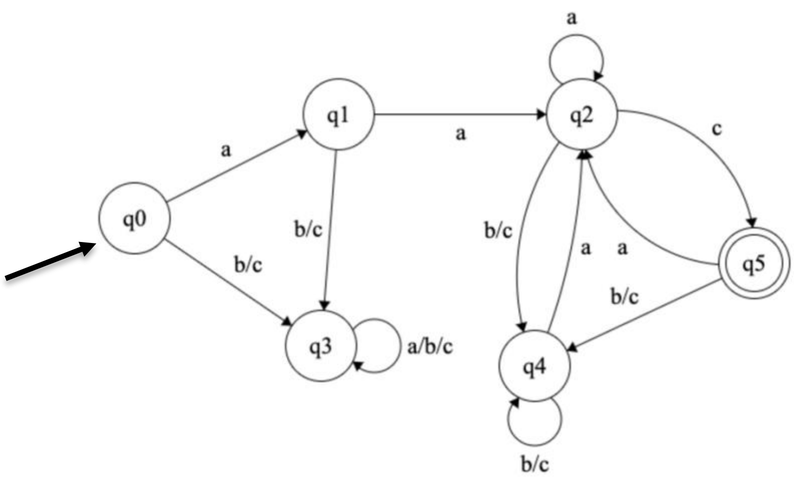
# Exercise 2

Submit by Wednesday 24/03/21

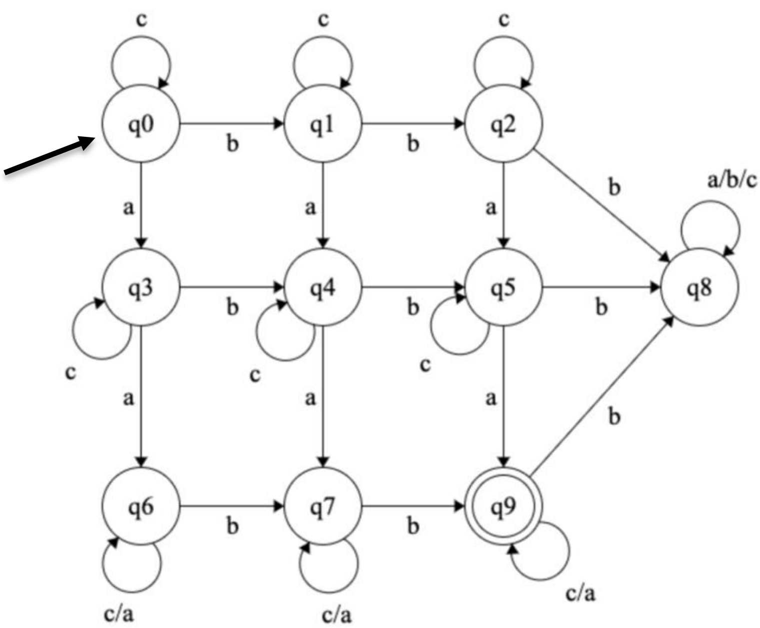
## **Question 1** (50 pts)

For each one of the following languages decide whether L is regular or not. If L is regular, define a DFA recognizing it. In all languages, if not defined otherwise, the language is defined over alphabet $\Sigma = \{a, b\}$. You just have to draw the state diagram of your DFA.

