Assignment 1

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Exercise 1

Given the problem (2), the request is to write the weak formulation. The problem is a 1D Helmholtz equation and it represent a steady equation with $u(x)$ amplitude of the wave.
In this problem we have non-null Dirichlet boundary conditions, so we introduce a new function $R_D \in H^1(0,L)$ s.t. $R_D(0) = \alpha$ and $R_D(L) = \beta$. The new variable are:

$$w = u - R_D$$
$$w(0) = w(L) = 0$$

with $w \in H_0^1(0,L)$ and $H^1(0,L) = \{v : (0,L) \to \mathbb{R} \text{ such that } v, v' \in L^2(0,L)\}$.
Substituting in the original problem:

$$\begin{cases} (\mathrm{w} + \mathrm{R}_D)'' + n^2\omega^2(w + R_D) = f \\ \mathrm{w}(0) = \mathrm{w}(\mathrm{L}) = 0 \end{cases}$$

Now we can take a test function $v \in C_0^1(0,L)$ multiple both side of the equation and integrate over the domain:

$$\int_0^L (w + R_D)'' v dx + \int_0^L n^2\omega^2 (w + R_D) v dx = \int_0^L f v dx \tag{1}$$

$$-\int_0^L w'v'dx - \int_0^L R_D v' dx + \int_0^L n^2\omega^2 w v dx + \int_0^L n^2\omega^2 R_D v dx = \int_0^L f v dx \tag{2}$$

If we assume the problem to be well posed, it becomes:

Find $w \in H_0^1(0,L)$ s.t.

$$-\int_0^L w'v'dx + n^2\omega^2 \int_0^L w v dx = \int_0^L f v dx + \int_0^L R_D v' dx - \int_0^L n^2\omega^2 R_D v dx v \in H_0^1(0,L) \tag{3}$$
$$\forall v \in H_0^1(0,L)$$

where:
$H^1(0,L) = \{v : (0,L) \to \mathbb{R} \text{ s.t. } v, v' \in L^2(0,L)\}$
$H_0^1(0,L) = \{v \in H^1(0,L) \text{ s.t. } v(0) = v(L) = 0\}$

If we introduce some bilinear forms the problem can be reformulated using a different notation. Let's introduce some bilinear as:

$$a : H_0^1(0,L) \times H_0^1(0,L) \to \mathbb{R} \tag{4}$$

$$a(w,v) = \int_0^L w'(x)v'(x)dx \quad \forall w,v \in H_0'(0,L) \tag{5}$$

and

$$m : H_0^1(0,L) \times H_0^1(0,L) \to \mathbb{R} \tag{6}$$

$$m(w,v) = \int_0^L w(x)v(x)dx \quad \forall w,v \in H_0'(0,L) \tag{7}$$

In the same way, we introduce F:

$$F : H_0^1(0,L) \to \mathbb{R} \tag{8}$$

$$F(v) = \int_0^L f(x)v(x)dx \qquad \forall v \in H_0'(0,L) \tag{9}$$

which is a linear form. In this way we can reformulate the problem:

Find $w \in H_0^1(0,L)$ s.t.

$$-a(w,v) + n^2w^2m(w,v) = F(v) + a(R_D,v) - n^2w^2m(R_D,v)$$

$\forall v \in H_0^1(0,L)$

Exercise 2

In order to write the Galerkin formulation of the problem, we have to discretize the domain in finite dimensional space V_h subset of V.
Considering the domain, we subdivided it in intervarls K_i and in $N+1$ equispaced gridpoints x_i with $i = 0, ..., N$
Now we can reformulate the problem into V_h:

Find $w_h \in V_h \subset H_0^1(0,L)$ s.t.

$$-a(w_h,v_h) + n^2w^2m(w_h,v_h) = F(v_h) + a(R_{Dh},v_h) - n^2w^2m(R_{Dh},v_h)$$

$\forall v_h \in V_h \subset H_0^1(0,L)$

Exercise 3

We know that :

$V_h = X_h^1 = \{v \in C^0(0,L) : v|_{k_i(h)} \in \mathbb{P}^1(k_i) \forall i = 1,..,N\}$

The basis function for V_h based on the grid is defined by:

$$\psi_i(x_j) = \delta_{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & otherwise \end{cases} \tag{10}$$

These functions are piece wise linear, and Lagrangian basis.

