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Minimal Steiner Trees for Rectangular Arrays of Lattice Points*

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We construct minimal Steiner trees for any square or rectangular array of integer lattice points on the Euclidean plane. © 1997 Academic Press

1. INTRODUCTION AND PRELIMINARIES

This paper answers a series of questions raised by Chung *et al.* in [3] on the length of the shortest network interconnecting a square or rectangular array of integer lattice points on the Euclidean plane. Such a network must clearly be a tree, and is known as a *minimal Steiner tree*. Minimal Steiner trees differ from minimal spanning trees in that they may contain vertices other than the initial given points. The original points being interconnected are usually referred to as *terminals* and the extra vertices as *Steiner points*. An example of a minimal Steiner tree, denoted X, for the vertices of a unit square is shown in Figure 1. A comprehensive discussion of minimal Steiner trees can be found in [5].

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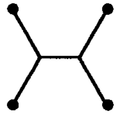


FIG. 1. The Steiner tree X.

In their paper, Chung *et al.* present a series of constructions for what they believed to be the minimal Steiner trees for all $n \times n$ square lattices. These constructions fall naturally into six classes based on the value of n modulo 6, except when n is a power of 2. The striking thing about all their constructions is that they contain large numbers of Xs as their basic building blocks. Recall that a Steiner tree (such as X) is referred to as *full* if each of its terminals has degree 1. The *full components* of a Steiner tree can be thought of as the smallest irreducible "blocks" from which the Steiner tree is composed (by union at the terminals). In each of Chung *et al.*'s constructions all but at most three of the full components are Xs and no full component contains more than 10 terminals. This contrasts markedly with the minimal Steiner trees for $2 \times n$ rectangular arrays which were shown in [4] to be full for all odd n.

Let T^* be a minimal Steiner tree on an $m \times n$ rectangular array of lattice points. For any $2^k \times 2^k$ square array, Chung *et al.* showed how to construct a Steiner tree all of whose full components are Xs, and conjectured that this Steiner tree is minimal. This was recently proved in [1], using the observation that, per terminal, X appears in some sense to be the most efficient possible full component of T^*. This idea is formalized by the concept of excess, which we define below.

Let T' be a subtree of T^* such that T' is a union of full components of T^* and spans r terminals. As in [1], let

$$\rho = \frac{|X|}{3} = \frac{1+\sqrt{3}}{3} = 0.91068\cdots$$

and define the *excess* of T' to be

$$e(T') = |T'| - (r-1)\rho.$$

Note that the excess is additive in the sense that if T' is a subtree of T^* such that $T' = \bigcup_{i=1}^{k} T_i$ where each T_i is a full component of T^* then $e(T') = \sum_{i=1}^{k} e(T_i)$. By definition $e(X) = 0$, and in [1, Theorem 3.1] it was shown that $e(T') \geqslant 0$ if T^* is a minimal Steiner tree for a $2^k \times 2^k$ square array. It immediately follows from the proof of that theorem that the full components of a minimal Steiner tree on any $m \times n$ rectangular array of lattice points have non-negative excess.

It is clear from the definition that for two or more Steiner trees on the same set of terminals, the tree with shortest length has the smallest excess. Hence the fact that T^* necessarily has non-negative excess immediately implies that the Steiner trees for $2^k \times 2^k$ square arrays composed solely of Xs are minimal.

It was proved in [3] that T^* is composed solely of Xs only if $m = n = 2^k$. So in other cases T^* must have one or more full components not equal to X. A crucial step in proving the minimality of the constructions of Chung *et al.* in these cases will be to show that only small full components of T^* have small excess. This follows from the results of [2], where we classify all possibly full components of a minimal Steiner tree for any nicely clustered set of lattice points. The classification, proved largely by geometric techniques, effectively reduces the problem to a purely combinatorial one, which we address in this paper.

In order to summarize and appreciate the significance of the results from [2] we first require some definitions. Consider an infinite square unit lattice on the Euclidean plane. A finite subset, P, of vertices of this lattice will be said to form a *Steiner-closed lattice set* if it satisfies the following conditions:

(i) there exists a spanning tree for P all of whose edges have length 1; and

(ii) given lattice points a and b such that $|ab| = 1$, if a minimal Steiner tree for P intersects the interior of ab then a and b are elements of P.

Note that if a set of lattice points P has the property that for any unit lattice edge meeting a lattice point not in P the interior of that edge lies entirely outside the convex hull of P, then P is Steiner-closed. It follows, for example, that any square or rectangular array of lattice points forms a Steiner-closed lattice set.

Given our infinite square unit lattice in the Euclidean plane, we define a *ladder* to be a finite sequence of adjacent unit squares all lying in the one row or column. A ladder is said to be *horizontal* if the squares all line in the same row, and *vertical* if they all lie in the same column. We define a *staircase* to be a finite sequence of adjacent right lattice triangles (each formed from two edges and a diagonal of a unit square) in the square lattice with the property that they are adjacent along unit edges and all the hypotenuses of the triangles are parallel. A staircase is said to be *ascending* if the hypotenuses lie at an angle of 45° from the horizontal and *descending* if they lie at an angle of 135° from the horizontal.

Let S be a finite alternating sequence of adjacent ladders and staircases, with the adjacencies occurring at the ends of the ladders and staircases.

A staircase in S is said to be *internal* if it is adjacent to two ladders, and *external* if it is adjacent to precisely one ladder. We say that S is a *strip* if it satisfies the following two conditions:

(i) Either all ladders in S are horizontal, or all ladders in S are vertical. Likewise, all staircases in S are ascending, or all are descending.

(ii) If S contains no ladders, then S contains exactly one or an even number of triangles. If S contains one or more ladders, then all internal staircases of S contain an even number of triangles, and all external staircases of S contain an odd number of triangles.

In [2, Theorem 5.6] it was proved that if T^* is a minimal Steiner tree for a Steiner-closed lattice set then each full component of T^* spans the vertices of a strip. It was then determined which strips have full minimal Steiner trees. To recall these results we need a little more notation. For any positive integers p and q, denote by a p-ladder a ladder formed from p adjacent unit squares, and similarly by a q-staircase a staircase formed from q adjacent right triangles. A $[2k, l]$-*strip* is defined to be a strip consisting of l $2k$-ladders separated by $l-1$ internal 2-staircases. Similarly, a $\langle 2k, l]$-*strip* is a $[2k, l]$-*strip* with an external 1-staircase (that is, a single triangle) on one end, while a $\langle 2k, l\rangle$-*strip* is a $[2k, l]$-strip with external 1-staircases on both ends. Where there is no possibility of ambiguity, we will sometimes also use this notation to describe a minimal Steiner tree spanning these lattice points.

THEOREM 1.1 [2]. *Let T^* be a minimal Steiner tree for a Steiner-closed lattice set. Let T be a full component of T^*, containing at least one Steiner point. Then the terminals of T are the vertices of either*

(i) *a right lattice triangle*;

(ii) *a unit square*;

(iii) *a $2k$-staircase*;

(iv) *a $\langle 2k, 1]$-strip*;

(v) *a $[2k, l]$-strip*; *or*

(vi) *a $\langle 2k, l\rangle$-strip.*

With k and l ranging over all positive integers this gives a complete irredundant classification of possible full components of T^*.

In proving the above theorem we showed precisely how to construct a minimal Steiner tree for each of these strips, and determined the length of T in terms of k and l [2, Table 1]. This description of $|T|$ immediately implies the following corollary, which states that for each of the above classes the excess of the minimal Steiner tree increases as k or l increases.

COROLLARY 1.2. *Let i be any positive integer. Then the excess of a minimal Steiner tree for a $2k$-staircase, $\langle 2k, 1]$-strip, $[2k, l]$-strip, or $\langle 2k, l\rangle$-strip is strictly less than that for a $(2k+i)$-staircase, $\langle 2k+i, 1]$-strip, $[2k+i, l]$-strip, or $\langle 2k+i, l\rangle$-strip respectively. Similarly, the excess of a minimal Steiner tree for a $[2k, l]$-strip, or $\langle 2k, l\rangle$-strip is strictly less than that for a $[2k, l+i]$-strip, or $\langle 2k, l+i\rangle$-strip respectively.*

Using this corollary it is now straightforward to systematically list all possible full components of T^* whose excess is less than some fixed constant. This now puts us in a good position to find the minimal Steiner trees for all $n \times n$ square arrays.

