

Desarrollo en fracción continua de $\sqrt{m^2 + k}$, con $\frac{2m}{k}$ entero

$$\begin{aligned}\sqrt{m^2 + k} &= m + (\sqrt{m^2 + k} - m) = m + \frac{1}{\frac{1}{\sqrt{m^2 + k} - m}} = m + \frac{1}{\frac{\sqrt{m^2 + k} + m}{k}} = m + \frac{1}{\frac{2m + \sqrt{m^2 + k} - m}{k}} = \\ &= m + \frac{1}{\frac{2m}{k} + \frac{\sqrt{m^2 + k} - m}{k}} = m + \frac{1}{\frac{2m}{k} + \frac{1}{\frac{k}{\sqrt{m^2 + k} - m}}} = \\ &= m + \frac{1}{\frac{2m}{k} + \frac{1}{2m + \frac{1}{\frac{k}{\sqrt{m^2 + k} - m}}}} = \left[m; \overline{\frac{2m}{k}, 2m} \right]\end{aligned}$$

Casos particulares:

m=3

$$k=1: \sqrt{m^2 + 1} = \left[m; \overline{2m, 2m} \right] = \left[m; \overline{2m} \right]$$

$$\sqrt{10} = \left[3; \overline{6} \right]$$

$$k=2: \sqrt{m^2 + 2} = \left[m; \overline{m, 2m} \right]$$

$$\sqrt{11} = \left[3; \overline{3, 6} \right]$$

$$k=m: \sqrt{m^2 + m} = \left[m; \overline{2, 2m} \right]$$

$$\sqrt{12} = \left[3; \overline{2, 6} \right]$$

$$k=2m: \sqrt{m^2 + 2m} = \sqrt{(m+1)^2 - 1} = \left[m; \overline{1, 2m} \right]$$

$$\sqrt{15} = \left[3; \overline{1, 6} \right]$$

Si m es par,

m=4

$$k=4: \sqrt{m^2 + 4} = \left[m; \overline{\frac{m}{2}, 2m} \right]$$

$$\sqrt{20} = \left[4; \overline{2, 8} \right]$$

$$k=m/2: \sqrt{m^2 + \frac{m}{2}} = \left[m; \overline{4, 2m} \right]$$

$$\sqrt{18} = \left[4; \overline{4, 8} \right]$$

Para $k = 1$, como la longitud del período es impar, las ecuaciones $x^2 - (m^2 + 1)y^2 = -1$ tienen soluciones.

Desarrollo en fracción continua de $\sqrt{(m+1)^2 - k}$, con $p = \frac{2(m+1)}{k}$ entero > 1

Haciendo $Q = (m+1)^2 - k = m^2 + 2m + 1 - k$, tenemos

$$\begin{aligned}
 \sqrt{Q} - m &= \frac{1}{\frac{1}{\sqrt{Q} - m}} = \frac{1}{\frac{\sqrt{Q} + m}{2m + 1 - k}} = \frac{1}{1 + \frac{\sqrt{Q} - (m+1-k)}{2m + 1 - k}} = \\
 &= \frac{1}{1 + \frac{1}{\frac{2m + 1 - k}{\sqrt{Q} - (m+1-k)}}} = \frac{1}{1 + \frac{1}{\frac{(2m + 1 - k)(\sqrt{Q} + (m+1-k))}{k(2m + 1 - k)}}} = \frac{1}{1 + \frac{1}{\frac{\sqrt{Q} + (m+1-k)}{k}}} = \\
 &= \frac{1}{1 + \frac{1}{(p-2) + \frac{\sqrt{Q} - (m+1-k)}{k}}} = \frac{1}{1 + \frac{1}{(p-2) + \frac{1}{\frac{k(\sqrt{Q} + (m+1-k))}{k(2m + 1 - k)}}}} = \\
 &= \frac{1}{1 + \frac{1}{(p-2) + \frac{1}{1 + \frac{1}{\frac{(2m + 1 - k)(\sqrt{Q} + m)}{2m + 1 - k}}}}} = \\
 &= \frac{1}{1 + \frac{1}{(p-2) + \frac{1}{1 + \frac{1}{\frac{(2m + 1 - k)(\sqrt{Q} + m)}{2m + 1 - k}}}}} \Rightarrow \sqrt{(m+1)^2 - k} = \left[m; 1, \frac{2(m+1)}{k} - 2, 1, 2m \right]
 \end{aligned}$$

