1..n} default 0;
param rhs{i in 1..n}=if i=start then 1 else (if i=end then -1 else 0);

```

```

var x{i in 1..n,j in 1..n:p[i,j]>0}>=0;
var outFlow{i in 1..n}=sum{j in 1..n:p[i,j]>0}x[i,j];
var inFlow{j in 1..n}=sum{i in 1..n:p[i,j]>0}x[i,j];
var logProb=sum{i in 1..n}sum{j in 1..n:p[i,j]>0}-log(p[i,j])*x[i,j];
var prob=2.718^logProb;

```

```

minimize z: sum {i in 1..n, j in 1..n:p[i,j]>0}-log(p[i,j])*x[i,j];
subject to limit {i in 1..n}: outFlow[i]-inFlow[i]=rhs[i];

```

```

data;
param n:=7;
param start:=4;
param end:=7;
```

```

param p:
1 2 3 4 5 6 7=:
1 .8 .3 .65 . .
2 . . . 9 .5 . .
3 . . . . 95 . .
4 . .85 . 7 . .
5 . . . . 5 .8
6 . . . . . 9;
```

```

solve;
display z,logProb,prob, x;
```

Set 6.4b

(a) Surplus capacities:

$$2-3: 40 - 0 = 40 \text{ units}$$

$$2-5: 30 - 20 = 10 \text{ units}$$

$$4-3: 5 - 0 = 5 \text{ units}$$

All other arcs have zero surplus capacities.

(b)

Flow through node 2 = 20 units

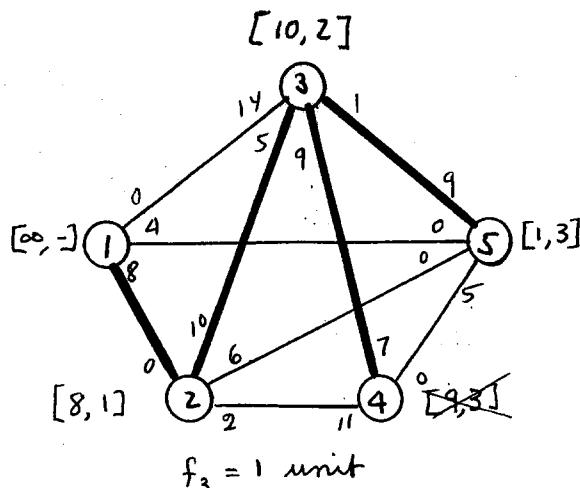
Flow through node 3 = 30 units

Flow through node 4 = 20 units

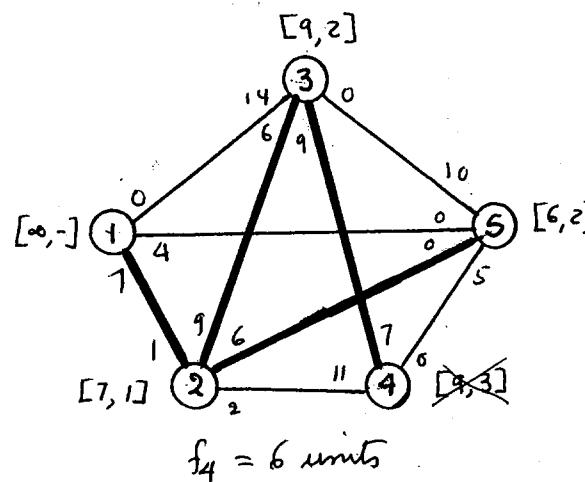
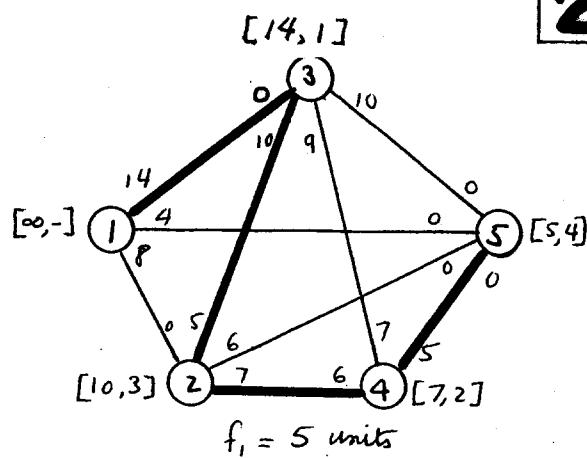
(c)

No, because the arcs out of node 1 have zero surplus capacity

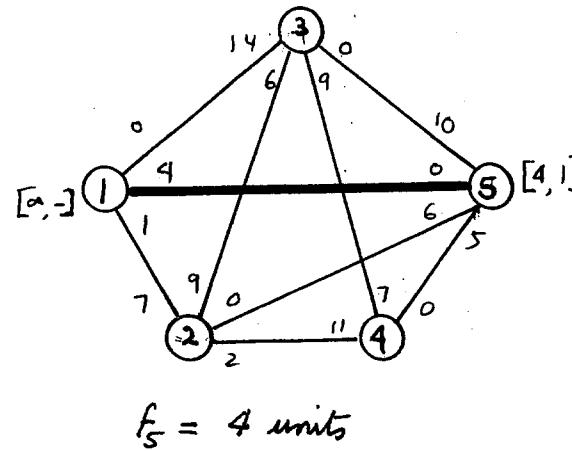
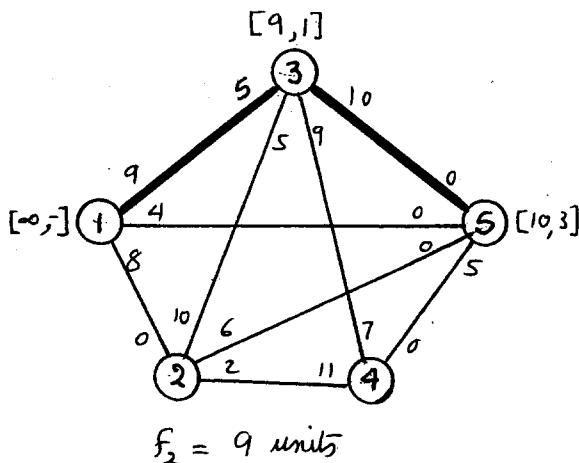
1



2

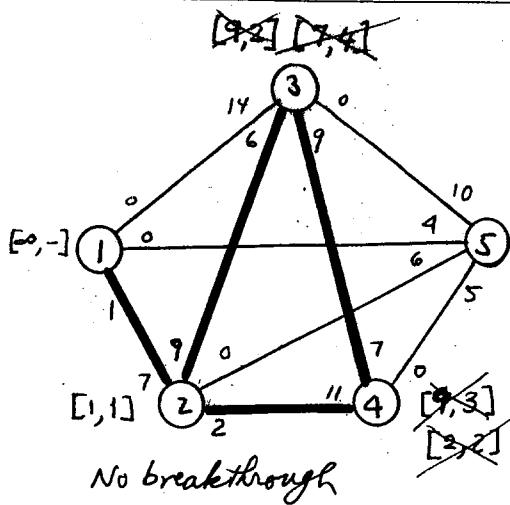


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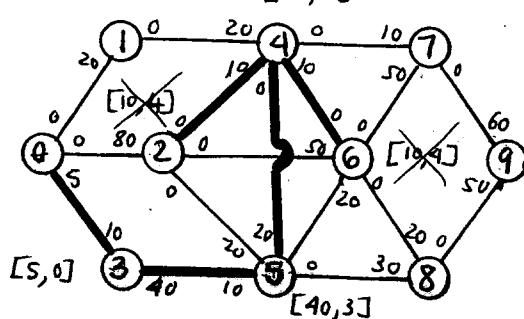
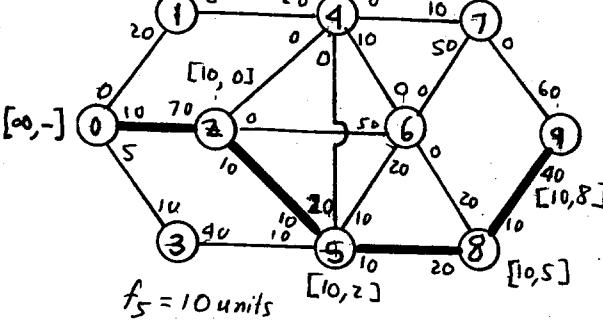
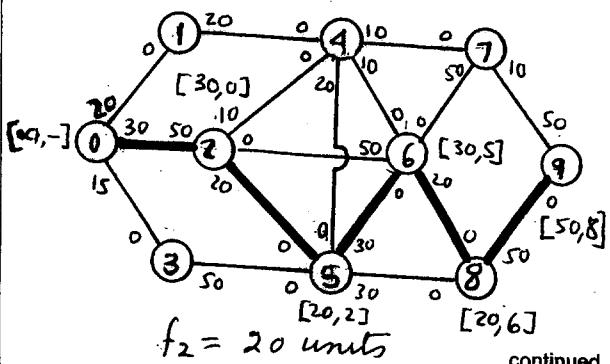
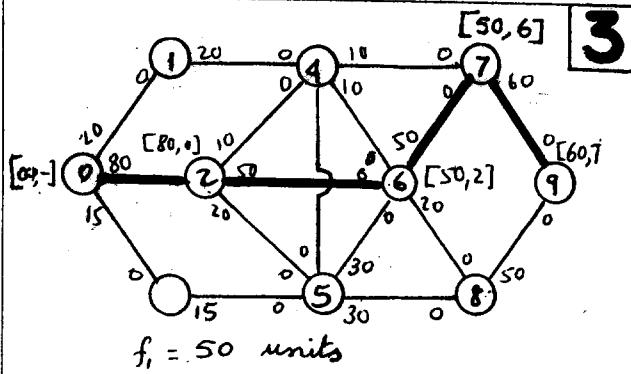
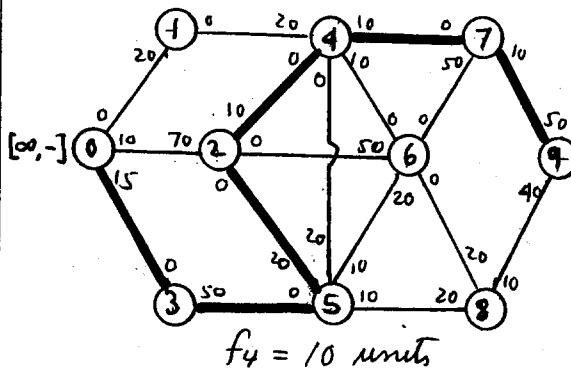
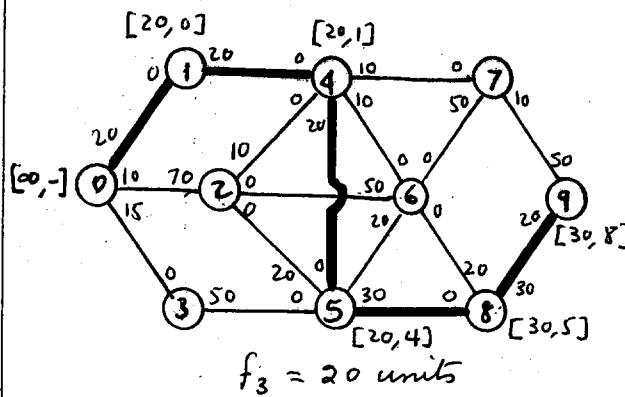
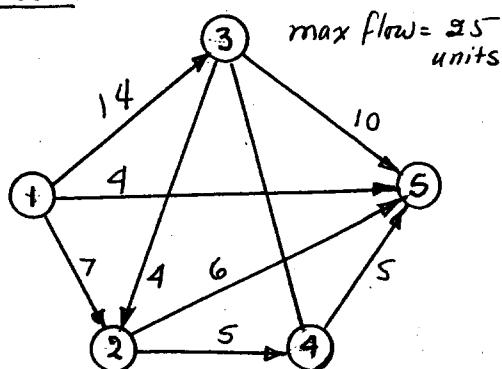


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Set 6.4b

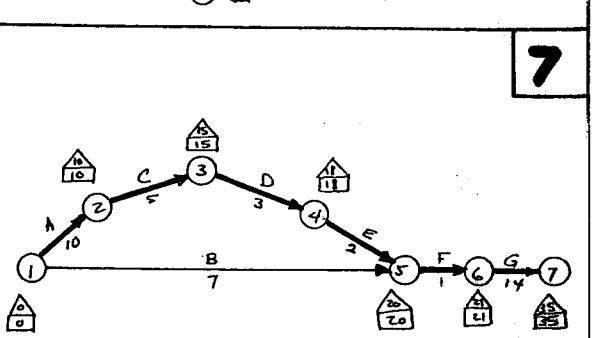
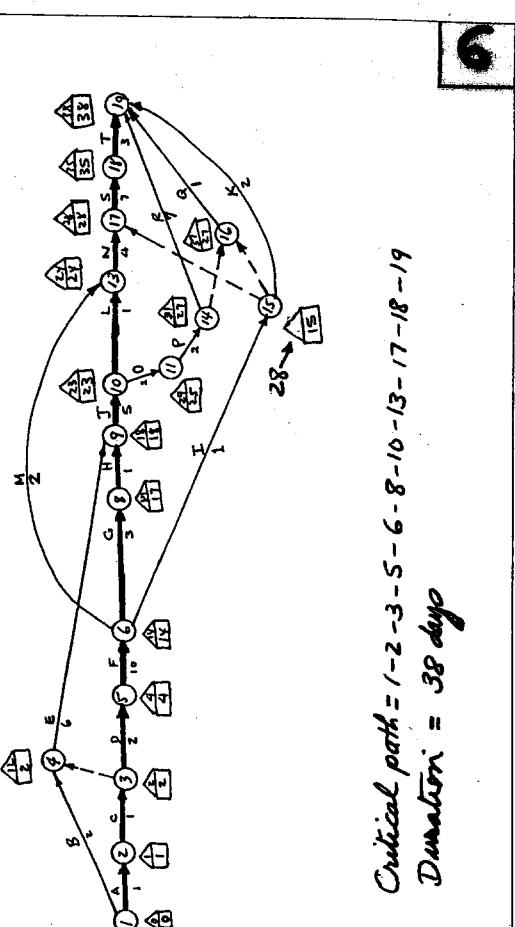
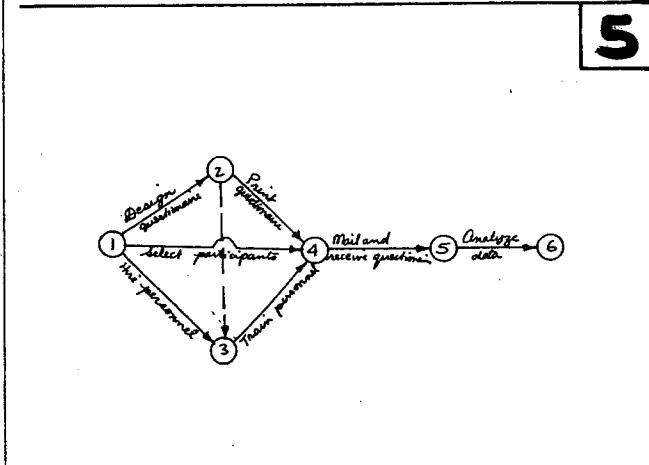
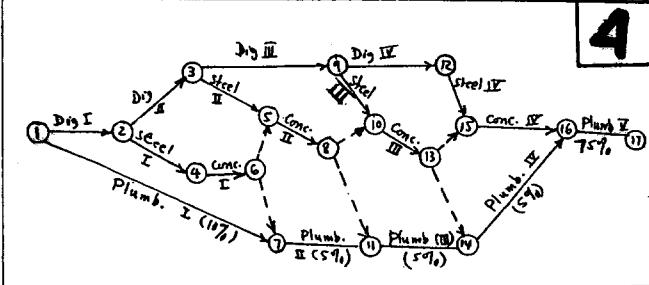
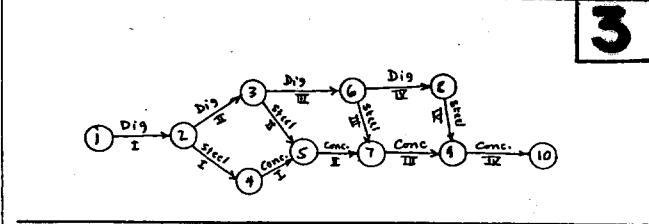
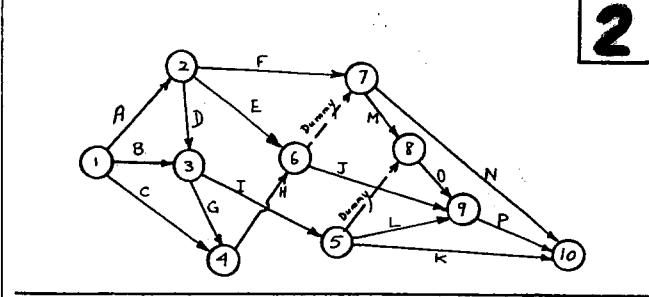
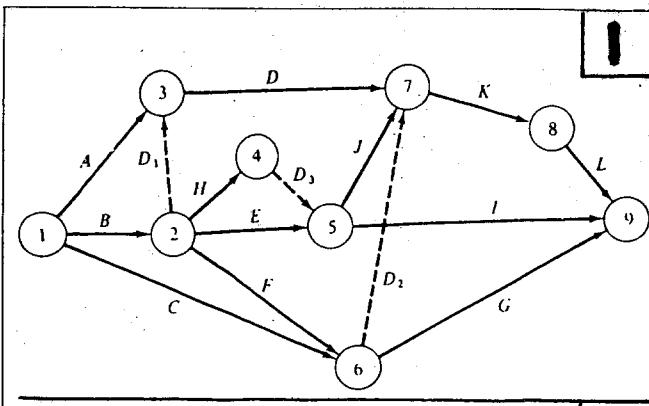


Solution:



continued...

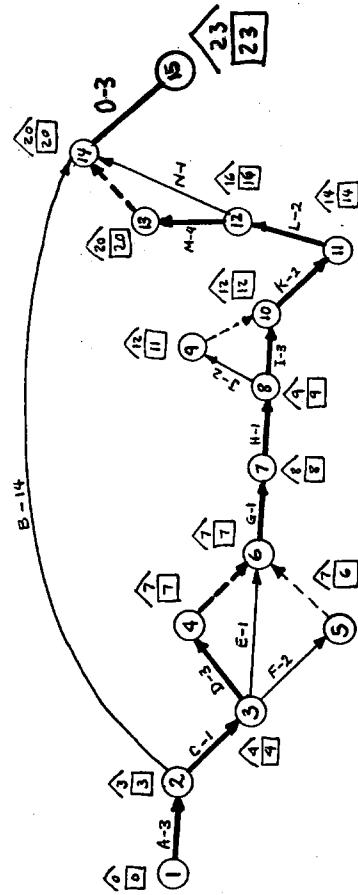
Set 6.5a



Critical path: 1-2-3-4-5-6-7
Duration: 35 days

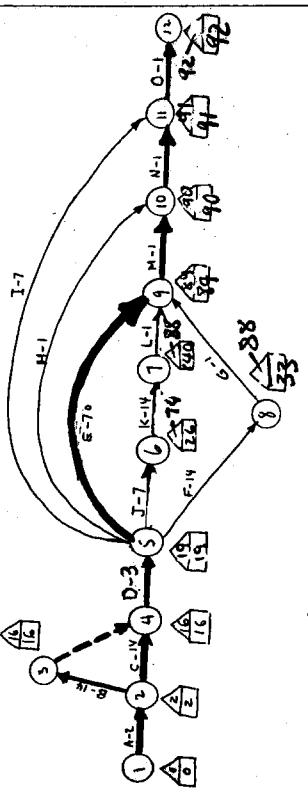
Set 6.5a

10



Critical path : 1-2-3-4-6-7-8-10-11-12-13-14-15
Duration: 20 days

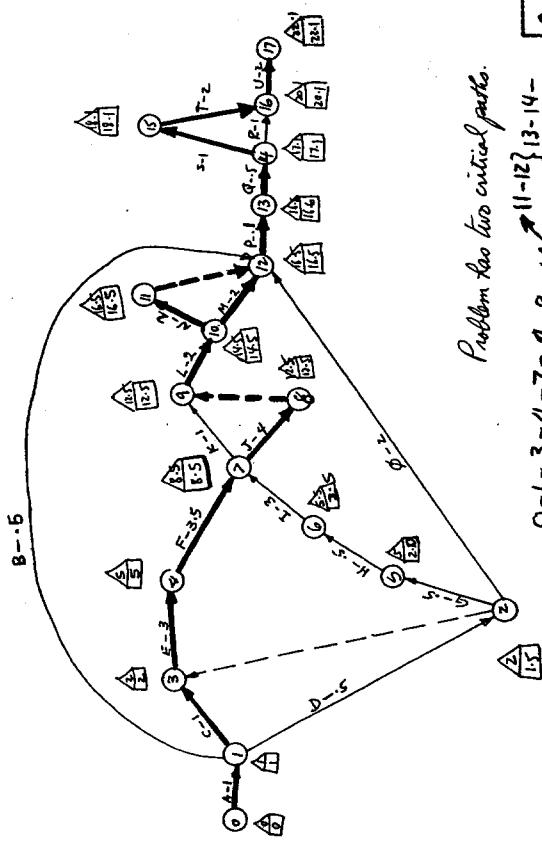
8



Critical paths:

1-2-3-4-5-9-10-11-12
1-2-4-5-9-10-11-12

9

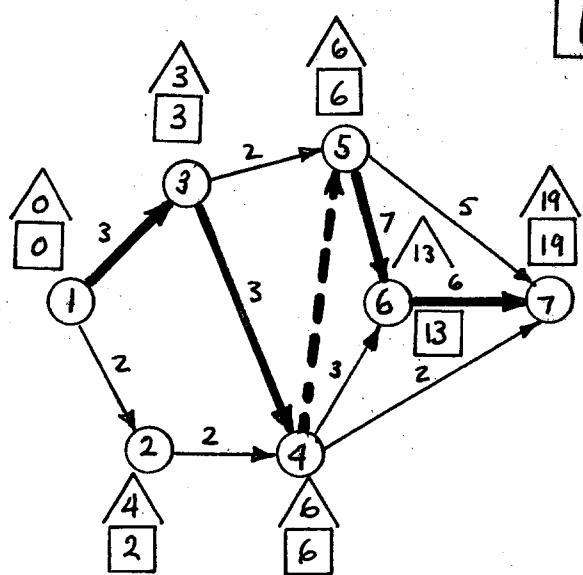


Problem has two critical paths.

0-1-3-4-7-8-9-10-11-12-13-14-15-16-17

6-29

Set 6.5b



1

See solution to Problem 6, Set 6.6a

3

See solution to Problem 8, Set 6.6a

4

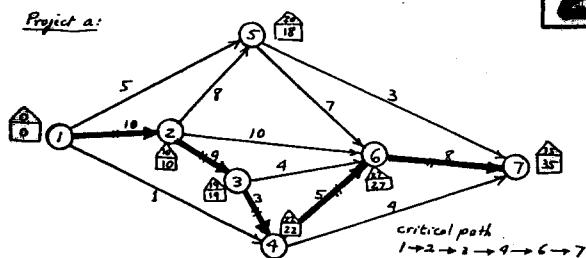
See solution to Problem 9, Set 6.6a

5

See solution to Problem 10, Set 6.6a

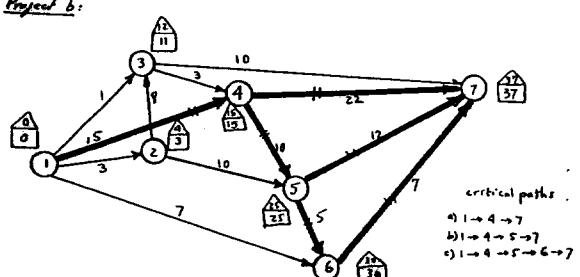
6

Project a:

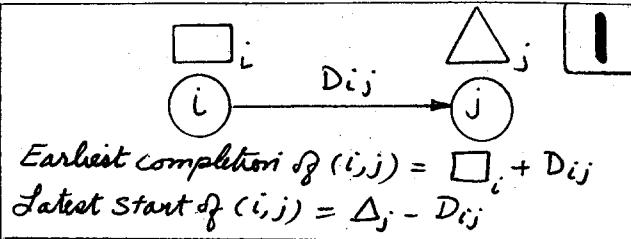


2

Project b:



5



Both floats are zero by definition

(a) $FF = 10, TF = 10, D = 4$
maximum delay = 10

(b) $FF = 5, TF = 10, D = 4$
maximum delay = 5
(c) $FF = 0, TF = 10, D = 4$
maximum delay = 0

(a) For B: $TF = 5, FF = 2$

Because $FF = 2$, a delay of 1 has no effect on succeeding activities.

For C: Starting at time 5 implies no delay. Thus, the earliest start time for E and F is time 8.

(b) For B: Delay = 3, $FF = 2$. Thus, the start of E and F must be delayed by at least $3-2 = 1$.

For C: Delay = $7-5 = 2$, $FF = 0$. Thus, start of E and F must be delayed by at least 2.

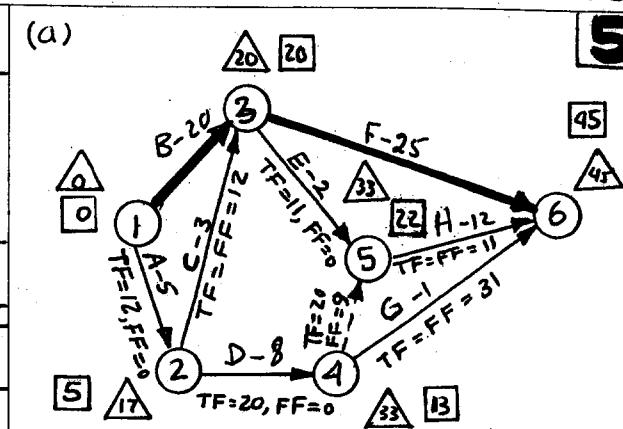
For B & C combined: Start of E and F must be delayed by $\max(1, 2) = 2$.

(c) Delay in B = 6. Because FF of B = 2, the start of E and F must be delayed by 4. Next, a delay of 4 in E will delay critical H by 1 because $FF_E = 3$. Also, a delay of 4 in E will not impact other activities in the project. Thus, the proposed delay in B will delay the entire project by 1 (because of the delay in critical H).

2

3

4



(b) Red-flagged activities are A, D, and E.

(c) $FF_A = 0$: Delay = 5 will delay each of C and D by 5.

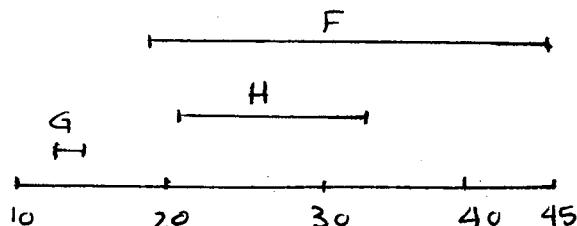
$FF_D = 0$: Delay = 5 will delay G by 5.

$FF_C = 12$: Delay = 5 does not affect other activities.

Conclusion: Start of C, D, and G is delayed by 5.

Note: If you use TORA to experiment with the effect of $Delay_A = 5$, the chart will only show a delay in C and D, but not in G. The effect of C and D on succeeding activities must be done manually. To effect that after $Delay_A = 5$ is implemented, select C with $Delay_C = 0$ and D with $Delay_D = 0$. $Delay_C = 0$ produces no action, but $Delay_D = 0$ will delay G and Duracy properly to match $Delay_A = 5$.

(d)



Two units of equipment are required.

Set 6.5c

6

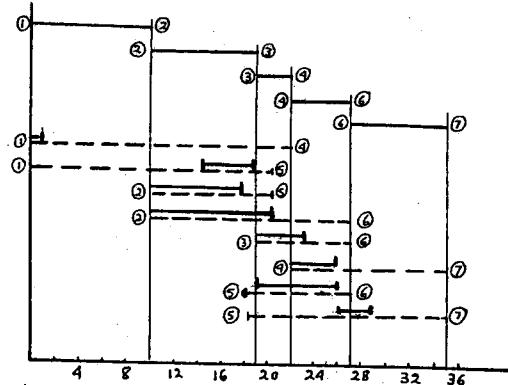
*** CPM SOLUTION ***

Title: (a)

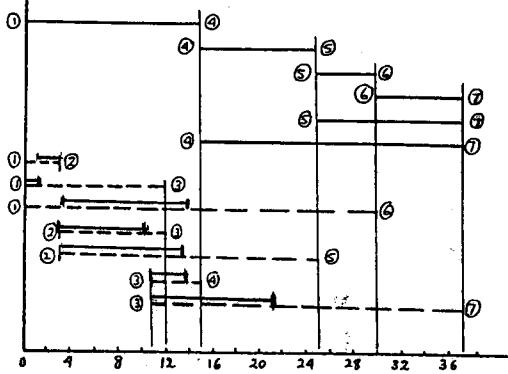
Size: 7 nodes x 13 activities

Activity	Duration	Earliest start	Compl.	Latest start	Total compl.	Total float	Free float
c 1-2	10.0	0.0	10.0	0.0	10.0	0.0	0.0
1-4	1.0	0.0	1.0	21.0	22.0	21.0	21.0
1-5	5.0	0.0	5.0	15.0	20.0	15.0	13.0
c 2-3	9.0	10.0	19.0	10.0	19.0	0.0	0.0
2-5	8.0	10.0	18.0	12.0	20.0	2.0	0.0
2-6	10.0	10.0	20.0	17.0	27.0	7.0	7.0
c 3-4	3.0	19.0	22.0	19.0	22.0	0.0	0.0
3-6	4.0	19.0	23.0	23.0	27.0	4.0	4.0
c 4-6	5.0	22.0	27.0	22.0	27.0	0.0	0.0
4-7	4.0	22.0	26.0	31.0	35.0	9.0	9.0
5-6	7.0	18.0	25.0	20.0	27.0	2.0	2.0
5-7	3.0	18.0	21.0	32.0	35.0	14.0	14.0
c 6-7	8.0	27.0	35.0	27.0	35.0	0.0	0.0

Project a:



Project b:



Project (a):

Red flagged activities:

(1-5), $TF = 15$, $FF = 13$

(2-5), $TF = 2$, $FF = 0$

In project (a), note the delay in the start of activity 5-6 to account for the effect of starting (1-5) at time 14.

Project (b):

The following activities are red-flagged:

Activity	TF	FF
1-2	1	0
1-3	11	10
2-3	1	0

continued...

Set 6.5d

	x_{12}	x_{13}	x_{24}	x_{34}	x_{35}	x_{45}	x_{46}	x_{47}	x_{56}	x_{57}	x_{67}	
Maximize z =	3	3	2	3	2	0	3	2	7	5	6	
Node 1	-1	-1										= -1
Node 2	1		-1									= 0
Node 3	1		-1	-1								= 0
Node 4		1	1		-1	-1	-1					= 0
Node 5			1	1			-1	-1				= 0
Node 6					1		1		-1			= 0
Node 7						1	1	1				= 1

Optimal:

$$x_{13} = x_{34}, x_{45}, x_{56} = x_{67} = 1$$

$$Z = 19$$

(a)

2

	x_{12}	x_{14}	x_{15}	x_{23}	x_{25}	x_{26}	x_{34}	x_{36}	x_{46}	x_{47}	x_{56}	x_{57}	x_{67}
Maximize z =	10	1	5	9	8	10	3	4	5	4	7	3	8
Node 1	-1	-1	-1										= -1
Node 2	1			-1	-1	-1							= 0
Node 3			1				-1	-1					= 0
Node 4		1					1		-1	-1			= 0
Node 5			1		-1			1		-1	-1		= 0
Node 6						1		1		1		-1	= 0
Node 7							1		1	1			= 1

$$\text{Optimum: } x_{12} = x_{23} = x_{34} = x_{46} = x_{67} = 1, Z = 35$$

(b)

	x_{12}	x_{13}	x_{14}	x_{16}	x_{23}	x_{25}	x_{34}	x_{37}	x_{45}	x_{47}	x_{56}	x_{57}	x_{67}
Maximize z =	3	1	15	7	8	10	3	10	10	22	5	12	7
Node 1	-1	-1	-1										= -1
Node 2	1			-1	-1	-1							= 0
Node 3		1			1		-1	-1					= 0
Node 4			1				1		-1	-1			= 0
Node 5						1		1		-1	-1		= 0
Node 6					1				1	1		-1	= 0
Node 7							1		1	1			= 1

$$\begin{aligned} \text{Optimum: } & x_{14} = x_{47} = 1 \\ & x_{14} = x_{45} = x_{57} = 1 \\ & x_{14} = x_{45} = x_{56} = x_{67} = 1 \end{aligned} \quad \left. \right\} \text{ alternative optima } Z = 37$$

Set 6.5e

Project (a)

Title:

Activity	Mean Duration	Variance
1 - 2	4.00	0.11
1 - 4	2.83	0.25
1 - 5	3.83	0.25
2 - 3	5.00	0.11
2 - 5	8.17	0.25
2 - 6	9.50	0.69
3 - 4	10.00	5.44
3 - 6	4.00	0.11
4 - 6	7.67	1.00
4 - 7	6.17	0.25
5 - 6	10.67	1.00
5 - 7	6.00	0.44
6 - 7	4.00	0.11

Title:

Node	Longest Path	Path Mean	Path Std. Dev.
2	1-2	4.00	0.33
3	1-2-3	9.00	0.47
4	1-2-3-4	19.00	2.38
5	1-2-5	12.17	0.60
6	1-2-3-4-6	26.67	2.58
7	1-2-3-4-6-7	30.67	2.60

Event	Latest occurrence time, LC	$P\{\text{occurrence} \leq \text{LC}\}$
2	4	.5
3	9	.5
4	19	.5
5	16	1.0
6	26.67	.5
7	30.67	.5

LC is determined by carrying out CPM calculations using average duration time

Example of probability calculations:

For node 5:

$$P\{T \leq 16\} = P\{Z \leq \frac{16 - 12.17}{6}\} \\ = P\{Z \leq 6.38\} \approx 1$$

continued...

Project (b)

Title:

Activity	Mean Duration	Variance
1 - 2	2.83	0.25
1 - 3	6.83	0.25
1 - 4	7.17	0.25
1 - 6	2.00	0.11
2 - 3	4.00	0.11
2 - 5	8.00	0.11
3 - 4	15.00	2.78
3 - 7	13.00	0.11
4 - 5	12.17	0.69
4 - 7	10.00	0.44
5 - 6	8.33	0.44
5 - 7	4.33	1.00
6 - 7	6.00	0.11

Title:

Node	Longest Path	Path Mean	Path Std. Dev.
2	1-2	2.83	0.50
3	1-3	6.83	0.50
4	1-3-4	21.83	1.74
5	1-3-4-5	34.00	1.93
6	1-3-4-5-6	42.33	2.04
7	1-3-4-5-6-7	48.33	2.07

Event	Latest occurrence time, LC	$P\{\text{occurrence} \leq \text{LC}\}$
2	2.83	.5
3	6.83	.5
4	21.83	.5
5	34.00	.5
6	42.33	.5
7	48.33	.5

All events happen to fall on the critical path (using average durations). This is the reason all probabilities = .5

CHAPTER 7

Advanced Linear programming

Sel 7.1a

$$Q = \{x_1, x_2 \mid x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$$

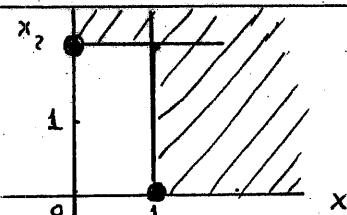
Let $(\bar{x}_1, \bar{x}_2) \geq 0$ and $(\tilde{x}_1, \tilde{x}_2) \geq 0$ be two distinct points in Q and define for $0 \leq \lambda \leq 1$:

$$(x_1, x_2) = \lambda(\bar{x}_1, \bar{x}_2) + (1-\lambda)(\tilde{x}_1, \tilde{x}_2) \geq 0$$

Then,

$$\begin{aligned} x_1 + x_2 &= \lambda\bar{x}_1 + (1-\lambda)\tilde{x}_1 + \lambda\bar{x}_2 + (1-\lambda)\tilde{x}_2 \\ &= \lambda(\bar{x}_1 + \tilde{x}_2) + (1-\lambda)(\bar{x}_1 + \tilde{x}_2) \\ &\leq \lambda(1) + (1-\lambda)(1) = 1 \end{aligned}$$

which shows that Q is convex.
The result is true even without the nonnegativity restrictions.



$$Q = \{x_1, x_2 \mid x_1 \geq 1 \text{ or } x_2 \geq 2\}$$

$$\begin{aligned} \text{Let } (\bar{x}_1, \bar{x}_2) &= (1, 0) \in Q \\ (\tilde{x}_1, \tilde{x}_2) &= (0, 2) \in Q \end{aligned}$$

Consider

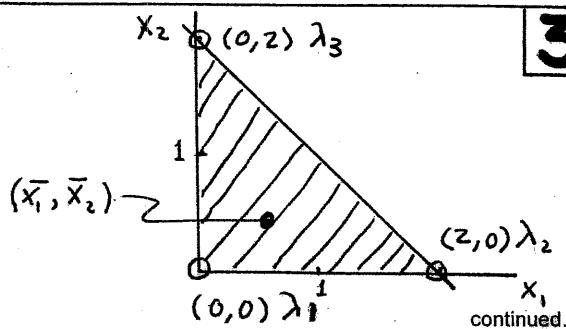
$$\begin{aligned} (x_1, x_2) &= \lambda(1, 0) + (1-\lambda)(0, 2) \\ &= (\lambda, 2-2\lambda) \quad 0 \leq \lambda \leq 1 \end{aligned}$$

For $0 < \lambda < 1$, we have

$$x_1 = \lambda < 1$$

$$x_2 = 2-2\lambda < 2$$

Thus, $(x_1, x_2) \notin Q$.



$$Q = \{x_1, x_2 \mid x_1 + x_2 \leq 2, x_1, x_2 \geq 0\}$$

$$\begin{aligned} (\bar{x}_1, \bar{x}_2) &= \lambda_1(0, 0) + \lambda_2(2, 0) + \lambda_3(0, 2) \\ &= (2\lambda_2, 2\lambda_3) \end{aligned}$$

$$\text{where } \lambda_1, \lambda_2, \lambda_3 \geq 0$$

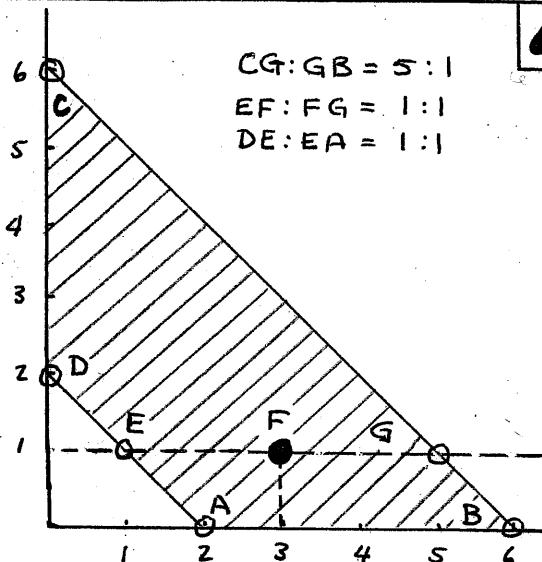
$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

4

$$CG : GB = 5 : 1$$

$$EF : FG = 1 : 1$$

$$DE : EA = 1 : 1$$



$$E = \frac{1}{2}A + \frac{1}{2}D$$

$$G = \frac{5}{6}B + \frac{1}{6}C$$

$$F = \frac{1}{2}E + \frac{1}{2}G$$

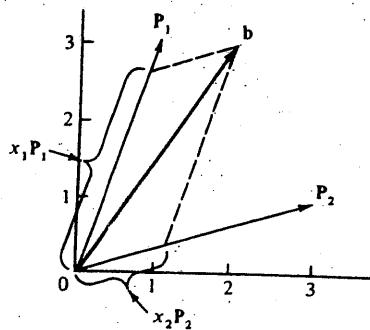
$$= \frac{1}{2}\left(\frac{1}{2}A + \frac{1}{2}D\right) + \frac{1}{2}\left(\frac{5}{6}B + \frac{1}{6}C\right)$$

$$= \frac{1}{4}A + \frac{1}{4}D + \frac{5}{12}B + \frac{1}{12}C$$

$$= \frac{1}{4}(2, 0) + \frac{1}{4}(0, 2) + \frac{5}{12}(6, 0) + \frac{1}{12}(0, 6)$$

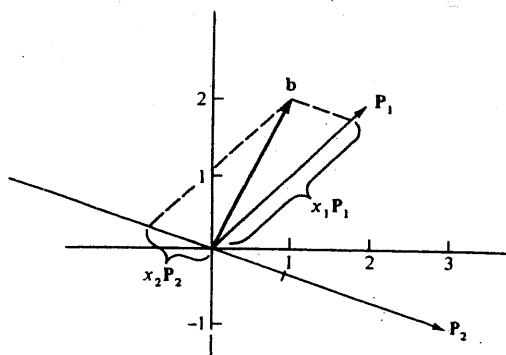
$$= (3, 1)$$

(a)



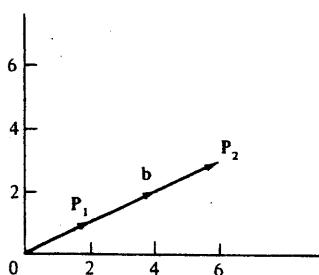
Unique solution:
 $(x_1, x_2) = (7/8, 3/8)$,
 left-side vectors P_1 and P_2
 are independent (basis)

(b)

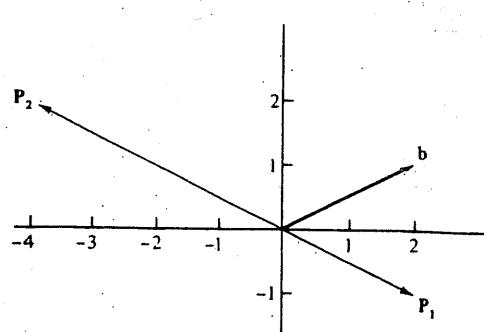


Unique solution:
 $(x_1, x_2) = (7/8, -1/4)$,
 P_1 and P_2 form a basis

(c)

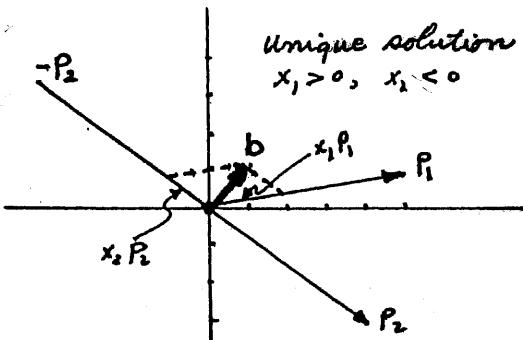


Infinity of solutions:
 P_1 and P_2 are dependent
 (no basis); b is also
 dependent



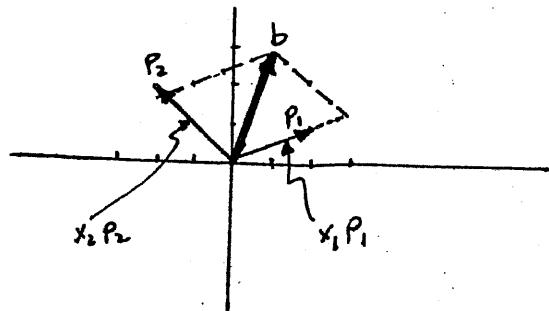
No solution: P_1 and P_2
 are dependent (no basis),
 but b is independent

(a) $\begin{pmatrix} 5 & 4 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



unique solution
 $x_1 > 0, x_2 < 0$

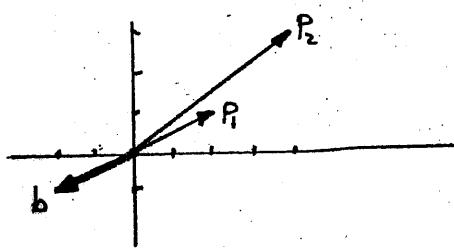
b) $\begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$



unique solution: $x_1, x_2 > 0$
 $x_1 > 1, x_2 < 1$

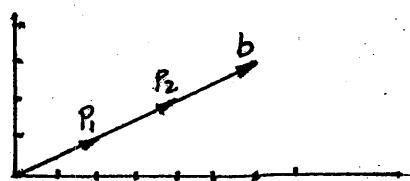
Set 7.1b

$$(c) \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$



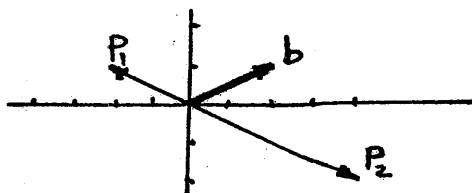
unique solution: $x_1 < 0, x_2 = 0$

$$(d) \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$



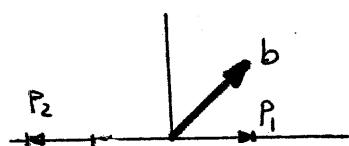
infinity of solutions

$$(e) \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



No solution

$$(f) \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



No solution

3

$$(a) \det(P_1, P_2, P_3) = \det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}$$

$= -4 \neq 0$, basis

$$(b) \det(P_1, P_2, P_4) = \det \begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}$$

$= -8 \neq 0$, basis

$$(c) \det(P_2, P_3, P_4) = \det \begin{pmatrix} 0 & 1 & 2 \\ 2 & 4 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

$= 0$, not a basis

(d) In this problem, a basis must include exactly 3 independent vectors.

4

(a) True

(b) True

(c) True

Set 7.1c

$$B = (P_3, P_4) = \begin{pmatrix} 2 & 4 \\ -2 & 6 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} .3 & -.2 \\ .1 & .1 \end{pmatrix}, \quad X_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, \quad C_B = (7, 5)$$

$$X_B = B^{-1}b = \begin{pmatrix} .3 & -.2 \\ .1 & .1 \end{pmatrix} \begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1.5 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$C_B B^{-1} = (7, 5) \begin{pmatrix} .3 & -.2 \\ .1 & .1 \end{pmatrix} = (2.6, -.9)$$

$$\{Z_j - c_j\}_{j=1,2} = (2.6, -.9) \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} - (1, 4) \\ = (1.5, -.5)$$

$$B^{-1}(P_1, P_2) = \begin{pmatrix} .3 & -.2 \\ .1 & .1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & .5 \\ .5 & 0 \end{pmatrix}$$

X_B is feasible but not optimal.

Tableau:

	x_1	x_2	x_3	x_4	
Z	1.5	-.5	0	0	$Z \leq 1.5$
x_3	0	.5	1	0	2
x_4	.5	0	0	1	1.5

2

$$\text{Maximize } Z = (5, 12, 4) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

subject to

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

$$\det(P_1, P_2) = \det \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

$$= -6 \neq 0 \Rightarrow \text{basis}$$

$$\det(P_2, P_3) = \det \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$$

$$= 0 \Rightarrow \text{not a basis}$$

$$\det(P_3, P_4) = \det \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$$= 1 \neq 0 \Rightarrow \text{basis}$$

$$X_B = (x_1, x_2, x_3)^T, \quad C_B = (2, 1, 0)$$

3

$$B^{-1} = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$C_B B^{-1} = (2, 1, 0) \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \\ = (2/5, 1/5, 0)$$

$$(Z_3 - c_3, Z_4 - c_4)$$

$$= (2/5, 1/5, 0) \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} - (0, 0) \\ = (-2/5, -1/5) \Rightarrow \text{optimal}$$

$$B^{-1}(P_1, P_2, P_3, P_4, P_5 | b)$$

$$= \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 4 & 3 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix} \mid \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -3/5 & 1/5 & 0 \\ 0 & 1 & 4/5 & -3/5 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \mid \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}$$

feasible ↑

$$Z = C_B B^{-1}b = (2, 1, 0) \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \mid \begin{pmatrix} 12/5 \\ 0 \\ 0 \end{pmatrix}$$

	x_1	x_2	x_3	x_4	x_5	Solution
Z	0	0	-2/5	-1/5	0	12/5
x_1	1	0	-3/5	1/5	0	3/5
x_2	0	1	4/5	-3/5	0	6/5
x_5	0	0	-1	1	1	0

$$X_B = (x_1, x_2, x_3)^T, \quad C_B = (0, c_2, c_1)$$

4

$$C_B B^{-1} = (0, c_2, c_1) \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = (0, c_2 - c_1, c_1)$$

For x_3, x_4 , and x_5 ,

$$\{Z_j - c_j\} = C_B B^{-1}(P_3, P_4, P_5) - (0, 0, 0) \\ = C_B B^{-1} = (0, c_2 - c_1, c_1)$$

From the tableau, we have

$$(0, c_2 - c_1, c_1) = (0, 3, 2)$$

which gives

$$c_1 = 2$$

$$c_2 = 5$$

Continued...

Set 7.1c

Hence,

$$\text{Optimum } Z = C_1 x_1 + C_2 x_2 + C_3 x_3 \\ = 2x2 + 5x6 + 0x2 = 34$$

To construct the original problem,

$$\bar{B}^{-1}(P_1 P_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus,

$$(P_1 P_2) = B \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \\ = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Similarly,

$$b = B \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}$$

Original model:

$$\text{Maximize } Z = 2x_1 + 5x_2$$

Subject to

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 6 \\ x_1 + x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

5

All that is needed is to show that the computations lead to the column under x_{II} .

For x_{II} , we have,

$$\{Z - C_{II}\} = C_B \bar{B}^{-1} I - C_{II} \\ = C_B \bar{B}^{-1} - C_{II}$$

Constraint coefficients

$$= \bar{B}^{-1} I = \bar{B}^{-1}$$

Set 7.2a

(a) current $B = (P_1, P_2)$
 P_1 must leave so that
 b is enclosed between P_2 and
 P_3 , hence yielding feasible values
of x_2 and x_3

(b) $B = (P_2, P_4)$ is a feasible basis

$$z_j - c_j = C_B B^{-1} P_j - c_j$$

Assume for convenience that

$$B = (P_1, P_2, \dots, P_m)$$

Then, for the basic vectors P_1, P_2, \dots , and P_m , we have

$$\begin{aligned} \{z_j - c_j\}_{j=1,2,\dots,m} &= C_B B^{-1}(P_1, \dots, P_m) - (c_1, \dots, c_m) \\ &= C_B B^{-1} B - C_B \\ &= C_B I - C_B = 0 \end{aligned}$$

Let NB represent the set of nonbasic variables at any iteration. Then

$$Z = Z^* - \sum_{j \in NB} (z_j - c_j) x_j$$

(a) since

$$z_j - c_j \begin{cases} > 0 & \text{for max} \\ < 0 & \text{for min} \end{cases}$$

it follows that all $x_j = 0, j \in NB$ because if any $x_j, j \in NB$ becomes positive $Z < Z^*$ for max and $Z > Z^*$ for min, which is not optimal. Thus, $X_B = B^{-1}b$ and $x_j = 0, j \in NB$ shows that the solution is unique.

Continued...

(b) If $z_j - c_j = 0$ for at least one $j \in NB$, then x_j can become basic at a value other than zero without changing the optimum value of Z . Thus, alternative optima exist.

Starting tableau (max):

	x_1	x_2	\dots	x_j	\dots	x_n	
Z	$-c_1$	$-c_2$	\dots	$-c_j$	\dots	$-c_n$	0

At the starting iteration:

$$B = I, C_B = 0$$

Hence

$$\begin{aligned} z_j - c_j &= C_B B^{-1} P_j - c_j \\ &= 0(P_j) - c_j \\ &= -c_j \end{aligned}$$

Starting tableau (assuming max):

	\dots	x_j	\dots	R_1	R_2	\dots	R_m	
	\dots	$-c_j$	\dots	M	M	\dots	M	0
R_1	\dots	P_j	\dots	I				
R_m	\dots							b

$$B = B^{-1} = I, C_B = (-M, -M, \dots, -M)$$

$$C_B B^{-1} = (-M, -M, \dots, -M)$$

$$\begin{aligned} \{z_j - c_j\} &= (-M, -M, \dots, -M)(P_1, \dots, P_n | I) \\ &\quad - (c_1, c_2, \dots, c_n, -M, \dots, -M) \end{aligned}$$

$$= ((-M, -M, \dots, -M)P_1 - c_1, \dots,$$

$$(-M, -M, \dots, -M)P_n - c_n, 0, \dots, 0)$$

which yields the following tableau

	x_1		R_1	\dots	R_m	
	$(-M, \dots, -M)P_j - c_j$		0	\dots	0	$(-M, \dots, -M)b$

Continued...

Set 7.2a

The vectors

$$\begin{pmatrix} c_k \\ p_k \end{pmatrix} \text{ and } \begin{pmatrix} -c_k \\ -p_k \end{pmatrix}$$

correspond to x_k^- and x_k^+ , respectively.

Assume that both x_k^- and x_k^+ are nonbasic, and let \mathbf{B} and \mathbf{c}_B correspond to the current solution. Then

$$z_k^- - c_k^- = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{P}_k - c_k$$

$$z_k^+ - c_k^+ = -\mathbf{c}_B \mathbf{B}^{-1} \mathbf{P}_k + c_k = -(z_k^- - c_k^-)$$

Thus, if x_k^- is a candidate for entering the basic solution, then x_k^+ cannot be an entering candidate, and vice versa.

If $z_k^+ - c_k^+ = (z_k^- - c_k^-) = 0$, then possibly one of the two variables may enter the basic solution to provide an alternative optimum. The two variables cannot be basic simultaneously because a basis \mathbf{B} cannot include two dependent vectors \mathbf{P}_k and $-\mathbf{P}_k$.

To show that the two variables cannot replace one another in alternative optima, assume that x_k^- is basic in the optimum solution. Then

$$\mathbf{B}^{-1} \mathbf{P}_k = (0, \dots, 1, \dots, 0)^T$$

$$\mathbf{B}^{-1} (-\mathbf{P}_k) = (0, \dots, -1, \dots, 0)^T$$

According to the feasibility condition, x_k^+ cannot replace x_k^- because the corresponding pivot element $\mathbf{B}^{-1}(-\mathbf{P}_k)$ is negative, unless $x_k^- = 0$, which is a trivial case.

6

Number of nonbasic variables = $n - m$. In the case of *nondegeneracy*, each entering nonbasic variable will be associated with a *distinct* adjacent extreme point. In the case of *degeneracy*, an entering nonbasic variable can result in a different basic solution without changing the extreme point itself. In this situation, the number of adjacent extreme points is less than $n - m$.

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Let $x_k = d_k (\geq 0)$ represent the current basic solution. Then, the new basic solution after x_j enters and x_k leaves is

$$x_j = \frac{d_r}{(\mathbf{B}^{-1} \mathbf{P}_j)_r} = \frac{0}{(\mathbf{B}^{-1} \mathbf{P}_j)_r} = 0, \text{ provided } (\mathbf{B}^{-1} \mathbf{P}_j)_r \neq 0$$

$$x_k = d_k - x_j (\mathbf{B}^{-1} \mathbf{P}_j)_k, \text{ all basic } x_k, k \neq j$$

The last equation is independent of $(\mathbf{B}^{-1} \mathbf{P}_j)_k$ for all k , because $x_j = 0$. Hence, x_k remains feasible for all k .

8

1. If the minimum ratio corresponds to more than one basic variable, the next iteration is degenerate.
2. If x_j is the entering variable and if the basic variable x_j is zero, the next iteration will continue to be degenerate if $(\mathbf{B}^{-1} \mathbf{P}_j)_k > 0$.
3. If for every zero basic variable, x_k , the pivot element $(\mathbf{B}^{-1} \mathbf{P}_j)_k \leq 0$, then the next iteration will not be degenerate.

Under nondegeneracy:

number of extreme points

= number of basic solutions

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Under degeneracy:

number of extreme points

< number of basic solutions

$$(a) x_j = \theta = \frac{x_n}{(\bar{B}^{-1}P_j)_n}, (\bar{B}^{-1}P_j)_n > 0$$

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For P_j , we have

$$\frac{\text{new } x_j}{\text{old } x_j} = \frac{\frac{x_n}{\alpha(\bar{B}^{-1}P_j)_n}}{\frac{x_n}{(\bar{B}^{-1}P_j)_n}} = \frac{1}{\alpha}$$

(b)

$$\frac{\text{new } x_j}{\text{old } x_j} = \frac{\frac{\beta(\bar{B}^{-1}b)_n}{\alpha(\bar{B}^{-1}P_j)_n}}{\frac{(\bar{B}^{-1}b)_n}{(\bar{B}^{-1}P_j)_n}} = \frac{\beta}{\alpha}$$

$$\text{New } (z_j - c_j) = C_B \left(\frac{1}{\alpha} \bar{B}^{-1} P_j \right) - \frac{1}{\alpha} c_j$$

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$$= \frac{1}{\beta} (\bar{B}^{-1} P_j - c_j)$$

$$= \frac{1}{\beta} (\text{old } z_j - c_j), \beta > 0$$

Conclusion: x_j remains nonbasic

A variable x_j can be made profitable either by increasing c_j or by decreasing z_j (which is the unit usage of resources by activity j). Of course, a combination of the two changes will work as well.

$$C_B = (c_1, c_2, \dots, c_m)$$

$$\bar{B} = (P_1, P_2, \dots, P_m)$$

For the basic variables

$$\begin{aligned} z_j - c_j &= C_B \bar{B}^{-1} (P_j, \dots, P_m) - (c_1, \dots, c_m) \\ &= \bar{B}^{-1} \bar{B} - C_B \\ &= C_B I - C_B = 0 \end{aligned}$$

Thus, for the basic variables, $z_j - c_j = 0$ regardless of the specific assignment to the vector C_B (e.g., D_B).

This result implies that changes in C_B cannot affect the optimality of the basic variables since these variables are already basic. It may, however, cause a nonbasic variable to become basic.

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Set 7.2b

	x_1	x_2	x_3	x_4	x_5	x_6	
Z	0	-2/3	5/6	0	0	0	20
x_1		2/3					4
x_4		4/3					2
x_5		5/3					5
x_6		1					2

(a) Starting iteration:

Let x_4 and x_5 be the slacks.

$$x_B = (x_4, x_5)^T, C_B = (0, 0), B = \bar{B} = I$$

First iteration:

$$C_B \bar{B}^{-1} = (0, 0)$$

$$(z_j - c_j)_{j=1,2,3} = (0, 0) \begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 4 \end{pmatrix} - (6, -2, 3) \\ = (-6, 2, -3) \Rightarrow x_1 \text{ enters}$$

$$x_B = \bar{B}^{-1} b = Ib = b = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\alpha^1 = \bar{B}^{-1} P_1 = P_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Theta = \min_{k=4,5} \{2/2, 4/1\} = 1 \Rightarrow x_4 \text{ leaves.}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}$$

$$x_B = (x_1, x_5)^T = (1, 3)^T$$

Second iteration:

$$C_B \bar{B}^{-1} = (6, 0) \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} = (3, 0)$$

$$(z_j - c_j)_{j=2,3,4} = (3, 0) \begin{pmatrix} -1 & 2 & 1 \\ 0 & 4 & 0 \end{pmatrix} - (-3, 3, 0) \\ = (-1, 3, 3) \Rightarrow x_2 \text{ enters}$$

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\alpha^2 = \bar{B}^{-1} P_2 = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$$

$$\Theta = \min_{k=1,5} \{-1, 3/2\} = 6 \Rightarrow x_6 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

$$x_B = (x_1, x_2)^T = (4, 6)^T, C_B = (6, -2)$$

continued...

7-10

Third iteration:

$$C_B \bar{B}^{-1} = (6, -2) \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = (2, 2)$$

$$(z_j - c_j)_{j=3,4,5} = (2, 2) \begin{pmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} - (3, 0, 0) \\ = (9, 2, 2) \Rightarrow \text{optimal}$$

Optimal solution:

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$Z = C_B x_B = 6x_4 + (-2)(6) = 12$$

(b)

Starting iteration: Let x_4, x_5 , and x_6 be the slack variables.

$$x_B = (x_4, x_5, x_6)^T, C_B = (0, 0, 0), B = \bar{B} = I$$

First iteration: $C_B \bar{B}^{-1} = (0, 0, 0)$

$$(z_j - c_j)_{j=1,2,3} = (0, 0, 0) \begin{pmatrix} 4 & 3 & 8 \\ 4 & 1 & 2 \\ -1 & 3 \end{pmatrix} - (2, 1, 2) \\ = (-2, -1, -2) \Rightarrow x_1 \text{ enters}$$

$$x_B = \bar{B}^{-1} b = Ib = b = (12, 8, 8)^T$$

$$\alpha^1 = \bar{B}^{-1} P_1 = P_1 = (4, 4, 4)^T$$

$$\Theta = \min_{k=4,5,6} \left\{ \frac{12}{4}, \frac{8}{4}, \frac{8}{4} \right\} = 2 \Rightarrow x_5 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$x_B = (x_4, x_1, x_6), C_B = (0, 2, 0)$$

Second iteration: $C_B \bar{B}^{-1} = (0, 1/2, 0)$

$$(z_j - c_j)_{j=2,3,5} = (0, 1/2, 0) \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1 & 1 \end{pmatrix} - (1, 2, 0) \\ = (-1/2, 4, 1/2) \Rightarrow x_2 \text{ enters}$$

$$x_B = \begin{pmatrix} x_4 \\ x_1 \\ x_6 \end{pmatrix} = \bar{B}'^{-1} b = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 12 \\ 8 \\ 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1/4 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2/4 \\ 1/4 \\ -2 \end{pmatrix}$$

$$\Theta = \min_{k=4,5,6} \left\{ \frac{4}{2}, \frac{2}{1/4}, -2 \right\} = 2, x_4 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 3 & 4 & 0 \\ 1 & 4 & 0 \\ -1 & 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/8 & 3/8 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

$$x_B = (x_2, x_1, x_6)^T, C_B = (1, 2, 0)$$

continued...

Set 7.2b

Third iteration: $\mathbf{G} \mathbf{B}^{-1} = (\frac{1}{4}, \frac{1}{4}, 0)$
 $(\mathbf{z}_j - \mathbf{c}_j)_{j=3,4,5} = (\frac{1}{4}, \frac{1}{4}, 0) \begin{pmatrix} 8 & 1 & 0 \\ 12 & 0 & 1 \\ 3 & 0 & 0 \end{pmatrix} - (2, 1, 0)$
 $= (3, \frac{1}{4}, \frac{1}{4}) \Rightarrow \text{optimal.}$

Optimal solution:

$$\mathbf{x}_B = \begin{pmatrix} x_2 \\ x_1 \\ x_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{8} & \frac{3}{8} & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 12 \\ 8 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{3}{2} \\ 4 \end{pmatrix}$$

$$z = 2x_3/4 + 1x_2 + 2x_0 = 5$$

(d)

Adding artificials, we get

$$\text{min } z = 2x_1 + x_2 + Mx_3 + Mx_5$$

$$\text{s.t. } \left(\begin{array}{ccc|ccc} \frac{3}{4} & \frac{1}{2} & 0 & 1 & 0 & 0 \\ \frac{3}{4} & \frac{3}{2} & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array} \right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$$

where x_3 and x_6 are slacks, and x_4 and x_5 are artificials.

Starting solution:

$$\mathbf{x}_B = (x_4, x_5, x_6), \mathbf{G} = (M, M, 0)$$

$$\mathbf{B} = \mathbf{B}' = I$$

First iteration: $\mathbf{G} \mathbf{B}^{-1} = (M, M, 0)$

$$(\mathbf{z}_j - \mathbf{c}_j)_{j=1,2,3} = (M, M, 0) \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & 0 \\ \frac{3}{4} & \frac{3}{2} & -1 \\ 1 & 2 & 0 \end{pmatrix} - (2, 1, 0)$$

$$= (-2 + 7M, -1 + 4M, -M)$$

Thus, x_1 enters.

$$\theta = \min_{K=4,5,6} \left\{ \frac{3}{3}, \frac{6}{4}, \frac{3}{1} \right\} = 1 \Rightarrow x_4 \text{ leaves}$$

$$\mathbf{B}'^{-1} = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

$$\mathbf{x}_B = (x_1, x_5, x_6)^T, \mathbf{G} = (2, M, 0)$$

Second iteration: $\mathbf{G} \mathbf{B}^{-1} = (\frac{2-4M}{3}, M, 0)$

$$(\mathbf{z}_j - \mathbf{c}_j)_{j=2,3,4} = \left(\frac{2-4M}{3}, M, 0 \right) \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} - (1, 0, 0)$$

$$= \left(\frac{5M-1}{3}, -M, \frac{2-4M}{3} \right) \Rightarrow x_2 \text{ enters}$$

$$\mathbf{x}_B = \begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{1}{2} \end{pmatrix}$$

$$\alpha^* = \begin{pmatrix} 1/2 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 5/3 \\ 5/3 \end{pmatrix}$$

$$\theta = \min_{k=1,5,6} \left\{ \frac{1}{1/3}, \frac{2}{5/3}, \frac{2}{5/3} \right\} \Rightarrow x_5 \text{ leaves}$$

Continued...

$$\mathbf{B}_{\text{next}}^{-1} = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3/5 & -4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\mathbf{x}_B = (x_1, x_2, x_6)^T, \mathbf{G} = (2, 1, 0)$$

Third iteration: $\mathbf{G} \mathbf{B}^{-1} = (\frac{2}{5}, \frac{1}{5}, 0)$

$$(\mathbf{z}_j - \mathbf{c}_j)_{j=3,4,5} = (2/5, 1/5, 0) \begin{pmatrix} 0 & 1 & 0 \\ -1/5 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - (0, M, M)$$

$$= (-1/5, 2/5 - M, 1/5 - M)$$

$$\Rightarrow \text{optimal solution.}$$

Optimal solution:

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}$$

$$z = 2x_3/5 + 1x_2/5 = 12/5$$

(d)

Minimize $Z = 5x_1 - 4x_2 + 6x_3 + 8x_4 + Mx_5$
 subject to

$$\begin{aligned} x_1 + 7x_2 + 3x_3 + 7x_4 + x_6 &= 46 \\ 3x_1 - x_2 + x_3 + 2x_4 + x_7 &= 20 \\ 2x_1 + 3x_2 - x_3 + x_4 - x_5 + x_8 &= 18 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0$$

Iteration 0:

$$\mathbf{x}_B = (x_6, x_7, x_8), \mathbf{G} = (0, 0, M), \mathbf{B}_0 = \mathbf{B}_0^{-1} = I$$

$$\{z_j - c_j\}_{j=1,2,3,4,5}$$

$$= (0, 0, M) \begin{pmatrix} 1 & 7 & 3 & 7 & 0 \\ 3 & -1 & 1 & 2 & 0 \\ 2 & 3 & -1 & 1 & -1 \end{pmatrix} - (5, -4, 6, 8, 0)$$

$$= (2M-5, [3M+4], -M-6, M-8, -M)$$

x_2 enters

$$\bar{\mathbf{B}}' P_2 = \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}, \bar{\mathbf{B}}' b = \begin{pmatrix} 46 \\ 20 \\ 18 \end{pmatrix}, \theta = \min \left\{ \frac{46}{7}, \frac{20}{-1}, \frac{18}{3} \right\}$$

x_8 leaves

$$\mathbf{B}_1 = \begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}, \bar{\mathbf{B}}_1^{-1} = \begin{pmatrix} 1 & 0 & -7/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1/3 \end{pmatrix}$$

$$x_{B_1} = \begin{pmatrix} x_6 \\ x_7 \\ x_2 \end{pmatrix} = \bar{\mathbf{B}}_1^{-1} b = \begin{pmatrix} 4 \\ 26 \\ 6 \end{pmatrix}$$

Continued...

Set 7.2b

Iteration 1:

$$x_B = (x_6, x_7, x_2)^T, c_B = (0, 0, -4)$$

$$c_B B^{-1} = (0, 0, -4/3)$$

$$\{z_j - c_j\}_{j=1,3,4,5}$$

$$= (0, 0, -4/3) \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{pmatrix} - (5, 6, 8, 0)$$

$$= (-23/3, -30/3, -28/3, 4/3)$$

x_5 enters

$$B_1^{-1} P_5 = \begin{pmatrix} 7/3 \\ -1/3 \\ -1/3 \end{pmatrix}, B_1^{-1} b = \begin{pmatrix} 4 \\ 26 \\ 6 \end{pmatrix}$$

x_6 leaves

Iteration 2:

$$x_B = (x_5, x_7, x_2)^T, c_B = (0, 0, -4)$$

$$B_2^{-1} = \begin{pmatrix} 0 & 0 & 7 \\ 0 & 1 & -1 \\ -1 & 0 & 3 \end{pmatrix}, B_2^{-1} b = \begin{pmatrix} 3/7 & 0 & 0 \\ 1/7 & 1 & 0 \\ 1/7 & 0 & 1 \end{pmatrix}$$

$$x_{B_2} = \begin{pmatrix} x_5 \\ x_7 \\ x_2 \end{pmatrix} = B_2^{-1} b = \begin{pmatrix} 12/7 \\ 186/7 \\ 46/7 \end{pmatrix}$$

$$c_B B_2^{-1} = (-4/7, 0, 0)$$

$$\{z_j - c_j\}_{j=1,3,4,6}$$

$$= (-4/7, 0, 0) \begin{pmatrix} 1 & 3 & 7 & 1 \\ 3 & 1 & 2 & 0 \\ 2 & -1 & 1 & 0 \end{pmatrix} - (5, 6, 8, 0)$$

$$= (-39/7, -54/7, -12, -4/7) \text{ optimum}$$

$$x_{B_2} = (x_5, x_7, x_2)^T = (12/7, 186/7, 46/7)$$

$$Z = -184/7$$

Iteration 0:

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$$x_{B_0} = (x_2, x_4, x_5)^T, c_B = (7, -10, 0)$$

$$B_0^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, B_0^{-1} b = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

continued...

$$x_B = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} = B_0^{-1} b = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$$

$$c_B B_0^{-1} = (7, -10, 0) \begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} = (17, 7, -17)$$

$$\{z_j - c_j\}_{j=1,3,6}$$

$$= (17, 7, -17) \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 3 \\ 1 & -3 & 0 \end{pmatrix} - (0, 11, 26)$$

$$= (-17, 16, 12) \quad x_3 \text{ enters}$$

$$B_0^{-1} b = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}, B_0^{-1} P_3 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \quad x_2 \text{ leaves}$$

Iteration 1:

$$x_B = (x_3, x_4, x_5)^T, c_B = (11, -10, 0)$$

$$B_1^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, B_1^{-1} b = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$x_B = B_1^{-1} b = \begin{pmatrix} 2 \\ 10 \\ 8 \end{pmatrix}$$

$$c_B B_1^{-1} = (11, -10, 0) \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} = (1, -9, -1)$$

$$\{z_j - c_j\}_{j=1,2,6}$$

$$= (1, -9, -1) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} - (0, 7, 26)$$

$$= (-1, -16, -52) \Rightarrow \text{optimum}$$

$$x_{B_1} = (x_3, x_4, x_5)^T = (2, 10, 8)^T$$

$$Z = -78$$

(a) Minimize $Z = 2x_1 + x_2 + M(x_4 + x_5)$

subject to

$$3x_1 + x_2 + x_4 = 3$$

$$4x_1 + 3x_2 - x_3 + x_5 = 6$$

$$x_1 + 2x_2 + x_6 = 3$$

4

Phase I: $x_1, \dots, x_6 \geq 0$

Iteration 0:

$$x_B = (x_4, x_5, x_6)^T, c_B = (1, 1, 0)$$

$$B_0^{-1} = I, c_B B_0^{-1} = (1, 1, 0)$$

continued...

Set 7.2b

$$\{z_j - c_j\}_{j=1,2,3}$$

$$= (1, 1, 0) \begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & -1 \\ 1 & 2 & 0 \end{pmatrix} - (0, 0, 0)$$

$\Rightarrow (7, 4, -1)$, x_1 enters

$$\bar{B}_0^{-1} P_1 = \bar{B}_0^{-1} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \bar{B}_0^{-1} b = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$$

$$\theta = \min \left\{ \frac{3}{3}, \frac{6}{4}, \frac{3}{1} \right\} \Rightarrow x_4 \text{ leaves}$$

Iteration 1:

$$X_B = (x_1, x_5, x_6)^T, C_B = (0, 1, 0)$$

$$B_1 = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, B_1^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ -4/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

$$X_B = \begin{pmatrix} x_1 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ z \end{pmatrix}$$

$$C_B B_1^{-1} = (-4/3, 1, 0)$$

$$\{z_j - c_j\}_{j=2,3,4}$$

$$= (-4/3, 1, 0) \begin{pmatrix} 1 & 0 & 1 \\ 3 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix} - (0, 0, 1)$$

$\Rightarrow (5/3, -1, -7/3)$ x_2 enters

$$\bar{B}_1^{-1} P_2 = \bar{B}_1^{-1} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 5/3 \\ 5/3 \end{pmatrix},$$

$$\bar{B}_1^{-1} b = \begin{pmatrix} 1 \\ z \\ z \end{pmatrix}$$

$$\theta = \min \left\{ \frac{1}{1/3}, \frac{2}{5/3}, \frac{2}{5/3} \right\}, x_5 \text{ leaves}$$

Iteration 2:

$$X_B = (x_1, x_2, x_6)^T, C_B = (0, 0, 0)$$

$$B_2 = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}, B_2^{-1} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Since X_B does not include the artificial x_4 and x_5 , we can use to start Phase II.

Continued...

Phase II: objective max $z = 2x_1 + x_2$

Iteration 0:

$$X_B = (x_1, x_2, x_6), C_B = (2, 1, 0)$$

$$\bar{B}_0^{-1} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$X_B = \begin{pmatrix} x_1 \\ x_2 \\ x_6 \end{pmatrix} = \bar{B}_0^{-1} b = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}$$

$$C_B \bar{B}_0^{-1} = (2, 1, 0) \begin{pmatrix} 3/5 & -1/5 & 0 \\ -4/5 & 3/5 & 0 \\ 1 & -1 & 1 \end{pmatrix} = (2/5, 1/5, 0)$$

$$\{z_j - c_j\}_{j=3} = (2/5, 1/5, 0) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - 0 = -1/5$$

x_3 enters

$$\bar{B}_0^{-1} P_3 = \begin{pmatrix} 1/5 \\ -3/5 \\ 1 \end{pmatrix}, \bar{B}_0^{-1} b = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix} x_6 \text{ leaves}$$

Iteration 1:

$$X_B = (x_1, x_2, x_3), C_B = (2, 1, 0)$$

$$B_1 = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 3 & -1 \\ 1 & 2 & 0 \end{pmatrix}, B_1^{-1} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\{z_j - c_j\}_{j=6}$$

$$= (3/5, 0, 1/5) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - 0 = 1/5 > 0$$

optimum!

$$X_B = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 6/5 \\ 0 \end{pmatrix}$$

$$z = 12/5$$

Minimize $z = 3x_1 + 2x_2$

subject to

$$-3x_1 - x_2 + x_3 = -3$$

$$-4x_1 - 3x_2 + x_4 = -6$$

$$x_1 + x_2 + x_5 = 3$$

5

Iteration 0:

$$X_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix}, B_0 = \bar{B}_0^{-1} = I$$

Continued...

Set 7.2b

$$x_B = \begin{pmatrix} -3 \\ -6 \\ 3 \end{pmatrix} \Rightarrow x_4 \text{ leaves}$$

$$C_B = (0, 0, 0), C_B B^{-1} = (0, 0, 0)$$

$$\{z_j - c_j\}_{j=1,2}$$

$$= (0, 0, 0) \begin{pmatrix} -3 & -1 \\ -4 & -3 \\ 1 & 1 \end{pmatrix} - (3, 2) = (-3, -2)$$

$$(\text{row 2 of } B_0^{-1})(P_1, P_2)$$

$$= (0, 1, 0) \begin{pmatrix} -3 & -1 \\ -4 & -3 \\ 1 & 1 \end{pmatrix} = (-4, -3)$$

$$\Theta = \min_{j=1,2} \left\{ \left| \frac{-3}{-4} \right|, \left| \frac{-2}{-3} \right| \right\} = 2/3 \Rightarrow x_2 \text{ enters}$$

Iteration 1:

$$x_B = \begin{pmatrix} x_3 \\ x_2 \\ x_5 \end{pmatrix}, B_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & 1 \end{pmatrix}, B_1^{-1} = \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & -1/3 & 0 \\ 0 & 1/3 & 1 \end{pmatrix}$$

$$x_B = \begin{pmatrix} x_3 \\ x_2 \\ x_5 \end{pmatrix} = B_1^{-1} b$$

$$= \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & -1/3 & 0 \\ 0 & 1/3 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -6 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} x_3 \text{ leaves}$$

$$C_B = (0, 2, 0)$$

$$C_B B^{-1} = (0, -2/3, 0)$$

$$\{z_j - c_j\}_{j=1,4} = (0, -2/3, 0) \begin{pmatrix} -3 & 0 \\ -4 & 1 \\ 1 & 0 \end{pmatrix} - (3, 0) \\ = (-1/3, -2/3)$$

$$(\text{row 1 of } B_1^{-1})(P_1, P_4)$$

$$= (1, -1/3, 0) \begin{pmatrix} -3 & 0 \\ -4 & 1 \\ 1 & 0 \end{pmatrix} = (-5/3, -1/3)$$

$$\Theta = \min_{j=1,4} \left\{ \left| \frac{-1/3}{-5/3} \right|, \left| \frac{-2/3}{-1/3} \right| \right\} = 1/5$$

x_1 enters

Iteration 2:

$$x_B = \begin{pmatrix} x_1 \\ x_2 \\ x_5 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} -3 & -1 & 0 \\ -4 & -3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$B_2^{-1} = \begin{pmatrix} -3/5 & 1/5 & 0 \\ 4/5 & -3/5 & 0 \\ -1/5 & 2/5 & 1 \end{pmatrix}$$

$$x_B = B_2^{-1} b$$

$$= \begin{pmatrix} -3/5 & 1/5 & 0 \\ 4/5 & -3/5 & 0 \\ -1/5 & 2/5 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -6 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3/5 \\ 6/5 \\ 6/5 \end{pmatrix}$$

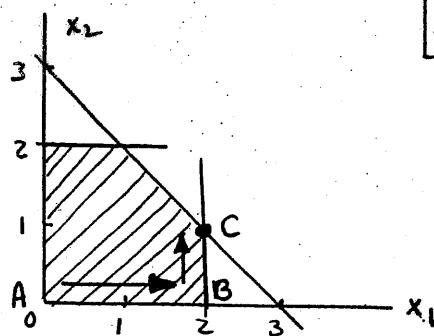
Feasible!

$$Z = 3 \times 3/5 + 2 \times 6/5 = 21/5$$

continued...

Set 7.3a

a)



b)

Iteration 1: x_1 enters

	x_1	x_2	x_3	solution
\bar{z}	-2	-1	0	0
x_3	1	1	1	3

$$\theta = \min \{3/1, -, 2\} = 2$$

Substitute x_1 at its upper bound: $x_1 = 2 - x_1'$

	x_1'	x_2	x_3	solution
\bar{z}	2	-1	0	2
x_3	-1	1	1	1

This solution ($x_1 = 2$, $x_2 = 0$) coincides with point B in the solution space above. The solution now has $x_1' = 0$, which implies that $x_1 = 2$, thus reducing the solution space to line segment BC.

Iteration 2: x_2 enters

$$\theta = \min \{1/1, -, 2\} = 1$$

	x_1'	x_2	x_3	solution
\bar{z}	1	0	1	3
x_2	-1	1	1	1

Optimum: $x_1' = 0 \Rightarrow x_1 = 2, x_2 = 1$ which is the same as point C.

c) As shown in (b) above, the substitution of the upper-bounding method recognizes the extreme point implicitly by using the substitution

$$x_j = \mu_j - x_j'$$

2

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
\bar{z}	-6	-2	-8	-4	-2	-10	0	0
x_7	8	1	8	2	2	4	1	13

x_6 enters: $\theta = \min \{13/4, -, 1\} = 1$

$$x_6 = 1 - x_6'$$

	x_1	x_2	x_3	x_4	x_5	x_6'	x_7	
\bar{z}	-6	-2	-8	-4	-2	10	0	10
x_7	8	1	8	2	2	-4	1	9

x_3 enters: $\theta = \min \{9/8, -, 1\} = 1$

$$x_3 = 1 - x_3'$$

	x_1	x_2	x_3'	x_4	x_5	x_6'	x_7	
\bar{z}	-6	-2	8	-4	-2	10	0	18
x_7	8	1	-8	2	2	-4	1	1

x_1 enters: $\theta = \min \{1/8, -, 1\} = 1/8$, x_7 leaves

	x_1	x_2	x_3'	x_4	x_5	x_6'	x_7	
\bar{z}	0	-5/4	2	-5/2	-1/2	7	3/4	18 3/4
x_1	1	1/8	-1	1/4	1/4	-1/2	1/8	1/8

x_7 enters: $\theta = \min \{1/8, -, 1\} = 1/8$, x_1 leaves

	x_1	x_2	x_3'	x_4	x_5	x_6'	x_7	
\bar{z}	10	0	-8	0	2	2	2	20
x_4	4	1/2	-4	1	1	-2	1/2	1/2

x_3' enters: $\theta = \min \{-, \frac{1/2-1}{(-4)}, 1\} = 1/8$

x_4 leaves, $x_4 = 1 - x_4'$

	x_1	x_2	x_3'	x_4'	x_5	x_6'	x_7	
\bar{z}	2	-1	0	2	0	6	1	21
x_3'	-1	-1/8	1	1/4	-1/4	1/2	-1/8	1/8

x_2 enters: $\theta = \min \{-, \frac{1/8-1}{-1/8}, 1\} = 1$

$$x_2 = 1 - x_2'$$

	x_1	x_2'	x_3'	x_4'	x_5	x_6'	x_7	
\bar{z}	2	1	0	2	0	6	1	22
x_3'	-1	1/8	1	1/4	-1/4	1/2	-1/8	1/4

Optimum solution:

$$x_1 = 0$$

$$x_2 = 1$$

$$x_3 = 3/4$$

$$Z = 22$$

$$x_4 = 1$$

$$x_5 = 0$$

$$x_6 = 1$$

Set 7.3a

(a) Minimize

	x_1	x_2	x_3	x_4	x_5	
\bar{z}	-6	2	3	0	0	0
x_4	2	4	2	1	0	8
x_5	1	-2	3	0	1	7

$$x_3 \text{ enters: } \theta = \min\left\{\frac{7}{3}, -\right\} = 1; x_3 = 1 - x_3'$$

	x_1	x_2	x_3'	x_4	x_5	
\bar{z}	-6	2	-3	0	0	-3
x_4	2	4	-2	1	0	6
x_5	1	-2	-3	0	1	4

$$x_2 \text{ enters: } \theta = \min\left\{\frac{6}{4}, -\right\} = 3/2; x_4 \text{ leaves}$$

	x_1	x_2	x_3'	x_4	x_5	
\bar{z}	-7	0	-2	-1/2	0	-6
x_2	1/2	1	-1/2	1/4	0	3/2
x_5	2	0	-4	1/2	1	7

$$\text{Optimum: } x_1 = 0, x_2 = 3/2, x_3 = 1, \bar{z} = -6$$

b) Maximize

	x_1	x_2	x_3	x_4	x_5	
\bar{z}	-3	-5	-2	0	0	0
x_4	1	2	2	1	0	10
x_5	2	4	3	0	1	15

$$x_2 \text{ enters: } \theta = \min\left\{\frac{15}{4}, -\right\} = 3; x_2 = 3 - x_2'$$

	x_1	x_2'	x_3	x_4	x_5	
\bar{z}	-3	5	-2	0	0	15
x_4	1	-2	2	1	0	4
x_5	2	-4	3	0	1	3

$$x_1 \text{ enters: } \theta = \min\left\{\frac{3}{2}, -\right\} = 1.5; x_5 \text{ leaves}$$

	x_1	x_2'	x_3	x_4	x_5	
\bar{z}	0	-1	5/2	0	3/2	39/2
x_4	0	0	1/2	1	-1/2	5/2
x_1	1	-2	3/2	0	1/2	3/2

$$x_2' \text{ enters: } \theta = \min\left\{-\frac{3/2-2}{-2}, 2\right\} = 1/4$$

	x_1'	x_2'	x_3	x_4	x_5	
\bar{z}	1/2	0	7/4	0	5/4	93/4
x_4	0	0	1/2	1	-1/2	5/2
x_5	1/2	1	-3/4	0	-1/4	5/4

$$\text{Optimum: } x_1 = 4, x_2 = 7/4, x_3 = 0, \bar{z} = 93/4$$

3

(a) Substitute $x_1 = 1 + y_1, x_3 = y_3 + 2$

Phase 1: $0 \leq y_2 \leq 2, 0 \leq x_2 \leq 3, y_3 \geq 0$

	y_1	x_2	y_3	x_4	x_5	R
\bar{z}	1	2	-1	-1	0	0
x_5	2	1	1	0	1	0
R_1	1	2	-1	-1	0	1
\bar{z}	0	0	0	0	0	-1
x_5	3/2	0	3/2	1/2	1	0
x_2	1/2	1	-1/2	-1/2	0	1

4

Phase 2:

	y_1	x_2	y_3	x_4	x_5	
\bar{z}	-2	0	1	-1	0	3
x_5	3/2	0	3/2	1/2	1	2
x_2	1/2	1	-1/2	-1/2	0	2

$$y_1 \text{ enters: } \theta = \min\left\{\frac{2}{3/2}, -\right\} = 4/3; x_5 \text{ leaves}$$

	y_1	x_2	y_3	x_4	x_5	
\bar{z}	0	0	3	-1/3	4/3	17/6
y_1	1	0	1	1/3	2/3	4/3
x_2	0	1	-1	-2/3	-1/3	4/3
x_4	0	1/2	3/2	1	1/2	5/2

$$x_4 \text{ enters: } \theta = \min\left\{\frac{4/3}{1/2}, \frac{4/3-3}{-2/3}, -\right\} = 5/2$$

x_2 leaves, $x_2 = 1 - x_2'$

	y_1	x_2'	y_3	x_4	x_5	
\bar{z}	0	1/2	7/2	0	3/2	13/2
y_1	1	-1/2	1/2	0	1/2	1/2
x_4	0	3/2	3/2	1	1/2	5/2

$$\text{Optimum: } x_1 = 3/2, x_2 = 3, x_3 = 2, \bar{z} = 13/2$$

b) Let $x_i = 1 + y_i, 0 \leq y_i \leq 2, 0 \leq x_2 \leq 1$

Phase 1:

	y_1	x_2	x_3	R	x_4	x_5	
\bar{z}	-1	2	0	0	0	0	1
R	-1	2	-1	1	0	0	1
x_4	3	2	0	0	1	0	7
x_5	-1	1	0	0	0	1	2
\bar{z}	-2	0	-1	1	0	0	0
x_2	-1/2	1	-1/2	1/2	0	0	1/2
y_1	4	0	1	-1	1	0	6
x_5	-1/2	0	1/2	-1/2	0	1	3/2

Phase 2:

	y_1	x_2'	x_3	x_4	x_5	
\bar{z}	0	4	1	0	0	4
y_1	1	2	-1	0	0	1
x_2'	0	-8	-3	1	0	1/2
x_5	0	1	1	0	1	1/2

$$\text{Optimum: } x_1 = 2, x_2 = 1, \bar{z} = 4$$

Set 7.3a

5

c) Let $x_1 = 1 + y_1$

$$0 \leq y_1 \leq 2, 0 \leq x_2 \leq 5, 0 \leq x_3 \leq 2$$

	x_1	x_2	x_3	x_4	x_5	x_6	
\bar{z}	-4	-2	-6	0	0	0	4
x_4	4	-1	0	1	0	0	5
x_5	-1	1	2	0	1	0	9
x_6	-3	1	4	0	0	1	15

$$x_3 \text{ enters: } \theta = \min \{ 15/4, -, 2 \} = 2; x_3 = 2 - x_3'$$

	y_1	x_2	x_3'	x_4	x_5	x_6	
\bar{z}	-4	-2	6	0	0	0	16
x_4	4	-1	0	1	0	0	5
x_5	-1	1	-2	0	1	0	5
x_6	-3	1	-4	0	0	1	7

$$y_1 \text{ enters: } \theta = \min \{ \frac{5}{4}, -, 2 \} = 5/4; x_4 \text{ leaves}$$

	y_1	x_2	x_3'	x_4	x_5	x_6	
\bar{z}	0	-3	6	1	0	0	21
y_1	1	-1/4	0	1/4	0	0	5/4
x_5	0	3/4	-2	1/4	1	0	25/4
x_6	0	1/4	-4	3/4	0	1	43/4

$$x_2 \text{ enters: } \theta = \min \{ \frac{25}{3}, \frac{5/4 - 2}{-1/4}, 5 \} = 3$$

$$y_1 \text{ leaves, } y_1 = 2 - y_1'$$

	y_1'	x_2	x_3'	x_4	x_5	x_6	
	12	0	6	-2	0	0	30
x_2	4	1	0	-1	0	0	3
x_5	-3	0	-2	1	1	0	4
x_6	-1	0	-4	1	0	1	10

$$x_4 \text{ enters: } \theta = \min \{ 4, \frac{3-5}{-1}, - \} = 2$$

$$x_2 \text{ leaves, } x_2 = 5 - x_2'$$

	y_1'	x_2'	x_3'	x_4	x_5	x_6	
\bar{z}	4	2	6	0	0	0	34
x_4	-4	1	0	1	0	0	2
x_5	3	-1	-2	0	1	0	2
x_6	1	-1	-4	0	0	1	8

Optimum Solution:

$$x_1 = 3$$

$$x_2 = 5$$

$$x_3 = 2$$

$$z = 34$$

Let X_u represent the basic and nonbasic variables in X that have been substituted at their upper bound. Also, let X_n be the remaining basic and nonbasic variables. Suppose that the order of the vectors of (A, I) corresponding to X_u and X_n are given by the matrices D_u and D_n , and let the vector C of the objective function be partitioned correspondingly to give (C_u, C_n) . The equations of the linear programming problem at any iteration then become

$$\begin{pmatrix} 1 & -C_u & -C_n \\ 0 & D_u & D_n \end{pmatrix} \begin{pmatrix} z \\ X_u \\ X_n \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Instead of dealing with two types of variables, X_u and X_n , X_u is put at zero level by using the substitution

$$X_u = U_u - X_n'$$

where U_u is a subset of U representing the upper bounds for the variables in X_u . This gives

$$\begin{pmatrix} 1 & -C_u & C_n \\ 0 & D_u & -D_n \end{pmatrix} \begin{pmatrix} z \\ X_n \\ X_n' \end{pmatrix} = \begin{pmatrix} C_n U_u \\ b - D_n U_u \end{pmatrix}$$

The optimality and the feasibility conditions can be developed more easily now, since all nonbasic variables are at zero level. However, it is still necessary to check that no basic or nonbasic variable will exceed its upper bound.

Define X_b as the basic variables of the current iteration, and let C_b represent the elements corresponding to X_b in C . Also, let B be the basic matrix corresponding to X_b . The current solution is determined from

$$\begin{pmatrix} 1 & -C_b & z \\ 0 & B & X_n \end{pmatrix} = \begin{pmatrix} C_n U_u \\ b - D_n U_u \end{pmatrix}$$

By inverting the partitioned matrix as in Section 4.1.3, the current basic solution is given by

$$\begin{pmatrix} z \\ X_n \end{pmatrix} = \begin{pmatrix} 1 & C_b B^{-1} \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} C_n U_u \\ b - D_n U_u \end{pmatrix} = \begin{pmatrix} C_n U_u + C_b B^{-1}(b - D_n U_u) \\ B^{-1}(b - D_n U_u) \end{pmatrix}$$

By using

$$b' = b - D_n U_u$$

the complete simplex tableau corresponding to any iteration is

Basic	X_z^T	X_n^T	Solution
z	$C_b B^{-1} D_u - C_u$	$-C_b B^{-1} D_n + C_n$	$C_b B^{-1} b' + C_n U_u$
X_n	$B^{-1} D_u$	$-B^{-1} D_n$	$B^{-1} b'$

$$(a) b' = b - D_n U_u$$

$$= \begin{pmatrix} 7 \\ 15 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} (3) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} X_n \\ X_1 \end{pmatrix} = \bar{B}^{-1} b' = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}$$

$$(b) X_B = \begin{pmatrix} X_4 \\ X_2' \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 0 & -4 \end{pmatrix}, \bar{B}^{-1} = \begin{pmatrix} 1 & -1/4 \\ 0 & -1/4 \end{pmatrix}$$

$$b' = b - D_n U_u$$

$$= \begin{pmatrix} 7 \\ 15 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$$X_B = \begin{pmatrix} X_4 \\ X_2' \end{pmatrix} = \begin{pmatrix} 1 & -1/4 \\ 0 & -1/4 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 5/4 \\ 5/4 \end{pmatrix}$$

6

Set 7.3a

$$\text{Minimize } Z = 6x_1 - 2x_2 - 3x_3$$

Subject to

$$2x_1 + 4x_2 + 2x_3 + x_4 = 8$$

$$x_1 - 2x_2 + 3x_3 + x_5 = 7$$

$$0 \leq x_1 \leq 2, 0 \leq x_2 \leq 2, 0 \leq x_3 \leq 1$$

We use the tableau developed in Problem 5 above.

Iteration 0:

$$x_B = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix}, B = B^{-1} = I$$

$$c_B = (0, 0), c_B B^{-1} = (0, 0)$$

$$\{Z_j - g_j\}_{j=1,2,3}$$

$$= (0, 0) \begin{pmatrix} 2 & 4 & 2 \\ 1 & -2 & 3 \end{pmatrix} - (6, -2, -3)$$

$$= (-6, 2, 3), x_3 \text{ enters}$$

$$B^{-1} P_3 = B^{-1} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = B^{-1} b = \begin{pmatrix} 8 \\ 7 \end{pmatrix} \Rightarrow \theta_1 = 7/3$$

$$\text{Since } B^{-1} P_3 > 0, \theta_2 = \infty$$

$$\theta = \min \{ 7/3, \infty, 1 \} = 1$$

Thus, x_3 becomes nonbasic at its upper bound.

New Solution: $x_Z = (x_1, x_2), x_u = x_3$

$$u_u = 1, c_u = -3$$

$$D_2 = \begin{pmatrix} 2 & 4 \\ 1 & -2 \end{pmatrix}, D_u = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, c'_2 = (6, -2)$$

$$b' = \begin{pmatrix} 8 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}(1) = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = B^{-1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, Z = -3$$

Iteration 1: $C_2 = (6, -2), c_u = c'_3 = 3$

$$P'_3 = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, B = B^{-1} = I, c_B = (0, 0), c_B B^{-1} = (0, 0)$$

$$\{Z_j - g_j\}_{j=1,2}$$

$$= (0, 0) \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} - (6, -2) = (-6, 2)$$

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$$\{Z_j - g_j\}_{u(j=3)}$$

$$= (0, 0) \begin{pmatrix} -2 \\ -3 \end{pmatrix} - (3) = -3$$

x_2 enters

$$B^{-1} P_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}, X_B = \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\Theta_1 = \frac{6}{4} = 3/2, \Theta_2 = \infty \text{ (because } U_5 = \infty)$$

$$\theta = \min \{ 3/2, \infty, 2 \} = 3/2$$

x_4 leaves

Iteration 2: $C_2 = (x_1, x_4), X_u = x_3$

$$X_B = \begin{pmatrix} x_2 \\ x_5 \end{pmatrix}, P'_3 = \begin{pmatrix} -2 \\ -3 \end{pmatrix}, b' = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$B = \begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix}, B^{-1} = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 1 \end{pmatrix}$$

$$c_B = (-2, 0), c_B B^{-1} = (-1/2, 0)$$

$$\{Z_j - g_j\}_{z(j=1,4)}$$

$$= (-1/2, 0) \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} - (6, 0) = (-7, 0)$$

$$\{Z_j - g_j\}_{u(j=3)}$$

$$= (-1/2, 0) \begin{pmatrix} -2 \\ -3 \end{pmatrix} - 3 = -2$$

Optimum!

$$X_B = \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 7 \end{pmatrix}$$

$$x_3 = 1 - 0 = 1$$

$$Z = -6$$

continued...

Set 7.3a

(a)

To convert the problem into a dual feasible solution, we use the following substitutions:

$$x_1 = 2 - x_1', \quad x_2 = 3 - x_2'$$

Thus,

$$\text{minimize } Z = 3x_1' + 2x_2' + 2x_3 - 12$$

Subject to

$$-2x_1' - x_2' + x_3 \leq 1$$

$$-x_1' + 2x_2' - x_3 \leq -9$$

$$0 \leq x_1' \leq 2, 0 \leq x_2' \leq 3, 0 \leq x_3 \leq 1$$

	x_1'	x_2'	x_3	x_4	x_5	
Z	-3	-2	-2	0	0	-12
x_4	-2	-1	1	1	0	1
x_5	-1	2	-1	0	1	-9

x_5 leaves and x_3 enters

	x_1'	x_2'	x_3	x_4	x_5	
Z	-1	-6	0	0	-2	6
x_4	-2	1	0	1	1	-8
x_3	1	-2	1	0	-1	9

x_3 above its upper bound, substitute $x_3 = 1 - x_3'$, then multiply the second row by -1.

	x_1'	x_2'	x_3'	x_4	x_5	
Z	-1	-6	0	0	-2	6
x_4	-2	1	0	1	1	-8
x_3'	-1	2	1	0	1	-8

x_3' leaves and x_1' enters

	x_1'	x_2'	x_3'	x_4	x_5	
Z	0	-8	-1	0	-3	14
x_4	0	-3	-2	1	-1	8
x_1'	1	-2	-1	0	-1	8

Substitute $x_1' = 2 - x_1$, and multiply second row by -1.

8

	x_1	x_2	x_3	x_4	x_5	
Z	0	-8	-1	0	-3	14
x_4	0	-3	-2	1	-1	8
x_1	1	2	1	0	1	-8

x_1 -row shows that the problem has no feasible solution

$$(b) \text{ Let } x_1 = 2 - x_1'$$

$$x_2 = 3 - x_2'$$

This substitution will result in a dual feasible starting solution

	x_1'	x_2'	x_3	x_4	x_5	
Z	1	5	2	0	0	17
x_4	-4	-2	2	1	0	12
x_5	1	3	-4	0	1	-6
Z	$3/2$	$13/2$	0	0	$1/2$	14
x_4	$-7/2$	$-1/2$	0	1	$1/2$	9
x_3	$-1/4$	$-3/4$	1	0	$-1/4$	$3/2$

Optimum!

$$x_1 = 2 - 0 = 2$$

$$x_2 = 3 - 0 = 3$$

$$x_3 = 3/2$$

$$Z = 14$$

Continued...

Primal:

Maximize $z = CX$
Subject to

$$AX = b \quad \leftarrow Y$$

$$X \geq 0$$

Dual:

Minimize $w = Yb$
Subject to

$$YA \geq C$$

Y unrestricted

Dual in equation form:

Minimize $w = Yb$
Subject to

$$YA - IS = C \quad \leftarrow X$$

Y unrestricted

$$S \geq 0$$

Dual of dual:

Maximize $z = CX$
Subject to

$$AX = b$$

$$-X \leq 0 \Rightarrow X \geq 0$$

The first set of constraints is
equation because Y is unrestricted

The last problem shows that
the dual of the dual is the primal

1

Primal in equation form:

Minimize $z = CX$
Subject to

$$AX - IS = b \quad \leftarrow Y$$

$$X \geq 0$$

$$S \geq 0$$

Dual:

Maximize $w = Yb$
Subject to

$$YA \leq C$$

$$-Y \leq 0 \Rightarrow Y \geq 0$$

2

Set 7.4b

Primal in equation form:

$$\text{Maximize } Z = x_1 + x_2$$

Subject to

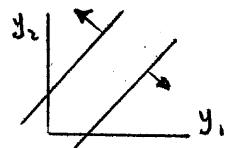
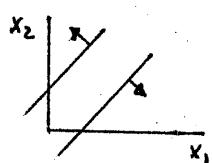
$$\begin{aligned} x_1 - x_2 + s_1 &= -1 & \leftarrow y_1 \\ -x_1 + x_2 + s_2 &= -1 & \leftarrow y_2 \end{aligned}$$

Dual:

$$\text{Minimize } w = -y_1 - y_2$$

Subject to

$$\begin{aligned} y_1 - y_2 &\geq 1 \\ -y_1 + y_2 &\geq 1 \\ y_1, y_2 &\geq 0 \end{aligned}$$



(a) Dual:

$$\text{Minimize } w = y_1 - 5y_2 + 6y_3$$

Subject to

$$\begin{aligned} 2y_1 + 4y_3 &\geq 50 \\ y_1 + 2y_2 &\geq 30 \\ y_3 &\geq 10 \\ y_1, y_2, y_3 &\text{ unrestricted.} \end{aligned}$$

(b) $2x_1 = -5 \Rightarrow x_1 < 0$, infeasible

(c) Inspection of the second dual constraint shows that y_2 can be increased indefinitely without violating any of the dual constraints.

Thus, $w = y_1 - 5y_2 + 6y_3$ is unbounded.

(d)

Primal infeasible $\Rightarrow \begin{cases} \text{dual infeasible} \\ \text{or} \\ \text{dual unbounded} \end{cases}$

Primal unbounded \Rightarrow dual infeasible

(a) Minimize $w = 2y_1 + 5y_2$
Subject to

$$\begin{aligned} 2y_1 + y_2 &\geq 5 \\ -y_1 + 2y_2 &\geq 12 \\ 3y_1 + y_2 &\geq 4 \\ y_2 &\geq 0 \end{aligned}$$

y_1 unrestricted

(b)

$$(i) B = (P_1 P_2) = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix}, \bar{B}^{-1} = \begin{pmatrix} -1/3 & 1 \\ 1/3 & 0 \end{pmatrix}$$

$$x_B = \begin{pmatrix} -1/3 & 1 \\ 1/3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 13/3 \\ 2/3 \end{pmatrix} \text{ feasible}$$

$$C_B = (0, 4)$$

$$Y = C_B \bar{B}^{-1} = (0, 4) \begin{pmatrix} -1/3 & 1 \\ 1/3 & 0 \end{pmatrix} = (4/3, 0)$$

Dual feasibility:

$$2y_1 + y_2 = 2 \times 4/3 + 1 \times 0 = 8/3 \neq 5$$

Dual infeasible \Rightarrow primal nonoptimal.

$$(ii) B = (P_2 P_3) = \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}, \bar{B}^{-1} = \begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{pmatrix}$$

$$x_B = \begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 13/7 \\ 9/7 \end{pmatrix} \text{ feasible}$$

Dual feasibility:

$$Y = C_B \bar{B}^{-1} = (12, 4) \begin{pmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{pmatrix} = (-4/7, 40/7)$$

$$2y_1 + y_2 = 2(-4/7) + 40/7 = \frac{32}{7} \neq 5$$

x_B is not optimal

$$(iii) B = (P_1 P_2) = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \bar{B}^{-1} = \begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix}$$

$$x_B = \begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 9/5 \\ 8/5 \end{pmatrix} \text{ feasible}$$

Dual feasibility:

$$Y = C_B \bar{B}^{-1} = (5, 12) \begin{pmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{pmatrix} = (-2/5, 29/5)$$

Y satisfies all dual constraints. Thus x_B is optimal.

continued...

3

Set 7.4b

(iv) $B = (P_1 P_4) = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

$$B^{-1} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}$$

$$X_B = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \text{ feasible}$$

Dual feasibility:

$$Y = G B^{-1} = (5, 0) \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix} = (5/2, 0)$$

Y does not satisfy second dual constraint. X_B is not optimum

(a) Dual:

Minimize $w = 4Y_1 + 8Y_2$

Subject to

$$\begin{cases} Y_1 + Y_2 \geq 2 \\ Y_1 + 4Y_2 \geq 4 \\ Y_1 \geq 4 \\ Y_2 \geq -3 \end{cases} \quad \begin{array}{l} \text{all } Y \\ \text{unrestricted} \end{array}$$

(b) $X_B = (x_2, x_3)^T$

$$B = \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix}, B^{-1} = \begin{pmatrix} 0 & -1/4 \\ 1 & -1/4 \end{pmatrix}$$

$$G_B = (4, 4), G_B B^{-1} = (4, 0)$$

$$Z_1 - c_1 = G_B B^{-1} P_1 - c_1$$

$$= (4, 0) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2 = 2 > 0$$

$$Z_4 - c_4 = (4, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - (-3) = 3 > 0$$

X_B optimal

(c) X_3 basic $\Rightarrow Z_3 - c_3 = 0$, or

$$Y P_3 - c_3 = (Y_1, Y_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 4 = 0, \text{ or}$$

$$Y_1 - 4 = 0 \Rightarrow Y_1 = 4 \quad ①$$

X_2 basic $\Rightarrow Z_2 - c_2 = 0$, or

$$Y P_2 - c_2 = (Y_1, Y_2) \begin{pmatrix} 1 \\ 4 \end{pmatrix} - 4 = 0, \text{ or}$$

$$Y_1 + 4Y_2 = 4. \text{ Given } ①, \text{ we get } Y_2 = 0.$$

$$B^{-1} b = X_B$$

$$\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} \Rightarrow \begin{array}{l} b_1 = 4 \\ b_2 = 6 \\ b_3 = 8 \end{array}$$

Dual objective value is

$$w = Y b = (0, 3, 2) \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} = 34$$

From the dual:

$$G_B B^{-1} = Y$$

$$(C_1, C_2, 0) \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} = (0, 3, 2)$$

$$\begin{array}{l} C_2 - C_1 = 3 \\ C_1 = 2 \end{array} \Rightarrow C_1 = 2, C_2 = 5$$

Primal objective value is

$$Z = G X_B = (2, 5, 0) \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix} = 34$$

$$\sum_{i=1}^m c_i (B^{-1} P_k)_i = (G_B B^{-1}) P_k \\ = Y P_k \\ = \sum_{i=1}^m Y_i a_{ik}$$

Minimize $w = Y b$

Subject to $YA = C$

Y unrestricted

Dual: Minimize $Y_b - Y_2 L + Y_3 U$

Subject to

$$Y_1 A - Y_2 + Y_3 \geq C$$

$$Y_1, Y_2, Y_3 \geq 0$$

Let $Y = Y_3 - Y_2 \Rightarrow Y$ unrestricted.
Hence $Y_1 A + (Y_3 - Y_2) \geq C$ can be written as $Y_1 A + Y \geq C$. Since Y is unrestricted, its value can always be selected such that $Y_1 A + Y \geq C$ is satisfied

5

6

7

8

Set 7.5a

For X_{B_0} :

$$\{z_j - c_j\}_{j=1,4,5}$$

$$= (4+14t, 1-t, 2+3t) \geq (0, 0, 0)$$

The inequalities are satisfied for

$$-2/7 \leq t \leq 1$$

$$(a) G(t) B_0^{-1} = (2, 5-6t, 0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \quad 2$$

$$= (1, 2-3t, 0)$$

$$X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$

$$\{z_j - c_j\}_{j=1,4,5} = (1, 2-3t, 0) \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3+3t, 0, 0)$$

$$= (4-12t, 1, 2-3t) \geq (0, 0, 0)$$

X_{B_0} remains optimal for $t \leq 1/3$

At $t = 1/3$, x_1 enters solution

$$\tilde{B}_0^{-1} P_1 = \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 3/2 \\ 2 \end{pmatrix}$$

x_6 leaves.

$$X_{B_1} = (x_2, x_3, x_1)^T$$

$$B_1 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 3 \\ 4 & 0 & 1 \end{pmatrix}$$

$$B_1^{-1} = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}$$

$$X_{B_1} = B_1^{-1} b = (25/4, 90/4, 5)^T$$

$$G(t) B_1^{-1} = (2, 5-6t, 3+3t) \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}$$

$$= (5-12t, 3t, -2+6t)$$

$$\{z_j - c_j\}_{j=4,5,6}$$

$$= (5-12t, 3t, -2+6t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (0, 0, 0)$$

$$= (5-12t, 3t, -2+6t)$$

X_{B_1} remains optimal for $1/3 \leq t \leq 5/12$

continued...

At $t = 5/12$, x_4 enters

$$\tilde{B}_1 P_4 = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 3/2 \\ -1 \end{pmatrix}$$

x_3 leaves

$$X_{B_2} = (x_2, x_4, x_1)^T$$

$$B_2 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 3 \\ 4 & 0 & 1 \end{pmatrix}$$

$$B_2^{-1} = \begin{pmatrix} 0 & -1/12 & 1/4 \\ 1 & -1/6 & -1/2 \\ 0 & 1/3 & 0 \end{pmatrix}$$

$$X_{B_2} = B_2^{-1} b = (5/2, 15, 20)^T$$

$$G(t) B_2^{-1} = (2, 0, 3+3t) B_2^{-1}$$

$$= (0, 5/6+t, 1/2)$$

$$\{z_j - c_j\}_{j=3,5,6}$$

$$= (0, 5/6+t, 1/2) \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (5-6t, 0, 0)$$

$$= (-10/3 + 8t, 5/6+t, 1/2)$$

X_{B_2} remains optimal for $5/12 \leq t < \infty$

$$(b) X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$

$$G(t) B_0^{-1} = (2+t, 5+2t, 0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

$$= (1+t/2, 2+3t/4, 0)$$

$$\{z_j - c_j\}_{j=1,4,5}$$

$$= (1+t/2, 2+3/4t, 0) \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3-2t, 0, 0)$$

$$= (4+19t/4, 1+t/2, 2+3t/4) \geq (0, 0, 0)$$

X_{B_0} is optimal for all $t \geq 0$

$$(c) X_{B_0} = (x_2, x_3, x_6)^T = (5, 30, 10)^T$$

$$G(t) B_0^{-1} = (2+2t, 5-t, 0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

$$= (1+t, 2-t, 0)$$

$$\{z_j - c_j\}_{j=1,4,5}$$

$$= (1+t, 2-t, 0) \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3+t, 0, 0)$$

$$= (4-3t, 1+t, 2-t) \geq (0, 0, 0) \text{ continued...}$$

Set 7.5a

x_{B_0} remains optimal for the range $t \leq 4/3$. At $t = 4/3$, x_1 enters solution. As in Part (a) above, x_6 leaves.

$$B_1^{-1} = \begin{pmatrix} 1/4 & -1/8 & 1/8 \\ 3/2 & -1/4 & -3/4 \\ -1 & 1/2 & 1/2 \end{pmatrix}, X_{B_1} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

$$X_{B_1} = B_1^{-1} b = (25/4, 90/4, 5)^T$$

$$\begin{aligned} C_B(t) B_1^{-1} &= (2+2t, 5-t, 3+t) B_1^{-1} \\ &= (5-2t, t/2, -2+3/2t) \end{aligned}$$

$$\{Z_j - c_j\}_{j=4,5,6}$$

$$\begin{aligned} &= (5-2t, t/2, -2+3/2t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - (0, 0, 0) \\ &= (5-2t, t/2, -2+3/2t) \geq (0, 0, 0) \end{aligned}$$

x_{B_1} remains optimal for

$$4/3 \leq t \leq 5/2$$

At $t = 5/2$, x_4 enters solution.

As in Part (a), we have x_3 leaving and

$$B_2^{-1} = \begin{pmatrix} 0 & -1/12 & 1/4 \\ 1 & -1/6 & -1/2 \\ 0 & 1/3 & 0 \end{pmatrix}, X_{B_2} = \begin{pmatrix} x_2 \\ x_4 \\ x_1 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

$$\begin{aligned} C_B(t) B_2^{-1} &= (2+2t, 0, 3+t) \begin{pmatrix} 0 & -1/12 & 1/4 \\ 1 & -1/6 & -1/2 \\ 0 & 1/3 & 0 \end{pmatrix} \\ &= (0, 5/6 + t/6, 1/2 + t/2) \end{aligned}$$

$$\{Z_j - c_j\}_{j=3,5,6}$$

$$\begin{aligned} &= (0, 5/6 + t/6, 1/2 + t/2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\quad - (5-t, 0, 0) \end{aligned}$$

$$\begin{aligned} &= (-10/3 + 4t/3, 5/6 + t/6, 1/2 + t/2) \\ &\geq (0, 0, 0) \end{aligned}$$

x_{B_2} remains optimal for $\frac{5}{2} \leq t < \infty$

$$\text{Minimize } Z = (4-t)x_1 + (1-3t)x_2 + (2-2t)x_3$$

Subject to

$$3x_1 + x_2 + 2x_3 = 3$$

$$4x_1 + 3x_2 + 2x_3 - x_4 = 6$$

$$x_1 + 2x_2 + 5x_3 + x_5 = 4$$

$$x_1, x_2, \dots, x_5 \geq 0$$

3

Continued...

$$X_{B_0} = (x_1, x_2, x_4)^T = (2/5, 9/5, 1)$$

$$B_0^{-1} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\begin{aligned} C_B(t) B_0^{-1} &= (4-t, 1-3t, 0) B_0^{-1} \\ &= \left(\frac{7+t}{5}, 0, -\frac{1+8t}{5} \right) \end{aligned}$$

$$\{Z_j - c_j\}_{j=3,5}$$

$$= \left(\frac{7+t}{5}, 0, -\frac{1+8t}{5} \right) \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 5 & 1 \end{pmatrix} - (2-2t, 0)$$

$$= \left(-\frac{1+28t}{5}, -\frac{1+8t}{5} \right) \leq (0, 0)$$

B_0 remains optimal for all $t \geq 0$.

The dual simplex method requires that the LP problem be put in the form:

$$\text{Minimize } Z = CX$$

Subject to

$$-AX \leq -b, \quad x \geq 0$$

Let B_i be the basis associated with critical value t_i in the parametric analysis. To obtain t_{i+1} , we consider

$$\{Z_j - c_j\} \text{ nonbasic } x_j$$

$$= C_B(t) B_i^{-1} (-P_j) - c_j(t) \leq 0$$

where P_j is the j^{th} column vector of A .

In the present problem, the first two constraints are of the type \geq . Hence, only the first two constraints are multiplied by -1 .

$$X_{B_0} = (x_3, x_2, x_6)^T = (3/2, 3/2, 0)$$

$$B_0^{-1} = \begin{pmatrix} -3/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, C_{B_0}(t) = (1+2+4t, 0)$$

Set 7.5a

$$C_B(t) B_0^{-1} = (-\frac{1}{2} + 2t, -\frac{1}{2} - 2t, 0)$$

$$\{z_j - c_j\}_{j=4,5}^2 = C_B(t) B_0^{-1} P_{\mathbb{R}} - G(t)$$

$$= (-\frac{1}{2} + 2t, -\frac{1}{2} - 2t, 0) \begin{pmatrix} -3 & 1 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} - (3+t, 0, 0)$$

$$= (-13t - 3, -\frac{1}{2} + 2t, 0) \leq (0, 0, 0)$$

Thus, $t_1 = 1/4 \Rightarrow x_{B_0}$ remains optimal for $0 \leq t \leq 1/4$.

At $t = 1/4$, x_4 enters and x_6 leaves.

$$x_{B_1} = (x_3, x_2, x_4)^T = (\frac{3}{2}, \frac{3}{2}, 0)^T$$

$$B_1^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \end{pmatrix}, C_{B_1}(t) = (1, 2+4t, 0)$$

$$C_{B_1}(t) B_1^{-1} = (0, -\frac{1}{2} - 2t, \frac{1}{2} - 2t)$$

$$\{z_j - c_j\}_{j=5,6}^2 = (0, -\frac{1}{2} - 2t, \frac{1}{2} - 2t) \begin{pmatrix} -3 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - (3+t, 0, 0)$$

$$= (-4 - 9t, -\frac{1}{2} - 2t, \frac{1}{2} - 2t) \leq (0, 0, 0)$$

conditions are satisfied for $t \geq 1/4$. Thus, x_{B_1} is optimal for all $t \geq 1/4$.

Summary:

$x_{B_0} = (x_3, x_2, x_6) = (\frac{3}{2}, \frac{3}{2}, 0)$ is optimal for $0 \leq t \leq 1/4$

$x_{B_1} = (x_3, x_2, x_4) = (\frac{3}{2}, \frac{3}{2}, 0)$ is optimal for $t \geq 1/4$

OR

$$\left. \begin{array}{l} x_1 = 0 \\ x_2 = \frac{3}{2} \\ x_3 = \frac{3}{2} \end{array} \right\} \text{for all } t \geq 0$$

$$x_{B_0} = (x_2, x_3, x_6)^T = (\frac{5}{2}, 3, 0)^T$$

5

$$C_B(t) = (2 - 2t^2, 5 - t, 0)$$

$$B_0^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

$$C_B(t) B_0^{-1} = (2 - 2t^2, 5 - t, 0) \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

$$= (1 - t^2, \frac{t^3}{2} - \frac{t}{2} + 2, 0)$$

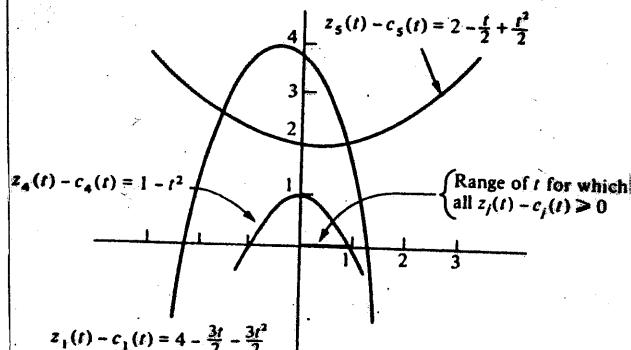
continued...

$$\{z_j - c_j\}_{j=1,4,5} =$$

$$= (1 - t^2, \frac{t^3}{2} - \frac{t}{2} + 2, 0) \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3 + 2t^2, 0, 0)$$

$$= (4 - \frac{3t}{2} - \frac{3t^2}{2}, 1 - t^2, 2 - \frac{t}{2} + \frac{t^2}{2}) \geq (0, 0, 0)$$

The graph below summarizes the optimality conditions.



x_{B_0} remains optimal for $0 \leq t \leq 1$.

Set 7.5b

$$(a) \quad X_{B_0} = (X_2, X_3, X_6)^T$$

$$= \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 40+2t \\ 60-3t \\ 30+6t \end{pmatrix}$$

$$= \begin{pmatrix} 5+t/4 \\ 30-3t/2 \\ 10-t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-20 \leq t \leq 10, \quad t_1 = 10$$

X_6 leaves at $t = 10$.

$$(\text{row of } B_0^{-1} \text{ associated with } X_6)(P_1 P_4 P_5) =$$

$$= (-2, 1, 1) \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (2, -2, 1)$$

$$\{z_j - c_j\}_{j=1,4,5}$$

$$= (2, 5, 0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3, 0, 0)$$

$$= (4, 1, 2)$$

$$\frac{x_1}{z_j - c_j} \quad x_1 \quad \frac{x_4}{1} \quad x_5 \quad \frac{x_5}{2}$$

$$\frac{x_6}{2} \quad -2 \quad 1$$

x_4 enters.

$$\text{new } B_1 = (P_2 P_3 P_4) = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

$$(b) \quad X_{B_0} = (X_2, X_3, X_6)^T$$

$$= \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 40-t \\ 60+2t \\ 30-5t \end{pmatrix}$$

$$= \begin{pmatrix} 5-t \\ 30+t \\ 10-t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-30 \leq t \leq 5 \quad t_1 = 5$$

X_2 leaves when $t = 5$.

$$(\text{row of } B_0^{-1} \text{ associated with } X_2)(P_1 P_4 P_5) =$$

$$= (1/2, -1/4, 0) \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= (-1/4, 1/2, -1/4)$$

$$\{z_j - c_j\}_{j=1,4,5}$$

$$= (2, 5, 0) \begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} - (3, 0, 0)$$

$$= (4, 1, 2)$$

$$\frac{x_1}{z_j - c_j} \quad x_1 \quad x_4 \quad \frac{x_5}{2}$$

$$\frac{x_6}{-1/4} \quad 1/2 \quad -1/4$$

x_5 enters

$$\text{new } B_1 = (P_5 P_3 P_6) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_{B_0} = (x_1, x_2, x_4)^T = (2/5, 9/5, 1)$$

x_4 = surplus in constraint 2

x_5 = slack in constraint 3

$$B_0^{-1} = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix}$$

$$X_{B_0}(t) = B_0^{-1} \begin{pmatrix} 3+3t \\ 6+2t \\ 4-t \end{pmatrix} = \begin{pmatrix} 2/5 + 7/5t \\ 9/5 - 6/5t \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Thus, } 0 \leq t \leq 3/2, \quad t_1 = 3/2$$

At $t = 3/2$, x_2 leaves the solution.

To determine the entering variable, we use the dual simplex computation.

$$(\text{row of } B_0^{-1} \text{ associated with } X_2)(P_3 P_5) =$$

$$= (-1/5, 0, 3/5) \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ 5 & 1 \end{pmatrix} = (13/5, 3/5)$$

Because $(13/5, 3/5) \geq 0$, the problem has no feasible solution for $t > 3/2$ (per dual simplex conditions).

Summary:

$$x_1 = 2/5, x_2 = 9/5, x_3 = 0, \text{ for } 0 \leq t \leq 3/2$$

No feasible solution for $t > 3/2$

Continued...

Set 7.5b

For the dual simplex, the feasibility condition is

$$\bar{B}_0^{-1} \bar{b}'(t) \geq 0$$

where $\bar{b}'(t)$ is modified such that the element $b_i(t)$ associated with \geq constraint is replaced with $-b_i(t)$.

$$x_{B_0} = (x_3, x_2, x_6)^T = (3/2, 3/2, 0)$$

$$\bar{B}_0^{-1} = \begin{pmatrix} -3/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\bar{b}'(t) = \begin{pmatrix} -3-2t \\ -6+t \\ 3-4t \end{pmatrix}$$

The top two elements appear with an opposite sign because the first two constraints are of the type ≥ 0 , hence reversing their signs in the dual simplex method.

$$\bar{B}_0^{-1} \bar{b}'(t) = \begin{pmatrix} -3/2 & -1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3-2t \\ -6+t \\ 3-4t \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 + 5/2t \\ 3/2 - 3/2t \\ -6t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus,

$$x_3 = 3/2 + 5/2t \geq 0 \text{ gives } t \geq -\frac{3}{5}$$

$$x_2 = 3/2 - 3/2t \geq 0 \text{ gives } t \leq 1$$

$$x_6 = -6t \text{ gives } t \leq 0$$

Thus, for $t \geq 0$, the solution

x_{B_0} is feasible for $t=0$ only.

Else, the problem has no feasible solution for $t > 0$

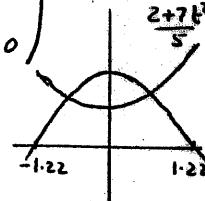
$$x_{B_0} = (x_1, x_2, x_3)^T$$

4

$$\begin{aligned} x_{B_t} &= \bar{B}_0^{-1} \bar{b}(t) = \begin{pmatrix} 2/5 & 0 & -1/5 \\ -1/5 & 0 & 3/5 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3+3t^2 \\ 6+2t^2 \\ 4-t^2 \end{pmatrix} \\ &= \begin{pmatrix} 2/5 + 7/5t^2 \\ 9/5 - 6/5t^2 \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$-1.22 \leq t \leq 1.22$$

x_2 leaves at $t = 1.22$



$$(\text{Row 2 of } \bar{B}_0^{-1})(P_4 P_5)$$

$$= (-1/5, 0, 3/5) \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} = (0, 3/5)$$

\Rightarrow no feasible solution exists
for $t > 1.22$

continued...

CHAPTER 8

Goal Programming

Set 8.1a

Additional constraint:

$$.075x_g \geq -1(550x_p + 35x_f + 55x_s + .075x_g)$$

|

The constraint simplifies to

$$55x_p + 3.5x_f + 5.5x_s - .0675x_g \leq 0$$

Thus,

$$55x_p + 3.5x_f + 5.5x_s - .0675x_g + S_5^- + S_5^+ = 0$$

G_5 : Minimize S_5^+

x_1 = Number of band concerts / yr

x_2 = number of art shows / yr

2

G_1 : Minimize S_1^-

G_2 : Minimize S_2^-

G_3 : Minimize S_3^-

Constraints:

$$1500x_1 + 3000x_2$$

$$200x_1 + S_1^- - S_1^+ \leq 1500$$

$$100x_1 + 400x_2 + S_2^- - S_2^+ = 1000$$

$$250x_2 + S_3^- - S_3^+ = 1200$$

$$250x_2 + S_3^- - S_3^+ = 800$$

All variables are ≥ 0

x_1 = in-state freshmen

3

x_2 = out-of-state freshmen

x_3 = international freshmen

$$(a) x_1 + x_2 + x_3 \geq 1200$$

$$(b) \frac{27x_1 + 26x_2 + 23x_3}{x_1 + x_2 + x_3} \geq 25$$

$$(c) \frac{x_3}{x_1 + x_2 + x_3} \geq .1$$

$$(d) \frac{1/2x_1 + 2/5x_2 + 1/9x_3}{1/2x_1 + 3/5x_2 + 8/9x_3} \geq .75$$

$$(e) \frac{x_2}{x_1 + x_2 + x_3} \geq .2$$

Goal program:

G_1 : Minimize S_1^-

G_2 : Minimize S_2^-

G_3 : Minimize S_3^-

G_4 : Minimize S_4^-

G_5 : Minimize S_5^-

Constraints:

$$x_1 + x_2 + x_3 + S_1^- - S_1^+ = 1200$$

$$2x_1 + x_2 - 2x_3 + S_2^- - S_2^+ = 0$$

$$-1x_1 - 1x_2 + 9x_3 + S_3^- - S_3^+ = 0$$

$$1/8x_1 - 1/20x_2 - 5/9x_3 + S_4^- - S_4^+ = 0$$

$$-2x_1 + .8x_2 - 2x_3 + S_5^- - S_5^+ = 0$$

All variables ≥ 0

x_1 = lb of limestone per day

x_2 = lb of corn per day

x_3 = lb of soybean meal per day

4

$$x_1 + x_2 + x_3 \geq 6000$$

$$.38x_1 + .001x_2 + .002x_3 \leq .012(x_1 + x_2 + x_3)$$

$$.38x_1 + .001x_2 + .002x_3 \geq .008(x_1 + x_2 + x_3)$$

$$.09x_2 + .5x_3 \geq .22(x_1 + x_2 + x_3)$$

$$.02x_2 + .08x_3 \leq .05(x_1 + x_2 + x_3)$$

Goals:

G_1 : Minimize S_1^-

G_2 : Minimize S_2^-

G_3 : Minimize S_3^-

G_4 : Minimize S_4^-

G_5 : Minimize S_5^-

Constraints:

$$x_1 + x_2 + x_3 + S_1^- - S_1^+ = 6000$$

$$.368x_1 - .01x_2 - .01x_3 + S_2^- - S_2^+ = 0$$

$$.372x_1 - .007x_2 - .006x_3 + S_3^- - S_3^+ = 0$$

$$-.22x_1 - .13x_2 + .28x_3 + S_4^- - S_4^+ = 0$$

$$-.05x_1 - .03x_2 + .03x_3 + S_5^- - S_5^+ = 0$$

All variables ≥ 0

Goal programming is not suitable for this problem because nutritional requirements must be met. However, goal programming can assist in deciding which nutritional requirements are "demanding" from the standpoint of optimization. The information may then be used to decide if alternative nutritional requirements can be specified in a manner that does not adversely impact cost minimization.

continued...

Set 8.1a

x_j = number of production runs in shift j , $j=1, 2, 3$

$$\frac{500x_1 + 600x_2 + 640x_3}{300x_1 + 280x_2 + 360x_3} = \frac{4}{2}$$

or

$$-100x_1 + 40x_2 - 80x_3 = 0$$

$$\text{Minimize } Z = S_1^- + S_1^+$$

Subject to

$$-100x_1 + 40x_2 - 80x_3 + S_1^- - S_1^+ = 0$$

$$4 \leq x_1 \leq 5, 10 \leq x_2 \leq 20, 3 \leq x_3 \leq 5$$

Constraints:

$$x_1 + S_1^- - S_1^+ = 80$$

$$x_2 + S_2^- - S_2^+ = 60$$

$$5x_1 + 3x_2 + S_3^- - S_3^+ = 480$$

$$6x_1 + 2x_2 + S_4^- - S_4^+ = 480$$

all variables ≥ 0

5

x_j = number of units of part j , $j=1, 2, 3, 4$

6

$$G_1: \text{minimize } S_1^+$$

$$G_2: \text{minimize } S_2^+$$

$$G_3: \text{minimize } S_3^+$$

$$G_4: \text{minimize } S_4^+$$

$$G_5: \text{minimize } S_5^-$$

$$G_6: \text{minimize } S_6^+$$

$$G_7: \text{minimize } S_7^-$$

$$G_8: \text{minimize } S_8^-$$

$$G_9: \text{minimize } S_9^+$$

Constraints:

$$5x_1 + 6x_2 + 4x_3 + 7x_4 + S_1^- - S_1^+ = 600$$

$$3x_1 + 2x_2 + 6x_3 + 4x_4 + S_2^- - S_2^+ = 600$$

$$2x_1 + 4x_2 - 2x_3 + 3x_4 + S_3^- - S_3^+ = 30$$

$$-2x_1 - 4x_2 + 2x_3 - 3x_4 + S_4^- - S_4^+ = 30$$

$$x_1 + S_5^- - S_5^+ = 10$$

$$x_2 + S_6^- - S_6^+ = 10$$

$$x_3 + S_7^- - S_7^+ = 10$$

$$x_4 + S_8^- - S_8^+ = 10$$

$$x_1 - x_2 + S_9^- - S_9^+ = 0$$

all variables ≥ 0

x_j = units of product j , $j=1, 2$

7

$$G_1: \text{minimize } S_1^+$$

$$G_2: \text{minimize } S_2^+$$

$$G_3: \text{minimize } S_3^+$$

$$G_4: \text{minimize } S_4^+$$

x_j = number of 1-day stays admitted on day j , $j=1, 2, 3, 4$

8

y_j = number of 2-day stays admitted on day j , $j=1, 2, 3, 4$

w_j = number of 3-day stays admitted on day j , $j=1, 2, 3, 4$

$$G_1: \text{minimize } S_1^+$$

$$G_2: \text{minimize } S_2^+$$

$$G_3: \text{minimize } S_3^+$$

$$G_4: \text{minimize } S_4^+$$

Subject to

$$x_1 + x_2 + x_3 + x_4 = 30$$

$$y_1 + y_2 + y_3 + y_4 = 25$$

$$w_1 + w_2 + w_3 + w_4 = 20$$

$$x_1 + y_1 + w_1 + S_1^- - S_1^+ = 20$$

$$x_2 + y_2 + y_3 + w_2 + w_3 + S_2^- - S_2^+ = 30$$

$$x_3 + y_3 + y_4 + w_3 + w_4 + S_3^- - S_3^+ = 30$$

$$x_4 + y_4 + w_4 + S_4^- - S_4^+ = 30$$

all variables ≥ 0

(x, y) = desired home location

9

$$G_1: \text{minimize } S_1^+$$

$$G_2: \text{minimize } S_2^+$$

$$G_3: \text{minimize } S_3^+$$

Subject to

$$\sqrt{(x-1)^2 + (y-1)^2} + S_1^- - S_1^+ = 25$$

$$\sqrt{(x-20)^2 + (y-15)^2} + S_2^- - S_2^+ = 10$$

$$\sqrt{(x-4)^2 + (y-7)^2} + S_3^- - S_3^+ = 1$$

all variables ≥ 0

Continued...

Set 8.1a

\hat{y}_i = estimated value of y_i
given the independent
values x_{ij} , $j = 1, 2, \dots, n$

$$= b_0 + b_1 x_{i1} + b_2 x_{i2} + \dots + b_n x_{in}$$

The parameters b_0, b_1, \dots, b_n are determined by minimizing

$$\sum_{i=1}^m |y_i - \hat{y}_i|$$

where m is the number of observed points.

The equivalent goal programming model is given as

$$\text{minimize } Z = \sum_{i=1}^m (s_i^- + s_i^+)$$

subject to

$$\hat{y}_i + s_i^- - s_i^+ = y_i, i = 1, 2, \dots, m$$

$$s_i^-, s_i^+ \geq 0, i = 1, 2, \dots, m$$

The values of the unknown parameters b_0, b_1, \dots, b_n are introduced in the optimization problem by using the substitution

$$\hat{y}_i = b_0 + b_1 x_{i1} + b_2 x_{i2} + \dots + b_n x_{in}$$

Thus, the variables of the model are s_i^- , s_i^+ , b_0, b_1, \dots, b_n .

Only s_i^- and s_i^+ are required to be nonnegative.

$$\text{minimize} \left[\max_{i=1,2,\dots,m} \{ |y_i - \hat{y}_i| \} \right]$$

II

Let

$$d = \max \{ |y_1 - \hat{y}_1|, |y_2 - \hat{y}_2|, \dots, |y_m - \hat{y}_m| \}$$

continued...

The problem reduces to the following goal program:

$$\text{minimize } Z = d$$

Subject to

$$\begin{cases} \hat{y}_i + d \geq y_i \\ \hat{y}_i - d \leq y_i \end{cases} \quad i = 1, 2, \dots, m$$

$$d \geq 0$$

Set 8.2a

Minimize $Z = S_1^- + S_2^- + S_3^- + S_4^+ + S_5^+$

S.t.

$$550X_p + 35X_f + 55X_s + .075X_g + S_1^- - S_1^+ = 16$$

$$55X_p - 31.5X_f + 5.5X_s + .0075X_g + S_2^- - S_2^+ = 0$$

$$110X_p + 7X_f - 44X_s + .015X_g + S_3^- - S_3^+ = 0$$

$$X_g + S_4^+ - S_4^- = 2$$

$$55X_p + 3.5X_f + 5.5X_s - .0675X_g + S_5^- - S_5^+ = 0$$

Solution: $X_p = .0201$, $X_f = .0457$, $X_s = -.0582$

$X_g = 2$ cents, $S_5^+ = 1.45$, all others = 0

Gasoline tax goal is $\$1.45$ million short of its $\$1.6$ million

Minimize $Z = S_1^- + 2S_2^- + S_3^-$

S.t. $1500X_1 + 3000X_2 + S_1^- + S_1^+ \leq 15000$

$$200X_1 + 400X_2 + S_2^- - S_2^+ = 1000$$

$$100X_1 + 400X_2 + S_2^- - S_2^+ = 1200$$

$$250X_2 + S_3^- - S_3^+ = 800$$

Solution: $Z = 175$, $X_1 = 5$, $X_2 = 2.5$.

$S_1^- = S_1^+ = 0$: Goal 1 satisfied

$S_2^+ = 300$: Goal 2 overachieved by 300 persons

$S_3^- = 175$: Goal 3 underachieved by 175 persons

(a) Minimize $Z = 2S_2^- + S_3^- + S_4^+ + S_5^+$

S.t. $X_1 + X_2 + X_3 \geq 1200$

$$2X_1 + X_2 - 2X_3 + S_2^- - S_2^+ = 0$$

$$125X_1 - .05X_2 - .556X_3 + S_3^- - S_3^+ = 0$$

$$-1X_1 - .1X_2 + .9X_3 + S_4^+ - S_4^- = 0$$

$$-2X_1 + .8X_2 - 2X_3 + S_5^- - S_5^+ = 0$$

Solution: $Z = 0$: all goals are satisfied

$X_1 \geq 80$, $X_2 \geq 240$, $X_3 = \approx 159$

$S_2^+ = 152.25$: ACT score overachieved by 1.27 pts/street

$S_3^+ = 38.59$: # of international students overachieved by 39 students

(b) Minimize $Z = 4S_1^- + 2S_2^- + S_3^- + S_5^-$

$$X_1 + X_2 + X_3 + S_1^- - S_1^+ = 1200$$

Solution in (a) remains the same

Minimize $Z = S_1^- + S_2^+ + S_3^- + S_4^+ + S_5^+$

S.t. $X_1 + X_2 + X_3 + S_1^- - S_1^+ = 6000$

$$.368X_1 - .011X_2 - .01X_3 + S_2^- - S_2^+ = 0$$

$$.372X_1 - .007X_2 - .006X_3 + S_3^- - S_3^+ = 0$$

$$-.22X_1 - .13X_2 + .28X_3 + S_4^+ - S_4^- = 0$$

$$-.05X_1 - .03X_2 + .08X_3 + S_5^- - S_5^+ = 0$$

$Z = 0$: All goals are satisfied

$X_1 = 166.08$ lb, $X_2 = 2778.56$ lb, $X_3 = 3055.36$ lb

$S_2^+ = 24$: G3 overachieved by $\frac{24}{6000} = .004$

$S_4^+ = 457.75$: G4 overachieved by $\frac{457.75}{6000} = .0763$

calcium % = 1.2

protein % = $22 + .763 = 29.63$, fiber % = 5

Minimize $Z = S_1^- + S_1^+$

S.t. $-100X_1 + 40X_2 - 80X_3 + S_1^- - S_1^+ = 0$

$$4 \leq X_1 \leq 5, 10 \leq X_2 \leq 20, 3 \leq X_3 \leq 5$$

Solution: $Z = 0$: all goal are satisfied

$X_1 = 4$, $X_2 = 16$, $X_3 = 3$

$S_1^- = S_1^+ = 0$: Production is balanced.

Min $Z = S_3^- + S_4^- + 2S_5^- + 2S_6^- + 2S_7^- + 2S_8^-$

S.t. $5X_1 + 6X_2 + 4X_3 + 7X_4 \leq 600$

$$3X_1 + 2X_2 + 6X_3 + 4X_4 \leq 600$$

$$2X_1 + 4X_2 - 2X_3 + 3X_4 + S_3^- - S_3^+ = 30$$

$$-2X_1 - 4X_2 + 2X_3 - 3X_4 + S_4^- - S_4^+ = 30$$

$$X_1 + S_5^- - S_5^+ = 10$$

$$X_2 + S_6^- - S_6^+ = 10$$

$$X_3 + S_7^- - S_7^+ = 10$$

$$X_4 + S_8^- - S_8^+ = 10$$

$$X_1 - X_2 + S_9^- - S_9^+ = 0$$

$$+ S_9^- - S_9^+ = 0$$

$Z = 0$: all goals are satisfied

$X_1 = 10$, $X_2 = 10$, $X_3 = 30$, $X_4 = 10$

Assign a relatively large weight to the quota constraint.

Min $Z = 100(S_1^- + S_2^-) + (S_3^+ + S_4^+)$

S.t. $X_1 + S_1^- - S_1^+ = 80$

$$X_2 + S_2^- - S_2^+ = 60$$

$$5X_1 + 3X_2 + S_3^- - S_3^+ = 480$$

$$6X_1 + 2X_2 + S_4^- - S_4^+ = 480$$

Solution: $X_1 = 80$, $X_2 = 60$, $S_3^+ = 100$, $S_4^+ = 120$ min

Production quota can be met with 100 min of overtime on machine 1 and 120 min on machine 2

Min $Z = S_1^- + S_2^+ + S_3^+ + S_4^+$

S.t. $X_1 + X_2 + X_3 + X_4 = 30$

$$Y_1 + Y_2 + Y_3 + Y_4 = 25$$

$$W_1 + W_2 + W_3 + W_4 = 20$$

$$X_1 + Y_1 + W_1 + S_1^- - S_1^+ = 30$$

$$X_2 + Y_2 + Y_3 + W_2 + S_2^- - S_2^+ = 30$$

$$X_3 + Y_3 + Y_4 + W_3 + W_4 + S_3^- - S_3^+ = 30$$

$$X_4 + Y_4 + W_4 + W_3 + W_4 + S_4^- - S_4^+ = 30$$

Solution: $Z = 0$: All goals are met

$X_1 = 5$, $X_2 = 15$, $X_3 = 10$, $X_4 = 0$

$\sum 1\text{-day stays} = 30$

$Y_1 = 10$, $Y_2 = 0$, $Y_3 = 15$, $Y_4 = 0$

$\sum 2\text{-day stays} = 25$

$W_1 = 5$, $W_2 = 0$, $W_3 = 0$, $W_4 = 15$

$\sum 3\text{-day stays} = 20$

The solution shows that:

continued...

Set 8.2a

Nbr. beds used on day 1

$$= x_1 + y_1 + w_1 = 20 \quad (= \text{availability } 20)$$

Nbr. beds used on day 2 = $x_2 + y_2 + w_2 = 15 < 30$

Nbr. beds used on day 3 = $x_3 + y_3 + w_3 = 25 < 30$

Nbr. beds used on day 4 = $x_4 + y_4 + w_4 = 15 < 30$

Conclusion: All 1-, 2-, and 3-day stays can be met without overbooking

$$J = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$

9

$$\text{Minimize } Z = \sum_{i=1}^5 (S_i^- + S_i^+)$$

Subject to

$$b_0 + 30b_1 + 4b_2 + 5b_3 + S_1^- - S_1^+ = 40$$

$$b_0 + 39b_1 + 5b_2 + 10b_3 + S_2^- - S_2^+ = 48$$

$$b_0 + 44b_1 + 2b_2 + 14b_3 + S_3^- - S_3^+ = 38$$

$$b_0 + 48b_1 + 18b_2 + S_4^- - S_4^+ = 36$$

$$b_0 + 37b_1 + 3b_2 + 9b_3 + S_5^- - S_5^+ = 41$$

$$S_i^- \geq 0, i=1, 2, \dots, 5$$

b_0, b_1, b_2, b_3 unrestricted

TOA Solution:

$$b_0 = .8571$$

$$b_1 = 1.0714$$

$$b_2 = 2.881$$

$$b_3 = -.9048$$

$$S_3^+ = 3.0952$$

$$\text{all other } S_i^- \text{ and } S_i^+ = 0$$

Thus, the least-square estimator is given as

$$J = .8571 + 1.0714x_1 + 2.881x_2 - .9048x_3$$

$$J = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$

10

minimize $Z = d$

Subject to

$$b_0 + 30b_1 + 4b_2 + 5b_3 + d \geq 40$$

$$b_0 + 39b_1 + 5b_2 + 10b_3 + d \geq 48$$

$$b_0 + 44b_1 + 2b_2 + 14b_3 + d \geq 38$$

$$b_0 + 48b_1 + 18b_2 + d \geq 36$$

$$b_0 + 37b_1 + 3b_2 + 9b_3 + d \geq 41$$

$$b_0 + 30b_1 + 4b_2 + 5b_3 - d \leq 40$$

$$b_0 + 39b_1 + 5b_2 + 10b_3 - d \leq 48$$

$$b_0 + 44b_1 + 2b_2 + 14b_3 - d \leq 38$$

$$b_0 + 48b_1 + 18b_2 - d \leq 36$$

$$b_0 + 37b_1 + 3b_2 + 9b_3 - d \leq 41$$

$$b_0, b_1, b_2, b_3 \text{ unrestricted}$$

$$d \geq 0$$

TOA Solution:

$$b_0 = 27.5536$$

$$b_1 = -.0893$$

$$b_2 = 3.2679$$

$$b_3 = .6429$$

$$d = 1.1607$$

Chebyshov estimator:

$$J = 27.5536 - .0893x_1 + 3.2679x_2 + 1.1607x_3$$

Set 8.2b

Minimize $G_1 = \bar{S}_1$
subject to

$$\begin{aligned} 4x_1 + 8x_2 + \bar{S}_1 - S_1^+ &= 45 \\ 8x_1 + 24x_2 + \bar{S}_2 - S_2^+ &= 110 \\ x_1 + 2x_2 &\leq 10 \\ x_1 &\leq 6 \\ x_1, x_2, \bar{S}_1, S_1^+, \bar{S}_2, S_2^+ &\geq 0 \end{aligned}$$

TORA Solution:

$$x_1 = 2.5, x_2 = 3.75, \bar{S}_1 = 5 \\ S_1^+ = \bar{S}_2 = S_2^+ = 0$$

Both goals are automatically satisfied.

$$G_1 > G_2 > G_3 > G_4 > G_5$$

2

G_1 -Problem Solution:

$$x_p = .01745, x_f = .0457, x_s = .0582 \\ x_g = 21.33$$

$$\begin{aligned} \bar{S}_1 &= S_1^+ = \bar{S}_2 = S_2^+ = \bar{S}_3 = S_3^+ = \bar{S}_4 \\ &= S_4^+ = 0 \\ S_4^+ &= 19.33 \end{aligned}$$

Goals G_1, G_2, G_3 , and G are satisfied.

G_4 -Problem:

$$\text{Minimize } Z = S_4^+$$

Subject to G_1 -constraint & $\bar{S}_1 = S_1^+ = \bar{S}_3 = S_3^+ = 0$

$$\text{Solution: } x_1 = .0201, x_2 = .0457, x_3 = .0582, x_4 = 2 \\ S_4^+ = 1.45. G_5 \text{ is not satisfied}$$

G_5 -Problem: Minimize $Z = S_5^+$ subject to same constraints in G_4 & $S_4^+ = 0$

Solution:

Same as in G_4 , which means that G_5 cannot be satisfied.

(a) $G_1 > G_2 > G_3$

G_1 -Problem:

$$\text{Minimize } G_1 = \bar{S}_1$$

$$\text{TORA Solution: } \bar{S}_1 = 0, \bar{S}_2 = 0, \bar{S}_3 = 362.5 \\ x_1 = 5, x_2 = 1.75$$

G_2 is satisfied

G_3 -Problem:

$$\text{Minimize } G_3 = \bar{S}_3$$

$$\bar{S}_1 = 0, \bar{S}_2 = 0$$

$$\text{TORA Solution: } \bar{S}_3 = 175 \\ x_1 = 5, x_2 = 2.5$$

G_3 remains unsatisfied.

(b) $G_3 > G_2 > G_1$

G_3 -Problem: minimize $G_3 = \bar{S}_3$

$$\text{TORA Solution: } \bar{S}_3 = 280, \bar{S}_2 = 0, \bar{S}_1 = 0 \\ x_1 = 3.6, x_2 = 3.2$$

G_2 is satisfied.

G_1 -Problem: minimize $G_1 = \bar{S}_1$
 $\bar{S}_2 = 0, \bar{S}_3 = 0$

$$\text{TORA Solution: } x_1 = 3.6, x_2 = 3.2, \bar{S}_1 = 280$$

G_1 is not satisfied.

Problem G_1 : minimize $G_1 = \bar{S}_1$

$$\text{TORA Solution: } x_1 = 0, x_2 = 1080, x_3 = 120 \\ S_4^+ = 309.33, \bar{S}_2 = \bar{S}_3 = 0$$

G_2 (minimize S_2) is satisfied.

G_3 -Problem: minimize $G_3 = S_4^+$
 $\bar{S}_2 = 0, \bar{S}_3 = 0$

$$\text{TORA Solution: } x_1 = 1080, x_2 = 0, x_3 = 120$$

$$S_4^+ = 93.33, \bar{S}_5^+ = 240$$

$$\text{G4-Problem: Minimize } G_4 = \bar{S}_5^+ \\ \bar{S}_2 = 0, \bar{S}_3 = 0, S_4^+ = 93.33$$

$$\text{TORA Solution: } x_1 = 1080, x_2 = 0$$

$$x_3 = 120$$

$$S_5^+ = 240$$

G_3 and G_4 are unsatisfied

CHAPTER 9

Integer Linear Programming

Set 9.1a

$$\text{Max } Z = 20x_1 + 40x_2 + 20x_3 + 15x_4 + 30x_5$$

subject to

$$\begin{pmatrix} 5 & 4 & 3 & 7 & 8 \\ 1 & 7 & 9 & 4 & 6 \\ 8 & 10 & 2 & 1 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{pmatrix} \leq \begin{pmatrix} 25 \\ 25 \\ 25 \end{pmatrix}$$

$$(a) \quad x_1 \leq x_5, \quad x_3 \leq x_5, \quad \text{all } x_j \text{ binary}$$

$$\text{Solution: } x_2 = x_3 = x_5 = 1, \quad Z = 90$$

$$(b) \quad x_2 + x_3 \leq 1, \quad \text{all } x_j \text{ binary}$$

$$\text{Solution: } x_2 = x_4 = x_5 = 1, \quad Z = 85$$

Note: When you use TORA, add the upper bound $x_j \leq 1$ for all binary variables.

x_i = number of units of item i ,
 $i = 1, 2, \dots, 5$

2

$$\text{Maximize } Z = 4x_1 + 7x_2 + 6x_3 + 5x_4 + 4x_5$$

Subject to

$$\begin{pmatrix} 5 & 8 & 3 & 2 & 7 \\ 1 & 8 & 6 & 5 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{pmatrix} \leq \begin{pmatrix} 112 \\ 109 \end{pmatrix}$$

$$x_j \geq 0 \text{ and integer, } j = 1, 2, \dots, 5$$

Solution: $x_1 = 14, x_4 = 19$, all others are zero, $Z = 15$

x_{ij} = number of bottles of type i assigned to individual j

3

where $i = \begin{cases} 1, & \text{full} \\ 2, & \text{half-full} \\ 3, & \text{empty} \end{cases}$

$$\text{Total available wine} = 7 + 3\frac{1}{2} = 10\frac{1}{2}$$

$$\text{Share per individual} = \frac{10\frac{1}{2}}{3} = 3\frac{1}{2} \text{ bottles}$$

Constraints:

$$x_{11} + x_{12} + x_{13} = 7 \quad \left. \begin{array}{l} \text{bottle} \\ \text{type} \end{array} \right\}$$

$$x_{21} + x_{22} + x_{23} = 7 \quad \left. \begin{array}{l} \text{bottle} \\ \text{type} \end{array} \right\}$$

$$x_{31} + x_{32} + x_{33} = 7 \quad \left. \begin{array}{l} \text{bottle} \\ \text{type} \end{array} \right\}$$

$$x_{11} + \frac{x_{21}}{2} = 3.5 \quad \left. \begin{array}{l} \text{amount of} \\ \text{wine per} \end{array} \right\}$$

$$x_{12} + \frac{x_{22}}{2} = 3.5 \quad \left. \begin{array}{l} \text{amount of} \\ \text{wine per} \end{array} \right\}$$

$$x_{13} + \frac{x_{23}}{2} = 3.5 \quad \left. \begin{array}{l} \text{amount of} \\ \text{wine per} \end{array} \right\}$$

continued...

$$\begin{aligned} x_{11} + x_{21} + x_{31} &= 7 && \left. \begin{array}{l} \text{bottles} \\ \text{per} \end{array} \right\} \\ x_{12} + x_{22} + x_{32} &= 7 && \left. \begin{array}{l} \text{individual} \end{array} \right\} \\ x_{13} + x_{23} + x_{33} &= 7 && \left. \begin{array}{l} \text{(redundant)} \end{array} \right\} \end{aligned}$$

$$x_{ij} \geq 0 \text{ and integer}$$

Use dummy objective function

$$\text{maximize } Z = 0x_{11} + 0x_{12} + \dots + 0x_{33}$$

Feasible solution: (Alternative solutions exist)

individual

	1	2	3	Sum
F	3	3	1	7
H	1	1	5	7
E	3	3	1	7
Sum	7	7	7	
Qty.	3.5	3.5	3.5	

x_i = number of camels to Tarek

4

x_2 = number of camels to Sharif

x_3 = number of camels to Maisa

x_4 = number of camels to charity (=1)

r = dummy integer variable ≥ 0 .

y = total number of camels in the will

Constraints:

$$y = x_1 + x_2 + x_3 + 1$$

$$y = 2r + 1 \Rightarrow y \text{ is odd}$$

$$x_1 \geq \frac{1}{2}y, \quad x_2 \geq \frac{1}{3}y, \quad x_3 \geq \frac{1}{9}y$$

Using a dummy objective function, the problem reduces to

	y	x_1	x_2	x_3	r	
min	0	0	0	0	0	
	1	-1	-1	-1	0	= 1
	1	0	0	0	-2	= 1
	1	-2	0	0	0	≤ 0
	1	0	-3	0	0	≤ 0
	1	0	0	-9	0	≤ 0

continued...

Set 9.1a

6

Solution: $y = 27$ camels. Tarek gets 14, Sharif gets 9, and Masa gets 3.

Note: If you enter the last two constraints in the original fractional form, make sure that $1/3$ and $1/9$ are accurate to six decimal points (.333333 and .111111). Else, TORA fails to find solution.

x_{ij} = number of apples belonging to child i and sold at price j .

$$i = \begin{cases} 1 \rightarrow \text{Jim} \\ 2 \rightarrow \text{Bill} \\ 3 \rightarrow \text{John} \end{cases} \quad j = \begin{cases} 1 \rightarrow \$1/\text{apple} \\ 2 \rightarrow \$3/\text{apple} \end{cases}$$

Allocation of apples to children:

$$x_{11} + x_{12} = 50 \quad (\text{Jim})$$

$$x_{21} + x_{22} = 30 \quad (\text{Bill})$$

$$x_{31} + x_{32} = 10 \quad (\text{John})$$

Allocate same money to each child:

$$\frac{x_{11}}{7} + 3x_{12} = \frac{x_{21}}{7} + 3x_{22}$$

$$\frac{x_{11}}{7} + 3x_{12} = \frac{x_{31}}{7} + 3x_{32}$$

Objective function:

$$\text{Maximize } Z = \frac{x_{11}}{7} + 3x_{12}$$

ILP:

$$\text{Maximize } Z = x_{11} + 21x_{12}$$

Subject to

$$x_{11} + x_{12} = 50$$

$$x_{21} + x_{22} = 30$$

$$x_{31} + x_{32} = 10$$

$$x_{11} + 21x_{12} - x_{21} - 21x_{22} = 0$$

$$x_{11} + 21x_{12} - x_{31} - 21x_{32} = 0$$

$x_{ij} \geq 0$ and integer

Solution:

$\$1/\text{apple}$ $\$3/\text{apple}$

			\$
Jim	42	8	30