2. MINIMAL NETWORKS FOR SQUARE ARRAYS

Before proving our central theorem, it will be useful to simplify our notation a little. Let A_{2k} be a minimal Steiner tree for a $[2k, 1]$-strip (i.e., a $2k$-ladder), let B_{2k+1} be a minimal Steiner tree for a $\langle 2k, 1]$-strip, and let C_{2k+2} be a minimal Steiner tree for a $\langle 2k, 1\rangle$-strip. Examples of each of these trees are given in Figure 2. Note that in each case the subscript corresponds to the number of unit squares entered by the tree. By Theorem 1.1, all these minimal Steiner trees are full. Also, let I denote a unit edge, and Y the minimal Steiner tree for a right lattice triangle.

Let n be a positive integer which is not equal to 6 or a power of 2. In [3] Chung *et al.* construct a series of Steiner trees for $n \times n$ checkerboards which they conjecture to be minimal. The lengths of these conjectured solutions are as shown in Table I, except when $n = 6k$, in which case their construction has length $(n^2 - 3)\rho + |Y|$ and excess $e(Y) \approx 0.11048$. (In the proof of Theorem 2.1 we will show how their conjectured solutions in this case can be improved for $n \neq 6$.) The usefulness of the concept of excess lies in the fact that the conjectured minimal Steiner trees for square arrays all have small bounded excess. Let T be a minimal Steiner tree for an $n \times n$ square array. We know that in each case $e(T)$ must be less than or equal to the excesses listed in Table I. Using Corollary 1.2 and the characterization of minimal Steiner trees for strips in [2] we can list all possible full components of T by listing all full minimal Steiner trees on strips with excess less than or equal to $3e(I)$. This list appears in Table II.

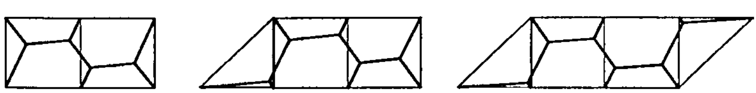


FIG. 2. The minimal Steiner trees A_2, B_3, and C_4.

TABLE I

Minimal Lengths and Excesses for Steiner Trees on $n \times n$ Square Arrays, for All n Not Equal to 6 or a Power of 2

n	Length of minimal Steiner tree	Excess
$6k$	$(n^2-4)\rho+\lvert A_2\rvert$	$e(A_2)\approx 0.07176$
$6k+1$	$(n^2-4)\rho+3$	$3e(I)\approx 0.26795$
$6k+2$	$(n^2-10)\rho+\lvert A_4\rvert$	$e(A_4)\approx 0.14897$
$6k+3$	$(n^2-3)\rho+2$	$2e(I)\approx 0.17863$
$6k+4$	$(n^2-10)\rho+\lvert A_4\rvert$	$e(A_4)\approx 0.14897$
$6k+5$	$(n^2-4)\rho+3$	$3e(I)\approx 0.26795$

We can now show that the minimal solutions in Table I are correct for all n not equal to 6 or a power of 2.

THEOREM 2.1. *Let T be a minimal Steiner tree for an $n\times n$ square lattice, where n is not a power of 2. If $n=6$, then $|T|=33\rho+|Y|$. For any other value of n, $|T|$ is as given in Table* I.

Proof. Let $\{T_i\}_{i\in I}$ be the set of full components of T, and let r_i be the number of terminals in each T_i. Clearly, $\Sigma_I(r_i-1)=n^2-1$. Hence, if $\{T_i\}_{i\in I'\subset I}$ is the set of full components of T which are not Xs, then

$$\sum_{I'}(r_i-1)\equiv n^2-1 \pmod 3.$$

So in order to check whether the conjectured solutions in Table I can be improved we only need to consider sets of trees in Table II whose total

TABLE II

All Possible Full Components of T

Full component	r_i-1 (mod 3)	Approximate excess
X	0	0
A_2	2	0.071764
I	1	0.089316
Y	2	0.110449
A_4	0	0.148967
B_3	0	0.156259
C_2 (i.e., a 2-staircase)	0	0.177262
[2, 2]-strip	2	0.227552
A_6	1	0.233649
B_5	1	0.236679
C_4	1	0.242209

excess is less than that conjectured, and whose full components other than X satisfy the above condition.

For $n=6k$, this means the only possibilities for improving the conjectured solution come from trees with one full component being an A_2, and the rest Xs. We first show that such a tree cannot cover the 6×6 square lattice. Assume, on the contrary, the 6×6 square lattice can be covered by such a tree. The two squares of the A_2 cannot lie along an outer edge of the lattice since the remaining 3 vertices on that edge cannot then be covered by Xs. Hence, there is an X in each corner square of the lattice. If one square of the A_2 lies along an outer edge of the lattice (as in Figure 3) then there is no way of connecting the two corner Xs on that edge to the rest of the tree. Hence, all the outer edge vertices are covered by Xs, three of which must lie on each outer edge. Wherever one places the A_2, however, one of its squares must share an edge with a boundary square containing an X, giving the desired contradiction.

For the case where $n=6k$ and $k>1$ the $n\times n$ square lattice can be covered by Xs and an A_2, as seen by taking the conjecture minimal tree in [3], deleting the Y, and a nearby X, and covering the 5 free vertices so created with an A_2. This is illustrated for $k=2$ in Figure 4. For $n=6k+2$ and $6k+4$, a single A_4 has the smallest excess possible for the required set of trees, and so these conjectures are true immediately.

The only other combinations of minimal trees with smaller excess than the conjectured values, and with the correct number of vertices modulo 3, are

(a) for $n=6k+3$, where $n^2-1\equiv 2 \pmod 3$, just one A_2 or Y, and

(b) for all $n=6k+1$ and $n=6k+5$, where $n^2-1\equiv 0 \pmod 3$, various combinations with small excess such that the sum of r_i-1 over all components is divisible by 3.

These values of n are all odd, so no boundary edge of the array can be covered entirely with Xs, which necessarily cover the edge terminals two at a time (since two squares containing Xs cannot share an edge). Hence, at

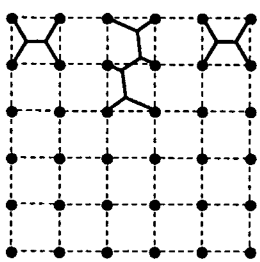


FIG. 3 The two corner Xs shown cannot be connected to the rest of the tree.

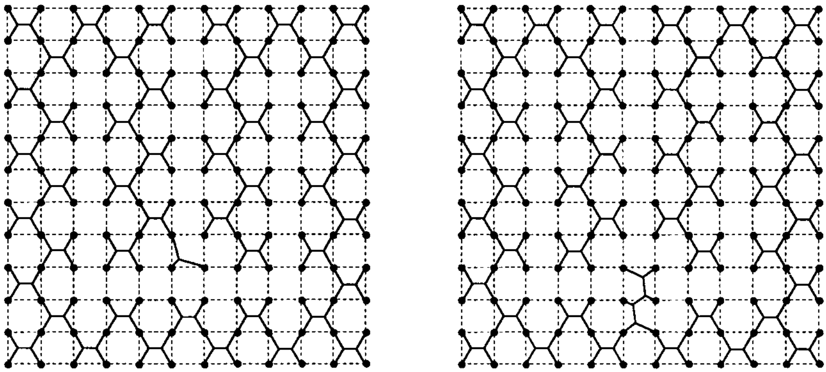


FIG. 4 The conjectured and actual minimal Steiner trees for the 12×12 array.

least one terminal along each boundary edge of the array must be contained in one of the non-X components. This condition immediately eliminates (a), and implies that in (b) at least two non-X components must appear. If only two appear and one is an A_2, then they must cover vertices at diagonally opposite corners of the board. But, in that case, if we consider the edge of the board with only two vertices covered by the A_2, it follows that the number of remaining terminals along this edge is odd and hence cannot be covered with Xs. For similar reasons we do not need to consider combinations with only two non-Xs, one of which is any of the configurations in Table II. The only remaining possibility with total excess less than $3e(I)$ is three A_2s. This is also eliminated by the same reasoning, and the proof is complete. ∎

3. MINIMAL NETWORKS FOR RECTANGULAR ARRAYS

In their concluding remarks, Chung *et al.* [3] briefly mention the problem of constructing minimal Steiner trees for $m \times n$ rectangular arrays of lattice points, and state, in effect, that for m and n sufficiently large the excess of the minimal Steiner tree is bounded. This problem is somewhat more difficult than that for square arrays. Here we present a complete solution to the problem for all m and n. Throughout this section we will assume $m \neq n$.

