Classical Physics 1

Ilay Wischnevsky Shlush

May 10, 2024

Contents

1. **רקע מתמטי:** וקטורים וסקלרים
2. **קינמטיקה:** תנועה במימד אחד, מהירות ותאוצה, תנועה ב-3 מימדים, תנועה מעגלית, קורדינטות קוטביות, מהירות ותאוצה
3. **כוחות וחוקי ניוטון:** מערכות אינרציאליות, טרנספורמציית גליליי, תנועה יחסית, כוחות חיכוך בין מוצקים, תנועה בגז ובנוזל, עוד תנועה מעגלית, מערכות לא אינרציאליות, כוחות מדומים, אפילו עוד תנועה מעגלית
4. **תנע קווי:** שימור תנע קווי, מרכז המסה, מערכות עם מסה משתנה
5. **עבודה ואנרגיה:** אנרגיה קינטית ופוטנציאלית, שימור אנרגיה, כוח מרכזי, הספק, כוחות לא משמרים, גראדיינט ורוטור
6. ~~**מערכות חלקיקים:** תנועה במערכת מרכז המסה, התנגשויות, מערכת דו-גופית~~
7. **תנע זוויתי:** מומנטים, גוף קשיח, ~~פרסציה, גלגול עם וללא החלקה~~
8. **תנועה הרמונית פשוטה:** מטוטלת, תנודה מרוסנת ומאולצת, תהודה
9. ~~**כבידה:** תנועה בשדה כובד, הבעיה הדו-גופית~~

Dr Yair Arcavi
arcavi@tauex.tau.ac.il
TA: Nadav Shrayer
nadavshrayer@gmail.com

0.1 Units

We will use only combinations of : length (L), time (T) and mass (M).
MKS / SI:
$[L] = m$
$[M] = kg$
$[T] = s$

Analysing units (example: pendulum period):
Relevant parameters: m, l, θ_0, g

$$T = m^a l^b \theta_0^c g^d$$

$$s^1 = kg^a m^b \left(\frac{m}{s^2}\right)^d = kg^a m^{b+d} s^{-2d}$$

By comparing the exponents in both sides of the equation we can say,

$$a = 0$$

$$b + d = 0$$

$$-2d = 1$$

$$\Downarrow$$

$$d = -1/2, b = 1/2, a = 0$$

$$\boxed{T = C\sqrt{\frac{l}{g}}}$$

1 Vectors

A scalar has *only* magnitude (mass, time, etc). Vectors have both magnitude and direction (more rigorously, a vector is defined by its contravariance under coordinate transformations. Mathematical objects can be defined by their behavior under transformation).

Notation: $\vec{v}$ - vector. v or $|\vec{v}|$ - vector magnitude (scalar). End of class 1.

Two vectors are equal when their direction and magnitude are equal ($v^i \vec{e_i} = u^i \vec{e'}_i$)

A unit vector is notated by a hat: $\hat{v}$.

1.1 Vector Operations

1.1.1 Vector Scalar Product

Magnitude of new vector: $\left|b\vec{A}\right| = |b| \left|\vec{A}\right|$
Direction is the same as A or opposite depending on $b > 0$.

1.1.2 Vector Addition/Subtraction

Addition: head to tail.
Subtraction is equivalent to $\vec{A} + \left(-1 \cdot \vec{B}\right)$.

1.1.3 Properties

The aforementioned operations satisfy the properties:
commutative, associative, distributive.

1.1.4 Dot Product

The dot product is an operation between two vectors $\vec{a}$, $\vec{b}$ that results in a scalar - the projection of $\vec{a}$ onto $\vec{b}$, times $\left|\vec{b}\right|$ (commutative).

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|cos\left(\alpha\right)$$

1.1.5 Cross Product

The cross product is an operation between two vectors that results in a perpendicular vector with the magnitude: area of the parallelogram created by the vectors (anti-commutative, non-associative, distributive with $+$).

$$|\vec{r_1} \times \vec{r_2}| = |\vec{r_1}||\vec{r_2}|sin\left(\alpha\right)$$

1.2 Coordinate Systems

A vector can be represented by a list of scalar components v_i. The scalar components are the lengths of the projections of $\vec{v}$ onto the basis vectors of the coordinate system.

The norm/magnitude of the vector is then

$$|\vec{v}| = \sqrt{\sum_i \left(v_i\right)^2}$$

Always work with a right handed system $\hat{x} \times \hat{y} = \hat{z}$.

End of class 2.[1]

[1] A little more in depth in Linear Algebra course notes (includes cartesian representations of operations). The dot/cross product cartesian formulas can be shown easily by simply writing out the product in cartesian coordinates and using the distributive property.

2 Kinematics

2.1 1 Dimension

Suppose a point object. We shall choose some time as $t = 0$ and write the object's position at any time as a continuous differentiable function $x(t)$ $[m]$.

Displacement: $\Delta x(t_1, t_2) = x(t_2) - x(t_1)$ $[m$

Average Velocity: $\bar{v} = \frac{\Delta x(t_1,t_2)}{t_2 - t_1} = \frac{x_2 - x_1}{t_2 - t_1}$ $[m/s]$

Velocity: the average velocity as t_2, t_1 gets closer to some t. $v(t) \equiv \frac{dx}{dt} \equiv \dot{x}$ $[m/s]$

Acceleration: $a(t) = \dot{v}(t) = \ddot{x}(t)$

According to the fundamental theorem of calculus:

$$v = \frac{dx}{dt} \leftrightarrow x = \int_0^t v(t)dt' + x_0$$

$$a = \frac{dv}{dt} \leftrightarrow v = \int_0^t a(t')dt' + v_0$$

2.2 n Dimensions

Generalizing to n dimensions, we now have a position vector $\vec{r}(t)$. In a cartesian coordinate system,

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{[x(t + \Delta t)\hat{x} + y(t + \Delta t)\hat{y} + z(t + \Delta t)\hat{z}] - [x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}]}{\Delta t}$$

$$= \hat{x} \lim_{\Delta t \to 0} \left[\frac{x(t + \Delta t) - x(t)}{\Delta t}\right] + \hat{y} \lim_{\Delta t \to 0} \left[\frac{y(t + \Delta t) - y(t)}{\Delta t}\right] + \hat{z} \lim_{\Delta t \to 0} \left[\frac{z(t + \Delta t) - z(t)}{\Delta t}\right]$$

$$= \boxed{\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}}$$

Only true when the basis vectors are not dependent on time.

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$$

The integration also occurs independently for every component (again, as long as the basis vectors aren't time dependent).