Casos particulares:

m=4

$$k=1: \sqrt{(m+1)^2 - 1} = [m; \overline{1, 2m, 1, 2m}] = [m; \overline{1, 2m}]$$

$$\sqrt{24} = [4; \overline{1, 8}]$$

$$k=2: \sqrt{(m+1)^2 - 2} = [m; \overline{1, m-1, 1, 2m}]$$

$$\sqrt{23} = [4; \overline{1, 3, 1, 8}]$$

$$k=m+1: \sqrt{(m+1)^2 - (m+1)} = [m; \overline{1, 0, 1, 2m}] = [m; \overline{2, 2m}]$$

$$\sqrt{20} = [4; \overline{2, 8}]$$

Si m es impar,

$$k=4: \sqrt{(m+1)^2 - 4} = \left[m; 1, \frac{m-3}{2}, 1, 2m \right];$$

$$m=5: \sqrt{32} = [5; \overline{1, 1, 1, 10}]$$

$$k=\frac{m+1}{2}: \sqrt{(m+1)^2 - \frac{m+1}{2}} = [m; \overline{1, 2, 1, 2m}];$$

$$m=5: \sqrt{33} = [5; \overline{1, 2, 1, 10}]$$

Desarrollo en fracción continua de $\sqrt{n^2 + 4}$, con $n = 2m + 1$

Haciendo $Q = (2m + 1)^2 + 4 = 4m^2 + 4m + 5$, tenemos

$$\begin{aligned}
 \sqrt{Q} - (2m + 1) &= \frac{1}{\frac{1}{\sqrt{Q} - (2m + 1)}} = \frac{1}{\frac{\sqrt{Q} + (2m + 1)}{4}} = \frac{1}{m + \frac{\sqrt{Q} - (2m - 1)}{4}} = \frac{1}{m + \frac{1}{\frac{4}{\sqrt{Q} - (2m - 1)}}} = \\
 &= \frac{1}{m + \frac{1}{\frac{4(\sqrt{Q} + (2m - 1))}{8m + 4}}} = \frac{1}{m + \frac{1}{\frac{\sqrt{Q} + (2m - 1)}{2m + 1}}} = \frac{1}{m + \frac{1}{1 + \frac{\sqrt{Q} - 2}{2m + 1}}} = \frac{1}{m + \frac{1}{1 + \frac{2m + 1}{\sqrt{Q} - 2}}} = \\
 &= \frac{1}{m + \frac{1}{1 + \frac{1}{\frac{(2m + 1)(\sqrt{Q} + 2)}{(2m + 1)^2}}}} = \frac{1}{m + \frac{1}{1 + \frac{1}{\frac{\sqrt{Q} + 2}{2m + 1}}}} = \frac{1}{m + \frac{1}{1 + \frac{1}{1 + \frac{\sqrt{Q} - (2m - 1)}{2m + 1}}}} = \\
 &= \frac{1}{m + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\frac{(2m + 1)(\sqrt{Q} + (2m - 1))}{8m + 4}}}}}} = \frac{1}{m + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\frac{\sqrt{Q} + (2m - 1)}{4}}}}}} = \\
 &= \frac{1}{m + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{m + \frac{\sqrt{Q} - (2m + 1)}{4}}}}}} = \frac{1}{m + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{m + \frac{1}{\sqrt{Q} - (2m + 1)}}}}}} =
 \end{aligned}$$

$$\frac{1}{m + \frac{1}{1 + \frac{1}{1 + \frac{1}{m + \frac{1}{4(\sqrt{Q} + (2m+1))}}}}} = \frac{1}{m + \frac{1}{1 + \frac{1}{1 + \frac{1}{m + \frac{1}{\sqrt{Q} + (2m+1)}}}}} =$$

$$\frac{1}{m + \frac{1}{1 + \frac{1}{1 + \frac{1}{m + \frac{1}{2(2m+1) + (\sqrt{Q} - (2m+1))}}}}} \Rightarrow$$

$$\sqrt{Q} = (2m+1) + \frac{1}{m + \frac{1}{1 + \frac{1}{1 + \frac{1}{m + \frac{1}{2(2m+1) + \frac{1}{m + \frac{1}{1 + \frac{1}{m + \frac{1}{2(2m+1) + \dots}}}}}}}}}$$

$$\sqrt{(2m+1)^2 + 4} = [2m+1; \overline{m, 1, 1, m, 2(2m+1)}]$$

$$m=1: \sqrt{(2 \cdot 1 + 1)^2 + 4} = \sqrt{13} = [3; \overline{1, 1, 1, 1, 6}]$$

$$m=2: \sqrt{(2 \cdot 2 + 1)^2 + 4} = \sqrt{29} = [5; \overline{2, 1, 1, 2, 10}]$$

$$m=3: \sqrt{(2 \cdot 3 + 1)^2 + 4} = \sqrt{53} = [7; \overline{3, 1, 1, 3, 14}]$$

Como la longitud del período es impar, las ecuaciones $x^2 - (n^2 + 4)y^2 = -1$ con n impar, tienen soluciones.

