**CS120: Intro. to Algorithms and their Limitations** Anshu & Hesterberg

# Lecture 20: NP and NP-completeness

*Harvard SEAS - Fall 2023* *2023-11-14*

## 1 Announcements

- PS7 due Nov 15
- PS8 out Nov 14
- Next SRE on Thu Nov 16

Recommended Reading:

- MacCormick §14, 17

## 2 Recap

**Definition 2.1.** A computational problem $\Pi = (\mathcal{I}, \mathcal{O}, f)$ is in $\mathsf{NP}_{\mathsf{search}}$ if the following conditions hold:

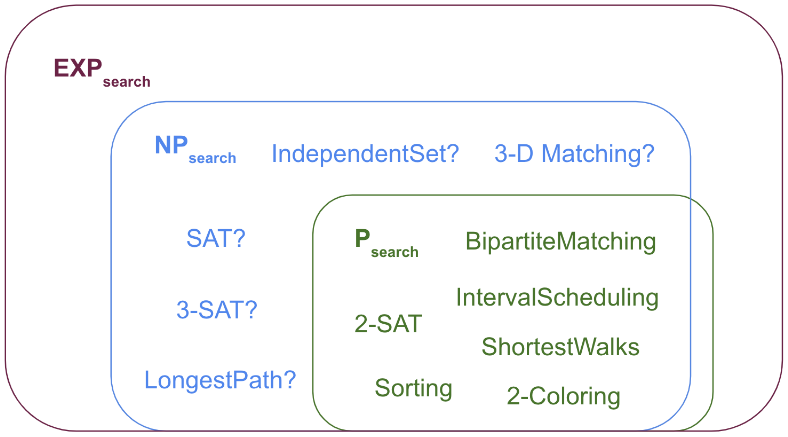
1. All solutions are of polynomial length: There is a polynomial $p$ such that for every $x \in \mathcal{I}$ and every $y \in f(x)$, we have $|y| \leq p(|x|)$, where $|z|$ denotes the bitlength of $z$.
2. All solutions are verifiable in polynomial time: There's a polynomial-time verifier $V$ that, given $x \in \mathcal{I}$ and a potential solution $y$, decides whether $y \in f(x)$.

Examples are - Satisfiability, Graph Coloring.
Every problem in $\mathsf{NP}_{\mathsf{search}}$ can be solved in exponential time:

**Proposition 2.2.** $\mathsf{NP}_{\mathsf{search}} \subseteq \mathsf{EXP}_{\mathsf{search}}$.

*Proof.* Exhaustive search! We can enumerate over all possible solutions and check if any is a valid solution. This has runtime $O(2^{p(n)} \cdot (n + p(n))^c)$ which is bounded by the exponential $O(2^{n^d})$. □

Our diagram of complexity classes looks like this:



**Definition 2.3** (NP-completeness, search version)**.** A problem $\Pi$ is $\mathsf{NP}_{\mathsf{search}}$-complete if:

1. $\Pi$ is in $\mathsf{NP}_{\mathsf{search}}$
2. $\Pi$ is $\mathsf{NP}_{\mathsf{search}}$-*hard*: For every computational problem $\Gamma \in \mathsf{NP}_{\mathsf{search}}$, $\Gamma \leq_p \Pi$.

There are natural NP-complete problems. The first one is CNF-Satisfiability:

**Theorem 2.4** (Cook–Levin Theorem)**.** *SAT is* $\mathsf{NP}_{\mathsf{search}}$*-complete.*

Once we have one $\mathsf{NP}_{\mathsf{search}}$-complete problem, we can get others via reductions from it.

## 3 Mapping Reductions

The usual strategy for proving that a problem $\Gamma \in \mathsf{NP}_{\mathsf{search}}$ is also $\mathsf{NP}_{\mathsf{search}}$-hard (and hence $\mathsf{NP}_{\mathsf{search}}$-complete) follows a standard structure:

This is referred to as a *mapping reduction* and used widely in the theory of NP-completeness.

Note that the above outline only proves $\mathsf{NP}_{\mathsf{search}}$-hardness; a proof that a problem is $\mathsf{NP}_{\mathsf{search}}$-complete should also check that it's in $\mathsf{NP}_{\mathsf{search}}$.

## 4 3-SAT is $\mathsf{NP}_{\mathsf{search}}$-complete.

We revisit the theorem from Lec 19:

**Theorem 4.1.** *3-SAT is* $\mathsf{NP}_{\mathsf{search}}$*-complete.*

*Proof.* 1. 3SAT is in $\mathsf{NP}_{\mathsf{search}}$.