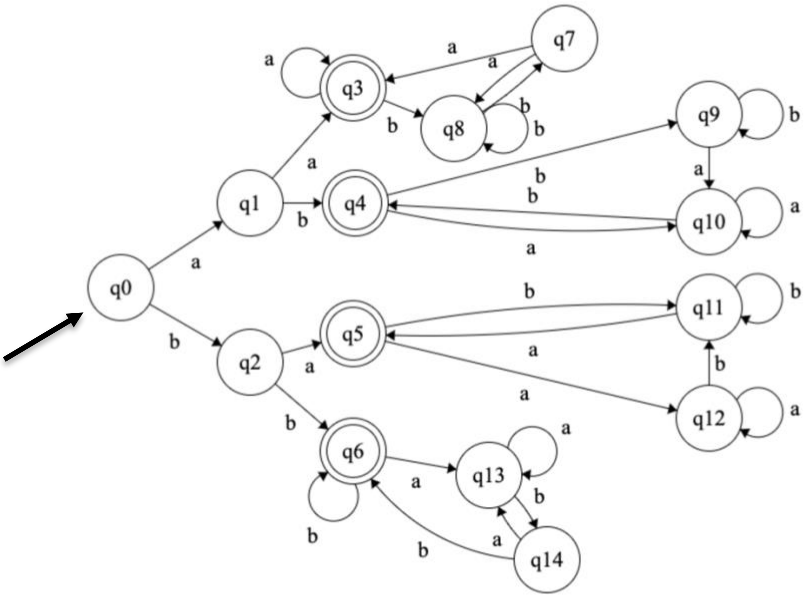
a. $L = \{w \in \{a, b, c\}^* \mid w\ starts\ with\ \boldsymbol{aa}\ and\ ends\ with\ \boldsymbol{ac}\}$



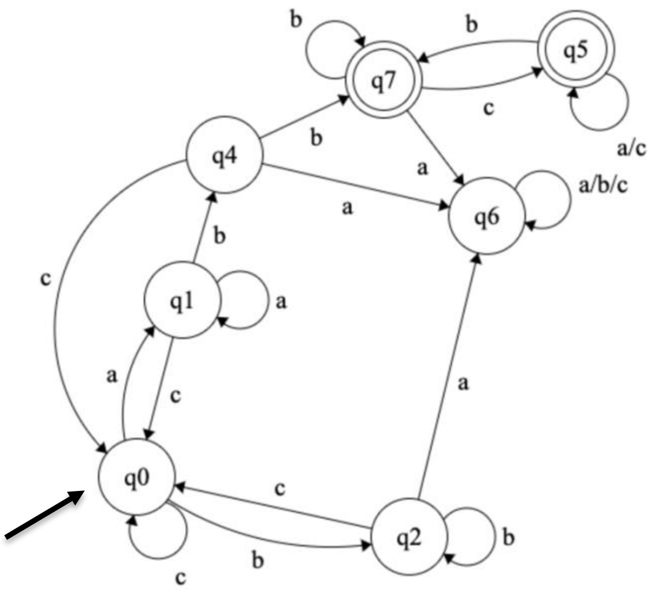

b. $L = \{w \in \{a, b, c\}^* \mid \#_a(w) > 2\ \ and\ \ \#_b(w) = 2\ \}$



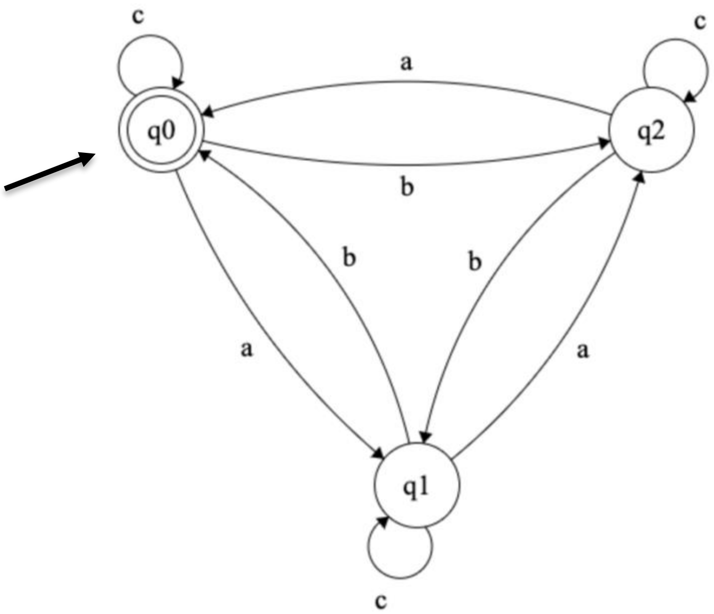
*c.* $L = \{w \in \{a,b\}^* |\ \ |w| \geq 2$ and the first two letters are identical to the last two letters$\}$



d. $L = \{w \in \{a,b,c\}^* \mid w\ has\ \boldsymbol{abb}\ but\ does\ not\ have\ \boldsymbol{ba}\ as\ substring\ \}$