Desarrollo en fracción continua de $\sqrt{n^2 - 4}$, con $n = 2m + 1$

Haciendo $Q = (2m + 1)^2 - 4 = 4m^2 + 4m - 3$, tenemos

$$\begin{aligned}
 \sqrt{Q} - 2m &= \frac{1}{\frac{1}{\sqrt{Q} - 2m}} = \frac{1}{\frac{\sqrt{Q} + 2m}{4m - 3}} = \frac{1}{1 + \frac{\sqrt{Q} - (2m - 3)}{4m - 3}} = \frac{1}{1 + \frac{1}{\frac{4m - 3}{\sqrt{Q} - (2m - 3)}}} = \\
 &= \frac{1}{1 + \frac{1}{\frac{(4m - 3)(\sqrt{Q} + (2m - 3))}{16m - 12}}} = \frac{1}{1 + \frac{1}{\frac{\sqrt{Q} + (2m - 3)}{4}}} = \frac{1}{1 + \frac{1}{(m - 1) + \frac{\sqrt{Q} - (2m - 1)}{4}}} = \\
 &= \frac{1}{1 + \frac{1}{(m - 1) + \frac{1}{\frac{4(\sqrt{Q} + (2m - 1))}{8m - 4}}}} = \frac{1}{1 + \frac{1}{(m - 1) + \frac{1}{\frac{\sqrt{Q} + (2m - 1)}{2m - 1}}}} = \\
 &= \frac{1}{1 + \frac{1}{(m - 1) + \frac{1}{2 + \frac{\sqrt{Q} - (2m - 1)}{2m - 1}}}} = \frac{1}{1 + \frac{1}{(m - 1) + \frac{1}{2 + \frac{1}{\frac{2m - 1}{\sqrt{Q} - (2m - 1)}}}}} = \\
 &= \frac{1}{1 + \frac{1}{(m - 1) + \frac{1}{2 + \frac{1}{\frac{(2m - 1)(\sqrt{Q} + (2m - 1))}{8m - 4}}}}} = \frac{1}{1 + \frac{1}{(m - 1) + \frac{1}{2 + \frac{1}{\frac{\sqrt{Q} + (2m - 1)}{4}}}}} = \\
 &= \frac{1}{1 + \frac{1}{(m - 1) + \frac{1}{2 + \frac{1}{(m - 1) + \frac{1}{\frac{1}{\sqrt{Q} - (2m - 3)}}}}}} =
 \end{aligned}$$

$$\frac{1}{1 + \frac{1}{(m-1) + \frac{1}{2 + \frac{1}{(m-1) + \frac{1}{4(\sqrt{Q} + (2m-3))}}}}} = \frac{1}{1 + \frac{1}{(m-1) + \frac{1}{2 + \frac{1}{(m-1) + \frac{1}{\sqrt{Q} + (2m-3)}}}}} =$$

$$\frac{1}{1 + \frac{1}{(m-1) + \frac{1}{2 + \frac{1}{(m-1) + \frac{1}{1 + \frac{\sqrt{Q} - 2m}{4m-3}}}}} = \frac{1}{1 + \frac{1}{(m-1) + \frac{1}{2 + \frac{1}{(m-1) + \frac{1}{1 + \frac{4m-3}{\sqrt{Q} - 2m}}}}} =$$

$$\frac{1}{1 + \frac{1}{(m-1) + \frac{1}{2 + \frac{1}{(m-1) + \frac{1}{1 + \frac{1}{(4m-3)(\sqrt{Q} + 2m)}}}}} = \frac{1}{1 + \frac{1}{(m-1) + \frac{1}{2 + \frac{1}{(m-1) + \frac{1}{1 + \frac{1}{4m + (\sqrt{Q} - 2m)}}}}} \Rightarrow$$

$$\sqrt{Q} = 2m + \frac{1}{1 + \frac{1}{(m-1) + \frac{1}{2 + \frac{1}{(m-1) + \frac{1}{1 + \frac{1}{4m + \frac{1}{1 + \frac{1}{(m-1) + \frac{1}{2 + \frac{1}{(m-1) + \dots}}}}}}}}}}$$

$$\sqrt{(2m+1)^2 - 4} = [2m; \overline{1, m-1, 2, m-1, 1, 4m}]$$

$$m=1: \sqrt{(2 \cdot 1 + 1)^2 - 4} = \sqrt{5} = [2; \overline{1, 0, 2, 0, 1, 4}] = [2; \overline{4, 4}] = [2; \overline{4}]$$

$$m=2: \sqrt{(2 \cdot 2 + 1)^2 - 4} = \sqrt{21} = [4; \overline{1, 1, 2, 1, 1, 8}]$$

$$m=3: \sqrt{(2 \cdot 3 + 1)^2 - 4} = \sqrt{45} = [6; \overline{1, 2, 2, 2, 1, 12}]$$

$$m=4: \sqrt{(2 \cdot 4 + 1)^2 - 4} = \sqrt{77} = [8; \overline{1, 3, 2, 3, 1, 16}]$$

$$m=5: \sqrt{(2 \cdot 5 + 1)^2 - 4} = \sqrt{117} = [10; \overline{1, 4, 2, 4, 1, 20}]$$

