Linear Algebra 1

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1 Vectors

1.1 Initial Definition (~~~)

A mathematical object with a magnitude and direction in space (as opposed to directionless scalars). Vectors can be described by a collection of n ordered numbers.

$$v_{col} = \begin{pmatrix} v_1 \\ v_2 \\ \ldots \\ v_n \end{pmatrix}, \qquad v_{row} = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix}$$

The vector's magnitude/norm is

$$|v| = \sqrt{\sum_i (v_i)^2}$$

1.2 Unit Vectors

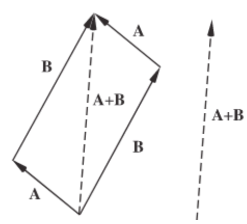
A vector with magnitude of 1. Usually notated with a "hat" ($\hat{x}$). A vector can be written as $\vec{r} = |\vec{r}|\hat{r}$. Some useful unit vectors are $\hat{x} = (1,0,0)$, $\hat{y} = (0,1,0)$, $\hat{z} = (0,0,1)$. In that case,

$$\vec{r} = \begin{pmatrix} x & y & z \end{pmatrix} = x\hat{x} + y\hat{y} + z\hat{z}$$

1.3 Addition

Vector additions is done from head to tail and is commutative. In a cartesian coordinate system:

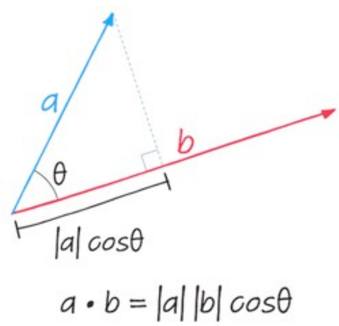
$$\vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z) = (A_x + B_x)\,\hat{x} + (A_y + B_y)\,\hat{y} + (A_z + B_z)\,\hat{z}$$



1.4 Dot Product

The dot product is an operation between two vectors $\vec{a}$, $\vec{b}$ that results in a scalar - the projection of $\vec{a}$ onto $\vec{b}$, times $\left|\vec{b}\right|$ (commutative).

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|cos\,(\alpha)$$



In a cartesian coordinate system: (*HW: show*)

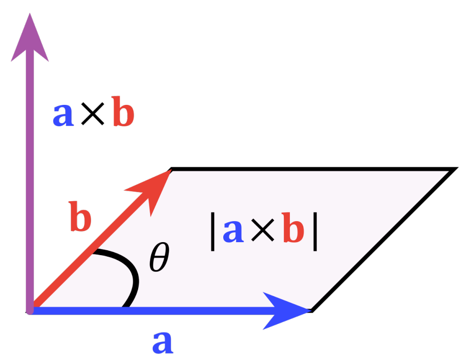
$$\vec{a} \cdot \vec{b} = (a_x, a_y, a_z) \cdot (b_x, b_y, b_z) = a_x b_x + a_y b_y + a_z b_z$$

1.5 Cross Product

The cross product is an operation between two vectors that results in a perpendicular vector with the magnitude: (area of the parallelogram created by the vectors)

$$|\vec{r_1} \times \vec{r_2}| = |\vec{r_1}||\vec{r_2}|sin\,(\alpha)$$

The direction is given the by the right hand rule.
The cross product is anti-commutative: $\vec{r_1} \times \vec{r_2} = -\vec{r_2} \times \vec{r_1}$



We can write the cross product using a 3x3 matrix determinant

$$\vec{r_1} \times \vec{r_2} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ x_1 z_2 - z_1 x_2 \\ x_1 y_2 - y_1 x_2 \end{pmatrix}$$

1.6 Determinant

A determinant of a 2x2 matrix is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

A determinant of a 3x3 matrix is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The minus sign *defines* the right hand rule.

1.7 Parallelepiped Volume

If we build a parallelepiped from vectors r1, r2, r3, we can find its volume by

$$V = \vec{r_3} \cdot (\vec{r_1} \times \vec{r_2}) = \begin{vmatrix} x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

This is important because if all three vectors are in the same plane, $V = 0$.