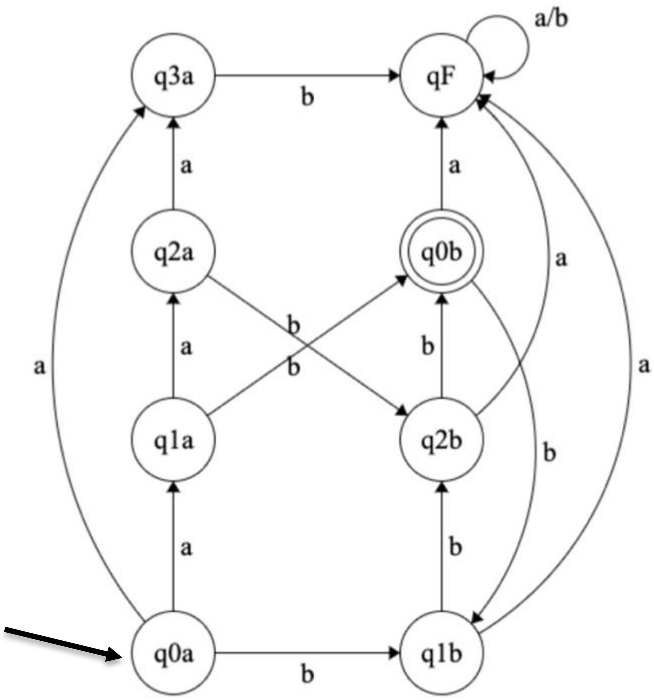
e. $L = \{w \in \{a, b, c\}^* \mid \ |\#_a(w) - \#_b(w)| \ mod\ 3 = 0\}$




f. $L = \{w \in \{a, b, c\}^* \mid \ |\#_a(w) - \#_b(w)| \ mod\ 3 = 1\}$

Non-regular

g. $L = \{a^i b^j \ \mid \ i\ mod\ 4 = j\ mod\ 3\ \}$




h. $L = \{w \in \{a, b, c\}^* \mid \ |\#_a(w) - \#_b(w)| < 3\}$

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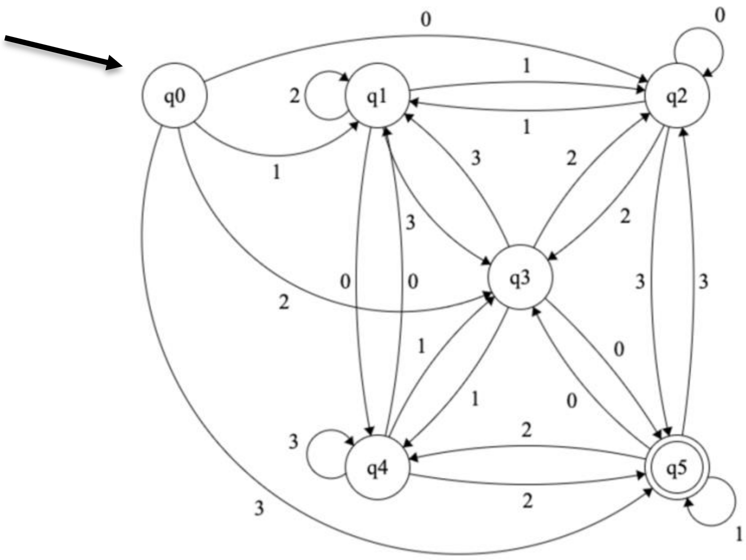
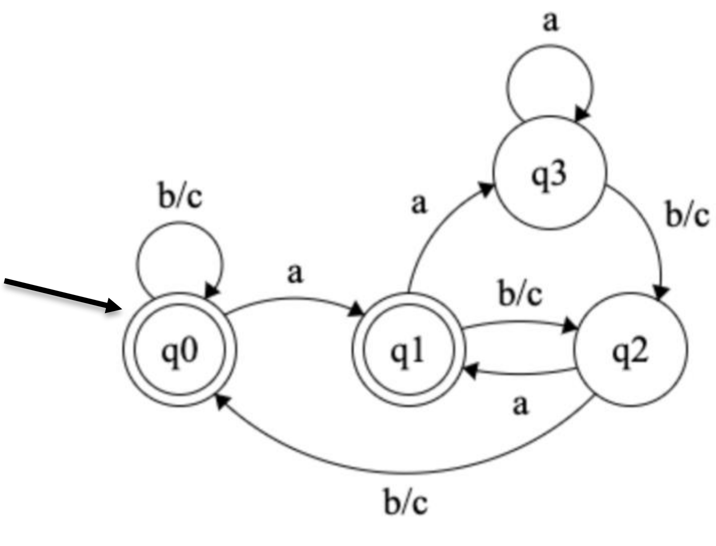
Non-regular

