Automata and Formal Languages, 2021

Exercise 1

Submit by Wednesday 17/03/21

Question 1 (30 pts)

Write the elements of the following sets:

a. $\{a, b, c\} \times \{2,3,4\} = \{(a, 2), (a, 3), (a, 4), (b, 2), (b, 3), (b, 4), (c, 2), (c, 3), (c, 4)\}$

b. $\{b, c, \{a\}, \{a, b\}\} - \{\{b, a\}, a, b, d\} = \{c, \{a\}\}$

c. $2^{\{1,2\}} \times (\{a, \{b, c\}\} \cup 2^{\phi}) = \{\phi, \{1\}, \{2\}, \{1,2\}\} * (\{a, \{b, c\}] \cup \{\phi\}) =$
$\{(\phi, a), (\phi, \{b, c\}), (\phi, \phi), (\{1\}, a), (\{1\}, \{b, c\}), (\{1\}, \phi), (\{2\}, a),$

d. $(\{2\}, \{b, c\}), (\{2\}, \phi), (\{1,2\}, a), (\{1,2\}, \{b, c\}), (\{1,2\}, \phi)\}$

e. $\{\{a, b, c, d\}\} - \{a, b, c, d\} = \{\{a, b, c, d\}\}$

f. $2^{\{a,b,c\}} - 2^{\{c,d\}} = \{\phi, (a), (b), (c), (a, b), (a, c), (b, c), (a, b, c)\} -$
$\{\phi, (c), (d), (c, d)\} = \{(a), (b), (a, b), (a, c), (b, c), (a, b, c)\}$

g. $2^{\phi} = \{\phi\}$

h. $2^{2^{\phi}} = \{\phi, \{\phi\}\}$

i. $\{a, b, \phi\}^2 = \{a, b, \phi\} * \{a, b, \phi\} =$
$\{(a, a), (a, b), (a, \phi), (b, a), (b, b), (b, \phi), (\phi, a), (\phi, b), (\phi, \phi)\}$

j. $\{1\} \times \phi \times \{1,2\} = \phi$

k. $\{a, b\} \times \{\phi\} = \{(a, \phi), (b, \phi)\}$

Question 2 (10 pts)

The following theorem is proven by a bi-directional inclusion. See below a proof for the 2nd direction. Provide the 1st direction –

Prove: $(L_1 \cup L_2)^* = (L_1^* \circ L_2^*)^*$

Proof: We prove by a bi-directional inclusion

2nd direction: $(L_1^* \circ L_2^*)^* \subseteq (L_1 \cup L_2)^*$:

Let $w \in (L_1^* \circ L_2^*)^*$.

This gives that w can be decomposed into $w = w_1 w_2 \dots w_k$

Where each $w_i \in L_1^* \circ L_2^*$. But each w_i can be decomposed again into $w_i = u_i v_i,\ u_i \in L_1^*$, $v_i \in L_2^*$

That gives that $\forall i\ \exists\ i_k \geq 0\ \ s.t\ \ u_i = u_{i_1} u_{i_2} \dots u_{i_k}\ \ where\ each\ u_{i_j} \in L_1$

Similarly, $\forall i\ \ \exists\ i_l \geq 0\ \ s.t\ \ v_i = v_{i_1} v_{i_2} \dots v_{i_l}\ \ where\ each\ v_{i_j} \in L_2$

Therefore, $\exists n\ s.t\ \ w = z_1 z_2 z_3 \dots . z_n$ where each $z_i \in L_1$ or $z_i \in L_2$

This gives (by the definition of Kleen Star) that $w \in (L_1 \cup L_2)^*$ q.e.d.

1st direction: $(L_1 \cup L_2)^* \subseteq (L_1^* \circ L_2^*)^*$: **provide a proof**.

