Memphis 1

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March 7, 2024

Contents

Part I
Differential Calculus of Single Variable Real Functions

1 Function

1.1 Definition

Function (~): A map between two sets. Mapping from domain to co-domain/image.
Set: A collection of elements (e.g numbers)

$$y = f(x)$$

y is the *dependent* variable, x is the *independent* variable.

1.2 Real

A real function is a function for which both the dependent (y) and independent (x) variable are *real* ($a \in \mathbb{R}$, $a = a^*$).

The real numbers include:

- Natural (1,2,3,4...)
- Integer (-2,-1,0,1,2...)
- Rational ($\frac{1}{2}$, 36.21...)
- Irrational (π, $\sqrt{2}$, e)

The real numbers *don't* include the imaginary and complex numbers.

1.3 Single Valued Function

Definition: For each set of value of the independent variables, there is a *single* corresponding value of the function.

A counter example $f(x) = \pm\sqrt{x}$ is *multi-valued*.

1.4 Domain

The domain of the function is the set of inputs for which the function is defined, meaning there exists $y_0 = f(x_0)$.

Any real x: $x \in \mathbb{R}$
$f(x) = \frac{g(x)}{h(x)} \quad \{x \mid h(x) \neq 0\}$
$f(x) = \sqrt{g(x)} \quad \{x \mid g(x) \geq 0\}$
$f(x) = log(g(x)) \quad \{x \mid g(x) > 0\}$

For example, $f(x) = \pm\sqrt{x}$ is defined (real) over $x \geq 0$.
For example, $f(x) = \frac{1}{x}$ is defined over $x \neq 0$.
For example, $f(x) = \pm\sqrt{4 - x^2}$ is defined (real) over $-2 \leq x \leq 2$.

1.5 Even/Odd

An even function satisfies $f(x) = f(-x)$
An odd function satisfies $-f(x) = f(-x)$

1.6 Limits

1.6.1 Definition

"The limit of $f(x)$ as x goes to x_0 is equal to A":

$$\lim_{x \to x_0} f(x) = A$$

Meaning (~): $f(x)$ can be arbitrarily close to A given a choice of x close enough to x_0.

Definition: For any arbitrarily small $\varepsilon > 0$, there is $\delta > 0$ so that if $|x - x_0| < \delta$ then $|f(x) - A| < \varepsilon$.
Notice: The limit *doesn't* depend on $f(x_0)$ and it isn't a requirement that $f(x_0)$ is defined.
The limit from the left $\lim_{x \to x_0^-} f(x) = A$: x goes to x_0 from numbers smaller than x_0.
The limit from the left $\lim_{x \to x_0^+} f(x) = B$: x goes to x_0 from numbers larger than x_0.

1.6.2 Existance

A limit "exists" if: there exists a limit both from the left and from the right and *they are equal* $A = B$ (or if one side is undefined).
For example:

$$lim_{x \to 0^-} \sqrt{x} = undefined$$

$$lim_{x \to 0^+} \sqrt{x} = 0$$

Therefore, the limit exists.

$$lim_{x \to 0} \sqrt{x} = 0$$

1.6.3 Properties

Basically, any arithmetic operation can be inserted into/extracted from the limit:

1. if $\quad f(x) = c \quad \Rightarrow \quad \lim_{x \to a} f(x) = c$
2. $\lim_{x \to \infty} x^\alpha = \begin{cases} \infty & \alpha > 0 \\ 1 & \alpha = 0 \\ 0 & \alpha < 0 \end{cases}$
3. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
5. $\lim_{x \to a} [f(x) \cdot g(x)] = [\lim_{x \to a} f(x)] \cdot [\lim_{x \to a} g(x)]$
6. $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \lim_{x \to a}, \quad g(x) \neq 0$
7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$
8. $\lim_{x \to a} log\,[f(x)] = log\,[\lim_{x \to a} f(x)]\,, \quad \lim_{x \to a} f(x) > 0$

Undefined quantities:
$\frac{\infty}{\infty}, \frac{0}{0}, 0 \cdot \infty, \infty - \infty, 1^\infty, \infty^0, 0$

1.6.4 Infinities

$$\lim_{x \to a} f(x) = +\infty$$

Meaning (~): as x goes to x_0, $f(x)$ can be larger than any number we pick.
Definition: For any arbitrarily large $M > 0$, there is $\delta > 0$ so that if $|x - x_0| < \delta$ then $f(x) > M$.

The negative case can be defined in a similar manner $\lim_{x \to a} f(x) = -\infty$.

For example:

$$\lim_{x\to 0^-}\frac{1}{x}=-\infty,\quad \lim_{x\to 0^+}\frac{1}{x}=+\infty$$

(the limit does *not* exist)

Similarly we can have limits of a variable going to infinity.

$$\lim_{x\to\infty} f(x)=A$$

Definition: For any arbitrarily large $N>0$, there is $\varepsilon>0$ so that if $x>N$ then $|f(x)-A|<\varepsilon$.

1.7 Continuity

A function is said to be continuous at point x_0 if

$$\lim_{x\to x_0} f(x)=f(x_0)$$

This is actually 3 conditions in 1:

1. $f(x_0)$ is defined
2. The limit is defined
3. The limit is equal to the function (as stated in the equation above)

For example, $f(x)=\frac{1}{x-2},\ \ x_0=2$
$f(2)$ is undefined $\Rightarrow$ the function is not continuous at $x=2$
(as well as the other reasons).

