CS120: Intro. to Algorithms and their Limitations Hesterberg & Vadhan

Sender–Receiver Exercise 4: Reading for Receivers

Harvard SEAS - Fall 2022 2022-11-01

The goals of this exercise are:

- to develop your skills at understanding, distilling, and communicating proofs and the conceptual ideas in them,
- to practice reductions to SAT, and in particular how logic is useful for modelling problems

To prepare for this exercise as a receiver, you should try to understand the theorem statement and definition in Section 1 below, and review the material on Logic and Satisifiability covered in class on October 28 and November 2. Your partner sender will communicate the proof of Theorem 1.1.

1 The Result

In class, we saw how the (seemingly hard) problem of Graph k-Coloring can be efficiently reduced to CNF-Satisfiability (SAT). Although SAT also seems to be a hard problem (as we'll formalize in the last part of the course), this allowed all the effort put into SAT Solvers to solve many large k-coloring instances in practice.

In this exercise, you'll see a similar reduction for the *Longest Path* problem.

Input : A digraph G = (V, E) and two vertices $s, t \in V$

Output: A longest path from s to t in G with no repeated vertices, or

 \perp if no path from s to t exists

Computational Problem LongestPath

Actually, it will be more convenient to consider a version where the desired path length is specified in the input.

Input : A digraph G = (V, E), two vertices $s, t \in V$, and a path-length $k \in \mathbb{N}$

Output: A path from s to t in G of length k and no repeated vertices, or

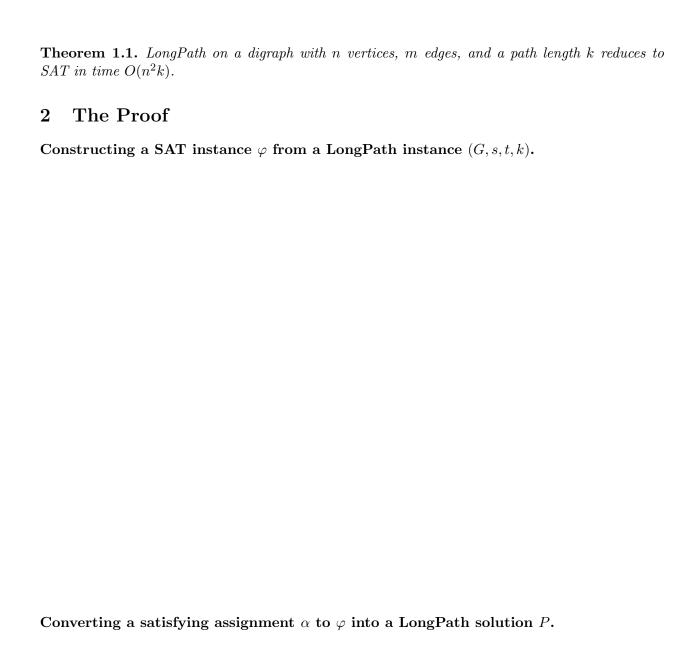
 \perp if no such path from s to t exists

Computational Problem LongPath

If we have an efficient algorithm for LongPath, then we can solve LongestPath by trying $k = n, n-1, \ldots, 0$ until we succeed in finding a path. The k=n case is known as the Hamiltonian Path problem, which is a special case of the notorious Travelling Salesperson Problem (TSP). In the TSP, we have a salesperson who wishes to visit n cities (to sell their goods) in the shortest travel time possible. If we model the possible travel between cities as a directed graph, then Hamiltonian Path corresponds to the special case where all pairs u, v of cities either have a travel time of 1 (edge (u, v) present) or a very large travel time (edge (u, v) not present). In such a case, the only way to visit all cities in travel time at most n-1 is via a Hamiltonian Path. ¹

The reduction from LongPath to SAT is given as follows.

 $^{^{1}}$ Often in the TSP, it is also required that the salesperson return back to their starting city s. If we add edges of travel time 1 from all cities to s, then we see that Hamiltonian Path is also a special case of this variant of the TSP.



Correctness of the Reduction.

Runtime of the Reduction.