The organization of this section is as follows. After first proving a useful technical lemma, we give constructions for the minimal Steiner trees where $\min\{m, n\} \leqslant 7$, but not equal to 4. Next, we consider the case where m and n are both greater than 7, first solving the case where m and n are even, then various cases where at least one of m and n is odd. Finally we examine the minimal Steiner trees for arrays in which $\min\{m, n\} = 4$. These trees can have very large excess, and hence require the development of special techniques to prove their optimality.

One of the most basic methods of constructing Steiner trees with small excess for large arrays is to incorporate in them long zigzagging sequences of Xs (this method is used extensively in [3]). We will refer to such a sequence as a *chain of* Xs. For example, the Xs in the lefthand diagram of Figure 4 form a single chain of Xs folded around on itself. The following lemma, which gives us a simple method of expanding certain arrays without increasing the excess of their minimal Steiner trees, will prove useful throughout this section.

LEMMA 3.1. *Let m be an even integer greater than or equal to* 6. *Let T be a minimal Steiner tree for an $m \times n$ rectangular array, and let T' be a minimal Steiner tree for an $m \times (n+6k)$ rectangular array* $(k \geqslant 0)$. *If T contains a sequence of five adjacent Xs, three of which lie on one of the edges of the array with m terminals, then $e(T') \leqslant e(T)$.*

Proof. We first prove the lemma for $k=1$. Suppose T contains a sequence of five adjacent Xs, three of which lie along the right edge of the array. Denote by X_2 the middle of these three Xs. Place an $m \times 6$ closed chain of Xs to the right of the array, as shown in Figure 5. Now delete X_2 and the closest X in the chain of Xs and add two neighbouring Xs as shown in the figure. The resulting construction is clearly a tree and has the same excess as T. This procedure can be repeated as often as required, proving the lemma for all $k > 0$. ∎

There are two other observations we will need throughout this section (both of which are mentioned in the proof of Theorem 2.1). First we should

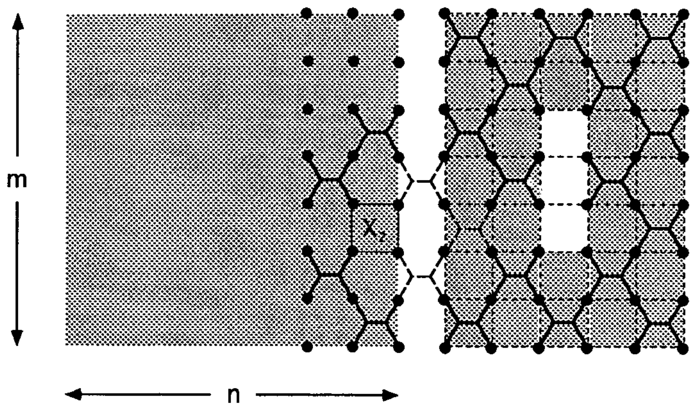


FIG. 5. Figure for Lemma 3.1.

note that, for all cases where T has sufficiently bounded excess, all full components of T must come from those listed in Table II, and satisfy the condition that $\sum (r_i - 1) \equiv mn - 1 \pmod 3$. Also observe that if m is odd then T must contain at least two non-X full components, except in a few cases where m or n is small.

3.1. *Small Cases*

Assume throughout this subsection that $m < n$. We will construct minimal Steiner trees, T, for all $m \times n$ rectangular lattices where $m \leqslant 7$ but $m \neq 4$. The case where $m = 4$ is discussed in Section 3.3. If $m = 1$ then T consists entirely of Is. If $m = 2$ then T is either the full tree A_{n-1} (if n is odd) or an alternating sequence of Xs and Is (if n is even). This was proved in [4]. Note that in each of these cases $e(T)$ is unbounded.

THEOREM 3.2. *Let T be a minimal Steiner tree for a $3 \times n$ rectangular array with $n > 3$. Then the set of full components of T consists of $(n-1)$ Xs and two Is.*

Proof. The constructions for odd and even values of n are simply straight chains of Xs with an I on either end, as shown in Figure 6. Since 3 is odd, there must be at least two non-X components and, by Table II, two Is is the combination with smallest excess having the correct number of vertices modulo 3. Hence the construction is minimal. ∎

THEOREM 3.3. *Let T be a minimal Steiner tree for a $5 \times n$ rectangular array with $n > 5$. Then the set of full components of T consists of Xs and*

(i) *two Is if $n \equiv 0 \pmod 3$; or*

(ii) *two Is and an A_2 if $n \equiv 1 \pmod 3$; or*

(iii) *three A_2s if $n = 8$; or*

(iv) *three Is if $n \equiv 2 \pmod 3$ and $n \geqslant 11$.*

Proof. Figures 7 and 8 show Steiner trees for $n = 6$, 7, 8 and 11 with the required set of full components. In each of the diagrams in Figure 7 we can increase n by 3 without changing the set of non-X components in the tree by deleting the I in the bottom left-hand corner, then adding five Xs to the left of the construction and an I in the top left-hand corner. Clearly this

FIG. 6. Minimal Steiner trees for $3 \times n$ arrays.

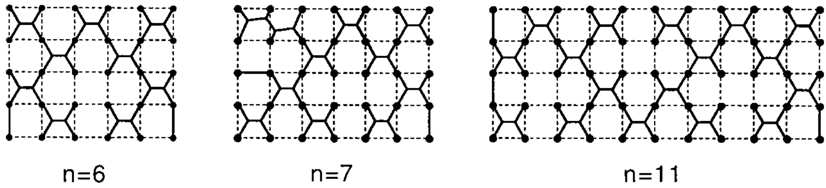


FIG. 7. Minimal Steiner trees for the $5 \times n$ arrays, where $n = 6$, 7, or 11.

procedure can be repeated as often as required, giving a construction for Steiner trees with the correct set of full components.

We next show that the Steiner trees constructed in each case are minimal. In Case (i) the excess is the smallest possible, so minimality immediately holds.

For Case (ii), note that the construction has the smallest possible excess for a tree with three non-X components spanning the correct number of vertices modulo 3. There are five possible sets of two non-X components with smaller excess: two $A_2 s$; an A_2 and a Y; two Ys; an I and an A_4; an I and a B_3. We will show that none of these can occur in T.

Supppose T contains two A_2s. Since the leftmost and rightmost edges of the array cannot be covered by Xs, one of the A_2s must lie along (that is, cover three terminals of) the leftmost edge of the array, and the other must lie along the rightmost edge of the array. Both clearly lie in corners of the array, and hence are each adjacent to at most one X. But if we consider the forest, T_0, obtained by deleting the two A_2s from T, a simple parity argument shows that none of the Xs lying along the top edge of the array can be in the same connected component of T_0 as any of the Xs lying along the bottom edge. Hence one of the A_2s must be adjacent to two Xs, giving the desired contradiction.

If T contains a Y, then that Y must lie on the leftmost or rightmost edge of the array in an adjacent square to an X. But in that case, the X and Y can be replaced by an A_2 to form a shorter tree. If T contains an A_4 it must lie along the leftmost or rightmost edge of the array and cannot be adjacent to any X.

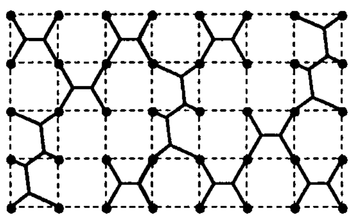


FIG. 8. A minimal Steiner tree for the 5×8 array.