2. 3SAT is $\mathsf{NP}_{\mathsf{search}}$-hard: Since every problem in NP reduces to SAT, all we need to show is SAT $\leq_p$ 3SAT (since reductions are transitive).

For part (2) we provide a reduction R:

$$\text{SAT instance } \varphi \xrightarrow{\text{polytime R}} \text{3SAT instance } \varphi'$$

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1 R(φ) :
  Input   : A CNF formula $\varphi$
  Output  : A 3-CNF formula $\varphi'$
2 $\varphi' = \varphi$
3 while $\varphi'$ has a clause $C = (\ell_0 \vee \ldots \vee \ell_{k-1})$ of length $k > 3$ do
4     Remove $C$
5     Add clauses
6 return $\varphi'$
```

*R* **runs in polynomial time:** At each iteration of the while loop, we take a clause of length $k$ and produce clauses of length 3 and $k-1$. Thus, the total length of a too-large clause goes down by 1 at each step. The number of iterations is bounded by $\sum_{C \in \varphi, |C|>3} |C| \leq nm$ where $|C|$ is the width of the clause.

**Claim 4.2.** *If $\varphi$ is satisfiable then $\varphi' = R(\varphi)$ is satisfiable.*

*Proof of claim.* Assume that $\varphi$ is satisfiable. Let $\varphi = \varphi_0, \varphi_1, \ldots, \varphi_t = R(\varphi)$ be the formula $\varphi'$ as it evolves through the $t$ loop iterations. We will prove by induction on $i$ that $\varphi_i$ is satisfiable for $i = 0, \ldots, t$. constructed through the $t$ loop iterations.

**Base case ($i = 0$):**

**Induction step:** By the induction hypothesis, we can assume that $\varphi_{i-1}$ is satisfiable, and now we need to show that $\varphi_i$ is satisfiable:

□

Finally, we need to show we can transform a satisyfing assignment $\alpha'$ to $\varphi'$ into a satisfying assignment $\alpha$ to $\varphi$. Our transformation simply discards all introduced dummy $y$ variables and takes the assignment to the variables originally in $\varphi$.

**Claim 4.3.** *If $\alpha'$ satisfies $R(\varphi)$, then $\alpha'|_\varphi$ also satisfies $\varphi$, where $\alpha'|_\varphi$ is the restriction of the assignment $\alpha'$ to the variables in $\varphi$.*

*Proof of claim.* We prove by "backwards induction" that $\alpha'$ satisfies $\varphi_i$ for $i = t, \ldots, 0$. We can then drop the extra $t$ variables that don't appear in $\varphi$ without changing the satisfiability. (We call this "backwards induction" since our base cases is $i = t$.)

The base case ($i = t$) follows because $\alpha'$ satisfies $R(\varphi) = \varphi_t$ by assumption.

For the induction step:

□

This completes the proof that 3-SAT is $\mathsf{NP}_{\mathsf{search}}$-complete. □

## 5 Independent Set is $\mathsf{NP}_{\mathsf{search}}$-complete

Next we turn to IndependentSet. (Formally the IndependentSet-ThresholdSearch version.)

**Theorem 5.1.** *IndependentSet is $\mathsf{NP}_{\mathsf{search}}$-complete.*

*Proof.* We'll do this proof less formally than we did the proof of $\mathsf{NP}_{\mathsf{search}}$-completeness of 3SAT.

1. In $\mathsf{NP}_{\mathsf{search}}$:

2. $\mathsf{NP}_{\mathsf{search}}$-hard: We will show 3SAT $\leq_p$ IndependentSet.

We've previously encoded many other problems in SAT, but here we're going in the other direction and showing a graph problem can encode SAT.

Our reduction $R(\varphi)$ takes in a CNF and produces a graph $G$ and a size $k$. We'll use as an example the formula

$$\varphi(z_0, z_1, z_2, z_3) = (\neg z_0 \vee \neg z_1 \vee z_2) \wedge (z_0 \vee \neg z_2 \vee z_3) \wedge (z_1 \vee z_2 \vee \neg z_3).$$

Our graph $G$ consists of:

- Variable gadgets:

- Clause gadgets:

We pick $k = m + n$. An algorithm $R$ can create this graph (and $k$) in polynomial time given $\varphi$. The graph for the formula $\varphi$ is depictedin Figure **??**.

Note that (analogously to the SAT to 3SAT case) the correspondence between 3SAT and ISET does not exactly preserve the set of satisfying solutions (they aren't even the same problem) but we can go from solutions to one to solutions to the other- see Figure **??**.