For example, $f(x)=3x^3-7x^2-4x-2$,
The functions is continuous for every x. (polynomials are always continuous everywhere)

For example, $f(x)=\frac{x+2}{x-1}$
f(1) is undefined $\Rightarrow$ non-continuous at x=1

For example, $f(x)=\lfloor x\rfloor$
is *defined* everywhere, but at every integer x the limit does *not exist* $\Rightarrow$ non-continuous at $x\in\mathbb{Z}$

A function that is continuous at every point in the region $a\le x\le b$ is said to be continuous over that region. End of class 1.

2 Derivative

2.1 Definition

The derivative of a function $y = f(x)$ at point x_0 is

$$\frac{dy}{dx}|_{x=x_0} \equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, \quad \Delta x \equiv x - x_0$$

Derivative from the right/left: $f'_{\pm}(x_0) = \lim_{\Delta x \to 0^{\pm}} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.

The derivative exists if and only if: $f'_{+}(x_0) = f'_{-}(x_0)$ (if the function is differentiable at x_0 it must also be continuous at x_0).

A function that is differentiable at every point in the region $a \le x \le b$ is said to be differentiable over that region.

Tangent line:

$$y = y_0 + \frac{dy}{dx}|_{x=x_0}(x - x_0)$$

2.2 Properties

Given a constant c and the functions of x - f, g, h:

Linearity

$\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$

$\frac{d}{dx}(cf) = c\frac{df}{dx}$

Product rule

$\frac{d}{dx}(f \cdot g) = \frac{df}{dx}g + f\frac{dg}{dx}$

$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$

Chain rule

$\frac{df(u)}{dx} = \frac{df}{du}\frac{du}{dx}$

Polynomial, trigonometric, exponential and logarithmic derivatives

$\frac{dx^n}{dx} = nx^{n-1}$

$\frac{d}{dx}sin(x) = cos(x)$

$\frac{d}{dx}cos(x) = -sin(x)$

$\frac{d}{dx}tan(x) = \frac{1}{cos^2(x)} = 1 + tan^2(x)$

$\frac{d}{dx}sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}a^x = log_e a \cdot a^x$

$\frac{d}{dx}log_a x = log_a e \cdot \frac{1}{x}$

The reciprocal/inverse function of $y = f(x)$ is $x = f^{-1}(y)$. $\frac{dy}{dx} = 1/\frac{dx}{dy}$.

2.3 Properties Proofs

Product rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\left[g(x+\Delta x) - g(x)\right] + g(x)\left[f(x+\Delta x) - f(x)\right]}{\Delta x}$$

$$= f(x) \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + g(x) \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= f'g + g'f$$

End of class 2.

Log derivative:

$$\frac{d}{dx}\left(log_a x\right) = \lim_{\Delta x \to 0} \frac{log_a(x+\Delta x) - log_a(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x}{x} \frac{log_a(\frac{x+\Delta x}{x})}{\Delta x} = \frac{1}{x} \lim_{\Delta x \to 0} log_a \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}$$

$$\frac{1}{x} log_a \left(\lim_{\Delta x \to 0} \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right) = \frac{1}{x} log_a\left(e\right)$$

2.4 Logarithmic Derivative

Sometimes we know how to differentiate the ln of a function but not the function itself. In that case we can use:

$$\frac{d}{dx}\left[lnf(x)\right] = \frac{1}{f(x)}\frac{df}{dx}$$

$$\Downarrow$$

$$\boxed{\frac{df}{dx} = f(x) \cdot \frac{d}{dx}\left[lnf(x)\right]}$$

2.5 l'Hopital's Rule

If $\lim_{x\to 0}\frac{f(x)}{g(x)} = "\frac{0}{0}"$ or $\frac{\infty}{\infty}$ (or similar) the limit isn't defined. We can replace the functions with their derivatives

$$\boxed{\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}}$$

2.6 Examples

$$y = sin^{-1}(x),\ x = sin(y)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(sin^{-1}(x)\right) = \frac{1}{dx/dy} = \frac{1}{cos(y)} = \frac{1}{\pm\sqrt{1 - sin^2 y}} = \frac{\pm 1}{\sqrt{1 - x^2}}$$

$$y = tan^{-1}x,\ x = tan\, y$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(tan^{-1}x\right) = \frac{1}{dx/dy} = \frac{1}{1 + tan^2 y} = \frac{1}{1 + x^2}$$

$$y = a^x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(a^x\right) = \frac{1}{dx/dy} = \frac{y}{log_a(e)} = \frac{a^x}{log_a(e)} = ln(a)a^x$$

$$y = x^{x-1}$$

$$\frac{d}{dx}\left(x^{x-1}\right) = x^{x-1} \cdot \frac{d}{dx}\left((x-1) \cdot lnx\right) = x^{x-1} \cdot \left(1 - \frac{1}{x} + lnx\right)$$

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{2e^{2x}}{1} = 2$$

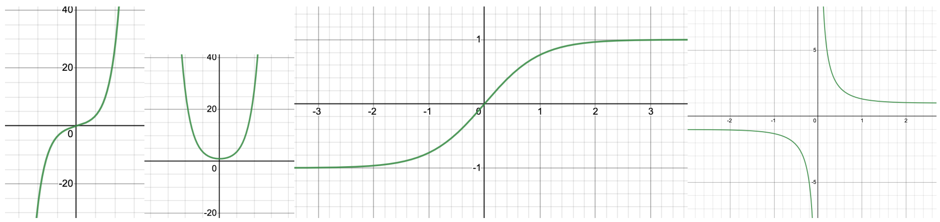
$$\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} 2xe^{-x} = \lim_{x \to \infty} 2e^{-x} = 0$$

$$\lim_{x \to 0^+} x^2\,(lnx) = \lim_{x \to 0^+} \frac{lnx}{1/x^2} = \lim_{x \to 0^+} \frac{1/x}{-2/x^3} = \lim_{x \to 0^+} \frac{x^2}{-2} = 0$$

