8. DIFFERENTIAL EQUATIONS

8.1. Introduction :

One of the branches of Mathematics conveyed clearly in the principal language of science called "Differential equations", plays an important role in Science, Engineering and Social Sciences. Let us analyse a few of the examples cited below.

(1) Suppose that there are two living species which depend for their survival on a common source of food supply. This fact results in a competition in consuming the available food. The phenomenon, is commonly noticed in the plant life having common supply of water, fertilizer and minerals. However, whenever the competition between two species begins, the growth rate of one is retarded and we can note that the rate of retardation is naturally proportional to the size of the other species present at time t. This situation can be expressed as a Mathematical model whose solution would help us to determine the time at which one species would become extinct.

(2) Several diseases are caused by spread of an infection. Suppose that the susceptible population of a town is p. One person gets the infection. Because of contact another susceptible person is also infected. This process continues to cover the entire susceptible population. With some assumptions to simplify the mathematical considerations this situation can be framed into a mathematical model and a solution can be determined which would provide informations regarding the spread of the epidemic in the town.

(3) If a dead body is brought for a medical examination at a particular time, the exact time of death can be determined by noting the temperature of the body at various time intervals, formulating it into a mathematical problem with available initial conditions and then solving it.

(4) The determination of the amount of a radioactive material that disintegrates over a period of time is yet another mathematical formulation which yield the required result.

(5) Several examples exist in which two nations have disputes on various issues. Each nation builds its own arms to defend the nation from attack. Naturally a spirit of race in building up arms persists between conflicting nations. A small grievance quite often creates a war-like

situation and adds to increasing the level of arms. These commonly experienced facts can be presented in a mathematical language and hence solved. It is a fact that such a model has been tested for some realistic situations that had prevailed in the First and Second World War between conflicting nations.

From the above examples it is found that the mathematical formulation to all situations turn out to be differential equations. Thus the latent significance of differential equations in studying physical phenomena becomes apparent. This branch of Mathematics called 'Differential Equations' is like a bridge linking Mathematics and Science with its applications. Hence it is rightly considered as the language of Sciences.

Galileo once conjectured that the velocity of a body falling from rest is proportional to the distance fallen. Later he decided that it is proportional to the time instead. Each of these statements can be formulated as an equation involving the rate of change of an unknown function and is therefore an example of what Mathematicians call a Differential Equation. Thus $\frac{ds}{dt} = kt$ is a differential equation which gives velocity of a falling body from a distance s proportional to the time t.

Definition: An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a Differential Equation.

If $y = f(x)$ is a given function, then its derivative $\frac{dy}{dx}$ can be interpreted as the rate of change of y with respect to x. In any natural process the variables involved and their rates of change are connected with one another by means of the basic scientific principles that govern the process. When this expression is written in mathematical symbols, the result is often a differential equation.

Thus a differential equation is an equation in which differential coefficients occur. Its importance can further be realised from the fact that every natural phenomena is governed by differential equations.

Differential equation are of two types.

(i) Ordinary and (ii) Partial.

In this chapter we concentrate only on Ordinary differential equations.

Definition : An ordinary differential equation is a differential equation in which a single independent variable enters either explicitly or implicitly.

For instance (i) $\frac{dy}{dx} = x + 5$ (ii) $(y')^2 + (y')^3 + 3y = x^2$ (iii) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

are all ordinary differential equations.

8.2 Order and degree of a differential equation :

Definition : The **order** of a differential equation is the order of the highest order derivative occurring in it. The **degree** of the differential equation is the degree of the highest order derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned.

The degree of a differential equation does not require variables r, s, t … to be free from radicals and fractions.

Example 8.1: Find the order and degree of the following differential equations:

(i) $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + y = 7$ (ii) $y = 4\frac{dy}{dx} + 3x\frac{dx}{dy}$

(iii) $\frac{d^2y}{dx^2} = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}$ (iv) $(1 + y')^2 = y'^2$

Solution : (i) The order of the highest derivative in this equation is 3. The degree of the highest order is 1. $\therefore$ (order, degree) = (3, 1)

(ii) $$y = 4\frac{dy}{dx} + 3x\frac{dx}{dy} \Rightarrow y = 4\left(\frac{dy}{dx}\right) + 3x\frac{1}{\left(\frac{dy}{dx}\right)}$$

Making the above equation free from fractions involving $\frac{dy}{dx}$ we get

$$y \cdot \frac{dy}{dx} = 4\left(\frac{dy}{dx}\right)^2 + 3x$$

Highest order = 1

Degree of Highest order = 2

(order, degree) = (1, 2)

(iii) $\frac{d^2y}{dx^2} = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}$

To eliminate the radical in the above equation, raising to the power 4 on both sides, we get $\left(\frac{d^2y}{dx^2}\right)^4 = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^3$. Clearly (order, degree) = (2, 4).

(iv) $(1 + y')^2 = y'^2 \Rightarrow 1 + y'^2 + 2y' = y'^2$ from which it follows that

$$2\frac{dy}{dx} + 1 = 0 \quad \therefore \text{(order, degree)} = (1, 1).$$

8.3 Formation of differential equations :

Let $f(x, y, c_1) = 0$ be an equation containing x, y and one arbitrary constant c_1. If c_1 is eliminated by differentiating $f(x, y, c_1) = 0$ with respect to the independent variable once, we get a relation involving x, y and $\frac{dy}{dx}$, which is evidently a differential equation of the first order. Similarly, if we have an equation $f(x, y, c_1, c_2) = 0$ containing two arbitrary constants c_1 and c_2, then by differentiating this twice, we get three equations (including f). If the two arbitrary constants c_1 and c_2 are eliminated from these equations, we get a differential equation of second order.

In general if we have an equation $f(x, y, c_1, c_2, \ldots c_n) = 0$ containing n arbitrary constants $c_1, c_2 \ldots c_n$, then by differentiating n times we get $(n + 1)$ equations in total. If the n arbitrary constants $c_1, c_2, \ldots c_n$ are eliminated we get a differential equation of order n.

Note : If there are relations involving these arbitrary constants then the order of the differential equation may reduce to less than n.

Illustration :

Let us find the differential equation of straight lines $y = mx + c$ where both m and c are arbitrary constants.

Since m and c are two arbitrary constants differentiating twice we get

$$\frac{dy}{dx} = m$$

$$\frac{d^2y}{dx^2} = 0$$

Both the constants m and c are seen to be eliminated. Therefore the required differential equation is $\frac{d^2y}{dx^2} = 0$

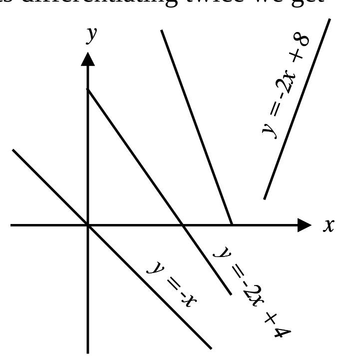

Fig. 8.1

Note : In the above illustration we have taken both the constants m and c as arbitrary. Now the following two cases may arise.

Case (i) : m is arbitrary and c is fixed. Since m is the only arbitrary constant in $y = mx + c$; … (1)

Differentiating once we get

$$\frac{dy}{dx} = m \qquad \text{… (2)}$$

Eliminating m between (1) and (2) we get the required differential equation

$$x\left(\frac{dy}{dx}\right) - y + c = 0$$

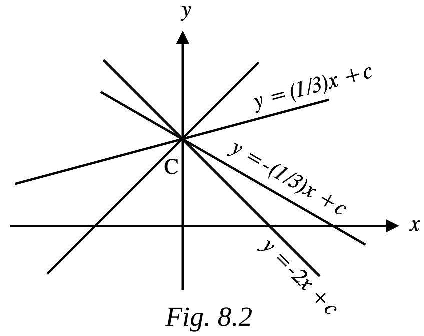

Fig. 8.2

Case (ii) : c is an arbitrary constant and m is a fixed constant.

Since c is the only arbitrary constant differentiating once we get $\frac{dy}{dx} = m$. Clearly c is eliminated from the above equation. Therefore the required differential equation is $\frac{dy}{dx} = m$.

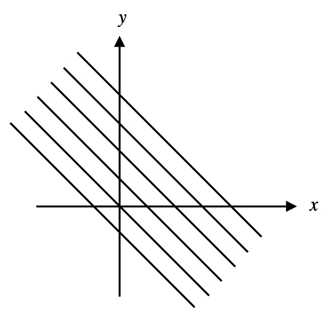

Fig. 8.3

Example 8.2: Form the differential equation from the following equations.

