# Particle Physics + QFT Course

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## Contents

## Part I

# Particle Phyiscs

## 1 Existing frameworks

- A framework is used for describing the evolution of a system.
- In a theory a chosen framework is applied to a context.
- A model is an effective theory that requires some inputs that are not predicted by theory itself.

| Newtonian | Relativistic |
|---|---|
| Low speeds | Any speed |
| 3D + absolute time | 3+1D spacetime |
| galilean principle | relativity principle |

| Newtonian | Quantum(non relativistic) |
|---|---|
| large decoherent systems | Any slow system |
| deterministic | probabilistic |
| $x(t)$ | $\psi(x, t)$ |

QFT is relativistic quantum theory.

## 2 Symmetries and Transformations

### 2.1 Transformations

- **Global vs local**
- **Discrete vs continuous**
- **Compact vs non-compact**. Compact means the transformation includes the boundary terms.
- **Spacetime vs internal.** Special relativity arrises from symmetries of spacetime, which concern spacetime coordinates. Internal transformations regard underlying physics, and from their symmetries arrise the forces.

### 2.2 Groups

A group G is a set of elements {A,B,$\cdots$} with composition $\cdot$ that satisfies:

1. **Closure** - $A, B \in G \Rightarrow A \cdot B \in G$.
2. **Identity** - There exists $I \in G$ so that $I \cdot A = I$ for any $A \in G$.
3. **Inverse** - For any $A \in G$ there exists $A^{-1} \in G$ so that $A^{-1} \cdot A = I$.
4. **Associativity** - $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

Not required - G is abelian $A \cdot B = B \cdot A$ (commutativity of composition).
A subset of G that satisfies the group properties is a subgroup.
A generator of the group is an element that by applying the composition to it can generate every element of the group.
Isomorphic groups are groups with the same “multiplication table”.
A group representation can be faithfull, or degenerate.

## 2.3 Invariance and Duals

(Dot product example)
Using some trasnformation given by $A$ on $r$ and the dual $\tilde{r}$,

$$A \in G, \;\; r \to Ar, \; \tilde{r} \to \left(A^{-1}\right)^T \tilde{r}$$

$\tilde{r}^T r$ is invariant:

$$\tilde{r}^T r = \left(\left(A^{-1}\right)^T \tilde{r}\right)^T Ar = \tilde{r}^T A^{-1} Ar = \tilde{r}^T r$$

The dual of a representation can be found using the metric tensor $\tilde{r} = gr$. Therefore

$$\tilde{r}^T r = (gr)^T r = r^T g^T r = r^T gr \to r^T A^T gAr$$

$$\boxed{A^T gA = g}$$

since the expression must be invariant.

## 2.4 Groups

- O(n): Group of nxn real matrices that don't change vector magnitude: $O^T O = I \to det(O) = \pm 1$. Contains rotations and reflections. $O(3) = SO(3) \times \mathbb{Z}_2$.
- SO(n): Group of nxn real matrices that represent rotations: $O^T O = I$, $det(O) = +1$.
- U(n): Group of nxn unitary matrices: $U^\dagger U = I \to det(U) = e^{i\theta}$.
- SU(n): Group of nxn unitary matrices that uniquely represent a rotation: $U^\dagger U = I, det(U) = 1$.
- $\mathbb{Z}_2 = \{P, I\}$. While $P = -I \to P^2 = I$

## 2.5 Note

Given the group, using the transformations one could find the metric with $A^T gA = g$. (at last we have minkowski metric proof yas)

Otherwise, given the metric and a representation $r$, one can find the transformations that leave $\tilde{r}^T r$ invariant.

# 3 Special Relativity

## 3.1 Intro

Special relativity discusses the SO(1,3) group with the Minkowski metric $\eta_{\mu\nu}$ $(-1,1,1,1)$ and the Lorentz transformations $\Lambda \in SO(1,3)$, $det\,(\Lambda) = +1$. The transformations are rotations in every plane of the 6 combinations - 3 rotations and 3 boosts (along time axis). Together they form the Lorentz group.

$$ds = \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} \quad \Lambda_{yz}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos\theta & sin\theta \\ 0 & 0 & -sin\theta & cos\theta \end{pmatrix} \quad \Lambda_{xt}(\beta) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With an invariant quantity accordingly $d\vec{s} = dx^{\mu}$, $ds^2 = dx_{\mu}dx^{\mu} = -c^2dt^2 + dx^2 + dy^2 + dz^2$. With index

notation we can write the vector and dual vector transformation rules $v^{\mu} \to \Lambda^{\mu'}_{\mu} v^{\mu} = v^{\mu'}$, $v_{\mu} \to \Lambda^{\mu}_{\mu'} v_{\mu} = v_{\mu'}$. Index clarification: $\Lambda^{\mu'}_{\ \mu} = \Lambda$, $\Lambda^{\mu}_{\ \mu'} = \Lambda^{-1}$, $\Lambda_{\mu}^{\ \mu'} = \Lambda^{T}$.