3 Hyperbolic Function

3.1 Definition

1. Hyperbolic Sine $sinh(x) = sh(x) \equiv \frac{e^x - e^{-x}}{2}$
2. Hyperbolic cosine $cosh(x) = ch(x) \equiv \frac{e^x + e^{-x}}{2}$
3. Hyperbolic tangent $tanh(x) = th(x) \equiv \frac{sinh(x)}{cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4. Hyperbolic cotangent $coth(x) = 1/tanh(x)$



3.2 Identities and Derivatives

- $cosh^2(x) - sinh^2(x) \left(= \frac{e^{2x} - e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4}\right) = 1$
- $\frac{d}{dx} sinh(x) = cosh(x)$
- $\frac{d}{dx} cosh(x) = sinh(x)$
- $\frac{d}{dx} tanh(x) \left(= \frac{d}{dx} \frac{sinh(x)}{cosh(x)} = \frac{cosh^2(x) - sinh^2(x)}{cosh^2(x)}\right) = \frac{1}{cosh^2(x)} = 1 - tanh^2(x)$
- $\frac{d}{dx} coth(x) = \frac{-1}{sinh^2(x)} = 1 - coth^2(x)$

3.3 Inverse Functions

- $\frac{d}{dx} sinh^{-1}(x) = \frac{1}{cosh(y)} = \frac{+1}{\sqrt{1 + sinh^2(y)}} = \frac{1}{\sqrt{1 + x^2}}$
- $\frac{d}{dx} cosh^{-1}(x) = \frac{1}{sinh(y)} = \frac{\pm 1}{\sqrt{cosh^2(y) - 1}} = \frac{\pm 1}{\sqrt{x^2 - 1}}$
- $\frac{d}{dx} tanh^{-1}(x) = \frac{1}{1 - x^2}$

4 Parametric Function Representation

4.1 Definition

Instead of the relation $y = f(x)$, we have a third variable acting as a *parameter*:

$$\begin{cases} x &= x(t) \\ y &= y(t) \end{cases}$$

4.2 Examples

For example,

1. an ellipse is mathematically described as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or parametrically,

$$\begin{cases} x &= acos(t) \\ y &= bsin(t) \end{cases}$$

Checking for equivalence

$$\frac{a^2 cos^2(t)}{a^2} + \frac{b^2 sin^2(t)}{b^2} = cos^2(t) + sin^2(t) \stackrel{\checkmark}{=} 1$$

2. A hyperbola is parametrically

$$\begin{cases} x &= acosh(t) \\ y &= bsinh(t) \end{cases}$$

or explicitly,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

4.3 Derivative

To differentiate a function with parametric representation:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy/dt}{dx/dt}$$

4.4 Examples

Differentiating the hyperbola parametrically

$$\frac{dy}{dt} = bcosh(t)$$

$$\frac{dx}{dt} = asinh(t)$$

$$\frac{dy}{dx} = \frac{bcosh(t)}{asinh(t)} = \frac{b}{a}\frac{\pm x}{a\sqrt{cosh^2(t) - 1}} = \frac{\pm bx}{a\sqrt{x^2 - a^2}}$$

or explicitly

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

$$\frac{dy}{dx} = \frac{\pm 2bx}{2a\sqrt{x^2 - a^2}} = \frac{\pm bx}{a\sqrt{x^2 - a^2}}$$

5 Asymptotes

An asymptote of $f(x)$ is a linear function ($y = kx + c$) which $f(x)$ tends to (only!) at $x \to \pm\infty$.

$$\boxed{\lim_{x\to\infty} [f(x) - (kx + c)] = 0}$$

To find the coefficients k, c - we know $\lim_{x\to\infty} \frac{1}{x} = 0$. Multiplying both equations,

$$\lim_{x\to\infty} \left[\frac{f(x)}{x} - k\right] = 0$$

Therefore

$$\boxed{k = \lim_{x\to\infty} \frac{f(x)}{x}}, \quad \boxed{c = \lim_{x\to\infty} [f(x) - kx]}$$

Bimkom asymptot anachit : mitbader / nekudat singulariut.

6 Taylor Series

6.1 Definition

The first approximation of $f(x)$ at the point $x = a$ is the tangent to the function, and has the same first derivative at the point a.

$$y_1(x) = f(a) + (x - a)\, f'(a)$$

The second approximation is a parabola, and has the same first and second derivatives at the point a.

$$y_2(x) = f(a) + (x - a)f'(a) + \frac{1}{2}(x - a)^2 f''(a)$$

Therefore the n-th approximation is

$$y_n = \sum_{i=0}^{n} \frac{(x - a)^i}{i!} \frac{d^i f}{dx^i}$$

And the taylor expansion is

$$\boxed{f(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{(x - a)^i}{i!} \frac{d^i f}{dx^i}}$$

6.2 Examples

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} ... = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} ... = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\boxed{e^{ix} = cos(x) + isin(x)}$$

7 Complex Numbers

7.1 Definition

A complex number is a sum of a real number a and an imaginary number bi.

$$z = a + bi \in \mathbb{C}$$

Where $i \equiv \sqrt{-1}$. (for example, $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$).