(i) $y = e^{2x}(A + Bx)$ (ii) $y = e^{x}(A \cos 3x + B \sin 3x)$

(iii) $Ax^2 + By^2 = 1$ (iv) $y^2 = 4a(x - a)$

Solution :

(i) $y = e^{2x}(A + Bx)$

$$ye^{-2x} = A + Bx \qquad \text{… (1)}$$

Since the above equation contains two arbitrary constants, differentiating twice, we get $y'e^{-2x} - 2y\, e^{-2x} = B$

$$\{y''e^{-2x} - 2y'\, e^{-2x}\} - 2\{y'e^{-2x} - 2y\, e^{-2x}\} = 0$$

$$e^{-2x}\{y'' - 4y' + 4y\} = 0 \qquad [\because e^{-2x} \neq 0]$$

$y'' - 4y' + 4y = 0$ is the required differential equation.

(ii) $\quad y = e^{x} (A \cos 3x + B \sin 3x)$

$ye^{-x} = A \cos 3x + B \sin 3x$

We have to differentiate twice to eliminate two arbitrary constants

$$y'e^{-x} - ye^{-x} = -3A \sin 3x + 3B \cos 3x$$

$$y'' e^{-x} - y'e^{-x} - y'e^{-x} + ye^{-x} = -9 (A \cos 3x + B \sin 3x)$$

$$\text{i.e., } e^{-x} (y'' - 2y' + y) = -9ye^{-x}$$

$$\Rightarrow y'' - 2y' + 10y = 0 \quad (\because e^{-x} \neq 0)$$

(iii) $Ax^2 + By^2 = 1$ … (1)

Differentiating, $2Ax + 2Byy' = 0$ i.e., $Ax + Byy' = 0$ … (2)

Differentiating again, $A + B (yy'' + y'^2) = 0$ … (3)

Eliminating A and B between (1), (2) and (3) we get

$$\begin{vmatrix} x^2 & y^2 & -1 \\ x & yy' & 0 \\ 1 & yy'' + y'^2 & 0 \end{vmatrix} = 0 \Rightarrow (yy'' + y'^2) x - yy' = 0$$

(iv) $y^2 = 4a(x - a)$ … (1)

Differentiating, $2yy' = 4a$ … (2)

Eliminating a between (1) and (2) we get

$$y^2 = 2yy' \left(x - \frac{yy'}{2}\right)$$

$$\Rightarrow (yy')^2 - 2xyy' + y^2 = 0$$

EXERCISE 8.1

(1) Find the order and degree of the following differential equations.

(i) $\dfrac{dy}{dx} + y = x^2$ (ii) $y' + y^2 = x$

(iii) $y'' + 3y'^2 + y^3 = 0$ (iv) $\dfrac{d^2y}{dx^2} + x = \sqrt{y + \dfrac{dy}{dx}}$

(v) $\dfrac{d^2y}{dx^2} - y + \left(\dfrac{dy}{dx} + \dfrac{d^3y}{dx^3}\right)^{\frac{3}{2}} = 0$ (vi) $y'' = (y - y'^3)^{\frac{2}{3}}$

(vii) $y' + (y'')^2 = (x + y'')^2$ (viii) $y' + (y'')^2 = x(x + y'')^2$

(ix) $\left(\dfrac{dy}{dx}\right)^2 + x = \dfrac{dx}{dy} + x^2$ (x) $\sin x\, (dx + dy) = \cos x\, (dx - dy)$

(2) Form the differential equations by eliminating arbitrary constants given in brackets against each

(i)	$y^2 = 4ax$	$\{a\}$
(ii)	$y = ax^2 + bx + c$	$\{a, b\}$
(iii)	$xy = c^2$	$\{c\}$
(iv)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\{a, b\}$
(v)	$y = Ae^{2x} + Be^{-5x}$	$\{A, B\}$
(vi)	$y = (A + Bx)e^{3x}$	$\{A, B\}$
(vii)	$y = e^{3x}\{C\cos 2x + D\sin 2x)$	$\{C, D\}$
(viii)	$y = e^{mx}$	$\{m\}$
(ix)	$y = Ae^{2x}\cos(3x + B)$	$\{A, B\}$

(3) Find the differential equation of the family of straight lines $y = mx + \frac{a}{m}$ when (i) m is the parameter ; (ii) a is the parameter ; (iii) a, m both are parameters

(4) Find the differential equation that will represent the family of all circles having centres on the x-axis and the radius is unity.

8.4 Differential equations of first order and first degree :

In this section we consider a class of differential equations, the order and degree of each member of the class is equal to one. For example,

(i) $yy' + x = 0$ (ii) $y' + xy = \sin x$ (iii) $y' = \frac{x+y}{x-y}$ (iv) $x\,dy + y\,dx = 0$

Solutions of first order and first degree equations:

We shall consider only certain special types of equations of the first order and first degree. viz., (i) Variable separable (ii) Homogeneous (iii) Linear.

8.4.1 Variable separable :

Variables of a differential equation are to be rearranged in the form

$f_1(x)\,g_2(y)\,dx + f_2(x)\,g_1(y)\,dy = 0$

i.e., the equation can be written as

$$f_2(x)g_1(y)dy = -f_1(x)\,g_2(y)\,dx$$

$$\Rightarrow \frac{g_1(y)}{g_2(y)}\,dy = -\frac{f_1(x)}{f_2(x)}\,dx$$

The solution is therefore given by $\int \frac{g_1(y)}{g_2(y)}\,dy = -\int \frac{f_1(x)}{f_2(x)}\,dx + c$

Example 8.3: Solve : $\frac{dy}{dx} = 1 + x + y + xy$

Solution : The given equation can be written in the form

$$\frac{dy}{dx} = (1 + x) + y(1 + x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x)(1 + y)$$

$$\Rightarrow \quad \frac{dy}{1 + y} = (1 + x)dx$$

Integrating, we have

$\log(1 + y) = x + \frac{x^2}{2} + c$, which is the required solution.

Example 8.4: Solve $3e^x \tan y\, dx + (1 + e^x) \sec^2 y\, dy = 0$

Solution : The given equation can be written in the form

$$\frac{3e^x}{1 + e^x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

Integrating, we have

$$3 \log(1 + e^x) + \log \tan y = \log c$$

$$\Rightarrow \quad \log[\tan y\,(1 + e^x)^3] = \log c$$

$\Rightarrow \quad (1 + e^x)^3 \tan y = c$, which is the required solution.

Note : The arbitrary constant may be chosen like c, $\frac{1}{c}$, $\log c$, e^c etc depending upon the problem.

Example 8.5: Solve $\frac{dy}{dx} + \left(\frac{1 - y^2}{1 - x^2}\right)^{\frac{1}{2}} = 0$

Solution : The given equation can be written as

$$\frac{dy}{dx} = -\left(\frac{1 - y^2}{1 - x^2}\right)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{dy}{\sqrt{1 - y^2}} = -\frac{dx}{\sqrt{1 - x^2}}$$

Integrating, we have $\sin^{-1} y + \sin^{-1} x = c$

$\Rightarrow \sin^{-1}\left[x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right] = c$

$\Rightarrow x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = C$ is the required solution.