2.3 Constant Acceleration

2.3.1 Proof

$$a(t) = a_0$$

Choosing a coordinate system with 1) origin at the object 2) acceleration in $\hat{y}$ direction 3) $\hat{z}$ axis perpendicular to $\vec{v_0}$:

$$\vec{r_0} = (0, 0, 0)$$

$$\vec{a} = (0, a_0, 0)$$

$$\vec{v_0} = (v_{0,x}, v_{0,y}, 0)$$

Now solving

$$v_z(t) = v_z(0) + \int_0^t a_z(t')dt' = 0$$

$$r_z(t) = r_z(0) + \int_0^t v_z(t')dt' = 0$$

$$v_y(t) = v_y(0) + \int_0^t a_y(t')dt' = v_{0,y} + a_0 t$$

$$r_y(t) = r_y(0) + \int_0^t v_y(t')dt' = v_{0,y}t + \frac{a_0}{2}t^2$$

$$v_x(t) = v_x(0) + \int_0^t a_x(t')dt' = v_{0,x}$$

$$r_x(t) = r_x(0) + \int_0^t v_x(t')dt' = v_{0,x}t$$

$$\boxed{\vec{v}(t) = v_{0,x}\,\hat{x} + (v_{0,y} + a_0 t)\,\hat{y}}$$

$$\boxed{\vec{r}(t) = v_{0,x}t\,\hat{x} + \left(v_{0,y}t + \frac{a_0}{2}t^2\right)\hat{y}}$$

$$t = \frac{x}{v_{0,x}}$$

$$\boxed{y(x) = \frac{v_{0,y}}{v_{0,x}}x + \frac{a_0}{2v_{0,x}^2}x^2}$$

End of class 3.

2.3.2 Angled throw

An object is thrown from height h with initial velocity $\vec{v}_0$ under the influence of gravitational acceleration g only. Where and at what velocity will it reach the ground?

We'll choose the same coordinate system (origin at the object, y axis in direction of acceleration, no z component to v_0). We are given:

$$\vec{a} = g\hat{y}$$

$$\vec{v}_0 = v_{0,x}\hat{x} + v_{0,y}\hat{y}$$

Therefore,

$$\vec{v}(t) = v_{0,x}\hat{x} + (v_{0,y} + gt)\,\hat{y}$$

$$\vec{r}(t) = v_{0,x}t\,\hat{x} + \left(v_{0,y}t + \frac{g}{2}t^2\right)\hat{y}$$

We want the velocity and position when the object reaches the floor: $r_y = h$.

$$-h + v_{0,y}t^* + \frac{1}{2}gt^{*2} = 0$$

$$\boxed{t^* = \frac{-v_{0,y} \pm \sqrt{v_{0,y}^2 + 2gh}}{g}}$$

$$\boxed{r^* = \vec{r}(t^*)}$$

$$\boxed{v^* = |\vec{v}(t^*)|}$$

Additionaly, if $v_{0,y} = 0$, $t^* = \sqrt{\frac{2h}{g}}$.

2.3.3 L'monk and the arrow

A zoo employee needs to shoot a sleeping arrow at a monkey. The monkey is on a tree, but the moment she shoots, the monkey will fall from the tree surprised by the noise. At what angle/velocity does the employee need to shoot the arrow for it to hit?

The tree is of height h and distance L from the employee.

$$\vec{r}_{arr}(t) = v_{0,x}t\,\hat{x} + \left(v_{0,y}t - \frac{g}{2}t^2\right)\hat{y}$$

$$\vec{r}_{monk}(t) = L\hat{x} + \left(h - \frac{g}{2}t^2\right)\hat{y}$$

At the moment of the arrow hitting t^* we have:

$$\vec{r}_{arr}(t) = \vec{r}_{monk}(t)$$

$$\begin{cases} v_{0,x}t^* & = L \\ v_{0,y}t^* - \frac{g}{2}t^{*2} & = h - \frac{g}{2}t^{*2} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} v_{0,x}t^* & = L \\ v_{0,y}t^* & = h \end{cases}$$

$$\Downarrow$$

$$tan\theta = \frac{v_{0,y}}{v_{0,x}} = \frac{h}{L}$$

Aim at the monk, and since the arrow and monk fall at the same speed it will hit.

There is still a possibility that the arrow will hit the ground, if the initial velocity of the arrow isn't large enough. We can solve this by setting the hit time to be smaller than the time it takes the monkey to fall to the ground.

$$t^* = \frac{L}{v_0, x} = \frac{h}{v_{0,y}} < t_{gr}$$

$$t_{gr} = \sqrt{\frac{2h}{g}}$$

Therefore,

$$\boxed{v_{0,x} \geq \frac{L}{\sqrt{\frac{2h}{g}}}}$$

$$\boxed{v_{0,y} \geq \frac{h}{\sqrt{\frac{2h}{g}}}}$$

2.4 Cartesian Circular Motion

2.4.1 Constant Radius

An object moves on a circle with radius R anti-clockwise with a constant speed v. Find EOMs.

Solution:
Using simple trigonometry with the origin of the coordinate system at the center of the circle,

$$\boxed{\vec{r}(t) = Rcos\left(\theta(t)\right)\hat{x} + Rsin\left(\theta(t)\right)\hat{y}}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -Rsin(\theta)\dot{\theta}\hat{x} + Rcos\left(\theta\right)\dot{\theta}\hat{y}$$

Defining angular velocity,

$$\omega \equiv \dot{\theta}, \quad [\omega] = \frac{rad}{s} = \frac{1}{s}$$

$$\boxed{\vec{v}(t) = -\omega Rsin(\theta)\hat{x} + \omega Rcos\left(\theta\right)\hat{y}}$$

We are given $|\vec{v}(t)| = v$.

$$v = |\vec{v}(t)| = \sqrt{\omega^2 R^2 sin^2\theta + \omega^2 R^2 cos^2\theta} = \omega R\sqrt{1} = \omega R$$

$$\boxed{v = \omega R}$$

$$\boxed{\theta(t) = \theta_0 + \omega t}$$

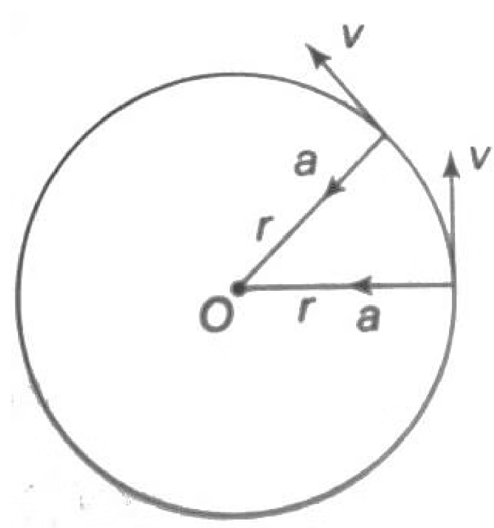
Notice, r and v are always perpendicular. r is radial, v is tangential.

$$\vec{r} \cdot \vec{v} = 0$$

Finally, the acceleration is always radial and towards the center of the circle.

$$\vec{a}(t) = -\omega^2 Rcos(\theta)\hat{x} - \omega^2 Rsin(\theta)\hat{y}$$

$$\boxed{\vec{a}(t) = -\omega^2\vec{r}(t)}$$



2.4.2 Changing Radius

The radius is now $R = ut$.We'll set $\theta_0 = 0$, that way $\theta = \omega t$.