Bill	21	9	30
John	0	10	30

Each child returns home with \$30.

$y = \text{original sum of money}$

$x_1 = \text{amount taken the first night}$

$x_2 = \text{amount taken the second night}$

$x_3 = \text{amount taken the third night}$

$x_4 = \text{amount given by first officer to each mariner}$

Minimize $Z = y$

subject to

$$x_1 = \frac{y-1}{3} + 1$$

$$x_2 = \frac{y-x_1-1}{3} + 1$$

$$x_3 = \frac{y-x_1-x_2-1}{3} + 1$$

$$x_4 = \frac{y-x_1-x_2-x_3-1}{3}$$

The ILP is given as

minimize $Z = y$

subject to

$$3x_1 - y = 2$$

$$x_1 + 3x_2 - y = 2$$

$$x_1 + x_2 + 3x_3 - y = 2$$

$$-x_1 - x_2 - x_3 - 3x_4 + y = 1$$

$x_1, x_2, x_3, x_4, y \geq 0$ and integer

Solution: $y = 79$ units

Resolve the problem after adding the constraint $y \geq 80$.

Solution: $y = 160$ units

Resolve the problem after adding the constraint $y \geq 161$

Solution: $y = 241$ units

General Solution: $y = 79 + 81n$,
 $n = 0, 1, 2, \dots$

Set 9.1a

Given $A=1$ and $Z=26$, let
 $x_j = 1$ if word j is selected and 0 if it is not selected.

$x_j = 1$ if word j is selected and 0 if it is not selected.

j	Word	L_{1j}	L_{2j}	L_{3j}	Score
1	AFT	1	6	20	27
2	FAR	6	1	18	25
3	TVA	20	22	1	43
4	ADV	1	4	22	27
5	JOE	10	15	5	30
6	FIN	6	9	14	29
7	OSF	15	19	6	40
8	KEN	11	5	14	30

$\sum_{j=1}^8 L_{1j}x_j < \sum_{j=1}^8 L_{2j}x_j$ implies that

$\sum_{j=1}^8 (L_{2j} - L_{1j}) > 0$, or $\sum_{j=1}^8 (L_{2j} - L_{1j}) \geq 1$

which translates to

$$5x_1 - 5x_2 + 2x_3 + 3x_4 + 5x_5 + 3x_6 + 4x_7 - 6x_8 \geq 1$$

Similarly, Constraint $\sum_{j=1}^8 L_{2j} < \sum_{j=1}^8 L_{3j}x_j$ translates to

$$14x_1 + 17x_2 - 21x_3 + 18x_4 - 14x_5 + 5x_6 - 13x_7 + 9x_8 \geq 1$$

ILP:

$$\text{Maximize } Z = 27x_1 + 25x_2 + 43x_3 + 27x_4 + 30x_5 + 29x_6 + 40x_7 + 30x_8$$

Subject to

$$5x_1 - 5x_2 + 2x_3 + 3x_4 + 5x_5 + 3x_6 + 4x_7 - 6x_8 \geq 1$$

$$14x_1 + 17x_2 - 21x_3 + 18x_4 - 10x_5 + 5x_6 - 13x_7 + 9x_8 \geq 1$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 5$$

$$x_j = (0, 1), j = 1, 2, \dots, 8$$

Solution: $x_1 = x_3 = x_4 = x_7 = x_8 = 1$

	L_{1j}	L_{2j}	L_{3j}	Score
AFT	1	6	20	27
TVA	20	22	1	43
ADV	1	4	22	27
OSF	15	19	6	40
KEN	11	5	14	30

$$\sum 48 \quad 56 \quad 63 \quad 167$$

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Because $\sum_{j=1}^8 L_{1j}x_j < \sum_{j=1}^8 L_{2j}x_j < \sum_{j=1}^8 L_{3j}x_j$, 8

the new objective function

$$\text{Maximize } Z = \sum_{j=1}^8 L_{2j}x_j$$

produces the desired result, including that of Problem 7.

$c_{ik} = \text{Nbr. of times letter } i \text{ is repeated in group } k, k = 1, 2$

$x_{ij} = \begin{cases} 1, & \text{if letter } i \text{ is assigned value } j \\ 0, & \text{otherwise} \end{cases}$

$$\text{Minimize } Z = \left| \sum_{i=1}^9 (c_{i1} - c_{i2}) \sum_{j=1}^9 j x_{ij} \right|$$

$$\text{s.t. } \sum_{j=1}^9 x_{ij} = 1, \text{ all } i$$

$$\sum_{i=1}^9 x_{ij} = 1, \text{ all } j$$

The objective function is equivalent to

$$\text{Minimize } Z = y$$

$$y \leq \sum_{j=1}^9 (c_{i1} - c_{i2}) \sum_{i=1}^9 j x_{ij} \leq Y$$

Solution: $Z = 0$

$A=8, E=3, F=7, H=2, O=1, P=4, R=6, S=9, T=5$

$x_{ij} = \begin{cases} 1, & \text{if song } i \text{ is on side } j \\ 0, & \text{if song } i \text{ is not on side } j \end{cases}$

$$\text{Minimize } Z = |S_1 - S_2|$$

Subject to

$$8x_{11} + 3x_{21} + 5x_{31} + 5x_{41} + 9x_{51} + 6x_{61} + 7x_{71} + 12x_{81} + S_1 = 30$$

$$8x_{12} + 3x_{22} + 5x_{32} + 5x_{42} + 9x_{52} + 6x_{62} + 7x_{72} + 12x_{82} + S_2 = 30$$

$$x_{i1} + x_{i2} = 1, \quad i = 1, 2, \dots, 8$$

$$\text{Let } y = |S_1 - S_2| \Rightarrow \begin{cases} S_1 - S_2 \leq y \\ S_1 - S_2 \geq -y \end{cases}$$

continued...

Set 9.1a

ILP:

$$\text{Minimize } Z = y$$

Subject to

$$8x_{11} + 3x_{21} + 5x_{31} + 5x_{41} + \\ + 9x_{51} + 6x_{61} + 7x_{71} + 12x_{81} + s_1 = 30$$

$$8x_{12} + 3x_{22} + 5x_{32} + 5x_{42} + 9x_{52} + \\ + 6x_{62} + 7x_{72} + 12x_{82} + s_2 = 30$$

$$x_{ij} + x_{i'j} = 1, \quad i=1,2,\dots,8$$

$$s_1 - s_2 - y \leq 0$$

$$s_1 - s_2 + y \geq 0$$

$$x_{ij} = (0,1), \quad i=1,2,\dots,8; \quad j=1,2$$

$$s_1, s_2, y \geq 0$$

Solution:

Side 1 : 5-6-8 (27 minutes)

Side 2 : 1-2-3-4-7 (28 minutes)

Problem has alternative optima.

Simpler Model:

$$\text{Minimize } Z = y$$

Subject to

$$8x_{11} + 3x_{21} + 5x_{31} + 5x_{41} + \\ + 9x_{51} + 6x_{61} + 7x_{71} + 12x_{81} \leq y$$

$$8x_{12} + 3x_{22} + 5x_{32} + 5x_{42} + \\ + 9x_{52} + 6x_{62} + 7x_{72} + 12x_{82} \leq y$$

$$x_{ij} + x_{i'j} = 1, \quad i=1,2,\dots,8$$

$$y \geq 0$$

Solution:

Side 1 : 3-4-6-8, time = 28 minutes

Side 2 : 1-2-5-7, time = 27 minutes

Add the constraints

$$x_{31} + x_{41} = 1$$

$$x_{32} + x_{42} = 1$$

use the simpler model in
Problem 10; that is,

continued...

$$\text{Minimize } Z = y$$

Subject to

$$8x_{11} + 3x_{21} + 5x_{31} + 5x_{41} + \\ + 9x_{51} + 6x_{61} + 7x_{71} + 12x_{81} \leq y$$

$$8x_{12} + 3x_{22} + 5x_{32} + 5x_{42} + \\ + 9x_{52} + 6x_{62} + 7x_{72} + 12x_{82} \leq y$$

$$x_{ij} + x_{i'j} = 1, \quad i=1,2,\dots,8$$

$$x_{31} + x_{41} = 1$$

$$x_{32} + x_{42} = 1$$

$$x_{ij} = (0,1) \text{ for all } i \text{ and } j$$

$$y \geq 0$$

Solution:

Side 1: 1-2-4-8, $\sum = 28$ min

Side 2: 3-5-6-7, $\sum = 27$ min

The tape must be at least
28 minutes.

$$x_{ij} = \begin{cases} 1, & \text{student } i \text{ selects course } j, \\ 0, & \text{otherwise} \end{cases}$$

P_{ij} = associated preference score

$$\text{Maximize } Z = \sum_{i=1}^{10} \sum_{j=1}^6 P_{ij} x_{ij}$$

$$\text{s.t. } \sum_{j=1}^6 x_{ij} = 2, \quad i=1,2,\dots,10$$

$$\sum_{i=1}^{10} x_{ij} \leq C_j, \quad j=1,2,\dots,6$$

Solution: Total score = 1775

Course	Students
1	2, 4, 9
2	2, 8
3	5, 6, 7, 9
4	4, 5, 7, 10
5	1, 3, 8, 10
6	1, 3

12

Set 9.1b

3

Route Delivery distance

1, 2, 3, 4	$10 + 32 + 14 + 15 + 9 = 80$
4, 3, 5	$9 + 15 + 18 + 8 = 50$
1, 2, 5	$10 + 32 + 20 + 8 = 70$
2, 3, 5	$12 + 14 + 18 + 8 = 52$
1, 4, 2	$10 + 17 + 21 + 12 = 60$
1, 3, 5	$10 + 8 + 18 + 8 = 44$

All routes start and end at ABC.

$$x_j = \begin{cases} 1, & \text{if route } j \text{ is selected} \\ 0, & \text{if otherwise} \end{cases}$$

$$\min Z = \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 80 & 50 & 70 & 52 & 60 & 44 \end{matrix}$$

Subject to

Customer ①	1	0	1	0	1	1	≥ 1
②	1	0	1	1	1	0	≥ 1
③	1	1	0	1	0	1	≥ 1
④	1	1	0	0	1	0	≥ 1
⑤	0	1	1	1	0	1	≥ 1

$$x_j = (0, 1), j = 1, 2, \dots, 6$$

Solution: $x_5 = x_6 = 1$, all others = 0
 $Z = 104$

Select routes (1, 4, 2) and (1, 3, 5). Customer 1 should be visited once using either route

Suppose that the 10 individuals are referred to by the code $k = a, b, \dots, j$. Let

$$x_k = \begin{cases} 1, & \text{individual } k \text{ included} \\ 0, & \text{individual } k \text{ not included.} \end{cases}$$

$$k = a, b, c, \dots, j.$$

$$\begin{matrix} x_a & x_b & x_c & x_d & x_e & x_f & x_g & x_h & x_i & x_j \end{matrix}$$

$\min Z = 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1$

Subject to

$$\begin{array}{lll} 1 & 1 & 1 & 1 & 1 & \geq 1 & (\text{Females}) \\ & & & & 1 & 1 & 1 & 1 & 1 & \geq 1 & (\text{males}) \\ 1 & 1 & 1 & & & 1 & \geq 1 & (\text{Students}) \\ & & 1 & & & & \geq 1 & (\text{admin}) \\ 1 & & 1 & 1 & 1 & \geq 1 & (\text{faculty}) \end{array}$$

Solution: Use individuals a, d, and f.

Problem has alternative optima

Station Towns it can serve

1	1, 3, 5
2	2, 4, 6
3	1, 3
4	2, 4
5	1, 5, 6
6	2, 5, 6

$$x_j = \begin{cases} 1, & \text{if station } j \text{ is selected} \\ 0, & \text{if station } j \text{ is not selected} \end{cases}$$

Assume that station j can be located in any of the towns it serves.

$$\text{Minimize } Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

Subject to

$$\begin{array}{lll} \text{Station 1: } x_1 & + x_3 & + x_5 \geq 1 \\ 2: & x_2 & + x_4 + x_6 \geq 1 \\ 3: & x_1 & + x_3 \geq 1 \\ 4: & x_2 & + x_4 \geq 1 \\ 5: & x_1 & + x_5 + x_6 \geq 1 \\ 6: & x_2 & + x_5 + x_6 \geq 1 \end{array}$$

$$x_j = (0, 1), j = 1, 2, \dots, 6$$

Constraints 3 and 4 are redundant

Solution: Select stations 1 and 2.

2

$x_{ij} = 1$ if guard i is posted between rooms j and i; zero otherwise.
 One constraint per room.

①	x_{15}	⑤
x_{12}		x_{56}
②	x_{24}	⑥
		x_{46}
③	x_{34}	⑦
		x_{47}

4

$$\text{Minimize } Z = x_{12} + x_{15} + x_{24} + x_{34} + x_{46} + x_{47} + x_{56} + x_{67}$$

Subject to

$$\begin{array}{lll} \text{Room 1: } x_{12} + x_{15} & \geq 1 \\ 2: & x_{12} + x_{24} & \geq 1 \\ 3: & x_{34} & \geq 1 & x_{ij} = (0, 1) \\ 4: & x_{24} + x_{34} + x_{46} + x_{47} & \geq 1 \\ 5: & x_{15} + x_{56} & \geq 1 \\ 6: & x_{46} + x_{56} + x_{67} & \geq 1 \\ 7: & x_{47} + x_{67} & \geq 1 \end{array}$$

$$\text{Solution: } x_{12} = x_{34} = x_{56} = x_{67} = 1$$

Alternative optima exist.

Set 9.1b

$$x_j = \begin{cases} 1, & \text{if town } j \text{ is selected} \\ 0, & \text{otherwise} \end{cases}$$

I_i = set of cities offering movie i

c_j = cost/show in city j

d_j = miles to city j

n_j = number of movies in city j

$$C_j = c_j n_j + d_j \times .75$$

$$\text{Minimize } Z = \sum_{j=1}^7 C_j x_j$$

s.t.

$$\sum_{j \in I_i} x_j \geq 1, \quad i = 1, 2, \dots, 7$$

Note: The formulation assumes that Bill will see all the movies in a visited town regardless of repetitions.

Solution: Cost = \$169.35

Visited town	movies
A	1, 6, 8
C	1, 8, 9
D	2, 4, 7
E	1, 3, 5, 10

Movie 1 will be seen 3 times and movie 8 twice. If Bill wants to see these movies only once, then movie 1 should be seen in City E (cost \$5.25) and movie 8 should be seen in city A (cost \$5.50)

$$\text{Net Cost} = 169.35 - (5.50 + 7.00) - 7.00 = \$149.85$$

$$x_j = \begin{cases} 1, & \text{if community } j \text{ is selected} \\ 0, & \text{otherwise} \end{cases}$$

P_j = population of community j

S_i = set of communities within 25 miles from community i

The idea of the model is that the larger the population of a community, the higher should be its preference for acquiring a new store. At the same time, we need to minimize the total number of new stores. Thus, using $1/P_j$ as a weight for x_j is an appropriate way for modeling the objective function

$$\text{minimize } Z = \sum_{j=1}^{10} \frac{1}{P_j} x_j$$

s.t.

$$\sum_{j \in S_i} x_j \geq 1$$

$$x_j = \{0, 1\}, \quad j = 1, 2, \dots, 10$$

Note: The determination of S_i can be automated in AMPL. See ampl9.1b-6.txt

Solution: New stores should be located in communities 6, 8, and 9

$$x_t = \begin{cases} 1, & \text{if transmitter } t \text{ is selected} \\ 0, & \text{otherwise} \end{cases}$$

7

C_t = construction cost of transmitter t

$$x_c = \begin{cases} 1, & \text{if community } c \text{ is covered by} \\ & \text{a transmitter} \\ 0, & \text{otherwise} \end{cases}$$

S_c = set of transmitters covering community c

P_c = population of community c

$$\text{Maximize } Z = \sum_{c=1}^{15} P_c x_c$$

s.t.

$$\sum_{t \in S_c} x_t \geq x_c, \quad c = 1, 2, \dots, 10$$

$$\sum_{t=1}^7 C_t x_t \leq 15$$

Examples of the determination of S_c :

$$S_1 = \{1, 3\}, \quad S_2 = \{1, 2\}, \quad S_3 = \{2\}, \quad S_4 = \{4\}$$

$$S_5 = \{2, 6\}, \quad S_6 = \{4, 5\}, \quad S_7 = \{3, 5, 6\}$$

Solution:

Build transmitters 2, 4, 5, 6, and 7. All communities, except community number 1, are covered.

Set 9.1b

$$x_j = \begin{cases} 1, & \text{if receiver } j \text{ is installed} \\ 0, & \text{otherwise, } j=1, 2, \dots, 8 \end{cases}$$

8

R_i = Set of receivers covering meter i ,
 $i = 1, 2, \dots, 10$

$$\begin{aligned} R_1 &= \{1, 6, 8\}, R_2 = \{1, 2\}, R_3 = \{1, 2, 5\}, \\ R_4 &= \{6, 7, 8\}, R_5 = \{3, 7\}, R_6 = \{3, 5\}, \\ R_7 &= \{3, 4, 6\}, R_8 = \{5, 8\}, R_9 = \{2, 4, 6\}, \\ R_{10} &= \{4\} \end{aligned}$$

$$\text{Minimize } Z = x_1 + x_2 + \dots + x_8$$

s.t.

$$\sum_{j \in R_i} x_j \geq 1, \quad i = 1, 2, \dots, 10$$

$$x_j = (0, 1), \quad j = 1, 2, \dots, 8$$

Solution: Install receivers 1, 4, 5,
 and 7.

$$x_{ij} = \begin{cases} 1, & \text{if meter } i \text{ uses receiver } j \\ 0, & \text{otherwise} \end{cases}$$

9

$$y_j = (0, 1), \quad i = 1, 2, \dots, 10, \quad j = 1, 2, \dots, 8$$

$$\text{Minimize } Z = y_1 + y_2 + \dots + y_8$$

s.t.

$$\sum_{i \in S_j} x_{ij} \leq 3 y_j, \quad j = 1, 2, \dots, 8$$

$$\sum_{i \notin S_j} x_{ij} = 0, \quad j = 1, 2, \dots, 8$$

$$\sum_{j=1}^8 x_{ij} \geq 1, \quad i = 1, 2, \dots, 10$$

where

S_j = Set of meters covered by
 receiver j

$$S_1 = \{1, 2, 3\}, S_2 = \{2, 3, 9\}, \text{ etc}$$

Solution:

Receiver	Covered meters
1	1, 2, 3
3	5, 6
4	7, 9, 10
8	4, 8

Install receivers 1, 3, 4, and 8.

continued...

Set 9.1c

x_j = Nbr. of units of product j , $j = 1, 2, 3$

$$y_j = \begin{cases} 1, & \text{if } x_j > 0 \\ 0, & \text{if } x_j = 0 \end{cases}$$

$$\text{Maximize } Z = (60 - 30)x_1 + (40 - 20)x_2 + (120 - 80)x_3 \\ - 100y_1 - 80y_2 - 150y_3$$

s.t.

$$5x_1 + 3x_2 + 8x_3 \leq 3000$$

$$4x_1 + 3x_2 + 5x_3 \leq 2500$$

$$x_1 \geq 100, x_2 \geq 150, x_3 \geq 200$$

$$x_1 \leq 5000, y_1, x_2 \leq 5000, y_2, x_3 \leq 5000, y_3$$

Solution: $Z = \$16670$

$$x_1 = 100, x_2 = 300, x_3 = 200$$

x_j = number of widget produced on machine j , $j = 1, 2, 3$

$$y_j = \begin{cases} 1, & \text{if machine } j \text{ is used} \\ 0, & \text{if machine } j \text{ is not used} \end{cases}$$

$$\text{Min } Z = 2x_1 + 10x_2 + 5x_3 + 300y_1 + 100y_2 + 200y_3$$

subject to

$$x_1 + x_2 + x_3 \geq 2000$$

$$x_1 - 600y_1 \leq 0$$

$$x_2 - 800y_2 \leq 0$$

$$x_3 - 1200y_3 \leq 0$$

$$x_1, x_2, x_3 \geq 500 \text{ and integer}$$

$$y_1, y_2, y_3 = (0, 1)$$

Solution: $x_1 = 600, x_2 = 500, x_3 = 900$

$$Z = \$11300$$

$$x_{ij} = \begin{cases} 1, & \text{if site } i \text{ is assigned to target } j \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Min } Z = 5y_1 + 6y_2 + 2x_{11} + x_{12} + 8x_{13} + 5x_{14} \\ + 4x_{21} + 6x_{22} + 3x_{23} + x_{24}$$

subject to

$$x_{11} + x_{21} = 1$$

$$x_{12} + x_{22} = 1$$

continued...

$$x_{13} + x_{23} = 1$$

$$x_{14} + x_{24} = 1$$

$$\left. \begin{array}{l} x_{11} + x_{12} + x_{13} + x_{14} \leq M y_1 \\ x_{21} + x_{22} + x_{23} + x_{24} \leq M y_2 \end{array} \right\} M \geq 4$$

$$y_i = (0, 1) \text{ for all } i$$

$$x_{ij} = (0, 1) \text{ for all } i \text{ and } j$$

Solution: $Z = 18$

site	assigned targets
1	1 and 2
2	3 and 4

The problem can be formulated as a regular transportation model. Since total supply = total demand, all three plants must work at full capacity and the setup cost is immaterial in this case. This will not be the case if total supply exceeds total demand.

The ILP formulation is

$$\text{Min } Z = 12,000y_1 + 11,000y_2 + 12,000y_3 \\ + 10x_{11} + 15x_{12} + \dots + 11x_{33}$$

subject to

$$\begin{aligned} x_{11} + x_{12} + x_{13} &\leq 1800y_1 & x_{ij} \geq 0 \text{ and integer} \\ x_{21} + x_{22} + x_{23} &\leq 1400y_2 \\ x_{31} + x_{32} + x_{33} &\leq 1300y_3 \end{aligned}$$

$$\begin{aligned} x_{11} + x_{21} + x_{31} &\geq 1200 & y_i = (0, 1) \\ x_{12} + x_{22} + x_{32} &\geq 1700 \\ x_{13} + x_{23} + x_{33} &\geq 1600 \end{aligned}$$

Solution: $x_{11} = 1200, x_{13} = 600, x_{22} = 1400, x_{32} = 300, x_{33} = 1000, y_1 = y_2 = y_3 = 1$

Total supply > Total demand.

Modified constraints:

$$x_{11} + x_{21} + x_{31} \geq 800$$

$$x_{12} + x_{22} + x_{32} \geq 800$$

Solution: $x_{11} = 1000, x_{13} = 800, x_{21} = 200, x_{22} = 800, y_1 = y_2 = 1, y_3 = 0$. Plant 3 is not used.

4

5

Set 9.1c

$$x_{ijt} = \begin{cases} 1, & \text{if product } i \text{ uses line } j \text{ in period } t \\ 0, & \text{otherwise} \end{cases}$$

6

$$v_{ijt} = \begin{cases} 1, & \text{if changeover is made to product } i \text{ on line } j \text{ in period } t \\ 0, & \text{otherwise} \end{cases}$$

$$I_{it} = \text{End inventory of product } i \text{ in period } t$$

$$I_{i0} = \text{Initial inventory of product } i$$

$$D_i = \text{Demand of product } i \text{ in period } t$$

$$r_{ij} = \text{production rate of product } i \text{ on line } j \text{ (units/month)}$$

$$s_{ij} = \text{Switching cost of product } i \text{ on line } j$$

$$c_{ij} = \text{Production cost of product } i \text{ on line } j (\$/unit)$$

$$h_i = \text{Holding cost/unit/month of product } i$$

$$\text{Minimize } Z = \sum_{i=1}^3 \sum_{j=1}^2 c_{ij} r_{ij} \left(\sum_{t=1}^6 x_{ijt} \right) + \sum_{i=1}^3 \sum_{j=1}^2 s_{ij} \left(\sum_{t=1}^6 v_{ijt} \right) + \sum_{i=1}^3 h_i \left(\sum_{t=1}^6 I_{it} \right)$$

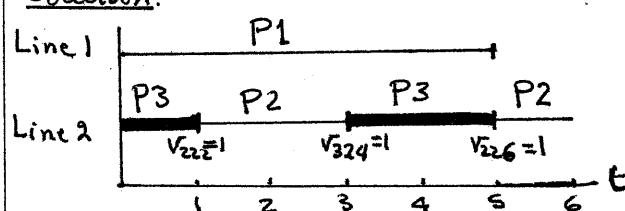
S.t.

$$\sum_{t=1}^3 x_{ijt} \leq 1, \quad i=1,2, \quad t=1,2, \dots, 6$$

$$v_{ijt} \geq x_{ijt} - x_{ijt-1} \quad \begin{cases} i=1,2,3 \\ j=1,2 \\ t=2,3, \dots, 6 \end{cases}$$

$$I_{it} = I_{i0} + \sum_{k=1}^t \left(\sum_{j=1}^2 r_{ij} x_{ijk} - D_{ik} \right), \quad i=1,2,3, \quad t=1,2, \dots, 6$$

Solution:



See file ampl9.1c-6.txt.

$$w_{ij} = \text{Line capacity in gal/hr from city } i \text{ to potential plant } j$$

7

$$F_i = \text{Fixed cost for plant located in city } i$$

$$P_i = \text{Population (in thousands) of city } i$$

$$y_i = \begin{cases} 1, & \text{if a plant is constructed in City } i \\ 0, & \text{Otherwise} \end{cases}$$

$$C_{ij} = \text{Construction cost of pipeline between cities } i \text{ and } j \text{ in } \$/1000 \text{ gal/hr}$$

$$\text{Minimize } Z = \sum_{i=1}^7 \left(\sum_{j=1}^7 C_{ij} \frac{w_{ij}}{1000} + F_i y_i \right)$$

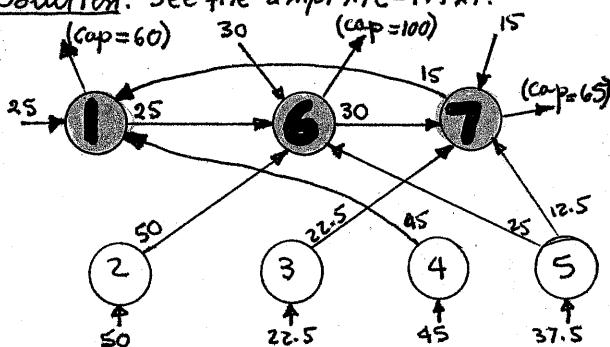
s.t.

$$\sum_{j=1}^7 w_{ij} \geq 500 P_i, \quad i=1,2, \dots, 7$$

$$\sum_{i=1}^7 w_{ij} \leq 100,000 y_j, \quad j=1,2, \dots, 7$$

$$\sum_{i=1}^7 y_i \leq 4$$

Solution: See file ampl9.1c-7.txt.



Plant 1 capacity = 60,000 gal/hr

6 capacity = 100,000 gal/hr

7 capacity = 65,000 gal/hr

Total cost = \$3,770,875

$$x_{tpc} = \text{gal of product } p \text{ in compartment } c \text{ on truck } t$$

8

$$y_{tpc} = \begin{cases} 1, & \text{if compartment } c \text{ on truck } t \text{ is used for product } p \\ 0, & \text{otherwise} \end{cases}$$

$$w_p = \text{Subcontracted gal of product } p$$

continued...

Set 9.1c

$$\text{Minimize } Z = 5w_1 + 12w_2 + 8w_3 + 10w_4$$

s.t.

$$\sum_{t=1}^4 \sum_{c=1}^5 x_{tpc} + w_p = \begin{cases} 10000, & p=1 \\ 15000, & p=2 \\ 12000, & p=3 \\ 8000, & p=4 \end{cases}$$

$$\sum_{p=1}^4 y_{tpc} = 1, \quad t=1,2,3,4, \quad c=1,2,\dots,5$$

$$\left. \begin{array}{l} x_{tp1} \leq 500 y_{tp1} \\ x_{tp2} \leq 750 y_{tp2} \\ x_{tp3} \leq 1200 y_{tp3} \\ x_{tp4} \leq 1500 y_{tp4} \\ x_{tp5} \leq 1750 y_{tp5} \end{array} \right\} \begin{array}{l} t=1,2,3,4 \\ p=1,2,3,4 \end{array}$$

Solution: See file ampl9.1c-8.txt

$$Z = \$148000$$

Truck	Product	500	750	1200	1500	1750
1	2	x		x	x	
4		x	x			
2	2		x			x
4		x		x	x	
3	2		x		x	x
4		x		x		
4	2	x	x		x	x
4			x			

Subcontracting:

$$\text{Product 1} = 10,000 \text{ gal}$$

$$3 = 12,000 \text{ gal}$$

$$4 = 2,000 \text{ gal}$$

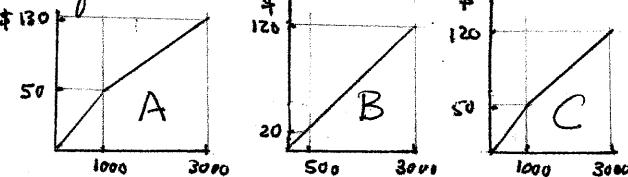
r_{ij} = weight i of cost function j,
 $i = 0, 1, 2; j = 1, 2, 3$

9

w_{ij} = (0, 1) $i = 0, 1, 2, j = 1, 2, 3$

y_j = {1, if Company j is used
 0, otherwise}

Cost functions:



continued...

$$\text{Minimize } Z = 50r_{11} + 130r_{21} + 20r_{12} + 120r_{22} + 50r_{13} + 120r_{23} + 10y_1 + 20y_2 + 25y_3$$

s.t.

$$\left. \begin{array}{l} r_{0j} \leq w_{0j} \\ r_{1j} \leq w_{0j} + w_{1j} \\ r_{2j} \leq w_{1j} \end{array} \right\} j = 1, 2, 3$$

$$r_{0j} + r_{1j} + r_{2j} = 1, \quad j = 1, 2, 3$$

$$w_{0j} + w_{1j} = 1, \quad j = 1, 2, 3$$

$$x_j \leq 3000 y_j, \quad j = 1, 2, 3$$

$$\sum_{j=1}^3 x_j \geq 3000$$

Solution: See file ampl9.1c-9

Use company A. Total cost = \$140

10

x_e = Nbr. of Eastern tickets

x_u = Nbr. of USAir tickets

x_c = Nbr. of Continental tickets

$$e_1, e_2 = (0, 1)$$

u, c = nonnegative integers

$$\text{Maximize } Z = 1000(x_e + 1.5x_u + 1.8x_c + 5e_1 + 5e_2 + 10u + 7c)$$

s.t.

$$x_e + x_u + x_c = 12$$

$$e_1 \leq \frac{x_e}{2}$$

$$e_2 \leq \frac{x_e}{6}$$

$$u \leq \frac{x_u}{6}$$

$$c \leq \frac{x_c}{5}$$

Solution: $Z = 39,000$ miles

$$x_e = 2 \text{ tickets}$$

$$x_u = 0$$

$$x_c = 10 \text{ tickets}$$

Set 9.1d

variables definitions :

x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	x_{23}
x_{31}	x_{32}	x_{33}

$$1 \leq x_{ij} \leq 9 \\ \text{and integer}$$

$$\sum_{j=1}^3 x_{ij} = 15, \quad i = 1, 2, 3$$

$$\sum_{i=1}^3 x_{ij} = 15, \quad j = 1, 2, 3$$

$$x_{11} + x_{22} + x_{33} = 15$$

$$x_{31} + x_{22} + x_{13} = 15$$

$$x_{11} \geq x_{12} + 1 \quad \text{or} \quad x_{11} \leq x_{12} - 1$$

$$x_{11} \geq x_{13} + 1 \quad \text{or} \quad x_{11} \leq x_{13} - 1$$

$$x_{12} \geq x_{13} + 1 \quad \text{or} \quad x_{12} \leq x_{13} - 1$$

$$x_{11} \geq x_{21} + 1 \quad \text{or} \quad x_{11} \leq x_{21} - 1$$

$$x_{11} \geq x_{31} + 1 \quad \text{or} \quad x_{11} \leq x_{31} - 1$$

$$x_{21} \geq x_{31} + 1 \quad \text{or} \quad x_{21} \leq x_{31} - 1$$

To remove "or" constraints, note that $x_{11} \geq x_{12} + 1$ or $x_{11} \leq x_{12} - 1$ can be replaced with the two simultaneous constraints:

$$\begin{aligned} -x_{11} + x_{12} + 15y_1 &\leq 14 \\ -x_{11} + x_{12} + 15y_1 &\geq 1 \end{aligned} \quad \{ y_1 = (0, 1)$$

Using a dummy objective function with all zero coefficients, the following solutions can be found

4	3	8
9	5	1
2	7	6

6	7	2
1	5	9
8	3	4

Other solutions exist.

Note:

If you use TORA to solve the problem, replace $y_j = (0, 1)$ with $0 \leq y_j \leq 1$ for all j

x_1 = daily units of product 1
 x_2 = daily units of product 2

2

$$\text{Maximize } Z = 10x_1 + 12x_2 \\ \text{Subject to}$$

$$x_1 + x_2 \leq 35$$

$$(x_1 \leq 20 \text{ and } x_2 \leq 10) \text{ or } (x_1 \leq 12 \text{ and } x_2 \leq 23)$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

$$\text{Maximize } Z = 10x_1 + 12x_2 \\ \text{Subject to}$$

$$x_1 + x_2 \leq 35$$

$$x_1 - 35y \leq 20$$

$$x_2 - 35y \leq 10$$

$$x_1 + 35y \leq 47$$

$$x_2 + 35y \leq 60$$

$$x_1, x_2, y \geq 0 \text{ and integer}$$

$$y = (0, 1) \quad M = 35$$

$$\text{Solution: } x_1 = 10, x_2 = 25, y = 1, Z = \$400 \\ \text{Select setting 2.}$$

x_j = daily number of units of product j

$y = \begin{cases} 0, & \text{if location 1 is selected} \\ 1, & \text{if location 2 is selected} \end{cases}$

$$\text{Maximize } Z = 25x_1 + 30x_2 + 22x_3$$

Subject to

$$\begin{aligned} (3x_1 + 4x_2 + 5x_3 \leq 100) \text{ or } (3x_1 + 4x_2 + 5x_3 \leq 90) \\ (4x_1 + 3x_2 + 6x_3 \leq 100) \text{ or } (4x_1 + 3x_2 + 6x_3 \leq 120) \end{aligned}$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer}$$

Let $M = 1000$, the "or" constraints are equivalent to

$$3x_1 + 4x_2 + 5x_3 \leq 100 + My$$

$$4x_1 + 3x_2 + 6x_3 \leq 100 + My$$

$$3x_1 + 4x_2 + 5x_3 \leq 90 + M(1-y)$$

$$4x_1 + 3x_2 + 6x_3 \leq 120 + M(1-y)$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer} \quad y = (0, 1)$$

$$\text{Solution: } x_1 = 26, x_2 = 3, x_3 = 0, y = 1 \\ \text{Use location 2. } Z = \$740$$

3

Set 9.1d

x_j = start time of job j , $j=1, 2, \dots, 10$

$$y_{ij} = \begin{cases} 1, & \text{if job } i \text{ precedes job } j \\ 0, & \text{otherwise} \end{cases}$$

$$w = (0, 1)$$

p_j = processing time of job j

d_j = due date of job j

$$\text{Minimize } Z = S_1^+ + S_2^+ + \dots + S_{10}^+$$

s.t.

$$\begin{aligned} M y_{ij} + x_i - x_j &\geq p_j & i = 1, 2, \dots, 10 \\ M(1 - y_{ij}) + x_j - x_i &\geq p_i & j = 1, 2, \dots, 10 \\ x_j + p_j + S_j^- - S_i^+ &= d_j \end{aligned}$$

$$\left. \begin{aligned} x_3 - (x_4 + p_4) &\leq M(1-w) - \epsilon \\ x_9 + p_9 - x_7 &\leq Mw \end{aligned} \right\} \epsilon \ll Mw$$

Solution: Total delay = 134 (see file ampl9.1d-4.txt)

Job	Start time
1	0
2	85
3	88
4	10
5	47
6	25
7	68
8	101
9	56
10	131

Optimal sequence: 1-4-6-5-9-7-2-3-8-10

Remove the last two constraints in Problem 4. Add the following constraints:

$$\begin{aligned} x_3 + p_3 &\leq x_4 \\ x_7 + p_7 &\geq x_8 - Mw \\ x_7 + p_7 &\leq x_8 + Mw \\ x_8 + p_8 &\geq x_7 - M(1-w) \\ x_8 + p_8 &\leq x_7 + M(1-w) \end{aligned} \quad \left. \begin{aligned} &\text{These four constraints} \\ &\text{translate} \\ &\text{either } x_7 + p_7 = x_8 \\ &\text{or } x_8 + p_8 = x_7 \end{aligned} \right\}$$

Solution: Total delay = 170

Optimal sequence: 1-3-4-5-6-9-2-7-8-10

x_j = Daily production of product j

$$\text{Max } Z = 25x_1 + 30x_2 + 45x_3$$

Subject to

$$3x_1 + 4x_2 + 5x_3 \leq 100$$

$$4x_1 + 3x_2 + 6x_3 \leq 100$$

$$x_3 \leq 0 \text{ or } x_3 \geq 5$$

$x_1, x_2, x_3 \geq 0$ and integer

Let $y = (0, 1)$ and $M = 100$. Then,

$$(x_3 \leq 0 \text{ or } x_3 \geq 5)$$

is equivalent to

$$(x_3 \leq My \text{ and } -x_3 \leq -5 + M(1-y))$$

which reduces to

$$x_3 - 100y \leq 0 \text{ and } -x_3 + 100y \leq 95$$

Solution: $x_1 = 0, x_2 = 11, x_3 = 11$

$y = 1 \Rightarrow$ produce product 3

$$Z = \$825$$

5

Set 9.1d

7

1. Straightforward formulation:

Let $x_{it} = 1$ if load i is assigned to trailer t , 0 otherwise

L_i = linear feet of load i

r_i = revenue from load i

Maximize $z = \sum_{i=1}^{10} \sum_{t=1}^2 r_i x_{it}$ subject to

$$\sum_{i=1}^{10} L_i x_{it} \leq 36, t = 1, 2$$

$$\sum_{t=1}^2 x_{it} \leq 1, i = 1, \dots, 10, x_{it} = (0, 1), i = 1, 2, \dots, 10$$

2. Formulation using if-then:

Let x_{it} = feet in trailer t assigned to load i

$y_i = (0, 1), i = 1, 2, \dots, 10, w_{it} = (0, 1), i = 1, 2, \dots, 10, t = 1, 2$

Maximize $z = \sum_{i=1}^{10} \sum_{t=1}^2 r_i x_{it}$ subject to

$$\sum_{i=1}^{10} x_{it} \leq 36, t = 1, 2$$

$$x_{i1} \leq L_i y_i, x_{i2} \leq L_i (1 - y_i), i = 1, 2, \dots, 10$$

(above constraint is not as efficient as $x_{i1} + x_{i2} \leq 1, i = 1, 2, \dots, 10$ in formulation 1)

(if $x_{it} > 0$ then $x_{it} = L_i$) translates to

$$x_{it} \leq M(1 - w_{it}), L_i - x_{it} \leq Mw_{it}, -L_i + x_{it} \leq Mw_{it}, i = 1, 2, \dots, 10, t = 1, 2$$

$$x_{it}, w_{it}, y_i = (0, 1), i = 1, 2, \dots, 10, t = 1, 2$$

Solution: $z = \$7929$. Problem has several alternative optima. (See file ampl9.1d-7.txt.)

Trailer	Load	Solution 1		Solution 2	
		Feet	Load	Feet	Load
1	1	5		1	5
	5	7		2	11
	6	9		6	9
	8	14		9	10
Total		35 ft		Total	35 ft
2	2	11		4	15
	4	15		5	7
	9	10		8	14
Total		36 ft		Total	36 ft

8

(a) Formulation 1:

$$\begin{cases} x_1 \leq 1, x_2 \leq 2 \\ \text{or} \\ x_1 + x_2 \leq 3, x_1 \geq 2 \end{cases} \equiv \begin{cases} x_1 - My \leq 1 \\ x_2 - My \leq 2 \\ x_1 + x_2 - M(1-y) \leq 3 \\ x_1 + M(1-y) \geq 2 \\ y = 0, 1, x_1, x_2 \geq 0 \end{cases} M \geq 3$$

Formulation 2:

$$\begin{cases} x_1 + x_2 \leq 3, x_2 \leq 2 \\ \text{and} \\ (x_1 \leq 1 \text{ or } x_1 \geq 2) \end{cases} \equiv \begin{cases} x_1 + x_2 \leq 3, x_2 \leq 2 \\ x_1 - My \leq 1 \\ x_1 + M(1-y) \geq 2 \\ y = 0, 1, x_1, x_2 \geq 0 \end{cases} M \geq 2$$

(b)

$$\begin{cases} x_1 + x_2 \leq 3 \\ \text{and} \\ (x_1 \geq 1 \text{ or } x_2 \geq 1) \end{cases} \equiv \begin{cases} x_1 + My \geq 1 \\ x_2 + M(1-y) \geq 1 \\ x_1 + x_2 \leq 3 \\ y = 0, 1, x_1, x_2 \geq 0 \end{cases} M \geq 3$$

(c)

$$\begin{cases} x_1 + x_2 \leq 3 \\ \text{and} \\ (x_1 + x_2 \geq 2 \text{ or } x_2 \leq 1) \end{cases} \equiv \begin{cases} x_1 + x_2 \leq 3 \\ x_1 + x_2 + My \geq 2 \\ x_2 - M(1-y) \leq 1 \\ y = 0, 1, x_1, x_2 \geq 0 \end{cases} M \geq 3$$

$$g_i(x_1, x_2, \dots, x_m) \leq b_{i1} + M y_i$$

9

$$i = 1, 2, \dots, m$$

$$y_1 + y_2 + \dots + y_m = k$$

$$y_i = (0, 1), i = 1, 2, \dots, m$$

$$g(x_1, x_2, \dots, x_m) \leq b_{11} y_1 + b_{21} y_2 + \dots + b_{m1} y_m$$

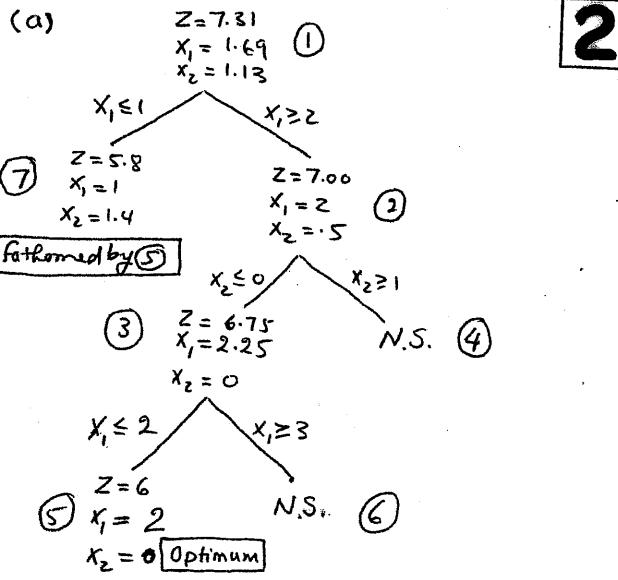
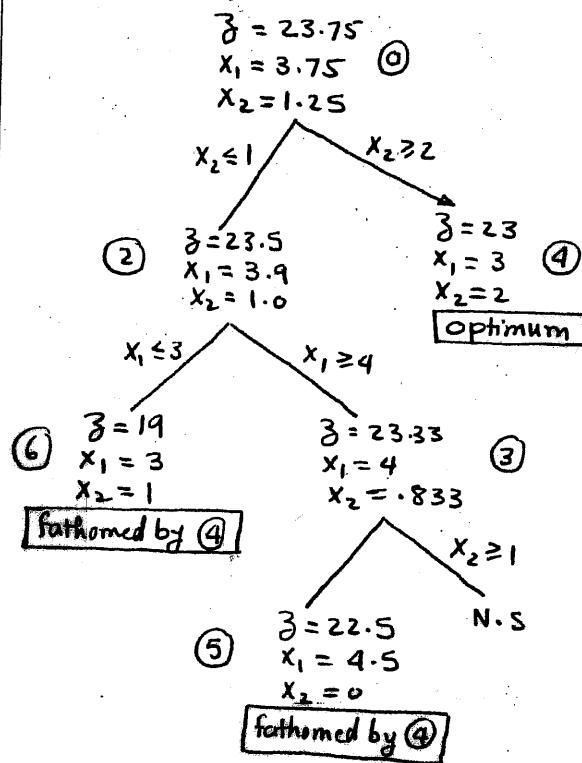
10

$$y_1 + y_2 + \dots + y_m = 1$$

$$y_i = (0, 1), i = 1, 2, \dots, m$$

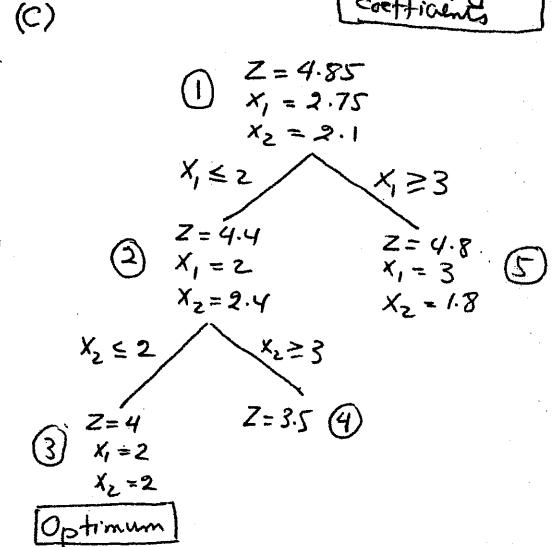
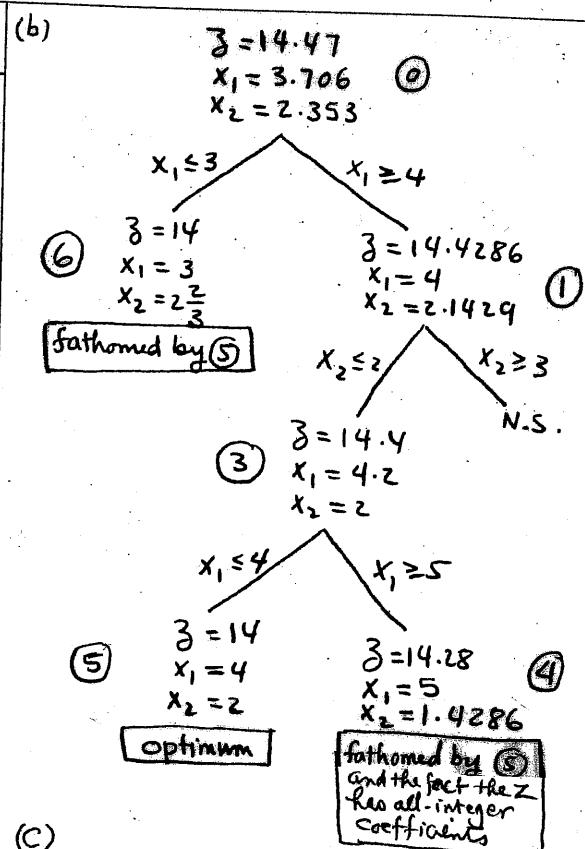
Set 9.2a

Note: all subproblems are solved by TORA using the MODIFY option to create each problem.



A different tree will result if branch $x_1 \leq 1$ at ① is investigated before $x_1 \geq 2$

continued...

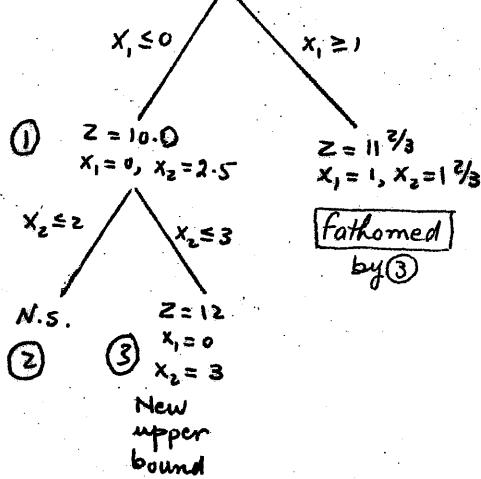


④ and ⑤ are fathomed by ③.
Fathoming of ⑤ requires the additional condition that the coefficients of Z are all-integer.

continued...

Set 9.2a

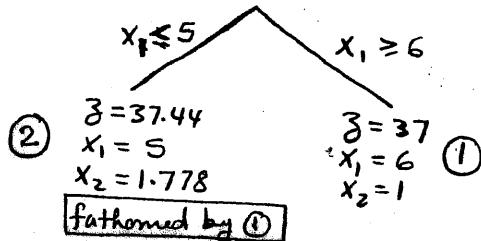
(d) $Z = 9.8$
 $x_1 = .2$
 $x_2 = 2.2$



Optimum solution: $x_1 = 0, x_2 = 3, Z = 12$

(e)

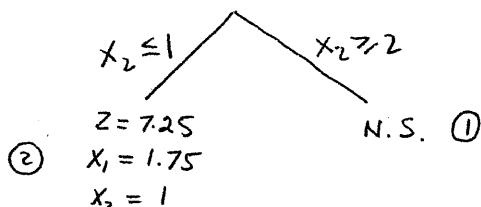
$\bar{Z} = 38.3846$
 $x_1 = 5.8462$ ⑥
 $x_2 = 1.3077$



Optimum: $x_1 = 6, x_2 = 1, \bar{Z} = 37$

(a)

$Z = 7.31$
 $x_1 = 1.69$ ⑥
 $x_2 = 1.13$

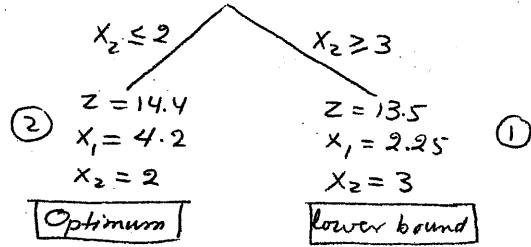


Optimum: $Z = 7.25, x_1 = 1.75, x_2 = 1$

3

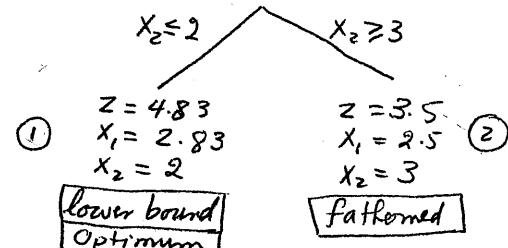
(b)

$Z = 14.47$
 $x_1 = 3.71$ ⑥
 $x_2 = 2.35$



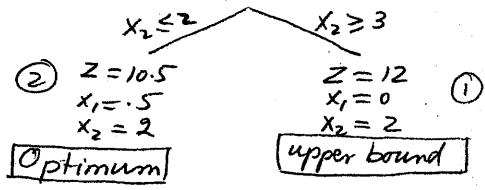
(c)

$Z = 4.85$
 $x_1 = 2.75$ ⑥
 $x_2 = 2.1$



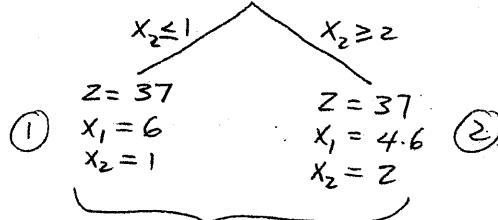
(d)

$Z = 9.8$ ⑥
 $x_1 = .2$
 $x_2 = 2.2$



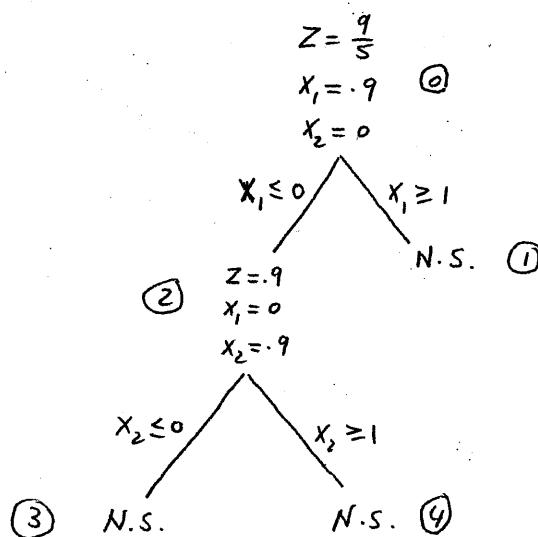
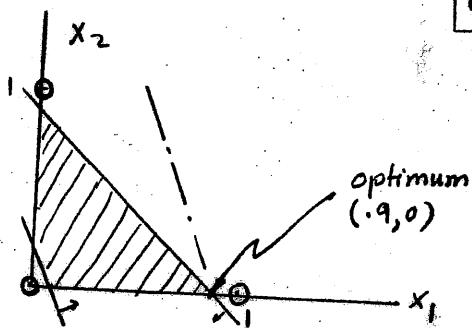
(e)

$Z = 3.8.38$ ⑥
 $x_1 = 5.85$
 $x_2 = 1.31$



continued...

Set 9.2a



Problem has no feasible solution.

4

$$\text{Max } Z = 18x_1 + 14x_2 + 8x_3 + 4x_4$$

Subject to

$$15x_1 + 12x_2 + 7x_3 + 4x_4 + x_5 \leq 37$$

$$0 \leq x_j \leq 1, j = 1, 2, \dots, 5$$

5

$$Z = 43 \quad (0)$$

$$x_4 = 0$$

$$x_1 = x_2 = x_3 = 1$$

1

$$x_4 = .75$$

$$x_4 \leq 0 \quad (1) \quad x_4 \geq 1 \Rightarrow x_4 = 1$$

$$(1) \quad Z = 40$$

$$x_1 = x_2 = x_3 = 1$$

$$x_4 = x_5 = 0$$

$$(2) \quad Z = 42.86$$

$$x_1 = x_2 = x_4 = 1$$

$$x_3 = .86$$

$$(3) \quad Z = 36$$

$$x_1 = x_2 = x_4 = 1$$

$$x_3 = x_5 = 0$$

$$(4) \quad Z = 42.83$$

$$x_1 = x_3 = x_4 = 1$$

$$x_2 = .92$$

$$(5) \quad \text{Fathomed}$$

$$x_3 = 0$$

$$x_3 = 1$$

$$(6) \quad Z = 30$$

$$x_1 = x_3 = x_4 = 1$$

$$x_2 = x_5 = 0$$

$$(7) \quad Z = 42.80$$

$$x_1 = .93$$

$$x_2 = x_3 = x_4 = x_5 = 0$$

$$(8) \quad \text{Fathomed}$$

$$x_1 = 0$$

$$x_1 = 1$$

$$(9) \quad Z = 26$$

$$x_1 = x_5 = 0$$

$$x_2 = x_3 = x_4 = 1$$

$$(10) \quad \text{Fathomed}$$

$$\text{Optimum: } Z = 40$$

$$x_1 = x_2 = x_3 = 1$$

$$x_4 = x_5 = 0$$

Set 9.2a

$$-x_1 + 10x_2 - 3x_3 \geq 15 \Rightarrow \begin{cases} -x_1 + 10x_2 - 3x_3 \geq 15 \\ \text{or} \\ -x_1 + 10x_2 - 3x_3 \leq -15 \end{cases}$$

The problem is

$$\text{Max } Z = x_1 + 2x_2 + 5x_3$$

Subject to

$$\begin{aligned} -x_1 + 10x_2 - 3x_3 + M\gamma &\geq 15 \\ -x_1 + 10x_2 - 3x_3 + M\gamma &\leq M - 15 \end{aligned} \quad (M=110)$$

$$2x_1 + x_2 + x_3 \leq 10$$

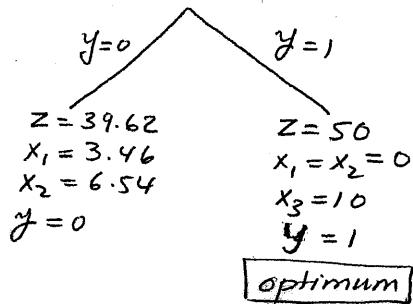
$$x_1, x_2, x_3 \geq 0, \gamma = (0, 1)$$

$$Z = 50$$

$$x_1 = x_2 = 0$$

$$x_3 = 10$$

$$= .45$$



(a) Replacing $x_j = (0, 1)$ with $0 \leq x_j \leq 1$

and $\gamma = (0, 1)$ with $0 \leq \gamma \leq 1$, TORA's ILP automated module determines the optimum in 9 subproblems and verifies optimality after examining 25,739 subproblems.

(b) See file solver9.2a-7b.xls. Solver examined over 25,000 subproblems before verifying optimality.

Number of examined subproblems with the objective function bound activated = 29

Number of examined subproblems without the objective bound activated = 35

6

Conversion to binary variables:

$$0 \leq x_1 \leq 2 \Rightarrow x_1 = y_{11} + 2y_{12}$$

$$0 \leq x_2 \leq 3 \Rightarrow x_2 = y_{21} + 2y_{22}$$

$$0 \leq x_3 \leq 6 \Rightarrow x_3 = y_{31} + 2y_{32} + 4y_{33}$$

$$\text{Max } Z = 18y_{11} + 36y_{12} + 14y_{21} + 28y_{22} + 8y_{31} + 16y_{32} + 32y_{33}$$

Subject to

$$15y_{11} + 30y_{12} + 12y_{21} + 24y_{22} + 7y_{31} + 14y_{32} + 28y_{33} \leq 43$$

$$\text{all } y_{ij} = (0, 1)$$

Optimum solution: $Z = 50$

$$y_{12} = y_{21} = 1 \Rightarrow x_1 = 2, x_2 = 1, x_3 = 0$$

The solution takes 6 iterations to find the optimum and 41 to verify it.

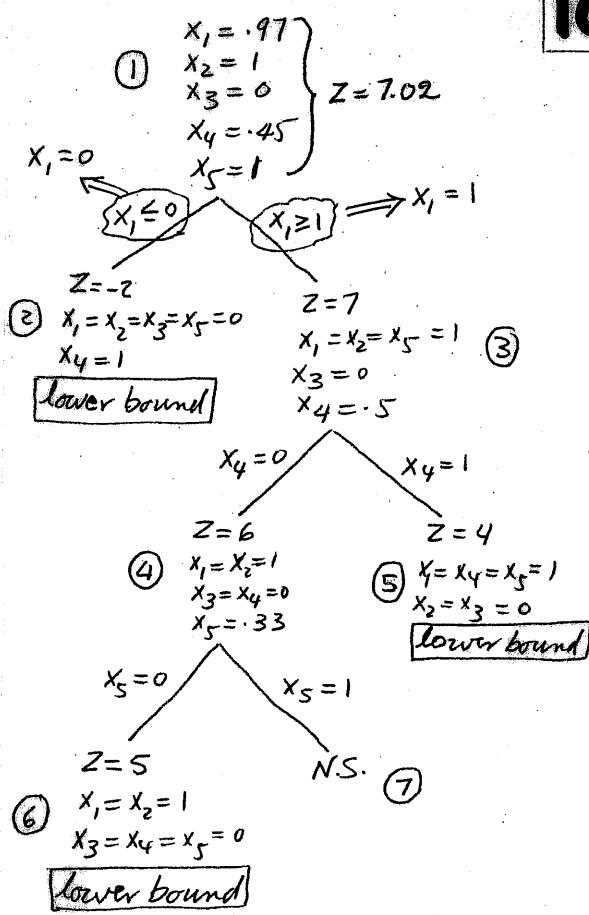
If the original problem is solved directly, it takes 4 iterations to find the optimum and 29 to verify optimality. This result points to the possibility that binary substitution may not offer any computational advantages.

7

8

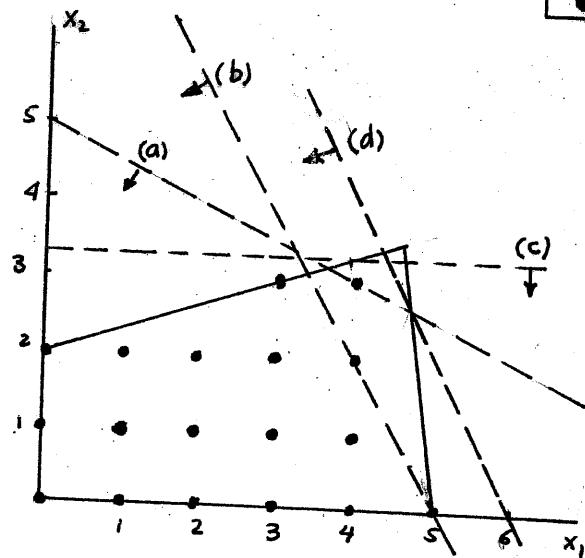
Set 9.2a

10



If the search sequence is ① → ② → ③ → ④ → ⑤ → ⑥, the lower bound will be successively updated as $Z = -2$ at ②, $Z = 4$ at ⑤ and $Z = 5$ at ⑥. In this case, only node ⑦ is fathomed without being investigated.

If the search sequence is ① → ③ → ④ → ⑥, the first lower bound will be $Z = 5$. However, even in this case, the remaining nodes ② and ⑤ must be examined because they have the potential of producing a better solution with $Z = 7$ (at ⑤, it could be an alternative solution with $Z = 7$). Only node ⑦ need not be examined.



(a) $x_1 + 2x_2 \leq 10$:

The cut is legitimate because it passes through an integer point and does not eliminate any feasible integer points.

(b) $2x_1 + x_2 \leq 10$:

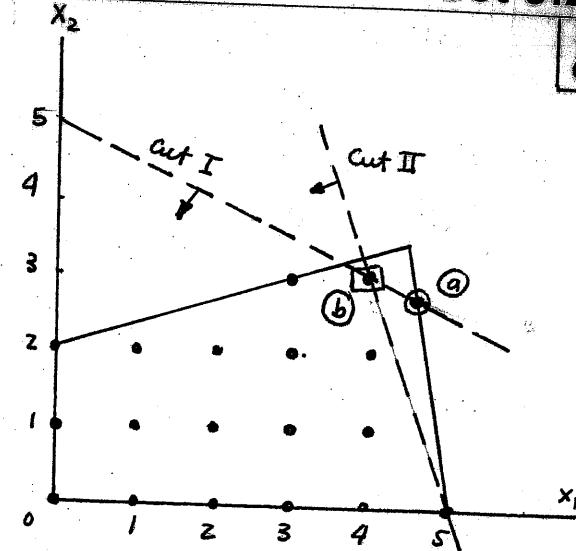
The cut is not legitimate because it eliminates a feasible integer point.

(c) $3x_2 \leq 10$:

The cut is not legitimate because it does not pass through an integer point.

(d) $2x_1 + x_2 \leq 12$:

The cut is legitimate because it passes through an integer point and does not exclude any feasible integer points. Note that it does not matter that the integer point through which the cut passes is itself infeasible [namely, (6, 0)].



Cut I produces the continuous optimum at point (a)

Cut II (together with I) produces the integer optimum at point (b).

Cut I:

$$-\frac{7}{22}x_3 - \frac{1}{22}x_4 \leq -\frac{1}{2}$$

From the original constraints,

$$x_3 = 6 + x_1 - 3x_2$$

$$x_4 = 35 - 7x_1 - x_2$$

Thus,

$$-\frac{7}{22}(6 + x_1 - 3x_2) - \frac{1}{22}(35 - 7x_1 - x_2) \leq -\frac{1}{2}$$

or

$$x_2 \leq 3$$

Cut II:

$$-\frac{1}{7}x_4 - \frac{6}{7}x_1 \leq -\frac{4}{7}$$

$$x_1 = -\frac{1}{2} + \frac{7}{22}x_3 + \frac{1}{22}x_4$$

or

$$-\frac{1}{7}(35 - 7x_1 - x_2) - \frac{6}{7}(-\frac{1}{2} + \frac{7}{22}x_3 + \frac{1}{22}x_4) \leq -\frac{4}{7}$$

or

$$x_1 + x_2 \leq 7$$

Set 9.2b

From the tableau of cut I, we have

$$x_3 + \frac{1}{7}x_4 - \frac{22}{7}s_1 = 1\frac{4}{7}$$

$$x_3 + \frac{1}{7}x_4 + (-4 + \frac{6}{7})s_1 = 1 + \frac{4}{7}$$

$$\text{Cut: } -\frac{1}{7}x_4 - \frac{6}{7}s_1 \leq -\frac{4}{7}$$

This cut happens to be the same as cut II in Example 9.2-2

Basic	x_1	x_2	x_3	Solution
Z	-1	-2	0	0
x_3	1	$\frac{1}{2}$	1	$13/4$
Z	3	0	4	13
x_2	2	1	2	$13/2$

4

The optimum constraint

$$2x_1 + x_2 + 2x_3 = 6\frac{1}{2}$$

produces the cut $s_1 = -\frac{1}{2}$, which is infeasible.

Next, convert the constraint to

$$4x_1 + 2x_2 \leq 13$$

The associated simplex tableaus are

Basic	x_1	x_2	x_3	Sol $\frac{1}{2}$
\bar{x}	-1	-2	0	0
x_3	4	2	1	13
I	3	0	1	13
x_2	2	1	$\frac{1}{2}$	$6\frac{1}{2}$

From the optimal constraint

$$2x_1 + x_2 + \frac{1}{2}x_3 = 6\frac{1}{2}$$

the cut is

$$s_1 - (0)x_1 - \frac{1}{2}x_3 = -\frac{1}{2}$$

The dual simplex produces the following iterations:

Basic	x_1	x_2	x_3	s_1	Sol $\frac{1}{2}$
\bar{x}	3	0	1	0	13
x_2	2	1	$\frac{1}{2}$	0	$6\frac{1}{2}$
s_1	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$
\bar{x}	3	0	0	2	12
x_2	2	1	0	$\frac{1}{2}$	6
x_3	0	0	1	-2	1

5

Optimum: $x_1 = 0, x_2 = 6, x_3 = 1, Z = 12$

(a) continuous optimum tableau:

6

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Sol $\frac{1}{2}$
Z	0	0	0	2	2	2	30
x_1	1			$\frac{3}{10}$	$\frac{1}{5}$	0	$2\frac{1}{2}$
x_2		1		$\frac{1}{20}$	$\frac{1}{5}$	0	$1\frac{1}{4}$
x_3			1	$\frac{1}{4}$	0	1	$6\frac{1}{4}$

From the x_1 -row

$$x_1 + \frac{3}{10}x_4 + \frac{1}{5}x_5 = 2\frac{1}{2},$$

the cut is

$$s_1 - \frac{3}{10}x_4 - \frac{1}{5}x_5 = -\frac{1}{2} \quad (\text{cut I})$$

Adding cut I and solving, we get

Basic	x_1	x_2	x_3	x_4	x_5	x_6	s_1	Sol $\frac{1}{2}$
Z	0	0	0	0	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{80}{3}$	
x_1	1				0	0	1	2
x_2		1			$\frac{1}{6}$	0	$\frac{1}{6}$	$1\frac{1}{6}$
x_3			1		$-\frac{1}{6}$	1	$\frac{5}{6}$	$5\frac{5}{6}$
x_4				1	$\frac{2}{3}$	0	$-\frac{10}{3}$	$1\frac{2}{3}$

From the x_3 -row

$$x_3 - \frac{1}{6}x_5 + x_6 + \frac{5}{6}s_1 = 5\frac{5}{6},$$

the cut is

$$s_2 - \frac{5}{6}x_5 - \frac{5}{6}s_1 = -\frac{5}{6} \quad (\text{cut I})$$

continued...

continued...

Set 9.2c

Cut II produces the following optimum tableau:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	s_1	s_2	Sol/Σ
Z	0	0	0	0	0	2	6	$4/5$	26
x_1	1					0	1	0	2
x_2		1				0	0	$1/5$	1
x_3			1			1	1	$-1/5$	6
x_4				1		0	-4	$4/5$	1
x_5					1	0	1	$-6/5$	1

which is all optimum and integer

Variable rounded Sol/Σ Integer Sol/Σ

x_1	2 (or 3)	2
x_2	1	1
x_3	6	6
Z	26 (or 30)	26

If x_1 is rounded to 3, the solution is infeasible

(b)

Continuous optimum tableau:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Sol/Σ
Z	0	0	0	2	3	5	29
x_3	0	0	1	$4/9$	$1/9$	$4/9$	$3 \frac{1}{3}$
x_2	0	1	0	$1/3$	$1/3$	$1/3$	3
x_1	1	0	0	$1/9$	$7/9$	$10/9$	$5 \frac{1}{3}$

From x_3 -row, we get cut I:

$$S_1 - \frac{4}{9}x_4 - \frac{1}{9}x_5 - \frac{4}{9}x_6 = -\frac{1}{3}$$

New tableau after cut I:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	s_1
Z				$5/2$	3	$9/2$	$55/2$
x_3	1		0	0	1		3
x_2		1		-1	0	$3/4$	$2 \frac{3}{4}$
x_1			1	$3/4$	1	$1/4$	$5 \frac{1}{4}$
s_1				1	$1/4$	- $9/4$	$3/4$

From x_2 -row, we get cut II:

$$S_2 - \frac{3}{4}s_1 = -\frac{3}{4}$$

New tableau after cut II is added:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	s_1	s_2	Sol/Σ	
Z	0	0	0	0	0	$5/2$	3	0	6	23
x_3	0	0	1	0	0	0	0	$4/3$		2
x_2	0	1	0	0	0	-1	0	0	1	2
x_1	1	0	0	0	$3/4$	1	0	$1/3$		5
x_4	0	0	0	1	$1/4$	1	0	-3		3
s_1	0	0	0	0	0	0	1	$-4/3$		1

Variable rounded solution integer Sol/Σ

x_1	5	5
x_2	3	2
x_3	3	2
Z	27	23

The rounded solution is infeasible.

continued...

Set 9.3a

The table gives the number of distinct employees who enter/leave manager's office when switch is made from project i to project j .

	1	2	3	4	5	6
1	-	4	4	6	6	5
2	4	-	6	4	6	3
3	4	6	-	4	8	7
4	6	4	4	-	6	5
5	6	6	8	6	-	5
6	5	3	7	5	5	-

$$x_{ij} = \begin{cases} 1, & \text{if project } j \text{ follows } i \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Minimize } Z = Mx_{11} + 4x_{12} + 4x_{13} + \dots + 5x_{64} + 5x_{65} + Mx_{66}$$

Subject to

$$\sum_{j=1}^6 x_{ij} = 1, \quad i = 1, 2, \dots, 6$$

$$\sum_{i=1}^6 x_{ij} = 1, \quad j = 1, 2, \dots, 6$$

Solution is a tour

$$x_{ij} = (0, 1)$$

Represent Basin, Wald, Bon, and Kiln by nodes 1, 2, 3, 4, and 5, respectively.

$$x_{ij} = \begin{cases} 1, & \text{if city } j \text{ follows city } i \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Minimize } Z = Mx_{11} + 120x_{12} + 220x_{13} + \dots + 185x_{53} + 190x_{54} + Mx_{55}$$

Subject to

$$\sum_{i=1}^5 x_{ij} = 1, \quad j = 1, 2, \dots, 5$$

$$\sum_{j=1}^5 x_{ij} = 1, \quad i = 1, 2, \dots, 5$$

Solution is a tour

$$x_{ij} = (0, 1)$$

continued...

1

$$x_{ij} = \begin{cases} 1, & \text{if hole } j \text{ follows hole } i \\ 0, & \text{otherwise} \end{cases}$$

3

$$\text{Minimize } Z = Mx_{11} + 1.2x_{12} + \dots + 1.9x_{65} + Mx_{66}$$

Subject to

$$\sum_{i=1}^6 x_{ij} = 1, \quad j = 1, 2, \dots, 6$$

$$\sum_{j=1}^6 x_{ij} = 1, \quad i = 1, 2, \dots, 6$$

$$x_{ij} = (0, 1)$$

Solution is a tour

Set 9.3b

(a)

	1	2	3	4
1	10	17	15	
2	20		19	18
3	50	44		25
4	45	40	20	

(b)

	1	2	3	4	5	6
1	4	4	6	6	6	5
2	4		6	4	6	3
3	4	6		4	8	7
4	6	4	4		6	5
5	6	6	8	6		5
6	5	3	7	5	5	

Solution summary:

Start city	Tour	Length
1	1-2-4-3-1	98
2	2-4-3-1-2	98
3	3-4-2-1-3	102
4	4-3-2-1-4	99
Reversals		
2-4	1-4-2-3-1	124
4-3	1-2-3-4-1	99
2-4-3	1-3-4-2-1	102

Solution: (1-2-4-3-1).

Min setup cost = \$98

Solution summary:

Start city	Tour	Length
1	1-3-4-2-6-5-1	26
2	2-6-5-4-3-1-2	26
3	3-4-2-6-5-1-3	26
4	4-3-1-2-6-5-4	26
5	5-6-2-4-3-1-5	26
6	6-2-4-3-1-5-6	26
Reversals		
3-4	1-4-3-2-6-5-1	30
4-2	1-3-2-4-6-5-1	30
2-6	1-3-4-6-2-5-1	28
6-5	1-3-4-2-5-6-1	28
3-4-2	1-2-4-3-6-5-1	30
4-2-6	1-3-6-2-4-5-1	30
2-6-5	1-3-4-5-6-2-1	26
3-4-2-6	1-6-2-4-3-5-1	30
4-2-6-5	1-3-5-6-2-4-1	30
3-4-2-6-5	1-5-6-2-4-3-1	26

Solution: (1-3-4-2-6-5-1). Alternative solutions
Min traffic = 26 employees

continued....

continued....

Set 9.3b

(c)

	1	2	3	4	5
1	120	220	150	210	
2	120	80	110	130	
3	220	80	160	15	
4	150	110	160	190	
5	210	130	185	190	

(d)

	1	2	3	4	5	6
1		1.2	0.5	2.6	4.1	3.2
2	1.2		3.4	4.6	2.9	5.2
3	0.5	3.4		3.5	4.6	6.2
4	2.6	4.6	3.5		3.8	0.9
5	4.1	2.9	4.6	3.8		1.9
6	3.2	5.2	6.2	0.9	1.9	

Solution summary:

Start city	Tour	Length
1	1-2-3-5-4-1	555
2	2-3-5-4-1-2	555
3	3-5-2-4-1-3	625
4	4-2-3-5-1-4	565
5	5-2-3-4-1-5	730
Reversals		
2-3	1-3-2-5-4-1	770
3-5	1-2-5-3-4-1	745
5-4	1-2-3-4-5-1	760
2-3-5	1-5-3-2-4-1	735
3-5-4	1-2-4-5-3-1	825
2-3-5-4	1-4-5-3-2-1	725

- Solutions: 1. (2-3-4-1-5-2)
 2. (5-2-3-4-1-5)
 3. (2-5-1-4-3-2)
 Length = 730 miles

Note: Tours 1 and 2 are the same. Tour 3 is the reverse order of tours 1 and 2 because the distance matrix is symmetrical.

Solution summary:

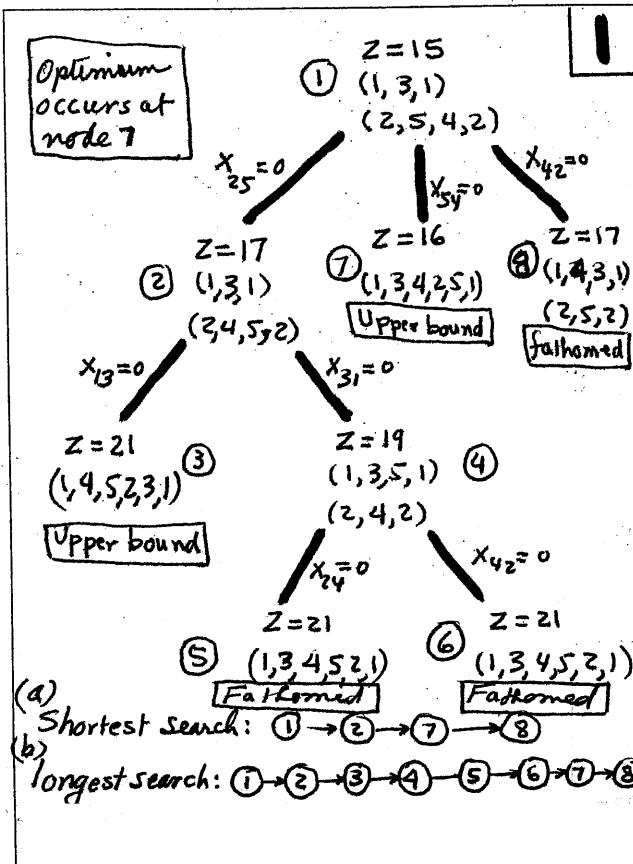
Start city	Tour	Length
1	1-3-2-5-6-4-1	12.2
2	2-1-3-4-6-5-2	10.9
3	3-1-2-5-6-4-3	10.9
4	4-6-5-2-1-3-4	10.9
5	5-6-4-1-3-2-5	12.2
6	6-4-1-3-2-5-6	12.2
Reversals		
1-3	2-3-1-4-6-5-2	12.2
3-4	2-1-4-3-6-5-2	18.3
4-6	2-1-3-6-4-5-2	15.5
6-5	2-1-3-4-5-6-2	16.1
1-3-4	2-4-3-1-6-5-2	16.6
3-4-6	2-1-6-4-3-5-2	16.3
4-6-5	2-1-3-5-6-4-2	13.7
1-3-4-6	2-6-4-3-1-5-2	17.1
3-4-6-5	2-1-5-6-4-3-2	15
1-3-4-6-5	2-5-6-4-3-1-2	10.9

- Solutions: 1. (2-1-3-4-6-5-2)
 2. (3-1-2-5-6-4-3)
 3. (4-6-5-2-1-3-4)
 4. (2-5-6-4-3-1-2)

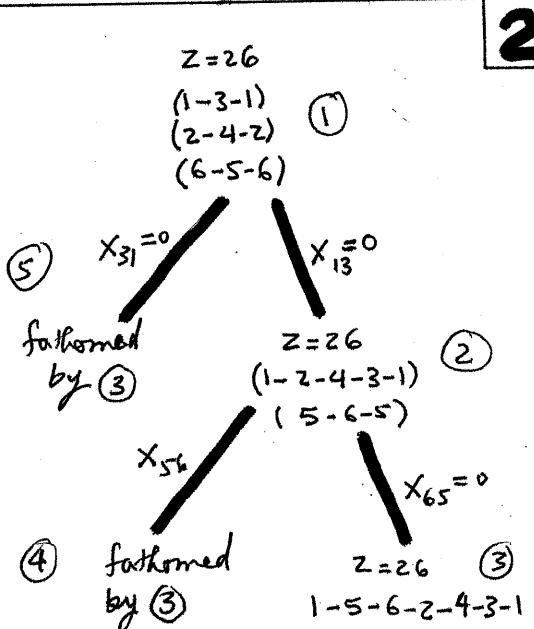
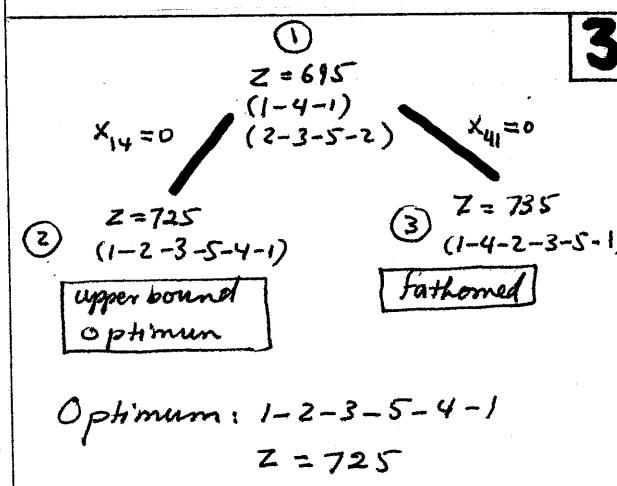
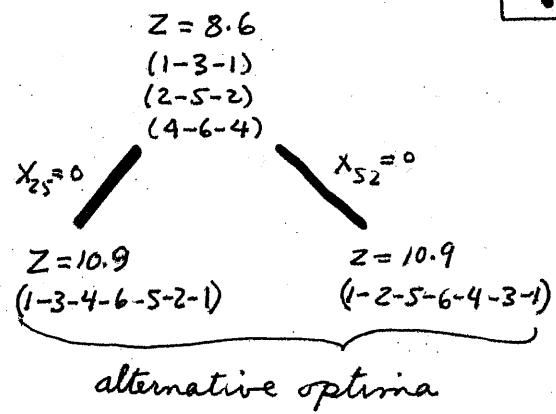
Length = 10.9 cm

Note: Tours 1 and 3 are the same. Tours 2 and 4 are the same (also, reverse order of 1 and 3).

Set 9.3c



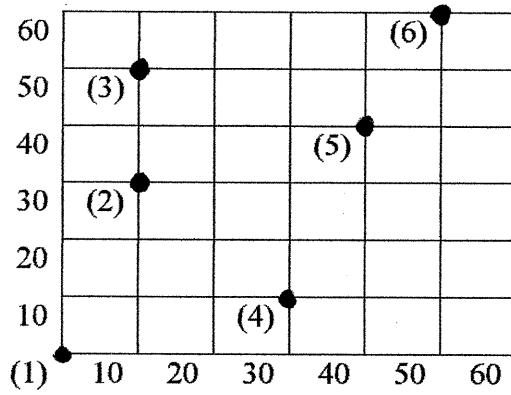
4



Set 9.3d

1

Layout:



Distance matrix:

	1	2	3	4	5	6
1		40	60	40	80	110
2	40		20	40	40	70
3	60	20		60	40	50
4	40	40	60		40	70
5	80	40	40	40		30
6	110	70	50	70	30	

Cuts:

$$6*x[2,3] + u[2] - u[3] \leq 5;$$

$$6*x[2,4] + u[2] - u[4] \leq 5;$$

$$6*x[2,5] + u[2] - u[5] \leq 5;$$

$$6*x[2,6] + u[2] - u[6] \leq 5;$$

$$6*x[3,2] - u[2] + u[3] \leq 5;$$

$$6*x[3,4] + u[3] - u[4] \leq 5;$$

$$6*x[3,5] + u[3] - u[5] \leq 5;$$

$$6*x[3,6] + u[3] - u[6] \leq 5;$$

$$6*x[4,2] - u[2] + u[4] \leq 5;$$

$$6*x[4,3] - u[3] + u[4] \leq 5;$$

$$6*x[4,5] + u[4] - u[5] \leq 5;$$

$$6*x[4,6] + u[4] - u[6] \leq 5;$$

$$6*x[5,2] - u[2] + u[5] \leq 5;$$

$$6*x[5,3] - u[3] + u[5] \leq 5;$$

$$6*x[5,4] - u[4] + u[5] \leq 5;$$

$$6*x[5,6] + u[5] - u[6] \leq 5;$$

$$6*x[6,2] - u[2] + u[6] \leq 5;$$

$$6*x[6,3] - u[3] + u[6] \leq 5;$$

$$6*x[6,4] - u[4] + u[6] \leq 5;$$

$$6*x[6,5] - u[5] + u[6] \leq 5;$$

Solution: See file ampl9.3d-1.txt.

1-2-3-6-5-4-1. Minimum length = 220 meters

2

Cuts:

subject to cut[2,3]: $5*X[2,3] + u[2] - u[3] \leq 4;$ subject to cut[2,4]: $5*X[2,4] + u[2] - u[4] \leq 4;$ subject to cut[2,5]: $5*X[2,5] + u[2] - u[5] \leq 4;$ subject to cut[3,2]: $5*X[3,2] - u[2] + u[3] \leq 4;$ subject to cut[3,4]: $5*X[3,4] + u[3] - u[4] \leq 4;$ subject to cut[3,5]: $5*X[3,5] + u[3] - u[5] \leq 4;$ subject to cut[4,2]: $5*X[4,2] - u[2] + u[4] \leq 4;$ subject to cut[4,3]: $5*X[4,3] - u[3] + u[4] \leq 4;$ subject to cut[4,5]: $5*X[4,5] + u[4] - u[5] \leq 4;$ subject to cut[5,2]: $5*X[5,2] - u[2] + u[5] \leq 4;$ subject to cut[5,3]: $5*X[5,3] - u[3] + u[5] \leq 4;$ subject to cut[5,4]: $5*X[5,4] - u[4] + u[5] \leq 4;$ **Solution:** 1-5-2-3-4-1, length = 45.**3**

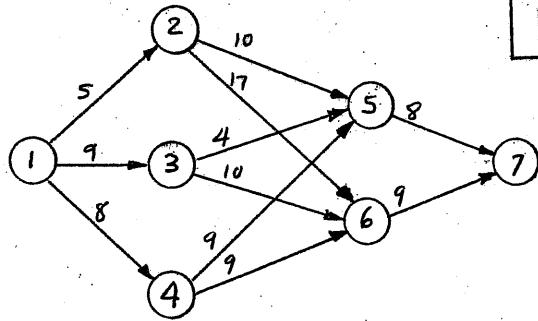
- (a) See file ampl9.3d-3a.txt.
- (b) See file ampl9.3d-3b.txt

...

CHAPTER 10

Deterministic Dynamic Programming

Set 10.1a



Stage 1:

To city	shortest distance	from city
2	5	1
3	9	1
4	8	1

Stage 2:

To city	shortest distance	from city
5	$\min\{5+10, 9+4, 8+9\} = 13$	3
6	$\min\{5+17, 9+10, 8+9\} = 17$	4

Stage 3:

To city	shortest distance	from city
7	$\min\{13+8, 17+9\} = 21$	5

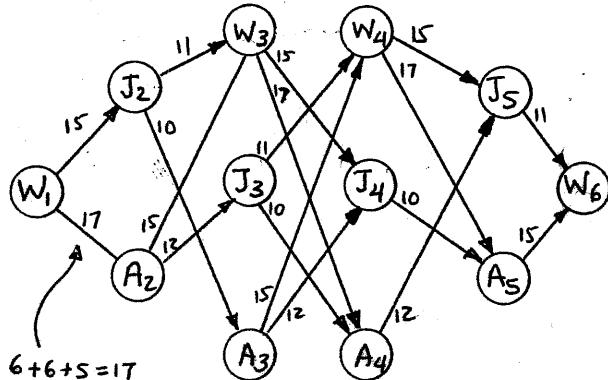
Optimum solution: shortest distance = 21 miles

Route: $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$

Define node N_i as:

$W \equiv W_1, J_1, \text{ and } A$ for Washington,
Jefferson, and Adams

$i = \text{day on which } N_i \text{ is visited}$



2

continued...

Stage 1:

TO	Longest distance	From
J_2	15	W_1
A_2	17	W_1

Stage 2:

TO	Longest distance	From
W_3	$\max\{15+11, 17+15\} = 32$	A_2
J_3	$17+12 = 29$	A_2
A_3	$15+10 = 25$	J_2

Stage 3:

TO	Longest distance	From
W_4	$\max\{29+11, 25+15\} = 40$	$J_3 \text{ or } A_3$
J_4	$\max\{32+15, 25+12\} = 47$	W_3
A_4	$\max\{32+17, 29+10\} = 49$	W_3

Stage 4:

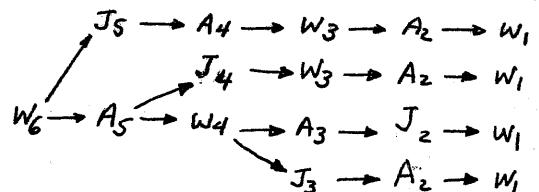
TO	Longest distance	From
J_5	$\max\{40+15, 49+12\} = 61$	A_4
A_5	$\max\{40+17, 47+10\} = 57$	$W_4 \text{ or } J_4$

Stage 5:

TO	Longest distance	From
W_6	$\max\{61+11, 57+15\} = 72$	$J_5 \text{ or } A_5$

Solution: 72 miles or 144 miles/day

To determine the optimum routes, start from Stage 5.



The routes can be summarized as:

Day	1	2	3	4	5
Route 1	W	A	W	A	J
Route 2	W	A	W	J	A
Route 3	W	J	A	W	A
Route 4	W	A	J	W	A

All routes visit J once and each of W and A twice

Set 10.2a

$$f_i(x_i) = \min_{\substack{\text{feasible} \\ (x_i, x_{i+1}) \\ \text{routes}}} \{d(x_i, x_{i+1}) + f_{i+1}(x_{i+1})\}, i=1,2$$

Stage 3:

$$f_3(x_3) = \min_{\substack{\text{feasible} \\ (x_3, x_4)}} \{d(x_3, x_4)\}$$

x_3	$d(x_3, x_4)$	Optimum Sol.	
	$x_4 = 7$	$f_3(x_3)$	x_4^*
5	8	8	7
6	9	9	7

Stage 2:

$$f_2(x_2) = \min_{\substack{\text{feasible} \\ (x_2, x_3)}} \{d(x_2, x_3) + f_3(x_3)\}$$

x_2	$d(x_2, x_3) + f_3(x_3)$		Opt. Sol.	
	$x_3 = 5$	$x_3 = 6$	$f_2(x_2)$	x_2^*
2	$10+8=18$	$17+9=26$	18	5
3	$4+8=12$	$10+9=19$	12	5
4	$9+8=17$	$9+9=18$	17	5

Stage 1:

$$f_1(x_1) = \min_{\substack{\text{feasible} \\ (x_1, x_2)}} \{d(x_1, x_2) + f_2(x_2)\}$$

x_1	$d(x_1, x_2) + f_2(x_2)$			Opt. Sol.	
	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	$f_1(x_1)$	x_1^*
1	$5+18=23$	$9+12=21$	$8+17=25$	21	3

Solution: distance = 21
route = 1 - 3 - 5 - 7

$$f_i(x_i) = \max_{\substack{\text{feasible} \\ (x_i, x_{i+1}) \\ \text{routes}}} \{d(x_i, x_{i+1}) + f_{i+1}(x_{i+1})\}, i=1, 2, 3, 4$$

$$\text{Stage 5: } f_5 = \max_{\substack{\text{feasible} \\ (x_5, x_6)}} \{d(x_5, x_6)\}$$

x_5	$d(x_5, x_6)$			Opt. Sol.	
	$x_6 = W_6$	$f_5(x_5)$	x_6^*		
J_5	11	11	W_6		
A_5	15	15	W_6		

continued...

Stage 4:

		$d(x_4, x_5) + f_5(x_5)$	Opt. Sol.
X_4	$X_5 = J_5$	$X_5 = A_5$	$f_4(x_4)$
W_4	$15+11=26$	$17+15=\textcircled{32}$	32
J_4	—	$10+15=\textcircled{25}$	25
A_4	$12+11=\textcircled{23}$	—	23

Stage 3:

		$d(x_3, x_4) + f_4(x_4)$	Opt. Sol.
X_3	$X_4 = W_4$	$X_4 = J_4$	$X_4 = A_4$
W_3	—	$15+25=\textcircled{40}$	$17+23=\textcircled{40}$
J_3	$11+22=\textcircled{43}$	—	$10+23=33$
A_3	$15+32=\textcircled{47}$	$17+25=42$	—

Stage 2:

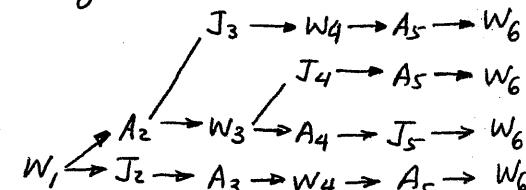
		$d(x_2, x_3) + f_3(x_3)$	Opt. Sol.
X_2	$X_3 = W_3$	$X_3 = J_3$	$X_3 = A_3$
J_2	$11+40=51$	—	$10+47=\textcircled{57}$
A_2	$15+40=\textcircled{55}$	$12+43=\textcircled{55}$	—

Stage 1:

		$d(x_1, x_2) + f_2(x_2)$	Opt. Sol.
X_1	$X_2 = J_2$	$X_2 = A_2$	$f_1(x_1)$
W_1	$15+57=\textcircled{72}$	$17+55=\textcircled{72}$	72

Solution:

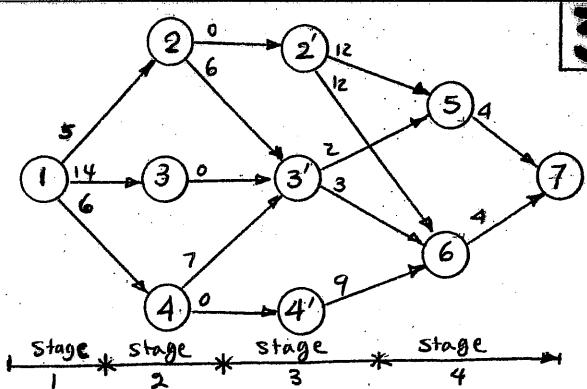
Longest distance = 72 miles



Routes:

	Day				
	1	2	3	4	5
Route 1:	W	A	J	W	A
Route 2:	W	A	W	J	A
Route 4:	W	A	W	A	J
Route 5:	W	J	A	W	A

Set 10.2a



3

Solution:

$$\text{Distance} = 17$$

Route: 1 - 2 - 3' - 5 - 7

Since ③ is the same as ③',
the optimal route is

1 - 2 - 3 - 5 - 7.

$$f_i(x_i) = \min_{\substack{\text{feasible} \\ (x_i, x_{i+1}) \\ \text{routes}}} \left\{ d(x_i, x_{i+1}) + f_{i+1}(x_{i+1}) \right\} \quad i = 1, 2, 3, 4$$

Stage 4:

x_4	$d(x_4, x_5)$	Opt. Sol.	
	$x_5 = 7$	$f_4(x_4)$	x_5^*
5	4	4	7
6	4	4	7

Stage 3:

x_3	$d(x_3, x_4) + f_4(x_4)$		Opt. Sol.	
	$x_4 = 5$	$x_4 = 6$	f_4	x_4^*
2'	$12 + 4 = 16$	$12 + 4 = 16$	16	5, 6
3'	$2 + 4 = 6$	$3 + 4 = 7$	6	5
4'	—	$9 + 4 = 13$	13	6

Stage 2:

x_2	$d(x_2, x_3) + f_3(x_3)$			Opt. Sol.	
	$x_3 = 2'$	$x_3 = 3'$	$x_3 = 4'$	f_3	x_3^*
2	$0 + 16 = 16$	$6 + 6 = 12$	—	12	3'
3	—	$0 + 6 = 6$	—	6	3'
4	—	$7 + 6 = 13$	$0 + 13 = 13$	13	3, 4

Stage 1:

x_1	$d(x_1, x_2) + f_2(x_2)$			Opt. Sol.	
	$x_2 = 2$	$x_2 = 3$	$x_2 = 4$	f_2	x_2^*
1	$5 + 12 = 17$	$14 + 6 = 20$	$6 + 13 = 19$	17	2

continued...

Set 10.3a

$$(X_1 = 3) \rightarrow m_1 = 0 \rightarrow (X_2 = 3) \rightarrow m_2 = 1 \rightarrow \\ (X_3 = 3 - 3 = 0) \rightarrow m_3 = 0.$$

Solution:

$$(m_1, m_2, m_3) = (0, 3, 0)$$

$$\text{Revenue} = 47$$

(a)

$$\text{Stage 3: } \max m_3 = \left[\frac{6}{2} \right] = 3$$

X ₃	40m ₃			Opt. Sol.	
	m ₃ = 0	m ₃ = 1	m ₃ = 2	f ₃	m ₃ *
0	0	-	-	0	0
1	0	-	-	0	0
2	0	40	-	40	1
3	0	40	-	40	1
4	0	40	80	80	2
5	0	40	80	80	2
6	0	40	80	120	3

$$\text{Stage 2: } \max m_2 = \left[\frac{6}{1} \right] = 6$$

X ₂	20m ₂ + f ₃ (X ₂ - m ₂)						Opt. Sol.		
	m ₂ = 0	1	2	3	4	5	6	f ₂	
0	0	-	-	-	-	-	0	0	
1	0	20	-	-	-	-	20	1	
2	40	20	40	-	-	-	40	0, 2	
3	40	60	40	60	-	-	60	1, 3	
4	80	60	80	60	80	-	80	2, 4, 0	
5	80	100	80	100	80	100	-	100	1, 3, 5
6	120	100	120	100	120	100	120	0, 2, 4, 6	

$$\text{Stage 1: } \max m_1 = \left[\frac{6}{4} \right] = 1$$

X ₁	70m ₁ + f ₂ (X ₁ - 4m ₁)			Opt. Sol.	
	m ₁ = 0	m ₁ = 1	f ₁	m ₁ *	
6	0 + 120	= 120	70 + 40 = 110	120	0

Optimum Solutions :

$$(m_1, m_2, m_3) = (0, 0, 3) \\ = (0, 2, 2) \\ = (0, 4, 1) \\ = (0, 6, 0)$$

$$\text{Value} = 120$$

continued...

$$(b) \text{ Stage 3: } \max m_3 = \left[\frac{4}{3} \right] = 1$$

X ₃	80m ₃		Opt. Sol.	
	m ₃ = 0	m ₃ = 1	f ₃	m ₃ *
0	0	-	-	0
1	0	-	-	0
2	0	-	-	0
3	0	80	80	1
4	0	80	80	1

$$\text{Stage 2: } \max m_2 = \left[\frac{4}{2} \right] = 2$$

X ₂	60m ₂ + f ₃ (X ₂ - 2m ₂)			Opt. Sol.	
	m ₂ = 0	m ₂ = 1	m ₂ = 2	f ₂	m ₂ *
0	0	-	-	-	0
1	0	-	-	-	0
2	0	60	-	-	60
3	80	60	-	-	80
4	80	60	120	120	2

$$\text{Stage 1: } \max m_1 = \left[\frac{4}{1} \right] = 4$$

X ₁	30m ₁ + f ₂ (X ₁ - m ₁)					Opt. Sol.	
	m ₁ = 0	1	2	3	4	f ₁	m ₁ *
4	120	90	120	90	120	120	0, 3, 4

Alternative optima:

$$(m_1, m_2, m_3) = (0, 2, 0) \quad \text{Value} \\ = (2, 1, 0) \\ = (4, 0, 0)$$

$$\text{Stage 3: } w_3 = 1, r_3 = 14, K_3 = -4$$

Dynamic Programming (Backward) Knapsack Model with Setup Cost									
Number of stages/Items	1	2	3	4	5	6	7	8	9
Stage 3	0	0	0	0	0	0	0	0	0
Optimal Solution	0	24	24	24	24	24	24	24	24
Setup	0	10	24	38	52	66	80	94	108
W ₃ m ₃	0	10	24	38	52	66	80	94	108
W ₂ m ₂	0	10	24	38	52	66	80	94	108
W ₁ m ₁	0	10	24	38	52	66	80	94	108
x ₃	0	1	2	3	4	5	6	7	8
r ₃	0	14	14	14	14	14	14	14	14
K ₃	0	-15	-15	-15	-15	-15	-15	-15	-15

$$\text{Stage 2: } w_2 = 3, r_2 = 47, K_2 = -15$$

Dynamic Programming (Backward) Knapsack Model with Setup Cost									
Number of stages/Items	1	2	3	4	5	6	7	8	9
Stage 2	0	0	0	0	0	0	0	0	0
Optimal Solution	2	24	24	24	24	24	24	24	24
Setup	0	10	24	38	52	66	80	94	108
W ₂ m ₂	0	10	24	38	52	66	80	94	108
W ₁ m ₁	0	10	24	38	52	66	80	94	108
x ₂	0	1	2	3	4	5	6	7	8
r ₂	0	47	47	47	47	47	47	47	47
K ₂	0	-15	-15	-15	-15	-15	-15	-15	-15

continued...

Set 10.3a

Stage 4:

x_4	$v(m_4)$							Opt. Sol.
	$m_4=1$	2	3	4	5	6	7	
1	10							10 1
2		20						20 2
3			30					30 3
4				40				40 4
5					50			50 5
6						60		60 6
7							70	70 7

Define the states as:

y_3 = number of width-feet assigned to corn

y_2 = number of width-feet assigned to corn and bean

y_1 = number of width-feet assigned to corn, bean, and tomato

$$y_1 = 10, \quad y_2 = 2, 3, \dots, 10, \quad y_3 = 0, 1, \dots, 7$$

$$\text{Stage 3: } f_3(y_3) = \max_{2x_3 \leq y_3} \{7x_3\}$$

Stage 3:

x_3	$v(m_3) + f_4(x_3 - m_3)$							Opt. Sol.
	$m_3=1$	2	3	4	5	6	7	
2	(50)	—	—	—	—	—	—	50 1
3	60	(70)	—	—	—	—	—	70 2
4	70	80	(90)	—	—	—	—	90 3
5	80	90	100	(110)	—	—	—	110 4
6	90	100	110	(120)	110	—	—	120 4
7	100	110	120	(130)	120	110	—	130 4
8	110	120	130	(140)	130	120	110	140 4

Stage 2: $v(m_2) + f_3(x_2 - m_2)$

x_2	$v(m_2) + f_3(x_2 - m_2)$							Opt. Sol.
	$m_2=1$	2	3	4	5	6	7	
3	(70)	—	—	—	—	—	—	70 1
4	90	(120)	—	—	—	—	—	120 2
5	110	(140)	(140)	—	—	—	—	140 2,3
6	130	(160)	(160)	150	—	—	—	160 2,3
7	140	(180)	(180)	170	150	—	—	180 2,3
8	150	190	(200)	190	170	150	—	200 3
9	160	200	(210)	(210)	190	170	150	210 3,4

Stage 1: $v(m_1) + f_2(x_1 - m_1)$

x_1	$v(m_1) + f_2(x_1 - m_1)$							Opt. Sol.
	$m_1=1$	2	3	4	5	6	7	
10	235	(250)	240	240	240	220	170	230 3

Solution: $m_1 = 2, m_2 = 3, m_3 = 4, m_4 = 1$

Total number of points = 250

x_1 = number of (2') rows of tomato

x_2 = number of (3') rows of bean

x_3 = number of (2') rows of corn

Maximize $Z = 10x_1 + 3x_2 + 7x_3$

Subject to

$$2x_1 + 3x_2 + 7x_3 \leq 10$$

$$0 \leq x_1 \leq 2, \quad x_2 \geq 1, \quad x_3 \geq 0$$

continued...

6

$$\text{Stage 1: } f_1(y_1) = \max_{\substack{2x_1 \leq y_1 \\ x_1 \leq 2}} \{10x_1 + f_2(y_1 - 2x_1)\}$$

y_1	$10x_1 + f_2(y_1 - 2x_1)$			Opt. Sol.
	$x_1 = 0$	$x_1 = 1$	$x_1 = 2$	
10	0 + 24 = 24	10 + 17 = 27	20 + 10 = 30	30 2

continued...

Set 10.3a

Solution:

$$(y_1 = 10) \rightarrow x_1 = 2 \rightarrow (y_2 = 10 - 4 = 6) \rightarrow x_2 = 1 \\ \rightarrow (y_3 = 6 - 3 = 3) \rightarrow x_3 = 1$$

Plant 2 rows of tomatoes, 1 row of beans, and 1 row of corn.

$x_j = 1$ if application j is selected,
and 0 if otherwise.

7

$$\text{maximize } Z = 78x_1 + 64x_2 + 68x_3 + 62x_4 + 85x_5$$

subject to

$$7x_1 + 4x_2 + 6x_3 + 5x_4 + 8x_5 \leq 23$$

$$x_j = (0, 1), \quad j = 1, 2, \dots, 5$$

$$\text{Stage 5: } f_5(y_5) = \max_{8x_5 \leq y_5} \{85x_5\}$$

y_5	$85x_5$		Opt. Sol.	
	$x_5 = 0$	$x_5 = 1$	f_5	x_5^*
0	0	—	0	0
1	0	—	0	0
⋮	⋮	⋮	⋮	⋮
7	0	—	0	0
8	0	85	85	1
9	0	85	85	1
⋮	⋮	⋮	⋮	⋮
23	0	85	85	1

Stage 4:

$$f_4(y_4) = \max_{5x_4 \leq y_4} \{62x_4 + f_5(y_4 - 5x_4)\}$$

y_4	$62x_4 + f_5(y_4 - 5x_4)$		Opt. Sol.	
	$x_4 = 0$	$x_4 = 1$	f_4	x_4^*
0	$0 + 0 = 0$	—	0	0
1	$0 + 0 = 0$	—	0	0
⋮	⋮	⋮	⋮	⋮
5	$0 + 0 = 0$	$62 + 0 = 62$	62	1
6	$0 + 0 = 0$	$62 + 0 = 62$	62	1
7	$0 + 0 = 0$	$62 + 0 = 62$	62	1
8	$0 + 85 = 85$	$62 + 0 = 62$	85	0
⋮	⋮	⋮	⋮	⋮
12	$0 + 85 = 85$	$62 + 0 = 62$	85	0
13	$0 + 85 = 85$	$62 + 85 = 147$	147	1
14	$0 + 85 = 85$	$62 + 85 = 147$	147	1
⋮	⋮	⋮	⋮	⋮
23	$0 + 85 = 85$	$62 + 85 = 147$	147	1

$$\text{Stage 3: } f_3(y_3) = \max_{6x_3 \leq y_3} \{68x_3 + f_4(y_3 - 6x_3)\}$$

y_3	$68x_3 + f_4(y_3 - 6x_3)$		Opt. Sol.	
	$x_3 = 0$	$x_3 = 1$	f_3	x_3^*
0	$0 + 0 = 0$	—	0	0
1	$0 + 0 = 0$	—	0	0
2	$0 + 0 = 0$	—	0	0
3	$0 + 0 = 0$	—	0	0
4	$0 + 0 = 0$	—	0	0
5	$0 + 62 = 62$	—	62	0
6	$0 + 62 = 62$	$68 + 0 = 68$	68	1
7	$0 + 62 = 62$	$68 + 0 = 68$	68	1
8	$0 + 85 = 85$	$68 + 0 = 68$	85	0
9	$0 + 85 = 85$	$68 + 0 = 68$	85	0
10	$0 + 85 = 85$	$68 + 0 = 68$	85	0
11	$0 + 85 = 85$	$68 + 62 = 130$	130	1
12	$0 + 85 = 85$	$68 + 62 = 130$	130	1
13	$0 + 147 = 147$	$68 + 62 = 130$	147	0
14	$0 + 147 = 147$	$68 + 85 = 153$	153	1
15	$0 + 147 = 147$	$68 + 85 = 153$	153	1
16	$0 + 147 = 147$	$68 + 85 = 153$	153	1
17	$0 + 147 = 147$	$68 + 85 = 153$	153	1
18	$0 + 147 = 147$	$68 + 85 = 153$	153	1
19	$0 + 147 = 147$	$68 + 147 = 215$	215	1
20	$0 + 147 = 147$	$68 + 147 = 215$	215	1
21	$0 + 147 = 147$	$68 + 147 = 215$	215	1
22	$0 + 147 = 147$	$68 + 147 = 215$	215	1
23	$0 + 147 = 147$	$68 + 147 = 215$	215	1

continued...

continued...

Set 10.3a

8

Stage 2:

$$f_2(y_2) = \max_{4x_2 \leq y_2} \{64x_2 + f_3(y_2 - 4x_2)\}$$

y_2	$x_2 = 0$	$x_2 = 1$	f_2	x_2^*	Opt. Sol.
0	$0+0=0$	-	0	0	
1	$0+0=0$	-	0	0	
2	$0+0=0$	-	0	0	
3	$0+0=0$	-	0	0	
4	$0+0=0$	$64+0=64$	64	1	
5	$0+62=62$	$64+0=64$	64	1	
6	$0+68=68$	$64+0=64$	68	0	
7	$0+68=68$	$64+0=64$	68	0	
8	$0+85=85$	$64+0=64$	85	0	
9	$0+85=85$	$64+62=126$	126	1	
10	$0+85=85$	$64+68=132$	132	1	
11	$0+130=130$	$64+68=132$	132	1	
12	$0+130=130$	$64+85=149$	149	1	
13	$0+147=147$	$64+85=149$	149	1	
14	$0+153=153$	$64+85=149$	153	0	
15	$0+153=153$	$64+130=194$	194	1	
16	$0+153=153$	$64+130=194$	194	1	
17	$0+153=153$	$64+147=211$	211	1	
18	$0+153=153$	$64+153=217$	217	1	
19	$0+215=215$	$64+153=217$	217	1	
20	$0+215=215$	$64+153=217$	217	1	
21	$0+215=215$	$64+153=217$	217	1	
22	$0+215=215$	$64+153=217$	217	1	
23	$0+215=215$	$64+215=279$	279	1	

Stage 1:

$$f_1(y_1) = \max_{7x_1 \leq y_1} \{78x_1 + f_2(y_1 - 7x_1)\}$$

y_1	$x_1 = 0$	$x_1 = 1$	f_1	x_1^*	Opt. Sol.
23	$0+279=279$	$78+194=272$	279	0	

Solution : $(y_1 = 23) \rightarrow x_1 = 0 \rightarrow (y_2 = 23) \rightarrow x_2 = 1 \rightarrow (y_3 = 23 - 4 = 19) \rightarrow x_3 = 1 \rightarrow (y_4 = 19 - 6 = 13) \rightarrow x_4 = 1 \rightarrow (y_5 = 13 - 5 = 8) \rightarrow x_5 = 1$

All but the first application are accepted.

$x_j = 1$ if precinct j is selected,
and 0 if otherwise.

Maximize $Z = 31x_1 + 26x_2 + 35x_3 + 28x_4 + 24x_5$
subject to

$$3.5x_1 + 2.5x_2 + 4x_3 + 3x_4 + 2x_5 \leq 10$$

$$x_j = (0, 1), j = 1, 2, \dots, 5$$

Stage 5: $f_5(y_5) = \max_{2x_5 \leq y_5} \{24x_5\}$
 $x_5 = (0, 1)$

y_5	$x_5 = 0$	$x_5 = 1$	f_5	x_5^*	Opt. Sol.
0	0	-	0	0	
.5	0	-	0	0	
1.	0	-	0	0	
1.5	0	-	0	0	
2.	0	24	24	1	
2.5	0	24	24	1	
↓	↓	↓	↓	↓	↓
10	0	24	24	1	

Stage 4:

$$f_4(y_4) = \max_{3x_4 \leq y_4} \{28x_4 + f_5(y_4 - 3x_4)\}$$

$$x_4 = 0, 1$$

y_4	$x_4 = 0$	$x_4 = 1$	f_4	x_4^*	Opt. Sol.
0	$0+0=0$	-	0	0	
.5	$0+0=0$	-	0	0	
1.	$0+0=0$	-	0	0	
1.5	$0+0=0$	-	0	0	
2.	$0+24=24$	-	24	0	
2.5	-	-	24	0	
3.	-	$28+0=28$	28	1	
3.5	-	$28+0=28$	28	1	
4.	-	$28+0=28$	28	1	
4.5	-	$28+0=28$	28	1	
5.	-	$28+24=52$	52	1	
↓	↓	↓	↓	↓	↓
10	$0+24=24$	$28+24=52$	52	1	

continued...

Set 10.3a

Stage 3:

$$f_3(y_3) = \max_{\substack{4x_3 \leq y_3 \\ x_3=0,1}} \{35x_3 + f_4(y_3 - 4x_3)\}$$

y_3	$35x_3 + f_4(y_3 - 4x_3)$		Opt. Sol.	
	$x_3 = 0$	$x_3 = 1$	f_3	x_3^*
0	$0+0=0$	—	0	0
.5	$0+0=0$	—	0	0
1.	$0+0=0$	—	0	0
1.5	$0+0=0$	—	0	0
2.	$0+24=24$	—	24	0
2.5	$0+24=24$	—	24	0
3.	$0+28=28$	—	28	0
3.5	$0+28=28$	—	28	0
4.	$0+28=28$	$35+0=35$	35	0
4.5	$0+28=28$	$35+0=35$	35	0
5.	$0+52=52$	$35+0=35$	52	0
5.5		$35+0=35$	52	0
6.		$35+24=59$	59	1
6.5		$35+24=59$	59	1
7.		$35+28=63$	63	1
7.5		$35+28=63$	63	1
8.		$35+28=63$	63	1
8.5		$35+28=63$	63	1
9.		$35+52=87$	87	1
9.5		$35+52=87$	87	1
10.		$35+52=87$	87	1

Stage 2:

$$f_2(y_2) = \max_{\substack{2.5x_2 \leq y_2 \\ x_2=0,1}} \{26x_2 + f_3(y_2 - 2.5x_2)\}$$

y_2	$26x_2 + f_3(y_2 - 2.5x_2)$		Opt. Sol.	
	$x_2 = 0$	$x_2 = 1$	f_2	x_2^*
0	$0+0=0$	—	0	0
.5	$0+0=0$	—	0	0
1.	$0+0=0$	—	0	0
1.5	$0+0=0$	—	0	0
2.	$0+24=24$	—	24	0
2.5	$0+24=24$	$26+0=26$	26	1
3.	$0+28=28$	$26+0=26$	28	0
3.5	$0+28=28$	$26+0=26$	28	0
4.	$0+35=35$	$26+0=26$	35	0
4.5	$0+35=35$	$26+24=50$	50	1
5.	$0+35=35$	$26+24=50$	50	1
5.5	$0+35=35$	$26+28=54$	54	1
6.	$0+59=59$	$26+28=54$	59	0
6.5	$0+59=59$	$26+35=61$	61	1
7.	$0+63=63$	$26+35=61$	63	0
7.5	$0+63=63$	$26+35=61$	63	0
8.	$0+63=63$	$26+35=61$	63	0
8.5	$0+63=63$	$26+59=85$	85	1
9.	$0+87=87$	$26+59=85$	87	0
9.5	$0+87=87$	$26+63=89$	89	1
10.	$0+87=87$	$26+63=89$	89	1

Stage 1:

$$f_1(y_1) = \max_{\substack{3.5x_1 \leq y_1 \\ x_1=0,1}} \{31x_1 + f_2(y_1 - 3.5x_1)\}$$

y_1	$31x_1 + f_2(y_1 - 3.5x_1)$	
	$x_1 = 0$	$x_1 = 1$
10	$0+89=89$	$31+61=92$

Solution:

$$(y_1 = 10) \rightarrow x_1 = 1 \rightarrow (y_2 = 10 - 3.5 = 6.5) \rightarrow x_2 = 1 \rightarrow (y_3 = 6.5 - 2.5 = 4) \rightarrow x_3 = 1 \rightarrow (y_4 = 4 - 4 = 0) \rightarrow x_4 = 0 \rightarrow (y_5 = 0) \rightarrow x_5 = 0.$$

Allocate funds to precincts 1, 2, and 3. Total population reached is $3100 + 2600 + 3500 = 9200$.

continued...

Set 10.3a

k_j = number of parallel units in component j , $j=1, 2, 3$

The problem can be written as

$$\text{maximize } r = r_1(k_1) \cdot r_2(k_2) \cdot r_3(k_3)$$

Subject to

$$c_1(k_1) + c_2(k_2) + c_3(k_3) \leq 10$$

where

$r_j(k_j)$ = reliability of component j given k_j parallel units

$c_j(k_j)$ = cost of component j given k_j parallel units

Define state as

y_j = capital assigned to components $j, j+1, \dots, 3$

Stage 3: $f_3(y_3) = \max_{k_3=1, 2, 3} \{R_3(k_3)\}$

y_3	$R_3(k_3)$			Optimal Solution	
	$k_3 = 1$	$k_3 = 2$	$k_3 = 3$	$f_3(y_3)$	k_3^*
2	.5	—	—	.5	1
3	.5	—	—	.5	1
4	.5	.7	—	.7	2
5	.5	.7	.9	.9	3
6	.5	.7	.9	.9	3

Stage 2: $f_2(y_2) = \max_{k_2=1, 2, 3} \{R_2(k_2) \cdot f_3[y_2 - c_2(k_2)]\}$

y_2	$R_2(k_2) \cdot f_3[y_2 - c_2(k_2)]$			Optimal Solution	
	$k_2 = 1$	$k_2 = 2$	$k_2 = 3$	$f_2(y_2)$	k_2^*
5	.7 × .5 = .35	—	—	.35	1
6	.7 × .5 = .35	—	—	.35	1
7	.7 × .7 = .49	.8 × .5 = .40	—	.49	1
8	.7 × .9 = .63	.8 × .5 = .40	.9 × .5 = .45	.63	1
9	.7 × .9 = .63	.8 × .7 = .56	.9 × .5 = .45	.63	1

Stage 1: $f_1(y_1) = \max_{k_1=1, 2, 3} \{R_1(k_1) \cdot f_2[y_1 - c_1(k_1)]\}$

y_1	$R_1(k_1) \cdot f_2[y_1 - c_1(k_1)]$			Optimal Solution	
	$k_1 = 1$	$k_1 = 2$	$k_1 = 3$	$f_1(y_1)$	k_1^*
6	.6 × .35 = .210	—	—	.210	1
7	.6 × .35 = .210	.8 × .35 = .280	—	.280	2
8	.6 × .49 = .294	.8 × .35 = .280	.9 × .35 = .315	.315	3
9	.6 × .63 = .378	.8 × .49 = .392	.9 × .35 = .315	.392	2
10	.6 × .63 = .378	.8 × .63 = .504	.9 × .49 = .441	.504	2

Solution: $(K_1^*, K_2^*, K_3^*) = (2, 1, 3)$

Composite $r = .504$

continued...

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State y_j = posterior of the quantity C allocated to variables $j, j+1, \dots, n$.

10

Stage n : $f_n(y_n) = \max_{x_n \leq y_n} \{x_n\}$

State	Opt. Sol.	
	f_n	x_n^*
y_n	y_n	y_n

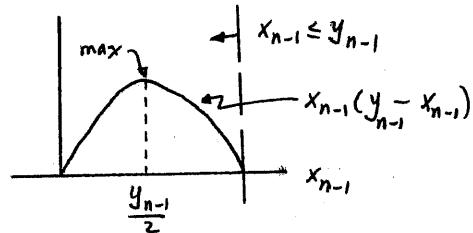
Stage $n-1$: $f_{n-1}(y_{n-1}) = \max_{x_{n-1} \leq y_{n-1}} \{x_{n-1}\}$

Given $f_n(y_n) = y_n$, then

$$f_n(y_{n-1} - x_{n-1}) = y_{n-1} - x_{n-1}$$

Thus,

$$f_{n-1}(y_{n-1}) = \max_{x_{n-1} \leq y_{n-1}} \{x_{n-1} (y_{n-1} - x_{n-1})\}$$



Opt. Sol.

State	f_{n-1}	x_{n-1}^*
y_{n-1}	$(y_{n-1}/2)^2$	$(y_{n-1}/2)$

Stage j

$$f_j(y_j) = \max_{x_j \leq y_j} \{x_j f_{j+1}(y_j - x_j)\}$$

Opt. Sol.

State	f_j	x_j^*
y_j	$\left(\frac{y_j}{n-j+1}\right)^{n-j+1}$	$\frac{y_j}{n-j+1}$

Solution: $(y_1 = q) \rightarrow x_1 = \frac{C}{n} \rightarrow (y_2 = \frac{n-1}{n}C) \rightarrow \dots \rightarrow y_j = \frac{n-j+1}{n}C \rightarrow x_j = \frac{C}{n}$

$$x_1 = x_2 = \dots = x_n = \frac{C}{n}, \quad z = \left(\frac{C}{n}\right)^n$$

Set 10.3a

$$f_n(y_n) = \min_{x_n=y_n} \{x_n^2\}$$

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$$f_i(y_i) = \min_{x_i > 0} \{x_i^2 + f_{i+1}\left(\frac{y_i}{x_i}\right)\}$$

z_j = amount of the resource allocated to variables $j, j+1, \dots, 4$.

$$\text{Stage 4: } f_4(z_4) = \max_{x_4 \leq y_4} \{x_4\}$$

Stage n:

$$f_n(y_n) = y_n^2, \quad x_n^* = y_n$$

Stage n-1:

$$f_{n-1}(y_{n-1}) = \min_{x_{n-1} > 0} \{x_{n-1}^2 + \left(\frac{y_{n-1}}{x_{n-1}}\right)^2\}$$

$$\frac{\partial \{ \cdot \}}{\partial x_{n-1}} = 2x_{n-1} - 2 \frac{y_{n-1}^2}{x_{n-1}^3} = 0$$

$$\text{or } x_{n-1}^* = \sqrt{y_{n-1}}, \quad f_{n-1}(y_{n-1}) = 2y_{n-1}$$

Stage n-2:

$$f_{n-2}(y_{n-2}) = \min_{x_{n-2} > 0} \{x_{n-2}^2 + 2\left(\frac{y_{n-2}}{x_{n-2}}\right)^2\}$$

$$\frac{\partial \{ \cdot \}}{\partial x_{n-2}} = 2x_{n-2} - 2 \frac{y_{n-2}}{x_{n-2}} = 0$$

$$\text{or } x_{n-2}^* = (y_{n-2})^{1/3}, \quad f_{n-2}(y_{n-2}) = 3y_{n-2}^{2/3}$$

Stage i:

Induction yields

$$x_i^* = y_i^{\frac{1}{n-i+1}}, \quad f_i(y_i) = (n-i+1)y_i^{\frac{2}{n-i+1}}$$

Stage 1:

$$x_1^* = C^n, \quad f_1(y_1) = n y_1^{2/n}$$

$$\text{Thus, } y_2 = \frac{y_1}{x_1} = C^{\frac{n-1}{n}} \Rightarrow x_2^* = C^{1/n}$$

$$\text{In general, } y_i = \sqrt[n]{C}$$

For proper decomposition, let

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = y_3 \text{ and } x_4 = y_3$$

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The problem is then written as

$$\text{Maximize } Z = (x_1+2)^2 + (x_2-5)^2 + x_3 x_4$$

Subject to

$$x_1 + x_2 + x_3 + x_4 \leq 5 \\ x_1, x_2, x_3, x_4 \geq 0 \text{ and integer}$$

Rearrangement of variables allows mixing multiplicative and additive decomposition

continued...

Z_4	x_4					Opt. Sol.		
	$x_4=0$	1	2	3	4	5	f_4	x_4^*
0	0	-	-	-	-	-	0	0
1	0	1	-	-	-	-	1	1
2	0	1	2	-	-	-	2	2
3	0	1	2	3	-	-	3	3
4	0	1	2	3	4	-	4	4
5	0	1	2	3	4	5	5	5

$$\text{Stage 3: } f_3(z_3) = \max_{x_3 \leq z_3} \{x_3 f_4(z_3 - x_3)\}$$

Z_3	$x_3 f_4(z_3 - x_3)$					Opt. Sol.		
	$x_3=0$	1	2	3	4	5	f_3	x_3^*
0	0x0=0	-	-	-	-	-	0	0
1	0x1=0	1x0=0	-	-	-	-	0	1
2	0x2=0	1x1=1	2x0=0	-	-	-	1	1
3	0x3=0	1x2=2	2x1=2	3x0=0	-	-	2	1,2
4	0x4=0	1x3=3	2x2=4	3x1=3	4x0=0	-	4	2
5	0x5=0	1x4=4	2x3=6	3x2=6	4x1=4	5x0=0	6	2,3

$$\text{Stage 2: } f_2(z_2) = \max_{x_2 \leq z_2} \{(x_2-5)^2 + f_3(z_2 - x_2)\}$$

Z_2	$(x_2-5)^2 + f_3(z_2 - x_2)$					Opt. Sol.		
	$x_2=0$	1	2	3	4	5	f_2	x_2^*
0	25+0=25	-	-	-	-	-	25	0
1	25+6=25	16+0=16	-	-	-	-	25	0
2	25+1=26	16+0=16	9+0=9	-	-	-	26	0
3	25+2=27	16+1=17	9+0=9	4+0=4	-	-	27	0
4	25+4=29	16+2=18	9+1=10	4+0=4	1+0=0	-	29	0
5	25+6=31	16+4=20	9+2=11	4+1=5	1+0=0	0+0=0	31	0

$$\text{Stage 1: } f_1(z_1) = \max_{x_1 \leq z_1} \{(x_1+2)^2 + f_2(z_1 - x_1)\}$$

Z_1	$(x_1+2)^2 + f_2(z_1 - x_1)$					Opt. Sol.		
	$x_1=0$	1	2	3	4	5	f_1	x_1^*
5	4+31	9+29	16+27	25+26	36+25	49+25	74	5

$$(Z_1=5) \rightarrow x_1=5 \rightarrow (Z_2=0) \rightarrow x_2=0 \rightarrow (Z_3=0) \rightarrow x_3=0 \rightarrow (Z_4=0) \rightarrow x_4=0$$

$$\text{Optimum: } (y_1, y_2, y_3, y_4) = (5, 0, 0, 0) \\ Z = 74$$

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Define state as

 y_i = amount of the resource allocated to variable $i, i+1, \dots$, and n

$$g_n(y_n) = \min_{x_3=y_3} \{ f_3(y_3) \}$$

$$g_i(y_i) = \min_{0 \leq x_i \leq y_i} \{ \max [f_i(x_i), g_{i+1}(y_i - x_i)] \}$$

$$\text{Stage 3: } g_3(y_3) = \min_{x_3=y_3} \{ x_3 - 2 \}$$

State	$g_3(y_3)$	x_3^*
y_3	$y_3 - 2$	y_3

$$\text{Stage 2: } \min_{0 \leq x_2 \leq y_2} \{ \max [5x_2 + 3, (y_2 - x_2 - 2)] \}$$

State	$g_2(y_2)$	x_2^*
$y_2 < 5$	0	3
$y_2 \geq 5$	$\frac{x_2 - 5}{6}$	$\frac{5}{6}x_2 - \frac{7}{6}$

$$\text{Stage 1: } g_1(y_1) = \min_{x_1 \leq y_1} \{ \max [x_1 + 5, g_2(y_1 - x_1)] \}$$

State	$g_1(y_1)$	x_1^*
$y_1 \leq \frac{37}{5}$	0	5
$y_1 > \frac{37}{5}$	$\frac{5y_1 - 37}{11}$	$\frac{5y_1 + 18}{11}$

$$(y_1 = 10) \rightarrow x_1 = \frac{50 - 37}{11} = \frac{13}{11} \rightarrow$$

$$(y_2 = \frac{97}{11}) \rightarrow x_2 = \frac{97/11 - 5}{6} = \frac{7}{11} \rightarrow$$

$$(y_3 = \frac{90}{11}) \rightarrow x_3 = \frac{90}{11}$$

$$g_1(10) = \frac{5 \times 10 + 18}{11} = \frac{68}{11}$$

Set 10.3b

(a) Stage 5: $b_5 = 8$

			Opt. Sol.	
x_4	$x_5 = 8$	f_5	x_5^*	
6	$0 + 4 + 2(2) = 8$	8	8	
7	$0 + 4 + 2(1) = 6$	6	8	
8	$0 + 0 = 0$	0	8	

(b) Stage 5: $b_5 = 2$

			Opt. Sol.	
x_4	$x_5 = 2$	f_5	x_5^*	
	$0 + 0$	0	2	

Stage 4: $b_4 = 8$

			Opt. Sol.	
x_3	$x_4 = 8$	f_4	x_4^*	
7	$0 + (4+2) + 1$	6	8	
8	$0 + 0 + 0$	0	8	

Stage 4: $b_4 = 6$

			Opt. Sol.	
x_3	$x_4 = 6$	$x_4 = 7$	$x_4 = 8$	f_4
3	$0 + (4+6) + 8$	$3 + (4+8) + 6$	$6 + (4+10) + 0$	18 6
4	$0 + (4+4) + 8$	$3 + (4+6) + 6$	$6 + (4+8) + 0$	16 6
5	$0 + (4+2) + 8$	$3 + (4+4) + 6$	$6 + (4+6) + 0$	14 6
6	$0 + 0 + 8$	$3 + (4+2) + 6$	$6 + (4+4) + 0$	8 6
7	$0 + 0 + 8$	$3 + 0 + 6$	$6 + (4+2) + 0$	8 6
8	$0 + 0 + 8$	$3 + 0 + 6$	$6 + 0 + 0$	6 8

Stage 3: $b_3 = 3$

			Opt. Sol.	
x_2	$x_3 = 3$	$x_3 = 4$	$x_3 = 5$	$x_3 = 6$
5	$0 + 0$	$3 + 0$	$6 + 0$	$9 + 4$
	$+ 18$	$+ 16$	$+ 14$	$+ 2 + 8$
				$+ 4 + 8$
				$+ 6 + 6$
				18 3
6	$0 + 0$	$3 + 0$	$6 + 0$	$9 + 0$
	$+ 18$	$+ 16$	$+ 14$	$+ 8$
				$+ 2 + 8$
				$+ 4 + 6$
				17 6
7	$0 + 0$	$3 + 0$	$6 + 0$	$9 + 0$
	$+ 18$	$+ 16$	$+ 14$	$+ 8$
				$+ 8$
				$+ 2 + 6$
				17 6
8	$0 + 0$	$3 + 0$	$6 + 0$	$9 + 0$
	$+ 18$	$+ 16$	$+ 14$	$+ 8$
				$+ 8$
				$+ 6$
				17 6

Stage 2: $b_2 = 5$

			Opt. Sol.	
x_1	$x_2 = 5$	$x_2 = 6$	$x_2 = 7$	$x_2 = 8$
6	$0 + 0 + 18$	$3 + 0 + 17$	$6 + 4 + 2 + 17$	$9 + 4 + 4 + 17$
				18 5
7	$0 + 0 + 18$	$3 + 0 + 17$	$6 + 0 + 17$	$9 + 4 + 2 + 17$
				18 5
8	$0 + 0 + 18$	$3 + 0 + 17$	$6 + 0 + 17$	$9 + 0 + 17$
				18 5

Stage 1: $b_1 = 6$

			Opt. Sol.	
x_0	$x_1 = 6$	$x_1 = 7$	$x_1 = 8$	x_1^*
0	$0 + (4 + 12)$	$3 + (4 + 14)$	$6 + (4 + 16)$	34 6
	$+ 18$	$+ 18$	$+ 18$	

Week i	b_i	x_i
1	6	6
2	5	5
3	3	3
4	6	6
5	8	8

continued...

Stage 4: $b_4 = 8$

			Opt. Sol.	
x_2	$x_3 = 7$	$x_3 = 8$	f_3	x_3^*
4	$0 + 4 + 6 + 6$	$3 + 4 + 8 + 0$	15	8
5	$0 + 4 + 4 + 6$	$3 + 4 + 6 + 0$	13	8
6	$0 + 4 + 2 + 6$	$3 + 4 + 4 + 0$	11	8
7	$0 + 0 + 6$	$3 + 4 + 2 + 0$	6	7
8	$0 + 0 + 6$	$3 + 0 + 0$	6	7

Stage 1: $b_1 = 6$

			Opt. Sol.	
x_0	$x_1 = 8$	f_1	x_1^*	
0	$0 + (4 + 2 \times 8) + 15$	35	8	

Optimum solution:

Week i	b_i	x_i	
1	8	8	Hire 8
2	4	7	Fire 1
3	7	7	—
4	8	8	Hire 1
5	2	2	Fire 6

Alternative optimum:

Week i	b_i	x_i	
1	8	8	Hire 8
2	4	4	Fire 4
3	7	8	Hire 4
4	8	8	—
5	2	2	Fire 6

Set 10.3b

Let

$C_3(x_{i-1} - x_i) = 100(x_{i-1} - x_i)$
be the severance cost of $x_{i-1} - x_i$ laborers, $x_{i-1} > x_i$.

$$f_i(x_i) = \min_{x_i \geq b_i} \{C_1(x_i - b_i) + C_2(x_i - x_{i-1}) + C_3(x_{i-1} - x_i) + f_{i+1}(x_i)\}$$

$$i = 1, 2, \dots, n$$

Stage 5 ($b_5 = 6$):

	$C_1(x_1 - 6) + C_2(x_1 - x_0) + C_3(x_0 - x_1)$	Optimum solution	
x_0	$x_1 = 6$	$f_0(x_0)$	x_0^*
4	$3(0) + 4 + 2(2) + 0 = 8$	8	6
5	$3(0) + 4 + 2(1) + 0 = 6$	6	6
6	$3(0) + 0 + 0 = 0$	0	6

Stage 4 ($b_4 = 4$):

	$C_1(x_1 - 4) + C_2(x_1 - x_0) + C_3(x_0 - x_1) + f_0(x_0)$	Optimum solution			
x_0	$x_1 = 4$	$x_1 = 5$	$x_1 = 6$	$f_0(x_0)$	x_0^*
8	$3(0) + 0 + 4 + 8 = 12$	$3(1) + 0 + 3 + 6 = 12$	$3(2) + 0 + 2 + 0 = 8$	8	6

Stage 3 ($b_3 = 8$):

	$C_1(x_1 - 8) + C_2(x_1 - x_0) + C_3(x_0 - x_1) + f_0(x_0)$	Optimum solution	
x_0	$x_1 = 8$	$f_0(x_0)$	x_0^*
7	$0 + 4 + 2(1) + 0 + 8 = 14$	14	8
8	$0 + 0 + 0 + 8 = 8$	8	8

Stage 2 ($b_2 = 7$):

	$C_1(x_1 - 7) + C_2(x_1 - x_0) + C_3(x_0 - x_1) + f_0(x_0)$	Optimum solution		
x_0	$x_1 = 7$	$x_1 = 8$	$f_0(x_0)$	x_0^*
5	$0 + 4 + 2(2) + 0 + 14 = 22$	$3(1) + 4 + 2(3) + 0 + 8 = 21$	21	8
6	$0 + 4 + 2(1) + 0 + 14 = 20$	$3(1) + 4 + 2(2) + 0 + 8 = 19$	19	8
7	$0 + 0 + 0 + 14 = 14$	$3(1) + 4 + 2(1) + 0 + 8 = 17$	14	7
8	$0 + 0 + 1 + 14 = 15$	$3(1) + 0 + 0 + 8 = 11$	11	8

Stage 1 ($b_1 = 5$):

	$C_1(x_1 - 5) + C_2(x_1 - x_0) + C_3(x_0 - x_1) + f_0(x_0)$	Optimum solution				
x_0	$x_1 = 5$	$x_1 = 6$	$x_1 = 7$	$x_1 = 8$	$f_0(x_0)$	x_0^*
0	$0 + 4 + 2(5) + 0 = 24$	$3(1) + 4 + 2(6) + 0 = 23$	$3(2) + 4 + 2(7) + 0 = 28$	$3(2) + 4 + 2(8) + 0 = 37$	35	5

The optimum solution is determined as
 $x_0 = 0 \rightarrow x_1^* = 5 \rightarrow x_2^* = 8 \rightarrow x_3^* = 8 \rightarrow x_4^* = 6 \rightarrow x_5 = 6$
The solution can be translated to the following plan:

Week i	Minimum Labor Force b_i	Actual Labor Force x_i	Decision
1	5	5	Hire 5 workers
2	7	8	Hire 3 workers
3	8	8	No change
4	4	6	Fire 2 workers
5	6	6	No change

Let

x_i = number of cars rented in week i

$C_i(x_i)$ = rental cost in week i

$$= \begin{cases} 220x_i, & \text{if } x_i \leq x_{i-1} \\ 500 + 220x_i, & \text{if } x_i > x_{i-1} \end{cases}$$

$$f_i(x_{i-1}) = \min_{x_i \geq b_i} \{C_i(x_i) + f_{i+1}(x_i)\}$$

$$i = 1, 2, 3, 4$$

continued...

3

Stage 4: $b_4 = 8$

	$x_3 = 8$	f_4	Opt. Sol.
7	$500 + 220 \times 8 = 2260$	2260	8
8	$220 \times 8 = 1760$	1760	8

Stage 3: $b_3 = 7$

	$x_3 = 7$	$x_3 = 8$	f_3	Opt. Sol.
4	$500 + 220(7) + 2260 = 4300$	$500 + 220(8) + 1760 = 4020$	4020	8
5	$500 + 220(7) + 2260 = 4300$	$500 + 220(8) + 1760 = 4020$	4020	8
6	$500 + 220(7) + 2260 = 4300$	$500 + 220(8) + 1760 = 4020$	4020	8
7	$220 \times 7 + 2260 = 3800$	$500 + 220(8) + 1760 = 4020$	3800	7
8	$220 \times 7 + 2260 = 3800$	$220 \times 8 + 1760 = 3520$	3520	8

Stage 2: $b_2 = 4$

	$x_2 = 4$	$x_2 = 5$	$x_2 = 6$	$x_2 = 7$	$x_2 = 8$	f_2	Opt. Sol.
7	$220(4) + 220(5) + 220(6) + 220(7) + 220(8) + 4900 = 5120$	$220(4) + 220(5) + 220(6) + 220(7) + 220(8) + 4900 = 5120$	$220(4) + 220(5) + 220(6) + 220(7) + 220(8) + 4900 = 5120$	$220(4) + 220(5) + 220(6) + 220(7) + 220(8) + 4900 = 5120$	$220(4) + 220(5) + 220(6) + 220(7) + 220(8) + 4900 = 5120$	4900	4
8	$220(4) + 220(5) + 220(6) + 220(7) + 220(8) + 4900 = 5120$	$220(4) + 220(5) + 220(6) + 220(7) + 220(8) + 4900 = 5120$	$220(4) + 220(5) + 220(6) + 220(7) + 220(8) + 4900 = 5120$	$220(4) + 220(5) + 220(6) + 220(7) + 220(8) + 4900 = 5120$	$220(4) + 220(5) + 220(6) + 220(7) + 220(8) + 4900 = 5120$	4900	4

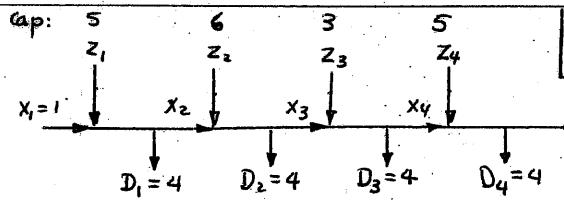
Stage 1: $b_1 = 7$

	$x_1 = 7$	$x_1 = 8$	f_1	Opt. Sol.
0	$500 + 220(7) + 4900 = 6940$	$500 + 220(8) + 4900 = 7160$	6940	4

Solution:

Week i	b_i	x_i	Rent
1	7	7	7 cars
2	4	4	Return 3
3	7	8	Rent 4
4	8	8	—

Set 10.3b



c/unit \$30 33 35 42
 h/unit \$2 3 4 -

Z_i = amount produced in period i

$$x_1 = 1, \quad 0 \leq Z_1 \leq 5$$

$$0 \leq x_2 \leq 2, \quad 0 \leq Z_2 \leq 6$$

$$0 \leq x_3 \leq 4, \quad 0 \leq Z_3 \leq 3$$

$$0 \leq x_4 \leq 3, \quad 0 \leq Z_4 \leq$$

Stage 4: $f_4(x_4) = \min_{\substack{Z_4 \geq 0 \\ Z_4+x_4=4}} \{42Z_4\}$

Stage 2:

$$f_2(x_2) = \min_{\substack{Z_2 \geq 0 \\ Z_2+x_2 \geq 4}} \{33Z_2 + 3(x_2+Z_2-4) + f_3(x_2+Z_2-4)\}$$

x_2	$Z_2=0$	1	2	3	4	5	6	Opt. Sol.
0	-	-	-	-	132 +308 =440	165 +3 +6 =270	198 +6 =232	$=438 =436$
1	-	-	-	99 +308 =407	132 +270 =405	165 +6 +9 =232	198 +9 +193	$=403 =400$
2	-	-	66 +308 =374	99 +270 =370	132 +232 =367	165 +193 +156 =366	198 +12 +156 =366	6

Stage 1:

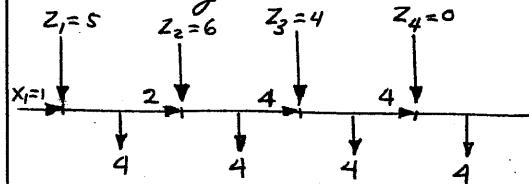
$$f_1(x_1) = \min_{\substack{Z_1 \geq 0 \\ Z_1+x_1 \geq 4}} \{30Z_1 + 2(x_1+Z_1-4) + f_2(x_1+Z_1-4)\}$$

x_1	$Z_1=0$	1	2	3	4	5	Opt. Sol.
1	-	-	-	90 +436 =526	120 +2 +400 =522	150 +4 +366 =520	520

Solution: Cost = \$520

$(x_1=1) \rightarrow Z_1=5 \rightarrow (x_2=1+5-4=2) \rightarrow Z_2=6 \rightarrow (x_3=2+6-4=4) \rightarrow Z_3=4 \rightarrow (x_4=4+4-4=4) \rightarrow Z_4=0$

Summary



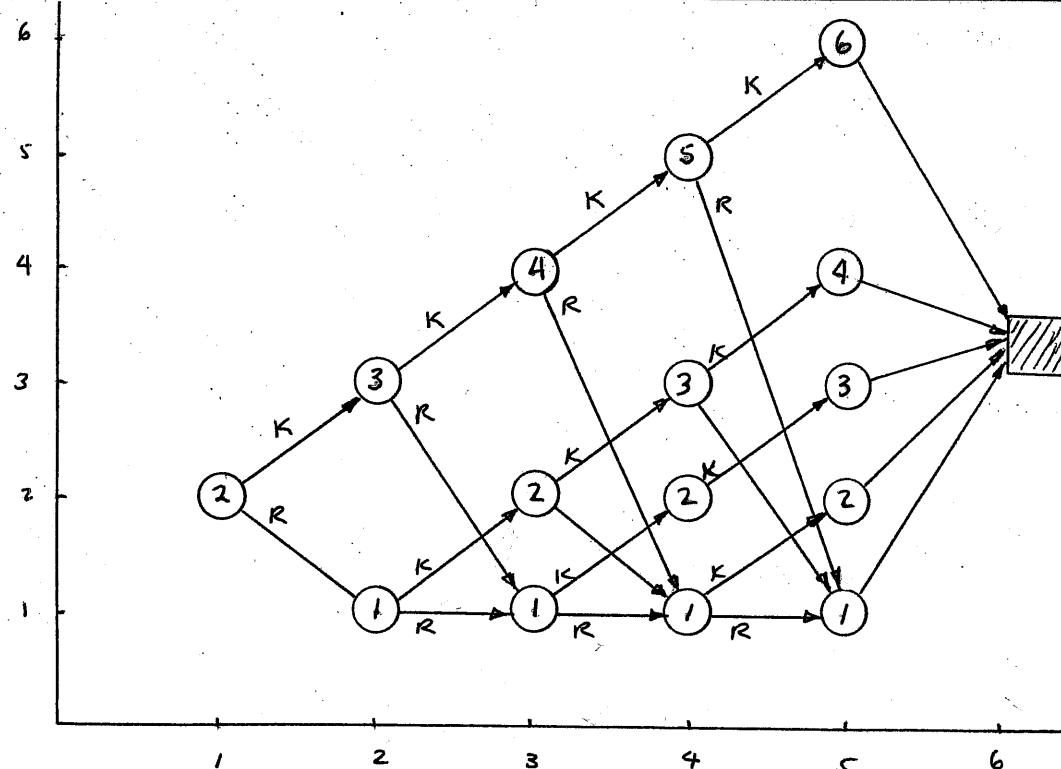
Stage 3: $f_3(x_3) = \min_{\substack{Z_3 \geq 0 \\ Z_3+x_3 \geq 4}} \{35Z_3 + 4(x_3+Z_3-4) + f_4(x_3+Z_3-4)\}$

x_3	$Z_3=0$	1	2	3	4	f_3	Z_3^*
0	-	-	-	-	140+0 +168 =308	308	4
1	-	-	-	105+0 +168 =273	140+4 +126 =270	270	4
2	-	-	70+0 +168 =238	105+4 +126 =235	140+8 +84 =232	232	4
3	-	35+0 +168 =203	70+4 +126 =200	105+8 +84 =194	140+12 +42 =193	193	3
4	0+0 +168 =168	35+4 +126 =162	70+8 +84 =162	105+12 +42 =159	140+16 +0 =156	156	4

continued...

Set 10.3c

(a)



Stage 4:

t			Opt. Sol.	
	K	R	f ₄	Dec.
1	$19 + 60 - .6 = 78.4$	$20 + 80 + 80 - 100 \cdot 2 = 79.8$	79.8	R
2	$18.5 + 50 - 1.2 = 67.3$	$20 + 60 + 80 - 100 \cdot 2 = 59.8$	67.3	K
3	$17.2 + 30 - 1.5 = 45.7$	$20 + 50 + 80 - 100 \cdot 2 = 49.8$	49.8	R
5	$14 + 10 - 1.8 = 22.2$	$20 + 10 + 80 - 100 \cdot 2 = 9.8$	22.2	K

Stage 3:

t			Opt. Sol.	
	K	R	f ₃	Dec.
1	$19 - .6 + 67.3 = 85.7$	$20 + 80 - 100 \cdot 2 + 79.8 = 79.6$	85.7	K
2	$18.5 - 1.2 + 49.8 = 67.1$	$20 + 60 - 100 \cdot 2 + 79.8 = 59.6$	67.1	K
4	$18.5 - 1.7 + 22.2 = 36$	$20 + 30 - 100 \cdot 2 + 79.8 = 29.6$	36	K

Stage 2:

t			Opt. Sol.	
	K	R	f ₂	Dec.
1	$19 - .6 + 67.1 = 85.5$	$20 + 80 - 100 \cdot 2 + 85.7 = 85.5$	85.5	K, R
3	$17.2 - 1.5 + 36 = 51.7$	$20 + 50 - 100 \cdot 2 + 85.7 = 55.5$	55.5	R

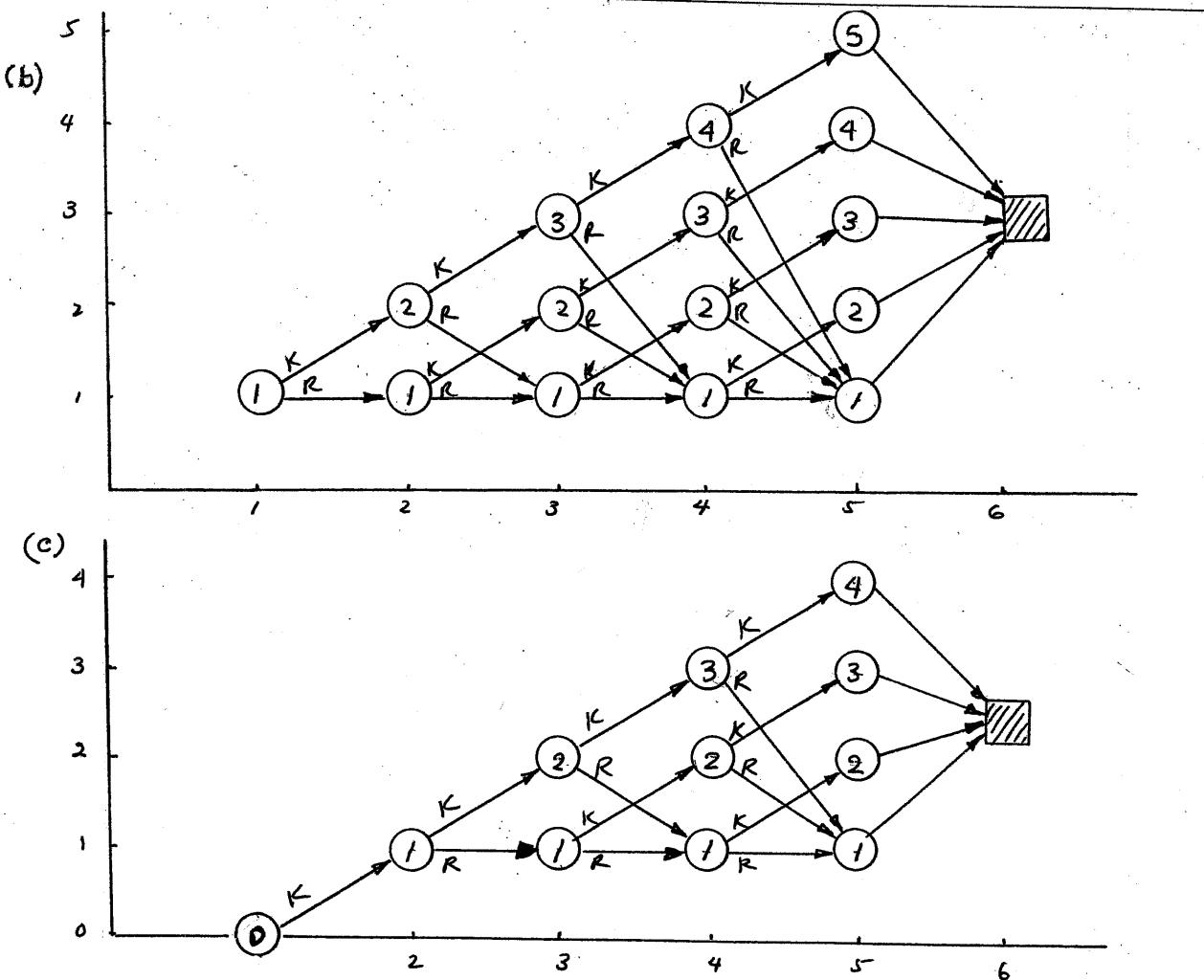
Stage 1:

t			Opt. Sol.	
	K	R	f ₁	Dec.
2	$18.5 - 1.2 + 55.5 = 72.8$	$20 + 60 - 100 \cdot 2 + 85.5 = 65.3$	72.8	K

Solution: $K \rightarrow R \rightarrow K \rightarrow K$, revenue = \$72,800

continued...

Set 10.3c



Since income from mowing is constant, it need not be taken into account.

2

$$f_4 \quad f_4(t) = \min \begin{cases} C(t) - S(t), & K \\ I(t) + c(1) - S(t), & R \end{cases}$$

$$f_i(t) = \min \begin{cases} C(t) + f_{i+1}(t+1), & K \\ I(t) + c(1) - S(t) + f_{i+1}(t), & R \end{cases}$$

where,

$C(t)$ = operating cost per year for a t -year old mower

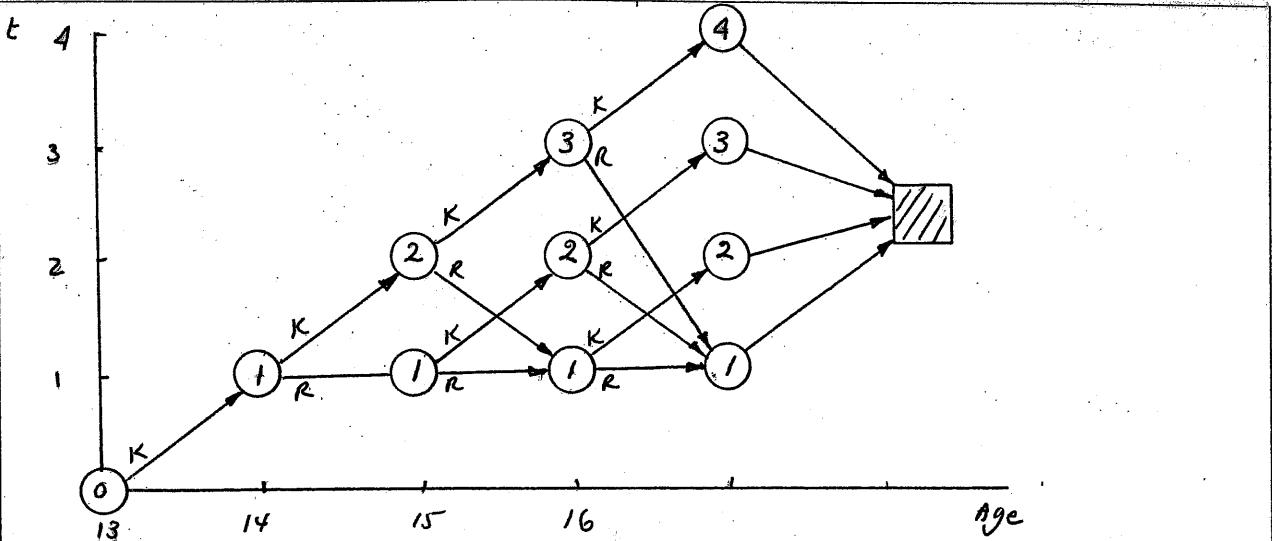
$I(t)$ = cost of a new mower after t years

$S(t)$ = salvage value of a t -year old mower

$f_i(t)$ = minimum cost for periods $i, i+1, \dots, 4$ given t -year mower.

continued...

Set 10.3c



Stage 4:

t			Opt. Sol.	
	K	R	f_4	Dec.
1	$144 - 130 = 14$	$260 + 120 - 150 - 150 = 80$	14	K
2	$168 - 110 = 58$	$260 + 120 - 135 - 150 = 95$	58	K
3	$192 - 90 = 102$	$260 + 120 - 120 - 150 = 110$	102	K

Stage 3:

t			Opt. Sol.	
	K	R	f_3	Dec.
1	$144 + 58 = 202$	$240 + 120 - 150 + 14 = 224$	202	K
2	$168 + 102 = 270$	$240 + 120 - 135 + 14 = 239$	239	R

Stage 2:

t			Opt. Sol.	
	K	R	f_2	Dec
1	$144 + 239 = 383$	$220 + 120 - 150 + 202 = 392$	383	K

Stage 1: The only option available at the start is K. Cost = $144 + 338 = 458$

Solution: $K \rightarrow K \rightarrow R \rightarrow K$, total cost = \$ 458

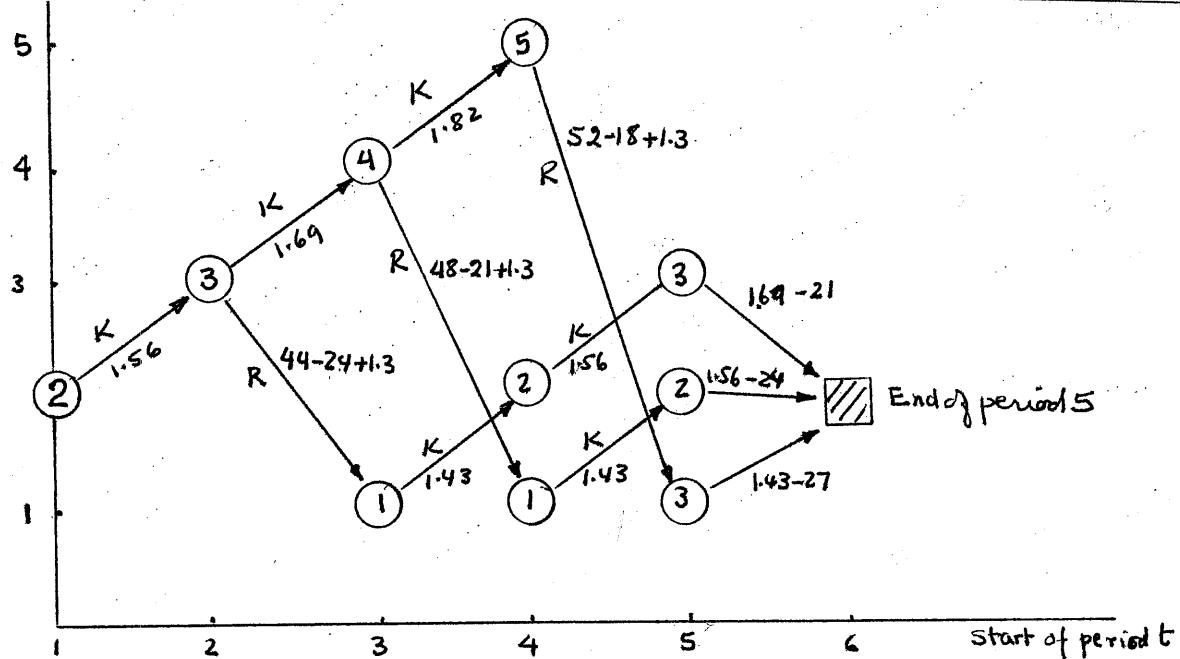
$$f_i(t) = \min \begin{cases} c(t) + f_{i+1}(t+1), & K \\ I(t) + c(1) - s(t) + f_{i+1}(1), & R \end{cases} \quad (2 \leq t \leq 5)$$

$$f_s(t) = \min \begin{cases} c(t) - s(t), & K \\ I(t) + c(1) - s(t), & R \end{cases}$$

3

continued...

Set 10.3c



Stage 5: (Start of year 5)

t			Optimum	
	K	R	f_5	Dec.
1	$1.43 - 27 = -25.57$	—	-25.57	K
2	$1.56 - 24 = -22.44$	—	-22.44	K
3	$1.69 - 21 = -19.31$	—	-19.31	K

Stage 4 (Start of year 4):

t			Optimum	
	K	R	f_4	Dec.
1	$1.43 + (-22.44) = -21.51$	—	-21.51	K
2	$1.56 + (-19.31) = -17.75$	—	-17.75	K
5	—	$52 - 18 + 1.3 + (-25.57) = 9.73$	9.73	R

Stage 3 (Start of year 3):

t			Optimum	
	K	R	f_3	Dec.
1	$1.43 + (-17.75) = -16.32$	—	-16.32	K
4	$1.82 + (9.73) = 11.55$	$48 - 21 + 1.3 + (-21.51) = 6.79$	6.79	R

Stage 2 (Start of year 2):

t			Optimum	
	K	R	f_2	Dec.
3	$1.69 + 6.79 = 8.48$	$44 - 24 + 1.3 - 16.32 = 4.98$	4.98	R

Stage 1 (Start of year 1): Keep is the only option. Cost = $1.56 + 4.98 = 6.54$

Solution:

$K \rightarrow R \rightarrow K \rightarrow K \rightarrow K$. Cost = \$ 6540

Set 10.3c

(a)

$$f_N(T_N) = \max_{T_N \leq N} \begin{cases} N - T_N^2 + N - (T_N + 1), & K \\ (N^2 - 0) + N - (0+1) - C + N - T_N, & R \end{cases}$$

$$f_i(T_i) = \max_{T_i \leq N} \begin{cases} (N - T_i^2) + f_{i+1}(T_i + 1), & K \\ (N^2 - 0) + (N - T_i) - C + f_{i+1}(1), & R \end{cases}$$

4

For $N = 3, C = 10,$

$$f_3(T_3) = \max_{T_3 \leq 3} \begin{cases} 11 - T_3 - T_3^2, & K \\ 4 - T_3, & R \end{cases}$$

$$f_i(T_i) = \max_{T_i \leq 3} \begin{cases} 9 - T_i^2 + f_{i+1}(T_i + 1), & K \\ 2 - T_i + f_{i+1}(1), & R \end{cases}, \quad i=1,2$$

(b)

Stage 3:

T_3	Optimum		f_3	Dec ³
	K	R		
1	9	3	9	K
2	5	2	5	K
3	-1	1	1	R

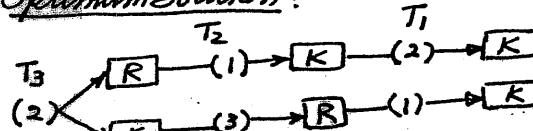
Stage 2:

T_2	Optimum		f_2	Dec ²
	K	R		
1	$8+5=13$	$1+9=10$	13	K
2	$5+1=6$	$0+9=9$	9	R
3	—	$-1+9=8$	8	R

Stage 1:

T_1	Optimum		f_1	Dec ¹
	K	R		
1	$8+9=17$	$1+13=14$	17	K
2	$5+8=13$	$0+13=13$	13	K, R
3	—	$-1+13=12$	12	R

Optimum solution:



Return = 13, (K, K, R) or (K, R, K)

5

$$f_4(T_4) = \max_{T_4 \leq 4} \begin{cases} \frac{4}{1+T_4} + 4 - (T_4 + 1), & K \\ \frac{4}{1+0} + 4 - (0+1) + 6 + (4-T_4), & R \end{cases}$$

continued...

$$= \max_{T_4 \leq 4} \begin{cases} \frac{4}{1+T_4} - T_4 + 3, & K \\ 5 - T_4, & R \end{cases}$$

$$f_i(T_i) = \max_{T_i \leq 4} \begin{cases} \frac{4}{1+T_i} + f_{i+1}(T_i + 1), & K \\ 2 - T_i + f_{i+1}(1), & R \end{cases}$$

Stage 4

T_4	K	R	f_4	Opt. Sol.
1	4.00	4	4	K, R
2	2.33	3	3	R
3	1.00	2	2	R
4	-0.20	1	1	R

Stage 3

T_3	K	R	f_3	Opt. Sol.
1	$2+3=5$	$1+4=5$	5	K, R
2	$1.33+2=3.33$	$0+4=4$	4	R
3	$1.00+1=2.00$	$-1+4=3$	3	R
4	$0.80+(-)=$	$-2+4=2$	2	R

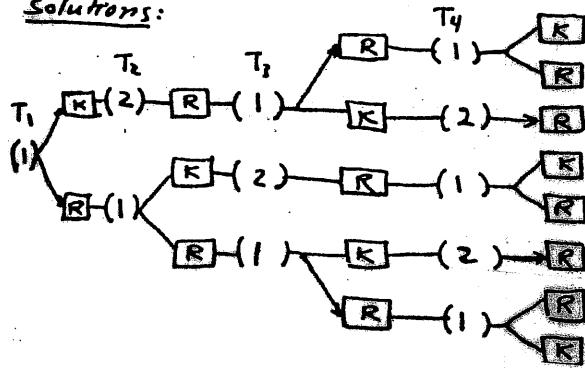
Stage 2:

T_2	K	R	f_2	Opt. Sol.
1	$2+4=6$	$1+5=6$	6	K, R
2	$1.33+3=4.33$	$0+5=5$	5	R
3	$1.00+2=3$	$-1+5=4$	4	R
4	$.80+(-)=$	$-2+5=3$	3	R

Stage 1:

T_1	K	R	f_1	Opt. Sol.
1	$2+5=7$	$1+6=7$	7	K, R
2	$1.33+4=5.33$	$0+6=6$	6	R
3	$1.00+3=4$	$-1+6=5$	5	R
4	$.80+(-)=$	$-2+6=4$	4	R

Solutions:



Set 10.3d

$$P_1 = 5, P_2 = 4, P_3 = 3, P_4 = 2$$

$$\alpha_1 = (1 + 0.085)$$

$$= 1.085$$

$$\alpha_2 = (1 + 0.08)$$

$$= 1.08$$

1	2
-0.18	0.023
0.017	0.022
0.021	0.026
0.025	0.030

q_{ij}

$$\text{Stage 4: } f_4(x_4) = \max_{0 \leq I_4 \leq x_4} \{S_4\}$$

$$S_4 = (1.085 + 0.025 - 1.08 - 0.03) I_4 + (1.08 + 0.03) x_4 \\ = 1.11 x_4$$

Opt. Sol.

State	$f_4(x_4)$	I_4^*
x_4	$1.11 x_4$	$0 \leq I_4 \leq x_4$

$$\text{Stage 3: } f_3(x_3) = \max_{0 \leq I_3 \leq x_3} \{S_3 + f_4(x_4)\}$$

$$S_3 = (1.085^2 - 1.08^2) I_3 + 1.08^2 x_3 \\ = 0.010825 I_3 + 1.1664 x_3$$

$$x_4 = P_4 + (q_{31} - q_{32}) I_3 + q_{32} x_3 \\ = 2000 + (-0.021 - 0.026) I_3 + 0.026 x_3 \\ = 2000 - 0.005 I_3 + 0.026 x_3$$

$$f_3(x_3) = \max_{0 \leq I_3 \leq x_3} \{0.010825 I_3 + 1.1664 x_3 + 1.11(2000 - 0.005 I_3 + 0.026 x_3)\}$$

$$= \max_{0 \leq I_3 \leq x_3} \{2220 + 0.005275 I_3 + 1.19526 x_3\}$$

Opt. Sol.

State	$f_3(x_3)$	I_3^*
x_3	$2220 + 1.200535 x_3$	x_3

$$\text{Stage 2: } f_2(x_2) = \max_{0 \leq I_2 \leq x_2} \{S_2 + f_3(x_3)\}$$

$$S_2 = (1.085^3 - 1.08^3) I_2 + 1.08^3 x_2 \\ = 0.0175771 I_2 + 1.259712 x_2$$

$$x_3 = 3000 + (-0.017 - 0.022) I_2 + 0.022 x_2 \\ = 3000 - 0.05 I_2 + 0.022 x_2$$

$$f_2(x_2) = \max_{0 \leq I_2 \leq x_2} \{0.0175771 I_2 + 1.259712 x_2 + \dots\}$$

$$1.259712 x_2 + 2220 + 1.200535(3000 - 0.05 I_2 + 0.022 x_2)\}$$

$$= \max_{0 \leq I_2 \leq x_2} \{5821.61 - 0.0424496 I_2 + 1.2861238 x_2\}$$

Opt.	Sol.
$f_2(x_2)$	I_2^*
5821.61	0
$1.2861238 x_2$	

$$\text{Stage 1: } f_1(x_1) = \max_{0 \leq I_1 \leq x_1} \{S_1 + f_2(x_1)\}$$

$$S_1 = (1.085^4 - 1.08^4) I_1 + 1.08^4 x_1$$

$$= 0.0253697 I_1 + 1.360489 x_1$$

$$x_2 = 4000 - 0.005 I_1 + 0.023 x_1$$

$$f_1(x_1) = \max_{0 \leq I_1 \leq x_1} \{-0.0253697 I_1 + 1.360489 x_1 + 5821.61 + 1.2861238(4000 - 0.005 I_1 + 0.023 x_1)\}$$

$$= \max_{0 \leq I_1 \leq x_1} \{10,966.11 + 0.018939 I_1 + 1.3900698 x_1\}$$

Opt.	Sol.
$f_1(x_1)$	I_1^*
10,966.11	5000
$1.3900698 x_1$	

$$x_2 = 4000 - 0.005 \times 5000 + 0.023 \times 5000 = \$4090$$

$$x_3 = 3000 - 0.005 \times 0 + 0.022 \times 4090 \cong \$3090$$

$$x_4 = 2000 - 0.005 \times 3090 + 0.026 \times 3090 = \$2065$$

Solution:

$$I_1 = x_1 = 5000 : \text{Invest \$5000 in FB}$$

$$I_2 = 0 : \text{Invest \$4090 in SB}$$

$$I_3 = 3090 : \text{Invest \$3090 in FB}$$

$$0 \leq I_4 \leq \$2065 : \text{Invest \$2065 in FB, SB, or both.}$$

continued...

Set 10.3d

x_i = cumulative amount available at the end of period i before a decision is made.

2

$$f_i(x_i) = \max_{y_i \leq x_i} \{ g(y_i) + f_{i+1}(x_i - y_i) \}$$

$$f_n(x_n) = \max_{y_n = x_n} \{ g(y_n) \}$$

where,

$$\alpha = 1.09, \quad g(y) = \sqrt{y}, \quad x_1 = 10,000\alpha$$

Stage n:

$$f_n(x_n) = \sqrt{x_n}, \quad y_n^* = x_n$$

Stage $n-1$:

$$f_{n-1}(x_{n-1}) = \max_{y_{n-1} \leq x_{n-1}} \{ \sqrt{y_{n-1}} + \sqrt{\alpha(x_{n-1} - y_{n-1})} \}$$

$$\frac{\partial f}{\partial y_{n-1}} = \frac{1}{2\sqrt{y_{n-1}}} - \frac{\alpha}{2\sqrt{\alpha(x_{n-1} - y_{n-1})}} = 0$$

$$y_{n-1}^* = \frac{x_{n-1}}{1+\alpha}$$

Because $\frac{\partial^2 f}{\partial y^2} < 0$, y_{n-1}^* is a maximum point.

$$f_{n-1}(x_{n-1}) = \sqrt{(1+\alpha)x_{n-1}}$$

Stage $n-2$:

$$f_{n-2}(x_{n-2}) = \max_{y_{n-2} \leq x_{n-2}} \{ \sqrt{y_{n-2}} + \sqrt{\alpha(1+\alpha)(x_{n-2} - y_{n-2})} \}$$

$$y_{n-2}^* = \frac{x_{n-2}}{1+\alpha+\alpha^2}$$

$$f_{n-2}(x_{n-2}) = \sqrt{(1+\alpha+\alpha^2)x_{n-2}}$$

Stage i:

By induction, we can show that

$$y_i^* = \frac{x_i}{(1+\alpha+\dots+\alpha^{n-i})}$$

continued...

$$f_i(x_i) = \sqrt{(1+\alpha+\dots+\alpha^{n-i})} x_i$$

Hence,

$$x_i = \alpha C, \quad C = \$10,000$$

$$y_1^* = \frac{\alpha C}{(1+\alpha+\dots+\alpha^{n-1})}$$

$$= \frac{C(1-\alpha)}{(1-\alpha^n)}$$

$$f_1(x_1) = \sqrt{(1+\alpha+\dots+\alpha^{n-1})} x_1$$

$$\text{Given } x_1 = \alpha C,$$

$$f_1(C) = \sqrt{\alpha(1+\alpha+\dots+\alpha^{n-1})} C$$

$$= \sqrt{\frac{\alpha(1-\alpha^n)}{(1-\alpha)}} C$$

$$x_2 = \bar{\alpha}(x_1 - y_1)$$

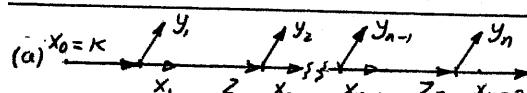
$$= \alpha^2 C \left(1 - \frac{1}{1+\alpha+\dots+\alpha^{n-1}} \right)$$

$$= \alpha^3 C \left(\frac{1-\alpha^{n-1}}{1-\alpha^n} \right)$$

$$y_2^* = \alpha^3 C \frac{(1-\alpha)}{1-\alpha^n}$$

In general, we have

$$y_i^* = \alpha^i C \left(\frac{1-\alpha}{1-\alpha^{n-i+1}} \right)$$



3

$$f_n(z_n) = \max_{y_n = z_n \leq 2K} \{ p_n y_n \}$$

$$f_i(z_i) = \max_{y_i \leq z_i \leq 2K} \{ p_i y_i + f_{i+1}(2[z_i - y_i]) \}$$

$i = 1, 2, \dots, n-1$

continued...

Set 10.3d

(b) Stage (year) 3:

Z_3	$y_3 = 0$	1	2	3	4	5	6	7	8	f_3	y_3^*	Optimum
0	0									0	0	
1		120								120	1	
2			240							240	2	
3				360						360	3	
4					480					480	4	
5						600				600	5	
6							720			720	6	
7								840		840	7	
8									960	960	8	

Stage (year) 2:

Z_2	$y_2 = 0$	1	2	3	4	f_2	y_2^*
0	$0 + 0 = 0$					0	0
1	$0 + 240 = 240$	$130 + 0 = 130$	—	—	—	240	0
2	$0 + 480 = 480$	$130 + 240 = 370$	$260 + 0 = 260$	—	—	480	0
3	$0 + 720 = 720$	$130 + 480 = 610$	$260 + 240 = 500$	$390 + 0 = 390$	—	720	0
4	$0 + 960 = 960$	$130 + 720 = 850$	$260 + 480 = 740$	$390 + 240 = 630$	$520 + 0 = 520$	960	0

Stage (year) 1:

Z_1	$y_1 = 0$	1	2	f_1	y_1^*	Optimum
0	—	—	—	—	—	
1	—	—	—	—	—	
2	$0 + 960 = 960$	$100 + 480 = 580$	$200 + 0 = 200$	960	0	

Solution:

$$Z_1 = 2 \rightarrow y_1 = 0 \rightarrow Z_2 = 4 \rightarrow y_2 = 0 \rightarrow Z_3 = 8 \rightarrow y_3 = 8$$

Revenue = \$960

Set 10.4a

(a)

$$f_2(v_2, w_2) = \max_{\substack{0 \leq 2x_2 \leq v_2 \\ 0 \leq 2x_2 \leq w_2}} \{ 14x_2 \}$$

$$= 14 \min \left\{ \frac{v_2}{2}, \frac{w_2}{2} \right\}$$

$$x_2^* = \min \left\{ \frac{v_2}{2}, \frac{w_2}{2} \right\}$$

$$f_1(v_1, w_1) = \max_{\substack{0 \leq 2x_1 \leq v_1 \\ 0 \leq 2x_1 \leq w_1}} \{ 4x_1 + f_2(v_1 - 2x_1, w_1 - 2x_1) \}$$

$$= \max \left(4x_1 + 14 \min \left\{ \frac{v_1 - 2x_1}{2}, \frac{w_1 - 2x_1}{2} \right\} \right)$$

For $v_1 = w_1 = 21$, $0 \leq x_1 \leq 3$,

$$f_1(21, 21) = \max \begin{cases} 42, & 0 \leq x_1 \leq 7/3 \\ 147 - 45x_1, & 7/3 \leq x_1 \leq 3 \end{cases}$$

$$= 42 \text{ for } 0 \leq x_1^* \leq 7/3$$

Next, $v_2 = v_1 - 2x_1 = 21 - 2x_1^*$

$$w_2 = w_1 - 7x_1^* = 21 - 7x_1^*$$

$$x_2^* = \min \left\{ \frac{21 - 2x_1^*}{7}, \frac{21 - 7x_1^*}{2} \right\}$$

$$= 3 - \frac{2}{7}x_1^*, \quad 0 \leq x_1^* \leq 7/3$$

Problem has infinite alternative solutions.

(b)

$$f_2(v_2, w_2) = \max_{\substack{0 \leq x_2 \leq v_2 \\ 0 \leq 2x_2 \leq w_2}} \{ 7x_2 \}$$

$$x_2 \text{ integer} \\ = 7 \min \left\{ \lfloor v_2 \rfloor, \lfloor \frac{w_2}{2} \rfloor \right\}$$

where $\lfloor a \rfloor = \text{largest integer } \leq a$.

$$f_1(v_1, w_1) = \max_{\substack{0 \leq 2x_1 \leq v_1 \\ 0 \leq 5x_1 \leq w_1}} \{ 8x_1 + f_2(v_1 - 2x_1, w_1 - 5x_1) \}$$

$$= \max \left\{ 8x_1 + 7 \min \left(\lfloor 8 - 2x_1 \rfloor, \lfloor \frac{15 - 5x_1}{2} \rfloor \right) \right\}$$

$$x_1 \leq \min \left\{ \lfloor \frac{v_1}{2} \rfloor, \lfloor \frac{w_1}{5} \rfloor \right\} = \min \left\{ \lfloor \frac{v_1}{2} \rfloor, \lfloor \frac{15}{5} \rfloor \right\} = 3$$

$$f_1(v_1, w_1) = \max_{x_1=0,1,2,3} \{ 8x_1 + 7 \min \left(\lfloor 8 - 2x_1 \rfloor, \lfloor \frac{15 - 5x_1}{2} \rfloor \right) \}$$

$$= \max_{x_1=0,1,2,3} \left\{ 8x_1 + 7 \left[\frac{15 - 5x_1}{2} \right] \right\}$$

$$= 49 \text{ at } x_1^* = 0$$

$$v_2 = v_1 - 2x_1 = v_1 = 8$$

$$w_2 = w_1 - 5x_1 = w_1 = 15$$

$$x_2^* = \min \left\{ \lfloor 8 \rfloor, \lfloor \frac{15}{2} \rfloor \right\} = 7$$

Optimum: $(x_1, x_2) = (0, 7)$, $Z = 49$

(c)

Forward formulation:

$$f_1(v_1, w_1) = \max_{\substack{0 \leq x_1 \leq v_1 \\ 0 \leq x_1 \leq w_1}} (7x_1^2 + 6x_1)$$

$$= \min \left\{ 7v_1^2 + 6v_1, 7w_1^2 + 6w_1 \right\}$$

$$\text{where } x_1^* = \min \{ v_1, w_1 \}$$

$$f_2(v_2, w_2) = \max_{0 \leq x_2 \leq 5} \left\{ 5x_2^2 + \min \left[7(v_2 - x_2)^2 + 6(v_2 - x_2), 7(w_2 - x_2)^2 + 6(w_2 - x_2) \right] \right\}$$

Now, $v_2 = 10$:

$$0 \leq v_1 = 10 - 2x_2 \Rightarrow 0 \leq x_2 \leq 5$$

$$0 \leq v_1 - x_1 \Rightarrow 0 \leq x_1 \leq v_1$$

$$w_2 = 9:$$

$$0 \leq w_1 = 9 + 3x_2 \Rightarrow x_2 \geq 0$$

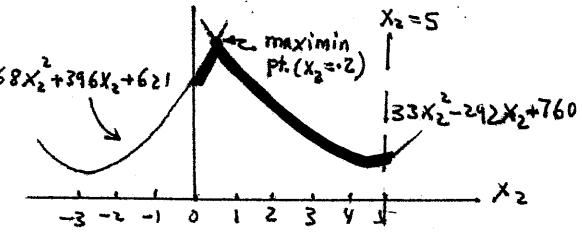
$$0 \leq w_1 - x_1 \Rightarrow 0 \leq x_1 \leq w_1$$

With $v_2 = 10$ and $w_2 = 9$, we get

$$f_2(v_2, w_2)$$

$$= \max_{0 \leq x_2 \leq 5} \left\{ 5x_2^2 + \min \left[28x_2^2 - 292x_2 + 760, 63x_2^2 + 396x_2 + 621 \right] \right\}$$

$$= \max_{0 \leq x_2 \leq 5} \left\{ \min \left[33x_2^2 - 292x_2 + 760, 68x_2^2 + 396x_2 + 621 \right] \right\}$$



continued...

continued...

Set 10.4a

Optimal solution :

$$V_2 = 10, W_2 = 9 \Rightarrow x_2^* = 2$$

$$\begin{aligned} V_1 &= 10 - 2x_2 = 9.6 \\ W_1 &= 9 + 3x_2 = 9.6 \end{aligned} \Rightarrow x_1^* = 9.6$$

$$\text{Optimal objective value} = 702.92$$

$$\text{Maximize } Z = r_1 x_1 + r_2 x_2 + \dots + r_n x_n$$

2

Subject to

$$w_1 x_1 + w_2 x_2 + \dots + w_n x_n \leq W$$

$$v_1 x_1 + v_2 x_2 + \dots + v_n x_n \leq V$$

$$x_j \geq 0 \text{ and integer}$$

where

x_j = number of units of item j

D.P. backward formulation :

Let

a_j = weight allocated to items $j, j+1, \dots$, and n

b_j = volume allocated to items $j, j+1, \dots$, and n

$f_j(a_j, b_j)$ = optimum revenue for items $j, j+1, \dots$, and n , given a_j and b_j

$$f_n(a_n, b_n) = \max_{\substack{0 \leq w_n x_n \leq a_n \\ 0 \leq v_n x_n \leq b_n}} \{r_n x_n\}$$

$$f_j(a_j, b_j) = \max_{\substack{0 \leq w_j x_j \leq a_j \\ 0 \leq v_j x_j \leq b_j}} \{r_j x_j + f_{j+1}(a_{j+1}, b_{j+1})\}$$

Order of computations

$$f_n \rightarrow f_{n-1} \rightarrow \dots \rightarrow f_1$$

$$a_1 = W$$

$$b_1 = V$$

CHAPTER 11

Deterministic Inventory Models

Set 11.3a

$$y^* = \sqrt{\frac{2KD}{h}}, t_0 = \frac{y^*}{D}, TCU(y^*) = \sqrt{2KDh}$$

a) $y^* = \sqrt{\frac{2 \times 100 \times 30}{.05}} = 346.4 \text{ units}$
 $t_0 = \frac{346.4}{30} = 11.55 \text{ days}$
 $TCU(y^*) = \frac{100 \times 30}{346.4} + \frac{.05 \times 346.4}{2} = \17.32

Policy: Order 346.4 units whenever inventory drops to 207.2 units
 Effective lead time = 6.91 days

b) $y^* = \sqrt{\frac{2 \times 50 \times 30}{.05}} \approx 245 \text{ units}$
 $t_0 = \frac{245}{30} = 8.16 \text{ days}$
 $L_e = 5.51 \text{ days}$
 $TCU(y^*) = \frac{50 \times 30}{245} + \frac{.05 \times 245}{2} = \12.25

Policy: Order 245 units whenever inventory drops to 165.15 units

c) $y^* = \sqrt{\frac{2 \times 100 \times 40}{.01}} = 894.4 \text{ units}$
 $t_0 = \frac{894.4}{40} = 22.36 \text{ days}$