Example 8.6: Solve : $e^x \sqrt{1 - y^2}\, dx + \frac{y}{x}\, dy = 0$

Solution : The given equation can be written as

$$xe^x dx = \frac{-y}{\sqrt{1 - y^2}}\, dy$$

Integrating, we have

$$\int xe^x\, dx = -\int \frac{y}{\sqrt{1 - y^2}}\, dy$$

$$\Rightarrow\ xe^x - \int e^x\, dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}} \text{ where } t = 1 - y^2 \text{ so that } -2y\, dy = dt$$

$$\Rightarrow\ xe^x - e^x = \frac{1}{2}\left(\frac{t^{\frac{1}{2}}}{1/2}\right) + c$$

$$\Rightarrow xe^x - e^x = \sqrt{t} + c$$

$$\Rightarrow xe^x - e^x - \sqrt{1 - y^2} = c \quad \text{which is the required solution.}$$

Example 8.7: Solve : $(x + y)^2 \frac{dy}{dx} = a^2$

Solution : Put $x + y = z$. Differentiating with respect to x we get

$$1 + \frac{dy}{dx} = \frac{dz}{dx} \quad \text{i.e.,} \quad \frac{dy}{dx} = \frac{dz}{dx} - 1$$

The given equation becomes $z^2\left(\frac{dz}{dx} - 1\right) = a^2$

$$\Rightarrow\ \frac{dz}{dx} - 1 = \frac{a^2}{z^2} \quad \text{or} \quad \frac{z^2}{z^2 + a^2}\, dz = dx$$

Integrating we have, $\int \frac{z^2}{z^2 + a^2}\, dz = \int dx$

$$\int \frac{z^2 + a^2 - a^2}{z^2 + a^2}\, dz = x + c \Rightarrow \int\left(1 - \frac{a^2}{z^2 + a^2}\right) dz = x + c$$

$$\Rightarrow\ z - a^2 . \frac{1}{a}\tan^{-1}\frac{z}{a} = x + c$$

$$\Rightarrow\ x + y - a\tan^{-1}\left(\frac{x + y}{a}\right) = x + c \qquad (\because z = x + y)$$

$$\text{i.e.,}\ \ y - a\tan^{-1}\left(\frac{x + y}{a}\right) = c, \ \text{which is the required solution.}$$

Example 8.8: Solve : $x\,dy = (y + 4x^5\, e^{x^4})dx$

Solution :

$$xdy - y\,dx = 4x^5\, e^{x^4} dx$$

$$\frac{xdy - ydx}{x^2} = 4x^3\, e^{x^4} dx$$

Integrating we have, $\displaystyle\int \frac{xdy - ydx}{x^2} = \int 4x^3\, e^{x^4} dx$

$$\Rightarrow \int d\left(\frac{y}{x}\right) = \int e^t\, dt \qquad \text{where } t = x^4$$

$$\Rightarrow \frac{y}{x} = e^t + c$$

i.e., $\dfrac{y}{x} = e^{x^4} + c$ which is the required solution.

Example 8.9: Solve: $(x^2 – y)dx + (y^2 – x)\, dy = 0$, if it passes through the origin.

Solution :

$$(x^2 - y)dx + (y^2 - x)\, dy = 0$$

$$x^2dx + y^2dy = xdy + ydx$$

$$x^2dx + y^2\, dy = d(xy)$$

Integrating we have, $\dfrac{x^3}{3} + \dfrac{y^3}{3} = xy + c$

Since it passes through the origin, $c = 0$

$\therefore$ the required solution is $\dfrac{x^3}{3} + \dfrac{y^3}{3} = xy$ or $x^3 + y^3 = 3xy$

Example 8.10 : Find the cubic polynomial in x which attains its maximum value 4 and minimum value 0 at $x = – 1$ and 1 respectively.

Solution : Let the cubic polynomial be $y = f(x)$. Since it attains a maximum at $x = –1$ and a minimum at $x = 1$.

$$\frac{dy}{dx} = 0 \text{ at } x = -1 \text{ and } 1$$

$$\frac{dy}{dx} = k\,(x + 1)\,(x - 1) = k(x^2 - 1)$$

Separating the variables we have $dy = k(x^2 – 1)\, dx$

$$\int dy = k\int (x^2 - 1)\, dx$$

$$y = k\left(\frac{x^3}{3} - x\right) + c \qquad \ldots (1)$$

when $x = -1$, $y = 4$ and when $x = 1$, $y = 0$

Substituting these in equation (1) we have

$$2k + 3c = 12 \;\; ; \;\; -2k + 3c = 0$$

On solving we have $k = 3$ and $c = 2$. Substituting these values in (1) we get the required cubic polynomial $y = x^3 - 3x + 2$.

Example 8.11 : The normal lines to a given curve at each point (x, y) on the curve pass through the point (2, 0). The curve passes through the point (2, 3). Formulate the differential equation representing the problem and hence find the equation of the curve.

Solution :

Slope of the normal at any point $P(x, y) = -\frac{dx}{dy}$

Slope of the normal $AP = \frac{y - 0}{x - 2}$ $\quad \therefore -\frac{dx}{dy} = \frac{y}{x - 2} \Rightarrow ydy = (2 - x)dx$

Integrating both sides, $\frac{y^2}{2} = 2x - \frac{x^2}{2} + c \qquad \ldots (1)$

Since the curve passes through (2, 3)

$\frac{9}{2} = 4 - \frac{4}{2} + c \Rightarrow c = \frac{5}{2}$; put $c = \frac{5}{2}$ in (1),

$\frac{y^2}{2} = 2x - \frac{x^2}{2} + \frac{5}{2} \Rightarrow y^2 = 4x - x^2 + 5$

EXERCISE 8.2

Solve the following :

(1) $\sec 2x\, dy - \sin 5x \sec^2 y dx = 0$

(2) $\cos^2 x dy + ye^{\tan x} dx = 0$

(3) $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$

(4) $yx^2 dx + e^{-x} dy = 0$

(5) $(x^2 + 5x + 7)\, dy + \sqrt{9 + 8y - y^2}\, dx = 0$

(6) $\frac{dy}{dx} = \sin(x + y)$

(7) $(x + y)^2 \frac{dy}{dx} = 1$

(8) $ydx + xdy = e^{-xy}\, dx$ if it cuts the y-axis.

8.4.2 Homogeneous equations :

Definition :

A differential equation of first order and first degree is said to be homogeneous if it can be put in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ or $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$

Working rule for solving homogeneous equation :

By definition the given equation can be put in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \dots (1)$$

To solve (1) put $y = vx$ … (2)

Differentiating (2) with respect to x gives

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \dots (3)$$

Using (2) and (3) in (1) we have

$$v + x\frac{dv}{dx} = f(v) \quad \text{or} \quad x\frac{dv}{dx} = f(v) - v$$

Seperating the variables x and v we have

$$\frac{dx}{x} = \frac{dv}{f(v) - v} \Rightarrow \log x + c = \int \frac{dv}{f(v) - v}$$

where c is an arbitrary constant. After integration, replace v by $\frac{y}{x}$.

Example 8.12: Solve : $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Solution : Put $y = vx$

$$\text{L.H.S.} = v + x\frac{dv}{dx} \quad ; \quad \text{R.H.S.} = v + \tan v$$

$$\therefore \quad v + x\frac{dv}{dx} = v + \tan v \quad \text{or} \quad \frac{dx}{x} = \frac{\cos v}{\sin v}\, dv$$

Integrating, we have $\log x = \log \sin v + \log c \Rightarrow x = c \sin v$

$$\text{i.e., } x = c\sin\left(\frac{y}{x}\right),$$

Example 8.13: Solve : $\left(2\sqrt{xy} - x\right) dy + ydx = 0$

Solution : The given equation is $\frac{dy}{dx} = \frac{-y}{2\sqrt{xy} - x}$

Put $y = vx$

$$\text{L.H.S.} = v + x\frac{dv}{dx}\ ;\ \text{R.H.S.} = \frac{-v}{2\sqrt{v}-1} = \frac{v}{1-2\sqrt{v}}$$

$$\therefore\ v + x\frac{dv}{dx} = \frac{v}{1-2\sqrt{v}}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\sqrt{v}}{1-2\sqrt{v}} \Rightarrow \left(\frac{1-2\sqrt{v}}{v\sqrt{v}}\right)dv = 2\frac{dx}{x}$$

$$\text{i.e., } \left(v^{-3/2} - 2.\frac{1}{v}\right)dv = 2\frac{dx}{x}$$

$$\Rightarrow -2v^{-1/2} - 2\log v = 2\log x + 2\log c$$

$$-v^{-1/2} = \log(v\,x\,c)$$

$$-\sqrt{\frac{x}{y}} = \log(cy) \Rightarrow \quad cy = e^{-\sqrt{x/y}} \text{ or } ye^{\sqrt{x/y}} = c$$

Note : This problem can also be done easily by taking $x = vy$

Example 8.14: Solve : $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$

Solution : $$\frac{dy}{dx} = -\frac{x^3+3xy^2}{y^3+3x^2y}$$

Put $y = \nu x$

$$\text{L.H.S.} = \nu + x\frac{d\nu}{dx}\ ;\ \text{R.H.S.} = -\frac{x^3+3xy^2}{y^3+3x^2y} = -\left(\frac{1+3\nu^2}{\nu^3+3\nu}\right)$$

$$\therefore\ \nu + x\frac{d\nu}{dx} = -\left(\frac{1+3\nu^2}{\nu^3+3\nu}\right)$$

$$\Rightarrow x\frac{d\nu}{dx} = -\frac{\nu^4+6\nu^2+1}{\nu^3+3\nu}$$

$$\Rightarrow \frac{4dx}{x} = -\frac{4\nu^3+12\nu}{\nu^4+6\nu^2+1}d\nu$$

Integrating, we have

$$4\log x = -\log(\nu^4+6\nu^2+1) + \log c$$

$$\log[x^4(\nu^4+6\nu^2+1)] = \log c$$

$$\text{i.e., } x^4(\nu^4+6\nu^2+1) = c \quad \text{or}$$

$$y^4 + 6x^2y^2 + x^4 = c$$

Note (i) : This problem can also be done by using variable separable method.

