

How Mortgage Repayments Are Calculated

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The derivation above is a cleaned-up version of the derivation by Gurdip Singh.

Definitions

Define the following quantities:

P is the principal amount borrowed
 I is the interest rate,
 p_i is the principal repayment at time i
 r_i is the remaining principal at time i
 n is the total number of repayments
 m is the monthly repayment amount

,

Such that you initially borrowed an amount, P , at an interest rate I (in fractional form, so e.g. 0.03 for 3%), to be paid back over n monthly payments, each repayment instalment of size m . At each monthly payment (indexed by i) you pay back p_i towards the principal amount (the rest of the monthly payment goes towards interest) and after that have an amount r_i of the principal amount left to be repaid.

Our goal is to calculate the monthly repayment, m , needed to pay back the loan in the time specified.

First payment

For convenience we will also define the amount of interest paid monthly (interest is charged annually, then this amount is just divided evenly across the months), as

$$j = I/12 ,$$

and the interest paid at month i we will denote as J_i

At each (monthly) payment, the amount paid towards interest is

$$J_i = r_i j$$

where r_i is the remaining principal to be paid as will be defined below, and clearly $r_1 = P$, such that for the first payment, $J_1 = r_1 j = Pj$.

Likewise, at each payment the amount paid towards repaying the principal is

$$p_i = m - J_i$$

so that at the first repayment this is $p_1 = m - J_1 = m - Pj$. towards the principal amount.

This means that after the i -th payment the remaining value of the principal to be paid off is given by

$$r_i = P - \sum_{j \leq i} p_j$$

.

So that after the first payment the remaining principal amount is

$$\begin{aligned} r_1 &= P - p_1 \\ &= P - (m - Pj) \\ &= P(1 + j) - m . \end{aligned}$$

Second payment

It is clear that at the next payment time, we have an interest payment of

$$J_2 = r_1 j = ((P + Pj) - m)j = (P(1 + j) - m)j ,$$

a principal payment of

$$p_2 = m - J_2 = m - (P(1 + j) - m)j ,$$

and a remaining principal amount left to pay

$$\begin{aligned} r_2 &= P - p_1 - p_2 \\ &= P - (m - Pj) - (m - (P(1 + j) - m)j) \\ &= P(1 + j) - m - (m - (P(1 + j) - m)j) \\ &= P(1 + j) - m - m + (P(1 + j) - m)j \\ &= P(1 + j) - 2m + Pj(1 + j) - mj \\ &= P(1 + j)(1 + j) - 2m - mj \\ &= P(1 + j)^2 - 2m - mj \\ &= P(1 + j)^2 - m(1 + j) - m \end{aligned}$$

where in the last line we have sought to write things in terms of factors of $(1 + j)$.

Third payment

Similarly for the third payment, we have an interest payment of

$$\begin{aligned} J_3 &= r_2 j \\ &= [P(1 + j)^2 - m(1 + j) - m]j , \end{aligned}$$

a contribution towards paying off the principal of

$$\begin{aligned} p_3 &= m - J_3 \\ &= m - [P(1 + j)^2 - m(1 + j) - m]j , \end{aligned}$$

and the remaining principal to be paid off after this payment is given by

$$\begin{aligned} r_3 &= P - p_1 - p_2 - p_3 \\ &= r_2 - p_3 \\ &= [P(1 + j)^2 - m(1 + j) - m] - [m - j(P(1 + j)^2 - m(1 + j) - m)] \\ &= P(1 + j)^2 - m(1 + j) - m - m + Pj(1 + j)^2 - mj(1 + j) - mj \\ &= P(1 + j)^3 - m(1 + j)^2 - 2m - mj \\ &= P(1 + j)^3 - m(1 + j)^2 - m(1 + j) - m . \end{aligned}$$

General Form

In the equations for r_i above we can already see the pattern which will generalise to further payment periods:

$$r_i = P(1+j)^i - m \sum_{k=0}^{i-1} (1+j)^k$$

The summation can be readily recognised to be a geometric series, for which we can use the well-known relation

$$\sum_{k=0}^l ar^k = a \left(\frac{1-r^{l+1}}{1-r} \right)$$

where in our case $a = m$, $r = 1 + j$, $l = i - 1$.

Considering the full set of payment times such that $i \rightarrow n$, we then have that the remaining principal amount to be paid after the last, n -th payment can be written as

$$\begin{aligned} r_n &= P(1+j)^n - m \left(\frac{1-(1+j)^n}{-j} \right) \\ &= P(1+j)^n + \frac{m[1-(1+j)^n]}{j} . \end{aligned}$$

We would like to find the monthly repayment, m , for which the debt has been fully repaid after the n payments. Therefore we set $r_i = 0$ in the equation above and solve for m , giving

$$m = \frac{Pj(1+j)^n}{(1+j)^n - 1} .$$

Validation

This final equation can be verified against the form given by Business Insider, but now with a full derivation of it to help understand it.

You can verify that the monthly repayments computed using the formula above numerically match those given by online calculators such as that from the Money Advice Service or the Money Saving Expert.

Implementation

Entering the interest rate in the usual, annual, percentage form, and the term of the mortgage in years, the monthly repayment required to repay the loan (including interest) in the desired term is given by the function below.