Let $w \in (L_1 \cup L_2)^*$, we will show that $w \in (L_1^* \circ L_2^*)^*$. We decompose w into $w = k_1 k_2 \dots k_i$, and each $w_i \in L_1 \cup L_2$. As we get that $w_i \in L_1\ or\ \ w_i \in L_2$ (this follows from the definition of union and closure of Kleen). In the case where $w_i \in L_2$, we have $w_i \in L_2^*$, by the definition of Kleen Star. Since we know that we have $\varepsilon \in L_1^*$ (by the definition), we see that $w_i = \varepsilon w_i \in L_1^* \circ L_2^*$. Similarly, we know that if $w_i \in L_1$, then $w_i \in L_1^*$, by definition. And from the definition of a Kleen star we also have that $\varepsilon \in L_2^*$, and we therefore know that $w_i = w_i \varepsilon \in L_1^* \circ L_2^*$. Therefore, $\exists k\ s.t\ \ w = w_1 w_2 w_3 \dots . w_k$ where each $w_i \in L_1^* \circ L_2^*$. So finally we get by the closure of Kleen Star that $w \in (L_1^* \circ L_2^*)^*$

Question 3 (20 pts)

Prove (by a formal and complete proof) or disprove (showing a counter example) the following claims:

a. If $L_1 \subseteq L_2\ \ then\ \ L_1^* \subseteq L_2^*$

True. To show this, let $L_1 \subseteq L_2$, and let w be a word in L_1^*. From the definition of closure of Kleen, we get that $w = w_1 w_2 \dots w_k$ where we have $w_i \in L_1$ for $1 \leq i \leq n$. Now for $w_i \in L_2$ for $1 \leq i \leq n$, by the same definition of Kleen closure, we get that $w \in L_2^*$. Therefore: $L_1^* \subseteq L_2^*$ by definition as required.

b. $(L_1^+ \circ L_2^+) \subseteq (L_1 \circ L_2)^+$

False.

Let $L_1 = \{a\}$, $L_2 = \{b\}$. Let $w = aabbb$, then we have that $w \in L_1^+ \circ L_2^+$, but we can see that $w \notin (L_1 \circ L_2)^+$, since $aa \notin L_1$.Therefore, $(L_1^+ \circ L_2^+) \nsubseteq (L_1 \circ L_2)^+$.

c. $L_1 \cap (L_2 \circ L_3) = (L_1 \cap L_2) \circ L_3$

False: Let $L_1 = \{ab\}$, $L_2 = \{a\}$, $L_3 = \{b\}$. We can see that we have $L_1 \cap (L_2 \circ L_3) = \{ab\} \cap \{ab\} = \{ab\}$, however, we can also see that $(L_1 \cap L_2) \circ L_3 = (\{ab\} \cap \{a\}) \circ \{b\} = \emptyset \circ \{b\} = \emptyset$. As $\emptyset \neq \{ab\}$ trivially, we get that in fact $L_1 \cap (L_2 \circ L_3) \neq (L_1 \cap L_2) \circ L_3$.

d. $(L_1 \circ L_2)^R = {L_2}^R \circ {L_1}^R$

True, by the definition of reverse words.

For the first direction: let $w \in (L_1 \circ L_2)^R$. We can decompose w into $w = (a \circ b)^R$, where $a \in L_1$ and $b \in L_2$. We can further decompose $a = a_1 a_2 \ldots a_k, b = b_1 b_2 \ldots b_l$. Where $a \circ b = a_1 a_2 \ldots a_k b_1 b_2 \ldots b_l$, therefore $w = b_l \ldots b_2 b_1 \circ a_k \ldots a_2 a_1 = b^R \circ a^R$

Therefore we showed that $(L_1 \circ L_2)^R \subseteq {L_2}^R \circ {L_1}^R$

For the second direction: let $w \in {L_2}^R \circ {L_1}^R$. We can decompose $w = b^R \circ a^R$ where $a \in L_1$ and $b \in L_2$. Therefore we have that $w = b_l \ldots b_2 b_1 a_k \ldots a_2 a_1 = b^R \circ a^R$. If a and b are decomposed as above then we have that $a = a_1 a_2 \ldots a_k, b = b_1 b_2 \ldots b_l$. As $a \circ b = a_1 a_2 \ldots a_k b_1 b_2 \ldots b_l$, trivially $(a \circ b)^R = b_l \ldots b_2 b_1 \circ a_k \ldots a_2 a_1$. As we have that $b_l \ldots b_2 b_1 \circ a_k \ldots a_2 a_1 = b_l \ldots b_2 b_1 a_k \ldots a_2 a_1$ and $w = b_l \ldots b_2 b_1 a_k \ldots a_2 a_1$, then $w = b_l \ldots b_2 b_1 \circ a_k \ldots a_2 a_1$ so $w \in (L_1 \circ L_2)^R$ we can see that ${L_2}^R \circ {L_1}^R \subseteq (L_1 \circ L_2)^R$.