Desarrollo en fracción continua de $\sqrt{(3k+1)^2 + (2k+1)}$

Haciendo $Q = (3k+1)^2 + (2k+1) = 9k^2 + 8k + 2$

$$\begin{aligned}
 \sqrt{Q} - (3k+1) &= \frac{1}{\sqrt{Q} - (3k+1)} = \frac{1}{\frac{\sqrt{Q} + (3k+1)}{2k+1}} = \frac{1}{2 + \frac{\sqrt{Q} - (k+1)}{2k+1}} = \frac{1}{2 + \frac{1}{\frac{2k+1}{\sqrt{Q} - (k+1)}}} \\
 &= \frac{1}{2 + \frac{1}{\frac{(2k+1)(\sqrt{Q} + (k+1))}{(2k+1)(4k+1)}}} = \frac{1}{2 + \frac{1}{\frac{\sqrt{Q} + (k+1)}{4k+1}}} = \frac{1}{2 + \frac{1}{1 + \frac{\sqrt{Q} - 3k}{4k+1}}} = \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{4k+1}{\sqrt{Q} - 3k}}}} \\
 &= \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{(4k+1)(\sqrt{Q} + 3k)}{2(4k+1)}}}} = \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{\sqrt{Q} + 3k}{2}}}} = \frac{1}{2 + \frac{1}{1 + \frac{1}{3k + \frac{\sqrt{Q} - 3k}{2}}}} \\
 &= \frac{1}{2 + \frac{1}{1 + \frac{1}{3k + \frac{1}{\frac{2(\sqrt{Q} + 3k)}{2(4k+1)}}}}} = \frac{1}{2 + \frac{1}{1 + \frac{1}{3k + \frac{1}{1 + \frac{\sqrt{Q} - (k+1)}{4k+1}}}}} \\
 &= \frac{1}{2 + \frac{1}{1 + \frac{1}{3k + \frac{1}{1 + \frac{1}{4k+1}}}}} = \frac{1}{2 + \frac{1}{1 + \frac{1}{3k + \frac{1}{1 + \frac{1}{\frac{(4k+1)(\sqrt{Q} + (k+1))}{(2k+1)(4k+1)}}}}} \\
 &= \frac{1}{2 + \frac{1}{1 + \frac{1}{3k + \frac{1}{1 + \frac{1}{2 + \frac{\sqrt{Q} - (3k+1)}{2k+1}}}}} = \frac{1}{2 + \frac{1}{1 + \frac{1}{3k + \frac{1}{1 + \frac{1}{2 + \frac{1}{\sqrt{Q} - (3k+1)}}}}} \\
 &= \frac{1}{2 + \frac{1}{1 + \frac{1}{3k + \frac{1}{1 + \frac{1}{2 + \frac{1}{\frac{(2k+1)(\sqrt{Q} + (3k+1))}{2k+1}}}}}
 \end{aligned}$$

$$= \frac{1}{2 + \frac{1}{1 + \frac{1}{3k + \frac{1}{1 + \frac{1}{2 + \frac{1}{2(3k+1) + (\sqrt{Q} - (3k+1))}}}}}} \Rightarrow$$

$$\sqrt{(3k+1)^2 + (2k+1)} = [3k+1; \overline{2, 1, 3k, 1, 2, 2(3k+1)}]$$

$$k=1: \sqrt{(3 \cdot 1 + 1)^2 + (2 \cdot 1 + 1)} = \sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$$

$$k=2: \sqrt{(3 \cdot 2 + 1)^2 + (2 \cdot 2 + 1)} = \sqrt{54} = [7; \overline{2, 1, 6, 1, 2, 14}]$$

$$k=3: \sqrt{(3 \cdot 3 + 1)^2 + (2 \cdot 3 + 1)} = \sqrt{107} = [10; \overline{2, 1, 9, 1, 2, 20}]$$

$$k=4: \sqrt{(3 \cdot 4 + 1)^2 + (2 \cdot 4 + 1)} = \sqrt{178} = [13; \overline{2, 1, 12, 1, 2, 26}]$$

$$k=5: \sqrt{(3 \cdot 5 + 1)^2 + (2 \cdot 5 + 1)} = \sqrt{267} = [16; \overline{2, 1, 15, 1, 2, 32}]$$