Exploiting the reference element technique we can obtain the element of the system, as

$$w_h(x) = \sum_{j=0}^{N} w_j \phi_j(x) \tag{11}$$

Substituting into the Galerkin formulation and considering $v_h(x) = \phi_i(x)$, we obtain:

$$-\sum_{j=0}^{N} w_j \int_0^L \phi_j' \phi_i' dx + n^2 w^2 \sum_{j=0}^{N} w_j \int_0^L \phi_j \phi_i dx = \int_0^L f_h \phi_i dx + \sum_{j=0}^{N} \int_0^L R_{D}{}'_h \phi_i' dx - n^2\omega^2 \int_0^L R_{Dh} \phi_i dx \tag{12}$$

We call the two integral on the left a_{ij} and m_{ij} and they are the entries of the matrix $\boldsymbol{A}$ and $\boldsymbol{M}$, and the one right F_i which are the entries of the vector $\mathbf{F}$.

$$-\sum_{j=0}^{N} w_j a(\phi_j, \phi_i) + n^2 w^2 \sum_{j=0}^{N} w_j m(\phi_j, \phi_i) = F(\phi_i) + a(R_D, \phi_i) - n^2\omega^2 m(R_D, \phi_i) \tag{13}$$

that leads to the algebraic formulation:

Find $\underline{w} = (w_1, ..., w_{N+1})^T \in R^N$ s.t. $-\mathbf{A}\underline{w} + n^2\omega^2\mathbf{M}\underline{w} = \mathbf{F} + \mathbf{A}\underline{R_D} - n^2\omega^2\mathbf{M}\underline{R_D}$ $\quad \mathbf{A} \in R^{N\times N}$, $\quad \mathbf{M} \in R^{N\times N}$, $\quad \mathbf{F} = (f1, f2, ...; f_{N+1})^T \in R^N \forall i = 1, ..., N+1$

Now we substitute $\underline{u}$ as $\underline{u} = \underline{w} + \underline{R_D}$, and we find :

$$-\mathbf{A}\underline{u} + n^2\omega^2\mathbf{M}\underline{u} = \mathbf{F} \tag{14}$$

Exercise 4

The implementation of the finite solution of the Exercise 3, starts from the Matlab code analyzed during the lessons. We need to complete the struct DATI, inserting the following:

domain	[0,1]
μ	1
ω	1
n	1
$u_{ex}(x,t)$	$sin(2\pi x)$
f	$sin(2\pi x) - 4\pi^2 sin(2\pi x)$
fem	P1

Then we have solve the linear system by running the simulation with $nRef = 5$, obtaining:

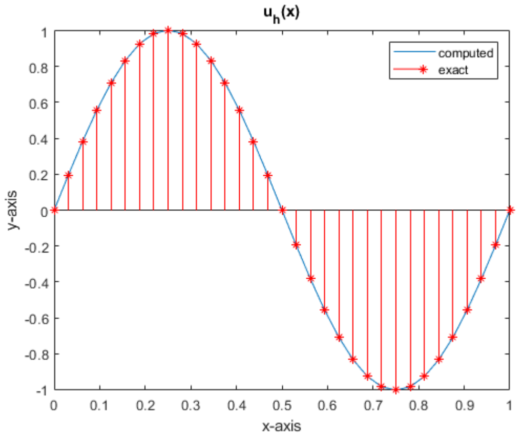

Figure 1: Solution for $nRef = 5$

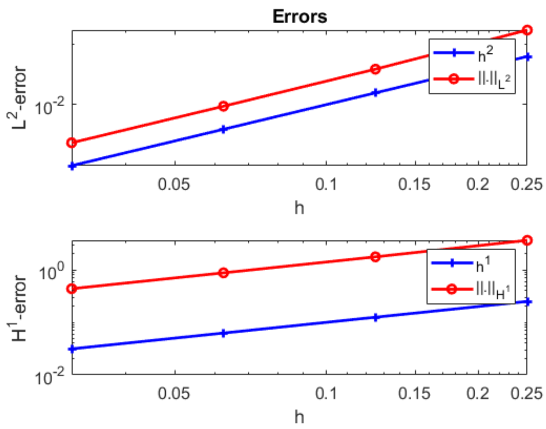

Figure 2: Convergence plot

The convergence test compute the solution in a sequence of mesh and then compute H^1 error and the L^2 error.