Finally, suppose T contains a B_3 and an I. If the B_3 lies along the leftmost or rightmost edge of the array then there must be an X in the adjacent square on the same edge, and again the length of T can be shortened by replacing the pair by an A_4. The only other possibility is that the B_3 lies along the top or bottom edge of the array with the external 1-staircase of the underlying strip located in one of the corner squares of the array. So, by symmetry, we can assume B_3 is in the bottom righthand corner of the array, as in Figure 9. If n is odd, it immediately follows from a parity argument that T must contain at least two non-X components other than the B_3, giving a contradiction. If n is even then, by a similar parity argument, there must be Xs in the squares in the top and bottom lefthand corners of the array, while the middle vertex of the lefthand edge of the array must be a terminal of the I (as in the figure). But there can now only be one X in the second column of Xs from the left, which immediately contradicts the fact that T is connected. This completes Case (ii).

For Cases (iii) and (iv), T must contain at least three non-X full components by a similar argument to that used in Case (ii). In Case (iii) the excess is the smallest possible, giving minimality. For Case (iv) we need to eliminate the possibility where T contains three A_2s.

So suppose T contains three A_2s. The three A_2s cannot all lie along outer edges of the array, as that would imply that an A_2 lies along exactly one of the top and bottom edges of the array, which is impossible by a simple parity argument. Let A' denote the A_2 not lying along the boundary of the array. Again let T_0 be the forest obtained by deleting all the A_2s from T. By the argument in Case (ii) A' must be adjacent to at least two Xs. It is easily checked that this is only possible if A' is vertical and adjacent to four Xs, as in Figure 8. Without loss of generality we may assume that there are at least five rows of Xs to the right of A'. Now if we consider all the Xs to the right of A', they must belong to exactly two components of T_0 each of which contain Xs of different parity. But only one of the Xs adjacent and to the right of A' can be adjacent to another X, hence there will be at least one X lying on the top or bottom edge unaccounted for, giving a contradiction. This completes the proof of the theorem. ∎

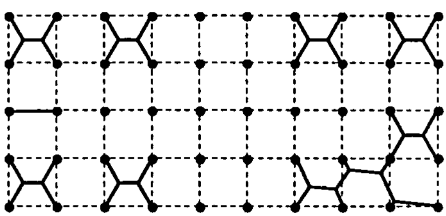


FIG. 9. A minimal Steiner tree for Case (ii) cannot contain a B_3 and an I.

THEOREM 3.4. *Let T be a minimal Steiner free for a $6 \times n$ rectangular array with $n > 6$. Then the set of full components of T consists of $(2n-1)$ Xs and two Is if n is odd, or $(2n-2)$ Xs and an A_2 if n is even.*

Proof. Suitable Steiner trees for $n = 7$ and $n = 8$ are shown in Figure 10. The first can be thought of as a chain of Xs with an I at each each end, and the second as two chains of Xs meeting at an A_2. We can generalize these diagrams to constructions for all n by increasing the length of one of the chains of Xs in each case. It is immediate that in each case the excess is the smallest possible. ∎

THEOREM 3.5. *Let T be a minimal Steiner tree for a $7 \times n$ rectangular array with $n > 7$. Then the set of full components of T consists of Xs and*

(i) *two Is if $n \equiv 0 \pmod 3$; or*
(ii) *two A_2s if $n = 8$; or*
(iii) *two Is and an A_2 if $n \equiv 2 \pmod 3$ and $n \geqslant 11$; or*
(iv) *three A_2s if $n = 10$ or 16; or*
(v) *three Is if $n \equiv 1 \pmod 3$ and $n \neq 10$ or 16.*

Proof. Much of the proof of this theorem is similar to the proof of Theorem 3.3. Constructions for $n = 8$, 9, 10, 11, 13 and 16 are shown in Figures 11 and 12. As in the proof of Theorem 3.3, the diagrams in Figure 11 can be extended, increasing n by any multiple of 3, to give constructions with the correct set of components for all remaining n.

In proving minimality, most of the proof follows closely that of Theorem 3.3. Only three cases are significantly more difficult, namely, showing that in Case (iii) the array cannot be covered by Xs and two A_2s or Xs, a B_3 and an I, and that in Case (v) the array cannot be covered by Xs and three A_2s. We will eliminate each of these possibilities in turn, by contradiction.

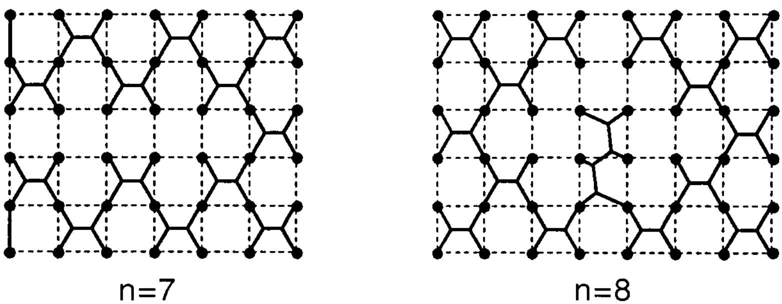


FIG. 10. Minimal Steiner trees for some $6 \times n$ arrays.

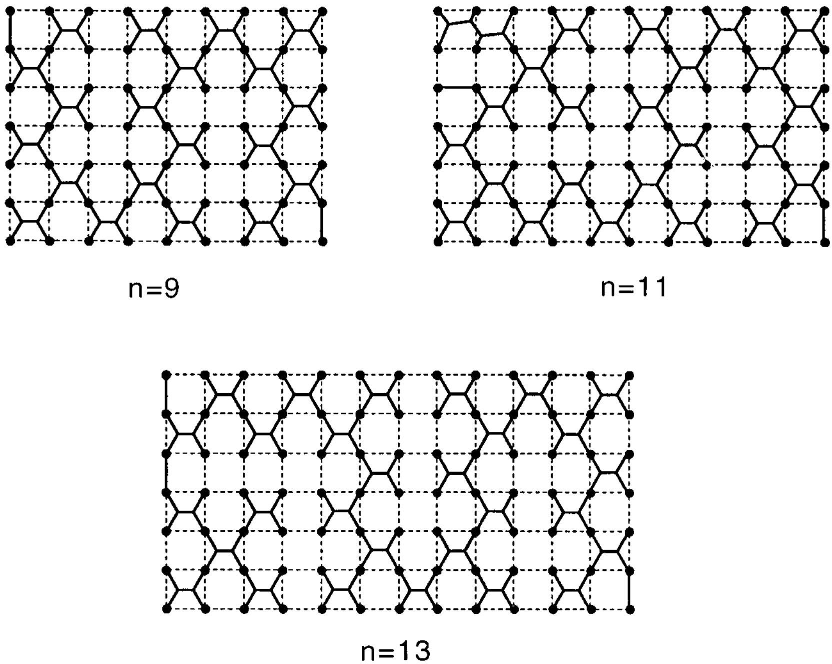


FIG. 11. Minimal Steiner trees for $7 \times n$ arrays when $n = 9$, 11, or 13.

(a) For the first possibility, let $n \equiv 2 \pmod 3$ and $n \geqslant 11$, and assume the corresponding minimal Steiner tree T contains Xs and two A_2s. Clearly n must be even and the A_2s must lie along the leftmost and rightmost edges of the array. As before, let T_0 be the forest obtained by deleting both A_2s from T. T_0 has at most three connected components, and on at least one side, say the left, an A_2, denoted A', is adjacent to two connected components of T_0 each of which reach more than three columns to the right (otherwise some of the boundary Xs of T are unreachable). Since A' is adjacent to two Xs it cannot be in a corner of the array, but must occur in the center of the leftmost column of unit squares, as in Figure 12(i). Also, as in the figure, the second column of squares from the left necessarily contains precisely two Xs (in the second and fifth squares from the top), and the third column from the left contains precisely three Xs, in the top square, the bottom square, and either the third or fourth square from the top. But it is now clear that since T is a tree the fourth column of squares from the left can contain at most one X, forcing one of the connected components of T_0 to terminate. This provides the desired contradiction.