Desarrollo en fracción continua de $\sqrt{(3k+2)^2 - (2k+1)}$

Haciendo $Q = (3k+2)^2 - (2k+1) = 9k^2 + 10k + 3$

$$\begin{aligned}
 \sqrt{Q} - (3k+1) &= \frac{1}{\sqrt{Q} - (3k+1)} = \frac{1}{\frac{\sqrt{Q} + (3k+1)}{2(2k+1)}} = \frac{1}{1 + \frac{\sqrt{Q} - (k+1)}{2(2k+1)}} \\
 &= \frac{1}{1 + \frac{1}{\frac{2(2k+1)}{\sqrt{Q} - (k+1)}}} = \frac{1}{1 + \frac{1}{\frac{2(2k+1)(\sqrt{Q} + (k+1))}{2(2k+1)^2}}} = \frac{1}{1 + \frac{1}{\frac{\sqrt{Q} + (k+1)}{2k+1}}} \\
 &= \frac{1}{1 + \frac{1}{2 + \frac{\sqrt{Q} - (3k+1)}{2k+1}}} = \frac{1}{1 + \frac{1}{2 + \frac{1}{\frac{2k+1}{\sqrt{Q} - (3k+1)}}}} \\
 &= \frac{1}{1 + \frac{1}{2 + \frac{1}{\frac{(2k+1)(\sqrt{Q} + (3k+1))}{2(2k+1)}}}} = \frac{1}{1 + \frac{1}{2 + \frac{1}{\frac{\sqrt{Q} + (3k+1)}{2}}}} \\
 &= \frac{1}{1 + \frac{1}{2 + \frac{1}{(3k+1) + \frac{\sqrt{Q} - (3k+1)}{2}}}} = \frac{1}{1 + \frac{1}{2 + \frac{1}{(3k+1) + \frac{1}{\frac{\sqrt{Q} - (3k+1)}{2}}}}} \\
 &= \frac{1}{1 + \frac{1}{2 + \frac{1}{(3k+1) + \frac{1}{\frac{2(\sqrt{Q} + (3k+1))}{2(2k+1)}}}}} = \frac{1}{1 + \frac{1}{2 + \frac{1}{(3k+1) + \frac{1}{2 + \frac{\sqrt{Q} - (k+1)}{2k+1}}}}} \\
 &= \frac{1}{1 + \frac{1}{2 + \frac{1}{(3k+1) + \frac{1}{2 + \frac{1}{2k+1}}}}} = \frac{1}{1 + \frac{1}{2 + \frac{1}{(3k+1) + \frac{1}{2 + \frac{1}{\frac{(2k+1)(\sqrt{Q} + (k+1))}{2(2k+1)^2}}}}} \\
 &= \frac{1}{1 + \frac{1}{2 + \frac{1}{(3k+1) + \frac{1}{2 + \frac{1}{2k+1}}}}} = \frac{1}{1 + \frac{1}{2 + \frac{1}{(3k+1) + \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{2(2k+1)}{\sqrt{Q} - (3k+1)}}}}}}}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1 + \frac{1}{2 + \frac{1}{(3k+1) + \frac{1}{2 + \frac{1}{1 + \frac{1}{\frac{2(2k+1)(\sqrt{Q} + (3k+1))}}{2(2k+1)}}}}}} \\
&= \frac{1}{1 + \frac{1}{2 + \frac{1}{(3k+1) + \frac{1}{2 + \frac{1}{1 + \frac{1}{2(3k+1) + (\sqrt{Q} - (3k+1))}}}}} \Rightarrow
\end{aligned}$$

$$\sqrt{(3k+2)^2 - (2k+1)} = [3k+1; \overline{1, 2, 3k+1, 2, 1, 2(3k+1)}]$$

$$k=1: \sqrt{(3 \cdot 1 + 2)^2 - (2 \cdot 1 + 1)} = \sqrt{22} = [4; \overline{1, 2, 4, 2, 1, 8}]$$

$$k=2: \sqrt{(3 \cdot 2 + 2)^2 - (2 \cdot 2 + 1)} = \sqrt{59} = [7; \overline{1, 2, 7, 2, 1, 14}]$$

$$k=3: \sqrt{(3 \cdot 3 + 2)^2 - (2 \cdot 3 + 1)} = \sqrt{114} = [10; \overline{1, 2, 10, 2, 1, 20}]$$

$$k=4: \sqrt{(3 \cdot 4 + 2)^2 - (2 \cdot 4 + 1)} = \sqrt{187} = [13; \overline{1, 2, 13, 2, 1, 26}]$$

$$k=5: \sqrt{(3 \cdot 5 + 2)^2 - (2 \cdot 5 + 1)} = \sqrt{278} = [16; \overline{1, 2, 16, 2, 1, 32}]$$