$$L_e = 7.64 \text{ days}$$

$$TCU(y^*) = \frac{100 \times 40}{894.4} + \frac{.01 \times 894.4}{2} = \$8.94$$

Policy: Order 894.4 units whenever inventory drops to 305.57 units.

d) $y^* = \sqrt{\frac{2 \times 100 \times 20}{.04}} = 316.23 \text{ units}$
 $t_0 = \frac{316.23}{20} = 15.81 \text{ days}$

$$L_e = 14.19 \text{ days}$$

$$TCU(y^*) = \frac{100 \times 20}{316.23} + \frac{.04 \times 316.23}{2} = \$12.65$$

Policy: Order 316.23 units whenever inventory drops to 283.8 units.

$$D = 300 \text{ lb/wk}, K = \$20, h = \$.03/\text{lb/day}$$

(a) $TC/\text{wk} = \frac{KD}{y} + \frac{hy}{2}$
 $= \frac{20 \times 300}{300} + \frac{7 \times .03 \times 300}{2} = \51.50

(b) $y^* = \sqrt{\frac{2 \times 20 \times 300}{(.03 \times 7)}} = 239 \text{ lb}$

$$t_0 = \frac{239}{300/7} = .8 \text{ wk}$$

$$TC/\text{wk} = \sqrt{2 \times 20 \times 300 \times .03 \times 7} = \$50.20$$

continued...

$$L_e = 0 \text{ days}$$

Policy: Order 239 lb whenever inventory drops to zero level.

c) Cost difference = \$51.50 - \$50.20
 $= \$1.30$

2) $h = \frac{.35}{7} = \$.05/\text{unit/day}$

D = 50 units/day, K = \$20

$y^* = \sqrt{\frac{2 \times 20 \times 50}{.05}} = 200 \text{ units}$

$t_0 = \frac{200}{50} = 4 \text{ days}$

$L = 7 \text{ days}, L_e = 3 \text{ days}$

R = 3 \times 50 = 150 units

Policy: Order 200 units whenever inventory drops to 150 units.

b) Optimum number of orders = $\frac{365}{4} \approx 91 \text{ orders}$

(a) Policy 1: D = $\frac{R}{L_e} = \frac{50}{10} = 5 \text{ units/day}$

$$\text{Cost/day} = \frac{KD}{y} + \frac{hy}{2}$$

$$= \frac{20 \times 5}{150} + \frac{.02 \times 150}{2} = \$2.17$$

Policy 2: D = $\frac{75}{15} = 5 \text{ units/day}$

$$\text{Cost/day} = \frac{20 \times 5}{200} + \frac{.02 \times 200}{2} = \$2.50$$

choose policy 1.

(b) K = \$20, D = 5 units/day
 h = \$.02, L = 22 days

$y^* = \sqrt{\frac{2 \times 20 \times 5}{.02}} = 100 \text{ units}$

$t_0 = \frac{100}{5} = 20 \text{ days}$

$L_e = 22 - 20 = 2 \text{ days}$

Reorder level = $2 \times 5 = 10 \text{ units}$

Order 100 units whenever the level drops to 10 units

$$\text{Cost/day} = \frac{20 \times 5}{100} + \frac{.02 \times 100}{2} = \$2.00$$

Set 11.3a

$D = 5 \text{ units/day}$

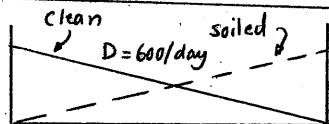
$h = \$0.10/\text{day}$

$K = \$100$

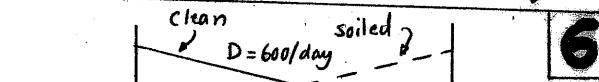
$$yt = \sqrt{\frac{2x5 \times 100}{h}} = 100 \text{ pallets}$$

$$t_0 = \frac{100}{5} = 20 \text{ days}$$

Pick up 100 pallets every 20 days.



5



$$TC/\text{day} = \frac{K}{y/D} + \frac{h_1 y}{2} + \frac{h_2 y}{2} + .6 D$$

$$= \frac{KD}{y} + (h_1 + h_2) \frac{y}{2} + .6 D$$

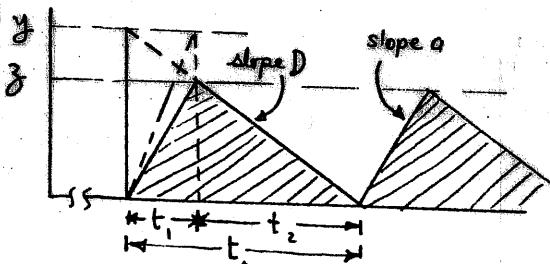
$$y^* = \sqrt{\frac{2KD}{(h_1 + h_2)}} = \sqrt{\frac{2 \times 81 \times 600}{(.01 + .02)}} = 1800 \text{ towels}$$

$$t_0 = 1800/600 = 3 \text{ days}$$

$$\text{Cost/day} = \frac{81 \times 600}{1800} + \frac{.03 \times 1800}{2} = \$54$$

Optimal policy: Pick up soiled towels and deliver an equal batch of 1800 towels every 3 days

6



a) From the geometry of the figure,

$$y = t_1(a - D) = \frac{y}{a}(a - D) = y\left(1 - \frac{D}{a}\right)$$

$$\begin{aligned} b) TCU(y) &= \frac{K + (\frac{3}{2})t_0 * h}{t_0} \\ &= \frac{KD}{y} + \frac{h}{2}\left(1 - \frac{D}{a}\right)y \end{aligned}$$

c) $\frac{\partial TCU(y)}{\partial y} = 0$ gives

$$-\frac{KD}{y^2} + \frac{h}{2}\left(1 - \frac{D}{a}\right) = 0$$

$$y^* = \sqrt{\frac{2KD}{h\left(1 - \frac{D}{a}\right)}}$$

$$(d) \lim_{a \rightarrow \infty} \sqrt{\frac{2KD}{h\left(1 - \frac{D}{a}\right)}} = \sqrt{\frac{2KD}{h}}$$

7

The basic assumption is that the employee will deposit sufficient funds in Europe to take advantage of the higher interest rate and periodically send lump sums to the US to take care of the obligations. This problem in the context of an application of the simple economic lot size formula with no shortages. The idea is that it may be more economical to hold funds longer in European banks to take advantage of their considerably higher interest rate. The cost of wiring funds from overseas ($= \$50$) may be regarded as the "setup" cost and the lost interest per dollar per year ($= .065 - .015 = .05$) can be treated as the "holding" cost. Using this information, the economic lot size formula will yield

$$\text{Deposit amount} = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \times 50 \times 12000}{.05}} = \$4899$$

$$\begin{aligned} \text{Time between deposits} &= t_0 = \frac{4899}{12000} = .408 \text{ year} \\ &= 4.9 \text{ months} \end{aligned}$$

Optimal policy: Send \$4899 ($\approx \5000) every 4.9 (≈ 5) months to the US. The first installment occurs at the start of the year

Alternative 1: Produce

$$\begin{aligned} y^* &= \sqrt{\frac{2KD}{h\left(1 - \frac{D}{a}\right)}} \\ &= \sqrt{\frac{2 \times 20 \times \frac{26000}{365}}{.02\left(1 - \frac{26000/365}{100}\right)}} = 703.7 \text{ units} \end{aligned}$$

Total cost / day

$$\begin{aligned} &= \frac{KD}{y^*} + \frac{h}{2}\left(1 - \frac{D}{a}\right)y^* \\ &= \frac{200 \times \frac{2600}{365}}{703.7} + \frac{.02}{2}\left(1 - \frac{26000}{100 \times 365}\right) \times 703.7 \end{aligned}$$

$$= \$4.05 \text{ per day}$$

9

continued...

Set 11.3a

alternative 2: Buy

$$y^* = \sqrt{\frac{2KD}{h}}$$

$$= \sqrt{\frac{2 \times 15 \times \frac{26000}{365}}{0.02}}$$

$$= 326.87 \text{ units}$$

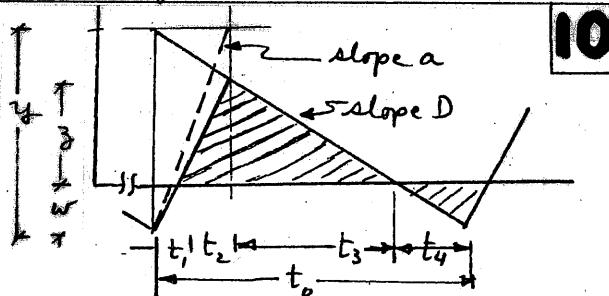
Total cost /day

$$= \frac{KD}{y^*} + \frac{h}{2} y^*$$

$$= \frac{15 \times \frac{26000}{365}}{326.87} + \frac{0.02}{2} \times 377.45$$

$$= \$6.54/\text{day}$$

The company should produce its own.



$$\bar{z} = y(1 - \frac{D}{a}) - w$$

$$TCU(y, w) = \left[K + \frac{h \{ y(1 - \frac{D}{a}) - w \}^2 + pw^2}{2D(1 - D/a)} \right] / t_0$$

$$= \frac{KD}{y} + \frac{h \{ y(1 - \frac{D}{a}) - w \}^2 + pw^2}{2y(1 - D/a)}$$

Partial derivatives = 0 give

$$-\frac{KD}{y^2} + h \left(\frac{1}{2} \left(1 - \frac{D}{a} \right) - \frac{w^2}{2y^2(1 - D/a)} \right) - \frac{pw^2}{2y^2(1 - \frac{D}{a})} = 0$$

$$h \left(\frac{w}{y(1 - \frac{D}{a})} - 1 \right) + \frac{pw}{y(1 - D/a)} = 0$$

This gives,

$$y^* = \sqrt{\frac{2KD(p+h)}{ph(1-D/a)}}, \quad w^* = \sqrt{\frac{2KDh(1-\frac{D}{a})}{p(p+h)}}$$

Set 11.3b

EOQ before quantity discount = 1800
towels per Problem 6, Set 11.2a.

$$\begin{aligned} \text{Total cost/day given batches of 1800 towels} \\ = DC_1 + \frac{KD}{y} + \frac{(h_1+h_2)y}{2} \\ = 600 \times .6 + \frac{81 \times 600}{1800} + \frac{.03 \times 1800}{2} = \$414 \end{aligned}$$

$$\begin{aligned} \text{Total cost/day given batches of 2500 towels} \\ = DC_2 + \frac{KD}{y} + \frac{(h_1+h_2)y}{2} \\ = 600 \times .5 + \frac{81 \times 600}{2500} + \frac{.03 \times 2500}{2} = \$356.94 \end{aligned}$$

Take advantage of price discount.

$$y_m = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \times 100 \times 30}{.05}} = 346.41$$

$y = 500$ units

Because $y_m < q$, we need to compute Q .

$$\begin{aligned} TCU_1(y_m) &= DC_1 + \frac{KD}{y_m} + \frac{hy_m}{2} \\ &= 30 \times 10 + \frac{100 \times 30}{346.41} + \frac{.05 \times 346.41}{2} \\ &= 317.32 \end{aligned}$$

The equation for computing Q is

$$Q^2 + \left(\frac{2(8 \times 30 - 317.32)}{.05} \right) Q + \frac{2 \times 100 \times 30}{.05} = 0$$

$$\text{or } Q^2 - 3092.82Q + 120000 = 0$$

This yields $Q = 3053.52$ units

$$\text{Because } y_m < q < Q \Rightarrow y^* = q = 500$$

$$t_o = \frac{500}{30} = 16.67 \text{ days} \Rightarrow L_C = 4.33$$

Order 500 units when inventory drops to 130.

$$y_m = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \times 50 \times 20}{.3}} = 81.65 \text{ units}$$

Because $q > y_m$, we need to compute Q .

$$\begin{aligned} TCU_1(y_m) &= 20 \times 25 + \frac{50 \times 20}{81.65} + \frac{.3 \times 81.65}{2} \\ &= 524.49 \end{aligned}$$

Q -equation:

$$Q^2 + \left(\frac{2(22.5 \times 20 - 524.49)}{.3} \right) Q + \frac{2 \times 50 \times 20}{.3} = 0$$

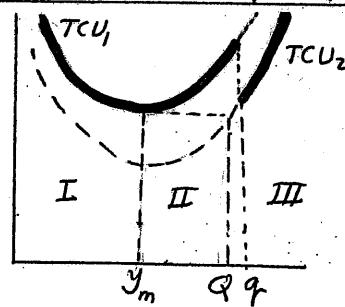
$$Q^2 - 496.63Q + 6666.67 = 0$$

continued...

Thus, $Q = 482.83$

Because $y_m < q < Q \Rightarrow y^* = 150$

Order 150 units when inventory drops to 0



4

From the preceding figure, the discount is not advantageous if

$$TCU_1(y_m) \leq TCU_2(q)$$

or

$$DC_1 + \frac{KD}{y_m} + \frac{hy_m}{2} \leq DC_2 + \frac{KD}{q} + \frac{hq}{2}$$

or

$$\begin{aligned} 20C_1 + \frac{50 \times 20}{81.65} + \frac{.3 \times 81.65}{2} \\ \leq 20C_2 + \frac{50 \times 20}{150} + \frac{.3 \times 150}{2} \end{aligned}$$

Thus, the condition reduces to

$$C_1 - C_2 \leq -23359$$

Let $d = \text{discount factor } (< 1)$.

$$\text{Then } C_2 = (1-d)C_1, \quad 0 < d < 1$$

Given $C_1 = 25$, we have

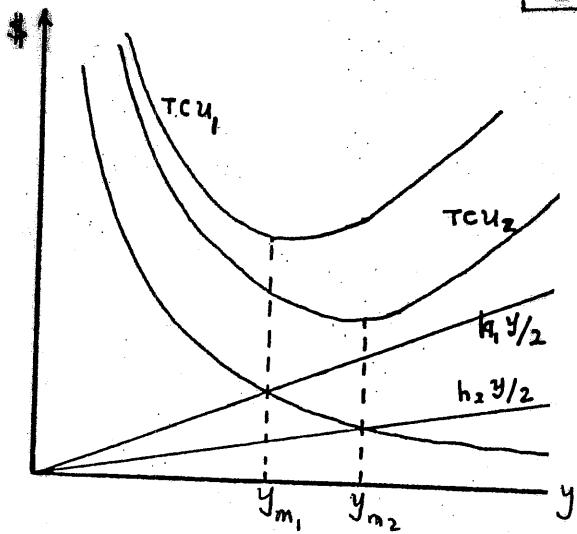
$$25d \leq -233588$$

$$\text{or } d \leq -0.09344$$

Thus, no advantage of the % discount is $\leq .9344\% (\approx 1\%)$

Set 11.3b

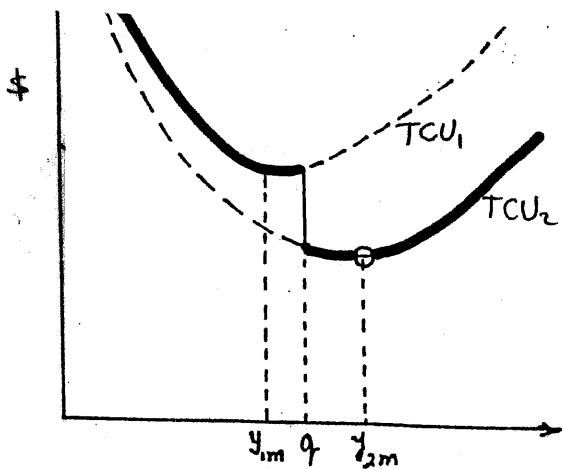
5



$$TCU_1(y) = \frac{KD}{y} + \frac{h_1 y}{2}$$

$$TCU_2(y) = \frac{KD}{y} + \frac{h_2 y}{2}$$

Case 1: $q < y_{2m}$



Solution:

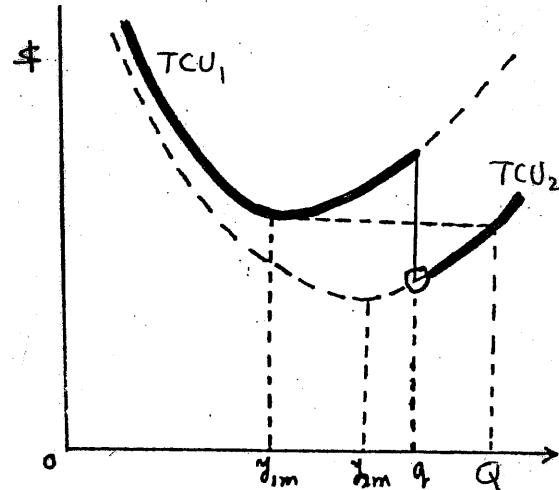
$$y^* = y_{2m}$$

$$TCU(y^*) = TCU_2(y_{2m})$$

Case 2: $y_{2m} < q \le Q$

The value of Q is determined from the equation:

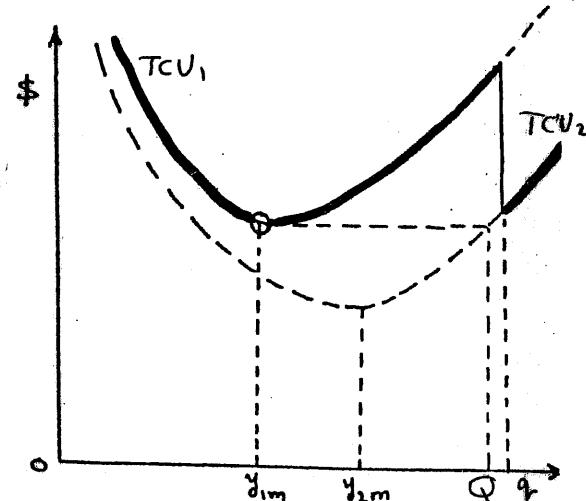
$$TCU_1(y_{im}) = TCU_2(Q)$$



Solution: $y^* = q$

$$TCU(y^*) = TCU_2(q)$$

Case 3: $y_{2m} < Q < q$



Solution: $y^* = y_{im}$, $TCU(y^*) = TCU_1(y_{im})$

$$TCU(y^*) = \begin{cases} TCU_2(y_{2m}), & q < y_{2m} \\ TCU_2(q), & y_{2m} < q \le Q \\ TCU_1(y_{im}), & y_{2m} < Q < q \end{cases}$$

continued...

Set 11.3c

See file ampl11.3c-1.txt.

AMPL model will not converge unless
 $K_i D_i / y_i$ is replaced with $K_i D_i / (y_i + \epsilon)$, where $\epsilon > 0$
and very small.

1

SOLUTION:

Total cost = 568.11

$y_1 = 4.42$

$y_2 = 6.87$

$y_3 = 4.12$

$y_4 = 7.20$

$y_5 = 5.80$

See file ampl11.3c-2.txt.

New constraint:

$$(1/2)(y_1 + y_2 + y_3) \leq 25$$

2

SOLUTION:

Total cost = 10.42

$y_1 = 10.83$

$y_2 = 16.85$

$y_3 = 22.32$

See file ampl11.3c-3.txt.

New constraint:

Average inventory for item i = $y_i / 2$.

$$(1/2)(100y_1 + 55y_2 + 100y_3) \leq 1000$$

3

SOLUTION:

Total cost = 14.31

$y_1 = 5.58$

$y_2 = 7.90$

$y_3 = 10.07$

See file ampl11.3c-4.txt.

AMPL model will not converge unless

$K_i D_i / y_i$ is replaced with $K_i D_i / (y_i + \epsilon)$, where $\epsilon > 0$
and very small.

4

New constraint:

$$365(10/y_1 + 20/y_2 + 5/y_3 + 10/y_4) \leq 150$$

SOLUTION:

Total cost = 54.71

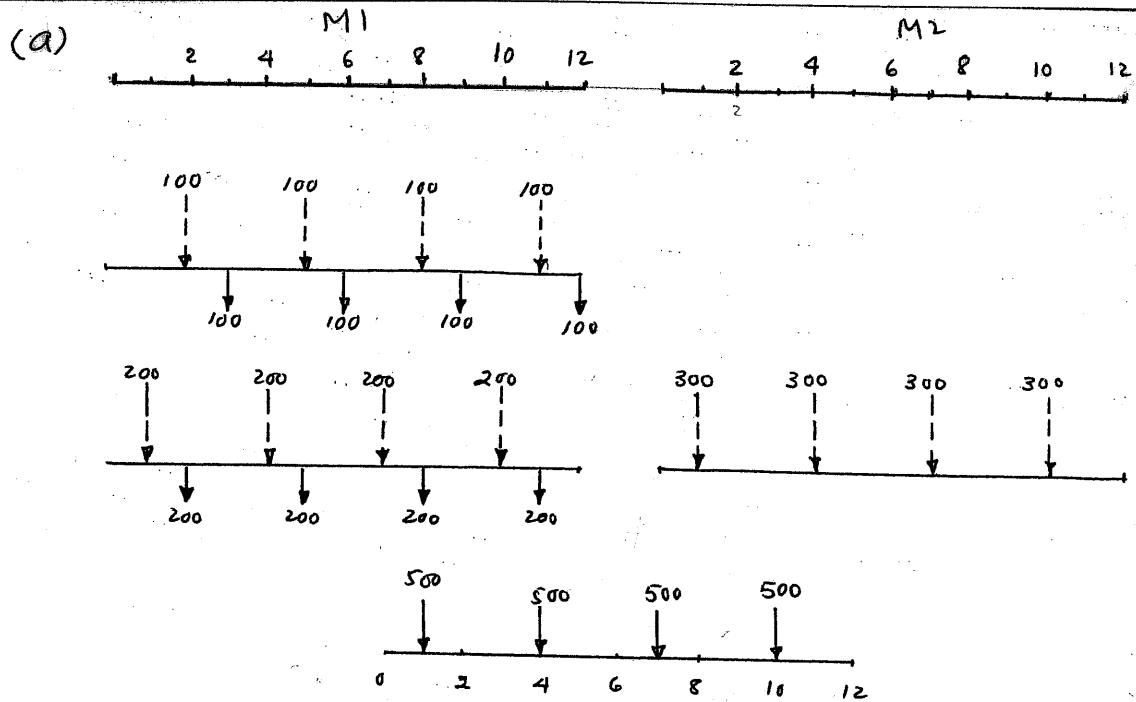
$y_1 = 155.30$

$y_2 = 118.81$

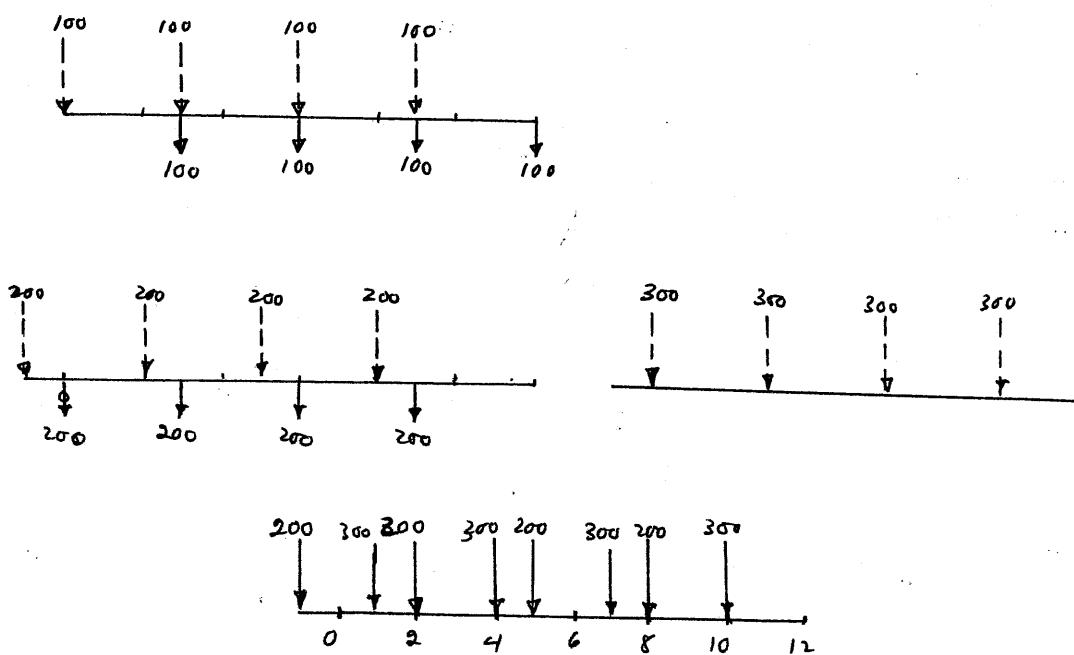
$y_3 = 74.36$

$y_4 = 90.09$

Set 11.4a



(b)



	1	2	3	4	Surplus		
R_1	90	5	5.1	5.25	5.37	0	
O_1	10	75	30	7.6	7.75	7.87	0
R_2			3	3.15	3.27	0	
O_2		100	4.5	4.65	4.77	0	
R_3			4	4.12	4.12	0	
O_3			6	6.12	6.12	0	
R_4			110	1.5	1.5	0	
O_4			50	20	20		
	100	190	210	160	20		

	1	2	3	4	5	Surplus	
R_1	100	4	4.5	5.	5.5	6.	0
O_1	50	6	6.5	7	7.5	8.	0
S	3	20	7	7.5	8	8.5	9
R_2		40	4	4.5	5.	6	0
O_2		60	6	6.5	7.	7.5	0
S		80	7	7.5	8	8.5	0
R_3			90	4	4.5	5	0
O_3			60	20	6.5	7	0
S			70	7	7.5	8	0
R_4				60	4	4.5	0
O_4				50	6	6.5	0
					7	7.5	20
						100	

(a)

	1	2	3	4	Surplus		
I	11	1	1.3	1.65	1.85	0	
II	2	1	2.3	3	2.65	2.85	0
III	5	5.3	5.65	5.85	5.85	0	
I		3	2	2.35	2.55	0	
II		11	4.35	1	4.55	0	
III		6	6.35	5	6.55	10(5)	0
I			3	2	2.2	0	
II			5	8	5.2	0	
III			7	7.2	4	0	
IV			10	10.2	10	0	
I				3	3	0	
II				8	4	0	
III				4	5	0	
IV				7	10	0	
	11	4	17	29	39		

	1	2	3	4	5	Surplus	
R_1	153	200	150	200	203	44	
O_1	70	9	0	70	6	0	
S	50	20	7	7.5	8	0	
R_2						20	20
O_2						50	50
S						100	100

Solution summary

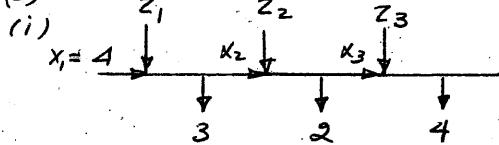
Demand

(b) Additional 10 units are produced as shown by the circled entries in period 4. The problem has alternative solutions.

Set 11.4c

(a) No, because inventory should not be held needlessly at the end of planning horizon

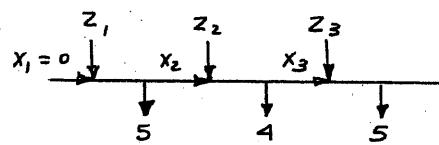
(b)



$$0 \leq z_1 \leq 5, 1 \leq z_2 \leq 5, 0 \leq z_3 \leq 4$$

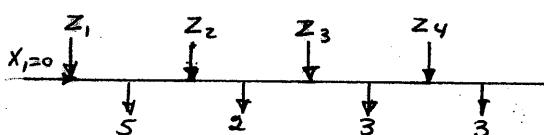
$$x_1 = 4, 1 \leq x_2 \leq 6, 0 \leq x_3 \leq 4$$

(ii)



$$5 \leq z_1 \leq 14, 0 \leq z_2 \leq 9, 0 \leq z_3 \leq 5$$

$$x_1 = 0, 0 \leq x_2 \leq 9, 0 \leq x_3 \leq 5$$



Stage 1: $f_1(x_2) = \min\{K_1 + C_1(z_1) + h_1 x_2\}$

$$z_1 = D_1 + x_2$$

where $C_i(z_i) = \begin{cases} 1z_i, & 0 \leq z_i \leq 6 \\ 2z_i, & z_i \geq 7 \end{cases}, i=1,2,\dots,4$

x_1	z_1	$K_1 = 5, h_1 = 1$	f_1	$Opt. Sol.$
0	5	10	10	5
1	12	12	12	6
2	15	15	15	7
3	18	18	18	8
4	21	21	21	9
5	24	24	24	10
6	27	27	27	11
7	30	30	30	12
8	33	33	33	13

Stage 2:

$$f_2(x_3) = \min_{0 \leq z_2 \leq D_2 + x_3} \{K_2 + C_2(z_2) + h_2 x_3 + f_1(x_3 + D_2 - z_2)\}$$

$$0 \leq z_2 \leq 8, 0 \leq x_3 \leq 6, D_2 = 2$$

$$K_2 = 7, h_2 = 1$$

x_3	z_2	f_2	$Opt. Sol.$
0	15	15	0
1	19	19	0
2	23	23	0, 4
3	27	27	5
4	31	31	6
5	35	35	6
6	39	39	7, 8

Stage 3: $0 \leq z_3 \leq 6, 0 \leq x_4 \leq 3, D_3 = 3$

x_4	z_3	f_3	$Opt. Sol.$
0	25	25	0
1	28	28	0
2	32	32	5
3	36	36	6

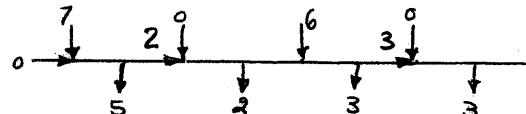
Stage 4: $0 \leq z_4 \leq 3, x_5 = 0, D_4 = 3$

x_5	z_4	f_4	$Opt. Sol.$
0	33	33	6

Solution:

$$(x_5 = 0) \rightarrow z_4 = 0 \rightarrow (x_4 = 3) \rightarrow z_3 = 6 \rightarrow (x_3 = 0) \rightarrow$$

$$z_2 = 0 \rightarrow (x_2 = 2) \rightarrow z_1 = 7$$



$$\text{Total cost} = \$33$$

continued...

Set 11.4c

$$f_i(x_2) = \min_{0 \leq z_i \leq D_i + x_2} \left\{ C_i(z_i) + K_i + h_i \left(\frac{x_i + z_i + x_2}{2} \right) \right\}$$

$$= \min_{0 \leq z_i \leq D_i + x_2} \left\{ K_i + C_i(z_i) + h_i \left(x_2 + \frac{D_i}{2} \right) \right\}$$

$$f_i(x_{i+1}) = \min_{0 \leq z_i \leq D_i + x_{i+1}} \left\{ K_i + C_i(z_i) + h_i \left(x_{i+1} + \frac{D_i}{2} \right) \right.$$

$$\left. + f_{i+1} \left(x_{i+1} + D_i - z_i \right) \right\}$$

3

Stage 1: $D_1 = 3$

								Opt. Sol.	
x_1	$z_1=2$	3	4	5	6	7	8	f_1	z_1
1	99	100	111	115	129	193	151	99	2

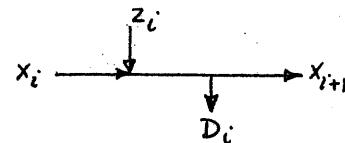
Solution:

$$(X_1=1) \rightarrow Z_1=2 \rightarrow (X_2=0) \rightarrow Z_2=3 \rightarrow$$

$$(X_3=1) \rightarrow Z_3=3$$

Cost = \$99

5



$$f_n(x_n) = \min_{z_n + x_n = D_n} \{ K_n + C_n(z_n) \}$$

4

$$f_i(x_i) = \min_{D_i \leq x_i + z_i \leq D_i + \dots + D_n} \{ K_i + C_i(z_i) + h_i(x_i + z_i - D_i) \}$$

$$+ f_{i+1}(x_{i+1} + z_i - D_i) \}$$

Stage 3: $D_3 = 4, 0 \leq x_3 \leq 4$

x_3						Opt. Sol.	
	$z_3=0$	1	2	3	4	f_3	z_3
0				56	56	4	
1			36		36	3	
2		26			26	2	
3			16		16	1	
4	0				0	0	

Stage 2: $D_2 = 2$

x_2							Opt. Sol.		
	$z_2=0$	1	2	3	4	5	6	f_2	z_2
0		83	76	89	102	109	76	3	
1	73	66	69	82	89		66	2	
2	56	56	59	62	69		56	0,1	
3	39	49	52	49			34	0	
4	32	42	39				32	0	
5	25	29					25	0	
6	12						12	0	

continued...

11-11

Set 11.4d

Period 1:

Wagner-Whitin (Forward) Dynamic Programming Inventory Model											
Number of periods, N		4		Current period		1		Optimum Solution Summary			
Period		1	2	3	4			x	f	z	
1	0	2	2	2							
2	114	114	114	114							
3	114	114	114	114							
4	0	22	90	67							
5	0	22	112	179							
6	0	142	322	466							
7	x ₁ = 0	0	1111111111111111					0	0	0	
8	x ₂ = 22	112	1111111111111111					0	179	635	179
9	x ₃ = 112	1111111111111111	454	11111111				164	22		
10	x ₄ = 179	1111111111111111	635					434	112		
11								635	179		

Stage 1: $D_1 = 150, X_1 = 50$

Opt. Sol.											f_1	Z_1
X_2	250	300	220	260	330	420	550	730	870	920	f_1	Z_1
0	700										1400	1400
100		1400									1400	1400
120			1540								1540	1200
160				1820							1820	1600
230					2310						2310	2300
320						2940					2940	9200
450							3850				3850	5500
630								5110			5110	7300
770									6090		6090	7700
920										6440	6440	9200

Period 2:

Wagner-Whitin (Forward) Dynamic Programming Inventory Model												
Number of periods, N		4		Current period		2		Optimum Solution Summary				
Period		1	2	3	4			x	f	z		
1	0	2	2	2								
2	114	114	114	114								
3	114	114	114	114								
4	0	22	90	67								
5	0	22	112	179								
6	0	142	322	466								
7	C2(2)= 0	0	150	339	472							
8	x ₁ = 0	164	152	1111111111111111				159	22	90	420	112
9	x ₂ = 90	524	1111111111111111	428	11111111			428	112	157	629	179
10	x ₃ = 157	792	1111111111111111	629				629	179			
11								635				

Stage 2: $D_2 = 100$

Opt. Sol.											f_2	Z_2
X_3	20	100	120	160	220	320	450	630	770	820	f_2	Z_2
0	1400	1400									1400	1400
20	1568		1540								1540	120
60	1880			1820							1820	160
120	2440				2310						2310	230
220	3160					2980					2980	320
350	4200						3850				3850	450
520	5694							5110			5110	630
670	6760								6090		6090	770
770	7160									6440	6440	9200

Period 3:

Wagner-Whitin (Forward) Dynamic Programming Inventory Model												
Number of periods, N		4		Current period		3		Optimum Solution Summary				
Period		1	2	3	4			x	f	z		
1	0	2	2	2								
2	114	114	114	114								
3	114	114	114	114								
4	0	22	90	67								
5	0	22	112	179								
6	0	142	322	466								
7	C3(3)= 0	0	365	498				428	0	179	635	179
8	x ₁ = 0	428	523	1111111111111111				428	0	179	635	179
9	x ₂ = 67	656	1111111111111111	724				656	0	period 3		
10								656	0			

Stage 3: $D_3 = 20$

Opt. Sol.											
X_4	20	60	130	220	350	530	670	720	f_3	Z_3	
0	1540	1580							1540	0	
90	1900		1820						1820	60	
120	2530			2240					2240	130	
200	3380				2780				2780	220	
380	4510					3560			3560	350	
510	6130						4640		4640	530	
650	7390							5480	5480	670	
780	8240								5780	5780	720

Stage 4: $D_4 = 40$

Opt. Sol.										
X_5	20	40	110	200	330	510	650	750	f_4	Z_4
0	1820	1900							1820	0
70	2310		2250						2250	110
160	2940			2700					2700	200
220	3550				3350				3350	330
370	5110					4250			4250	510
610	6010						4950		4950	650
660	6440							5200	5200	700
780	6380								6440	6440

Opt. Sol.										
X_6	25	50	70	160	290	470	610	660	f_5	Z_5
0	2250	2440							2250	0
90	2380		3160						2380	0
210	3790			4200					3790	0
400	6050				5640				5050	0
540	6030					6760			6030	0
590	6380						7160		6380	0

continued...

Set 11.4d

3

Stage 6: $D_6 = 90$

							Opt. Sol.	
x_7	26	90	220	400	540	590	f_6	Z_6
0	2880	3110					2880	0
130	4180		4600				4180	0
310	5980			6580			6980	0
450	7380				8120		7380	0
500	7880					8670	7880	0

Stage 7: $D_7 = 130$

							Opt. Sol.	
x_8	27	130	310	450	500	f_7	Z_7	
0	4180	3700		4600			3700	130
180	6160						4600	310
320	7700				5300		5300	450
370	8250					5550	5580	500

Stage 8: $D_8 = 180$

							Opt. Sol.	
x_9	28	180	320	370	f_8	Z_8		
0	4600	4720					4600	0
140	5860		5840				5840	220
190	6310			6240			6240	370

Stage 9: $D_9 = 140$

							Opt. Sol.	
x_{10}	29=0	140	190	f_9	Z_9			
0	5840	5180			5180		140	
50	6340		5380		5380		5380	190

Stage 10: $D_{10} = 50$

							Opt. Sol.	
x_{11}	$z_{10}=0$	50		f_{10}	Z_{10}			
0	5380		5780		5380		5380	0

Solution:

Period	Order Amount
1	100
2	120
3	0
4	200
5	0
6	0
7	310
8	0
9	190
10	0

Minimum cost = \$5380

Period 1:

Wagner-Whitin (Forward) Dynamic Programming Inventory Model									
Number of periods, N		5	Current period			1	Optimum Solution Summary		
Period	1	2	3	4	5		x	f	z
1	10	10	10	10	10				
2	70	60	60	60	60				
3	1	1	1	1	1				
4	50	70	100	30	60				
5									

Period 2:

Wagner-Whitin (Forward) Dynamic Programming Inventory Model									
Number of periods, N		6	Current period			2	Optimum Solution Summary		
Period	1	2	3	4	5		x	f	z
1	10	10	10	10	10				
2	70	60	60	60	60				
3	1	1	1	1	1				
4	50	70	100	30	60				
5									
6									

Period 3:

Wagner-Whitin (Forward) Dynamic Programming Inventory Model									
Number of periods, N		7	Current period			3	Optimum Solution Summary		
Period	1	2	3	4	5		x	f	z
1	10	10	10	10	10				
2	70	60	60	60	60				
3	1	1	1	1	1				
4	50	70	100	30	60				
5									
6									
7									

Period 4:

Wagner-Whitin (Forward) Dynamic Programming Inventory Model									
Number of periods, N		8	Current period			4	Optimum Solution Summary		
Period	1	2	3	4	5		x	f	z
1	10	10	10	10	10				
2	70	60	60	60	60				
3	1	1	1	1	1				
4	50	70	100	30	60				
5									
6									
7									
8									

Period 5:

Wagner-Whitin (Forward) Dynamic Programming Inventory Model									
Number of periods, N		9	Current period			5	Optimum Solution Summary		
Period	1	2	3	4	5		x	f	z
1	10	10	10	10	10				
2	70	60	60	60	60				
3	1	1	1	1	1				
4	50	70	100	30	60				
5									
6									
7									
8									
9									

continued...

Set 11.4d

Period 5:

Wagner-Whitin (Forward) Dynamic Programming Inventory Model						
Number of periods, N	6	Current period, t	5	6	7	8
Period	1	2	3	4	5	6
1	10	10	10	10	10	
2	10	10	10	10	10	
3	70	50	60	60		
4	1	1	1	1		
5	50	70	100	30	60	
6	0					
7						
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Set 11.4e

$i = 1, K_1 = \$250$:

Period, t	D_t	$TC(1, t)$	$TCU(1, t)$
1	60	250	$250/1 = 250$
2	70	$250 + 1 \times 70 = 320$	$320/2 = 160^*$
3	80	$320 + 2 \times 80 = 480$	$480/3 = 160^*$
4	90	$480 + 3 \times 90 = 750$	$750/4 = 187.50$

Produce $60 + 70 + 80 = 210$ for 1, 2, and 3

$i = 4, K_4 = \$300$:

Period, t	D_t	$TC(4, t)$	$TCU(4, t)$
4	90	300	$300/1 = 300$
5	85	$300 + 1 \times 85 = 385$	$385/2 = 192.5$
6	80	$385 + 2 \times 80 = 545$	$545/3 = 181.67$
7	75	$545 + 3 \times 75 = 770$	$770/4 = 192.5$

Produce $90 + 85 + 80 = 255$ for 4, 5, and 6

$i = 7, K_7 = \$250$:

Period, t	D_t	$TC(7, t)$	$TCU(7, t)$
7	75	250	$250/1 = 250$
8	70	$250 + 1 \times 70 = 320$	$320/2 = 160$
9	65	$320 + 2 \times 65 = 450$	$450/3 = 150$
10	60	$450 + 3 \times 60 = 630$	$630/4 = 157.50$

Produce $75 + 70 + 65 = 210$ for 7, 8, and 9

$i = 10, K_{10} = \$250$:

Period, t	D_t	$TC(10, t)$	$TCU(10, t)$
10	60	250	$250/1 = 250$
11	55	$250 + 1 \times 55 = 305$	$305/2 = 152.50$
12	50	$305 + 2 \times 50 = 405$	$405/3 = 135$

Produce $60 + 55 + 50 = 165$ for 10, 11, and 12

$i = 1, K = 200$:

t	D_t	$TC(1, t)$	$TCU(1, t)$
1	100	200	$200/1 = 200$
2	120	$200 + 1 \times 120 = 344$	$344/2 = 172$
3	50	$344 + 2 \times 50 = 464$	$464/3 = 154.67$
4	70	$464 + 3 \times 70 = 716$	$716/4 = 179$

$i = 4, K = \$200$:

t	D_t	$TC(4, t)$	$TCU(4, t)$
4	70	200	$200/1 = 200$
5	90	$200 + 1 \times 90 = 308$	$308/2 = 154$
6	105	$308 + 2 \times 105 = 560$	$560/3 = 186.67$

$i = 6, K = \$200$:

t	D_t	$TC(6, t)$	$TCU(6, t)$
6	105	200	$200/1 = 200$
7	115	$200 + 1.2 \times 115 = 338$	$338/2 = 169$
8	95	$338 + 2 \times 1.2 \times 95 = 566$	$566/3 = 188.67$

$i = 8, K = \$200$:

t	D_t	$TC(8, t)$	$TCU(8, t)$
8	95	200	$200/1 = 200$
9	80	$200 + 1.2 \times 80 = 296$	$296/2 = 148$
10	85	$296 + 2 \times 1.2 \times 85 = 500$	$500/3 = 166.67$

$i = 10, K = \$200$:

t	D_t	$TC(10, t)$	$TCU(10, t)$
10	85	200	$200/1 = 200$
11	100	$200 + 1.2 \times 100 = 320$	$320/2 = 160$
12	110	$320 + 2 \times 1.2 \times 110 = 584$	$584/3 = 194.67$

Schedule:

Produce	for periods
270	1, 2, and 3
160	4, and 5
220	6 and 7
175	8 and 9
185	10 and 11
110	12

2

Continued...

CHAPTER 12

Review of Probability Theory

Set 12.1a

	Eng'g	Non-Eng'g	Sum	
Math	150	250	400	
Non-math	29	571	600	Total = 1000
Sum	179	821		

(a) $P\{\text{Eng'g student had math}\} = \frac{150}{1000} = .15$

$P\{\text{Non-eng'g student had math}\} = \frac{250}{1000} = .25$

(b) $P\{\text{Non-eng'g had no math}\} = \frac{571}{1000} = .571$

(c) $P\{\text{Student is non-eng'g}\} = \frac{821}{1000} = .821$

Let

n = desired sample size

p_n = prob. n persons have distinct b'days

$$= \frac{365}{365} \cdot \frac{364}{365} \cdots \frac{365-n+1}{365}$$

$1-p_n$ = prob at least two persons out of n have the same b'day

Thus,

$$1-p_n > \frac{1}{2}$$

means $1-p_n$ is more likely to occur than p_n .

Now,

$$p_n < \frac{1}{2}$$

or $\frac{(365)(364)\cdots(365-n+1)}{(365)^n} < \frac{1}{2}$

A spreadsheet solution yields $n \geq 23$

12-2

1 $P\{\text{no one shares your b'day}\} = \frac{364}{365}$ 3

$P\{\text{no one among } n \text{ persons shares your b'day}\}$
 $= \left(\frac{364}{365}\right)^n$

$P\{\text{at least one person among } n \text{ shares your b'day}\}$
 $= 1 - \left(\frac{364}{365}\right)^n$

Thus, for two or more persons to share your b'day with more than 50% chance means

$$1 - \left(\frac{364}{365}\right)^n > \frac{1}{2}$$

or $n \ln\left(\frac{364}{365}\right) < \ln\left(\frac{1}{2}\right)$

or $n > \frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{364}{365}\right)} \approx 253$

The direction of the inequality has been reversed because $\ln x < 0$ for $0 < x < 1$

Set 12.1b

E = outcome of first toss

F = outcome of second toss

(a) Sum = 11:

$$(E \& F) = (5 \& 6) \text{ or } (6 \& 5)$$

$$P\{\text{sum} = 11\} = 2\left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{18}$$

(b) Sum = even value

$$\begin{aligned}(E \& F) &= (1 \& [1 \text{ or } 3 \text{ or } 5]) \text{ or} \\ &(2 \& [2 \text{ or } 4 \text{ or } 6]) \text{ or} \\ &(3 \& [1 \text{ or } 3 \text{ or } 5]) \text{ or} \\ &(4 \& [2 \text{ or } 4 \text{ or } 6]) \text{ or} \\ &(5 \& [1 \text{ or } 3 \text{ or } 5]) \text{ or} \\ &(6 \& [2 \text{ or } 4 \text{ or } 6])\end{aligned}$$

$$P\{E \& F\} = 6 \times \frac{1}{6} \left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} \right) = \frac{1}{2}$$

(c) Sum = odd value > 3

$$\begin{aligned}(E \& F) &= (1 \& [4 \text{ or } 6]) \text{ or} \\ &(2 \& [3 \text{ or } 5]) \text{ or} \\ &(3 \& [2 \text{ or } 4 \text{ or } 6]) \text{ or} \\ &(4 \& [1 \text{ or } 3 \text{ or } 5]) \text{ or} \\ &(5 \& [2 \text{ or } 4 \text{ or } 6]) \text{ or} \\ &(6 \& [1 \text{ or } 3 \text{ or } 5])\end{aligned}$$

$$P\{E \& F\} = 2 \times \frac{1}{6} \left(\frac{1}{6} + \frac{1}{6} \right) + 4 \times \frac{1}{6} \left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} \right) = \frac{4}{9}$$

$$(d) P\{(2 \text{ or } 4) \& (3 \text{ or } 5)\} = \left(2 \times \frac{1}{6}\right)^2 = \frac{1}{9}$$

$$\begin{aligned}(e) (E \& F) &= (3 \& [1 \text{ or } 2 \text{ or } 3]) \text{ or} \\ &(4 \& [1 \text{ or } 2 \text{ or } 3]) \text{ or} \\ &(5 \& [1 \text{ or } 2 \text{ or } 3]) \text{ or} \\ &(6 \& [1 \text{ or } 2 \text{ or } 3])\end{aligned}$$

$$P\{E \& F\} = 4 \times \frac{1}{6} \left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} \right) = \frac{1}{3}$$

$$(f) P\{4 \& [1 \text{ or } 3 \text{ or } 5]\} = \frac{1}{6} \left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} \right) = \frac{1}{12}$$

$$(a) (P\{2, 4, \text{ or } 6\})^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\begin{aligned}(b) P\{4 \& 6\} + P\{5 \& 5\} + P\{6 \& 4\} \\ = 3 \times \left(\frac{1}{6} \times \frac{1}{6}\right) \\ = \frac{1}{12}\end{aligned}$$

$$\begin{aligned}(c) P\{1 \& 4\} + P\{1 \& 5\} + P\{1 \& 6\} + \\ + P\{2 \& 5\} + P\{2 \& 6\} + P\{3 \& 6\} \\ + P\{4 \& 1\} + P\{5 \& 1\} + P\{6 \& 1\} \\ + P\{5 \& 2\} + P\{6 \& 2\} + P\{6 \& 3\}\end{aligned}$$

$$= 12 \times \frac{1}{6} \times \frac{1}{6} = \frac{12}{36} = \frac{1}{3}$$

Outcome	Probability
TTTH	$(\frac{1}{2})^4$
HTTH	$(\frac{1}{2})^5$
HHTTH	$2 \times (\frac{1}{2})^6$
THTTH	
HTHTTH	$4 \times (\frac{1}{2})^7$
THHTTH	
TTHTTH	
HHHTTH	

3

$$\begin{aligned}\text{Probability} &= \left(\frac{1}{2}\right)^4 \left[1 + \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 \right] \\ &= \frac{5}{32}\end{aligned}$$

p = probability Liz wins

we have

$$\begin{aligned}P\{\text{Liz, Jim, John, or Ann wins}\} \\ = p + 3p + 3p + 6p = 1\end{aligned}$$

$$\text{Thus, } p = \frac{1}{13}$$

$$(a) P\{\text{Jim wins}\} = 3p = \frac{3}{13}$$

$$\begin{aligned}(b) P\{\text{Liz or Ann wins}\} = p + 6p \\ = \frac{7}{13}\end{aligned}$$

$$(c) P\{\text{no woman wins}\}$$

$$= 1 - \frac{7}{13} = \frac{6}{13}$$

2

Set 12.1c

(a) $E = (2 \text{ or } 4)$
 $F = (1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5)$

$$P\{E|F\} = \frac{P\{EF\}}{P\{F\}} = \frac{P\{E\}}{P\{F\}} = \frac{2/6}{5/6} = 2/5$$

(b) $E = (3 \text{ or } 5)$

$$F = (1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5)$$

$$P\{E|F\} = \frac{P\{EF\}}{P\{F\}} = \frac{P\{E\}}{P\{F\}} = \frac{2/6}{5/6} = 2/5$$

Joint probabilities:

		WMS up	WMS down	Col. Sum
Door up	.6	.1	.7	
Door down	.05	.25	.3	
Row sum		.65	.35	

(a) $P\{\text{WMS up}\} = .6 + .05 = .65$

(b) $P\{\text{WMS up} | \text{Door up}\} = \frac{.6}{.7} = 6/7$

(c) $P\{\text{WMS down} | \text{Door down}\} = \frac{.25}{.3} = 5/6$

$$P\{A\} = .4 \quad P\{B\} = .25 \quad P\{AB\} = .15 \quad \boxed{3}$$

(a) $P\{B|A\} = \frac{P\{BA\}}{P\{A\}} = \frac{.15}{.4} = 3/8$

(b) $P\{A|B\} = \frac{P\{AB\}}{P\{B\}} = \frac{.15}{.25} = 3/5$

$$P\{A|B\} = \frac{P\{AB\}}{P\{B\}}$$

If $\frac{P\{AB\}}{P\{B\}} = P\{A\}$ then

$P\{AB\} = P\{A\} P\{B\}$, which shows that A and B must be independent.

$$P\{A|B\} = \frac{P\{AB\}}{P\{B\}}$$

$$= \frac{P\{B|A\} P\{A\}}{P\{B\}}$$

provided $P\{B\} > 0$.

1

(a) $P\{D\} = P\{D, A\} + P\{D, B\}$ 6

$$= P\{D|A\} P\{A\} + P\{D|B\} P\{B\}$$

$$= .1 \times .75 + .2 \times .25$$

$$= .125$$

(b) $P\{A|D\} = \frac{P\{D|A\} P\{A\}}{P\{D\}}$

$$= \frac{.1 \times .75}{.125} = .6$$

2

C = cancer

NC = no cancer

+ = test positive

$$P\{C|+\} = \frac{P\{C, +\}}{P\{+\}}$$

$$P\{+\} = P\{+, C\} + P\{+, NC\}$$

$$= P\{+|C\} P\{C\} + P\{+|NC\} P\{NC\}$$

$$= .9 \times .7 + .1 \times .3$$

$$= .66$$

Thus,

$$P\{C|+\} = \frac{P\{C, +\}}{P\{+\}} = \frac{P\{+|C\} P\{C\}}{P\{+\}}$$

$$= \frac{.9 \times .7}{.66}$$

$$\approx .9545$$

4

5

Set 12.2a

(a) $p(x) = kx$, $x = 1, 2, 3, 4, 5$

$$\sum_{x=1}^5 p(x) = k(1+2+3+4+5) = 15k = 1$$

Thus, $k = 1/15$, and

$$p(x) = \frac{x}{15}, x = 1, 2, \dots, 5$$

CDF:

$$P(x) = \sum_{y=1}^x \frac{y}{15} = \frac{x(x+1)}{30}, x = 1, 2, \dots, 5$$

(b) $P\{x=2 \text{ or } x=4\} = \frac{2+4}{15} = \frac{2}{5}$

(a) $\int_{10}^{20} \frac{k}{x^2} = 1$

$$k\left(\frac{1}{10} - \frac{1}{20}\right) = \frac{k}{20} = 1 \Rightarrow k = 20$$

$$f(x) = \frac{20}{x^2}, 10 \leq x \leq 20$$

(b) $F(x) = \int_{10}^x \frac{20}{t^2} dt$
 $= 2 - \frac{20}{x}$

(i) $P\{x > 12\} = P\{x \geq 12\}$

$$= 1 - \left(2 - \frac{20}{12}\right)$$

$$= \frac{2}{3}$$

(ii) $P\{13 \leq x \leq 15\}$

$$= P\{x \leq 15\} - P\{x \leq 13\}$$

$$= 2 - \frac{20}{15} - \left(2 - \frac{20}{13}\right)$$

$$= .205$$

1 $P\{\text{Demand} = d\} = \frac{1}{500}, 750 \leq d \leq 1250$

3

$$P\{d \geq 1100\} = 1 - P\{d \leq 1100\}$$

$$= 1 - \frac{1100 - 750}{500}$$

$$= .3$$

Set 12.3a

$$h(x) = \begin{cases} x-20, & x=21, 22, 23, 24 \\ 0, & x=10, 11, \dots, 20 \end{cases}$$

$$\begin{aligned} E\{h(x)\} &= \sum_{x=10}^{20} 0 \left(\frac{1}{15}\right) + \sum_{x=21}^{24} (x-20) \left(\frac{1}{15}\right) \\ &= \frac{2}{3} \text{ stamp} \end{aligned}$$

There is no inconsistency **2**
 because the two cases are mutually exclusive. There can be either surplus or shortage. When surplus occurs, its average value is $3\frac{2}{3}$ stamps. And when shortage occurs, its average value is $\frac{2}{3}$ stamp.

$$(a) P\{50 \leq x \leq 70\}$$

$$\begin{aligned} &= 1 - P\{35 \leq x \leq 49\} \\ &= 1 - \frac{15}{45} = \frac{2}{3} \end{aligned}$$

(b) Expected number of unsold copies

$$\begin{aligned} &= \sum_{x=35}^{49} (50-x) p(x) + \sum_{x=50}^{70} 0 p(x) \\ &= 50 \sum_{x=35}^{49} p(x) - \sum_{x=35}^{49} x p(x) \\ &= 50 \times \frac{15}{45} - \frac{1}{45} (35 + \dots + 49) \\ &= \frac{1}{45} (750 - 630) = 2.67 \end{aligned}$$

(c) Expected net profit

$$\begin{aligned} &= (50 - 2.67) \times 1 - 50 \times .5 \\ &= \$22.33 \end{aligned}$$

Set 12.3b

$$\begin{array}{cccccc} x & : & 1 & 2 & 3 & 4 & 5 \\ p(x) & : & \frac{1}{15} & \frac{2}{15} & \frac{3}{15} & \frac{4}{15} & \frac{5}{15} \end{array}$$

$$\begin{aligned} E\{x\} &= \sum_{x=1}^5 x p(x) \\ &= 1\left(\frac{1}{15}\right) + 2\left(\frac{2}{15}\right) + 3\left(\frac{3}{15}\right) + 4\left(\frac{4}{15}\right) + 5\left(\frac{5}{15}\right) \\ &= 3\frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{Var}\{x\} &= \sum_{x=1}^5 (x - \frac{11}{3})^2 p(x) \\ &= (1 - \frac{11}{3})^2 \left(\frac{1}{15}\right) + (2 - \frac{11}{3})^2 \left(\frac{2}{15}\right) + \\ &\quad (3 - \frac{11}{3})^2 \left(\frac{3}{15}\right) + (4 - \frac{11}{3})^2 \left(\frac{4}{15}\right) + \\ &\quad (5 - \frac{11}{3})^2 \left(\frac{5}{15}\right) \\ &\approx 1.556 \end{aligned}$$

$$\begin{aligned} E\{x\} &= \int_0^{20} \frac{20x}{x^2} dx \\ &= (\ln x \Big|_0^{20})(20) = 13.86 \end{aligned}$$

$$\begin{aligned} \text{Var}\{x\} &= 20 \int_0^{20} \frac{(x - 13.86)^2}{x^2} dx \\ &= 20 \left[x - 27.72 \ln x - \frac{197.10}{x} \right]_0^{20} \\ &= 7.81 \end{aligned}$$

$$(a) f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

$$\begin{aligned} E\{x\} &= \int_a^b \frac{x}{b-a} dx = \frac{x^2}{2(b-a)} \Big|_a^b \\ &= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2} \end{aligned}$$

$$\begin{aligned} (b) \int_a^b \frac{(x - \bar{x})^2}{b-a} dx &= \frac{1}{b-a} \left[\frac{x^3}{3} - \bar{x}x^2 + x\bar{x}^2 \right]_a^b \\ &= \frac{4b^2 + 4a^2 + 4ab - 3b^2 - 3a^2 - 6ab}{12} \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

$$\text{Var}\{x\} = \int_{-\infty}^{\infty} (x - E\{x\})^2 f(x) dx$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} x^2 f(x) dx - 2E\{x\} \int_{-\infty}^{\infty} x f(x) dx \\ &\quad + (E\{x\})^2 \int_{-\infty}^{\infty} f(x) dx \\ &= E\{x^2\} - 2(E\{x\})^2 - (E\{x\})^2 \\ &= E\{x^2\} - (E\{x\})^2 \end{aligned}$$

$$y = cx + d$$

$$\begin{aligned} E\{y\} &= \int (cx+d) f(x) dx \\ &= c \int x f(x) dx + d \int f(x) dx \\ &= c E\{x\} + d \end{aligned}$$

$$\begin{aligned} \text{Var}\{y\} &= E\{(cx+d)^2\} - E^2\{cx+d\} \\ &= E\{c^2 x^2 + d^2 + 2cdx\} \\ &\quad - [c E\{x\} + d]^2 \\ &= c^2 E\{x^2\} + d^2 + 2cd E\{x\} \\ &\quad - c^2 E^2\{x\} - d^2 - 2cd E\{x\} \\ &= c^2 (E\{x^2\} - E^2\{x\}) \\ &= c^2 \text{Var}\{x\} \end{aligned}$$

3

Set 12.3c

(a) $P(X_1)$

$$P(X_1, X_2) = \begin{bmatrix} 1 & .2 & 0 & .2 \\ 2 & 0 & .2 & 0 \\ 3 & .2 & 0 & .2 \end{bmatrix} \cdot \begin{bmatrix} .4 \\ .2 \\ .4 \end{bmatrix}$$

$$P(X_2) \quad .4 \quad .2 \quad .4$$

$$\begin{array}{c|ccc} X_1 & 1 & 2 & 3 \\ \hline P(X_1) & .4 & .2 & .4 \end{array}$$

$$\begin{array}{c|ccc} X_2 & 1 & 2 & 3 \\ \hline P(X_2) & .4 & .2 & .4 \end{array}$$

(b) No, because $P(X_1, X_2) \neq P(X_1)P(X_2)$

$$\begin{aligned} (c) E\{X_1 + X_2\} &= E\{X_1\} + E\{X_2\} \\ &= 2(1x.4 + 2x.2 + 3x.4) \\ &= 4 \end{aligned}$$

$$(d) \text{cov}(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2)$$

$$\begin{aligned} E(X_1 X_2) &= 1x.2 + 2x.0 + 3x.2 + 2x.0 \\ &\quad + 4x.2 + 6x.0 + 3x.2 + 6x.0 \\ &\quad + 3x.2 + 6x.0 + 9x.2 \\ &= 4.6 \end{aligned}$$

$$E\{X_1\} = 2, \quad E\{X_2\} = 2$$

$$\text{cov}(X_1, X_2) = 4.6 - 2 \times 2 = .6$$

$$(e) \text{Var}\{5X_1 - 6X_2\} = 25\text{Var}\{X_1\} + 36\text{Var}\{X_2\}$$

$$\begin{aligned} \text{Var}\{X_1\} &= \text{Var}\{X_2\} = E\{X_1^2\} - E^2\{X_1\} \\ &= 1x.4 + 4x.2 + 9x.4 - 2^2 \\ &= .8 \end{aligned}$$

$$\begin{aligned} \text{Var}\{5X_1 - 6X_2\} &= 25(.8) + 36(.8) \\ &\quad + 2(5)(-6)(.6) \\ &= 12.8 \end{aligned}$$

Set 12.4a

P{an even number in one throw}

$$= P\{2, 4, \text{ or } 6\}$$

$$= 3\left(\frac{1}{6}\right) = \frac{1}{2}$$

P{0 even number in 10 throws}

$$= C_0^{10} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{10}$$

Probability = P{One head in 5 throws}

$$+ P\{\text{one tail in 5 throws}\}$$

$$= 2 C_1^5 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4$$

$$= \frac{5}{16}$$

Being a fluke is equivalent to
a 50-50 chance of being correct.

$$P\{\text{a fluke}\} = C_8^{10} \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 +$$

$$C_9^{10} \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 +$$

$$C_{10}^{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$$

$$= \left(\frac{1}{2}\right)^{10} [45 + 10 + 1]$$

$$= .0547$$

Probability of a single match

$$= 6 \times \left(\frac{1}{6} \times \frac{1}{6}\right) = \frac{1}{6}$$

P{i matches out of 3 dice}

$$= C_i^3 \left(\frac{1}{6}\right)^i \left(\frac{5}{6}\right)^{3-i}, i=0, 1, 2, 3$$

i	0	1	2	3
P	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

$$\begin{aligned} \text{Expected payoff} &= -1\left(\frac{125}{216}\right) + 1\left(\frac{75}{216}\right) + \\ &2\left(\frac{15}{216}\right) + 3\left(\frac{1}{216}\right) \approx -.08 = -8 \text{ cents} \end{aligned}$$

Prob. of a match = $\frac{1}{6}$

Prob. of no match = $\frac{5}{6}$

Expected payoff = $50\left(\frac{1}{6}\right) - 10\left(\frac{5}{6}\right) = 0$

$E\{k\} = \sum_{k=0}^n k \binom{n}{k} p^k q^{n-k}$

$$= \sum_{k=1}^n k \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

$$= np \sum_{k=1}^n \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k}$$

$$= np \left(\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} p^j q^{n-1-j} \right)$$

$\text{Var}\{k\} = E\{k^2\} - E^2\{k\}$

$E\{k^2\} = \sum_{k=1}^n k^2 \binom{n}{k} p^k q^{n-k}$

$$= np \sum_{k=1}^{n-1} k \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k}$$

$$= np \sum_{k=0}^{n-1} (k+1) \frac{(n-1)!}{k!(n-k-1)!} p^k q^{n-1-k}$$

$$= np((n-1)p + 1)$$

$$= np(np + q)$$

$\text{Var}\{k\} = np(np + q) - (np)^2$

$$= npq$$

4

Set 12.4b

$$P\{n \geq 1 | t = 30 \text{ sec}\}$$

$$= \sum_{n=1}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!}$$

$$= 1 - e^{-\lambda t}$$

$$= 1 - e^{-4 \times 0.5} = 1 - e^{-2} = .8646$$

Case 1: $p = .1$

Binomial:

$$P\{0 \text{ or } 1 \text{ defective}\}$$

$$= C_0^{10} (.01)^0 (.99)^{10} + C_1^{10} (.01)^1 (.99)^9$$

$$= .99^{10} + 10 \times .01 \times .99^9 = .9957$$

Poisson:

$$\lambda = np = 10 \times .01 = .1$$

$$P_0 + P_1 = \frac{.1^0 e^{-0.1}}{0!} + \frac{.1^1 e^{-0.1}}{1!}$$

$$= e^{-0.1} (1 + .1) \approx .9953$$

Case 2: $p = .5$

Binomial:

$$P\{0 \text{ or } 1 \text{ defective}\}$$

$$= C_0^{10} (.5)(.5)^{10} + C_1^{10} (.5)^1 (.5)^9$$

$$= .5^{10} + 10 \times .5^{10} = .01074$$

Poisson:

$$\lambda = 10 \times .5 = 5$$

$$P_0 + P_1 = \frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!}$$

$$= .04043$$

1

$$\lambda = 20 \text{ customers/hr}$$

$$(a) P_0 = \frac{20^0 e^{-20}}{0!} \approx 0$$

$$(b) P_{n \geq 3} = 1 - P_0 - P_1 - P_2$$

$$= 1 - \frac{20^0 e^{-20}}{0!} - \frac{20^1 e^{-20}}{1!} - \frac{20^2 e^{-20}}{2!} \approx 1$$

Note: $n \geq 3 \Rightarrow (1 \text{ in service and at least 2 waiting})$

3

2

$$E\{X\} = \sum_{x=1}^{\infty} x \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

$$= \sum_{x=1}^{\infty} (\lambda t) \frac{(\lambda t)^{x-1} e^{-\lambda t}}{(x-1)!}$$

$$= (\lambda t) \sum_{x=0}^{\infty} \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

$$= \lambda t$$

$$\text{Var}\{X\} = E\{X^2\} - E\{X\}^2$$

$$E\{X^2\} = \sum_{x=1}^{\infty} x^2 \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

$$= \lambda t \sum_{x=1}^{\infty} x \frac{(\lambda t)^{x-1} e^{-\lambda t}}{(x-1)!}$$

$$= \lambda t \sum_{x=0}^{\infty} (x+1) \frac{(\lambda t)^x}{x!} e^{-\lambda t}$$

$$= \lambda t \left(\sum_{x=0}^{\infty} x \frac{(\lambda t)^x e^{-\lambda t}}{x!} + \sum_{x=0}^{\infty} \frac{(\lambda t)^x e^{-\lambda t}}{x!} \right)$$

$$= \lambda t (\lambda t + 1)$$

$$\text{Var}\{X\} = (\lambda t)^2 + \lambda t - (\lambda t)^2$$

$$= \lambda t$$

4

Set 12.4c

$$\lambda = 5 \text{ customers/min}$$

$$\lambda_{\text{rural}} = 7 \text{ customers/min}$$

$$\lambda = 5 + 7 = 12 \text{ customers/min.}$$

$$P\{t \leq \frac{5}{60}\} = 1 - e^{-12 \times \frac{5}{60}}$$

$$= 1 - .368$$

$$= .632$$

$$\begin{aligned} &= \int_0^\infty e^{-\lambda x} dx - x^2 e^{-\lambda x} \Big|_0^\infty - \frac{2}{\lambda} + \frac{1}{\lambda^2} \\ &= 2 \int_0^\infty x e^{-\lambda x} dx - x^2 e^{-\lambda x} \Big|_0^\infty - \frac{2}{\lambda} + \frac{1}{\lambda^2} \\ &= \frac{2}{\lambda} - \frac{2}{\lambda} + \frac{1}{\lambda^2} \\ &= \frac{1}{\lambda^2} \end{aligned}$$

$$E\{x\} = \int_0^\infty x \lambda e^{-\lambda x} dx$$

2

$$= - \int_0^\infty x d e^{-\lambda x}$$

$$= - \left[x e^{-\lambda x} - \int_0^\infty e^{-\lambda x} dx \right]$$

$$= - \left[x e^{-\lambda x} - \frac{1}{\lambda} \int_0^\infty \lambda e^{-\lambda x} dx \right]$$

$$= - \left[x e^{-\lambda x} - \frac{1}{\lambda} \right]_0^\infty$$

$$= \frac{1}{\lambda}$$

$$\text{Var}\{x\} = \int_0^\infty (x - E\{x\})^2 f(x) dx$$

$$= \int_0^\infty (x - \frac{1}{\lambda})^2 \lambda e^{-\lambda x} dx$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x} dx - 2 \int_0^\infty x e^{-\lambda x} dx + \frac{1}{\lambda^2} \int_0^\infty e^{-\lambda x} dx$$

$$= - \int_0^\infty x^2 d e^{-\lambda x} - \frac{2}{\lambda} + \frac{1}{\lambda^2}$$

$$= - \left[x^2 e^{-\lambda x} - \int_0^\infty e^{-\lambda x} dx^2 \right] - \frac{2}{\lambda} + \frac{1}{\lambda^2}$$

continued...

Set 12.4d

$$\begin{aligned}
 (a) P\{x \geq 26\} &= 1 - P\{x \leq 26\} \\
 &= 1 - P\left\{z \leq \frac{26-22}{2}\right\} \\
 &= 1 - P\{z \leq 2\} \\
 &= 1 - .9772 = .0228
 \end{aligned}$$

$$\begin{aligned}
 &= P\{z \geq .7072\} \\
 &= 1 - P\{z \leq .7072\} \\
 &\cong 1 - .760283 \\
 &\cong .239717
 \end{aligned}$$

$$\begin{aligned}
 (b) P\{x \leq 17\} &= P\left\{z \leq \frac{17-22}{2}\right\} \\
 &= P\{z \leq -2.5\} \\
 &= 1 - .9938 \\
 &= .0062
 \end{aligned}$$

Distribution of the weight of 5 individuals is normal with

$$\text{mean} = 5 \times 180 = 900 \text{ lb}$$

$$\text{standard deviation} = \sqrt{5 \times 15^2} = 33.54$$

$$\begin{aligned}
 P\{x \geq 1000\} &= 1 - P\left\{z \leq \frac{1000-900}{33.54}\right\} \\
 &= 1 - P\{z \leq 2.98\} \\
 &= 1 - .9986 \\
 &= .0014
 \end{aligned}$$

$$x_1 = N(.99, .01)$$

$$x_2 = N(1, .01)$$

$$P\{x_1 > x_2\} = P\{x_1 - x_2 \geq 0\}$$

$$\text{mean } \{x_1 - x_2\} = .99 - 1 = -.01$$

$$\text{standard deviation } \{x_1 - x_2\} = \sqrt{.01^2 + .01^2} = .01414$$

$$\begin{aligned}
 P\{x_1 - x_2 \geq 0\} &= P\left\{z \geq \frac{0 - (-.01)}{.01414}\right\} \\
 &\text{continued...}
 \end{aligned}$$

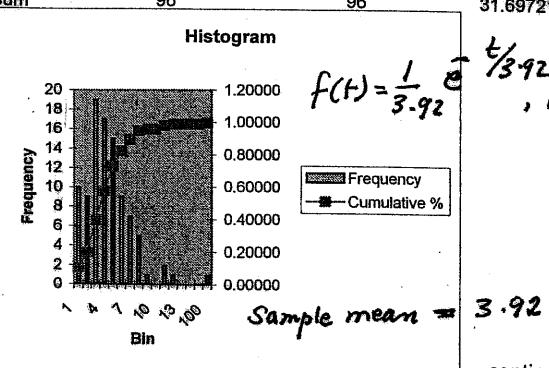
Set 12.5a

Step 1: Use ch12SampleMeanVar.xls to compute sample statistics and to prepare for creating the histogram as shown below

Sample Mean and Variance +Histogram						
Outputs						
Sample size	96	Mean	3.9218			
Minimum	0.1000	Variance	6.8809			
Maximum	15.9000	Std Dev.	2.6231			
Input						
Frequency distribution (A8:E100)						
4.3	0.9	5.8	2.7	0.5		
4.4	4.4	3.4	5.1	1		
0.1	4.9	15.9	2.1	1.5		
2.5	3.8	2.8	2.1	2		
3.4	0.4	0.9	4.5	2.5		
8.1	1.1	2.9	7.2	3		
2.6	4.9	4.1	11.5	3.5		
0.1	4.3	4.3	4.1	4		
2.2	5.2	1.1	2.1	4.5		
3.5	7.9	5.1	5.8	5		
0.5	6.4	2.1	3.2	5.5		
3.3	7.1	3.1	2.1	6		
3.4	0.7	3.4	7.8	6.5		
0.8	1.9	3.1	1.4	7		
4.1	4.8	6.7	2.3	7.5		
3.3	6.1	5.9	2.8	8		
3.1	2.7	2.9	3.8	8.5		
3.4	4.2	4.6	5.1	9		
0.9	2.4	5.1	2.6	9.5		
10.3	5.1	1.1	6.7	10		
2.9	8.2	3.3	7.3	10.5		
3.1	0.9	6.2	1.4	11		
4.5	1.2	10.7	2.3	11.5		
3.3	6.9	1.6	1.9			

Step 2: Apply Excel histogram to the sample above. The output below is for bin width of 1. Excel automatically provides the output below, less the columns titled n_i and Chi-value. You can then augment the spreadsheet with formulas to create the right-most column.

Bin	Oi	Cpi	ni	Chi-value	Revised χ^2
1	10	0.10417	21.60641	6.234669	
2	9	0.19792	16.74853	3.581217	
3	19	0.39583	12.97511	2.797605	
4	17	0.57292	10.05485	4.797204	
5	15	0.72917	7.791835	6.668217	
6	9	0.82292	6.038151	1.452853	25.50176
7	7	0.89583	4.679164	1.151112	1.643781
8	5	0.94792	3.626039	0.520614	
9	1	0.95833	2.809938	1.165818	2.02322
10	0	0.95833	2.177515	2.177515	
11	2	0.97917	1.687428	0.057899	
12	1	0.98958	1.307644	0.072378	2.498492
13	0	0.98958	1.013337	1.013337	
14	0	0.98958	0.785268	0.785268	
15	0	0.98958	0.608531	0.608531	
100	1	1.00000	2.095247	0.572518	
sum					
	96	96		31.69721	

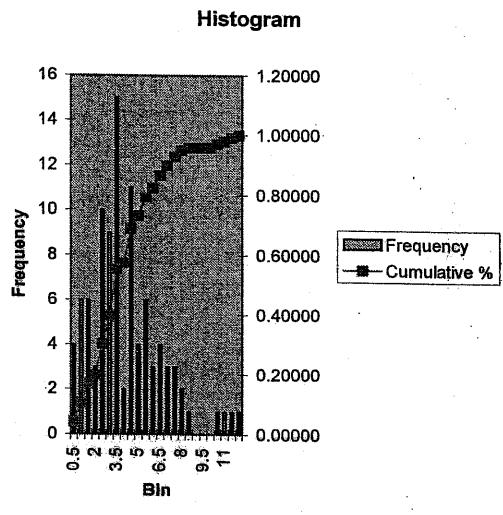


As can be seen from the output above, the spreadsheet can be modified to compute the χ^2 -value. Note that the grouping is necessary to guarantee that $n_i \geq 5$.

$$\chi^2\text{-value} = 31.69721, \chi^2_{9-1-1, .95} = 14.067, \text{Reject}$$

Binsize = .5 :

Bin	Oi	Cpi	ni	Chi-value	Revised χ^2
0.5	4	0.04167	11.49092	4.883325	
1	6	0.10417	10.11549	1.674389	
1.5	6	0.16667	8.904697	0.947507	
2	3	0.19792	7.83883	2.986961	
2.5	10	0.30208	6.900545	1.392154	
3	9	0.39583	6.07457	1.408847	
3.5	15	0.55208	5.347461	17.4235	13.29318
4	2	0.57292	4.707386	1.557114	1.944528
4.5	11	0.68750	4.143925	11.34329	
5	4	0.72917	3.647909	0.033983	1.438188
6	6	0.79167	3.211265	2.4218	
6.5	4	0.82292	2.826886	0.010601	0.533896
7	3	0.89583	2.190648	0.299021	0.819514
7.5	3	0.92708	1.928434	0.595434	
8	2	0.94792	1.697606	0.053865	
8.5	1	0.95833	1.494407	0.163569	1.541192
9	0	0.95833	1.315531	1.315531	
9.5	0	0.95833	1.158066	1.158066	
10	0	0.95833	1.019449	1.019449	
10.5	1	0.96875	0.897424	0.011725	
11	1	0.97917	0.790005	0.05582	
11.5	1	0.98958	0.695443	0.133375	
100	1	1.00000	5.114583	3.310103	3.310103
Sum					
	96	96		22.8806	



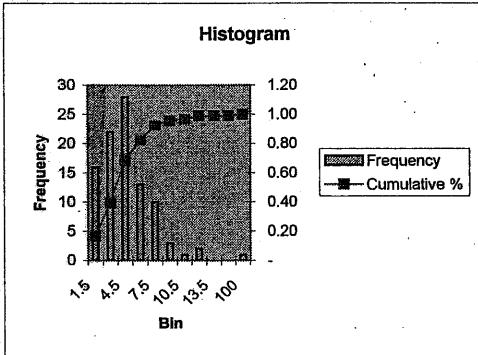
$$\chi^2\text{-value} = 22.88 \quad \chi^2_{12-1-1, .05} = 18.307 \quad \text{Reject.}$$

continued...

continued...

Set 12.5a

Bin	Oi	Cpi	ni	Chi-value	Revised χ^2
1.5	16	0.17	30.51111	6.901496	
3	22	0.40	20.81395	0.067586	
4.5	28	0.69	14.19877	13.41481	
6	13	0.82	9.686061	1.133814	
7.5	10	0.93	6.607598	1.741691	23.2594
9	3	0.96	4.507544	0.504197	1.692609
10.5	11	0.97	3.074938	1.400148	
12	2	0.99	2.097649	0.004546	1.963661
13.5	0	0.99	1.430966	1.430966	
15	0	0.99	0.97617	0.97617	
100	1	1.00	2.095247	0.572518	
Sum	96		96	26.91567	



$$\begin{aligned} \chi^2\text{-value} &= 26.92 \\ \chi^2_{7-1-1, .05} &= 11.07 \end{aligned} \quad \left. \begin{array}{l} \text{Reject.} \\ \text{Conclusion: accept hypothesis} \end{array} \right\}$$

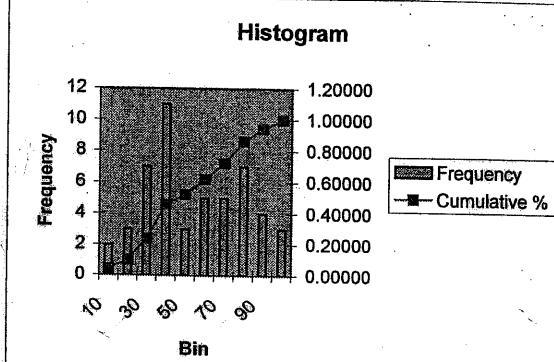
All three histograms call for rejecting the hypothesis that the sample is drawn from an exponential distribution with an estimated mean value of 3.92.

Note the effect of bin size on the χ^2 -value. The larger the bin size, the smaller the number of degrees of freedom for the χ^2 , and the tighter are the rejection limits.

Sample Mean and Variance + Histogram				
Output	50	Mean	50.7620	
Sample size	50	Variance	639.0783	
Minimum	5.6000	Std.Dev.	25.2800	
Printed				
Entered data in AS:E100				
25.8	67.3	35.2	36.4	58.7
47.9	94.8	61.3	59.3	93.4
17.8	34.7	56.4	22.1	48.1
48.2	35.8	65.3	30.1	72.5
5.8	70.8	88.9	76.4	17.3
77.4	66.1	23.9	23.8	36.8
5.6	36.4	93.5	36.4	76.7
89.3	39.2	78.7	51.9	63.6
89.5	58.6	12.8	28.6	82.7
38.7	71.3	21.1	35.9	29.2

$$(a) f(x) = \frac{1}{100}, 0 \leq x \leq 100$$

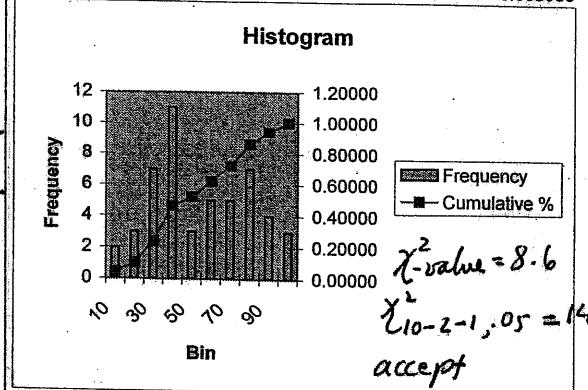
Bin	Oi	Cpi	ni	chi-value	revised chi
10	2	0.04000	5	1.8	
20	3	0.10000	5	0.8	
30	7	0.24000	5	0.8	
40	11	0.46000	5	7.2	
50	3	0.52000	5	0.8	
60	5	0.62000	5	0	
70	5	0.72000	5	0	
80	7	0.86000	5	0.8	
90	4	0.94000	5	0.2	
100	3	1.00000	5	0.8	
sum	50				13.2



$$\begin{aligned} \chi^2\text{-value} &= 13.2, \chi^2_{10-1, .05} = 16.9 \\ \text{Conclusion: accept hypothesis} \end{aligned}$$

$$(b) Hypothesis: f(x) = \frac{1}{94.8 - 5.6} = \frac{1}{89.2}$$

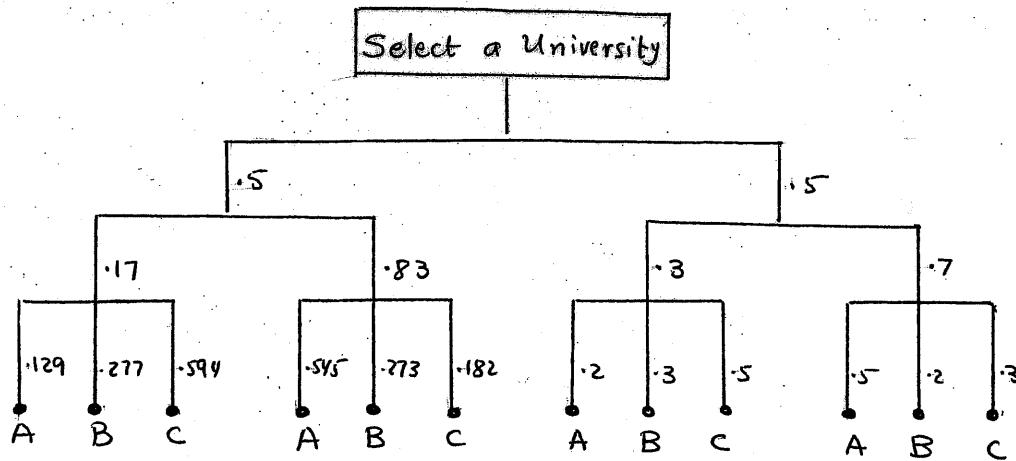
Bin	Oi	Cpi	ni	chi-value	revised chi
10	2	0.04000	2.466368	0.088186	1.168971
20	3	0.10000	5.605381	1.210981	
30	7	0.24000	5.605381	0.346981	
40	11	0.46000	5.605381	5.191781	
50	3	0.52000	5.605381	1.210981	7.227487
60	5	0.62000	5.605381	0.065381	
70	5	0.72000	5.605381	0.065381	
80	7	0.86000	5.605381	0.346981	
90	4	0.94000	5.605381	0.459781	0.202451
100	3	1.00000	2.690583	0.035583	
sum	50		50	9.022018	8.598909



CHAPTER 13

Decision Theory and Games

Set 13.1a



$$W_A = .5(17x_1 + 129 + .83x_2 - .5x_3) + .5(-.3x_1 + 2 + .7x_2 + .5x_3) = \boxed{.44214}$$

$$W_B = .5(.17x_1 + .277x_2 + .83x_3 + .273) + .5(.3x_1 + .3x_2 + .7x_3) = .25184$$

$$W_C = .5(.17x.594 + .83x.182) + .5(.3x.5 + .7x.3) = .30602$$

Select A.

Set 13.1b

K20	= \$M\$8 &= " &TEXT(\$E\$4*(S\$4*\$N8+\$N\$5*\$N13)+\$L\$5*(\$P\$4*\$P8+\$P\$5*\$P13), "####0.00000")											
cht4AHP-p1												
AHP-Analytic Hierarchy Process												
	Solution summary											
MJ:	MLR:	JLR:										
M J	0.5 0.5	L R	0.17 0.83	L R	0.3 0.7							
MUL:		JUL:										
UA UB UC	0.129 0.277 0.594	UA UB UC	0.2 0.3 0.5									
MUR:		JUR:										
UA UB UC	0.545 0.273 0.182	UA UB UC	0.5 0.2 0.3									
Final ranking												
UA = 0.44214												
UB = 0.26184												
UC = 0.30602												

← formula given on top

2

$$A = E \begin{bmatrix} 1 & 2 & .25 \\ .5 & 1 & .2 \\ 4 & 5 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} .182 & .25 & .172 \\ .091 & .125 & .138 \\ .727 & .625 & .690 \end{bmatrix}$$

$$A\bar{W} = \begin{bmatrix} 1 & 2 & .25 \\ .5 & 1 & .2 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} .201 \\ .118 \\ .681 \end{bmatrix} = \begin{bmatrix} .60725 \\ .3547 \\ 2.075 \end{bmatrix}$$

$$n_{max} = .60725 + .3547 + 2.075 = 3.037$$

$$CI = \frac{3.073 - 3}{3 - 1} = .0185$$

$$RI = \frac{1.98(1)}{3} = .66$$

$$CR = \frac{.0185}{3.037} = .028 < .1, \text{ acceptable}$$

$$\bar{W} = \begin{bmatrix} .632 & .333 & .769 \\ .211 & .111 & .038 \\ .158 & .556 & .192 \end{bmatrix} \begin{bmatrix} .578 \\ .120 \\ .302 \end{bmatrix}$$

$$A_I \bar{W} = \begin{bmatrix} 1 & 3 & 4 \\ .33 & 1 & .2 \\ .25 & 5 & 1 \end{bmatrix} \begin{bmatrix} .578 \\ .120 \\ .320 \end{bmatrix} = \begin{bmatrix} 2.146 \\ .373 \\ 1.0465 \end{bmatrix}$$

$$n_{max} = 2.14 + .373 + 1.0465 = 3.5655$$

$$CI = \frac{3.5655 - 3}{2} = .28275$$

$$RI = \frac{1.98(1)}{3} = .66$$

$$CR = \frac{.28275}{.66} = .428 > .1, \text{ not acceptable}$$

$$\bar{W} = \begin{bmatrix} .222 & .100 & .571 \\ .667 & .300 & .143 \\ .111 & .600 & .286 \end{bmatrix} \begin{bmatrix} .298 \\ .370 \\ .332 \end{bmatrix}$$

Continued...

Continued...

Set 13.1b

$$A_E \bar{W} = \begin{bmatrix} 1 & .33 & 2 \\ 3 & 1 & .5 \\ .5 & 2 & 1 \end{bmatrix} \begin{bmatrix} .298 \\ .370 \\ .332 \end{bmatrix} = \begin{bmatrix} 1.085 \\ 1.430 \\ 1.221 \end{bmatrix}$$

$$n_{max} = \frac{3.736}{3}$$

$$CI = \frac{3.736 - 3}{2} = .368, RI = .66$$

$$CR = \frac{.368}{.66} = .558 > 1, \text{not acceptable}$$

$$N_R = \begin{bmatrix} .25 & .143 & .400 \\ .50 & .286 & .200 \\ .25 & .571 & .400 \end{bmatrix} \begin{array}{c} \bar{W} \\ .264 \\ .329 \\ .407 \end{array}$$

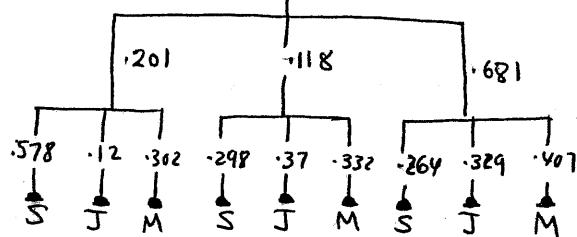
$$A_R \bar{W} = \begin{bmatrix} 1 & .5 & 1 \\ 2 & 1 & .5 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} .264 \\ .329 \\ .407 \end{bmatrix} = \begin{bmatrix} .8355 \\ 1.0605 \\ 1.329 \end{bmatrix}$$

$$n_{max} = 3.225$$

$$CI = \frac{3.225 - 3}{2} = .1125, RI = .66$$

$$CR = \frac{.1125}{.66} = .17 > 1, \text{not acceptable}$$

Select employee



$$W_S = .201 \times .578 + .118 \times .298 + .681 \times .264 = .331$$

$$W_J = .201 \times .12 + .118 \times .37 + .681 \times .329 = .292$$

$$W_M = .201 \times .302 + .118 \times .332 + .681 \times .407 = .377$$

Decision:

Select Maisa

$$N = \begin{bmatrix} .667 & .667 \\ .333 & .333 \end{bmatrix} \begin{array}{c} \bar{W} \\ .667 \\ .333 \end{array}$$

$$N_k = \begin{bmatrix} .25 & .25 \\ .75 & .75 \end{bmatrix} \begin{array}{c} .25 \\ .75 \end{array}$$

$$N_j = \begin{bmatrix} .8 & .8 \\ .2 & .2 \end{bmatrix} \begin{array}{c} .8 \\ .2 \end{array}$$

$$N_{ky} = \begin{bmatrix} .546 & .571 & .500 \\ .272 & .286 & .333 \\ .182 & .143 & .167 \end{bmatrix} \begin{array}{c} \bar{W} \\ .539 \\ .297 \\ .164 \end{array}$$

$$A_{ky} \bar{W} = \begin{bmatrix} 1 & 2 & 3 \\ .5 & 1 & 2 \\ .333 & .5 & 1 \end{bmatrix} \begin{bmatrix} .539 \\ .297 \\ .164 \end{bmatrix} = \begin{bmatrix} 1.625 \\ .8945 \\ .4922 \end{bmatrix}$$

$$n_{max} = 3.01167$$

$$RI = \frac{.01167/2}{.66} = .0088 < 1, \text{acceptable}$$

$$N_{kw} = \begin{bmatrix} .286 & .333 & .273 \\ .143 & .167 & .182 \\ .571 & .500 & .545 \end{bmatrix} \begin{array}{c} \bar{W} \\ .297 \\ .164 \\ .539 \end{array}$$

$$A_{kw} \bar{W} = \begin{bmatrix} 1 & 2 & .5 \\ .5 & 1 & .333 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} .297 \\ .164 \\ .539 \end{bmatrix} = \begin{bmatrix} .8945 \\ .4922 \\ 1.625 \end{bmatrix}$$

$$n_{max} = 3.0117$$

$$RI = \frac{.0117/2}{.66} = .008 < 1, \text{acceptable}$$

$$N_{jy} = \begin{bmatrix} .571 & .750 & .333 \\ .143 & .188 & .500 \\ .286 & .062 & .167 \end{bmatrix} \begin{array}{c} \bar{W} \\ .551 \\ .277 \\ .172 \end{array}$$

$$A_{jy} \bar{W} = \begin{bmatrix} 1 & 4 & 2 \\ .25 & 1 & 3 \\ .5 & .333 & 1 \end{bmatrix} \begin{bmatrix} .551 \\ .277 \\ .172 \end{bmatrix} = \begin{bmatrix} 2.003 \\ .93075 \\ .5398 \end{bmatrix}$$

$$n_{max} = 3.476$$

$$RI = \frac{.476/2}{.66} = .3576 > 1, \text{not acceptable}$$

$$N_{jw} = \begin{bmatrix} .308 & .273 & .500 \\ .615 & .546 & .375 \\ .077 & .182 & .125 \end{bmatrix} \begin{array}{c} \bar{W} \\ .360 \\ .512 \\ .128 \end{array}$$

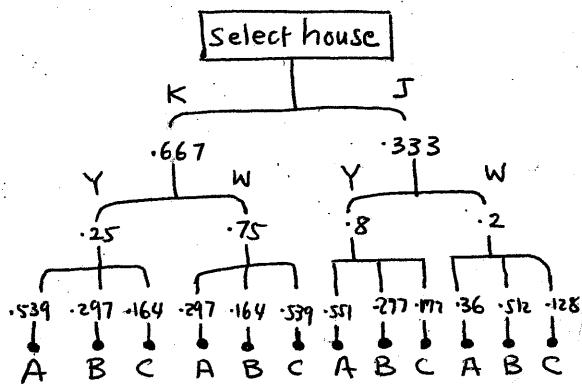
continued...

Set 13.1b

$$A\bar{W} = \begin{bmatrix} 1 & .5 & 4 \\ 2 & 1 & 3 \\ .25 & .33 & 1 \end{bmatrix} \begin{bmatrix} .360 \\ .512 \\ .128 \end{bmatrix} = \begin{bmatrix} 1.128 \\ 1.616 \\ .3887 \end{bmatrix}$$

$$n_{max} = 3.1333$$

$$RI = \frac{1.333/2}{.66} = .100, \text{ acceptable}$$



$$W_A = .667(.25 \times .539 + .75 \times .297) + .333(.8 \times .551 + .2 \times .36) = .4092$$

$$W_B = .667(.25 \times .297 + .75 \times .164) + .333(.8 \times .277 + .2 \times .512) = .2395$$

$$W_C = .667(.25 \times .164 + .75 \times .539) + .333(.8 \times .172 + .2 \times .128) = .3513$$

Select A.

$$N = \begin{bmatrix} .167 & .143 & .172 \\ .167 & .143 & .138 \\ .667 & .714 & .690 \end{bmatrix} \quad \bar{W} = \begin{bmatrix} .161 \\ .149 \\ .690 \end{bmatrix}$$

$$A\bar{W} = \begin{bmatrix} 1 & 1 & .25 \\ 1 & 1 & .20 \\ 4 & 5 & 1 \end{bmatrix} \begin{bmatrix} .161 \\ .149 \\ .690 \end{bmatrix} = \begin{bmatrix} .4825 \\ .4480 \\ 2.079 \end{bmatrix}$$

$$n_{max} = 3.0095$$

$$CR = \frac{.0095/2}{.66} = .0072 < .1, \text{ acceptable}$$

$$N_R = \begin{bmatrix} .667 & .667 \\ .333 & .333 \end{bmatrix} \quad \bar{W} = \begin{bmatrix} .667 & (H) \\ .333 & (P) \end{bmatrix}$$

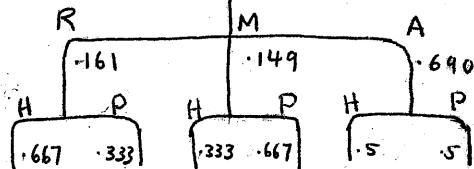
$$N_M = \begin{bmatrix} .333 & .333 \\ .667 & .667 \end{bmatrix} \quad \bar{W} = \begin{bmatrix} .333 & (H) \\ .667 & (P) \end{bmatrix}$$

$$N_A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \bar{W} = \begin{bmatrix} .5 & (H) \\ .5 & (P) \end{bmatrix}$$

N_R, N_M, N_A are consistent because they are 2-dimensional.

Continued...

Select publisher



$$W_H = .161 \times .667 + .149 \times .333 + .690 \times .5 = .502$$

$$W_P = .161 \times .333 + .149 \times .667 + .690 \times .5 = .498$$

Choose H.

$$\bar{W} = \begin{bmatrix} .286 & .25 & .294 \\ .143 & .125 & .118 \\ .571 & .625 & .588 \end{bmatrix} \quad \bar{W} = \begin{bmatrix} .277 \\ .128 \\ .595 \end{bmatrix}$$

$$A\bar{W} = \begin{bmatrix} 1 & 2 & .5 \\ 1 & 1 & .2 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} .277 \\ .128 \\ .595 \end{bmatrix} = \begin{bmatrix} .8305 \\ .3855 \\ 1.789 \end{bmatrix}$$

$$n_{max} = 3.005$$

$$RI = \frac{.005/2}{.66} = .0039 < .1, \text{ acceptable}$$

$$N_L = \begin{bmatrix} .3 & .429 & .273 \\ .1 & .142 & .182 \\ .6 & .429 & .546 \end{bmatrix} \quad \bar{W} = \begin{bmatrix} .334 \\ .141 \\ .525 \end{bmatrix}$$

$$A_L\bar{W} = \begin{bmatrix} 1 & 3 & .5 \\ .333 & 1 & .333 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} .334 \\ .141 \\ .525 \end{bmatrix} = \begin{bmatrix} 1.0195 \\ .427 \\ 1.6663 \end{bmatrix}$$

$$n_{max} = 3.06283$$

$$RI = \frac{.06283/2}{.66} = .04 < .1, \text{ acceptable}$$

$$N_C = \begin{bmatrix} .5 & .5 & .5 \\ .25 & .25 & .25 \\ .25 & .25 & .25 \end{bmatrix} \quad \bar{W} = \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} \quad \text{consistent}$$

$$N_R = \begin{bmatrix} .474 & .471 & .500 \\ .474 & .471 & .444 \\ .052 & .059 & .056 \end{bmatrix} \quad \bar{W} = \begin{bmatrix} .482 \\ .463 \\ .056 \end{bmatrix}$$

Continued...

Set 13.1b

$$A_R^W = \begin{bmatrix} 1 & 1 & 9 \\ 1 & 1 & 8 \\ 1/9 & 1/8 & 1 \end{bmatrix} \begin{bmatrix} .482 \\ .463 \\ .056 \end{bmatrix} = \begin{bmatrix} 1.449 \\ 1.393 \\ .167 \end{bmatrix}$$

$n_{\max} = 3.0094$

$$RI = \frac{.0094/2}{.66} = .0071 < 1, \text{ acceptable}$$

$$\begin{aligned} W_I &= .277(-.334x_1 + .141x_2 + .525x_3) \\ &+ .128(.5x_3 + .25x_5 + .25x_2) \\ &+ .595(.482x_7 + .463x_1 + .056x_3) \\ &= .3406 \end{aligned}$$

$$\begin{aligned} W_B &= .277(-.334x_5 + .141x_4 + .525x_2) \\ &+ .128(.5x_4 + .25x_2 + .25x_4) \\ &+ .595(.482x_1 + .463x_5 + .056x_2) \\ &= -.2813 \end{aligned}$$

$$\begin{aligned} W_S &= .277(-.334x_4 + .141x_4 + .525x_5) \\ &+ .128(.5x_3 + .25x_3 + .25x_4) \\ &+ .595(.482x_2 + .463x_5 + .056x_5) \\ &= .3798 \Rightarrow \text{Select Smith} \end{aligned}$$

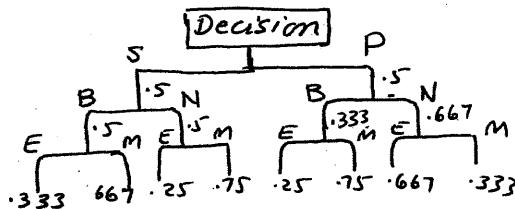
$$N_S = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$$

6

$$N_P = \begin{bmatrix} .667 & .667 \\ .333 & .333 \end{bmatrix}$$

$$N_{SB} = \begin{bmatrix} .333 & .333 \\ .667 & .667 \end{bmatrix}, N_{PB} = \begin{bmatrix} .25 & .25 \\ .75 & .75 \end{bmatrix}$$

$$N_{SN} = \begin{bmatrix} .25 & .25 \\ .75 & .75 \end{bmatrix}, N_{PN} = \begin{bmatrix} .667 & .667 \\ .333 & .333 \end{bmatrix}$$



$$W_E = .5(.5x_{.333} + .5x_{.25}) + .5(.333x_{.25} + .667x_{.667}) = .4097$$

$$W_M = .5(.5x_{.667} + .5x_{.333}) + .5(.333x_{.667} + .667x_{.333}) = .5903$$

Decision: Keep music program.

Car Model	PP/yr	MC	CD	RD
M1	6	1.8	4.5	1.5
M2	8	1.2	2.25	.75
M3	10	.6	1.125	.6
Sum	24	3.6	7.875	2.85

7

All the comparison matrices are developed based on the average costs

$$A = \begin{bmatrix} PP & MC & CD & RD \\ PP & 1 & \frac{24}{3.6} & \frac{24}{7.875} & \frac{24}{2.85} \\ MC & \frac{3.6}{24} & 1 & \frac{3.6}{7.875} & \frac{3.6}{2.85} \\ CD & \frac{7.875}{24} & \frac{7.875}{3.6} & 1 & \frac{7.875}{2.85} \\ RD & \frac{2.85}{24} & \frac{2.85}{3.6} & \frac{2.85}{7.875} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6.67 & 3.048 & 8.421 \\ .15 & 1 & .457 & 1.263 \\ .328 & 2.188 & 1 & 2.763 \\ .119 & .792 & .362 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} M1 & 6/8 & 6/10 \\ M2 & 8/6 & 8/10 \\ PP & 10/6 & 10/8 \end{bmatrix} = \begin{bmatrix} 1 & .75 & .6 \\ 1.33 & 1 & .8 \\ 1.67 & 1.25 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} M1 & 6/4 & 6/2 \\ M2 & 4/6 & 4/2 \\ MC & 2/6 & 2/4 \end{bmatrix} = \begin{bmatrix} 1 & 1.5 & 3 \\ .667 & 1 & 2 \\ .333 & .5 & 1 \end{bmatrix}$$

$$A_{CD} = \begin{bmatrix} M1 & \frac{4500}{2250} & \frac{4500}{1125} \\ M2 & 1 & \frac{2250}{1125} \\ M3 & \frac{1125}{4500} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ .5 & 1 & 2 \\ .25 & .5 & 1 \end{bmatrix}$$

$$A_{RD} = \begin{bmatrix} 1 & \frac{1500}{750} & \frac{1500}{600} \\ \frac{7500}{1500} & 1 & \frac{750}{600} \\ \frac{600}{1500} & \frac{600}{750} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2.5 \\ .5 & 1 & 1.25 \\ .4 & .8 & 1 \end{bmatrix}$$

continued...

Set 13.1b

$$N = \begin{bmatrix} .626 & .626 & .626 & .628 \\ .094 & .094 & .094 & .094 \\ .205 & .205 & .205 & .206 \\ .075 & .074 & .074 & .074 \end{bmatrix} \quad \bar{W}$$

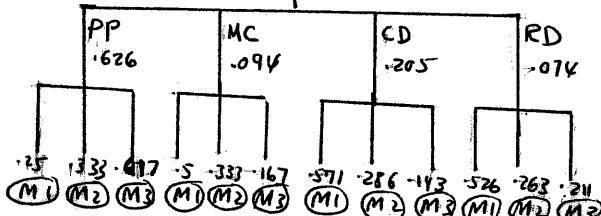
$$N_{PP} = \begin{bmatrix} .250 & .250 & .250 \\ .333 & .333 & .333 \\ .417 & .417 & .417 \end{bmatrix} \quad \bar{W}$$

$$N_{MC} = \begin{bmatrix} .500 & .500 & .500 \\ .333 & .333 & .333 \\ .167 & .167 & .167 \end{bmatrix} \quad \bar{W}$$

$$N_{CD} = \begin{bmatrix} .571 & .571 & .571 \\ .286 & .286 & .286 \\ .143 & .143 & .143 \end{bmatrix} \quad \bar{W}$$

$$N_{RD} = \begin{bmatrix} .526 & .526 & .526 \\ .263 & .263 & .263 \\ .211 & .211 & .211 \end{bmatrix} \quad \bar{W}$$

Select Model



$$W_{M_1} = .626 \times .25 + .094 \times .5 + .205 \times .571 \\ + .074 \times .526 = 3595$$

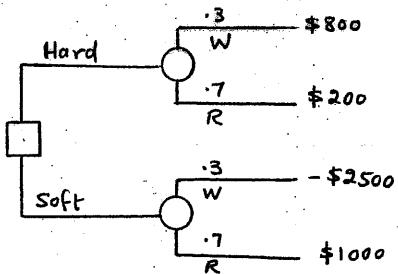
$$W_{M_2} = .626 \times .333 + .094 \times .333 + .205 \times .286 \\ + .074 \times .263 = 3185$$

$$W_{M_3} = .626 \times .417 + .094 \times .167 + .205 \times .143 \\ + .074 \times .211 = 3217$$

Since the comparison matrices are based on costs, the model with the smallest weight is selected.

Select M_2 .

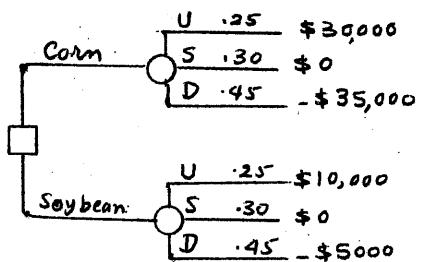
Set 13.2a



$$EV\{\text{Hard}\} = 800 \times 0.3 + 200 \times 0.7 = \$380$$

$$EV\{\text{Soft}\} = -2500 \times 0.3 + 1000 \times 0.7 = -\$50$$

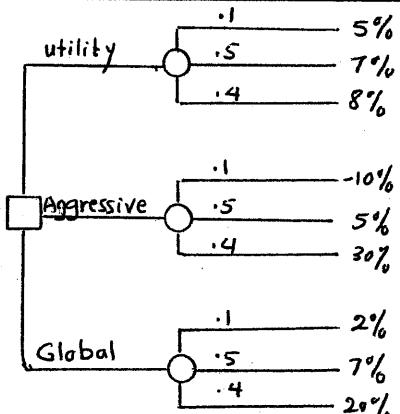
Select "Hard" button.



$$EV(\text{Corn}) = 30,000 \times 0.25 + 0 \times 0.3 + (-35,000) \times 0.45 = -\$8250$$

$$EV(\text{Soybean}) = 10,000 \times 0.25 + 0 \times 0.3 + (-5000) \times 0.45 = (\$250)$$

Select Soybean.



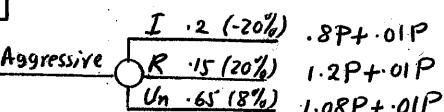
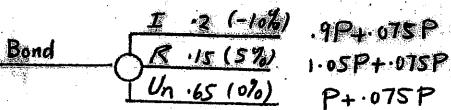
$$EV(\text{utility}) = 5 \times 0.1 + 7 \times 0.5 + 8 \times 0.4 = 7.2\%$$

$$EV(\text{aggressive}) = -10 \times 0.1 + 5 \times 0.5 + 30 \times 0.4 = 13.5\%$$

$$EV(\text{global}) = 2 \times 0.1 + 7 \times 0.5 + 20 \times 0.4 = 11.7\%$$

Select aggressive stock

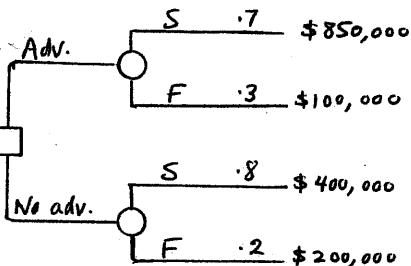
P = amount invested



$$EV(\text{Bond}) = P(0.975 \times 2 + 1.125 \times 1.5 + 1.075 \times 6.5) = 1.0625P$$

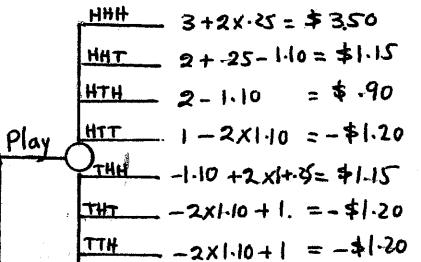
$$EV(\text{Aggressive}) = P(0.81 \times 2 + 1.21 \times 1.5 + 1.09 \times 6.5) = 1.052P$$

Select Bond



$$EV(\text{adv.}) = 850 \times 0.7 + 100 \times 0.3 = (\$625,000)$$

$$EV(\text{no adv.}) = 400 \times 0.8 + 200 \times 0.2 = \$360,000$$



Play \$0

$$EV(\text{play}) = \frac{1}{8} \left\{ 3.50 + 1.15 + .90 - 1.20 + 1.15 - 1.20 - 1.20 - 3.30 \right\}$$

$$= -\$0.025$$

$$EV(\text{no play}) = 0$$

Set 13.2a

$$(\text{even/even}) \equiv \{(2,2), (4,4), (6,6)\}$$

$$(\text{odd/odd}) \equiv \{(1,1), (3,3), (5,5)\}$$

(odd/even or even/odd)

$$\equiv \{(1,2), (1,4), (1,6), (3,2), (3,4), (3,6), (5,2), (5,4), (5,6), (2,1), (2,3), (2,5), (4,3), (4,5), (4,1), (6,1), (6,3), (6,5)\}$$

$$P\{e/e\} = 3 \times \left(\frac{1}{6}\right)^2 = \frac{1}{12} \quad (\text{outcome 1})$$

$$P\{O/O\} = 3 \times \left(\frac{1}{6}\right)^2 = \frac{1}{12} \quad (\text{outcome 2})$$

$$P\{e/O \text{ or } O/e\} = 18 \left(\frac{1}{6}\right)^2 = \frac{1}{2} \quad (\text{outcome 3})$$

$$P\{\text{others}\} = \frac{1}{3} \quad (\text{outcome 4})$$

Select outcomes (1,2)	1	$\frac{1}{12}$	\$2
	2	$\frac{1}{12}$	\$1.95
	.324	$\frac{5}{16}$	0

Select (1,3)	1	$\frac{1}{12}$	\$2
	3	$\frac{1}{2}$	\$1.95
	.284	$\frac{5}{12}$	0

Select (1,4)	1	$\frac{1}{12}$	\$2
	4	$\frac{1}{3}$	\$1.50
	.283	$\frac{7}{12}$	0

Don't play game	0		
Select (2,3)	2	$\frac{1}{12}$	\$1.95
	3	$\frac{1}{2}$	\$1.95

Select (2,4)	2	$\frac{1}{12}$	\$1.95
	4	$\frac{1}{3}$	\$1.50
	.183	$\frac{7}{12}$	0

Select (3,4)	3	$\frac{1}{12}$	\$1.95
	4	$\frac{1}{3}$	\$1.50
	.182	$\frac{1}{6}$	\$0

$$EV(1,2) = \frac{1}{12}(2 + 1.95) - 2 = -\$1.67$$

$$EV(1,3) = \frac{1}{12} \times 2 + \frac{1}{2} \times 1.95 - 2 = -\$0.86$$

$$EV(1,4) = \frac{1}{12} \times 2 + \frac{1}{3} \times 1.50 - 2 = -\$1.33$$

$$EV(2,3) = \frac{1}{12} \times 1.95 + \frac{1}{2} \times 1.95 - 2 = -\$0.86$$

$$EV(2,4) = \frac{1}{12} \times 1.95 + \frac{1}{3} \times 1.50 - 2 = -\$1.34$$

$$EV(3,4) = \frac{1}{12} \times 1.95 + \frac{1}{3} \times 1.50 - 2 = -\$0.53$$

Don't play the game

Continued...

7

8

Penalty matrix:

	Lot defective %			
	.8%	1%	1.2%	1.4%
A (.8%)	0	200	400	600
B (1.2%)	200	100	0	200
C (1.4%)	300	200	100	0

ship .8% .4	0
10% .3	\$200
12% .25	\$400
14% .05	\$600

.8% .4 \$200
10% .3 \$100
12% .25 0
14% .05 \$200

.8% .4 \$300
10% .3 \$200
12% .25 \$100
14% .05 0

$$EV(A) = 0 \times .4 + 200 \times .3 + 400 \times .25 + 600 \times .05 = -\$190$$

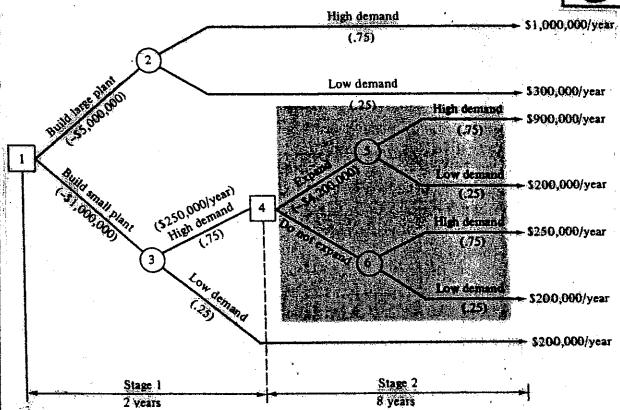
$$EV(B) = 200 \times .4 + 100 \times .3 + 0 \times .25 + 200 \times .05 = \$120$$

$$EV(C) = 300 \times .4 + 200 \times .3 + 100 \times .25 + 0 \times .05 = \$205$$

Select customer B

(a)

9



(b) $E\{\text{profit at node 4} | \text{expansion}\}$

$$= (900 \times .75 + 200 \times .25) \times 8 - 4200 = \$1,600,000$$

$E\{\text{profit at node 4} | \text{no expansion}\}$

$$= (250 \times .75 + 200 \times .25) \times 8$$

$$= (\$1,900,000)$$

Continued...

13-9

Set 13.2a

At node 4, no expansion
is recommended.

9 continued

$E(\text{profit at node 1 | large plant})$

$$= (1000x.75 + 300x.25) \times 10 - 5000 \\ = \$3,250,000$$

$E(\text{profit at node 1 | small plant})$

$$= (1900 + 2 \times 250) \times .75 + 10 \times 200 \times .25 - 1000 \\ = \$1,300,000$$

Decision: Start with large plant

Node 4:

$E(\text{annual profit | expansion})$

$$= 900x.75 + 200x.25 = \$725,000$$

$E(\text{annual profit | no expansion})$

$$= 250x.75 + 200x.25 = \$237,500$$

$$E(\text{profit | expansion}) = 725 [P/A]_8^{10\%} - 4200$$

$$= 725 \times 5.3349 - 4200$$

$$= \$332,198$$

$E(\text{profit | no expansion})$

$$= 237.5x [P/A]_8^{10\%}$$

$$= 237.5 \times 5.3349 = \$1,267,000$$

Decision at 4: no expansion

Node 1:

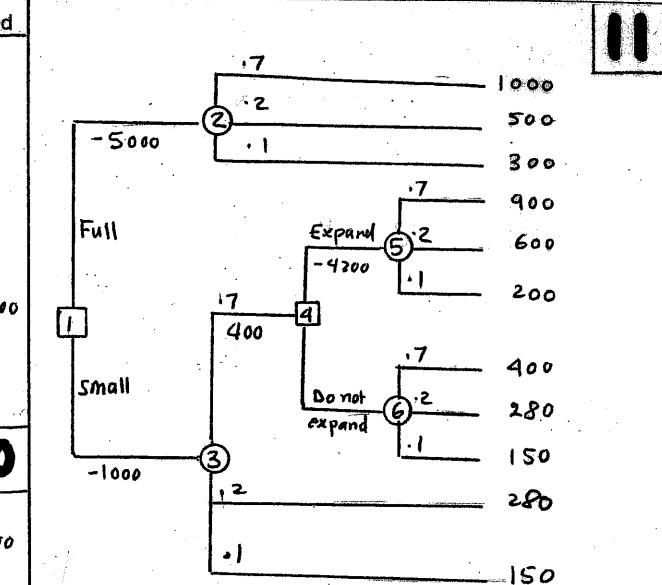
$E(\text{profit | large plant})$

$$= (1000x.75 + 300x.25) [P/A]_{10}^{10\%} - 5000 \\ = \$69,295$$

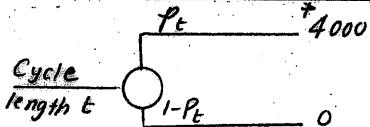
$E(\text{profit | small plant})$

$$= (1267 [P/S]_2^{10\%} + 250 [P/A]_2^{10\%}) \times .75 \\ + 200 [P/A]_{10}^{10\%} \times .25 - 1000 \\ = \$417,970$$

Decision: Construct a small plant
now and do not expand two years
from now.



Set 13.2a



$$E(\text{breakdown cost given } t) = 4000P_t$$

$$\underline{E=1:} \quad Cost = 20 \times 75 = \$1500$$

$$\underline{t = 2}: \\ \text{Exp. breakdown cost} = 4000x \cdot 03 \\ = \$120$$

$$\text{Av. cost/year} = \frac{1500 + 120}{2} = \$810$$

$$\underline{t = 3:} \quad \text{Exp. breakdown cost} = \\ 130 + 47777x \cdot .04 = \$280$$

$$\text{Av. cost/year} = \frac{1500 + 280}{3} = \$593.33$$

t = 4:
Exp. breakdown cost =

$$\text{Av. cost/year} = \frac{1500 * 480}{\$1} = \$495$$

$$\begin{aligned} t = 5: \\ \text{Exp. breakdown cost} = \\ 480 + 4000x.06 = \$720 \end{aligned}$$

$$\text{Av. cost/year} = \frac{1500 + 720}{5} = \$444$$

t=6: $\frac{t}{5} = 1$ $t = 5$

$$720 + 4000 \times .07 = \$1000$$

$$\text{Av. cost/year} = \frac{1300 / 1000}{6} = \$416.67$$

Eva H

$$1000 + 4000x \cdot 0.08 = 1320$$

$$\text{Av. cost/year} = \frac{1500 + 1320}{7} = \$402.86$$

$$\underline{t = 8:}$$

$$\text{Ar. Cost/yr} = \frac{1500 + 1320 + 4000x \cdot 0.09}{8} \\ = (\textcircled{\$397.50})$$

continued...

$$E=9; \\ \text{Av. cost/yr} = \frac{1500 + 1680 + 4000x \cdot 1}{9} = \$397.78$$

Decision:

Optimum cycle length = 8 years, Cost/yr = \$39750

Demand	Income	
.2 (100)	\$120	13
.25 (150)	120	
.3 (200)	120	
.15 (250)	120	
.1 (300)	120	
.2 (100)	$120 + .25 \times 50 = 132.50$	
.25 (150)	180	
.3 (200)	180	
.15 (250)	180	
.1 (300)	180	
.2 (100)	$120 + 100 \times .25 = 145$	
.25 (150)	$180 + 50 \times .25 = 192.5$	
.3 (200)	240	
.15 (250)	240	
.1 (300)	240	
.2 (100)	$120 + 150 \times .25 = \$157.50$	
.25 (150)	$180 + 100 \times .25 = \$205$	
.3 (200)	$240 + 50 \times .25 = \$252.50$	
.15 (250)	300	
.1 (300)	300	
.2 (100)	$120 + 200 \times .25 = 170$	
.25 (150)	$180 + 150 \times .25 = 217.50$	
.3 (200)	$240 + 100 \times .25 = 265$	
.15 (250)	$300 + 50 \times .25 = 312.50$	
.1 (300)	360	

$$E(\text{profit} | 100 \text{ loaves}) = 120 - 55 = \$65$$

$E(\text{profit} \mid 150 \text{ loans})$

$$= 132.50x.2 + 180x.8 - 82.50 = \$88$$

$$E(\text{profit} \mid 200 \text{ leaves}) = 145x_2 + 192.5x_3 - 25 + 240x_5 - 110 = 99.13$$

$$E(\text{profit} / 250 \text{ loaves}) = 157.50x.2 + 205x.25 + 252.50x.3 + 300x.25$$

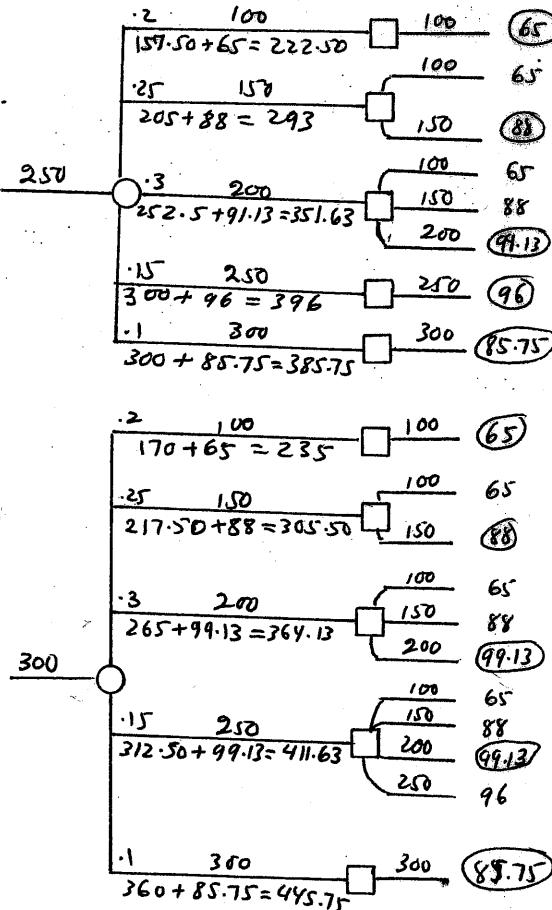
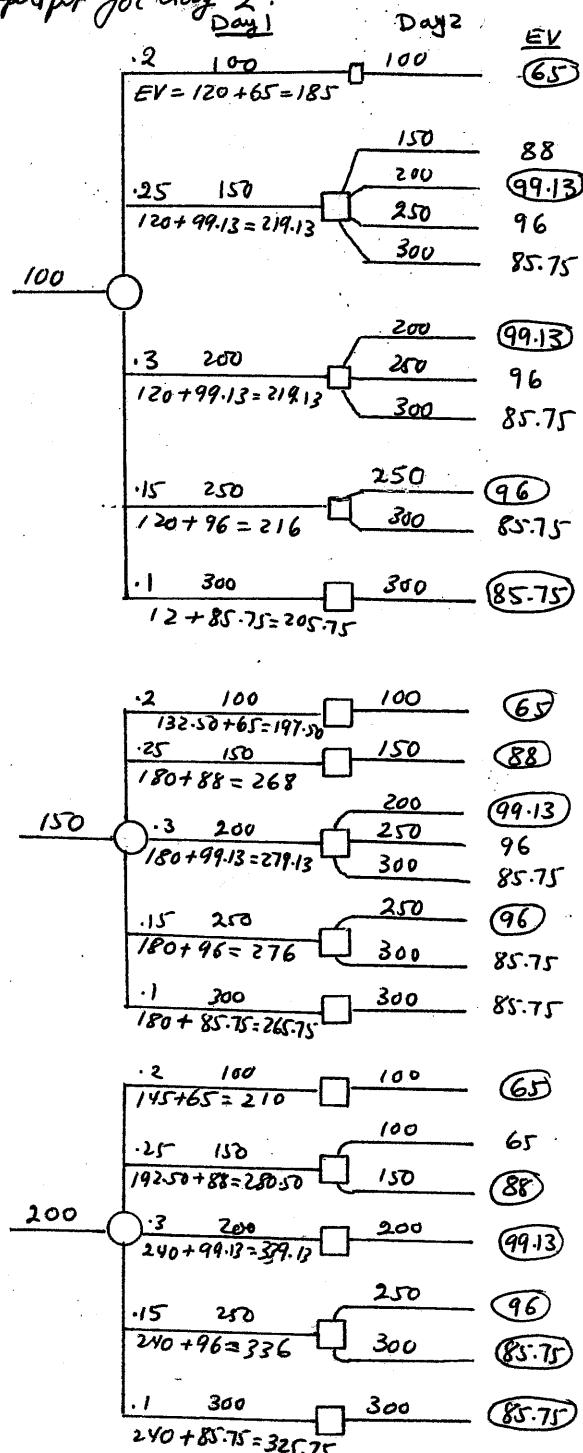
$$= \$96$$

$$E(\text{profit} | 300 \text{ loans}) = 170x_2 + 217.50x_3 + 265x_4 + 312.50x_5 + 360x_6 = \$85.75$$

Set 13.2a

Make use of the results in Problem to determine the expected profit for day 2.

14



$$E(\text{profit} | 100 \text{ loaves}) = 185x \cdot 2 + 219.13x \cdot .25 + 216x \cdot .15 + 205.75x \cdot .1 - 55 = \$155.50$$

$$E(\text{profit} | 150 \text{ loaves}) = 197.50x \cdot 2 + 268x \cdot .25 + 279.13x \cdot .3 + 276x \cdot .15 + 265.75x \cdot .1 - 82.50 = 175.71$$

$$E(\text{profit} | 200 \text{ loaves}) = 210x \cdot 2 + 280.5x \cdot .25 + 339.13x \cdot .3 + 336x \cdot .15 + 325.75x \cdot .1 - 110 = 186.84$$

$$E(\text{profit} | 250 \text{ loaves}) = 222.5x \cdot 2 + 293x \cdot .25 + 351.63x \cdot .3 + 396x \cdot .15 + 385.75x \cdot .1 - 137.50 = 183.71$$

$$E(\text{profit} | 300 \text{ loaves}) = 235x \cdot 2 + 305.5x \cdot .25 + 364.13x \cdot .3 + 411.63x \cdot .15 + 445.75x \cdot .1 - 165 = 173.93$$

Solution: Revenue = \$186.84

Day 1: Stock 200 loaves

Day 2: Stock level = demand

continued...

(a) Decision tree

$$\begin{array}{c} \square \\ \alpha \\ \xrightarrow{\quad} E\{\text{profit}|\alpha\} \end{array}$$

(b) Profit given α

$$\begin{aligned} &= r\alpha(1-p) - C\alpha p \\ &= \alpha(r - [C+r]p) \end{aligned}$$

$C = \$50$ is the loss per defective item
 $r = \$5$ is the profit per good item

$$E\{\text{profit}|\alpha\} = \alpha[r - (C+r)E\{p\}]$$

$$E\{p\} = \int_0^1 p \alpha p^{\alpha-1} dp = \frac{\alpha}{\alpha+1}$$

Hence

$$E\{\text{profit}|\alpha\} = \alpha r - (C+r) \frac{\alpha^2}{\alpha+1}$$

$$\begin{aligned} \frac{\partial E\{\text{profit}\}}{\partial \alpha} &= r - (C+r) \frac{2\alpha(\alpha+1) - \alpha^2}{(\alpha+1)^2} \\ &= r - (C+r) \frac{\alpha(\alpha+2)}{(\alpha+1)^2} \end{aligned}$$

Equating the derivative to zero,
we get

$$C\alpha^2 + 2Ca\alpha - r = 0$$

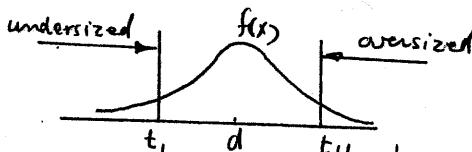
Using $C = \$50$ and $r = \$5$, we get

$$50\alpha^2 + 100\alpha - 5 = 0$$

Thus, $\alpha = 0.49$ or 49 pieces
per day.

Let N = number of cylinders

16



$$E\{\text{Cost}\} = N \left\{ C_1 \int_{t_U}^{\infty} f(x) dx + C_2 \int_{-\infty}^{t_L} f(x) dx \right\}$$

Let $\Phi(z)$ be the standard normal.

$$E\{\text{Cost}\} = N \left\{ C_1 \int_{\frac{t_U-d}{\sigma}}^{\infty} \Phi(z) dz + C_2 \int_{-\infty}^{\frac{t_L-d}{\sigma}} \Phi(z) dz \right\}$$

Continued...

(a) $\square \xrightarrow{d} E\{\text{Cost}\}$

(b)

$$\frac{\partial E\{\text{Cost}\}}{\partial d}$$

$$\begin{aligned} &= -\frac{C_2}{\sigma} \phi\left(\frac{t_L-d}{\sigma}\right) + \frac{C_2}{\sigma} \phi\left(\frac{t_U-d}{\sigma}\right) \\ &= 0 \end{aligned}$$

Thus,

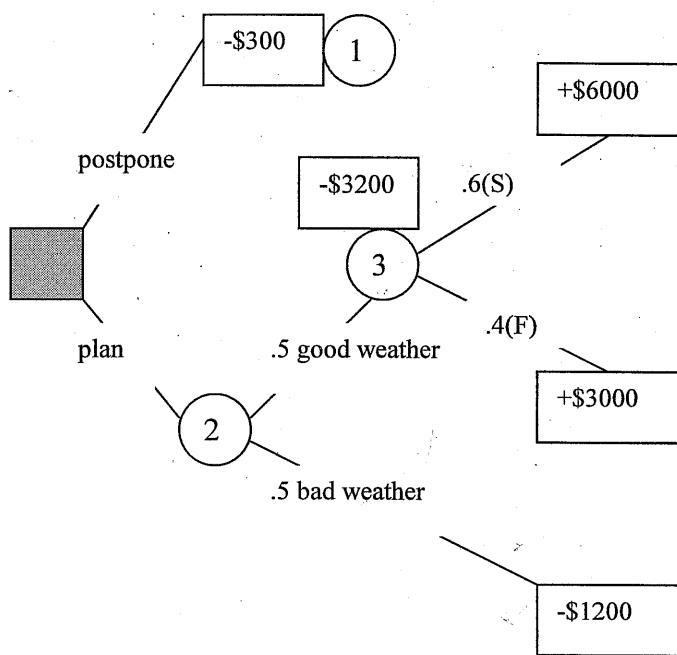
$$\begin{aligned} \frac{C_2}{C_1} &= \frac{\phi\left(\frac{t_U-d}{\sigma}\right)}{\phi\left(\frac{t_L-d}{\sigma}\right)} \\ &= \frac{1}{2} e^{-\frac{1}{2} \left(\frac{t_U-d}{\sigma}\right)^2} + \frac{1}{2} e^{-\frac{1}{2} \left(\frac{t_L-d}{\sigma}\right)^2} \end{aligned}$$

On simplification, we get

$$d^* = \frac{1}{2} \left(t_L + t_U - \frac{2\sigma^2}{t_L - t_U} \ln \frac{C_2}{C_1} \right)$$

Set 13.2a

17



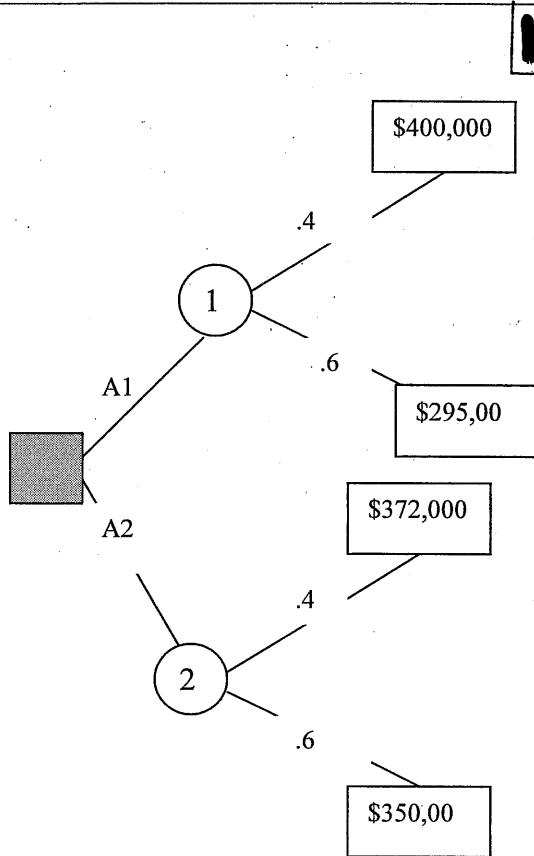
$$E\{\text{Plan}\} = .5(.6 \times 6000 + .4 \times 3000 - 3200) + .5(-1200) = \$200 > -\$300$$

Select "Plan".

P{good W}	Expected value				Decision
	Node 3	Node 2	Node 1		
0	\$4,800.00	-\$1,200.00	-\$300.00		postpone
0.1	\$4,800.00	-\$920.00	-\$300.00		postpone
0.2	\$4,800.00	-\$640.00	-\$300.00		postpone
0.3	\$4,800.00	-\$360.00	-\$300.00		postpone
0.4	\$4,800.00	-\$80.00	-\$300.00		plan
0.5	\$4,800.00	\$200.00	-\$300.00		plan
0.6	\$4,800.00	\$480.00	-\$300.00		plan
0.7	\$4,800.00	\$760.00	-\$300.00		plan
0.8	\$4,800.00	\$1,040.00	-\$300.00		plan
0.9	\$4,800.00	\$1,320.00	-\$300.00		plan
1	\$4,800.00	\$1,600.00	-\$300.00		plan

Set 13.2a

(a)



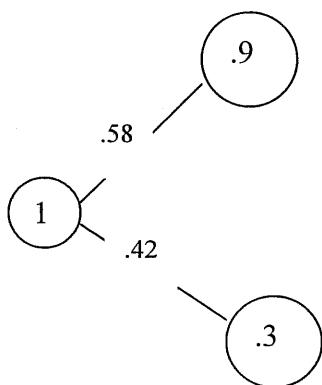
18

$$E\{A1\} = .4 \times 400 + .6 \times 295.5 = \$337,300$$

$$E\{A2\} = .4 \times 372 + .6 \times 350 = \$358,800$$

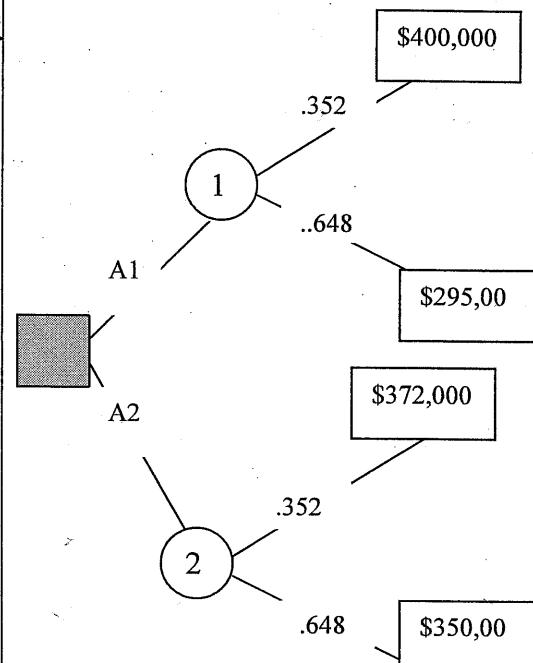
Use mix A2.

(b)



$$\text{Expected probability of price increase} = .58 \times .9 + .42 \times .3 = .648$$

continued...



$$E\{A1\} = .352 \times 400 + .648 \times 295.5 = \$332,284$$

$$E\{A2\} = .352 \times 372 + .648 \times 350 = \$357,744$$

Use mix A2. Decision remains the same. Hence, additional cost is not warranted.

19

$$E\{\text{shortage}\} = \int_I^{200} (x - I) \frac{200}{x^2} dx = 200 \left(\ln \frac{200}{I} + \frac{I}{200} - 1 \right) \leq 40$$

$$E\{\text{surplus}\} = \int_{100}^I (I - x) \frac{200}{x^2} dx = 200 \left(\ln \frac{100}{I} + \frac{I}{100} - 1 \right) \leq 20$$

Simplifying, we get

$$\ln I - \frac{I}{200} \geq 4.098 \quad (1)$$

$$\ln I - \frac{I}{100} \geq 3.505 \quad (2)$$

Using a spreadsheet, the two aspiration levels are satisfied for

$$99 \leq I \leq 151$$

Set 13.2b

States of nature:

m_1 = took calculus

m_2 = didn't take calculus

Outcomes:

v_1 : does well

v_2 : doesn't do well

$P\{m\}$

		v_1	v_2
.3	m_1	.75	
.7	m_2	.5	

$$P\{v\} = .3 \times .75 + .7 \times .5 \\ = .575$$

Prior probabilities:

$$P\{A\} = .75, P\{B\} = .25$$

Let \bar{z} represent the event of having one defective in a sample of size five.

$$P\{\bar{z}|A\} = C^5(.01)^1 (.99)^4 = .04803$$

$$P\{\bar{z}|B\} = C^5(.02)^1 (.98)^4 = .09224$$

$$P\{\bar{z}, A\} = .04803 \times .75 = .036022$$

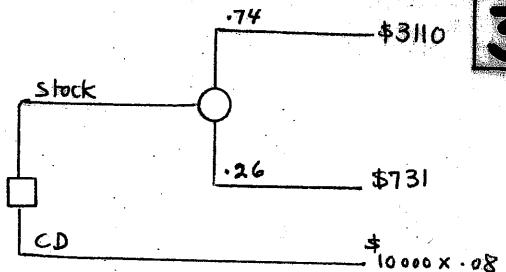
$$P\{\bar{z}, B\} = .09224 \times .25 = .023059$$

$$P\{\bar{z}\} = .036022 + .023059 = .059081$$

$$P\{A|\bar{z}\} = \frac{.036022}{.059081} = .6097$$

$$P\{B|\bar{z}\} = \frac{.023059}{.059081} = .3903$$

1



3

$$EV(\text{Stock}) = .74 \times 3110 + .26 \times 731 \\ = \$2491.46$$

$$EV(\text{CD}) = 10,000 \times .08 = \$800$$

Decision: invest in stock

(a) $P\{\text{success}\} = .7 \quad P\{\text{failure}\} = .3$

4

$$E\{\text{publisher offer}\} = 20,000 + .7(200,000x_1) + .3(10,000x_2) \\ = \$163,000$$

$E\{\text{revenue if you undertake publishing}\}$

$$= -90,000 + .7(200,000x_1) + .3(10,000x_2) = \$196,000$$

Decision: Publish it yourself.

2

(b) Define

m_1 = novel is a success

m_2 = novel is not a success

v_1 = survey predicts success

v_2 = survey does not predict success

$$P\{v_j | m_i\} = \begin{matrix} v_1 & v_2 \\ m_1 & [.8 & .2] \\ m_2 & [.15 & .85] \end{matrix}$$

Prior probabilities: $P\{m_1\} = .7 \quad P\{m_2\} = .3$

$$P\{m_i, v_j\} = \begin{matrix} v_1 & v_2 \\ m_1 & [.8 \times .7 & .2 \times .7] \\ m_2 & [.15 \times .3 & .85 \times .3] \end{matrix}$$

$$= \begin{matrix} v_1 & v_2 \\ m_1 & [.56 & .14] \\ m_2 & [.045 & .255] \end{matrix}$$

continued...

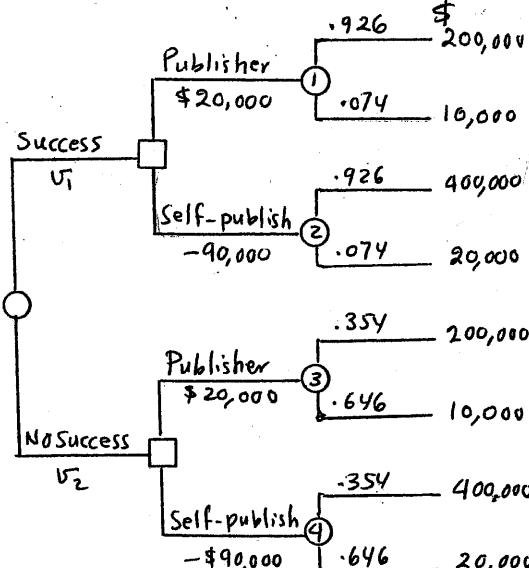
Set 13.2b

5

$$P\{V_1\} = .56 + .045 = .605$$

$$P\{V_2\} = .14 + .255 = .395$$

$$P\{m_i | V_j\} = \begin{matrix} m_1 & \begin{bmatrix} .56 & .14 \\ .605 & .395 \end{bmatrix} \\ m_2 & \begin{bmatrix} .045 & .255 \\ .605 & .395 \end{bmatrix} \\ = & \begin{bmatrix} .926 & .354 \\ .074 & .646 \end{bmatrix} \end{matrix}$$



$$E\{\text{revenue}|①\} = .926 \times 200 + .074 \times 10 + 20 = \$205,940$$

$$E\{\text{revenue}|②\} = .926 \times 400 + .074 \times 20 - 90 = \$281,880$$

$$E\{\text{revenue}|③\} = .354 \times 200 + .646 \times 10 + 20 = \$97,260$$

$$E\{\text{revenue}|④\} = .354 \times 400 + .646 \times 20 - 90 = \$64,520$$

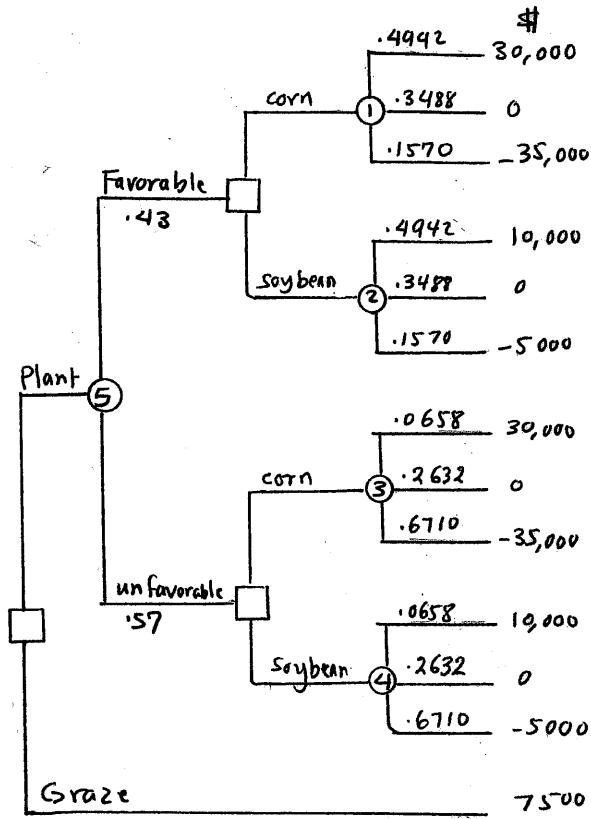
Decision: If survey predicts success, publish the book yourself. Otherwise, use the publisher.

$$\begin{matrix} a_1 & a_2 \\ s_1 & \begin{bmatrix} .85 & .15 \\ .5 & .5 \end{bmatrix} \\ s_2 & \begin{bmatrix} .5 & .5 \\ .15 & .85 \end{bmatrix} \\ s_3 & \begin{bmatrix} .25 & .30 \\ .45 & .45 \end{bmatrix} \end{matrix}$$

$$P\{S, a\} = \begin{bmatrix} .2125 & .0375 \\ .15 & .15 \\ .0675 & .3825 \end{bmatrix}$$

$$P\{a\} = (.43 \quad .57)$$

$$P\{S|a\} = \begin{matrix} a_1 & \begin{bmatrix} .4942 & .3488 & .1570 \\ .0658 & .2632 & .6710 \end{bmatrix} \\ a_2 & \begin{bmatrix} .4942 & .3488 & .1570 \\ .0658 & .2632 & .6710 \end{bmatrix} \end{matrix}$$



$$E\{\text{revenue}|①\} = 30x.4942 + 0x.3488 - 35x.1570 = \$9331$$

$$E\{\text{revenue}|②\} = 10x.4942 + 0x.3488 - 5x.1570 = \$4157$$

$$E\{\text{revenue}|③\} = 30x.0658 + 0x.2632 - 35x.6710 = -\$21,511$$

$$E\{\text{revenue}|④\} = 10x.0658 + 0x.2632 - 5x.6710 = -\$2697$$

$$E\{\text{revenue}|⑤\} = .43x.9331 + (-.2697)x.57 = 2478$$

Decision: Choose grazing

Set 13.2b

$$P\{a|v\} = \begin{bmatrix} v_1 & v_2 \\ .95 & .05 \\ .3 & .7 \end{bmatrix}, P\{v\} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$$

6

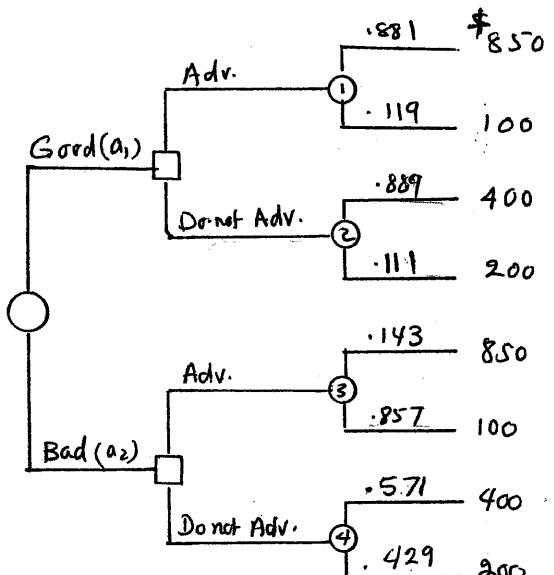
$$P\{v, a\} = \begin{bmatrix} v_1 & v_2 \\ .665 & .035 \\ .090 & .210 \end{bmatrix} P\{a\} = (.755, .245)$$

$$P\{v|a\} = \begin{bmatrix} v_1 & v_2 \\ .881 & .143 \\ .119 & .857 \end{bmatrix}$$

$$P\{a|w\} = \begin{bmatrix} w_1 & w_2 \\ .8 & .2 \\ .4 & .6 \end{bmatrix}, P\{w\} = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$$

$$P\{w, a\} = \begin{bmatrix} w_1 & w_2 \\ .64 & .16 \\ .08 & .12 \end{bmatrix} P\{a\} = (.72, .28)$$

$$P\{w|a\} = \begin{bmatrix} w_1 & w_2 \\ .889 & .571 \\ .111 & .429 \end{bmatrix}$$



$$E\{\text{revenue}|①\} = 850x.881 + 100x.119 = \$760.75$$

$$E\{\text{revenue}|②\} = 400x.889 + 200x.111 = \$377.80$$

$$E\{\text{revenue}|③\} = 850x.143 + 100x.857 = \$207.25$$

$$E\{\text{revenue}|④\} = 400x.571 + 200x.429 = \$314.70$$

Decision:

Advertise if test is good, else do not advertise

(a) $\theta_1 = \text{lot is good (4% defectives)}$
 $\theta_2 = \text{lot is bad (15% defectives)}$

$Z_1 = \text{both items of the sample are good}$
 $Z_2 = \text{one item is good}$
 $Z_3 = \text{both items are bad}$

$$P\{\theta_1\} = .95, P\{\theta_2\} = .05$$

$$P\{Z_1|\theta_1\} = C_2^2 (.96)^2 (.04)^0 = .922$$

$$P\{Z_2|\theta_1\} = C_1^2 (.96)^1 (.04)^1 = .0768$$

$$P\{Z_3|\theta_1\} = C_0^2 (.96)^0 (.04)^2 = .0016$$

$$P\{Z_1|\theta_2\} = C_2^2 (.85)^2 (.15)^0 = .7225$$

$$P\{Z_2|\theta_2\} = C_1^2 (.85)^1 (.15)^1 = .255$$

$$P\{Z_3|\theta_2\} = C_0^2 (.85)^0 (.15)^2 = .0225$$

$$P\{\theta, Z\} = \begin{bmatrix} \theta_1 & Z_1 & .8759 \\ \theta_1 & Z_2 & .07296 \\ \theta_2 & Z_1 & .00152 \\ \theta_2 & Z_2 & .036125 \\ \theta_2 & Z_3 & .01275 \\ \theta_2 & Z_4 & .001125 \end{bmatrix}$$

$$P\{Z\} = (.912025, .08571, .002645)$$

$$P\{\theta|Z\} = \begin{bmatrix} \theta_1 & .96039 & .85124 & .57467 \\ \theta_2 & .03961 & .14876 & .42533 \end{bmatrix}$$

Case 1: Two good items (Z_1)

$$E(\text{cost}/\text{customer A}) \quad \begin{array}{c} G \\ 5\% \\ 8\% \end{array} \quad \begin{array}{c} B \\ \$50 \\ \$200 \end{array} \quad \begin{array}{c} A \\ \$1000 \\ \$700 \end{array}$$

$$= 50x.96039 + 1000x.03961 = \$87.63$$

$$E(\text{cost}/\text{customer B})$$

$$= 200x.96039 + 700x.03961 = \$219.81$$

Decision: Ship lot to A

Case 2: One good item (Z_2)

$$E(\text{cost}/\text{customer A})$$

$$= 50x.85124 + 1000x.14876 = \$191.32$$

$$E(\text{cost}/\text{customer B})$$

$$= 200x.85124 + 700x.14876 = 274.38$$

Decision: Ship lot to A

Case 3: Both items bad (Z_3)

$$E\{\text{cost}|A\} = 50x.57467 + 1000x.42533 = \$454.06$$

$$E\{\text{cost}|B\} = 200x.57467 + 700x.42533 = \$412.67$$

Decision: Ship to B

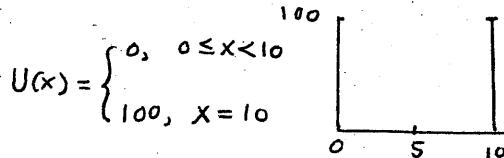
Set 13.2c

(a) $E\{\text{value of poker game}\}$

$$= .5 \times 10 + .5 \times 0 = \$5$$

No advantage

(b)



(c) Because $U(5) = 0$ and $U(10) = 100$, the decision is to play the poker game

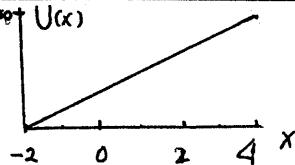
Worst condition cost = $900,000 + 350,000$
 $= \$1,250,000$

Best condition savings = $900,000$

Lottery:

$$\begin{aligned} U(x) &= p U(-1,250,000) + (1-p) U(900,000) \\ &= p(0) + (1-p)(100) \\ &= 100(1-p) = 100 - 100p \end{aligned}$$

(a) $U(x)$



$$\frac{U(0)}{U(4)} = \frac{0 - (-2)}{4 - (-2)} = \frac{2}{6} = \frac{1}{3}$$

$$U(0) = \frac{1}{3}(100) = 33.33$$

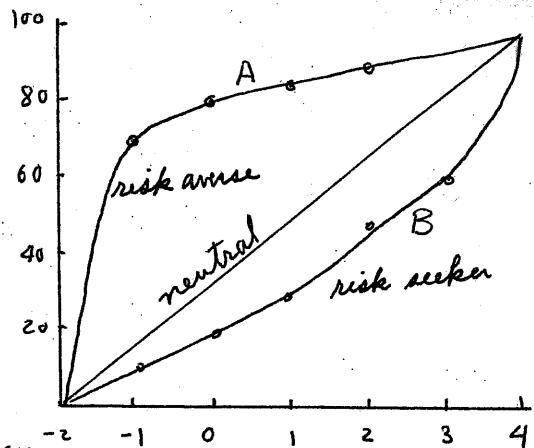
Now, $U(0) = p U(-2) + (1-p) U(4)$
 $= 100(1-p)$

Thus, for $U(0) = 33.33$, $-p = .6667$

b)	X	$U(x)_A$	$U(x)_B$
	-2	0	0
	-1	70	10
	0	80	20
	1	85	30
	2	90	50
	3	95	60
	4	100	100

Continued...

3 continued



(c) Venture I:

$$U_A(3000) = 95, U_A(-1000) = 70$$

$$EU(I) = .4 \times 95 + .6 \times 70 = 80$$

Venture II:

$$U_A(2000) = 90, U_A(0) = 80$$

$$EU(II) = .4 \times 90 + .6 \times 80 = (84)$$

Decision: Select II

$$E\{\$ \text{ venture II}\} = \frac{84-80}{85-80} = \frac{x-0}{1-0}$$

$$\Rightarrow x = .8 \text{ or } \$800$$

(d) Venture I:

$$U_B(3000) = 60, U_B(-1000) = 10$$

$$EU(I) = .6 \times 60 + .4 \times 10 = (40)$$

Venture II:

$$U_B(2000) = 50, U_B(0) = 20$$

$$EU(II) = .6 \times 50 + .4 \times 20 = 38$$

Decision: Select I.

$$E\{\$ \text{ venture I}\} = \$1500$$

Set 13.3a

(a)

Laplace:

$$E(a_1) = \frac{1}{3}(85+60+40) = 61.67$$

$$E(a_2) = \frac{1}{3}(92+85+81) = 86$$

$$E(a_3) = \frac{1}{3}(100+88+82) = 89$$

Study all night.

Maximin:

Because this is a reward matrix, we use maximin

$$\begin{bmatrix} 85 & 60 & 40 \\ 92 & 85 & 81 \\ 100 & 88 & 82 \end{bmatrix} \xrightarrow{\text{min}} \begin{bmatrix} 40 \\ 81 \\ 82 \end{bmatrix}$$

Decision: Study all night

Savage:

$$\text{"Cost" matrix} = \begin{bmatrix} -85 & -60 & -40 \\ -92 & -85 & -81 \\ -100 & -88 & -82 \end{bmatrix}$$

$$\text{Regret matrix} = \begin{bmatrix} 15 & 28 & 42 \\ 8 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row max}} \begin{bmatrix} 42 \\ 8 \\ 0 \end{bmatrix} \xleftarrow{\text{min}}$$

Decision: Study all night

Hurwicz:

$$\begin{array}{cccccc} \text{Row} & \text{Row} & & & & \text{At} \\ \text{min} & \text{max} & \alpha(\text{Row min}) + (1-\alpha)(\text{Row max}) & & & \alpha = .5 \\ \hline a_1 & -85 & -40 & -40 - 45\alpha & = -62.5 & \\ a_2 & -92 & -81 & -81 - 11\alpha & = -86.5 & \\ a_3 & -100 & -82 & -82 - 18\alpha & = -91 & \end{array}$$

Decision: Study all night

(b)

$$\text{"Cost" matrix} = \begin{bmatrix} -80 & -60 & 0 \\ -90 & -80 & -80 \\ -90 & -80 & -80 \end{bmatrix}$$

Laplace:

$$E(a_1) = \frac{1}{3}(80+60+0) = -46.67$$

$$E(a_2) = \frac{1}{3}(90+80+80) = -83.33$$

$$E(a_3) = \frac{1}{3}(90+80+80) = -83.33$$

Decision: Select second or third.

Minimax

$$\begin{bmatrix} -80 & -60 & 0 \\ -90 & -80 & -80 \\ -90 & -80 & -80 \end{bmatrix} \xrightarrow{\text{min}} \begin{bmatrix} 0 \\ -80 \\ -80 \end{bmatrix}$$

Select either the second or the third option

Savage:

$$\begin{bmatrix} 10 & 20 & 80 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{min}} \begin{bmatrix} 80 \\ 0 \\ 0 \end{bmatrix}$$

Select either the second or the third option.

Hurwicz:

	Row min	Row max	$\alpha(\text{Row min}) + (1-\alpha)(\text{Row max})$	At $\alpha = .5$
a_1	-80	0	-80α	-40
a_2	-90	-80	$-80 - 10\alpha$	-85
a_3	-90	-80	$-80 - 10\alpha$	-85

Select the second or the third option

Laplace:

$$E(a_1) = \frac{1}{4}(-20+60+30-5) = 16.25$$

$$E(a_2) = \frac{1}{4}(40+50+35+0) = 31.25$$

$$E(a_3) = \frac{1}{4}(-50+100+45-10) = 21.25$$

$$E(a_4) = \frac{1}{4}(12+15+15+10) = 13$$

Plant wheat

Minimax:

$$\begin{array}{cccccc} & & & & & \text{Row max} \\ & a_1 & \begin{bmatrix} 20 & -60 & -30 & 5 \end{bmatrix} & 20 \\ & a_2 & \begin{bmatrix} -40 & -50 & -35 & 0 \end{bmatrix} & 0 \\ & a_3 & \begin{bmatrix} 50 & -100 & -45 & 10 \end{bmatrix} & 50 \\ & a_4 & \begin{bmatrix} -12 & -15 & -15 & -10 \end{bmatrix} & -10 \end{array} \xrightarrow{\text{minimax}}$$

Recommend grazing.

Savage:

	Row max
a_1	60
a_2	50
a_3	90
a_4	85

Plant wheat

continued...

continued...

Set 13.3a

Hurwicz:

			at $\alpha = .5$
	(Row) min	(Row) max	$\alpha(\text{Row min}) + (1-\alpha)(\text{Row max})$
a_1	-60	20	$20 + 80 = -20$
a_2	-50	0	$-50 \alpha = -25$
a_3	-100	50	$50 - 150\alpha = -25$
a_4	-15	-10	$-10 - 5\alpha = -12.5$

Select wheat or soybeans.

Laplace:

$$\min_{a_i} \frac{\int_{Q^*}^{Q^{**}} (K_i + c_i Q) dQ}{Q^{**} - Q^*}$$

$$= \min_{a_i} \left\{ K_i + \frac{c_i}{2} (Q^{**} - Q^*) \right\}$$

$$E\{a_1\} = 100 + \frac{5}{2} (3000) = \$7600$$

$$E(a_2) = 40 + \frac{12}{2} (3000) = \$18,040$$

$$E(a_3) = 150 + \frac{3}{2} (3000) = \$4650$$

$$E(a_4) = 90 + \frac{8}{2} (3000) = \$12,090$$

Select machine 3

Minimax:

$$\min_{a_i} \max_{Q^* \leq Q \leq Q^{**}} \{ K_i + c_i Q \}$$

$$= \min_{a_i} \{ K_i + c_i Q^{**} \}$$

$$\text{machine } 3: \{ K_i + c_i Q^{**} \}$$

$$1 \quad 100 + 5 \times 4000 = \$20,100$$

$$2 \quad 40 + 12 \times 4000 = \$48,040$$

$$3 \quad 150 + 3 \times 4000 = (\$12,150)$$

$$4 \quad 90 + 8 \times 4000 = \$32,090$$

Select machine 3.

2 continued

Savage:

$$\min_{a_i} \max_{Q^* \leq Q \leq Q^{**}} \{ K_i + c_i Q - \frac{\min_{a_i} (K_i + c_i Q)}{a_i} \}$$

$$a_1 \quad \frac{\text{cost}}{100+5Q}$$

Regret

$$a_2 \quad 40+12Q$$

Regret

$$a_3 \quad 150+3Q$$

Regret

$$a_4 \quad 90+8Q$$

Regret

Smallest for $1000 \leq Q \leq 4000$

Hurwicz:

$$\min_{a_i} \{ \alpha (K_i + c_i Q^*) + (1-\alpha) (K_i + c_i Q^{**}) \}$$

$$= \min_{a_i} \{ K_i + c_i (\alpha Q^* + (1-\alpha) Q^{**}) \}$$

For $\alpha = 1/2$, we have

$$a_1: 100 + 5 \left(\frac{1000}{2} + \frac{4000}{2} \right) = \$12,600$$

$$a_2: 40 + 12 \times 2500 = \$30,040$$

$$a_3: 150 + 3 \times 2500 = (\$7600)$$

$$a_4: 90 + 8 \times 2500 = \$20,090$$

Select machine 3.

Continued...

Set 13.4a

(a)

$$\begin{bmatrix} 8 & 6 & 2 & 8 \\ 8 & 9 & 4 & 5 \\ 7 & 5 & 3 & 5 \\ 8 & 9 & 4 & 5 \end{bmatrix} \quad \boxed{4}$$

Saddle point solution at (2,3)

(b)

$$\begin{bmatrix} 4 & -4 & -5 & 6 \\ -3 & -4 & -9 & -2 \\ 6 & 7 & -8 & -9 \\ 7 & 3 & -9 & 5 \end{bmatrix} \quad \boxed{-5}$$

Saddle point solution at (1,3)

(a) $p \geq 5, q \leq 5$

(b) $p \leq 7, q \geq 7$

(a)

$$\begin{bmatrix} 1 & 9 & 6 & 0 \\ 2 & 3 & 8 & 4 \\ -5 & -2 & 10 & -3 \\ 7 & 4 & -2 & -5 \end{bmatrix} \quad \boxed{0} \quad \boxed{2}$$

$2 < v < 4$

(b)

$$\begin{bmatrix} -1 & 9 & 6 & 8 \\ -2 & 10 & 4 & 6 \\ 5 & 3 & 0 & 7 \\ 7 & -2 & 8 & 4 \end{bmatrix} \quad \boxed{-1} \quad \boxed{-2}$$

$0 < v < 7$

(c)

$$\begin{bmatrix} 3 & 6 & 1 \\ 5 & 2 & 3 \\ 4 & 2 & -5 \\ 5 & 6 \end{bmatrix} \quad \boxed{1} \quad \boxed{2}$$

$2 < v < 3$

(d)

$$\begin{bmatrix} 3 & 7 & 1 & 3 \\ 4 & 8 & 0 & -6 \\ 6 & -9 & -2 & 4 \\ 6 & 8 \end{bmatrix} \quad \boxed{1} \quad \boxed{-6}$$

$0 < v \leq 1$

Define the following strategies:

- 1 - no campaign
- 2 - TV
- 3 - Newspaper
- 4 - Radio
- 5 - TV + newspaper
- 6 - TV + radio
- 7 - Radio + newspaper
- 8 - TV + radio + newspaper

The payoff is the additional percentage of customers reached by Company A.

	1	2	3	4	5	6	7	8	
1	0	-50	-30	-20	-80	-70	-50	-100	-100
2	50	0	20	30	-30	-20	0	-50	-50
3	30	-20	0	10	-50	-40	-20	-70	-70
4	20	-30	-10	0	-60	-50	-30	-80	-80
5	80	30	50	60	0	10	30	-20	-20
6	70	20	40	50	-10	0	20	-30	-30
7	50	0	20	30	-30	-20	0	-50	-50
8	100	50	70	80	20	30	50	0	0
	100	50	70	80	20	30	50	0	0

The game has a saddle point at (8,8), meaning that both companies should advertise in all three media. The game is fair because its value equals zero.

$\min_j a_{ij} \leq a_{ij}, \text{ all } i, j$

$\max_i \min_j a_{ij} \leq \max_i a_{ij}, \text{ all } j$

$\leq \min_j \max_i a_{ij}$

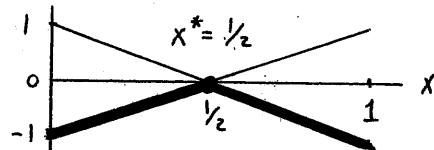
Set 13.4b

3

	y	$1-y$
	B_H	B_T
x	A_H	1 -1
$1-x$	A_T	-1 1

B' s pure strategy A's expected payoff

B_H	$x + (-1)(1-x) = -1+2x$
B_T	$-x + 1(1-x) = 1-2x$



B' s game:

$$y - (1-y) = -y + (1-y) \Rightarrow y^* = \frac{1}{2}$$

$$\text{Value of the game} = -1 + 2\left(\frac{1}{2}\right) = 0$$

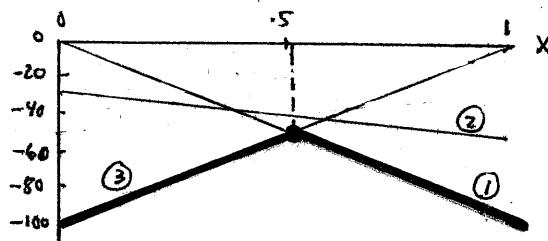
Robin's Payoff matrix:

	$100-A$	$50/50-B/B$	$100B$
x	A	$\begin{bmatrix} -100 & -50 & 0 \end{bmatrix}$	
$(1-x)$	B	$\begin{bmatrix} 0 & -30 & -100 \end{bmatrix}$	

2

Police strategy Robin's expected payoff

1	$-100x$
2	$-50x + (-30)(1-x) = -30-20x$
3	$-100 + 100x$



Robin's strategy: mix A and B 50-50.

$$\text{Game cost} = \$50$$

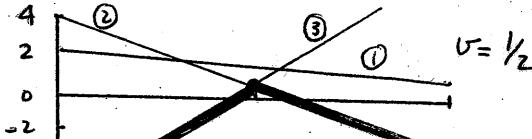
Police strategy:

$$-100y_1 = -100(1-y_1) \Rightarrow y_1 = .5$$

$$\text{Solution: } y_1 = .5, y_2 = 0, y_3 = .5$$

(a) B' s strategy A's exp. payoff

1	$-x + 2$
2	$-7x + 4$
3	$13x - 6$

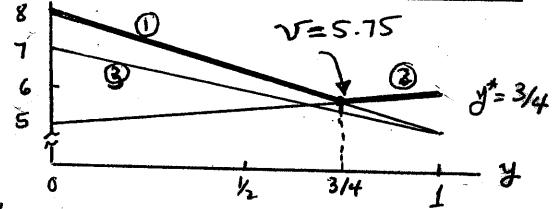


$$\begin{aligned} A's \text{ game: } x_1 = x_2 = .5, v = .5 \\ B's \text{ game: mix } B's \text{ (2) and (3)} \end{aligned}$$

$$-10y_2 + 7 = 10y_2 - 6 \Rightarrow y_2 = 13/20, y_3 = 7/20$$

(b) A's pure strategy B's exp. payoff

1	$-3y + 8$
2	$y + 5$
3	$-2y + 7$



B' s game: mix B' 's (1) and (2)

$$-x_1 + 6 = 3x_1 + 5 \Rightarrow x_1 = 1/4, x_2 = 3/4, x_3 = 0$$

(a) A's strategy B's exp. payoff

1	$5\left(\frac{49}{54}\right) + 50\left(\frac{5}{54}\right) + 50(0) = \frac{55}{6}$
2	$1\left(\frac{49}{54}\right) + 1\left(\frac{5}{54}\right) + 1(0) = 1$
3	$10\left(\frac{49}{54}\right) + 1\left(\frac{5}{54}\right) + 10(0) = \frac{55}{6}$

$$\max(\text{exp. payoffs}) = \boxed{\frac{55}{6}}$$

B' s strategy A's exp. payoff

1	$5\left(\frac{1}{6}\right) + 1(0) + 10\left(\frac{5}{6}\right) = \frac{55}{6}$
2	$50\left(\frac{1}{6}\right) + 1(0) + 1\left(\frac{5}{6}\right) = \frac{55}{6}$
3	$50\left(\frac{1}{6}\right) + 1(0) + 10\left(\frac{5}{6}\right) = \frac{100}{6}$

$$\min(\text{exp. payoffs}) = \boxed{\frac{55}{6}}$$

$$\text{value of the game} = \frac{55}{6}$$

(b)

$$\begin{aligned} v &= \left(5\left(\frac{1}{6}\right) + 1(0) + 10\left(\frac{5}{6}\right)\right)\left(\frac{49}{54}\right) \\ &\quad + \left(50\left(\frac{1}{6}\right) + 1(0) + 1\left(\frac{5}{6}\right)\right)\left(\frac{5}{54}\right) \\ &\quad + \left(50\left(\frac{1}{6}\right) + 1(0) + 10\left(\frac{5}{6}\right)\right)0 = \frac{55}{6} \end{aligned}$$

13-23

Set 13.4c

		Team 2					
		AB	AC	AD	BC	BD	CD
Team 1	AB	1	0	0	0	0	-1
	AC	0	1	0	0	-1	0
	AD	0	0	1	-1	0	0
	BC	0	0	-1	1	0	0
	BD	0	-1	0	0	1	0
	CD	-1	0	0	0	0	1

Team 1 LP:

$$\text{Maximize } Z = v \\ \text{s.t.}$$

