Laiba Fatima
22K-5195
BSE-3B

LA Assignment 3

Ex 6.1

9. $u = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$ $\quad v = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$
$= (3)(-1) + (-2)(3) + (4)(1) + (8)(1)$
$= -3 - 6 + 4 + 8$
$= 3$

15. $p = x + x^3 \quad q = 1 + x^2 \quad ; x_0 = -2, x_1 = -1, x_2 = 0, x_3 = 1$

$\langle p, q \rangle = (-10)(5) + (-2)(2) + (0)(1) + (2)(2)$
$= -50 + (-4) + 0 + 4$
$= -50$

21. $u = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$ $\quad v = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

$||u|| = \sqrt{(3)^2 + (-2)^2 + (4)^2 + (8)^2}$
$= \sqrt{93}$

$d(u, v) = ||u - v|| = \sqrt{(3+1)^2 + (-2-3)^2 + (4-1)^2 + (8-1)^2}$
$= 3\sqrt{11}$

Ex. 6.2

1.a) $u = (1, -3)$, $v = (2, 4)$

$$\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|}$$

$$\cos\theta = \frac{2 - 12}{(\sqrt{1+9})(\sqrt{4+16}} = \frac{-10}{(\sqrt{10})(\sqrt{20})} = \frac{-10}{\sqrt{200}}$$

$$\cos\theta = -\frac{1}{\sqrt{2}}$$

b) $u = (-1, 5, 2)$ $v = (2, 4, -9)$

$$\cos\theta = \frac{-2 + 20 - 18}{(\sqrt{1+25+4})(\sqrt{4+16+81}} = \frac{-20 + 20}{\sqrt{30}\sqrt{101}} = \frac{0}{\sqrt{30}\sqrt{101}}$$

$$\cos\theta = 0$$

$\cancel{\theta = 90}$

c) $u = (1, 0, 1, 0)$, $v = (-3, -3, -3, -3)$

$$\cos\theta = \frac{-3 - 3}{(\sqrt{2})(\sqrt{36})}$$

$$\cos\theta = \frac{-1}{\sqrt{2}}$$

4. $p = x - x^2$, $q = 7 + 3x + 3x^2$

$$\cos\theta = \frac{(0)(7) + (1)(3) + (-1)(3)}{\sqrt{2}\ \sqrt{67}}$$

$$\cos\theta = 0$$

6. $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

$$\cos\theta = \frac{(2)(-3) + (4)(1) + (-1)(4) + (3)(2)}{\sqrt{(2)^2 + (4)^2 + (-1)^2 + (3)^2}\ \sqrt{(-3)^2 + (1)^2 + (4)^2 + (2)}}$$

$\cos\theta = 0$

8. a) $u = (u_1, u_2, u_3)$, $v = (0, 0, 0)$

$\langle u, v \rangle = (u_1)(0) + (u_2)(0) + (u_3)(0) = 0$

$\langle u, v \rangle$ is orthogonal

b) $u = (-4, 6, -10, 1)$ $v = (2, 1, -2, 9)$

$\langle u, v \rangle = (-4)(2) + (6)(1) + (-10)(-2) + (1)(9)$

$= 27$ $\therefore$ not orthogonal

c) $u = (a, b, c)$ $v = (-c, 0, a)$

$\langle u, v \rangle = (a)(-c) + (b)(0) + (c)(a)$

$= -ac + ac = 0$

$\langle u, v \rangle$ is ~~not~~ orthogonal

12. $u = \begin{bmatrix} 5 & -1 \\ 2 & -2 \end{bmatrix}$ $v = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$

$\langle u, v \rangle = (5)(1) + (-1)(3) + (2)(-1) + (-2)(0)$

$= 0$

$\langle u, v \rangle$ is orthogonal

14. $u = (2,-4)$, $v = (0,3)$

$\langle u,v \rangle = 2(0) + (-4)(3)$

$= -12$

$\langle u,v \rangle$ is not orthogonal with respect to Euclidean inner product.

$\langle u,v \rangle = 2(2)(0) + k(-4)(3)$

$= -12k$ $(k=0)$ ✗

⇒ 4; There exists no such k that u and v are orthogonal with respect to $\langle u,v \rangle$ because $\langle u,v \rangle = 2u_1v_1$ doesn't represent inner product.

17. $p_1 = 2 + kx + 6x^2$, $p_2 = L + 5x + 3x^2$, $p_3 = 1 + 2x + 3x^2$

$\langle p_1, p_3 \rangle = (2)(1) + (k)(2) + 6(3)$

$= 2 + 2k + 18$

$= 2k + 20$

$k = -10$

$\langle p_2, p_3 \rangle = L + (5)(2) + (3)(3)$

$= L + 10 + 9$

$= L + 19$

$L = -19$

$\langle p_1, p_2 \rangle = 2(L) + k(5) + 6(3)$

$= 2L + 5k + 18$ — (1)

$= 2(-19) + 5(-10) + 18$

$= -70$

not equal to zero so there exists no such k & c where p_1, p_2, p_3 are mutually orthogonal.

