DEFINITION OF A MARKOV CHAIN

Let X_t be a random variable that characterizes the state of the system at discrete points in time $t = 1, 2, \ldots$ The family of random variables $\{X_t\}$ forms a **stochastic process**. The number of states in a stochastic process may be finite or infinite, as the following two examples demonstrate:

Example 17.1-1 (Machine Maintenance)

The condition of a machine at the time of the monthly preventive maintenance is characterized as fair, good, or excellent. For month t, the stochastic process for this situation can be represented as:

$$X_t = \left\{ \begin{array}{ll} 0, & \text{if the condition is poor} \\ 1, & \text{if the condition is fair} \\ 2, & \text{if the condition is good} \end{array} \right\}, t = 1, 2, \ldots$$

The random variable X_t is *finite* because it represents three states: poor (0), fair (1), and good (2).

Example 17.1-3 (The Gardener Problem)

Every year, at the beginning of the gardening season (March through September), a gardener uses a chemical test to check soil condition. Depending on the outcome of the test, productivity for the new season falls in one of three states: (1) good, (2) fair, and (3) poor. Over the years, the

A stochastic process is a **Markov process** if the occurrence of a future state depends only on the immediately preceding state. This means that given the chronological times $t_0, t_1, \ldots, t_n$, the family of random variables $\{X_{t_n}\} = \{x_1, x_2, \ldots, x_n\}$ is said to be a Markov process if it possesses the following property:

$$P\{X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1}, \ldots, X_{t_0} = x_0\} = P\{X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1}\}$$

In a Markovian process with n exhaustive and mutually exclusive states (outcomes), the probabilities at a specific point in time $t = 0, 1, 2, \ldots$ is usually written as

$$p_{ij} = P\{X_t = j | X_{t-1} = i\}, (i, j) = 1, 2, \ldots, n, t = 0, 1, 2, \ldots, T$$

This is known as the **one-step transition probability** of moving from state i at $t - 1$ to state j at t. By definition, we have

$$\sum_j p_{ij} = 1, i = 1, 2, \ldots, n$$

$$p_{ij} \geq 0, (i, j) = 1, 2, \ldots, n$$

A convenient way for summarizing the one-step transition probabilities is to use the following matrix notation:

$$\mathbf{P} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & p_{n3} & \cdots & p_{nn} \end{pmatrix}$$

The matrix **P** defines the so-called **Markov chain**. It has the property that all its transition probabilities p_{ij} are fixed (stationary) and independent over time. Although a Markov chain may include an infinite number of states, the presentation in this chapter is limited to finite chains only, as this is the only type needed in the text.

PROBLEM SET 17.1A

1. An engineering professor purchases a new computer every two years with preferences for three models: *M*1, *M*2, and *M*3. If the present model is *M*1, the next computer may be *M*2 with probability .2 or *M*3 with probability .15. If the present model is *M*2, the probabilities of switching to *M*1 and *M*3 are .6 and .25, respectively. And, if the present model is *M*3, then the probabilities of switching to *M*1 and *M*2 are .5 and .1, respectively. Represent the situation as a Markov chain.

States: Models M1, M2, and M3

	M1	M2	M3
M1	0.65	0.2	0.15
M2	0.6	0.15	0.25
M3	0.5	0.1	0.4

ABSOLUTE AND n-STEP TRANSITION PROBABILITIES

Given the initial probabilities $\mathbf{a}^{(0)} = \{a_j^{(0)}\}$ of starting in state j and the transition matrix $\mathbf{P}$ of a Markov chain, the absolute probabilities $\mathbf{a}^{(n)} = \{a_j^{(n)}\}$ of being in state j after n transitions ($n > 0$) are computed as follows:

$$
\begin{aligned}
\mathbf{a}^{(1)} &= \mathbf{a}^{(0)}\mathbf{P} \\
\mathbf{a}^{(2)} &= \mathbf{a}^{(1)}\mathbf{P} = \mathbf{a}^{(0)}\mathbf{P}\mathbf{P} = \mathbf{a}^{(0)}\mathbf{P}^2 \\
\mathbf{a}^{(3)} &= \mathbf{a}^{(2)}\mathbf{P} = \mathbf{a}^{(0)}\mathbf{P}^2\mathbf{P} = \mathbf{a}^{(0)}\mathbf{P}^3
\end{aligned}
$$

Continuing in the same manner, we get

$$\mathbf{a}^{(n)} = \mathbf{a}^{(0)}\mathbf{P}^n, n = 1, 2, \ldots$$

The matrix $\mathbf{P}^n$ is known as the ***n*-step transition matrix**. From these calculations we can see that

$$\mathbf{P}^n = \mathbf{P}^{n-1}\mathbf{P}$$

or

$$\mathbf{P}^n = \mathbf{P}^{n-m}\mathbf{P}^m, 0 < m < n$$

These are known as **Chapman-Kolomogorov** equations.