Note (ii) : Sometimes it becomes easier in solving problems of the type $\frac{dx}{dy} = \frac{f_1(x/y)}{f_2(x/y)}$. The following example explains this case.

Example 8.15:

Solve : $(1 + e^{x/y})dx + e^{x/y}(1 - x/y)\, dy = 0$ given that $y = 1$, where $x = 0$

Solution : The given equation can be written as

$$\frac{dx}{dy} = \frac{(x/y - 1)e^{x/y}}{1 + e^{x/y}} \qquad \dots (1)$$

Put $x = vy$

$$\text{L.H.S.} = v + y\frac{dv}{dy} \;;\; \text{R.H.S.} = \frac{(v-1)e^{v}}{1+e^{v}}$$

$$\therefore\; v + y\frac{dv}{dy} = \frac{(v-1)e^{v}}{1+e^{v}}$$

$$\text{or } y\frac{dv}{dy} = -\frac{(e^{v} + v)}{1+e^{v}}$$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{(e^{v}+1)}{e^{v}+v}\,dv$$

Integrating we have, $\log y = -\log(e^{v} + v) + \log c$

or $y(e^{v} + v) = c \;\Rightarrow ye^{x/y} + x = c$

Now $y = 1$ when $x = 0 \Rightarrow 1e^{0} + 0 = c \Rightarrow c = 1$

$\therefore\; ye^{x/y} + x = 1$

Example 8.16: Solve : $xdy - ydx = \sqrt{x^2 + y^2}\, dx$

Solution : From the given equation we have

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} \qquad \dots (1)$$

Put $y = vx$

$$\text{L.H.S.} = v + x\frac{dv}{dx} \;;\; \text{R.H.S.} = \frac{v + \sqrt{1+v^2}}{1}$$

$$\therefore\; v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \quad \text{or} \quad \frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}}$$

Integrating, we have, $\log x + \log c = \log \left[v + \sqrt{v^2 + 1}\right]$

$$\text{i.e., } xc = v + \sqrt{v^2 + 1} \Rightarrow x^2c = y + \sqrt{(y^2 + x^2)}$$

EXERCISE 8.3

Solve the following :

(1) $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$ (2) $\frac{dy}{dx} = \frac{y(x - 2y)}{x(x - 3y)}$ (3) $(x^2 + y^2)\, dy = xy\, dx$

(4) $x^2 \frac{dy}{dx} = y^2 + 2xy$ given that $y = 1$, when $x = 1$.

(5) $(x^2 + y^2)\, dx + 3xy\, dy = 0$

(6) Find the equation of the curve passing through (1, 0) and which has slope $1 + \frac{y}{x}$ at (x, y).

8.4.3 Linear Differential Equation :

Definition :

A first order differential equation is said to be linear in y if the power of the terms $\frac{dy}{dx}$ and y are unity.

For **example** $\frac{dy}{dx} + xy = e^x$ is linear in y, since the power of $\frac{dy}{dx}$ is one and also the power of y is one. If a term occurs in the form $y\frac{dy}{dx}$ or y^2, then it is not linear, as the degree of each term is two.

A differential equation of order one satisfying the above condition can always be put in the form $\frac{dy}{dx} + Py = Q$, where P and Q are function of x only. Similarly a first order linear differential equation in x will be of the form $\frac{dx}{dy} + Px = Q$ where P and Q are functions of y only.

The solution of the equation which is linear in y is given as

$y e^{\int Pdx} = \int Q e^{\int Pdx}\, dx + c$ where $e^{\int Pdx}$ is known as an integrating factor and it is denoted by I.F.

Similarly if an equation is linear in x then the solution of such an equation becomes

$$x\, e^{\int P dy} = \int Q\, e^{\int P dy}\, dy + c \qquad \left(\text{where } e^{\int P dy} \text{ is I.F.}\right)$$

We frequently use the following properties of logarithmic and exponential functions :

(i) $e^{\log A} = A$ (ii) $e^{m \log A} = A^m$ (iii) $e^{-m \log A} = \dfrac{1}{A^m}$

Example 8.17 : Solve : $\dfrac{dy}{dx} + y \cot x = 2 \cos x$

Solution : The given equation is of the form $\dfrac{dy}{dx} + Py = Q$. This is linear in y.

Here $P = \cot x$ and $Q = 2 \cos x$

I.F. $= e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

$\therefore$ The required solution is

$$y \text{ (I.F.)} = \int (Q \text{ (I.F.)})\, dx + c \Rightarrow y(\sin x) = \int 2 \cos x \sin x\, dx + c$$

$$\Rightarrow y \sin x = \int \sin 2x\, dx + c$$

$$\Rightarrow y \sin x = -\frac{\cos 2x}{2} + c$$

$$\Rightarrow 2y \sin x + \cos 2x = c$$

Example 8.18 : Solve : $(1 - x^2)\dfrac{dy}{dx} + 2xy = x\sqrt{(1 - x^2)}$

Solution: The given equation is $\dfrac{dy}{dx} + \left(\dfrac{2x}{1 - x^2}\right) y = \dfrac{x}{\sqrt{(1 - x^2)}}$. This is linear in y

$$\text{Here } \int P dx = \int \frac{2x}{1 - x^2}\, dx = -\log(1 - x^2)$$

$$\text{I.F.} = e^{\int P dx} = \frac{1}{1 - x^2}$$

The required solution is

$$y \cdot \frac{1}{1 - x^2} = \int \frac{x}{\sqrt{(1 - x^2)}} \times \frac{1}{1 - x^2}\, dx. \quad \text{Put } 1 - x^2 = t \Rightarrow -2x dx = dt$$

$$\therefore \frac{y}{1 - x^2} = \frac{-1}{2} \int t^{-3/2}\, dt + c$$

$$\Rightarrow \quad \frac{y}{1-x^2} = t^{-1/2} + c$$

$$\Rightarrow \quad \frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c$$

Example 8.19 : Solve : $(1 + y^2)dx = (\tan^{-1}y - x)dy$

Solution : The given equation can be written as $\dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1}y}{1+y^2}$.

This is linear in x. Therefore we have

$$\int P dy = \int \frac{1}{1+y^2}\, dy = \tan^{-1}y$$

$$\text{I.F.} = e^{\int P dy} = e^{\tan^{-1}y}$$

The required solution is

$$x e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \left(\frac{\tan^{-1}y}{1+y^2}\right) dy + c \qquad \left\{\begin{array}{l}\text{put } \tan^{-1}y = t \\ \therefore \dfrac{dy}{1+y^2} = dt\end{array}\right.$$

$$\Rightarrow \quad x e^{\tan^{-1}y} = \int e^t \,.\, t \;\, dt + c$$

$$\Rightarrow \quad x e^{\tan^{-1}y} = te^t - e^t + c$$

$$\Rightarrow \quad x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

Example 8.20 : Solve : $(x + 1)\dfrac{dy}{dx} - y = e^x(x + 1)^2$

Solution : The given equation can be written as $\dfrac{dy}{dx} - \dfrac{y}{x+1} = e^x(x+1)$

This is linear in y. Here $\int P dx = -\int \dfrac{1}{x+1}\, dx = -\log(x+1)$

$$\text{So} \quad \text{I.F.} = e^{\int P dx} = e^{-\log(x+1)} = \frac{1}{x+1}$$

$\therefore$ The required solution is

$$y \cdot \frac{1}{x+1} = \int e^x (x+1) \frac{1}{x+1}\; dx + c$$

$$= \int e^x\, dx + c$$

$$\text{i.e.,} \quad \frac{y}{x+1} = e^x + c$$

Example 8.21 : Solve : $\dfrac{dy}{dx} + 2y\ \tan x = \sin x$

Solution : This is linear in y. Here $\int P dx = \int 2 \tan x\, dx = 2 \log \sec x$

$$\text{I.F.} = e^{\int P dx} = e^{\log \sec^2 x} = \sec^2 x$$

The required solution is

$$y \sec^2 x = \int \sec^2 x \cdot \sin x\, dx$$
$$= \int \tan x\ \sec x\ dx$$
$$\Rightarrow\ y \sec^2 x = \sec x + c \quad \text{or} \quad y = \cos x + c \cos^2 x$$

