Assignment 4

Linear Algebra (National University of Computer and Emerging Sciences)

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Course:- Linear Algebra
~~87~~ Section:- A
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M. Hassan

Question 1

$v_1 = (1,0,2),\ v_2 = (-1,1,0),\ v_3 = (0,0,1)$

$\lambda_1 = -1 \quad , \lambda_2 = 0 \quad , \lambda_3 = 1$

i) $P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ $\quad$ $\text{Adj } P = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^t$

$|P| = 1(1-0) - (-1)(0-0) + 0(0-2)$

$|P| = 1 + 0 + 0$

$|P| = 1 \neq 0$ so eigenvectors are L.I

$\text{Adj } P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$ $\quad$ $P^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$

$PAP^{-1} = D$

$PAP^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$

$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 1 \end{bmatrix}$

$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 1 \end{bmatrix}$

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ii) A^{20}

$n = 20$

$A^{20} = P D^{20} P^{-1}$

D is diagnol matrix so non diagnol values are 0 and we can just ~~square~~ take exponential of diagnol values

$$D^{20} = \begin{bmatrix} -1^{20} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1^{20} \end{bmatrix}$$

$$D^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^{20} = P^{-1} D^{20} P$

$$A^{20} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} P \Rightarrow A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} P$$

$$A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \Rightarrow A^{20} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

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Question 2

$$A = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{vmatrix} 7-\lambda & 1 & -2 \\ -3 & 3-\lambda & 6 \\ 2 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$7-\lambda\left((3-\lambda)(2-\lambda) - 12\right) - 1\left(-3(2-\lambda) - 12\right) - 2\left(-6 - 2(3-\lambda)\right) = 0$$

$$7-\lambda(6 - 5\lambda + \lambda^2 - 12) - 1(-6 + 3\lambda - 12) - 2(-6 - 6 + 2\lambda) = 0$$

$$(7-\lambda)(\lambda^2 - 5\lambda - 6) + 6 - 3\lambda + 12 + 12 + 12 - 4\lambda = 0$$

$$7\lambda^2 - 35\lambda - 42 - \lambda^3 + 5\lambda^2 + 6\lambda + 18 + 24 - 7\lambda = 0$$

$$-\lambda^3 + 7\lambda^2 + 5\lambda^2 - 35\lambda + 6\lambda - 7\lambda - 42 + 42 = 0$$

$$-\lambda^3 + 12\lambda^2 - 36\lambda = 0$$

$$\lambda(-\lambda^2 + 12\lambda - 36) = 0$$

$$\lambda(-\lambda^2 + 6\lambda + 6\lambda - 36) = 0$$

$$\lambda(\lambda(-\lambda + 6) - 6(-\lambda + 6)) = 0$$

$$\lambda(-\lambda + 6)(\lambda - 6) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = 6, \quad \lambda_3 = 6$$

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a) $\lambda_1 = 0$

$$\begin{bmatrix} 7-\lambda & 1 & -2 \\ -3 & 3-\lambda & 6 \\ 2 & 2 & 2-\lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

row operations

$R_2 + (3/7)R_1$, $R_3 - (2/7)R_1$

$$\begin{bmatrix} 7 & 1 & -2 \\ 0 & 24/7 & 36/7 \\ 0 & 12/7 & 18/7 \end{bmatrix}$$ $R_1 = \frac{R_1}{7}$
$R_3 = R_3 - R_2 \frac{1}{2}$

$$\begin{bmatrix} 1 & 1/7 & -2/7 \\ 0 & 24/7 & 36/7 \\ 0 & 0 & 0 \end{bmatrix}$$ $R_2 = \frac{7}{24} \times R_2 \Rightarrow$ $$\begin{bmatrix} 1 & 1/7 & -2/7 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 = R_1 - \frac{1}{7} R_2$

$-\frac{3-4}{14} = \frac{-7}{14}$

$$\begin{bmatrix} 1 & 0 & -\frac{3}{14} - \frac{2}{7} \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

let $v_3 = t$

$v_1 = \frac{1}{2} t$, $v_2 = \frac{-3}{2} t$, $v_3 = t$

$$V = \begin{bmatrix} 1/2 \\ -3/2 \\ 1 \end{bmatrix} t$$

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b) $\lambda = 6$

$$\begin{bmatrix} 7-\lambda & 1 & -2 \\ -3 & 3-\lambda & 6 \\ 2 & 2 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix}$$