The monthly repayment and total, final payment amount (including interest) for a 460k GBP principal value, with an interest rate of 1.78, over a term of 30 years, is given by:

```
repaymentCalc = function(principal, interestrate, term){
  i = 0.01*1/12*interestrate
  n = 12*term
  monthlyrepayment = principal*i*(1+i)^n/((1+i)^n-1)
  totalrepayment = n*monthlyrepayment
  return(list(monthlytotal=monthlyrepayment, finaltotal=totalrepayment))
}

repaymentCalc(460000, 1.78, 30)
```

```
## $monthlytotal
## [1] 1650.09
##
## $finaltotal
## [1] 594032.5
```

The breakdown for each payment of payment towards principal and towards interest, as well as the remaining principal to be paid, and the monthly repayment made (with a minimum payment to complete the principal repayment within the term as given by the function above), is given by the function below, following exactly the same algorithm as detailed in the sections above to compute the relevant quantities.

```
repaymentCalcDetailed = function(principal, interestrate, term, monthlypayment){

  monthlyinterest = 0.01*interestrate/12
  months = seq(12*term)
  principalrepayments = c()

  interestpayments = c()
  remainingprincipals = c()
  remainingprincipal = principal

  for(i in months){
    interestpayment = remainingprincipal*monthlyinterest
    principalrepayment = monthlypayment - interestpayment
    principalrepayments = c(principalrepayments, principalrepayment)
    remainingprincipal = principal - sum(principalrepayments)
    if(remainingprincipal < 0){
      principalrepayments = principalrepayments[-length(principalrepayments)]
      break
    }
    interestpayments = c(interestpayments, interestpayment)
    remainingprincipals = c(remainingprincipals, remainingprincipal)
  }

  #stopifnot(abs(sum(principalrepayments) - principal) < 1.0)

  return(list(interest=interestpayments, principal=principalrepayments, remainingprincipal=remainingprincipal))
}

repaymentCalcDetailed(460000, 1.78, 30, 1650.09)
```

```
## $interest
##      [1] 682.333333 680.897828 679.460193 678.020425 676.578522 675.134480
##      [7] 673.688296 672.239967 670.789489 669.336860 667.882076 666.425134
##     [13] 664.966031 663.504764 662.041329 660.575724 659.107944 657.637988
##     [19] 656.165851 654.691530 653.215022 651.736324 650.255433 648.772345
##     [25] 647.287057 645.799566 644.309869 642.817961 641.323841 639.827505
##     [31] 638.328949 636.828170 635.325165 633.819930 632.312463 630.802760
##     [37] 629.290817 627.776631 626.260200 624.741519 623.220585 621.697396
##     [43] 620.171947 618.644235 617.114257 615.582010 614.047490 612.510693
##     [49] 610.971617 609.430258 607.886613 606.340678 604.792450 603.241925
##     [55] 601.689100 600.133972 598.576538 597.016793 595.454734 593.890358
##     [61] 592.323662 590.754642 589.183295 587.609617 586.033604 584.455254
##     [67] 582.874562 581.291526 579.706141 578.118405 576.528314 574.935864
##     [73] 573.341052 571.743875 570.144328 568.542409 566.938113 565.331438
```

```

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## [103] 524.380734 522.710932 521.038653 519.363893 517.686649 516.006918
## [109] 514.324695 512.639976 510.952758 509.263038 507.570812 505.876075
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## [133] 473.194164 471.448436 469.700117 467.949206 466.195697 464.439587
## [139] 462.680872 460.919549 459.155612 457.389060 455.619887 453.848089
## [145] 452.073664 450.296606 448.516913 446.734579 444.949602 443.161977
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## [241] 268.877652 266.828853 264.777016 262.722135 260.664206 258.603225
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## [283] 180.158992 177.978594 175.794962 173.608091 171.417976 169.224613
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## [295] 153.779700 151.560173 149.337354 147.111237 144.881819 142.649093
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## [355] 14.610007 12.184045 9.754485 7.321320 4.884547 2.444159
##
## $principal
## [1] 967.7567 969.1922 970.6298 972.0696 973.5115 974.9555 976.4017
## [8] 977.8500 979.3005 980.7531 982.2079 983.6649 985.1240 986.5852
## [15] 988.0487 989.5143 990.9821 992.4520 993.9241 995.3985 996.8750
## [22] 998.3537 999.8346 1001.3177 1002.8029 1004.2904 1005.7801 1007.2720
## [29] 1008.7662 1010.2625 1011.7611 1013.2618 1014.7648 1016.2701 1017.7775

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```

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```

```

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## [251] 165959.309937 164555.392914 163149.393413 161741.308347 160331.134621
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## [316]	70235.064370	68689.156382	67140.955297	65590.457715	64037.660227
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## [331]	46804.114824	45223.450928	43640.442380	42055.085703	40467.377414
## [336]	38877.314023	37284.892039	35690.107962	34092.958289	32493.439511
## [341]	30891.548113	29287.280576	27680.633375	26071.602981	24460.185859
## [346]	22846.378468	21230.177263	19611.578692	17990.579201	16367.175227
## [351]	14741.363203	13113.139559	11482.500716	9849.443092	8213.963099
## [356]	6576.057144	4935.721629	3292.952949	1647.747496	0.101655