## 3.2 Derivatives

The time derivative $\frac{ds}{dt}$ does not form as a tensor and therefore is not a valid generalization of a time derivative, and does not make a useful 4-velocity. Instead we use the length of the path, "proper time", which is the time from passed in the rest system.

$$cd\tau = \sqrt{-ds^2} = \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}$$

since $ds^2 = -c^2d\tau^2$. We can also now say $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\beta^2}} = \gamma$.

The generalization of the space derivative $\vec{\nabla}$ is the 4-derivative $\partial_{\mu} = \frac{\partial}{\partial X^{\mu}} = \left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$. As the index suggests this derivative results in a covariant (dual space) vector. We have a 4-gradient dual vector $\partial_{\mu}\Phi$ and a 4-divergence scalar $\partial_{\mu}A^{\mu}$.

Finally a covariant derivative as a derivative along tangent vectors on a manifold. More precisely, given a point $p$ on a manifold $M$, a vector field $\mathbf{u}$: $M \to T_pM$ and a tangent vector $\mathbf{v} \in T_pM$, the covariant derivative of $\mathbf{u}$ at p along $\mathbf{v}$ is a tangent vector $(\nabla_{\mathbf{v}}\mathbf{u})_p$ that is linear in v, additive in u and obeys the product rule.

$\nabla_{\mathbf{v}}\mathbf{u} = \nabla_{v^{\mu}\mathbf{e}_{\mu}} u^{\sigma}\mathbf{e}_{\sigma}$
Linearity in v: $= v^{\mu}\nabla_{\mathbf{e}_{\mu}} u^{\sigma}\mathbf{e}_{\sigma}$
Product rule: $= v^{\mu}u^{\sigma}\nabla_{\mathbf{e}_{\mu}}\mathbf{e}_{\sigma} + v^{\mu}\mathbf{e}_{\sigma}\nabla_{\mathbf{e}_{\mu}}u^{\sigma}$
Connection definition, der. of scalar: $\boxed{\nabla_{\mathbf{v}}\mathbf{u} = \left(v^{\mu}u^{\sigma}\Gamma^{\rho}_{\ \sigma\mu} + v^{\mu}\frac{\partial u^{\sigma}}{\partial x^{\mu}}\right)\mathbf{e}_{\rho}}$

## 3.3 Velocity, Momentum and Energy

The 4-velocity can is defined as $U^{\mu} = \frac{dX^{\mu}}{d\tau} \to U = (\gamma c, \gamma\vec{v})$, with an obviously invariant scalar (using $U_{\mu}U^{\mu}$) of $-c^2$.

The 4-momentum is defined as $P^{\mu} = mU^{\mu} \to P = (\gamma mc, \gamma m\vec{v})$ with $m$ being the rest mass. A useful low-speeds-approximation ($\beta \ll 1$) is $\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$. Applying it to $P_0$ we find the standard newtonian kinetic energy, in addition to a constant minimum energy term, called rest energy.

$$\gamma mc = mc + \frac{1}{2}mv^2/c = \frac{E_0 + E_{kinetic}}{c} = \frac{E}{c}$$

The rest energy is $E = mc^2$, and the total energy $E = \gamma mc^2$. We can say $P = (E/c, \vec{p})$.

Finding another invariant quantity, $P_\mu P^\mu = -m^2c^2 = -\frac{E^2}{c^2} + p^2$. This is the *mass-shell* condition:

$$E^2 = m^2c^4 + p^2c^2$$

| | | |
|---|---|---|
| $P_\mu P^\mu < 0$ | timelike | massive: $m^2 > 0$ |
| $P_\mu P^\mu = 0$ | lightlike | massless: $m = 0$ |
| $P_\mu P^\mu > 0$ | spacelike | tachyonic: $m^2 < 0$ |

## 3.4 Electromagnetism

In 3D we have a scalar electric potential and a vector magnetic potential, defining the electric and magnetic vector fields.

$$\vec{E} = \nabla\phi + \frac{\partial\vec{A}}{c\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

We can combine the vector and scalar to make a EM 4-potential dual vector.

$$A_\mu = \begin{pmatrix} \phi & \vec{A} \end{pmatrix}$$

Then we define a (0,2) antisymmetric tensor combining E and B:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_x & -B_y \\ E_y & -B_x & 0 & B_z \\ E_z & B_y & -B_z & 0 \end{pmatrix}$$

$$F_{\mu\nu}F^{\mu\nu} = -2\left(\frac{E^2}{c^2} - B^2\right)$$

# 4 Lie Groups, Lie Algebras

## 4.1 Lie Groups

$<$ Notice the possibility for comparing different continuous groups is missing (unlike discrete groups, which have multiplication tables). $>$

A Lie group is defined as group which forms a manifold with a differentiable structure: meaning, you can create a space with dimensionality corresponding to the number of free parameters, and each transformation in the group will be uniquely represented by a point in said space.