The complex conjugate of z, notated z^* or $\bar{z}$ is

$$z^* = a - bi = r\left[cos\left(-\varphi\right) + isin\left(-\varphi\right)\right] = re^{-i\varphi}$$

It has the properties

$$z + z^* = 2Re(z)$$

$$z - z^* = 2Im(z)$$

7.2 Polar and Cartesian

We can define some basic function of the complex numbers

$$Re(z) \equiv a, \quad Im(z) \equiv b$$

$$r = |z| \equiv zz^* = \sqrt{a^2 + b^2}, \quad \varphi = arg(z) \equiv arctan\left(\frac{b}{a}\right)$$

That way complex numbers can be written both in polar and cartesian representation

$$z = a + bi = r\left(cos\varphi + isin\varphi\right) = re^{i\varphi}$$

7.3 Polar Product

We can write multiplication of complex numbers in polar form:

$$z_1 z_2 = r_1\left(cos\varphi_1 + isin\varphi_2\right) r_2\left(cos\varphi_2 + isin\varphi_2\right)$$

$$= r_1 r_2\left[\left(cos\varphi_1 cos\varphi_2 - sin\varphi_1 sin\varphi_2\right) + i\left(cos\varphi_1 sin\varphi_2 + sin\varphi_1 cos\varphi_2\right)\right]$$

$$= r_1 r_2\left[cos\left(\varphi_1 + \varphi_2\right) + isin\left(\varphi_1 + \varphi_2\right)\right]$$

We get that complex number multiplication is equivalent to radius r multiplication with argument addition φ (de Moivre).

$$z^n = r^n\left[cos\left(n\varphi\right) + isin\left(n\varphi\right)\right] = r^n e^{in\varphi}$$

If instead of n we have some fraction $1/n$ we need to have n solutions.

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[cos\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right) + isin\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right)\right] = r^{\frac{1}{n}} e^{i\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right)}$$

For example, $\sqrt[3]{1} = e^{i\frac{2\pi k}{3}}, k \in \mathbb{Z}$.

7.4 Hyperbolic and Trig. Functions

Using Euler's equation, we can write cos and sin as

$$cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$sin\varphi = \frac{e^{i\varphi} - e^{i\varphi}}{2i}$$

We now see we have

$$cosh(i\varphi) = cos(\varphi)$$

$$sinh(i\varphi) = isin(\varphi)$$

and

$$cos(i\varphi) = cosh(\varphi)$$

$$sin(i\varphi) = isinh(\varphi)$$

7.5 Additional Operations

Raising e to a complex number

$$e^z = e^{x+yi} = e^x e^{iy}$$

natural log of a complex number is

$$\ln z = \ln\left(re^{i(\varphi+2\pi n)}\right) = \ln(r) + i(\varphi + 2\pi n)$$

Complex number raised to a complex number

$$z_1^{z_2} = e^{\ln(z_1)z_2} = ...$$

8 Differentials

Given $y = f(x)$, the differentials of x and y are

$$dx = \lim_{\Delta x \to 0} \Delta x$$

$$dy = f'(x)dx$$

dy is the first order expansion for the change in y as a function of x.

Differentials satisfy the properties:

$d(f+g) = df + dg$

$d(c \cdot f) = c \cdot df$

$d(uv) = u \cdot dv + du \cdot v$

$d\left(\frac{u}{v}\right) = \frac{v \cdot du - u \cdot dv}{v^2}$

$d(\sin x) = (\cos x)\,dx$

etc.

Part II
Integral Calculus of Single Variable Real Functions

9 Integrals

9.1 Definition

We'll take the interval $[a, b]$ and divide it to n parts. Summing the rectangles under the graph to get the area we have

$$A = \sum_{k=1}^{n} f(\xi_n)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(\xi_n) \cdot \Delta x_k$$

As $\Delta x_k \to 0$:

$$\boxed{\int_a^b f(x)\,dx = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} f(\xi_n) \cdot \Delta x_k}$$

If $f(x) \geq 0$ within the integration interval $[a, b]$, $\int_a^b f(x)\,dx$ is the area between the graph and the x axis.
If $f(x) < 0$ somewhere in the interval, $\int_a^b f(x)\,dx$ is just the sum of "positive and negative areas".
$\int_a^b f(x)\,dx$ gives a scalar. x is a dummy variable.
$f(x)\,dx$ is the integrand.

9.2 Indefinite Integral

A function $F(x)$ that satisfies $F'(x) = f(x)$ is the anti derivative/indefinite integral of $f(x)$.
If $F(x)$ is an anti derivative of $f(x)$ then $F(x) + c$ is also an anti derivative.
We'll notate the anti derivative as

$$F(x) = \int f(x)dx$$

9.3 The Fundemental Theorem of Calculus

If $F'(x) = f(x)$ then $\boxed{\int_a^b f(x)dx = F(b) - F(a)}$.

9.3.1 The Mean Value Theorem Proof

If $f(x)$ is continuous over $[a, b]$, then there exists some $a \leq \xi \leq b$ so that $\frac{\int_a^b f(x)dx}{b-a} = f(\xi)$.
Proof:

Let m and M be the minimum and maximum values of $f(x)$ in the interval. Then $\forall x \in [a, b] : m \leq f(x) \leq M$.
Therefore $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$. Dividing by $(b-a)$ we get $m \leq \frac{\int_a^b f(x)dx}{b-a} \leq M$.
Since $f(x)$ is continuous there must exist some ξ so that $f(\xi) = \frac{\int_a^b f(x)dx}{b-a}$.