EXERCISE 8.4

Solve the following :

(1) $\dfrac{dy}{dx} + y = \text{x}$

(2) $\dfrac{dy}{dx} + \dfrac{4x}{x^2+1}\ y = \dfrac{1}{(x^2+1)^2}$

(3) $\dfrac{dx}{dy} + \dfrac{x}{1+y^2} = \dfrac{\tan^{-1} y}{1+y^2}$

(4) $(1+x^2)\dfrac{dy}{dx} + 2xy = \cos x$

(5) $\dfrac{dy}{dx} + \dfrac{y}{x} = \sin(x^2)$

(6) $\dfrac{dy}{dx} + xy = x$

(7) $dx + xdy = e^{-y} \sec^2 y\, dy$

(8) $(y-x)\dfrac{dy}{dx} = a^2$

(9) Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is $y = 2(e^x - x - 1)$

8.5 Second order linear differential equations with constant coefficients :

A general second order non-homogeneous linear differential equation with constant coefficients is of the form

$a_0 y'' + a_1 y' + a_2 y = X$ … (1),

where a_0, a_1, a_2 are constants $a_0 \neq 0$, and X is a function of x. The equation

$a_0 y'' + a_1 y' + a_2 y = 0,\ a_0 \neq 0$ … (2)

is known as a homogeneous linear second order differential equation with constant coefficients,

To solve (1), first we solve (2). To do this we proceed as follows :

Consider the function $y = e^{px}$, p is a constant.

$$\text{Now } y' = pe^{px} \text{ and } y'' = p^2 e^{px}$$

Note that the derivatives look similar to the function $y = e^{px}$ itself and if $L(y) = a_0y'' + a_1y' + a_2y$ then

$$\begin{aligned} L(y) &= L(e^{px}) \\ &= (a_0p^2e^{px} + a_1pe^{px} + a_2\, e^{px}) \\ &= (a_0p^2 + a_1p + a_2)e^{px} \end{aligned}$$

Hence if $L(y) = 0$ then it follows that $(a_0p^2 + a_1p + a_2)e^{px} = 0$.

Since $e^{px} \neq 0$ we get that $a_0p^2 + a_1p + a_2 = 0$ … (3)

Note that e^{px} satisfies the equation $L(y) = a_0y'' + a_1y' + a_2y = 0$ then p must satisfy $a_0p^2 + a_1p + a_2 = 0$. Moreover if the various derivatives of a function look similar in form to the function itself then e^{px} will be an ideal candidate to solve $a_0y'' + a_1y' + a_2y = 0$. Hereafter we will consider only those set of differential equations which admits e^{px} as one of the solutions. Hence we have the following :

Theorem : If λ is a root of $a_0p^2 + a_1p + a_2 = 0$, then $e^{\lambda x}$ is a solution of $a_0y'' + a_1y' + a_2y = 0$

8.5.1 Definition : The equation $a_0p^2 + a_1p + a_2 = 0$ is called the characteristic equation of (2).

In general the characteristic equation has two roots say λ_1 and λ_2. Then the following three cases do arise.

Case (i) : λ_1 and λ_2 are real and distinct.

In this case, by the above theorem $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are solutions of (2), and the linear combination $y = c_1\, e^{\lambda_1 x} + c_2e^{\lambda_2 x}$ is also a solution of (2).

For $L(y) = a_0(c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x})'' + a_1(c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x})' + a_2(c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x})$

$= c_1(a_0\lambda_1^2 + a_1\lambda_1 + a_2)e^{\lambda_1 x} + c_2(a_0\lambda_2^2 + a_1\,\lambda_2 + a_2)e^{\lambda_2 x} = c_1\,.\,0 + c_2\,.\,0 = 0.$

and the solution $c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$ is known as the complementary function.

Case (ii) : λ_1 and λ_2 are complex $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$

In this case as the two roots λ_1 and λ_2 are complex from theory of equations $e^{\lambda_1 x} = e^{(a + ib)x} = e^{ax}\,.\,e^{ibx} = e^{ax}(\cos bx + i\sin bx)$ and

$e^{\lambda_2 x} = e^{ax}(\cos bx - i \sin bx)$

Hence the solution

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = e^{ax}\left[(c_1 + c_2)\cos bx + i(c_1 - c_2)\sin bx\right]$$

$$= e^{ax}[A \cos bx + B \sin bx] \text{ where } A = c_1 + c_2 \text{ and } B = (c_1 - c_2)i$$

and the complementary function is $e^{ax}[A \cos bx + B \sin bx]$.

Case (iii) : The roots are real and equal $\lambda_1 = \lambda_2$ (say)

Clearly $e^{\lambda_1 x}$ is one of the solutions of (2). By using the double root property, we will obtain $xe^{\lambda_1 x}$ as the other solution of (2). Now the linear combination $c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$ becomes the solution. i.e., $y = (c_1 + c_2 x)e^{\lambda_1 x}$ is the solution or C.F.

The above discussion is summarised as follows :

Given $a_0 y'' + a_1 y' + a_2 y = 0$

Determine its characteristic equation

$a_0 p^2 + a_1 p + a_2 = 0$ … (3).

Let λ_1, λ_2 be the two roots of (3), then the solution of (2) is

$$y = \begin{cases} Ae^{\lambda_1 x} + Be^{\lambda_2 x} & \text{if } \lambda_1 \text{ and } \lambda_2 \text{ are real and distinct} \\ e^{ax}(A \cos bx + B \sin bx) & \text{if } \lambda_1 = a + ib \text{ and } \lambda_2 = a - ib \\ (A + Bx)e^{\lambda_1 x} & \text{if } \lambda_1 = \lambda_2 \text{ (real)} \end{cases}$$

A and B are arbitrary constants.

General solution :

The general solution of a linear equation of second order with constant co-efficient consists of two parts namely the complementary function and the particular integral.

Working rule :

To obtain the complementary function (C.F.) we solve the equation

$a_0 \dfrac{d^2y}{dx^2} + a_1 \dfrac{dy}{dx} + a_2 y = 0$ and obtain a solution $y = u$ (say). Then the general solution is given by $y = u + \nu$ where ν is called the particular integral of (1).

The function u, the complementary function is associated with the homogeneous equation and v, the particular integral is associated with the term X. If $X = 0$ then the $C.F.$ becomes the general solution of the equation.

Note : In this section we use the differential operators

$$D \equiv \frac{d}{dx} \text{ and } D^2 \equiv \frac{d^2}{dx^2} \quad ; \quad Dy = \frac{dy}{dx} \quad ; \quad D^2y = \frac{d^2y}{dx^2}$$

8.5.2 Method for finding Particular Integral :

(a) Suppose *X* is of the form $e^{\alpha x}$, α *a* constant

$$D(e^{\alpha x}) = \alpha e^{\alpha x} \quad ; \quad D^2(e^{\alpha x}) = \alpha^2 e^{\alpha x} \ \ldots$$

$$D^n(e^{\alpha x}) = \alpha^n e^{\alpha x}, \text{ then } f(D)\, e^{\alpha x} = f(\alpha)\, e^{\alpha x} \quad \ldots (1)$$

Note that $\frac{1}{f(D)}$ is the inverse operator to $f(D)$.

Operating both sides of (1) by $\frac{1}{f(D)}$ we have,

$$f(D) \cdot \frac{1}{f(D)} e^{\alpha x} = \frac{1}{f(D)} f(\alpha) e^{\alpha x}$$

$$\Rightarrow e^{\alpha x} = \frac{1}{f(D)} f(\alpha) e^{\alpha x} \quad \left(\because f(D) \cdot \frac{1}{f(D)} = I\right)$$

$$\text{then } \frac{1}{f(\alpha)} e^{\alpha x} = \frac{1}{f(D)} e^{\alpha x}$$

Thus the *P.I.* is given by $\frac{1}{f(D)} e^{\alpha x} = \frac{1}{f(\alpha)} e^{\alpha x}$ represented symbolically. …(2)

(2) holds when $f(\alpha) \neq 0$.

If $f(\alpha) = 0$ then $D = \alpha$ is a root of the characteristic equation for the differential equation $f(D) = 0 \Rightarrow D - \alpha$ is a factor of $f(D)$.