$$\vec{r} = ut\left[cos(\omega t)\hat{x} + sin(\omega t)\hat{y}\right]$$

$$\vec{v} = u\left[cos(\omega t)\hat{x} + sin(\omega t)\hat{y}\right] + ut\left[-\omega sin(\omega t)\hat{x} + \omega cos(\omega t)\hat{y}\right]$$

$$\vec{a} = \left[-2u\omega sin(\omega t) - ut\omega^2 cos(\omega t)\right]\hat{x}$$

$$+\left[2u\omega cos(\omega t) - ut\omega^2 sin(\omega t)\right]\hat{y}$$

2.5 Polar Coordinates

2.5.1 Introduction

Vectors can be written in cartesian coordinates

$$\vec{r} = r_x\hat{x} + r_y\hat{y}$$

Certain systems (like circular motion) can be described more easily using polar coordinates.

$$\begin{cases} r &= \sqrt{r_x^2 + r_y^2} \\ \theta &= arctan\left(\frac{r_y}{r_x}\right) \end{cases} \iff \begin{cases} x &= rcos(\theta) \\ y &= rsin(\theta) \end{cases}$$

We can write the basis vectors using $\hat{r} = \frac{\partial x}{\partial r}\hat{x} + \frac{\partial y}{\partial r}\hat{y}$ and $\hat{\theta} = \frac{\partial x}{\partial \theta}\hat{x} + \frac{\partial y}{\partial \theta}\hat{y}$ and normalizing $\hat{\theta}$

$$\begin{cases} \hat{r} &= cos\theta\hat{x} + sin\theta\hat{y} \\ \hat{\theta} &= -sin\theta\hat{x} + cos\theta\hat{y} \end{cases}$$

We can write $\vec{r}$ as

$$\boxed{\vec{r} = r\hat{r}}$$

These new basis vectors are dependent on the position in space, and therefore while in motion they change in time.

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(cos\theta\hat{x} + sin\theta\hat{y}\right) = \dot{\theta}\left(-sin\theta\hat{x} + cos\theta\hat{y}\right) = \dot{\theta}\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = \frac{d}{dt}\left(-sin\theta\hat{x} + cos\theta\hat{y}\right) = -\dot{\theta}\left(cos\hat{x} + sin\theta\hat{y}\right) = -\dot{\theta}\hat{r}$$

Using this we can derive the velocity and acceleration vectors.

$$\frac{d\vec{r}}{dt} = \boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}$$

$$\frac{d\vec{v}}{dt} = \boxed{\vec{a} = \left(\ddot{r} - \omega^2 r\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}}$$

2.5.2 Constant Radius

Now with polar coordinates it is much simpler to solve. Setting $\dot{r} = 0$, $\dot{\theta} = \omega$:

$$\vec{v} = \omega r\hat{\theta}$$

$$\vec{a} = -\omega^2 r\hat{r}$$

2.5.3 Changing Rdaius

Setting $\dot{r} = u$, $\dot{\theta} = \omega$:

$$\vec{v} = u\hat{r} + \omega r\hat{\theta}$$

$$\vec{a} = -\omega r^2\hat{r} + 2u\omega\hat{\theta}$$

2.5.4 Cylindrical Coordinates

We can use the 2D polar coordinates on a plane and extend it to 3D by just adding a third coordinate z:

$$(r, \theta, z)$$

3 Dynamics

3.1 Newton's Laws of Motion

3.1.1 First Law

A body remains at rest, or in motion at a constant speed in a straight line, unless acted upon by a force.

$$\Sigma F = 0 \iff \ddot{x} = 0$$

This is true only for inertial systems.

3.1.2 Second Law

The net force on a body is equal to the body's acceleration multiplied by its mass (*only* if $\dot{m} = 0$) or, equivalently, the rate at which the body's momentum changes with time.

$$\vec{F} = \frac{d\vec{p}}{dt} \overset{\text{const. mass}}{=} m\vec{a}$$

3.1.3 Third Law

If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

$$\vec{F}_{1,2} = -\vec{F}_{2,1}$$

3.2 Different Forces

3.2.1 Gravity

Between every two masses acts a force of attraction. Generally the force is

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r}$$

Near the surface of the earth it can be described as

$$\vec{F} = m\vec{g}$$

with $\vec{g}$ pointing toward the center of the earth ("down"). The approximation is given by $\vec{g} = \frac{Gm_e}{r_e^2}\hat{r}$.

3.2.2 Normal Force (N)

Electromagnetic force acting between touching surfaces. Always perpendicular to the surface.

3.2.3 Tension (T)

Pulling force along a string. If a string is massless $\Sigma F = ma = 0$ and therefore must be in equillibrium. Thus the tension along the string is constant.

At the atomic level, when atoms or molecules are pulled apart from each other and gain potential energy with a restoring force still existing, the restoring force might create what is also called tension.

3.2.4 Friction Between Solids (f)

a) If the solids have a non zero relative velocity $\vec{V}$, the friction is kinetic (f_k).

$$f_k = -\mu_k M$$

Where μ_k is the kinetic friction coefficient.
b) When, $\vec{V} = 0$, the friction is static (f_s). Because $\vec{a} = 0$,

$$f_s = -\Sigma F_{\text{non\_friction}}$$

At some point when f_s is big enough, the friction becomes kinetic.

$$f_{s,max} = \mu_s N$$

$$\mu_s \geq \mu_k$$

3.2.5 Spring

The force a spring applies on an object attached to it, proportional to the length of contraction/extension.

$$F = -k\Delta x$$

Where k is the spring constant $[k] = \frac{N}{m}$. k is the answer to "how much force is needed extend the spring".

3.2.6 Buoyancy (B)

$$F_B = \rho V g$$

Where ρ is the density of the fluid and V is the volume of the object that is sunken.

3.2.7 Solid-Fluid Friction

For a spherical object at low speeds (stokes' law)

$$F = 6\pi R\eta\vec{v}$$

R is the radius
η is the viscosity
$\vec{v}$ relative speed to fluid

3.3 Linear Momentum

3.3.1 Definition

We'll define momentum as $\vec{p} \equiv m\vec{v}$. Newton's second law is $F = \frac{d\vec{p}}{dt}$.

3.3.2 Conservation

One of the pros of working with momentum is solving systems with a large number of particles.

Suppose N particles and on each of them are acting forces.

$$\vec{F}_i = \vec{F}_{i,ext} + \sum_j \vec{F}_{i,j}$$

The net force acting on the system of particles is $\sum_i \vec{F}_i = \sum_i \vec{F}_{i,ext} + \sum_{ij} \vec{F}_{i,j} = \sum_i \vec{F}_{i,ext}$ (because of the 3rd law). Using the 2nd law we get $\sum_i \frac{d\vec{p}_i}{dt} = \sum_i \vec{F}_{i,ext} \equiv \vec{F}_{ext}$. We'll define $\vec{p} \equiv \sum_i \vec{p}_i$.

$$\boxed{\frac{d\vec{p}}{dt} = \vec{F}_{ext}}$$

If $F_{ext} = 0$ momentum is conserved.

