Changes in extremes

Detection and consequences

Ilaria Prosdocimi

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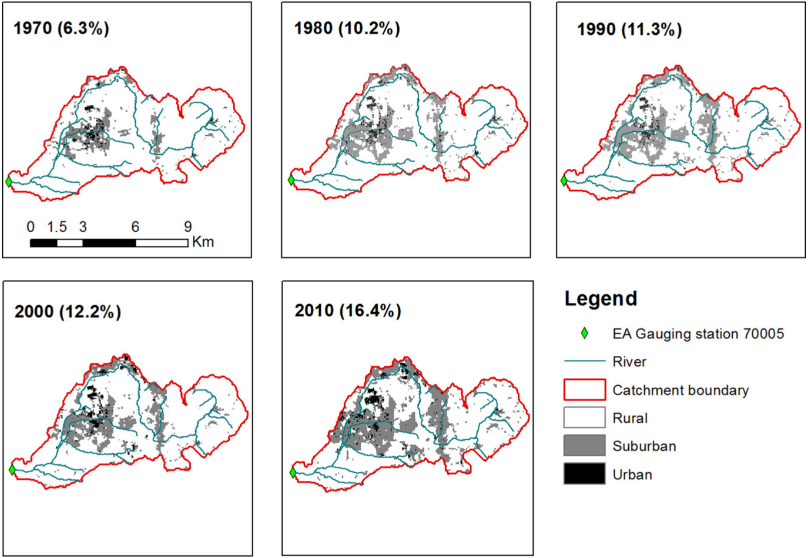
Change (?)

Increasing interest in assessing changes in extremes related to natural hazards.

Many studies investigate changes in extreme rainfall and extreme flows.

Changes in magnitude/frequencies: infrastructures are designed to withstand extreme events of some magnitude. Problematic if these become more (or less!) frequent.

What causes change



from Prosdocimi et al. (2015), WRR, doi:10.1002/2015WR017065

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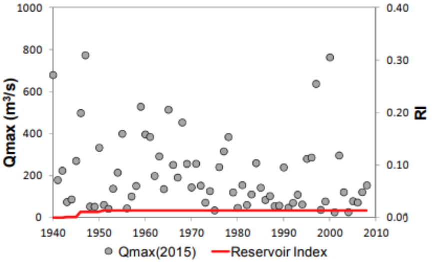
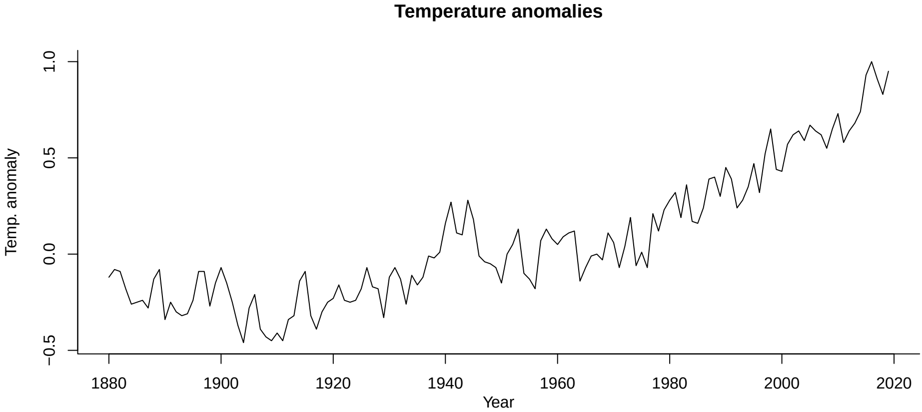


from Lopez Frances (2013), HESS, doi:10.5194/hess-17-3189-2013

What causes change

Implicit assumption:



NOAA National Centers for Environmental information, Climate at a Glance:
Global Time Series, published June 2020, retrieved on July 5, 2020 from https://www.ncdc.noaa.gov/cag/

Why study change?