$$\begin{aligned} v - x_1 & \leq 0 \\ v - x_2 & \leq 0 \\ v - x_3 + x_4 & \leq 0 \\ v + x_3 - x_4 & \leq 0 \\ v + x_2 - x_5 & \leq 0 \\ v + x_1 - x_6 & \leq 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 & = 1 \end{aligned}$$

v unrestricted, $x_j \geq 0$

Team 1 Solution: $x_1 = x_6 = .5$, all others = 0
Team 2 Solution: $y_1 = y_6 = .5$, all others = 0 $\forall v \in [0, 1]$

(a) Maximize $Z = v$

s.t.

$$\begin{aligned} v - 3x_1 - 2x_2 + x_3 + x_4 & \leq 0 \\ v + 2x_1 - 3x_2 - 2x_3 + 2x_4 & \leq 0 \\ v - x_1 + 3x_2 + 2x_3 - 4x_4 & \leq 0 \\ v - 2x_1 - 2x_3 - x_4 & \leq 0 \\ x_1 + x_2 + x_3 + x_4 & = 1 \end{aligned}$$

(b) v unrestricted, all $x_j \geq 0$

Solution:

value of game = .5 in favor of UA

UA Strategy: $x_2 = x_4 = .5$, all others = 0

DU Strategy: $x_1 = .58$, $x_3 = .42$,
all others = 0

(c) Expected number of points

$$= 60 \times .5 = 30$$

in favor of UA

1 $(n_1, n_2) = \text{Bello's allocation between the two ports}$

$$= \{(2, 0), (1, 1), (0, 2)\}$$

Enemy's allocation = $\{(3, 0), (2, 1), (1, 2), (0, 3)\}$

(a) $(3, 0) \quad (2, 1) \quad (1, 2) \quad (0, 3)$

(2, 0)	-1	-1	0	0
(1, 1)	0	-1	-1	0
(0, 2)	0	0	-1	-1

Maximize $Z = v$

s.t.

$$\begin{aligned} v + x_1 & \leq 0 \\ v + x_1 + x_2 & \leq 0 \\ v + x_2 + x_3 & \leq 0 \\ v + x_3 & \leq 0 \\ x_1 + x_2 + x_3 & = 1 \end{aligned}$$

(b) v unrestricted, $x_1, x_2, x_3 \geq 0$

Solution: $v = -\frac{1}{2} \Rightarrow \text{enemy wins}$

$$\begin{aligned} x_1 & = .5, x_2 = 0, x_3 = .5 \\ y_1 & = .5, y_2 = y_3 = -y_4 = \end{aligned}$$

(a, b) = (Nbr shown, Nbr guessed)

2 $(1, 1) \quad (1, 2) \quad (2, 1) \quad (2, 2)$

(1, 1)	0	2	-3	0
(1, 2)	-2	0	0	3
(2, 1)	3	0	0	-4
(2, 2)	0	-3	4	0

Maximize $Z = v$

s.t.

$$\begin{aligned} v - 2x_2 - 3x_3 & \leq 0 \\ v - 2x_1 + 3x_4 & \leq 0 \\ v + 3x_1 - 4x_4 & \leq 0 \\ v - 3x_2 + 4x_3 & \leq 0 \\ x_1 + x_2 + x_3 + x_4 & = 1 \end{aligned}$$

Solution: v unrestricted, $x_j \geq 0$

Player A:

$$x_1 = 0, x_2 = .571, x_3 = .429, x_4 = 0$$

Player B:

$$y_1 = 0, y_2 = .571, y_3 = .429, y_4 = 0$$

value of the game = 0

Chapter 14

Probabilistic Inventory Models

Set 14.1a

(a) Effective lead time L

$$= 15 - 10 = 5 \text{ days}$$

$$M_L = 100 \times 5 = 500 \text{ units}$$

$$\sigma_L = \sqrt{10^2 \times 5} = 22.36 \text{ units}$$

$$B \geq 22.36 \times 1.645 \approx 37 \text{ units}$$

Order 1000 units whenever the inventory level drops to 537 units

(b) Effective lead time $L = 23 - 20 = 3 \text{ days}$

$$M_L = 100 \times 3 = 300 \text{ units}$$

$$\sigma_L = \sqrt{10^2 \times 3} = 17.32 \text{ units}$$

$$B \geq 17.32 \times 1.645 \approx 29 \text{ units}$$

Order 1000 units whenever the inventory level drops to 329 units

(c) Effective lead time = 8 days

$$M_L = 100 \times 8 = 800 \text{ units}$$

$$\sigma_L = \sqrt{10^2 \times 8} = 28.28 \text{ units}$$

$$B \geq 28.28 \times 1.645 \approx 47 \text{ units}$$

(d) Effective lead time = 0

$$M_L = \sigma_L = 0, B \geq 0$$

Order 1000 units whenever the inventory level drops to 0 unit.

Demand/day = $N(200, 20)$

$h = \$0.04/\text{day/unit}$, $K = \$100$, $L = 7 \text{ days}$

$$\text{order quantity} = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \times 100 \times 200}{0.04}} = 1000 \text{ units}$$

$$\text{Cycle length} = \frac{1000}{200} = 5 \text{ days}$$

Effective lead time = $7 - 5 = 2 \text{ days}$

$$M_L = 200 \times 100 = 2000 \text{ units } K = \frac{100}{0.02} = 5000$$

$$\sigma_L = \sqrt{20^2 \times 2} = 28.28$$

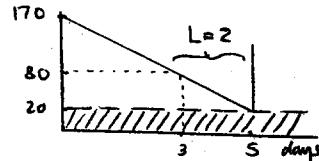
$$B \geq 28.28 \times 2.06 = 58.27 = 59 \text{ discs}$$

Order 1000 discs whenever the inventory level drops to 459 units.

Demand/day = $N(30, 5)$

$h = \$0.02/\text{day/unit}$, $K = \$30$

$$(a) L = \frac{80 - 20}{30} = 2 \text{ days}$$



$$M_L = 60 \text{ units}$$

$$\sigma_L = \sqrt{5^2 \times 2} \approx 7.07 \text{ units}$$

$P\{\text{demand during } L \geq 80\}$

$$= P\{Z \geq \frac{80 - 60}{7.07}\}$$

$$= P\{Z \geq 2.83\}$$

$$= 1 - .9977 = .0023$$

$$(b) y = \sqrt{\frac{2 \times 30 \times 30}{0.02}} = 300 \text{ rolls}$$

$$\text{Cycle length} = \frac{300}{30} = 10 \text{ days}$$

Lead time = 2 days

$$M_L = 2 \times 30 = 60 \text{ units}$$

$$\sigma_L = \sqrt{5^2 \times 2} = 7.07 \text{ units}$$

$$K_1 = 1.28$$

$$B \geq 7.07 \times 1.28 \approx 10$$

Order 300 rolls whenever the inventory level drops to 70 rolls.

Set 14.1b

(a) $D/y = \frac{1000}{320} = 3.125$ set-ups |

(b) $\frac{KD}{y} = 100 \times 3.125 = \$312.50/\text{month}$

(c) $h\left(\frac{y}{2} + R - E\{x\}\right) = 2\left(\frac{320}{2} + 94 - 50\right) = \408

(d) $pS = 10 \times .20397 \approx \2.04

(e) $\int_R^{\infty} f(x) dx = \int_{94}^{100} \frac{1}{100} dx = \frac{100-94}{100} = .06$

$D = 1000$ gallons per month

$K = \$100$, $h = \$2/\text{gal}/\text{month}$

$p = \$10/\text{gal}$.

$$f(x) = \frac{1}{50}, \quad 0 \leq x \leq 50, \quad E\{x\} = 25$$

$$\hat{y} = \sqrt{\frac{2 \times 1000(100+10 \times 25)}{2}} = 591.6$$

$$\tilde{y} = \frac{PD}{h} = \frac{10 \times 1000}{2} = 5000$$

$\tilde{y} > \hat{y} \Rightarrow$ unique solution exists

$$S = \int_R^{50} (x-R) \frac{1}{50} dx = \frac{R^2}{100} - R + 25$$

$$y_i = \sqrt{\frac{2 \times 1000(100+10S)}{2}} = \sqrt{100,000 + 10,000S}$$

$$\int_{R_i}^{50} \frac{1}{50} dx = \frac{2y_i}{5000} \Rightarrow R_i = 50 - \frac{y_i}{100}$$

Iteration 1:

$$S = 0$$

$$y_1 = \sqrt{100,000} = 316.23 \text{ gal}$$

$$R_1 = 50 - \frac{316.23}{100} = 46.84 \text{ gal}$$

Iteration 2:

$$S = \frac{46.84}{100} - 46.84 + 25 = .099856$$

$$y_2 = \sqrt{100,000 + 10,000 \times .099856} = 317.80$$

$$R_2 = 50 - \frac{317.80}{100} = 46.82$$

Iteration 3:

$$S = \frac{46.82}{100} - 46.82 + 25 = .101124$$

continued...

$y_3 = \sqrt{100,000 + 10,000 \times .101124} = 317.82$ | 2 continued

$$R_3 = 50 - \frac{317.82}{100} = 46.82$$

Optimum solution:

$$y^* \approx 318 \text{ gal}, \quad R^* \approx 47 \text{ gal}$$

3

$$f(x) = \frac{1}{20}, \quad 40 \leq x \leq 60, \quad E\{x\} = 50$$

$$\hat{y} = \sqrt{\frac{2 \times 1000(100+10 \times 50)}{2}} = 774.6 \text{ gal}$$

$$\tilde{y} = \frac{10 \times 1000}{2} = 5000 \text{ gal}$$

$\tilde{y} > \hat{y} \Rightarrow$ unique solution exists

$$S = \int_R^{60} (x-R) \frac{1}{20} dx = \frac{1}{20} \left[\frac{x^2}{2} - RX \right]_R^{60} = \frac{R^2}{40} - 3R + 90$$

$$y_i = \sqrt{100,000 + 10,000S}$$

$$\int_{R_i}^{60} \frac{1}{20} dx = \frac{2y_i}{10 \times 1000} \Rightarrow R_i = 60 - \frac{y_i}{250}$$

Iteration 1:

$$S = 0$$

$$y_1 = \sqrt{100,000} = 316.23 \text{ gal}$$

$$R_1 = 60 - \frac{316.23}{250} = 58.735$$

Iteration 2:

$$S = \frac{58.735}{40} - 3 \times 58.735 + 90 = .04$$

$$y_2 = \sqrt{100,000 + 10,000 \times .04} = 316.823$$

$$R_2 = 60 - \frac{316.823}{250} = 58.733 \text{ gal}$$

Optimum solution:

$$y^* = 316.85 \approx 317 \text{ gal.}$$

$$R^* = 58.73 \approx 59 \text{ gal.}$$

R^* in the present model is smaller than R^* in Example because $f(x)$ has a smaller variance, and hence less uncertainty.

Set 14.1b

For the normal distribution, it can be shown that the following approximation holds

$$S = \int_R^\infty (x-R) f(x) dx \\ \approx \sqrt{\text{Var}\{x\}} L(R_s) \quad ①$$

where

$\text{Var}\{x\}$ = variance of x given $f(x)$

$$R_s = \frac{R - E\{x\}}{\sqrt{\text{Var}\{x\}}} \quad ②$$

$L(R_s)$ = standard normal loss integral

$$= \int_{R_s}^{\infty} (z-R_s) \Phi(z) dz$$

$\Phi(z)$ is $N(0, 1)$. The values of $L(\cdot)$ can be found in standard statistical tables

$$\int_R^\infty f(x) dx = \frac{hy}{PD}$$

$$\text{or } \int_R^a \Phi(z) dz = \frac{hy}{PD} \quad ③$$

The steps of the solution algorithm are:

1. Compute first trial

$$y = \sqrt{\frac{2KD}{h}}$$

2. Compute R_s from ③ using the current value of y and the standard normal tables

3. Compute R from ② using the current value of R_s ; that is,

$$R = E\{x\} + R_s \sqrt{\text{Var}\{x\}}$$

4

If two successive values of R are approximately equal, stop. Otherwise, go to step 4

4 continued

4. Compute S from ① using standard normal loss integral tables. Then find

$$y = \sqrt{\frac{2D(K+RS)}{h}}$$

Go to step 2.

Set 14.2a

$$E\{C(y)\} = h \sum_{D=0}^y (y-D) f(D) + p \sum_{D=y+1}^{\infty} (D-y) f(D)$$

Consider $E\{C(y)\} \leq E\{C(y-1)\}$:

$$\begin{aligned} E\{C(y-1)\} &= h \sum_{D=0}^{y-1} (y-1-D) f(D) \\ &\quad + p \sum_{D=y}^{\infty} (D-y+1) f(D) \\ &= h \sum_{D=0}^{y-1} (y-D) f(D) \\ &\quad + p \sum_{D=y}^{\infty} (D-y) f(D) \\ &\quad - h \sum_{D=0}^{y-1} f(D) + p \sum_{D=y}^{\infty} f(D) - c \\ &= E\{C(y)\} + p - (h+p) \sum_{D=0}^{y-1} f(D) \end{aligned}$$

Thus,

$$E\{C(y-1)\} - E\{C(y)\} = p - (h+p) P\{D \leq y\} \geq 0$$

Hence

$$P\{D \leq y-1\} \leq \frac{p}{p+h}$$

Similarly, it can be shown that

$$P\{D \leq y\} \geq \frac{p}{p+h}$$

Thus, y^* must satisfy

$$P\{D \leq y^*-1\} \leq \frac{p}{p+h} \leq P\{D \leq y^*\}$$

$$f(D) = \frac{1}{5}, \quad 10 \leq D \leq 15$$

$$\int_0^y f(D) dD \leq .1:$$

$$\int_{10}^y \frac{1}{5} dD = \frac{y-10}{5} \leq .1 \Rightarrow y \leq 10.5$$

$$\int_0^y f(D) dD \leq .1:$$

$$\int_0^y \frac{1}{5} dD = \frac{15-y}{5} \leq .1 \Rightarrow y \geq 14.5$$

The two conditions cannot be satisfied simultaneously.

1

$$q = \frac{p}{p+h} = \frac{p}{p+1}$$

y	0	1	2	3	4	5	6
$P\{D \leq y\}$.05	.15	.25	.45	.7	.85	.9

3

From the CDF,

$$P\{D \leq 4-1\} = .45$$

$$P\{D \leq 4\} = .7$$

Thus,

$$.45 \leq \frac{p}{p+1} \leq .7$$

$$\text{or } .43 \leq p \leq .82$$

Maximize expected revenue.

$$\begin{aligned} E\{\text{revenue}\} &= -10y + \int_{200}^{250} 25D f(D) dD \\ &\quad + \int_{200}^{250} 25y f(D) dD \\ &= -10y + \left[\frac{25D^2}{100} \right]_{200}^y + \left[\frac{25y^2}{50} \right]_{200}^{250} \\ &= -.25y^2 + 115y - 10,000 \end{aligned}$$

$$\frac{\partial E\{\text{revenue}\}}{\partial y} = -.5y + 115 = 0$$

$$y = 230 \text{ copies}$$

2

$$\begin{aligned} E\{\text{revenue}\} &= -7y + \int_{90}^y [25D + 5(y-D)] f(D) dD \\ &\quad + \int_y^{150} 25y f(D) dD \\ &= -\frac{y^2}{6} + 48y - 1350 \end{aligned}$$

$$\frac{\partial E\{\text{revenue}\}}{\partial y} = -\frac{y}{3} + 48$$

$$y = 144 \text{ donuts}$$

Decision: Stock 12 dozens

4

5

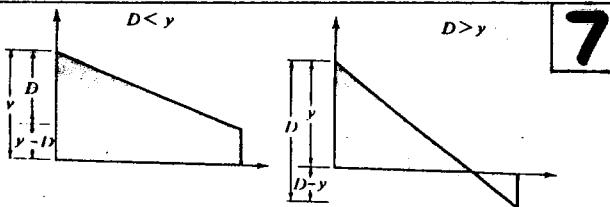
Set 14.2a

Use continuous pdf as an approximation 6

$$\begin{aligned}
 & -50y + \int_{20}^{30} [110D + 55(y-D)] f(D) dD \\
 & + \int_y^{\infty} 110y f(D) dD \\
 & = -50y + \frac{1}{10} \left[55yD + \frac{55D^2}{2} \right] \Big|_{20}^y + 110y \left[\frac{D}{10} \right] \Big|_y^{\infty} \\
 & = -2.75y^2 + 175y - 1100
 \end{aligned}$$

$$\frac{\partial E\{\text{revenue}\}}{\partial y} = -5.5y + 175 = 0$$

$$y \approx 32 \text{ coals}$$



$$\text{Average holding inventory} = y - \frac{D}{2} \quad \text{Average holding inventory} = \frac{y^2}{2D}$$

$$\text{Average shortage inventory} = 0 \quad \text{Average shortage inventory} = \frac{(D-y)^2}{2D}$$

$$\begin{aligned}
 E\{C(y)\} &= c(y-x) + h \left\{ \int_0^y (y-\frac{D}{2}) f(D) dD \right. \\
 &\quad \left. + \int_{\frac{y}{2}}^{\infty} \frac{y^2}{2D} f(D) dD \right\} + p \int_y^{\infty} \frac{(D-y)^2}{2D} f(D) dD
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial E\{C(y)\}}{\partial y} &= c + \left(\int_0^y f(D) dD + \int_y^{\infty} \frac{y}{D} f(D) dD \right) \\
 &\quad - p \int_y^{\infty} \left(\frac{D-y}{D} \right) f(D) dD = 0
 \end{aligned}$$

$$\int_{y^*}^{\infty} f(D) dD + y^* \int_{y^*}^{\infty} \frac{f(D)}{D} dD = \frac{p-c}{p+h}$$

$$f(D) = \frac{1}{100}, \quad 0 \leq D \leq 100$$

$$\int_0^y f(D) dD + y \int_y^{\infty} \frac{f(D)}{D} dD = \frac{p-c}{p+h}$$

$$\int_0^y \frac{1}{100} dD + y \int_y^{\infty} \frac{1}{100D} dD = \frac{p-c}{p+h}$$

$$\frac{y}{100} + \frac{y}{100} (\ln 100 - \ln y) = \frac{p-c}{p+h}$$

$$.056y - .01y \ln y = \frac{45-30}{45+25} = .2143$$

Trial and error yield $y^* = 5.5$ units

8

Set 14.2b

$$E\{C(s)\} = K + E\{C(S)\}$$

$$.25s^2 - 4.5s + 22.5 = 5 + .25S^2 - 4.5S + 22.5$$

$$.25s^2 - 4.5s + 15.25 = 0 \quad (\text{for } S=9)$$

Solution: $s = (4.53 \text{ or } 13.47)$

Policy: If $x < 4.53$, order $9-x$

$x \geq 4.53$, do not order

$$E\{R(y)\} = -c(y-x) +$$

$$\int_0^y [rD - h(y-D)] f(D) dD + \int_y^\infty [ry - p(D-y)] f(D) dD$$

$$\frac{\partial E\{R(y)\}}{\partial y} = -c - \int_0^y h f(D) dD + ry f(D) + \int_y^\infty (r+p) f(D) dD - ry f(D) = 0$$

Thus,

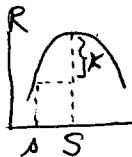
$$\int_0^{y^*} f(D) dD = \frac{r+p-c}{r+p-h}$$

In the presence of setup cost, we have an $s-S$ policy. Define s such that

$$E\{R(s)\} = E\{R(S)\} - K$$

For the numeric problem,

$$E\{R(y)\} = .4y^2 + 5y - 20 - 2x$$



$$\int_0^S f(D) dD = \frac{3+4-2}{3+4-1} = .625$$

$$\text{Thus, } S = 6.25$$

$$\text{Next, } -.4s^2 + 5s - 5.625 = 0$$

$$\text{Thus, } s = 1.25$$

Policy:

If $x < 1.25$, order $6.25-x$

$x \geq 1.25$, do not order

$$-\frac{s^2}{6} + 4s - 1350$$

$$= -10 - \frac{144^2}{6} + 48 \times 144 - 1350$$

Thus,

$$s^2 - 288s + 20676 = 0$$

$$s = \begin{cases} 136.25 \\ 151.25 \end{cases}$$

Optimal policy

If $x < 136$, order $144-x$

$x \geq 136$, do not order

2

3

Set 14.3a

$$L(y_i) = \int_0^{y_i} (2D - h(y_i - D)) f(D) dD + \int_{y_i}^{\infty} (ry_i + (\alpha r' - p)(D - y_i)) f(D) dD \quad |$$

$i=1, 2$

where

$$r' = \begin{cases} r & i=1 \\ r-c & i=2 \end{cases}$$

$$g_2(x_2) = \max_{y_2 \geq x_2} \{-c(y_2 - x_2) + L(y_2)\}$$

$$g_1(x_1) = \max_{y_1 \geq x_1} \left\{ -c(y_1 - x_1) + L(y_1) + \alpha E\{g_2(y_2 - D)\} \right\}$$

For period 2:

$$\frac{\partial f_2(y_2 | x_2)}{\partial y_2} = -c + L'(y_2^*) = 0$$

or

$$\int_0^{y_2^*} f(D) dD = \frac{r+p-c-\alpha(r-c)}{r+p+h-\alpha(r-c)}$$

$$g_2(y_2 - D) = \begin{cases} L_2(y_2 - D), & D \leq y_2 - y_2^* \\ -c(y_2^* - y_2 + D) + L(y_2^*), & D \geq y_2 - y_2^* \end{cases}$$

$$E\{g_2(y_2 - D)\} = \int_0^{y_1 - y_2^*} L_2(y_2 - D) f(D) dD + \int_{y_1 - y_2^*}^{\infty} (-c(y_2^* - y_2 + D) + L(y_2^*)) f(D) dD$$

This, when substituted in the expression for $g_1(x_1)$, will yield total expected profit in terms of y_1 . Hence, the value of y_1^* can be obtained.

In terms of the given numerical example, we have

$$\frac{1}{10} \int_0^{y_2^*} dD = \frac{2+3-1-0.8(2-1)}{2+3+1-0.8(2-1)} = 0.75$$

$$\text{Thus, } y_2^* = 7.5$$

$$L(\bar{z}) = \frac{1}{10} \left\{ \int_0^3 (2D - 1(\bar{z}-D)) dD + \int_3^{\infty} (2\bar{z} + (0.8r' - 3)(D - \bar{z})) dD \right\}$$

| continued

$$= (0.4r' - 0.255)\bar{z}^2 + (5 - 0.8r')\bar{z} + (4r' - 15)$$

Hence

$$L(y_2) = [0.4(2-1) - 0.255]y_2^2 + [5 - 0.8(2-1)]y_2 + [4(2-1) - 15] \\ = -0.215y_2^2 + 4.2y_2 - 11$$

$$L(y_2^*) = L(7.5) = 8.4$$

$$g_2(y_2 - D) = \begin{cases} -0.215(y_2 - D)^2 + 4.2(y_2 - D) - 11, & D \leq y_2 - 7.5 \\ 0.9 - y_2 + D, & D \geq y_2 - 7.5 \end{cases}$$

$$E\{g_2(y_2 - D)\} = \frac{1}{10} \left\{ \int_0^{y_1 - 7.5} [-0.215(y_2 - D)^2 + 4.2(y_2 - D) - 11] dD + \int_{y_1 - 7.5}^{\infty} (0.9 - y_2 + D) dD \right\}$$

$$= \frac{1}{10} (-0.072y_1^3 + 2.115y_1^2 - 11y_1 - 5y_1^2 - 5 \cdot 4y_1 - 19.625) \\ = \frac{1}{10} (-0.072y_1^3 + 1.115y_1^2 - 16.4y_1 - 24.625)$$

$$L(y_1) = (0.4x_2 - 0.255)y_1^2 + (5 - 0.8x_2)y_1 + (4x_2 - 15)$$

$$= -0.175y_1^2 + 3.4y_1 - 7$$

$$g_1(x_1) = \max_{y_1 \geq x_1} \left\{ -1(y_1 - x_1) - 0.175y_1^2 + 3.4y_1 + 7 + \frac{8}{10} (-0.072y_1^3 + 1.115y_1^2 - 16.4y_1 - 24.625) \right\}$$

$$= \max_{y_1 \geq x_1} \left\{ -0.00576y_1^3 - 0.075y_1^2 + 0.89y_1 - 8.97 + x_1 \right\}$$

$$\frac{\partial \{ \cdot \}}{\partial y_1} = -0.01728y_1^2 - 0.15y_1 + 0.89 = 0 \\ y_1^* = 9.02$$

continued...

Set 14.3a

optimal policy:

1 continued

$$\text{Period 1} \begin{cases} \text{order } 9.02 - x_1, & x_1 \leq 9.02 \\ \text{order 0,} & x_1 \geq 9.02 \end{cases}$$

$$\text{Period 2} \begin{cases} \text{order } 7.5 - x_2, & x_2 \leq 7.5 \\ \text{order 0,} & x_2 \geq 7.5 \end{cases}$$

For the infinite model:

$$\frac{1}{10} \int_0^{y^*} dD = \frac{3 + .2(2-1)}{3 + .1 + .2 \times 2} = .915$$

$$y^* = 9.15 > y_2^* > y_1^*$$

$$\int_0^{y^*} f(D) dD = .08 \int_0^{y^*} D dD$$

2

$$= .04 y^{*2}$$

Thus,

$$\begin{aligned} .04 y^{*2} &= \frac{p + (1-\alpha)(r-c)}{p + h + (1-\alpha)r} \\ &= \frac{10 + .1 \times 2}{10 + 1 + .1 \times 10} = .85 \end{aligned}$$

$$\text{Thus, } y^* = 4.61$$

Policy:

$$\text{order } 4.61 - x, \quad \text{if } x \leq 4.61$$

$$\text{order 0,} \quad \text{if } x \geq 4.61$$

$$\begin{aligned} g(x) = \min_{y \geq x} \{ &c(y-x) + \\ &h \int_0^y (y-D)^2 f(D) dD + \\ &p \int_y^\infty (D-y)^2 f(D) dD + \\ &\alpha \int_0^\infty g(y-D) f(D) dD \} \end{aligned}$$

3

$$\begin{aligned} \frac{\partial \{ \cdot \}}{\partial y} &= c + 2h \int_0^y (y-D) f(D) dD \\ &- 2p \int_y^\infty (D-y) f(D) dD \\ &+ \alpha E\{g'(y-D)\} \end{aligned}$$

Continued...

3 continued

Since this is a cost function,
 $g'(y-D) = -c$.

Now, $\frac{\partial \{ \cdot \}}{\partial y} = 0$ yields,

$$\begin{aligned} &\{(1-\alpha)c + 2h y^*\} \int_0^{y^*} f(D) dD \\ &- 2h \int_0^{y^*} D f(D) dD \\ &+ 2p y^* (1 - \int_0^{y^*} f(D) dD) \\ &- 2p E\{D\} \\ &+ 2p \int_0^{y^*} D f(D) dD \} = 0 \end{aligned}$$

This simplifies to

$$(h-p) \left\{ y^* \int_0^{y^*} f(D) dD - \int_0^{y^*} D f(D) dD \right\} + p y^* = \frac{2p E\{D\} - (1-\alpha)c}{2} \quad (1)$$

$$\text{or } y^* \left\{ \frac{1}{h-p} + \int_0^{y^*} f(D) dD - \int_0^{y^*} D f(D) dD \right\} = \frac{2p E\{D\} - (1-\alpha)c}{2(h-p)}$$

y^* can be determined from the last equation. When $h=p$, (1) yields

$$y^* = \frac{2p E\{D\} - (1-\alpha)c}{2p}$$

This result is independent of $f(D)$ except in so far as $E\{D\}$ is concerned.

Chapter 15

Queuing Systems

Set 15.1a

(a) Efficiency = $100 - 29 = 71\%$

(b) For average waiting time ≤ 3 minutes, at least 5 cashiers are needed

For efficiency $\geq 90\%$, the associated idleness percentage is $\leq 10\%$. The corresponding number of cashiers is at most 2.

Conclusion:

The two conditions cannot be satisfied simultaneously.
At least one of the two conditions must be relaxed.

$C_A = \$18 \text{ per hour}$

$C_B = \$25 \text{ per hour}$

Length of queue A = 4 jobs

Length of queue B = $7 \times 4 = 2.8$ jobs

Cost of A = $\$18 + 4 \times \$10 = \$58 \text{ per hour}$

Cost of B = $\$25 + 2.8 \times \$10 = \$53 \text{ per hour}$

Decision:

Select Model B.

1

2

Set 15.2a

3

Situation	Customer	Server
a	Plane	Runway
b	Passenger	Taxi
c	Machinist	Clerk at tool crib
d	Letter	Clerk
e	Student	Registrar's office
f	Cases	Judge
g	Shopper	Cashier
h	Car	Parking space

2

Situation	Calling Source	Customers arrival
a	∞	Individual
b	∞	Individual
c	∞	Individual
d	∞	Bulk
e	∞	Individual
f	∞	Individual
g	∞	Individual
h	∞	Individual

Situation	Interarrival time	Service time
a	Probabilistic	Time to clear runway
b	Probabilistic	Ride time
c	Probabilistic	Time to receive tool
d	Deterministic	Time to process letter
e	Probabilistic	Time to process registr.
f	Probabilistic	Trial time
g	Probabilistic	Check-out time
h	Probabilistic	Parking time.

Situation	Queue Capacity	Queue Discipline
a	∞	FIFO
b	∞	FIFO
c	∞	FIFO
d	∞	Random
e	∞	FIFO
f	∞	FIFO
g	∞	FIFO
h	0	None

#	Queueing situation	Customer
1	Arrival of orders	Orders
2	Processing (single machine)	Rush orders
3	Processing (single machine)	Regular jobs
4	Processing (Prod. line)	Rush jobs
5	Processing (Prod. line)	Regular jobs
6	Receipt of completed jobs	Completed orders
7	Tool crib	Tools
8	Machine breakdown	machines

#	Servers	Discipline	Service time	Queue length	Source
1	Foreman	Priority	Sorting time	∞	∞
2	Machine	FIFO	Prod. time	∞	∞
3	machine	FIFO	Prod. time	∞	∞
4	Prod. line	FIFO	Prod. time	∞	∞
5	Prod. line	FIFO	Prod. time	∞	∞
6	Shipping facilities	FIFO	Loading time	finite	∞
7	Tool crib	Priority	Exchange time	finite	finite
8	Repair persons	Priority	Repair time	finite	finite

(a) T. (b) T. (c) T.

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- (a) None.
- (b) None.
- (c) None.
- (d) None.
- (e) Jockey or balk
- (f) None
- (g) Jockey
- (h) None

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Set 15.3a

(a) Av. interarrival time (in time units)

$$= \frac{1}{\text{arrival rate } \lambda} \quad |$$

(b) Let \bar{I} = av. interarrival time

(i) $\lambda = \frac{60}{10} = 6 \text{ arrivals/hr}$
 $\bar{I} = 10 \text{ minutes} = \frac{1}{6} \text{ hour}$

(ii) $\lambda = \frac{60}{3} = 20 \text{ arrivals/hr}$
 $\bar{I} = \frac{6}{2} = 3 \text{ minutes} = \frac{1}{20} \text{ hr}$

(iii) $\lambda = \frac{10}{30} \times 60 = 20 \text{ arrivals/hr}$
 $\bar{I} = \frac{30}{10} = 3 \text{ minutes} = \frac{1}{20} \text{ hour}$

(iv) $\lambda = 1/5 = 2 \text{ arrivals/hour}$
 $\bar{I} = .5 \text{ hour}$

(c) Let \bar{S} = av. service time

(i) $\mu = \frac{60}{12} = 5 \text{ services/hour}$
 $\bar{S} = 12 \text{ minutes} = .2 \text{ hour}$

(ii) $\mu = \frac{60}{7.5} = 8 \text{ services/hr}$
 $\bar{S} = 7.5 \text{ min} = .125 \text{ hr}$

(iii) $\mu = \frac{5}{30} \times 60 = 10 \text{ services/hr}$
 $\bar{S} = \frac{30}{5} = 6 \text{ min} = \frac{1}{10} \text{ hr}$

(iv) $\mu = \frac{1}{.3} = 3.33 \text{ services/hr}$
 $\bar{S} = .3 \text{ hour}$

(a) $\lambda_{\text{hour}} = .2 \text{ failures/hr}$

$\lambda_{\text{week}} = .2 \times 24 \times 7 = 33.6 \text{ failures/week}$

(b) $P\{\text{at least one failure in 2 hours}\}$

$= P\{\text{time betw. failures} \leq 2\}$

$= P\{t \leq 2\} = 1 - e^{-2 \times 2} \approx .33$

(c) $P\{t > 3 \text{ hrs}\} = 1 - P\{t \leq 3\} = e^{-3 \times 2} \approx .55$

(d) $P\{t \leq 1 \text{ hour}\} = 1 - e^{-2 \times 1} = .18$

$\lambda = \frac{1}{.05} = 20 \text{ arrivals/hr}$

(a) $f(t) = \lambda e^{-\lambda t}$
 $= 20 e^{-20t}, t > 0$

(b) $P\{t > \frac{15}{60}\} = P\{t > .25\}$
 $= e^{-20 \times .25}$
 $= .00674$

(c) $P\{t \leq \frac{3}{60}\} = P\{t \leq .05\}$
 $= 1 - e^{-20 \times .05} = .632$

$P\{t > \frac{5}{60}\} = e^{-20 \times \frac{5}{60}} = .189$

(d) $t = 45 - 10 = 35 \text{ minutes}$

Av. # of arrivals in 35 min.
 $= 20 \times \frac{35}{60} = 11.67 \text{ arrivals}$

$\lambda = \frac{1}{6} \text{ arrivals/hr}$

$P\{t \geq 1\} = e^{-\frac{1}{6} \times 1} = .846$

$P\{t \leq .5\} = 1 - e^{-\frac{1}{6} \times .5}$
 $= 1 - e^{-1/12} = .08$

(a) $\lambda = \frac{60}{10} = 6 \text{ arrivals/hr}$

(b) $P\{t \geq \frac{15}{60}\} = e^{-6 \times \frac{15}{60}} = .223$

(c) $P\{t \leq \frac{20}{60}\} = 1 - e^{-6 \times \frac{20}{60}} = .865$

(a) $P\{t \leq \frac{2}{60}\} = 1 - e^{-35(2/60)} = .6886$

(b) $P\{\frac{2}{60} \leq t \leq \frac{3}{60}\}$
 $= P\{t \leq \frac{3}{60}\} - P\{t \leq \frac{2}{60}\}$
 $= (1 - e^{-35 \times 3/60}) - (1 - e^{-35 \times 2/60})$
 $= e^{-70/60} - e^{-105/60} = .1376$

(c) $P\{t > \frac{3}{60}\} = e^{-35(3/60)} = .1738$

Set 15.3a

$$\lambda = \frac{60}{1.5} = 40 \text{ arrivals/hr}$$

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Jim's Payoff	-2¢	+2¢	
Prob.	$P\{t \geq 1\}$	$P\{t \leq 1\}$	

$$P\{t \geq 1\} = e^{-40(1/60)} = .5134$$

$$P\{t \leq 1\} = 1 - .5134 = .4866$$

$$\begin{aligned} \text{Jim's exp. payoff / arriving customer} \\ = -2 \times .5134 + 2 \times .4866 \end{aligned}$$

$$= -.0536 \text{ cent}$$

Jim's exp. payoff / 8 hours

$$= -.0536(8\lambda)$$

$$= -.0536 \times 8 \times 40$$

$$\approx -17.15 \text{ cent}$$

Conclusion: Jim will pay Ann an average of 17 cents every 8 hrs

Jim's payoff	2	0	-2
Probability	$P\{t \leq 1\}$	$P\{1 \leq t \leq 1.5\}$	$P\{t \geq 1.5\}$
$P\{t \leq 1\}$.4866		
$P\{t \geq 1.5\}$	$e^{-40(1.5/60)}$		
	= .3679		
	<hr/>	<hr/>	<hr/>
	2	0	-2
	.4866	.1455	.3679

Jim's expected payoff / 8 hours

$$= [2 \times .4866 + 0 \times .1455 - 2 \times .3679] \times 40 \times 8$$

$$\approx 76 \text{ cents}$$

$2\text{\$}$	$3\text{\$}$	$-5\text{\$}$	$-6\text{\$}$
$t \leq 1$	$1 \leq t \leq 1.5$	$1.5 \leq t \leq 2$	$t \geq 2$

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$$\lambda = 40 \text{ arrivals/hr}$$

$$-40/60$$

$$P\{t \leq 1\} = 1 - e^{-40(1/60)} = .4866$$

$$P\{1 \leq t \leq 1.5\} = e^{-40(1.5/60)} - e^{-40(1/60)}$$

$$= .1455$$

$$P\{1.5 \leq t \leq 2\} = e^{-40(2/60)} - e^{-40(1.5/60)} = .1043$$

$$P\{t \geq 2\} = e^{-40(2/60)} = .2636$$

Jim's exp. payoff / 8 hours

$$= 8 \times 40 (2 \times .4866 + 3 \times .1455 - 5 \times .1043 - 6 \times .2636) \approx -2.22 \text{ cents}$$

Jim pays Ann an average of \$2.22 / 8 hours.

$$(a) \lambda = \frac{60}{6} = 10 \text{ customers/hr}$$

$$P\{t \leq 4 \text{ min}\} = 1 - e^{-10(4/60)} = .4866$$

(b)

$$\% \text{ discount} = \begin{cases} 10\%, & \text{if } t \leq 4 \\ 6\%, & \text{if } 4 < t \leq 5 \\ 2\%, & \text{if } t > 5 \end{cases}$$

$$P\{t \leq 4\} = .4866$$

$$P\{4 < t \leq 5\} = e^{-10(4/60)} - e^{-10(5/60)}$$

$$= .0788$$

$$P\{t > 5\} = e^{-10(5/60)} =$$

$$= .4346$$

Expected % discount

$$= 10 \times .4866 + 6 \times .0788 + 2 \times .4346$$

$$= 6.208\%$$

Set 15.3a

$$\lambda = \frac{365 \times 24}{9000} = .973 \text{ failure / yr}$$

$$P\{t \leq 1\} = 1 - e^{-0.973 \times 1} = .622$$

Lack-of-memory property applies.

(a) The waiting time for the green bus is exponential with mean 10 minutes:

$$f(t) = .1 e^{-0.1t}, t \geq 0$$

(b) The waiting time for the red bus is exponential with mean 7 minutes:

$$f(t) = \frac{1}{7} e^{-t/7}, t \geq 0$$

$$E\{t\} = \int_0^\infty t \lambda e^{-\lambda t} dt$$

$$= - \int_0^\infty t d e^{-\lambda t}$$

$$= - \left(t e^{-\lambda t} - \int_0^\infty e^{-\lambda t} dt \right)$$

$$= - \left(t e^{-\lambda t} - \frac{1}{\lambda} e^{-\lambda t} \right) \Big|_0^\infty$$

$$= \frac{1}{\lambda}$$

$$E\{t^2\} = \lambda \int_0^\infty t^2 e^{-\lambda t} dt$$

$$= - \int_0^\infty t^2 d e^{-\lambda t}$$

$$= - \left[t^2 e^{-\lambda t} - \int_0^\infty 2t e^{-\lambda t} dt \right]$$

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$$= - \left[t^2 e^{-\lambda t} - \frac{2}{\lambda} \int_0^\infty t \lambda e^{-\lambda t} dt \right] \Big|_0^\infty$$

$$= + \frac{2}{\lambda^2}$$

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$$\text{Var}\{t\} = E\{t^2\} - E\{t\}^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}$$

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continued...

Set 15.4a

TORA input = $(5, 0, 0, \infty, \infty)$

$$\begin{aligned} P_{n \geq 5}(t=1 \text{ hr}) &= 1 - [P_0(1) + \dots + P_4(1)] \\ &= 1 - e^{-5} (1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}) \\ &= 1 - .44049 = .55951 \end{aligned}$$

$$\lambda = 1 \text{ trip/month}$$

(a) $\lambda t = 3$: TORA input = $(3, 0, 0, \infty, \infty)$

$$P_0(3) = \frac{(1 \times 3)^0 e^{-1 \times 3}}{0!} = .049787$$

(b) $\lambda t = 12$: TORA input = $(12, 0, 0, \infty, \infty)$

$$\begin{aligned} P_{n \leq 8}(t=12) &= P_0(12) + \dots + P_8(12) \\ &= \frac{12^0 e^{-12}}{0!} + \frac{12^1 e^{-12}}{1!} + \dots + \frac{12^8 e^{-12}}{8!} \\ &= .15503 \end{aligned}$$

$$(c) P_0(t) = \frac{1^0 e^{-1}}{0!} = e^{-1} = .3679$$

TORA input = $(1, 0, 0, \infty, \infty)$

$$\lambda = 2 \text{ arrivals/minute}$$

(a) $\lambda t = 2 \times 5 = 10 \text{ arrivals}$

$$(b) \lambda t = 2 \times 5 = 1$$

TORA input = $(1, 0, 0, \infty, \infty)$

$$P_0(t=.5) = e^{-2 \times .5} = .3679$$

$$(c) 1 - P_0(t=.5) = 1 - .3679 = .6321$$

$$(d) \lambda t = 2 \times 3 = 6 \text{ arrivals}$$

TORA input = $(6, 0, 0, \infty, \infty)$

$$P_0(t=3) = \frac{(2 \times 3)^0 e^{-2 \times 3}}{0!} = .00248$$

$$\lambda = 1/5 = .2 \text{ arrival/min}$$

$$(a) P_2(t=7) = \frac{(2 \times 7)^2 e^{-2 \times 7}}{2!} = .24167$$

TORA input = $(1.4, 0, 0, \infty, \infty)$

$$(b) P_1(t=5) = \frac{(2 \times 5)^1 e^{-2 \times 5}}{1!} = .36788$$

$$\lambda = 25 \text{ books per day}$$

$$(a) \lambda t = 25 \times 30 = 750 \text{ books} = 7.5 \text{ shelves}$$

$$(b) 10 \text{ bookcases} = 10 \times 5 \times 100 = 5000 \text{ books}$$

$$\begin{aligned} P_{n \geq 5000}(t=30) &= 1 - [P_0(30) + \dots + P_{5000}(30)] \\ &\approx 0 \end{aligned}$$

$$(a) \lambda_{\text{green}} = .1 \text{ stop/min}, \lambda_{\text{red}} = 1/7 \text{ stop/min}$$

$$\lambda_{\text{combined}} = .1 + \frac{1}{7} = .24286 \text{ stop/min}$$

$$P_2(S) = \frac{(.24286 \times 5)^2}{2!} e^{-24286 \times 5} = .219$$

The two buses could be 2G, 2R or 1G and 1R.

$$(b) P\{t \leq 2\} = 1 - e^{-243 \times 2} = .3849$$

$$E\{n|t\} = \sum_{n=1}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= \lambda t + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t + e^{-\lambda t} e^{\lambda t} = \lambda t$$

$$E\{n^2|t\} = \sum_{n=0}^{\infty} n^2 \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= \sum_{n=1}^{\infty} n^2 \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

$$= \lambda t + e^{-\lambda t} \sum_{n=1}^{\infty} \frac{n(\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t e^{-\lambda t} \frac{\partial}{\partial \lambda t} \left(\lambda t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \right)$$

$$= \lambda t e^{-\lambda t} \frac{d}{d \lambda t} (\lambda t e^{\lambda t})$$

$$= \lambda t e^{-\lambda t} (\lambda t e^{\lambda t} + e^{\lambda t})$$

$$= (\lambda t)^2 + \lambda t$$

Thus,

$$\text{var}\{n|t\} = (\lambda t)^2 + \lambda t - (\lambda t)^2$$

$$= \lambda t$$

Set 15.4a

$$p'_0(t) = -\lambda p_0(t) \quad (1)$$

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$$p'_n(t) = -\lambda p_n(t) + \lambda p_{n-1}(t) \quad (2)$$

From (1)

$$dp_0(t) = -\lambda p_0(t) dt$$

which yields

$$p_0(t) = A e^{-\lambda t}$$

$$\text{Because } p_0(0) = 1 \Rightarrow A = 1, p_0(t) = e^{-\lambda t}$$

For $n=1$:

$$\begin{aligned} p'_1(t) &= -\lambda p_1(t) + \lambda p_0(t) \\ &= -\lambda p_1(t) + \lambda e^{-\lambda t} \end{aligned}$$

or

$$p'_1(t) + \lambda p_1(t) = \lambda e^{-\lambda t}$$

This yields the solution:

$$\begin{aligned} p_1(t) &= e^{\int \lambda dt} \left\{ \int \lambda e^{-\lambda t} e^{-\int \lambda dt} dt + C \right\} \\ &= \lambda t e^{-\lambda t} + C \end{aligned}$$

$$\text{Because } p_1(0) = 0, C = 0, \text{ and}$$

$$p_1(t) = \frac{\lambda t e^{-\lambda t}}{1!}$$

Induction proof:

Given

$$p_i(t) = \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

then

$$p'_{i+1}(t) + \lambda p_{i+1}(t) = \lambda \frac{(\lambda t)^{i+1} e^{-\lambda t}}{i+1!}$$

The solution is

$$\begin{aligned} p_{i+1}(t) &= e^{\int \lambda dt} \left\{ \frac{\int \lambda (\lambda t)^i e^{-\lambda t} e^{\int \lambda dt} dt}{i!} + C \right\} \\ &= \frac{e^{-\lambda t} (\lambda t)^{i+1}}{(i+1)!} + C \end{aligned}$$

Because $p_{i+1}(0) = 0, C = 0$, and

$$p_{i+1}(t) = \frac{e^{-\lambda t} (\lambda t)^{i+1}}{(i+1)!}$$

continued...

Set 15.4b

$M = 3 \text{ dozens/day}$, $N = 18$
 TORA input data = $(0, Mt, 1, 18, 18)$

$$(a) \mu = 3 \times 3 = 9$$

$$P_0(t=3) = .00532 \text{ (from TORA)}$$

$$(b) Mt = 3 \times 2 = 6$$

$$\sum_{n=0}^{18} n P_n(2) = 11.955$$

(c) This part can be solved using Poisson or exponential distributions:

$$\text{Poisson: } Mt = 3 \times 1 = 3$$

$$\text{Probability} = P_0(1) + P_1(1) + \dots + P_{17}(1) \\ = .9502 \text{ (from TORA)}$$

$$\text{Exponential: mean} = 1/3 \text{ day}$$

$$P\{\text{purchasing at least one dozen in 1 day}\} \\ = P\{\text{time between purchases} \leq 1\} \\ = 1 - e^{-3 \times 1} = .9502$$

$$(d) \text{Exponential: } P\{t \leq .5\} = 1 - e^{-3 \times .5} = .7769$$

$$\text{Poisson: } P_0(.5) + P_1(.5) + \dots + P_{17}(.5) = .7769$$

$$(e) P_0(1) = 0 \quad (Mt = 3 \times 1 = 3)$$

$$N = 40, M = 10 \text{ calls/h}$$

TORA input $(0, Mt, 1, 40, 40)$

$$(a) P_{n>0}(t=4) = 1 - P_0(4) \\ = 1 - .521 = .479$$

$$(b) E\{n|t=4\} = \sum_{n=0}^{40} n P_n(4) \approx 2.5 \text{ blocks} \\ \approx 25 \text{ tickets}$$

$$N = 48, M = \frac{4 \times 10}{8} = 5 \text{ calls/h}$$

$$Mt = 5 \times 4 = 20 \text{ calls}$$

$$P_0(4) \approx .000005 \text{ (from TORA)}$$

$$N = 48, Mt = 5 \times 8 = 40, P_0(8) = .11958$$

$$M = 1/1 = 1 \text{ withdrawal/week}$$

$$N = 5, Mt = 4$$

$$P_0(4) = .37116$$

2

$N = 80 \text{ items}, M = 5 \text{ items/day}$

6

$$(a) Mt = 5 \times 2 = 10 \text{ items}$$

$$P_0(2) = .1251$$

$$(b) Mt = 5 \times 4 = 20 \text{ items}$$

$$P_0(4) = .00001$$

$$(c) Mt = 5 \times 4 = 20 \text{ items}$$

$$E\{n|4 \text{ days}\} = \sum_{n=0}^{80} n P_n(4) \approx 60 \text{ items}$$

$$\text{Av. # of withdrawals} = 80 - 60 \\ = 20 \text{ items}$$

$$M = 1/1 = 1 \text{ breakdown/day}$$

$$N = 10, Mt = 1 \times 2 = 2$$

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$$\text{From TORA, } P_0(2) = .00005$$

$$(a) N = 25, M = 3/day$$

$$t = 6 \text{ days, } Mt = 18$$

$$\text{Av. stock remaining after 6 days} \\ = E\{n|t=6\} = 7.11$$

$$\text{Av. order size} = 25 - 7.11$$

$$\approx 18 \text{ items}$$

$$(b) t = 4, Mt = 3 \times 4 = 12$$

$$P_0(4) = .00069$$

$$(c) t = 6, Mt = 3 \times 6 = 18$$

$$P_{n \leq 14}(6) = P_0(6) + \dots + P_{14}(6) = .9696$$

3

4

5

$$P\{\text{time betw. departures} > T\}$$

$$= P\{\text{no departures during } T\}$$

$$= P\{N \text{ left after time } T\}$$

$$= P_N(T)$$

$$P\{t > T\} = P_N(T) = \frac{(MT)^0 e^{-MT}}{0!} \\ = e^{-MT}$$

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Set 15.4b

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$$p'_N(t) = -\mu p_N(t) \quad (1)$$

$$p'_n(t) = -\mu p_n(t) + \mu p_{n+1}(t), \quad 0 \leq n < N \quad (2)$$

From (1), we get

$$p_N(t) = C e^{-\mu t}$$

Given $p_N(0) = 1$, then $C = 1$ and

$$p_N(t) = e^{-\mu t}$$

Next, consider (2) for $n = N-1$

$$p'_{N-1}(t) = -\mu p_{N-1}(t) + \mu p_N(t)$$

$$= -\mu p_{N-1}(t) + \mu e^{-\mu t}$$

Thus,

$$p_{N-1}(t) = e^{-\int_{0}^{Mt} \{ \int_{0}^{t} \mu e^{-\mu u} e^{\int_{0}^{u} \mu dt} du \} dt} + C$$

$$= e^{-Mt} Mt + C$$

Because $p_{N-1}(0) = 0$, $C = 0$ and $p_{N-1}(t) = (Mt)e^{-Mt}$

Induction proof:

Given $p_{n+1}(t) = \frac{(Mt)^{N-n-1} e^{-Mt}}{(N-n-1)!}$, then

$$p'_n(t) = -\mu p_n(t) + \frac{\mu (Mt)^{N-n-1} e^{-Mt}}{(N-n-1)!}$$

Solution gives

$$p_n(t) = \frac{(Mt)^{N-n} e^{-Mt}}{(N-n)!}$$

Set 15.5a

$$(a) P\{0 \text{ counter open}\} = P_0 = \frac{1}{55}$$

$$P\{1 \text{ counter open}\} = P_1 + P_2 + P_3 \\ = \frac{1}{55} (2+8+8) = \frac{14}{55}$$

$$P\{2 \text{ counters open}\} = P_4 + P_5 + P_6 \\ = \frac{1}{55} (8+8+8) = \frac{24}{55}$$

$$P\{3 \text{ counters open}\} = P_7 + P_8 + \dots \\ = 1 - (P_0 + \dots + P_6) \\ = 1 - (\frac{1}{55} + \frac{14}{55} + \frac{24}{55}) = \frac{16}{55}$$

$$(b) \text{Av. # busy counters} \\ = 0 \times \frac{1}{55} + 1 \times \frac{14}{55} + 2 \times \frac{24}{55} + 3 \times \frac{16}{55} \\ = 2 \text{ counters}$$

$$(c) \text{Av. # idle counters} = 3 - 2 = 1$$

$$\lambda = 1/5 = .2 \text{ arrival/min} \\ = 12 \text{ arrivals/hr}$$

$$(a) M_n = \begin{cases} 5 \text{ customers/hr}, & n=0,1,2 \\ 10 \text{ customers/hr}, & n=3,4 \\ 15 \text{ customers/hr}, & n=5,6 \\ 20 \text{ customers/hr}, & n \geq 7 \end{cases}$$

$$P_0 = \frac{12}{5} P_0 = 2.4 P_0$$

$$P_1 = \left(\frac{12}{5}\right)^2 P_0 = 5.76 P_0$$

$$P_2 = \left(\frac{12}{5}\right)^3 \left(\frac{12}{10}\right) P_0 = 6.912 P_0$$

$$P_3 = \left(\frac{12}{5}\right)^4 \left(\frac{12}{10}\right)^2 P_0 = 8.2944 P_0$$

$$P_4 = \left(\frac{12}{5}\right)^5 \left(\frac{12}{10}\right)^3 \left(\frac{12}{15}\right) P_0 = 6.63552 P_0$$

$$P_5 = \left(\frac{12}{5}\right)^6 \left(\frac{12}{10}\right)^4 \left(\frac{12}{15}\right)^2 P_0 = 5.308416 P_0$$

$$P_{n \geq 7} = \left(\frac{12}{5}\right)^7 \left(\frac{12}{10}\right)^5 \left(\frac{12}{15}\right)^3 \left(\frac{12}{20}\right)^2 P_0 = 5.308416 \cdot 6 P_0$$

$$\text{From } \sum_{n=0}^{\infty} P_n = 1, \text{ we get } P_0 = .002587$$

$$P_1 = .05421, P_2 = .13010, P_3 = .15612$$

$$P_4 = .18735, P_5 = .14988, P_6 = .1199$$

$$P_{n \geq 7} = .1199 \cdot (.6)^{n-6}$$

$$(b) P_{n \geq 7} = 1 - (P_0 + P_1 + \dots + P_6) = .8$$

$$(c) P\{0 \text{ counter}\} = P_0 = .002587$$

$$P\{1 \text{ counter}\} = P_1 + P_2 = .18431$$

$$P\{2 \text{ counters}\} = P_3 + P_4 = .34347$$

$$P\{3 \text{ counters}\} = P_5 + P_6 = .26978$$

$$P\{4 \text{ counters}\} = P_7 + P_8 + \dots = .199853$$

Av. # idle counters

$$= 4 - (1 \times 18431 + 2 \times 34347 + 3 \times 26978 + 4 \times 199853) \approx 1.52$$

$$M_n = \begin{cases} 5^n, & n=1,2 \\ 15, & n=3,4, \dots \end{cases}$$

$$P_1 = \left(\frac{10}{5}\right) P_0 = 2 P_0$$

$$P_2 = \left(\frac{10}{5}\right) \left(\frac{10}{10}\right) P_0 = 2 P_0$$

$$P_{n \geq 3} = \left(\frac{10}{5}\right) \left(\frac{10}{10}\right) \left(\frac{10}{15}\right)^{n-2} P_0 = 2 \left(\frac{2}{3}\right)^{n-2} P_0$$

Thus,

$$P_0 + 2P_0 + 2P_0 + \left[2 \left(\frac{2}{3}\right) + 2 \left(\frac{2}{3}\right)^2 + \dots\right] P_0 = 1$$

which gives $P_0 = .1111$

(a) Prob that 3 counters are in use

$$= P_{n \geq 3} = 1 - (P_0 + P_1 + P_2) \\ = 1 - (.1111 + .2222 + .2222) \\ = .4445$$

$$(b) P_{n \leq 2} = P_0 + P_1 + P_2 = .5555$$

$$\lambda_n = \begin{cases} 12 \text{ cars/hr}, & n=0,1,\dots,10 \\ 0, & n \geq 11 \end{cases}$$

$$M_n = 60/6 = 10 \text{ cars/hr}$$

$$P_n = \left(\frac{12}{10}\right)^n P_0, \quad n=1,2,\dots,10 \\ = 0, \quad n \geq 11$$

$$P_0 \left(1 + 1.2 + 1.2^2 + \dots + 1.2^{10}\right) = P_0 \frac{1 - 1.2^{11}}{1 - 1.2}$$

$$\text{Thus, } P_0 = .0311$$

Continued...

Continued...

Set 15.5a

$$(a) P_{10} = \left(\frac{12}{10}\right)^{10} P_0 = .19259$$

$$(b) P_{n \geq 1} = 1 - P_0 = 1 - .0311 = .9689$$

(c) Av. length of the line

$$= 0P_0 + 1P_1 + \dots + 10P_{10}$$

$$= 1x.03732 + 2x.04479 + 3x.05375 + 4x.0645 + 5x.0774 + 6x.09288 + 7x.11145 + 8x.13374 + 9x.16049 + 10x.19259 = 6.71071$$

$$\lambda_n = 6 \text{ arrivals/hr, } n=0, 1, \dots, 8$$

$$= 5 \text{ arrivals/hr, } n=9, 10, \dots, 11, 12$$

$$M_n = n/5 = 2n/\text{hr}, n=1, 2, 3, 4$$

$$= 10/\text{hr}, n \geq 5$$

$$P_1 = \frac{6}{2} P_0 = 3P_0$$

$$P_2 = \frac{6}{2} \cdot \frac{6}{4} P_0 = 4.5 P_0$$

$$P_3 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} P_0 = 4.5 P_0$$

$$P_4 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} P_0 = 3.375 P_0$$

$$P_5 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} \cdot \frac{6}{10} P_0 = 2.025 P_0$$

$$P_6 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} \cdot \frac{6}{10} P_0 = 1.215 P_0$$

$$P_7 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} \cdot \frac{6}{10} \cdot \frac{6}{10} P_0 = .729 P_0$$

$$P_8 = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} \cdot \frac{(6)}{10}^4 P_0 = .4374 P_0$$

$$P_{n \geq 9} = \frac{6}{2} \cdot \frac{6}{4} \cdot \frac{6}{6} \cdot \frac{6}{8} \cdot \frac{(6)}{10}^4 \cdot \left(\frac{5}{10}\right)^{n-8} P_0 = .4374 \cdot (5)^{n-8} P_0$$

$$\text{From } \sum_{n=0}^{12} P_n = 1, \text{ we get } P_0 = .0495$$

$$(a) P_{12} = .4374 \times .5^4 \times .0495 = .00135$$

$$(b) P_{n \geq 5} = 1 - (P_0 + P_1 + \dots + P_4) = .2385$$

$$(c) \text{Av. # busy tables} = 0P_0 + 1P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_{n \geq 5} = 2.9768$$

4 continued

$$(d) 1P_6 + 2P_7 + \dots + 7P_{12}$$

5 continued

$$= 1x.0602 + 2x.0361 + 3x.0217 + 4x.0108 + 5x.0054 + 6x.0027 + 7x.00135 \\ = .2935 \text{ pair}$$

$$\lambda = 4 \text{ customers/hr}$$

6

$$\lambda_n = \begin{cases} 4, & n=0, 1, \dots, 4 \\ 0, & n \geq 5 \end{cases}$$

$$M_n = \frac{60}{15} = 4 \text{ customers/hr}$$

$$(a) P_1 = \frac{4}{4} P_0$$

$$P_2 = \left(\frac{4}{4}\right)^2 P_0$$

$$P_3 = \left(\frac{4}{4}\right)^3 P_0$$

$$P_4 = \left(\frac{4}{4}\right)^4 P_0$$

$$P_0 + P_1 + \dots + P_4 = 1 \Rightarrow P_0 = 1/5$$

$$P_0 = P_1 = P_2 = P_3 = P_4 = 1/5$$

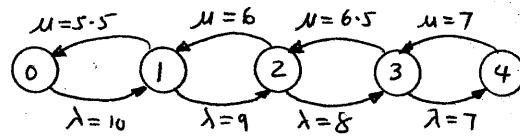
(b) expected # in shop =

$$0P_0 + 1P_1 + 2P_2 + 3P_3 + 4P_4$$

$$= \frac{1}{5}(1+2+3+4) = 2$$

$$(c) P_4 = .2$$

7



$$(a) 5.5 P_1 = 10 P_0$$

$$10P_0 + 6P_2 = (5.5+9)P_1$$

$$9P_1 + 6.5P_3 = (6+8)P_2$$

$$8P_2 + 7P_4 = (6.5+7)P_3$$

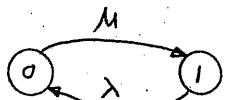
$$(b) P_1 = 1.82 P_0 \Rightarrow P_2 = 2.727 P_0$$

$$P_3 = 3.3566 P_0, P_4 = 3.3566 P_0$$

$$P_0 + P_1 + \dots + P_4 = 1 \Rightarrow P_0 = .088882$$

$$P_1 = .1614, P_2 = .2422, P_3 = .2981, P_4 = .33566$$

8



$$(a) \mu P_1 = \lambda P_0$$

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$(b) P_0 + \frac{\lambda}{\mu} P_0 = 1$$

$$P_0 = \frac{1}{1+\rho}, \quad \rho = \lambda/\mu$$

$$P_1 = \frac{\rho}{1+\rho}$$

$$(c) L_S = 0P_0 + 1P_1 = \frac{\rho}{1+\rho}$$

$$(d) \lambda_{eff} = \lambda P_0 = \frac{\lambda}{1+\rho}$$

$$(e) W_f = \frac{L_S}{\lambda_{eff}} - \frac{1}{\mu} \\ = \frac{\rho/(1+\rho)}{\lambda/(1+\rho)} - \frac{1}{\mu} = 0$$

9

$$\lambda_{n-1} P_{n-1} + M_{n+1} P_{n+1} =$$

$$\lambda_{n-1} \left(\frac{\lambda_0}{M_1} \cdot \frac{\lambda_1}{M_2} \cdots \frac{\lambda_{n-2}}{M_{n-1}} \right) +$$

$$M_{n+1} \left(\frac{\lambda_0}{M_1} \cdot \frac{\lambda_1}{M_2} \cdots \frac{\lambda_n}{M_{n+1}} \right)$$

$$= M_n \left(\frac{\lambda_0}{M_1} \frac{\lambda_1}{M_2} \cdots \frac{\lambda_{n-1}}{M_n} \right) +$$

$$\lambda_n \left(\frac{\lambda_0}{M_1} \frac{\lambda_1}{M_2} \cdots \frac{\lambda_{n-1}}{M_n} \right)$$

$$= M_n P_n + \lambda_n P_n$$

$$= (M_n + \lambda_n) P_n$$

Set 15.6a

$$\begin{aligned}
 (a) L_q &= \sum_{n=6}^8 (n-5) P_n \\
 &= 1P_6 + 2P_7 + 3P_8 \\
 &= 1 \times 0.05847 + 2 \times 0.03508 + 3 \times 0.02105 \\
 &= .19177
 \end{aligned}$$

$$\begin{aligned}
 (b) W_q &= \frac{L_q}{\lambda_{\text{eff}}} \\
 &= \frac{.19177}{5.8737} = .03265 \text{ hour} \\
 W_s &= W_q + \frac{1}{\mu} \\
 &= .03265 + \frac{1}{2} = .53265 \text{ hour}
 \end{aligned}$$

$$\begin{aligned}
 (c) \lambda_{\text{lost}} &= \lambda P_8 \\
 &= 6 \times 0.02105 = .1263 \text{ car/hr}
 \end{aligned}$$

$$\text{Number lost}/8 \text{ hrs} = .1263 \times 8 = 1.01 \text{ cars}$$

$$\begin{aligned}
 (d) \text{Average number of empty spaces} \\
 &= C - (L_s - L_q) \\
 &= C - \sum_{n=0}^8 n P_n + \sum_{n=c+1}^8 (n-C) P_n \\
 &= (C \sum_{n=0}^8 P_n - C \sum_{n=c+1}^8 P_n) \\
 &\quad - \left(\sum_{n=0}^8 n P_n - \sum_{n=c+1}^8 n P_n \right) \\
 &= C \sum_{n=0}^c P_n - \sum_{n=0}^c n P_n \\
 &= \sum_{n=0}^{c-1} (C-n) P_n
 \end{aligned}$$

$$\begin{aligned}
 (e) \lambda_n &= 6 \text{ cars/hr}, n=0, 1, \dots, 6 \\
 \mu_n &= \begin{cases} \left(\frac{4}{3}\right)n, & n=1, 2, \dots, 6 \\ 8, & n=7, 8, 9, 10 \end{cases} \\
 P_n &= \left(\frac{6}{413}\right)^n \frac{1}{n!} P_0, n=0, 1, \dots, 6
 \end{aligned}$$

2

2 continued

$$P_n = \frac{\left(\frac{6}{413}\right)^n}{6!} P_0, n=7, 8, 9, 10$$

$$\begin{aligned}
 P_0 \left(1 + \frac{9/2}{1!} + \frac{(9/2)^2}{2!} + \frac{(9/2)^3}{3!} + \frac{(9/2)^4}{4!} + \frac{(9/2)^5}{5!} + \frac{(9/2)^6}{6!} \right. \\
 \left. + \frac{(9/2)^7}{6!6^1} + \frac{(9/2)^8}{6!6^2} + \frac{(9/2)^9}{6!6^3} + \frac{(9/2)^{10}}{6!6^4} \right) = 1
 \end{aligned}$$

$$\text{Thus, } P_0 = .0004$$

n	P_n	n	P_n
1	.0004	6	.10027
2	.001141	7	.12534
3	.002852	8	.15667
4	.005348	9	.19584
5	.008022	10	.24480

$$\begin{aligned}
 (b) \lambda_{\text{eff}} &= \lambda (1 - P_{10}) = 10 (1 - 0.2448) \\
 &= 7.552 \text{ cars/hr}
 \end{aligned}$$

$$\begin{aligned}
 (c) L_s &= 0P_0 + 1P_1 + 2P_2 + \dots + 10P_{10} \\
 &= 7.6941 \text{ cars}
 \end{aligned}$$

$$(d) W_s = \frac{L_s}{\lambda_{\text{eff}}} = \frac{7.6941}{7.552} = 1.0155 \text{ cars}$$

$$W_q = 1.0155 - \frac{1}{4/3} = .2655$$

$$\begin{aligned}
 (e) L_q &= \lambda_{\text{eff}} W_q \\
 &= .2655 \times 7.552 \\
 &= 2.005 \text{ cars}
 \end{aligned}$$

$$\begin{aligned}
 \text{Average number of occupied} \\
 \text{spaces} &= L_s - L_q \\
 &= 7.6941 - 2.005
 \end{aligned}$$

$$= 5.6891 \text{ spaces}$$

continued...

Set 15.6b

(a) % utilization = $100(1-P_0)$

$$= 100 \cdot \frac{\lambda}{\mu}$$

$$= 100 \left(\frac{4}{6}\right) = 66.67\%$$

(b) $P_{n \geq 1} = 1 - P_0 = \frac{\lambda}{\mu} = \frac{4}{6} = .6667$

(c) $P_{n \leq 7} = P_0 + P_1 + \dots + P_7$

$$= 1 - \left(\frac{\lambda}{\mu}\right)^8 = 1 - \left(\frac{4}{6}\right)^8 = .961$$

(d) $P_0 + P_1 + \dots + P_K \geq .99$

From Figure 17-6, $K = 11$

Also, we can determine K from

$$1 - P^{K+1} \geq .99$$

$$(K+1) \geq \frac{\ln .01}{\ln (4/6)} = 11.$$

or $K \geq 11.350 - 1 = 10.358$

Thus, $K \geq 11$

Note that the desired number of parking spaces is almost doubled (from 5 to 11) to accommodate the increase in the acceptance percentage from 90% to 99%.

$\lambda = 1/5 = .2 \text{ job/day}$

$\mu = 1/4 = .25 \text{ job/day}$

From the TOR A output on the next column,

(a) $P_0 = .2$

(b) Av. income/month = \$50 μt

$$= 50 \times .25 \times 30 \\ = \$375$$

(c) Av. number of jobs awaiting completion = $L_q = 3.2 \text{ jobs}$

$$\text{cost} = 3.2 \times \$40 = \$128$$

Continued...

Lambda = 0.20000	Mu = 0.25000
Lambda eff = 0.20000	Rho/c = 0.80000
Ls = 4.00000	Lq = 3.20000
Ws = 20.00000	Wq = 16.00000

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.20000	0.20000	23	0.00118	0.99528
1	0.16000	0.38000	24	0.00094	0.99622
2	0.12800	0.48800	25	0.00075	0.99693
3	0.10240	0.59040	26	0.00060	0.99758
4	0.08192	0.67232	27	0.00048	0.99807
5	0.06554	0.73786	28	0.00039	0.99845
6	0.05243	0.79028	29	0.00031	0.99876
7	0.04194	0.83833	30	0.00025	0.99901
8	0.03355	0.86578	31	0.00020	0.99921
9	0.02684	0.89263	32	0.00016	0.99937
10	0.02147	0.91410	33	0.00013	0.99949
11	0.01718	0.93128	34	0.00010	0.99957
12	0.01374	0.94502	35	0.00008	0.99965
13	0.01100	0.95602	36	0.00006	0.99974
14	0.00880	0.96482	37	0.00005	0.99979
15	0.00704	0.97185	38	0.00004	0.99983
16	0.00563	0.97748	39	0.00003	0.99987
17	0.00450	0.98199	40	0.00003	0.99989
18	0.00360	0.98559	41	0.00002	0.99991
19	0.00288	0.98847	42	0.00002	0.99993
20	0.00231	0.99078	43	0.00001	0.99995
21	0.00184	0.99262	44	0.00001	0.99996
22	0.00148	0.99410			

$\lambda = 1/4 = .25 \text{ case/wk}$

$\mu = 1/1.5 = .66667 \text{ case/wk}$

M/M/c/GD/N/K Queueing Model

Input Data		Output Results	
$\lambda =$	0.25	$\lambda_{eff} =$	0.66667
c =	1	Source limit, K =	infinity
Sys. Lim., N =	infinity	infinity	
n	Pn	CPn	1-CPn
0	0.625002	0.625002	0.374998
1	0.234375	0.859376	0.140624
2	0.087890	0.947266	0.052734
3	0.032959	0.980225	0.019775
4	0.012359	0.992584	0.007416
5	0.004635	0.997219	0.002781
6	0.001738	0.999857	0.001043
7	0.000652	0.999609	0.000391
8	0.000244	0.999853	0.000147
9	0.000092	0.999945	0.000055
10	0.000034	0.999979	0.000021
11	0.000013	0.999992	0.000008
12	0.000005	0.999997	0.000003
13	0.000002	0.999999	0.000001
14	0.000001	1.000000	0.000000

(a) $L_q = .225 \text{ case}$

(b) $1 - P_0 = 1 - .625 = .375 \text{ or } 37.5\%$

(c) $W_s = 2.4 \text{ weeks}$

Present situation :

$\lambda = 90 \text{ cars/hr}$

$\mu = \frac{3600}{38} = 94.7368 \text{ cars/hr}$

New situation:

$\lambda = 90 \text{ cars per hour}$

$\mu = \frac{3600}{30} = 120 \text{ cars per hour}$

Continued...

Set 15.6b

Comparative Analysis

Scenario	c	Lambda	Mu	L/da eff	p0	Ls	Lq	Ws	Wq
1	1	90.00000	94.73680	90.00000	0.05000	19.00017	18.00017	0.21111	0.20056
2	1	90.00000	120.00000	90.00000	0.25000	3.00000	2.25000	0.03333	0.02500

$$L_s (\text{present}) = 19 \text{ cars}$$

$$\% \text{ of idle time (new)} = p_0 (\text{new}) \times 100 \\ = 100 \times .25 = 25\%$$

The device can be justified based on the number of waiting customers, L_q , in the present system, but not on the basis of % idle time in the new one.

Scenario 1-(MM/1):(GD/infinity/infinity)

5

Lambda = 0.40000	Mu = 0.66667
Lambda eff = 0.40000	Rho/c = 0.60000
Ls = 1.49998	Lq = 0.89998
Ws = 3.74995	Wq = 2.24996

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.40000	0.40000	11	0.00145	0.99782
1	0.24000	0.64000	12	0.00087	0.99869
2	0.14400	0.78400	13	0.00052	0.99922
3	0.08640	0.87040	14	0.00031	0.99953
4	0.05184	0.92224	15	0.00019	0.99972
5	0.03110	0.95335	16	0.00011	0.99983
6	0.01866	0.97201	17	0.00007	0.99990
7	0.01210	0.98320	18	0.00004	0.99994
8	0.00672	0.98992	19	0.00002	0.99996
9	0.00403	0.99395	20	0.00001	0.99998
10	0.00242	0.99637			

$$(a) p_0 = .4$$

$$(b) L_q = .9 \text{ car}$$

$$(c) W_q = 2.25 \text{ minutes}$$

$$(d) P_{n \geq 11} = 1 - CP_{10} = 1 - .99637 = .0036$$

Scenario 1-(MM/1):(GD/infinity/infinity)

6

Lambda = 10.00000	Mu = 12.00000
Lambda eff = 10.00000	Rho/c = 0.83333
Ls = 5.00000	Lq = 4.16667
Ws = 0.50000	Wq = 0.41667

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.16667	0.16667	27	0.00121	0.99393
1	0.13889	0.30556	28	0.00101	0.99494
2	0.11574	0.42130	29	0.00084	0.99579
3	0.09565	0.51775	30	0.00070	0.99649
4	0.08030	0.59812	31	0.00059	0.99707
5	0.06569	0.66510	32	0.00049	0.99756
6	0.05582	0.72092	33	0.00041	0.99797
7	0.04651	0.76743	34	0.00034	0.99831
8	0.03876	0.80619	35	0.00028	0.99859
9	0.03230	0.83849	36	0.00024	0.99882
10	0.02692	0.86541	37	0.00020	0.99902
11	0.02243	0.88784	38	0.00016	0.99918
12	0.01869	0.90654	39	0.00014	0.99932
13	0.01558	0.92211	40	0.00011	0.99943
14	0.01298	0.93509	41	0.00009	0.99953
15	0.01082	0.94591	42	0.00008	0.99961
16	0.00901	0.95403	43	0.00007	0.99967
17	0.00751	0.96244	44	0.00005	0.99973
18	0.00626	0.96870	45	0.00005	0.99977
19	0.00522	0.97392	46	0.00004	0.99981
20	0.00435	0.97826	47	0.00003	0.99984
21	0.00362	0.98189	48	0.00003	0.99987
22	0.00302	0.98491	49	0.00002	0.99989
23	0.00252	0.98742	50	0.00002	0.99991
24	0.00210	0.98952	51	0.00002	0.99992
25	0.00175	0.99126	52	0.00001	0.99994
26	0.00146	0.99272	53	0.00001	0.99995

$$(a) p_0 + p_1 + p_2 = .4213$$

continued...

$$(b) 1 - CP_2 = 1 - .4213 = .5787$$

$$(c) W_q = .417 \text{ hour}$$

(d) Let $N = \text{spaces (including car being served)}$

$$CP_{N-1} \geq .9$$

Because $CP_{11} = .88784$ and $CP_{12} = .90654$, $N-1 \geq 12 \Rightarrow N \geq 13$.

In general, $L_s < L_q + 1$. The reason is that $p > 0$, usually. Consider

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n \\ = \sum_{n=1}^{\infty} n p_n - \sum_{n=1}^{\infty} p_n \\ = L_s - (1-p)$$

The closer p is to zero, the more likely $L_s \approx L_q + 1$ will hold.

Consider

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n \\ = \sum_{n=1}^{\infty} (n-1)(1-p)^{n-1} \\ = (1-p) \sum_{n=1}^{\infty} \frac{d}{dp} (1-p)^{n-1} \\ = (1-p) \sum_{n=1}^{\infty} \frac{d}{dp} \sum_{n=0}^{\infty} p^n \\ = (1-p) \sum_{n=1}^{\infty} \frac{p^2}{(1-p)^2} \left(\frac{1}{1-p} \right) \\ = p^2 (1-p) \frac{1}{(1-p)^2} \\ = \frac{p^2}{1-p}$$

$$(a) P\{j \text{ in queue} | j \geq 1\}$$

$$= P\{n \text{ in system} | n \geq 2\}$$

$$= \frac{P_n}{\sum_{j=2}^{\infty} P_j}$$

Thus,

$$\text{expected number} = \sum_{n=2}^{\infty} (n-1) \frac{P_n}{\sum_{j=2}^{\infty} P_j}$$

$$= \frac{\sum_{n=2}^{\infty} np_n - \sum_{n=2}^{\infty} P_n}{\sum_{n=2}^{\infty} P_n}$$

$$= \frac{\sum_{n=1}^{\infty} np_n - P_1}{\sum_{n=2}^{\infty} P_n} - 1$$

$$= \frac{\frac{P}{1-P} - P(1-P)}{1 - [(1-P) + P(1-P)]} - 1$$

$$= \frac{1}{1-P}$$

(b) Exp. number in queue given the system is not empty

$$= \sum_{n=1}^{\infty} (n-1) \left(\frac{P_n}{\sum_{j=1}^{\infty} P_j} \right)$$

$$= \frac{\sum_{n=1}^{\infty} np_n - \sum_{n=1}^{\infty} P_n}{\sum_{j=1}^{\infty} P_j}$$

$$= \frac{\left(\frac{P}{1-P}\right) - P}{P}$$

$$= \frac{P}{1-P}$$

Thus,

Exp. waiting time in queue for those who must wait

$$= \frac{P/(1-P)}{\lambda}$$

$$= \frac{1}{M-\lambda}$$

Continued...

Set 15.6c

1

$$w(\tau) = (\mu - \lambda) e^{-(\mu - \lambda)\tau}, \tau > 0$$

$$\begin{aligned} \lambda &= 1/4 = .25/\text{hr} \\ \mu &= 1/1.5 = .667/\text{hr} \end{aligned} \quad \left. \begin{aligned} (\mu - \lambda) &= .417 \\ f &= \lambda/\mu = \frac{1.5}{4} = .375 \end{aligned} \right\}$$

$$w(\tau) = .417 e^{-0.417\tau}, \tau > 0$$

$$P\{\tau > 1\} = e^{-0.417 \times 1} = .659$$

(a) Standard deviation = $\frac{1}{\mu - \lambda} = \frac{1}{0.417} = 2.4$ 2

(b) $w(\tau) = (\mu - \lambda) e^{-(\mu - \lambda)\tau}, \tau > 0$

$$P\left\{\frac{1}{2(\mu - \lambda)} \leq \tau \leq \frac{3}{2(\mu - \lambda)}\right\}$$

$$= (1 - e^{-1.5}) - (1 - e^{-0.5})$$

$$= e^{-0.5} - e^{-1.5}$$

$$= .3834$$

$W_s \leq 10$ minutes, $\lambda = 4/\text{hr}$ 3

$$\frac{1}{(\mu - \lambda)} \leq \frac{10}{60} \text{ hr}$$

or $\mu - \lambda \geq 6$

or $\mu \geq 6 + \lambda = 10/\text{hr}$

$P\{\tau > \frac{10}{60}\} \leq .1$, or

$$e^{-\frac{1}{6}(\mu - 4)} \leq .1$$

$\mu - 4 \geq 13.8$

$\mu \geq 17.8/\text{hr}$

$P\{\tau > 5\} = e^{-(\mu - \lambda)t} = e^{-0.267 \times 5} = .2636$ 5

where $\lambda = .4/\text{min}$, $\mu = .667/\text{min}$

Exp. # customers in a 12-hr day
 $= \lambda \times 12 \times 60 = .4 \times 12 \times 60 = 288$ cust.

Exp. cost = $288 \times .2636 \times .5 = \37.95

6

Let $w_{n+1}(t|n)$ = conditional pdf for waiting in queue given there are n customers ahead

= n -fold convolution of the exponential pdf

$$= \frac{\mu(\mu t)^{n-1} e^{-\mu t}}{(n-1)!}$$

$w(t)$ = absolute pdf of waiting time in queue

$g(t, n)$ = joint pdf of t and n

$$= w_{n+1}(t|n) p_n$$

$$= \frac{\mu(\mu t)^{n-1} e^{-\mu t}}{(n-1)!} f^n(1-p)$$

(a) For $t > 0$

$$w(t) = \sum_{n=1}^{\infty} g(t, n)$$

$$= \frac{\mu p e^{-\mu t} (1-p)}{e^{-\mu p t}} \sum_{n=1}^{\infty} \frac{(\mu p t)^{n-1}}{(n-1)!} e^{\mu p t}$$

$$= \mu p (1-p) e^{\mu(1-p)t}, t > 0$$

For $t = 0$, $w(0) = p_0 = (1-p)$

$$w(t) = \begin{cases} 1-p, & t=0 \\ \mu p (1-p) e^{\mu(1-p)t}, & t>0 \end{cases}$$

(b) $W_q = E\{t\}$

$$= \int_0^\infty t w(t) dt$$

$$= 0 \cdot w(0) + \int_0^\infty t w(t) dt$$

$$= \int_{0^+}^\infty \mu p t (1-p) e^{\mu(1-p)t} dt$$

$$= \frac{p}{\mu(1-p)}$$

Set 15.6d

(a) $p_0 = .3654$

(b) $W_q = .207$ hour

(c) Average number of empty spaces = $4 - L_q$

$$= 4 - .788$$

= 3.212 spaces

(d) $\frac{p_0}{5} = .04812$

(e) $W_s \leq 10$ minutes

Title: 17.6d-1
Comparative Analysis

Scenario	c	Lambda	Mu	L/da eff	p0	Ls	Lq	Ws	Wq
1	1	4.00000	6.00000	3.80752	0.36541	1.42256	0.78797	0.37362	0.20695
2	1	6.00000	9.00000	3.91779	0.44003	1.1691	0.56034	0.28659	0.14413
3	1	4.00000	6.00000	3.83742	0.36541	1.42256	0.78797	0.37362	0.20695
4	1	4.00000	6.00000	3.95118	0.56987	0.75340	0.31257	0.19520	0.10484
5	1	4.00000	10.00000	3.97532	0.60247	0.64199	0.24446	0.16149	0.06149

M (cars/hr)	W_s (hrs)	W_s (min)
6	.3736	22.4
7	.287	17.16
8	.23	13.80
9	.19	11.40
10	.16	9.60

Desired service rate = 10 cars/hr
Thus, the service time must be reduced from $\frac{60}{10} = 6$ minutes to $\frac{60}{15} = 4$ minutes, a 40% reduction

m = number of parking spaces

An arriving car will not find a space

if there are $m+1$ cars in the system. Thus, find m such that $P_{m+1} \leq .01$

TORA input = (4, 6, 1, $m+1$, ∞)

m	$N=m+1$	P_N
4	5	.04812
5	6	.0311
6	7	.0203
7	8	.01335
8	9	.009

Select the number of parking spaces $m \geq 8$

Continued...

m = number of seats.

The $N = m+1$, and

$$\lambda_{eff} = \frac{\lambda}{N} = \frac{\lambda}{m+1} \text{ customers/hr}$$

TORA input = (6, 5, 1, N , ∞)

Title: 17.6d-3
Comparative Analysis

Scenario	c	Lambda	Mu	L/da eff	p0	Ls	Lq	Ws	Wq
1	1	6.00000	6.00000	3.62337	0.27473	1.12088	0.36560	0.30909	0.10809
2	1	6.00000	9.00000	4.00000	0.27473	1.25278	0.31207	0.42418	0.22418
3	1	6.00000	5.00000	4.32310	0.13438	2.12188	0.27278	0.32278	0.14278
4	1	6.00000	5.00000	4.49647	0.10071	3.02117	2.12188	0.67190	0.47190
5	1	6.00000	5.00000	4.51288	0.07742	3.70584	2.78728	0.80423	0.50423

$$m \quad N = m+1 \quad \lambda_{eff} (\text{customers/hr})$$

$$1 \quad 2 \quad 3.63 \\ 2 \quad 3 \quad 4.07$$

Use two seats or less

$\lambda = 10$ generators per hour

$$M = \frac{60}{15} = 4 \text{ generators per hour}$$

$$N = 7 + 1 = 8$$

Title: 17.6d-4
Scenario 1—(M/M/1):(GD/8/Infinity)

Lambda = 10.00000	Mu = 4.00000
Lambda eff = 3.99843	Rho/c = 2.50000
Ls = 7.33569	Lq = 6.33609
Ws = 1.83464	Wq = 1.58464

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.00039	0.00039	5	0.03841	0.06375
1	0.00098	0.00138	6	0.09603	0.15978
2	0.00246	0.00383	7	0.24006	0.39984
3	0.00615	0.00998	8	0.60016	1.00000
4	0.01536	0.02534			

$$(a) P_8 \approx .6$$

$$(b) Lq = 6.34 \text{ generators}$$

(c) Let C = belt capacity. Thus, $N = C + 1$. The assembly department is kept in operation so long as at least one empty space remains on the belt; that is,

$$\begin{aligned} P\{\text{empty space on belt}\} &= P_0 + P_1 + \dots + P_C \\ &= \frac{1-\rho}{1-\rho^{C+2}} \sum_{n=0}^C \rho^n \\ &= \frac{1-\rho}{1-\rho^{C+2}} \cdot \frac{1-\rho^{C+1}}{1-\rho} \\ &= \frac{1-\rho^{C+1}}{1-\rho^{C+2}} \end{aligned}$$

Continued...

Continued...

Set 15.6d

$$\lim_{c \rightarrow \infty} \frac{1 - p^{c+1}}{1 - p^{c+2}} = \lim_{c \rightarrow \infty} \frac{-(c+1)p^c}{-(c+2)p^{c+1}}$$

$$= \lim_{c \rightarrow \infty} \frac{c+1}{(c+2)p}$$

$$= \lim_{c \rightarrow \infty} \frac{(1 + 1/c)}{(1 + 2/c)} \frac{1}{p}$$

$$= \frac{1}{p}$$

In the present example, $p = 10/4$ and $1/p = .4$. Thus,

$$\lim_{c \rightarrow \infty} (p_0 + p_1 + \dots + p_c) = 1/p = .4$$

This result means that regardless of how large the belt is, the probability of finding an empty space cannot exceed .4. Thus, achieving a 95% utilization for the assembly dept. is impossible.

The result makes sense because the arrival rate λ ($= 10/\text{hr}$) is 2½ times larger than the service rate ($= 4$). The only way we can accomplish the desired result is to reduce λ and/or increase M .

(a) $P_{50} \approx .00002$

(b) $P\{\text{w/ish is not fulfilled}\}$
 $= P\{\text{48 or more in restaurant}\}$

$$= P_{48} + P_{49} + P_{50}$$

$$= 1 - (P_0 + P_1 + \dots + P_{47})$$

$$= 1 - .99993$$

$$= .00007$$

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Title: 17.6d-5
Scenario 1-(M/M/1):(GD/50/infinity)

TORA input = (10, 12, 1, 50, ∞)

Lambda = 10.00000 Mu = 12.00000
Lambda eff = 9.99982 Rho/c = 0.83333
Ls = 4.99533 Lq = 4.16201
Ws = 0.49954 Wq = 0.41621

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.16668	0.16668	26	0.00146	0.99281
1	0.13890	0.30558	27	0.00121	0.99402
2	0.11575	0.42133	28	0.00101	0.99504
3	0.09646	0.51779	29	0.00084	0.99584
4	0.08038	0.59818	30	0.00070	0.99558
5	0.06599	0.66516	31	0.00059	0.99717
6	0.05592	0.72098	32	0.00049	0.99765
7	0.04652	0.76759	33	0.00041	0.99806
8	0.03876	0.80627	34	0.00034	0.99840
9	0.03230	0.83257	35	0.00028	0.99868
10	0.02692	0.85549	36	0.00024	0.99892
11	0.02243	0.88792	37	0.00020	0.99911
12	0.01869	0.90662	38	0.00016	0.99928
13	0.01558	0.92220	39	0.00014	0.99941
14	0.01298	0.93518	40	0.00011	0.99952
15	0.01082	0.94600	41	0.00009	0.99962
16	0.00902	0.95501	42	0.00008	0.99970
17	0.00751	0.96253	43	0.00007	0.99976
18	0.00626	0.96879	44	0.00006	0.99982
19	0.00522	0.97401	45	0.00005	0.99986
20	0.00435	0.97835	46	0.00004	0.99990
21	0.00362	0.98198	47	0.00003	0.99993
22	0.00302	0.98500	48	0.00003	0.99995
23	0.00252	0.98751	49	0.00002	0.99998
24	0.00210	0.98961	50	0.00002	1.00000
25	0.00175	0.99136			

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TORA input = (20, 7.5, 1, 15, ∞)

Title: 17.6d-6
Scenario 1-(M/M/1):(GD/15/infinity)

Lambda = 20.00000 Mu = 7.50000
Lambda eff = 7.50000 Rho/c = 2.66667
Ls = 14.40000 Lq = 13.40000
Ws = 1.92000 Wq = 1.78667

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.00000	0.00000	8	0.00065	0.00104
1	0.00000	0.00000	9	0.00174	0.00278
2	0.00000	0.00000	10	0.00463	0.00742
3	0.00000	0.00001	11	0.01236	0.01678
4	0.00001	0.00002	12	0.03296	0.05273
5	0.00003	0.00005	13	0.08789	0.14062
6	0.00009	0.00015	14	0.23438	0.37500
7	0.00024	0.00039	15	0.62500	1.00000

- (a) $P_0 \approx 0$
(b) $P_{n \leq 14} = P_0 + \dots + P_{14} = .375$
(c) $W_s = 1.92 \text{ hours}$

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$$(a) P_{n \leq 4} = P_0 + P_1 + \dots + P_4 = .962$$

$$(b) \lambda_{lost} = \lambda P_5 = 5 \times .038 = .19 \text{ cust./hr}$$

$$(c) L_s = 0 \times .399 + 1 \times .249 + 2 \times .156 + 3 \times .097 + 4 \times .061 + 5 \times .038 = 1.286$$

continued...

continued...

$$(d) W_q = W_s - \frac{1}{\mu}$$

$$\lambda_{eff} = 5(1 - 0.38) = 4.81 \text{ cust/hr}$$

$$W_s = \frac{L_s}{\lambda_{eff}}$$

$$= \frac{1.286}{4.81}$$

$$= .2675 \text{ hour}$$

$$W_q = .2675 - \frac{1}{8}$$

$$= .1424 \text{ hour}$$

$$P_n = \frac{(1-p)p^n}{1-p^{N+1}}$$

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$$\begin{aligned} \lim_{p \rightarrow 1} p_n &= \lim_{p \rightarrow 1} \frac{p^n - p^{n+1}}{1 - p^{N+1}} \\ &= \lim_{p \rightarrow 1} \frac{n p^{n-1} - (n+1)p^n}{-(N+1)p^N} \\ &= \frac{1}{N+1} \end{aligned}$$

Thus,

$$\begin{aligned} L_s &= \sum_{n=0}^N n p_n \\ &= \frac{1}{N+1} \sum_{n=0}^N n \\ &= \frac{N(N+1)}{2(N+1)} = \frac{N}{2} \end{aligned}$$

$$W_s = W_q + \frac{1}{\mu}$$

$$\lambda_{eff} W_s = \lambda_{eff} W_q + \frac{\lambda_{eff}}{\mu}$$

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Thus,

$$L_s = L_q + \frac{\lambda_{eff}}{\mu}$$

or

$$\lambda_{eff} = \mu(L_s - L_q)$$

Set 15.6e

TOA input = (8, 5, 2, 00, 00)

Title: 17.6e-1
Scenario 1-(M/M/2):(GD/infinity/infinity)

Lambda = 8.00000 Mu = 5.00000
Lambda eff = 8.00000 Rho/c = 0.80000
Ls = 4.44444 Lq = 2.84444
Ws = 0.55556 Wq = 0.35556

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.11111	0.11111	23	0.00131	0.99475
1	0.17778	0.28889	24	0.00105	0.99580
2	0.14222	0.43111	25	0.00084	0.99524
3	0.11378	0.54489	26	0.00067	0.99731
4	0.09102	0.63591	27	0.00054	0.99785
5	0.07282	0.70873	28	0.00043	0.99828
6	0.05825	0.76698	29	0.00034	0.99862
7	0.04660	0.81359	30	0.00028	0.99890
8	0.03728	0.85087	31	0.00022	0.99912
9	0.02983	0.88070	32	0.00018	0.99930
10	0.02386	0.90456	33	0.00014	0.99944
11	0.01909	0.92365	34	0.00011	0.99955
12	0.01527	0.93892	35	0.00009	0.99954
13	0.01222	0.95113	36	0.00007	0.99971
14	0.00977	0.96091	37	0.00006	0.99977
15	0.00782	0.96873	38	0.00005	0.99982
16	0.00625	0.97498	39	0.00004	0.99985
17	0.00500	0.97998	40	0.00003	0.99988
18	0.00400	0.98399	41	0.00002	0.99991
19	0.00320	0.98719	42	0.00002	0.99992
20	0.00256	0.98975	43	0.00002	0.99994
21	0.00205	0.99180	44	0.00001	0.99995
22	0.00164	0.99344			

TOA input = (16, 5, 4, 00, 00)

Title: 17.6e-1
Scenario 2-(M/M/4):(GD/infinity/infinity)

Lambda = 16.00000 Mu = 5.00000
Lambda eff = 16.00000 Rho/c = 0.80000
Ls = 5.58573 Lq = 2.38573
Ws = 0.34911 Wq = 0.14911

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.02730	0.02730	24	0.00138	0.99450
1	0.08737	0.11467	25	0.00110	0.99560
2	0.13979	0.25446	26	0.00088	0.99548
3	0.14911	0.40357	27	0.00070	0.99718
4	0.11929	0.52285	28	0.00056	0.99778
5	0.09543	0.61828	29	0.00045	0.99820
6	0.07634	0.69463	30	0.00036	0.99856
7	0.06107	0.75570	31	0.00029	0.99885
8	0.04886	0.80458	32	0.00023	0.99908
9	0.03939	0.84585	33	0.00018	0.99926
10	0.03127	0.87492	34	0.00015	0.99941
11	0.02502	0.89994	35	0.00012	0.99953
12	0.02001	0.91995	36	0.00009	0.99962
13	0.01601	0.93596	37	0.00008	0.99970
14	0.01281	0.94877	38	0.00006	0.99990
15	0.01025	0.95901	39	0.00005	0.99981
16	0.00820	0.96721	40	0.00004	0.99985
17	0.00656	0.97377	41	0.00003	0.99988
18	0.00525	0.97901	42	0.00002	0.99990
19	0.00420	0.98321	43	0.00002	0.99992
20	0.00336	0.98657	44	0.00002	0.99994
21	0.00269	0.98926	45	0.00001	0.99995
22	0.00215	0.99140	46	0.00001	0.99996
23	0.00172	0.99312			

(a) $C=2:$

$$P\{\text{all servers are busy}\} = \left(\sum_{n \geq 2} p_n\right)^2 \\ = (1 - .29)^2 \\ = .504$$

$C=4:$

$$P\{\text{all servers are busy}\} = 1 - P_{n \leq 3} \\ = 1 - .404 \\ = .596$$

$C=4$ yields a higher probability that all servers are busy.

continued...

(b)

Title: 17.6e-1
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	4	16.00000	5.00000	16.00000	0.02730	5.58573	2.38573	0.34911	0.14911
2	5	16.00000	5.00000	16.00000	0.02715	3.71299	0.51299	0.23208	0.03208
3	6	16.00000	5.00000	16.00000	0.02777	3.34328	0.14520	0.20908	0.00903

for $C = 5$, $Wq = .032$ hour ≈ 2 min

$C = 4$, $Wq = .149$ hour ≈ 9 min

Select $C \geq 5$

$C=2: \lambda = 8 \text{ calls/hr}$

$$M = \frac{60}{14.5} = 4.1379 \text{ calls/hr}$$

$C=4: \lambda = 16 \text{ calls/hr}$

$$M = 4.1379 \text{ calls per hour}$$

$$\text{utilization} = \lambda/M = .967$$

Title: 17.6e-1
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	2	16.00000	8.00000	16.00000	0.01685	28.49905	27.58471	3.67228	3.44559

$Wq = \begin{cases} 3.446 \text{ hours for } C=2 \\ 1.681 \text{ hours for } C=4 \end{cases}$

Consolidation reduces the waiting time by more than 51%.

(a) $\lambda = \frac{60}{5} = 12 \text{ per hour}$

$$M = 10 \text{ per hour}$$

$$C > \frac{\lambda}{M} = 1.2 \Rightarrow C \geq 2$$

(b) $\lambda = \frac{60}{2} = 30 \text{ per hour}$

$$M = \frac{60}{6} = 10 \text{ per hour}$$

$$C > \frac{\lambda}{M} = \frac{30}{10} = 3 \Rightarrow C \geq 4$$

(c) $\lambda = 30 \text{ per hour}$, $M = 40 \text{ per hr}$.

$$C > \frac{30}{40} = .75 \Rightarrow C \geq 1$$

$\lambda = 45 \text{ customers/hr}$

$$M = \frac{60}{5} = 12 \text{ customers/hr}$$

$$C > \frac{45}{12} \text{ or } C \geq 4$$

Desired $Wq \leq 30 \text{ seconds} = .0083 \text{ hr}$

Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	4	45.00000	12.00000	45.00000	0.00355	18.72545	12.57545	0.37193	0.28154
2	5	45.00000	12.00000	45.00000	0.01855	5.12537	3.35537	0.11412	0.03879
3	6	45.00000	12.00000	45.00000	0.02208	4.12903	0.37903	0.03176	0.00842
4	7	45.00000	12.00000	45.00000	0.02309	3.86873	0.11873	0.08597	0.00264

Select $C \geq 7$.

Set 15.6e

TOA input: (20, 12, 3, ∞, ∞)

Title: 15.6e-5
Scenario 1- (M/M/3):(GD/infinity/infinity)

Lambda = 20.00000	Mu = 12.00000
Lambda eff = 20.00000	Rho/c = 0.55556
Ls = 2.04137	Lq = 0.37470
Ws = 0.10207	Wq = 0.01874

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.17266	0.17266	10	0.00218	0.99728
1	0.28777	0.46043	11	0.00121	0.99849
2	0.23981	0.70024	12	0.00067	0.99916
3	0.13323	0.83347	13	0.00037	0.99953
4	0.07401	0.90748	14	0.00021	0.99974
5	0.04112	0.94860	15	0.00012	0.99986
6	0.02284	0.97144	16	0.00006	0.99992
7	0.01269	0.98414	17	0.00004	0.99996
8	0.00705	0.99119	18	0.00002	0.99998
9	0.00392	0.99510	19	0.00001	0.99999

$m = \text{size of waiting room.}$

$$P_0 + P_1 + \dots + P_{m+2} \geq .999 \Rightarrow m \geq 10$$

$$C=2, \lambda_{\text{windows}} = .8 \times \frac{60}{3} = 16 \text{ /hr}$$

$$\mu = \frac{60}{5} = 12 \text{ per hour}$$

Title: 15.6e-6
Scenario 1- (M/M/2):(GD/infinity/infinity)

Lambda = 16.00000	Mu = 12.00000
Lambda eff = 16.00000	Rho/c = 0.66667
Ls = 2.40000	Lq = 1.06667
Ws = 0.15000	Wq = 0.06667

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.20000	0.20000	14	0.00137	0.99726
1	0.26667	0.46667	15	0.00091	0.99817
2	0.17778	0.64444	16	0.00061	0.99878
3	0.11852	0.76296	17	0.00041	0.99919
4	0.07901	0.84198	18	0.00027	0.99946
5	0.05267	0.89465	19	0.00018	0.99964
6	0.03512	0.92977	20	0.00012	0.99976
7	0.02341	0.95318	21	0.00008	0.99984
8	0.01611	0.96879	22	0.00005	0.99989
9	0.01040	0.97919	23	0.00004	0.99993
10	0.00694	0.98613	24	0.00002	0.99995
11	0.00462	0.99075	25	0.00002	0.99997
12	0.00308	0.99383	26	0.00001	0.99998
13	0.00206	0.99589			

$$(a) P_{n \geq 2} = 1 - (P_0 + P_1)$$

$$= 1 - .46667$$

$$= .5333$$

$$(b) P_0 = .2$$

$$(c) L_q = 1.067$$

(d) NO, because $\lambda > \mu$. The minimum number of windows should $\geq \frac{\lambda}{\mu} = \frac{16}{12} = 1.33$
Number of windows ≥ 2

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$$\lambda = 25 \times \frac{60}{15} = 100 \text{ jobs/hour}$$

$$\mu = \frac{60}{2} = 30 \text{ jobs/hour}, C = 4$$

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Title: 15.6e-7
Scenario 1- (M/M/4):(GD/infinity/infinity)

Lambda = 100.00000	Mu = 30.00000
Lambda eff = 100.00000	Rho/c = 0.83333
Ls = 6.62194	Lq = 3.28861
Ws = 0.06622	Wq = 0.03289

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.02131	0.02131	28	0.00138	0.99811
1	0.07103	0.09234	29	0.00115	0.99825
2	0.11839	0.21073	30	0.00096	0.99851
3	0.13154	0.34228	31	0.00080	0.99880
4	0.09202	0.45190	32	0.00066	0.99901
5	0.09135	0.54325	33	0.00055	0.99923
6	0.07813	0.61937	34	0.00046	0.99945
7	0.06344	0.68281	35	0.00038	0.99968
8	0.05286	0.73568	36	0.00032	0.99980
9	0.04045	0.77973	37	0.00027	0.99989
10	0.03671	0.81644	38	0.00022	0.99993
11	0.03059	0.84703	39	0.00019	0.99997
12	0.02549	0.87253	40	0.00015	0.99999
13	0.02125	0.89377	41	0.00013	0.99996
14	0.01770	0.91148	42	0.00011	0.99994
15	0.01475	0.92623	43	0.00009	0.99995
16	0.01229	0.93853	44	0.00007	0.99993
17	0.01025	0.94877	45	0.00006	0.99996
18	0.00854	0.95731	46	0.00005	0.99999
19	0.00711	0.96443	47	0.00004	0.99997
20	0.00593	0.97035	48	0.00004	0.99992
21	0.00494	0.97530	49	0.00003	0.99995
22	0.00412	0.97941	50	0.00002	0.99998
23	0.00343	0.98294	51	0.00002	0.99999
24	0.00286	0.98570	52	0.00002	0.99999
25	0.00238	0.98869	53	0.00001	0.99999
26	0.00199	0.99007	54	0.00001	0.99994
27	0.00165	0.99173	55	0.00001	0.99995

$$(a) P_{n \geq 4} = 1 - C P_3$$

$$= 1 - .34228 = .65772$$

$$(b) W_S = .06622 \text{ hour}$$

$$(c) L_q = 3.29 \text{ jobs}$$

$$(d) P_0 = .021 \Rightarrow 2.1\% \text{ idleness}$$

$$(e) \text{Av. # of idle computers} = 4 - (L_S - L_q)$$

$$= 4 - (6.62 - 3.29) = .67$$

$$\lambda = 15 + 10 + 20 = 45 \text{ customers/hour}$$

$$\mu = \frac{60}{6} = 10 \text{ customers/hour}$$

$$C > 45/10 = 4.5 \Rightarrow C \geq 5$$

8

Title: 15.6e-8
Comparative Analysis

Scenario	c	Lambda	Mu	Ls eff	p0	Ls	Lq	Ws	Wq
1	5	45.00000	10.00000	45.00000	0.00496	11.36244	6.86244	0.25250	0.16250
2	6	45.00000	10.00000	45.00000	0.00496	5.76465	1.28665	0.02311	0.01621
3	7	45.00000	10.00000	45.00000	0.01046	4.89100	0.39100	0.10369	0.00969

$$(a) W_S \leq 15/60 = .25 \text{ hour} \Rightarrow C \geq 6$$

$$(b) \% \text{ idle} = \frac{C - (L_S - L_q)}{C} \times 100$$

$$\frac{C - L_S}{C} \quad \frac{L_q - C}{C} \quad \% \text{ idle}$$

$$\begin{array}{lll} 5 & 11.362 & 6.862 \\ 6 & 5.765 & 1.265 \\ 7 & 4.891 & 0.391 \end{array} \quad \begin{array}{ll} .5 & 1.5 \\ 1.5 & 2.5 \end{array} \quad \begin{array}{ll} 10\% & \\ 25\% & \end{array}$$

Select C = 5

(c) C	5	6	7
P0	.00496	.00914	.01046

Select C ≤ 6

Set 15.6e

- Limited space inside a bank or a grocery store.
- Multiple queues appear to offer more courteous service.

9

For C parallel servers:

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$$L_q = \frac{\rho}{C-\rho}, \text{ provided } \frac{\rho}{C} \rightarrow 1$$

Thus,

$$W_{q_c} = \frac{1}{\lambda_c} \frac{\rho}{C-\rho} = \frac{1}{(C\mu - \lambda_c)}$$

For a single server

$$W_{q_1} = \frac{\lambda_1}{\mu(C\mu - \lambda_1)}$$

Because $\lambda_c = C\lambda_1$, we have

$$\begin{aligned} \frac{W_{q_c}}{W_{q_1}} &= \left(\frac{\frac{1}{C(C\mu - \lambda_1)}}{\frac{\lambda_1}{\mu(C\mu - \lambda_1)}} \right) = \frac{1}{C(\frac{\lambda_1}{\mu})} \\ &= \frac{1}{C(\frac{\lambda c / \mu}{c})} \\ &= \frac{1}{C(S/c)} \end{aligned}$$

$$\lim_{\frac{\rho}{c} \rightarrow 1} \frac{W_{q_c}}{W_{q_1}} = \frac{1}{C}$$

Determination of p_0 involves the finite series sum

$$\sum_{n=c}^{\infty} \left(\frac{\rho}{c}\right)^{n-c} = \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu c}\right)^j$$

The series will diverge if $\lambda \geq \mu c$. The condition requires that customers be serviced at a rate faster than the rate at which they arrive at the facility. Else, the queue will build up to infinity.

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$$\begin{aligned} L_q &= \sum_{n=c}^{\infty} (n-c) p_n \\ &= \sum_{n=c}^{\infty} n p_n - c \sum_{n=c}^{\infty} p_n + \sum_{n=0}^{c-1} n p_n - \\ &\quad \sum_{n=0}^{c-1} n p_n + c \sum_{n=0}^{c-1} p_n - c \sum_{n=0}^{c-1} p_n \\ &= \sum_{n=0}^{\infty} n p_n - c \sum_{n=0}^{\infty} p_n + \sum_{n=0}^{c-1} (c-n) p_n \\ &= L_s - c + (\text{number of idle servers}) \\ &= L_s - \bar{C} \end{aligned}$$

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Now, by definition

$$L_s = L_q + \frac{\lambda_{\text{eff}}}{\mu}$$

$$\text{It follows that } \bar{C} = \frac{\lambda_{\text{eff}}}{\mu}$$

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$$p_n = \begin{cases} \frac{\lambda^n}{n! \mu^n} p_0, & n \leq c \\ \frac{\lambda^n}{c! c^{n-c} \mu^n} p_0, & n \geq c \end{cases}$$

for $c = 1$,

$$p_n = \begin{cases} \frac{\lambda}{\mu} p_0, & n=1 \\ \left(\frac{\lambda}{\mu}\right)^n p_0, & n \geq 1 \end{cases}$$

Thus,

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0, \quad n=1, 2, \dots$$

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$$\begin{aligned} L_q &= p_0 \frac{1}{c!} \sum_{n=c+1}^{\infty} (n-c) \frac{(\lambda/\mu)^n}{c^{n-c}} \\ &= p_0 \frac{(\lambda/\mu)^c}{c!} \sum_{n=c+1}^{\infty} (n-c) \left(\frac{\lambda}{\mu c}\right)^{n-c} \\ &= p_0 \frac{(\lambda/\mu)^c}{c!} \sum_{j=1}^{\infty} j \left(\frac{\lambda}{\mu c}\right)^j \\ &= p_0 \frac{(\lambda/\mu)^c}{c!} \frac{\lambda}{\mu c} \frac{d}{d(\lambda/\mu c)} \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu c}\right)^j \\ &= p_0 \frac{(\lambda/\mu)^c}{c!} \left\{ \frac{\lambda/\mu c}{(1 - \lambda/\mu c)^2} \right\} \\ &= p_0 \frac{\rho/c}{(1 - \rho/c)^2} = \frac{\rho}{(C-\rho)^2} p_0 \end{aligned}$$

(a) $P\{\text{a customer is waiting}\}$

$$\begin{aligned}
 &= P\{\text{at least } c+1 \text{ in system}\} \\
 &= \sum_{n=c+1}^{\infty} p_n \\
 &= \sum_{n=c}^{\infty} p_n - p_c \\
 &= p_0 \frac{\rho^c}{c!} \frac{1}{1-\frac{\rho}{c}} - p_c \\
 &= p_c \left\{ \frac{1}{1-\frac{\rho}{c}} - 1 \right\} \\
 &= p_c \left(\frac{\rho}{c-\rho} \right)
 \end{aligned}$$

(b) Expected number in queue given the queue is not empty

$$\begin{aligned}
 &= \sum_{i=c+1}^{\infty} (i-c) \frac{p_i}{\sum_{j=c+1}^{\infty} p_j} \\
 &= \frac{L_q}{\sum_{j=c+1}^{\infty} p_j} = \frac{L_q}{p_c \left(\frac{\rho}{c-\rho} \right)}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } L_q &= \frac{p_0}{c!} \sum_{n=c+1}^{\infty} (n-c) \frac{\rho^n}{c^{n-c}} \\
 &= p_0 \frac{\rho^c}{c!} \sum_{j=1}^{\infty} j \left(\frac{\rho}{c}\right)^j \\
 &= p_0 \frac{\rho^c}{c!} \left(\frac{\rho/c}{(1-\rho/c)^2}\right), \frac{\rho}{c} < 1 \\
 &= p_c \left\{ \frac{c\rho}{(c-\rho)^2} \right\}, \frac{\rho}{c} < 1
 \end{aligned}$$

Substitution for L_q yield the desired result.(c) Exp. waiting time for those who must wait = Exp. waiting time given there are c in the system.

$$\begin{aligned}
 &= \frac{1}{\lambda} \sum_{i=c+1}^{\infty} (i-c) \frac{p_i}{\sum_{n=0}^{\infty} p_n} \\
 &= \frac{L_q/\lambda}{p_c/(1-\rho/c)} = \frac{1}{\mu(c-\rho)}
 \end{aligned}$$

15

16

First convert the c -channel case into an equivalent single channel. Let the customer just arriving be the j th in queue. Because there are c channels in parallel, the service time, t_i , of each of the other $j-1$ customers and the (one) customer in service are determined as follows: let t_1, t_2, \dots, t_c be the actual service times in the c channels. Then,

$$\begin{aligned}
 P\{t > T\} &= P\{\min_{1 \leq i \leq c} t_i > T\} \\
 &= (e^{-\mu c T})^c = e^{-\mu c T}
 \end{aligned}$$

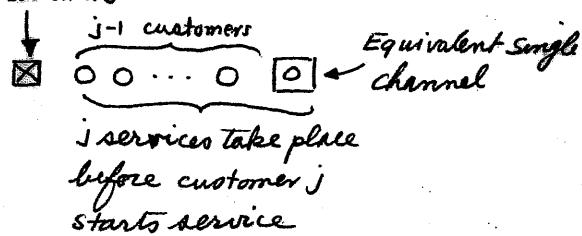
This is true because if $\min_i t_i > T$, then every t_i must be $> T$.

Now,

$$\begin{aligned}
 F_t(T) &= 1 - P\{t > T\} \\
 &= 1 - e^{-\mu c T}, \quad T > 0
 \end{aligned}$$

Thus,

$$f(t) = \frac{\partial F_t(T)}{\partial T} = \mu c e^{-\mu c T}, \quad T > 0$$

which is exponential with mean $\frac{1}{\mu c}$.The c channels can be converted into an equivalent single channel as customer j 

Before customer j starts service, j other customers each with a service time T must be processed first.

Continued...

Set 15.6e

The assumption here is that all c channels are busy. If there are any idle servers, arriving customer j will have zero waiting time in queue and the special case is treated separately.

Let τ be the waiting time in queue given there are j other customer yet to be serviced. Then

$$\tau = T_1' + T_2' + \dots + T_j'$$

where T_1', T_2', \dots, T_j' are exponential with mean $1/\mu c$. T_i' represents the remaining service time for the customer already in service. The lack of memory property indicate that T_i' is also exponential with mean $1/\mu c$. Thus,

$$W_q(\tau|j) = \frac{\mu c (\mu c \tau)^{j-1} e^{-\mu c \tau}}{(j-1)!}, \tau > 0$$

Let $W_q(\tau)$ be the absolute pdf, then

$$W_q(\tau) = \sum_{j=1}^{\infty} W_q(\tau|j) q_j.$$

where

$$q_j = \begin{cases} \sum_{k=0}^{c-1} p_k, & j=0 \\ p_{c+j-1}, & j>0 \end{cases}$$

Hence, for $\tau > 0$

$$\begin{aligned} W_q(\tau) &= \sum_{j=1}^{\infty} \frac{\mu c (\mu c \tau)^{j-1} e^{-\mu c \tau}}{(j-1)!} \frac{\rho^{c+j-1}}{c! c^{j-1}} p_0 \\ &= \frac{\rho^c \mu c e^{-\mu c \tau}}{c!} p_0 \sum_{j=0}^{\infty} \frac{(\rho \mu c \tau/c)^j}{j!} \\ &= \frac{\rho^c \mu c e^{-\mu c \tau}}{c!} p_0 e^{-\lambda \tau} \\ &= \frac{\rho^c \mu c e^{-\mu c(c-\rho)\tau}}{(c-1)!} p_0 \end{aligned}$$

For $\tau = 0$, the corresponding probability is $\sum_{k=0}^{c-1} p_k$, or

$$\begin{aligned} 1 - \sum_{k=c}^{\infty} p_k &= 1 - \sum_{j=0}^{\infty} p_{c+j} \\ &= 1 - \sum_{j=0}^{\infty} \frac{\rho^{c+j}}{c! c^j} p_0 \\ &= 1 - \frac{\rho^c}{c!} \left(\frac{p_0}{1 - \frac{\rho}{c}} \right) \\ &= 1 - \left\{ \frac{\rho^c p_0}{(c-1)! (c-\rho)} \right\} \end{aligned}$$

Hence,

$$W_q(\tau) = \begin{cases} 1 - \frac{\rho^c p_0}{(c-1)! (c-\rho)}, & \tau = 0 \\ \frac{\rho^c e^{-\mu(c-\rho)\tau}}{(c-1)!} p_0, & \tau > 0 \end{cases}$$

17

$$\begin{aligned} P\{\tau > y\} &= \int_y^{\infty} W_q(\tau) d\tau \\ &= \frac{c \mu \rho^c p_0}{c!} \int_y^{\infty} e^{-(c\mu - \lambda)\tau} d\tau \\ &= \frac{\rho^c c \mu}{c! (c\mu - \lambda)} e^{-(c\mu - \lambda)y} p_0 \\ &= \frac{\rho^c p_0}{c! (1 - \frac{\rho}{c})} e^{-(c\mu - \lambda)y} \\ &= P\{\tau > 0\} e^{-(c\mu - \lambda)y} \end{aligned}$$

where $P\{\tau > 0\} = 1 - P\{\tau = 0\}$

continued...

From Problem 16, the waiting time in the system is computed as

$$T = T_1 + T_2 + \dots + T_j + t_j$$

where

t_j = actual service time for customer j .

t_j is exponential with mean $1/\mu$

Thus, T is the convolution of the waiting time in queue and the actual service time of customer j . This means that $w(T)$ is the convolution of $w_q(\tau)$ and $g(t)$; that is,

$$w(T) = w_q(\tau) * g(t)$$

where

$$g(t) = \mu e^{-\mu t}, \quad t > 0$$

$$w(T) = w_q(0)g(T)$$

$$+ \int_{0^+}^T w_q(\tau)g(T-\tau)d\tau$$

$$= \left(1 - \frac{\rho^c p_0}{(c-1)! (c-p)}\right) \mu e^{-\mu T}$$

$$+ P_0 \int_{0^+}^T \frac{\mu p^c e^{-\mu(c-p)\tau}}{(c-1)!} \mu e^{-\mu(T-\tau)} d\tau$$

$$= \left(1 - \frac{\rho^c p_0}{(c-1)! (c-p)}\right) \mu e^{-\mu T}$$

$$+ \frac{\mu p^c e^{-\mu T}}{(c-1)! (c-1-p)} P_0 \left\{ 1 - e^{-\mu(c-1-p)T} \right\}$$

$$= \mu e^{-\mu T} \frac{\rho^c p_0 N e^{-\mu T}}{(c-1)! (c-1-p)} \frac{(c-p-1)}{(c-p)}$$

$$+ \frac{\mu p^c e^{-\mu T} P_0}{(c-1)! (c-1-p)} - \frac{\mu p^c e^{-\mu T} e^{-\mu(c-1-p)T}}{(c-1)! (c-1-p)}$$

Continued...

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$$= \mu e^{-\mu T} + \frac{\rho^c p_0 N e^{-\mu T}}{(c-1)! (c-1-p)} \left\{ \frac{1}{c-p} - e^{-N(c-1-p)T} \right\}$$

$T > 0$

Set 15.6f

(a) $C - (L_s - L_q) = 4 - (4.24 - 1.54)$
 $= 1.3 \text{ cars}$

(b) $P_q = .04468$

(c) Title: 6t-1
Comparative Analysis

Scenario	c	Lambda	Mu	U/ds eff	p ₀	L _s	L _q	W _s	W _q
1	4	16.00000	5.00000	15.2815	0.03121	4.2398	1.1542	0.27481	0.07481
2	4	16.00000	5.00000	15.2869	0.03238	4.0283	0.57450	0.26387	0.06387
3	4	16.00000	5.00000	15.2854	0.03393	3.7847	0.77903	0.25184	0.05184
4	4	16.00000	5.00000	15.2849	0.03531	3.51216	0.87078	0.23881	0.04881
5	4	16.00000	5.00000	14.24151	0.03531	3.20550	0.35719	0.22508	0.02508

m = length of waiting list

N = m + 4

m	N	W _q (hr)	W _q (min)
6	10	.075	4.5
5	9	.064	3.83
4	8	.052	3.12
3	7	.039	2.33
2	6	.025	1.5

Select m ≤ 3

C = 2, λ = 20/hr, N = 5

μ = 60/6 = 10/hr

(a) P₅ = .1818 or 18.18%

(b) P₁ = .1818 or 18.18%

(c) % utilization = 100 $\left(\frac{L_s - L_q}{C}\right)$
 $= \frac{2.727 - 1.091}{2} \times 100$
 $= 81.8\%$

(d) Probability = P₂ + P₃ + P₄ = .54546

(e) P_N ≤ .1

N	5	...	8	9	10
P _N	.1818		.1176	.1053	.0952

N ≥ 10 spaces (including the pumps)

(f) P₀ ≤ .05

N	5	...	8	9	10
P _N	.0909		.0588	.0526	.0476

N ≥ 10

λ = 60/10 = 6/hr

μ = 60/30 = 2/hr, N = 18

3

(a) # idle mechanics

$$= C - (L_s - L_q)$$

$$= 3 - (9.54 - 6.71) = .17$$

(b) P₁₈ = .0559

$$\lambda_{lost} = .0559 \times 6 = .3354 \text{ job/hr}$$

lost jobs in 10 hrs = 3.354 jobs

(c) P_{n≤17} = P₀ + P₁ + ... + P₁₇
 $= .9441$

(d) P_{n≤2} = P₀ + P₁ + P₂ = .10559

(e) L_q = 6.7081 mower

(f) $\frac{L_s - L_q}{C} = \frac{9.54 - 6.71}{3} = .944$

N = 40, C = 30, λ = 20/hr

μ = 60/60 = 1/hr

4

(a) P₄₀ = .00014

(b) P₃₀ + P₃₁ + ... + P₃₉ = P_{n≤39} - P_{n≤29}
 $= .99986 - .97533$
 $= .02453$

(c) P₂₉ = .01248

(d) L_s - L_q = 20.043 - 0.96 ≈ 20 spaces

(e) L_q = .046

continued...

continued...

(f) If there are 30 cars or more in the lot, the student will not make it to class. Thus,

$P\{\text{not finding a parking space}\}$

$$= P_{30} + P_{31} + \dots + P_{40} = 1 - P_{n \leq 29}$$

$$= 1 - .97533 = .02467$$

No. of students who cannot park during an 8-hr period = $20 \times .02467 \times 8$
 ≈ 4 students

$$1 = p_0 \left\{ \sum_{n=0}^{c-1} \frac{p^n}{n!} + \frac{p^c}{c!} \sum_{n=c}^N \left(\frac{p}{c}\right)^{n-c} \right\} \quad 5$$

$$= p_0 \left\{ \sum_{n=0}^{c-1} \frac{p^n}{n!} + \frac{p^c}{c!} \frac{1 - (\frac{p}{c})^{N-c+1}}{1 - \frac{p}{c}} \right\}$$

$$p_0 = \left\{ \sum_{n=0}^{c-1} \frac{p^n}{n!} + \frac{p^c}{c!} \left(\frac{1 - (\frac{p}{c})^{N-c+1}}{1 - \frac{p}{c}} \right) \right\}$$

$$\bar{c} = L_s - L_q$$

6

$$= \lambda_{\text{eff}} (w_s - w_q)$$

$$= \lambda_{\text{eff}} \left(\frac{1}{M} \right)$$

$$1 = \frac{p_0}{c!} \sum_{n=c}^N \frac{p^n}{c^{n-c}} + p_0 \sum_{n=0}^{c-1} \frac{p^n}{n!} \quad 7$$

$$= \frac{p_0 p^c}{c!} \sum_{n=0}^{N-c} \left(\frac{p}{c}\right)^n + p_0 \sum_{n=0}^{c-1} \frac{p^n}{n!}$$

$$= \frac{p_0 p^c}{c!} (N-c+1) + p_0 \sum_{n=0}^{c-1} \frac{p^n}{n!}$$

Thus,

$$p_0 = \left\{ \sum_{n=0}^{c-1} \frac{p^n}{n!} + \frac{p^c}{c!} (N-c+1) \right\}^{-1}$$

$$L_q = \sum_{n=c}^N (n-c) p_n$$

$$= \sum_{j=0}^{N-c} j p_j + c$$

$$= \frac{p}{c!} \frac{p^c}{c} \sum_{j=0}^{N-c} j \left(\frac{p}{c}\right)^{j-1} p_0$$

continued...

$$= \frac{p^c}{c!} \sum_{j=0}^{N-c} j p_0 \quad (\text{because } \frac{p}{c} = 1)$$

$$= \frac{p^c}{c!} \frac{(N-c)(N-c+1)}{2} p_0$$

$$= \frac{p^c (N-c)(N-c+1)}{2c!} p_0$$

8

$$\lambda_n = \begin{cases} \lambda, & n=0, 1, 2, \dots, c-1 \\ 0, & n=c \end{cases}$$

$$M_n = nM, \quad n=0, 1, \dots, c$$

Thus,

$$p_n = \frac{p^n}{n!} p_0, \quad n=0, 1, 2, \dots, c$$

$$\sum_{n=0}^c p_n = \sum_{n=0}^c \frac{p^n}{n!} p_0 = 1$$

$$p_0 = \left\{ \sum_{n=0}^c \frac{p^n}{n!} \right\}^{-1}$$

Set 15.6g

(a) $P_0 = 0$

1

(b) $P_{n \geq 10} = 1 - P_{n \leq 9} = 1$

(c) $P_{n \leq 40} - P_{n \leq 29} = .7771 - .13787$
 $= .63923$

(d) $L_S = 36$

Net annual equity

$$= \$1000 \times 36 \{ .1(1-.3) + .9(1+.15) \}$$

$$= \$39,780$$

$\lambda = \frac{100}{8} = 12.5 / \text{hr}$

2

$M = \frac{60}{30} = 2 / \text{hr}$

(a) $L_S = 6.25 \approx 7 \text{ seats}$

(b) $P_{n \geq 8} = 1 - (P_0 + P_1 + \dots + P_7)$
 $= 1 - .7089 = .2911$

(c) $P_0 = .00193$

$P = .1$

3

c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
2	1.00000	10.00000	1.00000	0.90476	0.10025	0.00025	0.10025	0.00025
4	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
10	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
20	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
50	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000
9999	1.00000	10.00000	1.00000	0.90484	0.10000	0.00000	0.10000	0.00000

$P = .9$

4

c	λ	M	λ_{eff}	P_0	L_s	L_q	W_s	W_q
10	9.00000	1.00000	9.00000	0.00007	15.01858	6.01858	1.66873	0.66873
15	9.00000	1.00000	9.00000	0.00012	9.07235	0.07235	1.00004	0.00004
25	9.00000	1.00000	9.00000	0.00012	9.00000	0.00000	1.00000	0.00000
50	9.00000	1.00000	9.00000	0.00012	9.00000	0.00000	1.00000	0.00000
9999	9.00000	1.00000	9.00000	0.00012	9.00000	0.00000	1.00000	0.00000

- For very small P_0 , $(M/M/\infty) : (GD/\infty)$ provides reliable estimates for $(M/M/c) : (GD/\infty/\infty)$.
- For large P_0 , $(M/M/\infty)$ gives reliable estimates only if c is large

Set 15.6h

$$(a) R=1: \lambda_{eff} = \lambda(22-L_S) \\ = .5(22 - 12.004) \\ = 4.998$$

$$R=4: \lambda_{eff} = .5(22 - 2.1) = 9.95$$

$$(b) \text{No. of idle repair persons} \\ = 4 - (L_S - L_q) \\ = 4 - (2.1 - .11) = 2.01$$

$$(c) P_0 = .10779$$

$$(d) R = 3:$$

$$P\{2 \text{ or } 3 \text{ are idle}\} = P_0 + P_1 \\ = .34492$$

Title: 6h-1
Scenario 3-(MM/3):(GD/22/22)

Lambda = 0.50000	Mu = 5.00000
Lambda eff = 9.76696	Rho/c = 0.03333
Ls = 2.46596	Lq = 0.51257
Ws = 0.25248	Wq = 0.05248

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.10779	0.10779	8	0.00953	0.99244
1	0.23713	0.34492	9	0.00445	0.99689
2	0.24899	0.59390	10	0.00193	0.99881
3	0.16599	0.75980	11	0.00077	0.99959
4	0.10513	0.86502	12	0.00028	0.99987
5	0.06308	0.92810	13	0.00009	0.99996
6	0.03574	0.96384	14	0.00003	0.99999
7	0.01906	0.98291			

Productivity of repair persons

= Av. # busy repair persons

R	Repair prod.	Shop prod.
1	100%	45.44%
2	88.2%	80.15%
3	65.1%	88.7%
4	49.7%	90.45%

R=2 yield 80.15% shop productivity and also maintain repair productivity at 88.2%

Increasing R, in effect, increases the number of machines that remain operative, and hence the chance of additional breakdowns. Stated differently, if all machines remain broken, there will be no new calls for repair service, and $\lambda_{eff} = 0$

3

$$\lambda = \frac{60}{45} = 1.33 \text{ machines/hr}$$

$$\mu = \frac{60}{8} = 7.5 \text{ machines/hr}$$

$$R=1, K=5$$

4

Title: 6h-4
Scenario 1-(MM/1):(GD/5/5)

Lambda = 1.33333	Mu = 7.50000
Lambda eff = 4.99939	Rho/c = 0.17778
Ls = 1.25045	Lq = 0.58386
Ws = 0.25012	Wq = 0.11679

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.33341	0.33341	3	0.11240	0.95293
1	0.29637	0.62978	4	0.03996	0.99290
2	0.21075	0.84053	5	0.00710	1.00000

$$(a) L_S = 1.25 \text{ machines}$$

$$(b) P_0 = .33341$$

$$(c) W_S = .25 \text{ hour}$$

$$\lambda = 60/45 = 1.33/\text{hr}$$

$$\mu = 60/20 = 3/\text{hr}$$

$$R=4, K=4$$

5

Title: 6h-5
Scenario 1-(MM/4):(GD/4/4)

Lambda = 1.33333	Mu = 3.00000
Lambda eff = 3.69230	Rho/c = 0.11111
Ls = 1.23077	Lq = 0.00000
Ws = 0.33333	Wq = 0.00000

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.22972	0.22972	3	0.08067	0.99104
1	0.40839	0.63811	4	0.00896	1.00000
2	0.27226	0.91037			

$$(a) L_S = 1.23 \text{ workers}$$

$$(b) P_0 = .22972$$

Set 15.6h

$$\lambda = \frac{60}{30} = 2 \text{ calls/hr/baby}$$

6

$$\mu = \frac{60}{120} = .5 \text{ /hr}$$

$$R = 5, K = 5$$

Title: 6h-6
Scenario 1- (M/M/5); (GD/5/5)

Lambda = 2.00000	Mu = 0.50000
Lambda eff = 2.00000	Rho/c = 0.80000
Ls = 4.00000	Lq = 0.00000
Ws = 2.00000	Wq = 0.00000

n	Probability, pn	Cumulative, Pn	n	Probability, pn	Cumulative, Pn
0	0.00032	0.00032	3	0.20480	0.26272
1	0.00640	0.00672	4	0.40960	0.67232
2	0.05120	0.05792	5	0.32768	1.00000

(a) No. "awake" babies

$$= 5 - L_S = 5 - 4 = 1 \text{ baby}$$

$$(b) P_S = .32768$$

$$(c) P_{n \leq 2} = P_0 + P_1 + P_2 = .05792$$

$$\bar{R} = L_S - L_Q$$

8

$$= \lambda_{\text{eff}} (W_S - W_Q)$$

$$= \lambda_{\text{eff}} \left(\frac{1}{\mu} \right)$$

$$\text{Hence } \lambda_{\text{eff}} = \mu \bar{R}$$

$$P_n = \begin{cases} C_n^k \rho^n n! P_0, & n=0,1 \\ C_n^k n! \rho^n P_0, & n=1,2,\dots,K \end{cases}$$

9

$$= \frac{K!}{(K-n)!} \rho^n P_0, \quad n=0,1,2,\dots,K$$

$$L_S = \sum_{n=0}^K n P_n = P_0 K! \sum_{n=0}^K \frac{n \rho^n}{(K-n)!}$$

$$= K - \left(\frac{1-P_0}{\rho} \right)$$

7

$$P_n = \begin{cases} \frac{K\lambda}{\mu} \frac{(K-1)\lambda}{2\mu} \dots \frac{(K-n)\lambda}{n\mu} P_0, & 0 \leq n \leq R \\ \frac{K\lambda}{\mu} \frac{(K-1)\lambda}{2\mu} \dots \frac{(K-R)\lambda}{R\mu} \dots \frac{K-n}{R\mu} P_0, & R \leq n \leq K \end{cases}$$

Thus,

$$P_n = \begin{cases} \frac{K(K-1)\dots(K-n)}{1 \times 2 \times \dots \times n} \left(\frac{\lambda}{\mu} \right)^n P_0, & 0 \leq n \leq R \\ \frac{C_n^k n!}{R!} \left(\frac{\lambda}{\mu} \right)^n P_0, & R \leq n \leq K \end{cases}$$

$$= \begin{cases} C_n^k \rho^n P_0, & 0 \leq n \leq R \\ C_n^k \frac{n! \rho^n}{R! R^{n-R}} P_0, & R \leq n \leq K \end{cases}$$

Set 15.7a

$$\begin{aligned}
 \% \text{ idle} &= \frac{1 - (L_s - L_q)}{1} \times 100 \\
 &= [1 - (L_s - L_q)] \times 100 \\
 &= (1 - 1.333 + .667) \times 100 \\
 &= 33.3\%
 \end{aligned}$$

1 $\lambda = .3 \text{ job/day}$

3

Service time distribution:

$$f(t) = .5, \quad 2 \leq t \leq 4 \text{ days}$$

$$E\{t\} = 3 \text{ days}$$

$$\text{Var}\{t\} = \frac{4}{12} = .333 \text{ days}^2$$

(a) $E\{t\} = 14 \text{ min}$

2

$$\text{Var}\{t\} = \frac{(20-8)^2}{12} = 12 \text{ min}^2$$

$$\lambda = 4/\text{hr} = .0667/\text{min}$$

$$L_s = 7.867 \text{ cars}$$

$$W_s = 118 \text{ min} = 1.967 \text{ hours}$$

$$L_q = 6.933 \text{ cars}$$

$$W_q = 104 \text{ min} = 1.733 \text{ hours}$$

(b) $E\{t\} = 12 \text{ min}$

$$\text{Var}\{t\} = 9 \text{ min}^2$$

$$\lambda = .0667/\text{min}$$

$$L_s = 2.5 \text{ cars}$$

$$W_s = 37.5 \text{ min} = .625 \text{ hours}$$

$$L_q = 1.7 \text{ cars}$$

$$W_q = 25.5 \text{ min} = .425 \text{ hr}$$

(c) $E\{t\} = 4x.2 + 8x.6 + 15x.2 = 8.6 \text{ min}$

$$\text{Var}\{t\} = (4-8.6)^2(.2) + (8-8.6)^2(.6)$$

$$+ (15-8.6)^2(.2) = 12.64 \text{ min}^2$$

$$L_s = 1.0244 \text{ cars}$$

$$W_s = 15.3657 \text{ min} = .256 \text{ hr}$$

$$L_q = .451 \text{ car}$$

$$W_q = 6.765 \text{ min} = .113 \text{ hr}$$