Units: $[\vec{p}] = \left[kg \cdot \frac{m}{s}\right]$

3.3.3 Impulse

The impulse is defined as

$$\boxed{\vec{J} \equiv \int_{t_1}^{t_2} \vec{F}(t)dt = \Delta\vec{p}}$$

Units: $\left[\vec{J}\right] = [N \cdot s]$ or $\left[kg \cdot \frac{m}{s}\right]$.

Because of Newton's 3rd law,

$$\boxed{\vec{J}_{1,2} = -\vec{J}_{2,1}}$$

3.3.4 Plastic Collision

Plastic: after collision the bodies have equal velocity: $v_1 = v_2$

3.3.5 Rocket

A system (rocket of mass M) that loses mass in a continuous manner. The mass is "thrown" at a velocity $\vec{u}(t)$ in the rocket's system.

$\vec{p}(t) = (M + \Delta m)\vec{V}$

$\vec{p}(t + \Delta t) = M\left(\vec{V} + \Delta\vec{V}\right) + \Delta m\left(\vec{u} + \vec{V}\right)$

$$\vec{F}_{ext} = \frac{d\vec{p}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{p}(t+\Delta t) - \vec{p}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{M\Delta\vec{V} + \Delta m\vec{u}}{\Delta t} \to M\frac{d\vec{V}}{dt} + \frac{dm}{dt}\vec{u}$$

$\frac{dm}{dt} = -\frac{dM}{dt}$ because it's just the lost mass of the rocket.

The rocket equation is then:

$$\boxed{\vec{F}_{ext} = M(t)\frac{d\vec{V}}{dt} - \frac{dM}{dt}\vec{u}}$$

I. Without external forces, and with $\frac{d\vec{u}}{dt} = 0$:

$$\frac{d\vec{V}}{dt} = \frac{1}{M(t)}\frac{dM}{dt}\vec{u}$$

$$\vec{V}(t_f) - \vec{V}(t_i) = \vec{u}\ln\frac{M_f}{M_i}$$

So if the rocket has $\vec{V}(0) = 0$ and the mass is thrown backwards:

$$\vec{V}(t) = -u\ln\left(\frac{M(t)}{M_0}\right)\hat{x}$$

II. With gravity, rocket going up, and with $\frac{d\vec{u}}{dt} = 0$:

$$\vec{F}_{ext} = -M(t)g = M(t)\frac{d\vec{V}}{dt} + \frac{dM}{dt}u = \frac{d\vec{p}}{dt}$$

$$\frac{d\vec{V}}{dt} = -u\frac{1}{M(t)}\frac{dM}{dt} - g$$

$$V(t) = V - u\ln\left(\frac{M(t)}{M_0}\right) - gt$$

3.4 Center of Mass

For a collection of n masses m_i with positions $\vec{r}_i$ with respect to some coordinate system (notating $M \equiv \sum_i^n m_i$), we'll define the center of mass (CM) as the following:

$$\boxed{\vec{R}_{CM} \equiv \frac{1}{M}\sum_i^n m_i \vec{r}_i}$$

We can differentiate it to get the center of mass velocity:

$$\boxed{\vec{V}_{CM} = \frac{1}{M}\sum_i^n m_i \vec{v}_i = \frac{\vec{p}_{tot}}{M}}$$

and again, to get the acceleration:

$$\boxed{\vec{A}_{CM} = \frac{1}{M}\sum_i^n m_i \vec{a}_i = \frac{\vec{F}_{ext}}{M}}$$

(*if $\frac{dm_i}{dt} = 0$).

At the center of mass coordinate system $\vec{p}_{tot}' = 0$.

4 Galilean Relativity and Intertial Forces

4.1 Introduction

Galilean tranformation transforms between coordinate frames.

B in a A's frame:

$$\vec{r}_{BA} = \vec{r}_B - \vec{r}_A$$

$$\vec{v}_{BA} = \vec{v}_B - \dot{\vec{r}}_A$$

$$\vec{a}_{BA} = \vec{a}_B - \ddot{\vec{r}}_A$$

B is inertial $\iff$ $\vec{a}_{BA} = 0$.
If B isn't inertial we can't use newton's laws. To make them still work, we add an imaginary force:

$$m\vec{a}_{CB} = F - m\vec{a}_{BA}$$

4.2 General Imaginary Force

Let's find a more general imaginary force.

Differentiating:

$$\vec{r}' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$$

$$\dot{\vec{r}}' = (\dot{x}'\hat{x}' + \dot{y}'\hat{y}' + \dot{z}'\hat{z}') + x'\dot{\hat{x}}' + y'\dot{\hat{y}}' + z'\dot{\hat{z}}'$$

$$\ddot{\vec{r}}' = (\ddot{x}'\hat{x}' + \ddot{y}'\hat{y}' + \ddot{z}'\hat{z}) + 2\left(\dot{x}'\dot{\hat{x}}' + \dot{y}'\dot{\hat{y}}' + \dot{z}'\dot{\hat{z}}'\right) + \left(x'\ddot{\hat{x}}' + y'\ddot{\hat{y}}' + z'\ddot{\hat{z}}'\right)$$

using the galilean transformation $\vec{r}' = \vec{r} - \vec{R}$.

$$\vec{a} = \vec{A} + \ddot{\vec{r}}'$$

not substituting

$$\vec{a} = \vec{A} + \vec{a}' + 2\left(\dot{x}'\dot{\hat{x}}' + \dot{y}'\dot{\hat{y}}' + \dot{z}'\dot{\hat{z}}'\right) + \left(x'\ddot{\hat{x}}' + y'\ddot{\hat{y}}' + z'\ddot{\hat{z}}'\right)$$

therefore

$$\vec{a}' = \vec{a} - \vec{A} - 2\left(\dot{x}'\dot{\hat{x}}' + \dot{y}'\dot{\hat{y}}' + \dot{z}'\dot{\hat{z}}'\right) - \left(x'\ddot{\hat{x}}' + y'\ddot{\hat{y}}' + z'\ddot{\hat{z}}'\right)$$

and

$$m\vec{a}' = m\vec{a} - m\vec{A} - 2m\left(\dot{x}'\dot{\hat{x}}' + \dot{y}'\dot{\hat{y}}' + \dot{z}'\dot{\hat{z}}'\right) - m\left(x'\ddot{\hat{x}}' + y'\ddot{\hat{y}}' + z'\ddot{\hat{z}}'\right)$$

Continuing using rotating vectors section to simplify:

$$\dot{x}'\dot{\hat{x}}' + \dot{y}'\dot{\hat{y}}' + \dot{z}'\dot{\hat{z}}' = \vec{\omega} \times (\dot{x}'\hat{x}' + \dot{y}'\hat{y}' + \dot{z}'\hat{z}') = \vec{\omega} \times \vec{v}'$$

$$x'\ddot{\hat{x}}' + y'\ddot{\hat{y}}' + z'\ddot{\hat{z}}' = \vec{\omega} \times (\vec{\omega} \times \vec{r}')$$

Finally, we get (assuming $A = 0$)

$$\boxed{m\vec{a}' = m\vec{a} + [-2m(\vec{\omega} \times \vec{v}')] + [-m\vec{\omega} \times (\vec{\omega} \times \vec{r}')]}$$

Where:

$$\boxed{\vec{F}_{\text{coriolis}} = -2m\left(\vec{\omega} \times \vec{v}'\right)}$$

$$\boxed{\vec{F}_{\text{centrifugal}} = -m\vec{\omega} \times \left(\vec{\omega} \times \vec{r}'\right)}$$

The centrifugal force is what makes us feel like we're pulled outside.
Coriolis force is only for moving objects in rotating systems.