- Understand if process of interest (river flow, rainfall, etc) is evolving in time
- Understand how process of interest is affected by external drivers
- Assess risk connected to a certain hazard and its evolution
- If this is changing, how to account for this

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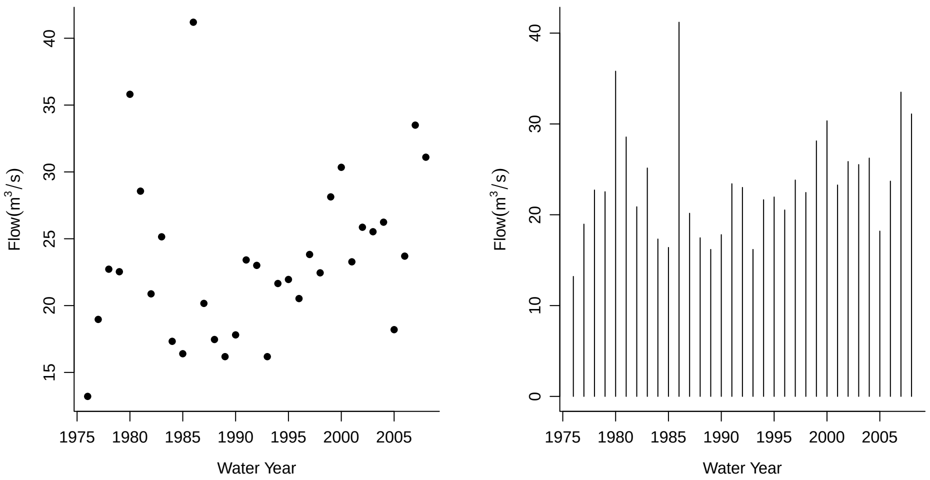
Detection, attribution and management.

The Lostock at Littlewood Bridge



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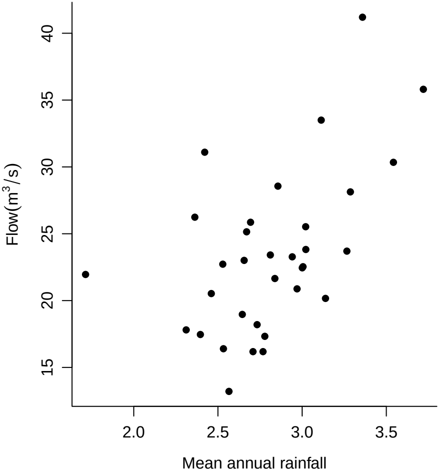

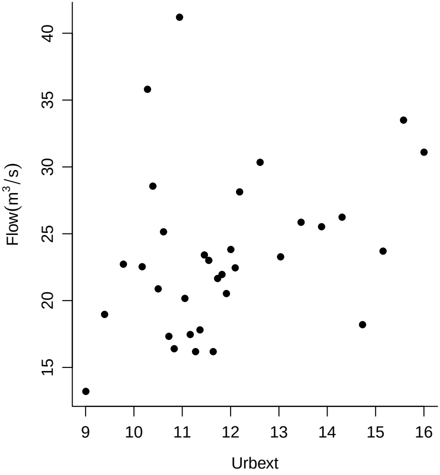

Statistical tools

We assume that $\boldsymbol{y} = (y_1, \ldots, y_n)$ is a random sample from a population.

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Inference framework:

- Parametric: assume that y_i is a realisation of some distribution described by **parameters** $\boldsymbol{\theta}$ ($f(y_i; \boldsymbol{\theta})$)
- Non-parametric: no assumption on the distribution of $f(y)$ is made (well, less assumptions...)

Parameteric framework

Advantage of parametric framework:

- Describe the whole distribution (including, for example, quantiles)
- A very general framework
- Easy to extend to very complex models (but estimation can be complicated)

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The parametric framework:

- Assume that each member of the sample y_i comes from some distribution Y_i
- Often assumed: $(Y_1, \ldots, Y_n)$ are independent and identically distributed (iid)
- Assume that Y_i follows a known distribution parametrised by $\boldsymbol{\theta}$
- (for example $Y_i \sim N(\mu, \sigma)$, with $\boldsymbol{\theta} = (\mu, \sigma)$)
- Find estimates $\hat{\boldsymbol{\theta}}$ based on the sample

Estimation methods

- Method of moments
- Maximum likelihood
- Bayesian approaches

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Choice of framework and estimation method should depend on:

- Actual data properties
- Main inferential question (and importance of uncertainty assessment)
- Computational hurdle
- Model complexity
- Presence of prior information (which can be formalised)

Maximum likelihood estimation

The likelihood function is defined as

$$L(\boldsymbol{\theta}; \boldsymbol{y}) = \prod_{i=1}^{n} f(y_i, \boldsymbol{\theta}),$$

but calculations typically employ the log-likelihood

$$l(\boldsymbol{\theta}; \boldsymbol{y}) = \sum_{i=1}^{n} \log f(y_i, \boldsymbol{\theta}).$$

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$\hat{\boldsymbol{\theta}}_{ML}$ is the value that maximises $l(\boldsymbol{\theta}; \boldsymbol{y})$.