Desarrollo en fracción continua de $\sqrt{(6k-1)^2 + (4k-1)}$

Haciendo $Q = (6k-1)^2 + (4k-1) = 4k(9k-2)$

$$\begin{aligned}
 \sqrt{Q} - (6k-1) &= \frac{1}{\sqrt{Q} - (6k-1)} = \frac{1}{\frac{\sqrt{Q} + (6k-1)}{4k-1}} = \frac{1}{3 + \frac{\sqrt{Q} - (6k-2)}{4k-1}} \\
 &= \frac{1}{3 + \frac{1}{\frac{4k-1}{\sqrt{Q} - (6k-2)}}} = \frac{1}{3 + \frac{1}{\frac{(4k-1)(\sqrt{Q} + (6k-2))}{4(4k-1)}}} = \frac{1}{3 + \frac{1}{(3k-1) + \frac{\sqrt{Q} - (6k-2)}{4}}} \\
 &= \frac{1}{3 + \frac{1}{(3k-1) + \frac{1}{\frac{4}{\sqrt{Q} - (6k-2)}}}} = \frac{1}{3 + \frac{1}{(3k-1) + \frac{1}{\frac{4(\sqrt{Q} + (6k-2))}{4(4k-1)}}}} \\
 &= \frac{1}{3 + \frac{1}{(3k-1) + \frac{1}{3 + \frac{1}{\frac{4k-1}{\sqrt{Q} - (6k-1)}}}}} = \frac{1}{3 + \frac{1}{(3k-1) + \frac{1}{3 + \frac{1}{\frac{4k-1}{\sqrt{Q} - (6k-1)}}}}} \\
 &= \frac{1}{3 + \frac{1}{(3k-1) + \frac{1}{3 + \frac{1}{\frac{4k-1}{\sqrt{Q} - (6k-1)}}}}} = 3 + \frac{1}{(3k-1) + \frac{1}{3 + \frac{1}{\frac{4k-1}{\sqrt{Q} - (6k-1)}}}} \Rightarrow
 \end{aligned}$$

$$\sqrt{(6k-1)^2 + (4k-1)} = [6k-1; \overline{3, 3k-1, 3, 2(6k-1)}]$$

$$k=1: \sqrt{(6 \cdot 1 - 1)^2 + (4 \cdot 1 - 1)} = \sqrt{28} = [5; \overline{3, 2, 3, 10}]$$

$$k=2: \sqrt{(6 \cdot 2 - 1)^2 + (4 \cdot 2 - 1)} = \sqrt{128} = [11; \overline{3, 5, 3, 22}]$$

$$k=3: \sqrt{(6 \cdot 3 - 1)^2 + (4 \cdot 3 - 1)} = \sqrt{300} = [17; \overline{3, 8, 3, 34}]$$

$$k=4: \sqrt{(6 \cdot 4 - 1)^2 + (4 \cdot 4 - 1)} = \sqrt{544} = [23; \overline{3, 11, 3, 46}]$$

$$k=5: \sqrt{(6 \cdot 5 - 1)^2 + (4 \cdot 5 - 1)} = \sqrt{860} = [29; \overline{3, 14, 3, 58}]$$

Desarrollo en fracción continua de $\sqrt{(5k+1)^2 + (4k+1)}$

Haciendo $Q = (5k+1)^2 + 4k+1 = 25k^2 + 14k + 2$

$$\begin{aligned}
 \sqrt{Q} - (5k+1) &= \frac{1}{\sqrt{Q} - (5k+1)} = \frac{1}{\frac{\sqrt{Q} + (5k+1)}{4k+1}} = \frac{1}{2 + \frac{\sqrt{Q} - (3k+1)}{4k+1}} \\
 &= \frac{1}{2 + \frac{1}{\frac{4k+1}{\sqrt{Q} - (3k+1)}}} = \frac{1}{2 + \frac{1}{\frac{(4k+1)(\sqrt{Q} + (3k+1))}{(4k+1)^2}}} = \frac{1}{2 + \frac{1}{2 + \frac{\sqrt{Q} - (5k+1)}{4k+1}}} \\
 &= \frac{1}{2 + \frac{1}{2 + \frac{1}{\frac{4k+1}{\sqrt{Q} - (5k+1)}}}} = \frac{1}{2 + \frac{1}{2 + \frac{1}{\frac{(4k+1)(\sqrt{Q} + (5k+1))}{4k+1}}}} \\
 &= \frac{1}{2 + \frac{1}{2 + \frac{1}{2(5k+1) + (\sqrt{Q} - (5k+1))}}} \Rightarrow
 \end{aligned}$$