(a) $L_q = 4.2 \text{ homes}$

(b) $W_s = 17 \text{ days}$

(c) $E\{t\} = 1.5, \text{ Var}\{t\} = \frac{1}{12} = .0833$

$$L_q = .191 \text{ home}$$

$$W_s = 2.14 \text{ days}$$

4 $\lambda = \frac{30}{8 \times 60} = .0625 \text{ prescr./min}$

$$E\{t\} = 12 + 3 = 15 \text{ min}$$

$$\text{Var}\{t\} = 9 + \frac{(4-2)^2}{12} = 9.333 \text{ min}^2$$

(a) $P_0 = .0625$

(b) $L_q = 7.3 \text{ prescriptions}$

(c) $W_s = 132.17 \text{ min} = 2.2 \text{ hours}$

5 $\lambda = 1/45/\text{min} = .0222/\text{min}$

$$E\{t\} = 28 + 4.5 = 32.5 \text{ min}$$

$$\text{Var}\{t\} = \frac{(6-3)^2}{12} = .75$$

(a) $L_q = .9395 \text{ item}$

(b) $P_0 = .278$

(c) $W_s = 74.78 \text{ min}$

6 $L_s = \lambda E\{t\} + \frac{\lambda^2(E^2\{t\} + \text{Var}\{t\})}{2(1-\lambda E\{t\})}$

$$= \lambda E\{t\} + \frac{(\lambda E\{t\})^2}{2(1-\lambda E\{t\})}$$

$$= P + \frac{P^2}{2(1-P)}$$

Set 15.7a

7

$$\begin{aligned}
 L_S &= \frac{m\lambda}{\mu} + \frac{\lambda^2 \left(\frac{m^2}{\mu^2} + \frac{n}{\mu^2} \right)}{2(1 - \frac{m\lambda}{\mu})} \\
 &= mp + \frac{m^2 p^2 + mp^2}{2(1 - mp)} \\
 &= mp + \frac{m(m+1)p^2}{2(1 - mp)}
 \end{aligned}$$

8

$$\begin{aligned}
 E\{t\} &= \frac{1}{\mu}, \quad \text{Var}\{t\} = \frac{1}{\mu^2} \\
 L_S &= \frac{\lambda}{\mu} + \frac{\lambda^2 \left(\frac{1}{\mu^2} + \frac{1}{\mu^2} \right)}{2(1 - \lambda/\mu)} \\
 &= p + \frac{p^2}{1-p} \\
 &= \frac{p}{1-p}
 \end{aligned}$$

9

(a) Because each server receives every c^{th} customer and the interarrival time at the channel is exponential with mean $1/\lambda$, the interarrival time at each server is the convolution of c exponential distributions each with mean $\frac{1}{\lambda}$. This means that the interarrival time is gamma with mean c/λ and variance c/λ^2 .

(b) The interarrival time at the i^{th} server is exponential with mean $\frac{1}{\alpha_i \lambda}$. This means that the arrivals at server i is Poisson with mean $\alpha_i \lambda$, $i=1, 2, \dots, c$

Set 15.9a

$$(a) M_2 = \frac{24}{(\frac{1000}{36}) \times \frac{1}{60}} = 5.184 \text{ jobs/day}$$

$$M_3 = \frac{24}{(\frac{1000}{50}) \times \frac{1}{60}} = 7.2 \text{ jobs/day}$$

$$M_4 = \frac{24}{(\frac{1000}{66}) \times \frac{1}{60}} = 9.5 \text{ jobs/day}$$

$$(b) ETC_i = 24 C_{ii} + 80 Lq_i$$

i	λ_i	M_i	Lq_i	C_{ii}	ETC_i
1	4	4.32	11.57	\$15	\$1285.60
2	4	5.18	2.62	20	689.60
3	4	7.20	.69	24	631.20
4	4	9.50	.31	27	672.80

Select model 3.

$$\lambda = 3 \text{ /hr}$$

$$M_1 = 5 \text{ /hr}, \quad C_1 = \$15$$

$$M_2 = 8 \text{ /hr}, \quad C_2 = \$20$$

Cost/Broken machine = \$50/hr

(M/M/1) : (GD/10/10):

$$\lambda = 3, \mu = 5 \Rightarrow L_S = 8.33$$

(M/M/1) : (GD/10/10):

$$\lambda = 3, \mu = 8 \Rightarrow L_{S_2} = 7.33$$

$$TC_1 = 50L_{S_1} + 15 = 50 \times 8.33 + 15 \\ = \$431.50/\text{hr}$$

$$TC_2 = 50L_{S_2} + 20 = 50 \times 7.33 + 20 \\ = \$386.50/\text{hr}$$

Here second repair person.

$$\lambda = 10/\text{hr} = .167/\text{min}$$

Scanner A:

Service time distribution:

$$f_A(t) = \frac{1}{(\frac{35}{10}) - (\frac{25}{10})} = 1, 2.5 \leq t \leq 3.5$$

Continued...

$$E_A\{t\} = 3 \text{ min}$$

$$Var_A\{t\} = \frac{1}{12} \text{ min}^2$$

Scanner B:

$$f_B(t) = \frac{1}{\frac{35}{15} - \frac{25}{15}} = 1.5, \quad \frac{5}{3} \leq t \leq \frac{7}{3}$$

$$E_B\{t\} = 2 \text{ min}$$

$$Var_B\{t\} = \frac{(2/3)^2}{12} = \frac{1}{27} \text{ min}^2$$

From Excel file PKFormula.xls,

$$L_{SA} = .755 \text{ customer}$$

$$L_{SB} = .419 \text{ customer}$$

$$TC_A = .2L_{SA} + C_A \\ = (-.2 \times .755 + \frac{10}{10 \times 60}) \times 60 = \$10.06/\text{hr}$$

$$TC_B = .2L_{SB} + C_B \\ = (-.2 \times .419 + \frac{15}{10 \times 60}) \times 60 = \$6.53/\text{hr}$$

Select scanner B

(a) $M = \text{number of filled orders/hr}$

$\lambda = \text{number of requested orders/hr}$

$C_1 = \text{cost/unit increase in production rate}$

$C_2 = \text{cost of waiting/unit waiting time/cust.}$

$TC(M) = \text{Total cost/unit waiting time}$
given μ

$$= C_1 \mu + C_2 L_S$$

$$= C_1 \mu + C_2 \frac{\lambda}{M-\lambda}$$

$$\frac{\partial TC(M)}{\partial M} = C_1 - C_2 \frac{\lambda}{(M-\lambda)^2} = 0$$

$$M = \lambda + \sqrt{\frac{C_2}{C_1} \lambda}$$

$$(c) \lambda = 3, C_1 = .1 \times 500 = \$50, C_2 = \$100$$

$$M = 3 + \sqrt{\frac{100}{50} \times 3} = 5.45 \text{ orders/hr}$$

Optimum production rate

$$= 500 \times 5.45 \approx 2725 \text{ pieces/hr}$$

Set 15.9a

$$\lambda = 80 \text{ jobs/wk}$$

$$C_1 = \$250/\text{wk} \quad C_2 = \$500/\text{job/wk}$$

$$M = \lambda + \sqrt{\frac{C_2}{C_1}} \lambda$$

$$= 80 + \sqrt{\frac{500}{250} \times 80} = 92.65 \text{ jobs/wk}$$

5

$$\lambda = 25 \text{ groups/hr}$$

$$\underline{\text{Model A: }} M = 26/\text{hr}, N = 20$$

$$\text{Operating cost } C_A = \$12000/\text{month}$$

$$\text{From TORA: } P_{20} = .03128$$

$$L_q = 7.65 \text{ groups}$$

6

$$\text{Cost/hr} = \text{operating cost/hr} + \text{waiting cost/hr} + \text{cost of lost customers/hr}$$

$$= \frac{C_A}{30 \times 10} + 10L_q + \lambda P_N \times 15$$

$$= \frac{12000}{30 \times 10} + 10 \times 7.65 + 25 \times .03128 \times 15$$

$$= \$128.23/\text{hr}$$

$$\underline{\text{Model B: }} M = 29/\text{hr}, N = 30$$

$$C_B = \$16000/\text{month}$$

$$\text{From TORA: } P_{30} = .0016$$

$$L_q = 5.07 \text{ groups}$$

$$\text{Cost/hr} = \frac{\$16000}{30 \times 10} + 10 \times 5.07 + 25 \times .0016 \times 15$$

$$= \$104.63$$

Select model B

Let

$$C_3 = \text{cost/unit time/additional capacity unit.}$$

The cost model in Problem 6 is modified by adding the term $C_3 N$ to the cost equation.

7

P_0 is the probability of running out of stock. Thus,

8

$$\begin{aligned} \text{Cost of lost sales per hour} &= C_1 \lambda P_0 \\ E\{\text{cost}\}/\text{unit time} &= E\{\text{lost sales cost}\}/\text{unit time} \\ &\quad + E\{\text{holding cost}\}/\text{unit time} \\ &= C_1 \lambda P_0 + C_2 L_S \end{aligned}$$

For (M/M/1) : (GD/oo/oo)

$$P_0 = (1 - \rho)$$

$$L_S = \frac{\rho}{1 - \rho}$$

Thus,

$$E\{\text{cost}\}/\text{unit time} = C_1 \lambda (1 - \rho) + C_2 \frac{\rho}{1 - \rho}$$

$$\frac{\partial E\{\text{cost}\}}{\partial \rho} = -C_1 \lambda + \frac{C_2}{(1 - \rho)^2} = 0$$

Thus,

$$\rho = 1 \pm \sqrt{\frac{C_1 \lambda}{C_2}}$$

Under steady state, ρ must be less than 1. Thus,

$$\rho = 1 - \sqrt{\frac{C_1 \lambda}{C_2}}$$

The solution requires $\sqrt{\frac{C_1 \lambda}{C_2}} < 1$ in order for ρ not to assume a negative value. Note that

$$\rho = \frac{\lambda}{M}, \text{ where } M \text{ is a constant.}$$

This means that M is the actual optimization variable.

Set 15.9b

$$C_1 = \$20, C_2 = \$45,$$

$$\lambda = 17.5/\text{hr}, \quad N = 10/\text{hr}$$

Title: 15.9b-1 (M/M/c/GD/infinity/infinity)

Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	2	17.50000	10.00000	17.50000	0.00000	7.49597	5.71667	0.45667	0.32667
2	3	17.50000	19.00000	17.50000	0.15654	1.47172	0.45715	0.25250	0.00000
3	4	17.50000	10.00000	17.50000	0.17038	1.84206	0.09206	0.10528	0.00628
4	5	17.50000	10.00000	17.50000	0.17314	1.76982	0.01962	0.10112	0.00112

$$ETC(c) = 20c + 45L_s$$

C	$L_s(c)$	$ETC(c)$
2	7.467	$20 \times 2 + 45 \times 7.467 = \376.07
→ 3	2.217	$20 \times 3 + 45 \times 2.217 = \159.77
4	1.842	$20 \times 4 + 45 \times 1.842 = \162.89
5	1.770	$20 \times 5 + 45 \times 1.770 = \179.65

Use three clerks

$$Cost/\text{hr} = C_1 L_s + C_2 c$$

$$C_1 = \$30, \quad C_2 = \$18$$

$$(M/M/c):(GD/10/10): \lambda = 1/20 = 0.05/\text{hr}$$

$$M = 1/3 = 0.333/\text{hr}$$

Title: 15.9b-2
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	2	0.05000	0.33300	0.41603	0.21439	1.67942	0.43010	4.03663	1.03883
2	3	0.05000	0.33300	0.43167	0.24269	1.36246	0.06554	3.15478	0.15173

$$(Cost/\text{hr} \text{ for } C=2) = 30 \times 1.68 + 18 \times 2 = \$86.40$$

$$(Cost/\text{hr} \text{ for } C=3) = 30 \times 1.36 + 18 \times 3 = \$94.80$$

(a) No, because the cost is higher

(b) Schedule loss/breakdown = $C_1 W_s$

$$C=2: W_s = 4.037 \text{ hours}$$

$$\text{Schedule loss} = 30 \times 4.037 = \$121.11$$

$$C=3: W_s = 3.155 \text{ hours}$$

$$\text{Schedule loss} = 30 \times 3.155 = \$94.65$$

The problem is similar to the machine repair model. The executives are the "machines" and the WATS line is the "server".

Arrival rate/executive = 2 calls/day

$$\text{Service rate} = \frac{480}{6}$$

$$= 80 \text{ calls/day}$$

Continued...

TOVA input:

$$R=1: (2, 80, 1, 100, 100)$$

$$R=2: (2, 80, 2, 100, 100)$$

Title: 15.9b-3
Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	2	2.00000	80.00000	80.00000	0.00200	59.99881	59.99881	0.74999	0.73749

(a) No WATS:

$$\begin{aligned} \text{Cost/month} &= (2 \text{ calls}/8 \text{ hrs/exec}) \times \\ &(100 \text{ exec}) \times (6 \text{ min/call}) \times \\ &(50 \text{¢/min}) \times (200 \text{ hrs/month}) \\ &= \$15,000/\text{month} \end{aligned}$$

One WATS Line: $L_q = 59$

$\text{Cost/month} = \text{Cost of WATS line} +$

$$\begin{aligned} C_1 L_q \\ = \$2000/\text{month} + 59 \left(\frac{14}{100} \times 60 \times 200 \right) \\ = \$9080 \end{aligned}$$

$$\text{Savings} = 15,000 - 9080$$

$$= \$5920/\text{month}$$

(b) Two WATS lines: $L_q = 18.4$

$$\begin{aligned} \text{Cost/month} &= 2 \times 2000 + \\ &18.4 \left(\frac{14}{100} \times 200 \times 60 \right) \\ &= \$6200 \end{aligned}$$

Additional savings

$$= 9080 - 6200 = \$2880$$

Lease a second WATS line

Set 15.9b

Rate of breakdown/machine, λ
 $= \frac{57.8}{8 \times 20} = .36125/\text{hr}$

$$M = \frac{60}{6} = 10/\text{hr}$$

TORA model: (M/M/3):(GD/20/20)

W_s = lost time per breakdown

λ = number of breakdowns/hr/mach

lost time per mach/hr = λW_s

From TORA, $W_s = .10118 \text{ hr}$

Lost revenue/machine/hr

$$= 25 \times (.36125 \times .10118) \times \$2 \\ = \$1.83$$

Lost revenue for all machines

$$= 20 \times 1.83 = \$36.50$$

Cost of 3 repairpersons/hr

$$= 3 \times 20 = \$60.$$

$$TC(c) = C_1 + C_2 L_s(c)$$

$$TC(c-1) = (c-1) C_1 + C_2 L_s(c-1)$$

$$TC(c+1) = (c+1) C_1 + C_2 L_s(c+1)$$

$$TC(c-1) - TC(c) \\ = -C_1 + C_2 \{L_s(c-1) - L_s(c)\}$$

$$TC(c+1) - TC(c) \\ = C_1 - C_2 \{L_s(c) - L_s(c+1)\}$$

At a minimum point, we must have

$$TC(c-1) \geq TC(c)$$

$$TC(c+1) \geq TC(c)$$

Thus,

$$L_s(c-1) - L_s(c) \geq \frac{C_1}{C_2}$$

$$L_s(c) - L_s(c+1) \leq \frac{C_1}{C_2}$$

4

or

$$L_s(c) - L_s(c+1) \leq \frac{C_1}{C_2} \leq L_s(c-1) - L_s(c)$$

$$\frac{C_1}{C_2} = \frac{12}{50} = .24$$

C	$L_s(c)$	$L_s(c) - L_s(c+1)$	
2	7.467	-	
3	2.217	5.25	
4	1.842	.375	$\leftarrow \frac{C_1}{C_2} = .24$
5	1.764	.078	

$$C^* = 4$$

5

continued...

$$\lambda = 1/7 = .1428 \text{ breakdown/hr}$$

$M = .95$ repair per hour

TORA model: $(M/M/R) : (G.D/10/10)$

Comparative Analysis

Scenario	c	Lambda	Mu	L'da eff	p0	Ls	Lq	Ws	Wq
1	1	0.14280	0.25000	0.25000	0.00001	8.24932	7.24934	32.99772	28.99772
2	1	0.14280	0.25000	0.25000	0.00001	8.24932	7.24934	32.99772	28.99772
3	3	0.14280	0.25000	0.71071	0.00548	5.02302	2.18017	7.06758	3.06758
4	4	0.14280	0.25000	0.83618	0.00949	4.14443	0.79972	4.95641	0.95641
5	5	0.14280	0.25000	0.91089	0.01408	3.31048	0.20000	4.26587	0.26587
6	6	0.14280	0.25000	0.94047	0.01081	3.26859	0.05272	4.05931	0.05931
7	7	0.14280	0.25000	0.98007	0.01089	3.64098	0.00867	4.00955	0.00955
8	8	0.14280	0.25000	0.98578	0.01091	3.63662	0.00091	4.00100	0.00100

(a) From TORA's output

$$L_s < 4 \Rightarrow R \geq 5$$

(b) From TORA's output

$$W_q < 1 \Rightarrow R \geq 4$$

$$C_1 = \$12$$

2

C	L_s
2	7.467
3	2.217
4	1.842

$$2.217 - 1.842 \leq \frac{12}{C_2} \leq 7.467 - 2.217$$

$$.375 \leq \frac{12}{C_2} \leq 5.25$$

or

$$\$2.29 \leq C_2 \leq \$32$$

Chapter 16

Simulation Modeling

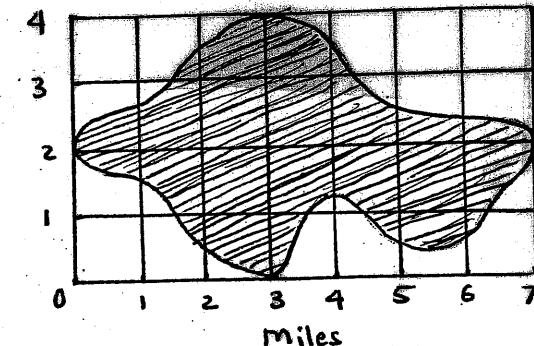
Set 16.1a

R1	R2	X	Y	$(X-1)^2 + (Y-2)^2 \leq 1$	in?
0.0589	0.6733	3.411	3.733	22.46021	1
0.4799	0.9486	0.799	6.486	20.164597	1
0.6139	0.5933	2.139	2.933	2.16781	1
0.9341	0.1782	5.341	-1.218	29.199805	0
0.3473	0.5644	-0.527	2.644	2.746465	1
0.3529	0.3646	-0.471	0.646	3.997157	1
0.7676	0.8931	3.676	5.931	22.613737	1
0.3919	0.7876	-0.061	4.876	9.439937	1
0.5199	0.6358	1.199	3.358	1.883765	1
0.7472	0.8954	3.472	5.954	21.7449	1
				Total=	9
				Area estimate=	90

Exact area = 78.54 cm^2 . Estimate from Figure 18-2 is 78.56 cm^2 for a sample size of $n=30,000$. Current estimate = 90 cm^2 , which is unreliable because the sample size is too small.

1

3

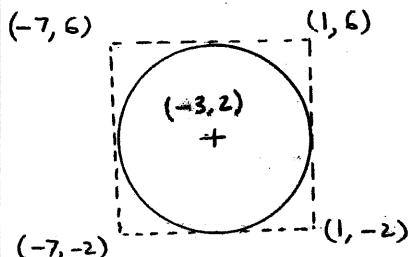


$$(a) X = -7 + 8R_1$$

$$Y = -2 + 8R_2$$

$$f(x) = \frac{1}{8}, \quad -7 \leq x \leq 1$$

$$f(y) = \frac{1}{8}, \quad -2 \leq y \leq 6$$



(b)

Monte Carlo Estimation of the Area of a Circle						
Input Data						
No. Replications, N =	10					
Sample size, n =	100,000					Steps = 1
Radius, r = 5	4					
Center, cx =	-3					
Center, cy =	2					
Output results						
Exact area =	50.265					
Press to Execute Monte Carlo						
Monte Carlo Calculations:						
n=100000						
Replication 1	50.223					
Replication 2	50.378					
Replication 3	50.113					
Replication 4	50.260					
Replication 5	50.244					
Replication 6	50.330					
Replication 7	50.327					
Replication 8	50.252					
Replication 9	50.236					
Replication 10	50.467					
Mean =	50.283					
Std. Deviation =	0.099					
95% lower conf. limit =	50.212					
95% upper conf. limit =	50.354					

2

R1	R2	X	Y	in?
0.0589	0.6733	4.123	2.6932	No
0.4799	0.9486	3.3593	3.7944	Yes
0.6139	0.5933	4.2973	2.3732	Yes
0.9341	0.1782	6.5387	.7128	No
0.3473	0.5644	2.4311	2.9576	Yes
0.3529	0.3646	2.4703	1.4584	Yes
0.7676	0.8931	5.3732	3.5724	No
0.3919	0.7876	2.7433	3.1504	Yes
0.5199	0.6358	3.6393	2.5432	No
0.7472	0.8954	5.2304	3.5816	No

points in = 5

$$\text{Area estimate} = \frac{5}{10} \times (4 \times 7) = 14 \text{ miles}^2$$

(4) $P\{H\} = .5 \quad P\{T\} = .5$
If $0 \leq R \leq .5$, Juni gets \$10
.5 < R ≤ 1, Jan gets \$10

R	Jan's pay	R	Jan's pay
0.0589	-10	0.3529	-10
0.6733	10	0.3646	-10
0.4799	-10	0.7676	10
0.9486	10	0.8931	10
0.6139	10	0.3919	-10
0.5933	10	0.7876	10
0.9341	10	0.5199	10
0.1782	-10	0.6358	-10
0.3473	-10	0.7472	10
0.5644	10	0.8954	10

$$\bar{X}_1 = \$2 \quad \bar{X}_2 = \$4$$

continued...

Set 16.1a

<u>R</u>	Jan's pay	<u>R</u>	Jan's pay	<u>R</u>	Jan's pay
.5861	10	.3455	-10	.7900	10
.1281	-10	.4871	-10	.7698	10
.2867	-10	.8111	10	.2871	-10
.8216	10	.8912	10	.9534	10
.3866	-10	.4291	-10	.1394	-10
.7125	10	.2302	-10	.9025	10
.2108	-10	.5423	10	.1605	-10
.3575	-10	.4208	-10	.3567	-10
.2926	-10	.6975	10	.3670	-10
.8261	10	.5954	10	.5513	10

$$\bar{X}_3 = -\$2$$

$$\bar{X}_4 = \$0$$

$$\bar{X}_5 = \$0$$

(b) Av. Jan's pay based on 5 rep'ts.

$$= 2 + 4 - 2 + 0 + 0$$

$$= \$.8$$

$$S = \sqrt{\frac{(2 - .8)^2 + (4 - .8)^2 + (-2 - .8)^2 + 2(0 - .8)^2}{5-1}}$$

$$= \sqrt{\frac{80.8}{4}} = 2.28$$

Confidence interval:

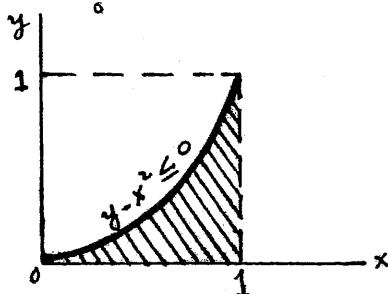
$$.8 - \frac{2.28}{\sqrt{5}} t_{.025, 4} \leq \mu \leq .8 + \frac{2.28}{\sqrt{5}} t_{.025, 4}$$

Given $t_{.025, 4} = 2.776$, the 95% confidence interval is

$$-2.03 \leq \mu \leq 3.63$$

(c) Theoretical Jan's payoff = \$0.

Estimate $\int_0^1 x^2 dx$



5

Continued...

(a)

Let $x=R1$ and $y=R2$.

Experiment: If $R2 < R1^2$, count point "in".

Estimate of integral = $(1x1)(\text{Points "in"})/5$

(b)

	R1	R2	1=in, 0=out
Rep 1	0.0589	0.6733	0
	0.4799	0.9486	0
	0.6139	0.5933	0
	0.9341	0.1782	1
	0.3473	0.5644	0
Integral estimate =			0.2
Rep 2	0.3529	0.3646	0
	0.7676	0.8931	0
	0.3919	0.7876	0
	0.5199	0.6358	0
	0.7472	0.8954	0
Integral estimate =			0
Rep 3	0.5869	0.1281	1
	0.2867	0.8216	0
	0.8261	0.3866	1
	0.7125	0.2108	1
	0.3575	0.2926	0
Integral estimate =			0.6
Rep 4	0.3455	0.4871	0
	0.8111	0.8912	0
	0.4291	0.2302	0
	0.5954	0.5423	0
	0.4208	0.6975	0
Integral estimate =			0
overall integral estimate =			0.2
Std. Deviation =			0.244949
95% lower confidence limit =			-0.189714
95% upper confidence limit =			0.5485706
Exact integral value =			0.3333

The given estimate is not "good" when compared with the exact value because sample size ($n = 5$) is too small.

$$7 = (6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)$$

6

Monte Carlo experiment:

<u>R</u>	outcome
$0 \leq R \leq \frac{1}{6}$	1
$\frac{1}{6} < R \leq \frac{1}{3}$	2
$\frac{1}{3} < R \leq \frac{1}{2}$	3
$\frac{1}{2} < R \leq \frac{2}{3}$	4
$\frac{2}{3} < R \leq \frac{5}{6}$	5
$\frac{5}{6} < R \leq 1$	6

$0 \leq R \leq .167$	1
$.167 < R \leq .333$	2
$.333 < R \leq .5$	3
$.5 < R \leq .667$	4
$.667 < R \leq .833$	5
$.833 < R \leq 1$	6

Continued...

16-3

Set 16.1a

R_1	R_2	Sum	Payoff
.0589	.6733	1+5 = 6 point	
.4799	.9486	3+6 = 9	
.6139	.5933	4+4 = 8	
.9341	.1782	6+2 = 8	
.3473	.5644	3+4 = 7 \rightarrow	-\$10
.3529	.3646	3+3 = 6 point	
.7676	.8931	5+6 = 11	
.3919	.7876	3+5 = 8	
.5199	.6358	4+4 = 8	
.7472	.8954	5+6 = 11	
.5869	.1281	4+1 = 5	
.2867	.8216	2+5 = 7 \rightarrow	-\$10
.8261	.3866	5+3 = 8 point	
.7125	.2108	5+2 = 7 \rightarrow	-\$10
.3575	.2926	3+2 = 5 point	
.3455	.4871	3+3 = 6	
.8111	.8912	5+6 = 11	
.4291	.2302	3+2 = 5 \rightarrow	\$10
.5954	.5423	4+4 = 8 point	
.4208	.6975	3+5 = 8 \rightarrow	\$10

Lead time:

$$0 \leq R \leq .5, \quad L = 1 \text{ day} \\ .5 < R \leq 1, \quad L = 2 \text{ days}$$

Demand/day:

$$0 \leq R \leq .2, \quad d = 0 \text{ unit} \\ .2 < R \leq .9, \quad d = 1 \text{ unit} \\ .9 < R \leq 1, \quad d = 2 \text{ units}$$

Let $p(d, L)$ be the joint pdf of demand and lead time. The procedure calls for constructing a frequency table of demand and lead time.

The maximum demand during lead time is $2 \times 2 = 4$ units, so that the demand $d = 0, 1, 2, 3, 4$. We will use the random numbers in Table 16-1 in the following manner: First use a random number to generate a lead time. If $L=1$ day, use one

random number to generate 7 continued
the demand in that day. If $L=2$ days, use two random numbers to generate the demands for the two days. For example, $R = .058962$ yields $L=1$. Next, $R = .6733$ gives $d=1$. Thus, we update the frequency table by increasing the frequency of the entry $(d=1, L=1)$ by one. The frequency table using the first two columns of R in Table 16-1 is

		0	1	2	3	4	
		1	1	111	11	0	0
L		2	11	0	1111	1111	0

		0	1	2	3	4	
		1	1	7	2	0	0
L		2	2	0	7	4	0

$$\text{Total } n = 23$$

Relative frequency table: $P(L)$

		0	1	2	3	4	
L		1	1/23	7/23	2/23	0	0
L		2	2/23	0	7/23	4/23	0

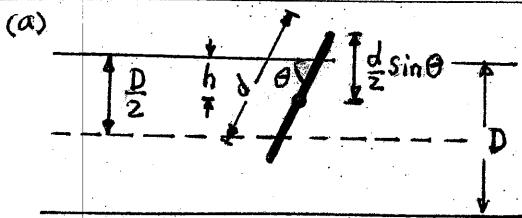
$$p(d) = 3/23 \quad 7/23 \quad 9/23 \quad 4/23 \quad 0$$

Notice that

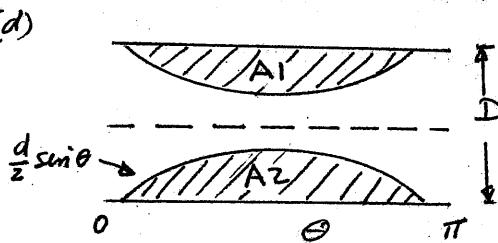
$$P(d) = \sum_L P(d, L)$$

$$P(L) = \sum_d P(d, L)$$

Set 16.1a



8



From graph, needle will touch line or cross it if

$$h \leq \frac{d}{2} \sin \theta$$

(b) Generate $h = R_1 \times D/2$

$$\theta = \pi \times R_2$$

If $h \leq \frac{d}{2} \sin \theta$, needle touches. Else it doesn't.

$$\text{Probability estimate} = \frac{\# \text{ touches}}{\text{sample size}}$$

(c)

A	B	C	D	E
D=	20	d=	10	
(RAND()*.5+.5 RAND()*.PI() SE51*.5*SIN(C4)) IF(B4<=D4,1,0)				
h	theta	$\frac{d}{2} \sin(\theta)/2$	1=touch, 0=else	

Rep 1	8.396953573	1.3165558	4.839272983	0
	7.107859045	2.9048959	1.172463622	0
	0.27542965	0.8440783	3.736795168	1
	1.267504547	2.8354706	1.506816139	1
	9.237262421	0.7436482	3.38488765	0
	2.495379696	2.9719552	0.844125326	0
	4.253169953	2.8396976	1.486650397	0
	8.516662244	1.4161445	4.940326141	0
	4.224254495	0.7887632	3.547410981	0
	3.690266876	3.0811598	0.301979787	0
	Estimate of probability=			0.2
Rep 2	0.712918949	1.5238102	4.994481772	1
	9.381794079	2.5979258	2.586388239	0
	1.360072144	2.0189288	4.506289193	1
	8.477675064	1.9724771	4.60202594	0
	0.99443686	1.300734	4.81877136	1
	5.170438974	1.4568612	4.967582038	0
	5.056822846	1.6844549	4.967739087	0
	5.864264693	0.0683356	0.341412027	0
	6.87137267	2.6283793	2.454895584	0
	1.092023022	2.6522347	2.350296303	1
	Estimate of probability=			0.4
Rep 3	9.712756211	1.694489	4.961799031	0
	6.686447356	1.2243834	4.702983326	0
	6.436673778	2.4581589	3.157296664	0
	1.324134345	2.2441568	3.908652279	1
	1.775706228	2.255079	3.874363448	1
	0.090587765	2.7080167	2.100592855	1
	4.979938633	2.5138689	2.936520016	0
	8.678634219	2.7348178	1.978247037	0
	2.179672677	1.8339609	4.827857959	1
	9.640572895	1.2431615	4.734030551	0
	Estimate of probability=			0.4
Rep 4	8.227016322	2.6999829	2.136976805	0
	8.757368267	2.1537385	4.174233356	0
	4.203914479	0.1860064	0.92467824	0
	6.098369885	2.1672345	4.13670754	0
	4.960185836	0.7841548	3.531135292	0
	3.899078191	1.8047989	4.863730557	1
	5.840727605	0.727722	3.325852126	0
	6.645324046	0.498725	2.391531067	0
	5.361422671	0.89898	3.913462424	0
	3.223016816	1.6715052	4.974665749	1
	Estimate of probability=			0.2
	Mean value =			0.3
	Std. Deviation =			0.1155
	95% LCL =			0.1163
	95% UCL =			0.4837

(c) From (c),

$$\tilde{p} = .3$$

Thus,

$$\frac{2d}{\pi D} = .3$$

$$\text{or } \pi \approx \frac{2d}{.3D}$$

$$\approx \frac{2 \times 10}{.3 \times 20}$$

$$\approx 3.33$$

Set 16.2a

- (a) Discrete
- (b) Continuous
- (c) Discrete

1

In discrete simulation, there
are two main events: arrivals and
departures. An arrival event may
experience delay before starting
service. When service has been
completed, customer leaves the
facility.

2

The description of the discrete
simulation situation by arrival
and departure events is the
reason discrete simulation
is associated with queues.

Events:

1

 A_1 = rush job arrives A_2 = regular job arrives D_1 = rush job departs D_2 = regular job departs A_0 = job arrives at carousel

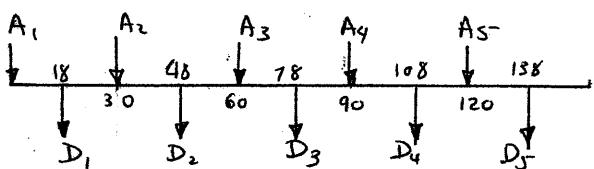
2

 A_1 = job arrives at station 1 A_2 = job arrives at station 2 A_3 = job arrives at station 3 D_1 = job departs station 1 D_2 = job departs station 2 D_3 = job departs station 3 A_1 = car enters lane 1

3

 A_2 = car enters lane 2 A_3 = car goes elsewhere D_1 = car departs lane 1 D_2 = car departs lane 2.

4



Set 16.3b

$$t = -\frac{1}{\lambda} \ln(1-R)$$

$\lambda = 4 \text{ customers/hr}$

Customer R t(hrs) Arrival time

Customer	R	t(hrs)	Arrival time
1	-	-	0
2	.0589	.015	$0 + .015 = .015$
3	.6733	.280	$.015 + .28 = .295$
4	.4799	.163	$.295 + .163 = .458$
A ₁ A ₂		A ₃	A ₄
↓		↓	↓
0 .015		.295	.458

$$f(t) = \frac{1}{b-a}, \quad a \leq t \leq b$$

$$F(t) = \int_0^t \frac{1}{b-a} dx = \frac{t-a}{b-a}, \quad a \leq t \leq b$$

$$R = \frac{t-a}{b-a}$$

$$t = a + (b-a)R$$

$$f_1(t_1) = .5 e^{-.5t_1}, \quad \lambda = 1/2 \text{ arrival/hr}$$

$$f_2(t) = \frac{1}{.9}, \quad 1.1 < t < 2$$

$$R = .0589, \quad a_1 = -2 \ln(1-.0589) = .12 \text{ hr}$$

$$R = .6733, \quad d_1 = 1.1 + .9 \times .6733 = 1.71 \text{ hrs}$$

$$R = .4799, \quad a_2 = -2 \ln(1-.4799) = 1.31 \text{ hrs}$$

$$R = .9486, \quad a_3 = -2 \ln(1-.9486) = 5.94 \text{ hrs}$$

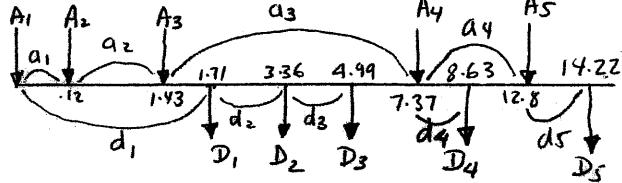
$$R = .6139, \quad d_2 = 1.1 + .9 \times .6139 = 1.65 \text{ hrs}$$

$$R = .5933, \quad d_3 = 1.1 + .9 \times .5933 = 1.63 \text{ hrs}$$

$$R = .9341, \quad a_4 = -2 \ln(1-.9341) = 5.44 \text{ hrs}$$

$$R = .1782, \quad d_4 = 1.1 + .9 \times .1782 = 1.26 \text{ hrs}$$

$$R = .3473, \quad d_5 = 1.1 + .9 \times .3473 = 1.41 \text{ hrs}$$



- (a) $0 \leq R < .2, \quad d = 0$
 $.2 \leq R < .5, \quad d = 1$
 $.5 \leq R < .9, \quad d = 2$
 $.9 \leq R \leq 1., \quad d = 3$

4

Day	R	Demand d	Stock level
0	-	-	5
1	.0589	0	5
2	.6733	2	3
3	.4799	1	2

Replenish stock on day 3

Repair/.2, Package/.8:

5

$0 \leq R < .2, \quad \text{goto Repair}$

$.2 \leq R \leq 1., \quad \text{goto Package}$

Package/.8, Repair/.2:

$0 \leq R < .8, \quad \text{goto Package}$

$.8 \leq R \leq 1, \quad \text{goto Repair}$

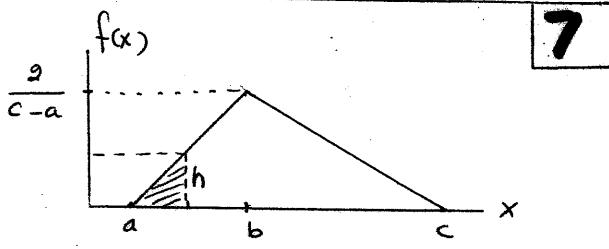
Example: $R = .1$ leads to Repair in the first case and to Package in the second case

$0 \leq R < .5 : H$

$.5 \leq R \leq 1. : T$

n R outcome Payoff

1	.0589	H	\$2
—	—	—	—
1	.6733	T	0
2	.4799	H	$2^2 = \$4$



continued...

Set 16.3b

(a) $F(x) = \begin{cases} \frac{(x-a)^2}{(b-a)(c-a)}, & a \leq x \leq b \\ 1 - \frac{(c-x)^2}{(c-b)(c-a)}, & b \leq x \leq c \end{cases}$ [7 continued]

For $R = \frac{(x-a)^2}{(b-a)(c-a)}$,

$$x = a + \sqrt{R(b-a)(c-a)}, \quad 0 \leq R \leq \frac{b-a}{c-a}$$

For $R = 1 - \frac{(c-x)^2}{(c-b)(c-a)}$,

$$x = c - \sqrt{(c-b)(c-a)(1-R)}, \quad \frac{b-a}{c-a} \leq R \leq 1$$

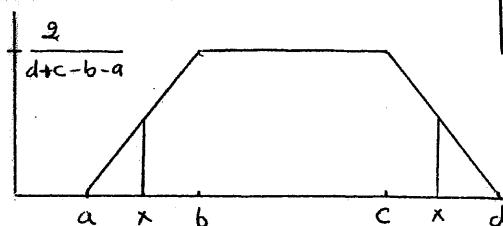
(b) $a = 1, b = 3, c = 7$

$$\frac{b-a}{c-a} = \frac{3-1}{7-1} = -0.333$$

Thus,

$$X = \begin{cases} 1 + \sqrt{(3-1)(7-1)R} \\ = 1 + \sqrt{12R}, \quad 0 \leq R \leq 0.333 \\ 7 - \sqrt{(7-3)(7-1)(1-R)} \\ = 7 - \sqrt{24(1-R)}, \quad -0.333 \leq R \leq 1 \end{cases}$$

R	X
.0589	1.84
.6733	4.20
.4799	3.47
.9486	5.89
.6139	3.96



8

(a) $F(x) = \begin{cases} \frac{(x-a)^2}{(b-a)(d+c-b-a)}, & a \leq x \leq b \\ \frac{1}{(b-a)(d+c-b-a)} + \frac{2(x-a)}{(d+c-b-a)}, & b \leq x \leq c \\ 1 - \frac{(d-x)^2}{(d-c)(d+c-b-a)}, & c \leq x \leq d \end{cases}$

Continued...

$$R = \frac{(x-a)^2}{(b-a)(d+c-b-a)} \text{ gives}$$

$$x = a + \sqrt{(b-a)(d+c-b-a)R}, \quad 0 \leq R \leq \frac{b-a}{(d+c-b-a)}$$

$$R = \frac{1}{(b-a)(d+c-b-a)} + \frac{2(x-b)}{(d+c-b-a)} \text{ gives}$$

$$x = \frac{1}{2} \left(R - \frac{1}{(b-a)(d+c-b-a)} \right) (d+c-b-a),$$

$$\frac{b-a}{d+c-b-a} \leq R \leq 1 - \frac{d-c}{(d+c-b-a)}$$

$$R = 1 - \frac{(d-x)^2}{(d-c)(d+c-b-a)}$$

$$x = d - \sqrt{(d-c)(d+c-b-a)(1-R)},$$

$$1 - \frac{d-c}{(d+c-b-a)} \leq R \leq 1$$

(b) $a = 1, b = 2, c = 4, d = 6$

$$1 + \sqrt{(2-1)(6+4-2-1)R} = 1 + \sqrt{7R}, \quad 0 \leq R \leq .143$$

$$2 + \frac{6+4-2-1}{2}(R - \frac{1}{(2-1)(6+4-2-1)})$$

$$= 2 + 3.5(R - .143),$$

$$.143 \leq R \leq .714$$

$$6 - \sqrt{(6-4)(6+4-2-1)(1-R)}$$

$$= 6 - \sqrt{14(1-R)}$$

$$.714 \leq R \leq 1$$

$R \quad X$

.0589	1.64
.6733	3.86
.4799	3.18
.9486	5.15
.6139	3.65

$f(x) = pq^x, \quad x = 0, 1, 2, \dots$
 $(p+q) = 1$

$$F(x) = p \sum_{t=0}^x q^t$$

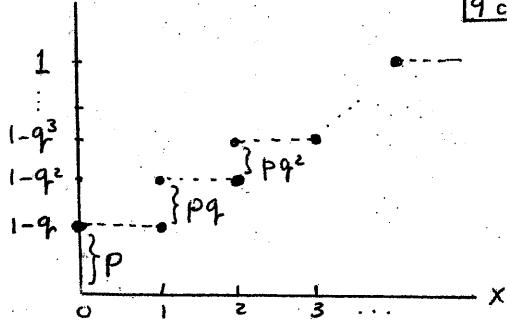
$$= 1 - q^{x+1}, \quad x = 0, 1, 2, \dots$$

9

Continued...

Set 16.3b

9 continued



Sampling procedure:

if $0 \leq R \leq p$, then $x = 0$.

For $p < R \leq 1$, we have

$$1 - q^n \leq R \leq 1 - q^{n+1}$$

or $n \leq \frac{\ln(1-R)}{\ln q} \leq n+1$

Thus, for $p \leq R \leq 1$, compute

$$x = \left[\frac{\ln(1-R)}{\ln q} \right]$$

where $[a]$ is the largest integer less than or equal to a .

For $p = .6$, $q = .4$, we have

R	$\frac{\ln(1-R)}{\ln q}$	x
.0589	—	0
.6733	1.22	1
.4799	—	0
.9486	3.24	3
.6139	1.03	1

$$f(x) = \alpha \beta^{-\alpha} x^{\alpha-1} e^{-(x/\beta)^{\alpha}}, \quad x > 0$$

$$= \frac{\alpha}{\beta} \left(\frac{x}{\beta} \right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad x > 0$$

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}, \quad x > 0$$

Thus, $R = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$

or $x = \beta \left[-\ln(1-R) \right]^{1/\alpha}$

Set 16.3c

1

$$y = -\frac{1}{5} \ln \{ (0.0589x \cdot 6733x \cdot 4799x \cdot 9486x) \}$$

$$= .803 \text{ hour}$$

2

$$\lambda = 5 \text{ events/hr}, t = 1$$

$$e^{-5x_1} = e^{-5} = .00673$$

$i R_1 R_2 \dots R_4$

1	.0589
2	.0589x.6733 = .0397
3	.0397x.4799 = .0190
4	.0190x.9486 = .0181
5	.0181x.6139 = .0111
6	.0111x.5933 = .00656
7	.00656x.9341 = .00614

Hence $n = 6$

$\mu = 8, \sigma = 1, N(8, 1)$

2

3

Convolution method:

$$X = R_1 + R_2 + \dots + R_6 = 6.1094$$

$$Y = 8 + 1(6.1094 - 6) = 8.1094$$

Box-Miller method:

$$X = \sqrt{-2 \ln R_1} \cos(2\pi R_2)$$

$$= \sqrt{-2 \ln 0.0589} \cos(2\pi x \cdot 6733)$$

$$\approx -1.103$$

$$Y = 8 + 1(-1.103) = 6.897$$

4

$$\lambda = 6 \text{ / day} \quad m = 5$$

$$Y = -\frac{1}{6} \ln (0.0589x \cdot 6733x \cdot 4799x \cdot 9486x \cdot 6139) = .751 \text{ hour}$$

$N(27, 3): \mu = 27, \sigma = 3$

5

Given R_1 and R_2 , we have

$$X_1 = \sqrt{-2 \ln R_1} \cos(2\pi R_2)$$

$$X_2 = \sqrt{-2 \ln R_1} \sin(2\pi R_2)$$

$$Y_1 = \mu + \sigma X_1$$

$$Y_2 = \mu + \sigma X_2$$

Continued...

J	K	L	M	N	O
Mean = 27	Std. Dev. = 3				
R1	R2	x1	x2	y1	y2
5 0.0589	0.6733	-1.1030306	-2.108827	23.69091	20.67352
6 0.4799	0.9486	1.149111	-0.384576	30.44733	25.84627
7 0.6139	0.5933	-0.8229152	-0.546495	24.53125	25.36051
			mean y=	25.09163	
			Sy	3.197533	

Formulas:

$$L5 = \text{SQRT}(-2 * \text{LN}(J5)) * \text{COS}(2 * \text{PI}() * K5)$$

$$M4 = \text{SQRT}(-2 * \text{LN}(J5)) * \text{SIN}(2 * \text{PI}() * K5)$$

$$N4 = \$K\$1 + L4 * \$M\$1$$

$$O4 = \$K\$1 + M4 * \$M\$1$$

$$X_i = 10 + (20 - 10) R_i$$

$$= 10 + 10 R_i, i = 1, 2, 3, 4$$

$$t = X_1 + X_2 + X_3 + X_4$$

$$= 40 + 10(R_1 + R_2 + R_3 + R_4)$$

R ₁	R ₂	R ₃	R ₄	t (sec)	Σt
1 0.0589	0.6733	-1.103	-2.1088	61.61	61.60
2 0.6139	0.5933	1.1491	-0.3846	63.20	124.81
3 0.3473	0.7676	0.8931	0.3919	64.00	188.81
4 0.7876	0.5199	0.6358	0.7472	66.91	255.72
5 0.8954	0.5869	0.1281	0.2867	58.94	314.69

The number of mice that exit the maze in 300 seconds is 4

Let X_1, X_2, \dots, X_n be n successive random deviates obtained from the geometric distribution as given in Problem 9, Set 18.3b. Then

$$X_i = \left[\frac{\ln R_i}{\ln(1-p)} \right], i = 1, 2, \dots, n$$

Because the negative binomial is the convolution of r independent geometric random variables, it follows that a random negative binomial sample can be determined as

$$X = \sum_{i=1}^n \left[\frac{\ln R_i}{\ln(1-p)} \right]$$

Note that $[a]$ represents the largest integer $\leq a$

8

Set 16.3d

Step 1: $R = .6139$
 $X = .6139$

Step 2: $R = .5933$

Step 3: $\frac{f(.6139)}{g(.6139)} = .948 > .5933$

Step 1: $R = .9341$, $X = .9341$ Reject X

Step 2: $R = .1782$

Step 3: $\frac{f(.9341)}{g(.9341)} = \frac{.3693}{1.5} = .246 > .1782$

Reject X

Step 1: $R = .3473$, $X = .3473$

Step 2: $R = .5644$

Step 3: $\frac{f(.3473)}{g(.3473)} = \frac{.9067}{1.5} = .6044 > .5644$

Reject X

Step 1: $R = .3529$, $X = .3529$

Step 2: $R = .3646$

Step 3: $\frac{f(.3529)}{g(.3529)} = \frac{.913}{1.5} = .6086 > .3646$

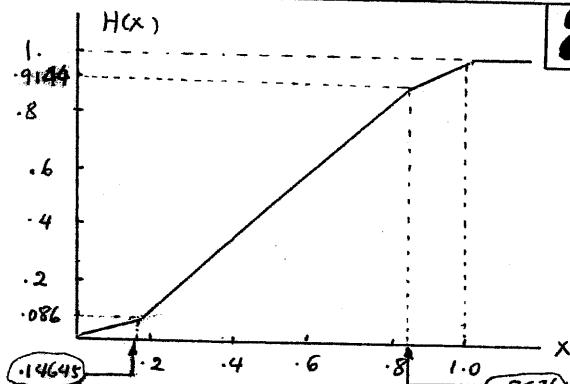
Reject X

Step 1: $R = .7676$, $X = .7676$

Step 2: $R = .8931$

Step 3: $\frac{f(.7676)}{g(.7676)} = \frac{.7135}{1.5} = .4756 < .8931$

Accept $X = .7676$



Step 1: $R = .4799$, $X = .4831$

Step 2: $R = .9486$

Step 3: $\frac{f(.4831)}{g(.4831)} = \frac{.9988}{1.5} = .6659 > .9486$

Reject X

Step 1: $R = .6139$, $X = .5974$

Step 2: $R = .5933$

continued...

Step 3: $\frac{f(.5974)}{g(.5974)} = \frac{.962}{1.5} = .6413 < .5933$ 2 continued
 reject X

Step 1: $R = .9341$, $X = .8804$

Step 2: $R = .1782$

Step 3: $\frac{f(.8804)}{g(.8804)} = \frac{.842}{1.5} = .5613 > .1782$

Reject X

Step 1: $R = .3529$, $X = .375$

Step 2: $R = .3646$

Step 3: $\frac{f(.375)}{g(.375)} = \frac{.937}{1.5} = .6247 > .3646$

Reject X

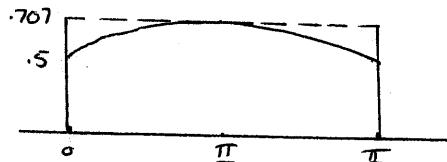
Step 1: $R = .7676$, $X = .7286$

Step 2: $R = .8931$

Step 3: $\frac{f(.7286)}{g(.7286)} = \frac{.791}{1.5} = .5273 < .8931$

Accept X

3



$f(x) = \frac{\sin(x) + \cos(x)}{2}$ $0 \leq x \leq \frac{\pi}{2}$

$\max f(x) = .707 \text{ at } x = \frac{\pi}{4}$

$g(x) = .707 \quad 0 \leq x \leq \pi/2$

$h(x) = \frac{g(x)}{\text{area under } g(x)}$

$= \frac{.707}{.707 \times \frac{\pi}{2}} = .637 \quad 0 \leq x \leq \frac{\pi}{2}$

$\int_{12}^{20} \frac{K_1}{t} dt = K_1 \ln \frac{20}{12} = 1$

Thus, $K_1 = 1.96$

$\int_{18}^{22} \frac{K_2}{t^2} dt = K_2 \left(\frac{1}{18} - \frac{1}{22} \right) = 1$

Thus, $K_2 = 99$

$f_1(t) = \frac{1.96}{t}, \quad 12 \leq t \leq 20$

$f_2(t) = \frac{99}{t^2}, \quad 18 \leq t \leq 22$

4

continued...

Set 16.3d

.163

.098

.31

.205

$$h_1(t) = \frac{.261 - .008125t}{1.044}$$

$$= .25 - .007783t$$

$$H_1(t) = .025t - .00778 \left. \frac{t^2}{2} \right|_{12}$$

$$= .25t - .003892t^2 - 2.44$$

$$h_2(t) = \frac{.7825 - .02625t}{1.03}$$

$$= .76 - .0255t$$

$$H_2(t) = .76t - .01275t^2 - 9.55$$

Sample computations from $H_2(t)$:

Step 1: $R_1 = .0589$

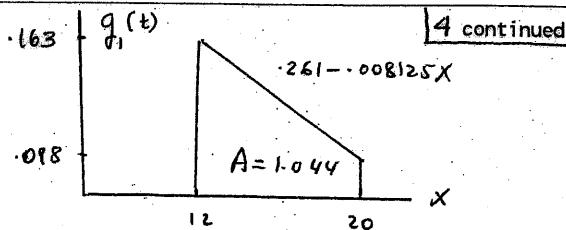
$$.76t - .01275t^2 - 9.55 = .0589$$

$$t^2 - 59.6t + 753.64 = 0$$

$$t = \frac{59.6 \pm \sqrt{(-59.6)^2 - 4 \times 753.64}}{2}$$

$$= 18.2$$

Step 2: $R = .6733$



$$\text{Step 3: } \frac{f_2(18.21)}{g_2(18.21)} = \frac{\left(\frac{99}{18.21^2}\right)}{.7825 - .02625 \times 18.21}$$

$$= .98 > .6733$$

Reject t.

continued...

Set 16.4a

1

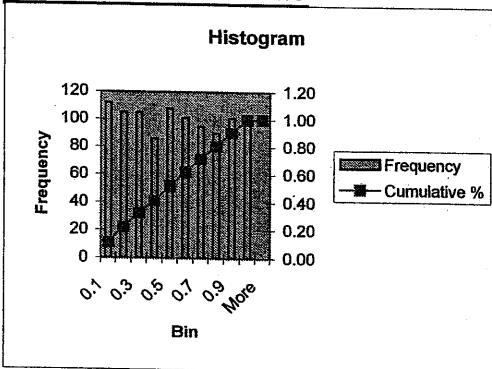
Multiplicative Congruential Method	
Input data	
$a_1 =$	17
$c_1 =$	111
$m_1 =$	7
$m_1^2 =$	103
How many numbers?	50
Output results	
Press to Generate Sequence	
Generated random numbers:	

1	0.23301
2	0.03883
3	0.73786
4	0.62136
5	0.64078
6	0.97087
7	0.58252
8	0.98058
9	0.74757
10	0.78641
11	0.44660
12	0.66990
13	0.46602
14	0.00000
15	0.07767
16	0.39806
17	0.84466
18	0.43689
19	0.50485
20	0.66019
21	0.30097
22	0.19417
23	0.37864
24	0.51456
25	0.82524
26	0.10680
27	0.89320
28	0.26214
29	0.53398
30	0.15534
31	0.71845
32	0.29126
33	0.02913
34	0.57282
35	0.81553
36	0.94175
37	0.08738
38	0.56311
39	0.65049
40	0.13592
41	0.38835
42	0.67961
43	0.63107
44	0.80583
45	0.77670
46	0.28155
47	0.86408
48	0.76699
49	0.11650
50	0.05825

R=RAND()	Bin
0.813455	0.1
0.21757	0.2
0.937991	0.3
0.840823	0.4
0.19536	0.5
0.681599	0.6
0.829291	0.7
0.377723	0.8
0.149187	0.9
0.965781	1
0.808752	
0.957601	
0.502469	
0.620944	
0.992405	
0.97218	
0.051905	
0.144368	
0.129308	
0.676603	
0.140868	
0.486705	
0.12415	
0.821802	
0.954853	
0.301267	
0.827929	
0.917179	
0.07369	
0.462159	
0.333902	
0.390604	
0.723163	
0.041401	
0.805603	
0.556012	

Bin	Frequency	Cumulative %
0.1	112	0.11
0.2	105	0.22
0.3	105	0.32
0.4	86	0.41
0.5	108	0.52
0.6	101	0.62
0.7	95	0.71
0.8	90	0.80
0.9	101	0.90
1	97	1.00
More	0	1.00

Sample
Size = 1000



Set 16.5a

$C = 2$ barbers

$$f_1(t) = .1 e^{-0.1t}, \quad t > 0$$

$$f_2(t) = \frac{1}{15}, \quad 15 \leq t \leq 30$$

$$t_1 = -12 \ln R$$

$$t_2 = 15 + 15R$$

A_1 at $T=0$:

$$T(A_1) = 0 + (-10 \ln .0589) = 28.3$$

$$T(D_2) = 0 + (15 + 15 \times .6733) = 25.1$$

Barber 1 busy

D_2 at $T=25.1$:

Barber 1 idle

A_2 at $T=28.3$:

$$T(A_2) = 28.3 - 10 \ln .4799 = 35.6$$

$$T(D_2) = 28.3 + (15 + 15 \times .9486) = 57.5$$

Barber 1 busy $A_3 \ D_2$

A_3 at $T=35.6$:

$$T(A_3) = 35.6 - 10 \ln .6139 = 40.5$$

$$T(D_3) = 35.6 + (15 + 15 \times .5933) = 59.5$$

Barber 2 busy $A_4 \ D_2 \ D_3$

A_4 at $T=40.5$:

$$T(A_4) = 40.5 - 10 \ln .9341 = 41.2$$

A_4 waits in queue

 $A_5 \ D_2 \ D_3$
 A_4 ← queue

A_5 at $T=41.2$:

$$T(A_5) = 41.2 - 10 \ln .1782 = 58.4$$

A_5 waits in queue

 $D_2 \ A_6 \ D_3$
 $A_4 \ A_5$ ← queue

D_2 at $T=57.5$:

Barber 1 idle

Take A_4 out of queue

$$T(D_4) = 57.5 + 15 + 15 \times .3473 = 77.7$$

Barber 1 busy

 $A_6 \ D_3 \ D_4$
 A_5 ← queue

A_6 at $T=58.4$:

$$T(A_7) = 58.4 - 10 \ln .5644 = 64.1$$

Put A_6 in queue $D_3 \ A_7 \ D_4$

D_3 at $T=59.5$:

 $A_5 \ A_6$ ← queue

Barber 2 idle

Take A_5 out of queue

$$T(D_5) = 59.5 + 15 + 15 \times .3529 = 79.8$$

Barber 2 busy

 $A_7 \ D_4 \ D_5$
 A_6 ← queue

A_7 at $T=64.1$:

$$T(A_8) = 64.1 - 10 \ln .3646 = 74.2$$

Put A_7 in queue

 $A_8 \ D_4 \ D_5$
 $A_6 \ A_7$ ← queue

A_8 at $T=74.2$:

$$T(A_9) = 74.2 + (-10 \ln .7676)$$

$$= 76.8$$

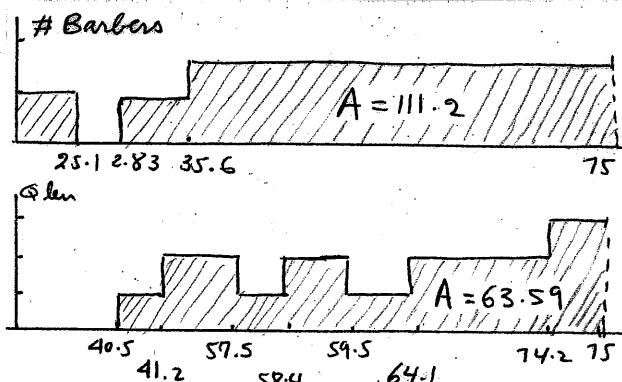
Place A_8 in queue.

 $A_9 \ D_4 \ D_5$
 $A_6 \ A_7 \ A_8$ ← queue

continued...

continued...

Set 16.5a



$$\text{Av. facility utilization} = \frac{111.2}{75}$$

$$= 1.48 \text{ barbers}$$

$$\text{Av. queue length} = \frac{63.59}{75} = .8 \text{ customer}$$

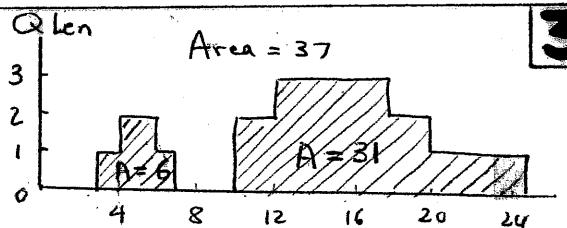
$$\text{Av. waiting time in queue} = \frac{63.59}{8}$$

$$= 7.95 \text{ min}$$

$$\text{Av. waiting time for those who must wait} = \frac{63.59}{5} = 12.72 \text{ min}$$

- (a) Observation.
- (b) Time.
- (c) Observation.
- (d) Observation.
- (e) Observation.
- (f) Time.

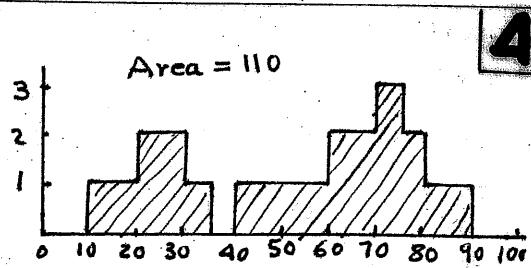
2



$$(a) \bar{Q} = \frac{37}{25} = 1.48 \text{ customers}$$

$$(b) \text{Number of waiting customers} = 5$$

$$\bar{W} = \frac{37}{5} = 7.4 \text{ hours}$$



(a) Average utilization

$$= \frac{110}{100} = 1.1 \text{ barber}$$

(b) Average idle time

$$= \frac{10 + (40 - 35) + (100 - 90)}{3}$$

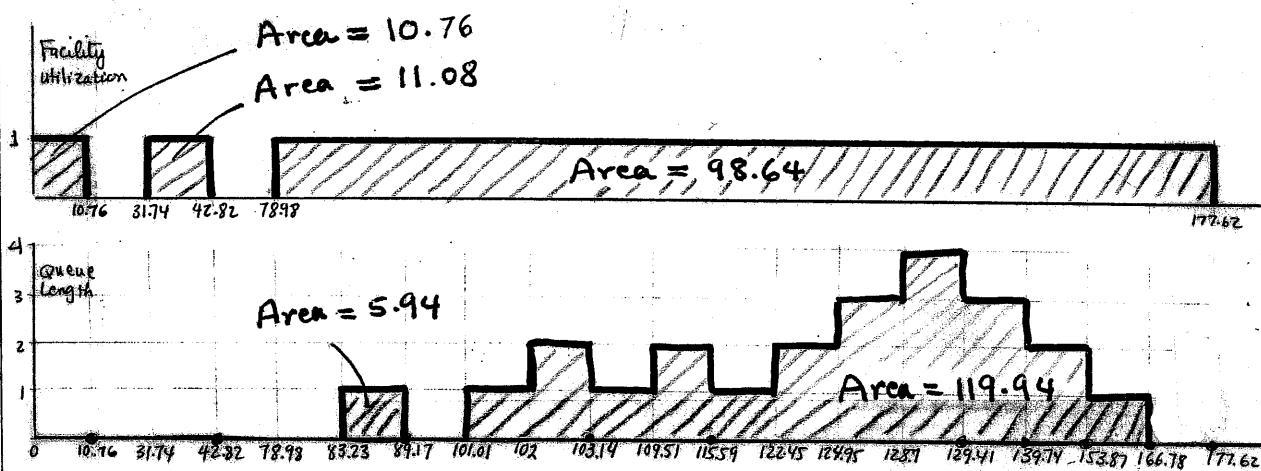
$$= \frac{25}{3}$$

$$= 8.33 \text{ minutes}$$

3

Set 16.5b

Simulation of a Single-Server Queueing Model							
		Simulation Calculations					
Nbr of arrivals =	10	InterArrTime	ServiceTime	ArrivTime	DepartTime	Wq	Ws
Enter x in column A to select interarrival pdf:							
Constant =		1	31.74	10.76	0.00	10.76	0.00
Exponential: $\lambda =$	0.0667	2	47.24	11.07	31.74	42.82	0.00
Uniform: $a =$	8	b =	9	4.25	10.19	78.98	89.17
Triangular: $a =$		b =	c =	17.78	13.96	83.23	103.14
Enter x in column A to select service time pdf:				0.99	12.45	101.01	115.59
Constant =		4	7.51	13.82	102.00	129.41	13.59
Exponential: $\mu =$		5	12.94	10.33	109.51	139.74	19.90
Uniform: $a =$	10	b =	15	2.51	14.13	122.45	153.87
Triangular: $a =$		b =	c =	3.74	12.90	124.95	166.78
		10	9.02	10.84	128.70	177.62	38.08
Output Summary							
Av. facility utilization =	0.68						
Percent idleness (%) =	32.17						
Maximum queue length =	4						
Av. queue length, Lq =	0.71						
Av. nbr in system, Ls =	1.39						
Av. queue time, Wq =	12.58						
Av. system time, Ws =	24.63						
Sum(ServiceTime) =	120.47						
Sum(Wq) =	125.88						
Sum(Ws) =	246.32						



From the graph:

$$\sum \text{Service times} = 10.76 + 11.08 + 98.64 = 120.48$$

$$\sum \text{queue waiting times} = 5.94 + 119.94 = 125.88$$

(The small difference between these answers and the simulation output is because of roundoff error.)

$$\text{Av. facility utilization} = \frac{120.48}{177.62} = .6783$$

$$\text{Av. queue length} = \frac{125.88}{177.62} = .7087$$

$$\text{Av. waiting time in queue} = \frac{125.88}{10} = 12.588$$

$$\text{Av. waiting time in system} = \frac{120.48 + 125.88}{10} = 24.636$$

Set 16.5b

Nbr of arrivals = 500 <<Maximum 500

Enter x in column A to select interarrival

Constant =				
Exponential: <i>x</i> =			4	
Uniform: <i>a</i> =			<i>b</i> =	
Triangular: <i>a</i> =			<i>b</i> =	

Enter x in column A to select service

Constant =				
Exponential: <i>x</i> =			6	
Uniform: <i>a</i> =			<i>b</i> =	
Triangular: <i>a</i> =			<i>b</i> =	

Output Summary

Av. facility utilization =	0.66
Percent idleness (%) =	33.84
Maximum queue length=	0
Av. queue length, Lq =	1.42
Av. nbr in system, Ls =	2.08
Av. queue time, Wq =	0.37
Av. system time, Ws =	0.54

Av. facility utilization =	0.61
Percent idleness (%) =	38.65
Maximum queue length=	0
Av. queue length, Lq =	0.91
Av. nbr in system, Ls =	1.52
Av. queue time, Wq =	0.24
Av. system time, Ws =	0.40

Av. facility utilization =	0.65
Percent idleness (%) =	35.11
Maximum queue length=	0
Av. queue length, Lq =	0.91
Av. nbr in system, Ls =	1.56
Av. queue time, Wq =	0.22
Av. system time, Ws =	0.38

Av. facility utilization =	0.68
Percent idleness (%) =	31.70
Maximum queue length=	0
Av. queue length, Lq =	1.35
Av. nbr in system, Ls =	2.03
Av. queue time, Wq =	0.32
Av. system time, Ws =	0.48

Av. facility utilization =	0.60
Percent idleness (%) =	39.83
Maximum queue length=	0
Av. queue length, Lq =	1.14
Av. nbr in system, Ls =	1.74
Av. queue time, Wq =	0.30
Av. system time, Ws =	0.46

Summary:

	utiliz	Lq	Ls	Wq	Ws
mean	.64	1.146	1.786	.29	.452
Std. Dev.	.0339	.2388	.2598	.0608	.0642

95% confidence limits:

$$t_{4,025} = 2.776$$

$$UCL = \bar{X} + \frac{2.776S}{\sqrt{n}} = \bar{X} + 1.245$$

$$LCL = \bar{X} - 1.245$$

	utiliz	Lq	Ls	Wq	Ws
LCL	.598	.850	1.464	.215	.372
UCL	.682	1.442	2.108	.365	.531

Poisson queue output:

Scenario 1—(M/M/1)-(GD/infinity/infinity)

Lambda = 4.00000 Mu = 6.00000
Lambda eff = 4.00000 Rho/c = 0.66667

Ls = 2.00000 Lq = 1.33333
Ws = 0.50000 Wq = 0.33333

3

S = 200 <<Maximum 500

Jump A to select interarrival pdf:

=	11.5
rat:	
a =	b =
r:	a = b = c =

Jump A to select service time pdf:

=	
rat:	
a =	b =
r:	a = b = c = 9.5 15

Av. facility utilization =	0.96
Percent idleness (%) =	4.20
Maximum queue length=	2
Av. queue length, Lq =	0.12
Av. nbr in system, Ls =	1.08
Av. queue time, Wq =	1.36
Av. system time, Ws =	12.38

continued...

continued...

Set 16.5b

	Av. facility utilization =	0.96
	Percent idleness (%) =	3.85
	Maximum queue length=	2
(2)	Av. queue length, Lq =	0.12
	Av. nbr in system, Ls =	1.08
	Av. queue time, Wq =	1.33
	Av. system time, Ws =	12.39
	Av. facility utilization =	0.97
	Percent idleness (%) =	2.98
	Maximum queue length=	2
(3)	Av. queue length, Lq =	0.19
	Av. nbr in system, Ls =	1.16
	Av. queue time, Wq =	2.14
	Av. system time, Ws =	13.33
	Av. facility utilization =	0.96
	Percent idleness (%) =	3.58
	Maximum queue length=	2
(4)	Av. queue length, Lq =	0.16
	Av. nbr in system, Ls =	1.13
	Av. queue time, Wq =	1.88
	Av. system time, Ws =	12.97
	Av. facility utilization =	0.97
	Percent idleness (%) =	3.39
	Maximum queue length=	2
(5)	Av. queue length, Lq =	0.17
	Av. nbr in system, Ls =	1.14
	Av. queue time, Wq =	2.00
	Av. system time, Ws =	13.12

utilization:

$$\text{mean} = \frac{.96 + .96 + .97 + .96 + .97}{5}$$

$$= .964$$

$$\text{st.dev.} = .0311$$

Set 16.6a

$$W_1 = \frac{14}{3} = 4.67 \text{ (time units)}$$

$$W_2 = \frac{10}{4} = 2.5$$

$$W_3 = \frac{11}{3} = 3.67$$

$$W_4 = \frac{6}{3} = 2$$

$$W_5 = \frac{15}{4} = 3.75$$

$$\bar{W} = \frac{4.67 + 2.5 + 3.67 + 2 + 3.75}{5}$$

$$= 3.32 \text{ time units}$$

Discard observations during the transient period (0, 100)

$$W_1 = \frac{12 + 30 + 10 + 14 + 16}{5} = 16.4 \text{ time units}$$

$$W_2 = \frac{15 + 17 + 20 + 22}{4} = 18.5$$

$$W_3 = \frac{10 + 20 + 30 + 15 + 25 + 31}{6} = 21.83$$

$$W_4 = \frac{15 + 17 + 20 + 14 + 13}{5} = 15.8$$

$$W_5 = \frac{25 + 30 + 15}{3} = 23.33$$

$$\bar{W} = 19.17 \quad S = 3.3$$

Confidence interval

$$\bar{W} \pm t_{.025, 4} \frac{S}{\sqrt{n}}$$

$$= 19.17 \pm 2.776 \frac{3.3}{\sqrt{5}}$$

or

$$15.07 \leq \mu \leq 23.27$$

Batch	a_i	b_i	y_i
1	6	7	.869
2	10	7	1.369
3	6	9	.584

$$\bar{a} = 7.33 \quad \bar{b} = 7.67 \quad \bar{y} = .941$$

$$Sy = .397$$

continued...

$$y_i = \frac{3x7.33 - (3-1)(3x7.33 - a_i)}{7.67 - 3x7.67 - b_i}$$

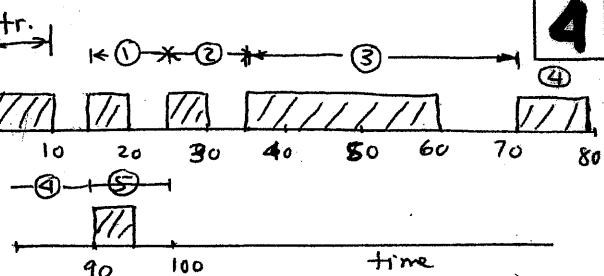
$$= 2.867 - \frac{43.98 - 2a_i}{23.01 - b_i}$$

95% confidence interval:

$$.941 - 2.776 \frac{.397}{\sqrt{3}} \leq \mu \leq .941 + 2.776 \frac{.397}{\sqrt{3}}$$

$$.305 \leq \mu \leq 1.577$$

3 continued



(a) Start points are 15, 25, 35, 70, 90

(b)

Batch	a_i	b_i	y_i
1	5	10	.54
2	5	10	.54
3	25	35	.94
4	10	20	.45
5	5	10	.54

$$\bar{a} = 10 \quad \bar{b} = 17 \quad \bar{y} = .602$$

$$Sy = .193$$

$$y_i = \frac{5x10 - 4(5x10 - a_i)}{17 - 5x17 - b_i}$$

$$= 2.94 - \frac{200 - 4a_i}{85 - b_i}$$

$$.602 - 2.776 \frac{.193}{\sqrt{5}} \leq \mu \leq .602 + 2.776 \frac{.193}{\sqrt{5}}$$

$$\text{or} \quad .36 \leq \mu \leq .84$$

$$(c) t = \frac{90}{5} = 18$$

i	1	2	3	4	5
A	8	13	14	10	5
a_i	.44	.72	.78	.56	.28

Mean = .556, Std. Dev. = .2042

Chapter 17

Markov Chains

Set 17.1a

1

States: Models M1, M2, and M3

	M1	M2	M3
M1	0.65	0.2	0.15
M2	0.6	0.15	0.25
M3	0.5	0.1	0.4

2

- s1: car on patrol
- s2: car responding to a call
- s3: car at call scene
- s4: apprehension made.
- s5 transport to police station

S1	S2	S3	S4	S5
S1	0.4	0.6	0	0
S2	0.1	0.3	0.6	0
S3	0.1	0	0.5	0.4
S4	0.4	0	0	0
S5	1	0	0	0

3

States: Q0, Q1, Q2, Q3, Q4, paid, bad debt
 $p\{\text{bad,bad}\}=P\{\text{paid,paid}\}=1$
 $P\{\text{Q0,paid}\}=2000/10000, P\{\text{Q0,Q1}\}=3000/10000,$
 $P\{\text{Q0,Q2}\}=3000/10000, P\{\text{Q0,Q3}\}=2000/10000,$
 $P\{\text{Q1,paid}\}=4000/25000, P\{\text{Q1,Q2}\}=12000/25000,$
 $P\{\text{Q1,Q3}\}=6000/25000, P\{\text{Q1,Q4}\}=3000/25000,$

Input Markov chain:

	Q0	Q1	Q2	Q3	Q4	PAID	BAD
Q0	.00	.30	.30	.20	.00	.20	.00
Q1	.00	.00	.48	.24	.12	.16	.00
Q2	.00	.00	.00	.30	.55	.15	.00
Q3	.00	.00	.00	.00	.16	.84	.00
Q4	.00	.00	.00	.00	.00	.50	.50
PAID	.00	.00	.00	.00	.00	1.00	.00
BAD	.00	.00	.00	.00	.00	.00	1.00

4

States: dialysis, cadaver transplant,
living donor transplant, >1year survivors, death

	Dialysis	CTransp	LTransp	>1yrS	Death
Dialysis	0.5	0.3	0.1	0	0.1
CTransp	0.3	0	0	0.5	0.2
LTransp	0.15	0	0	0.75	0.1
>1yrS	0.05	0	0	0.9	0.05
Death	0	0	0	0	1

Set 17.2a

1

Input Markov chain:		
M1	M2	M3
0.65	0.2	0.15
0.6	0.15	0.25
0.5	0.1	0.4

Output (2-step or 4 yrs.) transition matrix P^2						
M1	M2	M3				
0.6175	0.175	0.2075				
0.605	0.1675	0.2275				
0.585	0.155	0.26				

$P\{M1|M1\} = .6175$
 $P\{M2|M2\} = .1675$
 $P\{M3|M3\} = .26$

3

Initial probabilities:						
Q0	Q1	Q2	Q3	Q4	PAID	BAD
0.2	0.1	0.3	0.2	0	0	0.2

Input Markov chain:							
Q0	Q1	Q2	Q3	Q4	PAID	BAD	
.00	.30	.30	.20	.00	0.20	.00	
Q1	.00	.00	.48	.24	.12	0.16	.00
Q2	.00	.00	.00	.30	.55	0.15	.00
Q3	.00	.00	.00	.00	.16	0.84	.00
Q4	.00	.00	.00	.00	.00	0.50	.50
PAID	.00	.00	.00	.00	.00	1.00	.00
BAD	.00	.00	.00	.00	.00	0.00	1.

2

Initial probabilities:				
S1	S2	S3	S4	S5
0	0	1	0	0

Input Markov chain:				
S1	S2	S3	S4	S5
0.4	0.6	0	0	0
0.1	0.3	0.6	0	0
0.1	0	0.5	0.4	0
0.4	0	0	0	0.6
1	0	0	0	0

Output (2-step or 2 patrols) transition matrix P^2

S1	S2	S3	S4	S5
0.22	0.42	0.36	0	0
0.13	0.15	0.48	0.24	0
0.25	0.06	0.25	0.2	0.24
0.76	0.24	0	0	0
0.4	0.6	0	0	0

Absolute 2-step probabilities = $(0 \ 0 \ 1 \ 0 \ 0)P^2$

Absolute	
State	(2-step)
S1	0.25
S2	0.06
S3	0.25
S4	0.2
S5	0.24

$P\{\text{apprehension, S4, in 2 patrols}\} = .2$

Absolute		
State	(2-step)	\$500,000p
Q0	0	0
Q1	0	0
Q2	0.0288	14400
Q3	0.0468	23400
Q4	0.09124	45620
PAID	0.52866	264330
BAD	0.3045	152250

\$500,000

Set 17.2a

4

(a)

Initial probabilities:

Dialy	CTrans	LTrans	>1yrS	Death
1	0	0	0	0

Input Markov chain:

	Dialy	CTrans	LTrans	>1yrS	Death
Dialy	0.5	0.3	0.1	0	0.1
CTrans	0.3	0	0	0.5	0.2
LTrans	0.15	0	0	0.75	0.1
>1yrS	0.05	0	0	0.9	0.05
Death	0	0	0	0	1

Output (2-step) transition matrix

	Dialy	1stYrC	1stYrL	>1yrS	Death
Dialy	0.355	0.15	0.05	0.225	0.22
CTrans	0.175	0.09	0.03	0.45	0.25
LTrans	0.1125	0.045	0.015	0.675	0.15
>1yrS	0.07	0.015	0.005	0.81	0.1
Death	0	0	0	0	1

**Absolute
(2-step)**

State	Absolute (2-step)
Dialy	0.355
CTrans	0.15
LTrans	0.05
>1yrS	0.225
Death	0.22

$$P\{\text{transplant}\} = .15 + .05 = .2$$

(b)

Initial probabilities:

Dialy	CTrans	LTrans	>1yrS	Death
0	0	0	1	0

Input Markov chain:

Dialy	CTrans	LTrans	>1yrS	Death
Dialy	0.5	0.3	0.1	0
CTrans	0.3	0	0	0.5
LTrans	0.15	0	0	0.75
>1yrS	0.05	0	0	0.9
Death	0	0	0	1

Output (4-step) transition matrix

	Dialy	CTrans	LTrans	>1yrS	Death
Dialy	0.1737	0.0724	0.024	0.363	0.37
CTrans	0.1128	0.0425	0.014	0.465	0.37
LTrans	0.0967	0.0317	0.011	0.602	0.26
>1yrS	0.0847	0.0242	0.008	0.682	0.2
Death	0	0	0	0	1

**Absolute
(4-step)**

State	Absolute (4-step)
Dialy	0.08474
CTrans	0.02423
LTrans	0.00807
>1yrS	0.68197
Death	0.20099

$$P\{\text{surviving 4 more years}\} = .68197$$

continued...

Set 17.3a

1

- (a) Using excelMarkovChains.xls, all the states of the chain are periodic with period 3.

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \mathbf{P}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{P}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{P}^4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- (b) States 1, 2, and 3 are transient, State 4 is absorbing.
(c) State 1 is transient. States 2 and 3 form a closed set. State 4 is absorbing.
States 5 and 6 form a closed set.
(d) All the states communicate and the chain is ergodic.

Set 17.4a

(b)

$$\text{Population} = 150,000 \times P\{\text{1-step}\}$$

Absolute		Population
State	(1-step)	
inner	0.106667	16000
sub	0.54	81000
rural	0.353333	53000

$$\text{Population} = 150,000 \times P\{\text{2-step}\}$$

Absolute		Population
State	(2-step)	
inner	0.098667	14800
sub	0.417667	62650
rural	0.483667	72550

(c)

$$\text{Long-run population} = 150,000 \times \pi_i$$

Steady		Population
State	state	
inner	0.073892	11084
sub	0.275862	41379
rural	0.650247	97537

(c)

State	Steady state	No. of cars	
		12	98
Phx	0.0311	12	
Den	0.2442	98	
Chi	0.4139	166	>110
Atl	0.3108	124	>110

$$\text{total} = 400$$

Chicago and Atlanta will have space availability problem

(d)

State	Mean return time (wks)	
Phx	32.17	
Den	4.09	
Chi	2.42	
Atl	3.22	

8

(a)

Initial probabilities:

Equal initial probabilities

Phx	Den	Chi	Atl
0.25	0.25	0.25	0.25

Input Markov chain:

Phx	Den	Chi	Atl
0.7	0.06	0.18	0.06
0	0.7	0.18	0.12
0	0.15	0.7	0.15
0.03	0.03	0.24	0.7

(b)

State	(2-step)	No. of cars	
		Absolute	cars
Phx	0.1355	54	
Den	0.2319	93	
Chi	0.3645	146	
Atl	0.2681	107	

$$\text{total} = 400$$

9

(a)

Tally of i followed by j

	0	1	2	3	sum
0	2	2	1	3	8
1	2	1	2	2	7
2	2	3	1	1	7
3	2	0	4	1	7

Input Markov chain:

	0	1	2	3
0	0.25	0.25	0.125	0.375
1	0.28571	0.142857	0.285714	0.28571
2	0.28571	0.428571	0.142857	0.14286
3	0.28571	0	0.571429	0.14286

(b)

Output Results

State	Steady state	Mean return time	
		time	
0	0.275862	3.6249995	
1	0.215779	4.6343799	
2	0.270638	3.6949792	
3	0.237722	4.2065916	

$$\pi_0 = 0.275862$$

continued...

continued...

Set 17.4a

14

(a) State=(i,j,k)=(# in yr -2, # in yr-1, # in cur yr)
 $i, j, k = (0 \text{ or } 1)$

Example: (1-0-0) this yr links to (0-0-1) if a contract is secured next yr.

0	1-	0-	0-	1-	1-	0-	1-
0-	0-	1-	0-	1-	0-	1-	1-
0	0	0	1	0	1	1	1
0-0-0	.1	0	0	.9	0	0	0
1-0-0	.2	0	0	.8	0	0	0
0-1-0	0	.2	0	0	0	.8	0
0-0-1	0	0	.2	0	0	0	.8
1-1-0	0	.3	0	0	0	.7	0
1-0-1	0	0	.3	0	0	0	.7
0-1-1	0	0	0	0	.3	0	0
1-1-1	0	0	0	0	.5	0	0

(b)

State	Steady state
0-0-0	.014859
1-0-0	.066865
0-1-0	.066865
0-0-1	.066865
1-1-0	.178306
1-0-1	.178306
0-1-1	.178306
1-1-1	.249629

Expected # contracts in 3 yrs =

$$1(.066865 + .066865 + .066865) + \\ 2(.178306 + .178306 + .178306) + \\ 3(.249629) = 2.01932$$

Expected # contracts/yr=2.01932/3=.67311

(a) States: 0, 1, 2, 3, 4

15

Input Markov chain

	0	1	2	3	4
0	.5	.5	0	0	0
1	0	.6	.4	0	0
2	0	0	.7	.3	0
3	0	0	0	.8	.2
4	1	0	0	0	0

continued...

17-12

Set 17.5a

1

(a) Initial probabilities:

1	2	3	4	5
1	0	0	0	0

Input Markov chain:

1	2	3	4	5
0	.3333	.3333	.3333	0
.3333	0	.3333	0	.3333
.3333	.3333	0	0	.3333
.5	0	0	0	.5
0	.3333	.3333	.3333	0

State	Absolute	Steady
	(3-step)	state
1	.07407	.214286
2	.2963	.214286
3	.2963	.214286
4	.25926	.142857
5	.07407	.214286

(b) $\alpha_5 = .07407$

(c) $\pi_5 = .214286$

(d)

Matrix I:

1	2	3	4	5
1	0	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1
5	0	0	0	1

Matrix P:

1	2	3	4	5
0	.3333	.3333	.3333	0
.3333	0	.3333	0	.333
.3333	.3333	0	0	.333
.5	0	0	0	.5
0	.3333	.3333	.3333	0

Perform first passage time calculations below:

I-N

i=5	1	2	3	4
1	1	-.333	-.333	.3333
2	-.333	1	-.333	0
3	-.25	-.25	1	-.25
4	-.333	0	-.333	1

continued...

inv(I-N)

1	2	3	4
2	1	1	.6667
2	1	1.625	.875
3	1	.875	1.625
4	1	.5	.5

Mu

5
4.6666
3.8333
3.8333
3.3333

$$\mu_{15} = 4.6666$$

2

Matrix I:

1	2	3	4	5
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1
5	0	0	0	1

Matrix P:

1	2	3	4	5
0	.3333	.3333	.3333	0
.3333	0	.3333	0	.333
.25	.25	0	.25	.25
.3333	0	.3333	0	.333
0	.3333	.3333	.3333	0

Perform first passage time calculations below:

I-N

1	2	3	4
1	-.333	-.333	-.3333
2	1	-.333	0
3	-.25	1	-.25
4	0	-.333	1

inv(I-N)

1	2	3	4	5
2	1	1.3333	5.3333	5.3333
2	1	1.0667	4.2666	4.2666
3	1	.8	1.8667	4.4666
4	1	.6	1.0667	4.2666

$$\mu_{15} = 5.3333$$

(as opposed to 4.6666 in Part (d) of Problem 1)

Set 17.5a

3

(a)

Initial probabilities: (Jim-Joe)=(i-j)

3-2	2-3	1-4	4-1	0-5	5-0
1	0	0	0	0	0

Input Markov chain:

3-2	2-3	1-4	4-1	0-5	5-0
3-2	.0	.5	0	.5	0
2-3	.5	0	.5	0	0
1-4	0	.5	0	0	.5
4-1	.5	0	0	0	.5
0-5	.3	0	0	0	.7
5-0	.3	0	0	0	.7

(b)

Output (3-step) transition matrix

3-2	2-3	1-4	4-1	0-5	5-0
.075	.375	0	.25	.125	.175
.45	0	.25	0	.175	.125
.105	.325	0	.2	.37	0
.355	.075	.125	.075	0	.37
.297	.105	.075	.105	.343	.075
.297	.105	.075	.105	0	.418

$$P\{\text{Joe wins in 3 tosses}\} = P\{3-2 \rightarrow 0-5\} = .125$$

$$P\{\text{Jim wins in 3 tosses}\} = P\{3-2 \rightarrow 5-0\} = .175$$

(c)

Output Results

State	Absolute (3-step)	Steady state	Mean return time
3-2	.075	.257143	3.8888891
2-3	.375	.171429	5.8333335
1-4	0	.085714	11.666665
4-1	.25	.128571	7.77777801
0-5	.125	.142857	7.0000019
5-0	.175	.214286	4.6666665

$$P\{\text{game ends in Jim's favor}\} = \pi_{5-0} = .214$$

$$P\{\text{game ends in Joe's favor}\} = \pi_{5-0} = .143$$

(d)

Matrix I:

	1	2	3	4	5	6
1	1	0	0	0	0	0
2	0	1	0	0	0	0
3	0	0	1	0	0	0
4	0	0	0	1	0	0
5	0	0	0	0	1	0
6	0	0	0	0	0	1

Matrix P:

3-2	2-3	1-4	4-1	0-5	5-0
0	.5	0	.5	0	0
.5	0	.5	0	0	0
0	.5	0	0	.5	0
.5	0	0	0	0	.5
.3	0	0	0	.7	0
.3	0	0	0	0	.7

$i=0-5$

I-N

3-2	1	-1	0	-.5	0
2-3	-.5	1	-.5	0	0
1-4	0	-1	1	0	0
4-1	-.5	0	0	1	-1
5-0	-.3	0	0	0	.3

inv(I-N)

3-2	6	4	2	3	5
2-3	4	4	2	2	3.3
1-4	2	2	2	1	1.7
4-1	6	4	2	4	6.7
5-0	6	4	2	3	8.3

Mu

0-5

3-2	20	←expected number of tosses till Joe wins
2-3	15.3	
1-4	8.7	
4-1	22.7	
5-0	23.3	

$i=5-0$

I-N

3-2	1	-1	0	-.5	0
2-3	-.5	1	-.5	0	0
1-4	0	-1	1	0	-1
4-1	-.5	0	0	1	0
0-5	-.3	0	0	0	.3

continued...

continued...

Set 17.5a

inv(I-N)					Mu
	pink	red	orange	white	
pink	2.5	0	0	2.5	
red	2.36111	1.66667	.22222	4.25	
orange	1.66667	0	1.33333	3	

It takes 4.25 years from red to white

5

(a) **Input Markov chain:**

A	B	C	
A	.75	.1	.15
B	.2	.75	.05
C	.125	.125	.75

(b) **Steady state**

State	state
A	.394737
B	.307018
C	.298246

A: 39.5%, B: 30.7%, C: 29.8%

(c) **I-N**

i = 2 (B)	A	C
A	.25	-.15
C	-.125	.25

4

(a) **Input Markov chain:**

pink	red	orange	white	
pink	.6	0	0	.4
red	.5	.4	.1	0
orange	.5	0	.25	.25
white	.5	0	0	.5

(b) **Initial probabilities:**

pink	red	orange	white
.25	.25	.25	.25

Absolute Steady

State	(5-step)	state
pink	0.55555	0.555556
red	0.00256	0
orange	0.00179	0
white	0.4401	0.444445

After 5 years, 56% pink, 44% white.
Red and orange will vanish. Approximately same result in the long run.

(c) **I-N**

j=4(white)	pink	red	orange
pink	.4	0	0
red	-.5	.6	-.1
orange	-.5	0	.75

continued...

Set 17.6a

$$(I-N)^{-1} = \begin{pmatrix} 1.07 & 1.02 & .98 & 0.93 \\ 0.07 & 1.07 & 1.03 & 0.98 \\ 0 & 0 & 1.07 & 1.02 \\ 0 & 0 & 0.07 & 1.07 \end{pmatrix}$$

$$(I-N)^{-1}A = \begin{pmatrix} .16 & .84 \\ .12 & .88 \\ .08 & .92 \\ .04 & .96 \end{pmatrix}$$

Labor cost = $\{ \$20 \times [1.07(30/60) + .98(20/60)] + \$18[1.02(10/60) + .93(10/600)] \} / (.84)$
 $= \$27.48$

1

(a) Matrix P:

3

	1	2	3	4	5	6	0
1	0	4	0	0	0	0	.6
2	.6	0	4	0	0	0	0
3	0	6	0	.4	0	0	0
4	0	0	.6	0	.4	0	0
5	0	0	0	.6	0	.4	0
6	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1

inv(I-N)

	1	2	3	4	5
1	1.5865	0.9774	0.5714	0.3008	0.1203
2	1.4662	2.4436	1.4286	0.7519	0.3008
3	1.2857	2.1429	2.7143	1.4286	0.5714
4	1.0150	1.6917	2.1429	2.4436	0.9774
5	0.6090	1.0150	1.2857	1.4662	1.5865

MU P{i to j}

	Absorption	6	0
1	3.556391	1	0.048 0.952
2	6.390977	2	0.12 0.88
3	8.142857	3	0.229 0.771
4	8.270677	4	0.391 0.609
5	5.962406	5	0.635 0.365

(b) Average # of bets to termination = 8.14286

(c) P{win double}=.229, P{lose all}=.771

2

(a)

States: 1wk, 2wk, 3wk, Library

Matrix P:

	1	2	3	lib
1	0	0.3	0	0.7
2	0	0	0.1	0.9
3	0	0	0	1
lib	0	0	0	1

(b)

	1	2	3	lib
1	1	0.3	.03	1
2	0	1	.01	2
3	0	0	1	3

I keep the book 1.33 wks on the average.

4

(a) Matrix P:

	1	2	3	4	5(D)	M
1	0.5	0.5	0	0	0	0
2	0	0.5	0.5	0	0	0
3	0	0	0.2	0.5	0	0.3
4	0	0	0	0.5	0.5	0
5(D)	0	0	0	0	1	0
M	0	0	0	0	0	1

(b)

	1	2	3	4	Mu absorption
1	2	2	1.25	1.25	1 6.5
2	0	2	1.25	1.25	2 4.5
3	0	0	1.25	1.25	3 2.5
4	0	0	0	2	4 2

Years as a student = 6.5 years

continued...

Set 17.6a

(c)

$$P\{i \text{ to } j\} = \text{inv}(I-N)A$$

	D	M
1	0.625	0.375
2	0.625	0.375
3	0.625	0.375
4	1	0

$$P\{\text{Master}\} = .375$$

(d)

Expected pay=

$$\$15,000(5 \times .625 + 3 \times .375) = \$63,750$$

5

(a) States: 55, 56, ..., 62, quit

Matrix P

55	56	57	58	59	60	61	62	Q
0	.9	0	0	0	0	0	0	.1
0	0	.89	0	0	0	0	0	.11
0	0	0	.88	0	0	0	0	.12
0	0	0	0	.87	0	0	0	.13
0	0	0	0	0	.86	0	0	.14
0	0	0	0	0	0	.85	0	.15
0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1	0
Q	0	0	0	0	0	0	0	1

(b)

inv(I-N)

55	56	57	58	59	60	61
1	.9	.8	.7	.61	.53	.448
0	1	.89	.78	.68	.59	.498
0	0	1	.88	.77	.66	.56
0	0	0	1	.87	.75	.636
0	0	0	0	1	.86	.731
0	0	0	0	0	1	.85
0	0	0	0	0	0	1

Mu P{i to j}

62/Q	62	Q
4.99	.448	.552
4.44	.498	.502
3.86	.56	.44
3.25	.636	.364
2.59	.731	.269
1.85	.85	.15
1	1	0

$$P\{\text{retire at 62}\} = .448$$

$$(c) P\{\text{quit at 57}\} = .44$$

$$(d) P\{\text{off payroll}\} = 3.25 \text{ years}$$

6

(a)

Matrix P

	Q0	Q1	Q2	Q3	Q4	PAID	BAD
Q0	0	.3	.3	.2	0	.2	0
Q1	0	0	.48	.24	.12	.16	0
Q2	0	0	0	.3	.55	.15	0
Q3	0	0	0	0	.16	.84	0
Q4	0	0	0	0	0	.5	.5
PAID	0	0	0	0	0	1	0
BAD	0	0	0	0	0	0	1

inv(I-N) Mu

	Q0	Q1	Q2	Q3	Q4
Q0	1	.3	.44	.41	.35
Q1	0	1	.48	.38	.45
Q2	0	0	1	.3	.6
Q3	0	0	0	1	.16
Q4	0	0	0	0	1

$$\text{Expected # qrtrs till absorption} = 2.49$$

(b)

P{i to j}

PAID	BAD
.83	.17
.78	.22
.7	.3
.92	.08
.5	.5

$$P\{Q0 \rightarrow \text{bad}\} = .17$$

$$P\{Q0 \rightarrow \text{Paid}\} = .83$$

(c)

Mu

	Q0	Q1	Q2	Q3	Q4
Q0	2.49				
Q1	2.31				
Q2	1.9				
Q3	1.16				
Q4	1				

$$\text{Nbr. of qrtrs till settled} = 1.9$$

continued...

Set 17.6a

7

(a) State (i-j)=(Sets won by Andre-Sets won by John)

Matrix P:

0-0	0-1	0-2	1-0	1-1	1-2	2-0	2-1	2-2	2-3	3-0	0-3	1-3	3-1	3-2
0	.4	0	.6	0	0	0	0	0	0	0	0	0	0	0
0-1	0	0	.4	0	.6	0	0	0	0	0	0	0	0	0
0-2	0	0	0	0	0	.6	0	0	0	0	0	.4	0	0
1-0	0	0	0	0	.4	0	.6	0	0	0	0	0	0	0
1-1	0	0	0	0	0	.4	0	.6	0	0	0	0	0	0
1-2	0	0	0	0	0	0	0	.6	0	0	0	.4	0	0
2-0	0	0	0	0	0	0	0	.4	0	0	.6	0	0	0
2-1	0	0	0	0	0	0	0	.4	0	0	0	0	.6	0
2-2	0	0	0	0	0	0	0	0	.4	0	0	0	0	.6
2-3	0	0	0	0	0	0	0	0	1	0	0	0	0	0
3-0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0-3	0	0	0	0	0	0	0	0	0	0	1	0	0	0
1-3	0	0	0	0	0	0	0	0	0	0	0	1	0	0
3-1	0	0	0	0	0	0	0	0	0	0	0	0	1	0
3-2	0	0	0	0	0	0	0	0	0	0	0	0	0	1

(b)

inv(I-N)

0-0	0-1	0-2	1-0	1-1	1-2	2-0	2-1	2-2
0	1	.4	.16	.6	.48	.3	.4	.4
0-1	0	1	.4	0	.6	.5	0	.4
0-2	0	0	1	0	0	.6	0	0
1-0	0	0	0	1	.4	.2	.6	.5
1-1	0	0	0	0	1	.4	0	.6
1-2	0	0	0	0	0	1	0	0
2-0	0	0	0	0	0	0	1	.4
2-1	0	0	0	0	0	0	0	.4
2-2	0	0	0	0	0	0	0	1

P{i to j}

MU	2-3	3-0	0-3	1-3	3-1	3-2	P{A}	P{J}
0-0	4.07	0-0	.1	.22	.06	.12	.26	.21
0-1	3.27	0-1	.2	0	.16	.19	.22	.26
0-2	1.96	0-2	.1	0	.4	.24	0	.22
1-0	2.93	1-0	.1	.36	0	.06	.29	.17
1-1	2.48	1-1	.2	0	0	.16	.36	.29
1-2	1.6	1-2	.2	0	0	.4	0	.36
2-0	1.56	2-0	.1	.6	0	0	.24	.1
2-1	1.4	2-1	.2	0	0	0	.6	.24
2-2	1	2-2	.4	0	0	0	0	.6

Average # of sets till end of match= 4.07

Probability Andre will win = sum of ($P_{3-0} + P_{3-1} + P_{3-2}$) given 0-0 start= .69

(c) $P\{\text{Andre wins} \mid \text{current score } 1-2\}=.36$.

(d) The average number of sets till termination is 1.6. In **ONE** set the termination score can be 1-3 (J's favor), or in **TWO** sets it can be 2-3 (J's favor) or 3-2 (A's favor). The average number of sets to termination is thus more than 1 and less than 2 (= 1.6).

Set 17.6a

(a)

Matrix P:

	1	2	3	4	F
1	.2	.8	0	0	0
2	0	.22	.78	0	0
3	0	0	.25	.75	0
4	0	0	0	.3	.7
F	0	0	0	0	1

(b)

inv(I-N)

	1	2	3	4	F
1	1.25	1.282	1.333	1.429	1 5.29
2	0	1.282	1.333	1.429	2 4.04
3	0	0	1.333	1.429	3 2.76
4	0	0	0	1.429	4 1.43

(c) To be able to take Cal II, the student must finish in 16 weeks (4 transitions) or less. Average number of transitions needed = 5.29. Hence, an average student will not be able to finish Cal I on time.

(d) No!

8

(a)

Matrix P:

	0	1	2	3	D
0	.5	.5	0	0	0
1	.4	0	.6	0	0
2	.3	0	0	.7	0
3	.2	0	0	0	.8
D	0	0	0	0	1

States: 0, 1, 2, 3, Delete

10

inv(I-N)

	0	1	2	3	D
0	5.952	2.976	1.786	1.25	0 12
1	3.952	2.976	1.786	1.25	1 9.96
2	2.619	1.31	1.786	1.25	2 6.96
3	1.19	.595	.357	1.25	3 3.39

(b)

A new customer stays 12 years on the list

(c) 6.96 years

(a)

9

States: 0, 1, 2, 3, 4, 5, promotion

Matrix P:

	0	1	2	3	4	5	P
0	.2	.7	.1	0	0	0	0
1	0	.2	.7	.1	0	0	0
2	0	0	.2	.7	.1	0	0
3	0	0	0	.2	.7	.1	0
4	0	0	0	0	.2	.7	.1
5	0	0	0	0	0	0	1
P	0	0	0	0	0	0	1

(b)

inv(I-N)

	0	1	2	3	4	5	P
0	1.25	1.094	1.113	1.11	1.1	.89	0 6.57
1	0	1.25	1.094	1.11	1.1	.89	1 5.46
2	0	0	1.25	1.09	1.1	.89	2 4.35
3	0	0	0	1.25	1.1	.89	3 3.23
4	0	0	0	0	1.3	.88	4 2.13
5	0	0	0	0	0	1	5 1

It takes 6.57 on the averages to be promoted.

(b)

inv(I-N)

	108	109	110	111	112
108	2.5	2	1.5	1	.5
109	2	4	3	2	1
110	1.5	3	4.5	3	1.5
111	1	2	3	4	2
112	.5	1	1.5	2	2.5

continued...

Set 17.6a

MU absorb		P{i to j}	
		107	113
108	7.5	.83	.17
109	12	.67	.33
110	13.5	.5	.5
111	12	.33	.67
112	7.5	.17	.83

The last two columns (low=107, high=113) provide the answer as a function of the current voltage. For example, if current voltage is 109, $P\{\text{low}\}=.67$, $P\{\text{high}\}=.33$

(c)

Setting current voltage at 110 guarantees an average time to failure of $13.5(15)= 202.5$ minutes.

12

	Dialysis	1stYrC	1stYrL	>1yrS	Death
Dialysis	.5	3	.1	0	.1
1stYrC	3	0	0	.5	.2
1stYrL	.15	0	0	.75	.1
>1yrS	.05	0	0	.9	.05
Death	0	0	0	0	1

	inv(I-N)	Mu			
	Dialysis	1stYrC	>1yrS	1stYrL	death
Dialysis	3.5398	1.0619	7.96	.354	12.92
1stYrC	1.9469	1.5841	9.38	.1947	13.11
1stYrL	1.8584	.5575	11.71	1858	15.28
>1yrS	1.7699	.531	14	.177	16.46

- (a) # years on dialysis=3.54 years.
- (b) Longevity = 12.92 years.
- (c) Life expectancy = 16.46 years
- (d) 14 years.
- (e) >1yrSurvivor has the highest longevity (= 16.46 years) and the least number of years on dialysis (= 1.7699 years).

Chapter 18

Classical Optimization Theory

Set 18.1a

$$(a) \frac{\partial f}{\partial x} = 3x^2 + 1 = 0$$

$$x = \pm \sqrt{-1/3}$$

The necessary condition yields imaginary roots. The problem has no stationary points.

$$(b) \frac{\partial f}{\partial x} = 4x^3 + 2x = 0$$

$$x = 0, x = \pm \sqrt{-1/2}$$

For $x = 0$,

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 2 = 2 > 0 \Rightarrow \text{min}$$

$$(c) \frac{\partial f}{\partial x} = 16x^3 - 2x = 0$$

$$x = 0, .353, -.353$$

$$\frac{\partial^2 f}{\partial x^2} = 48x^2 - 2$$

$$x = 0: \frac{\partial^2 f}{\partial x^2} = -2 \Rightarrow \text{max}$$

$$x = .353: \frac{\partial^2 f}{\partial x^2} = 6 \Rightarrow \text{min}$$

$$x = -.353: \frac{\partial^2 f}{\partial x^2} = 6 \Rightarrow \text{min}$$

$$(d) f(x) = (3x-2)^2(2x-3)^2 \\ = (6x^2 - 13x + 6)^2$$

$$\frac{\partial f}{\partial x} = 2(6x^2 - 13x + 6)(12x - 13) = 0$$

$$x = 2/3, 3/2, 13/12$$

$$\frac{\partial^2 f}{\partial x^2} = 2(216x^2 - 468x + 241)$$

$$x = 2/3: \frac{\partial^2 f}{\partial x^2} = 50 \Rightarrow \text{min}$$

$$x = 3/2: \frac{\partial^2 f}{\partial x^2} = 50 \Rightarrow \text{min}$$

$$x = 13/12: \frac{\partial^2 f}{\partial x^2} = -25 \Rightarrow \text{max}$$

$$(e) \frac{\partial f}{\partial x} = 30x^4 - 12x^2 = 0 \Rightarrow x = (0, \pm \sqrt{6}/3)$$

$$\frac{\partial^2 f}{\partial x^2} = 120x^3 - 24x$$

$$x = 0: \frac{\partial^2 f}{\partial x^2} = 0, \frac{\partial^2 f}{\partial x^3} = 360x^2 - 24 |_{x=0} = -24 \Rightarrow \text{inflexion}$$

$$x = \sqrt{6}/3: \frac{\partial^2 f}{\partial x^2} = 14.88 \Rightarrow \text{min}$$

$$x = -\sqrt{6}/3: \frac{\partial^2 f}{\partial x^2} = -14.88 \Rightarrow \text{max}$$

$$(a) \frac{\partial f}{\partial x_1} = 3x_1^2 - 3x_2 = 0$$

$$\frac{\partial f}{\partial x_2} = 3x_2^2 - 3x_1 = 0$$

$$(x_1, x_2) = (0, 0), (1, 1)$$

$$H = \begin{pmatrix} 6x_1 & -3 \\ -3 & 6x_2 \end{pmatrix}$$

$$(x_1, x_2) = (0, 0):$$

principal minor determinants
 $= (0, -9) \Rightarrow \text{indefinite}$
 $\Rightarrow (0, 0) \text{ is not an extreme point}$

$$(x_1, x_2) = (1, 1):$$

Principal minor determinants
 $= (6, 27) \Rightarrow \text{positive definite}$
 $\Rightarrow (1, 1) \text{ is a minimum point.}$

$$(b) \frac{\partial f}{\partial x_1} = 4x_1 + 6 + 2x_2 x_3 = 0 \quad (1)$$

$$\frac{\partial f}{\partial x_2} = 2x_2 + 6 + 2x_1 x_3 = 0 \quad (2)$$

$$\frac{\partial f}{\partial x_3} = 2x_3 + 6 + 2x_1 x_2 = 0 \quad (3)$$

$$(3) - (2) \text{ yields } (x_3 - x_2) - x_1(x_3 - x_2) = 0 \\ \text{or } (x_3 - x_2)(1 - x_1) = 0$$

$$\text{Thus, } x_3 = x_2 \text{ or } x_1 = 1$$

$$\text{For } x_1 = 1:$$

$$\text{from (1), } 10 + 2x_2 x_3 = 0 \quad (4)$$

$$\text{from (2), } 2x_2 + 2x_3 + 6 = 0 \quad (5)$$

Hence, $x_2 = -(3+x_3)$. Substituting in (4), then

$$10 - 2x_3(3+x_3) = 0$$

$$\text{or } x_3^2 + 3x_3 - 5 = 0$$

$$\text{Thus, } x_3 = 1.2 \text{ or } x_3 = -4.2$$

$$\text{or, } x_2 = -4.2 \text{ or } x_2 = 1.2$$

$$(x_1, x_2, x_3) = \begin{cases} (1, -4.2, 1.2) \\ (1, 1.2, -4.2) \end{cases}$$

$$\text{For } x_2 = x_3:$$

$$\text{from (2), } 2x_2 + 6 + 2x_1 x_2 = 0$$

$$\text{or, } (1+x_1) = \frac{-3}{x_2}$$

continued...

Set 18.1a

From (1), $2x_1 + 3 + x_2^2 = 0$

2 continued

Substituting $(1+x_1) = -3/x_2$, then

$$-\frac{3}{x_2} + \frac{1}{2} + \frac{x_2^2}{2} = 0$$

or

$$x_2^3 + x_2 - 6 = 0$$

This gives the solution $x_2 \approx 1.65$.

(The remaining two roots are

imaginary.) Thus, $x_1 = \frac{-3}{1.65} - 1 = -2.82$

and $(x_1, x_2, x_3) = (-2.82, 1.65, 1.65)$

$$H = \begin{pmatrix} 4 & 2x_3 & 2x_2 \\ 2x_3 & 2 & 2x_1 \\ 2x_2 & 2x_1 & 2 \end{pmatrix}$$

$$\underline{x} = (1, -4.2, 1.2):$$

Principal minor determinants (PMD)
 $= (4, 2.24, -223) \Rightarrow$ indefinite

$$\underline{x} = (1, 1.2, -4.2):$$

PMD = $(4, -62.56, -155.5) \Rightarrow$ indefinite

$$\underline{x} = (-2.82, 1.65, 1.65):$$

PMD = $(4, 2.25, -67.4) \Rightarrow$ indefinite

$$\frac{\partial f}{\partial x_1} = 2x_2x_3 - 4x_3 + 2x_1 - 2 = 0$$

3

$$\frac{\partial f}{\partial x_2} = 2x_1x_3 - 2x_3 + 2x_2 - 4 = 0$$

$$\frac{\partial f}{\partial x_3} = 2x_1x_2 - 4x_1 - 2x_2 + 2x_3 + 4 = 0$$

Solutions: $(0, 3, 1), (0, 1, -1),$
 $(2, 1, 1), (1, 2, 0), (2, 3, -1)$

$$H = \begin{pmatrix} 2 & 2x_3 & 2x_2 - 4 \\ 2x_3 & 2 & 2x_1 - 2 \\ 2x_2 - 4 & 2x_1 - 2 & 2 \end{pmatrix}$$

$$\text{PMD}_{(0,3,1)} = (2, 0, -32) \quad \text{indefinite}$$

$$\text{PMD}_{(0,1,-1)} = (2, 0, -32) \quad \text{indefinite}$$

$$\text{PMD}_{(2,1,1)} = (2, 0, -32) \quad \text{indefinite}$$

$$\text{PMD}_{(1,2,0)} = (2, 4, 8) \quad \text{positive def} \Rightarrow \min$$

$$\text{PMD}_{(2,3,-1)} = (2, 0, -32) \quad \text{indefinite}$$

The problem is equivalent to

4

$$\text{Minimize } Z = (x_2 - x_1^2)^2 + (x_2 - x_1 - 2)^2$$

$$\frac{\partial Z}{\partial x_1} = 2(x_2 - x_1^2)(-2x_1) + 2(x_2 - x_1 - 2)(-1) = 0$$

$$\frac{\partial Z}{\partial x_2} = 2(x_2 - x_1^2) + 2(x_2 - x_1 - 2) = 0$$

Thus, solve

$$2x_1^3 - 2x_1x_2 + x_1 - x_2 + 2 = 0 \quad (1)$$

$$x_1^2 + x_1 - 2x_2 + 2 = 0 \quad (2)$$

From (2),

$$x_2 = \frac{x_1^2 + x_1 + 2}{2}$$

From (1), we get

$$2x_1^3 - 3x_1^2 - 3x_1 + 2 = 0$$

Solutions: $(x_1, x_2) = (2, 4)$ and $(-1, 1)$

Note: The given method complicates a simple problem. Nevertheless the idea is interesting

From Taylor's theorem

$$f(y_0 + h) = f(y_0) + f'(y_0)h + \frac{f''(y_0)h^2}{2!} + \dots + \frac{f^{(n)}(y_0 + \theta h)h^n}{n!}$$

Let $f'(y_0) = f''(y_0) = \dots = f^{(n-1)}(y_0) = 0$
according to the assumption. Then

$$f(y_0 + h) - f(y_0) = \frac{f^{(n)}(y_0 + \theta h)h^n}{n!}$$

Because $f^{(n)}(y_0 + \theta h)$ has the same sign as $f^{(n)}(y_0)$, then

(1) If n is even: $h^n > 0$ and $f(y_0 + h) - f(y_0)$ has the same sign as $f^{(n)}(y_0) \Rightarrow y_0$ is maximum if $f^{(n)}(y_0) < 0$, and y is min $f^{(n)}(y_0) > 0$.

(2) If n is odd: $h^n < 0$ or > 0 , depending on whether $h < 0$ or > 0 , respectively. Thus, at y_0 , $f(y_0 + h) - f(y_0)$ will change sign from negative (positive) to positive (negative) depending on whether $f^{(n)}(y_0) > 0$ (< 0). Thus, y_0 is an inflection point.

Set 18.1b

$$f(x) = 4x^4 - x^2 + 5$$

$$\frac{\partial f}{\partial x} = 16x^3 - 2x = 0$$

Cell C3 formula: $(16*A3^3 - 2*A3)/(48*A3^2 - 2)$

Solution:

(1) Initial $x_0 = 1 \Rightarrow x^* = 0$

(2) Initial $x_0 = 10 \Rightarrow x^* = -35355$

(3) Initial $x_0 = -10 \Rightarrow x^* = -35355$

C3		
$= (16*A3^3 - 2*A3)/(48*A3^2 - 2)$		
=chn2newtonraphson		
Newton-Raphson (One-Variable) Method		
Input data: Type [A3]/[A3] in C3 where A3 represents x in f(x)	A	#VALUE!
Initial x0=	0.1	
Solution:		
x* =	0.00000	
Calculations:		
x(k)	x(k+1)	f(x(k))/f'(x(k))
0.100000	-0.021053	0.121052632
-0.021053	0.000161	-0.02120563
0.000151	0.000000	0.000150098
0.000000	0.000000	5.49757E-11

C3		
$= (16*A3^3 - 2*A3)/(48*A3^2 - 2)$		
=chn2newtonraphson		
Newton-Raphson (One-Variable) Method		
Input data: Type [A3]/[A3] in C3 where A3 represents x in f(x)	A	#VALUE!
Initial x0=	10	
Solution: (Excel error may result if selected x0 causes division by zero)		
x* =	0.35355	
Calculations:		
x(k)	x(k+1)	f(x(k))/f'(x(k))
10.000000	6.669446	3.330554398
6.669446	4.450466	2.218979699
4.450466	2.973232	1.477233933
2.973232	1.991542	0.981690459
1.991542	1.341790	0.64975121
1.341790	0.915719	0.42607095
0.915719	0.642400	0.273319294
0.642400	0.476363	0.166036563
0.476363	0.389003	0.087360433
0.389003	0.357876	0.031127068
0.357876	0.353630	0.004245528
0.353630	0.353553	7.705E-05

C3		
$= (16*A3^3 - 2*A3)/(48*A3^2 - 2)$		
=chn2newtonraphson		
Newton-Raphson (One-Variable) Method		
Input data: Type [A3]/[A3] in C3 where A3 represents x in f(x)	A	#VALUE!
Initial x0=	-10	
Solution: (Excel error may result if selected x0 causes division by zero)		
x* =	-0.35355	
Calculations:		
x(k)	x(k+1)	f(x(k))/f'(x(k))
-10.000000	-6.669446	3.330554398
-6.669446	-4.450466	2.218979699
-4.450466	-2.973232	1.477233933
-2.973232	-1.991542	0.981690459
-1.991542	-1.341790	0.64975121
-1.341790	-0.915719	0.42607095
-0.915719	-0.642400	0.273319294
-0.642400	-0.476363	0.166036563
-0.476363	-0.389003	0.087360433
-0.389003	-0.357876	0.031127068
-0.357876	-0.353630	0.004245528
-0.353630	-0.353553	7.705E-05

$$f(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + x_3^2 +$$

$$6(x_1 + x_2 + x_3) + 2x_1 x_2 x_3$$

2

$$\frac{\partial f}{\partial x_1} = 4x_1 + 2x_2 x_3 + 6 = 0 \quad (=F_1)$$

$$\frac{\partial f}{\partial x_2} = 2x_2 + 2x_1 x_3 + 6 = 0 \quad (=F_2)$$

$$\frac{\partial f}{\partial x_3} = 2x_3 + 2x_1 x_2 + 6 = 0 \quad (=F_3)$$

$$\nabla F_1 = (4, 2x_3, 2x_2)$$

$$\nabla F_2 = (2x_3, 2, 2x_1)$$

$$\nabla F_3 = (2x_2, 2x_1, 2)$$

Thus,

$$B = \begin{pmatrix} 4 & 2x_3 & 2x_2 \\ 2x_3 & 2 & 2x_1 \\ 2x_2 & 2x_1 & 2 \end{pmatrix}$$

(note that B is the Hessian matrix)

$$A = \begin{pmatrix} 4x_1 + 2x_2 x_3 + 6 \\ 2x_2 + 2x_1 x_3 + 6 \\ 2x_3 + 2x_1 x_2 + 6 \end{pmatrix}$$

Let $X = (0, 0, 0)$ be the starting point.

$$X' = (0, 0, 0) - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$$

$$= (-1.5, -3, -3)$$

$$X^2 = (-1.5, -3, -3) \begin{pmatrix} 4 & -6 & -6 \\ -6 & 2 & -3 \\ -6 & -3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 18 \\ 9 \\ 9 \end{pmatrix}$$

$$= (-2.68, -4.89, -4.89)$$

We continue in the same manner until $X^k \approx X^{k+1}$. If the present sequence does not converge, choose another starting point.

Set 18.2a

(a) $\partial_c f = -46 \partial x_2$

$$= -.046 \text{ for } \partial x_2 = .001$$

$$\begin{pmatrix} \partial x_1 \\ \partial x_3 \end{pmatrix} = - J^{-1} C \partial x_2$$

$$= \begin{pmatrix} 2.83 \\ -2.50 \end{pmatrix} \times .001$$

$$= \begin{pmatrix} -.00283 \\ -.00250 \end{pmatrix}$$

$$x^0 + \partial x = (1 - .00283, 2 + .001, 3 + .0025) \\ = (.99717, 2.001, 3.0025)$$

$$f(x^0 + \partial x) = 57.9538$$

$$\partial_c f = 58 - 57.9538 = -.04618$$

The approximation is better.

(b) $\partial x_1 = 2.83 \partial x_2$

$$\partial x_3 = -2.5 \partial x_2$$

(c) $\nabla_y f = (6x_2, 10x_1, x_3)$

$$\nabla_z f = (2x_1 + 5x_3^2)$$

$$J = \begin{pmatrix} 2x_2 & x_1 \\ 2x_1 & 2x_3 \end{pmatrix}$$

$$C = \begin{pmatrix} x_3 \\ 2x_1 + 2x_2 \end{pmatrix}$$

At $x^0 = (1, 2, 3)$,

$$J^{-1} C = \begin{pmatrix} 6 & 1 \\ 2 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 6/34 & -1/34 \\ -2/34 & 6/34 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} .353 \\ .882 \end{pmatrix}$$

$$\partial_c f = [47 - (12, 30) \begin{pmatrix} .353 \\ .882 \end{pmatrix}] \partial x, \\ = 16.316 \partial x,$$

For $\partial_c f = -.46$, we have

$$16.316 \partial x_1 = -.46$$

$$\text{or } \partial x_1 = -.0282$$

I continued

continued...

Set 18.2b

(a) No, the necessary and sufficient conditions are the same in both methods.

(b) The Jacobian method computes the constrained gradient of the objective function directly. The new method computes the constrained objective function from which we can compute the constrained gradient.

$$Y = (x_2, x_3) \quad Z = (x_1)$$

$$\nabla f(Y) = (6x_2, 10x_1 x_3)$$

$$\nabla f(Z) = (2x_1 + 5x_3^2)$$

$$J = \begin{pmatrix} 2x_2 & x_1 \\ 2x_1 & 2x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 1 \\ 2 & 6 \end{pmatrix} \text{ at } X = (1, 2, 3)$$

$$C = \begin{pmatrix} x_3 \\ 2x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \text{ at } X = (1, 2, 3)$$

$$J^{-1} C = \begin{pmatrix} 6/14 & -1/14 \\ -2/14 & 6/14 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 6/17 \\ 15/17 \end{pmatrix}$$

$$\nabla f(Z) = 47 \quad \nabla f(Y) = (12, 30)$$

$$\partial_C f = \left[(47 - (12, 30)) \begin{pmatrix} 6/17 \\ 15/17 \end{pmatrix} \right] \partial x_1$$

$$= 16.316 \partial x_1$$

From Example 20.3-1, given
 $\partial x_2 = .01$, then $\partial x_1 = -.0283$
and

$$\partial_C f = 16.316 \times (-.0283) \approx -.46$$

$$Y = X_n$$

$$Z = (x_1, x_2, \dots, x_{n-1})$$

$$\nabla f(Y) = 2x_n$$

$$\nabla f(Z) = (2x_1, 2x_2, \dots, 2x_{n-1})$$

$$J = \nabla g(Y) = \prod_{i=1}^{n-1} x_i = \frac{C}{x_n},$$

$$C = \nabla g(Z) = \left(\frac{C}{x_1}, \frac{C}{x_2}, \dots, \frac{C}{x_{n-1}} \right).$$

$$x_i \neq 0, i = 1, 2, \dots, n$$

$$\nabla_C f = (2x_1, \dots, 2x_{n-1}) - 2x_n \left(\frac{x_n}{C} \right) \left(\frac{C}{x_1}, \dots, \frac{C}{x_{n-1}} \right)$$

$$= 0$$

$$i = 1, 2, \dots, n-1$$

Thus, necessary conditions are

$$2x_i - \frac{2x_n^2}{x_i} = 0, i = 1, 2, \dots, n-1$$

The solution of these equations yields

$$x_1 = x_2 = \dots = x_n$$

Hence, from the constraint

$$x_i^* = \sqrt[n]{C}, i = 1, 2, \dots, n$$

Sufficient conditions:

$$\frac{\partial_C f}{\partial x_i} = 2x_i - \frac{2x_n^2}{x_i}, i = 1, 2, \dots, n-1$$

$$\frac{\partial^2 f}{\partial x_i^2} = 2 + \frac{2x_n^2}{x_i^2} = 4 \text{ at } x_i^* \text{ for all } i$$

Hence,

$$H = \begin{pmatrix} 4 & & & \\ & 4 & & \\ & & 0 & \\ & & & 4 \end{pmatrix}$$

which is positive definite $\Rightarrow \min$

$$\frac{\partial f}{\partial g} = \nabla f(Y) J^T \text{ at } X^0$$

$$= 2\sqrt[n]{C} \frac{\sqrt[n]{C}}{2} = 2\sqrt[n]{C^{2-n}}$$

$$\text{For } \partial g = \delta, \quad \partial f = 2\sqrt[n]{C^{2-n}} = 2\delta \left(C^{\frac{2-n}{n}} \right)$$

Set 18.2b

$$Z = x_1, Y = x_2$$

$$\nabla f(Z) = 10x_1 + 2x_2$$

$$\nabla f(Y) = 2x_1 + 2x_2$$

$$J = \nabla g(Y) = x_1$$

$$C = \nabla g(Z) = x_2$$

$$\begin{aligned} \nabla_c f &= (2x_2 + 10x_1) - (2x_1 + 2x_2) \frac{1}{x_1} x_2 \\ &= \frac{-2}{x_1} (x_2^2 - 5x_1^2) \end{aligned}$$

$$\nabla_c f = 0 \Rightarrow x_2 = \pm \sqrt{5} x_1$$

$$g(x) = 0 \Rightarrow x_1^2 = 10/15$$

The stationary points are

$$(2.115, 4.729), (-2.115, -4.729)$$

Sufficiency condition:

$$\frac{\partial}{\partial Z} \nabla_c f = 10 + 2 \left(\frac{x_2^2}{x_1^2} \right)$$

Thus, both stationary points are min.

$$\begin{aligned} (a) \quad \partial f &= \nabla f(Y) J^{-1} \partial g \\ &= (2x_1 + 2x_2) \left(\frac{1}{x_1} \right) \partial g \end{aligned}$$

$$\partial g = -0.01, \text{ thus, } \partial f = -0.0647$$

$$\begin{aligned} (b) \quad \partial f &= \nabla f(Y) J^{-1} \partial g + \nabla_c f \partial Z \\ &= 14 \left(\frac{1}{2} \right) (-0.01) + \\ &\quad [30 - 14] \left(\frac{1}{2} \right) (5) (-0.01) \\ &= -0.12 \end{aligned}$$

$$Y = (x_2, x_3), Z = x_1$$

$$\text{at } X^0 = (1, 1, 1)$$

$$\begin{aligned} \nabla f(Y^0) &= (4x_2 + 5x_1, 20x_3) \\ &= (9, 20) \end{aligned}$$

4

$$\nabla g(Y^0) = \begin{pmatrix} 2x_2 + 3x_3 & 3x_2 \\ 5x_1 & 2x_3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 \\ 5 & 2 \end{pmatrix}$$

$$\nabla g(Z^0) = \begin{pmatrix} 1 \\ 2x_1 + 5x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

$$\partial f = \nabla_c f(Y^0) J^{-1} \partial g + \nabla_c f(Y^0, Z^0) \partial Z$$

$$\nabla_c f(Y^0) J^{-1} = (9, 20) \begin{pmatrix} -2/5 & 3/5 \\ 1 & -1 \end{pmatrix}$$

$$= (82/5, -73/5)$$

$$\nabla_c f(Y^0, Z^0) = \left[7 - (9, 20) \begin{pmatrix} -2/5 & 3/5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \end{pmatrix} \right]$$

$$\partial_c f = (82/5, -73/5) \begin{pmatrix} \partial g_1 \\ \partial g_2 \end{pmatrix} + 92.8 \partial x_1$$

$$\text{For } (\partial g_1, \partial g_2) = (-0.01, 0.02), \partial x_1 = -0.01$$

$$\partial_c f = -\frac{82}{5} - \frac{1.46}{5} + .928 = .472$$

(a)

$$Y = (x_2, x_3), Z = (x_3, x_4)$$

6

$$J = \nabla g(Y) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \text{ which}$$

is singular. We must

(b) select a new set Y and Z

Let

$$Y = (x_2, x_4), Z = (x_1, x_3)$$

$$\nabla f(Z) = (2x_1, 2x_3)$$

$$\nabla f(Y) = (2x_2, 2x_4)$$

$$\nabla g(Y) = \begin{pmatrix} 2 & 5 \\ 2 & 6 \end{pmatrix}, J^{-1} = \begin{pmatrix} 3 & -5/2 \\ -1 & 1 \end{pmatrix}$$

$$\nabla g(Z) = \begin{pmatrix} 1 & 3 \\ 1 & 5 \end{pmatrix}$$

$$\begin{aligned} \nabla_c f &= (2x_1, 2x_3) - (2x_2, 2x_4) \begin{pmatrix} 3 & -5/2 \\ -1 & 1 \end{pmatrix} x \\ &\quad \begin{pmatrix} 1 & 3 \\ 1 & 5 \end{pmatrix} \end{aligned}$$

$$= (2x_1 - x_2, 2x_3 + 7x_2 - 4x_4)$$

$$\nabla_c f = 0 \text{ yields}$$

continued...

continued...

Set 18.2b

$$2x_1 - x_2 = 0 \quad (1)$$

$$2x_3 + 7x_2 - 4x_4 = 0 \quad (2)$$

$$x_1 + 2x_2 + 3x_3 + 5x_4 - 10 = 0 \quad (3)$$

$$x_1 + 2x_2 + 5x_3 + 6x_4 - 15 = 0 \quad (4)$$

From (1), $2x_1 = x_2$

Substitution in (3) and (4) yields

$$5x_1 + 3x_3 + 5x_4 = 10$$

$$5x_1 + 5x_3 + 6x_4 = 15$$

$$14x_1 + 2x_3 - 4x_4 = 0$$

The solution is

$$(x_1, x_2, x_3, x_4) = \left(\frac{-5}{74}, \frac{-10}{74}, \frac{155}{74}, \frac{60}{74} \right)$$

$$H = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \text{positive definite}$$

Thus, the stationary point is a minimum point.

$$\text{For } Y^0 = (-10/74, 60/74)$$

$$\nabla f(Y^0) = (-10/37, 60/37)$$

$$\frac{\partial f}{\partial q} = \nabla f(Y^0) J^{-1} = \left(\frac{-10}{37}, \frac{60}{37} \right) \begin{pmatrix} 3 & -5/2 \\ -1 & 1 \end{pmatrix} = \left(-\frac{90}{37}, \frac{85}{37} \right)$$

$$\partial f = \nabla f(Y^0) J^{-1} \frac{\partial q}{\partial g}$$

$$= \left(-\frac{90}{37}, \frac{85}{37} \right) \begin{pmatrix} -0.01 \\ -0.02 \end{pmatrix} = -0.07$$

For the LP problem,

7

indep. vars = nonbasic variables

dep. vars = basic variables

$$\nabla f(Y) = (c_1, c_2, \dots, c_m) = C_B$$

$$\nabla f(Z) = (c_{m+1}, c_{m+2}, \dots, c_n)$$

$$\nabla g(Y) = J = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mm} \end{pmatrix} = B$$

$$\nabla g(Z) = \begin{pmatrix} a_{1,m+1} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m,m+1} & \dots & a_{mn} \end{pmatrix}$$

continued...

6 continued

7 continued

$$= (P_{m+1}, P_{m+2}, \dots, P_n)$$

$$\nabla f = \{(C_{m+1}, \dots, C_n) -$$

$$(C_1, \dots, C_m) \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mm} \end{pmatrix}^{-1} X$$

$$(P_{m+1}, \dots, P_n)\}$$

$$= \{C_j - C_B B^{-1} P_j\}, j = m+1, \dots, n$$

$$= \{C_j - Z_j\}, \text{ provided } B^{-1} \text{ exists}$$

The Jacobian method cannot be applied to LP directly without first accounting for the nonnegativity constraints. This is accomplished by using the substitution $x_j = w_j^2$.

Set 18.2c

$$f(\underline{w}) = 5w_1^2 + 3w_2^2$$

s.t.

$$\begin{aligned} g_1(\underline{w}) &= w_1^2 + 2w_2^2 + w_3^2 - 6 = 0 \\ g_2(\underline{w}) &= 3w_1^2 + w_2^2 + w_4^2 - 9 = 0 \end{aligned}$$

$$Y = (w_1, w_2), Z = (w_3, w_4)$$

$$\nabla f(Y) = (10w_1, 6w_2)$$

$$\nabla f(Z) = (0, 0)$$

$$\nabla g(Y) = \begin{pmatrix} 2w_1 & 4w_2 \\ 6w_1 & 2w_1 \end{pmatrix}$$

$$\nabla g(Z) = \begin{pmatrix} 2w_3 & 0 \\ 0 & 2w_4 \end{pmatrix}$$

$$\begin{aligned} J^{-1} &= \frac{1}{-20w_1w_2} \begin{pmatrix} 2w_2 & -4w_2 \\ -6w_1 & 2w_1 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} -1/w_1 & 2/w_1 \\ 3/w_2 & -1/w_1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} J^{-1}C &= \frac{1}{10} \begin{pmatrix} -1/w_1 & 2/w_1 \\ 3/w_2 & -1/w_2 \end{pmatrix} \begin{pmatrix} 2w_2 & 0 \\ 0 & 2w_4 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} -2w_3 & 4w_4 \\ w_1 & w_1 \\ 6w_3 & -2w_4 \\ w_2 & w_2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \nabla_C f &= (0, 0) - (10w_1, 6w_2) \begin{pmatrix} -w_3 & 2w_4 \\ 3w_3 & -w_4 \\ 5w_1 & 5w_2 \end{pmatrix} \\ &= \left(-\frac{8}{5}w_3, -\frac{14}{5}w_4 \right) = 0 \end{aligned}$$

$$w_3 = w_4 = 0$$

From the constraints,

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} w_1^2 \\ w_2^2 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix} \Rightarrow w_1^2 = \frac{12}{5}, w_2^2 = \frac{9}{5}$$

$$f(\underline{w}_0) = \left(5 \times \frac{12}{5} + 3 \times \frac{9}{5} \right) = 17.4$$

To check if the point is a max,

consider

$$H_{w_0} = \begin{pmatrix} -8/5 & 0 \\ 0 & -14/5 \end{pmatrix} \Rightarrow \text{negative def.}$$

Thus,

$$\underline{x}_0 = \left(\frac{12}{5}, \frac{9}{5}, 0, 0 \right)$$

is a maximum point.

Sensitivity coefficients:

$$\nabla f(\underline{y}_0) J^{-1} = (10w_1, 6w_2) \begin{pmatrix} -1/10w_1 & 2/10w_1 \\ 3/10w_2 & -1/10w_2 \end{pmatrix} = (-8, 1.4)$$

Dual values:

$$GB^{-1} = (5, 3) \begin{pmatrix} -1/5 & 2/5 \\ 3/5 & -1/5 \end{pmatrix} = (-8, 1.4)$$

$$\text{Dual obj value} = 6 \times 8 + 9 \times 1.4 = 17.4$$

Lagrangian Method:

$$\begin{aligned} L(\underline{w}, \lambda) &= 5w_1^2 + 3w_2^2 - \lambda_1(w_1^2 + 2w_2^2 + w_3^2 - 6) \\ &\quad - \lambda_2(3w_1^2 + w_2^2 + w_4^2 - 9) \end{aligned}$$

$$\frac{\partial L}{\partial w_1} = 10w_1 - 2\lambda_1 w_1 - 6\lambda_2 w_2 = 0$$

$$\frac{\partial L}{\partial w_2} = 6w_2 - 4\lambda_1 w_2 - 2\lambda_2 w_4 = 0$$

$$\frac{\partial L}{\partial w_3} = -2\lambda_1 w_3 = 0$$

$$\frac{\partial L}{\partial w_4} = -2\lambda_2 w_4 = 0$$

$$g_1(\underline{w}) = 0$$

$$g_2(\underline{w}) = 0$$

$$\text{The solution is } (\underline{w}^0, \lambda^0) = \left(\frac{12}{5}, \frac{9}{5}, 0, 0, -8, 1.4 \right)$$

Sufficiency condition:

$$B = H = \begin{array}{|cc|ccccc|} \hline & & 2w_1 & 2w_2 & 2w_3 & 0 \\ \hline 0 & 0 & 6w_1 & 2w_2 & 0 & 2w_4 \\ \hline 2w_1 & 6w_1 & 10-2\lambda_1-6\lambda_2 & 0 & 0 & 0 \\ 2w_2 & 2w_2 & 0 & 6-4\lambda_1-2\lambda_2 & 0 & 0 \\ 2w_3 & 0 & 0 & 0 & -2\lambda_1 & 0 \\ 0 & 2w_4 & 0 & 0 & 0 & -2\lambda_2 \\ \hline & & & & & & \underline{w}, \lambda^0 \\ \end{array}$$

$$= \begin{bmatrix} 0 & 0 & 3 & 2.64 & 0 & 0 \\ 0 & 0 & 9 & 2.64 & 0 & 0 \\ 3 & 9 & 0 & 0 & 0 & 0 \\ 2.64 & 2.64 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.6 & 0 \\ 0 & 0 & 0 & 0 & -1.6 & -2.8 \end{bmatrix}$$

continued...

continued...

Set 18.2c

$$2m+1 = 5$$

I continued

The value of the 5th principal minor determinant = -427 and that of the 6th principal minor determinant is 1130, following the signs of $(-1)^{m+1}$ and $(-1)^{m+2}$ (-, +, respectively). Hence λ^0, λ^0 is a maximum point

$$\frac{\partial}{\partial x_1} = 2x_1 - \lambda_1 - \lambda_2 = 0 \quad ① \quad \boxed{2}$$

$$\frac{\partial}{\partial x_2} = 4x_2 - 2\lambda_1 x_2 - 5\lambda_2 = 0 \quad ②$$

$$\frac{\partial}{\partial x_3} = 20x_3 - \lambda_1 - \lambda_2 = 0 \quad ③$$

$$\frac{\partial}{\partial \lambda_1} = -(x_1 + x_2^2 + x_3 - 5) = 0 \quad ④$$

$$\frac{\partial}{\partial \lambda_2} = -(x_1 + 5x_2 + x_3 - 7) = 0 \quad ⑤$$

From ① and ③, $x_1 = 10x_3$.

Substitution in ④ and ⑤ yields

$$x_2^2 + 11x_3 = 5 \quad ⑥$$

$$5x_2 + 11x_3 = 7 \quad ⑦$$

⑥ and ⑦ give

$$x_2^2 - 5x_2 + 2 = 0$$

Solution:

$$x_1^0 = (-14.4, 4.56, -1.44)$$

$$x_2^0 = (4.4, -4.4, .44)$$

For x_1^0 , from ② and ③

$$\lambda_1^1 = 38.5, \lambda_2^1 = -67.3$$

For x_2^0 , from ② and ③

$$\lambda_1^2 = 10.2, \lambda_2^2 = -1.4$$

Stationary points:

$$(x_1^0, \lambda_1^0) = (-14.4, 4.56, -1.44, 38.5, -67.3)$$

$$(x_2^0, \lambda_2^0) = (4.4, -4.4, .44, 10.2, -1.4)$$

Both points are minima

$$L(X, \lambda) = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

$$-\lambda_1(x_1 + 2x_2 + 3x_3 + 5x_4 - 10)$$

$$-\lambda_2(x_1 + 2x_2 + 5x_3 + 6x_4 - 15)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - 2\lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 2x_3 - 3\lambda_1 - 5\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_4} = 2x_4 - 5\lambda_1 - 6\lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + 2x_2 + 3x_3 + 5x_4 - 10) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -(x_1 + 2x_2 + 5x_3 + 6x_4 - 15) = 0$$

Solution:

$$(X^0, \lambda^0) = \left(\frac{-5}{74}, \frac{-10}{74}, \frac{60}{74}, \frac{-90}{37}, \frac{85}{37} \right)$$

The values of λ^0 are the same as the sensitivity coefficients obtained in Problem 20.26-6.

3

By definition

$$\lambda = \frac{\partial f}{\partial g}$$

If the right-hand side of $g(x) \geq 0$ is changed to $\partial g \geq 0$, the constraints become more restrictive. This means that the value of $f(x)$ can never improve. Thus,

$$\frac{\partial f}{\partial g} \leq 0 \text{ or } \lambda \leq 0$$

Replace $g(x) = 0$ with

$$g(x) \leq 0$$

$$-g(x) \leq 0$$

Thus,

$$L(x, \lambda_1, \lambda_2) = f(x) - \lambda_1(g(x) + S_1^2) - \lambda_2(-g(x) + S_2^2)$$

The K-T conditions are then given by,

$$\lambda_1 \geq 0, \lambda_2 \geq 0 \quad (1)$$

$$\frac{\partial L}{\partial x} = \nabla f(x) - (\lambda_1 - \lambda_2) \nabla g(x) = 0 \quad (2)$$

$$\frac{\partial L}{\partial S_1} = -2\lambda_1 S_1 = 0 \quad (3)$$

$$\frac{\partial L}{\partial S_2} = -2\lambda_2 S_2 = 0 \quad (4)$$

$$\frac{\partial L}{\partial \lambda_1} = g(x) + S_1^2 = 0 \quad (5)$$

$$\frac{\partial L}{\partial \lambda_2} = -g(x) + S_2^2 = 0 \quad (6)$$

$$\text{From (5) and (6), } S_1^2 + S_2^2 = 0$$

Because $S_1^2 + S_2^2 \geq 0$, then

$$S_1^2 = S_2^2 = 0$$

as should be expected. This means that conditions (3) and (4) are trivial and conditions (5) and (6) reduce to $g(x) = 0$.

$$\text{Let } \lambda = \lambda_1 - \lambda_2$$

Because $\lambda_1, \lambda_2 \geq 0$, λ is unrestricted in sign.

The K-T conditions become

$$(i) \lambda \text{ unrestricted in sign}$$

$$(ii) \nabla f(x) - \lambda \nabla g(x) = 0$$

$$(iii) g(x) = 0$$

2

$$(A) \max f(x) = x_1^3 - x_2^2 + x_1 x_3^2$$

3

$$\begin{aligned} & \text{s.t.} \\ & x_1 + x_2^2 + x_3 = 5 \\ & -5x_1^2 + x_2^2 + x_3 \leq -2 \\ & -x_1 \leq 0 \\ & -x_2 \leq 0 \\ & -x_3 \leq 0 \end{aligned}$$

$$\begin{aligned} L(x, \lambda) = & f(x) - \lambda_1(x_1 + x_2^2 + x_3 - 5) \\ & - \lambda_2(-5x_1^2 + x_2^2 + x_3 + 2) \\ & - \lambda_3(-x_1 + S_1^2) \\ & - \lambda_4(-x_2 + S_2^2) \\ & - \lambda_5(-x_3 + S_3^2) \end{aligned}$$

The K-T conditions are

$$(1) \lambda_1 \text{ unrestricted}$$

$$(2) \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0$$

$$(3) (3x_1^2 + x_3^2, -2x_2, 2x_1, x_3)$$

$$-(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \begin{pmatrix} 1 & 2x_2 & 1 \\ -10x_1 & 2x_2 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= (0, 0, 0, 0, 0)$$

$$(4) (\lambda_2, \lambda_3, \lambda_4, \lambda_5) \begin{pmatrix} -5x_1^2 + x_2^2 + x_3 + 2 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} = 0$$

continued...

continued...

Set 18.2d

$$\textcircled{5} \quad g(x) = 0$$

$$(b) \max -f(x) = -x_1^4 - x_2^2 - 5x_1 x_2 x_3 \\ \text{s.t.}$$

$$x_1 - x_2^2 + x_3^3 - 10 \leq 0$$

$$-x_1^3 - x_2^2 - 4x_3^2 + 20 \leq 0$$

$$\textcircled{1} \quad \lambda_1, \lambda_2 \geq 0$$

$$\textcircled{2} \quad (-4x_1^3 - 5x_2 x_3, -2x_2 - 5x_1 x_3, -5x_1 x_2)$$

$$-(\lambda_1, \lambda_2) \begin{pmatrix} 1 & -2x_2 & 3x_3^2 \\ -3x_1^2 & -2x_2 & -8x_3 \end{pmatrix} = (0, 0)$$

$$\textcircled{3} \quad (\lambda_1, \lambda_2) \begin{pmatrix} x_1 - x_2^2 + x_3^3 - 10 \\ -x_1^3 - x_2^2 - 4x_3^2 + 20 \end{pmatrix} = 0$$

$$\textcircled{4} \quad x_1 - x_2^2 + x_3^3 - 10 \leq 0 \\ -x_1^3 - x_2^2 - 4x_3^2 + 20 \leq 0$$

3 continued

$$L(X, \lambda_1, \lambda_2, \lambda_3)$$

5 continued

$$= f(x) - \lambda_1(g_1(x) + s_1^2)$$

$$- \lambda_2(g_2(x))$$

$$- \lambda_3(g_3(x) + s_3^2)$$

K-T conditions:

$$\textcircled{1} \quad \lambda_1 \geq 0, \lambda_2 \text{ unrestricted}, \lambda_3 \geq 0$$

$$\textcircled{2} \quad \frac{\partial L}{\partial X} = \nabla f(x) + \lambda_1 \nabla g_1(x)$$

$$- \lambda_2 \nabla g_2(x)$$

$$- \lambda_3 \nabla g_3(x)$$

$$\textcircled{3} \quad \frac{\partial L}{\partial s_1} = 2\lambda_1 s_1 = 0$$

$$\textcircled{4} \quad \frac{\partial L}{\partial s_3} = -2\lambda_3 s_3 = 0$$

$$\textcircled{5} \quad \frac{\partial L}{\partial \lambda_1} = g_1(x) - s_1^2 = 0$$

$$\textcircled{6} \quad \frac{\partial L}{\partial \lambda_2} = -g_2(x) = 0$$

$$\textcircled{7} \quad \frac{\partial L}{\partial \lambda_3} = -(g_3(x) + s_3^2) = 0$$

Sufficient conditions:

$f(x)$ concave

$g_1(x)$ concave

$g_2(x)$ linear or $\lambda_2 g_2(x)$ convex

$g_3(x)$ convex

Consider

4

$$L(X, \lambda) = f(x) - \lambda g(x)$$

Because all the constraints are equations, the elements of λ are unrestricted. However, because $g(x)$ is a linear function, $g(x)$ can be either convex or concave. Thus, for $\lambda_i > 0$, we take $g_i(x)$ as a convex function so that

$-\lambda_i g_i(x)$ is concave. Similarly, if $\lambda_i < 0$, $g_i(x)$ is assumed concave, in which case $-\lambda_i g_i(x)$ is also concave. Given $f(x)$ is concave, hence $L(X, \lambda)$ is concave. If $g(x)$ is nonlinear, it cannot be both convex and concave, a central argument in the case of linear $g(x)$.