Asymptotically ($n \to \infty$) we have that $\hat{\boldsymbol{\theta}}_{ML} \sim N(\boldsymbol{\theta}, I_E(\boldsymbol{\theta})^{-1})$ where $I_E(\boldsymbol{\theta})$ is the expected information matrix, with elements

$$e_{i,j}(\theta) = E\left[-\frac{d^2 l(\theta)}{d\theta_i d\theta_j}\right]$$

Typically $I_E(\boldsymbol{\theta})$ is unknown: use the observed information matrix evaluated at $\hat{\boldsymbol{\theta}}$.

Parametric models for change

- Assume Y_i comes from a distribution $f(\boldsymbol{\theta}_i, y_i)$
- Assume $\boldsymbol{\theta}_i = g(\boldsymbol{x}_i)$
- So $Y_i = (Y|X = x_i)$ with $f(g(\boldsymbol{x}_i), y_i)$

Example. Linear regression (with two explanatory variables):

- $Y_i \sim N(\mu_i, \sigma)$; $\boldsymbol{\theta}_i = (\mu_i, \sigma)$
- $\mu_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}$ - linear relationship
- σ is constant
- As a consequence: $E[Y_i] = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}$, $V[Y_i] = \sigma^2$

Parametric models for change

Linear regression likelihood:

$$l(\boldsymbol{\theta}; \boldsymbol{y}) = \sum_{i=1}^{n} \log f(y_i, \boldsymbol{\theta}) \propto -n\log(\sigma) - \frac{(y - \beta_0 - \beta_1 x_{1i} - \beta_2 x_{2i})^2}{2\sigma^2}$$

ML estimates can be derived analytically: $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\sigma})$.

And we have, for example, $\hat{\beta}_i \sim N(\beta_i, \hat{\sigma}_{\beta_i})$.

From this once can construct confidence intervals for β_i or perform a test such as:

$$H_0 : \beta_0 \geq \tilde{\beta} \qquad VS \qquad H_0 : \beta_0 < \tilde{\beta}$$

By default $\tilde{\beta} = 0$, but one can test for any value $\tilde{\beta}$ and any type of statistical test (equality, larger than or equal, smaller than or equal).

Notice that if x_j is a factor one can account for step changes (change points).

Parametric models of change in extremes

Describing extremes is a different task than describing the typical behaviour.

$(y_1, \dots, y_n)$ is a sample of extremes: what is a reasonable assumption for Y?

Extreme Value Theory gives theoretical derivation, but practice is often different.

Regardless of the choice of $f(y, \boldsymbol{\theta})$ - parametric models of change for extremes can be easily constructed assuming $Y_i = (Y|X = x_i)$ and $\boldsymbol{\theta}_i = g(\boldsymbol{x}_i)$.

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What is an extreme?

- Largest event over a certain amount of time (eg water year, season)
- Events larger than a certain high threshold (independent events?)

Parametric models in extremes

Traditional (asymptotic) results based on extremes of stationary series:

- Block maxima: $Y \sim GEV(\mu, \sigma, \xi)$
- Threshold exceedance magnitude: $Y \sim GP(\sigma, \xi)$
- Threshold exceedance frequency: $N \sim Pois(\lambda)$

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In practice other distributions are often assumed for Flow maxima

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Generalised Extreme Value distribution

$$\text{The GEV CDF:} \quad F(y, \boldsymbol{\theta}) = \exp\left\{ -\left(1 + \xi \frac{y-\mu}{\sigma}\right)^{-1/\xi} \right\}$$

$\boldsymbol{\theta} = (\mu, \sigma, \xi)$:

- $\mu \in \mathbb{R}$: location parameter
- $\sigma > 0$; scale parameter
- $\xi \in \mathbb{R}$: shape parameter.

