Cam Scanner 05-05-2021 04

Linear Algebra (National University of Computer and Emerging Sciences)

Second Order Linear PDE's With Constant co-efficient :-

→ Obtain general Solution of each of the following Linear homogeneous PDE with constant co-efficients:-

Question 1:

$$Z_{xx} - 5Z_{xy} - 6Z_{yy} = 0$$

Solution:-

$$Z_{xx} - 5Z_{xy} + 6Z_{yy} = 0$$

The given PDE is equivalent to:

$$D = \frac{\partial}{\partial x} \quad , \quad D' = \frac{\partial}{\partial y}$$

$$(D^2 - 5DD' + 6)Z = 0$$

with $Z = \phi(y + mx)$

Replace $D = m$ & $D' = 1$

we obtain the auxillary equation

$$\Rightarrow m^2 - 5m + 6 = 0$$

$$\Rightarrow m^2 - 2m - 3m + 6 = 0 \quad \text{:: factorization}$$

$$\Rightarrow m(m-2) - 3(m-2) = 0$$

$$\Rightarrow (m-2)(m-3) = 0$$

$$\Rightarrow m = 2, 3$$

The general solution of the given PDE is:

$$Z = \phi_1(y + 2x) + \phi_2(y + 3x)$$

Question 2:-

$$(2D^2 - D'^2 + 2D)z = 0.$$

Solution:

Here $(D, D') = (2D^2 - D'^2 + 2D)$ is not homogeneous in D & D'

let

$$z = A\exp(\alpha x + \beta y)$$

thus

$$(2\alpha^2 - \beta^2 + 2\alpha)\, A\,ex(\alpha x + \beta y) = 0$$

Since

$$\exp(\alpha x + \beta y) \neq 0 \quad \& \quad A \neq 0$$

then they are ~~Obtaine~~ arbitrary constant & it follow that;

$$\Rightarrow 2\alpha^2 - \beta^2 + 2\alpha = 0$$

$$\beta^2 = 2\alpha^2 - 2\alpha$$

$$\sqrt{\beta^2} = \sqrt{2\alpha(\alpha+1)}$$

$$\beta = \sqrt{2\alpha(\alpha+1)}$$

for choice of each α we will be find β -

So that:-

$$\beta_i = \sqrt{2\alpha_i(\alpha_i+1)}$$

Thus the general solution of the PDE's is.

$$Z = \sum_i A_i \exp\left[\alpha_i \alpha + \sqrt{2\alpha_i(\alpha_i+1)}\, y\right]$$

where A_i and α_i are arbitrary constant.

Question 3:-

$$(D - D' - 1)(D - D' - 2) z = 0$$

Solution:-

$$(D - D' - 1)(D - D' - 2) z = 0$$

The given PDE is in linear factor. For the first & second factors on the left side of eq: ① we have

$$a_1 = 1 \quad , \quad a_2 = +1$$
$$b_1 = -1 \quad , \quad b_2 = -1$$
$$c_1 = -1 \quad , \quad c_2 = -2$$

Now using the formula (Here $a_1 \neq 0$ & $a_2 \neq 0$)

$$Z = \exp\left(-\frac{c_1 x}{a_1}\right) \phi_1 (b_1 x + a_1 y)$$

$$+ \exp(-c_2 x / c_2) \cdot \phi_2 (b_2 x - a_2 y)$$

The general solution of the given PDE's is

$$Z = \exp(x)\, \phi_1 (-x - y) + \exp(2x)\, \phi_2 (-x - y).$$

Question 4:-

$$Z_{xx} - 6Z_{xy} + 9Z_{yy} = 0$$

Solution:-

$$Z_{xx} - 6Z_{xy} + 9Z_{yy} = 0$$

The given PDE's is equivalent to:

$$\left(D = \frac{\partial}{\partial x} \quad , \quad D' = \frac{\partial}{\partial y} \right)$$

$$(D^2 - 6DD' + 9D'^2)Z = 0$$

with $Z = \phi(y + mx)$

Replace $D = m$, $D' = 1$

We have auxillary equation:

$\Rightarrow m^2 - 6m + 9 = 0$

$\Rightarrow m^2 - 3m - 3m + 9 = 0$

$m(m-3) - 3(m-3)$

$(m-3)(m-3) = 0$

$m = 3, 3$

Then the general solution of the PDE's is :-

$$Z = \phi_1(y + 3x) + x\,\phi_2(y + 3x)$$

Question 5:-

$$(2D + D' + 1)^2 Z = 0$$

Solution:-

$$(2D + D' + 1)^2 Z = 0$$

The given PDE $f(D, D')$ has

repeated factors therefore using the formula -

$$Z = \exp\left(-\frac{\alpha x}{a}\right)\left(\phi(bx - ay) + \frac{x}{a}\phi(bx - ay)\right)$$

we obtain the general solution as -

$$Z = \exp\left(\frac{-1}{2}x\right)\left[\phi(x-2y) + \frac{x}{2}\phi(x-2y)\right]$$

Question 6:-

$$(D^2 + DD' + D + D' + 1)Z = 0$$

Solution:-

$$(D^2 + DD' + D + D' + 1)Z = 0$$

Here $f(D, D') = (D^2 + DD' + D + D' + 1)$ is not homogeneous in D & D'

Suppose that $Z = A\exp(\alpha x + \beta y)$ then substitution we obtain -

$$(\alpha^2 + \alpha\beta + \alpha + \beta + 1)A\exp(\alpha x + \beta y) = 0 \longrightarrow (A)$$

Since $\exp(\alpha x + \beta y) \neq 0$ & $A \neq 0$.

it follows that -

$$\Rightarrow \alpha^2 + \alpha\beta + \alpha + \beta + 1 = 0$$

$$\Rightarrow \alpha\beta + \beta = -\alpha^2 - \alpha - 1$$

$$\Rightarrow \beta(\alpha + 1) = -(\alpha^2 + \alpha + 1)$$

$$\beta = \frac{-(\alpha^2 + \alpha + 1)}{(\alpha + 1)}$$

for each choice of α we can find a value of B

Thus there ~~ex~~ are finite numbers of pairs (α_i, B_i)

Connected through:-

$$\beta_i = -\frac{(\alpha_i^2 + \alpha_i + 1)}{(\alpha_i + 1)}$$

therefore we can write general solution of the given PDE as

$$Z = \sum_i A_i \exp(\alpha_i x + \beta_i y)$$

$$B_i = -(\alpha^2{}_i + \alpha_i + 1)$$