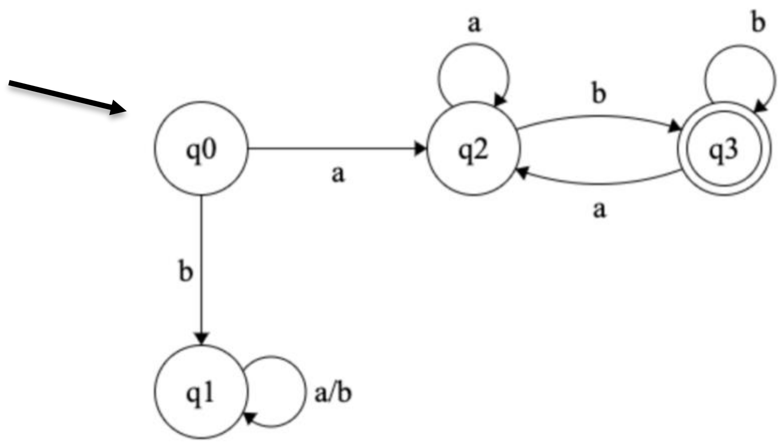
i. $L = \{w \in \{0,1,2,3\}^+ \mid\ w\ is\ a\ number\ in\ basis\ 4,\ and\ (w\ mod\ 5) = 3,$
$w\ can\ include\ leading\ 0's\}$




j. $L = \Sigma^* - \{w \in \Sigma^* \mid\ \ w = ua\sigma,\ u \in \Sigma^*, \sigma \in \Sigma\}\ where\ \Sigma = \{a, b, c\}$



k. $L = \{ w \mid w = a^n u b^n, \quad n > 0 \, , \, u \in \Sigma^* \}$



## **Question 2** (10 pts)

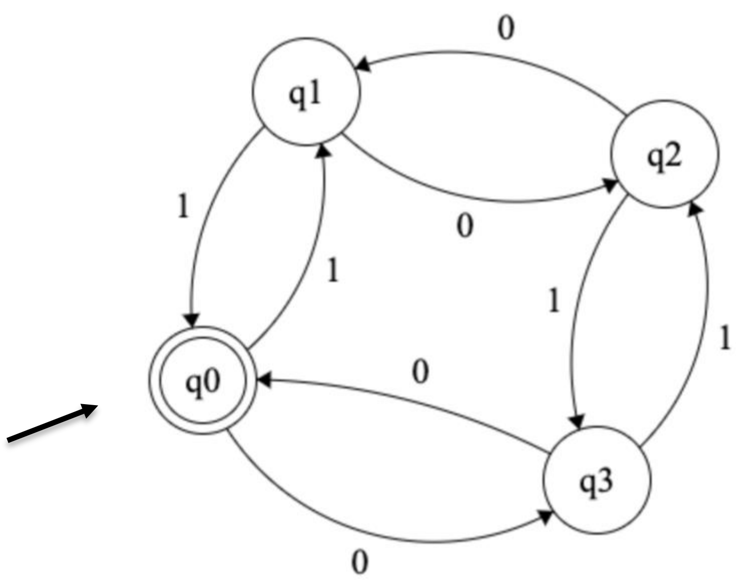
The formal definition of an automaton A is