Example 17.2-1

The following transition matrix applies to the gardener problem with fertilizer (Example 17.1-3):

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} .30 & .60 & .10 \\ .10 & .60 & .30 \\ .05 & .40 & .55 \end{pmatrix} \end{array}$$

The initial condition of the soil is good—that is $\mathbf{a}^{(0)} = (1, 0, 0)$. Determine the absolute probabilities of the three states of the system after 1, 8, and 16 gardening seasons.

$$\mathbf{P}^2 = \begin{pmatrix} .30 & .60 & .10 \\ .10 & .60 & .30 \\ .05 & .40 & .55 \end{pmatrix}\begin{pmatrix} .30 & .60 & .10 \\ .10 & .60 & .30 \\ .05 & .40 & .55 \end{pmatrix} = \begin{pmatrix} .1550 & .5800 & .2650 \\ .1050 & .5400 & .3550 \\ .0825 & .4900 & .4275 \end{pmatrix}$$

$$\mathbf{P}^4 = \begin{pmatrix} .1550 & .5800 & .2650 \\ .1050 & .5400 & .3550 \\ .0825 & .4900 & .4275 \end{pmatrix}\begin{pmatrix} .1550 & .5800 & .2650 \\ .1050 & .5400 & .3550 \\ .0825 & .4900 & .4275 \end{pmatrix}$$

$$= \begin{pmatrix} .10679 & .53295 & .36026 \\ .10226 & .52645 & .37129 \\ .09950 & .52193 & .37857 \end{pmatrix}$$

$$\mathbf{P}^8 = \begin{pmatrix} .10679 & .53295 & .36026 \\ .10226 & .52645 & .37129 \\ .09950 & .52193 & .37857 \end{pmatrix} \begin{pmatrix} .10679 & .53295 & .36026 \\ .10226 & .52645 & .37129 \\ .09950 & .52193 & .37857 \end{pmatrix}$$

$$= \begin{pmatrix} .101753 & .525514 & .372733 \\ .101702 & .525435 & .372863 \\ .101669 & .525384 & .372863 \end{pmatrix}$$

$$\mathbf{P}^{16} = \begin{pmatrix} .101753 & .525514 & .372733 \\ .101702 & .525435 & .372863 \\ .101669 & .525384 & .372863 \end{pmatrix} \begin{pmatrix} .101753 & .525514 & .372733 \\ .101702 & .525435 & .372863 \\ .101669 & .525384 & .372863 \end{pmatrix}$$

$$= \begin{pmatrix} .101659 & .52454 & .372881 \\ .101659 & .52454 & .372881 \\ .101659 & .52454 & .372881 \end{pmatrix}$$

Thus,

$$\mathbf{a}^{(1)} = (1 \quad 0 \quad 0) \begin{pmatrix} .30 & .60 & .10 \\ .10 & .60 & .30 \\ .05 & .40 & .55 \end{pmatrix} = (.30 \quad .60 \quad .1)$$

$$\mathbf{a}^{(8)} = (1 \quad 0 \quad 0) \begin{pmatrix} .101753 & .525514 & .372733 \\ .101702 & .525435 & .372863 \\ .101669 & .525384 & .372863 \end{pmatrix} = (.101753 \quad .525514 \quad .372733)$$

$$\mathbf{a}^{(16)} = (1 \quad 0 \quad 0) \begin{pmatrix} .101659 & .52454 & .372881 \\ .101659 & .52454 & .372881 \\ .101659 & .52454 & .372881 \end{pmatrix} = (.101659 \quad .52454 \quad .372881)$$

The rows of $\mathbf{P}^8$ and the vector of absolute probabilities $\mathbf{a}^{(8)}$ are almost identical. The result is more pronounced for $\mathbf{P}^{16}$. It demonstrates that, as the number of transitions increases, the absolute probabilities are independent of the initial $\mathbf{a}^{(0)}$. In this case the resulting probabilities are known as the **steady-state probabilities.**

PROBLEM SET 17.2A

1. Consider Problem 1, Set 17.1a. Determine the probability that the professor will purchase the current model in four years.

Input Markov chain:

	M1	M2	M3
M1	0.65	0.2	0.15
M2	0.6	0.15	0.25
M3	0.5	0.1	0.4

Output (2-step or 4 yrs.) transition matrix P^2

	M1	M2	M3
M1	0.6175	0.175	0.2075
M2	0.605	0.1675	0.2275
M3	0.585	0.155	0.26

P{M1|M1}=.6175
P{M2|M2}=.1675
P{M3|M3}=.26

17.3 CLASSIFICATION OF THE STATES IN A MARKOV CHAIN

The states of a Markov chain can be classified based on the transition probability p_{ij} of **P**.

1. A state j is **absorbing** if it returns to itself with certainty in one transition—that is $p_{jj} = 1$.
2. A state j is **transient** if it can reach another state but cannot itself be reached back from another state. Mathematically, this will happen if $\lim_{n\to\infty} p_{ij}^{(n)} = 0$, for all i.
3. A state j is **recurrent** if the probability of being revisited from other states is 1. This can happen if, and only if, the state is not transient.
4. A state j is **periodic** with period $t > 1$ if a return is possible only in $t, 2t, 3t, \ldots$ steps. This means that $p_{jj}^{(n)} = 0$ whenever n is not divisible by t.