Remember that an IndependentSet problem consists of 1) a graph $G$ and 2) a minimum size $k$ of an independent set. How can we choose the size $k$ for this reduction? Intuitively, we might think about assigning True to the variables whose corresponding vertices are selected as part of the independent set. Then, we'll choose $k = n + m$, where $n$ is the number of variables and $m$ is the number of clauses in the original Boolean formula. The hope is that the clause edges will force exactly $n$ of the variable gadgets to be set to True, and at least one vertex in each of the $m$ clause gadget is also True. We'll now prove that this claim is true.

**Claim 5.2.** *$G$ has an independent set of size $k = n + m$ if and only if $\varphi$ is satisfiable. Moreover, we can map independent sets of size $k$ to satisfying assignments of $\varphi$ in polynomial time.*

*Proof of claim.*

□

This completes the proof that IndependentSet is $\mathsf{NP}_{\mathsf{search}}$-complete. □

## 6 Optional read: Three-Dimensional Matching

A month ago, we saw algorithms to find maximum matchings in a bipartite graph: that is, given a graph whose vertices are in two sets $V_0$ and $V_1$, and a set $E$ of edges each of which contains exactly one vertex from $V_0$ and one vertex from $V_1$, we can find (in polynomial time) a maximum-size matching, a subset of $E$ in which no edges overlap (no edges share an endpoint).

Just as changing 2SAT to 3SAT turns a polynomial-time solvable problem $\mathsf{NP}_{\mathsf{search}}$-complete, we'll see now that changing "two" to "three" makes that efficiently solvable problem $\mathsf{NP}_{\mathsf{search}}$-complete. (A similar example which we won't prove in CS 120: changing 2-coloring to 3-coloring also turns a polynomial-time solvable problem $\mathsf{NP}_{\mathsf{search}}$-complete). We note here that a "hypergraph" is like a graph, but its (hyper)edges can consist of any number of vertices (not necessarily exactly two).

| | |
|---|---|
| **Input** | **:** A hypergraph $G = (V, E)$ where $V$ is partitioned into three sets $V_0$, $V_1$, and $V_2$, and each hyperedge contains exactly three vertices, one from each of $V_0$, $V_1$, and $V_2$. |
| **Output** | **:** A set of hyperedges which are disjoint and cover all the vertices, if one exists. |

**Computational Problem** ThreeDimensionalMatching

**Theorem 6.1.** *ThreeDimensionalMatching (AKA 3DM) is* $\mathsf{NP}_{\mathsf{search}}$*-complete.*

*Proof.* There are two requirements for a problem to be $\mathsf{NP}_{\mathsf{search}}$-complete: (1) it's in $\mathsf{NP}_{\mathsf{search}}$ (2) other problems in $\mathsf{NP}_{\mathsf{search}}$ reduce to it in polynomial time.

**$\mathsf{NP}_{\mathsf{search}}$ membership of 3DM.** To show that ThreeDimensionalMatching is in NP, we need to show that there exists a polynomial-time algorithm that, given a potential solution $y$ (that is, a set of triples of vertices denoting a hyperedge), checks whether it's a set of hyperedges which is disjoint and covers all the vertices.

To do so,

**$\mathsf{NP}_{\mathsf{search}}$-hardness of 3DM: reduction from 3SAT** To show that ThreeDimensionalMatching is $\mathsf{NP}_{\mathsf{search}}$-hard, we need to show that every problem in $\mathsf{NP}_{\mathsf{search}}$ reduces to it in polynomial time. We'll again use the fact that every problem in $\mathsf{NP}_{\mathsf{search}}$ reduces to 3SAT in polynomial time, so if we can reduce from $3SAT$ to ThreeDimensionalMatching, transitivity of polynomial-time reductions means that every problem in $\mathsf{NP}_{\mathsf{search}}$ reduces to ThreeDimensionalMatching in polynomial time. We'll use the reductions framework mentioned earlier: we'll take an input to 3SAT (that is, a 3CNF formula), make it an input to 3DM (that is, a graph), call a 3DM oracle, and use the resulting matching to make a satisfying assignment to 3SAT.

The full reduction is complicated (see the end of these notes), so we'll work our way up to it, building 3DM gadgets that simulate gradually more of 3SAT.

