Introduction to Property Testing

Lecture 6

Lecture by Dr. Reut Levi
Typeset by Steven Karas

2020-05-27
Last edited 18:10:46 2020-05-27

Disclaimer These notes are based on the lectures for the course Introduction to Property Testing, taught by Dr. Reut Levi at IDC Herzliyah in the spring semester of 2019/2020. Sections may be based on the lecture slides prepared by Dr. Reut Levi.

1 Agenda

- Recap of last week

An extension was granted for homework #2 until Sunday. The next homework will be due in 2 weeks from today: exercises 8.5 and 8.7.

2 Testing Bipartite-ness

We define that an edge disturbs (u_1, u_2) if both endpoints are in the same set $N(u_i)$ for some $i \in 1, 2$.

Claim 2 For any good U and any 2-partition of U at least $\frac{\varepsilon k^2}{6}$ edges disturb (u_1, u_2).

G is ε-far from bipartite thus each 2-partition of $[k]$ has at elast $\frac{\varepsilon k^2}{6}$ violating edges. This is also true for $V_1' = N(U_2)$ and $V_2' = [k] \setminus V_1'$

There is a bound on the number of edges that have an endpoint that is not in $N(u)$: the number of edges incident to high-degree vertices not in $N(u)$. Recall that u is good. u is good defined such that all but $\leq \frac{\varepsilon k}{6}$ high degree vertices are in $N(u)$.

The number of edges incident at vertices that are not high degree $\leq k\frac{\varepsilon k}{6}$. Thus, $\leq \frac{\varepsilon k^2}{3}$ edges that do not have both endpoints in $N(u)$. Thus, $\geq \frac{\varepsilon k^2}{2} - \frac{\varepsilon k^2}{3} = \frac{\varepsilon k^2}{6}$ violating edges with respect to (v_1', v_2') with both endpoints in $N(u)$. These edges disturb (u_1, u_2) since $V_1' \cap N(u) = N(u_2)$ and $V_2' \cap N(u) \subseteq N(u_1)$.

Proof follows from $G([R])$ is bipartite only if either:

1. u is not good
2. u is good but there exists a 2-partition of u_1 such that none of the edges disturbing it appear in $G([R])$.

By claim 1, $\Pr[\text{Event } 1] \leq 1/6$, and $\Pr[\text{Event } 2] \leq \Pr[\exists$ 2-partition of u such that none of the disturbing edges are i
By claim 2, each 2-partition of u has $\geq \frac{\varepsilon k^2}{6}$ disturbing edges.

Pair the m vertices of S into $\frac{m}{2}$ pairs. By union bound over 2^t 2-partitions of u:

$$\Pr[\text{Event } 2] \leq 2^t \cdot \left(1 - \frac{\varepsilon k^2/6}{k^2/2}\right)^{m/2} < \frac{1}{6}$$

because $m = \Omega(t/\varepsilon)$ and $\left(1 - \frac{1}{x}\right)^x < \frac{1}{e}$.

3 External Graph Theory

This theory concerns itself with the effects of local properties on global properties of graphs.

3.1 Testing subgraph freeness

We will focus on 3-clique freeness (triangles).

3.1.1 Szemerédi's regularity lemma

Notation:

If A, B are disjoint we denote $E(A, B)$ as the edges with one endpoint in A and the other in B. If $A, B \subseteq V$ are disjoint and nonempty, we define the edge-density of (A, B) as $d(A, B) \stackrel{\text{def}}{=} \frac{E(A,B)}{|A|\cdot|B|}$.

We say that (A, B) is γ-regular if for every $A' \subseteq A; B' \subseteq B$ such that $|A'| \geq \gamma|A|$ and $|B'| \geq \gamma|B|$ it holds that $|d(A', B') - d(A, B)| \leq \gamma$.

For every $l \in \mathbb{N}$ and $\gamma > 0$, there exists $T = T(l, \gamma)$ such that for every sufficiently large $G = (V, E)$, there exists $t = [l, T]$ and t-partition of V, denoted $V_1, \ldots, V_t$ such that:

1. for all $i \in [t]$, it holds that:
$$\left\lfloor \frac{|V|}{t} \right\rfloor \leq |V_i| \leq \left\lceil \frac{|V|}{t} \right\rceil$$
2. for all but at most γ-fraction of $\{i, j\} \in \binom{[t]}{2}$, it holds that v_i, v_j is γ-regular.

Theorem: There are at least $\rho(\varepsilon) \cdot k^3$ triangles in a graph which is ε-far from being triangle free, where k is the number of vertices in the graph.

4 Next week

We will prove this theorem next week.

References

[1] Oded Goldreich. *Introduction to property testing.* Cambridge University Press, 2017.