(b) Again letting $n \equiv 2 \pmod 3$, assume the non-X full components of T are a B_3 and an I. By the proof of Theorem 3.3(ii), we can assume that

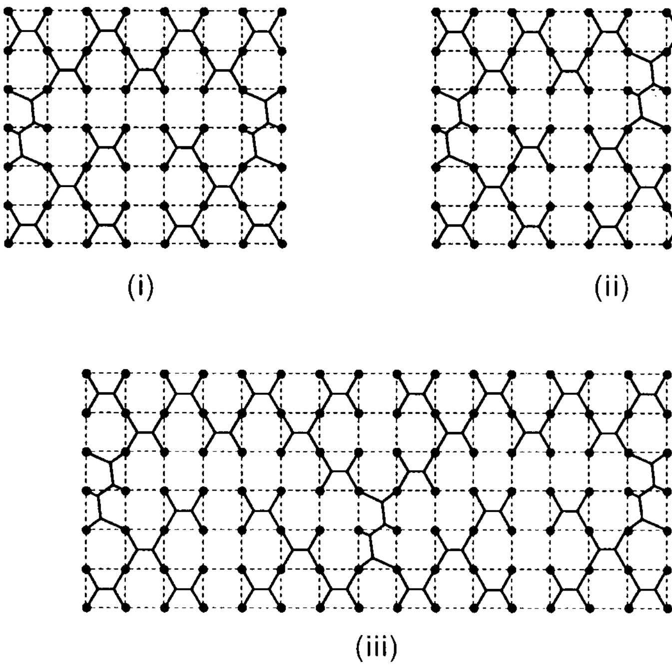


FIG. 12. Minimal Steiner trees for $7 \times n$ arrays when $n = 8$, 10, or 16.

the B_3 lies in the bottom righthand corner of the array positioned so that it meets only one terminal on the righthand edge of the array (namely the bottom terminal), and that n is even. Again there must be Xs in the top and bottom lefthand corners of the array, while the third vertex from the top or bottom of the lefthand edge of the array must be a terminal of the I. But, by the same argument as in (a), the fourth column of squares from the left can contain at most one X, which makes it impossible for T to connect the Xs in the top and bottom lefthand corners of the array.

(c) For the final possibility, let $n \equiv 1 \pmod 3$ and $n \neq 10$ or 16, and assume the corresponding minimal Steiner tree T contains Xs and three A_2s. Again, as in the proof of Theorem 3.3, n must be even and exactly two of the A_2s must lie along the leftmost and rightmost edges of the array. Let A' denote the remaining A_2 and again let T_0 be the forest obtained by deleting all three A_2s from T. Without loss of generality we may assume A' intersects or lies to the left of the central column of unit squares of the array. It follows, as in (a), that the connected components of T_0 containing the Xs in the top and bottom edges of the array immediately to the right of A' must reach more than three columns to the right of A'. We will first show that this does not occur if A' lies in the fourth and fifth squares from the top of the kth column from the left where k is even (as in Figure 12(iii)). The argument is very similar to that in (a). It is easily checked that, as in the figure, Xs must occur as follows: in the top, third and bottom squares of the $(k+1)$st column; in the second and fourth squares of the $(k+2)$nd column; and in the top, bottom and either third or fourth squares of the $(k+3)$rd column. As in (a), there can now be only one X in the $(k+4)$th column, giving a contradiction. If A' is in any other (horizontal or vertical) position in the array a similar or easier argument again contradicts the required condition on T_0. ∎

3.2. *Large Cases*

The strategy here is very similar to that for squares. Let T be a minimal Steiner tree for an $m \times n$ rectangular array. As in [3] we think of T as consisting of a chain of Xs winding around a central core, which is a Steiner tree for an $a \times b$ rectangular array (a and b even) connected to the chain in a suitable way. This is illustrated in Figure 13, for m and n both even. If m or n is odd we can delete the part of the chain of Xs running along the bottom or leftmost edge of the array. This construction gives us a Steiner tree whose excess is determined by the values of m and n mod 6. We then show that each such tree is minimal using a similar method of proof to Theorem 2.1.

There are a number of essentially different cases to consider. The first is where m and n are both even.

THEOREM 3.6. *Let m and n be distinct even integers greater than 7 such that $m \bmod 6 \leqslant n \bmod 6$. Let T be a minimal Steiner tree for an $m \times n$ rectangular array. Then the set of full components of T consists entirely of Xs and*

(i) *an A_2 if $(m, n) \equiv (0, 0)$, $(0, 2)$ or $(0, 4)$ (mod 6); or*

(ii) *an A_4 if $(m, n) \equiv (2, 2)$ or $(4, 4)$ (mod 6); or*

(iii) *an I if $(m, n) \equiv (2, 4)$ (mod 6).*

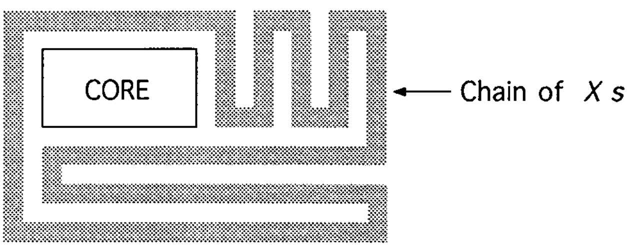


FIG. 13. A core and its surrounding chain of Xs.

Proof. To prove this theorem we have to show how to construct each of the above solutions, then prove minimality. The size of the core in each case is shown in Table III. We consider each of the cases in turn.

(i) In this case the core can be covered by Xs and a Y as shown in Figure 14. To attach the core correctly to the surrounding chain of Xs we follow the same procedure as for the $6k \times 6k$ square, that is, we attach the core to the chain of Xs as in the proof of Lemma 3.1, then we delete the Y of the core and a nearby X of the chain of Xs and cover the free vertices with an A_2 (as in Figure 4). The excess here is the smallest possible for the required set of trees, so the construction gives a minimal Steiner tree.

(ii) A suitable construction for the 8×14 array is shown in Figure 15. This can be extended to a construction for any $8 \times (14 + 6k)$ array using Lemma 3.1. The 10×10 and 14×14 cores are shown in [3]. Again these can be extended by multiples of 6 in one direction using Lemma 3.1 or can be attached to a surrounding chain of Xs in a similar manner. The excess in each case is the smallest possible, so the solution is minimal.

(iii) The chain of Xs can be suitably attached to the 4×8 core as shown in Figure 16. Again the excess, and hence length, is the smallest possible. ▮

We next consider cases where at least one of m and n is odd, and m and n are both greater than 7. In each case T must contain at least two non-X full components.

THEOREM 3.7. *Let m and n be distinct integers at least one of which is odd, such that $m \equiv 0 \pmod 3$. Let T be a minimal Steiner tree for an $m \times n$ rectangular array. Then the set of full components of T consists entirely of Xs and two Is.*

Proof. We can construct T as a single chain of Xs with an I at each end. The excess is the smallest possible. ▮

THEOREM 3.8. *Let m and n be distinct odd integers, such that m and n are both greater than 7, neither is a multiple of 3 and $m \bmod 6 \geqslant n \bmod 6$.*

TABLE III

Core Sizes for Rectangular Arrays where m and n Are Even

$(m, n) \pmod 6$	Core size	$(m, n) \pmod 6$	Core size
(0, 0)	6×6	(2, 2)	14×14
(0, 2)	8×6	(2, 4)	4×8
(0, 4)	4×6	(4, 4)	10×10

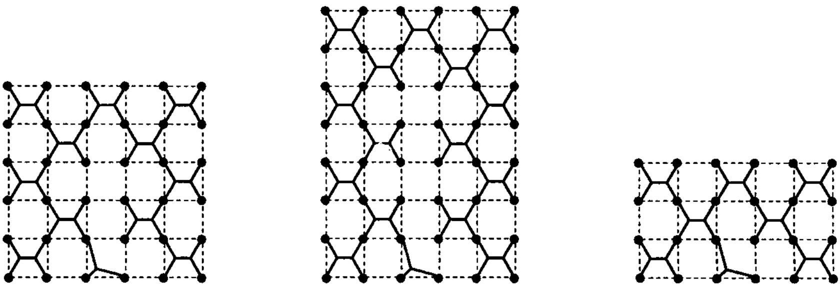


FIG. 14. The 6×6, 8×6, and 4×6 cores.

Let T be a minimal Steiner tree for an $m\times n$ rectangular array. Then the set of full components of T consists entirely of Xs and

(i) *three Is if* $(m, n)\equiv(1, 1)$ *or* $(5, 5)$ (mod 6); *or*

(ii) *an A_2 and two Is if* $(m, n)\equiv(5, 1)$ (mod 6).