A general element of a Lie group can be written as the exponential map

$$A = e^{ig_A v^A}$$

where $g_A$ is the generator associated with A and $v^A$ parametrizes it.

yz rotations $R_{yz}(\theta) \in SO(3)$ can serve as an example. Using a taylor expansion with the following generator we get

$$g_{yz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$R_{yz}\left(\theta\right) = e^{ig_{yz}\theta} = I + ig_{yz}\theta + \frac{1}{2!}\left(ig_{yz}\theta\right)^2 + \frac{1}{3!}\left(ig_{yz}\theta\right)^3 + ...$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{1}{2!}\theta^2 + ... & \theta - \frac{1}{3!}\theta^3 + ... \\ 0 & -\theta + \frac{1}{3!}\theta^3 - ... & 1 - \frac{1}{2!}\theta^2 + ... \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\left(\theta\right) & sin\left(\theta\right) \\ 0 & -sin\left(\theta\right) & cos\left(\theta\right) \end{pmatrix}$$

and similarly the same could be done for xz and xy rotations using the $g_{xz}$ and $g_{xy}$ generator. This could be shown at once with the generator and parameters $g_A v^A = g_{xy}\alpha + g_{xz}\beta + g_{yz}\gamma$. This set of 3 infinititesimal generators is not unique, because it is based on the selection of a coordinate system. The generators form a basis in the tangent space of the group manifold at the origin, so the analogy to coordinates is quite deep.

## 4.2 Lie Algebras

By finding the commutation relation of the generators of the Lie group, we can have a full description of the group that is independent of the choice of coordinate system. $f^{ijk}$ are called the structure constants.

$$\boxed{[g_i, g_j] = f^{ijk} g_k}$$

By solving this equation we can get the all sets of generators of the Lie group corresponding to the structure constants used. With this abstraction, we can find new relations between groups (for example, SO(3) and SU(2)).

# 5 Spinors

## 5.1 3D

For SO(3), we have $[g_i, g_j] = i\epsilon^{ijk} g_k$ and equivalently $\boxed{\sigma_k = -\frac{i}{2}[\sigma_i, \sigma_j]}$. Solving this, we can get back SO(3). Additionaly, we can get the following 2x2 generators ($\sigma_i$ being the pauli matrices)

$$g_{yz} = \frac{1}{2}\sigma_x, \;\; g_{zx} = \frac{1}{2}\sigma_y, \;\; g_{xy} = \frac{1}{2}\sigma_z$$

$$U_{yz}(\theta) = e^{ig_{yz}\theta} = \begin{pmatrix} cos\left(\frac{\theta}{2}\right) & isin\left(\frac{\theta}{2}\right) \\ isin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

This satisfies $U^\dagger U = 1$, $det\, U = +1$. and therefore belong to SU(2) which act on complex two-component objects called *spinors.* They are two-component objects but reflect rotations in three dimensions, they share the same Lie algebra SO(3). Only SO(3) is degenerate as

$$U(2\pi) = -I, \; R(2\pi) = I, \; U(4\pi) = R(4\pi) = I.$$

The generators satisfy the Clifford algebra $\boxed{\{\sigma_i, \sigma_j\} = 2\delta_{ij} I_{2\times 2}}$.

The spinors of 3D space transform as $\boxed{\chi \rightarrow \chi' = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}}\chi}$.

## 5.2 Relativistic 4D

### 5.2.1 Lorentz Generators

The six generators for Special Relativity corresponding to the three rotations $R_{yz}$ , $R_{zx}$, $R_{xy}$ and the three boosts $B_{xt}$, $B_{yt}$, $B_{zt}$ are usually called $J_1$, $J_2$, $J_3$ and $K_1$, $K_2$, $K_3$. The rotations can be easily generalized from the 3D case simply by padding with zeros. For example, similarly to $g_{yz}$,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

If one takes the 3 boosts and considers their Taylor expansion, then using the exponential map $B = e^{iK\beta}$ gives the K matrices (the boost generators). The algebra for SO(1,3) becomes

$$[J_i, J_j] = i\epsilon^{ijk} J_k$$

$$[K_i, K_j] = -i\epsilon^{ijk} J_k$$

$$[J_i, K_j] = i\epsilon^{ijk} K_k$$

The rotations build a subgroup, but the boosts do not, and the mixed commutators $[J_i, K_j]$ are not zero.

### 5.2.2 Mixed SO(3) Generators

We can build a transformation that is a mix of boosts and rotations that is still SO(1,3) but can be seperated to subgroups. $J_{\pm i} = \frac{1}{2}(J_i \pm iK_i)$ gets us the algebra

$$[J_{+i}, J_{+j}] = i\epsilon^{ijk} J_{+k}$$

$$[J_{-i}, J_{-j}] = i\epsilon^{ijk} J_{-k}$$

$$[J_{+i}, J_{-j}] = 0$$

We now see we can divide $SO(1,3)$ into two subgroups that satisfy the $SO(3)$ algebra. This means $SO(1,3) \sim SO(3) \times SO(3)$ or at least near the identity.