2 Systems of Equations

2.1 Homogenous

A *linear* system of equations can be represented using vectors
*(an equation without a degree of freedom (=0) is homogenous).
Usually we write:

$$a_{11}x + a_{12}y + a_{13}z = 0$$

$$a_{21}x + a_{22}y + a_{23}z = 0$$

$$a_{31}x + a_{32}y + a_{33}z = 0$$

Now with vectors:

$$\vec{x} = (x, y, z)$$

$$\vec{R}_1 = (a_{11}, a_{12}, a_{13})$$

$$\vec{R}_2 = (a_{21}, a_{22}, a_{23})$$

$$\vec{R}_3 = (a_{31}, a_{32}, a_{33})$$

$$\vec{R}_1\vec{x} = \vec{R}_2\vec{x} = \vec{R}_3\vec{x} = 0$$

We're given a geometric meaning - the variable vector $\vec{x}$ is perpendicular to the coefficient vectors R1, R2, R3. Therefore the coefficient vectors must all be in the same plane for there to be a solution, and the determinant of the 3 coefficient vectors (as shown in 1.7) must be *zero*.

If there is a solution, it can be found using the cross product:

$$\boxed{\vec{x} = \alpha\vec{R}_1 \times \vec{R}_2}$$

2.2 Non-homogenous

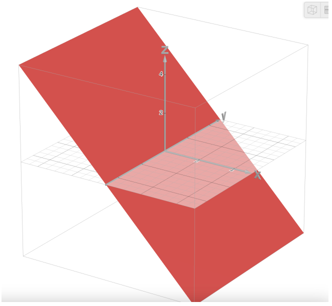
$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Each equation here represents a plane.
The solution to the equations is any point that is *on all three planes.* End of class 1.

For example,
$z = 0$ corresponds to the xy plane.
$y = 3$ translation of the xz plane.
$x + z = 0$ diagonal plane through y axis



$$\vec{c_1} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \quad \vec{c_2} = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \quad \vec{c_3} = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x\vec{c_1} + y\vec{c_2} + z\vec{c_3} = \vec{b}$$

Linear combination of $\vec{c}_i$. Left side of the equation spans a space (line, plane or volume). There is an answer if to the set of the equation, $\vec{b}$ is inside the spanned space.
If the vectors aren't in the same plane, there *must* be a solution (since they span the entire 3D space).

$$\det \begin{vmatrix} \vec{c_1} \\ \vec{c_2} \\ \vec{c_3} \end{vmatrix} \neq 0 \quad \Rightarrow \quad \checkmark$$

2.3 Matrixes

2.3.1 Vector-Matrix Product

A system of equations with n equations and m variables:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1m}x_m = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2m}x_m = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nm}x_m = b_n$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

For now, a matrix is simply a way to organize our numbers:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

$$\boxed{A\vec{x} = \vec{b}}$$

! A matrix's order is the number of rows and columns. $(n \times m)$ for a matrix with n rows and m columns.
Using the above example we can define matrix multiplication, so that the boxed equation is satisfied:

$$\sum_j A_{ij}x_j = b_i$$

Notice the length of x must equate to the number of columns of A.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1*2+0*3 \\ 0*2+1*3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

I is the identity matrix:

$$I_{2\times 2} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2.3.2 Vector-Matrix Product Interpertation

Seperating the matrix's rows and columns into vectors:

$$\vec{R}_i = (a_{i1}, a_{i2}, \cdots, a_{im}), \quad \vec{C}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

$$A = \begin{pmatrix} \vec{R}_1 \\ \vec{R}_2 \\ \vdots \\ \vec{R}_n \end{pmatrix} = \begin{pmatrix} \vec{C}_1 & \vec{C}_2 & \cdots & \vec{C}_m \end{pmatrix}$$

1. The component b_i is the dot product $\vec{R}_i \cdot \vec{x}$.

2. The vector $\vec{b}$ is a linear combination of $\vec{C}_j$ with coefficients x_j

2.3.3 Gauss Elimination

Given the matrix of coefficients, use arithmetic manipulation between the different rows to make the matrix of the coefficients into the identity matrix. That way, the vector of $\vec{b}$ becomes the solution.

If there is more than one solution, there must be the diagonal of ones with zeros underneath, but above it there could be numbers.

The three allowed operations on rows are:

1. Changing the order of equations

2. Multiplying an equation by a constant

3. Adding equations

2.3.4 Matrix Operations

1. Matrix Addition

(a vector is also a matrix with a dimension of length 1)

- A, B and C must be of the same order.
- Can also be written with components: $c_{ij} = a_{ij} + b_{ij}$

$$A + B = C$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1}+a_{n1} & \cdots & a_{nm}+b_{nm} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nm} \end{pmatrix}$$

- The zero matrix satisfies $A + 0_{n\times m} = 0_{n\times m} + A = A$ and has $0_{ij} = 0$.