Proof. We first show that we can construct the given solutions in each case. Again we think of T as consisting of a chain of Xs winding around a central core, as in Figure 13, with the part of the chain running along the bottom and left-hand edge of the array deleted. For (i), let the core be the minimal Steiner tree for a 4×4 or 2×2 square array respectively. By putting an I at each end of the chain and using a third I to attach the core to the chain we obtain the given solution.

For (ii), let the core be the minimal Steiner tree for the 2×4 rectangular array, as in Figure 17. To attach the core to the chain of Xs, we delete the I in the top lefthand corner of the array, the two nearby Xs of the chain and the I of the core, and replace them by an A_2, an I and X, as shown in the figure. This gives the desired construction.

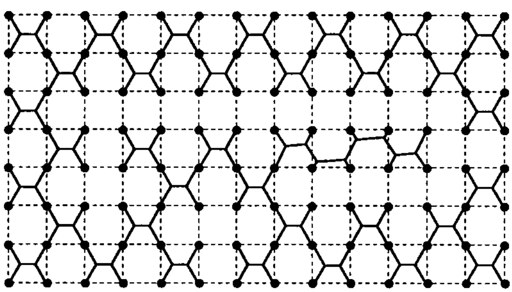


FIG. 15. A minimal Steiner tree for the 8×14 array.

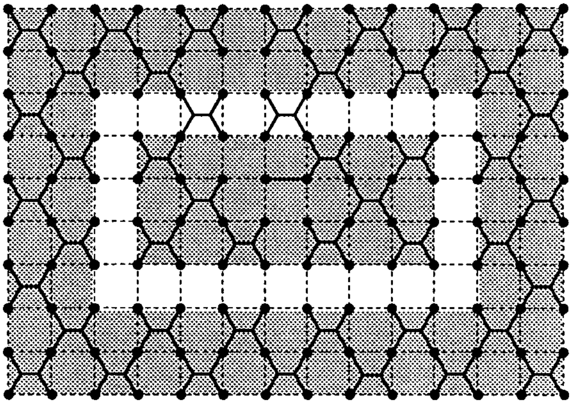


FIG. 16. Attaching a 4×8 core to a chain of Xs.

Finally, we show that these constructions are minimal. Since all four boundary edges of the array contain an odd number of terminals, if T contains only two non-X components they must both be Is, whereas if T contains three non-X components at least one must be an I. It follows that the constructions given have the smallest possible excess. ▮

THEOREM 3.9. *Let m be an odd integer and n an even integer, such that m and n are both greater than 7 and neither is a multiple of 3. Let T be a minimal Steiner tree for an $m \times n$ rectangular array. Then the set of full components of T consists entirely of Xs and*

(i) *an A_2 and an I if $(m, n) \equiv (5, 2)$ or $(1, 4) \pmod 6$; or*

(ii) *two A_2s if $(m, n) \equiv (5, 4)$ or $(1, 2) \pmod 6$.*

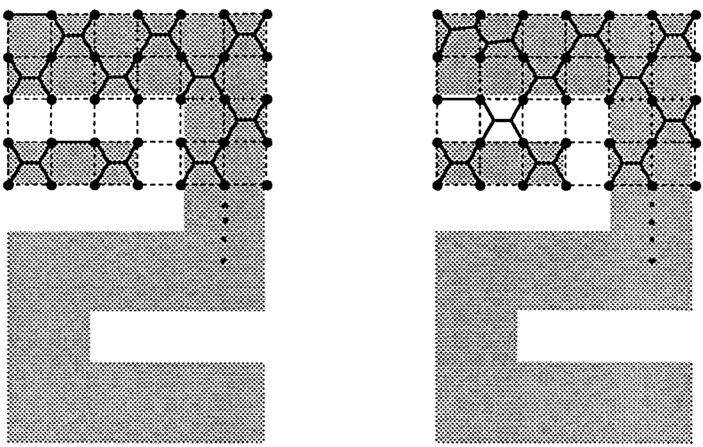


FIG. 17. Attaching a 2×4 core to a chain of Xs.

Proof. For this theorem the core and chain method of construction does not give a minimal solution, so a somewhat different method of constructing solutions for all m and n is required. It is clear in each case that if we can construct a Steiner tree with the set of full components given in the statement of the theorem then the Steiner tree has smallest possible excess and hence is minimal.

(i) Let $(m, n) \equiv (5, 2) \pmod 6$. A construction for the case where $(m, n) = (11, 8)$ is given in Figure 18(a). We can generalize this construction to a Steiner tree for an $11 \times (8 + 6k_2)$ array by extending the tree to the right as follows: extend the part of the Steiner tree in the top four rows of squares by deleting the I and adding $10k_2$ new Xs and an I to the top four rows (as in the proof of Theorem 3.4); then extend the part of the tree in the bottom five rows of squares using Lemma 3.1. This new Steiner tree has the correct set of full components, and can now be extended to a Steiner tree for an $(11 + 6k_1) \times (8 + 6k_2)$ array with no extra non-X components by again applying Lemma 3.1.

If $(m, n) \equiv (1, 4) \pmod 6$, the construction is almost identical. We begin with the Steiner tree for the 13×10 array given in Figure 18(b), extend the part of the tree in the top six rows of squares as in the proof of Theorem 3.5, extend the part of the tree in the bottom five rows by Lemma 3.1, then increase m, again by Lemma 3.1.

(ii) Let $(m, n) \equiv (5, 4) \pmod 6$. A construction for the smallest case, $(m, n) = (11, 10)$, with the correct set of components is given in Figure 19(a). We can extend this six rows to the right without changing the excess as follows: in the far right row of squares of the 11×10 array delete

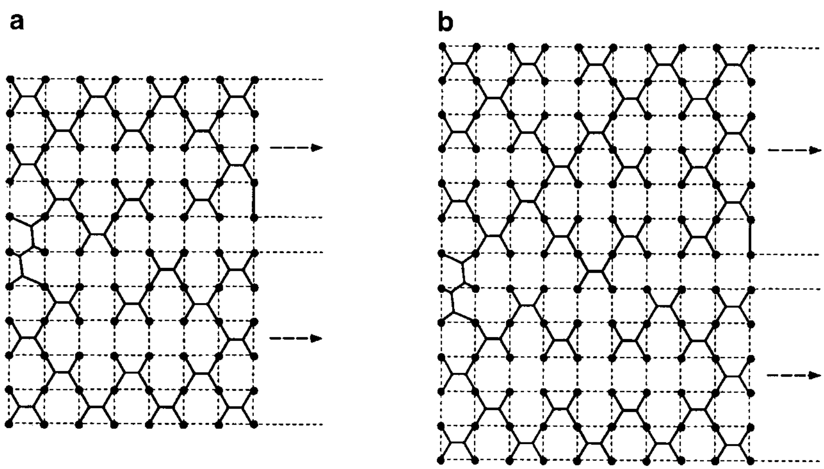


FIG. 18. Minimal Steiner treees for the 11×8 and 13×10 arrays.

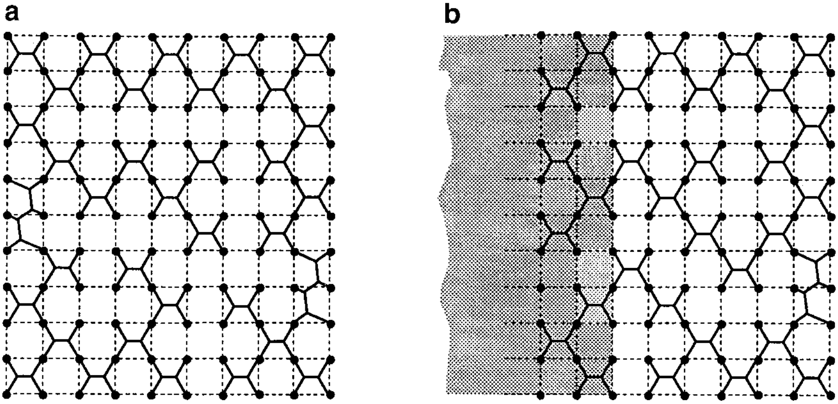


FIG. 19. A minimal Steiner tree for the 11×10 array, and a scheme for horizontally extending this tree.

the X in the third square from the top and replace the A_2 by an X in the third square from the bottom; then add Xs and an A_2 to the six rows to the right of the array, as shown in Figure 19(b). This extension to the right can be repeated arbitrarily often, giving solutions for all $11 \times (10 + 6k_2)$ arrays. We can then increase the height of the solution by multiples of 6 using Lemma 3.1.