For the group $SO(3)$ of rotations in space the corresponding spinors are based on $SU(2)$, and therefore for the group $SO(1,3) \sim SO(3) \times SO(3)$ of Lorentz transformations the corresponding spinors are based on $SU(2) \times SU(2)$.

When using cartesian product the number of degrees of freedom adds up. This means we get 6 degrees of freedom (as expected by 3 rots and 3 boosts) and a spinor with 4 components (that have nothing to do with spacetime components, as when we divided $SO(1,3)$ to two subgroups we mixed space and time).

### 5.2.3 Generalizing Spinors to 4D

Generalizing the Clifford algebra of $SO(3)$ to 4 dimensions we get $\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4\times 4}}$.

To write the spinor transformation rule we can no longer use the dot product notation we used for 3D, since the rotation can no longer be described by an axis of rotation but only by a plane of rotation.

So to continue the analogy to 3D we will define $\boxed{\sigma^{\mu\nu} = \frac{-i}{4}[\gamma^\mu, \gamma^\nu]}$, together with the angles $\omega_{\mu\nu}$. Clearly, $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ and $\omega^{\mu\nu} = -\omega^{\nu\mu}$. Again, since the generator is asymmetric we get 6 degrees of freedom.

The spinors of 4D space transform as $\boxed{\psi \to \psi' = e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi = S[\Lambda]^{a'}_{a}\psi^a}$.

The new matrices we have defined are the gamma/Dirac matrices:

$$\gamma^0 = -i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = -i\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

The gamma matrices have the properties:

- Clifford algebra: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_{4\times 4}$.
- $\left(\gamma^0\right)^2 = -I, \ \left(\gamma^i\right)^2 = I$

The quantity $\tilde{\psi}^\dagger\psi$ is invariant for $\tilde{\psi} = i\gamma^0\psi$. This means we are working with the metric $g = i\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Defining the spinor adjoint: $\bar{\psi} \equiv \tilde{\psi}^\dagger = i\psi^\dagger\gamma^0$, and of course the quantity $\bar{\psi}\psi$ is invariant.

The gamma matrices have one spacetime index and 2 spin space indices $\gamma^{\mu a}{}_b$, but as they are always found between spinors they are easy to transform. $\bar{\psi}\gamma^\mu\psi$ transforms as a vector.

# 6 Mechanics

## 6.1 Lagrangian Framework

Lagrangian mechanics is based on the principle of least action $S = \int L(q, \dot{q})dt$, $\delta S = 0$. We can find the equations of motion of a system using the Euler-Lagrange equation $\frac{\partial L}{\partial q} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$ (derived from the principle of least action.

*<the following is just the usual functional minimizing problem to find the euler-lagrange equation>*

In QFT the action is replaced by

$$S = \int \mathcal{L}\left(\phi, \partial_\mu \phi\right) d^4x$$

Using the least action principle,

$$\delta S = \int \delta \mathcal{L}\left(\phi, \partial_\mu \phi\right) d^4x = \int \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\delta\partial_\mu\phi\right] d^4x$$
$$= \int \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\delta\phi\right] d^4x + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\delta\phi|_{\partial M}$$

As we require boundary conditions $\phi(x^\mu|_{\partial M})$ for a spacetime region M in which the field exists, the variation of the field at the boundary must be zero ($\delta\phi|_{\partial M} = 0$). Now using the fundemental lemma of variational calculus,

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\right]\delta\phi \; d^4x = 0$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} = 0}$$

We have the Lagrangian

$$\mathcal{L} = \mathcal{L}_{kinetic} + \mathcal{L}_{interactions}$$

A Lagrangian with only a kinetic but no interaction term is called the free-field Lagrangian.

## 6.2 Examples and Eqs. of Motion

### 6.2.1 Showcase

Consider the free Lagrangians for scalar (spin-0), spinor (spin-1/2) and vector (spin-1) relativistic fields.

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2\phi^2$$

$$\mathcal{L}_{1/2} = (\hbar c)\,\bar{\psi}\gamma^\mu\partial_\mu\psi + mc^2\bar{\psi}\psi$$

$$\mathcal{L}_1 = \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A_\mu A^\mu$$

Notice the free Lagrangian contains a kinetic energy-like term (derivative squared) but with no mass, and an additional term with mass. This allows massless particles with rest energy.

Plugging those into the Euler-Lagrange equation we get the following equations of motion:

### 6.2.2 Proofs

Starting with the mass-shell condition and using the QM momentum and energy operators,

$$\hat{E} = i\hbar\partial_t, \;\; \hat{p} = -i\hbar\nabla$$

$$(mc)^2 = \left(\frac{E}{c}\right)^2 - p^2$$

Substituting the QM operators we get the Klein Gordon equation of motion.

$$\left(\frac{mc}{\hbar}\right)^2 = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$$

$$\left[\partial_\mu\partial^\mu - \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0$$

From this one can find the appropriate Lagrangian that would yield this equation of motion, the spin 0 Lagrangian.

