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Honors Course
Homework 9

Definition: The thickness of a shape in a given direction is the distance between two parallel lines tangent to its circumference in that direction.

Question 1

Calculate the area of a Releaux triangle.

The Reuleaux triangle is formed by connecting the vertices of an equilateral triangle with arcs, where the arc between any two vertices has its center at the third vertex. The width of a convex set in the plane is defined as the minimum distance between any two parallel lines that enclose it and the two minimum-distance lines are both necessarily tangent lines to the convex set. A curve of constant width is the boundary of a convex set and has the property that for every direction of parallel lines, the two tangent lines with that direction that are tangent to opposite sides of the curve are at a distance equal to the width, that is to say the distance between any two parallel tangent lines is constant (this is also a property of a circle). A curve of constant width constructed by drawing arcs from each vertex of an equilateral triangle between the other two vertices.

Let the arc radius be r. Since the area of each meniscus-shaped portion of the Reuleaux triangle is a circular segment with opening angle $\theta = \frac{\pi}{3}$, so we get that the area of each segment is in total $A_s = \frac{1}{2}r^2(\theta - sin\theta) = (\frac{\pi}{6} - \frac{\sqrt{3}}{4})r^2$ as we remove the area of an equilateral triangle from each segment. The area of the central equilateral triangle with $a = r$ is $A_t = \frac{1}{4}\sqrt{3}r^2$, and the total area is $A = 3A_s + A_t = \frac{\pi - \sqrt{3}}{2}r^2$.

Question 2

Prove that Releaux triangles have constant width.

Let A, B, C be three points that form the triangle $\triangle ABC$, and let this triangle be a Releaux triangle where the curved arcs are D, E, F. Let r be the radius of curvature in the triangle. Any tangent to the triangle must make contact with the triangle either at a vertex, such as A, or a curved arc, such as D (in the case that the point A is opposite to the arc D). If the tangent line does indeed touch vertex A, the perpendicular to the tangent at A meets the opposite side of the triangle at D, at a distance of r. As D is a point on a circular arc, the perpendicular to AD at $D$is a tangent of the triangle. Both tangents are perpendicular to AD, and they are parallel, separated by distance r. If the tangent meets the curved arc at D, its perpendicular at D must pass through its centre of curvature at A. AD is thus a radius of the curved arc. The perpendicular to AD at A is parallel to the tangent at D, separated by a distance of r. So any pair of parallel tangents that is opposite and meets the triangle must be separated by a distance of r, and so the triangle has constant width.

Question 3

a. Show that it is not possible to find in a plane 4 points that the distance between any two is equal.

Let P, Q, R be points and let them exist in a two-dimensional plane. Furthermore, let the distance between these points be equal, that is to say let them be equidistant. The arrangement of these points in our two dimensional plane must be an equilateral triangle. If P, Q, R is not a triangle, then the points would not in fact all be equidistant from each other. If P, Q, R are on a single line, then the perpendicular bisectors are parallel and there would be no equidistant point, a contradiction. Therefore, we have as a condition that in order for there to be an equidistant point from our three distinct points P, Q, R that the points don't lie on a single line. Then, $\triangle PQR$ is a non-degenerate triangle (where a non-degenerate triangle is defined as a triangle formed by picking three points at random and these points are in a straight line). Therefore, there is exactly one point equidistant from the three given points, which is the circumcentre of $\triangle PQR$, however the points are not equidistant to each other and this circumcentre. If the three points lie on a line then the triangle degenerates into a line and no point on the plane will be equidistant from all three points, except if $R = P$ or $R = Q$ or $P = Q$. Adding another point on the two-dimensional plane would not produce any system of equidistant points as if we were to take four distinct points and plot them on a two-dimensional plane and connect each pair of vertices with an edge, we would always have ${}_4C_2 = 6$ edges created, no matter the geometry. However, the lengths of these edges would be of different lengths. Therefore the only remaining solution would be to place a point centre of triangle at a depth of $\sqrt{\frac{2}{3}}d$, where d is the distance between all points, therefore leaving the two-dimensional plane and creating a tetrahedron in a three-dimensional space, a contradiction to the points being in a two-dimensional plane.

b. Show that it is not possible to find 5 points in space that the distance between any two is equal.

We proved above that the configurations for 3 and 4 points are the equilateral triangle (in a two-dimensional plane) and the equilateral tetrahedron in $\mathbb{R}^3$ (a three-dimensional space). We can add another point to the tetrahedron but the only one that has the same distance to all vertices is the centroid, however, the distance from any point to another (that is not the centroid) is not equal to the distance of any such point to the centroid, therefore there is no such configuration for 5 points.