A = ( $Q = \{q_0, q_1, q_2, q_3\},\ \sigma = \{0,1\},\ \delta\ ,\ q_0,\ F = \{q_0\}\}$

and $\delta$ is defined as follows:

| $\delta$ | 0 | 1 |
|---|---|---|
| $q_0$ | $q_3$ | $q_1$ |
| $q_1$ | $q_2$ | $q_0$ |
| $q_2$ | $q_1$ | $q_3$ |
| $q_3$ | $q_0$ | $q_2$ |

a. Draw the automaton.




b. Define L(A). Explain your answer shortly.

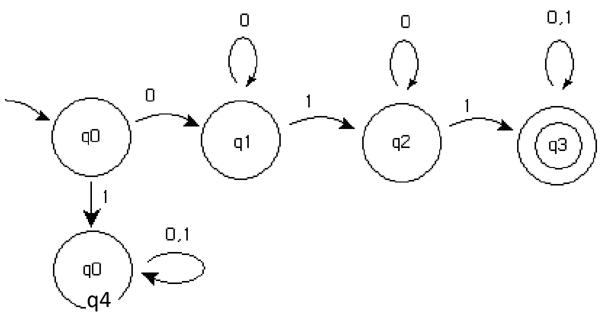
L(A) = $\{w \in \{0,1\}^* \mid \#_0(w) = even,\ \ \#_1(w) = even\}$

The language of A is comprised of words where the number of 0s and 1s is even (and trivially the number of characters in the string is even). q0 is our starting and accepting state, that represents the number of 0s and 1s being even when it accepts a string, where q1 and q3 represent an uneven number of 1s **or** 0s (respectively) and q2 represents an uneven number of 0s ***and*** 1s.

## Question 3 (10 pts)

Write **formal definitions** for the following DFAs (5 tuple – don't forget the transition table). For each DFA, define also the language it recognizes.

a.




A = ( $Q = \{q_0, q_1, q_2, q_3, q_4\},\ \sigma = \{0,1\},\ \delta\ ,\ q_0,\ F = \{q_3\}\}$

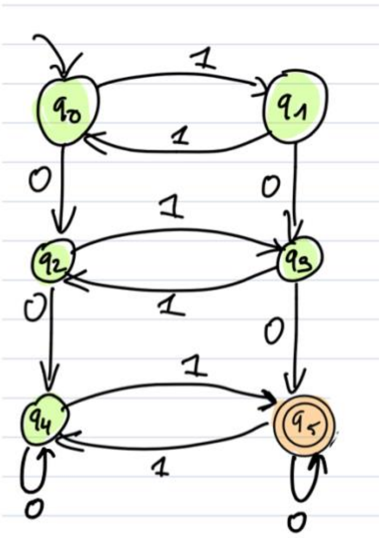
and δ is defined as follows:

| δ | 0 | 1 |
|---|---|---|
| $q_0$ | $q_1$ | $q_4$ |
| $q_1$ | $q_1$ | $q_2$ |
| $q_2$ | $q_2$ | $q_3$ |
| $q_3$ | $q_3$ | $q_3$ |
| $q_4$ | $q_4$ | $q_4$ |

L(A) = $\{w \in \{0,1\}^* |\ w\ starts\ with\ 0,\quad \#_1(w) \geq 2\}$

The language of A is comprised of words that begin with 0, any word beginning with 1 immediately goes to a sink at $q_4$. For words that begin with 0, the words in the language of A must have at least 2 1s in order to be accepted by the accepting state at $q_3$. The states $q_1$ and $q_2$ represent words that begin with 0, but do not have at least two 1s, $q_1$ represents seeing a 0 (at least 1) and it was the beginning, while $q_2$ represents everything that was at $q_1$, and exactly one 1.

b.