$Y \sim GEV(\mu, \sigma, \xi)$ is defined on $y : 1 + \xi(y - \mu)/\sigma > 0$, this means:

- $y \in [\mu - \sigma/\xi, \infty)$, if $\xi > 0$ (Frechet)
- $y \in (-\infty, \mu - \sigma/\xi]$, if $\xi < 0$ (Weibull)
- $y \in (-\infty, \infty)$, if $\xi = 0$ (Gumbel)

BUT! In engineering/hydrology $Y \sim GEV(\xi, \alpha, \kappa)$ and $\kappa = -\xi$. Software can use different parametrisation.

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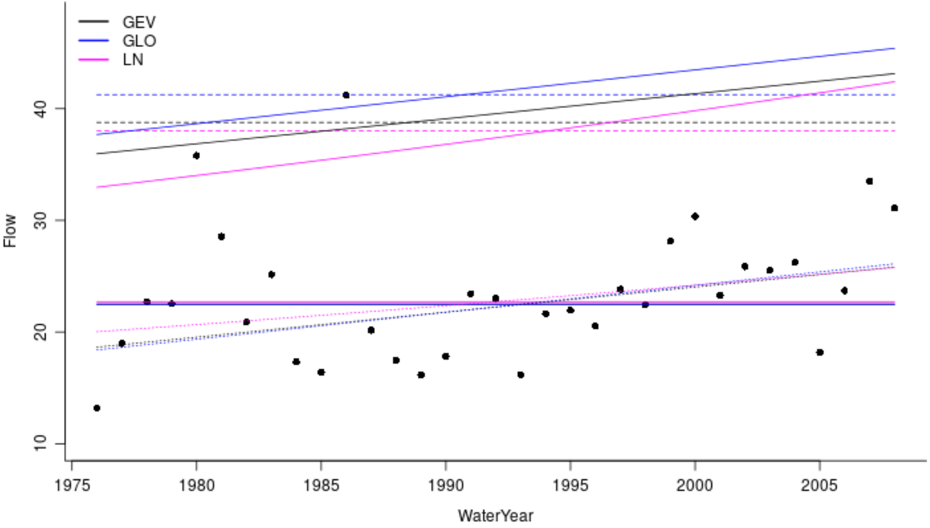
Quantile function (for $\xi \neq 0$):

$$q(y, \theta) = \mu + \frac{\sigma}{\xi}\left[(-\log(1-p))^{-\xi} - 1\right]$$

Changes in annual maxima - choice of distribution

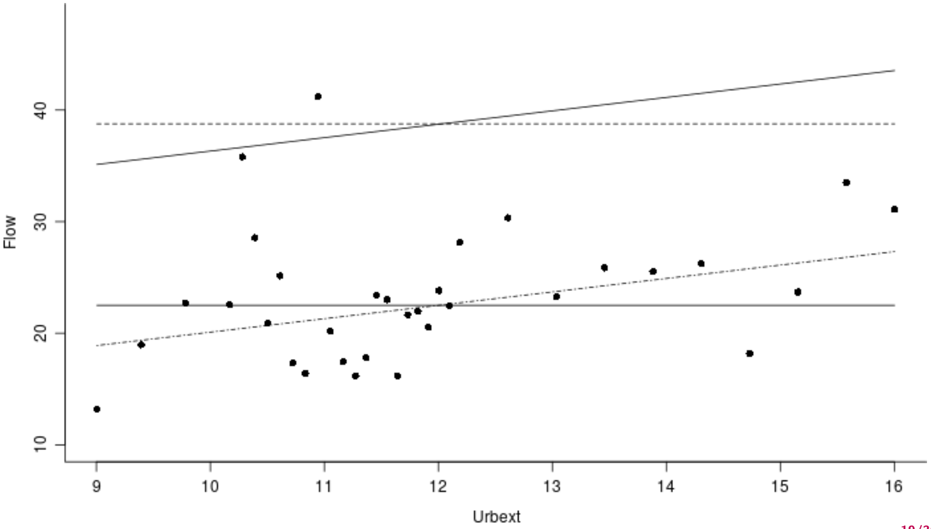
The Lostock at Littlewood Bridge: median and effective 50-yrs event.