**Simple variable gadget:** In 3SAT, each variable can be set to true or false. To simulate that in 3DM, we need some ability to make a binary choice.

**Expanded variable gadget** In the simple variable gadget, the rest of the graph we're constructing can only interface with the gadget at two vertices, $v_2$ and $v_3$. It may be useful to have more than two vertices to interface with, so we create a bigger variable gadget:

**Assignment gadget:** An assignment to 3SAT consists of an assignment of all $n$ variables, each of which can be independently set either true or false. To simulate this, we

**Clause gadget** A clause like $C = (\neg z_{120} \vee z_{121} \vee \neg z_{124})$ can be satisfied in at most[1] three ways: in that case, by having $z_{120}$ set false, by having $z_{121}$ set true, or by having $z_{124}$ set false. To simulate this, we

**Cleanup gadget** The gadgets above guarantee that if the formula is not satisfiable, there's no matching that covers all the vertices. In the other direction, we have some cleanup to do: if the formula is satisfiable, our described use of the gadgets covers most of the vertices: the clause vertices $u_{j,k}$ and some of the variable-gadget vertices $v_{i,j,0}$ and $v_{i,j,1}$. However, vertices $v_{i,j,2}$ and $v_{i,j,3}$ may not be covered yet, even if we have a valid solution to 3SAT: maybe variable $i$ isn't used in clause $j$, or clause $j$ had more than one true literals so a perfect matching doesn't use some vertices corresponding to some literals that satisfied it. To use up any extra vertices $v_{i,j,2}$ and $v_{i,j,3}$, we

[1]A 3SAT clause may have fewer than three literals.

So, all together, the reduction is:

- Given a 3SAT problem $\varphi$ with $n$ variables $z_0, z_1, \ldots, z_{n-1}$ that are used in clauses[2] and $m$ clauses $C_0, \ldots, C_{m-1}$, we'll make a graph $R(\varphi)$ with $12nm + 6m$ vertices.
  - Name $12nm$ of the vertices $v_{i,j,k,\ell}$, where $i \in [n]$, $j \in [m]$, $k \in [4]$, and $\ell \in [3]$.
  - Name the other $6m$ vertices $u_{j,k,\ell}$ where $j \in [m]$, $k \in [2]$, and $\ell \in [3]$.
- We include the following hyperedges:
  - For each $i \in [n]$, $j \in [m]$, and $\ell \in [3]$, add the hyperedge $(v_{i,j,0,\ell}, v_{i,j,1,\ell}, v_{i,j,2,\ell})$. (Call these "True hyperedges".)
  - For each $i \in [n]$, $j \in [m]$, and $\ell \in [3]$, add the hyperedge $(v_{i,j+1,0,\ell}, v_{i,j,1,\ell}, v_{i,j,3,\ell})$. (Call these "False hyperedges".) Consider $j \bmod m$: that is, $j + 1$ should wrap back around to 0.
  - For each $i \in [n]$, $j \in [m]$, and $k \in \{2, 3\}$, add the hyperedge $(v_{i,j,k,0}, v_{i,j,k,1}, v_{i,j,k,2})$. (Call these "cleanup hyperedges".)
  - For each $j \in [m]$ and $\ell \in [3]$ and positive literal $x_i \in C_j$, add the hyperedge $(u_{j,0,\ell}, u_{j,1,\ell}, v_{i,j,3,\ell})$. (Call these "positive clause-satisfying edges".)
  - For each $j \in [m]$ and $\ell \in [3]$ and negative literal $\neg x_i \in C_j$, add the hyperedge $(u_{j,0,\ell}, u_{j,1,\ell}, v_{i,j,2,\ell})$. (Call these "negative clause-satisfying hyperedges".)
- After we generate the graph $R(\varphi)$ as above, call the 3DM oracle on it. If it returns $\perp$, return $\perp$. If it returns a 3DM, assign each variable $z_i$ to be true if the hyperedge $(v_{i,0,0,0}, v_{i,0,1,0}, v_{i,0,2,0})$ was picked, and false otherwise.

**3-partition** Note that the definition of 3DM requires the graph's vertices to be divisible into three sets, where each hyperedge contains one vertex from each set. This is true for the graph we've constructed by putting the vertices $v_{i,j,k,\ell}$ or $u_{j,k,\ell}$ where $\ell + \min(k, 2) \in \{0, 3\}$ into one set $V_0$, the vertices where $\ell + \min(k, 3) \in \{1, 4\}$ into another set $V_1$, and the vertices where $\ell + \min(k, 3) = 2$ into another set $V_2$. You can check that every hyperedge defined in the reduction has one of each.

**$\mathsf{NP}_{\mathsf{search}}$-hardness of 3DM: runtime of the reduction** The reduction from 3SAT to 3DM is an algorithm that takes as input a 3SAT formula $\varphi$ and produces a hypergraph $R(\varphi)$ as above, then does some faster steps (calls the oracle and reads out an answer). To produce the graph, the algorithm does nothing more complicated than run some loops (over $i \in [n]$, $j \in [m]$, etc., or over clauses in the input), adding one thing to the graph in each instance of the loop, so the runtime of the algorithm is just proportional to the size of the graph it outputs.

That graph has size polynomial in the size of the input: the size of the input formula is $\Theta(m)$, and the size of the produced graph is $O(nm)$. Since we threw out variables not in any clauses, $n < m$ so $O(nm) = O(m^2)$, so the runtime is $O(m^2)$.