A = ( $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\},\ \sigma = \{0,1\},\ \delta\ ,\ q_0,\ F = \{q_5\}\}$

and δ is defined as follows:

| δ | 0 | 1 |
|---|---|---|
| $q_0$ | $q_2$ | $q_1$ |
| $q_1$ | $q_3$ | $q_0$ |
| $q_2$ | $q_4$ | $q_3$ |
| $q_3$ | $q_5$ | $q_2$ |
| $q_4$ | $q_4$ | $q_5$ |
| $q_5$ | $q_5$ | $q_4$ |

L(A) I s defined as follows:
L(A) = $\{w \in \{0,1\}^* |\ \#_0(w) \geq 2,\ \#_1(w) = odd\ \}$

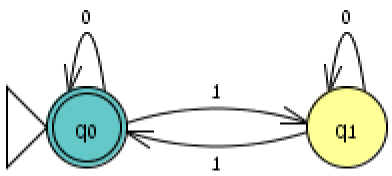
The language of A is comprised of words containing the characters 0 and 1, where the number of 0s in the string is at least 2, and the number of 1s in the string is odd. The starting state is $q_0$, at which point you have seen either no characters or an even number of 1s, at $q_1$ we have seen an odd number of 1s but no 0s, at $q_2$ we have seen exactly one 0, but may also have seen an even number of 1s, at $q_3$ we have seen exactly one 0, and an odd number of 1s, at $q_4$ we have seen at least two 0s, and an even number of 1s (including no 1s), and at $q_5$ we have seen at least two 0s, and an odd number of 1s, at which point we can accept the string. ☺

## Question 4 (30 pts)

The language $L$ is defined as follows:

$L = \{w \in \{0,1\}^* \mid \#_1(w) \; is \; even\}$

And denote the automaton A:




**Claim:** *L=L(A)*

Complete the proof of the above claim.

**<u>Proof</u>**:

Define: $g(w) = \#_1(w) \; mod \; 2$

**Lemma 1:** $g(w) = i \quad \Leftrightarrow \quad \delta'(q_0, w) = q_i$

**Proof (of lemma 1):**

**<u>1st direction</u>**: $\delta'(q_0, w) = q_i \Rightarrow g(w) = i$

We will prove by induction over the length of w:

**Induction basis:**

For $|w| = 0, \; i.e. \; w = \varepsilon$

According to the definition of A: $\delta'(q_0, \varepsilon) = q_0$ and indeed $g(w) = 0$

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**Induction Hypothesis:**

For any $u \in \{0,1\}^*\ s.t.\ |u| = n$ , we assume $\delta'(q_0, u) = q_i \ \Rightarrow\ g(u) = i$

**Induction step:**

we prove the 1st direction for $w = u\sigma\ ,\ where\ \sigma \in \{0,1\}$ and $|u| = n$

According the definition of $\delta'$: $\delta'(q_0, u\sigma) = \delta(\delta'(q_0, u), \sigma) = q_i$

But, $|u| = n,$ and according to the hypothesis of the induction $\delta'(q_0, u) = q_j \ \Rightarrow\ g(u) = j$

The letter $\sigma$ can be 0 or 1, we will check all possible pairs of $\sigma\ , q_j$ :

a. Assume $\delta'(q_0, u) = q_0$ $\Longrightarrow$ (induction hypo.) $g(u) = 0$

   Also, $\delta'(q_0, w) = \delta'(q_0, u\sigma) = \delta(\delta'(q_0, u), \sigma) = \delta(q_0, \sigma)$