# Part II

# -

## 7 Variational Calculus

### 7.1 Functionals

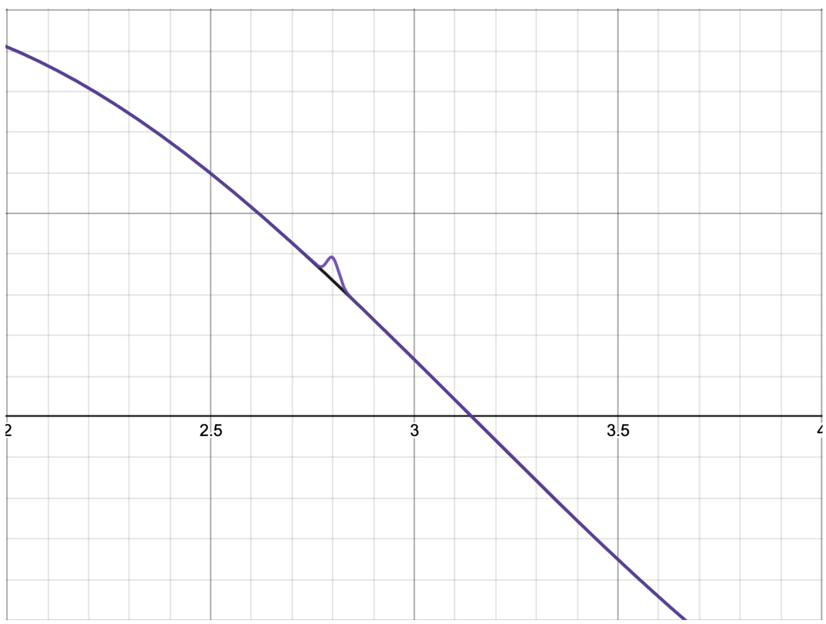
A functional $F[f]$ is a function that maps a function to a scalar value, a "function of a function". The function in brackets is simply the syntax to say the functional depends on the function.

For example, $I[f] = \int_{-1}^{1} f(x)dx$ is a functional. Evaluated with $f(x) = 3x^2$, its value is $I[3x^2] = 2$.

We can define a **functional derivative** as follows:

$$\frac{\delta F}{\delta f(x)} \equiv \lim_{\epsilon \to 0} \frac{F[f(u) + \epsilon\delta(u-x)] - F[f(u)]}{\epsilon}$$

*Quite beautiful :)* This is equivalent to the difference when evaluating the functional with some function f(u) and with the same function but with an infinitesimal increase ("bump") at u=x.




Some rules of functional differentiation (that can be proved using the above definition):

$J[f] = \int [f(y)]^p \phi(y)dy$ . $\frac{\delta J}{\delta f(x)} = pf(x)^{p-1}\phi(x)$

$H[f] = \int g[f(x_0)]dx$ . $\frac{\delta H}{\delta f(x)} = \frac{dg}{dx}|_{x=x_0}$

$\frac{\delta f(u)}{\delta f(x)} = \delta(u-x)$

$\frac{\delta f'(u)}{\delta f(x)} = \dot{\delta}(u-x)$

A general functional derivative can be written as:

$$J[f] = \int_{x_1}^{x_2} L(x, f, f')dx$$

$$\frac{\delta J}{\delta f} = \frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'}$$

# 8 Lagrangian Mechanics

## 8.1 Introduction

We define the Lagrangian as the difference of kinetic and potential energy

$$\boxed{L = V - T}$$

And the action S as the time integral of the lagrangian.

$$S = \int_{t_1}^{t_2} L dt$$

Let us motivate this: the average kinetic and potential energy along any trajectory is

$$\bar{T}[x(t)] = \frac{1}{\tau}\int_0^\tau \frac{1}{2}m\left(\dot{x}\right)^2 dt$$

$$\bar{V}[x(t)] = \frac{1}{\tau}\int_0^\tau V\left[x(t)\right] dt$$

Therefore we can write

$$\frac{\delta\bar{T}}{\delta x(t)} = \frac{-m\ddot{x}}{\tau}, \qquad \frac{\delta\bar{V}}{\delta x(t)} = \frac{V'}{\tau}$$

Given newton's second law $m\ddot{x} = -V'$. Therefore, we can say,

$$\frac{\delta\bar{T}}{\delta x(t)} - \frac{\delta\bar{V}}{\delta x(t)} = 0$$

$$\frac{\delta}{\delta x(t)}\left(\bar{T} - \bar{V}\right) = 0$$

Given the action $S = \int T - V dt = \tau\left(\bar{T} - \bar{V}\right)$, we get the **principle of least action**.