For $(m, n) \equiv (1, 2) \pmod 6$, we apply exactly the same argument using the initial diagram and extension shown in Figure 20. ∎

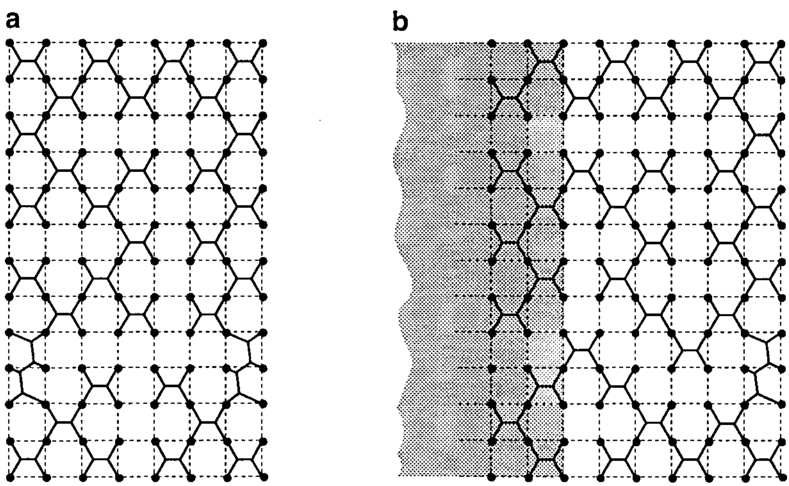


FIG. 20. A minimal Steiner tree for the 13×8 array, and a scheme for horizontally extending this tree.

TABLE IV

Summary of All Non-X Full Components of T for m and n Sufficiently Large

		n (mod 6)					
		0	1	2	3	4	5
m (mod 6)	0	A_2	I, I	A_2	I, I	A_2	I, I
	1		I, I, I	A_2, A_2	I, I	A_2, I	A_2, I, I
	2			A_4	I, I	I	A_2, I
	3				I, I	I, I	I, I
	4					A_4	A_2, A_2
	5						I, I, I

A summary of the non-X components of T for $m \times n$ rectangular arrays where m and n are sufficiently large is given in Table IV.

3.3. *The Case* $m = 4$

The difficulty in finding minimal Steiner trees for all $4 \times n$ rectangular arrays lies in the fact that the excess appears to be unbounded, which means that, in theory, an arbitrarily large number of collections of non-X components must be considered. In this subsection we will show that with a little careful analysis of exactly how full components of T fit into the array, we can reduce this to a finite problem by showing that the part of T in any long sequence of consecutive columns of the array has relatively large excess. This will allow us to prove the optimality of the minimal Steiner trees for these arrays, first for $n = 4k$ and then, with a little more case analysis, for the other values of n.

We begin by establishing some new definitions for this subsection. Let T be a given Steiner tree for a $4 \times n$ rectangular array, and label the columns of squares of the array, from left to right, 1 to $n-1$. We say that a full component of T is in column i if it has points lying strictly between the leftmost and rightmost edges of column i. (So it is possible some components may be in more than one column.) A vertical I is deemed to be in column i if it lies on the lefthand edge of that column.

We can partition the columns of the array into a set of *groups* $\{g_i\}$ as follows. Passing through the columns from 1 to $n-1$, we begin a new group (whose subscript is one greater than that of the previous group) every time we encounter a column containing a non-X component of T such that the previous column only contains Xs. The first (leftmost) group is denoted g_0 if it contains only Xs or g_1 if its first column contains a non-X component. It is easy to verify that consecutive runs of columns containing no non-X must have length at most 3. Hence g_0 contains at most three columns.

It will also be useful to define the notion of a tail of a group. Suppose column j is the rightmost column of a group g_i. Let l be the smallest non-negative integer such that columns $j-l$ to j inclusive completely contain a non-X component of T. Then the columns $j-l$ to j are said to comprise the *tail* of g_i.

Define the *excess of a column* to be the total contribution to excess of the non-Xs in that column, where any non-X which lies in k columns is deemed to contribute $1/k$ of its excess to each column it lies in. For a group g_i, define $f(i)$ to be the average excess of its columns.

Finally, define the *surplus* of a column to be the excess of that column minus $e(I)/4$. The surplus of a group g_i, denoted $s(g_i)$, is the sum of the surpluses of the columns of g_i, that is, $s(g_i)=a(f(i)-e(I)/4)$ where a is the number of columns in g_i. Note that, like excess, surplus is additive over columns and groups. We can define the surplus of T, $s(T)$, to be $\sum s(g_i)$ where the sum is taken over all groups of the array with $i \geqslant 1$.

The usefulness of the concepts of groups and their surplus will become clear when we examine some constructions for T. Let T_4 denote the minimal Steiner tree for a 4×4 square array. This consists of fives Xs, one in the central square, and one in each corner. Similarly, let T_i denote a minimal Steiner tree for a $4\times i$ rectangular array, where $i=5$, 6, 7 or 9. Suitable trees T_i are depicted in Figure 21. In each case the minimality of these trees can easily be checked using excess. Now for $n=8$ or $n>9$ we construct T as follows: for $n\equiv 0$, 1, 2 or 3 (mod 4) let T be comprised of a T_4, T_9, T_6 or T_7 respectively followed by an alternating sequence of (horizontal) Is and T_4s. Our aim is to show that in each case these constructions are

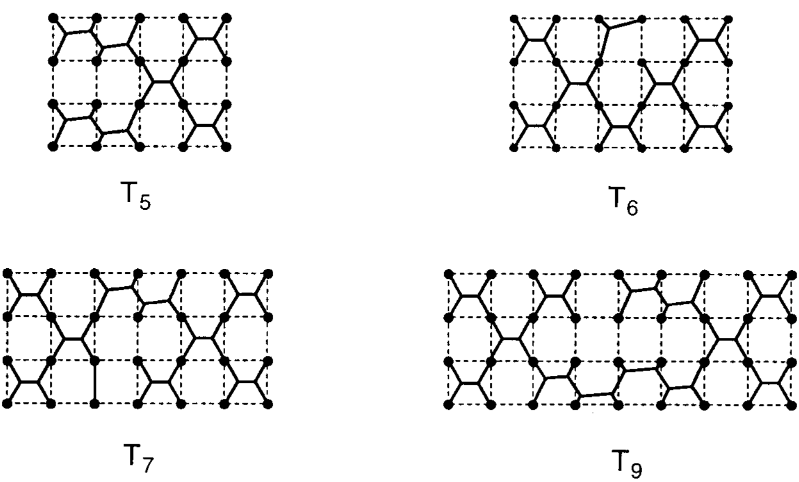


FIG. 21. Minimal Steiner trees for the $4\times n$ arrays when $n=5$, 6, 7, or 9.

minimal. Now note that for all $i > 1$ we have $s(g_i) = 0$. This means that for each residue of n modulo 4 $s(T)$ is a constant. This allows us to write down upper bounds for the surplus of a minimal Steiner tree for a $4 \times n$ array. Our general strategy now is to show that in such a minimal Steiner tree the groups g_i $(i > 0)$ not consisting of an I followed by a T_4 have positive surplus, and furthermore such groups can only contain a small number of columns because of the upper bound on $s(T)$.

The proofs of our main theorems involve considering the average excess of certain components of T in each of the squares of the array they enter. In this context the following technical lemma will be useful. The proof of the lemma follows immediately from the formulae for the lengths of the possible full components of T given in [2, Table 1].