9.3.2 Integral is Anti Derivative Proof

Let $F(x) = \int_a^x f(t)dt + C$. We'll prove $F'(x) = f(x)$.
Proof:

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right] = \frac{1}{h}\left[\int_x^{x+h} f(t)dt\right]$$

Now using the mean value theorem for $x \leq \xi \leq x + h$:

$$= \frac{1}{h}\left[h \cdot f(\xi)\right] = f(\xi)$$

And since $f(x)$ is continuous $\lim_{h\to 0} f(x+h) = f(x)$. Finally,

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} f(\xi) \overset{\checkmark}{=} f(x)$$

9.3.3 F(b)-F(a) Proof

Let $F(x) = \int_a^x f(t)dt + C$.

$$F(a) = 0 + C = C$$

$$F(b) = \int_a^b f(t)dt + C = \int_a^b f(t)dt + F(a)$$

Finally,

$$\int_a^b f(t)dt = F(b) - F(a)$$

9.4 Properties

$\int_a^b (f \pm g)\, dx = \int_a^b f dx \pm \int_a^b g dx$
$\int_a^b cf(x)dx = c\int_a^b f(x)dx$
$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$
$\int_a^b f dx = -\int_b^a f dx$
$\frac{d}{dx}\int_a^x f(t)dt = f(x)$
$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f\left[v(x)\right]\frac{dv}{dx} - f\left[u(x)\right]\frac{du}{dx}$

$\int \cos u \cdot du = -\sin u + C$
$\int a^u du = \frac{a^u}{\ln a} + c$
$\int \sinh u \cdot du = \cosh u + C$
$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$
$\int u^n du = \frac{u^{n+1}}{n+1} + C$
$\int \frac{du}{u} = \ln|u| + C$

Integration by parts: $\boxed{\int fg'dx = fg - \int f'gdx}$

Partial Fractions: $\frac{P(x)}{Q(x)}$ where $\lim_{x\to\infty}\frac{P(x)}{Q(x)} = 0$. We'll define the fraction as a sum of $\frac{A}{(ax+b)^r}, \frac{Bx+C}{(ax^2+bx+c)^r}$

9.5 Examples

For example,

1)

$$\int cos^2 x dx = \int \cos x \cos x dx = \sin x \cos x + \int \sin^2(x)\,dx = \sin x \cos x + \int 1 - \cos^2(x)\,dx$$

$$I = \sin x \cos x + x - I$$

$$I = \frac{\sin x \cos x}{2} + \frac{x}{2} + C$$

2)

$$\int (x+2)\sin\left(x^2+4x-6\right)dx = \int (x+2)\sin u \frac{du}{2x+4} = \int \frac{\sin u}{2}du = -\frac{\cos\left(x^2+4x-6\right)}{2} + C$$

3)

$$\int \frac{x^2}{\sqrt{a^2-x^2}}dx = \frac{1}{a}\int \frac{x^2}{\sqrt{1-\left(\frac{x}{a}\right)^2}}dx \overset{u=\frac{x}{a}}{=} a^2 \int \frac{u^2}{\sqrt{1-u^2}}du \overset{u=\sin v}{=} a^2 \int \frac{\sin^2 v \cdot \cos v}{\cos v}dv = a^2 \int \sin^2 v dv$$

$$= \frac{a^2}{2}\left(v - \sin v \cos v\right) + C = \frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} - \frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right) + C$$

4)

$$\int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{(x-1)^2+4}} = \int \frac{dx}{2\sqrt{\left(\frac{x-1}{2}\right)^2+1}} \overset{u=\frac{x-1}{2}}{=} \int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1}u + C$$

$$= \sinh^{-1}\left(\frac{x-1}{2}\right) + C$$

5)

$$\int \frac{6-x}{2x^2-x-15}dx$$

We'll write the fraction as following:

$$\frac{6-x}{(x-3)(2x+5)} = \frac{A}{x-3} + \frac{B}{2x+5}$$

$$2Ax + 5A + Bx - 3B = 6 - x$$

$$\Downarrow$$
$$2A + B = -1$$
$$5A - 3B = 6$$
$$\Downarrow$$
$$A = \tfrac{3}{11}, B = \tfrac{17}{11}$$

Finally, substituting back we get

$$\int \frac{6-x}{2x^2-x-15}dx = \int \frac{3/11}{x-3} + \frac{17/11}{2x+5}dx = \frac{3}{11}\ln|x-3| + \frac{17}{11}\frac{1}{2}\ln|2x+5| + C$$

6)

$$\int \frac{2x^2+3}{\left(x^2+1\right)^2}dx$$

$$\frac{2x^2+3}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2}$$

$$\Downarrow$$

$$2x^2+3 = Ax^3+Bx^2+(A+C)x+(B+D)$$

$$\Downarrow$$

$$A=0$$
$$B=2$$
$$C=0$$
$$D=1$$

$$\int \frac{2x^2+3}{\left(x^2+1\right)^2}dx = \int \frac{2}{x^2+1} + \frac{1}{\left(x^2+1\right)^2}dx = 2\tan^{-1}(x) + ...$$

$$\int \frac{1}{\left(x^2+1\right)^2}dx \overset{x=\tan u}{=} \int \frac{\frac{1}{\cos^2 u}}{\frac{1}{\cos^4 u}}du = \int \cos^2 u du = \frac{1}{2}\left(u + \frac{\tan u}{\tan^2 u+1}\right) = \frac{1}{2}\left(\tan^{-1}x + \frac{x}{x^2+1}\right)$$

finally,

$$\int \frac{2x^2+3}{\left(x^2+1\right)^2}dx = \frac{5}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{x^2+1} + C$$