18. $u = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $v = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$Au = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$Av = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$\langle Au, Av \rangle = (9 \times 2) + (6 \times -3) = 0$ $\therefore$ orthogonal

Ex. 6.3

3. a) $p_1(x) = 2/3 - 2/3x + 1/3x^2$,

$p_2(x) = 2/3 + 1/3x - 2/3x^2$,

$p_3(x) = 1/3 + 2/3x + 2/3x^2$

$$\langle p_1(x), p_2(x) \rangle = (2/3)(2/3) + (-2/3)(1/3) + (1/3)(-2/3)$$
$$= 0$$

$$\langle p_2(x), p_3(x) \rangle = (2/3)(1/3) + (1/3)(2/3) + (-2/3)(2/3)$$
$$= 0$$

$$\langle p_1(x), p_3(x) \rangle = (2/3)(1/3) + (-2/3)(2/3) + (1/3)(2/3)$$
$$= 0$$

∴ ~~orthonormal~~

∴ orthogonal

6. $A = \begin{bmatrix} 1/5 & -1/2 & 1/3 \\ 1/5 & 1/2 & 1/3 \\ 1/5 & 0 & -2/3 \end{bmatrix}$

let $S = \left\{ \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right), (-1/2, 1/2, 0), \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) \right\}$

be the columnspace

$v_1 = (1/5, 1/5, 1/5)$, $v_2 = \left(-1/2, \frac{1}{2}, 0\right)$, $v_3 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right)$

be the column vectors

$$\langle v_1, v_2 \rangle = \left(\frac{1}{5}\right)(-1/2) + \left(\frac{1}{5}\right)\left(\frac{1}{2}\right) + 0$$
$$= 0$$

$$\langle v_1, v_3 \rangle = \left(\frac{1}{5}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{5}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{5}\right)(-2/3)$$
$$= 0$$

$$\langle v_2, v_3 \rangle = (-1/2)(1/3) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + 0$$
$$= 0$$

$\therefore$ A forms an orthogonal basis of column space.

Now finding unit vector to make orthonormal

$$\hat{v}_1 = \frac{v_1}{\|v_1\|} = \frac{(1/5, 1/5, 1/5)}{\sqrt{(1/5)^2 + (1/5)^2 + (1/5)^2}}$$

$$\hat{v}_1 = \frac{5}{\sqrt{3}}\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\hat{v}_2 = \frac{v_2}{\|v_2\|} = \frac{(-1/2, 1/2, 0)}{\sqrt{(-1/2)^2 + (1/2)^2}} = \sqrt{2}\left(-1/2, 1/2, 0\right)$$
$$= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\hat{v}_3 = \frac{v_3}{\|v_3\|} = \frac{(1/3, 1/3, -2/3)}{\sqrt{(1/3)^2 + (1/3)^2 + (-2/3)^2}} = \frac{\sqrt{3}}{\sqrt{6}}\left(\frac{1}{3}, \frac{1}{3}, \frac{-2}{3}\right)$$
$$= \left(+1/\sqrt{6}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

set of orthonormal bases:

$$\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)\right\}$$

13. $v_1 = (2,-2,1)$, $v_2 = (2,1,-2)$, $v_3 = (1,2,2)$ $u = (-1,0,2)$

$\langle v_1, v_2 \rangle = (2)(2) + (-2)(1) + (1)(-2)$

$= 0$

$\langle v_1, v_3 \rangle = 2(1) + (-2)(2) + (1)(2)$

$= 0$

$\langle v_2, v_3 \rangle = 2(1) + 1(2) + (-2)(2)$

$= 0$

v_1, v_2, v_3 are orthogonal vectors to each other

$$\therefore u = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2 \rangle v_2}{\|v_2\|^2} + \frac{\langle u, v_3 \rangle}{\|v_3\|^2} v_3$$

$$\frac{\langle u, v_1 \rangle}{\|v_1\|} v_1 = \frac{(-1,0,2)(2,-2,1)}{\left(\sqrt{2^2 + (-2)^2 + 1^2}\right)^2} = 0$$

$$\frac{\langle u, v_2 \rangle}{\|v_2\|} = \frac{(-1,0,2)(2,1,-2)}{\left(\sqrt{2^2 + 1^2 + (-2)^2}\right)^2} = \frac{-6}{9} = \frac{-2}{3}$$

$$\frac{\langle u, v_3 \rangle}{\|v_3\|} = \frac{(-1,0,2)(1,2,2)}{\left(\sqrt{1^2 + 2^2 + 2^2}\right)^2} = \frac{1}{3}$$