[2]If any variables are not used in any clauses, ignore them: they can be set arbitrarily.

**$\mathsf{NP}_{\mathsf{search}}$-hardness of 3DM: proof of correctness of the reduction** As we built up the reduction gadget by gadget, we proved properties of each gadget which, combined, constitute a proof of correctness. However, we'll write a proof of correctness here separately for two reasons:

1. To make clear the distinction between a reduction (just the algorithm described in bullet points above) and a proof of correctness (statements like "a perfect matching must/can pick certain edges").

2. To see how all the things we proved about the gadgets fit together into a full proof of correctness.

To prove the reduction is correct, we need to prove it's correct on all inputs: that is, for every input 3SAT formula $\varphi$ that's unsatisfiable, the output is $\perp$, and for every input $\varphi$ that's satisfiable, the output of the reduction is a satisfying assignment. When a proof of correctness is divided into those two pieces, they're called "soundness" and "completeness", respectively. In the reductions framework from the start of lecture, Item **??** is a proof of completeness and Item **??** is a proof of soundness.

**$\mathsf{NP}_{\mathsf{search}}$-hardness of 3DM: proof of soundness of the reduction** To prove soundness, we need to prove that if $\varphi$ is unsatisfiable, the reduction returns $\perp$. It's easier (here and often) to prove the equivalent contrapositive statement: if the reduction returns an assignment, then it satisfies the 3SAT formula. If the reduction returns an assignment, it did so in the last bullet point, and the 3DM oracle found a matching; in particular, each clause vertex $u_{j,0,0}$ is covered. Only (at most) three hyperedges contain that vertex, each of which also uses a vertex like $v_{i,j,3,0}$ or $v_{i,j,2,0}$ from some expanded variable gadget corresponding to a literal in the clause. We proved when we described the expanded variable gadgets that the variable gadget leaves $v_{i,j,3,0}$ or $v_{i,j,2,0}$, respectively, uncovered only if it picked the "True hyperedges" or "False hyperedges", respectively, for the variable gadget representing $x_i$. The last step of the reduction sets $x_i$ to be true or false, respectively, in those cases, so clause $C_j$ is satisfied by that value of $x_i$. That's true for each clause, so the whole formula is satisfied.

**$\mathsf{NP}_{\mathsf{search}}$-hardness of 3DM: proof of completeness of the reduction** To prove completeness, we need to prove that if $\varphi$ has some satisfying assignment $\alpha$, the reduction returns a satisfying assignment (not necessarily $\alpha$). This is mostly a matter of saying that our gadgets can be used as intended:

1. For each variable $x_i$ that's true in $\alpha$, pick all the "true" hyperedges in $G$'s variable gadgets for $x_i$.

2. For each variable that's false in $\alpha$, pick all the "false" hyperedges in $G$'s variable gadgets for $x_i$.

3. For each clause $C_j$, choose $\alpha$'s first true literal in it (one exists because $\alpha$ is a satisfying assignment), and pick the corresponding hyperedges.

4. Finally, choose whatever cleanup hyperedges are unused.

Together, these show that the 3DM problem has a solution. So the oracle returns a solution (not necesssarily the one described above), so the reduction returns an assignment, and the soundness proof above guarantees that the returned assignment is in fact a satisfying assignment. □