7)

$$\int \frac{dx}{5+4\sin x} \overset{u=\tan\frac{x}{2}}{=} \int \frac{2du}{(1+u^2)\left(5+4\frac{2u}{1+u^2}\right)} = \int \frac{2du}{5+5u^2+8u} = \frac{2}{5}\int \frac{du}{u^2+\frac{8}{5}u+1} = \frac{2}{5}\int \frac{du}{\left(u+\frac{4}{5}\right)^2+\frac{9}{25}}$$

$$\frac{2}{5}\frac{25}{9}\int \frac{du}{\left[\frac{5}{3}\left(u+\frac{4}{5}\right)\right]^2+1} \overset{z=\frac{5}{3}\left(u+\frac{4}{5}\right)}{=} \frac{2}{5}\frac{25}{9}\frac{3}{5}\int \frac{dz}{z^2+1} = \frac{2}{3}\tan^{-1}z = \frac{2}{3}\tan^{-1}\left(\frac{5}{3}\left(u+\frac{4}{5}\right)\right)$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{5}{3}\left(\tan\frac{x}{2}+\frac{4}{5}\right)\right) + C$$

8)

$$\int_0^{\pi/2} \frac{dx}{5+4\sin x} \overset{u=\tan\frac{x}{2}}{=} \frac{2}{5}\int_0^1 \frac{du}{\left(u+\frac{4}{5}\right)^2+\frac{9}{25}} \overset{z=\frac{5}{3}\left(u+\frac{4}{5}\right)}{=} \frac{2}{3}\int_{\frac{4}{3}}^3 \frac{dz}{z^2+1} = \left(\frac{2}{3}\tan^{-1}z\right)|_{4/3}^{3}...$$

9)

$$\int_0^a f(x)dx \overset{y=a-x}{=} -\int_a^0 f(a-y)dy = \int_0^a f(a-y)dy = \int_0^a f(a-x)dx$$

wow so surprisingg

10)

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \pi/2$$

$$I = \frac{\pi}{4}$$

9.6 Improper Integrals

When the integrand $f(x)$ is integrated over an infinite interval, or $f(x)$ isn't defined somewhere/goes to ∞ in the interval.

For example,

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{M\to\infty} \int_0^M \frac{dx}{1+x^2} = \lim_{M\to\infty} \left[\tan^{-1}(x)\right]_0^M = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0^+} \left(2\sqrt{x}\right)|_\varepsilon^1 = \lim_{\varepsilon\to 0^+} \left(2-2\sqrt{\varepsilon}\right) = 2$$

$$\int_0^1 \frac{dx}{x} = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{dx}{x} = \lim_{\varepsilon\to 0^+} \ln|x||_\varepsilon^1 = \lim_{\varepsilon\to 0^+} (-\ln\varepsilon) = \infty$$

$$\int_{-1/2}^1 \frac{dx}{x} = \ln|x||_{-1/2}^1 = \left(-\ln\frac{1}{2}\right) \quad \text{nah bro. it aint defined. dont even try.}$$

There are integrals with no analytic solution. For example, $\int e^{x^2} dx$.

Then, either it can be approximated numerically (on a computer), or it can be solved using a taylor expansion.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ...$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} ...$$

$$\int_0^1 e^{x^2} dx = \int_0^1 \sum_{k=0}^\infty \frac{x^{2k}}{k!} = \sum_{k=0}^\infty \frac{x^{2k+1}}{(2k+1)\,k!}|_0^1 = \sum_{k=0}^\infty \frac{1}{(2k+1)\,k!} \approx 0.463$$

$$\int_0^1 \frac{\sin x}{x} dx \approx \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right) dx = ...$$

9.7 Area Between Curves

The area between two curves y_1 and y_2 in the interval $[a, b]$ is $\boxed{A = \int_a^b |y_1 - y_2|\, dx}$.

9.8 Curve Length

The length of the curve $f(x)$ over a *small* Δx, is $\Delta\ell = \sqrt{\Delta x^2 + \Delta y^2} = \Delta x\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \to d\ell = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

Integrating over the interval $[a, b]$, we get

$$\boxed{l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{f(a)}^{f(b)} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy = \int_{u(a)}^{u(b)} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} du}$$

We'll use the formula that is most convenient as they are all equal.

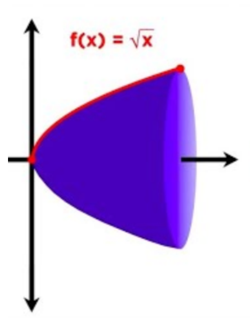
<u>For example,</u>

The length of $y = x^2$ from $x = 0$ to $x = 1$ is

$$\ell = \int_0^1 \sqrt{1+4x^2} dx = \frac{1}{2}\int_0^1 \sqrt{1+u^2} du = \int_0^1 \sqrt{1+\sinh^2 v} ...$$

9.9 Solid of Revolution

We can create a rotationally symmetric 3D body by taking any function $f(x)$ and revolving its curve around an axis (x or y).



To calculate the volume of a solid around the x axis:

$$V_{ab} = \int_a^b \pi f^2(x)dx$$

To calculate the volume of a solid around the y axis:

$$V_{ab} = \int_{f(a)}^{f(b)} \pi x^2 df = \int_a^b \pi x^2 f'(x)dx$$

we can also divide it (solid around y axis) to hollow cylinders to get the volume on $x \in [-b, b]$:

$$V_{ab} = \int_0^b 2\pi x \left[f(b) - f(x)\right] dx = \pi f(b)b^2 - 2\pi \int_0^b x f(x)dx$$

Surface area of body around x:

$$S = \int_a^b 2\pi f(x)d\ell = 2\pi \int_a^b f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Surface area of body around y:

$$S = \int_a^b 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Generally it is recommended to just create these equations yourself, and not remember them.