4.3 Rotating Vectors

Suppose $\vec{B}$ rotating around $\hat{z}$ with $\omega = \frac{d\theta}{dt}$. In cylindrical coordinates, $\vec{B} = B_r\hat{r} + B_z\hat{z}$. Using $\dot{\hat{r}} = \omega\hat{\theta}$, $B_r = B\sin\alpha$.

$$\frac{d\vec{B}}{dt} = B_r\dot{\hat{r}} = B\sin\alpha \cdot \omega\hat{\theta}$$

We'll define the vector $\vec{\omega}$ with magnitude ω and direction perpendicular to the rotation (using right hand rule).

Then we got:

$$\boxed{\dot{\vec{B}} = \vec{\omega} \times \vec{B}}$$

Meaning, differentiating rotating vector is equivalent to to crossing it with it $\vec{\omega}$.

For example for the unit vectors,

$$\ddot{\hat{x}} = \vec{\omega} \times \left(\vec{\omega} \times \hat{x}\right)$$
$$\ddot{\hat{y}} = \vec{\omega} \times \left(\vec{\omega} \times \hat{y}\right)$$
$$\ddot{\hat{z}} = \vec{\omega} \times \left(\vec{\omega} \times \hat{z}\right)$$

5 Energy

5.1 Kinetic Energy and Work

Using $\vec{F} = ma$ requires knowing $\vec{F}(t)$, but sometimes we only know $\vec{F}(\vec{r})$. We need to solve

$$m\frac{d\vec{v}(t)}{dt} = \vec{F}(\vec{r})$$

We'll integrate the equation along the path from $x_1$to x_2.

$$m\int_{x_1}^{x_2} \frac{dv}{dt}dx = \int_{x_1}^{x_2} F dx$$

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \int_{x_1}^{x_2} F dx$$

We'll notate kinetic energy $E_k = \frac{1}{2}mv^2$ and work $W = \int_{x_1}^{x_2} F dx$. We get the following 1D equation:

$$\boxed{\Delta E_k = W}$$

5.2 Generalizing to more Dimensions

Now generalizing to more than one dimension. For an infinitesimal movement $\Delta\vec{r} \to 0$:

$$\vec{F}\cdot\Delta\vec{r} = m\frac{d\vec{v}}{dt}\cdot\Delta\vec{r} = m\frac{d\vec{v}}{dt}\cdot\vec{v}\Delta t$$

$$\frac{d\vec{v}}{dt}\cdot\vec{v} = \frac{1}{2}\left(\frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt}\right) = \frac{1}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{1}{2}\frac{d\left(v^2\right)}{dt}$$

Then we get

$$\vec{F}\cdot\Delta\vec{r} = \frac{1}{2}m\frac{d\left(v^2\right)}{dt}\Delta t$$

We now need to sum over the entire path, and take the limit as the steps become smaller ($\Delta\vec{r}_i \to 0$).

$$\sum_i \vec{F}(\vec{r}_i)\cdot\Delta\vec{r}_i = \frac{1}{2}m\sum_i \frac{d\left(v_i^2\right)}{dt}\Delta t$$

$$\boxed{W = \int_A^B \vec{F}(\vec{r})\cdot d\vec{r}_i = \Delta E_k}$$

Then the work is better defined as $W = \int_A^B \vec{F}(\vec{r})\cdot d\vec{r}_i$.

5.3 Power

Power has units of $\left[\frac{J}{s}\right] = [Watt]$ and is defined as

$$P = \frac{dW}{dt} = \lim_{\Delta t\to 0}\frac{\vec{F}\cdot\Delta\vec{r}}{\Delta t} = \vec{F}\cdot\frac{d\vec{r}}{dt} = \vec{F}\cdot\vec{v}$$

5.4 Potential Energy

5.4.1 Definition

The work of some forces isn't dependent on the path taken. They are called conservative forces. In those cases we can write:

$$\boxed{U(x,y,z) = -\int_{(x_0,y_0,z_0)}^{(x,y,z)} \vec{F}\cdot d\vec{r} + U(x_0,y_0,z_0)}$$

The work of such forces in terms of the potential energy U is:

$$W_{A\to B} = \int_A^B \vec{F}\cdot d\vec{r} = -\int_{(x_0,y_0,z_0)}^{A} \vec{F}\cdot d\vec{r} + \int_{(x_0,y_0,z_0)}^{B} \vec{F}\cdot d\vec{r} = -(U_0 - U_A) + (U_0 - U_B) = U_A - U_B$$

We'll define the total energy as $E = E_k + U$. Substituting into the work-energy relation we get.

$$\Delta E = \Delta E_k + \Delta U = 0$$

This is conservation of energy.

5.4.2 Multiple Forces

When there are multiple forces, we get that the sum of works of each forces is the total work.

$$W = \Delta E_k = \int_c \Sigma\vec{F}\cdot d\vec{r} = \sum_i \int_c \vec{F}_i \cdot d\vec{r} = \sum W$$

If only conservative forces are doing work in the system, we can use $E_A = E_B$. Otherwise we have to use $\Delta E_k = \sum W$, though we can still calculate the work of the conservative forces as $W_c = -\Delta U$.

5.4.3 Spatial Derivatives

The total change in some function Δf is given by:

$$\begin{aligned}\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y) &= [f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)] + [f(x, y+\Delta y) - f(x,y)] \\ &= \Delta_x f + \Delta_y f\end{aligned}$$

With a limit to 0 we get:

$$\boxed{df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$$

Applying this to the total work, we get the following:

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$

and we know from the work-energy relation:

$$\Delta U = U(r+\Delta r) - U(r) = -\int_r^{r+\Delta r} \vec{F}\cdot d\vec{r} = -F\int_r^{r+\Delta r} d\vec{r} = -\vec{F}\cdot\Delta\vec{r} = -F_x\Delta x - F_y\Delta y - F_z\Delta z$$

Therefore we get for each direction:

$$F_x = -\frac{\partial U}{\partial x}, F_y = -\frac{\partial U}{\partial y}, F_z = -\frac{\partial U}{\partial z}$$

We'll now define the operator "nabla" (upside down triangle) as the vector of derivative operators:

$$\boxed{\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}$$

Using it we can create different vector or scalar derivatives:

1. Gradient. A scalar field→vector field derivative. We also have $df = \vec{\nabla} f \cdot d\vec{r}$.

$$\vec{\nabla} U = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \Rightarrow \boxed{F = -\vec{\nabla} U}$$

1. Divergence. A vector field→scalar field derivative.

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial \vec{V}}{\partial x} + \frac{\partial \vec{V}}{\partial y} + \frac{\partial \vec{V}}{\partial z}$$

2. Curl. A vector field→vector field derivative.

$$\vec{\nabla} \times \vec{V} = ...$$

5.4.4 Is conservative?

Let c_1 be some path $A \to B$, c_2 the same path but with opposite direction. Then $W_{A\to B} = \int_{c_1} \vec{F} d\vec{r} = -\int_{c_2} \vec{F} d\vec{r} = -W_{B\to A}$.
Let c_3 be some other path. If $\vec{F}$ is conservative: $\int_{c_3} \vec{F} d\vec{r} = \int_{c_1} \vec{F} d\vec{r} = -\int_{c_2} \vec{F} d\vec{r}$. Therefore

$$\oint_{c_3+c_2} \vec{F} d\vec{r} = \int_{c_3} \vec{F} d\vec{r} + \int_{c_2} \vec{F} d\vec{r} = 0$$

Meaning, the work of a conservative force over a closed path is zero.

$$\boxed{\oint \vec{F}_{cons} d\vec{r} = 0}$$

But how can we check if $\oint \vec{F} d\vec{r} = 0$ for any closed path? We can keep splitting a closed path into multiple close paths so that their sum is equal to the original path.

$$\oint_c \vec{F} d\vec{r} = \sum_i \oint_{c_i} \vec{F} d\vec{r}$$

We continue splitting infinitely.

Then we get that if each infinitesimal loop satisfies $\oint_{c_i} \vec{F} d\vec{r} = 0$ the total work is zero for every closed path, and F is conservative.

If $\exists i : \oint_{c_i} \vec{F} d\vec{r} \neq 0$ then $\vec{F}$ is not conservative.
We'll now look at the work of an infinitesimal loop. We can say it is a rectangle:

$$\oint_{c_i} \vec{F} d\vec{r} = \int_{(x,y)}^{(x+\Delta x,y)} F_x dx + \int_{(x+\Delta x,y)}^{(x+\Delta x,y+\Delta y)} F_y dy + \int_{(x+\Delta x,y+\Delta y)}^{(x,y+\Delta y)} F_x dy + \int_{(x,y+\Delta y)}^{(x,y)} F_x dy$$

F is constant throughout the each section of the path of the rectangle because the rectangle is infinitesimal.

$$= F_x\left(x + \frac{\Delta x}{2}, y\right)\Delta x + F_y\left(x + \Delta x, y + \frac{\Delta y}{2}\right)\Delta y - F_x\left(x + \frac{\Delta x}{2}, y + \Delta y\right)\Delta x - F_y\left(x, y + \frac{\Delta y}{2}\right)\Delta y$$

$$= -\frac{\Delta x \Delta y \left[F_x\left(x+\frac{\Delta x}{2}, y+\Delta y\right) - F_x\left(x+\frac{\Delta x}{2}, y\right)\right]}{\Delta y} + \frac{\Delta y \Delta x \left[F_y\left(x+\Delta x, y+\frac{\Delta y}{2}\right) - F_y\left(x, y+\frac{\Delta y}{2}\right)\right]}{\Delta x}$$

We'll now take the limit as $\Delta x \to 0, \Delta y \to 0$:

$$= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy$$

Hence:

$$\boxed{\vec{F} \text{ cons} \iff \oint\limits_{c_i} \vec{F} d\vec{r} = 0 \iff \frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}}$$

Additionaly we got

$$\oint\limits_{c} \vec{F} d\vec{r} = \sum_i \oint\limits_{c_i} \vec{F} d\vec{r} = \int\int \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy$$

This is an integral over the area encompassed by the curve c. This is Stoke's theorem.

5.4.5 Is conservative? 3D

$\vec{F}$ is conservative if:

$$\begin{aligned} \tfrac{\partial F_y}{\partial x} - \tfrac{\partial F_x}{\partial y} &= 0 \\ \tfrac{\partial F_z}{\partial x} - \tfrac{\partial F_x}{\partial z} &= 0 \\ \tfrac{\partial F_y}{\partial z} - \tfrac{\partial F_z}{\partial y} &= 0 \end{aligned}$$

Or more simply written,

$$\boxed{\vec{F} \text{ cons} \iff \vec{\nabla} \times \vec{F} = \vec{0}}$$

The more general stokes's theorem is

$$\int_S \vec{F} \cdot d\vec{S} = \int_V \left(\vec{\nabla} \times \vec{F}\right) dV$$

6 Angular Momentum

6.0.1 Definition

We'll define angular momentum as

$$\boxed{\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}}$$

With units $\left[\vec{L}\right] = \frac{kg \cdot m^2}{s}$

It is dependent on the choice of coordinate system, since $\vec{r}$ is dependent on it.

Only the component of $\vec{r}$ that is perpendicular to $\vec{v}$ is important for calculating $\vec{L}$. $\vec{L} = m\left(\vec{r}_\perp + \vec{r}_\parallel\right) \times \vec{v} = m\vec{r}_\perp \times \vec{v}$.

6.0.2 Torque

For an object in circular motion with constant radius and speed, relative to the center of the circle, $\vec{L} = m\vec{r} \times \vec{v} = mR\omega R\vec{z} = m\omega R^2 \hat{z}$ (constant).

We'll check when the angular momentum is a constant.

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \left(\vec{v} \times \vec{p}\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right) = \vec{r} \times \vec{F}$$

In that case we'll define torque as

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}}$$

With units $[\vec{\tau}] = Nm$.

It is possible that $\sum \vec{F} = 0$ but $\sum \vec{\tau} \neq 0$. In that case the center of mass is constant, but the object spins/rotates.

It is also possible that $\sum \vec{\tau} = 0$ but $\sum \vec{F} \neq 0$. In that case the center of mass moves but the object doesn't rotate.

In constant circular motion $\vec{\tau} = 0$ since $\vec{F}$ is in the same direction as $\vec{r}$.
Similar to linear momentum, we have.

$$\boxed{\frac{d\vec{L}_{tot}}{dt} = \vec{\tau}_{tot,ext}}$$

6.0.3 Center of Mass

We can write some useful equations by considering the angular momentum and torque of the center of mass coorinate system

$$\vec{L}_{tot} = \vec{L}_{CM} + \sum_i \vec{L}_i'$$

where of course $\vec{L}_{CM} \equiv M\vec{R}_{CM} \times \vec{V}_{CM}$.
and additionaly $\vec{\tau}_{CM} = \vec{R}_{CM} \times \vec{F}_{tot}$.

$$\vec{\tau}_{tot} = \vec{\tau}_{CM} + \vec{\tau}_{tot}'$$

but since we know only externals matter

$$\frac{d\vec{L}_{tot}}{dt} = \vec{R}_{CM} \times \vec{F}_{ext} + \vec{\tau}_{tot}'$$

(notice imaginary forces cancel out) here ^ because $\sum_i \left(\vec{r}_i' \times \vec{F}_i\right) = \sum_i \left(\vec{r}_i' \times m_i \vec{A}_{CM}\right) = \sum_i m_i \vec{r}_i' \times \vec{A}_{CM} = 0$

7 Rigid Body

Doesn't change its shape: for each two points i, j on the body: $\frac{d|r_{ij}|}{dt} = 0$.

