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Honors Course
Homework 1

Question 1

For each question determine $f = O(g)$, $f = \Omega(g)$ or both, that is to say, $f = \Theta(g)$.

a. $f = \Theta(g)$

b. $f = O(g)$

c. $f = \Theta(g)$

d. $f = \Theta(g)$

e. $f = \Theta(g)$

f. $f = \Theta(g)$

g. $f = \Omega(g)$

h. $f = \Omega(g)$

i. $f = \Omega(g)$

j. $f = \Omega(g)$

k. $f = \Omega(g)$

l. $f = O(g)$

m. $f = O(g)$

n. $f = \Theta(g)$

o. $f = \Omega(g)$

p. $f = O(g)$

q. $f = \Theta(g)$

Question 2

Show that if c is a positive real number, that $g(n) = 1{+}c + c^2 + ... + c^n$

a. $c < \mathit{1}$ if $\Theta(1)$

Assume that c is indeed a value which is less than 1. From the question we can identify that we have a geometric series, the formula for a partial geometric series is below:

$g(n) = \mathit{1}{+}c + c^2 + ... + c^n = \frac{c^n - \mathit{1}}{c - \mathit{1}}$

If $c < \mathit{1}$, then using the formula the equation can be written as $g(n) = \frac{\mathit{0}-\mathit{1}}{c-\mathit{1}} = \frac{\mathit{1}}{\mathit{1}-c}$ as n approaches infinity, and since the value of the limit of $c^{n+\mathit{1}}$ is 0, we get that the limit as n approaches infinity of $g(n) = \frac{\mathit{1}}{\mathit{1}-c}$, $\frac{1}{1-c} > g(n) > 1$. Since we determined that the value of $c < \mathit{1}$ is decreasing, the notation is indeed $\Theta(1)$ as the $g(n)$ must be larger than 1 and has to be less than the constant $\frac{1}{1-c}$ multiplied by some number (both of which are O(1), so we have $\Theta(1)$.

b. $c = \mathit{1}$ if $\Theta(n)$

Let $c = \mathit{1}$. We will take our $g(n)$, where $g(n) = 1{+}c + c^2 + ... + c^n$ and replace our c with the number 1.

$1{+}c + c^2 + ... + c^n = 1 + 1 + 1 + ... + 1$ (n times) = n = Θ(n)

c. $c > \mathit{1}$ if $\Theta(c^n)$

Here we can expland the geometric series formula:

$\frac{c^n-1}{c-1} = \frac{c^n}{c-1} + \frac{-1}{c-1} = \frac{1}{c-1} \cdot c^n + \frac{-1}{c-1} = \Theta(c^n)$

Question 3

Fibonacci numbers $F_{\mathit{0}}, F_1, F_2, ...$ are defined as follows:

$F_{\mathit{0}} = \mathit{0}, F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$

In this question we will show that the elements in the series are growing exponentially and calculate impediments to the growth rate.

a. Prove by induction that for $n \geq \mathit{6}, F_n \geq 2^{0.5n}$

For the base case, we know $F_6 = 13$ and $2^{0.5 \cdot 6} = 2^3 = 8$, therefor it holds.

For our inductive step, for $n \geq \mathit{6}$, we assume $F_n \geq 2^{0.5\mathrm{n}}$ is true for some integer n, so $F_{n-\mathit{1}} \geq 2^{0.5(\mathrm{n}-1)}$. We will show $F_{n+\mathit{1}} \geq 2^{0.5(\mathrm{n}+1)}$ by applying the Fibonaci definition that we learned in class: $F_n + F_{n-\mathit{1}} \geq 2^{0.5\mathrm{n}+0.5} \rightarrow F_n + F_{n-\mathit{1}} \geq 2^{0.5\mathrm{n}} \cdot \sqrt{2}$.

Now we can use our previous assumptions:

$2^{0.5n} + 2^{0.5(n-1)} \geq 2^{0.5n} \cdot \sqrt{2} \rightarrow 2^{0.5n} + 2^{0.5n} \cdot 2^{-0.5} \geq 2^{0.5n} \cdot \sqrt{2} \rightarrow 2^{0.5n}(1 + \frac{1}{\sqrt{2}}) \geq 2^{0.5n} \cdot \sqrt{2}$

Now we can divide by our $2^{0.5n}$ for $1 + \frac{1}{\sqrt{2}} \geq \sqrt{2}$

Our assumptions hold :)

b. Find a constant $c < 1$, s.t $F_n \leq 2^{cn}$ for every $n \geq 0$. Prove the validity of the claim.