e. $(L_1 \cup L_2)^* = (L_1^* \circ L_2)^* \cup (L_1 \circ L_2^*)^*$

Counterexample - we can define the following:

$$\Sigma = \{0,1\} \quad L_1 = \{0\} \quad L_2 = \{1\} \quad w = 10$$

We have on the left-hand side that $w \in (L_1 \cup L_2)^* = \{0,1\}$, but according to the right-hand side, w cannot be contained in $(L_1^* \circ L_2)^* \cup (L_1 \circ L_2^*)^*$ since one expression cannot end with 0, and one cannot start with 1, so therefore there is no such equality and w does not belong.

Question 4 (20 pts)

Show concrete examples for the following languages:

a. Languages L_1 and L_2 over {a,b} where $(L_1 \cup L_2)^* \neq {L_1}^* \cup {L_2}^*$

Let $L_1 = \{a\}$, $L_2 = \{b\}$, for the word $w = abab$, $w \in (L_1 \cup L_2)^*$ but we have that $w \notin L_1{}^* \cup L_2{}^*$.

b. Two infinite languages L_1 and L_2 where $L_1 \not\subseteq L_1 \circ L_2$ and $L_2 \not\subseteq L_1 \circ L_2$

Let $L_1 = \{a\}^+$, $L_2 = \{b\}^+$, we define the words $w_1 = aa$ and $w_2 = bb$, $w_1 \in L_1$ however, we can see that $w_1 \notin L_1{}^\circ L_2$, which means that trivially $L_1 \not\subseteq L_1 \circ L_2$, by definition, similarly we can see that $w_2 \in L_2$ but that $w_2 \notin L_1{}^\circ L_2$, which means that we have $L_2 \not\subseteq L_1 \circ L_2$, by definition.

c. Languages $L_1$and L_2 over {a,b} where $\overline{L_1 \cup L_2} \neq \overline{L_1} \cup \overline{L_2}$

Let $L_1 = \{a\}$, and $L_2 = \{b\}$, and let our word be $w = b$, then we have that $\overline{L_1 \cup L_2} = \overline{\{a\} \cup \{b\}} = \overline{\{a, b\}} = \{a, b\}^* - \{a, b\}$, and we can see that $\overline{L_1} \cup \overline{L_2} = \overline{\{a\}} \cup \overline{\{b\}} = \{a, b\}^*$, so we get that $w \in \overline{L_1} \cup \overline{L_2}$ but $w \notin \overline{L_1 \cup L_2}$.

d. Language L over {a,b} where $L \neq L \circ L$, but $L^* = L \circ L$

Let $L = \{\varepsilon, a, aaa \dots\}$, then $L \circ L = \{\varepsilon, a, aa, aaa, \dots\} = L^*$. But we can see then that $L \neq L \circ L = L^*$

Question 5 (20 pts)

Given are the following languages:

$L_1 = \{\varepsilon\}$; $L_2 = \phi$; $L_3 = \{ab, b, a\}$; $L_4 = \{aa, \varepsilon\}$; $L_5 = \{\varepsilon, bb, aa, b\}$

Define the languages below. No need to explain or prove but give the simplest expression (for example, if $L = \{1{,}10\}$ and $L' = \{0\}$ and $L'' = (L \cup L')^*$ then $L'' = \{0{,}1\}^*$ is a simpler expression than $L'' = \{0{,}1{,}10\}^*$)

a. $L_1 \circ L_5 = \{\varepsilon\} \circ \{\varepsilon, bb, aa, b\} = \{\varepsilon, bb, aa, b\}$

b. $L_2 \circ L_3 = \phi \circ \{ab, b, a\} = \phi$

c. $L_2^* \circ L_5 = \{\varepsilon\} \circ \{\varepsilon, bb, aa, b\} = \{\varepsilon, bb, aa, b\}$

d. $\mathrm{L}_3^* - L_5 = \{a, b\}^+ - \{bb, aa, b\}$

e. $(L_3 \cup L_5)^R = \{\varepsilon, a, b, ab, aa, bb\}^R = \{\varepsilon, a, b, ab, aa, bb\}$