For each point i (α between $\vec{r}_i$ and $\vec{\omega} = \omega\hat{z}$):

$$\left|\vec{L}_i\right| = m_i \omega \left|\vec{r}_i\right|^2 \sin\alpha$$

z component is:

$$L_{i,z} = |L_i| \sin\alpha = m_i \omega \left|\vec{r}_i\right|^2 \sin^2\alpha = m_i \omega R_i^2$$

so for the entire body

$$L_z = \omega \sum_i m_i R_i^2$$

we'll define the moment of inertia as

$$\boxed{I = \sum_i m_i R_i^2}$$

basically the meaning of I is how far from a certain axis of rotation a body is.
or for a goof ratsif

$$I = \int r^2 dm = \int r^2 \rho dV$$

and now we have

$$\boxed{L_z = I_z \omega}$$

also

$$\boxed{\tau_z = I\dot{\omega}}$$

8 Harmonic Oscillator

8.1 Spring

A mass m moves under the influence of the spring force $F = -kx$. We get the equation of motion $\ddot{x}+\omega^2 x = 0$ where we have defined $\omega = \sqrt{\frac{k}{m}}$. We call a system that has this behavior a simple harmonic oscillator.

8.2 General Case

Suppose some potential energy $U(\vec{r})$. $F = -\vec{\nabla} U$, or in one dimension $U(x)$ and $F = -\frac{dU}{dx}$. We have 3 types of equilibrium points, where the $F = 0$ (in 3D we have more, such as saddle points):

1. Minimum (stable)
2. Maximum (unstable)
3. Inflection ($\frac{d^2U}{dx^2} = 0$)

Using a taylor approximation around a *minimum* equillibrium point x_0.

$$U(x) \approx U(x_0) + \frac{dU}{dx}|_{x_0} (x - x_0) + \frac{1}{2}\frac{d^2U}{dx^2}|_{x_0} (x - x_0)^2$$

Since x_0 is a minimum, $\frac{dU}{dx}|_{x_0} = 0$. Definining a coordinate system with $x_0 = 0$, potential $U(0) = 0$, and defining $k \equiv \frac{d^2U}{dx^2}|_{x_0}$, we get:

$$U(x) \approx \frac{1}{2} k x^2$$

Under this potential we get the force $F = -\frac{dU}{dx} = -kx$.

So *every* system with a stable equilibrium is approximately a harmonic oscillator near that point! and therefore can be described by the equation of motion $\ddot{x} + \omega^2 x = 0$.

8.3 EOM Solution

$\ddot{x} + \omega_0^2 x = 0$ is a linear second order (homogenous) differential equation. Therefore it has two independent solutions. We need a function that when differentiated twice gets a minus sign and is multiplied by ω^2. *sin* and *cos* are easy guesses. There *cannot* exist any other independent solutions to this equation (this can be proven). Our general solution must be a linear combination of the two:

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

To get a specific solution, we substitute our initial conditions. We can write $A = \sqrt{C_1^2 + C_2^2}$ and $\phi = \arctan\left(\frac{-C_1}{C_2}\right)$ and then

$$\boxed{x(t) = A\cos(\omega_0 t + \phi)}$$

8.4 Energy

Let us consider the energy of a simple harmonic oscillator.

$$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$$

$$E_k = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega_0^2 A^2 \sin^2(\omega_0 t + \phi)$$

and then it is easy to see

$$E = U + E_k = \frac{1}{2}kA^2$$

The total energy is constant as expected.

8.5 Example with Moment of Inertia

Solid body with moment of inertia I_o around point o distance l from the center of mass, and mass M.

$$\sum \tau_o = -lMg\sin\theta = I_0\ddot{\theta}$$

Using taylor approximation we get

$$\ddot{\theta} + \frac{lMg}{I_0} = 0$$

and therefore $\theta(t)$ oscillates harmonically with $\omega_0 = \sqrt{\frac{lMg}{I_o}}$.

8.6 Example With Additional Constant Force

Consider a mass on a vertical spring with gravity.

$$-mg - ky = m\ddot{y}$$

Let $\tilde{y} = y + \frac{mg}{k}$. $\tilde{y} = 0$ when $ky = -mg$ (equilibrium). Substituting we get a harmonic oscillator around the equilibrium point, with $\omega_0 = \sqrt{\frac{k}{m}}$ (same angular velocity).

$$\ddot{\tilde{y}} + \frac{k}{m}\tilde{y} = 0$$

8.7 Damped Harmonic Oscillator

We'll add another force $f = -\beta\hat{x}$. We have forces $-kx - \beta\dot{x} = m\ddot{x}$ and therefore

$$\ddot{x} + \frac{\beta}{m}\dot{x} + \frac{k}{m}x = 0$$

Let $\tau = 2\frac{m}{\beta}$. It has units of time and therefore must be related to the effect of friction over time.
Let and $\omega_0^2 = \frac{k}{m}$. $\frac{1}{\omega_0}$ is proportional to the period of the oscillation.
We now have:

$$(*)\ddot{x} + \frac{2}{\tau}\dot{x} + \omega_0^2 x = 0$$

1. If $\tau >> \frac{1}{\omega_0}$ (equiv. $\frac{1}{\tau} << \omega_0$) : $\ddot{x} + \omega_0^2 x = 0$ - the usual harmonic oscillator equation of motion.
2. If $\tau << \frac{1}{\omega_0}$ ("risun hazak" , equiv. $\frac{1}{\tau} >> \omega_0$): $\ddot{x} + \frac{2}{\tau}\dot{x} = 0$ - simply the equation of motion of a body under friction We know thse solution to this equation is $x = Ae^{\alpha t}$.