We will show that for $c = 1.5$ this holds.

$F_0 = 1, 2^{1.5 \cdot 0} = 1$
$F_1 = 1, 2^{1.5 \cdot 1} = 2.83$

We now will assume that $F_n \leq 2^{1.5 \cdot n}$

$F_{n+1} = F_n + F_{n-1} \leq 2^{1.5n} + 2^{1.5(n-1)} = 2^{1.5n} + 2^{1.5n} \cdot 2^{-1.5} = 2^{1.5n}(1 + \frac{1}{2^{1.5}})$
$2^{1.5n}(1 + \frac{1}{2^{1.5}}) \leq 2^{1.5(n+1)}$
$(1 + \frac{1}{2^{1.5}}) \leq 2^{1.5}$
$1.35 \leq 2.83$

Our assumption holds.

c. What is the largest c for which we can prove that $F_n = \Omega(2^{cn})$?

From the definition we show, $F_n \geq 2^{cn}$,

$F_n = F_{n-1} + F_{n-2} \geq 2^{c(n-1)} + 2^{c(n-2)}$

To satisfy $F_n \geq 2^{cn}$, we show $2^{cn} \leq 2^{c(n-1)} + 2^{c(n-2)}$

Solving the inequality as before:
$2^{cn} \leq 2^{cn} \cdot 2^{-c} + 2^{cn-2c}$
$2^{cn} \leq 2^{cn}(2^{-c} + 2^{-2c})$
$1 \leq 2^{-c} + 2^{-2c}$

We can now replace 2^{-c} with t.

$1 \leq t + t^2$
$0 \leq t^2 + t - 1$
$t \geq 0.618$
$\frac{1}{2^c} \geq 0.618$
$c \leq 0.694$

We get c $\approx$0.694, so as c gets larger the right hand side gets smaller, so in order to satisfy $(2^{-c} + 2^{-2c}) \geq 1$, $c \leq 0.694$.

So, the largest value for c when $F_n = \Omega(2^{cn})$ is 0.694.

Question 4

Show that the length of the binary representation of a number (that is to say at base 2) is at most four times the length of its decimal representation (i.e. at base 10). What is the approximate ratio of lengths for very large numbers?

Let n be some number.

The length of n in binary digits (the number of binary digits required to represent n) is $\lfloor log_2(n) \rfloor + 1$, and the length of n in decimal digits (the number of decimal digits required to represent n) is $\lfloor log_{10}(n) \rfloor + 1$, this can be further generalized to the number of digits required to represent a positive integer n at base k is $k(n) := \lfloor logkn \rfloor + \mathit{1}$.

Therefore, the ratio of the length of a binary representation of a number to its decimal length is
$\frac{\lfloor log_2(n) \rfloor + 1}{\lfloor log_{10}(n) \rfloor + 1}$

As we can see, the constants do not affect the ratio, and neither do the floors, so the ratio for very large numbers is is:
$\frac{log_2(n)}{log_{10}(n)} = log_2 10 \approx 3.321$

Question 5

Write in your favorite programming language and implement with it addition, subtraction, and multiplication of two 32-bit numbers and address why the subtraction output size is 33 bits. Use logical operations on bits only (NOT, XOR, OR, AND, you are also allowed to use SHIFT if necessary).

Addition:

Input: $x, y \in \{0, 1\}^{32}$ Output: $x + y \in \{0, 1\}^{33}$

Subtraction:

Input: $x, y \in \{0, 1\}^{32}$ Output: $x - y \in \{0, 1\}^{33}$

Multiplication:

Input: $x, y \in \{0, 1\}^{32}$ Output: $xy \in \{0, 1\}^{33}$

Bonus: Utilize the multiplication operation so that for every $x, y \in \{0, 1\}^{32}$ you do not use more than 16 multiplication operations.

I have provided question 5 in a seperate document. Thank you.