Let $f(D) = (D - \alpha)\,\theta(D)$, where $\theta(\alpha) \neq 0$ then

$$\frac{1}{f(D)} e^{\alpha x} = \frac{1}{(D-\alpha)\,\theta(D)} \cdot e^{\alpha x}$$

$$= \frac{1}{D-\alpha} \cdot \frac{1}{\theta(\alpha)} e^{\alpha x}$$

$$= \frac{1}{\theta(\alpha)} \frac{1}{D-\alpha} e^{\alpha x} \quad \ldots (3)$$

$$\text{Put } \frac{1}{(D-\alpha)} e^{\alpha x} = y \Rightarrow (D-\alpha)y = e^{\alpha x}$$

$$\text{then } y e^{-\int \alpha\, dx} = \int e^{\alpha x} \cdot e^{-\int \alpha\, dx} \cdot dx$$

$$\text{i.e., } y e^{-\alpha x} = \int e^{\alpha x} e^{-\alpha x}\, dx \Rightarrow y = e^{\alpha x} x$$

Substituting in (3) we have

$$\frac{1}{f(D)} e^{\alpha x} = \frac{1}{\theta(\alpha)} x e^{\alpha x}$$

If further, $\theta(\alpha) = 0$, then $D = \alpha$ is a repeated root for $f(D) = 0$.

$$\text{Then } \frac{1}{f(D)} e^{\alpha x} = \frac{x^2}{2} e^{\alpha x}$$

Example 8.22 : Solve : $(D^2 + 5D + 6)y = 0$ or $y'' + 5y' + 6y = 0$

Solution : To find the *C.F.* solve the characteristic equation

$$p^2 + 5p + 6 = 0$$

$$\Rightarrow (p + 2)(p + 3) = 0 \Rightarrow p = -2 \text{ and } p = -3$$

The *C.F.* is $Ae^{-2x} + Be^{-3x}$.

Hence the general solution is $y = Ae^{-2x} + Be^{-3x}$ where A and B are arbitrary constants.

Example 8.23 : Solve : $(D^2 + 6D + 9)y = 0$

Solution : The characteristic equation is

$$p^2 + 6p + 9 = 0$$

$$\text{i.e., } (p + 3)^2 = 0 \Rightarrow p = -3, -3$$

The C.F. is $(Ax + B)e^{-3x}$

Hence the general solution is $y = (Ax + B)e^{-3x}$

where A and B are arbitrary constants.

Example 8.24 : Solve : $(D^2 + D + 1)y = 0$

Solution : The characteristic equation is $p^2 + p + 1 = 0$

$$\therefore p = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

Hence the general solution is $y = e^{-x/2}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right]$

where A and B are arbitrary constant.

Example 8.25 : Solve : $(D^2 - 13D + 12)y = e^{-2x}$

Solution : The characteristic equation is $p^2 - 13p + 12 = 0$

$$\Rightarrow (p - 12)(p - 1) = 0 \Rightarrow p = 12 \text{ and } 1$$

The C.F. is $Ae^{12x} + Be^{x}$

Particular integral $\quad P.I. = \dfrac{1}{D^2 - 13D + 12} e^{-2x}$

$$= \frac{1}{(-2)^2 - 13(-2) + 12} e^{-2x} = \frac{1}{4 + 26 + 12} e^{-2x}$$

$$= \frac{1}{42} e^{-2x}$$

Hence the general solution is $y = CF + PI \Rightarrow y = Ae^{12x} + Be^{x} + \frac{1}{42} e^{-2x}$

Example 8.26 : Solve : $(D^2 + 6D + 8)y = e^{-2x}$

Solution : The characteristic equation is $p^2 + 6p + 8 = 0$

$$\Rightarrow (p + 4)(p + 2) = 0 \Rightarrow p = -4 \text{ and } -2$$

The C.F. is $Ae^{-4x} + Be^{-2x}$

Particular integral $\quad P.I. = \frac{1}{D^2 + 6D + 8} e^{-2x} = \frac{1}{(D+4)(D+2)} e^{-2x}$

Since $f(D) = (D + 2)\,\theta(D))$

$$= \frac{1}{\theta(-2)} xe^{-2x} = \frac{1}{2} xe^{-2x}$$

Hence the general solution is $y = Ae^{-4x} + Be^{-2x} + \frac{1}{2} xe^{-2x}$

Example 8.27 : Solve : $(D^2 - 6D + 9)y = e^{3x}$

Solution : The characteristic equation is $p^2 - 6p + 9 = 0$

$$\text{i.e., } (p - 3)^2 = 0 \Rightarrow p = 3, 3$$

The C.F. is $(Ax + B)e^{3x}$

Particular integral $\quad P.I. = \frac{1}{D^2 - 6D + 9} e^{3x}$

$$= \frac{1}{(D-3)^2} e^{3x} = \frac{x^2}{2} e^{3x}$$

Hence the general solution is $y = (Ax + B)e^{3x} + \frac{x^2}{2} e^{3x}$

Example 8.28 : Solve : $(2D^2 + 5D + 2)y = e^{-\frac{1}{2}x}$

Solution : The characteristic equation is $2p^2 + 5p + 2 = 0$

$$\therefore p = \frac{-5 \pm \sqrt{25 - 16}}{4} = \frac{-5 \pm 3}{4}$$

$$\Rightarrow\ p = -\frac{1}{2} \text{ and } -2$$

The C.F. is $Ae^{-\frac{1}{2}x} + Be^{-2x}$

Particular integral $\quad P.I. = \dfrac{1}{2D^2 + 5D + 2} e^{-\frac{1}{2}x} = \dfrac{1}{2\left(D + \frac{1}{2}\right)(D + 2)} e^{-\frac{1}{2}x}$

$$= \frac{1}{\theta\left(-\frac{1}{2}\right) . 2} xe^{-\frac{1}{2}x} = \frac{1}{3} x e^{-\frac{1}{2}x}$$

Hence the general solution is $y = Ae^{-\frac{1}{2}x} + Be^{-2x} + \frac{1}{3} x e^{-\frac{1}{2}x}$

Caution : In the above problem we see that while calculating the particular integral the coefficient of D expressed as factors is made unity.

(b) When *X* is of the form sin *ax* or cos *ax*.

Working rule :

Formula 1: Express $f(D)$ as function of D^2, say $\phi(D^2)$ and then replace D^2 by $-a^2$. If $\phi(-a^2) \neq 0$. Then we use the following result.

$$P.I. = \frac{1}{f(D)} \cos ax = \frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax$$

For example $\quad PI = \dfrac{1}{D^2 + 1} \cos 2x = \dfrac{1}{-2^2 + 1} \cos 2x = -\dfrac{1}{3} \cos 2x$

Formula 2 : Sometimes we cannot form $\phi(D^2)$. Then we shall try to get $\phi(D, D^2)$, that is, a function of D and D^2. In such cases we proceed as follows :

For example : $\quad P.I. = \dfrac{1}{D^2 - 2D + 1} \cos 3x$

$= \dfrac{1}{-3^2 - 2D + 1} \cos 3x \qquad$ Replace D^2 by -3^2

$= \dfrac{-1}{2(D + 4)} \cos 3x$

$= \dfrac{-1}{2} \dfrac{D - 4}{D^2 - 4^2} \cos 3x \qquad$ Multiply and divide by $D - 4$

$$= \frac{-1}{2} \frac{1}{-3^2-4^2} (D-4) \cos 3x$$

$$= \frac{1}{50} (D-4) \cos 3x$$

$$= \frac{1}{50} [D \cos 3x - 4 \cos 3x] = \frac{1}{50} [-3 \sin 3x - 4 \cos 3x]$$

Formula 3 : If $\phi(-a^2) = 0$ then we proceed as shown in the following example:

Example : $\quad P.I. = \frac{1}{\phi(D^2)} \cos ax = \frac{1}{D^2 + a^2} \cos ax$

$$= \frac{1}{(D+ia)(D-ia)} \cos ax$$

$$= \text{R.P.} \left[\frac{1}{(D+ia)(D-ia)} e^{iax} \right] = R.P. \left[\frac{1}{\theta(ia)} x e^{iax} \right]$$

$$= \text{Real part of} \left[\frac{x e^{iax}}{2ia} \right] \text{ as } \theta(ia) = 2ia$$

$$= \frac{-x}{2a} [\text{Real part of } i [\cos ax + i \sin ax]]$$

$$= \frac{-x}{2a} (-\sin ax) = \frac{x \sin ax}{2a}$$

Note : If $X = \sin ax$

Formula 1 : $\frac{1}{\phi(-a^2)} \sin ax$

Formula 2 : Same as $\cos ax$ method

Formula 3 : $\frac{1}{D^2 + a^2} \sin ax = \text{I.P.} \left[\frac{1}{(D+ia)(D-ia)} e^{iax} \right] = \frac{-x}{2a} \cos ax$