LEMMA 3.10. *Let T_0 be one of the possible full components of T, other than an X, as listed in Theorem* 1.1. *Let a be the number of unit squares of the array entered by T_0 (setting $a = 1$ if $T_0 = I$). Then $e(T_0)/a > e(I)/4 + (e(I) - e(A_2))/a \approx 0.0226 + 0.018/a$.*

LEMMA 3.11. *Let T be a minimal Steiner tree for a $4 \times n$ rectangular array. For each group g_i of the array we have $s(g_i) \geqslant 0$ whenever $i > 0$, with equality occurring only if the part of T in g_i consists of a single I followed by a T_4.*

Proof. Fix a minimal Steiner tree T, and let g_i be a group of the array such that $i > 0$. It immediately follows from Lemma 3.10 that if the tail of g_i has positive surplus then g_i itself has positive surplus. To show that g_i has non-negative surplus we consider two cases.

(i) Suppose the tail of g_i contains a non-X component other than an A_2. Since no four consecutive columns can contain only Xs, g_i must have at least one non-X in all but the last three columns. It is now an easy exercise to verify that the full components of the tail of g_i contribute too much to the excess of any column they are in to get an average less than or equal to $e(I)/4$ when adding only three columns with excess 0. Hence $s(g_i) \geqslant 0$, with equality only occurring when the only non-X component in g_i is a single edge.

(ii) Suppose on the other hand that all the non-X components of the tail of g_i are A_2s. It is easy to see that the tail of g_i has at most four columns. If the tail of g_i has three or less columns or contains more than one A_2 then the average excess per column is at least $e(A_2)/3 > e(I)/4$ as required. If the tail of g_i has four columns and exactly one A_2, then the A_2 must be spread across the top or bottom two squares of two columns and the pattern of X's is that of a T_4 minus one of its corner Xs (as in the group

G_4 illustrated in Table 5). But it is clear that in this case the column immediately preceding the tail of g_i must contain a non-X component of T, implying that g_i is not equal to its tail. By Lemma 3.10 it follows that the average excess per column of the columns of g_i not in the tail of g_i is greater than $e(I)/4 + (e(I) - e(A_2))/a$ where $a + 4$ is the number of columns in g_i. Hence

$$f(i) > \frac{1}{a+4}\left(a\left(\frac{e(I)}{4} + \frac{e(I) - e(A_2)}{a}\right) + e(A_2)\right) = e(I)/4$$

implying that $s(g_i) > 0$.

Note that in the above analysis $s(g_i) = 0$ only when g_i consists of a single I followed by a T_4, as required. ∎

THEOREM 3.12. *Let T be a minimal Steiner tree for a $4 \times 4k$ rectangular array. Then the set of full components of T consists of $5k$ Xs and $(k-1)$ Is.*

Proof. This is an immediate corollary of the previous lemma. Let T be an alternating sequence of T_4s and Is. Since g_0 consists of three columns and $s(T) = 0$, it follows that the excess, and hence length, of T is as small as possible. It also follows from Lemma 3.11 that this choice of full components for the minimal Steiner tree is unique. ∎

If n is not a multiple of 4 the situation is a little more complicated as it involves categorizing groups with small surplus.

THEOREM 3.13. *Let T be a minimal Steiner tree for a $4 \times n$ rectangular array where n is greater than 9 and is not a multiple of 4. Then the set of full components of T consists of*:

(i) $(5k+2)$ *Xs, $(k-1)$ Is and a Y if $n = 4k+2$; or*

(ii) $(5k+2)$ *Xs, k Is and an A_2 if $n = 4k+3$; or*

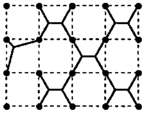
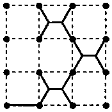
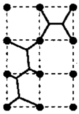
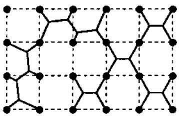
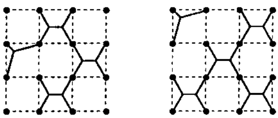
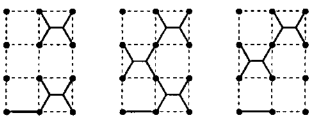
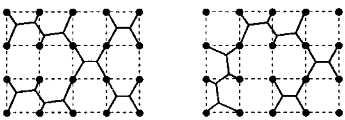
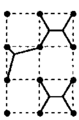
(iii) $(5k+3)$ *Xs, $(k-1)$ Is an A_2 and an A_4 if $n = 4k+5$.*

Proof. As mentioned above, we can construct trees with the above sets of components by taking a T_6, T_7 or T_9 followed by an alternating series of (horizontal) Is and T_4s. We will prove these constructions are minimal by contradiction.

Suppose there exists a minimal Steiner tree T for a $4 \times n$ rectangular array (n not a multiple of 4) with a set of full components other than those given in the statement of the theorem. We may assume n is the smallest integer greater than 9 for which such a T exists. Since T_6, T_7 and T_9 are minimal, it follows that no group of the array contains a horizontal I followed by a T_4. Thus each g_i, with $i > 0$, has strictly positive surplus.

TABLE V

All Groups with Surplus at Most $e(Y) - e(I)/2$, Listed in Order of the Size of Their Surpluses

G_1		$s(G_1) \approx 0.021$
G_2		$s(G_2) \approx 0.022$
G_3		$s(G_3) \approx 0.027$
G_4		$s(G_4) \approx 0.032$
G_5	 (a) (b)	$s(G_5) \approx 0.043$
G_6	 (a) (b) (c)	$s(G_6) \approx 0.044$
G_7	 (a) (b)	$s(G_7) \approx 0.054$
G_8		$s(G_8) \approx 0.0658$

Now n is of the form $4k+2$, $4k+3$ or $4k+5$ $(k>1)$. We will examine the first of these possibilities in detail. The other two possibilities can then be eliminated by an entirely staightforward extension of this argument. So assume $n=4k+2$.

Since g_0, if it exists, may consist of up to three columns of Xs, we have

$$s(T) \leqslant e(T_6) - \frac{5e(I)}{4} + \frac{3e(I)}{4} = e(Y) - \frac{e(I)}{2} \approx 0.0658.$$

Now it follows from the lengths of the possible full components of T given in [2, Table 1] that groups containing full components which span a large number of columns have a large surplus. Hence it is straightforward to systematically find and list every possible group whose surplus lies within a given bound. In Table V we list (and illustrate) all possible groups with surplus at most $e(Y)-e(I)/2$. The groups as illustrated are unique up to some minor rearrangement of their full components which does not change the number of columns involved, or the position of Xs in the rightmost column.

Since $n \geqslant 10$, the groups g_i, $i>0$, must collectively contain at least 6 columns. Hence there must be at least two groups other than g_0 in the array. This immediately eliminates the groups G_7 and G_8 as the surplus of either of these groups combined with that of any other group is too high. The rightmost group of the array must be either G_1, G_4, $G_5(b)$ or $G_6(b)$. If the rightmost group is G_1 or $G_5(b)$ then the group preceding it must be a $G_6(a)$ or G_2 respectively. In each case, the pair of groups has a combined surplus of exactly $e(Y)-e(I)/2$ but also results in a tree for the 4×10 array containing the same set of full components as in the statement of the theorem. If the rightmost group is a G_4 or $G_6(\mathrm{b})$ then the group preceding it must be a G_3 or a $G_6(\mathrm{c})$. The latter possibility immediately causes $S(T)$ to be too large, while G_3 causes a problem as it must also be preceded by a G_3 or $G_6(\mathrm{c})$, again resulting in too large a surplus. This eliminates the possibility $n=4k+2$.

We similarly eliminate the cases $n=4k+3$ and $n=4k+5$, contradicting the existence of T, and hence proving the theorem. ▮

4. CONCLUDING REMARKS

It is clear that the basic approach used in this paper can be generalized to other Steiner-closed lattice sets. In particular, for any specific Steiner-closed lattice set one can apply the following algorithm: find a Steiner tree, T, for the set of terminals such that the full components of T come from those listed in Theorem 1.1 and such that T contains as many Xs as possible; then consider all possible sets of full non-X components with the correct number of vertices modulo 3 whose combined excess is less than the excess of T. If the minimal Steiner tree has small excess compared to the number of terminals this procedure will generally be very efficient. Similar methods to those employed in this paper should also allow one to compute the minimal Steiner trees for other large families of Steiner-closed lattice sets which have bounded excess.

It should also be noted that the proofs of all the geometric lemmas required for these results only assume that the underlying space behaves like the Euclidean plane in any bounded neighbourhood. Hence the techniques of this paper can also be applied to sets of lattice points on a flat torus, for example, or other locally Euclidean surfaces.

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