STEADY-STATE PROBABILITIES AND MEAN RETURN TIMES OF ERGODIC CHAINS

In an ergodic Markov chain, the steady-state probabilities are defined as

$$\pi_j = \lim_{n\to\infty} a_j^{(n)}, \quad j = 0, 1, 2, \ldots$$

These probabilities, which are independent of $\left\{a_j^{(0)}\right\}$, can be determined from the equations

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$$

$$\sum_j \pi_j = 1$$

(One of the equations in $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ is redundant.) What $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ says is that the probabilities $\boldsymbol{\pi}$ remain unchanged after one transition, and for this reason they represent the steady-state distribution.

A direct by-product of the steady-state probabilities is the determination of the expected number of transitions before the systems returns to a state j for the first time. This is known as the **mean first return time** or the **mean recurrence time,** and is computed in an n-state Markov chain as

$$\mu_{jj} = \frac{1}{\pi_j}, j = 1, 2, \ldots, n$$

Example 17.4-1

To determine the steady-state probability distribution of the gardener problem with fertilizer (Example 17.1-3), we have

$$(\pi_1\ \pi_2\ \pi_3) = (\pi_1\ \pi_2\ \pi_3)\begin{pmatrix} .3 & .6 & .1 \\ .1 & .6 & .3 \\ .05 & .4 & .55 \end{pmatrix}$$

which yields the following set of equations:

$$\pi_1 = .3\pi_1 + .1\pi_2 + .05\pi_3$$
$$\pi_2 = .6\pi_1 + .6\pi_2 + .4\pi_3$$
$$\pi_3 = .1\pi_1 + .3\pi_2 + .55\pi_3$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$

Recalling that one (any one) of the first three equations is redundant, the solution is $\pi_1 = 0.1017$, $\pi_2 = 0.5254$, and $\pi_3 = 0.3729$. What these probabilities say is that, in the long run, the soil condition approximately will be good 10% of the time, fair 52% of the time, and poor 37% of the time.

Arranged the equation

$.7\Pi_1=.1\ \Pi_2 +0.05\Pi_3$---------(1)

$.4\Pi_2=.6\ \Pi_1 +0.4\Pi_3$ ----------(2)

$.45\Pi_3=.1\ \Pi_1 + 0.3\Pi_2$---------(3)

$\Pi_1=1- \Pi_2 -\Pi_3$-------------------(4)

From 1 and 4

$.8\ \Pi_2 +0.75\Pi_3 =.7$-------------(5)

From 2 and 4

$\Pi_2 +0.2\Pi_3=.6$------------------(6)

From5 and 6

$\Pi_3 =0.3729$

From6

$\Pi_2 =0.5254$

From4

$\Pi_1 =0.1017$

The mean first return times are computed as

$$\mu_{11} = \frac{1}{.1017} = 9.83, \mu_{22} = \frac{1}{.5254} = 1.9, \mu_{33} = \frac{1}{.3729} = 2.68$$

This means that, depending on the current state of the soil, it will take approximately 10 gardening seasons for the soil to return to a *good* state, 2 seasons to return to a *fair* state, and 3 seasons to return to a *poor* state. These results point to a more "bleak" than "promising" outlook for the soil condition under the proposed fertilizer program. A more aggressive program should improve the picture. For example, consider the following transition matrix in which the probabilities of moving to a good state are higher than in the previous matrix:

$$\mathbf{P} = \begin{pmatrix} .35 & .6 & .05 \\ .3 & .6 & .1 \\ .25 & .4 & .35 \end{pmatrix}$$

In this case, $\pi_1 = 0.31, \pi_2 = 0.58,$ and $\pi_3 = 0.11$, which yields $\mu_{11} = 3.2, \mu_{22} = 1.7$, and $\mu_{33} = 8.9$, a reversal of the "bleak" outlook given previously.

Example 17.4-2 (Cost Model)

Consider the gardener problem with fertilizer (Example 17.1-3). Suppose that the cost of the fertilizer is \$50 per bag and the garden needs two bags if the soil is good. The amount of fertilizer is increased by 25% if the soil is fair and 60% if the soil is poor. The gardener estimates the annual yield to be worth \$250 if no fertilizer is used and \$420 if fertilizer is applied. Is it worthwhile to use the fertilizer?

Using the steady state probabilities in Example 17.4-1, we get

$$\begin{aligned}\text{Expected annual cost of fertilizer} &= 2 \times \$50 \times \pi_1 + (1.25 \times 2) \times \$50 \times \pi_2 \\ &\quad + (1.60 \times 2) \times \$50 \times \pi_3 \\ &= 100 \times .1017 + 125 \times .5254 + 160 \times .3729 \\ &= \$135.51\end{aligned}$$

$$\text{Increase in the annual value of the yield} = \$420 - \$250 = \$170$$

The results show that, on the average, the use of fertilizer nets 170 − 135.51 = \$34.49. Hence the use of fertilizer is recommended.