Changes in annual maxima

Time is not a cause for change, but land cover changes impact peak flow.

$$\mu = \mu_0 + \mu_{urb} urb \quad \text{while } (\sigma, \xi) \text{ constant}$$



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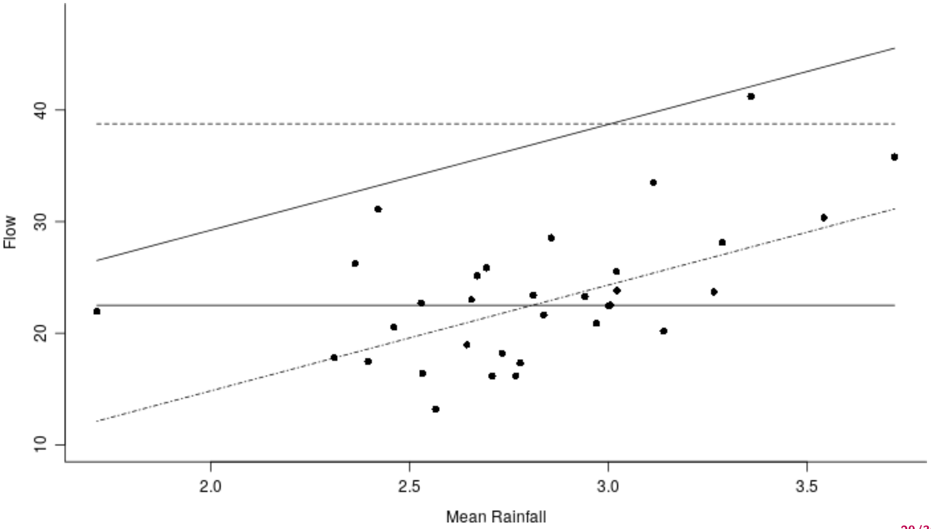

Changes in amax - effect of rain given Urbext

Separate effect of rain and urbanisation:

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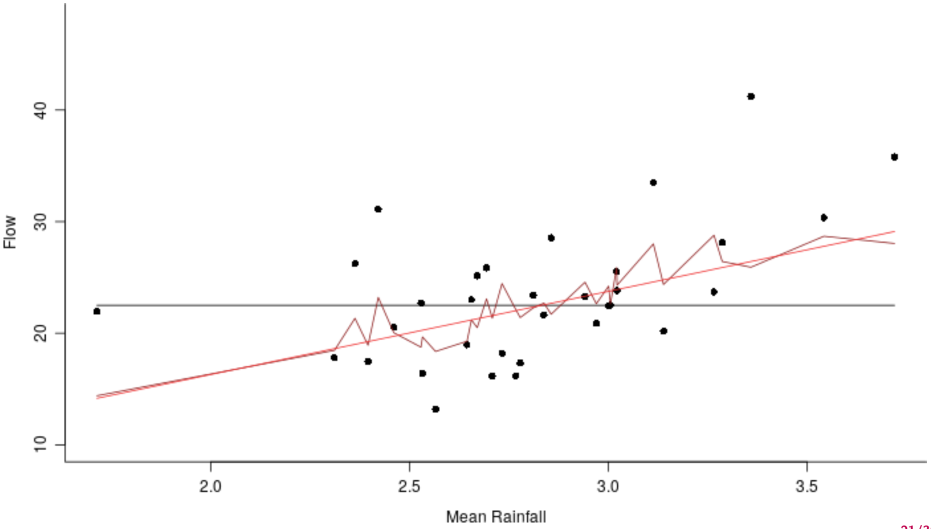

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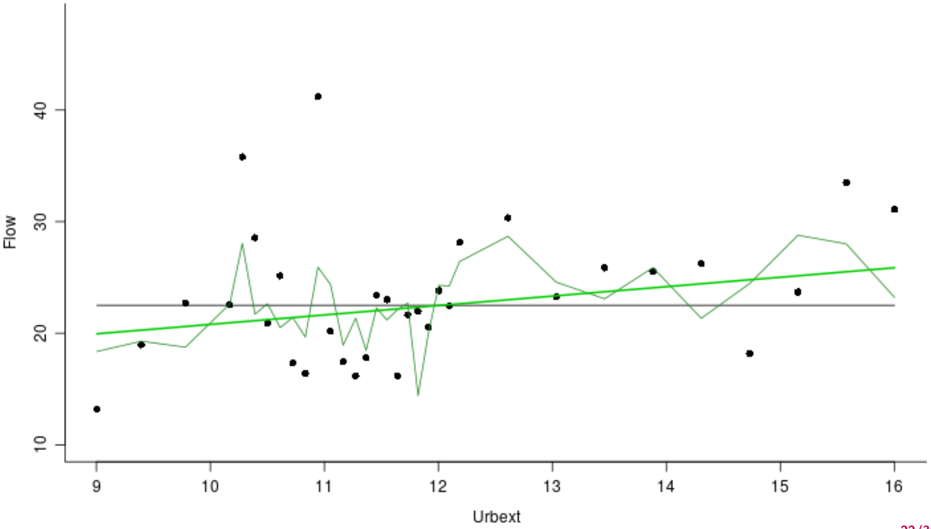