Maximize $f(x)$

5

$$\text{s.t. } g_1(x) \geq 0$$

$$g_2(x) = 0$$

$$g_3(x) \leq 0$$

continued...

Chapter 19

Nonlinear Programming Algorithms

Set 19.1a

Dichotomous/Golden Section Search					
Input data: Type (C3) in E3 where C3 represents x in f(x)					
A =	0.01	C3		#VALUE!	
Minimum x =	0	Maximum x =	3		
Solution:	Enter x to select: Dichotomous		GoldenSection	x	
x* =	2.00443	f(x*) =	5.99912		
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
0.000000	3.000000	1.145898	1.854102	3.437694	5.862306
1.145898	3.000000	1.854102	2.291796	5.862306	5.902735
1.854102	3.000000	2.291796	2.562306	5.902735	5.812565
2.291796	3.000000	2.124612	2.291796	5.958453	5.902735
2.124612	3.000000	2.291796	2.124612	5.958453	5.958463
2.124612	2.157428	2.021286	2.124612	5.872283	5.958463
2.157428	2.124612	2.021286	2.060753	5.992905	5.99749
2.157428	2.080753	1.996894	2.021286	5.990683	5.992905
1.996894	2.080753	2.021286	2.036361	5.992905	5.987800
1.996894	2.036361	2.011969	2.021286	5.996010	5.982005
1.996894	2.021286	2.006211	2.011969	5.997930	5.996010
1.996894	2.011969	2.002653	2.006211	5.999116	5.997930
1.996894	2.006211	2.000453	2.002653	5.999649	5.999116

(a)

2

Dichotomous:

Dichotomous/Golden Section Search					
Input data: Type (C3) in E3 where C3 represents x in f(x)					
A =	0.05	C3		#VALUE!	
Minimum x =	2	Maximum x =	4		
Solution:	Enter x to select: Dichotomous	x2	GoldenSection	x	
x* =	3.00000	f(x*) =	64000.00000		
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
2.000000	4.000000	2.975000	3.025000	64000.000000	64000.000000
2.975000	3.025000	2.975000	3.025000	64000.000000	64000.000000

Golden section:

Dichotomous/Golden Section Search					
Input data: Type (C3) in E3 where C3 represents x in f(x)					
A =	0.05	C3		#VALUE!	
Minimum x =	2	Maximum x =	4		
Solution:	Enter x to select: Dichotomous		GoldenSection	x	
x* =	3.00000	f(x*) =	64000.00000		
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
2.000000	4.000000	2.763932	3.236068	76.013156	76.013156
2.763932	3.236068	2.944272	3.056729	5777.595627	5777.595627
2.944272	3.056729	2.956044	3.013156	439204.000002	439204.000002
2.956044	3.013156	2.996894	3.003106	64000.00000	64000.00000

continued...

19-2

(b)

Dichotomous:

Dichotomous/Golden Section Search					
Input data: Type (C3) in E3 where C3 represents x in f(x)					
A =	0.05	C3		#VALUE!	
Minimum x =	0	Maximum x =	3.14159		
Solution:	Enter x to select: Dichotomous	x2	GoldenSection	x	
x* =	0.86027	f(x*) =	0.56045		
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
0.000000	3.141590	1.545795	1.595795	0.038643	-0.038649
0.000000	1.595795	0.772698	0.822698	0.553310	0.553312
0.772698	1.595795	1.159346	1.209346	0.463668	0.427652
0.772698	1.209346	0.966122	1.016122	0.549235	0.531657
0.772698	1.016122	0.869510	0.919510	0.561009	0.557416
0.772698	0.919510	0.81204	0.871204	0.559519	0.550973
0.81204	0.919510	0.845357	0.895357	0.560664	0.552813
0.81204	0.895357	0.833200	0.883200	0.560341	0.560447
0.833200	0.895357	0.839318	0.889318	0.560640	0.560219
0.833200	0.889318	0.866299	0.886299	0.560499	0.560392
0.833200	0.886299	0.834790	0.884790	0.560422	0.560447
0.834790	0.886299	0.835544	0.885544	0.560461	0.560453
0.834790	0.885544	0.835167	0.885167	0.560442	0.560453
0.835167	0.885544	0.835356	0.885356	0.560452	0.560443
0.835167	0.885356	0.836261	0.886261	0.560447	0.560449
0.835261	0.886261	0.835309	0.885309	0.560449	0.560445
0.835261	0.885309	0.835265	0.885265	0.560448	0.560447
0.835261	0.885265	0.835273	0.885273	0.560447	0.560447
0.835261	0.885273	0.835267	0.885267	0.560447	0.560447
0.835267	0.885273	0.835270	0.885270	0.560447	0.560447
0.835267	0.885270	0.835269	0.885269	0.560447	0.560447

Golden Section:

Dichotomous/Golden Section Search					
Input data: Type (C3) in E3 where C3 represents x in f(x)					
A =	0.05	C3		#VALUE!	
Minimum x =	0	Maximum x =	3.14159		
Solution:	Enter x to select: Dichotomous	x2	GoldenSection	x	
x* =	0.84194	f(x*) =	0.56098		
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
0.000000	3.141590	1.199861	1.941609	0.434844	-0.705858
0.000000	1.941609	0.741629	1.199861	0.546854	0.434844
0.000000	1.199861	0.458352	0.741629	0.411042	0.546854
0.458352	1.199861	0.741629	0.916704	0.546854	0.467759
0.741629	1.199861	0.916704	1.024906	0.557759	0.432110
0.741629	1.024906	0.849831	0.916704	0.560982	0.397759
0.741629	0.916704	0.806501	0.849831	0.566337	0.560982
0.806501	0.916704	0.849831	0.875374	0.560982	0.560081
0.806501	0.875374	0.834044	0.849831	0.560363	0.560082
0.834044	0.875374	0.849831	0.855568	0.560982	0.561036

continued...

(c)

Dichotomous:

A	B	C	D	E	F
Dichotomous/Golden Section Search					
Input data: Type (C3) in E3, where C3 represents x in f(x)					
Δ = 0.05	C3			#VALUE!	
Minimum x = 1.5	Maximum x = 2.5				
Solution: Enter x to select: Dichotomous: x GoldenSection: x					
x* = 2.47500	f(x*) = 2.50000				
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
1.500000	2.500000	1.975000	2.025000	-0.154967	0.158069
1.975000	2.500000	2.212500	2.262500	1.369735	1.561395
2.212500	2.500000	2.331250	2.381250	2.011242	2.217450
2.331250	2.500000	2.390625	2.440625	2.250974	2.399295
2.390625	2.500000	2.420313	2.470313	2.344860	2.459576
2.420313	2.500000	2.435156	2.485156	2.384798	2.462454
2.435156	2.500000	2.442578	2.492578	2.402393	2.491900
2.442578	2.500000	2.446289	2.496289	2.411543	2.496119
2.446289	2.500000	2.448145	2.498145	2.415728	2.498102
2.448145	2.500000	2.449072	2.499072	2.417791	2.499062
2.449072	2.500000	2.449536	2.499536	2.418915	2.499533
2.449536	2.500000	2.449768	2.499768	2.419325	2.499767
2.449768	2.500000	2.449884	2.499884	2.419580	2.499884
2.449884	2.500000	2.449942	2.499942	2.419707	2.499942
2.449942	2.500000	2.449971	2.499971	2.419770	2.499971
2.449971	2.500000	2.449986	2.499986	2.419802	2.499986
2.449986	2.500000	2.449993	2.499993	2.419818	2.499993
2.449993	2.500000	2.449996	2.499996	2.419826	2.499996
2.449996	2.500000	2.449998	2.499998	2.419830	2.499998

Golden section:

A	B	C	D	E	F
Dichotomous/Golden Section Search					
Input data: Type (C3) in E3, where C3 represents x in f(x)					
Δ = 0.05	C3			#VALUE!	
Minimum x = 1.5	Maximum x = 2.5				
Solution: Enter x to select: Dichotomous: x GoldenSection: x					
x* = 2.47214	f(x*) = 2.47317				
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
1.500000	2.500000	1.881996	2.118034	-0.881986	0.767511
1.881996	2.500000	2.118034	2.263932	0.767511	1.689344
2.118034	2.500000	2.263932	2.354102	1.669344	2.111112
2.263932	2.500000	2.354102	2.409830	2.111112	2.313781
2.354102	2.500000	2.409830	2.444272	2.313781	2.406906
2.409830	2.500000	2.444272	2.465558	2.406905	2.451137
2.444272	2.500000	2.465558	2.478714	2.451137	2.473172
2.465558	2.500000	2.478714	2.486844	2.473172	2.484720

(d)

Dichotomous:

A	B	C	D	E	F
Dichotomous/Golden Section Search					
Input data: Type (C3) in E3, where C3 represents x in f(x)					
Δ = 0.05	C3			#VALUE!	
Minimum x = 2	Maximum x = 4				
Solution: Enter x to select: Dichotomous: x GoldenSection: x					
x* = 3.00000	f(x*) = 0.00062				
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
2.000000	4.000000	2.975000	3.025000	0.000625	0.000625
2.975000	3.025000	2.975000	3.025000	0.000625	0.000625

Golden section:

A	B	C	D	E	F
Dichotomous/Golden Section Search					
Input data: Type (C3) in E3, where C3 represents x in f(x)					
Δ = 0.05	C3			#VALUE!	
Minimum x = 2	Maximum x = 4				
Solution: Enter x to select: Dichotomous: x GoldenSection: x					
x* = 3.00000	f(x*) = 0.00017				
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
2.000000	4.000000	2.763932	3.236068	0.055728	0.055728
2.763932	3.236068	2.944272	3.055728	0.053108	0.053108
2.944272	3.055728	2.966844	3.013166	0.000173	0.000173
2.966844	3.013166	2.996844	3.003106	0.000010	0.000010

(e)

Dichotomous:

A	B	C	D	E	F
Dichotomous/Golden Section Search					
Input data: Type (C3) in E3, where C3 represents x in f(x)					
Δ = 0.05	C3			#VALUE!	
Minimum x = 0	Maximum x = 4				
Solution: Enter x to select: Dichotomous: x GoldenSection: x					
x* = 1.97500	f(x*) = 7.99999				
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
0.000000	4.000000	1.975000	2.025000	7.900000	7.975000
0.000000	2.025000	0.987500	1.037500	3.950000	4.150000
0.987500	2.025000	1.481250	1.531250	5.925000	6.125000
1.481250	2.025000	1.728125	1.770125	6.912500	7.112500
1.728125	2.025000	1.851563	1.901563	7.406250	7.606250
1.851563	2.025000	1.913281	1.963281	7.653125	7.853125
1.913281	2.025000	1.944141	1.994141	7.776563	7.976563
1.944141	2.025000	1.955700	2.009570	7.833281	1.980671
1.944141	2.009570	1.951655	2.001655	7.807422	1.996145
1.944141	2.001655	1.947998	1.997998	7.791992	1.991992
1.947998	2.001655	1.949927	1.999927	7.799707	1.999707
1.949927	2.001655	1.950891	2.000891	7.803564	1.999109
1.949927	2.000891	1.950409	2.004409	7.801636	1.999561
1.949927	2.000409	1.950168	2.000168	7.800671	1.999652
1.949927	2.000168	1.950407	2.000047	7.800169	1.999953
1.949927	2.000047	1.949987	1.999987	7.799948	1.999948
1.949987	2.000047	1.950017	2.000017	7.600069	1.999933
1.949987	2.000017	1.949992	2.000002	7.600008	1.999999
1.949995	2.000002	1.949998	1.999998	7.799978	1.999976
1.949998	2.000002	1.950000	2.000000	7.800001	2.000000

Golden section:

=F(C3<=2.4*C3,4-C3)					
ch21DichotomousGoldenSection					
A	B	C	D	E	F
Dichotomous/Golden Section Search					
Input data: Type (C3) in E3, where C3 represents x in f(x)					
Δ = 0.05	C3			#VALUE!	
Minimum x = 0	Maximum x = 4				
Solution: Enter x to select: Dichotomous: x GoldenSection: x					
x* = 2.00000	f(x*) = 7.97516				
Calculations:					
xL	xR	x1	x2	f(x1)	f(x2)
0.000000	4.000000	1.527664	2.472136	6.111456	1.527664
0.000000	2.472136	0.944272	1.527664	3.777088	6.111456
0.944272	2.472136	1.527664	1.886544	6.111456	7.554175
1.527664	2.472136	1.114566	1.750776	1.886544	7.003106
1.114566	2.472136	1.750776	1.886544	1.973689	7.554175
1.750776	2.111456	1.886544	1.973689	7.554175	7.694755
1.886544	2.111456	1.973689	2.026311	2.026311	7.894755
1.886544	2.026311	1.941166	1.973689	1.973689	7.764666
1.941166	2.026311	1.973689	1.993769	1.993769	7.947558
1.973689	2.026311	1.993769	2.005211	2.005211	7.975165
1.973689	2.005211	1.986111	1.993769	1.993769	7.993769
1.986111	2.005211	1.993769	2.044446	2.044446	7.975164

Continued...

Set 19.1b

Because $f(X)$ is strictly concave, a sufficient condition for optimality is $\nabla f(X) = 0$.

To solve $\nabla f(X) = 0$ by the Newton-Raphson method, consider Taylor's expansion about an initial X^0 ,

$$\nabla f(X) = \nabla f(X^0) + H(X - X^0)$$

The Hessian matrix H is independent of X because $f(X)$ is quadratic. The given expansion is exact because higher-order derivatives are zero.

Given $\nabla f(X) = 0$, we get

$$X = X^0 - H^{-1} \nabla f(X^0)$$

Because X satisfies $\nabla f(X) = 0$, X must be optimum regardless of the choice of initial X^0

$$\nabla f(X) = (4 - 4X_1 - 2X_2, 6 - 2X_1 - 4X_2)$$

$$\text{Let } X^0 = (5, 5) \Rightarrow \nabla f(X^0) = (-26, -24)^T$$

$$H = \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix}, H^{-1} = \begin{pmatrix} -1/3 & 1/6 \\ 1/6 & -1/3 \end{pmatrix}$$

Thus, the optimum is

$$X = (5) - \begin{pmatrix} -1/3 & 1/6 \\ 1/6 & -1/3 \end{pmatrix} \begin{pmatrix} -26 \\ -24 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 4/3 \end{pmatrix}$$

(a) 2 $f(X) = (X_2 - X_1^2)^2 + (1 - X_1)^2$

$$\nabla f(X) = [4(X_1^3 - X_1 X_2) + 2(X_1 - 1), 2(X_2 - X_1^2)]$$

$$X^0 = (0, 0)$$

$$\nabla f(X^0) = (-2, 0)^T$$

$$X = (-2r, 0)^T$$

$$h(r) = 16r^4 + 4r^2 + 4r + 1$$

$$r^* = -.2949$$

$$X' = (0, 0) + (-.2949)(-2, 0) = (-5898, 0)$$

continued...

(b) 2 continued $\nabla f(X) = C + 2X^TA$

$$= (1 - 10X_1 - 6X_2 - X_3,)$$

$$3 - 6X_1 - 4X_2,)$$

$$5 - X_1 - X_3,)$$

$$X^0 = (0, 0, 0)^T$$

$$\nabla f(X^0) = (1, 3, 5)$$

$$X = (1, 3, 5)r$$

$$h(r) = 35r + r^2(1, 3, 5)A\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$\text{Optimal } r = .299145$$

$$X' = (.299145, .897436, 1.495726)$$

$$\nabla f(X') = (-15.88, 2.84614, 3.205129)$$

$$X = X' + r \nabla f(X')$$

Set 19.2a

$$f_1(x_1) = e^{-x_1} + x_1, \quad g_1'(x_1) = x_1^2$$

$$f_2(x_2) = (x_2+1)^2, \quad g_2'(x_2) = x_2$$

$$k_1 \quad a_i^{k_1} \quad f_1(a_i^{k_1}) \quad g_1'(a_i^{k_1}) \quad \text{Var}$$

1	0	1	0	t_1'
2	.5	1.1	.25	t_1^2
3	1.	1.37	1.	t_1^3
4	1.5	1.72	2.25	t_1^4
5	1.732	1.91	3.00	t_1^5

k_2	$a_i^{k_2}$	$f_2(a_i^{k_2})$	$g_2'(a_i^{k_2})$	Var
1	0	1.	0	t_2'
2	.5	2.25	.5	t_2^2
3	1.	4.	1.	t_2^3
4	1.5	6.25	1.5	t_2^4
5	2.	9.	2.	t_2^5
6	2.5	12.25	2.5	t_2^6
7	3.	16.	3.	t_2^7

$$\begin{aligned} \text{maximize } Z &\cong t_1' + 1.1t_1^2 + 1.37t_1^3 + \\ &1.72t_1^4 + 1.91t_1^5 + \\ &t_2' + 2.25t_2^2 + 4t_2^3 + \\ &6.25t_2^4 + 9t_2^5 + 12.25t_2^6 + \\ &16t_2^7 \end{aligned}$$

subject to

$$\begin{aligned} .25t_1^2 + t_1^3 + 2.25t_1^4 + 3t_1^5 + \\ .5t_2^2 + t_2^3 + 1.5t_2^4 + 2t_2^5 + 2.5t_2^6 + \\ + 3t_2^7 &\leq 3 \end{aligned}$$

$$\begin{array}{ll} 0 \leq t_1' \leq y_1' & 0 \leq t_2' \leq y_2' \\ 0 \leq t_1^2 \leq y_1' + y_1^2 & 0 \leq t_2^2 \leq y_2' + y_2^2 \\ 0 \leq t_1^3 \leq y_1^2 + y_1^3 & 0 \leq t_2^3 \leq y_2^2 + y_2^3 \\ 0 \leq t_1^4 \leq y_1^3 + y_1^4 & 0 \leq t_2^4 \leq y_2^3 + y_2^4 \\ 0 \leq t_1^5 \leq y_1^4 & 0 \leq t_2^5 \leq y_2^4 + y_2^5 \\ 0 \leq t_1^6 \leq y_1^5 & 0 \leq t_2^6 \leq y_2^5 + y_2^6 \\ 0 \leq t_1^7 \leq y_1^6 & 0 \leq t_2^7 \leq y_2^6 \end{array}$$

$$\begin{aligned} t_1' + t_1^2 + t_1^3 + t_1^4 + t_1^5 + t_1^6 + t_1^7 &= 1 \\ t_2' + t_2^2 + t_2^3 + t_2^4 + t_2^5 &= 1 \end{aligned}$$

$$y_1^i = (0, 1) \quad i = 1, 2, \dots, 5$$

$$y_2^i = (0, 1) \quad i = 1, 2, \dots, 7$$

Use the formulation in Problem 1, less all the constraints in \mathcal{J}_1 . We use S_1, t_1^1 , and t_2^1 as the starting basic solution mainly for simplicity and to avoid using artificial starting basic variables. This can be achieved by substituting out t_1^1 in the Z-equation using 2

$t_1^1 = 1 - t_1^2 - t_1^3 - t_1^4 - t_1^5$	$t_1^1 \quad t_1^2 \quad t_1^3 \quad t_1^4 \quad t_1^5 \quad t_2^1 \quad t_2^2 \quad t_2^3 \quad t_2^4 \quad t_2^5 \quad t_2^6 \quad t_2^7 \quad S_1$
$Z = -1 - 37 - 72 - 91$	$0 - 1.25 - 3 - 5.85 - 8 - 11.75 - 15 \quad 0$
$S_1 = 0.25$	$1.225 \quad 3 \quad 0 \quad 0.5 \quad 1 \quad 1.5 \quad 2.25 \quad 3 \quad 1 \quad 3$
$t_1^1 = 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1$	
$t_2^1 = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1$	
$Z = 0 - 1 - 37 - 72 - 91 \quad 15 \quad 13.75 \quad 12 \quad 9.75 \quad 7 \quad 3.75 \quad 0 \quad 0 \quad 17$	
$S_1 = 0.25$	$1 \quad 2.25 \quad 3 \quad -3 \quad -2.5 \quad -2 \quad -1.5 \quad -1 \quad -0.5 \quad 0 \quad 1 \quad 0$
$t_1^1 = 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1$	
$t_2^1 = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1$	
$Z = 0 \quad 0 \quad -0.3 \quad -1.8 \quad -2.9 \quad 13.8 \quad 12.75 \quad 11.2 \quad 9.15 \quad 6.6 \quad 3.55 \quad 0 \quad -4 \quad 17$	
$t_2^1 = 0 \quad 1 \quad 4 \quad 9 \quad 12 \quad -12 \quad -10 \quad -8 \quad -6 \quad -4 \quad -2 \quad 0 \quad 4 \quad 0$	
$t_1^1 = 1 \quad 0 \quad -3 \quad -8 \quad -11 \quad 12 \quad 10 \quad 8 \quad 6 \quad 4 \quad 2 \quad 0 \quad -4 \quad 1$	
$t_2^1 = 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1$	

$$t_1^1 = 1, \quad t_2^1 = 1$$

Optimal solution: $X_1 = 0, X_2 = 3, Z = 17$

Let $y = x_1 x_2 x_3$. Because this is a maximization problem, $y > 0$.

$$\ln y = \ln x_1 + \ln x_2 + \ln x_3$$

Maximize $Z = y$

subject to

$$-\ln y + \ln x_1 + \ln x_2 + \ln x_3 = 0$$

$$x_1^2 + x_2 + x_3 \leq 4$$

which is separable.

$$f_1(y) = y \quad g_1(y_1) = -\ln y$$

$$f_2(y_2) = \ln x_2, \quad g_2^2(y_2) = \ln x_2$$

$$f_3(y_3) = x_3, \quad g_3^3(y_3) = x_3$$

use $0 \leq y \leq 7$ and $0 \leq x_i \leq 4$ to determine the breaking points; then solve using restricted basis

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Set 19.2a

Separability requires using the **4**
ln function to separate the products
into single-variable functions. That is,
 $y_1 = x_1 x_2$ and $y_2 = x_1 x_3$. However,
to ensure that $\ln(0)$ will not be
encountered, we use the substitution

$$\begin{cases} w_1 = x_1 + 1 \\ w_2 = x_2 + 1 \\ w_3 = x_3 + 1 \end{cases} \Rightarrow w_1, w_2, w_3 > 0$$

Thus,

$$\begin{aligned} x_1 x_2 &= w_1 w_2 - w_1 - w_2 + 1 \\ x_1 x_3 &= w_1 w_3 - w_1 - w_3 + 1 \end{aligned}$$

Let $v_1 = w_1, v_2 = w_2, v_3 = w_3$. Hence,

$$x_1 x_2 = v_1 - w_1 - w_2 + 1$$

$$x_1 x_3 = v_2 - w_1 - w_3 + 1$$

where $\ln(v_1) = \ln(w_1) + \ln(w_2)$

$$\ln(v_2) = \ln(w_1) + \ln(w_3)$$

The problem is expressed as

$$\text{Maximize } z = v_1 + v_3 - 2w_1 - w_2 + 1$$

Subject to

$$v_1 + v_2 - 2w_1 - w_2 \leq 9$$

$$\ln(v_1) - \ln(w_1) - \ln(w_2) = 0$$

$$\ln(v_2) - \ln(w_1) - \ln(w_3) = 0$$

$$v_1, v_2, w_1, w_2, w_3 \geq 0$$

$$\text{Let } y = e^{2x_1 + x_2^2} > 0$$

$$\ln y = 2x_1 + x_2^2$$

$$\text{Maximize } z = y + (x_3 - 2)^2$$

Subject to

$$\ln y - 2x_1 - x_2^2 = 0$$

$$x_1 + x_2 + x_3 \leq 6$$

$$y, x_1, x_2, x_3 \geq 0$$

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6

$$w_1 = x_1 + 1$$

$$w_2 = x_2 + 1$$

$$w_3 = x_3 + 1$$

$$\text{Next, } y_1 = e^{x_1 x_2}$$

$$\ln y_1 = x_1 x_2$$

Now,

$$x_1 x_2 = w_1 w_2 - w_1 - w_2 + 1$$

$$= y_2 - w_1 - w_2 + 1$$

where

$$\ln y_2 = \ln w_1 + \ln w_2$$

Thus,

$$\ln y_1 = y_2 - w_1 - w_2 + 1 \quad (1)$$

$$\ln y_2 = \ln w_1 + \ln w_2$$

Next,

$$x_2^2 x_3 = (w_2 - 1)^2 (w_3 - 1)$$

$$= w_2^2 w_3^2 + w_2^2 + w_3^2 - 2w_2 w_3 - w_2 + 2w_2^2 + 1$$

Let

$$y_3 = w_2^2 w_3, y_4 = w_2 w_3$$

$$\ln y_3 = 2 \ln w_2 + \ln w_3$$

$$\ln y_4 = \ln w_2 + \ln w_3$$

and

$$x_2^2 x_3 = y_3 + w_3 - y_4 - w_2^2 + 2w_2 + 1 \quad (2)$$

$$\ln y_3 = 2 \ln w_2 + \ln w_3$$

$$\ln y_4 = \ln w_2 + \ln w_3$$

also,

$$x_2 x_3 = w_2 w_3 - w_2 - w_3 + 1 \quad (3)$$

$$= y_3 - w_2 - w_3 + 1$$

$$\ln y_5 = \ln w_2 + \ln w_3$$

Finally,

$$x_3 x_4 = x_3 x_4^+ - x_3 x_4^-, x_4^+, x_4^- \geq 0$$

$$\text{Put } y_6 = x_3 x_4^+ \text{ and } y_7 = x_3 x_4^-$$

$$\text{and let } w_4^+ = 1 + x_4^+$$

$$w_4^- = 1 + x_4^-$$

Thus,

$$x_3 x_4^+ = y_6 - w_3 + w_4^+ + 1 \quad (4)$$

$$\ln y_6 = \ln w_3 + \ln w_4^+$$

continued...

Set 19.2a

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$$\begin{aligned} x_3 x_4^- &= y_9 - w_3 - w_4^- + 1 \\ \ln y_9 &= \ln w_3 + \ln w_4^- \end{aligned} \quad (5)$$

From (1) through (5), the problem becomes:

$$\begin{aligned} \text{Maximize } Z = & y_1^+ y_3^+ w_2 - 2y_4^- w_2^2 + 2w_1 + 1 \\ & + w_4^+ - w_4^- \end{aligned}$$

Subject to

$$\ln y_1^+ = y_2^+ - w_1 - w_2 + 1$$

$$\ln y_2^+ = \ln w_1 + \ln w_2$$

$$\ln y_3^+ = 2 \ln w_2 + \ln w_3$$

$$\ln y_4^+ = \ln w_2 + \ln w_3^- \quad \text{same}$$

$$\ln y_5^+ = \ln w_2 + \ln w_3^- \quad \text{same}$$

$$\ln y_8^+ = \ln w_3 + \ln w_4^+$$

$$\ln y_9^+ = \ln w_3 + \ln w_4^-$$

$$w_1 + y_3^+ - w_2 - w_3 + y_4^- - y_9^+ - w_4^+ - w_4^- \leq 10$$

$$y_i \geq 0, w_i \geq 0, \text{ all } i \text{ and } j$$

$$b = a_{k-1,i} - a_{k-2,i}$$

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$$\delta = \min \{ b - x_{k-1,i}, x_{k,i} \}$$

It is feasible to subtract δ from $x_{k,i}$ and add it to $x_{k-1,i}$. The net change in the value of the objective function is

$$\Delta = \delta (P_{k-1,i} - P_{k,i}) > 0$$

Because $P_{k-1,i} < P_{k,i}$ (minimization), $\Delta < 0$. Thus, adding δ to $x_{k-1,i}$ leads to a smaller value of the objective function.

The end result is that it is never optimal to have positive $x_{k,i}$ if $x_{k-1,i}$ has not attained its upper limit $a_{k-1,i} - a_{k,i}$.

$$\text{Minimize } Z = x_1^4 + 2x_2^+ - 2x_2^- + x_3^2$$

Subject to

$$x_1^2 + x_2^+ - x_2^- + x_3^2 \leq 4$$

$$x_1^+ + x_2^+ - x_2^- \leq 3$$

$$-x_1^+ - x_2^+ + x_2^- \leq 3$$

$$x_1^+, x_2^+, x_2^-, x_3 \geq 0$$

$$f_1(x_1) = x_1^4; g_1^1(x_1) = x_1^2, g_1^2(x_1) = x_1,$$

$$g_1^3(x_1) = -x_1,$$

$$f_3(x_3) = x_3^2; g_3^1(x_3) = x_3^2$$

k_1	a_{k_1}	$f_1(a_{k_1})$	P_{k_1}	g_1^1	$P_{k_1}^1$	g_2^1	$P_{k_1}^2$	g_3^1	$P_{k_1}^3$
0	0	0	-	0	-	0	-	0	-
1	1	1	1	1	1	1	1	1	1
2	2	16	15	4	3	2	1	2	1
3	3	81	65	90	5	3	1	3	1

k_3	a_{k_3}	$f_3(a_{k_3})$	P_{k_3}	g_3^1	$P_{k_3}^1$
0	0	0	0	0	0
1	1	1	1	1	1
2	2	4	3	4	3
3	3	9	5	9	5

$$\begin{aligned} \text{Min } Z = & x_{11} + 15x_{12} + 6.5x_{13} + 2x_2 + \\ & x_{13} + 3x_{23} + 5x_{33} \end{aligned}$$

Subject to

$$x_{11} + 3x_{12} + 5x_{13} + x_2^+ - x_2^- + x_{13} + 3x_{23} + 5x_{33} \leq 4$$

$$x_{11} + x_{12} + x_{13} + x_2^+ - x_2^- \leq 3$$

$$-x_{11} - x_{12} - x_{13} - x_2^+ - x_2^- \leq 3$$

$$0 \leq x_{ij} \leq 1; i=1,3, j=1,2,3$$

$$x_2^+, x_2^- \geq 0$$

Use simplex with upper bounding to determine the approximate optimum solution.

Set 19.2b

$$Z = (6, 3) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (x_1, x_2) \begin{pmatrix} -2 & -2 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$D = \begin{pmatrix} -2 & -2 \\ -2 & -3 \end{pmatrix}$$

Principal minor determinants: -2, +2

Negative definite $\Rightarrow Z$ is concave

Constraints:

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} X - \begin{pmatrix} 1 \\ 4 \\ 0 \\ 0 \end{pmatrix} \leq 0, \lambda S = UX = 0$$

X^T	λ^T	U^T	S^T	RHS
4	4	1	2	-1 0 0 0 6
4	6	1	3	0 -1 0 0 3
1	1	0	0	0 1 0 0 1
2	3	0	0	0 0 1 1 4

Basis	R	x_1	x_2	λ_1	λ_2	U_1	U_2	R_1	R_2	S_1	S_2	Sol.
R	①	8	10	2	5	-1	-1	0	0	0	0	9
R_1	0	4	4	1	2	-1	0	①	0	0	0	6
R_2	0	4	6	1	3	0	-1	0	①	0	0	3
S_1	0	1	1	0	0	0	0	0	0	①	0	1
S_2	0	2	3	0	0	0	0	0	0	0	①	4
R	①	9/3	0	1/3	0	-1	2/3	0	-5/3	0	0	4
R_1	0	4/3	0	1/3	0	-1	2/3	①	-2/3	0	0	9
R_2	0	2/3	①	1/6	1/2	0	-1/6	0	1/6	0	0	1/2
S_1	0	1/3	0	-1/6	-1/2	0	1/6	0	-1/6	①	0	1/2
S_2	0	0	0	-1/2	-3/2	0	1/2	0	-1/2	0	①	5/2
R	①	0	-2	0	-1	-1	1	0	-2	0	0	3
R_1	0	0	-2	0	-1	-1	1	①	-1	0	0	3
R_2	0	①	3/2	1/4	3/4	0	-1/4	0	1/4	0	0	3/4
S_1	0	0	-1/2	-1/4	-3/4	0	1/4	0	-1/4	①	0	1/4
S_2	0	0	0	-1/2	-3/2	0	1/2	0	-1/2	0	①	5/2
R	①	0	0	1	2	-1	0	0	-1	-4	0	2
R_1	0	0	0	1	2	-1	0	①	0	-4	0	2
R_2	0	①	1	0	0	0	0	0	0	1	0	1
U_1	0	0	-2	-1	-3	0	①	0	-1	4	0	1
S_2	0	0	1	0	0	0	0	0	0	-2	①	2
R	①	0	0	0	0	0	0	-1	-1	0	0	0
λ_1	0	0	0	①	2	-1	0	1	0	-4	0	2
λ_2	0	①	1	0	0	0	0	0	0	1	0	1
U_2	0	0	-2	0	-1	-1	①	1	-1	0	0	3
S_2	0	0	1	0	0	0	0	0	0	-2	①	2

continued...

Optimum solution:

1 continued

$$x_1 = 1, \lambda_1 = 2, M_1 = 0, S_1 = 0$$

$$x_2 = 0, \lambda_2 = 0, M_2 = 3, S_2 = 0$$

$$Z = 4$$

Let $w = -Z$. Then, the problem becomes

Maximize

$$w = (-1, 3, 5)X + X^T \begin{pmatrix} -2 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -3 \end{pmatrix} X$$

Subject to

$$\begin{pmatrix} -1 & -1 & -1 \\ 3 & 2 & 1 \end{pmatrix} X \leq \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$D = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -3 \end{pmatrix}$$

Principal minor determinants =

-2, 3, -7 \Rightarrow negative definite

$\Rightarrow w$ is concave

Necessary conditions:

$$\begin{bmatrix} 4 & 2 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ 2 & 4 & 2 & -1 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 6 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ \lambda \\ U \\ S \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \\ -1 \\ 6 \end{bmatrix}$$

$$\lambda S = 0 \quad UX = 0$$

Optimal Solution:

$$x_1 = 0, x_2 = 0.4, x_3 = 0.7$$

Set 19.2c

Transformed problem:

$$\text{Maximize } Z = x_1 + 2x_2 + 5x_3$$

Subject to

$$2x_1 + 3x_2 + 5x_3 + 1.28y \leq 10$$

$$9x_1^2 + 16x_3^2 - y^2 = 0$$

$$7x_1 + 5x_2 + x_3 \leq 12.4$$

$$x_1, x_2, x_3, y \geq 0$$

1

Transformed problem:

$$\text{Maximize } Z = x_1 + x_2^2 + x_3$$

Subject to

$$x_1^2 + 5x_2^2 + 2\sqrt{x_3} + 1.28y \leq 10$$

$$16x_2^2 + 25x_3 - y^2 = 0$$

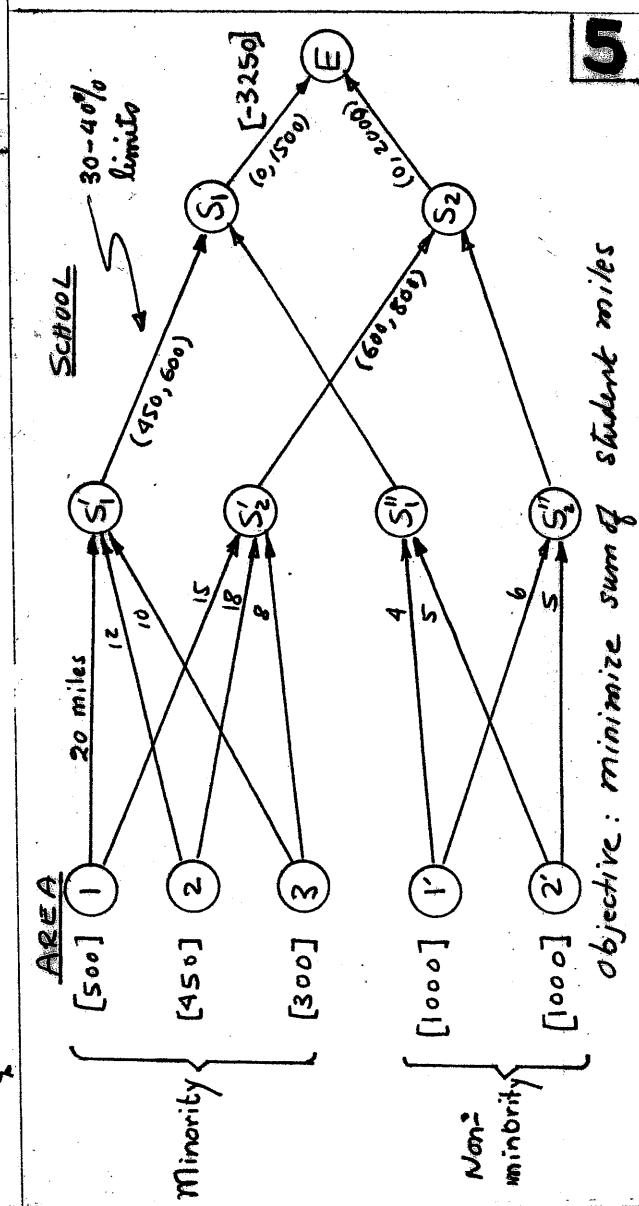
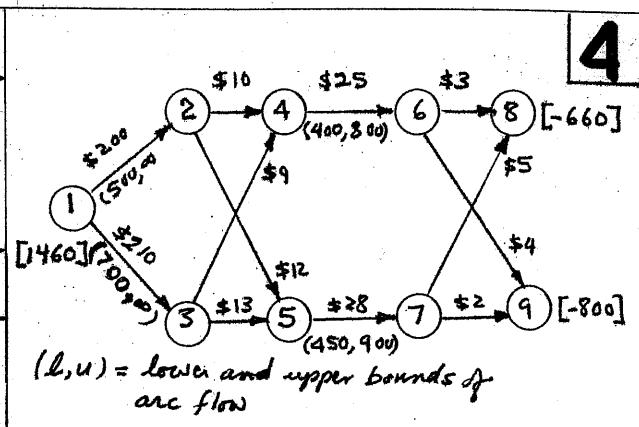
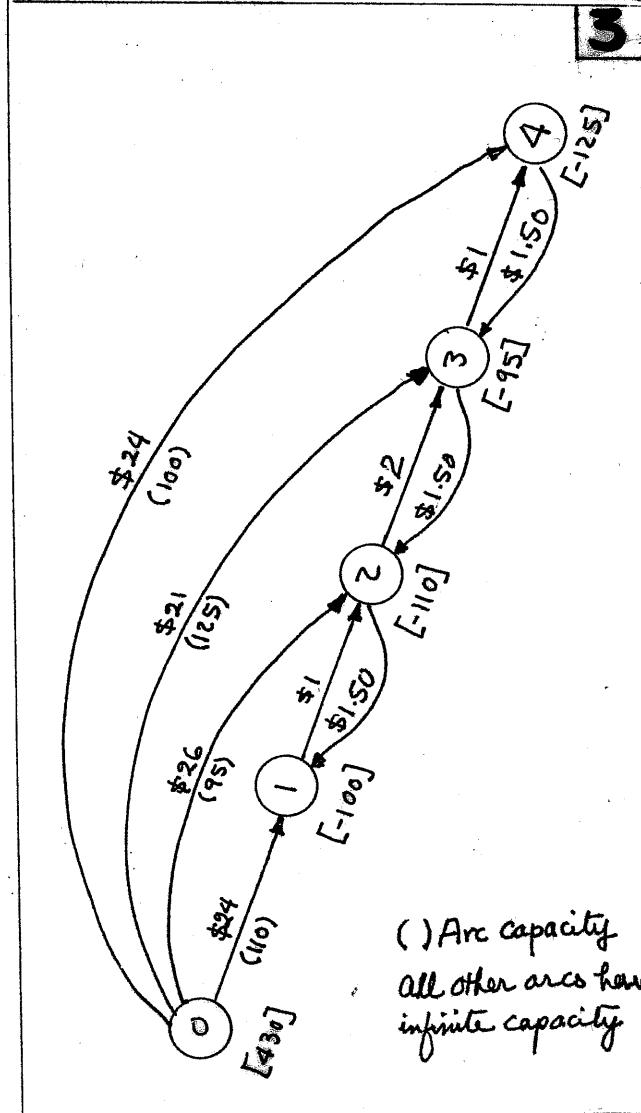
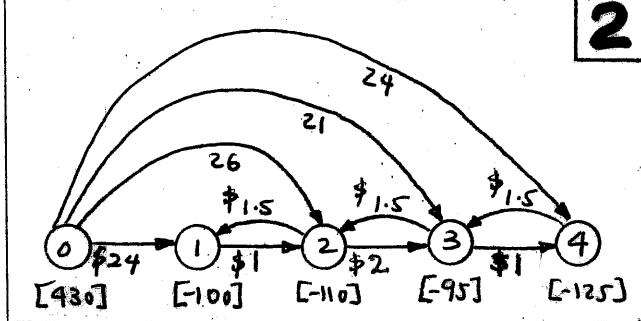
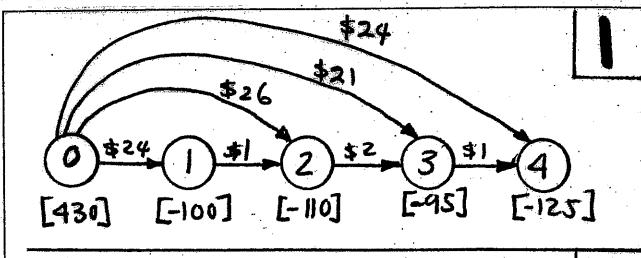
$$x_1, x_2, x_3, y \geq 0$$

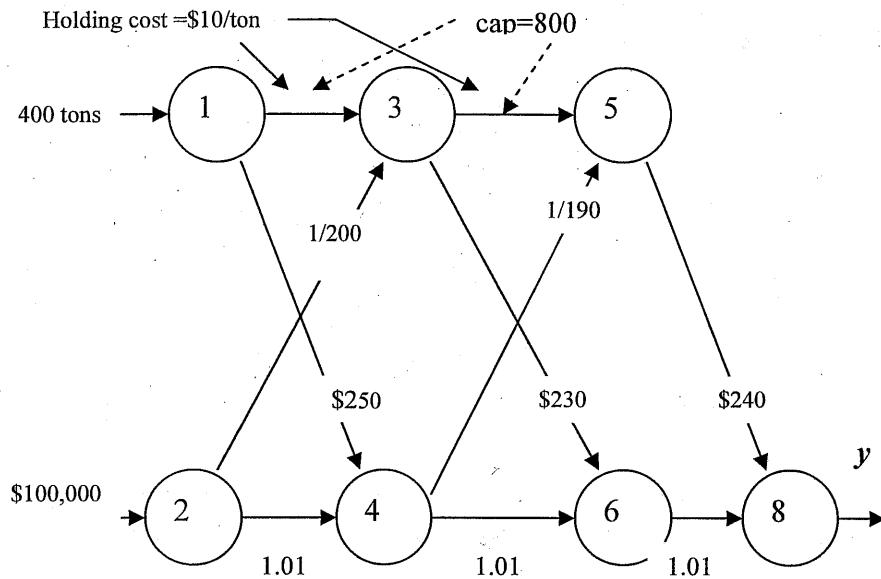
2

Chapter 20

Additional Network and LP algorithms

Set 20.1a





Let

$$x_{ij} = \begin{cases} \text{Tons from node } i \text{ to node } j, & i = 1, 3, 5, j = 4, 6, 8 \\ \text{Dollars from node } i \text{ to node } j, & i = 2, 4, 6, 8, j = 3, 5 \end{cases}$$

y = Total revenue

The associated LP is

$$\text{Maximize } z = y - 10(x_{13} + x_{35} + x_{57})$$

subject to

$$x_{13} + x_{14} = 400$$

$$x_{13} + x_{23} / 200 = x_{35} + x_{36}$$

$$x_{35} + x_{45} / 190 = x_{58}$$

continued...

Set 20.1a

$$x_{23} + x_{24} = 100000$$

$$1.01x_{24} + 250x_{14} = x_{45} / 190 + x_{46}$$

$$1.01x_{46} + 230x_{36} = x_{68}$$

$$1.01x_{68} + 240x_{58} = y$$

$$x_{13} \leq 800$$

$$x_{35} \leq 800$$

$$x_{ij} \geq 0, \text{ for all } i \text{ and } j$$

Optimum solution: $z = \$48,240,000$

$$x_{14} = 400 \text{ tons}, x_{58} = 20,100 \text{ tons}$$

$$x_{24} = \$100,000, x_{45} = \$38,190,000$$

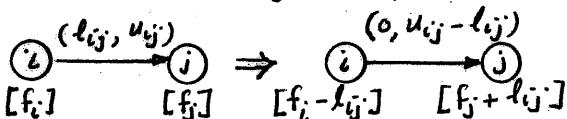
Set 20.1b

$x_{i,j}$ = Flow amount from node i to node j
Case 1: No substitution of lowerbounds

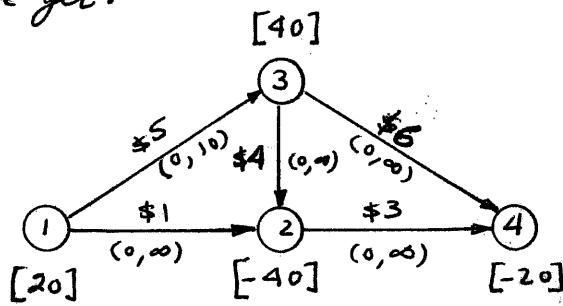
	x_{12}	x_{13}	x_{24}	x_{32}	x_{34}	
min Z	1	5	3	4	6	
Node 1:	1	1				= 50
Node 2:	-1		1	-1		= -40
Node 3:		-1	1	1	1	= 20
Node 4:			-1	-1	-1	= -30
lower bd:	0	30	10	10	0	
upper bd:	∞	40	00	00	00	

Optimum: $x_{12} = 20, x_{13} = 30, x_{24} = 10, x_{32} = 30, x_{34} = 20$
 $Z = \$440$

Case 2: lower bounds substituted directly on network using the following rule:



Applying this rule to the network we get:



	x'_{12}	x'_{13}	x'_{24}	x'_{32}	x'_{34}	
min Z	1	5	3	4	6	
node 1	1	1				= 20
node 2	-1		1	-1		= -40
node 3		-1	1	1	1	= 40
node 4			-1	-1	-1	= -20
upper bd	∞	10	∞	∞	∞	

where $x_{12} = x'_{12}, x_{13} = x'_{13} + 30$

$x_{24} = x'_{24} + 10, x_{32} = x'_{32} + 10$

$x_{34} = x'_{34}$

Optimum: $Z = \$440$

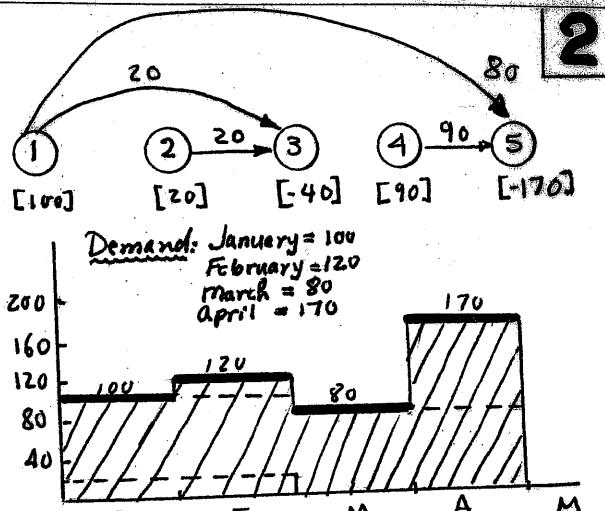
$x_{12} = x'_{12} = 20$

$x_{32} = x'_{32} + 10 = 10 + 10 = 20$

$x_{13} = x'_{13} + 30 = 30$

$x_{34} = x'_{34} = 20$

$x_{24} = x'_{24} + 10 = 10$



$x_{13} = 20$: hire 20 at the start of January for termination at start of March

$x_{23} = 20$: hire 20 at the start of February for termination at the start of March

$x_{15} = 80$: hire 80 at the start of January for termination at the start of May

$x_{45} = 90$: hire 90 at the start of April for termination at the start of May.

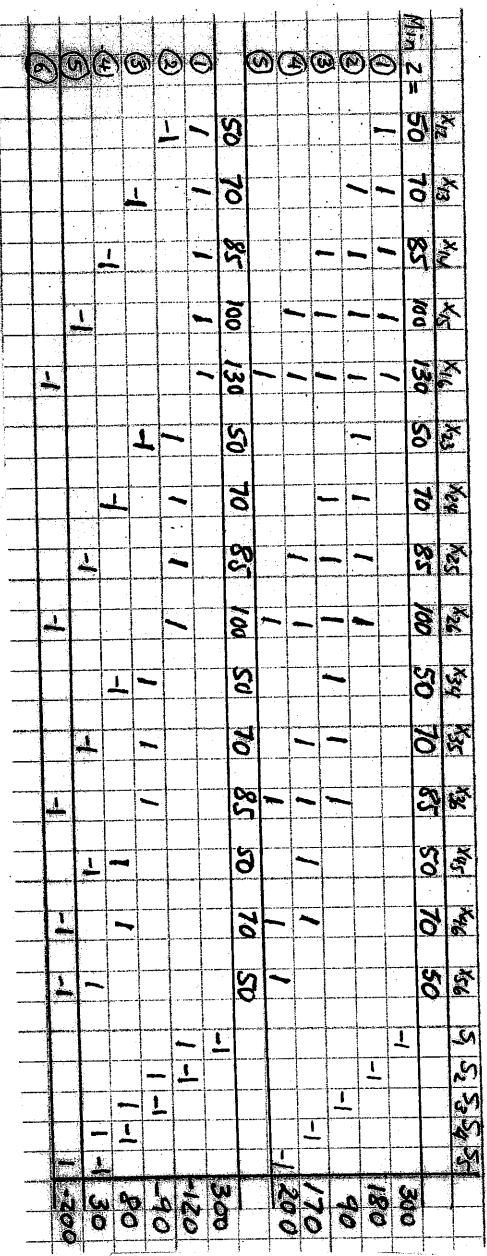
3

$x_{i,j} =$ hired at the start of i and terminated at the start of j
 $j \geq i+1$

	x_{13}	x_{14}	x_{15}	x_{24}	x_{25}	x_{34}	x_{35}	S_1	S_2	S_3	S_4	
Jan	1	1	1					-1				= 100
Feb	1	1	1	1	1			-1				= 120
Mar	1	1	1	1	1	1		-1				= 80
Apr.	1	1	1	1	1	1	1	-1				= 170
May												
Jan	1	1	1					-1				= 100
Feb				1	1	1	1	-1				= 20
Mar	-1							1	1	1		= 40
Apr.		-1							1	1	1	= 90
May			-1						-1			= 170

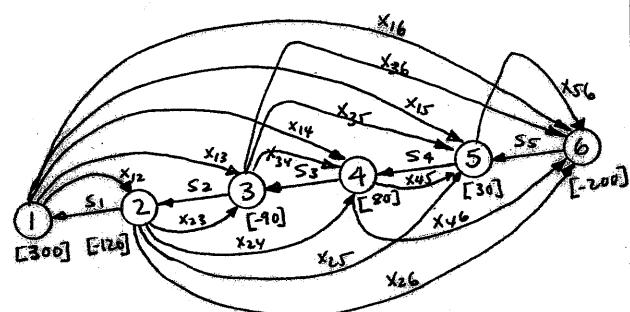
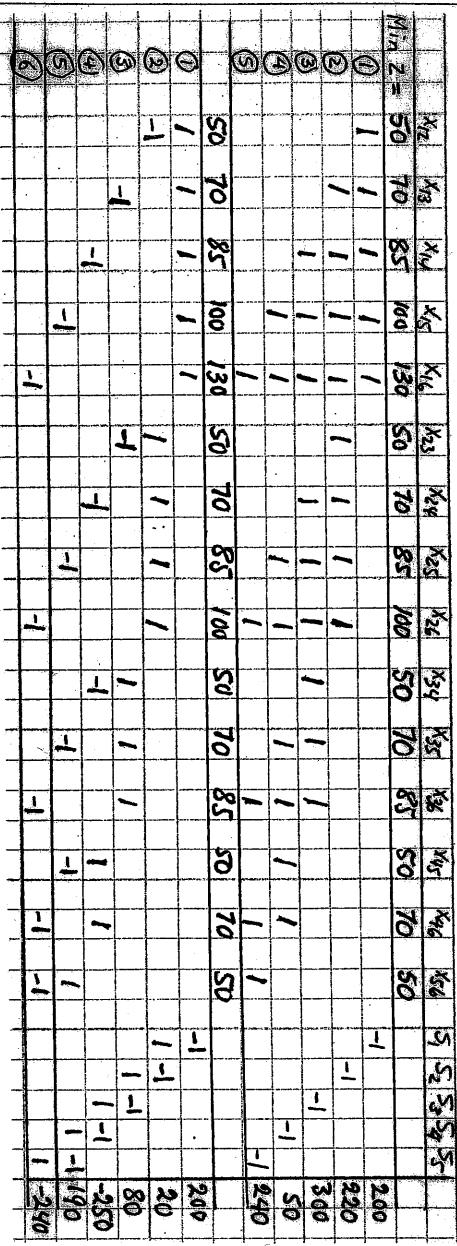
Set 20.1b

(a)

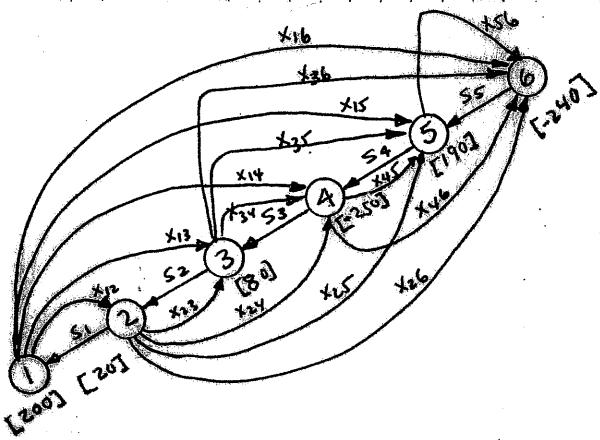


4

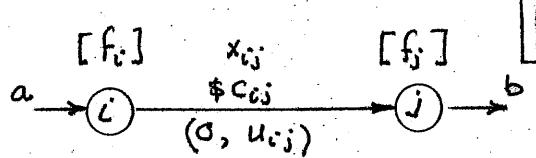
(b)



continued...



Set 20.1b



Let a represent the total flow of incoming arcs at node i and b the total flow of outgoing arcs at node j .

$$x_{ij} \leq u_{ij} \Rightarrow x_{ij} + x'_{ij} = u_{ij}, x'_{ij} \geq 0 \quad (1)$$

Node i :

$$a + f_i = x_{ij} \quad (2)$$

Node j :

$$x_{ij} + f_j = b \quad (3)$$

$$\text{Thus, } u_{ij} - x'_{ij} - b = -f_j.$$

or

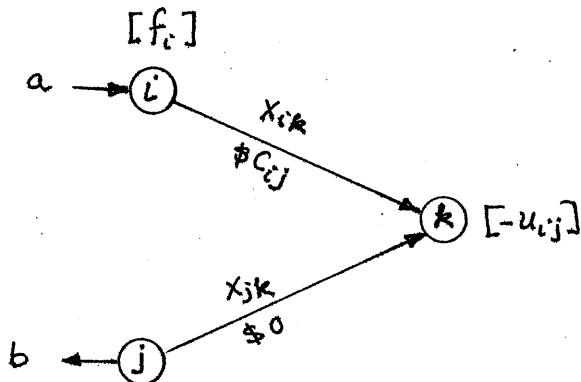
$$x'_{ij} + b = u_{ij} + f_j. \quad (4)$$

Letting

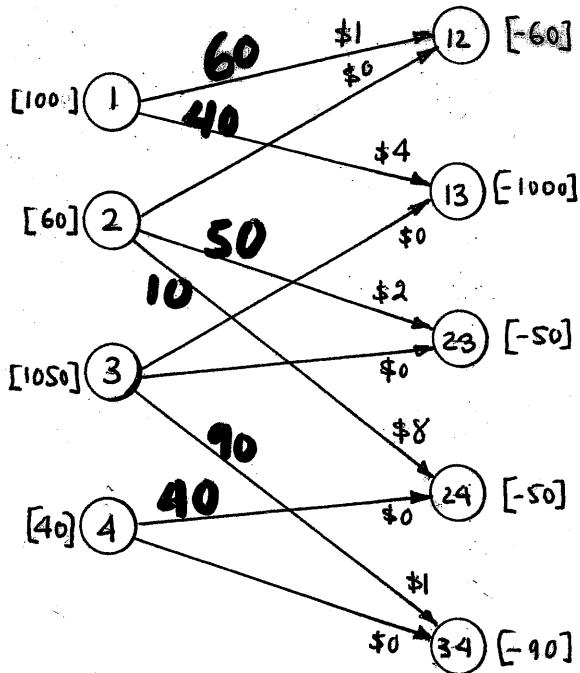
$$x_{ik} = x_{ij}$$

$$x_{jk} = x'_{ij}$$

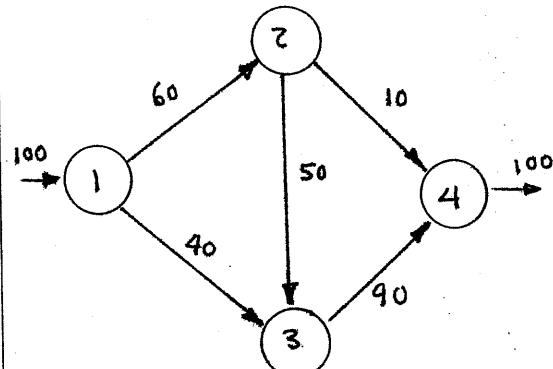
equations (1), (2), and (4) produce the following equivalent network



Application of the transformation to the network in figure 6-42, we get



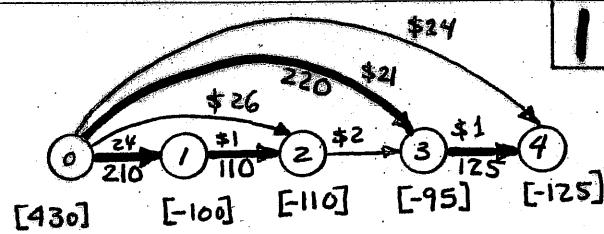
Optimum solution is obtained by using TORA's transportation model, and is shown in bold on the arcs. This solution is translated in terms of the original network as follows:



Total cost = \$490

continued...

Set 20.1c



The spanning tree shown by heavy arcs gives a starting basic feasible solution. We compute the dual values w_i , $i = 0, 1, \dots, 4$ as follows:

$$w_0 \equiv 0$$

$$w_0 - w_1 = 24 \implies w_1 = -24$$

$$w_1 - w_2 = 1 \implies w_2 = -25$$

$$w_0 - w_3 = 21 \implies w_3 = -21$$

$$w_3 - w_4 = 1 \implies w_4 = -22$$

Evaluation of the nonbasic arcs.

$$Z_{02} - C_{02} = w_0 - w_2 - c_{02}$$

$$= 0 - (-25) - 26 = -1$$

$$Z_{04} - C_{04} = 0 - (-22) - 24 = -2$$

$$Z_{23} - C_{23} = -25 - (-21) - 2 = -6$$

The given spanning tree solution is optimal.

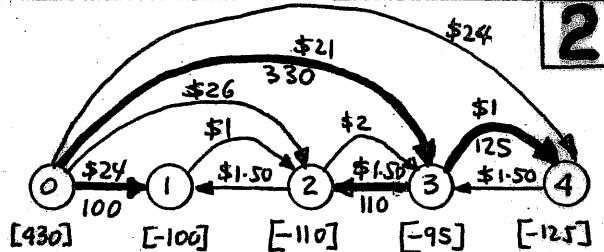
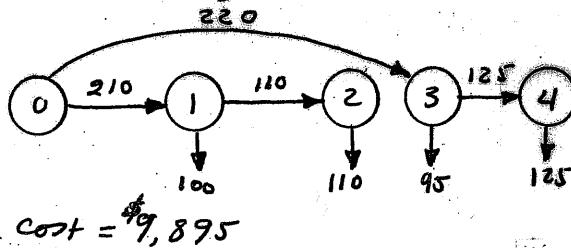
Transshipment Solution:

Since there are no finite upper bounds, the problem can be solved directly as a transshipment model

	1	2	3	4	B = 430
0	24 (210)	26 0	21 (220)	24 M	430
1	320 B	110 B	M	M	
2	M	430	2	M	
3	M	M	0 305	125 B	
	100+B	110+B	95+B	125+B	

continued...

Solution Summary:



$$w_0 \equiv 0$$

$$w_1 = -24, w_2 = -22.5, w_3 = -21, w_4 = -22$$

$$Z_{02} - C_{02} = 0 - (-22.5) - 26 = -3.5$$

$$Z_{04} - C_{04} = 0 - (-22) - 24 = -2$$

$$Z_{12} - C_{12} = -24 - (-22.5) - 1 = -2.5$$

$$Z_{21} - C_{21} = -22.5 - (-24) - 1.5 = 0$$

$$Z_{23} - C_{23} = -22.5 - (-21) - 2 = -3.5$$

Optimum!

Transshipment Solution:

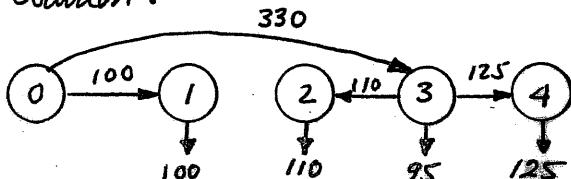
	1	2	3	4
0	24 (100)	26 0	21 (330)	24 M
1	430		M	M
2	1.5	430	2	M
3	M	1.5	0 110	125 B
4	M	M	1.5 95	0 125 B

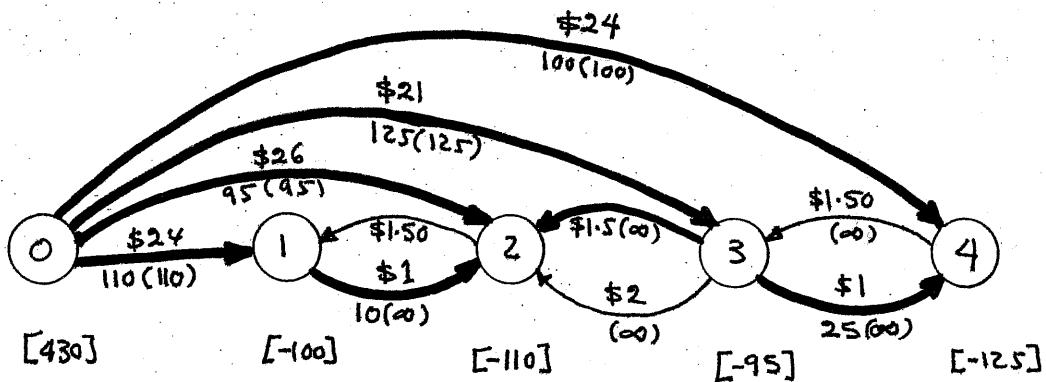
430

B = 430

$$100+B \quad 110+B \quad 95+B \quad 125+B \quad \text{Cost} = \$9,620$$

Summary of the optimum solution:

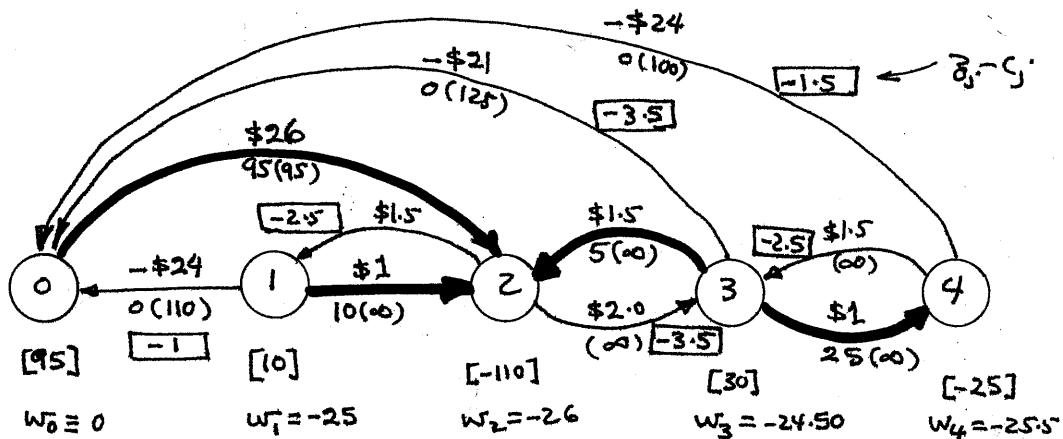




This solution is not basic because it does not comprise a spanning tree. To convert it into a spanning tree, substitute cut arcs 0-1, 0-3, and 0-4 at upper bound—that is,

$$x_{01} = 110 - x_{10}, \quad x_{03} = 125 - x_{30}, \quad x_{04} = 100 - x_{40}$$

Arcs x_{10} , x_{30} & x_{40} are now nonbasic at zero level.



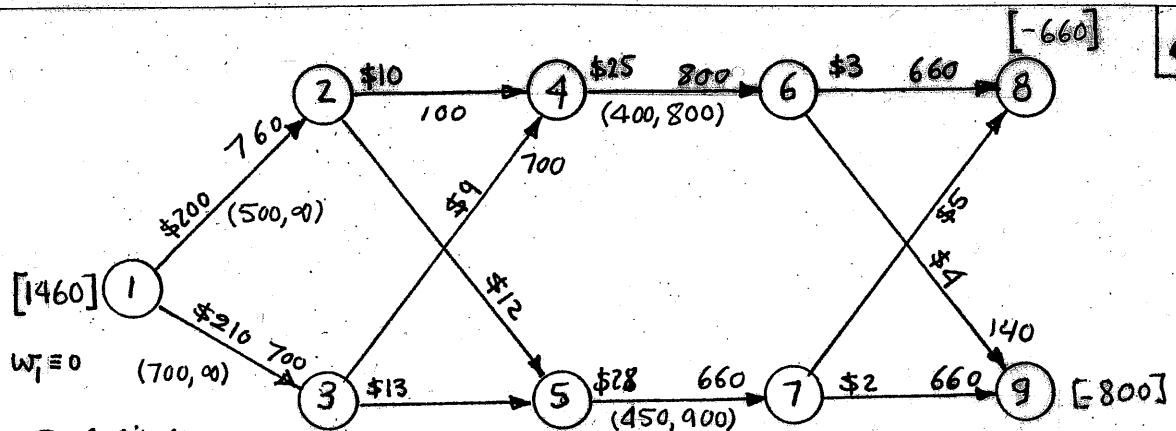
Optimum Solution because all $Z_j - C_j \leq 0$

Optimum Solution summary:

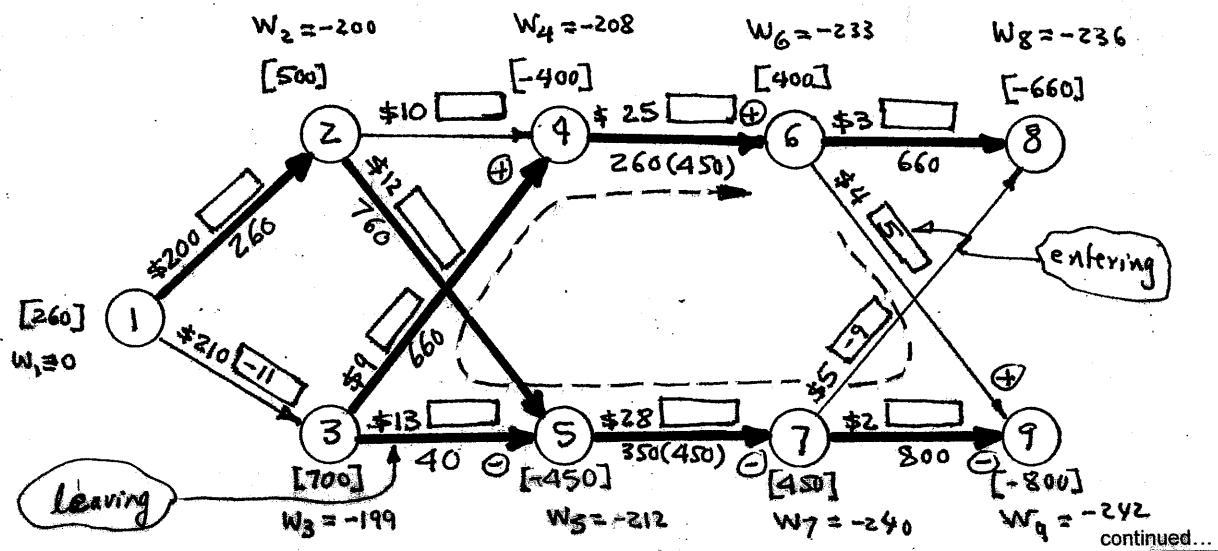
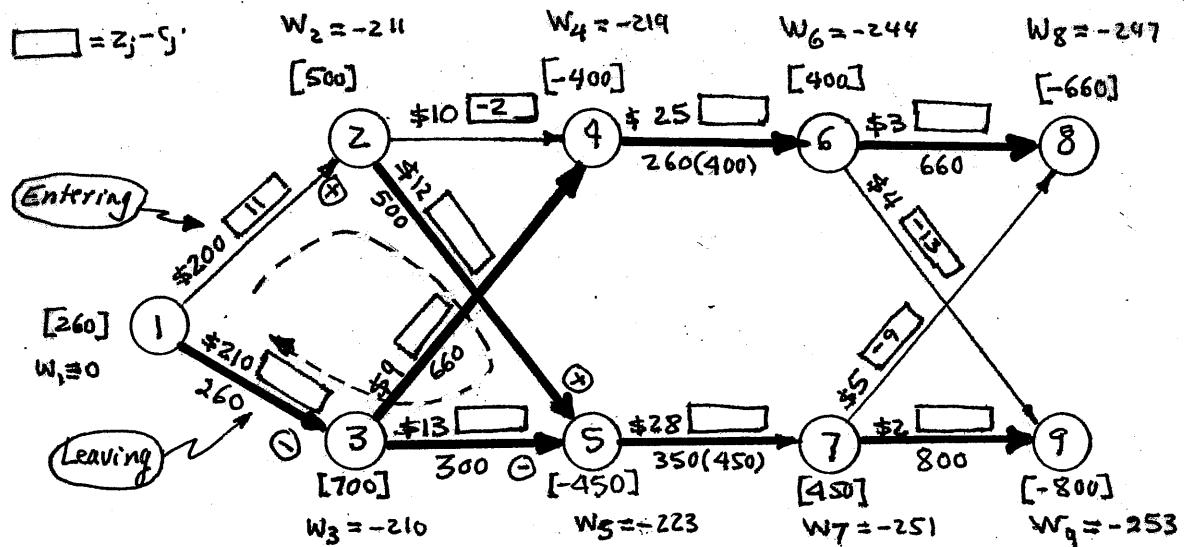
Period	Production	Demand	Surplus
1	110	100	10
2	95	110	-15
3	125	95	30
4	100	125	-25

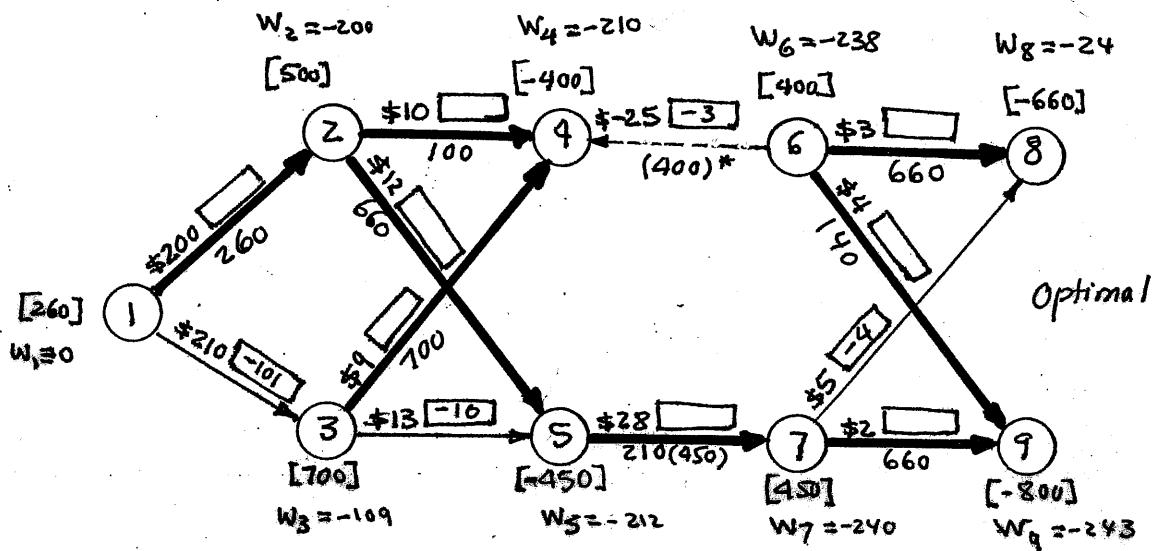
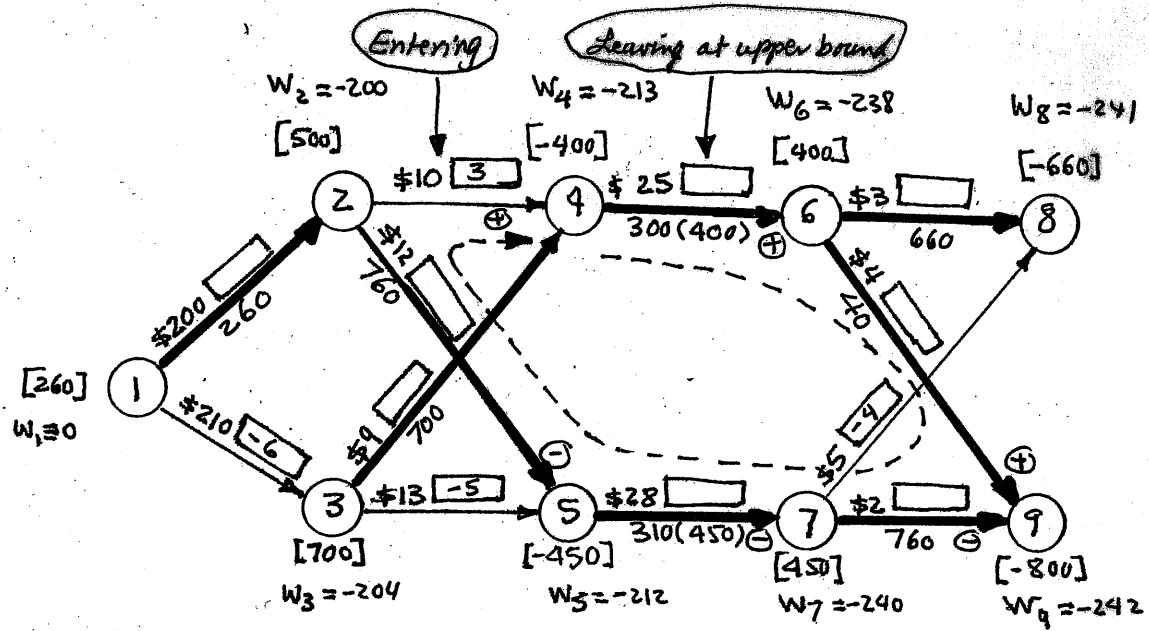
Total cost = \$10,177.50

Set 20.1c



Substituting out the lower bounds, the following iterations will result:





Optimal Solution:

$$x_{12} = 260 + 500 = 760, \quad x_{13} = 0 + 700 = 700$$

$$x_{24} = 100, \quad x_{25} = 660$$

$$x_{34} = 700 > x_{35} = 0$$

$$x_{46} = 400 + 400 = 800, \quad x_{56} = 210, \quad x_{57} = 210 + 450 = 660$$

$$x_{68} = 660, \quad x_{69} = 140$$

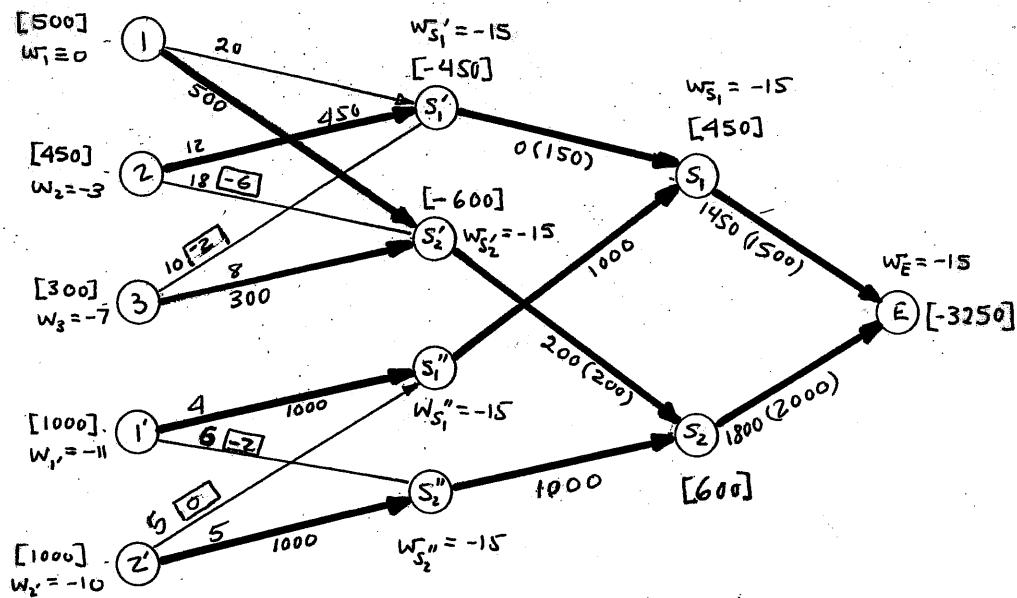
$$x_{78} = 0, \quad x_{79} = 660$$

$$\text{Total cost} = \$356,560$$

Set 20.1c

Network after substituting out lower bounds:

5

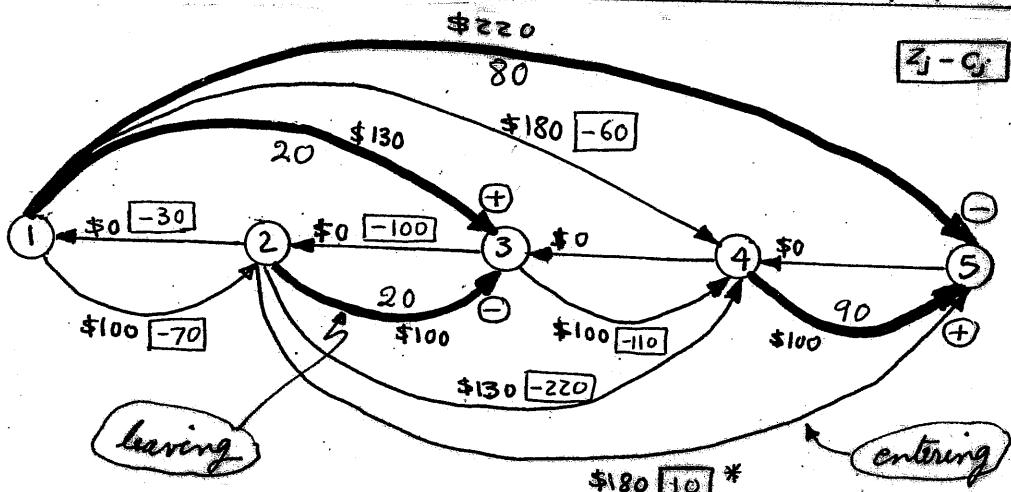


The given solution is optimal. Total student miles = 24,300

From Area	To School	Number of Students
1	S_2	500
2	S_1	450
3	S_2	300
$1'$	S_1	1000
$2'$	S_2	1000

Problem has alternative optimum

6

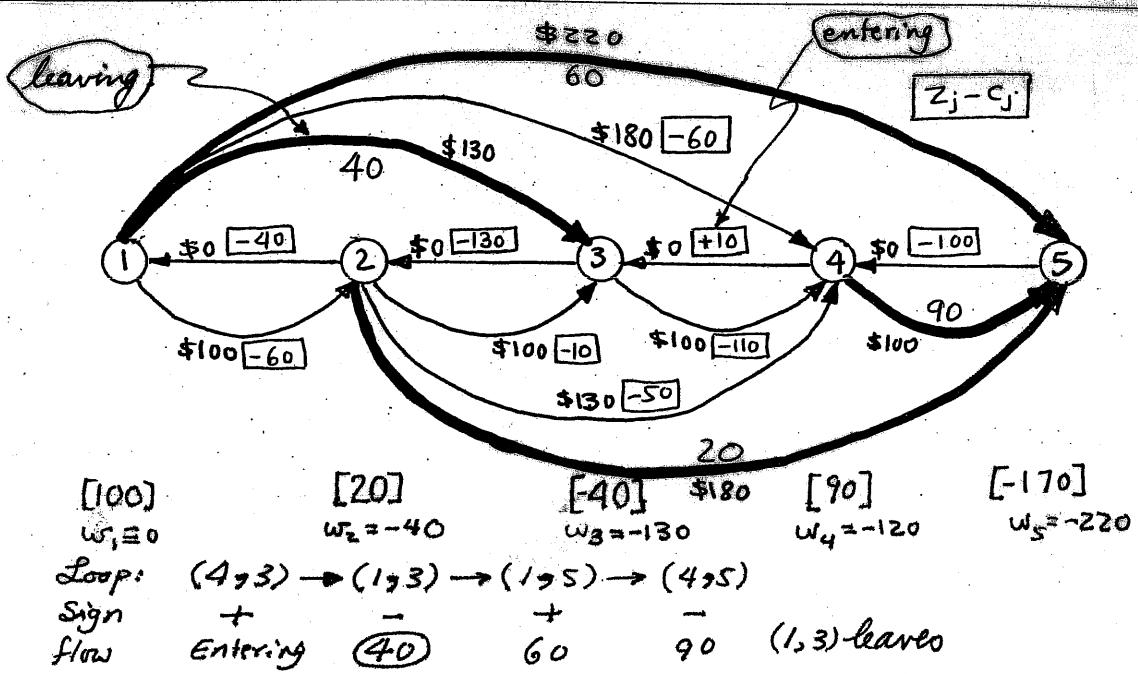


$$\text{Loop: } (2,5) \rightarrow (1,5) \rightarrow (1,3) \rightarrow (2,3) \leftarrow \text{leaving}$$

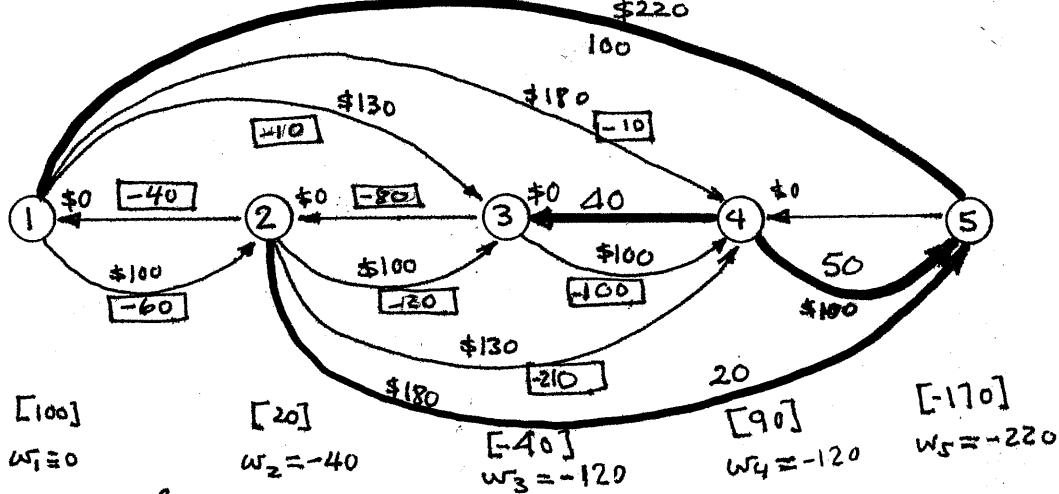
$$\begin{array}{ll} \text{Sign:} & + \\ \text{Flow: Entering} & -80 \end{array}$$

continued...

Set 20.1c



Sign flow Entering $(4,0)$ 60 90 $(1,3)$ -leaves



Optimum solution:

$x_{15} = 100$: Hire 100 at the start of period 1 for the entire horizon

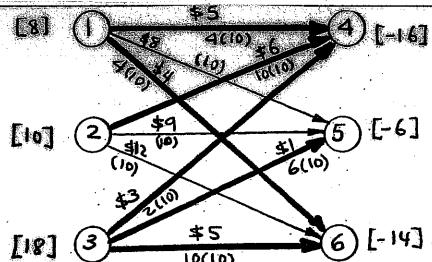
$x_{25} = 20$: Hire 20 at the start of period 2 for the end of the horizon

$x_{45} = 50$ Hire 50 at the start of period 4 for one period only

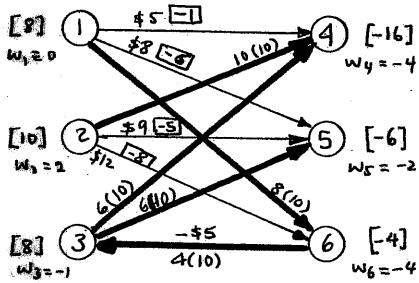
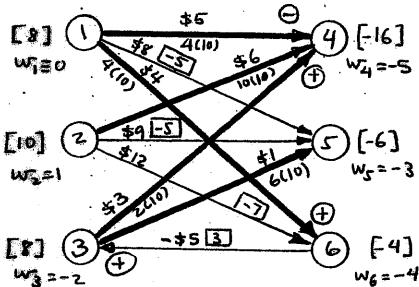
$x_{43} = 40$ means that period 4 (March) will carry a surplus of 40 workers ($= 120 - 80 = 40$).

Total cost = \$30,600

Set 20.1c



Substitute $x_{36} = 10 - x_{63}$ to produce a spanning tree. $C_{63} = -\$5$



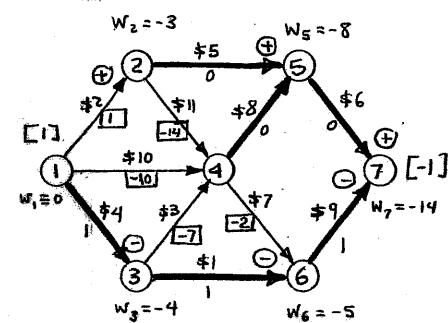
Optimum Solution: Cost \$146

$$x_{16} = 8,$$

$$x_{24} = 10$$

$$x_{34} = 6$$

$$x_{35} = 6, x_{36} = 10 - x_{63} = 10 - 4 = 6$$

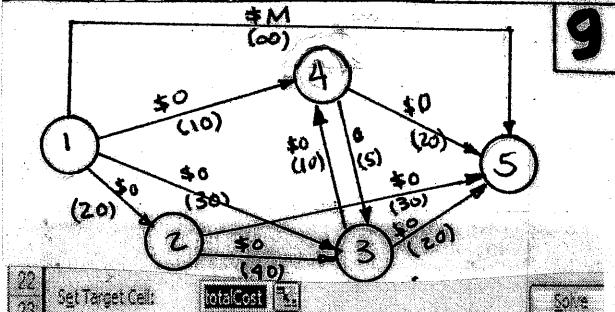
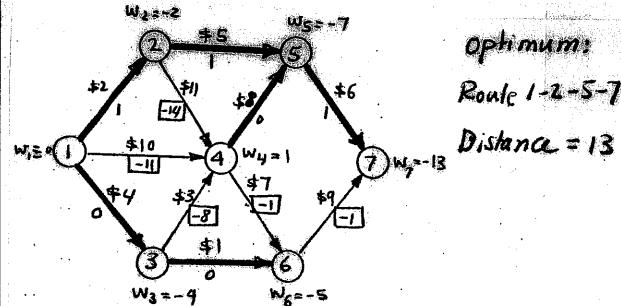


x_{12} enters the train

$x_{13}, x_{36}, \text{ or } x_{67}$ can leave. Select x_{67}

continued...

20-14



22	Set Target Cell:	TotalCost	Min	Solve
23	Equal To:	(1)	Max	Close
24	By Changing Cells:			
25	solution			Guess
26	Subject to the Constraints:			Options
27	netFlow = 0			Add
28	solution <= capacity			Remove
29	solution >= 0			
30				

Cell	Formula	Copy to
F15	=SUMIF(B3:E3,>0,B15:E15)-F3	F16:F18
B20	=SUMIF(B9:B12,>0,B15:B18)-B7	C20:E20
G16	=OFFSET(A\$20,0,ROW(A1))	G17:G19
H15	=F15-G15	H18:H19
G20	=SUMPRODUCT(unitCost,solution)	

Solver Minimum-Cost Capacitated Model (Problem 6.5c-9)							
capacity	N2	N3	N4	N5	Range	Cells	
N1	20	30	10	999999	1000	Capacity	B3:E6
N2		40		30		unitCost	B9:E12
N3			10	20		solution	B15:E18
N4		5		20		netFlow	H15:H19
					1000	totalCost	G20
unitCost	N2	N3	N4	N5			
N1	0	0	0	999999			
N2	0	0	0	0			
N3	0	0	0	0			
N4	0	0	0	0			
solution	N2	N3	N4	N5	outFlow	inFlow	netFlow
N1	20	30	10	940	1E-07	0	9.8E-08
N2	0	0	0	20	20	20	-1E-11
N3	0	0	10	20	30	30	-8E-10
N4	0	0	0	20	20	20	3.2E-09
N5					0	1E-07	-1E-07
	20	30	20	1E-07	totalCos	9.4E+08	

continued...

Set 20.1c

Optimal solution:

$$\begin{aligned} N1-N2 &= 20, N1-N3 = 30, N1-N4 = 10, \\ N2-N5 &= 20, N3-N4 = 10, N3-N5 = 20, \\ N4-N5 &= 20. \text{ Maximum flow} = 60 \end{aligned}$$

(a) AMPL: See file amplProb6.5c-10a.txt.

10

Solver:

totalCost	=SUMPRODUCT(unitCost,solution)				F	G	H
A	B	C	D	E	F	G	H
Solver Minimum-Cost Capacitated Model (Example 6.5-4)							
1 capacity	N1	N2	N3	N4			
2	110	95	125	100	430		
3 N0							
4 N1		9999					
5 N2	9999		9999				
6 N3		9999		9999			
7 N4			9999				
8	100	110	95	125			
9 unitCost	N1	N2	N3	N4			
10 N0	24	26	21	24			
11 N1		1					
12 N2	1.5		1				
13 N3		1.5		1			
14 N4			1.5				
15							
16 solution	N1	N2	N3	N4	outFlow	inFlow	netFlow
17 N0	110	95	125	100	-7.5E-10	0	8E-10
18 N1	0	10	0	0	10	10	5.1E-11
19 N2	0	0	0	0	0	-8E-11	7.8E-11
20 N3	0	5	0	25	30	30	4.5E-10
21 N4	0	0	0	0	0	-2E-10	1.7E-10
22	10	-8E-11	30	-2E-10	totalCost=	10177.5	

Solver Parameters

Set Target Cell:	totalCost	Solve
Equal To:	<input type="radio"/> Max <input checked="" type="radio"/> Min <input type="radio"/> Value of: 0	Close
By Changing Cells:	solution	Guess Options
Subject to the Constraints:	netFlow = 0 solution <= Capacity solution >= 0	
	Add	Change

Optimum solution:

Period 1: Produce 110, surplus 10

Period 2: Produce 95, shortage 15

Period 3: Produce 125, surplus 30

Period 4: Produce 100, shortage 25

Total shortage = 15+25=40

Total surplus = 10+30=40

Cost = \$10,177.50

(c) Solver

	A	B	C	D	E
1 Problem 6.5c-10(c)					
2 capacity	n4	n5	n6		
3 n1	10	10	10	8	
4 n2	10	10	10	10	
5 n3	10	10	10	18	
6	16	6	14		
7 unitCost	n4	n5	n6		
8 n1	5	8	4		
9 n2	6	9	12		
10 n3	3	1	5		
11 solution	n4	n5	n6	rowSum	
12 n1	0	0	8	0	
13 n2	10	0	0	0	
14 n3	6	6	6	0	
15 colSum	0	0	0		
16 totalCost	146				

Solver Parameters

Set Target Cell:	totalCost	
Equal To:	<input type="radio"/> Max <input checked="" type="radio"/> Min <input type="radio"/> Value of: 0	
By Changing Cells:	solution	
Subject to the Constraints:	colSum = 0 rowSum = 0 solution <= capacity solution >= 0	

Optimum solution:

x16 = 8

x24 = 10

x34 = 6

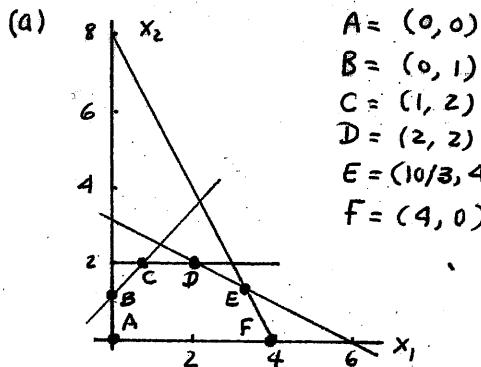
x35 = 6

x36 = 6

Cost = \$146

continued...

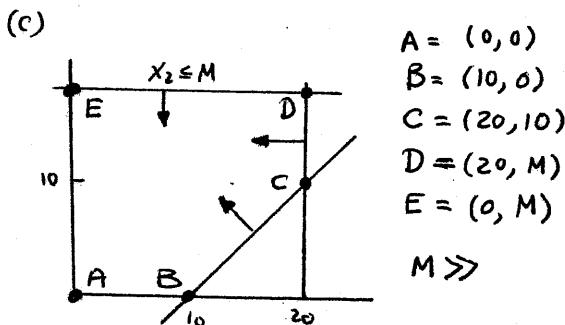
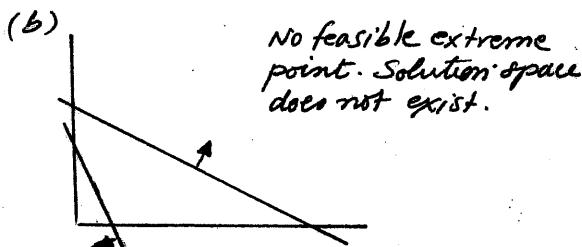
Set 20.2a



$$(x_1, x_2) = \alpha_1(0,0) + \alpha_2(0,1) + \alpha_3(1,2) + \alpha_4(2,2) + \alpha_5(10/3, 4/3) + \alpha_6(4,0)$$

$$\alpha_1 + \alpha_2 + \dots + \alpha_6 = 1$$

$$\alpha_j \geq 0, j=1,2,\dots,6$$

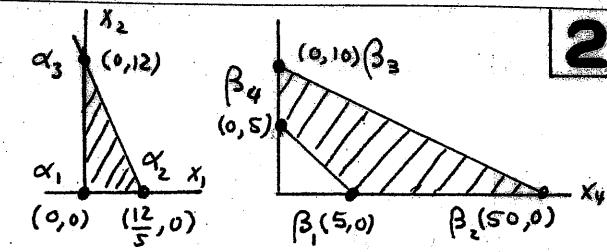


$$(x_1, x_2) = \alpha_1(0,0) + \alpha_2(10,0) + \alpha_3(20,10) + \alpha_4(20,M) + \alpha_5(0,M)$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1$$

$$\alpha_j \geq 0, j=1,2,\dots,5$$

continued...



$$(x_1, x_2) = \alpha_1(0,0) + \alpha_2(12/5,0) + \alpha_3(0,12) \\ = (12/5 \alpha_2 + 12 \alpha_3)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \alpha_1, \alpha_2, \alpha_3 \geq 0$$

$$(x_4, x_5) = \beta_1(5,0) + \beta_2(50,0) + \beta_3(0,10) \\ + \beta_4(0,5) \\ = (5\beta_1 + 50\beta_2, 10\beta_3 + 5\beta_4)$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1, \beta_1, \beta_2, \beta_3, \beta_4 \geq 0$$

Thus, the original problem can be expressed in terms of the α 's and the β 's as

$$\text{Max } Z = (0, \frac{36}{5}, 60, 5, 50, 10, 5)(\alpha_1, \alpha_2, \alpha_3, \beta_1, \dots, \beta_4)^T$$

subject to

$$\begin{pmatrix} 0 & 12/5 & 12 & 5 & 50 & 10 & 5 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ S_1 \end{pmatrix} = \begin{pmatrix} 40 \\ 1 \\ 1 \end{pmatrix}$$

$$\alpha_i, \beta_j, S_i \geq 0$$

M-method: $M = 1000$

Iteration 1:

Iterat	alfa1	alfa2	alfa3	beta1	beta2	beta3	x6
z (max)	-1000.00	-1007.20	(1060.00)	-1005.00	(-1050.00)	(-1010.00)	
sx8	0.00	2.40	12.00	5.00	50.00	10.00	
Rx9	1.00	1.00	1.00	1.00	0.00	0.00	
Rx10	0.00	0.00	0.00	1.00	1.00	1.00	
beta4							
	x7	sx8	Rx9	Rx10			Solution
Basic	x1	x2	x3	x4	x5	x6	
z (max)	60.00	52.80	0.00	-1005.00	(-1050.00)	(-1010.00)	
x3	-12.00	-9.60	0.00	5.00	50.00	10.00	
x8	1.00	1.00	1.00	0.00	0.00	0.00	
Rx9	0.00	0.00	1.00	0.00	1.00	1.00	
Rx10	1.00	0.00	0.00	1.00	1.00	1.00	

$$\min \{z_j - c_j\} = -1060 = -60 - M, \text{ corresponds to } \alpha_3$$

$$\min \{z_j - c_j\} = -1050 = -50 - M, \text{ corresponds to } \beta_2$$

Thus, α_3 enters solution (its extreme pt is $(0,12)$)

Iteration 2:

Iterat	alfa1	alfa2	alfa3	beta1	beta2	beta3	x6
z (max)	60.00	52.80	0.00	-1005.00	(-1050.00)	(-1010.00)	
x3	-12.00	-9.60	0.00	5.00	50.00	10.00	
x8	1.00	1.00	1.00	0.00	0.00	0.00	
Rx9	0.00	0.00	1.00	0.00	1.00	1.00	
Rx10	1.00	0.00	0.00	1.00	1.00	1.00	
beta4							
	x7	sx8	Rx9	Rx10			Solution
Basic	x1	x2	x3	x4	x5	x6	
z (max)	-1005.00	0.00	1060.00	0.00	540.00		
sx8	5.00	1.00	-12.00	0.00	28.00		
x3	0.00	0.00	1.00	0.00	1.00		
Rx9	1.00	0.00	0.00	1.00	1.00		

$$\min \{z_j - c_j\} = 0, \text{ corresponds to } \alpha_3$$

continued...

Set 20.2a

$$\min \{z_j - c_j\} = -1050 = -50M, \text{ corresponds to } \beta_2$$

Thus, β_2 enters solution (into extreme point is $(50, 0)$)

Iteration 3:

Iterat	Basic	alfa1	alfa2	alfa3	beta1	beta2	beta3	x6
Z (max)	x1	-192.00	-148.00	0.00	-900.00	0.00	-800.00	
x5	-0.24	-0.18	0.00		0.18	1.00	0.20	
x3	1.00	1.00	1.00		0.00	0.00	0.00	
Rx10	0.24	0.19	0.00		0.90	0.00	0.80	
Basic	x7	x8			Rx9	Rx10	Solution	
Z (max)	-900.00	21.00			808.00	0.00	-552.00	
x5	0.10	0.02			-0.24	0.00	0.56	
x3	0.00	0.00			1.00	0.00	1.00	
Rx10	0.90	-0.02			0.24	1.00	0.44	

$$\min \{z_j - c_j\} = -192 = -\frac{13M}{50} + 48, \text{ corresponds to } \alpha_i$$

$$\min \{z_j - c_j\} = -900 = -\frac{9M}{10}, \text{ corresponds to } \beta_1$$

$$(z_j - c_j) \text{ for } x_8 = 21$$

Iteration 4: optimum.

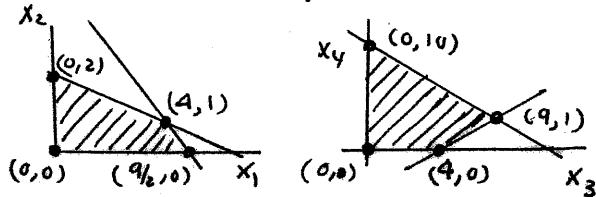
Iterat	Basic	alfa1	alfa2	alfa3	beta1	beta2	beta3	x6
Z (max)	x2	48.00	43.00	0.00	0.00	0.00	0.00	0.00
x5	-0.27	-0.27	0.00		0.00	1.00	0.11	
x3	1.00	1.00	1.00		0.00	0.00	0.00	
x4	0.27	0.21	0.00		1.00	0.00	0.89	
Basic	x7	x8			Rx9	Rx10	Solution	
Z (max)	0.00	-1.00			1048.00	1000.00	88.00	
x5	0.00	0.02	-0.27	-0.11	0.51			
x3	0.68	0.00	1.00	0.00	1.00			
x4	1.00	-0.02	0.27	1.11	0.49			

Optimum solution: $Z = 88$

Variable	Associated extreme point
$x_1 = \alpha_1 = 0$	$(0, 0)$
$x_2 = \alpha_2 = 0$	$(12/5, 0)$
$x_3 = \alpha_3 = 1$	$(0, 12)$
$x_4 = \beta_1 = .4889$	$(5, 0)$
$x_5 = \beta_2 = .5111$	$(50, 0)$
$x_6 = \beta_3 = 0$	$(0, 10)$
$x_7 = \beta_4 = 0$	$(0, 5)$

$$D_1 = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}, \underline{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, b = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

$$D_2 = \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}, \underline{x}_2 = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$$



3

Continued...

$$(x_1, x_2) = \alpha_2(9/2, 0) + \alpha_3(4, 1) + \alpha_4(0, 2)$$

$$= \left(\frac{9}{2}\alpha_2 + 4\alpha_3, \alpha_3 + 2\alpha_4 \right)$$

$$(x_3, x_4) = \beta_2(4, 0) + \beta_3(9, 1) + \beta_4(0, 10)$$

$$= (4\beta_2 + 9\beta_3, \beta_3 + 10\beta_4)$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0, \beta_1, \beta_2, \beta_3, \beta_4$$

$$Z = 0 \quad \frac{9}{2} \quad 7 \quad 6 \quad 0 \quad 20 \quad 47 \quad 20 \\ 0 \quad 22 \frac{1}{2} \quad 23 \quad 6 \quad 0 \quad 16 \quad 36 \quad 0 \quad \geq 10$$

$$1 \quad 1 \quad = 1$$

$$\alpha_i \geq 0, \beta_i \geq 0, i = 1, 2, 3, 4$$

TORA optimum solution:

$$\alpha_3 = 1, \beta_3 = 1$$

all other variables = 0

Thus,

$$(x_1, x_2) = (4, 1)$$

$$(x_3, x_4) = (9, 1) \quad Z = 54$$

Subproblem j=1:

$$x_1 = (x_1, x_2), \underline{c}_1 = (1, 3), A_1 = (5, 3)$$

$$D_1 = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}, b_1 = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

Subproblem j=2:

$$x_2 = (x_3, x_4), \underline{c}_2 = (5, 2), A_2 = (4, 0)$$

Starting solution:

Use R_1, R_2, R_3 as starting artificial vars.

$$\underline{B} = (R_1, R_2, R_3)^T = (10, 1, 1)^T$$

$$B = B^{-1} = I, \underline{c}_B = (-M, -M, -M)$$

Iteration 1:

$$j=1: \text{Min } \underline{w}_1 = -(5M+1)x_1 - (3M+3)x_2 - M$$

$$\text{Optimum: } \hat{x}_{11} = (4, 1), \underline{w}_1^* = -24M - 7$$

$$j=2: \text{Min } \underline{w}_2 = -(4M+5)x_3 - 2x_4 - M$$

$$\text{Optimum: } \hat{x}_{21} = (9, 1), \underline{w}_2^* = -37M - 47$$

$$Z_S - C_S = (-M, -M, -M) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = M > 0$$

Continued...

20-17

Set 20.2a

Thus, β_3 associated with $\hat{X}_{31} = (9, 1)$ enters the solution:

$$P_{21} = \begin{pmatrix} A_2 \hat{X}_{21} \\ 0 \end{pmatrix} = \begin{pmatrix} 36 \\ 0 \\ 1 \end{pmatrix}, \bar{B}^{-1} P_{21} = \begin{pmatrix} 36 \\ 0 \\ 1 \end{pmatrix}$$

$$\theta = \min \left\{ \frac{10}{36}, -\frac{1}{1} \right\} = \frac{10}{36}, R_1 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1/36 & 0 & 0 \\ 0 & 1 & 0 \\ -1/36 & 0 & 1 \end{pmatrix}$$

$$x_B = (\beta_3, R_2, R_3)^T = \bar{B}_{\text{next}}^{-1} (10, 1, 1)^T = (10/36, 1, 26/36)^T.$$

$$C_{21} = C_2 \hat{X}_{21} = (5, 2) \begin{pmatrix} 9 \\ 1 \end{pmatrix} = 47$$

Iteration 2:

$$C_B = (47, -M, -M)$$

j=1:

$$\text{Min } w_1 = \left(\frac{199}{36} + \frac{5M}{36} \right) x_1 + \left(\frac{33}{36} + \frac{3M}{36} \right) x_3 - M$$

$$\text{Optimum: } \hat{X}_{12} = (0, 0), w_1^* = -M$$

j=2:

$$\text{Min } w_2 = \left(\frac{8}{36} + \frac{4M}{36} \right) x_3 - 2x_4$$

$$\text{Optimum: } \hat{X}_{22} = (0, 10), w_2^* = -20 - M$$

$$Z_S - C_S = (47, -M, -M) \bar{B}^{-1} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = -\frac{47}{36} - \frac{M}{36}$$

Thus, β_4 associated with $\hat{X}_{22} = (0, 10)$ enters the solution:

$$P_{22} = \begin{pmatrix} A_2 \hat{X}_{22} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \bar{B}^{-1} P_{22} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\theta = \min \left\{ -\frac{1}{1}, \frac{26/36}{1} \right\} = \frac{26}{36}, R_3 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1/36 & 0 & 0 \\ 0 & 1 & 0 \\ -1/36 & 0 & 1 \end{pmatrix}$$

$$x_B = (\beta_3, R_2, \beta_4)^T = \bar{B}^{-1} \begin{pmatrix} 10 \\ 1 \\ 1 \end{pmatrix} = \left(\frac{10}{36}, 1, \frac{26}{36} \right)^T$$

$$C_{22} = C_2 \hat{X}_{22} = (5, 2) \begin{pmatrix} 0 \\ 10 \end{pmatrix} = 20$$

Iteration 3:

$$C_B = (47, -M, 20)$$

$$j=1: \text{Min } w_1 = \frac{99}{36} x_1 - \frac{27}{36} x_2 - M$$

$$\text{Optimum: } \hat{X}_{13} = (0, 2), w_1^* = \frac{3}{2} - M$$

$$j=2: \text{Min } w_2 = -2x_3 - 2x_4 + 20$$

$$\text{Optimum: } \hat{X}_{23} = (9, 1)^T, w_2^* = 0$$

$$Z_S - C_S = (47, -M, 20) \bar{B}^{-1} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = -\frac{3}{4}$$

Thus, α_3 associated with $\hat{X}_{13} = (0, 2)$ enters the solution:

$$P_{13} = \begin{pmatrix} A_1 \hat{X}_{13} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (5, 3) \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}$$

$$\bar{B}^{-1} P_{13} = (1/6, 1, -1/6)^T$$

$$\theta = \min \left\{ \frac{10/36}{1/6}, \frac{1}{1}, -\frac{1}{6} \right\} = 1, R_2 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1/36 & -1/6 & 0 \\ 0 & 1 & 0 \\ -1/36 & 1/6 & 1 \end{pmatrix}$$

$$x_B = (\beta_3, \alpha_4, \beta_4)^T = \left(\frac{4}{36}, 1, \frac{32}{36} \right)^T$$

$$C_{13} = C_1 \hat{X}_{13} = (1, 3) \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 6$$

Iteration 4:

$$C_B = (47, 6, 20)$$

$$j=1: \text{Min } w_1 = \frac{11}{4} x_1 - \frac{3}{4} x_2 + \frac{3}{4}$$

$$\text{Optimum: } \hat{X}_{14} = (0, 2), w_1^* = 0$$

$$j=2: \text{Min } w_2 = -2x_3 - 2x_4 + 20$$

$$\text{Optimum: } \hat{X}_{24} = (0, 10), w_2^* = 0$$

$$Z_S - C_S = (47, 6, 20) \bar{B}^{-1} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = -\frac{3}{4}$$

Thus, S enters the solution.

$$\bar{B}^{-1} P_S = \bar{B}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \left(-\frac{1}{36}, 0, \frac{1}{36} \right)^T$$

$$\theta = \min \left\{ -\frac{1}{36}, \frac{32/36}{1/36} \right\} = 32, \beta_4 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 6 & 36 \end{pmatrix}$$

$$x_B = (\beta_3, \alpha_4, S)^T = \bar{B}^{-1} \begin{pmatrix} 10 \\ 1 \\ 1 \end{pmatrix} = (1, 1, 32)^T$$

Iteration 5:

$$C_B = (47, 6, 0)$$

$$j=1: \text{Min } w_1 = -x_1 - 3x_2 + 6$$

$$\text{Optimum: } \hat{X}_{15} = (4, 1)^T, w_1^* = -1$$

$$j=2: \text{Min } w_2 = -5x_3 - 2x_4 + 47$$

$$\text{Optimum: } \hat{X}_{25} = (9, 1), w_2^* = 0$$

Thus, α_3 associated with $\hat{X}_{15} = (4, 1)^T$ enters the solution:

$$P_{15} = \begin{pmatrix} (5, 3) \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 23 \\ 1 \\ 0 \end{pmatrix}$$

$$\bar{B}^{-1} P_{15} = (0, 1, -17)^T$$

continued...

continued...

Set 20.2a

$$\theta = \min \left\{ -\frac{1}{1}, -\frac{1}{1} \right\} = 1, \alpha_4 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 23 & 36 \end{pmatrix}, C_{15} = (1, 3) \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = 7$$

$$X_B = (\beta_3, \alpha_3, S)^T = \bar{B}^{-1} \begin{pmatrix} 10 \\ 1 \\ 1 \end{pmatrix} = (1, 1, 49)$$

Iteration 6:

$$g = (47, 7, 0)$$

$$j=1: \min w_1 = -x_1 - 3x_2 + 7$$

$$\text{Optimum: } \hat{X}_{16} = (4, 1), w_1^* = 0$$

$$j=2: \min w_2 = -5x_3 - 2x_4 + 47$$

$$\text{Optimum: } \hat{X}_{26} = (9, 1), w_2^* = 0$$

Optimum is reached!

$(\beta_3, \alpha_3, S) = (1, 1, 49)$ translates
to $(x_1, x_2) = (4, 1)$ and $(x_3, x_4) = (9, 1)$

$$Z = 54$$

j=1:

$$\underline{X}_1 = (x_1, x_2)^T$$

$$C_1 = (6, 7)$$

$$D_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\underline{A}_1 = (1, 1)$$

$$\underline{b}_1 = \begin{pmatrix} 10 \\ 8 \end{pmatrix}$$

j=2:

$$\underline{X}_2 = (x_3, x_4)^T$$

$$C_2 = (3, 5)$$

$$D_2 = (5, 1)$$

$$\underline{A}_2 = (1, 1)$$

$$\underline{b}_2 = (12)$$

j=3:

$$\underline{X}_3 = (x_5, x_6)^T$$

$$C_3 = (1, 1)$$

$$D_3 = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$$

$$\underline{A}_3 = (1, 1)$$

$$\underline{b}_3 = \begin{pmatrix} 5 \\ 50 \end{pmatrix}$$

Starting solution:

$$X_B = (S_1, R_1, R_2, R_3)^T = (50, 1, 1, 1)^T$$

S_1 is the slack of the common constraint.

Iteration 0:

$$G_B = (0, -M, -M, -M)$$

$$B = \bar{B}^{-1} = I$$

5

Iteration 1:

j=1:

$$\text{minimize } w_1 = -6x_1 - 7x_2 - M$$

$$\text{Solution: } \hat{X}_{11} = (2, 8)^T$$

$$w_1^* = -68 - M$$

j=2:

$$\text{minimize } w_2 = -3x_3 - 5x_4 - M$$

$$\text{Solution: } \hat{X}_{21} = (0, 12)^T$$

$$w_2^* = -60 - M$$

j=3:

$$\text{minimize } w_3 = -x_5 - x_6 - M$$

$$\text{Solution: } \hat{X}_{31} = (50, 0)^T$$

$$w_3^* = -50 - M$$

β_{1r} associated with \hat{X}_{11} enters the solution

$$\bar{P}_{11} = \begin{pmatrix} (1, 1)(2, 8) \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{B}^{-1} \bar{P}_{11} = (10, 1, 0, 0)^T$$

$$\theta = \min \left\{ \frac{50}{10}, \frac{1}{1}, -, - \right\} = 1, R_1 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1 & -10 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X_B = (S_1, \beta_{11}, R_2, R_3) = (40, 1, 1, 1)$$

$$C_{11} = \underline{C}_1 \hat{X}_{11} = 68$$

$$\text{Iteration 2: } G_B = (0, 68, -M, -M)$$

j=1:

$$\text{minimize } w_1 = -6x_1 - 7x_2 - M$$

$$\text{Solution: } \hat{X}_{12} = (2, 8)^T$$

$$w_1^* = 0$$

j=2:

$$\text{minimize } w_2 = -3x_3 - 5x_4 - M$$

Continued...

Continued...

Set 20.2a

Solution: $\hat{Y}_{22} = (0, 12)^T$
 $w_2^* = -60 - M$

j=3:

minimize $w_3 = -x_5 - x_6 - M$

Solution: $\hat{X}_{32} = (50, 0)^T$
 $w_3^* = -50 - M$

β_{22} associated with \hat{Y}_{22} enters
the solution.

$$P_{22} = \begin{pmatrix} (1,1)(0,12)^T \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\bar{B}^{-1} P_{22} = (12, 0, 1, 0)^T$$

$$\theta = \min \left\{ \frac{40}{12}, -\frac{1}{1}, -\frac{1}{1} \right\}; R_2 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1 & -10 & -12 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X_B = (s_1, \beta_{11}, \beta_{22}, R_3)^T = (28, 1, 1, 1)^T$$

$$C_{22} = C_2 \hat{X}_{22} = 60$$

Iteration 3: $B = (0, 68, 60, -M)$

j=1:

minimize $w_1 = -6x_1 - 7x_2 - M$

Solution: $\hat{X}_1 = (2, 8)^T$

$$w_1^* = 0$$

j=2:

minimize $w_2 = -3x_3 - 5x_4 - M$

Solution: $\hat{X}_{23} = (0, 12)^T$

$$w_2^* = 0$$

j=3:

minimize $w_3 = -x_5 - x_6 - M$

Solution: $\hat{X}_{33} = (50, 0)^T$

$$w_3^* = -50 - M$$

β_{33} associated with \hat{Y}_{33} enters solution

$$P_{33} = \begin{pmatrix} (1,1)(50, 0)^T \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 1 \end{pmatrix}$$

$$\bar{B}^{-1} P_{33} = (50, 0, 0, 1)^T$$

$$\theta = \min \left\{ \frac{28}{50}, -\frac{1}{1}, -\frac{1}{1}, \frac{1}{1} \right\}; S_1 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1/50 & -10/50 & -12/50 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/50 & 10/50 & 12/50 & 1 \end{pmatrix}$$

$$X_B = (\beta_{33}, \beta_{11}, \beta_{22}, R_3)^T = \left(\frac{14}{25}, 1, 1, \frac{11}{25} \right)^T$$

$$C_{33} = C_3 \hat{X}_{33} = 50$$

Iteration 4: $B = (50, 68, 60, -M)$

j=1:

minimize $w_1 = \left(\frac{M}{50} - 5 \right) x_1 + \left(\frac{M}{50} - 6 \right) x_2 + 50 - \frac{M}{5}$

Solution: $\hat{X}_{41} = (0, 0)^T$

$$w_1^* = 50 - M/5$$

j=2:

minimize $w_2 = \frac{50+M}{50} (x_3 + x_4) - 540$

Solution: $\hat{X}_{42} = (0, 0)^T$

$$w_2^* = -540$$

j=3:

minimize $w_3 = \frac{M}{50} (x_5 + x_6) - M$

Solution: $\hat{X}_{43} = (5, 0)^T$

$$w_3^* = -9M$$

S₁:

$$z_{S_1} - c_{S_1} = B^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0 = 1 + M/50$$

β_{43} associated with \hat{X}_{43} enters solution

$$P_{43} = \begin{pmatrix} (1,1)(5,0)^T \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\bar{B}^{-1} P_{43} = \left(\frac{1}{10}, 0, 0, \frac{9}{10} \right)^T$$

continued...

continued...

Set 20.2a

$$\theta = \min \left\{ \frac{28/50}{1/10}, \dots, \frac{22/50}{9/10} \right\}$$

$= 22/45$, R_3 leaves

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1/45 & -10/45 & -12/45 & -5/45 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/45 & 10/45 & 12/45 & 50/45 \end{pmatrix}$$

$$X_B = (\beta_{33}, \beta_{11}, \beta_{22}, \beta_{43})$$

$$= (23/45, 1, 1, 22/45)$$

$$C_{43} = (1, 1, 0, 0)(5, 0, 0, 45)^T = 5$$

It can be shown that iteration 5 will prove optimality.

Optimum Solution:

$$(x_1, x_2) = I(2, 8) = (2, 8)$$

$$(x_3, x_4) = I(0, 12) = (0, 12)$$

$$(x_5, x_6) = \frac{23}{45}(50, 0) + \frac{22}{45}(5, 0) = (28, 0)$$

$$Z = g X_B = (50, 68, 60, 50) \left(\frac{23}{45}, 1, 1, \frac{22}{45} \right) = 156$$

Since the original problem is minimization, we must maximize w_j for each subproblem j. 6

j=1:

$$X_1 = (x_1, x_2)^T$$

$$C_1 = (5, 3)$$

$$A_1 = (1, 1)$$

$$D_1 = \begin{pmatrix} 5 & 1 \\ 5 & -1 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} 20 \\ 5 \end{pmatrix}$$

j=2:

$$X_2 = (x_3, x_4)$$

$$C_2 = (8, -5), A_2 = (1, 1)$$

$$D_2 = (1, 1), b_2 = 20$$

Iteration 0:

$$X_B = (R_1, R_2, R_3)^T = (25, 1, 1)^T$$

$$B = B^{-1} = I, C_B = (M, M, M)$$

Iteration 1:

j=1:

$$\text{maximize } w_1 = (-5+M)x_1 + (M-3)x_2 + M$$

$$\text{Solution: } \hat{X}_1 = (5/2, 15/2)^T, w_1^* = 11M - 35$$

j=2:

$$\text{maximize } w_2 = (M-8)x_3 + (M+5)x_4 + M$$

$$\text{Solution: } \hat{X}_{21} = (0, 20)^T$$

$$w_2^* = 21M + 100$$

$$Z_{S_1} - C_{S_1} = (M, M, M) I \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = -M$$

β_{21} associated with \hat{X}_{21} enters solution

$$P_{21} = \begin{pmatrix} (1, 1) & (0, 20)^T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 1 \end{pmatrix}$$

$$\bar{B} P_{21} = (20, 0, 1)^T$$

$$\theta = \min \left\{ \frac{25}{20}, -, \frac{1}{1} \right\} = 1, R_3 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1 & 0 & -20 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_B = \bar{B}^{-1}(25, 1, 1)^T = (5, 1, 1)^T$$

$$= (R_1, R_2, \beta_{21})^T$$

$$C_{21} = \sum_{j=2} C_j \hat{X}_{21} = (8, -5)(0, 20)^T = -100$$

Iteration 2: $\bar{B} = (M, M, -100)$

j=1:

$$\text{maximize } w_1 = (-5+M)x_1 + (M-3)x_2 + M$$

$$\text{Solution: } \hat{X}_{12} = (5/2, 15/2)^T$$

$$w_1^* = 11M - 35$$

j=2:

$$\text{maximize } w_2 = (M-8)x_3 + (M+5)x_4 - 20M - 100$$

$$\text{Solution: } \hat{X}_{22} = (0, 20)^T$$

$$w_2^* = 0$$

$$Z_{S_1} - C_{S_1} = -M$$

β_{12} associated with $\hat{X}_{12} (5/2, 15/2)$ enters the solution.

$$P_{12} = \begin{pmatrix} (1, 1) & (5/2, 15/2)^T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$$

$$\bar{B} P_{12} = (10, 1, 0)^T$$

$$\theta = \min \left\{ \frac{5}{10}, \frac{1}{1}, - \right\} = \frac{1}{2}, R_1 \text{ leaves}$$

Continued...

Set 20.2a

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1/10 & 0 & -2 \\ -1/10 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_B = (\beta_{12}, R_2, \beta_{21})^T = (1/2, 1/2, 1)$$

$$C_{12} = C_1 \hat{X}_{12} = (5, 3) (5/2, 15/2)^T = 35$$

$$\underline{\text{Iteration 3: }} G_B = (35, M, -100)$$

j=1:

$$\text{maximize } w_1 = -\left(\frac{M}{10} + \frac{3}{2}\right)x_1 - \left(\frac{M}{10} - \frac{1}{2}\right)x_2 + M$$

$$\underline{\text{Solution: }} \hat{X}_{13} = (1, 0)^T$$

$$w_1^* = .9M - 3/2$$

j=2:

$$\text{maximize } w_2 = -\left(\frac{9}{2} + \frac{M}{10}\right)x_3 - \left(\frac{M}{10} - \frac{17}{2}\right)x_4 - 800$$

$$\underline{\text{solution: }} \hat{X}_{23} = (0, 20)^T$$

$$w_2^* = -630 - 2M$$

$$Z_{S_1} - C_{S_1} = (35, M, -100) \bar{B}^{-1} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = M/10 - 1/2$$

β_{13} associated with \hat{X}_{13} enters solution

$$P_{13} = \begin{pmatrix} (1, 1)(1, 0)^T \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\bar{B}^{-1} P_{13} = (1/10, 9/10, 0)^T$$

$$\Theta = \min \left\{ \frac{1/2}{1/10}, \frac{1/2}{9/10}, -3 \right\} = 1/9, R_2 \text{ leaves}$$

$$\bar{B}_{\text{next}}^{-1} = \begin{pmatrix} 1/9 & -1/9 & -20/9 \\ -1/9 & 10/9 & 20/9 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_B = (\beta_{12}, \beta_{13}, \beta_{21})^T = \bar{B}^{-1} (25, 1, 1)^T = (4/9, 5/9, 1)^T$$

$$C_{13} = C_1 \hat{X}_{13} = (5, 3) (1, 0)^T = 5$$

$$\underline{\text{Iteration 4: }} G_B = (35, 5, -100)$$

j=1:

$$\text{maximize } w_1 = -5/3x_1 + 1/3x_2 + 5$$

$$\underline{\text{solution: }} \hat{X}_{14} = (5/2, 15/2)^T, w_1^* = 10/3$$

continued...

j=2:

$$\text{maximize } w_2 = -\frac{14}{3}x_3 + \frac{25}{3}x_4 - 800$$

$$\underline{\text{solution: }} \hat{X}_{24} = (0, 20)^T$$

$$w_2^* = -633\frac{1}{3}$$

$$Z_{S_1} - C_{S_1} = (35, 5, -100) \bar{B}^{-1} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} - 0 = -\frac{30}{9}$$

β_{14} associated with \hat{X}_{14} enters the solution
After one iteration, we get.

$$(x_1, x_2) = 1 \left(\frac{5}{2}, \frac{15}{2} \right) + 0(1, 0) = \left(\frac{5}{2}, \frac{15}{2} \right)$$

$$(x_3, x_4) = 1(0, 20) = (0, 20), Z = 195$$

Dual Problem:

$$\text{maximize } Z = 8x_1 + 2x_2 + 4x_3 + 10x_4$$

s.t.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &\leq 10 \\ 4x_1 + x_2 &\leq 2 \\ -x_1 + x_2 &\leq 4 \\ x_3 + 2x_4 &\leq 8 \\ x_3 - x_4 &\leq 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

j=1:

$$\underline{X}_1 = (x_1, x_2)^T$$

$$\underline{C}_1 = (8, 2), \quad \underline{A}_1 = (1, 2)$$

$$\underline{D}_1 = \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix}, \quad \underline{b}_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

j=2:

$$\underline{X}_2 = (x_3, x_4)^T$$

$$\underline{C}_2 = (4, 10), \quad \underline{A}_2 = (3, 1)$$

$$\underline{D}_2 = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad \underline{b}_2 = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

Iteration 0:

$$X_B = (S_1, R_1, R_2)^T = (10, 1, 1)$$

$$B = B^{-1} = I$$

Iteration 1: $G_B = (0, -M, -M)$

j=1:

$$\text{minimize } w_1 = -8x_1 - 2x_2 - M$$

$$\underline{\text{solution: }} \hat{X}_{11} = (1/2, 0), \text{ or}$$

$$\hat{X}_{11} = (0, 2)$$

$$w_1^* = -4 - M$$

continued...

Set 20.2a

j=2:

$$\text{minimize } w_2 = -4x_3 - 10x_4 - M$$

Solution: $\hat{X}_{21} = (0, 4, 0, 5)^T$

$$w_2^* = -40 - M$$

β_{21} associated with \hat{Y}_{21} enters

$$P_{21} = \begin{pmatrix} (3, 1)(0, 4)^T \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$$\bar{B}^{-1} P_{21} = (4, 0, 1)^T$$

$$\theta = \min \left\{ \frac{10}{4}, -\frac{1}{1} \right\} = 1, R_2 \text{ leaves}$$

$$\bar{B}_{new}^{-1} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_B = \bar{B}^{-1}(10, 1, 1)^T = (6, 1, 1)^T = (S_1, R_1, \beta_{21})$$

$$C_{21} = C_2 \hat{X}_{21} = (4, 10, 0, 0)(0, 4, 0, 5) = 40$$

Iteration 2: $\hat{g} = (0, -M, 40)$

j=1:

$$\text{minimize } w_1 = -8x_1 - 2x_2 - M$$

Solution: $\hat{X}_{12} = (1/2, 0)^T \quad \text{or}$

$$\hat{X}_{12} = (0, 2)^T$$

$$w_1^* = -4 - M$$

j=2:

$$\text{minimize } w_2 = -4x_3 - 10x_4 + 40$$

Solution: $\hat{X}_{22} = (0, 4)^T$

$$w_2^* = 0$$

β_{12} associated with $\hat{Y}_{12} = (0, 2)^T$

[or $(1/2, 0, 0, 9/2)$] enters the solution.

$$P_{12} = \begin{pmatrix} (1/2)(0, 2)^T \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

$$\bar{B}^{-1} P_{12} = (4, 1, 0)^T$$

$$\theta = \min \left\{ \frac{6}{4}, \frac{1}{1}, -3 \right\} = 1, R_1 \text{ leaves}$$

Continued...

$$\bar{B}_{new}^{-1} = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_B = (S_1, \beta_{12}, \beta_{21}) = (2, 1, 1)^T$$

$$C_{12} = C_1 \hat{X}_{12} = (8, 2)(0, 2)^T = 4$$

Iteration 3: $\hat{g} = (0, 4, 40)$

j=1:

$$\text{minimize } w_1 = -8x_1 - 2x_2 + 4$$

Solution: $\hat{X}_{13} = (1/2, 0)^T$

$$w_1^* = 0$$

j=2:

$$\text{minimize } w_2 = -4x_3 - 10x_4 + 40$$

Solution: $\hat{X}_{23} = (0, 4)^T$

$$w_2^* = 0$$

Solution is optimum!

$$(x_1, x_2) = 1(0, 2) = (0, 2)$$

$$(x_3, x_4) = 1(0, 4) = (0, 4)$$

The problem has an alternative solution, which can be determined using $\hat{Y}_{12} = (Y_2, 0)$ in place of $(0, 2)$ in iteration 2. The alternative solution is

$$x_1 = 1/2, x_2 = 0, x_3 = 0, x_4 = 4$$

To determine the primal solution, note that the basic dual variables as given above are

$$(x_2, x_4, S_1, S_3, S_5)$$

$$\text{or } (x_1, x_4, S_1, S_3, S_5)$$

where S_1 is the slack associated with the common constraint, S_3 is the slack for constraint 3, and S_5 is the slack for constraint 5. Thus,

Dual variable	Primal constraint equation
x_2	$2y_1 + y_2 + y_3 = 2$
x_4	$y_1 + 2y_4 - y_5 = 10$
S_1	$y_1 = 0$
S_3	$y_3 = 0$
S_5	$y_5 = 0$

Continued...

Set 20.2a

Solution: $y_1 = 0, y_2 = 2, y_3 = 0, y_4 = 5, y_5 = 0$

Consider the second alternative solution

Dual
variable Primal constraint equation

$$\begin{array}{lllll} x_1 & y_1 + 4y_2 - y_3 & & & = 8 \\ x_2 & y_1 + 2y_4 - y_5 & & & = 10 \\ s_1 & y_1 & & & = 0 \\ s_3 & y_3 & & & = 0 \\ s_5 & y_5 & & & = 0 \end{array}$$

Solution: $\bar{y}_1 = 0, \bar{y}_2 = 2, \bar{y}_3 = 0, \bar{y}_4 = 5, \bar{y}_5 = 0$

$$\text{objective value} = 2x_2 + 5x_4 \\ = 44$$

8

Let B be the current basis of the master problem and C_B the vector of the corresponding coefficients in the objective function. Thus, according to the revised simplex method, the current solution is optimal if for all nonbasic P_j^k ,

$$z_j^k - c_j^k = C_B B^{-1} P_j^k - c_j^k \geq 0$$

where, from the definition of the master problem,

$$c_j^k = C_j \hat{X}_j^k \quad \text{and} \quad P_j^k = \begin{pmatrix} r_0 & \left(\begin{array}{c} A_j \hat{X}_j^k \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{array} \right) \\ n & \end{pmatrix} \rightarrow (r_0 + j)\text{th place}$$

The expression for $z_j^k - c_j^k$ can be simplified as follows. Let

$$B^{-1} = \overbrace{(R_0 \mid V_1, V_2, \dots, V_j, \dots, V_n)}^{r_0 \quad n}$$

where R_0 is the matrix of size $(r_0 + n) \times r_0$ consisting of the first r_0 columns of B^{-1} , and V_j is the $(r_0 + j)$ th column of the same matrix B^{-1} . Thus

$$\begin{aligned} z_j^k - c_j^k &= (C_B R_0 A_j \hat{X}_j^k + C_B V_j) - C_j \hat{X}_j^k \\ &= (C_B R_0 A_j - C_j) \hat{X}_j^k + C_B V_j \end{aligned}$$

Set 20.3a

TORA optimal solution : ($M = 10$)

Final Iteration No.: 10
Objective Value = 0

Variable	Value
y	x1: 0.18182
	x2: 0.00000
	x3: 0.00000
w	x4: 0.09091
	x5: 0.00000
	x6: 0.63636
	x7: 0.09091
x	x8: 0.00000

$$y_1 = (M+1)x_1 = 11 \times 0.18182 \approx 2$$

$$y_2 = (M+1)x_2 = x_0 = 0$$

$$y_3 = (M+1)x_3 = 11x_0 = 0$$

$$w_1 = (M+1)x_4 = 11 \times 0.09091 \approx 1$$

$$w_2 = (M+1)x_5 = 11x_0 = 0$$

Primal:

$$\text{Maximize } z = 2y_1 + y_2$$

Subject to

$$y_1 - y_2 \leq 2$$

$$y_1 + 2y_2 \leq 4$$

$$y_1, y_2 \geq 0$$

Dual:

$$\text{Minimize } w = 2w_1 + 4w_2$$

Subject to

$$w_1 + w_2 \geq 2$$

$$-w_1 + 2w_2 \geq 1$$

$$w_1, w_2 \geq 0$$

Thus, conversion of the primal and dual constraints to equations yields:

$$2y_1 + y_2 - 2w_1 - 4w_2 = 0$$

$$y_1 - y_2 + y_3 = 2 \quad w_1 + w_2 + w_3 = 2$$

$$y_1 + 2y_2 + y_4 = 4 \quad -w_1 + 2w_2 - w_4 = 1$$

all variables ≥ 0

Next,

$$y_1 - y_2 + y_3 - 2S_2 = 0 \quad w_1 + w_2 - w_3 - 2S_2 = 0$$

$$y_1 + 2y_2 + y_4 - 4S_2 = 0 \quad -w_1 + 2w_2 + w_4 - S_2 = 0$$

$$2y_1 + y_2 - 2w_1 - 4w_2 = 0$$

$$y_1 + y_2 + y_3 + y_4 + w_1 + w_2 + w_3 + w_4 - M S_2 + S_1 = 0$$

$$y_1 + y_2 + y_3 + y_4 + w_1 + w_2 + w_3 + w_4 + S_1 + S_2 = M+1$$

Let

$$y_j = (M+1)x_j, \quad j = 1, 2, 3, 4$$

$$w_j = (M+1)x_j, \quad j = 5, 6, 7, 8$$

$$S_1 = (M+1)x_9$$

$$S_2 = (M+1)x_{10}$$

Thus, the equations become

$$x_1 - x_2 + x_3 - 2x_{10} = 0$$

$$x_1 + 2x_2 + x_4 - 4x_{10} = 0$$

$$x_5 + x_6 - x_7 - 2x_{10} = 0$$

$$-x_5 + 2x_6 - x_8 - x_{10} = 0$$

$$2x_1 + x_2 - 2x_5 - 4x_6 = 0$$

$$x_1 + x_2 + \dots + x_9 - Mx_{10} = 0$$

$$x_1 + x_2 + \dots + x_{10} = 1$$

all $x_j \geq 0$

The complete problem thus becomes:

$$\text{minimize } z = x_{11}$$

Subject to

$$x_1 - x_2 + x_3 - 2x_{10} + x_{11} = 0$$

$$x_1 + 2x_2 + x_4 - 4x_{10} + 0x_{11} = 0$$

$$x_5 + x_6 - x_7 - 2x_{10} + x_{11} = 0$$

$$-x_5 + 2x_6 - x_8 - x_{10} + x_{11} = 0$$

$$2x_1 + x_2 - 2x_5 - 4x_6 + 3x_{11} = 0$$

$$x_1 + x_2 + \dots + x_9 - Mx_{10} + (M-9)x_{11} = 0$$

$$x_1 + x_2 + \dots + x_{11} = 1$$

all $x_j \geq 0$

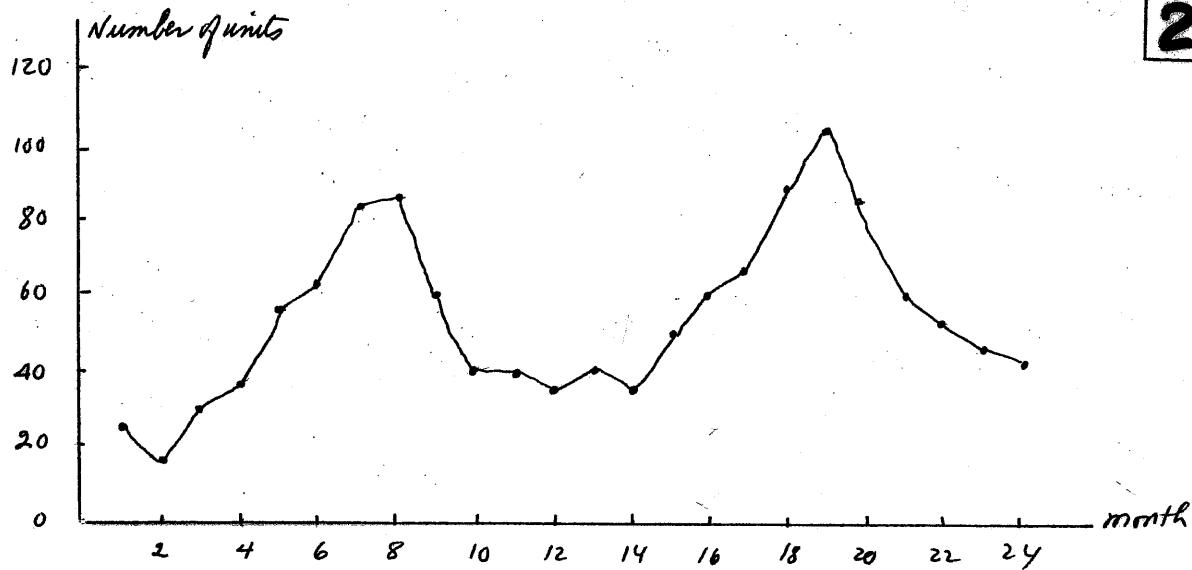
Chapter 21

Forecasting models

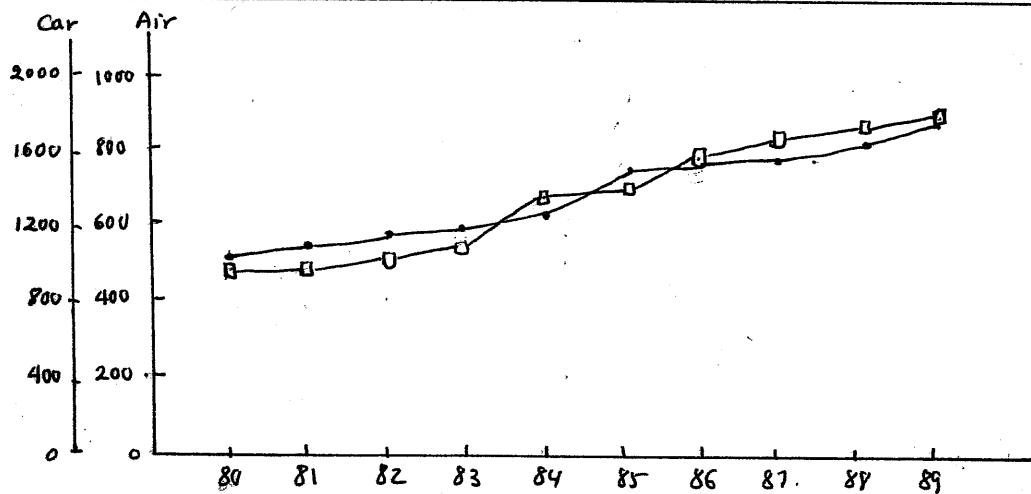
Set 21.1a

$$\bar{y}_{25}^* = \frac{54+42+64+60+70+66+57+55+52+62+70+72}{12} = 60.33$$

Larger n suppresses the fluctuations in data



The seasonal nature of the data makes the moving average unsuitable as a prediction model. We need to select n small; e.g., $n=3$ yields $\bar{y}_{25}^* = 50$.



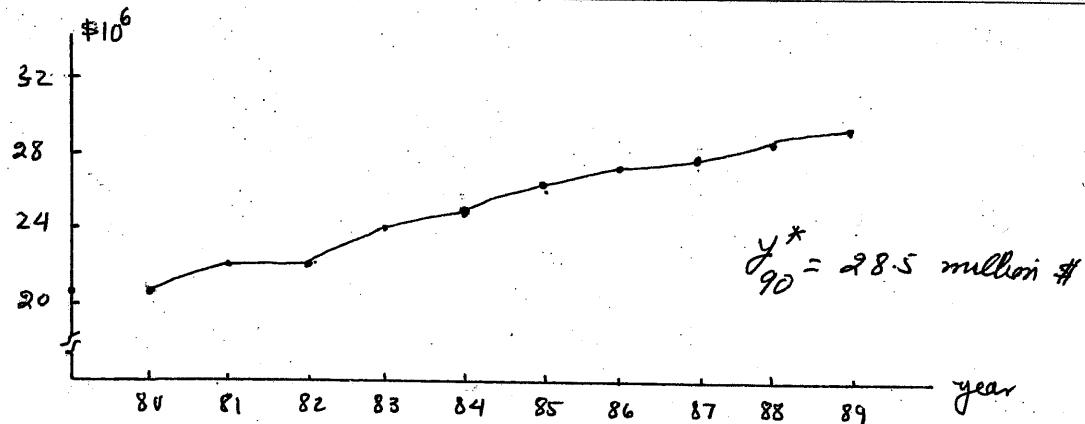
Data show an upward trend. Select small $n=3$ for the moving average.

Car: $\bar{y}_{90}^* = 1791.3$ individuals

Air: $\bar{y}_{90}^* = 938.33$ individuals

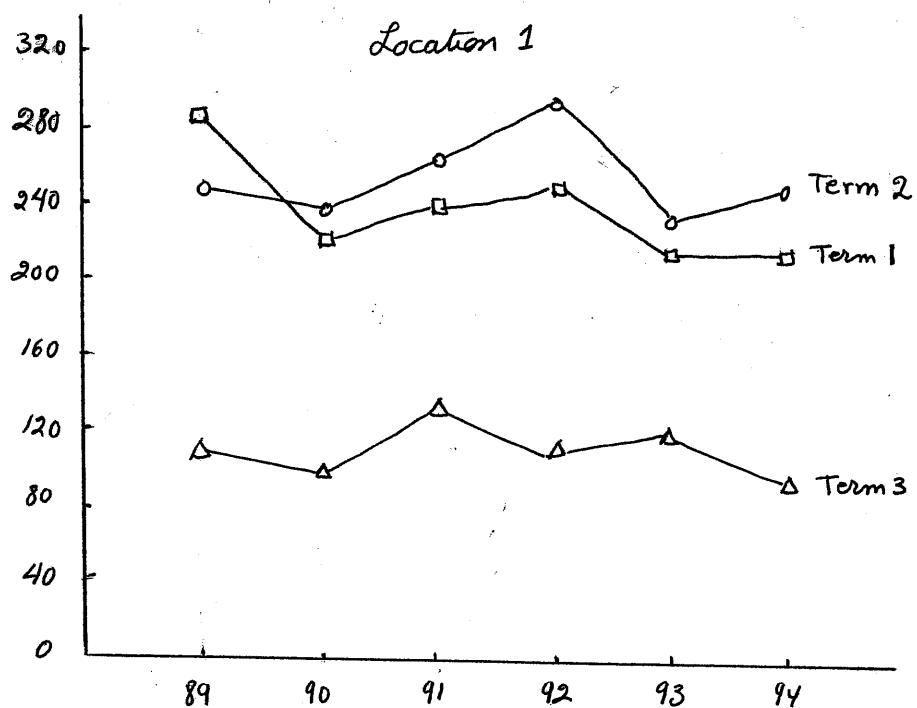
Set 21.1a

4



Data show linear trend. Use small $n=3$ for the moving average.

5



The data appear stable. The moving average should apply nicely to this case. Similar analysis can be carried out for the remaining locations. Use $n=5$

- At location 1:
- Term 1 : $y^*_{95} = 238.6$ students
 - Term 2 : $y^*_{95} = 260.2$ students
 - Term 3 : $y^*_{95} = 117$ students

Set 21.2a

1
 $\alpha = .2, \gamma_{25}^* = 59.63$

2
Car: $\alpha = .2, \gamma_{90}^* = 1577.71$ individuals

Orn: $\alpha = .2, \gamma_{90}^* = 797.75$ individuals

3
 $\alpha = .2 \quad \gamma_{90}^* = \26.27 million

4
For location 1: $\alpha = .2$.

Term 1: $\gamma_{95}^* = 254.33$

Term 2: $\gamma_{95}^* = 256.13$

Term 3: $\gamma_{95}^* = 116.38$

Set 21.3a

The data have both seasonal variations and a trend. Regression analysis can be used to detect the trend.

1

$$y = 39.23 + 1.262X, \quad r = .394$$

$$y_{25}^* = 70.77 \text{ units}$$

Car: $y = 977.4 + 92.69X, \quad r = .9928$

2

$$y_{90}^* = 1997 \text{ individuals}$$

Avi.: $y = 407.73 + 59.21X, \quad r = .9895$

$$y_{90}^* = 1059.07 \text{ individuals}$$

Y = $20.6 + .873X, \quad r = .991$

3

$$y_{90}^* = \$30.2 \text{ million}$$

Location 1:

4

Term 1: $y = 272.73 - 7.4X, \quad r = -.538$

Term 2: $y = 247.4 + 3.03X, \quad r = .247$

Term 3: $y = 119.2 - .6286X, \quad r = -.0836$

$$\sum_{i=1}^n (y_i - bx_i - a) = \sum_{i=1}^n y_i - b \sum_{i=1}^n x_i - na$$

$$= n\bar{y} - nb\bar{x} - na$$

$$= n(\bar{y} - b\bar{x} - a)$$

$$= n(\bar{y} - b\bar{x} - \bar{y} + b\bar{x})$$

5

$$= 0$$

Chapter 22

Probabilistic Dynamic Programming

Set 22.1a

$$f_6(j) = 2j$$

$$f_i(j) = \max \begin{cases} \text{end: } 2j \\ \text{spin: } \frac{1}{8} \sum_{k=1}^8 f_{i+1}(k) \end{cases}$$

$$f_i(0) = \frac{1}{8} \sum_{k=1}^8 f_2(k) \quad i = 2, 3, 4, 5$$

		Opt. Soln.	
Spin 5 outcome	f ₅ (j)	Decision	
1	2	End	
2	4	End	
3	6	End	
4	8	End	
5	10	End	
6	12	End	
7	14	End	
8	16	End	

$$\begin{aligned} \text{Stage 5: } f_5(j) &= \max \left\{ 2j, \frac{1}{8} (f_6(1) + f_6(2) + \dots + f_6(8)) \right\} \\ &= \max \left\{ 2j, \frac{72}{8} \right\} \\ &= \max \{ 2j, 9 \} \end{aligned}$$

		Opt. Sol.	
Spin 4 outcome	Spin	f ₅ (j)	Decision
1	2	9	Spin
2	4	9	Spin
3	6	9	Spin
4	8	9	Spin
5	10	9	10
6	12	9	12
7	14	9	14
8	16	9	16

Stage 4:

$$\begin{aligned} f_4(j) &= \max \left\{ 2j, \frac{1}{8} (9+9+9+9+10+12+14+16) \right\} \\ &= \max \{ 2j, 11 \} \end{aligned}$$

Spin 3 outcome	End	Spin	Opt. Sol.	
			f ₄ (j)	Decision
1	2	11	11	Spin
2	4	11	11	Spin
3	6	11	11	Spin
4	8	11	11	Spin
5	10	11	11	Spin
6	12	11	12	End
7	14	11	14	End
8	16	11	16	End

$$\text{Stage 3: } f_3(j) = \max \{ 2j, 12.125 \}$$

Spin 2 outcome	End	Spin	Opt. Sol.	
			f ₃ (j)	Decision
$1 \leq j \leq 6$	$2j$	12.125	12.125	Spin
$j = 7, 8$	$2j$	12.125	$2j$	End

$$\text{Stage 2: } f_2(j) = \max \{ 2j, 12.84375 \}$$

Spin 1 outcome	End	Spin	Opt. Sol.	
			f ₂ (j)	Decision
$1 \leq j \leq 6$	$2j$	12.84375	12.84375	Spin
$j = 7, 8$	$2j$	12.84375	$2j$	End

$$\text{Stage 1: } f_1(0) = \frac{1}{8} (6 \times 12.84375 + 14 + 16) = 13.38$$

Solution:

Spin #	Strategy
1	Continue to spin
2	Continue if #1 produces 1 → 6, else end
3	Continue if #2 produces 1 → 6, else end
4	Continue if #3 produces 1 → 5, else end
5	Continue if #4 produces 1 → 4, else end
6	End

Expected return = \$13.38

continued...

Set 22.1a

Let O_j represent the best offer at the end of day j , where

2

$$j = \begin{cases} 1, \text{high offer} \\ 2, \text{medium offer} \\ 3, \text{low offer} \end{cases}$$

$$j = 1, 2, 3$$

$$f_4(j) = O_j$$

$$f_i(j) = \max \left\{ \begin{array}{l} \text{accept: } O_j \\ \text{continue: } \frac{1}{3} (f_1(i+1) + f_2(i+1) + f_3(i+1)) \end{array} \right.$$

$$f_1(0) = \frac{1}{3} \{ f_2(1) + f_2(2) + f_3(2) \}$$

Stage 4:

Day 3 best offer j	Opt. Sol.	
	$f_4(j)$	Decision
1	1050	Accept
2	1900	Accept
3	2500	Accept

Stage 3:

$$f_3(j) = \max \left\{ O_j, \frac{1}{3} (f_1(j+1) + f_2(j+1) + f_3(j+1)) \right\}$$

$$= \max \{ O_j, 1816.67 \}$$

Day 2 best offer j	Opt. Sol.			
	Accept	Continue	$f_3(j)$	Decision
1	1050	1816.67	1816.67	Continue
2	1900	1816.67	1900	Accept
3	2500	1816.67	2500	Accept

Stage 2:

$$f_2(j) = \max \left\{ O_j, \frac{1}{3} (f_1(j+1) + f_2(j+1) + f_3(j+1)) \right\}$$

$$= \max \{ O_j, 2072.33 \}$$

Day 1 best offer j	Opt. Sol.			
	Accept	Continue	$f_2(j)$	Decision
1	1050	2072.33	2072.33	Continue
2	1900	2072.33	2072.33	Continue
3	2500	2072.33	2072.33	Accept

Stage 1:

$$f_1(0) = \frac{1}{3} (2 \times 2072.33 + 2500)$$

$$= \$2214.82$$

Solution:

Day 1: Accept if offer is high

Day 2: Accept if offer is medium or high

Day 3: Accept any offer.

Set 22.2a

Stage 4:

$$f_4(x_4) = x_4(1 + .8x_6 + .4x_2 + .2x_2) \\ = 1.6x_4$$

State	Opt. sol.	
	$f_4(x_4)$	y_4
x_4	$1.6x_4$	x_4 (invest all)

Stage 3:

$$f_3(x_3) = \max \left\{ .2 \times 1.6(x_3 + 4y_3) \right. \\ \left. 0 \leq y_3 \leq x_3 + .4 \times 1.6(x_3 - y_3) \right. \\ \left. + .4 \times 1.6(x_3 - y_3) \right\} \\ = \max \left\{ 1.6x_3 \right\} \\ 0 \leq y_3 \leq x_3$$

State	Optimum	
	$f_3(x_3)$	y_3
x_3	$1.6x_3$	$0 \leq y_3 \leq x_3$

Stage 2:

$$f_2(x_2) = \max \left\{ .4x_2 \cdot 1.6(x_2 + y_2) + .4x_2 \cdot 1.6x_2 \right. \\ \left. + .2 \times 1.6(x_2 - y_2) \right\} \\ = \max \left\{ 1.6x_2 + .32y_2 \right\} \\ 0 \leq y_2 \leq x_2$$

State	Opt. sol.	
	$f_2(x_2)$	y_2
x_2	$1.92x_2$	x_2

$$f_1(x_1) = \max \left\{ .1x_1 \cdot 1.92(x_1 + 2y_1) \right. \\ \left. + .4x_1 \cdot 1.92(x_1 + y_1) + .5x_1 \cdot 1.92(x_1 + 5y_1) \right\} \\ = \max \left\{ 1.92x_1 + 1.632y_1 \right\} \\ 0 \leq y_1 \leq x_1$$

State	Opt. sol.	
	$f_1(x_1)$	y_1
x_1	$3.552x_1$	x_1

Solution: accumulation = \$35,520

Invest \$10,000 in year 1, all in year 2, none in year 3, and all in year 4.

C_i = penalty cost/shortage unit of item i

2

Z_i = number of units of item i

V_i = volume per unit of item i

X_i = m^3 assigned to items i, \dots, n

P_{ij} = probability of j demand units of item i

$f_i(x_i)$ = minimum expected penalty cost for items $i, i+1, \dots, n$, given x_i

$$f_i(x_i) = \min_{0 \leq Z_i \leq \left[\frac{X_i}{V_i} \right]} \left\{ C_i \sum_{j > Z_i} (j - Z_i) P_{ij} + f_{i+1}(x_i - Z_i V_i) \right\} \\ i = 1, 2, \dots, n$$

$$f_{n+1}(\cdot) = 0$$

Table for exp. shortage cost:

item i	$Z_i = 1$	$Z_i = 2$	$Z_i = 3$
1	$8(1 \times 5) = 40$	0	0
2	$10(1 \times 4 + 2x_2 + 3 \times 1) = 110$	$10(1 \times 2 + 2x_2 - 1) = 40$	$10(1 \times 1) = 10$
3	$15(1 \times 2 + 2x_2 \times 5) = 180$	$15(1 \times 5) = 75$	0

$$\text{Stage 3: } f_3(x_3) = \min_{Z_3 \leq \left[\frac{x_3}{V_3} \right]} \left\{ 15 \sum_{j > Z_3} (j - Z_3) P_{3j} \right\}$$

x_3	$V_3 = 3$			Opt. sol.	
	$Z_3 = 1$	$Z_3 = 2$	$Z_3 = 3$	$f_3(x_3)$	Z_3
3	18	—	—	18	1
4	18	—	—	18	1
5	18	—	—	18	1
6	18	7.5	—	7.5	2
7	18	7.5	—	7.5	2
8	18	7.5	—	7.5	2
9	18	7.5	0	0	3
10	18	7.5	0	0	3

$$\text{Stage 2: } V_2 = 1$$

$$f_2(x_2) = \min_{Z_2 \leq \left[\frac{x_2}{V_2} \right]} \left\{ 10 \sum_{j > Z_2} (j - Z_2) P_{2j} + f_3(x_2 - Z_2) \right\}$$

Set 22.2a

2 continued

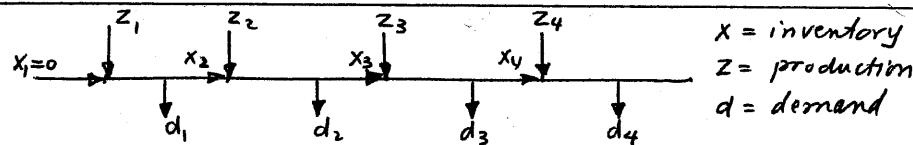
X_2	$Z_2 = 1$	2	3	4	5	6	7	f_3	Z_3
4	$11+18=29$	—	—	—	—	—	—	29	1
5	$11+18=29$	$4+18=22$	—	—	—	—	—	22	2
6	$11+18=29$	$4+18=22$	$1+18=19$	—	—	—	—	19	3
7	$11+7.5=18.5$	$4+18=22$	$1+18=19$	$0+18=18$	—	—	—	18	4
8	$11+7.5=18.5$	$4+7.5=11.5$	$1+18=19$	$0+18=18$	$0+18=18$	—	—	11.5	2
9	$11+7.5=18.5$	$4+7.5=11.5$	$1+7.5=8.5$	$0+18=18$	$0+18=18$	$0+18=18$	—	8.5	3
10	$11+0=11$	$4+7.5=11.5$	$1+7.5=8.5$	$0+7.5=7.5$	$0+18=18$	$0+18=18$	$0+18=18$	7.5	4

Stage 1: $f_1(x_1) = \min_{Z_1 \leq \left[\frac{x_1}{v_1} \right]} \left\{ 8 \sum_{j > Z_1} (j - Z_1) P_{1,j} + f_2(x_1 - 2Z_1) \right\}$

x_1	$Z_1 = 1$	$Z_1 = 2$	$Z_1 = 3$	f_1	Z_1
10	$4+11.5=15.5$	$0+19=19$	$0+29=29$	15.5	1

Solution:

$$(x_1 = 10) \rightarrow Z_1 = 1 \rightarrow (x_2 = 8) \rightarrow Z_2 = 2 \rightarrow (x_3 = 6) \rightarrow Z_3 = 2$$



$$f_n(x_n) = \min_{Z_n} \{ C(x_n) \}$$

$$f_i(x_i) = \min_{Z_i} \left\{ C(x_i) + \sum_{d=0}^3 f_{i+1}(x_i + Z_i - d) P(d) \right\}, \quad C(x_i) = \begin{cases} x_i, & x_i \geq 0 \\ -2x_i, & x_i \leq 0 \end{cases}$$

$$i = 1, 2, \dots, n-1$$

Stage 4:

X_4	$Z_4 = 0$	1	2	3	f_4	Z_4
-3	—	—	—	6	6	3
-2	—	—	4	—	4	2
-1	—	2	—	—	2	1
0	0	—	—	—	0	0
1	1	—	—	—	1	0
2	2	—	—	—	2	0
3	3	—	—	—	3	0

Notice that negative x_4 allows for the possibility of backordering by producing for year 3 in period 4.

continued...

Set 22.2a

$$\text{Stage 3: } f_3(x_3) = \min_{Z_3} \{C(x_3) + .5f_4(x_3+Z_3-1) + .3f_4(x_3+Z_3-2) + .2f_4(x_3+Z_3-3)\} \quad [3 \text{ continued}]$$

x_3	$Z_3 = 0$	1	2	3	Opt. Sol.	
					f_3	Z_3
-3	—	—	—	$6 + .5x2 + .3xy + .2x6 = 9.4$	9.4	3
-2	—	—	$4 + .5x2 + .3xy + .2x6 = 7.4$	$4 + .5x0 + .3x2 + .2xy = 5.4$	5.4	3
-1	—	$2 + .5x2 + .3xy + .2x6 = 5.4$	$2 + .5x0 + .3x2 + .2xy = 3.4$	$2 + .5x1 + .3x0 + .2x2 = 2.9$	2.9	3
0	$0 + .5x2 + .3xy + .2x6 = 3.4$	$0 + .5x0 + .3x2 + .2xy = 1.4$	$0 + .5x1 + .3x0 + .2x2 = .9$	$0 + .5x2 + .3x1 + .2x0 = 1.3$.9	2
1	$1 + .5x0 + .3x2 + .2xy = 2.4$	$1 + .5x1 + .3x0 + .2x2 = 1.9$	$1 + .5x2 + .3x1 + .2x0 = 2.3$	—	1.9	1
2	$2 + .5x1 + .3x0 + .2x2 = 2.9$	$2 + .5x2 + .3x1 + .2x0 = 3.3$	—	—	2.9	0
3	$3 + .5x2 + .3x1 + .2x0 = 4.3$	—	—	—	4.3	0

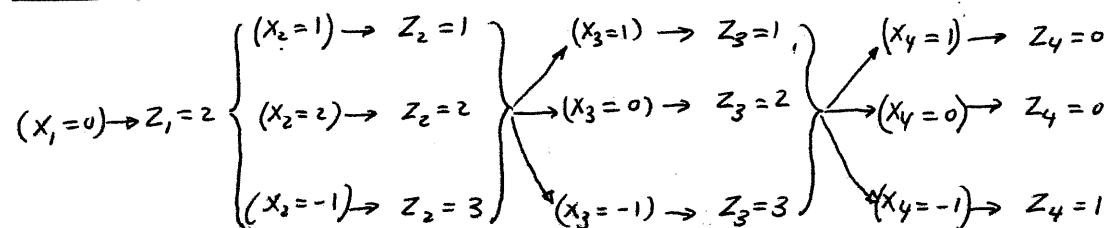
$$\text{Stage 2: } f_2(x_2) = \min_{Z_2} \{C(x_2) + .5f_3(x_2+Z_2-1) + .3f_3(x_2+Z_2-2) + .2f_3(x_2+Z_2-3)\}$$

x_2	$Z_2 = 0$	1	2	3	Opt. Sol.	
					f_2	Z_2
-3	—	—	—	$6 + .5x7.9 + .3x5.4 + .2x9.4 = 10.94$	10.94	3
-2	—	—	$4 + .5x2.9 + .3x5.4 + .2x9.4 = 8.94$	$4 + .5x.9 + .3x2.9 + .2x5.4 = 6.4$	6.4	3
-1	—	$2 + .5x2.9 + .3x5.4 + .2x9.4 = 6.94$	$2 + .5x.9 + .3x2.9 + .2x5.4 = 4.4$	$2 + .5x1.9 + .3x1.9 + .2x2.9 = 3.8$	3.8	3
0	$0 + .5x2.9 + .3x5.4 + .2x9.4 = 4.94$	$0 + .5x.9 + .3x2.9 + .2x5.4 = 2.4$	$0 + .5x1.9 + .3x1.9 + .2x2.9 = 1.8$	$0 + .5x2.9 + .3x1.9 + .2x1.9 = 2.2$	1.8	2
1	$1 + .5x.9 + .3x2.9 + .2x5.4 = 3.4$	$1 + .5x1.9 + .3x1.9 + .2x2.9 = 2.8$	$1 + .5x2.9 + .3x1.9 + .2x1.9 = 3.2$	—	2.8	1
2	$2 + .5x1.9 + .3x1.9 + .2x2.9 = 3.8$	$2 + .5x2.9 + .3x1.9 + .2x1.9 = 4.2$	$2 + .5x4.3 + .3x2.9 + .2x1.9 = 5.4$	—	3.8	0
3	$3 + .5x2.9 + .3x1.9 + .2x1.9 = 5.2$	$3 + .5x4.3 + .3x2.9 + .2x1.9 = 6.4$	—	—	5.2	0

$$\text{Stage 1: } f_1(x_1) = \min_{Z_1} \{C(x_1) + .5f_2(x_1+Z_1-1) + .3f_2(x_1+Z_1-2) + .2f_2(x_1+Z_1-3)\}$$

x_1	$Z_1 = 0$	1	2	3	Opt. Sol.	
					f_1	Z_1
0	$0 + .5x3.8 + .3x6.4 + .2x10.94 = 6.008$	$0 + .5x1.8 + .3x3.8 + .2x6.4 = 3.32$	$0 + .5x2.8 + .3x1.8 + .2x3.8 = 2.7$	$0 + .5x3.8 + .3x2.8 + .2x1.8 = 3.1$	2.7	2

Solution:



continued...

Set 22.2a

4

Stage i = center i

Alternative y_i = number of bikes assigned to center i

State x_i = number of bikes assigned to centers $i, i+1, \dots$, and n

d_i = demand in center i

$f_i(x_i)$ = maximum expected revenue for stages $i, i+1, \dots$, and n given x_i .

$$f_n(x_n) = \max_{y_n \leq x_n} \{ C_n E\{d_n | y_n\}\}$$

$$f_i(x_i) = \max_{y_i \leq x_i} \{ C_i E\{d_i | y_i\} + f_{i+1}(x_i - d_i)\}, \quad i=1, 2, \dots, n-1$$

where

$E\{d_i | y_i\}$ = Average demand at center i given y_i bikes are allocated to center i

$$= 0P_0 + 1P_1 + \dots + y_i P_{y_i} + y_i (P_{y_i} + P_{y_{i+1}} + \dots + P_8)$$

Example calculations:

$$\begin{aligned} E\{d_i | y_i = 2\} &= 0P_0 + 1P_1 + 2(P_2 + P_3 + \dots + P_8) \\ &= 0 + 1 \times 1.5 + 2 \times .85 = 1.85 \end{aligned}$$

Table for $C_i \cdot E\{d_i | y_i\}$:

i	C_i	$y_i = 0$	1	2	3	4	5	6	7	8
1	5	0	5.40	9.60	12.00	13.20	13.80	13.80	13.80	13.8
2	7	0	6.86	13.51	19.46	23.66	25.76	26.81	27.51	27.86
3	6	0	5.00	9.25	18.25	19.75	20.50	20.75	20.875	20.875

Stage 3: $f_3(x_3) = \max_{y_3 \leq x_3} \{ C_3 E\{d_3 | y_3\}\}$

x_3	$y_3 = 0$	Opt. Sol.								
		1	2	3	4	5	6	7	8	f_3
0	0	—								0
1	0	5	—							5.00
2	0	5	9.25	—						9.25
3	0	5	9.25	18.25	—					18.25
4	0	5	9.25	18.25	19.75	—				19.75
5	0	5	9.25	18.25	19.75	20.50	—			20.50
6	0	5	9.25	18.25	19.75	20.50	20.75	—		20.75
7	0	5	9.25	18.25	19.75	20.50	20.75	20.875	—	20.875
8	0	5	9.25	18.25	19.75	20.50	20.75	20.875	20.875	20.85

Continued...

Set 22.2a

$$\text{Stage 2: } f_2(x_2) = \max_{y_2 \leq x_2} \{ C_2 E\{d_2 | y_2\} + f_3(x_2 - y_2)\}$$

x_2	$y_2 = 0$	1	2	3	4	5	6	7	8	f_2	y_2	Opt^m
0	$0+0=0$	—								0	0	
1	$0+5=5$	$6.86+0$ $=6.86$	—							6.86	1	
2	$0+9.25$ $=9.25$	$6.86+5$ $=11.86$	$13.51+0$ $=13.51$	—						13.51	2	
3	$0+18.25$ $=18.25$	$6.86+9.25$ $=16.11$	$13.51+5$ $=18.51$	$19.46+0$ $=19.46$	—					19.46	3	
4	$0+19.75$ $=19.75$	$6.86+18.25$ $=25.11$	$13.51+9.25$ $=22.76$	$19.46+5$ $=24.46$	$23.66+0$ $=23.66$	—				25.11	1	
5	$0+20.50$ $=20.50$	$6.86+19.75$ $=26.61$	$13.51+18.25$ $=31.76$	$19.46+9.25$ $=28.71$	$23.66+5$ $=28.66$	$23.76+0$ $=23.76$	—			31.76	2	
6	$0+20.75$ $=20.75$	$6.86+20.5$ $=27.36$	$13.51+19.75$ $=33.26$	$19.46+18.25$ $=37.71$	$23.66+9.25$ $=32.91$	$23.76+5$ $=30.76$	$26.81+0$ $=26.81$	—		37.71	3	
7	$0+20.875$ $=20.875$	$6.86+20.75$ $=27.61$	$13.51+20.5$ $=34.01$	$19.46+19.75$ $=39.21$	$23.66+18.25$ $=41.91$	$23.76+9.25$ $=35.01$	$26.81+5$ $=31.81$	$27.51+0$ $=27.51$	—	41.91	4	
8	$0+20.875$ $=20.875$	$6.86+21.875$ $=27.735$	$13.51+20.75$ $=34.70$	$19.46+20.5$ $=39.96$	$23.66+19.75$ $=43.41$	$23.76+18.25$ $=44.01$	$26.81+9.25$ $=36.06$	$27.51+5$ $=32.51$	$27+86$ $=27.86$	44.01	5	

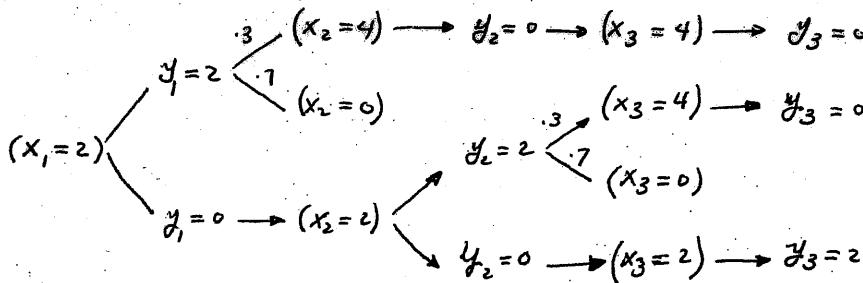
$$\text{Stage 1: } f_1(x_1) = \max_{y_1 \leq x_1} \{ C_1 E\{d_1 | y_1\} + f_2(x_1 - y_1)\}$$

x_1	$y_1 = 0$	1	2	3	4	5	6	7	8	f_1	y_1	Opt
8	$0+44.01$ $=44.01$	$5.4+41.9$ $=47.3$	$9.6+37.71$ $=47.31$	$12+31.76$ $=43.76$	$13.2+25.11$ $=38.31$	$13.8+19.46$ $=33.26$	$13.8+13.5$ $=27.3$	$13.8+6.86$ $=20.66$	$13.8+0$ $=13.8$	47.31	2	

Optimum solution:

$$(x_1 = 8) \rightarrow y_1 = 2 \rightarrow (x_2 = 6) \rightarrow y_2 = 3 \rightarrow (x_3 = 3) \rightarrow y_3 = 3$$

Set 22.3a



Stage 3:

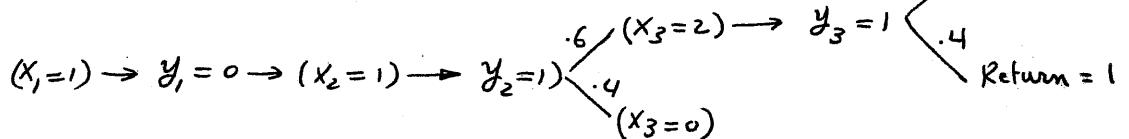
X_3	$y_3 = 0$	$.6 P\{X_3 + Y_3 \geq 3\} + .4 P\{X_3 - Y_3 \geq 3\}$				Opt. Sol.	y_3
		1	2	3	4		
0	$.6x0 + .4x0 = 0$	—	—	—	—	0	0
1	$.6x0 + .4x0 = 0$	$.6x0 + .4x0 = 0$	—	—	—	0	0
2	$.6x0 + .4x0 = 0$	$.6x1 + .4x0 = .6$	$.6x1 + .4x0 = .6$	—	—	.6	1
3	$.6x1 + .4x1 = 1$	$.6x1 + .4x0 = .6$	$.6x1 + .4x0 = .6$	$.6x1 + .4x0 = .6$	—	1	0
4	$.6x1 + .4x1 = 1$	$.6x1 + .4x1 = 1$	$.6x1 + .4x0 = .6$	$.6x1 + .4x0 = .6$	$.6x1 + .4x0 = .6$	1	0

Stage 2:

X_2	$.6 f_3 (X_3 + Y_3) + .4 f_3 (X_3 - Y_3)$			Opt. Sol.	
	$y_2 = 0$	1	2		
0	$.6x0 + .4x0 = 0$	—	—	0	0
1	$.6x0 + .4x0 = 0$	$.6x0 - .6 + .4x0 = .36$	—	.36	1
2	$.6x0 + .4x0 = 0$	$.6x1 + .4x0 = .6$	$.6x1 + .4x0 = .6$.6	1

Stage 1:

X_1	$.6 f_2 (X_2 + Y_2) + .4 f_2 (X_2 - Y_2)$		f_1	y_1	Opt. Sol.
	$y_1 = 0$	$y_1 = 1$			
1	$.6x - .36 + .4x - .36 = -.36$	$.6x - .6 + .4x = -.36$	-.36	0	



Set 22.3a

$$\text{Stage 3: } f_3(x_3) = \max_{y_3 \leq x_3} \{ .25 P\{x_3 + 2y_3 \geq 4\} + .75 P\{x_3 - y_3 \geq 4\} \}$$

3

x_3	$y_3=0$	1	2	3	4	5	6	7	8	9	f_3	y_3	Opt. Sol.
0	0	—	—	—	—	—	—	—	—	—	0	0	
1	0	0	—	—	—	—	—	—	—	—	0	0	
2	0	.25	.25	—	—	—	—	—	—	—	.25	1,2	
3	0	.25	.25	.25	—	—	—	—	—	—	.25	1,2,3	
4	1	.25	.25	.25	.25	—	—	—	—	—	1	0	
5	1	1	.25	.25	.25	.25	—	—	—	—	1	0,1	
6	1	1	1	.25	.25	.25	.25	—	—	—	1	0,1,2	
7	1	1	1	1	.25	.25	.25	.25	—	—	1	0,1,2,3	
8	1	1	1	1	1	.25	.25	.25	.25	—	1	0,1,2,3	
9	1	1	1	1	1	1	.25	.25	.25	.25	1	0 → 4	
											1	0 → 5	

$$\text{Stage 2: } f_2(x_2) = \max_{y_2 \leq x_2} \{ .25 f_3(x_2 + 2y_2) + .75 f_3(x_2 - y_2) \}$$

x_2	$y_2=0$	1	2	3	f_2	y_2	Opt. Sol.
0	0	—	—	—	0	0	
1	0	.25 × .25 = .0625	—	—	.0625	1	
2	.25	.25 × 1 + .75 × 0 = .25	.25 × 1 + .75 × 0 = .25	—	.25	0 → 2	
3	.25	.25 × 1 + .75 × .25 = .4375	.25 × 1 + .75 × 0 = .25	.25 × 1 + .75 × 0 = .25	.4375	1	

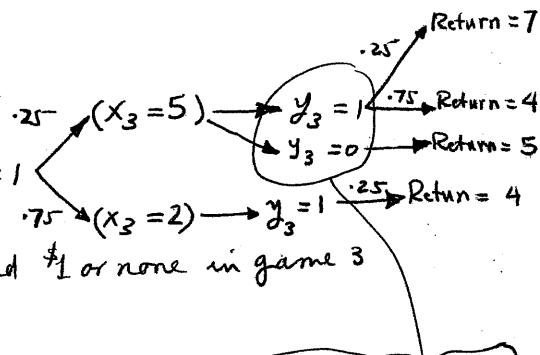
$$\text{Stage 1: } f_1(x_1) = \max_{y_1 \leq x_1} \{ .25 f_2(x_1 + 2y_1) + .75 f_2(x_1 - y_1) \}$$

x_1	$y_1=0$	1	f_1	y_1	Opt. Sol.
1	$.25 \times .0625 + .75 \times .0625 = .25 \times .4375 + .75 \times 0$ = .0625	=	-109375	1	

Solution: max probability = .109375

$$(x_1=1) \rightarrow y_1=1 \xrightarrow{.25} (x_2=3) \rightarrow y_2=1$$

Bet \$1 in game 1, \$1 in game 2, and \$1 or none in game 3



alternative optima

Chapter 23

Markovian Decision Process

Set 23.1a

1

$$P = \begin{bmatrix} .2 & .5 & .3 \\ .1 & .6 & .3 \\ .05 & .4 & .55 \end{bmatrix}$$

$$R = \begin{bmatrix} 7 & 6 & 3 \\ 7 & 4 & 0 \\ 6 & 3 & -2 \end{bmatrix}$$

2

<u>Stationary policy</u>	<u>Action</u>
1	Do not fertilize
2	Fertilize when in state 1
3	Fertilize when in state 2
4	Fertilize when in state 3
5	Fertilize when in 1 or 2
6	Fertilize when in 1 or 3
7	Fertilize when in 2 or 3
8	Fertilize regardless of state

Set 23.2a

$$V_1^1 = .9x_2 + .1x_1 - 1 = 1.7$$

$$V_2^1 = .6x_1 + .4x_2 - 3 = -.6$$

$$V_1^2 = .7x_4 + .3x_1 = 3.1$$

$$V_2^2 = .2x_2 + .8x_1 = -.4$$

$$f_N(k) = \max_k \{ V_i^k \}$$

$$f_m(i) = \max_k \{ V_i^k + \sum_j p_{ij} f_{n+1}(j) \}$$

Stage 3:

		V_i^k	$f_m(i)$	k^*	Optimal
i	$k=1$	$k=2$	$f_m(i)$	k^*	
1	1.7	3.1	3.1	2	
2	-.6	-.4	-.4	2	

Stage 2:

		$V_i^k + p_{11}^k f_3(1) + p_{12}^k f_3(2)$	$f_m(i)$	k^*	Optimal
i	$k=1$	$k=2$	$f_m(i)$	k^*	
1	$1.7 + .9x_3.1 + .1x_2 - .4$ = 4.45	$3.1 + .7x_3.1 + .3x_2 - .4$ = 5.15	5.15	2	
2	$-.6 + .6x_3.1 + .4x_2 - .4$ = 1.1	$-.4 + .2x_3.1 + .8x_2 - .4$ = -.1	-.1	1	

Stage 1:

		$V_i^k + p_{11}^k f_2(1) + p_{12}^k f_2(2)$	$f_m(i)$	k^*	Optimal
i	$k=1$	$k=2$	$f_m(i)$	k^*	
1	$1.7 + .9x_5.15 + .1x_1.1 = 6.445$	$3.1 + .7x_5.15 + .3x_1.1 = 7.035$	7.035	2	
2	$-.6 + .6x_5.15 + .4x_1.1 = 2.93$	$-.4 + .2x_5.15 + .8x_1.1 = 1.51$	1.51	1	

Solution:

Years 1 & 2: Don't advertise if product is successful; Otherwise, advertise

Year 3 : Don't advertise

$$V_1^R = 400x_4 + 520x_5 + 600x_1 - 200 = \$280$$

$$V_2^R = 300x_1 + 400x_7 + 700x_2 - 200 = \$250$$

$$V_3^R = 200x_1 + 250x_2 + 500x_7 - 200 = \$220$$

$$V_1^T = 1000x_7 + 1300x_2 + 1600x_1 - 900 = \$220$$

$$V_2^T = 800x_3 + 1000x_6 + 1700x_1 - 900 = \$110$$

$$V_3^T = 600x_1 + 700x_7 + 1100x_2 - 900 = -\$130$$

continued...

$$V_1^N = 400x_2 + 530x_3 + 4.3x_7/10 - 300 = \$258$$

$$V_2^N = 450x_1 + 7 + 800x_3 - 300 = \$255$$

$$V_3^N = 400x_2 + 650x_8 - 300 = \$300$$

Stage 3:

i	R	T	N	$f_3(i)$	Optimal
1	280	220	258	280	R
2	250	110	255	255	N
3	220	-130	300	300	N

Stage 2:

i	R	T	N	$f_2(i)$	Optimal
1	$280 + .4x_280 + .5x_255 + .1x_300 = 549.5$	$220 + .7x_280 + .2x_255 + .1x_300 = 497$	$258 + .2x_280 + .5x_255 + .3x_300 = 531.5$	549.5	R
2	$250 + .1x_280 + .7x_255 + .2x_300 = 516.5$	$110 + .3x_280 + .6x_255 + .1x_300 = 377$	$255 + .7x_255 + .3x_300 = 523.5$	523.5	N
3	$220 + .1x_280 + .3x_255 + .7x_300 = 534.5$	$-130 + .1x_280 + .2x_255 + .2x_300 = 136.5$	$300 + .0x_280 + .8x_255 + .8x_300 = 591$	591	N

Stage 1:

i	R	T	N	$f_1(i)$	Optimal
1	$280 + .4x_549.5 + .5x_523.5 + .1x_591 = 820.65$	$220 + .7x_549.5 + .2x_523.5 + .1x_591 = 748.75$	$258 + .2x_549.5 + .5x_523.5 + .3x_591 = 806.95$	820.65	R
2	$250 + .1x_549.5 + .8x_523.5 + .7x_591 = 845.7$	$110 + .3x_549.5 + .6x_523.5 + .1x_591 = 648.05$	$255 + .7x_523.5 + .3x_591 = 798.75$	798.75	N
3	$220 + .1x_549.5 + .3x_523.5 + .7x_591 = 845.7$	$-130 + .1x_549.5 + .2x_523.5 + .2x_591 = 409.6$	$300 + .0x_549.5 + .8x_523.5 + .8x_591 = 877.5$	877.5	N

Optimum Solution:

Use radio advertisement if the sales volume is poor;
Otherwise use newspaper advertisement

Set 23.2a

(a) P^k = transition matrix given i refrigerators on order. State $j \leq 2-i$

$$P^0 = \begin{bmatrix} 1 & 0 & 0 \\ .8 & .2 & 0 \\ .3 & .5 & .2 \end{bmatrix} \quad P^1 = \begin{bmatrix} .8 & .2 & 0 \\ -3 & .5 & .2 \\ - & - & - \end{bmatrix}$$

$$P^2 = \begin{bmatrix} .3 & .5 & .2 \\ - & - & - \end{bmatrix}$$

(b)

Let

$d_{i,k}$ = expected inventory cost/month given state i and decision k .

$$d_0^0 = (0x.2 + 1x.5 + 2x.3) \times 150 = \$165$$

$$d_1^0 = 5(1x.2 + 0x.5) + (0x.5 + 1x.3) \times 150 = \$46$$

$$d_2^0 = 5(2x.2 + 1x.5 + 0x.3) + (0x.3) \times 150 = \$4.5$$

$$d_0^1 = 100 + (1x.2 + 0x.5) \times 5 + 1x.3 \times 150 = \$146$$

$$d_1^1 = 100 + (2x.2 + 1x.5 + 0x.3) \times 5 + 0x.3 \times 150 = \$104.5$$

$$d_0^2 = 100 + (2x.2 + 1x.5 + 0x.3) \times 50 = \$104.5$$

(c)

Stage 3:

i	Opt ³		
	K=0	K=1	K=2
0	165	146	104.5
1	46	104.5	-
2	4.5	-	4.5

Stage 2:

i	Opt ²		
	K=0	K=1	K=2
0	$165 + 1x.104.5 = 269.5$	$146 + .8x104.5 + .2x46 = 238.8$	$104.5 + .3x104.5 + .5x46 + .2x4.5 = 159.75$
1	$46 + .8x104.5 + .2x46 = 138.8$	$104.5 + .3x104.5 + .5x46 + .2x4.5 = 159.75$	-
2	$4.5 + .3x104.5 + .5x46 + .2x4.5 = 59.75$	-	59.75

Stage 1:

$A_i^k + P_{i,j}^k f_j(0) + P_{i,j}^k f_j(1) + P_{i,j}^k f_j(2)$				
$K=0$	$K=1$	$K=2$	$f_i(i)$	K
0	$165 + 1x159.75 = 324.75$	$146 + .8x159.75 + .2x138.8 = 301.56$	$104.5 + .3x159.75 + .5x138.8 + .2x59.75 = 233.75$	2
1	$46 + .8x159.75 + .2x138.8 = 201.56$	$104.5 + .3x159.75 + .5x138.8 + .2x59.75 = 233.75$	-	0
2	$4.5 + .3x159.75 + .5x138.8 + .2x59.75 = 13.75$	-	-	0

Optimum solution: If stock level at beginning of month is zero, order two refrigerators; otherwise order none.

Let

$d_{i,j,k}$ = expected inventory cost given state i , decision k and month j .

P_j^k = transition matrix given month j and decision alternative k .

$$P_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ .9 & .1 & 0 \\ -.5 & .4 & .1 \end{bmatrix} \quad d_{0,1}^0 = (1x.4 + 2x.5) \times 150 = 210 \\ d_{1,1}^0 = (1x.1) \times 5 + (1x.5) \times 150 = 75.5 \\ d_{2,1}^0 = (2x.1 + 1x.4) \times 5 + (0x.5) \times 150 = 3$$

$$P_1^1 = \begin{bmatrix} .9 & .1 & 0 \\ .5 & .4 & .1 \\ - & - & - \end{bmatrix} \quad d_{0,1}^1 = 100 + (1x.1) \times 5 + (1x.5) \times 150 = 175.5 \\ d_{1,1}^1 = 100 + (2x.1 + 1x.4) \times 5 = 103$$

$$P_1^2 = \begin{bmatrix} .5 & .4 & .1 \\ - & - & - \end{bmatrix} \quad d_{0,1}^2 = 100 + (2x.1 + 1x.4) \times 5 = 103$$

$$P_2^0 = \begin{bmatrix} 1 & 0 & 0 \\ .7 & .3 & 0 \\ -.2 & .5 & .3 \end{bmatrix} \quad d_{0,2}^0 = (1x.5 + 2x.2) \times 150 = 135 \\ d_{1,2}^0 = (1x.3) \times 5 + (1x.2) \times 150 = 31.5 \\ d_{2,2}^0 = (2x.3 + 1x.5) \times 5 = 5.5$$

$$P_2^1 = \begin{bmatrix} .7 & .3 & 0 \\ .2 & .5 & .3 \\ - & - & - \end{bmatrix} \quad d_{0,2}^1 = 100 + (1x.3) \times 5 + (1x.2) \times 150 = 131.5 \\ d_{1,2}^1 = 100 + (2x.3 + 1x.5) \times 5 = 105.5$$

$$P_2^2 = \begin{bmatrix} .2 & .5 & .3 \\ - & - & - \end{bmatrix} \quad d_{0,2}^2 = 100 + (2x.3 + 1x.5) \times 5 = 105.5$$

continued...

continued...

Set 23.2a

$$P_3^0 \begin{bmatrix} 1 & 0 & 0 \\ -8 & 2 & 0 \\ -4 & -4 & 2 \end{bmatrix} d_{0,3}^0 = (1x.5 + 2x.4) \times 150 = 195$$

$$d_{1,3}^0 = (1x.2)x5 + 1x.-4 \times 150 = 61$$

$$d_{2,3}^0 = (2x.2 + 1x.-4) \times 5 = 4$$

$$P_3^1 \begin{bmatrix} -8 & -2 & 0 \\ -4 & -4 & -2 \\ - & - & - \end{bmatrix} d_{0,3}^1 = 100 + (1x.-2)x5 + (1x.-4) \times 150 = 161$$

$$d_{1,3}^1 = 100 + (2x.-2 + 1x.-4)x5 = 104$$

$$P_3^2 \begin{bmatrix} -4 & -4 & -2 \\ - & - & - \end{bmatrix} d_{0,3}^2 = 100 + (2x.-2 + 1x.-4)x5 = 104$$

Stage 3:

i	$d_{i,3}^k$			$f_{i,3}^k$	Optim
	$k=0$	$k=1$	$k=2$		
0	210	175.5	103	103	2
1	75.5	103	—	75.5	0
2	3	—	—	3	0

i	$d_{i,3}^k + P_{i,2,i}^k f_i^k(0) + P_{i,1,i}^k f_i^k(1) + P_{i,0,i}^k f_i^k(2)$			$f_{i,3}^k$	Optim
	$k=0$	$k=1$	$k=2$		
0	$195 + 103 = 298$	$131.5 + .7 \times 103 + .3 \times 75.5 = 226.25$	$105.5 + .2 \times 103 + .5 \times 75.5 = 164.75$	164.75	2
1	$131.5 + .7 \times 103 + .3 \times 75.5 = 126.25$	$105.5 + .2 \times 103 + .5 \times 75.5 + .3 \times 3 = 164.75$	—	126.25	0
2	$5.5 + .2 \times 103 + .5 \times 75.5 + .3 \times 3 = 64.75$	—	—	64.75	0

i	$d_{i,1}^k + P_{i,2,1}^k f_i^k(0) + P_{i,1,1}^k f_i^k(1) + P_{i,0,1}^k f_i^k(2)$			$f_{i,1}^k$	Optim
	$k=0$	$k=1$	$k=2$		
0	$195 + 103 = 298$	$161 + .8 \times 164.75 + .2 \times 126.25 = 318.05$	$104 + .4 \times 164.75 + .4 \times 126.25 = 233.35$	233.35	2
1	$61 + .8 \times 164.75 + .2 \times 126.25 = 218.05$	$104 + .4 \times 164.75 + .4 \times 126.25 + .2 \times 64.75 = 233.35$	—	218.05	0
2	$4 + .4 \times 164.75 + .4 \times 126.25 + .2 \times 64.75 = 133.35$	—	—	133.35	0

Solution: Order 2 if in state 0;
otherwise, don't order.

Set 23.3a

Def

$S=1$: do not advertise

$S=2$: advertise regardless of stock level.

$S=3$: Advertise whenever in state 1

$S=4$: Advertise whenever in state 2

$$P^1 = \begin{bmatrix} .9 & .1 \\ .6 & .4 \end{bmatrix} \quad R^1 = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} .7 & .3 \\ .2 & .8 \end{bmatrix} \quad R^2 = \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix} \quad R^3 = \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} .9 & .1 \\ .2 & .8 \end{bmatrix} \quad R^4 = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

S	$V_{i,j}$	
	$i=1$	$i=2$
1	1.7	-6
2	3.1	-4
3	3.1	-6
4	1.7	-4

$$\underline{S=1}: .9\pi_1 + .6\pi_2 = \pi_1 \quad \left. \right\} \pi_1 = 6/7 \\ \pi_1 + \pi_2 = 1 \quad \left. \right\} \pi_2 = 1/7$$

$$\underline{S=2}: .7\pi_1 + .2\pi_2 = \pi_1 \quad \left. \right\} \pi_1 = 2/5 \\ \pi_1 + \pi_2 = 1 \quad \left. \right\} \pi_2 = 3/5$$

$$\underline{S=3}: .7\pi_1 + .6\pi_2 = \pi_1 \quad \left. \right\} \pi_1 = 2/3 \\ \pi_1 + \pi_2 = 1 \quad \left. \right\} \pi_2 = 1/3$$

$$\underline{S=4}: .9\pi_1 + .2\pi_2 = \pi_1 \quad \left. \right\} \pi_1 = 2/3 \\ \pi_1 + \pi_2 = 1 \quad \left. \right\} \pi_2 = 1/3$$

S	π_1^s	π_2^s	E^s
1	6/7	1/7	1.3714
2	2/5	3/5	1.
3	2/3	1/3	1.8667
4	2/3	1/3	1.

Optimum decision:

Advertise whenever in state 2.

Appendix A

AMPL Modeling Language

Set A.2a