$R_2 + 3R_1 \,,\ R_3 - 2R_1$

$$= \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

let $v_2 = s \,,\ v_3 = t$

$v_1 = 2t - s$

$$V = \begin{bmatrix} 2t-s \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s$$

Eigenvectors

$P_1 = (1/2, -3/2, 1) \,,\ P_2 = (2, 0, 1) \,,$

$P_3 = (-1, 1, 0)$

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$$P = \begin{bmatrix} 1/2 & 2 & -1 \\ -3/2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad P^{-1} = \frac{1}{|P|} \text{adj} P$$

$$\text{adj} P = \begin{bmatrix} -1 & 1 & -3/2 \\ -1 & 1 & 3/2 \\ 2 & 1 & 3 \end{bmatrix}^t = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & 1 \\ -3/2 & 3/2 & 3 \end{bmatrix}$$

$$|P| = \begin{vmatrix} 1/2 & 2 & -1 \\ -3/2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \frac{1}{2}(-1) - 2(-1) - 1\left(-\frac{3}{2}\right)$$

$$= -\frac{1}{2} + 2 + \frac{3}{2}$$

$$= \frac{2}{2} + 2$$

$$= 1 + 2 = 3$$

$$P^{-1} = \begin{bmatrix} -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \\ \frac{-1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A = PDP^{-1}$$

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a) $A + \cancel{3I}\ 4I$

let $B = A + 4I$

~~The eigen values of B will be same as A+3.~~

So

$\lambda_1 = 3,\ \lambda_2 = 9,\ \lambda_3 = 9$

$B = PDP^{-1}$

$B = P(D+3I)P^{-1}$

$$B = P\begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 10 \end{bmatrix}P^{-1}$$

$\lambda_1 = 4,\ \lambda_2 = 6,\ \lambda_3 = 10$

b) A^{-3}

As λ_1 of A is 0 so the ~~eig~~ A^{-1} can't exist Hence A^{-3} won't have eigen values.

If A has eigen val λ then A^{-1} will have $1/\lambda$ and $1/0$ doesn't exist.

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Question 3:

$$\frac{dx}{dt} = 2x + y \qquad x(0) = 1$$

$$\frac{dy}{dt} = x + 2y \qquad y(0) = 5$$

let system of DE be

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} \Rightarrow (2-\lambda)^2 - 1 = 0$$

$$4 + \lambda^2 - 4\lambda - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda(\lambda - 3) - 1(\lambda - 3) = 0$$

$$\lambda_1 = 3, \lambda_2 = 1$$

Eigen vectors:

i) $\lambda_1 = 3$: $\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$: $-v_1 + 1 = 0 \Rightarrow v_1 = 1$

$1 - v_2 = 0 \Rightarrow v_2 = 1$

ii) $\lambda_2 = 1$: $\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$: ~~$v_1 + 1 = 0$~~ ~~$v_1 = -1$~~ ~~$1 + v_2 = 0$~~ ~~$v_2 = -1$~~ $\quad x + 2y = y \quad x = -y$

$2x + 2y = x \Rightarrow$

Then $\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{pmatrix} c_1 e^{3t} - c_2 e^{-t} \\ c_1 e^{3t} + c_2 e^{-t} \end{pmatrix}$

$x(t) = c_1 e^{3t} - c_2 e^{-t}$	$y(t) = c_1 e^{3t} - c_2 e^{-t}$
$x(0) = 1$	$y(0) = 5$
$1 = c_1 e^0 - c_2 e^0$	$5 = c_1 + c_2$
$c_1 - c_2 = 1$	

$C_1 = 1 + C_2$

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$5 = C_1 + C_2$

$5 = 1 + C_2 + C_2$

$2C_2 = 6$

$C_2 = 3$

$C_1 - C_2 = 1$

$C_1 = 1 + C_2$

$C_1 = 1 + 3$

$C_1 = 4$

$C_1 = 4, \quad C_2 = 3$

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Question 4:-

$X' = AX$

$$X' = \begin{bmatrix} 7 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

I solved this ~~matr~~ matrix in Question 2 and will be picking data from there.

$$\det(A - I\lambda) = \begin{vmatrix} 7-\lambda & 1 & -2 \\ -3 & 3-\lambda & 6 \\ 2 & 2 & 2-\lambda \end{vmatrix} = \lambda(-\lambda+6)(\lambda-6)$$

$$\lambda_1 = 0, \lambda_2 = 6, \lambda_3 = 6$$

i) Eigenvector for λ_1:-

$$\begin{bmatrix} 7-0 & 1 & -2 \\ -3 & 3-0 & 6 \\ 2 & 2 & 2-0 \end{bmatrix} = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