$$\boxed{\frac{\delta S}{\delta x(t)} = 0}$$

No infinitesimal change in the trajectory x(t) can change the action S. Therefore for any physically realizable trajectory, the action is at a minimum. Using this principle along with the definitions of the action and the Lagrangian, we can find the **euler-lagrange equation**[1], which can affectively replace $\vec{F} = m\vec{a}$ as the equation that describes the entire mechanics of a given system, but using scalar quantities (energies) instead of vectors (forces). The comma indices notate partial derivatives.

$$\frac{\delta S}{\delta x(t)} = \int \frac{\delta L}{\delta x(u)}\frac{\delta x(u)}{\delta x(t)} + \frac{\delta L}{\delta x_{,t}(u)}\frac{\delta x_{,t}(u)}{\delta x(t)} du$$

$$= \int \frac{\delta L}{\delta x(u)}\delta(u-t) + \frac{\delta L}{\delta x_{,t}(u)}\delta_{,t}(u-t) du$$

$$\frac{\delta L}{\delta x(t)} - \frac{d}{dt}\frac{\delta L}{\delta\dot{x}(t)} = 0$$

Depending on the which variables the Lagrangian is a function of $\left(t, y, \frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\ldots\right)$, you get the more general case of the EL equation.

[1] EL equation, for short.

$$\boxed{\frac{\delta L}{\delta q(t)} - \sum_i \frac{d}{dq_i}\frac{\delta L}{\delta q_{,i}(t)} = 0}$$

For a field, we can define a *Lagrangian density* for every point in space. The lagrangian density is comprised of the difference of the kinetic energy density and potential energy density.

$$\boxed{L = \int \mathcal{L} dx}$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0}$$

## 8.2 Wave Function Example

We can find the wave equation for waves along a string, by writing the Lagrangian density for a string and using euler's equation.

$$\mathcal{L}\left(\psi, \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial x}\right) = \frac{\rho}{2}\left(\frac{\partial \psi}{\partial t}\right)^2 - \frac{\mathcal{T}}{2}\left(\frac{\partial \psi}{\partial x}\right)^2$$

$$0 = \frac{\delta S}{\delta x(t)} = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_x \left( \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \right) - \partial_t \left( \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}, \quad c = \sqrt{\frac{\mathcal{T}}{\rho}}$$

## 8.3 Klein-Gordon Equation

Consider the Lagrangian density for a real scalar field $\phi(\vec{x}, t)$.[2]

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2$$

$$= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\nabla\phi\right)^2 - \frac{1}{2}m^2\phi^2$$

We have the following energy densities: kinetic energy $\frac{1}{2}\dot{\phi}^2$, gradient energy $\frac{1}{2}\left(\nabla\phi\right)^2$, and potential energy $\frac{1}{2}m^2\phi^2$. To determine the equations of motion for this field we write the EL equation:

$$\partial_\mu \left(\partial_\mu \phi\right) + m^2\phi = 0$$

This is the Klein-Gordon equation. Laplacian in Minkowski space(-time) can be denoted by $\Box$, called the d'Alembertian, or the wave operator.

$$\Box\phi = \partial^\mu \phi \partial_\mu \phi = \ddot{\phi} - \nabla^2\phi$$

The Klein-Gordon equation reads

$$\Box\phi + m^2\phi = 0$$

[2] $(\partial_\mu \phi)^2 = \partial^\mu \phi \partial_\mu \phi = g^{\mu\nu}\partial_\nu \phi \partial_\mu \phi$. We are refering to minkowski's metric: $g^{\mu\nu} = \eta^{\mu\nu} \to (+,-,-,-)$

# 9 Harmonic Oscillator

## 9.1 Lagrangian

Lagrangian of harmonic oscillator using ladder operators.

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

$$H = p\dot{x} - L(x,\dot{x}) = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\omega\hbar}}, \qquad a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\omega\hbar}}$$

$L = p\dot{x} - H = \frac{-i}{2}\sqrt{2m\omega\hbar}\left(a - a^\dagger\right)\sqrt{\frac{\hbar}{2m\omega}}\left(\dot{a} + \dot{a}^\dagger\right) - \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$
$= \frac{-i\hbar}{2}\left(a - a^\dagger\right)\left(\dot{a} + \dot{a}^\dagger\right) - \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$
$= i\hbar a^\dagger \dot{a} - \hbar\omega a^\dagger a$

Total time derivatives were removed since when integrating to find the action they are left as constants which do not change depending on the path (only the end points) and therefore do not affect the minimum action principle which is used to find the equations of motion.