Changes in annual maxima - estimated parameters

Rain as covariate (log-lik: -98.39)

	μ_0	μ_{urb}	μ_{rain}	σ	ξ
MLE	-5.604	-	9.479	4.042	0.003
se	8.158	-	2.830	0.622	0.168

Urbext as covariate (log-lik: -100.0004)

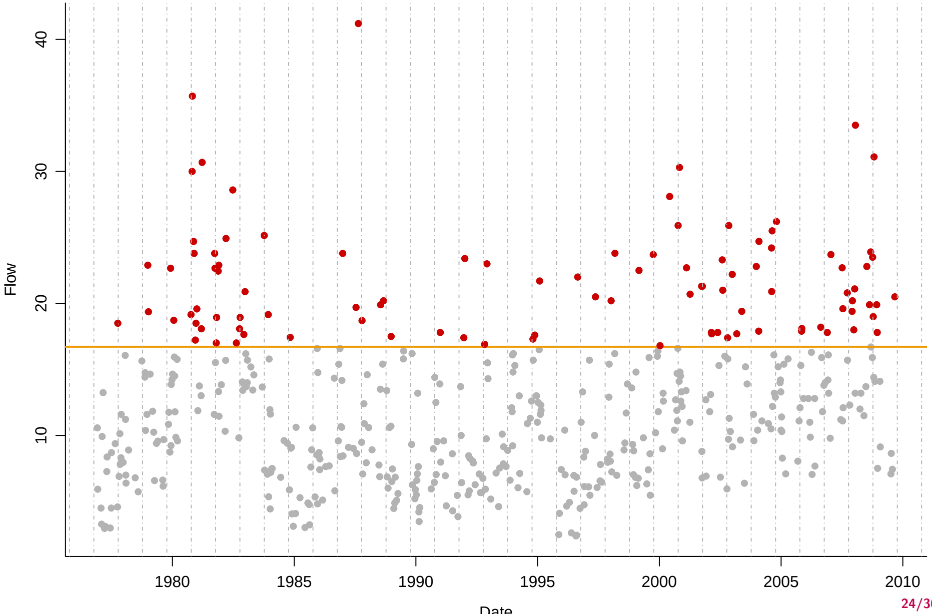
	μ_0	μ_{urb}	μ_{rain}	σ	ξ
MLE	6.53	1.20	-	4.17	0.04
se	4.79	0.40	-	0.60	0.13

Rain and urbext as covariate (log-lik: -96.47)

	μ_0	μ_{urb}	μ_{rain}	σ	ξ
MLE	-9.767	0.845	7.449	3.862	-0.016
se	7.344	0.422	2.668	0.580	0.153

Changes is peaks over threshold

Extract observations above a high threshold



Generalised Pareto Distribution

Y is taken to be the observations above a high threshold u ($Y = (X|X > u)$).

GP is the limiting distribution for the magnitude of exceedances.

$$F(y, u, \boldsymbol{\theta}) = 1 - \left(1 + \xi \frac{y - u}{\tilde{\sigma}}\right)^{-1/\xi}$$

u is a constant, $\boldsymbol{\theta} = (\sigma, \xi)$:

- $\sigma > 0$; scale parameter
- $\xi \in \mathbb{R}$: shape parameter.

The domain changes depending on the sign of ξ:

- $y \in [u, \infty)$, if $\xi \geq 0$
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$$\text{Quantile function:} \quad q(p, u, \boldsymbol{\theta}) = u + \frac{\sigma}{\xi}(p^{-\xi} - 1)$$

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Modelling change: $\sigma_0 + \sigma_1 x$

Frequency of exceedances: typically modelled as a Poisson (indep. of magnitude).

Point Process representation of extremes

$N = \{$no. Exceedance in a Year$\}$. $N \sim Pois(\lambda)$

$P($no. Exceedance in a Year$)$ is linked to magnitudes.