After doing row operations we reduce this matrix to

$$\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

let $z = s$

$x = 1/2\, s$

$y = -3/2\, s$

$$\begin{bmatrix} 1/2 \\ -3/2 \\ 1 \end{bmatrix} s$$

Eigenvector for λ_2, λ_3:- M. Hassan

$$\begin{bmatrix} 7-6 & 1 & -2 \\ -3 & 3-6 & 6 \\ 2 & 2 & 2-6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix}$$

using row operations:-

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} y &= s \\ z &= k \\ x &= 2k - s \end{aligned}$$

$$V = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} k + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s$$

$$X(t) = c_1 e^{\lambda t} \begin{bmatrix} 1/2 \\ -3/2 \\ 1 \end{bmatrix} + C_2 e^{\lambda t} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + C_3 e^{\lambda t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$x(t) = \frac{1}{2} c_1 + 2C_2 e^{6t} + - C_3 e^{6t}$$

$$y(t) = \frac{-3}{2} c_1 + (0) C_2 e^{6t} + C_3 e^{6t}$$

$$z(t) = c_1 + C_2 e^{6t} + (0) C_3 e^{6t}$$

$$x(t) = \frac{1}{2} c_1 + 2 C_2 e^{6t} + - C_3 e^{6t}$$

$$y(t) = c_1 + C_3 e^{6t}$$

$$z(t) = c_1 + C_2 e^{6t}$$

Question 5:- M. Hassan

$T: V \rightarrow V$ $[a,b]$

$T(f) = 5f(x) + 3\int_a^x f(t)\,dt$

Prove T is linear operator:-

suppose $f, g \in V$

$T(f,g) = 5[f(x)+g(x)] + 3\int_a^x (f(x)+g(x))\,dt$

$T(f,g) = 5f(x) + 5g(x) + 3\int_a^x f(x)\,dt + 3\int_a^x g(x)\,dt$

$T(f,g) = 5f(x) + 3\int_a^x f(x)\,dt + 5g(x) + 3\int_a^x g(x)\,dt$

$T(f,g) = T(f) + T(g)$

Hence first condition satisfies

let k be any scalar

$T(kf) = 5(k \cdot f(x)) + 3\int_a^x k \cdot f(t)\,dt$

$T(kf) = \cancel{5k}\, k(5 \cdot f(x)) + k\,3\int_a^x k \cdot f(t)\,dt$

$T(kf) = k\left(5f(x) + 3\int_a^x f(t)\,dt\right)$

$T(kf) = k\,T(f)$

Hence This conditions holds true as well.

Therefore T is a linear operator.

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Question 6:

$T_1 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ reflection by $\theta = 90°$ about y-axis

$T_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ reflection about xy plane.

Standard rotation matrix

$$A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

putting $\theta = 90°$

$$T_1(A) = \begin{bmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

The standard matrix for reflection about xz plane

$$T_2(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T_2 \circ T_1 = BA$

$$T_2 \circ T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$T_2 \circ T_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

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$$T_1 \circ T_2 = AB$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$T_1 \circ T_2 = T_2 \circ T_1$$

Hence they are commutative.

Question 7:   M Hassan

$T(1,2,1) = (1,5)$
$T(2,9,0) = (-1,1)$
$T(3,3,4) = (2,4)$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} + c\begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

$x_1 = a + 2b + 3c \quad -(i)$
$x_2 = 2a + 9b + 3c \quad -(ii)$
$x_3 = a + 4c$

$9(i) - 2(ii)$

$$\begin{array}{l} 9a + \cancel{18b} + 27c \\ \underline{\pm 4a \pm \cancel{18b} \pm 6c} \\ 5a + 21c \end{array}$$

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -23x_1 + 5x_2 + 14x_3 \\ -139x_1 + 31x_2 + 82x_3 \end{bmatrix}$$

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 23 & 5 & 14 \\ -139 & 31 & 82 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ Formula

Verifying $T(v_1)$, $T(v_2)$ M. Hassan

$$T(1,2,1) = \begin{bmatrix} -23 & 5 & 14 \\ -139 & 31 & 82 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$T(2,9,0) = \begin{bmatrix} -23 & 5 & 14 \\ -139 & 31 & 82 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$