```

#-----sets
set paint;
set resource;
#-----parameters
param unitprofit{paint};
param rhs {resource};
param aij {resource,paint};
#-----variables
var product{paint} >= 0;
#-----model
maximize profit: sum{j in paint} unitprofit[j]*product[j];
subject to
limit{i in resource}:
    sum{j in paint} aij[i,j]*product[j] <= rhs[i];
marineDemand1: product["marine"] <= 5;
marineDemand2: product["marine"] >= 5;

data;
set paint := exterior interior marine;
set resource := m1 m2 demand market;
param unitprofit :=
        exterior 5
        interior 4
        marine 3.5;
param rhs:=
        m1      24
        m2      6
        demand  1
        market   2;
param aij: exterior interior marine :=
        m1      6      4      5
        m2      1      2      7.5
        demand -1     1      0
        market   0     1      0;
solve;
#-----output results
display profit, product;

```

1

```

#-----sets
set paint;
#-----parameters
param m;
param unitprofit{paint};
param rhs {1..m};
param aij {1..m,paint};
#-----variables
var product{paint} >= 0;
#-----model
maximize profit: sum{j in paint} unitprofit[j]*product[j];
subject to limit{i in 1..m}:
    sum{j in paint} aij[i,j]*product[j] <= rhs[i];

```

continued...

A-2

```

data;
param m=4;
set paint := exterior interior;
param unitprofit :=
        exterior 5
        interior 4;
param rhs:=
        1      24
        2      6
        3      1
        4      2;
param aij: exterior interior :=
        1      6      4
        2      1      2
        3      -1     1
        4      0      1;
solve;
#-----output results
display profit, product;

```

3

```

#-----sets
set paint;
set resource;
#-----parameters
param upper{paint};
param lower{paint};
param unitprofit{paint};
param rhs {resource};
param aij {resource,paint};
#-----variables
var product{i in paint} >= lower[i] <= upper[i];
#-----model
maximize profit: sum{j in paint} unitprofit[j]*product[j];
subject to limit{i in resource}:
    sum{j in paint} aij[i,j]*product[j] <= rhs[i];
data;
set paint := exterior interior;
set resource := m1 m2 demand market;
param upper = "exterior" 2 "interior" 2.5;
param lower = "exterior" 1 "interior" 0;
param unitprofit :=
        exterior 5
        interior 4;
param rhs:=
        m1      24
        m2      6
        demand  1
        market   2;
param aij: exterior interior :=
        m1      6      4
        m2      1      2
        demand -1     1
        market   0     1;
solve;
#-----output results
display profit, product;

```

Set A.2a

4

```

#-----sets
set paint;
set resource;
#-----parameters
param unitProfit{paint};
param rhs {resource};
param aij {resource,paint};
#-----variables
var product{paint} >=0;
var varProfit{j in paint} unitProfit[j]*product[j];
var resourceUse{i in resource,j in paint} aij[i,j]*product[j];
#-----model
maximize profit: sum{j in paint} varProfit[j];
subject to limit{i in resource}:
    sum{j in paint} resourceUse[i,j] <= rhs[i];
data;
set paint := exterior interior;
set resource := m1 m2 demand market;
param unitProfit :=
    exterior 5
    interior 4;
param rhs:-
    m1      24
    m2      6
    demand  1
    market   2;
param aij: exterior interior :=
    m1      6      4
    m2      1      2
    demand -1     1
    market   0     1;
solve;
#-----output results
display profit, product, varProfit, resourceUse;
display {j in paint}(resourceUse["m1",j],resourceUse["m2",j]);

```

5

```

#-----sets
set input;
set output;
#-----parameters
param unitCost{input};
param yield{output,input};
param specs{output};
param minNeeds;
#-----variables
var feedStuff{input} >=0;
var farmUse=sum{j in input} feedStuff[j];
#-----model
minimize cost:sum{j in input}unitCost[j]*feedStuff[j];
subject to
aa: farmUse>=minNeeds;
bb{i in output}:
    sum{j in input}yield[i,j]*feedStuff[j]<=specs[i]*farmUse;
data;
set input := corn soy;
set output := protein fiber;
param minNeeds:=800;
param unitCost := corn .3 soy .9;
param specs:= protein -.3 fiber .05; #negative because of <=
param yield: corn soy :=
    protein   -.09    -.6
    fiber     .02     .06;
solve;
#-----output results
display cost,feedStuff, feedStuff.rc>a.txt;
display aa.dual,bb.dual>a.txt;

```

OUTPUT

cost = 437.647

```

: feedStuff feedStuff.rc :=
corn 470.588 8.32667e-17
soy 329.412 -1.11022e-16
;
```

aa.dual = 0.547059

```

bb.dual [*] :=
fiber -2.05116e-15
protein -1.17647

```

Reduced cost shows that both corn and soy assume positive values in the optimum solution.

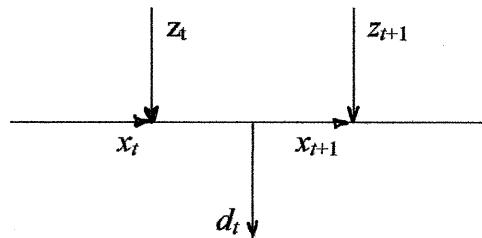
Dual price for constraint aa shows that a 1 unit increase in minNeeds increases the total cost by \$.55, approximately.

Set A.3a

1

```
param n;
param c{1..n};
var x{1..n};
rest{i in 1..n}:(if i<=n-1 then x[i]+x[i+1] else
x[1]+x[n])>=c[i];
```

2



$$x_1 = c, \quad x_{T+1} = 0$$

```
param T;
param c{1..T};
var x{1..T};
subject to
Period{t in 1..T}:
  (if t=1 then c else x[t]) + z[t] - d[t] -
  (if t<T then x[t+1] else 0)=0;
```

Set A.4a

(a)

```

param m;
param n;
param k;
param p;
param q;
param c
#.....method 1
set S1={1..m union m+k..n union n+p..q}
var x{S1};
subject to limit: sum{j in S1}x[j]>=c;
#.....method 2
set S2={1..q diff (m+1..m+k-1 union
n+1..n+p-1)}
var x{S2};
subject to limit: sum{j in S2}x[j]>=c;
(b)
param m;
param n;
param c;
param k;
var x{i in m..2*n+k};
#.....method 1
subject to CC:
  sum{i in m..2*n+k diff n+1..n+k-1}
  x[i]<=c;
#.....method 2
subject to CC:
  sum{i in m..2*n+k: i<=n or i>=n+k}x[i]
  <=c;

```

(See file a.4a-2.txt)

```

set productsUsingComp{1..5};
param c{1..5}; #component cost
param a{1..5}; #min availability
param d; #maximum demand for each product
var x{1..10,1..5}>=0; # units of product i that use component j

minimize z: sum{j in 1..5}(c[j]*(sum{i in
productsUsingComp[j]}x[i,j]));
subject to
  C{j in 1..5}:sum{i in productsUsingComp[j]}x[i,j]>=a[j];
  D{i in 1..10}: sum{j in 1..5}x[i,j]<=d;
data;
set productsUsingComp[1]:=1 2 5 10;
set productsUsingComp[2]:=3 6 7 8 9;
set productsUsingComp[3]:=1 2 3 5 6 7 9;
set productsUsingComp[4]:=2 4 6 8 10;
set productsUsingComp[5]:=1 3 4 5 6 7 9 10;

param a:=1 500 2 400 3 900 4 700 5 100;
param c:=1 1 2 3 3 2 4 6 5 4 6 9 7 2 8 5 9 10 10 7;
param d:=300;
display productsUsingComp,componentsInProduct;
solve;display x;

```

1

3

In the following code, the indexed set componentsInProduct is determined directly from the original data, which precludes the need to determine the elements of componentsInProduct[i], $i = 1, 2, \dots, 10$, manually.