Express this using GEV-parameters:

$$\log \lambda = -\frac{1}{\xi} \log \left[1 + \xi \frac{u - \mu}{\sigma} \right]$$

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Express this using GEV-parameters:

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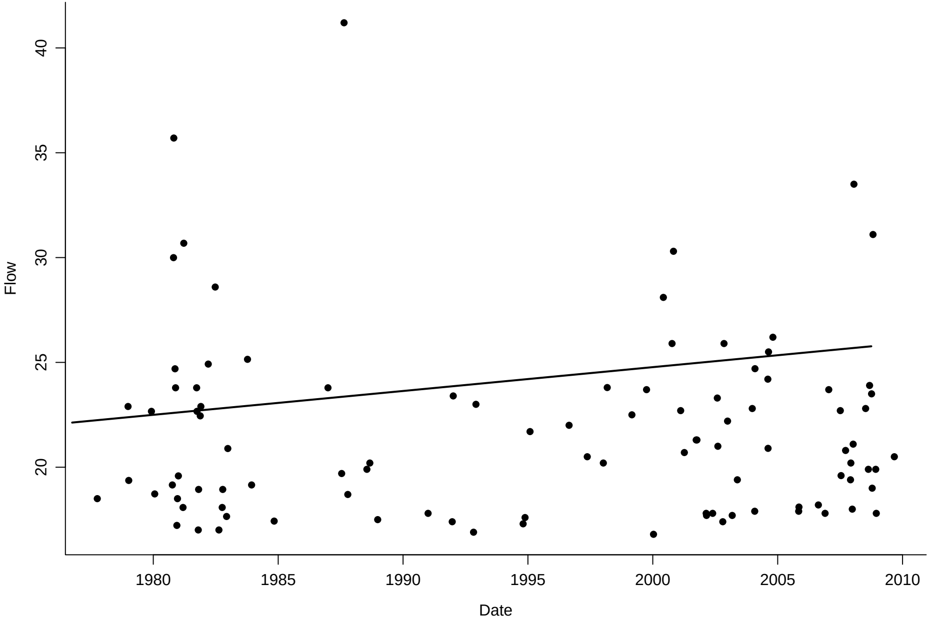
Express changes in magnitude and frequency in the same model

Same meaning as GEV models of change

Changes in Peaks - Point Process



Changes in extremes - comparing the models

Rain and urbext as covariate - GEV:

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MLE	-9.767	0.845	7.449	3.862	-0.016
se	7.344	0.422	2.668	0.580	0.153

Rain and urbext as covariate - PP:

	μ_0	μ_{urb}	μ_{rain}	σ	ξ
MLE	-12.139	0.930	8.007	4.622	-0.184
se	6.757	0.320	1.723	0.368	0.064

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Larger sample size leads to more precise estimation (statistically)

Tail estimate is quite different

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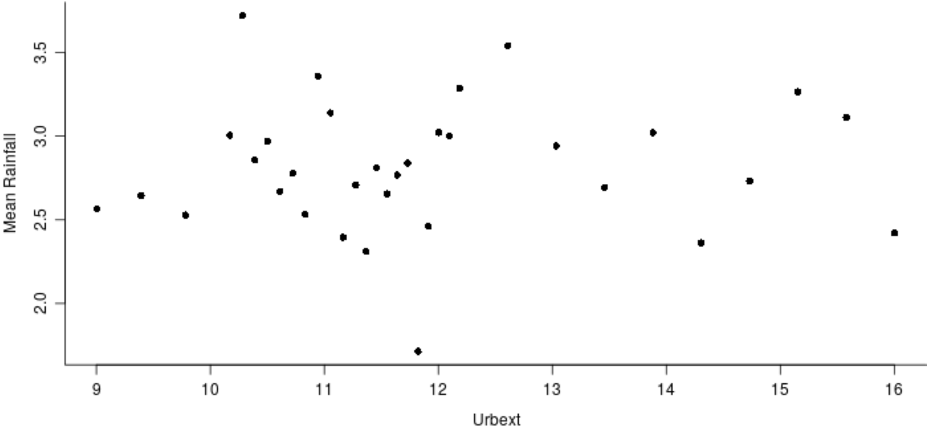
The assumption is that $Y_i = (Y|X = x_i)$ follows $f(y; \boldsymbol{\theta})$ - goodness of fit should be carried out on **residuals**

Statistical EVT and practice are not aligned

Changes in extremes - attribution

Kendall's $\hat{\tau}$(Urbext, Rain) $= 0.068$.



