2b.21

martedì 7 dicembre 2021 13:52

ESERCIZIO 2b.21 – SIMILE AL 2b.27.

come 2b.27

$$f(x,y) = \lambda^2 e^{-(\lambda x + \lambda y + a x y)} \qquad x, y \in (0, +\infty)$$

1) calcolare le $f_X(x)$ e $f_Y(y)$ e trovare il valore di a affinché X e Y siano stocasticamente indipendenti.

Per il valore di a trovato e $\lambda = 7$:

2) trovare la densità e la f.d.d di $Z = \frac{X}{X+Y}$, si tratta di una legge nota?

3) $E(Z)$ e $Var(Z)$

4) $P(-\sqrt{2} \leq Z < \frac{1}{3})$

SVLG

1)
$$f_X(x) = \int_0^{+\infty} \lambda^2 e^{-(\lambda x + \lambda y + a x y)} dy$$

$$= \int_0^{+\infty} \lambda^2 e^{-\lambda x} \cdot e^{-\lambda y} \cdot e^{-a x y} dy = \lambda^2 e^{-\lambda x} \int_0^{+\infty} e^{-\lambda y} e^{-a x y} dy$$

$$= \lambda^2 e^{-\lambda x} \int_0^{+\infty} e^{-y(\lambda + a x)} dy = \lambda e^{-\lambda x} \int_0^{+\infty} \lambda e^{-(\lambda + a x) y} dy$$

$$= \lambda^2 e^{-\lambda x} \int_0^{+\infty} \frac{\lambda + a x}{\lambda + a x} \cdot e^{-(\lambda + a x) y} dy = \frac{\lambda^2 \cdot e^{-\lambda x}}{\lambda + a x} \int_0^{+\infty} \underbrace{\lambda + a x \; e^{-(\lambda + a x) y}}_{= 1} dy$$

$$= \boxed{\frac{\lambda^2 e^{-\lambda x}}{\lambda + a x}}$$

$$f_Y(y) = \int_0^{+\infty} \lambda^2 e^{-(\lambda x + \lambda y + a x y)} dx$$

$$\lambda^2 \int_0^{+\infty} e^{-\lambda x} \cdot e^{-\lambda y} \cdot e^{-a x y} dx = \lambda^2 e^{-\lambda y} \int_0^{+\infty} e^{-\lambda x} \cdot e^{-a x y} dx$$

$$\lambda^2 e^{-\lambda y} \int_0^{+\infty} e^{-x(\lambda + a y)} dx = \lambda^2 e^{-\lambda y} \int_0^{+\infty} \frac{\lambda + a y}{\lambda + a y} \cdot e^{-x(\lambda + a y)} dx$$

$$= \frac{\lambda^2 e^{-\lambda y}}{\lambda + a y} \int_0^{+\infty} \underbrace{\lambda + a y \; e^{-(\lambda + a y) x}}_{=} dx = \boxed{\frac{\lambda^2 e^{-\lambda y}}{\lambda + a y}}$$

$$= \frac{\lambda e}{\lambda + a y} \int_0 \underbrace{\lambda + a y \quad e}_{1} \, dx = \boxed{\frac{\lambda^2 e}{\lambda + a y}}$$

$$\boxed{\frac{\lambda^2 e^{-\lambda x}}{\lambda + a x}} \quad \boxed{\frac{\lambda^2 e^{-\lambda y}}{\lambda + a y}} = \boxed{\lambda^2 e^{-(\lambda x + \lambda y + a x y)}}$$

determiniamo il valore di a affinché $f_x(x)$ e $f_y(y)$ siano indipendenti!

$$a = 0$$

2) $\lambda = 7$, $a = 0$!

$$\boxed{Z = \frac{X}{X + Y}}$$

$$\phi = \begin{cases} X = x \\ z = \frac{x}{x+y} \end{cases} \qquad \phi^{-1} = \begin{cases} X = x \\ y = x\left(\frac{1}{z} - 1\right) \end{cases} \qquad J\phi^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{1}{z} - 1 & -\frac{x}{z^2} \end{bmatrix} \quad |\Delta| = \frac{x}{z^2}$$

$$\cdot f_{x,z}(x, z) = f_{x,y}\left(x, x\left(\frac{1}{z} - 1\right)\right) \cdot \frac{x}{z^2}$$

ora, andiamo a studiare le funzioni indicatrici!

$$\boxed{0 < X < +\infty} \qquad 0 < x\left(\frac{1}{z} - 1\right) < +\infty$$

$$\boxed{0 < X < +\infty}$$

$$= \int_0^{+\infty} \lambda^2 e^{-(\lambda x + \lambda(x(\frac{1}{z} - 1))} \cdot \frac{x}{z^2} \, dx$$

$$= \frac{1}{z^2} \int_0^{+\infty} x \cdot \lambda^2 \cdot e^{-(\lambda x + \lambda(x(\frac{1}{z} - 1)))} \cdot dx$$

$$= \frac{1}{z^2} \int_0^{+\infty} x \cdot \lambda^2 \cdot e^{-\lambda x} \cdot e^{-\lambda(x(\frac{1}{z} - 1))} \, dx$$

$$= \frac{1}{z^2} \int_0^{+\infty} x \cdot \lambda^2 \cdot e^{-\lambda x} \cdot e^{-\lambda \boxed{\left(\frac{1}{z} - 1\right) x}} \, dx \quad !$$

$$= \frac{1}{z^2} \int^{+\infty} x \, \lambda^2 \, e^{\cancel{-\lambda x}} \quad \cancel{+\lambda x} \quad -\frac{\lambda}{z} x$$

$$= \frac{1}{z^2} \int_0^{+\infty} x \cdot \lambda^2 \cdot \cancel{e^{-\lambda x}} \cdot \cancel{e^{+\lambda x}} \cdot e^{-\frac{\lambda}{z} x} \, dx$$

$$= \frac{\lambda}{z} \int_0^{+\infty} \underbrace{\frac{\lambda}{z} e^{-\frac{\lambda}{z} x} \cdot x}_{f(x) \cdot X} \cdot dx$$

$$f(x) \cdot X = \frac{1}{\frac{\lambda}{z}} = \frac{z}{\lambda}$$

$= \frac{\lambda}{z} \cdot \frac{z}{\lambda} = \boxed{1}$ quindi z è uniformemente distribuita in $[0, 1]$

• la sua f.d.d:

$$F_z(z) = \begin{cases} 0 & \text{se } z < 0 \\ z & \text{se } z \in [0, 1] \\ 1 & \text{se } z > 1 \end{cases}$$

3) calcoliamo:

$$E(z) = \int_0^1 1 \cdot z \, dx = \left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2}$$

$$Var(z) = E(z^2) - E^2(z) = \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \boxed{\frac{1}{12}}$$

$$E(z^2) = \int_0^1 1 \cdot z^2 = \left[\frac{z^3}{3}\right]_0^1 = \frac{1}{3}$$

4) $P\left(0 \leq z \leq \frac{1}{3}\right) = \frac{1}{3}$ se $z \in [0, 1]$ ✓