$$\sqrt{(5k+1)^2 + (4k+1)} = [5k+1; \overline{2, 2, 2(5k+1)}]$$

$$k = 1: \sqrt{(5 \cdot 1 + 1)^2 + (4 \cdot 1 + 1)} = \sqrt{41} = [6; \overline{2, 2, 12}]$$

$$k = 2: \sqrt{(5 \cdot 2 + 1)^2 + (4 \cdot 2 + 1)} = \sqrt{130} = [11; \overline{2, 2, 22}]$$

$$k = 3: \sqrt{(5 \cdot 3 + 1)^2 + (4 \cdot 3 + 1)} = \sqrt{269} = [16; \overline{2, 2, 32}]$$

$$k = 4: \sqrt{(5 \cdot 4 + 1)^2 + (4 \cdot 4 + 1)} = \sqrt{458} = [21; \overline{2, 2, 42}]$$

$$k = 5: \sqrt{(5 \cdot 5 + 1)^2 + (4 \cdot 5 + 1)} = \sqrt{697} = [26; \overline{2, 2, 52}]$$

Como la longitud del período es impar, las ecuaciones $x^2 - Qy^2 = -1$ tienen soluciones.

Desarrollo en fracción continua de $\sqrt{(17k+2)^2 + (8k+1)}$

Haciendo $Q = (17k+2)^2 + 8k+1 = 289k^2 + 76k + 5$

$$\begin{aligned}
 \sqrt{Q} - (17k+2) &= \frac{1}{\sqrt{Q} - (17k+2)} = \frac{1}{\frac{\sqrt{Q} + (17k+2)}{8k+1}} = \frac{1}{4 + \frac{\sqrt{Q} - (15k+2)}{8k+1}} \\
 &= \frac{1}{4 + \frac{1}{\frac{8k+1}{\sqrt{Q} - (15k+2)}}} = \frac{1}{4 + \frac{1}{\frac{(8k+1)(\sqrt{Q} + (15k+2))}{(8k+1)^2}}} = \frac{1}{4 + \frac{1}{4 + \frac{\sqrt{Q} - (17k+2)}{8k+1}}} \\
 &= \frac{1}{4 + \frac{1}{4 + \frac{1}{\frac{8k+1}{\sqrt{Q} - (17k+2)}}}} = \frac{1}{4 + \frac{1}{4 + \frac{1}{\frac{(8k+1)(\sqrt{Q} + (17k+2))}{8k+1}}}} \\
 &= \frac{1}{4 + \frac{1}{4 + \frac{1}{2(17k+2) + (\sqrt{Q} - (17k+2))}}} \Rightarrow
 \end{aligned}$$

$$\sqrt{(17k+2)^2 + (8k+1)} = [17k+2; \overline{4, 4, 2(17k+2)}]$$

$$k = 1: \sqrt{(17 \cdot 1 + 2)^2 + (8 \cdot 1 + 1)} = \sqrt{370} = [19; \overline{4, 4, 38}]$$

$$k = 2: \sqrt{(17 \cdot 2 + 2)^2 + (8 \cdot 2 + 1)} = \sqrt{1313} = [36; \overline{4, 4, 72}]$$

$$k = 3: \sqrt{(17 \cdot 3 + 2)^2 + (8 \cdot 3 + 1)} = \sqrt{2834} = [53; \overline{4, 4, 106}]$$

$$k = 4: \sqrt{(17 \cdot 4 + 2)^2 + (8 \cdot 4 + 1)} = \sqrt{4933} = [70; \overline{4, 4, 140}]$$

$$k = 5: \sqrt{(17 \cdot 5 + 2)^2 + (8 \cdot 5 + 1)} = \sqrt{7610} = [87; \overline{4, 4, 174}]$$

Como la longitud del período es impar, las ecuaciones $x^2 - Qy^2 = -1$ tienen soluciones.