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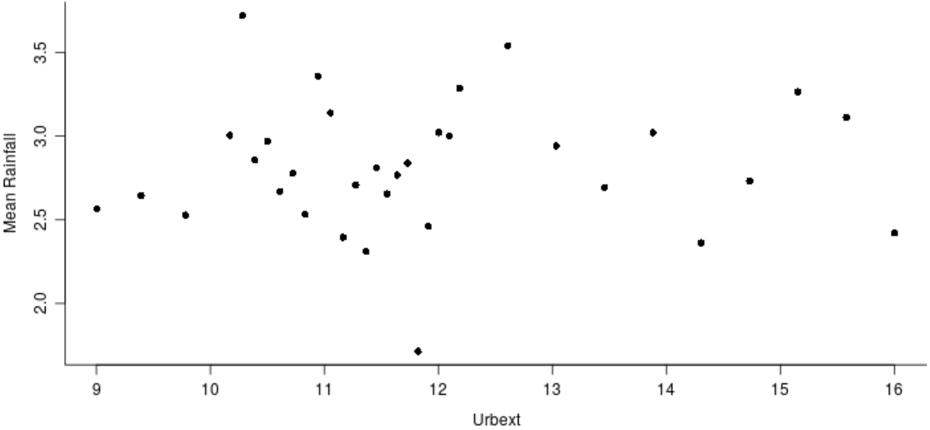

Reality is complex: linear models are a (over-simplified!) representation.

Detection

Methods sometimes chosen because of data availability

Statistical models rely on assumption of iid random observations

Short records: hard to identify complex evolutions

Short records: hard to observe a good range of the explantory variable

When detecting “change”: what are we detecting?[2]

[2] Merz et al, HESS, doi:10.5194/hess-16-1379-2012

Attribution

Golden standard of causality is randomised trials: what about observational studies?

Climate sciences reproduce the treatment/placebo framework with numerical experiments (how good for extremes?).

Some numerical experiments done in hydrology - but systems are complex.

Causality: a cascade of impacts (with feedback[3])

[3] Zhang et al, Nature, doi:10.1038/s41586-018-0676-z

Changes in annual maxima - uncertainity

Structures are designed for the "T-Year" event: estimated as the 1-1/T quantile.

If the distribution is changing so is the quantile.

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	Q100	95% lb	95% ub	width
no-change	30.514	41.837	53.159	11.322
Rain = max(Rain)	33.676	48.403	63.130	14.727

Adding parameters adds variation to the estimates - is it worth it?

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Bias-variance trade-off and parsimonious models.

Changes in extremes - consequences

How to quantify risk under change?[4]

Choice of distribution has an impact on estimates of rare events

Today I used "effective design events": $q(p; \hat{\boldsymbol{\theta}})$. So at $X = x^*$: $q(p; \hat{\boldsymbol{\theta}}(x^*))$.

Choice of distribution/model has an impact on estimates of rare events.

Choice of model has an impact of description of change. Compare effective return levels for x^* and x_0:

$$q(p; \hat{\boldsymbol{\theta}}(x^*)) - q(p; \hat{\boldsymbol{\theta}}(x_0)) = \mu_1(x^* - x_0)$$

[4] Volpi, Wires Water, doi:10.1002/wat2.1340

(Statistical) reccomended reading

Coles, S (2001), An introduction to statistical modeling of extreme values, Springer

Katz, R.W., Parlange, M.B. and Naveau, P., 2002. Statistics of extremes in hydrology. Advances in water resources, 25(8-12), pp.1287-1304.

Katz, Richard (2013) Statistical Methods for Nonstationary Extremes, Chapter 2 in A. AghaKouchak et al. (eds.), Extremes in a Changing Climate, Water Science and Technology Library 65, DOI 10.1007/978-94-007-4479-0 2,

Doing science the right way

Reproducibility crisis in several fields - open science movement as a results.

Replicability (i.e. being able to re-run the analysis) should be a given.

Start any project in a replicable way: literate programming and programmatic interfaces In R (and Python) this is increasingly feasible

Slides code at github.com/ilapros