10 Distributions

10.1 Mean and Expected Value

The mean value of a function over the interval $[a, b]$:

$$\bar{f} = \frac{\sum f_i}{\sum 1} = \frac{\sum f_i \Delta x}{\sum 1 \Delta x} \to \frac{\int_a^b f(x)dx}{\int_a^b dx} = \boxed{\frac{\int_a^b f(x)dx}{b-a}}$$

The expected value of $g(x)$, "$\langle g(x)\rangle$" under the distribution $N(x)$ is as following:

$$\bar{g} = \boxed{\langle g\rangle = \frac{\int_a^b g(x)N(x)dx}{\int_a^b N(x)dx}}$$

For example, for $N(x) = e^{-x^2}$, $\langle x \rangle = 0$, $\langle x^2 \rangle = \frac{1}{2}$ (just by calculating the integrals).
$\langle x^n \rangle$ is the n-th moment of x.

10.2 Gaussian Function

A general gaussian function is written as

$$\boxed{f(x) = e^{-\frac{(x-x_0)^2}{2\sigma^2}}}$$

σ is the width, x_0 the maximum. Its moments are $\langle x \rangle = 0$, $\langle x^2 \rangle = \sigma$, $\langle x^3 \rangle = 0$, $\langle x^4 \rangle = 3\sigma^4$...

Part III

Differential Calculus of Multivariable Real Functions

11 Introduction

11.1 Multivariable Functions

We can define functions dependent on many variables.

$$y = f(x, y, z, ...)$$

For example $z = \sqrt{1-(x^2+y^2)}$ is a sphere of radius 1.

11.2 Limits

Let $\varepsilon > 0$. There exists some δ that satisfies $(x-x_0)^2 + (y-y_0)^2 < \delta^2$ such that $|f(x,y) - A| < \varepsilon$:

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x,y) = A$$

The limit exists if for any path $(x,y) \to (x_0, y_0)$, $f(x,y) \to A$

For example we'll look at $\begin{matrix} x \to 0^- \\ y \to 1 \end{matrix}$ with $\tan^{-1}\left(\frac{y}{x}\right)$. We get $-\frac{\pi}{2}$. The limit does not exist because $\lim_{\substack{x \to 0^+ \\ y \to 1}} \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2}$.

11.3 Continuity

A function $f(x_1, x_2, ...)$ is continuous at at $(x_{1,0}, x_{2,0}, ...)$ if $\lim_{x_i \to x_{i,0}} f(x_1, x_2, ...) = f(x_{1,0}, x_{2,0}, ...)$.

This is actually 3 conditions:

1. The limit exists at that point (x_0, y_0)
2. The function is defined at that point (x_0, y_0)
3. The function and the limit are equal at that point (x_0, y_0)

12 Derivatives and Differentials

12.1 Derivative Definition

The partial derivative of a multivariable function is defined as:

$$\boxed{\frac{\partial f}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z...) - f(x, y, z...)}{\Delta x}}$$

Other notations for partial derivatives are $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial x}|_y=f_x=\partial_x f$.

For example,

1. $f(x,y)=2x^3+3xy^2$. $\frac{\partial f}{\partial x}=6x^2+3x$. $\frac{\partial f}{\partial y}=6xy$.
2. $f(x,y,z)=xy+yz+xz$. $f_x=y+z$. $f_y=x+z$. $f_z=x+y$.

Just like with the regular derivative operator, the partial derivative operator can be applied multiple times $\frac{\partial^n f}{\partial x^n}$. It also has multiply notation, for example $\frac{\partial^2 f}{\partial x^2}=f_{xx}$. Or we can even differentiate $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=f_{yx}=f_{xy}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$.

12.2 Differentials and Chain Rule

The perfect differential of some function $f(x_1,...,x_n)$ is

$$\boxed{df=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}dx_i}$$

we can prove this by $\Delta f=f(x+\Delta x,y+\Delta y)-f(x,y)$
$=[f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)]+[f(x,y+\Delta y)-f(x,y)]$
$=\frac{[f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)]}{\Delta x}\Delta x+\frac{[f(x,y+\Delta y)-f(x,y)]}{\Delta y}\Delta y$
$\rightarrow\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$

* df the pefect differential is the first order (linear) approximation of Δf with respect to x_i.

We can say each of the independent variables are actually function $x_1(t)$, $x_2(t)$... Hence the new chain rule is:

$$\boxed{\frac{df}{dt}=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{dx_i}{dt}}$$

For example,

Given some function $y=[f_1(x)]^{f_2(x)}$, $\ln y=f_2\ln f_1$. Differentiating we get, $y'=f_1^{f_2}\left(f_2'\ln f_1+\frac{f_2f_1'}{f_1}\right)$.

Or using the chain rule, $\frac{dy}{dt}=\frac{\partial y}{\partial f_1}\frac{df_1}{dx}+\frac{\partial y}{\partial f_2}\frac{df_2}{dx}=\left(f_2f_1^{f_2-1}\right)(f_1')+\left(\ln f_1\cdot f_1^{f_2}\right)f_2'=f_1^{f_2}\left(\frac{f_2f_1'}{f_1}+f_2'\ln f_1\right)$

Suppose we have a function $z(x,y)$ and $x(r,s)$, $y(r,s)$. Then

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$
$dz=\frac{\partial z}{\partial r}dr+\frac{\partial z}{\partial s}ds$
$dx=\frac{\partial x}{\partial r}dr+\frac{\partial x}{\partial s}ds$
$dy=\frac{\partial y}{\partial r}dr+\frac{\partial y}{\partial s}ds$

We can substitute dx,dy from (3) and (4) into (1). We'll get $dz=\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)dr+\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right)ds$.

Using (2), we got an even more general chain rule, $\boxed{\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}}$ (and similarly for $\frac{\partial z}{\partial s}$).