$$||u - u_h||_{H^1(0,1)} = [\int_0^1 (u - u_h)^2 dx + \int_0^1 (u' - u_h')^2 dx]^{1/2}$$

$$||u - u_h||_{L^2(0,1)} = [\int_0^1 (u - u_h)^2 dx]^{1/2}$$

The representation (2) is the behavior of the errors with respect to the mesh size. Theoretically, H^1 have to go to 0 like h, while L^2 as h^2, we can see as the numerical error computed respect the theoretical one.

Exercise 5

The problem we are considering now is the (1) of the homework. It can be rewritten as:

$$M\ddot{\underline{U}}(t) + A\dot{\underline{U}}(t) = \underline{F}(t) \tag{15}$$

where M,A $\in \mathbb{R}^{N-1,N-1}$, $\underline{U}$ is a vector containing the unknown coefficients $\underline{U}(t) = (u_1(t), ..., u_{N-1}(t))^T$ and $\underline{F}$ is the vector of the right hand side $\underline{F}(t) = (\int_0^L f\psi_1, ..., \int_0^L f\psi_{N-1})^T$.
It is a second order ordinary differential equation and considering the all problem it is:

$$\begin{cases} M\ddot{\underline{U}}(t) + A\underline{U}(t) = F(t) \quad t \in (0, T] \\ \underline{U}(0) = u_0(x) \\ \dot{\underline{U}}(0) = v_0(x) \end{cases} \tag{16}$$

The system can be transformed in a first order ODE:

$$\begin{cases} \dot{\underline{U}} = \underline{v} \\ M\dot{\underline{v}} + A\underline{U} = \underline{F} \\ \underline{U}(0) = u_0(x) \\ \underline{v(0)} = v_0(x) \end{cases} \tag{17}$$

We have now to integrate in time the ODE system, using the $Leap - Frog$ scheme : considering the time line, we discretize it, with equi-spaced intervals Δt and we approximate the derivative with a central finite difference:

$$\ddot{u}(t_k) \approx \frac{u(t_{k+1}) - 2u(t_k) + u(t_{k-1})}{\Delta t^2} \tag{18}$$

In order to compute u_{k+1} we need to know u_k and u_{k-1}, exploiting the $leap - frog$ scheme we compute a first step to calculate, with the initial condition u_0, u at t_1:

$$M\underline{U}_1 = (M - \frac{\Delta t^2}{2}A)\underline{u}_0(x) + \Delta t M\underline{u_1}(x) + \frac{\Delta t^2}{2}\underline{F_0} \tag{19}$$

and then we go on following this rule,

$$M\underline{U}_{k+1} = (2M - \Delta t^2 A)\underline{U}_k - M\underline{U}_{k-1} + \Delta t^2 \underline{F_k} \tag{20}$$

This scheme is second order accurate and it is explicit, and consequently conditionally stable which means that the scheme is stable if $\Delta t \leq costant\ \frac{h}{c}$. To implement it in Matlab we exploit the code CG_FEM_1D_WAVE_start, which compute the discretization in time with the leap-frog scheme and in space with the FEM. We have to modify the entry data as asked from the problem and the results are:

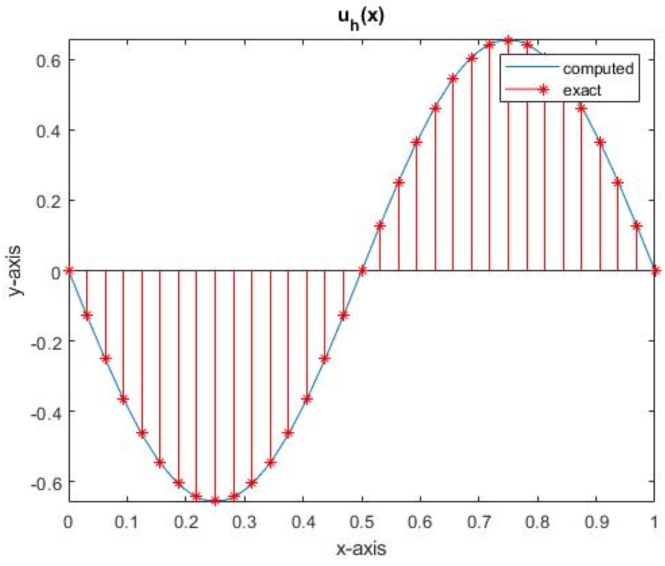

Figure 3: Solution with linear finite element discretization coupled with the leap-frog scheme

