Series aritmético-geométricas

$$S_k = \sum_{n=1}^{\infty} n^k r^n, \qquad |r| < 1, \;\; S_0 = \sum_{n=1}^{\infty} r^n \; = \; \frac{r}{1-r}, \qquad |r| < 1,$$

$$rS_k = \sum_{n=1}^{\infty} n^k r^{n+1} = \sum_{n=2}^{\infty} (n-1)^k r^n = \sum_{n=1}^{\infty} (n-1)^k r^n$$

$$(1-r)S_k = \sum_{n=1}^{\infty} (n^k - (n-1)^k) r^n = \sum_{n=1}^{\infty} \left(\binom{k}{1} n^{k-1} - \binom{k}{2} n^{k-2} + \cdots + \binom{k}{k} (-1)^{k+1} \right) r^n$$

$$S_k = \frac{\binom{k}{1} S_{k-1} - \binom{k}{2} S_{k-2} + \cdots + \binom{k}{k} (-1)^{k+1} S_0}{(1-r)}$$

$$S_1 = \sum_{n=1}^{\infty} n r^n \; = \frac{\binom{1}{1} S_0}{(1-r)} = \frac{r}{(1-r)^2}$$

$$S_2 = \frac{\binom{2}{1} S_1 - S_0}{(1-r)} = \frac{2r - r(1-r)}{(1-r)^3} = \frac{r(r+1)}{(1-r)^3}$$

$$S_3 = \frac{\binom{3}{1} S_2 - \binom{3}{3} S_1 + S_0}{(1-r)} = \frac{3r(r+1) - 3r(1-r) + r(1-r)^2}{(1-r)^4} = \frac{r(r^2 + 4r + 1)}{(1-r)^4}$$

$$S_4 = \frac{\binom{4}{1} S_3 - \binom{4}{2} S_2 + \binom{4}{1} S_1 - S_0}{(1-r)} = \frac{4r(r^2 + 4r + 1) - 6r(r+1)(1-r) + 4r(1-r)^2 - r(1-r)^3}{(1-r)^5}$$
$$= \; \frac{r(r^3 + 11r^2 + 11r + 1)}{(1-r)^5}$$

$$S_5 = \; \frac{r(r^4 + 26r^3 + 66r^2 + 26r + 1)}{(1-r)^6}$$

$$S_6 = \; \frac{r(r^5 + 57r^4 + 302r^3 + 302r^2 + 57r + 1)}{(1-r)^7}$$

$$S_7 = \; \frac{r(r^6 + 120r^5 + 1191r^4 + 2416r^3 + 1191r^2 + 120r + 1)}{(1-r)^8}$$

$$S_8 = \; \frac{r(r^7 + 247r^6 + 4293r^5 + 15619r^4 + 15619r^3 + 4293r^2 + 247r + 1)}{(1-r)^9}$$

$$S_9 = \; \frac{r(r^8 + 502r^7 + 14608r^6 + 88234r^5 + 156190r^4 + 88234r^3 + 14608r^2 + 502r + 1)}{(1-r)^{10}}$$

$$S_{10} = \; \frac{r(r^9 + 1013r^8 + 47840r^7 + 455192r^6 + 1310354r^5 + 1310354r^4 + 455192r^3 + 47840r^2 + 1013r + 1)}{(1-r)^{11}}$$