12.3 Implicit Functions

An implicit function is a function defined by an implicit equation:

$$\boxed{f(x_1,...,x_n)=\text{const.}}$$

To differentiate an implicit function we use the chain rule. For example for $f(x,y)=c$ we have $\frac{df}{dx}=\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$. Therefore $\boxed{\frac{dy}{dx}=-\frac{\partial f}{\partial x}/\frac{\partial f}{\partial y}}$. This is true for any number of variables as long as there is only one dependent variable (only with a partial derivative instead).

For example, for $f(x,y)=x^3+y^3-3axy=0$, we have $\frac{dy}{dx}=\frac{-3x^2+3ay}{3y^2-3ax}$.

12.4 Extrema

A necessary condition for an extremum point is that $\frac{\partial f}{\partial x_j}=0$ for all j (still it isn't necessarily an extremum point, could be a flat area). Equivalently, $df=0$.

Additionaly: $\Delta\equiv f_{xx}f_{yy}-(f_{xy})^2>0$ and also $f_{xx}<0$ (maximum) or $f_{xx}>0$ (minimum).
If $\Delta<0$ the point is a saddle point.

12.5 Lagrange Multipliers

Startegy of finding local maxima and minima of a function $f(x_1,...,x_n)$ under m constraints $\phi_j(x_1,...,x_n)=0$.

Let $G\equiv f+\sum_{i=1}^{m}\lambda_i\phi_i$ be the Lagrangian function.
We'll now find the stationary points of G: $\frac{\partial G}{\partial x_j}=0$ for all j, and additionaly $\frac{\partial G}{\partial\lambda_j}=\phi_j=0$ for all j.
We get $n+m$ variables - $x_1,...,x_n$ and $\lambda_1,...,\lambda_m$.

13 Transformations and Jacobians

We can have multiple dependent variables given a transformation (a bijection).

For example $x(r,\theta)=r\cos\theta$ and $y(r,\theta)=r\sin\theta$ define a transformation (specifically, from polar to cartesian coordinates).

For any transformation we can define a Jacobian.

$$\boxed{\left|J\left(\frac{x,y}{r,\theta}\right)\right|\equiv\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\equiv\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\frac{\partial x}{\partial r}\frac{\partial y}{\partial\theta}-\frac{\partial x}{\partial\theta}\frac{\partial y}{\partial r}}$$

For example, we get $\left|J\left(\frac{x,y}{r,\theta}\right)\right|=r$. and similarly we get $\left|J\left(\frac{r,\theta}{x,y}\right)\right|=\frac{1}{r}$.

Under a transformation $(x,y)\to(r,\theta)$, the area $dxdy$ becomes $rdrd\theta$, because:

$$\boxed{dA=dxdy=\left|J\left(\frac{x,y}{r,\theta}\right)\right|drd\theta}$$

We can write a chain rule for Jacobians. Given $\begin{cases}u(r,s) & x(u,v)\\ v(r,s) & y(u,v)\end{cases}$, (we can easily show this by simply carrying out the matrix product):

$$\boxed{J\left(\frac{x,y}{r,s}\right)=J\left(\frac{x,y}{u,v}\right)J\left(\frac{u,v}{r,s}\right)}$$

We immediately get from this that the Jacobian of the inverse transformation is the inverse of the Jacboian.

$$\boxed{\left|J\left(\frac{x,y}{r,\theta}\right)\right|=\frac{1}{\left|J\left(\frac{r,\theta}{x,y}\right)\right|}}$$

The same could be done similarly with a volume element $dV = dxdydz$ and higher dimensions.

Spherical coordinates are given by:

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$

and their Jacobian is

$$\left| J\left(\frac{x, y, z}{r, \theta, \varphi}\right) \right| = r^2 \sin\theta$$

steradians 4π
$\frac{dA}{r^2} = d\Omega = \sin\theta d\theta d\varphi = d\varphi d\left(\cos\theta\right)$ and $dV = r^2 \sin\theta dr d\varphi d\theta$

13.1 Differentiation under the Integral Sign

Leibniz rule:

$$\frac{d}{d\alpha} \int_{x_1}^{x_2} f\left(x, \alpha\right) dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha} dx$$

Only allowed when $\frac{\partial f}{\partial a}$ and f are continuous.

Part IV
Multivariable Integral Calculus

14 Multivariable Integrals

14.1 Definition

We can divide the volume under $F(x, y)$ similarly to how we divided the area under $f(x)$.
The integral is then

$$\boxed{I = \lim_{n\to\infty} \sum_{k=1}^{n} F(\xi_k, \eta_k)\Delta A_k}$$

where (ξ_k, η_k) is a point in the area and ΔA_k is the base area of the volume.

14.2 someth

We can divide the areas evenly so that $\Delta A_k = \Delta x_i \Delta y_j$. We'll write the sum from $x = a$ to $x = b$. And in that section, we'll write the sum on y from $y = f_1(x)$ to $y = f_2(x)$ for each x. Then

$$I = \lim_{n\to\infty} \sum_{i=1,n|j=1,m} F(\xi_k, \eta_k)\Delta x_i \Delta y_j$$

We can split the sum into two so that each time we sum all y's in a row, and then add the entire row.

$$= \lim_{n\to\infty} \sum_i \sum_j F(\xi_k, \eta_k)\Delta x_i \Delta y_j = \lim_{n\to\infty} \sum_i \Delta x_i \sum_j F(\xi_k, \eta_k)\Delta y_j$$

Now this is a regular one variable integral. So we now have

$$= \int_a^b dx \int_{f_1(x)}^{f_2(x)} F(x, y)dy = \int_a^b \int_{f_1(x)}^{f_2(x)} F(x, y)dxdy$$

We see that we can change the order of summing. This is Fubini's Theorem.