In particular report the errors obtained at the final observation time T = 4 are:

Error_{L2}	0.0031
Error_{SEMIH1}	0.2861
Error_{H1}	0.2861
Error_{∞}	0.0043

Exercise 6a

To solve the problem (2) of the homework with linear finite elements we exploit the same code of the exercise 4, considering the following set of data:

- $\Omega = (0,\, 1)$ and $n = 1$, $w = 2\pi$ and $u_{ex}(x,t) = sin(2\pi x)$
- $\Omega = (0,\, 1)$ and $n = 1$, $w = 1000\pi$ and $u_{ex}(x,t) = sin(2\pi x)$

The forcing term of the first set of data is: $f(x,t) = 0$. We also add the dependency on time to the solution, and the obtained results are:

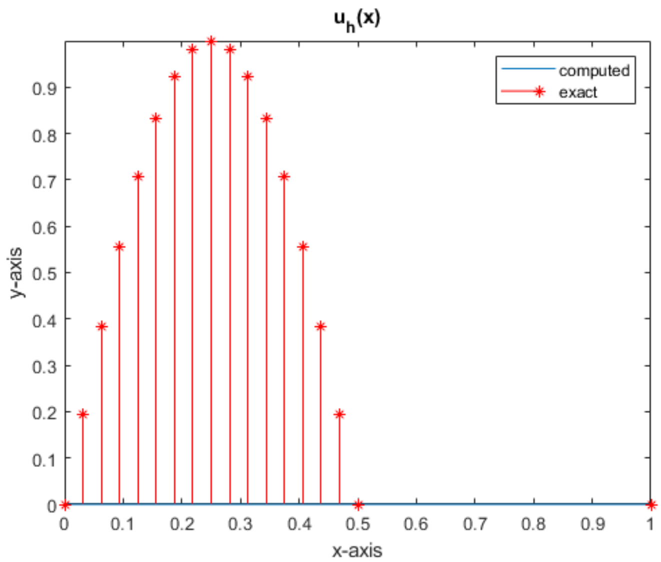

Figure 4: Plot for w=2π

e_{L2}	0.7071
e_{SH1}	4.4429
e_{H1}	4.4988
e_{∞}	1.0000

Table 1: Errors for $\omega = 2\pi$

Exercise 6b

With the second set of input data we can compute again the forcing term that this time is different from zero. The results:

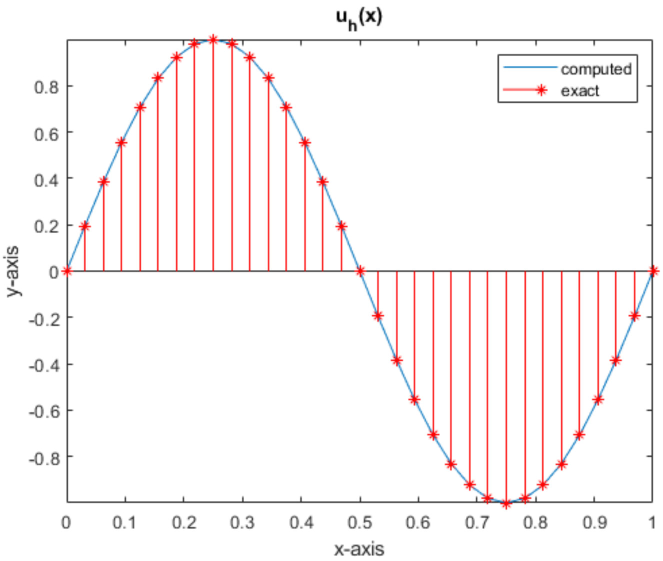

Figure 5: Plot for $w = 1000\pi$

e_{L2}	9.0754×10^{-9}
e_{SH1}	0.4357
e_{H1}	0.4357
e_{∞}	1.2835×10^{-8}

Table 2: Errors for $\omega = 2\pi$