Example 8.29 : Solve : $(D^2 - 4)y = \sin 2x$

Solution : The characteristic equation is $p^2 - 4 = 0 \Rightarrow p = \pm 2$

$$C.F. = Ae^{2x} + Be^{-2x} \ ;$$

$$\text{P.I.} = \frac{1}{D^2 - 4} (\sin 2x) = \frac{1}{-4-4} (\sin 2x) = -\frac{1}{8} \sin 2x$$

Hence the general solution is $y = C.F. + P.I. \Rightarrow y = Ae^{2x} + Be^{-2x} - \frac{1}{8} \sin 2x$

Example 8.30 : Solve : $(D^2 + 4D + 13)y = \cos 3x$

Solution : The characteristic equation is $p^2 + 4p + 13 = 0$

$$p = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm i6}{2} = -2 \pm i3$$

$$C.F. = e^{-2x}(A \cos 3x + B \sin 3x)$$

$$P.I. = \frac{1}{D^2 + 4D + 13}(\cos 3x)$$

$$= \frac{1}{-3^2 + 4D + 13}(\cos 3x) = \frac{1}{4D+4}(\cos 3x)$$

$$= \frac{(4D-4)}{(4D+4)(4D-4)}(\cos 3x) = \frac{4D-4}{16D^2 - 16}(\cos 3x)$$

$$= \frac{4D-4}{-160}(\cos 3x) = \frac{1}{40}(3 \sin 3x + \cos 3x)$$

The general solution is y = C.$F.$ + $P.I.$

$$y = e^{-2x}(A \cos 3x + B \sin 3x) + \frac{1}{40}(3 \sin 3x + \cos 3x)$$

Example 8.31 : Solve $(D^2 + 9)y = \sin 3x$

Solution : The characteristic equation is $p^2 + 9 = 0 \Rightarrow p = \pm 3i$

$$C.F. = (A \cos 3x + B \sin 3x)$$

$$P.I. = \frac{1}{D^2 + 9} \sin 3x$$

$$= \frac{-x}{6} \cos 3x \quad \text{since } \frac{1}{D^2 + a^2} \sin ax = \frac{-x}{2a} \cos ax$$

Hence the solution is $y = C.F. + P.I.$

$$\text{i.e., } y = (A \cos 3x + B \sin 3x) - \frac{x \cos 3x}{6}$$

(c) When X is of the form x and x^2

Working rule : Take the P.I. as $c_0 + c_1x$ if $f(x) = x$ and $c_0 + c_1x + c_2x^2$ if $f(x) = x^2$. Since P.I. is also a solution of $(aD^2 + bD + c)_y = f(x)$, take $y = c_0 + c_1x$ or $y = c_0 + c_1x + c_2x^2$ according as $f(x) = x$ or x^2. By substituting y value and comparing the like terms, one can find c_0, c_1 and c_2.

Example 8.32 : Solve : $(D^2 - 3D + 2)y = x$

Solution : The characteristic equation is $p^2 - 3p + 2 = 0 \Rightarrow (p - 1)(p - 2) = 0$

$p = 1, 2$

The *C.F.* is $(Ae^x + Be^{2x})$

$$\text{Let } P.I. = c_0 + c_1 x$$

$\therefore\ c_0 + c_1 x$ is also a solution.

$\therefore\ (D^2 - 3D + 2)(c_0 + c_1 x) = x$

i.e., $(-3c_1 + 2c_0) + 2c_1 x = x$

$$\Rightarrow 2c_1 = 1 \quad \therefore \quad c_1 = \frac{1}{2}$$

$$(-3c_1 + 2c_0) = 0 \Rightarrow c_0 = \frac{3}{4}$$

$$\therefore \text{ P.I.} = \frac{x}{2} + \frac{3}{4}$$

Hence the general solution is $y = C.F. + P.I.$

$y = Ae^x + Be^{2x} + \dfrac{x}{2} + \dfrac{3}{4}$

Example 8.33 :

Solve : $(D^2 - 4D + 1)y = x^2$

Solution : The characteristic equation is $p^2 - 4p + 1 = 0$

$$\Rightarrow p = \frac{4 \pm \sqrt{16 - 4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$C.F. = Ae^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x}$$

Let $P.I. = c_0 + c_1 x + c_2 x^2$

But P.I. is also a solution.

$\therefore\ (D^2 - 4D + 1)(c_0 + c_1 x + c_2 x^2) = x^2$

i.e., $(2c_2 - 4c_1 + c_0) + (-8c_2 + c_1)x + c_2 x^2 = x^2$

$c_2 = 1$

$-8c_2 + c_1 = 0 \Rightarrow c_1 = 8$

$2c_2 - 4c_1 + c_0 = 0 \Rightarrow c_0 = 30$

P.I. $= x^2 + 8x + 30$

Hence the general solution is $y = C.F. + P.I.$

$$y = Ae^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x} + (x^2 + 8x + 30)$$

EXERCISE 8.5

Solve the following differential equations :

(1) $(D^2 + 7D + 12)y = e^{2x}$

(2) $(D^2 - 4D + 13)y = e^{-3x}$

(3) $(D^2 + 14D + 49)y = e^{-7x} + 4$

(4) $(D^2 - 13D + 12)y = e^{-2x} + 5e^x$

(5) $(D^2 + 1)\, y = 0$ when $x = 0$, $y = 2$ and when $x = \frac{\pi}{2}$, $y = -2$

(6) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}$ when $x = \log 2$, $y = 0$ and when $x = 0$, $y = 0$

(7) $(D^2 + 3D - 4)\, y = x^2$

(8) $(D^2 - 2D - 3)y = \sin x \cos x$

(9) $D^2y = -9 \sin 3x$

(10) $(D^2 - 6D + 9)\, y = x + e^{2x}$

(11) $(D^2 - 1)y = \cos 2x - 2 \sin 2x$

(12) $(D^2 + 5)y = \cos^2 x$

(13) $(D^2 + 2D + 3)y = \sin 2x$

(14) $(3D^2 + 4D + 1)y = 3e^{-x/3}$

8.6 Applications :

In this section we solve problems on differential equations which have direct impact on real life situation. Solving of these types of problems involve

(i) Construction of the mathematical model describing the given situation

(ii) Seeking solution for the model formulated in (i) using the methods discussed earlier.

Illustration :

Let A be any population at time t. The rate of change of population is directly proportional to initial population i.e.,

$\frac{dA}{dt} \propto A$ i.e., $\frac{dA}{dt} = kA$ where k is called the constant of proportionality

(1) If $k > 0$, we say that A grows exponentially with growth constant k (growth problem).

(2) If $k < 0$ we say that A decreases exponentially with decreasing constant k (decay problem).

In all the practical problems we apply the principle that the rate of change of population is directly proportional to the initial population i.e., $\frac{dA}{dt} \alpha A$ or $\frac{dA}{dt} = kA$

(Here k may be positive or negative depends on the problem). This linear equation can be solved in three ways i.e., (i) variable separable (ii) linear (using I.F.) (iii) by using characteristic equation with single root k. In all the ways we get the solution as $A = ce^{kt}$ where c is the arbitrary constant and k is the constant of proportionality. In general we have to find out c as well as k from the given data. Sometimes the value of k may be given directly as in 8.35. $\frac{dA}{dt}$ is directly given in 8.38.