2. Product with Scalar

$$\beta A = B$$

$$\beta \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} \beta a_{11} & \cdots & \beta a_{1m} \\ \vdots & \ddots & \vdots \\ \beta a_{n1} & \cdots & \beta a_{nm} \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}$$

$$b_{ij} = \beta a_{ij}$$

3. Transpose

Transposing a matrix of order $(n \times m)$ gives the same matrix but with dimensions $(m \times n)$ by switching between rows and columns. $B = A^T$, $b_{ij} = a_{ji}$

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}^T = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1m} & \cdots & a_{nm} \end{pmatrix}$$

For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

- A transposed column vector becomes a row vector.
- Transposing twice gives back the original matrix $\left(A^T\right)^T = A$.
- $(A+B)^T = A^T + B^T$
- $A + A^T$ is symmetric.
- $A - A^T$ is anti-symmetric.

4. Trace

The sum of diagonal elements (defined only for square matrices)

$$Tr\left(A\right) = \sum_{i=1}^{n} a_{ii}$$

5. Matrix-Matrix Product

Proof:

Knowing the matrix-vector product, we can write:

$$y = Ax, \quad z = By$$

Substituting y,

$$z = By = BAx \equiv Cx$$

the composition of A, B can be used to define matrix multiplication.[1]

$$\sum_m c_{km} x_m \equiv z_k = \sum_n b_{kn} y_n = \sum_n b_{kn} \left(\sum_m a_{nm} x_m \right) = \sum_m \left(\sum_n b_{kn} a_{nm} \right) x_m$$

Therefore,

Definition

$$\boxed{c_{km} \equiv \sum_n b_{kn} a_{nm}}$$

! Each component of the final matrix is the dot product of the corresponding row and column of the multiplied matrices. It is like solving multiple matrix-vector products, where each column of the second matrix is a vector.

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta & a\gamma + b\delta \\ c\alpha + d\beta & c\gamma + d\delta \\ e\alpha + f\beta & e\gamma + f\delta \end{pmatrix}$$

$$I_{2\times 2} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad AI = IA = A$$

- Associative: $A(BC) = (AB)C$
- Distributive: $A(B + C) = AB + AC$
- Non-commutative: $AB \neq BA$
- $AB = 0$ does not mean $A = 0$ or $B = 0$.

6. Outer Product

The outer product between two matrices is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{n1}B \\ \vdots & \ddots & \vdots \\ a_{1m}B & \cdots & a_{nm}B \end{pmatrix}$$

2.3.5 Additional Matrix Properties/Definitions

- A matrix's order is the number of rows and columns. $(n \times m)$ for a matrix with n rows and m columns.
- If it is a square matrix $(n \times n)$ it said to have order n.
- A diagonal matrix has nonzero elements only on its diagonal ($a_{ij} = \delta_{ij} a_{ij}$).[2]

[1] Here k,n,m are indices and not the actual dimensions of the matrices.

[2] $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

- A symmetric matrix satisfies $a_{ij} = a_{ji}$.
- An anti-symmetric matrix satisfies $a_{ij} = -a_{ji}$ and therefore has a zero diagonal.
- Any matrix can be written as a sum of a symmetric and anti-symmetric matrices $A = \frac{\left(A+A^T\right)+\left(A-A^T\right)}{2}$.
- !The inverse of a matrix A is the matrix A^{-1} that satisfies $AA^{-1} = A^{-1}A = I$.
- !An orthogonal matrix is a matrix that satisfies $A^T A = I \iff A^T = A^{-1}$.
- !A unitary matrix is a complex matrix that satisfies $A^{\dagger} A = \left(A^T\right)^* A = I$.

2.3.6 Inverse Matrix

$$A^{-1}A = I$$

We can use gauss elmination on $(A \mid I)$ to get to $\left(I \mid A^{-1}\right)$.
A general 2x2 matrix solution is

$$A^{-1} = \frac{1}{ad-cb}\left(\begin{array}{cc} d & -b \\ -c & a \end{array}\right) = \frac{1}{detA}\left(\begin{array}{cc} d & -b \\ -c & a \end{array}\right)$$