   $a.1.\ \ if\ \ \sigma = 0\ \ then\ \delta'(q_0, w) = \delta(q_0, 0) =^A q_0$ and indeed $g(w) = g(u0) = g(u) = 0$

   $a.2.\ \ if\ \sigma = 1\ \ then\ \delta'(q_0, w) = \delta(q_0, 1) =^A q_1$ and indeed $g(w) = g(u1) = (g(u) + 1)\ mod\ 2\ = 1$

b. Assume $\delta'(q_0, u) = q_1$ $\Longrightarrow$ (induction hypo.) $g(u) = 1$

   Also, $\delta'(q_0, w) = \delta'(q_0, u\sigma) = \delta(\delta'(q_0, u), \sigma) = \delta(q_1, \sigma)$

   $b.1.\ \ if\ \ \sigma = 0\ \ then\ \delta'(q_0, w) = \delta(q_1, 0) =^A q_1$ and indeed $g(w) = g(u0) = g(u) = 1$

   b.2. $if\ \sigma = 1\ \ then\ \delta'(q_0, w) = \delta(q_0, 1) =^A q_0$ and indeed $g(w) = g(u1) = (g(u) + 1)\ mod\ 2\ = 0$

Q.E.D. (first direction)

**<u>2nd direction</u>** $g(w) = i \ \Longrightarrow\ \delta'(q_0, w) = q_i$

**Prove the 2nd direction and complete the main claim using Lemma 1.**

Base case:

For $|w| = 0,\ \ i.e.\ \ w = \varepsilon$

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According to the definition of A: we have $g(w) = 0$ and $\delta'(q_0, \varepsilon) = q_0$

Induction hypothesis:

For any $u \in \{0,1\}^*$ $s.t.$ $|u| = n$ , we assume $g(u) = i \Rightarrow \delta'(q_0, u) = q_i$

Induction step:

We prove the 2nd direction for $w = u\sigma$ , $where\ \sigma \in \{0,1\}$ and $|u| = n$

According the definition of $\delta'$: $\delta'(q_0, u\sigma) = \delta(\delta'(q_0, u), \sigma) = q_i$

But, $|u| = n,$ and according to the hypothesis of the induction $g(u) = j \Rightarrow \delta'(q_0, u) = q_j$

The letter $\sigma$ can be 0 or 1, we will check all possible pairs of $\sigma$ , $q_j$ :

a. Assume $g(u) = 0$ $\Longrightarrow$ (induction hypo.) $\delta'(q_0, u) = q_0$

   If $\sigma = 0$, then $g(w) = g(u\sigma) = g(u0) = g(u) = 0$ and $\delta'(q_0, w) = \delta(q_0, 0)$ and indeed $\delta(q_0, 0) = q_0$

   If $\sigma = 1$, then $g(w) = g(u\sigma) = g(u1) = (g(u) + 1)mod2 = 1$, and $\delta'(q_0, w) = \delta(q_0, 1) = q_1$

b. Assume $g(u) = 1$ $\Longrightarrow$ (induction hypo.) $\delta'(q_0, u) = q_1$

   If $\sigma = 0$, then $g(w) = g(u\sigma) = g(u0) = g(u) = 1$, and $\delta'(q_0, w) = \delta(q_1, 0) = q_1$

   If $\sigma = 1$, then $g(w) = g(u\sigma) = g(u1) = (g(u) + 1)mod2 = 0$, and $\delta'(q_0, w) = \delta(q_1, 1) = q_0$

   ☺

To prove **Claim:** *L=L(A), we use* the bidirectional containment argument

⇒ Let w be a word in L, by definition, the number of 1s in w is even, therefore g(w)=0 therefore by Lemma 1, $\delta'(q_0, w) = q_0$, meaning we accept the word and w is in L(A).

⇐ Let w be a word in L(A), then $\delta'(q_0, w) = q_0$, that is we accept the word w, then by Lemma 1 g(w)=0, in which case the number of 1s in the word is even and w is also a word in L.