```

set productsUsingComp{1..5};
set componentsInProduct{i in 1..10}=
  {j in 1..5:i in productsUsingComp[j]};
param c{1..10}; #component installation cost
param a{1..5}; #min availability
param d; #maximum demand for each product
var x{1..10,1..5}>=0;# units of product i that use component j

minimize z: sum{i in 1..10}c[i]*(sum{j in
componentsInProduct[i]}x[i,j]);
subject to
  C{j in 1..5}:sum{i in productsUsingComp[j]}x[i,j]>=a[j];
  D{i in 1..10}: sum{j in 1..5}x[i,j]<=d;
data;
set productsUsingComp[1]:=1 2 5 10;
set productsUsingComp[2]:=3 6 7 8 9;
set productsUsingComp[3]:=1 2 3 5 6 7 9;
set productsUsingComp[4]:=2 4 6 8 10;
set productsUsingComp[5]:=1 3 4 5 6 7 9 10;

```

```

param a:=1 500 2 400 3 900 4 700 5 100;
param c:=1 1 2 3 3 2 4 6 5 4 6 9 7 2 8 5 9 10 10 7;
param d:=300;
display productsUsingComp,componentsInProduct;
solve;display x;

```

2

Set A.5a

1

File RM3x.dat: The first row gives unitprofit. The first column in the remaining 4 rows defines rhs, and the second and third columns give aij.

5	4	
24	6	4
6	1	2
1	-1	1
2	0	1

2

File RM3xx.dat: Column 1 gives rhs. Column 2 repeats unitprofit[1] as many times as the number of constraints. Column 3 repeats unitprofit[2] as many times as the number of constraints. Columns 3 and 5 give aij. Convolved data file!

24	5	6	4	4
6	5	1	4	2
1	5	-1	4	1
2	5	0	4	1

Set A.5b

1

```
#-----sets
set paint;
set resource;
#-----parameters
param unitprofit{paint};
param rhs {resource};
param aij {resource,paint};
#-----variables
var product{paint} >= 0;
#-----model
maximize profit: sum{j in paint} unitprofit[j]*product[j];
subject to limit{i in resource}:
    sum{j in paint} aij[i,j]*product[j] <=rhs[i];
data;
set paint := exterior interior;
set resource := m1 m2 demand market;
param unitprofit :=
    exterior 5
    interior 4;
param rhs:=
    m1      24
    m2      6
    demand  1
    market  2;
param aij: exterior interior :=
    m1      6      4
    m2      1      2
    demand -1     1
    market  0      1;
solve;
#-----output results
printf "Objective value = %5.2f\n" profit>a1.txt;
printf "-----\n">a1.txt;
printf "Product    Quantity    Profit($)\n">a1.txt;
printf "-----\n">a1.txt;
printf "%s %s %s %s\n" product, quantity, profit, dual>a1.txt;
printf "Constraint  Slack amount  Dual price\n">a1.txt;
printf "-----\n">a1.txt;
printf "%s %s %s %s\n" constraint, slack, limit, dual>a1.txt;
printf "-----\n">a1.txt;
```

OUTPUT:

Objective value = 21.00		
<hr/>		
Product	Quantity	Profit(\$)
exterior	3.00	15.00
interior	1.50	6.00
<hr/>		
Constraint	Slack amount	Dual price
m1	0.00	0.75
m2	0.00	0.50
demand	2.50	0.00
market	0.50	0.00
<hr/>		

Set A.5c

Sets paint and resource cannot be read from the double-subscripted table RM4aij, and hence will not be defined for unitprofit and rhs.

```

#-----sets
set resource;
set paint;
#-----parameters
param unitprofit{paint};
param rhs {resource};
param aij {resource,paint};
#-----variables
var product{paint} >= 0;
#-----model objective
maximize profit: sum {j in paint}
unitprofit[j]*product[j];
#-----model constraints
subject to limit {i in resource}:
    sum {j in paint} aij[i,j]*product[j] <= rhs[i];
#-----read tables
table RM4profit IN: paint<-[COL1], unitprofit~COL2;
table RM4rhs IN: resource<-[constName],
rhs~availability;
table RM4aij IN: [resource,paint], aij;





```

1

(a)

```

let rhs["m1"]:=20;
for {i in 1..100000}
{
    solve;
    display rhs["m1"],product;
    if rhs["m1"]==35 then break;
    let rhs["m1"]:=rhs["m1"]+5;
}

```

(b)

```

let rhs["m1"]:=20;
repeat while rhs["m1"]<=35
{
    solve;
    display rhs["m1"],product;
    let rhs["m1"]:=rhs["m1"]+5;
}

```

(c)

```

let rhs["m1"]:=20;
repeat until rhs["m1"]>35
{
    solve;
    display rhs["m1"],product;
    let rhs["m1"]:=rhs["m1"]+5;
}

```

1

Chapter 2 Cases

2-1

The cost distribution as figured out by the Accounting Department is not suitable for economic analysis. The purchasing cost of 3,000,000 lb of oranges ($= .19 \times 3,000,000 = \$570,000$) is fixed, and hence plays no role in the development of the model. The variable cost per 5-gallon can should be recomputed to exclude the fixed cost of purchasing the oranges--that is,

For jam:

$$\begin{aligned} \text{Sales price/can} &= \$15.50 \\ \text{Variable cost/can} &= 9.85 - 5 \times (25.61/100) \\ &= \$8.57 \\ \text{Gross profit per lb} &= (15.50 - 8.57)/5 \\ &= \$1.39 \end{aligned}$$

For concentrate:

$$\begin{aligned} \text{Sales price/can} &= \$30.25 \\ \text{Variable cost/can} &= 21.05 - 30 \times (21.22/100) \\ &= \$14.68 \\ \text{Gross profit per lb} &= (30.25 - 14.68)/30 \\ &= \$0.52 \end{aligned}$$

For juice:

$$\begin{aligned} \text{Sales price/can} &= \$20.75 \\ \text{Variable cost/can} &= 13.28 - 30 \times (21.22/100) \\ &= \$10.10 \\ \text{Gross profit per lb} &= (20.75 - 10.10)/15 \\ &= \$0.71 \end{aligned}$$

LP Model:

$$\begin{aligned} x_1 &= \text{lb oranges used in jam} \\ x_2 &= \text{lb oranges used in concentrate} \\ x_3 &= \text{lb oranges used in juice} \end{aligned}$$

$$\text{Maximize } z = 1.39x_1 + .52x_2 + .71x_3 - \$570,000$$

subject to

$$\begin{aligned} x_1 &\leq .3 \times 3,000,00 \\ x_1/5 &\leq 10,000 \\ x_2/30 &\leq 12,000 \\ x_3/15 &\leq 40,000 \\ x_2/30 &\geq 2(x_3/15) \\ x_2 + x_3 &\leq .6 \times 3,000,000 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The second constraint dominates the first constraint. This means that there is a definite surplus of at least $900,000 - 5 \times 10,000 = 850,000$ lb of Grade I oranges. Because concentrate and juice can use Grade I, the last constraint should be changed to

$x_2 + x_3 \leq 1,800,000 + 850,000 = 2,650,000$
This means that the extra 850,000 lb of grade I can be used to produce concentrate and juice, if necessary.

Solution:

Final Iteration No: 5
Objective value (max) = 320603.4688

Variable	Value	Obj Coeff	Obj Val Contrib
x_1	-50000.0000	1.3900	69500.0000
x_2	360003.5938	0.5200	187201.8594
x_3	90002.2500	0.7100	63901.5938

2-2

Let

$$\begin{aligned} x_{ij} &= \text{Rolls produced of type } i \text{ in month } j \\ s_{ij} &= \text{Rolls purchased of type } i \text{ in month } j \end{aligned}$$

The objective function can be formulated to minimize the total cost of producing the rolls internally and/or purchasing them externally.

The maximum demand from the two mills can be summarized as:

	Roll 1	Roll 2	Roll 3
Month 1	700	300	400
Month 2	300	500	700
Month 3	100	400	500

LP Model:

$$\begin{aligned} \text{Maximize } z &= 90(x_{11} + x_{12} + x_{13}) + 130(x_{21} + x_{22} + x_{23}) + 180(x_{31} + x_{32} + x_{33}) + 108(s_{11} + s_{12} + s_{13}) + 145(s_{21} + s_{22} + s_{23}) + 194(s_{31} + s_{32} + s_{33}) \end{aligned}$$

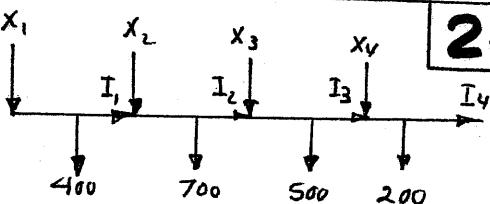
subject to

$$\begin{aligned} x_{11} + x_{12} + x_{13} + 5(x_{21} + x_{22} + x_{23}) + 7(x_{31} + x_{32} + x_{33}) &\leq 3 \times 320 \times 10 \\ 4(x_{21} + x_{22} + x_{23}) + 6(x_{31} + x_{32} + x_{33}) &\leq 3 \times 310 \times 8 \\ 6(x_{11} + x_{12} + x_{13}) + 3(x_{21} + x_{22} + x_{23}) &\leq 3 \times 300 \times 9 \\ 3(x_{11} + x_{12} + x_{13}) + 6(x_{21} + x_{22} + x_{23}) + 9(x_{31} + x_{32} + x_{33}) &\leq 3 \times 320 \times 10 \\ x_{11} + s_{11} &= 700, x_{12} + s_{12} = 300, x_{13} + s_{13} = 400 \\ x_{21} + s_{21} &= 300, x_{22} + s_{22} = 500, x_{23} + s_{23} = 700 \\ x_{31} + s_{31} &= 100, x_{32} + s_{32} = 400, x_{33} + s_{33} = 500 \\ \text{all } x_{ij}, s_{ij} &\geq 0 \end{aligned}$$

The solution of this model by TORA will show that it is cheaper to purchase all the rolls from outside source than to produce them locally. On the other hand, if we try to limit outside purchases, s_{ij} , to 5% of the total demand, as specified by the company, the problem will have no feasible solution.

These results point to the possibility that the company should be studying whether or not its present operation is economically viable.

2-3



$$\begin{aligned} \text{Minimize } Z &= 100(I_1 + I_2 + I_3 + I_4) \\ &+ 60(\bar{y}_{12} + \bar{y}_{23} + \bar{y}_{34}) \\ &+ 50(\bar{y}_{12}^+ + \bar{y}_{23}^+ + \bar{y}_{34}^+) \end{aligned}$$

Subject to

$$x_1 - I_1 = 400$$

continued...

Chapter 2 Cases

2-4

$$\begin{aligned}
 x_2 + I_1 - I_2 &= 700 \\
 x_3 + I_2 - I_3 &= 500 \\
 x_4 + I_3 - I_4 &= 200 \\
 x_1 - x_2 - y_{12}^+ + y_{12}^- &= 0 \\
 x_2 - x_3 - y_{23}^+ + y_{23}^- &= 0 \\
 x_3 - x_4 - y_{34}^+ + y_{34}^- &= 0 \\
 \text{all variables } &\geq 0
 \end{aligned}$$

Title: Comprehensive Problem 2-3
 Final Iteration No: 8
 Objective value (min) = 32500.0000

Variable	Value	Obj Coeff	Obj Val Contrib
x1 x1	550.0000	0.0000	0.0000
x2 x2	550.0000	0.0000	0.0000
x3 x3	200.0000	0.0000	0.0000
x4 x4	200.0000	0.0000	0.0000
x5 I1	150.0000	100.0000	15000.0000
x6 I2	0.0000	100.0000	0.0000
x7 I3	0.0000	100.0000	0.0000
x8 I4	0.0000	100.0000	0.0000
x9 y12+	0.0000	50.0000	0.0000
x10 y12-	0.0000	60.0000	0.0000
x11 y23+	350.0000	50.0000	17500.0000
x12 y23-	0.0000	60.0000	0.0000
x13 y34+	0.0000	50.0000	0.0000
x14 y34-	0.0000	60.0000	0.0000

Constraint	RHS	Slack(-)/Surplus(+)
1 (=)	400.0000	0.0000
2 (=)	700.0000	0.0000
3 (=)	200.0000	0.0000
4 (=)	200.0000	0.0000
5 (=)	0.0000	0.0000
6 (=)	0.0000	0.0000
7 (=)	0.0000	0.0000

$i = \begin{cases} 1 & \text{represent chairs} \\ 2 & \text{represent tables} \\ 3 & \text{represent bookshelves} \end{cases}$

x_{ij} = units of type i produced in period j
 s_{ij} = units of type i sold in period j
 z_{ij} = inventory units of type i left at the end of period j

The objective function includes three components:

1. Sales revenue
2. Production cost
3. Inventory cost

The constraints include

1. Production capacity
2. Demand limit
3. Labor requirement
4. Inventory balance equations

$$\begin{aligned}
 \text{Maximize } Z = & \sum_{j=1}^3 (45s_{1j} + 100s_{2j} + 20s_{3j}) \\
 & - (25x_{1j} + 65x_{2j} + 10x_{3j}) \\
 & - .02(25z_{1j} + 65z_{2j} + 10z_{3j})
 \end{aligned}$$

subject to

$$x_{1j} \leq 3000, x_{2j} \leq 1000, x_{3j} \leq 580, j=1,2,3$$

$$s_{11} \leq 2800, s_{12} \leq 2300, s_{13} \leq 3350$$

$$s_{21} \leq 500, s_{22} \leq 800, s_{23} \leq 1400$$

$$s_{31} \leq 320, s_{32} \leq 300, s_{33} \leq 600$$

$$20x_{1j} + 40x_{2j} + 15x_{3j} \leq \underbrace{150 \times 5 \times 4}_{j=1,2,3} \times 2 \times 8 \times 60$$

$$z_{ij} = z_{i,j-1} + x_{ij} - s_{ij}, i=1,2,3$$

$$z_{10} = 30, z_{20} = 100, z_{30} = 50$$

all variables ≥ 0

The effect of leaves can be investigated by changing the right-hand side of labor requirement constraint.

Chapter 3 Cases

3-1

Although LP theory says that in a simplex method solution the number of positive variables cannot exceed the number of constraints, the real issue in the presentation by the OR analyst is that the model is not complete. The fact that the manager insists that all five products must be produced indicates that some important restrictions are missing. In particular, the impact of the competition appears to be important, at least from the manager's standpoint. Such restrictions must then be taken into account.

Although the analyst is correct in stating that LP theory requires more constraints to produce a desired product mix, the argument should be made from the standpoint of formulating the model properly and realistically. Once this task is done, it would not be necessary to "bargain" with the manager about the need to add more constraints. Rather, the realistic model will indicate whether or not the production system is operating optimally.

Of course, when all the restrictions of the problem are taken into account, it may well be that the resulting model would not be a linear program at all.

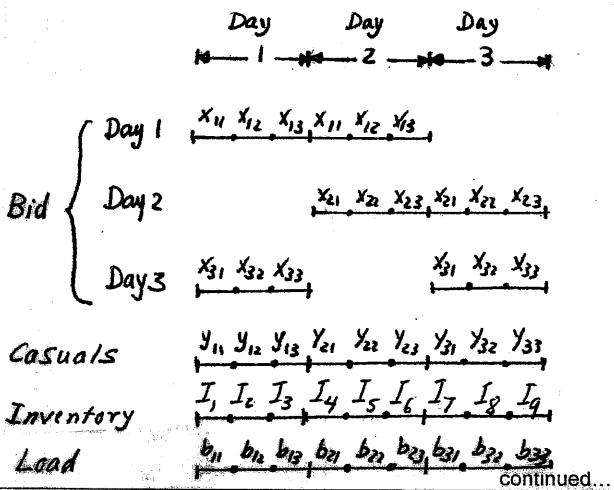
The conclusion from the analysis of this situation is that one must always aim at formulating the model to represent the original system as realistically as possible.

Consider a scaled-down model:

3-2

1. 3-day week
2. Each worker works 2 consecutive days per 3-day week
3. Loads are assumed constant over each 8-hour shift

The situation can be depicted as follows:



continued...

The situation assumes that the number of casuials that can be hired is limited; that is,

$$y_{ij} \leq c_{ij} \text{ for all } i \text{ and } j$$

The inventory variables I_i represent is the load amount left unfinished from the preceding period $i-1$

For the purpose of defining the objective function, we want to minimize the number of bid and casual workers used, we will assign the bid workers half the weight we assign to the casual workers. Thus, the objective function is

$$\text{Minimize } Z = \sum_{i=1}^3 \sum_{j=1}^3 x_{ij} + 2 \sum_{i=1}^3 \sum_{j=1}^3 y_{ij}$$

The constraints take into account the fact that a load is allowed to stay on the dock up to 16 hours (or two shifts). Thus, (See the chart)

$$\begin{aligned}
 (x_{11} + x_{12}) + (x_{31} + x_{32}) + y_{11} &\geq b_{11} \\
 (x_{12} + x_{13}) + (x_{32} + x_{33}) + y_{12} &\geq b_{12} \\
 (x_{11} + x_{33}) + (x_{13} + x_{21}) + y_{13} &\geq b_{13} \\
 (x_{11} + x_{12}) + (x_{21} + x_{22}) + y_{21} &\geq b_{21} \\
 (x_{12} + x_{13}) + (x_{22} + x_{23}) + y_{22} &\geq b_{22} \\
 (x_{13} + x_{21}) + (x_{23} + x_{31}) + y_{23} &\geq b_{23} \\
 (x_{21} + x_{22}) + (x_{31} + x_{32}) + y_{31} &\geq b_{31} \\
 (x_{22} + x_{23}) + (x_{32} + x_{33}) + y_{32} &\geq b_{32} \\
 (x_{11} + x_{33}) + (x_{23} + x_{31}) + y_{33} &\geq b_{33} \\
 x_{11} + x_{31} + y_{11} - I_1 + I_2 &= b_{11} \\
 x_{12} + x_{32} + y_{12} - I_2 + I_3 &= b_{12} \\
 x_{13} + x_{33} + y_{13} - I_3 + I_4 &= b_{13} \\
 x_{11} + x_{21} + y_{21} - I_4 + I_5 &= b_{21}
 \end{aligned}$$

continued...

Chapter 3 Cases

$$x_{12} + x_{22} + y_{22} - I_5 + I_6 = b_{22}$$

$$x_{13} + x_{23} + y_{23} - I_6 + I_7 = b_{23}$$

$$x_{21} + x_{31} + y_{31} - I_7 + I_8 = b_{31}$$

$$x_{22} + x_{32} + y_{32} - I_8 + I_9 = b_{32}$$

$$x_{23} + x_{33} + y_{33} - I_9 + I_1 = b_{33}$$

$$y_{ij} \leq c_{ij}$$

all variables are nonnegative.

The model has been tested for a 7-day week, three shifts per day, with the loads held constant over 4-hour intervals (instead of 8). The results were successfully implemented.

x_{ij} = bbl from line item i allocated to bidder j

3-3

b_{ij} = Bid for line item i by bidder j

c_i = capacity (bbl) of line item i

$$\text{Maximize } Z = \sum_{i=1}^6 \sum_{j=1}^8 b_{ij} x_{ij}$$

$$\text{s.t. } \sum_{i=1}^6 x_{ij} \leq 36,000, j=1,3,\dots,8 \quad (1)$$

$$\sum_{j=1}^8 x_{ij} \leq c_i, i=1,2,\dots,6 \quad (2)$$

$$x_{ij} \geq 0$$

Solution: (See file amp Case 3-3.txt)

1 2 3 4 5 6 7 8

1	20						
2		13	17				
3			6	19			
4	4				36	40	
5	19	16					
6			30				

Totals 4 19 36 0 13 36 36 36

$$\text{Revenue} = \$202,180$$

Ranking solution:

	1	2	3	4	5	6	7	8
1			20					
2				16				
3					35			
4	4							
5						30		
6							22	
Total	34	35	36	0	0	3	36	36

$$\text{Revenue} = \$201,550$$

LP solution is better than the ranking solution by \$630.

To investigate the effect of setting the 20% limit, the associated constraints (1) yield the following dual prices:

Constraint	Dual price
1	0
2	0
3	.03
4	0
5	.01
6	.02
7	.02
8	.02

The fact that some of the dual prices are positive shows that there are advantages in raising the limit above 20%, as this will increase the total revenue. Indeed, the following table shows the change in revenue with the limit:

Percentage	Revenue (\$)
20	202,180
30	203,110
40	203,320
≥ 48	203,450

Thus, it is advantageous in this case to raise the limit to 48%, as this will raise the revenue from \$202,180 at 20% to \$203,450. The best percentage is data dependent.

continued...

Chapter 4 Cases

$$\text{Maximize } Z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$(F1) \quad x_1 + 2x_2 + x_3 \leq 430 \quad (1)$$

$$(F2) \quad 3x_1 + 2x_3 \leq 460 \quad (2)$$

$$(R1) \quad x_1 + 4x_2 \leq 420 \quad (3)$$

$$(R2) \quad x_1 + x_2 + x_3 \leq 300 \quad (4)$$

$$x_3 \leq 240 \quad (5)$$

$$x_1, x_2, x_3 \geq 0$$

Optimum Solution (excluding (5))

Title: Comprehensive Problem 4-1

Final Iteration No: 3

Objective value (max) = 1290.0000

Variable	Value	Obj Coeff	Obj Val Contrib
x1 P1	0.0000	3.0000	0.0000
x2 P2	70.0000	2.0000	140.0000
x3 P3	230.0000	5.0000	1150.0000

Constraint	RHS	Slack(-)/Surplus(+)
1 (<=)	430.0000	60.0000-
2 (<=)	460.0000	0.0000-
3 (<=)	420.0000	140.0000-
4 (<=)	300.0000	0.0000-

*** SENSITIVITY ANALYSIS ***

Objective coefficients -- Single Changes:

Variable	Current Coeff	Min Coeff	Max Coeff	Reduced Cost
x1 P1	3.0000	-infinity	6.5000	3.5000
x2 P2	2.0000	0.0000	5.0000	0.0000
x3 P3	5.0000	2.6667	infinity	0.0000

Right-hand Side -- Single Changes:

Constraint	Current RHS	Min RHS	Max RHS	Dual Price
1 (<=)	430.0000	370.0000	infinity	0.0000
2 (<=)	460.0000	390.0000	600.0000	1.5000
3 (<=)	420.0000	280.0000	infinity	0.0000
4 (<=)	300.0000	230.0000	330.0000	2.0000

The given optimum solution satisfies constraint (3). Hence (3) is redundant.

The associated optimal inverse B^{-1} is

$$X_B = \begin{pmatrix} x_4 \\ x_3 \\ x_6 \\ x_2 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & -1/2 & 0 & -2 \\ 0 & 1/2 & 0 & 0 \\ 0 & 2 & 1 & -4 \\ 0 & -1/2 & 0 & 1 \end{pmatrix}$$

Question 1:

The additional constraint

$$x_3 \leq 210$$

is not satisfied by the given optimal solution. We use the dual simplex method to recover the new feasible

continued...

4-1

Solution. Then, we study the effect of the 20% increase in profitability on the optimality of the resulting feasible solution.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Solution
Z	7/2	0	0	0	3/2	0	2	0	1290
x_4	1/2	0	0	1	1/2	0	-2	0	60
x_3	3/2	0	1	0	1/2	0	0	0	230
x_6	3	0	0	0	2	1	-4	0	140
x_2	-1/2	1	0	0	-1/2	0	1	0	70
x_8	0	0	1	0	0	0	0	1	210
Z	7/2	0	0	0	3/2	0	2	0	1290
x_4	1/2	0	0	1	1/2	0	-2	0	60
x_3	3/2	0	1	0	1/2	0	0	0	230
x_6	3	0	0	0	2	1	-4	0	140
x_2	-1/2	1	0	0	-1/2	0	1	0	70
x_8	-3/2	0	0	0	-1/2	0	0	1	-20
Z	0	0	0	0	1/3	0	2	7/3	1243/3
x_4	0	0	0	1	1/3	0	-2	1/3	53 1/3
x_3	0	0	1	0	0	0	0	0	210
x_6	0	0	0	0	1	1	-4	2	100
x_2	0	1	0	0	-1/3	0	1	-1/3	76 2/3
x_1	1	0	0	0	1/3	0	0	-2/3	13 1/3

Next, we study the effect of changing the unit profit by +20% for variable x_3 .

New dual values

$$Y = (0, 6, 0, 2, 3) \begin{bmatrix} 1 & -1/2 & 0 & -2 & 1/3 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 2 & 1 & -4 & 1 \\ 0 & -1/2 & 0 & 1 & -1/3 \\ 0 & 1/3 & 0 & 0 & -2/3 \end{bmatrix} = (0, 1/3, 0, 2, 10/3)$$

continued...

Chapter 4 Cases

The new objective row thus reads as

	x_1	x_2	s_3	s_4	x_5	x_6	x_7	x_8	
z	0	0	0	0	$\frac{1}{3}$	0	2	$\frac{10}{3}$	1453.33

Thus, the last basic solution remains optimal.

Conclusions:

1. The new constraint $x_3 \leq 210$ reduces the profit from \$1290 to \$1243.33.
2. The 20% increase in the unit profit of x_3 increases the total profit to \$1453.33.
3. The proposal is acceptable because, in the end, the total profit increases from \$1290 to \$1453.33.

Question 2:

From TORA's printout,
worth per unit of $R_2 = \$2$

Because the unit price of additional units of R_2 is \$3 higher than the present supplier, the additional cost is not justifiable economically.

Question 3:

Objective Coefficients -- Simultaneous Changes d:

Nonbasic Var Optimality Condition

x_1 : P1	3.5000 +	1.5000 d3 +	-0.5000 d2 - d1 >= 0
s_5	1.5000 +	0.5000 d3 +	-0.5000 d2 >= 0
s_7	2.0000 +	1.0000 d2 >= 0	

Right-hand Side Ranging -- Simultaneous Changes D:

Basic Var Value/Feasibility Condition

s_4	60.0000 +	1.0000 d1 +	0.5000 d2 +	-2.0000 d4 ≥ 0
x_3 : P3	250.0000 +	0.5000 d2 >= 0		
s_6	140.0000 +	2.0000 d2 +	1.0000 d3 +	-4.0000 d4
x_2 : P2	70.0000 +	-0.5000 d2 +	1.0000 d4 >= 0	

TORA printout provides the effect of simultaneous changes in the availability of the resources. We can get the same result by considering

continued...

$$\begin{pmatrix} x_4 \\ x_3 \\ x_6 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & 0 & -2 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & 1 & -4 \\ 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 430 + D_1 \\ 460 + D_2 \\ 420 \\ 300 \end{pmatrix}$$

$$= \begin{pmatrix} 60 + D_1 + .5 D_2 \\ 230 + .5 D_2 \\ 140 + 2 D_2 \\ 70 - .5 D_2 \end{pmatrix}$$

Thus, for $D_1 = D_2 = 40$, the new solution is

$$x_1 = 0, x_2 = 50, x_3 = 250,$$

$$x_4 = 120, x_5 = 0, x_6 = 220, x_7 = 0$$

which is feasible.

$$\text{New profit} = 3x_0 + 2x_50 + 5x_{250} \\ = \$1350$$

$$\text{Increase in profit} = 1350 - 1290 = \$60$$

This result can also be computed from the dual objective function as

$$\Delta Z = D_1 y_1 + D_2 y_2 \\ = 40x_0 + 40x_{\frac{3}{2}} = \$60$$

Because the worth per unit of F_1 is zero, it means that resource F_1 is already abundant (indeed, $x_4 = 60$ minutes). Hence, we need to increase F_2 only by $D_2 = 40$ min at the cost of \$35, and the proposal is justifiable economically.

Question 4:

This question can be analyzed by adding the constraint $x_2 \geq 100$ and applying the dual simplex method.

The new optimum tableau is

continued...

Chapter 4 Cases

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
Z	2	0	0	3	0	0	0	5	1200
x_5	0	0	0	1	1	0	0	-1	30
x_6	1	0	0	-2	0	1	0	-2	60
x_7	1	0	0	4	0	0	1	0	20
x_3	1	0	1	1	0	0	0	1	200
x_2	0	1	0	-1	0	0	0	0	100

The new restriction reduces the profit by \$90.

We should have expected this result even before the new tableau is computed. The reason is that the present solution does not satisfy the new constraint. Hence, the value of the objective function must deteriorate.

Questions

Decrease in the unit processing time of P1 on F2 will produce the following reduced cost:

$$\begin{aligned} 1y_1 + (3-1)y_2 + 1y_3 + 1y_4 - \$3 \\ = 1 \times 0 + 2 \times \frac{3}{2} + 1 \times 0 + 1 \times 2 - 3 \\ = 2 \end{aligned}$$

Thus, the reduction in the processing still would not make P1 profitable

The first proposal should **4-2** not produce the desired results because it is based on an averaging procedure that does not have a valid theoretical basis. The second proposal may produce the desired result provided that the remaining two constraints are not violated.

We can check both proposals by computing

$$x_B = B^{-1} b^*$$

where b^* is the new right-hand side; that is

$$b^* = \begin{pmatrix} 32.4 \\ 14.4 \\ 1 \\ 2 \end{pmatrix} \text{ for proposal 1}$$

$$= \begin{pmatrix} 30 \\ 7.5 \\ 1 \\ 2 \end{pmatrix} \text{ for proposal 2}$$

Proposal 1:

$$x_B = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 32.4 \\ 14.4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ 6.75 \\ -4.85 \\ -4.75 \end{pmatrix}$$

The proposal does not result in 25% increase in x_1 and x_2 . Moreover, the resulting solution is infeasible.

Proposal 2:

$$x_B = \begin{pmatrix} 1/4 & -1/2 & 0 & 0 \\ -1/8 & 3/4 & 0 & 0 \\ 3/8 & -5/4 & 1 & 0 \\ 1/8 & -3/4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 30 \\ 7.5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3.75 \\ 1.875 \\ 2.875 \\ 1.125 \end{pmatrix}$$

New solution is feasible. It also results in the desired 25% increase in x_1 and x_2 .

continued...

Chapter 5 Cases

First, we compute the percent distribution of commodities to the different sites:

5-1

Site	Percent
N1	.6 x .85 = .51
N2	.6 x .15 = .09
C1	.15 x .6 = .09
C2	.15 x .4 = .06
S1	.25 x .8 = .20
S2	.25 x .2 = .05

Next, instead of dealing with three different types of products, we convert all them to equivalent returnables bottles by using the given transportation cost factors of 60% for cans and 70% for nonreturnables. The supply amount at the new factory equals the difference between the demand and supply at the existing plant.

Yr.	Returnables	Cans	Non-returnables	Equivalent returnables
	Demand	2400	1750	490
	Supply			3795
1	current	1800	1250	350
	new	600	500	140
2	Demand	2450	2000	500
	Supply			4000
	current	1850	1350	380
	new	600	650	120
3	Demand	2600	2300	600
	Supply			4400
	current	1900	1400	400
	new	700	900	200
4	Demand	2800	2650	650
	Supply			4845
	current	2050	1500	400
	new	750	1150	250
5	Demand	3100	3050	720
	Supply			5434
	current	2150	1800	450
	new	950	1250	270

continued...

Let P_1 , P_2 , and P_3 represent the locations of the existing plant, the central plant, and the south plant. The generic transportation model for each period is given as

	N1	N2	C1	C2	S1	S2	
P_1	.8	1.2	1.5	1.6	2.9	2.1	a_1
P_2	1.3	1.9	1.05	.8	1.5	1.7	a_2
P_3	1.9	2.9	1.2	1.6	.9	.8	a_3

b_{N1} b_{N2} b_{C1} b_{C2} b_{S1} b_{S2}

a_1 = Supply of equivalent returnables at existing plant

a_2 = Supply of equivalent returnables at new central plant
= 0, if new plant is located south

a_3 = Supply of equivalent returnables at new south plant
= 0, if plant is located center

b_j = Total (equivalent returnables) demand for the year \times allocated proportion for the site. ($j = N1, N2, C1, C2, S1, S2$)

For example, for year 1, we have

$a_1 = 2795$

$a_2 = \begin{cases} 998, & \text{if new plant is in center} \\ 0, & \text{if new plant is in south} \end{cases}$

$a_3 = \begin{cases} 998, & \text{if new plant is in south} \\ 0, & \text{if new plant is in center} \end{cases}$

$b_{N1} = .51 \times 3795 \approx 1939$

$b_{N2} = .09 \times 3795 \approx 342$

$b_{S2} = .05 \times 3795 \approx 190$

The following table gives all a_i and b_j

continued...

Appendix E

Case Studies

Chapter 5 Cases

Yr	a_1	5-1 continued						
		a_2 or a_3						
1	2795	998						
2	2926	1074						
3	3020	1380						
4	3230	1615						
5	3545	1889						
Total demand	b_{N1}	b_{N2}	b_{C1}	b_{C2}	b_{S1}	b_{S2}		
1	3793	1934	341	341	228	759	190	
2	4000	2040	360	360	240	800	200	
3	4400	2244	396	396	264	880	220	
4	4845	2471	436	436	291	969	242	
5	5434	2771	489	489	326	1087	272	

From the transportation model, we obtain the following summary:

Yr	Minimum cost (\$) given P_2	Minimum cost (\$) given P_3
1	4182.85	3653.10
2	3828.80	4039.20
3	4632.60	4341.20
4	5005.60	4665.70
5	4807.40	4922.80
Totals	22,457.25	21,622.00

By locating the plant in the south, we save

$22,457.25 - 21,622.00 = \835.25
over a period of 5 years, or
approximately $\$167.05$ per year.

The result shows that the transportation cost is not an important factor in the selection of the location (a saving of $\$167.05$ is not significant). Thus, other factors must be considered in the determination of the site of the new plant.

(a) Optimum tableau

5-2

	1	2	3	4	5	6	7	8	9	
1		192	444	216					108	960
2	62					7		90	42	201
3		21			50					71
4		4					20			24
5			99							99
	62	217	444	315	50	7	20	90	150	

The optimum solution corresponds to $1,886,300$ ($m^3 \times 100m$). Thus,
Cost savings $= (2,495,000 - 1,886,300) \times \6.65
 $= \$395,655$

(b) Divide the model into two phases:
Phase 1 is dedicated to building the perimeter road, and Phase 2 is used to build the roads that can be constructed only after the perimeter road is built.

We cannot use the transportation model, but must use a regular linear program that permits building the perimeter road in Phase 1 and the cross-roads in Phase 2.

Phase 1 distances (d_{ij1})

	1	2	3	4	5	6	7	8	9	
1			12	10		18	11	85	20	
2			14	12		20	13	10	22	
3		16	20		20	15		6	22	18
4		20	22		22	6		14		18
5			10	4				14	21	

All empty squares have a distance $M = 999999$
continued...

Chapter 5 Cases

The LP model is thus developed as follows: Define

$x_{ijk} = \text{amount in } (m^3 \times 100m)$
 moved from source i to
 destination j during phase k

$$\text{Minimize } Z = \sum_i \sum_j \sum_k d_{ijk} x_{ijk}$$

Subject to

$$\sum_j \sum_k x_{ijk} = a_i, \quad i=1, 2, \dots, 5$$

$$\sum_i \sum_{jk} x_{ijk} = b_j, \quad j=1, 2, \dots, 9$$

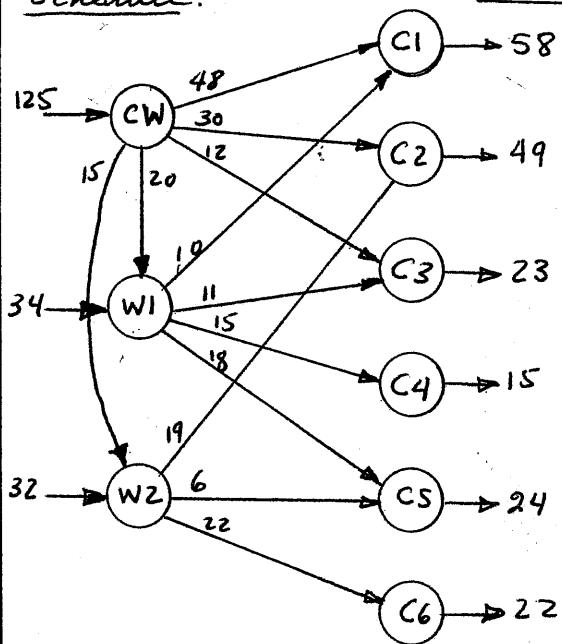
$$x_{ijk} \geq 0 \text{ for all } i, j, k$$

Optimum Solution (Phase I movements)

	1	2	3	4	5	6	7	8	9
1			444	216		7			150
2								90	90
3		21			50				7
4		4					20		2
5				99					9
	0	25	444	315	50	7	20	90	150

Current transportation
Schedule:

5-3



Associated van-miles

$$= 20 \times 5 + 15 \times 45 + \\ 48 \times 50 + 30 \times 30 + 12 \times 30 + \\ 10 \times 38 + 11 \times 30 + 15 \times 8 + 18 \times 10 + \\ 19 \times 35 + 6 \times 7 + 22 \times 90$$

$$= 8132$$

Next, we solve the problem as a transshipment model, using the same supply/demand amounts. continued...

Chapter 5 Cases

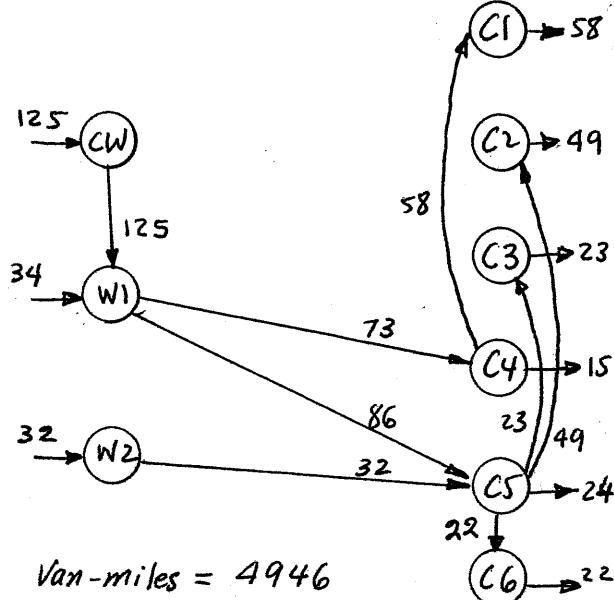
B = 191

5-3 continued

	C1	C2	C3	C4	C5	C6			
1	0	5	45	50	30	30	60	75	80
2	5	0	80	38	70	30	8	10	60
3	45	80	0	85	35	60	55	7	90
4	50	38	85	0	20	90	25	30	70
5	30	70	35	20	0	40	90	15	10
6	30	30	60	40	40	0	10	6	90
7	60	8	55	25	90	10	0	80	40
8	75	10	7	30	15	6	80	0	15
9	80	60	90	70	10	90	40	15	0

B B B $\frac{58}{B}$ $\frac{49}{B}$ $\frac{23}{B}$ $\frac{15}{B}$ $\frac{24}{B}$ $\frac{22}{B}$

Optimum Solution from TORA



Chapter 5 Cases

5-4

Layover times are computed depending on whether the roundtrip starts from A or from B. If the departure time from one city is not at least 90 minutes later than the arrival time of the crew at the same city, the crew must wait till the next day. For example, flight 1 arrives in City B at 8:30 and flight 10 leaves B at 7:30. If the crew of flight 1 is to return on flight 10, it must have a layover of 23 hours.

1. Layover time when roundtrip starts at A

Flight	10	20	30	40
1	23	24.75	8.00	11.50
2	20.75	22.50	5.75	9.25
3	15.50	17.25	24.50	4.00
4	14.00	15.75	23.00	2.50

2. Layover time when roundtrip starts at B

	10	20	30	40
1	20.5	18.75	11.50	8.00
2	22.75	21	15	
3	4.00	1.75	19	15.50
4	5.50	3.75	20.50	17.00

The two tables are combined such that the base with the smaller layover time is used. The result is the following combined matrix:

continued...

Chapter 5 Cases

= roundtrip starts in B

5-4 continued

	10	20	30	40	
1	20.5	18.75	8	8	1
2	20.75	21	5.75	9.25	1
3	4	1.75	19	4	1
4	5.5	3.25	20.5	2.5	1

1 1 1 1

TOA optimum solution:

1-40, 2-30, 3-20, 4-10

The solution is interpreted as

$B \rightarrow A(40)$, $A \rightarrow B(1)$

$A \rightarrow B(2)$, $B \rightarrow A(30)$

$B \rightarrow A(20)$, $A \rightarrow B(3)$

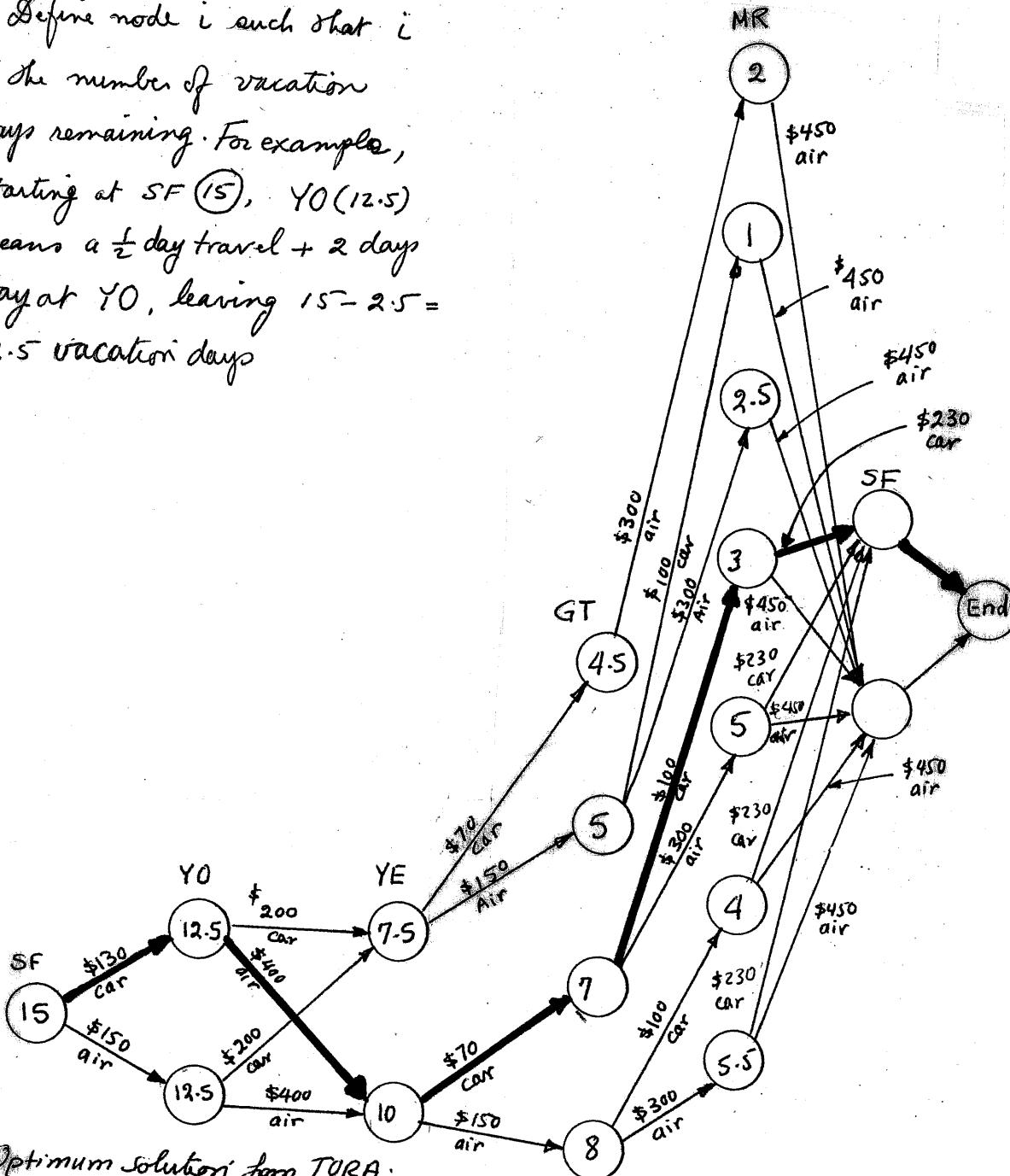
$B \rightarrow A(10)$, $A \rightarrow B(4)$

The optimum solution calls for stationing 1 crew in A and 3 crews in B.

Chapter 6 Cases

6-1

Define node i such that i is the number of vacation days remaining. For example, starting at SF (15), $Y0(12.5)$ means a $\frac{1}{2}$ day travel + 2 days stay at Y0, leaving $15 - 2.5 = 12.5$ vacation days

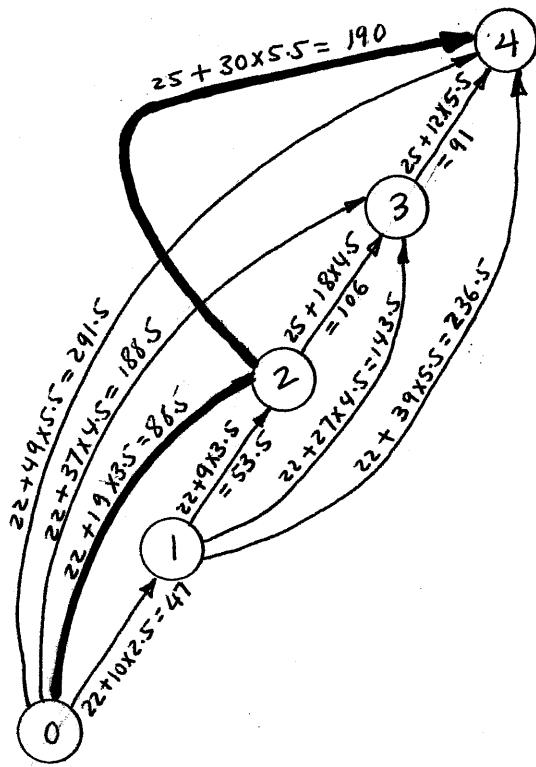


$$\text{Trip time} = (.5 + .5 + 1 + 2 + 3) + 4 \times 2 = 15 \text{ days}$$

$$\text{Trip cost} = \$930$$

Chapter 6 Cases

Arrange the books in ascending order so that node 1 represent the 6" books and node 4 represent the 12" books. The network starts from node 0. An arc from node i to node j , $i < j$, signifies that all the books of heights h_i, h_{i+1}, \dots, h_j are placed in a shelf of height h_j . The length of the arc equals the associated fixed plus variable costs. The optimum solution is given by the shortest route from node 0 to node 4.



$$\text{Total cost} = \$278.50$$

Solution:

Produce 19 ft of height 8" and 30 ft of height 12"

6-2

6-3

Shipment	Shipping Route	Delivery Date
1	A to D	10
2	A to E	15
3	B to D	4
4	B to E	5
5	C to E	18

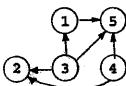
A	B	C	D	E
A			3	4
B			3	2
C			3	5
D	2	2	2	
E	3	1	4	

The ships schedules can be summarized as shown below.

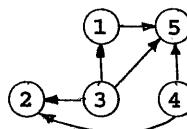
0--1--2--3--4--5--6--7--8--9--10--11--12--13--14--15--16--17--18--19--20--21--22

1> A-----B-----A,B,C
2> B-----D-----A,B,C
3> B-----E-----A-C
4> B-----E-----A-C
5> C-----E-----A-C

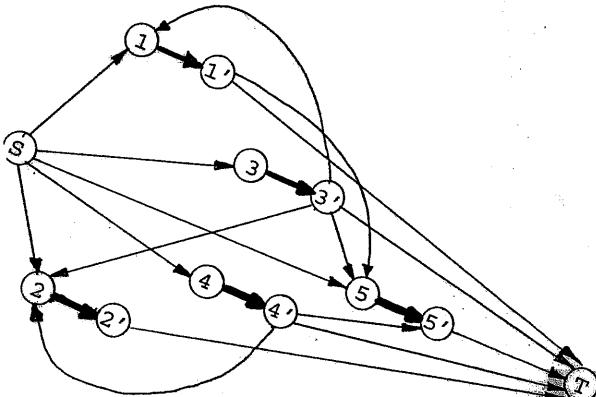
Precedence Relationships



Precedence Relationships



Flow Model



continued...

Chapter 6 Cases

In the flow model, the flow in arcs $(i - i')$, $i = 1, 2, \dots, 5$ must equal 1 to realize a feasible solution. The different arcs of the model represent the precedence relationships of the feasible schedule. The minimum flow from node S to node E will provide the minimum number of ships required to meet the proposed schedule.

The procedure for determining the minimum flow in the network consists of the following steps:

Step 1: Determine a feasible solution to the flow model from $S \rightarrow E$

Step 2: Determine the residue network of the feasible solution

Step 3: Determine the maximum flow in the residue network from $E \rightarrow S$; that is, from the end node E to the start node S.

Step 4: Determine the minimum flow from $S \rightarrow E$ as
feasible flow $S \rightarrow E - \text{maxflow } E \rightarrow S$

Step 1:

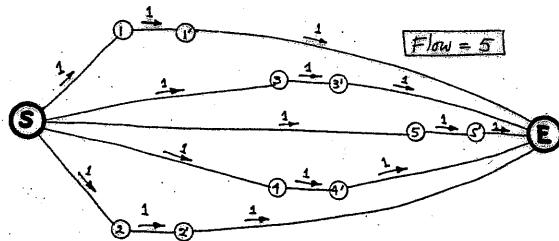
The easiest way to finding a feasible solution is to assume that each route is served by a separate ship. The network below provides such a solution.

continued...

E-16

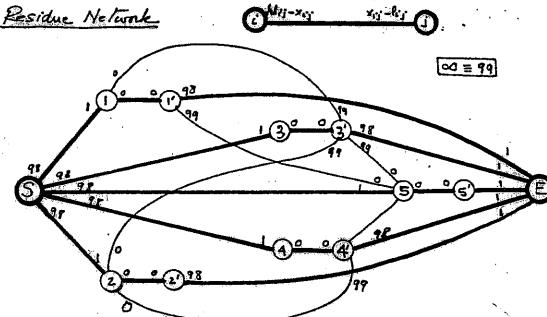
6-3 continued

6-3 continued

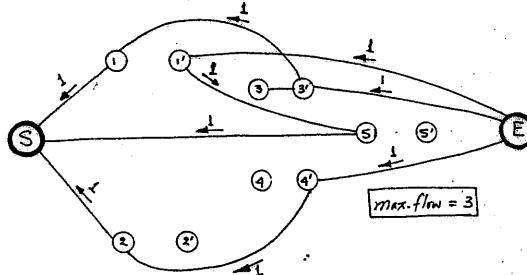


Step 2:

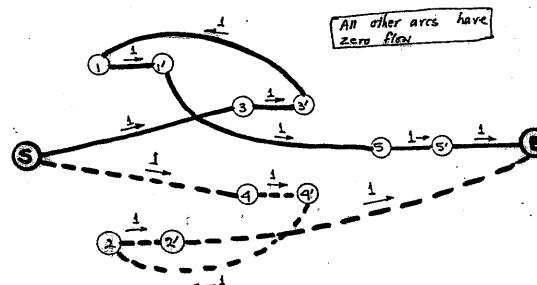
Residue Network



Step 3: Maximum flow in residue network (all missing arcs have zero flow) from $E \rightarrow S$



Step 4:



Solution:

Two ships are needed.

Chapter 6 Cases

6-4

Consider the case of four brokers. The actual situation for which the problem was analyzed included a total of 254 brokers.

Let

P_i = payables by broker i , $i=1,2,3,4$

R_{ij} = receivables by i from j

$i,j = 1,2,3,4$

A_i = assets of broker i , $i=1,2,3,4$

The data of the problem may be summarized as:

	1	2	3	4	assets
1	P_1	R_{12}	R_{13}	R_{14}	A_1
2	R_{21}	P_2	R_{23}	R_{24}	A_2
3	R_{31}	R_{32}	P_3	R_{34}	A_3
4	R_{41}	R_{42}	R_{43}	P_4	A_4

A broker is solvent if its net debt does not exceed its assets - that is, broker i is solvent if

$$P_i \leq A_i + \sum_{j \neq i} R_{ij}$$

The problem deals with the brokers whose total assets are less than their debt - that is,

$$P_i > A_i + \sum_{j \neq i} R_{ij}$$

The proposed solution calls for prorating all debts such that for broker i

$$\text{prorated payables} - \text{prorated receivables} = \text{assets}$$

Let

α_i = prorating factor for broker i

continued...

The value of α_i may be determined by solving 6-4 continued

$$P_i \alpha_i - \sum_{j \neq i} R_{ij} \alpha_j = A_i \quad \text{for all } i$$

Consider the following hypothetical illustration:

	α_1	α_2	α_3	α_4	assets	Deficit
1	35	-5	-4	-1	20	5
2	-10	40	-5	-7	15	3
3	-3	-4	20	-1	5	7
4	-5	-5	-5	30	10	5
	17	26	6	21		← Payables

Equations:

$$\begin{pmatrix} 35 & -5 & -4 & -1 \\ -10 & 40 & -5 & -7 \\ -3 & -4 & 20 & -1 \\ -5 & -5 & -5 & 30 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 20 \\ 15 \\ 5 \\ 10 \end{pmatrix}$$

Solution: $\alpha_1 = .76$, $\alpha_2 = .75$, $\alpha_3 = .55$, $\alpha_4 = .68$

The net result of the prorating is that the total assets ($= 20 + 15 + 5 + 10 = 50$) will be distributed to investors

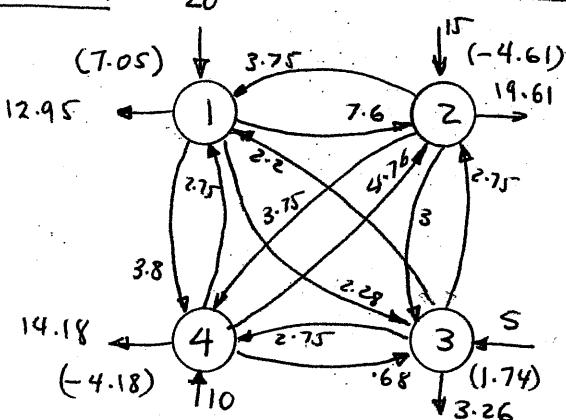
The following table gives the prorated cash:

	.76	.75	.55	.68	assets	Deficit
1	26.6	-3.75	-2.2	-0.68	20	0
2	-7.6	30	-2.75	-4.76	15	0
3	-7.28	-3	11	-0.68	5	0
4	-3.8	-3.75	-2.75	20.4	10	0
	12.95	19.61	3.26	14.28		← Payables

continued...

Chapter 6 Cases

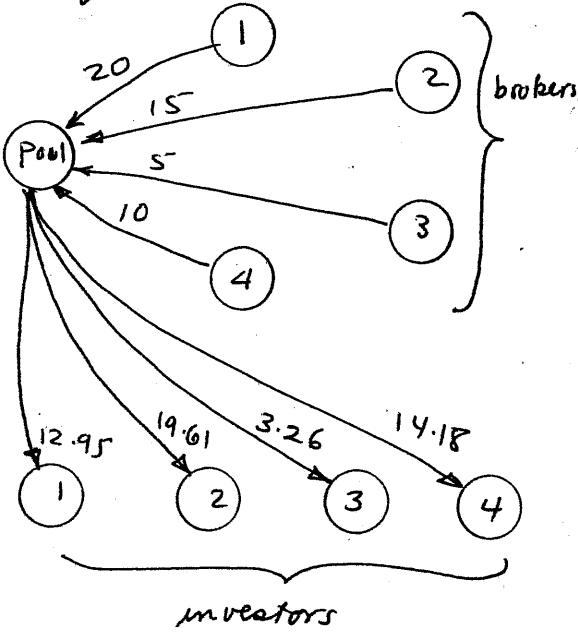
Solution:



6-4 continued

The next phase of the analysis calls for eliminating the "loops" from the given solution. For example, 1 owes 2 \$7.6 and 2 owes 1 \$3.75. The net result is that 1 owes 2 \$7.6 - 3.75 = \$3.85. The idea is to eliminate all higher level loops.

I first proposed solving the problem of the "loops" by pooling all the assets and distributing them to outside investors based on the determined value of α_c — that is



The proposed solution did not satisfy the legal requirements of the case. As a result, I proposed the following network-based LP:

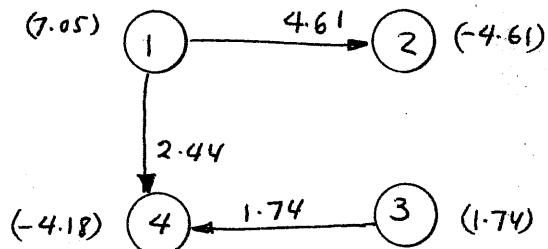
Let

x_{ij} = flow from node i to node j .

The minimization of $\sum x_{ij}$ subject to the following flow constraint should produce the minimum number of loops (see the network in opposite column):

	x_{12}	x_{13}	x_{14}	x_{21}	x_{23}	x_{24}	x_{31}	x_{32}	x_{34}	x_{41}	x_{42}	x_{43}	RHS VALUE
OBJ(MIN)	1	1	1	1	1	1	1	1	1	1	1	1	
NODE1	1	1	1	-1		-1		-1					7.05
NODE2		-1		1	1			-1		-1			-4.61
NODE3			-1		-1		1	1			-1		1.74
NODE4				-1				-1	1	1	1		-4.18
BOUND	7.6	2.28	3.8	3.75	3.75	3.75	2.2	2.25	2.25	6.8	4.76	6.8	

The optimum LP solution is summarized below.



continued...

Comprehensive Problems

The problem reduces to finding
a feasible solution of

$$\alpha_1 A + \alpha_2 B + \alpha_3 C = b$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

$$\alpha_1, \alpha_2, \alpha_3 \geq 0$$

$$(a) \text{ maximize } Z = 0\alpha_1 + 0\alpha_2 + 0\alpha_3$$

subject to

$$\alpha_1 \begin{pmatrix} 6 \\ 4 \\ 6 \\ -2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 4 \\ 12 \\ -4 \\ 8 \end{pmatrix} + \alpha_3 \begin{pmatrix} -4 \\ 0 \\ 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \\ 2 \end{pmatrix}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \quad \alpha_j \geq 0 \text{ all } j$$

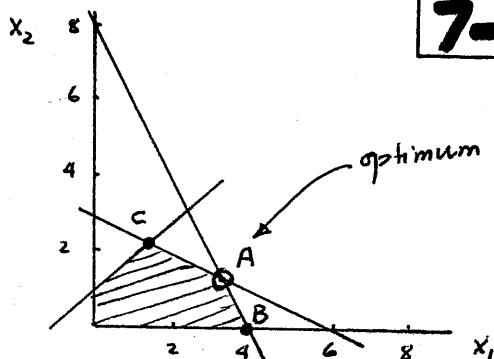
Solution: $\alpha_1 = 1/2, \alpha_2 = 1/4, \alpha_3 = 1/4$

$$(b) \text{ maximize } Z = 0\alpha_1 + 0\alpha_2 + 0\alpha_3$$

subject to

$$\alpha_1 \begin{pmatrix} 6 \\ 4 \\ 6 \\ -2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 4 \\ 12 \\ -4 \\ 8 \end{pmatrix} + \alpha_3 \begin{pmatrix} -4 \\ 0 \\ 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 9 \\ 9 \end{pmatrix}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \quad \alpha_j \geq 0 \text{ all } j$$



7-2

x_1	x_2	x_3	x_4	x_5	Solution
2	0	0	1/3	4/3	0
x_2	0	1	2/3	-1/3	0
x_1	1	0	-1/3	2/3	0
x_5	0	0	-1	1	3

The optimum solution occurs at A.
The adjacent extreme points B and C
are determined by making x_3 and
 x_4 basic, one at a time

continued...

E-19

Adjacent extreme pt B:
introduce x_3 into the basic vector

	x_1	x_2	x_3	x_4	x_5	
Z	0	-1/2	0	3/2	0	12
x_3	0	3/2	1	-1/2	0	2
x_1	1	1/2	0	1/2	0	4
x_5	0	3/2	0	1/2	1	5

Adjacent extreme point C: introduce
 x_4 into the basic vector

	x_1	x_2	x_3	x_4	x_5	
Z	0	0	5/3	0	-4/3	26/3
x_2	0	1	1/3	0	1/3	7/3
x_1	1	0	1/3	0	-2/3	4/3
x_4	0	0	-1	1	1	3

The next best extreme point is B
with $Z = 12$.

7-3

Iteration 0:

$$B_0 = I$$

$$B_0^{-1} = I$$

$$x_B = (y_1, y_2, y_3)^T = (150, 200, 300)^T$$

$$c_B = (1, 1, 1)$$

$$d = c_B B_0^{-1} = (1, 1, 1)I = (1, 1, 1)$$

Interactive AMPL solution (amplProb7-3.txt):

solve; display zj, a;

$$zj = 4$$

a [*] :=

$$1 \quad 4$$

$$2 \quad 0$$

$$3 \quad 0$$

Iteration 1:

$$P_1 = (4, 0, 0)^T \text{ -- associated variable } x_1$$

$$B_1^{-1} P_1 = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \Rightarrow y_1 \text{ leaves}$$

$$B_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_1^{-1} = \begin{pmatrix} .25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_B = (y_1, y_2, y_3)^T = (37.5, 200, 300)^T$$

$$c_B = (1, 1, 1)$$

$$d = c_B B_1^{-1} = (.25, 1, 1)$$

Interactive AMPL solution:

let d[1]:= .25;

solve; display zj, a;

$$zj = 2.25$$

a [*] :=

$$1 \quad 1$$

$$2 \quad 2$$

$$3 \quad 0$$

continued...

Comprehensive Problems

Iteration 2:

$\mathbf{P}_2 = (1, 2, 0)^T$ -- associated variable x_2

$$\mathbf{B}_1^{-1} \mathbf{P}_2 = \begin{pmatrix} .25 \\ 2 \\ 0 \end{pmatrix} \Rightarrow y_2 \text{ leaves}$$

$$\mathbf{B}_2 = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{B}_2^{-1} = \begin{pmatrix} .25 & -.125 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{x}_B = (x_1, x_2, y_3)^T = (150, 200, 300)^T$$

$$\mathbf{c}_B = (1, 1, 1)$$

$$\mathbf{d} = \mathbf{c}_B \mathbf{B}_2^{-1} = (.25, .375, 1)$$

Interactive AMPL solution:

let d[2]:= .375;

solve; display zj, a;

$$zj = 2$$

$$a[*] :=$$

$$1 \ 0$$

$$2 \ 0$$

$$3 \ 2$$

Iteration 3:

$\mathbf{P}_3 = (0, 0, 2)^T$ -- associated variable x_3

$$\mathbf{B}_2^{-1} \mathbf{P}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \Rightarrow y_3 \text{ leaves}$$

$$\mathbf{B}_3 = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \mathbf{B}_3^{-1} = \begin{pmatrix} .25 & -.125 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{pmatrix}$$

$$\mathbf{x}_B = (x_1, x_2, y_3)^T = (12.5, 100, 150)^T$$

$$\mathbf{c}_B = (1, 1, 1)$$

$$\mathbf{d} = \mathbf{c}_B \mathbf{B}_3^{-1} = (.25, .375, .5)$$

Interactive AMPL solution:

let d[3]:= .5;

solve; display zj, a;

No feasible solution — Process ends

Optimal solution: Cut 12.5 rolls using setting (4,0,0),
100 rolls using (1,2,0), and 150 rolls using (0,0,2).

Maximize $Z = CX$

subject to

$$AX \geq L$$

$$AX \leq U$$

$$X \geq 0$$

Let $Y = U - AX \geq 0$, the problem becomes

Maximize $Z = CX$

subject to

$$AX + Y = U$$

$$Y \leq U - L$$

$$X \geq 0 \quad Y \geq 0$$

7-4

Maximize $Z = 5x_1 + 4x_2 + 6x_3$

Subject to

$$x_1 + 7x_2 + 3x_3 + y_1 = 46$$

$$3x_1 - x_2 + x_3 + y_2 = 20$$

$$2x_1 + 3x_2 - x_3 + y_3 = 35$$

$$y_1 \leq 26, y_2 \leq 20, y_3 \leq 17$$

$$x_1, x_2, x_3, y_1, y_2, y_3 \geq 0$$

Optimum solution:

$$x_1 = 6.18, x_2 = 3.55, x_3 = 5$$

$$Z = 46.73$$

$$\begin{pmatrix} x_2 \\ x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 10/3 + \theta \\ 3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

7-5

The increase in x_1 can be translated to increasing the right-hand side of the constraints by D_1 , D_2 , and D_3 . The values of D_1 , D_2 , and D_3 can be computed from $B_2 b = X_B$; that is

$$\begin{pmatrix} x_2 \\ x_1 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 + D_1 \\ 8 + D_2 \\ 1 + D_3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 10/3 + \theta \\ 3 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \theta \\ 0 \end{pmatrix}$$

$$\text{or } \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \theta \\ 0 \end{pmatrix} = \begin{pmatrix} \theta \\ 2\theta \\ -\theta \end{pmatrix}$$

$$\theta = -10/3$$

	x_1	x_2	x_3	x_4	x_5	Sol^n
Z	0	0	1/3	4/3	0	8/3
x_2	0	1	2/3	-1/3	0	4/3
x_1	1	0	-1/3	2/3	0	0
x_5	0	0	-1	1	1	3
Z	1	0	0	2	0	8/3
x_2	2	1	0	1	0	4/3
x_3	-3	0	1	-2	0	0
x_5	-3	0	0	-1	1	3

continued...

continued...

Comprehensive Problems

$$\begin{pmatrix} X_2 \\ X_3 \\ X_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 6+\theta \\ 8+2\theta \\ 1-\theta \end{pmatrix} = \begin{pmatrix} 8+2\theta \\ -10-3\theta \\ -7-3\theta \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4 \leq \theta \leq -1/3$$

For $\theta < -4$, no feasible solution exists
Summary:

$-\infty \leq \theta \leq -4$: No feasible solution

$-4 \leq \theta \leq -1/3$: $X_1 = 0, X_2 = 8+2\theta$

$$Z = 16+4\theta$$

$-1/3 \leq \theta < \infty$: $X_1 = 10/3 + \theta, X_2 = 4/3$

$$Z = \frac{38}{3} + 3\theta$$

At $t=2$:

$$B = (P_2 P_3) = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, B^{-1} = \begin{pmatrix} 0 & 1/2 \\ 1 & -1 \end{pmatrix}$$

$$CB = (4t-8, 0)$$

$$CB^{-1} = (4t-8, 0) \begin{pmatrix} 0 & 1/2 \\ 1 & -1 \end{pmatrix} = (0, 2t-4)$$

$$Z_1 - C_1 = (0, 2t-4) \begin{pmatrix} 2 \\ 4 \end{pmatrix} - (10t-4) \\ = -2t-12$$

$$Z_4 - C_4 = (0, 2t-4) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 \\ = 2t-4$$

$$\begin{pmatrix} X_2 \\ X_3 \end{pmatrix} = B^{-1}b = \begin{pmatrix} 0 & 1/2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 8 \\ 6-2t \end{pmatrix} = \begin{pmatrix} 3-t \\ 3+2t \end{pmatrix}$$

At $t=2$: $Z_4 - C_4 = 0$
 at $t > 2$: $Z_4 - C_4 > 0$ } X_4 enters basis

P_4 replaces P_2 :

$$B = (P_3 P_4) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B^{-1}$$

$$CB = (0, 0) \quad CB^{-1} = (0, 0)$$

$$Z_1 - C_1 = (0, 0) P_1 - (10t-4) = -10t-4 \leq 0$$

$$Z_2 - C_2 = (0, 0) P_2 - (4t-8) = -4t+8 \leq 0$$

$$t \geq 5/2$$

$$\begin{pmatrix} X_4 \\ X_2 \end{pmatrix} = B^{-1}b = \begin{pmatrix} 8 \\ 6-2t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow t \leq 3$$

continued...

Solution becomes infeasible for $t > 3$. However, no feasible solution exists for $t > 3$

Summary:

$2 \leq t \leq 3$: optimal basis $B = (P_3 P_4)$

$t > 3$: no feasible solution exists

7-7

Let

$$J^+(J^-) = \{j \in NB \mid a_{ij} > 0 (< 0)\}$$

Then $x_i \leq c_i$ yields

$$x_i = c_i + f_i - \sum_{j \in J^+} a_{ij}x_j - \sum_{j \in J^-} a_{ij}x_j$$

Because $x_i - c_i \leq 0$, it follows that

$$-\sum_{j \in J^+} a_{ij}x_j - \sum_{j \in J^-} a_{ij}x_j \leq -f_i$$

Adding this inequality to the simplex tableau and applying the simplex method, then, under the assumption of no change in basis, the decrease in the value of Z is at least

$$P_d = \min_{j \in J^+} \left\{ \frac{(Z_j - g_j)f_i}{a_{ij}} \right\}$$

The corresponding upper bound on the value of Z is $C_0 - P_d$.

In a similar manner, $x_i \geq d_i$ gives

$$x_i - d_i = f_i - 1 - \sum_{j \in J^+} a_{ij}x_j - \sum_{j \in J^-} a_{ij}x_j$$

Thus,

$$\sum_{j \in J^+} a_{ij}x_j + \sum_{j \in J^-} a_{ij}x_j \leq f_i - 1$$

and

$$P_u = \min_{j \in J^-} \left\{ \frac{(Z_j - g_j)(f_i - 1)}{a_{ij}} \right\}$$

The associated upper bound on Z is $C_0 - P_u$

Chapter 8 Cases

$X_{ij} = \text{Acres from site } i \text{ using alternative } j$

8-1

Sawlogs constraint:

$$7X_{15} + 6X_{16} + 5X_{17} + \\ 5X_{25} + 4X_{26} + \\ 4X_{33} + 3X_{34} + 5X_{35} + S_1 - S_1^+ = 350,000$$

Plywood constraint:

$$6X_{13} + 7X_{14} + 5X_{23} + 4X_{24} + 4X_{32} + S_2 - S_2^+ = 150,000$$

Pulpwood constraint:

$$1 X_{11} + 10X_{12} + 5X_{13} + 4X_{14} + 3X_{15} + 2X_{16} + \\ 3X_{17} + 9X_{21} + 8X_{22} + 2X_{23} + 3X_{24} + 2X_{25} + \\ 2X_{26} + 7X_{31} + 6X_{32} + 2X_{33} + 2X_{34} + \\ X_{35} + S_3 - S_3^+ = 200,000$$

Reforestation Constraint:

$$1000X_{11} + 800X_{12} + \dots + 1500X_{17} + \\ 1000X_{21} + 800X_{22} + \dots + 1200X_{26} + \\ 1000X_{31} + 800X_{32} + 1500X_{33} + \\ 1200X_{34} + 1300X_{35} + S_4 - S_4^+ = 2,500,000$$

Rotation Constraints:

$$20X_{11} + 25X_{12} + 40X_{13} + 15X_{14} + \\ 40X_{15} + 40X_{16} + 50X_{17} \leq 100,000 \\ 20X_{21} + 25X_{22} + 40X_{23} + 15X_{24} + \\ 40X_{25} + 40X_{26} \leq 180,000 \\ 30X_{31} + 25X_{32} + 40X_{33} + 15X_{34} + \\ 40X_{35} \leq 200,000$$

Goal for total return from stumpage

$$= 100(1000,000 + 180,000 + 200,000)$$

$$= \$48,000,000$$

Total return constraint:

$$(20 \times 160)X_{11} + (117 \times 25)X_{12} + (140 \times 40)X_{13} + \\ (195 \times 15)X_{14} + (182 \times 40)X_{15} + (180 \times 40)X_{16} + \\ (135 \times 50)X_{17} + (102 \times 20)X_{21} + (55 \times 25)X_{22} + \\ (95 \times 40)X_{23} + (120 \times 15)X_{24} + (100 \times 40)X_{25} + \\ (90 \times 40)X_{26} + (60 \times 0)X_{31} + (48 \times 25)X_{32} + \\ (60 \times 40)X_{33} + (65 \times 15)X_{34} + (35 \times 40)X_{35} + \\ S_5 - S_5^+ = 48,000,000$$

continued...

Goals: $\min S_1$

$\min S_2$

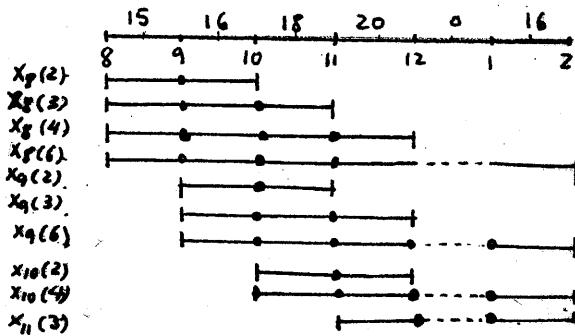
$\min S_3$

$\min S_4$

$\min S_5$

$X_{hj} = \text{number of volunteers starting at hour } h \text{ and continuing for } j \text{ successive hours}$

8-2



Constraints:

$$X_g(2) + X_g(3) + X_g(4) + X_g(6) + S_1 - S_1^+ = 15$$

$$X_g(2) + X_g(3) + X_g(4) + X_g(6) + S_2 - S_2^+ + \\ X_g(2) + X_g(3) + X_g(5) + S_2 - S_2^+ = 16$$

$$X_g(3) + X_g(4) + X_g(6) + X_g(2) + \\ X_g(3) + X_g(5) + X_{10}(2) + X_{10}(4) + S_3 - S_3^+ = 18$$

$$X_g(4) + X_g(6) + X_g(3) + X_g(5) + \\ X_{10}(2) + X_{10}(4) + X_{11}(3) + S_4 - S_4^+ = 20$$

$$X_g(6) + X_g(3) + X_{10}(4) + X_{11}(3) + S_5 - S_5^+ = 16$$

Objective function:

$$\text{Minimize } Z = S_1 + S_2 + S_3 + S_4 + S_5$$

continued...

Chapter 9 Cases

$$x_{ij} = \begin{cases} 1, & \text{building } i \text{ opens in yr } j \\ 0, & \text{otherwise} \end{cases} \quad \boxed{9-1}$$

R_{ij} = ft² of building i rented in year j and thereafter

I_{ij} = operating income per ft² of building i in year j

C_{ij} = construction cost of building i in year j

θ = inflation rate applied to operating income and construction cost

r = discount rate

D_{1j} = demand for high-rise ft² in yr j

D_{2j} = demand for garden space ft² in year j

K = capitalization rate used for determining property value in the year of sale

B_i = maximum capacity of building i

The sale value of building i at the end of 7 years is estimated as

$$R_{ij} \times \frac{I_{i7}}{K}$$

This means that the sale is estimated based on the net operating income for year 7.

Model:

$$\begin{aligned} \text{Maximize } Z = & \sum_{j=1}^7 \left\{ \left(\frac{1}{1+r} \right)^j \left(\sum_{i=1}^7 (I_{ij} R_{ij} - C_{ij} x_{ij}) \right) \right\} \\ & + \sum_{j=1}^7 \sum_{i=1}^7 \left(\frac{1}{1+r} \right)^j \left(\frac{I_{i7}}{K} \right) R_{ij} \end{aligned}$$

Subject to

continued...

$$\sum_{i=1}^3 R_{ij} \leq D_{1j}, \quad j=1, 2, \dots, 7$$

$$\sum_{i=1}^3 R_{ij} \leq D_{2j}, \quad j=1, 2, \dots, 7$$

$$\sum_{j=1}^y D_{1j} x_{ij} \geq R_{ij}, \quad i=1, 2, 3 \quad j=1, 2, \dots, 7$$

$$\sum_{j=1}^y D_{2j} x_{ij} \geq R_{ij}, \quad i=3, 4, \dots, 7 \quad j=1, 2, \dots, 7$$

$$\sum_{j=1}^7 R_{ij} \leq B_i, \quad i=1, 2, \dots, 7$$

$$\sum_{j=1}^7 x_{ij} = 1, \quad i=1, 2, \dots, 7$$

The rental income and expenses as given in the problem apply to year 1 of the planning horizon. These values must be adjusted for year j to allow for inflation. Assuming an inflation rate θ , the amount for year j is determined for year j by multiplying the values for year 1 by $(1+\theta)^{j-1}$.

$$S_{ij} = \begin{cases} \text{expected score of gymnast } i \text{ in event } j, & i=1, 2, \dots, N, \\ & j=1, 2, 3, 4. \end{cases} \quad \boxed{9-2}$$

$$x_{ij} = \begin{cases} 1, & \text{if gymnast } i \text{ is assigned to event } j \\ 0, & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1, & \text{if gymnast } i \text{ is an all-rounders} \\ 0, & \text{otherwise} \end{cases}$$

Model:

$$\text{Maximize } Z = \sum_{i=1}^N \sum_{j=1}^4 S_{ij} x_{ij} + \sum_{i=1}^N \left(\sum_{j=1}^4 S_{ij} \right) y_i$$

Subject to

$$\sum_{i=1}^N x_{ij} + y_i \leq 6, \quad j=1, 2, 3, 4$$

$$x_{ij} + y_i \leq 1, \quad i=1, 2, \dots, N \quad j=1, 2, 3, 4$$

$$\sum_{i=1}^N y_i \geq 4,$$

continued...

Chapter 9 Cases

$$\sum_{j=1}^n x_{ij} \leq 3, \quad i = 1, 2, \dots, N$$

$y_i, x_{ij} = (0, 1)$ for all i and j

x_{ij} = fraction of traffic originating from area code i and handled by center j , $i = 1, 2, \dots, 8$; $j = 1, 2, \dots, 7$

$y_j = \begin{cases} 1, & \text{if center } j \text{ is chosen} \\ 0, & \text{otherwise} \end{cases}$

c_{ij} = communication cost/hr between area i and area j

Define:

i	1	2	3	4	5	6	7	8
Area	501	918	316	417	314	816	502	606

j	1	2	3	4	5	6	7
-----	---	---	---	---	---	---	---

center Dallas Atlanta L'ville Denver LR Memphis St. Louis

Communication traffic constraints

$$x_{11} + x_{14} + x_{16} = 1 \quad (\text{Area 501})$$

$$x_{21} + x_{23} = 1 \quad (918)$$

$$x_{31} + x_{33} + x_{36} = 1 \quad (316)$$

$$x_{41} + x_{43} + x_{45} + x_{46} = 1 \quad (417)$$

$$x_{52} + x_{53} + x_{55} + x_{57} = 1 \quad (314)$$

$$x_{62} + x_{63} + x_{65} + x_{67} = 1 \quad (816)$$

$$x_{72} + x_{74} + x_{75} + x_{77} = 1 \quad (502)$$

$$x_{82} + x_{84} + x_{86} + x_{87} = 1 \quad (606)$$

Centers constraints:

$$x_{11} + x_{21} + x_{31} + x_{41} \leq M y_1$$

$$x_{52} + x_{62} + x_{72} + x_{82} \leq M y_2$$

$$x_{23} + x_{33} + x_{43} + x_{53} + x_{63} \leq M y_3$$

$$x_{14} + x_{74} + x_{84} \leq M y_4$$

$$x_{45} + x_{55} + x_{65} + x_{75} \leq M y_5$$

$$x_{16} + x_{36} + x_{46} + x_{86} \leq M y_6$$

$$x_{57} + x_{67} + x_{77} + x_{87} \leq M y_7$$

continued...

Limit on number of centers

$$3 \leq y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 \leq 4$$

Objective function: Minimize $Z =$

$$500,000 y_1 + 800,000 y_2 + \dots + 550,000 y_7 \\ + 14x_{11} + 24x_{14} + 19x_{16} \\ + 35x_{21} + 25x_{23} \\ + \dots \\ + 15x_{82} + 30x_{84} + 12x_{86} + 22x_{87}$$

Let

$$x_{ij} = \begin{cases} 1, & \text{if cluster } i \text{ is served by CSL in location } j \\ 0, & \text{otherwise} \end{cases}$$

$$y_j = \begin{cases} 1, & \text{if candidate loc. } j \text{ is selected} \\ 0, & \text{otherwise} \end{cases}$$

p = number of facilities

w_i = number of customers in cluster i

d_{ij} = distance between cluster i and CSL location j .

Model: Given p , determine

$$\text{Min } Z = \sum_{i=1}^5 \sum_{j=1}^5 w_i \cdot d_{ij} \cdot x_{ij} \cdot y_j$$

Subject to

$$\sum_{j=1}^5 y_j = p$$

$$\sum_{j=1}^5 x_{ij} = 1, \quad i = 1, 2, \dots, 5$$

$$\sum_{i=1}^5 x_{ij} \leq M y_j, \quad j = 1, 2, \dots, 5$$

$$y_j \text{ and } x_{ij} = (0, 1)$$

The idea of the algorithm is to specify a value for $p = 1, 2, \dots, 5$. Then the model is optimized to determine where the specified p CSL centers should be located.

continued...

Chapter 9 Cases

For example, if $p=1$, the optimum solution of the model (using TORA) will specify that the CSL should be located at $j=4$. This means that all 5 clusters will be served by the CSL located in location $j=4$. For this arrangement, the average traveled distance from $j=4$ to all 5 clusters is

$$\bar{D} = \frac{50 + 30 + 80 + 60 + 110}{5} = \frac{330 \text{ miles}}{5} = 66 \text{ miles}$$

Given that the truck travels at 45 miles per hour, the average time to reach a customer will be $\frac{66}{45} = 1.47 \text{ hour} = 88 \text{ minutes}$, which is less than the desired 90-minute response time.

Another way of looking at the solution is to consider the maximum travel distance from location $j=4$; namely,

$$D_4 = \max \{50, 30, 80, 60, 110\} = 110 \text{ miles}$$

The associated truck travel time is 2.44 hours or 147 minutes. Because it exceeds the limit of 90 minutes, the new

criterion calls for trying $p=2$. TORA will give two locations:

$j=3$ serving clusters 1 and 5

$j=4$ serving clusters 2, 3, and 4

$$\text{Thus, } D_3 = \max \{20, 40\} = 40 \text{ and}$$

$D_4 = \max \{30, 80, 60\} = 80 \text{ miles. The new solution is within the desired 90-mile limit.}$

18 possible configurations:

9-5

	Testers configurations
1	4C-A-4D
2	4C- A -C
3	4C-A-W
4	4C-S-4D
5	4C-S-C
6	4C-S-W
7	6C-A-4D
8	6C-A-C
9	6C-A-W
10	6C-S-4D
11	6C-S-C
12	6C-S-W
13	8C-A-4D
14	8C-A-C
15	8C-A-W
16	8C-S-4D
17	8C-S-C
18	8C-S-W

Let $T_i = \text{Set of testers using configuration } i$
 $i = 1, 2, \dots, 18$

$$T_1 = T_2 = T_3 = \emptyset \quad T_{14} = \{2, 4\}$$

$$T_4 = T_5 = \{1\} \quad T_{15} = T_{16} = \{4\}$$

$$T_6 = \{1, 5\} \quad T_{17} = \{2, 4\}$$

$$T_7 = T_8 = \{6\} \quad T_{18} = \{4, 5\}$$

$$T_9 = \{3, 6\}$$

$$T_{10} = T_{11} = \emptyset$$

$$T_{12} = \{3, 5\}$$

$$T_{13} = \{4\}$$

continued...

continued...

Chapter 9 Cases

$P_i = \text{Set of prototypes covering tester } i, i=1, 2, \dots, 6$

$P_1 = \{4, 5, 6\}, P_2 = \{14, 17\}, P_3 = \{9, 12\},$

$P_4 = \{13, 14, 15, 16, 17, 18\}$

$P_5 = \{6, 12, 18\}, P_6 = \{7, 8, 9\}$

$X_{ij} = \begin{cases} 1, & \text{if tester } i \text{ is covered by prototype } j \\ 0, & \text{otherwise} \end{cases}$

$\gamma_j = \begin{cases} 1, & \text{if any tester uses prototype } j \\ 0, & \text{otherwise} \end{cases}$

Minimize $Z = \sum_{i=1}^{18} \gamma_i$
s.t.

$$\sum_{j \in P_i} X_{ij} = 1, i=1, 2, \dots, 6$$

$$\sum_{i \in T_j} X_{ij} \leq M * \gamma_j, j=1, 2, \dots, 18$$

Solution: See file amplCase 9-5.txt

Prototype	Nbr. made	testers
6	2	1, 5
9	2	3, 6
14	2	2, 4

$F_{ij} = \text{feasible pairing } j \text{ of crew } i$
expressed in flight numbers

9-6

Example: Pairing $(C_3, C_6, C_4, C_8, C_3)$ of crew 1 is expressed as

$$F_{11} = \{10, 15, 12, 18\}$$

Pairing (C_3, C_2, C_8, C_3) of crew 1 is expressed as

$$F_{12} = \{9, 7, 18\}$$

$X_{ij} = \begin{cases} 1, & \text{if pairing } j \text{ of crew } i \text{ is used} \\ 0, & \text{otherwise} \end{cases}$

$\gamma_k = \text{Nbr. of crews overallocated to flight } k (\geq 0)$

$\text{Card}(F_{ij}) = \text{Nbr. of elements of } F_{ij}$

$N_i = \text{Nbr. of pairings for crew } i$

continued...

Minimize $Z = \sum_{(i,j)} \text{Card}(F_{ij}) X_{ij}$

$$\text{s.t. } \sum_{j=1}^{N_i} X_{ij} \leq 1, i=1, 2, \dots, 10 \quad (1)$$

$$\sum_{\substack{\text{defined}(i,j) \\ k \in F_{ij}}} X_{ij} - \gamma_k = 1, k=1, 2, \dots, 18 \quad (2)$$

Constraints (1) allow at most one pairing per crew and constraints (2) will give $\gamma_k \geq 0$ if flight k is covered by at least one crew. If flight k is cannot be covered by a crew, (2) is infeasible.

Solution: See file amplCase 9-6.txt.

crew	pairing
1	None
2	$1(C_3, C_6, C_4, C_8, C_3)$
3	$1(C_4, C_8, C_3, C_2, C_4)$
4	$1(C_1, C_8, C_3, C_6, C_4, C_1)$
5	$3(C_2, C_7, C_4, C_1, C_2)$
6	None
7	$1(C_5, C_2, C_8, C_3, C_1, C_5)$
8	$1(C_6, C_1, C_3, C_6)$
9	None
10	None

Flight, k	overallocation (γ_k)
3	1
10	1
11	2
17	1
18	2

All other flights are allocated one crew each.

If the pairings does not produce at least one crew allocation/flight, the problem will not have a feasible solution.

Chapter 9 Cases

$D_k = \text{Demand for module } k, k = 1, 2, 3$

$I_{ij} = \text{initial inventory of device } j, j = 1, 2, \dots, 5$

$C_{ij} = \begin{cases} 1, & \text{if device } i \text{ can be used in module } j \\ 0, & \text{otherwise} \end{cases}$

$$\|C_{ij}\| = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$P = \text{Total number of wafers produced}$

$x_j = r_j \cdot p_j, \text{ Nbr of binned devices of type } j$

$y_{jk} = \text{Nbr. of units of device } j \text{ in module } k$

Minimize $Z = P$

s.t.

$$I_j + x_j - \sum_{k=1}^3 C_{jk} y_{jk} \geq 0, \quad j = 1, 2, \dots, 5$$

$$\sum_{j=1}^5 C_{jk} y_{jk} \geq D_k, \quad k = 1, 2, 3$$

Solution: See file amplCase9-7.txt

PRODUCTION SCHEDULE:

Produced wafers = 85 units

Device	Initial inventory	Produced units	Total available	End inventory
1	10	17	27	0
2	4	16	20	0
3	8	8	16	13
4	0	25	25	0
5	3	17	20	0

Module	Demand units	Devices used	Nbr. of units
1	20	2	20
2	30	1	27
3	45	4	25
		5	20

9-7

$x_i = \begin{cases} 1, & \text{if check } i \text{ is cleared} \\ 0, & \text{otherwise} \end{cases}$

9-8

Constraints:

$$200x_1 + 75x_2 + 900x_3 + 25x_4 + 525x_5 + 100x_6 + 675x_7 \leq 1200 \quad (1)$$

If residual balance \geq amt of check j then
check j can be cleared (2)

Constraints (2) translate mathematically to

If $1200 - \sum_{i=1}^7 c_i x_i \geq c_j$ then $x_j = 1$

or If $1200 - \sum_{i=1}^7 c_i x_i - c_j \geq 0$ then $x_j \geq 1$

or If $1200 - \sum_{i=1}^7 c_i x_i - c_j \geq 0$ then $-x_j + 1 \leq 0$

or $1200 - \sum_{i=1}^7 c_i x_i - c_j \leq Mx_j - 0.0001 \quad (2a)$
 $-x_j + 1 \leq M(1 - x_j) \quad (2b)$

Actually (2a) implies (2b) in this case because (2a) requires x_j to equal 1 whenever the left-hand side allows it.

(a) Minimize $Z = \sum_{j=1}^7 x_j$.

Solution: See file amplCase9-8.txt

Clear checks 5 and 7 ($= 525 + 675 = 1200$)

(b) Maximize $Z = \sum_{j=1}^7 x_j$

Solution: See same file

Clear checks 1, 2, 4, 5, and 6

$(= 200 + 75 + 25 + 525 + 100 + 675) = 925$ Remaining balance
 $= 1200 - 925 = \$275$, which is less than the amount of any of the uncleared checks (3 and 7).

Chapter 9 Cases

C_i = capacity of line i (1000 bbl)

X_{ij} = 1000 bbl allocated to bidder j from line i

r_{ij} = min 1000 bbl from line i by bidder j

$$y_{ij} = \{0, 1\}$$

b_{ij} = bonus bid by bidder i on line j

$$\text{Maximize } Z = \sum_{i=1}^6 \sum_{j=1}^8 b_{ij} x_{ij}$$

s.t.

$$\sum_{i=1}^6 x_{ij} \leq 2 \sum_{k=1}^m C_k, j = 1, 2, \dots, 8$$

$$\sum_{j=1}^8 x_{ij} \leq C_i, i = 1, 2, \dots, 6$$

$$x_{ij} \leq M y_{ij}, \quad i = 1, 2, \dots, 6, j = 1, 2, \dots, 8$$

$$x_{ij} \geq r_{ij} y_{ij}$$

Solution: See file amplCase 9-9.txt

$$Z = \$201,750$$

Allocation:

	Bidder							
1		20						
2			12		18			
3				25				
4	11				17	12		
5		35						
6				11		19		

All quantities are in 1000 bbl.

9-9

S_{ij} = intensity measure for manager i working on project j

$$= (t_{ij} + 1) \times 6 \times \log(C_j) + 1$$

where

t_{ij} = travel time in hours by manager i to project j

C_j = cost in $10^6 \$$ of project j

$x_{ij} = \begin{cases} 1, & \text{if manager } i \text{ is assigned to project } j \\ 0, & \text{otherwise} \end{cases}$

$$\text{Minimize } Z = \sum_{i=1}^5 \sum_{j=1}^8 S_{ij} x_{ij}$$

s.t.

$$\sum_{i=1}^5 x_{ij} = 1, j = 1, 2, \dots, 8$$

$$\sum_{j=1}^8 x_{ij} \geq 1, i = 1, 2, \dots, 6$$

Each manager is assigned at least one project.

Solution: See file amplCase 9-10.txt, or file solverCase 9-10.xls

Manager	Assigned projects
a	6
b	3, 4
c	2, 7, 8
d	1
e	5

Alternative solution from Solver:

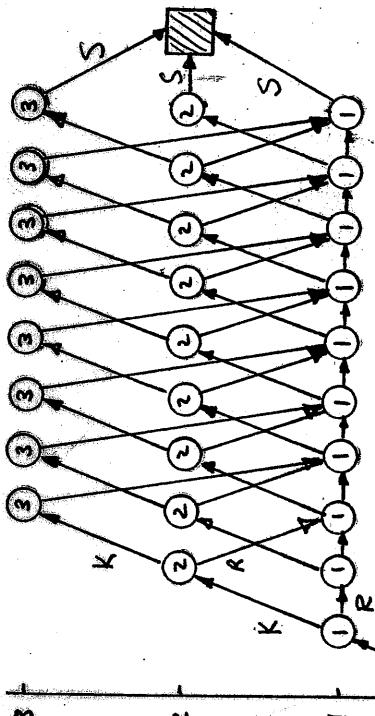
b	3
d	1, 4

9-10

Chapter 10 Cases

$$f_i(t) = \min \begin{cases} C(t) + f_{i+1}(t+1), & K \\ I(t) + c(0) - S(t) + f_{i+1}(1), & R \end{cases}$$

$$f_N(t) = \min \begin{cases} C(t) + S(t+1), & K \\ I(N) + c(0) - S(t), & R \end{cases}$$



10 - 1

Stage 8:

t	K		R		Optimun
	C(t) + f_8(t+1)	I(8) + c(0) - S(t) + f_8(1)	f_8	Dec	
1	.67 - 12.84 = -12.17	15.2 + .43 - 14 - 13.84 = -12.21	-12.21	R	-12.21
2	.7 - 12.14 = -12.07	15.2 + .43 - 13.2 - 13.84 = -11.41	-11.41	K	-12.07
3	must replace	15.2 + .43 - 12 - 13.84 = -10.21	-10.21	R	-10.21

Stage 7:

t	K		R		Optimun
	C(t) + f_7(t+1)	I(7) + c(0) - S(t) + f_7(1)	f_7	Dec	
1	.6 - 12.07 = -11.47	14.8 + .41 - 13.5 - 12.21 = -10.5	-10.5	K	-11.47
2	.62 - 10.21 = -9.59	14.8 + .41 - 12.9 - 12.21 = -9.9	-9.9	R	-9.9
3	must replace	14.8 + .41 - 11.9 - 12.21 = -8.9	-8.9	R	-8.9

Stage 6:

t	K		R		Optimun
	C(t) + f_6(t+1)	I(6) + c(0) - S(t) + f_6(1)	f_6	Dec	
1	.62 - 9.9 = -9.28	14.2 + .39 - 12.5 - 11.47 = -9.38	-9.38	R	-9.38
2	.7 - 8.9 = -8.2	14.2 + .39 - 12.0 - 9.9 = -7.31	-7.31	K	-8.2
3	must replace	14.2 + .39 - 11.2 - 8.9 = -5.51	-5.51	R	-5.51

Stage 5:

t	K		R		Optimun
	C(t) + f_5(t+1)	I(5) + c(0) - S(t) + f_5(1)	f_5	Dec	
1	.59 - 8.2 = -7.61	13.8 + .35 - 12 - 9.38 = -7.23	-7.61	K	-7.61
2	.63 - 5.51 = -4.88	13.8 + .35 - 11.8 - 9.38 = -7.03	-7.03	R	-7.03
3	—	13.8 + .35 - 11.2 - 9.38 = -6.43	-6.43	R	-6.43

Stage 4:

t	K		R		Optimun
	C(t) + f_4(t+1)	I(4) + c(0) - S(t) + f_4(1)	f_4	Dec	
1	.65 - 7.03 = -6.38	13.5 + .32 - 12 - 7.61 = -5.79	-6.38	K	-6.38
2	.7 - 6.46 = -5.76	13.5 + .32 - 11.5 - 7.61 = -5.29	-5.76	K	-5.76
3	—	13.5 + .32 - 11 - 7.61 = -4.79	-4.79	R	-4.79

continued...

continued...

Chapter 10 Cases

Stage 3:

<u>K</u>	<u>R</u>	Opt \Rightarrow	
t	$I(3) + c(t) - S(t) + f_4(1)$	f_3	Δe
1	$.55 - 5.76 = -5.21$	$13 + .28 - 12 - 6.38 = -5.1$	-5.21 K
2	$.60 - 4.79 = -4.19$	$13 + .28 - 11 - 6.38 = -4.1$	-4.19 K
3	—	$13 + .28 - 10 - 6.38 = -3.1$	-3.1 R

Stage 2:

<u>K</u>	<u>R</u>	Opt \Rightarrow	
t	$I(2) + c(t) - S(t) + f_3(1)$	f_2	Δe
1	$.6 - 4.19 = -3.59$	$12 + .25 - 11 - 5.21 = -3.96$	-3.96 R
2	$.68 - 3.1 = -2.42$	$12 + .25 - 8 - 5.21 = -2.46$	-2.46 R
3	—	$12 + .25 - 8 - 5.21 = -.96$	-.96 R

Stage 1:

$$t=0, \quad I(1) + c(t) + f_1(t+1) = 10 + 2 - 3.96 = 6.26.$$

Decision: K

Policy:

$$\begin{aligned} K &\rightarrow R \rightarrow K \rightarrow K \rightarrow R \rightarrow R \rightarrow \\ &K \rightarrow R \rightarrow K \rightarrow S \end{aligned}$$

Chapter 11 Cases

Economic lot size formula:

11-1

$$y = \sqrt{\frac{2KD}{h}}$$

h = annual holding cost/unit

K = setup cost

D = annual demand

Given h is a fixed proportion of the unit cost C , we have

$$y = \sqrt{\frac{2KD}{lC}}$$

Let

T = average time period needed to consume the average supply on hand, $\frac{y}{2}$

S = annual dollar usage of the item.

Then,

$$T = \frac{y/2}{D} = \frac{y}{2D}$$

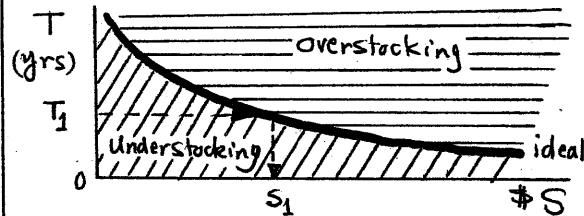
$$S = DC$$

Under optimal conditions, we have

$$T = \sqrt{\frac{2KD}{4D^2lC}} = \sqrt{\frac{K}{2lC}} \sqrt{\frac{1}{DC}} = \alpha \sqrt{\frac{1}{S}}$$

where α is a constant.

The relationship between T and S for a typical inventory can be graphed as



Policy: If the annual dollar usage is S_1 , order the quantity y every $2T_1$ time units

11-1

Inventory control should

11-2

be based on the data for the final product, because the demand for the purchased component is independent on the demand for the final product. Separate treatments of the two parts may result in shortage.

For the final product, we have

$$\begin{aligned} y &= \sqrt{\frac{2KD(p+h)}{ph}} \\ &= \sqrt{\frac{2 \times 100 \times 20(5+8)}{5 \times 8}} \\ &\approx 36 \text{ units} \end{aligned}$$

$$\text{time betw. orders} = \frac{30}{20} = 1.8 \text{ weeks} \approx 12 \text{ days}$$

Ordering policy:

Order $36 \times 2 = 72$ units of purchased components every 12 days. This policy leads to producing 36 units of the final product every 12 days.

11-3

Month	5-yr Av. Demand (rounded)
1	11
2	53
3	10
4	107
5	111
6	108
7	129
8	76
9	52
10	146
11	254
12	42

$$\bar{X} \approx 91$$

$$S_{\bar{X}} = 67.6$$

The fluctuations in the average demand per month suggests that the

continued...

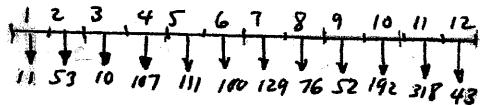
Chapter 11 Cases

use of the EOQ based on the average monthly demand for the past 2 years may lead to gross underestimation or overestimation of demand. The given data yields $\bar{x} = 91$ units with $s_x = 67.6$ units, which reflects extreme variations in demand.

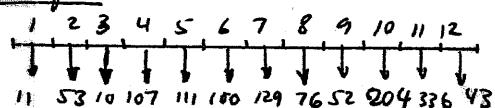
A study of the data shows that with the exception of Nov. and Dec. (and possibly April), the average monthly demand taken over the 5 year span is a good representation of the demand during the month. As for Nov. and Dec., there is a trend for increase in demand approximately equal to 12 units/year for Nov. and 18 units/year for Dec. A planning horizon of 12 months may thus be used to solve the problem. For each new year, the demand per month is taken equal to the averages given in the preceding table. In the cases of Nov. and Dec., the demands are increased by approximately 12 and 18 units, respectively, for each new year.

The following charts apply to the next two years

Next year:



Year after:

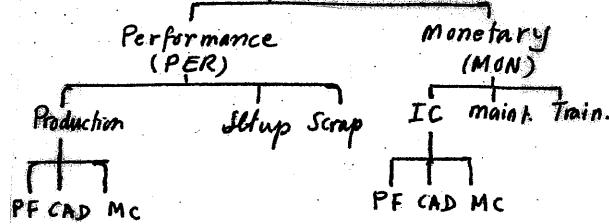


The problem can be solved by DP

Chapter 13 Cases

13-1

Select an alternative



Comparison matrices:

$$\begin{matrix}
 & \text{PER} & \text{MON} \\
 \text{PER} & \begin{bmatrix} 1 & 2/3 \\ 3/2 & 1 \end{bmatrix} \\
 \text{MON} &
 \end{matrix}$$

Performance:

$$\begin{matrix}
 & \text{PRO} & \text{SET} & \text{SCR} \\
 \text{PRO} & \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\
 \text{SET} & \begin{bmatrix} 1/2 & 1 & 4 \end{bmatrix} \\
 \text{SCR} & \begin{bmatrix} 1/3 & 1/4 & 1 \end{bmatrix}
 \end{matrix}$$

$$\begin{matrix}
 & \text{Monetary:} & \text{I} & \text{M} & \text{T} \\
 \text{I} & \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} \\
 \text{M} & \begin{bmatrix} .5 & 1 & 1 \end{bmatrix} \\
 \text{T} & \begin{bmatrix} .5 & 1 & 1 \end{bmatrix}
 \end{matrix}$$

Production (PRO):

$$\begin{matrix}
 & \text{PF} & \text{CAD} & \text{MC} \\
 \text{PF} & \begin{bmatrix} 1 & 8/14 & 8/40 \end{bmatrix} \\
 \text{CAD} & \begin{bmatrix} 14/8 & 1 & 14/40 \end{bmatrix} \\
 \text{MC} & \begin{bmatrix} 40/8 & 40/14 & 1 \end{bmatrix}
 \end{matrix}$$

Note: The higher is the production, the more desirable is the alternative.

Scrap (SCR):

$$\begin{matrix}
 & \text{PF} & \text{CAD} & \text{MC} \\
 \text{PF} & \begin{bmatrix} 1 & 165/440 & 44/440 \end{bmatrix} \\
 \text{CAD} & \begin{bmatrix} 440/165 & 1 & 44/165 \end{bmatrix} \\
 \text{MC} & \begin{bmatrix} 440/44 & 165/44 & 1 \end{bmatrix}
 \end{matrix}$$

Setup (SET):

$$\begin{matrix}
 & \text{PF} & \text{CAD} & \text{MC} \\
 \text{PF} & \begin{bmatrix} 1 & 20/30 & 3/30 \end{bmatrix} \\
 \text{CAD} & \begin{bmatrix} 30/20 & 1 & 3/20 \end{bmatrix} \\
 \text{MC} & \begin{bmatrix} 30/3 & 20/3 & 1 \end{bmatrix}
 \end{matrix}$$

continued...

Chapter 13 Cases

Initial cost (IC):

13-1 continued

	PF	CAD	MC
PF	1	25/12	120/12
CAD	12/25	1	120/25
MC	12/120	25/120	1

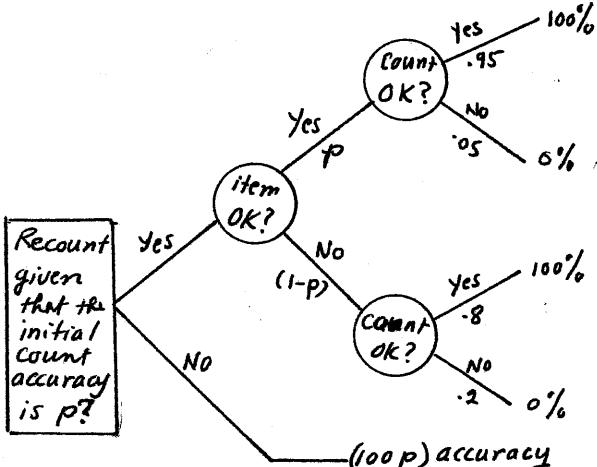
Maintenance cost (MA):

	PF	CAD	MC
PF	1	4/2	15/2
CAD	2/4	1	15/4
MC	2/15	4/15	1

Training (TR)

	PF	CAD	MC
PF	1	8/3	20/3
CAD	3/8	1	20/8
MC	3/20	8/20	1

13-2



Expected accuracy given "recount" is made

$$= (10 \times .95 + 0 \times .05) \times p + (100 \times .8 + 0 \times .2) (1-p)$$

$$= 15p + 80$$

Expected accuracy given "recount" is not made = 100p

continued...

thus, "recount" is warranted if

13-2 continued

$$15p + 80 > 100p$$

$$\text{or } p \leq .12\%$$

The present policy of using **13-3**
 2-707 or 1-747 crews between 5:00 and 17:00 and between 11:00 and 23:00 represents an overlap between 11:00 and 17:00. We can define the periods per day as

Period	Number of crews
1 (5:00-11:00)	2 (service 4-707 departures)
2 (11:00-17:00)	4 (Service 2-707 & 2-747)
3 (17:00-23:00)	2 (Service 1-747)

The corresponding probability of overnight delay is computed as follows. Let

C_i = number of crews called in period i , $i=1, 2, 3$

Period 1:

$$P\{C_1 > 4\} = 0, \text{ because there are 4 departures only}$$

Period 2:

$$\sum_{x=0}^4 P\{C_2 > 4-x | C_1 = x\} P\{C_1 = x\}$$

Examples of computations:

$$P\{C_2 > 0 | C_1 = 4\}$$

$$= 1 - [P\{0 \text{ call from 707 category 2}\} x P\{0 \text{ call from 707 category 4}\} x P\{0 \text{ call from 747 category 4}\} x P\{0 \text{ call from 747 category 6}\}]$$

$$= 1 - [(1 - .019)(1 - .006)(1 - .016)(1 - 0)]$$

$$= .042406848$$

$$P\{C_1 = 4\} = P\{1 \text{ call 707-3}\} x P\{1 \text{ call 707-6}\} x P\{1 \text{ call 707-2}\} x P\{1 \text{ call 707-3}\}$$

continued...

Chapter 13 Cases

13-3 continued

$$= .006x \cdot .003x \cdot .019x \cdot .006 \\ = .000000002052$$

$$P\{C_2 > 1 | C_1 = 3\}$$

$$= P\{C_2 = 1 | C_1 = 3\} + P\{C_2 > 0 | C_1 = 4\} \\ = P\{1, 707-2 \cap 0, 707-4 \cap 0, 707-6 \cap \\ 0, 747-4\} \\ + P\{0, 707-2 \cap 1, 707-4 \cap 0, 747-6 \cap \\ 0, 747-4\} \\ + P\{C_2 > 0 | C_1 = 4\} \\ = (.019x \cdot .994x \cdot .998x \cdot .984) + \\ (.006x \cdot .981x \cdot .998x \cdot .984) + \\ .042406848 \\ \approx .01808$$

$$P\{C_1 = 3\} = P\{1, 707-3 \cap 1, 707-6 \cap \\ 1, 707-2 \cap 0, 707-3\} \\ + P\{1, 707-3 \cap 1, 707-6 \cap \\ 0, 707-2 \cap 1, 707-3\} \\ + P\{1, 707-3 \cap 0, 707-6 \cap \\ 1, 707-2 \cap 1, 707-3\} \\ + P\{0, 707-3 \cap 1, 707-6 \cap \\ 1, 707-2 \cap 1, 707-3\} \\ = (.006x \cdot .003x \cdot .019x \cdot .994) + \\ (.006x \cdot .003x \cdot .981x \cdot .006) + \\ (.006x \cdot .997x \cdot .019x \cdot .006) + \\ (.994x \cdot .003x \cdot .019x \cdot .006) \approx .000002$$

The probabilities for period (conservatively rounded up) are given as

	$P\{C_2 > 4-x C_1 = x\}$	$P\{C_1 = x\}$	Product
0	.000001	.966356	.00000097
1	.000035	.033291	.00000116
2	.000478	.000351	.000000877
3	.01808	.000002	.00000004
4	.042407	.00000002	.0000000008
Total		.0000234	

continued...

13-3 continued

In a similar manner, we compute the probabilities for period 3, which must depend now on what happens in both periods 1 and 2. These computations yield
 $P\{\text{overnight delay for period 3}\}$
 $= .00001513$

Thus,

Total overnight delay probability

$$= 0 + .0000234 + .00001513 \\ = .000017$$

Average number of delays per year

$$= 365 \times .000017 = .006$$

This is equivalent to having one delay every 166 years.

Associated annual cost of reserve crews

$$= \$30,000 \times 28 \text{ members} \\ = \$840,000 \text{ per year}$$

Expected cost of delay per year

$$= \$50,000 \times .006 = \$350/\text{year}$$

These costs indicate that the use of four reserve crews is perhaps unwarranted. The idea is to attempt to reduce the number of crews with possible reallocation to the hours of the day.

The following policies are typical of the new proposals made for this situation:

continued...

Chapter 14 Cases

	Present Policy	Policy A1	Policy A2	Policy A3
Crews	4-B707	3-B707	2-B707	3B707
Allocation	2(5:00-17:00) 2(11:00-23:00)	1(5:00-17:00) 2(11:00-23:00)	2(10:00-22:00)	1(6:30-14:30) 2(14:30-22:30)
Cost/year	\$840,000	\$618,000	\$412,000	\$490,000
Av. nbr delays	1 day/166 years	1 day/9 years	1 day/6 years	1 day/9 years
Delay cost	\$350	\$5,500	\$8,500	\$5,500
Total cost	\$840,350	\$623,500	\$420,500	\$495,500

Policy A2 has the least total expected cost. the decision is based on adopting a long-term policy. If the data of the situation are changed, computations must be revised to see if the optimal policy changes.

Let

14-4

$P(S)$ = probability of an individual being schizophrenic

$P(\bar{S})$ = probability of an individual not being schizophrenic

$P(BA | S)$ = probability of schizophrenia given brain atrophy

$P(BA | \bar{S})$ = probability of brain atrophy given schizophrenia

$P(BA | \bar{S})$ = probability of brain atrophy given no schizophrenia

In terms of the data, we have

$$P(S) = .015$$

$$P(\bar{S}) = .985$$

$$P(BA | S) = .3$$

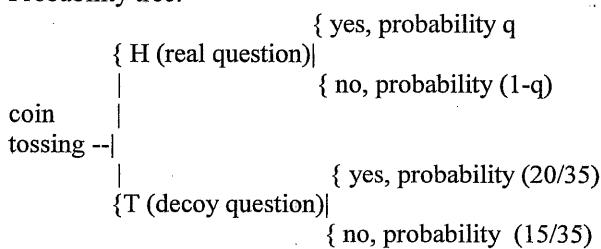
$$P(BA | \bar{S}) = .02$$

$$\text{It thus follows that } P(S | BA) = \frac{.3 \times .015}{.3 \times .015 + .02 \times .985} = .186$$

The result shows that, even though Hinkley's CAT scan shows brain atrophy, there is less than 20% chance that he is schizophrenic. This is not a strong argument in support of Hinkley's claim of mental illness.

Probability tree:

14-5



Per the results of the experiment, we have $P\{\text{yes vote}\}=18/35$

From the probability tree, we have

$$P\{\text{yes vote}\} = (1/2) \times q + (1/2) \times (20/35)$$

Thus,

$$(1/2) \times q + (1/2) \times (20/35) = (18/35)$$

Solving for q, we get

$$q = [(18/35) - (1/2) \times (20/35)] \times 2 = 16/35 = .457$$

Chapter 14 Case

14-1

The frequency histogram for the demand is given below. To be on the conservative side, we ignore the frequency of zero demand.

Nbr. of units	Frequency	Relative frequency	Cumulative relative frequency
1	89	.7807	.7807
2	20	.17544	.9561
3	4	.3509	.9912
4	1	.00877	1.00

Assuming that the demand stays stationary for at least the next year (that is, no appreciable trend), the company's requirement that the demand be met 95% of the time is satisfied with two units in stock.

Chapter 15 Cases

15-1

Because the teller is busy only 40% of the time, it is possible that one teller could attend to more than one customer. In fact, arriving customers may be served by a pool of tellers. The problem with this proposal is that a teller will not have a fixed station, which may create administrative problems in the bank operation.

The data show that the number of calls reaches a peak between 12:00 and 17:00 daily. The design of the system should be based on this extreme condition, rather

15-2

than on the overall average number of arrivals per day. Thus, for the daily period from 12:00 to 17:00, we have

$$\bar{x} = 9.11 \text{ arrival/hr}, s^2 = 7.81$$

There is no reason to believe that arriving calls will follow anything but a Poisson distribution. Notice that $\bar{x} = 9.11$ arrival/hr and $s^2 = 7.81$ are approximately equal, which supports the Poisson claim. (In general, we should use the chi-square to validate the Poisson assumption.)

Regarding the service time distribution (length of calls), the lack of data together with the principle of insufficient reason suggest once again that the service time distribution may also be exponential with mean 7 minutes.

We are now dealing with a Poisson queue with $\lambda = 9.11$ calls per hour and $\mu = 60/7 = 8.75$ phone answers per hour. The telephone lines represent the servers. Given $\lambda/\mu = 1.06$, the system needs at least 2 lines. We know, however, that λ must be larger than 9.11 because the available data do not reflect the calls that are lost when the lines are busy. We thus need to run a type of sensitivity analysis to give us some idea about the "adequacy" of the telephone service under extreme conditions. We must remember that the number of lost calls must be reduced to an absolute minimum because the facility deals with situations that could affect the well-being of a abused child.

The following table provides the measures of performance given $\lambda = 9.11$, 13.7, and 18.22 calls per hour. These values represent 100%, 150%, and 200% of the estimated arrival rate.

Lambda	9.11				13.7				18.22			
Nbr. of lines	2	3	4	5	2	3	4	5	2	3	4	5
L_q	.4	.06	.009	.001	2.8	.3	.06	.01	1.2	.23	.05	.01
W_q (sec)	162	21.6	3.6	.36	720	72	14.4	3.6	216	36	10.8	2.52
$P\{n > c\}$.2	.03	.01	.002	.6	.15	.04	.01	.36	.11	.04	.01

L_q could not be used as proper measure in this case. For example, for $\lambda = 9.11$, $L_q = .4$ waiting calls for $c = 2$. This may appear quite small, but if we examine W_q , the average

continued...

Chapter 15 Cases

waiting time until a call is acknowledged is 162 sec (about 3 min). This is a long waiting time for an anxious person reporting an abuse case. A waiting time of about 10 seconds is the most that can be tolerated in these situations. For example, for $\lambda = 18.22$ calls per hour, 5 lines are needed.

An initial analysis of the situation can be made by comparing the rate of arrival of calls for truck service with the service rate. From the data

15-3

$$\lambda = (0 \times 30 + 1 \times 90 + \dots + 12 \times 4) / (0 + 1 + 2 + \dots + 12) = 4.1 \text{ calls per hr}$$

The average service time per call is computed from the second table as

$$t_{\bar{B}} = (5 \times 61 + 15 \times 34 + \dots + 95 \times 2) / (61 + 34 + 15 + \dots + 2) = 20.2 \text{ min per call}$$

Thus, the service rate is

$$\mu = 1 / 20.2 = .05 \text{ services per min per truck} = 2.97 \text{ services per hr per truck}$$

Given three trucks are in service, the total service rate is $3 \times 2.97 = 8.9$ services per hr. Thus, the utilization of the trucks is computed as

$$\text{utilization} = \lambda / (3\mu) = 4.1 / 8.9 = .46$$

The low utilization shows that the three trucks are sufficient to service the six departments adequately. The main drawback with the current setup is that the trucks do not have a "home" station, a basic assumption is calculation the utilization factor. In other words, the 46% utilization assumes that the trucks are available in one service pool. This difficulty is rectified by placing all calls for service to a common dispatcher who is in constant contact with the drivers of the trucks.

From the data of the problem, the average rate of breakdown per machine per hour is computed as

15-4

$$\lambda = (7 + 8 + 8) / (3 \text{ mach} \times 8 \text{ hr}) = .9583 \text{ per machine per hr}$$

The difference between the failure time and the completion of repair gives the amount of time a broken machine spends in the repair system. Thus,

$$W_s = [(10+12+10+13+10+12+9) + (8+8+13+8+9+13+12+10) + (13+11+10+12+8+11+8+10)] / 24 \\ = 10.46 \text{ min}$$

We can also estimate the number of machines in the system L_s from the information in the second table. For simplicity, we take the average of all the given data points. Normally, we should treat L_s as a time-based variable. However, this would require a complete history of the number of broken machines at all hours of the day.

$$L_s = (6 + 6 + 9 + 6 + \dots + 8 + 8 + 6) / (8 \text{ data points} \times 5 \text{ days}) = 6.73 \text{ machines}$$

Chapter 15 Cases

If the data are correct, and if the situation behaves per the Poisson assumptions, then L_s and W_s must satisfy the formulas

$$L_s = \lambda_{\text{eff}} W_s \\ = \lambda(N - L_s)W_s$$

From the data, we have

$$\lambda = .9583 \text{ calls/machine/hr}$$

$$N = 30 \text{ machines}$$

$$L_s = 6.73 \text{ machines}$$

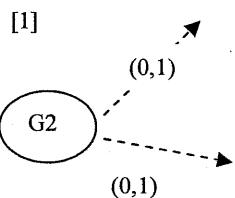
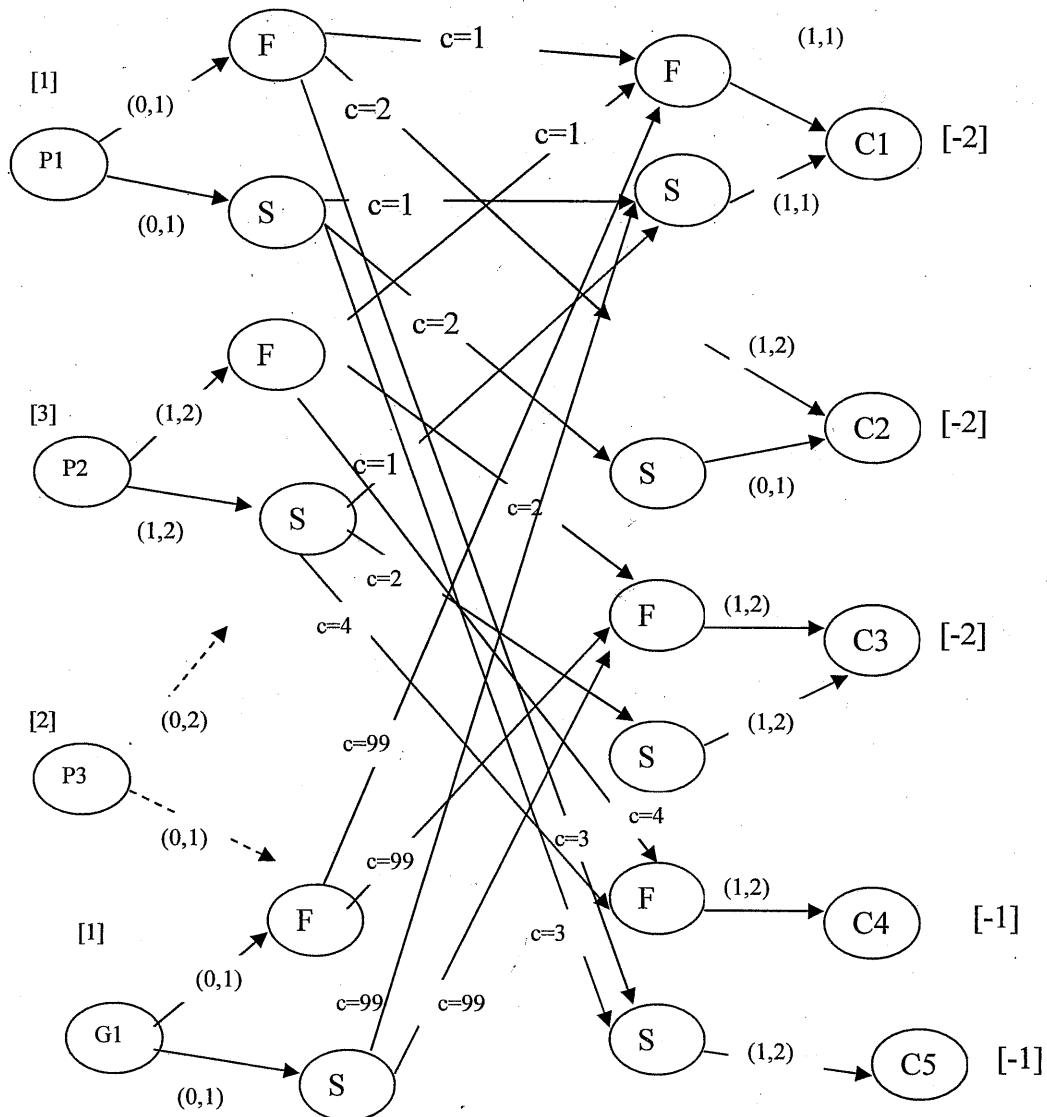
$$W_s = 10.46/60 = .1743 \text{ hr}$$

Thus,

$$\lambda(N - L_s)W_s = .9583(30 - 6.73)(.1743) \\ = 3.887 \text{ machines}$$

This result shows that the data for computing λ and L_s are not consistent. Hence is the conclusion reached by the manager.

Chapter 20 Case



P3 and G2 are treated in a similar manner

Chapter 22 Case

Assume that the machine starts new, and define,

22-1

n = planning horizon ($= 6$ years)

I = initial purchase price

TW_i = trade-in value of a working machine whose age just turned i years

TF_i = trade-in value of a failed machine whose age just turned i years

P_i = probability that an i -year old machine in working order at the start of a year fails at the end of the year.

SW_i = Salvage value at the end of the planning horizon of a working machine of age i .

SF_i = Salvage value at the end of the planning horizon of a failed machine of age i .

$f_K(i)$ = Minimum expected cost of the remaining periods of the horizon given that we start year K with a machine of age i and in working order

$K = 1, 2, \dots, n; i = 1, 2, \dots, K-1$

C_i = expected operating cost of a working machine of age i
that a working machine

$$f_K(i) = \min \left\{ \begin{array}{l} R: I - TW_i + C_0 + P_0 \{ I - TW_i + f_{K+1}(0) \} \\ \quad + (1-P_0) f_{K+1}(1) \\ K: C_i + P_i \{ I - TW_{i+1} + f_{K+1}(0) \} \\ \quad + (1-P_i) f_{K+1}(i+1) \end{array} \right.$$

$K = 1, 2, \dots, n-1$
 $i > 0$ continued...

$$f_K(0) = C_0 + P_0 \{ I - TW_i + f_{K+1}(0) \} + (1-P_0) f_{K+1}(1)$$

$$\left. \begin{array}{l} R: I - TW_i + C_0 - P_0 TF_i \\ \quad - (1-P_0) TW_i \end{array} \right\}$$

$$f_n(i) = \min \left\{ \begin{array}{l} K: C_i - P_i TF_{i+1} - (1-P_i) TW_{i+1} \end{array} \right\}$$

$$f_n(0) = C_0 - P_0 TF_1 - (1-P_0) TW_1$$