– u as a linear combination is written as

$$u = 0v_1 \frac{-2}{3} v_2 + \frac{1}{3} v_3$$

hence $(u) = \left(0, \frac{-2}{3}, \frac{1}{3}\right)$

30. $u_1 = (1,0,0)$, $u_2 = (3,7,-2)$, $u_3 = (0,4,1)$

$v_1 = (1,0,0)$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = u_2 - \frac{(3,7,-2)(1,0,0)}{(\sqrt{1^2})^2}(1,0,0)$$

$$= (3)(1,0,0) = u_2 - (3,0,0)$$

$$\cancel{v_2} \neq (3,7,-2) - (3,0,0)$$

$$v_2 = (0,7,-2)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$$

$$u_3 - \frac{\langle (0,4,1),(1,0,0) \rangle}{(\sqrt{1^2+0+0})^2}(1,0,0) - \frac{\langle 0,4,1),(0,7,-2) \rangle}{(\sqrt{0+7^2+(-2)^2})^2}(0,7,-2)$$

$$u_3 - \left[\frac{0+0+0}{1}(1,0,0)\right] - \frac{0+28-2}{53}(0,7,-2)$$

$$u_3 - 0 - \frac{26}{53}(0,7,-2)$$

$$v_3 = (0,4,1) - \frac{26}{53}(0,7,-2)$$

$$= \frac{15}{53}(0,2,7) = \left(0, \frac{30}{53}, \frac{105}{53}\right)$$

orthonormal Bases (q_1, q_2, q_3)

$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,0,0)}{\sqrt{1^2+0+0}} = (1,0,0)$

$q_2 = \frac{v_2}{\|v_2\|} = \frac{(0,7,-2)}{\sqrt{0+7^2+(-2)^2}} = \frac{1}{\sqrt{53}}(0,7,-2)$
$= (0, \frac{7}{\sqrt{53}}, \frac{-2}{\sqrt{53}})$

$q_3 = \frac{(0, 30/53, 105/53)}{\sqrt{0+(\frac{30}{53})^2+(\frac{105}{53})^2}} = \frac{1}{\sqrt{53}}(0,2,7)$
$= (0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}})$

$\left\{ (1,0,0), \cancel{\frac{1}{\sqrt{53}}} (0, \frac{7}{\sqrt{53}}, \frac{-2}{\sqrt{53}}), \cancel{\frac{1}{\sqrt{53}}} (0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}) \right\}$

49. $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ $u_1 = (1,-1,1,-1)$
$u_2 = (0,1,0,1)$
$u_3 = (1,1,1,1)$

$v_1 = (1,-1,1,-1)$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|u_1\|^2} v_1 = u_2 - \frac{\langle (0,1,0,1),(1,-1,1,-1) \rangle}{(\sqrt{1^2+(-1)^2+1^2+(-1)^2})^2} (1,-1,1,-1)$

$v_2 = (0,1,0,1) - (-1/2, 1/2, -1/2, 1/2) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

$v_3 = (1,1,1,1) - \frac{\langle (1,1,1,1),(1,-1,1,-1) \rangle}{(\sqrt{1^2+(-1)^2+1^2+(-1)^2})^2} (1,-1,1,-1)$

$\frac{\langle (1,1,1,1), (1/2, 1/2, 1/2, 1/2) \rangle}{(\sqrt{(1/2)^2+(1/2)^2+(1/2)^2+(1/2)^2})^2} (1/2, 1/2, 1/2, 1/2)$

$v_3 = (0,0,0,0)$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{(1,-1,1,-1)}{\sqrt{1^2+(-1)^2+1^2+(-1)^2}} = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{(1/2, 1/2, 1/2, 1/2)}{\sqrt{(1/2)^2+(1/2)^2+(1/2)^2+(1/2)^2}} = (1/2, 1/2, 1/2, 1/2)$$

$$q_3 = \frac{v_3}{\|v_3\|} = (0,0,0,0)$$ cannot be found as

v_3 completely becomes zero so no solution

47. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$

$$A = QR$$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}$$

$\langle u_1, q_1 \rangle = \langle (1,0,1), (1/\sqrt{2}, 0, 1/\sqrt{2}) \rangle = \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}}$
$= \sqrt{2}$

$\langle u_2, q_1 \rangle = \langle (0,1,2), (1/\sqrt{2}, 0, 1/\sqrt{2}) = 0 + 0 + 2/\sqrt{2}$
$= \sqrt{2}$

$\langle u_3, q_1 \rangle = \langle (2,1,0), (1/\sqrt{2}, 0, 1/\sqrt{2}) = 2/\sqrt{2} + 0 + 0$
$= \sqrt{2}$

$\langle u_2, q_2 \rangle = \langle (0,1,2), (-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \rangle = 0 + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}$
$= \cancel{0}\ \sqrt{3}$

$\langle u_3, q_2 \rangle = \langle (2,1,0), (-1/\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \rangle = \frac{-2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0$
$= \frac{1}{\sqrt{3}}$

$\langle u_3, q_3 \rangle = \langle (2,1,0), (1/\sqrt{2}, 2/\sqrt{6}, -1/\sqrt{6}) \rangle$
$= \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} + 0 = \frac{4}{\sqrt{6}}$

$$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -1/\sqrt{3} \\ 0 & 0 & 4/\sqrt{6} \end{bmatrix}$$

$A = QR$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -1/\sqrt{3} \\ 0 & 0 & 4/\sqrt{6} \end{bmatrix}$$