Solution : $\frac{dA}{dt} = kA$

(i) $\frac{dA}{A} = kdt \Rightarrow \log A = kt + \log c$

$\Rightarrow A = e^{kt + \log c} \Rightarrow A = ce^{kt}$

(ii) $\frac{dA}{dt} - kA = 0$ is linear in A

$$I.F. = e^{-kt}$$

$$Ae^{-kt} = \int e^{-kt} O\, dt + c \Rightarrow Ae^{-kt} = c$$

$$A = ce^{kt}$$

(iii) $(D - k)A = 0$

Chr. equation is $p - k = 0 \Rightarrow p = k$

The C.F. is ce^{kt}

But there is no $P.I.$

$\therefore A = ce^{kt}$

(iv) In the case of Newton's law of cooling (i.e., the rate of change of temperature is proportional to the difference in temperatures) we get the equation as

$$\frac{dT}{dt} = k(T - S)$$

[T– cooling object temperature, S – surrounding temperature]

$$\frac{dT}{T - S} = kdt \Rightarrow \log (T - S) = kt + \log c \Rightarrow T - S = ce^{kt}$$

$$\Rightarrow T = S + ce^{kt}$$

Example 8.34 : In a certain chemical reaction the rate of conversion of a substance at time t is proportional to the quantity of the substance still untransformed at that instant. At the end of one hour, 60 grams remain and at the end of 4 hours 21 grams. How many grams of the substance was there initially?

Solution :

Let A be the substance at time t

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \Rightarrow A = ce^{kt}$$

$$\text{When } t = 1, \; A = 60 \Rightarrow ce^{k} = 60 \qquad \ldots (1)$$

$$\text{When } t = 4, A = 21 \Rightarrow ce^{4k} = 21 \qquad \ldots (2)$$

$$(1) \Rightarrow c^4e^{4k} = 60^4 \qquad \ldots (3)$$

$$\frac{(3)}{(2)} \Rightarrow c^3 = \frac{60^4}{21} \Rightarrow c = 85.15 \text{ (by using log)}$$

Initially i.e., when $t = 0$, $A = c = 85.15$ gms (app.)

Hence initially there was 85.15 gms (approximately) of the substance.

Example 8.35 : A bank pays interest by continuous compounding, that is by treating the interest rate as the instantaneous rate of change of principal. Suppose in an account interest accrues at 8% per year compounded continuously. Calculate the percentage increase in such an account over one year. [Take $e^{.08} \approx 1.0833$]

Solution : Let A be the principal at time t

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \Rightarrow \frac{dA}{dt} = 0.08\,A, \text{ since } k = 0.08$$

$$\Rightarrow A(t) = ce^{0.08t}$$

$$\text{Percentage increase in 1 year} = \frac{A(1) - A(0)}{A(0)} \times 100$$

$$= \left(\frac{A(1)}{A(0)} - 1\right) \times 100 = \left(\frac{c.\, e^{0.08}}{c} - 1\right) \times 100 = 8.33\%$$

Hence percentage increase is 8.33%

Example 8.36 :

The temperature T of a cooling object drops at a rate proportional to the difference $T - S$, where S is constant temperature of surrounding medium. If initially $T = 150°C$, find the temperature of the cooling object at any time t.

Solution :

Let T be the temperature of the cooling object at any time t

$\frac{dT}{dt} \alpha (T - S) \Rightarrow \frac{dT}{dt} = k(T - S) \Rightarrow T - S = ce^{kt}$, where k is negative

$\Rightarrow \quad T = S + ce^{kt}$

When $t = 0$, $T = 150 \Rightarrow 150 = S + c \Rightarrow c = 150 - S$

$\therefore$ The temperature of the cooling object at any time is

$T = S + (150 - S)e^{kt}$

Note : Since k is negative, as t increases T decreases.

It is a decay problem. Instead of k one may take $-k$ where $k > 0$. Then the answer is $T = S + (150 - S)e^{-kt}$. Again, as t increases T decreases.

Example 8.37 : For a postmortem report, a doctor requires to know approximately the time of death of the deceased. He records the first temperature at 10.00 a.m. to be 93.4°F. After 2 hours he finds the temperature to be 91.4°F. If the room temperature (which is constant) is 72°F, estimate the time of death. (Assume normal temperature of a human body to be 98.6°F).

$\left[\log_e \frac{19.4}{21.4} = -0.0426 \times 2.303 \text{ and } \log_e \frac{26.6}{21.4} = 0.0945 \times 2.303\right]$

Solution :

Let T be the temperature of the body at any time t

By Newton's law of cooling $\frac{dT}{dt} \alpha (T - 72)$ since $S = 72°F$

$$\frac{dT}{dt} = k(T - 72) \Rightarrow T - 72 = ce^{kt}$$

$$\text{or} \quad T = 72 + ce^{kt}$$

At $t = 0$, $T = 93.4 \Rightarrow c = 21.4$ [First recorded time 10 a.m. is $t = 0$]

$$\therefore T = 72 + 21.4e^{kt}$$

When $t = 120$, $T = 91.4 \Rightarrow e^{120k} = \frac{19.4}{21.4} \Rightarrow k = \frac{1}{120} \log_e\left(\frac{19.4}{21.4}\right)$

$= \frac{1}{120}(-0.0426 \times 2.303)$

Let t_1 be the elapsed time after the death.

When $t = t_1$; $T = 98.6 \Rightarrow 98.6 = 72 + 21.4\, e^{kt_1}$

$$\Rightarrow t_1 = \frac{1}{k}\ \log_e\left(\frac{26.6}{21.4}\right) = \frac{-120 \times 0.0945 \times 2.303}{0.0426 \times 2.303} = -266 \text{ min}$$

[For better approximation the hours converted into minutes]

i.e., 4 hours 26 minutes before the first recorded temperature.

The approximate time of death is 10.00 hrs – 4 hours 26 minutes.

$\therefore$ Approximate time of death is 5.34 A.M.

Note : Since it is a decay problem, we can even take $\frac{dT}{dt} = -k(T-72)$ where $k > 0$. The final answer will be the same.

Example 8.38 : A drug is excreted in a patients urine. The urine is monitored continuously using a catheter. A patient is administered 10 mg of drug at time $t = 0$, which is excreted at a Rate of $-3t^{1/2}$ mg/h.

(i) What is the general equation for the amount of drug in the patient at time $t > 0$?

(ii) When will the patient be drug free?

Solution :

(i) Let A be the quantum of drug at any time t

The drug is excreted at a rate of $-3t^{\frac{1}{2}}$

$$\text{i.e., } \frac{dA}{dt} = -3t^{\frac{1}{2}} \Rightarrow A = -2t^{\frac{3}{2}} + c$$

$$\text{When } t = 0,\ A = 10 \Rightarrow c = 10$$

At any time t $\qquad A = 10 - 2t^{\frac{3}{2}}$

(ii) For drug free, $A = 0 \Rightarrow 5 = t^{\frac{3}{2}} \Rightarrow t^3 = 25 \Rightarrow t = 2.9$ hours.

Hence the patient will be drug free in 2.9 hours or 2 hours 54 min.

Example 8.39 :

The number of bacteria in a yeast culture grows at a rate which is proportional to the number present. If the population of a colony of yeast bacteria triples in 1 hour. Show that the number of bacteria at the end of five hours will be 3^5 times of the population at initial time.

Solution : Let A be the number of bacteria at any time t

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \Rightarrow A = ce^{kt}$$

Initially, i.e., when $t = 0$, assume that $A = A_0$

$$\therefore\ A_0 = ce^{\circ} = c$$

$$\therefore\ A = A_0 e^{kt}$$

$$\text{when } t = 1,\ A = 3A_0 \Rightarrow 3A_0 = A_0 e^k \Rightarrow e^k = 3$$

$$\text{When } t = 5,\ A = A_0 e^{5k} = A_0 (e^k)^5 = 3^5 . A_0$$

$\therefore$ The number of bacteria at the end of 5 hours will be 3^5 times of the number of bacteria at initial time

EXERCISE 8.6

(1) Radium disappears at a rate proportional to the amount present. If 5% of the original amount disappears in 50 years, how much will remain at the end of 100 years. [Take A_0 as the initial amount].

(2) The sum of Rs. 1000 is compounded continuously, the nominal rate of interest being four percent per annum. In how many years will the amount be twice the original principal? ($\log_e 2 = 0.6931$).

(3) A cup of coffee at temperature 100°C is placed in a room whose temperature is 15°C and it cools to 60°C in 5 minutes. Find its temperature after a further interval of 5 minutes.

(4) The rate at which the population of a city increases at any time is proportional to the population at that time. If there were 1,30,000 people in the city in 1960 and 1,60,000 in 1990 what population may be anticipated in 2020. $\left[\log_e \left(\frac{16}{13}\right) = .2070\ ;\ e^{.42} = 1.52\right]$

(5) A radioactive substance disintegrates at a rate proportional to its mass. When its mass is 10 mgm, the rate of disintegration is 0.051 mgm per day. How long will it take for the mass to be reduced from 10 mgm to 5 mgm. [$\log_e 2 = 0.6931$]