Desarrollo en fracción continua de $\sqrt{(37k+3)^2 + (12k+1)}$

Haciendo $Q = (37k+3)^2 + 12k+1 = 1369k^2 + 234k + 10$

$$\begin{aligned}
 \sqrt{Q} - (37k+3) &= \frac{1}{\sqrt{Q} - (37k+3)} = \frac{1}{\frac{\sqrt{Q} + (37k+3)}{12k+1}} = \frac{1}{6 + \frac{\sqrt{Q} - (35k+3)}{12k+1}} \\
 &= \frac{1}{6 + \frac{1}{\frac{12k+1}{\sqrt{Q} - (35k+3)}}} = \frac{1}{6 + \frac{1}{\frac{(12k+1)(\sqrt{Q} + (35k+3))}{(12k+1)^2}}} = \frac{1}{6 + \frac{1}{6 + \frac{\sqrt{Q} - (37k+3)}{12k+1}}} \\
 &= \frac{1}{6 + \frac{1}{6 + \frac{1}{\frac{12k+1}{\sqrt{Q} - (37k+3)}}}} = \frac{1}{6 + \frac{1}{6 + \frac{1}{\frac{(12k+1)(\sqrt{Q} + (37k+3))}{12k+1}}}} \\
 &= \frac{1}{6 + \frac{1}{6 + \frac{1}{2(37k+3) + (\sqrt{Q} - (37k+3))}}} \Rightarrow
 \end{aligned}$$

$$\sqrt{(37k+3)^2 + (12k+1)} = [37k+3; \overline{6, 6, 2(37k+3)}]$$

$$k = 1: \sqrt{(37 \cdot 1 + 3)^2 + (12 \cdot 1 + 1)} = \sqrt{1613} = [40; \overline{6, 6, 80}]$$

$$k = 2: \sqrt{(37 \cdot 2 + 3)^2 + (12 \cdot 2 + 1)} = \sqrt{5954} = [77; \overline{6, 6, 154}]$$

$$k = 3: \sqrt{(37 \cdot 3 + 3)^2 + (12 \cdot 3 + 1)} = \sqrt{13033} = [114; \overline{6, 6, 228}]$$

$$k = 4: \sqrt{(37 \cdot 4 + 3)^2 + (12 \cdot 4 + 1)} = \sqrt{22850} = [151; \overline{6, 6, 302}]$$

$$k = 5: \sqrt{(37 \cdot 5 + 3)^2 + (12 \cdot 5 + 1)} = \sqrt{35405} = [188; \overline{6, 6, 376}]$$

Como la longitud del período es impar, las ecuaciones $x^2 - Qy^2 = -1$ tienen soluciones.

Desarrollo en fracción continua de $\sqrt{((4m^2 + 1)k + m)^2 + (4mk + 1)}$

Es una generalización de los tres últimos apartados.

Haciendo $Q = ((4m^2 + 1)k + m)^2 + (4mk + 1)$

$$\begin{aligned}
 \sqrt{Q} - ((4m^2 + 1)k + m) &= \frac{1}{\frac{1}{\sqrt{Q} - ((4m^2 + 1)k + m)}} \\
 &= \frac{1}{\frac{\sqrt{Q} + ((4m^2 + 1)k + m)}{4mk + 1}} = \frac{1}{2m + \frac{\sqrt{Q} - ((4m^2 - 1)k + m)}{4mk + 1}} \\
 &= \frac{1}{2m + \frac{1}{\frac{4mk + 1}{\sqrt{Q} - ((4m^2 - 1)k + m)}}} = \frac{1}{2m + \frac{1}{(4mk + 1) \left(\frac{\sqrt{Q} + ((4m^2 - 1)k + m)}{(4mk + 1)^2} \right)}} \\
 &= \frac{1}{2m + \frac{1}{2m + \frac{1}{2m + \frac{\sqrt{Q} - ((4m^2 + 1)k + m)}{4mk + 1}}}} \\
 &= \frac{1}{2m + \frac{1}{2m + \frac{1}{\frac{4mk + 1}{\sqrt{Q} - ((4m^2 + 1)k + m)}}}} \\
 &= \frac{1}{2m + \frac{1}{2m + \frac{1}{(4mk + 1) \left(\frac{\sqrt{Q} + ((4m^2 + 1)k + m)}{4mk + 1} \right)}}} \\
 &= \frac{1}{2m + \frac{1}{2m + \frac{1}{2((4m^2 + 1)k + m) + (\sqrt{Q} - ((4m^2 + 1)k + m))}}} \Rightarrow
 \end{aligned}$$

$$\sqrt{((4m^2 + 1)k + m)^2 + (4mk + 1)} = [(4m^2 + 1)k + m; \overline{2m, 2m, 2((4m^2 + 1)k + m)}]$$

$$k = 1, m = 1: \sqrt{41} = [6; \overline{2, 2, 12}]$$

$$k = 1, m = 2: \sqrt{370} = [19; \overline{4, 4, 38}]$$

$$k = 1, m = 3: \sqrt{1613} = [40; \overline{6, 6, 80}]$$

$$k = 1, m = 4: \sqrt{4778} = [69; \overline{8, 8, 138}]$$

$$k = 1, m = 5: \sqrt{11257} = [106; \overline{10, 10, 212}]$$

Como la longitud del período es impar, las ecuaciones $x^2 - Qy^2 = -1$ tienen soluciones.