## 9.2 Excited States

$$N = a^\dagger a, \quad N|n\rangle = n|n\rangle$$

$$[N,a] = -a, \quad [N,a^\dagger] = a^\dagger$$

$$N\left(a|n\rangle\right) = (n-1)\left(a|n\rangle\right), \quad N\left(a^\dagger|n\rangle\right) = (n+1)\left(a^\dagger|n\rangle\right)$$

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$\langle n|m\rangle = \delta_{mn}$$

## 9.3 Uncertainty

With the units $m = \omega = \hbar = 1$, $a = \frac{1}{\sqrt{2}}(x + ip)$.

$$\langle n|x|n\rangle = \langle n|\frac{a + a^\dagger}{\sqrt{2}}|n\rangle = 0$$

$$\langle n|p|n\rangle = \langle n|\frac{-i\left(a - a^\dagger\right)}{\sqrt{2}}|n\rangle = 0$$

$$\langle n|x^2|n\rangle = \langle n|\frac{\left(a + a^\dagger\right)^2}{2}|n\rangle = \langle n|\frac{a^\dagger a + aa^\dagger}{2}|n\rangle = \langle n|\frac{2a^\dagger a + 1}{2}|n\rangle = n + \frac{1}{2}$$

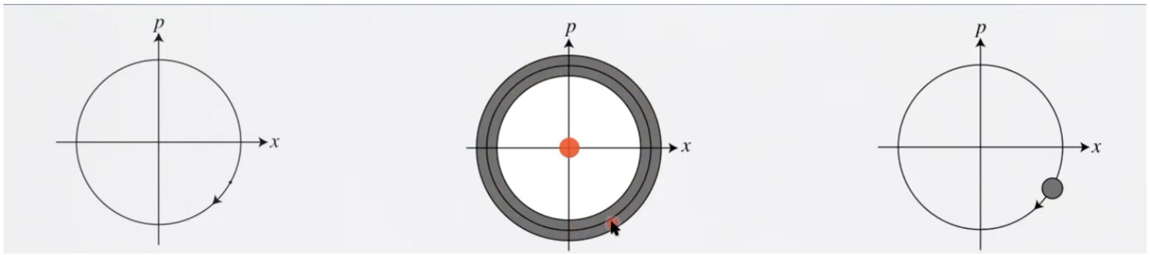
$$\langle n|p^2|n\rangle = \langle n|\frac{\left(a - a^\dagger\right)^2}{-2}|n\rangle = \langle n|\frac{a^\dagger a + aa^\dagger}{2}|n\rangle = \langle n|\frac{2a^\dagger a + 1}{2}|n\rangle = n + \frac{1}{2}$$

$$\sigma_x^2\sigma_p^2 = \left(n + \frac{1}{2}\right)^2 \geq \left(\frac{\hbar}{2}\right)^2$$

How do we take the classical limit if there is a greater uncertainty for greater n?

### 9.4 Coherent State

Classical hamiltonian: $\frac{1}{2}\left(x^2+p^2\right)=n+\frac{1}{2}$
Number eigenstate: $n \leq \frac{1}{2}\left(x^2+p^2\right) \leq n+1$
Coherent state: $0 \leq \frac{1}{2}\left((x-x_0)^2+(p-p_0)^2\right) \leq 1$




To find the coherent state we will find the eigenstate of the annihilation operator.

$$a|f\rangle = f|f\rangle, \quad a=\frac{1}{\sqrt{2}}\left(x+ip\right), \quad f\in\mathbb{C}$$

$$\langle f|x|f\rangle = \langle f|\frac{a+a^\dagger}{\sqrt{2}}|f\rangle = \frac{f+f^*}{\sqrt{2}}$$

$$\langle f|p|f\rangle = \langle f|\frac{-i\left(a+a^\dagger\right)}{\sqrt{2}}|f\rangle = \frac{-i\left(f+f^*\right)}{\sqrt{2}}$$

Therefore

$$f=\frac{1}{\sqrt{2}}\left(\langle x\rangle + i\langle p\rangle\right)$$

We can say the the annihilation operator is $a=\frac{\partial}{\partial a^\dagger}$ (just like $p \propto \frac{\partial}{\partial x}$). Then the eigenstate can be written as (using taylor expansion)

$$|f\rangle \equiv Ae^{fa^\dagger}|0\rangle = e^{-f^2/2}\sum_{n=0}^{\infty}\frac{f^n}{\sqrt{n!}}|n\rangle$$

Now checking the uncertainty in the coherent state we get

$$\langle f|x^2|f\rangle = \langle f|\frac{1}{2}\left(a+a^\dagger\right)^2|f\rangle = \frac{1}{2}\left(f^2+f^{*2}+2f^*f+1\right)$$

$$\langle f|x^2|f\rangle = \langle f|\frac{-1}{2}\left(a-a^\dagger\right)^2|f\rangle = \frac{1}{2}\left(f^2+f^{*2}-2f^*f-1\right)$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{1}{2}$$

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle = \frac{1}{2}$$

$$\boxed{\sigma_x\sigma_p = \frac{\hbar}{2}}$$

The coherent state has the minimum uncertainty.