Substituting $x = Ce^{\alpha t}$ (to see if it satisfies it) into the original differential equation (*) we get

$$\alpha^2 + \frac{2}{\tau}\alpha + \omega_0^2 = 0$$

solving for α (quadratic formula) we get

$$\alpha_{1,2} = -\frac{1}{\tau} \pm \sqrt{\frac{1}{\tau^2} - \omega_0^2}$$

1. The root is positive when $\tau < \frac{1}{\omega_0}$. We then have:

$$\boxed{x(t) = C_1 e^{\left(-\frac{1}{\tau}+\sqrt{\frac{1}{\tau^2}-\omega_0^2}\right)t} + C_2 e^{\left(-\frac{1}{\tau}-\sqrt{\frac{1}{\tau^2}-\omega_0^2}\right)t}}$$

In this solution there are no oscillations, only friction "risun yeter". If $\tau << \frac{1}{\omega_0}$ we can use a taylor approximation to get $\sqrt{\frac{1}{\tau^2} - \omega_0^2} = \frac{1}{\tau}\sqrt{1 - (\omega_0 t)^2} \approx \frac{1}{\tau}\left(1 - \frac{1}{2}(\omega_0 t)^2\right)$. and then

$$x(t) \overset{t\to\infty}{\to} C_1 e^{-\frac{1}{2}\omega_0^2\tau^2 t}$$

2. The root is negative when $\tau > \frac{1}{\omega_0}$. We'll define $\omega_1^2 = \omega_0^2 - \frac{1}{\tau^2}$. We then have

$$\alpha_{1,2} = -\frac{1}{\tau} \pm i\omega_1$$

$$x(t) = e^{-t/\tau}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$$

If a complex number is a solution to a homogenous (= 0) differential equation, then both the real and the imaginary parts seperately solve the equation. Meaning we have the two independent solutions

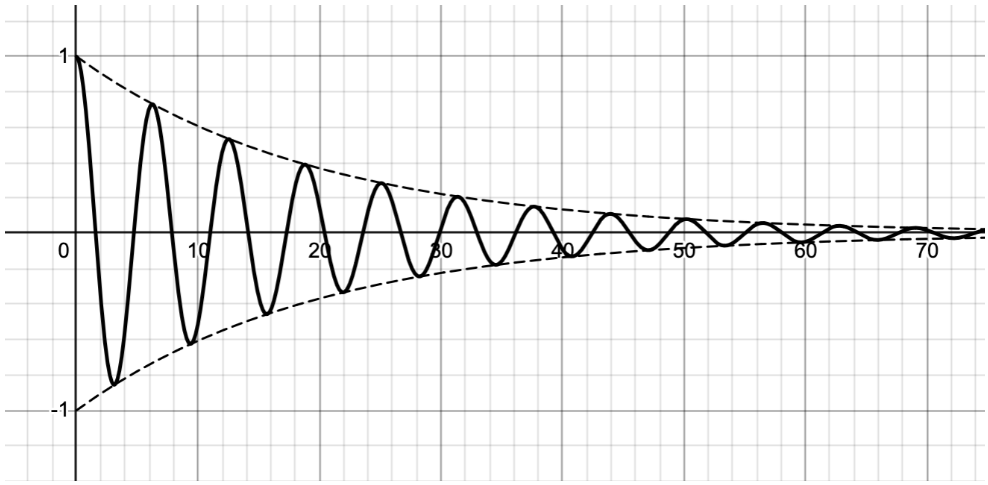
$$\hat{x}_1(t) = e^{-t/\tau}\left[(C_1 + C_2)\cos(\omega_1 t)\right]$$
$$\hat{x}_2(t) = e^{-t/\tau}\left[(C_1 - C_2)\sin(\omega_1 t)\right]$$

Defining $A = C_1 + C_2$, $B = C_1 - C_2$, we get the general solution

$$x(t) = e^{-t/\tau}\left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right]$$

or writing it with a phase (different A)

$$\boxed{x(t) = Ae^{-t/\tau}\cos(\omega_1 t + \phi)\,,\ \omega_1 = \sqrt{\omega_0^2 - \frac{1}{\tau^2}}}$$



3. the root is zero if $\tau = \frac{1}{\omega_0}$ “risun kriti” . We then have

$$\alpha = -\frac{1}{\tau} \Rightarrow x(t) = Ae^{-t/\tau}$$

This is already included in the 2nd solution we had, with $\omega_1 = 0, \phi = 0$. But a differential equation of 2nd order must have two independent solutions. We can guess it should be of the form $x(t) = f(t)Ae^{-t/\tau}$. Differentiating:

$$\dot{x}(t) = \dot{f}(t)e^{-t/\tau} - \tfrac{1}{\tau}f(t)e^{-t/\tau}$$
$$\ddot{x}(t) = \ddot{f}(t)e^{-t/\tau} - \tfrac{2}{\tau}\dot{f}(t)e^{-t/\tau} + \tfrac{1}{\tau^2}f(t)e^{-t/\tau}$$

Substituting this into the differential equation we get:

$$\ddot{f}(t) = 0 \Rightarrow f(t) = At + B$$

meaning

$$\boxed{x(t) = (At + B)\, e^{-t/\tau}}$$

8.8 Damped and Meulatz

We add a periodic force $F_0 \cos(\omega t)$ in addition to $-kx$ and $-\beta v$.

$$-kx - \beta\dot{x} + F_0 \cos(\omega t) = m\ddot{x}$$

$$\ddot{x} + \frac{2}{\tau} + \omega_0^2\dot{x} = \frac{F_0}{m}\cos(\omega t)$$

This is a non homogenous equation. The solution to such an equation is the same as the homegenous's , with an additional solution. We'll solve the following equation and then take the real part:

$$\ddot{x} + \frac{2}{\tau} + \omega_0^2\dot{x} = \frac{F_0}{m}e^{i\omega t}$$

We'll guess a solution

$$x(t) = Be^{i\omega t}$$

$$\dot{x}(t) = i\omega Be^{i\omega t}$$

$$\ddot{x} = -\omega^2 Be^{i\omega t}$$

Substituting into the original equation:

$$-\omega^2 B\cancel{e^{i\omega t}} + \frac{2}{\tau} i\omega B\cancel{e^{i\omega t}} + \omega_0^2 B\cancel{e^{i\omega t}} = \frac{F_0}{m} B\cancel{e^{i\omega t}}$$

$$B = \frac{F_0/m}{\omega_0^2 - \omega^2 + i\frac{2\omega}{\tau}} = \frac{F_0/m}{\left(\omega_0^2 - \omega^2\right)^2 + \frac{4\omega^2}{\tau^2}} \cdot \left(\omega_0^2 - \omega^2 - i\frac{2\omega}{\tau}\right)$$

We can write this in euler form: $B = a + bi = Ae^{i\phi}$, where $\tan\phi = \frac{b}{a}$ and $A^2 = a^2 + b^2$. So in our case $a = \omega_0^2 - \omega^2$ and $b = -\frac{2\omega}{\tau}$. We then get:

$$B = \frac{F_0/m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \frac{4\omega^2}{\tau^2}}} e^{-i\frac{2\omega}{\tau(\omega_0^2 - \omega^2)}}$$

Taking the real part, finally:

$$\boxed{x(t) = \frac{F_0}{m} \frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \frac{4\omega^2}{\tau^2}}} \cos\left(\omega t + \phi\right), \quad \tan\phi = \frac{-2\omega}{\tau\left(\omega_0^2 - \omega^2\right)}, \ \tau = \frac{2m}{\beta}, \ \omega_0 = \sqrt{\frac{k}{m}}}$$

There's a maximum of the amplitude (resonance) at $\omega = \sqrt{\omega_0^2 + \frac{2}{\tau^2}}$, with $A = \frac{\tau/2}{\sqrt{\omega_0^2 - \frac{1}{\tau^2}}} \frac{F_0}{m}$.