### 9.5 Coherent State Time Evolution

$$e^{-iHt/\hbar}|f\rangle = e^{-f^2/2}\sum_{n=0}^{\infty}\frac{f^n}{\sqrt{n!}}e^{-i\hbar\omega(n+\frac{1}{2})t/\hbar}|n\rangle = e^{-i\omega t/2}e^{-f^2/2}\sum_{n=0}^{\infty}\frac{\left(fe^{-i\omega t}\right)^n}{\sqrt{n!}}|n\rangle = \boxed{e^{-i\omega t/2}|fe^{-i\omega t}\rangle}$$

Adding time evolution we fe find that the eigenvalue of the coherent state as a function of time is $fe^{-i\omega t}$. Remembering $f = \frac{1}{\sqrt{2}}\left(\langle x\rangle + i\langle p\rangle\right)$ we get a motion corresponding to a circle on the x-p state space.

$$\langle x\rangle = \sqrt{2}Re\left(f_0\right)cos\left(\omega t\right)$$

$$\langle p\rangle = \sqrt{2}Im\left(f_0\right)sin\left(\omega t\right)$$

### 9.6 Number-Phase Uncertainty

Instead of using cartesian coordinates in x-p state space with $a = \frac{1}{\sqrt{2}}\left(x + ip\right)$ we can use polar coordinates: $r = |a| = \sqrt{a^\dagger a} = \sqrt{N}$. (Notice the phase $\theta$)

$$a = e^{i\theta}\sqrt{N},\ \ a^\dagger = \sqrt{N}e^{-i\theta}$$

$$[N, e^{i\theta}\sqrt{N}] = -e^{i\theta}\sqrt{N} \ \ \rightarrow \ \ N = i\frac{\partial}{\partial\theta} \ \ \rightarrow \ \ [N,\theta] = i$$

$$\sigma_N\sigma_\theta \geq \frac{\hbar}{2}$$

## 10 1D Lattice

We have a 1D lattice of independent harmonic oscillators. In a 1D lattice $a_n^\dagger$ is the creation operator, and $a_n$ is the annihilation operator that create/annihilate a particle at position n of the discrete lattice. Of course, $a_n|0\rangle = 0$. This creates a sort of a boson field - multiple particles can be created in the same position.

The Lagrangian is

$$L = \sum_n\left(i\hbar a_n^\dagger\dot{a_n} - \hbar\omega a_n^\dagger a_n\right)$$

But they cannot move from one position to another. To do that we have to add a kinetic term to the hamiltonian allowing movement

$$H \ni -K\left(a_{n+1}^\dagger a_n + a_n^\dagger a_{n+1}\right)$$

$$H = \sum_n\left(\hbar\omega a_n^\dagger a_n - K\left(a_{n+1}^\dagger a_n + a_n^\dagger a_{n+1}\right)\right)$$

And then the Lagrangian is

$$L = \sum_n\left(i\hbar a_n^\dagger\dot{a_n} - \hbar\omega a_n^\dagger a_n + K\left(a_{n+1}^\dagger a_n + a_n^\dagger a_{n+1}\right)\right)$$

given the state x§

$$|k\rangle = \sum_m e^{iknd}a_m^\dagger|0\rangle$$

# 11 Spinors Intuitively

## 11.1 Pauli Spinors

Spinors are in a way a rank 1/2 tensor, a square root of a vector. We can take a 3D vector and turn it into a (zero determinant) Pauli vector.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x - yi \\ x + yi & -z \end{pmatrix}$$

With the Pauli matrices being

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Which we can then factor into a pair of rank 1/2 tensors - Pauli spinors. Pauli spinors transform with $SU(2)$. Rotating each of the spinors by $\theta/2$ is equivalent to rotating the original vector by $\theta$.

$$\begin{pmatrix} z & x - yi \\ x + yi & -z \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \begin{pmatrix} -\xi_2 & \xi_1 \end{pmatrix}$$

## 11.2 Weyl Spinors

For a 4D vector, we add $ctI$ where I is the identity matrix. We then get the Weyl spinor, that transform with $SL(2, \mathbb{C})$.

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} ct + z & x - yi \\ x + yi & ct - z \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix}$$

## 11.3 Jones Vectors - EM Polarization

A Jones vector contains all the information needed to know about the polarization of a given light wave. The coordinates can be chosen as such that the z axis is the direction of travel. The wave can be represented mathematically as a product of an amplitude, a phase and a plane wave.

$$E = \begin{pmatrix} A_x e^{i\phi_x} \\ A_y e^{i\phi_y} \\ 0 \end{pmatrix} e^{i(\omega t - kz)}$$

The Jones vector is then

$$\vec{J} = \begin{pmatrix} A_x e^{i\phi_x} \\ A_y e^{i\phi_y} \end{pmatrix} = A_x e^{i\phi_x} \vec{H} + A_y e^{i\phi_y} \vec{V}$$

$$\vec{H} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \vec{V} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with H and V being the basis vectors representing horizontal polarization and vertical polarization.

## 11.4 Complex Projective Line (CP1)