2b.27

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COME 2b. 21

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$$f(x,y) = \begin{cases} \mu^2 e^{-(\mu x + \mu y + \beta x y)} & \text{SE } x>0, y>0 \\ 0 & \text{ALTRIMENTI} \end{cases}$$

1) $f_X(x)$ $f_Y(y)$ e TROVARE IL valore di β AFFINCHÉ X e Y SIANO STOCASTI. IND.

PER IL valore di β trovato e $\mu = 7$:

2) Trovare la densità e la f.d.d di $Z = \frac{X}{X+Y}$; si tratta di una legge NOTA?

3) $E(Z)$, $Var(Z)$

4) $P(-2\sqrt{2} \le Z < \frac{1}{7})$

SVLG

1) $$f_X(x) = \int_0^{+\infty} \mu^2 e^{-(\mu x + \mu y + \beta x y)} dy$$

$$= \mu^2 \int_0^{+\infty} e^{-(\mu x + \mu y + \beta x y)} dy = \mu^2 \int_0^{+\infty} e^{-\mu x} \cdot e^{-\mu y} \cdot e^{-\beta x y} dy =$$

$$= \mu^2 \cdot e^{-\mu x} \int_0^{+\infty} e^{-\mu y - \beta x y} dy = \mu^2 e^{-\mu x} \int_0^{+\infty} e^{-y(\mu + \beta x)} dy =$$

$$= \mu^2 e^{-\mu x} \int_0^{+\infty} e^{-(\mu + \beta x) y} \cdot \frac{\mu + \beta x}{\mu + \beta x} dy = \frac{\mu^2 e^{-\mu x}}{\mu + \beta x} \int_0^{+\infty} \underbrace{\mu + \beta x \; e^{-(\mu + \beta x) y}}_{\exp(\mu + \beta x) = 1} dy$$

$$= \boxed{\frac{\mu^2 e^{-\mu x}}{\mu + \beta x}}$$

$$f_Y(y) = \int_0^{+\infty} \mu^2 e^{-(\mu x + \mu y + \beta x y)} dx = \mu^2 \int_0^{+\infty} e^{-\mu x} \cdot e^{\mu y} \cdot e^{\beta x y} dy$$

$$= \mu^2 e^{-\mu y} \int_0^{+\infty} e^{-\mu x} \cdot e^{-\beta x y} dx = \mu^2 e^{\mu y} \int_0^{+\infty} e^{-x(\mu + \beta y)} dx =$$

$$= \mu^2 e^{-\mu y} \int_0^{+\infty} \frac{\mu + \beta y}{\mu + \beta y} e^{-(\mu + \beta y) x} dx = \frac{\mu^2 e^{\mu y}}{\mu + \beta y} \int_0^{+\infty} \underbrace{\mu + \beta y \; e^{-(\mu + \beta y) x}}_{\exp(\mu + \beta y) = 1} dx$$

$$= \boxed{\frac{\mu^2 e^{-\mu y}}{\mu + \beta y}}$$

$\mu + \beta y \qquad 0 \qquad \exp(\mu + \beta y) = 1 \checkmark$

$$= \boxed{\frac{\mu^2 e^{-\mu y}}{\mu + \beta y}}$$

Per essere indipendenti: $\boxed{f_x(x) \cdot f_y(y) = f(x,y)}$

$$\left[\frac{\mu^2 e^{-\mu x}}{\mu + \beta x}\right] \cdot \left[\frac{\mu^2 e^{-\mu y}}{\mu + \beta y}\right] = \mu^2 e^{-(\mu x + \mu y + \beta x y)} \rightarrow \underline{\beta = 0}$$

Se $\beta = 0$:

$$\frac{\mu^2 e^{-\mu x}}{\cancel{\mu}} \cdot \frac{\mu^2 e^{-\mu y}}{\cancel{\mu}} = \mu \cdot e^{-\mu x} \cdot \mu \cdot e^{-\mu y} = \boxed{\mu^2 e^{-(\mu x + \mu y)}}$$

$$\underline{\overset{\shortparallel}{f(x,y)}} \quad \checkmark$$

2) $z = \dfrac{x}{x+y}$

$$\phi = \begin{cases} X = x \\ z = \dfrac{x}{x+y} \end{cases} \qquad \phi^{-1}: \begin{cases} X = x \\ y = x\left(\dfrac{1}{z} - 1\right) \end{cases} = \quad J\phi^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{1}{z} - 1 & -\frac{x}{z^2} \end{bmatrix} \quad |\Delta| = \frac{x}{z^2}$$

$$f_{x,z}(x,z) = f_{x,y}\left(x, x\left(\tfrac{1}{z} - 1\right)\right) \cdot \frac{x}{z^2} \cdot 1_{[0,+\infty]}(x) \cdot 1_{[0,+\infty]}\left(x\left(\tfrac{1}{z} - 1\right)\right)$$

Ora studiamo le funzioni indicatrici:

$$\underline{0 < x < +\infty} \qquad 0 < x\left(\frac{1}{z} - 1\right) < +\infty$$

$$\underline{0 < x < +\infty}$$

$$f_z(z) = \int_0^{+\infty} f\left(x, x\left(\frac{1}{z} - 1\right)\right) \cdot \frac{x}{z^2}\, dx$$

$$= \int_0^{+\infty} \mu^2 e^{-(\mu x)} \cdot e^{-\left(\mu\left(x\left(\frac{1}{z} - 1\right)\right)\right)} \cdot \frac{x}{z^2}\, dx \qquad \mu^2 e^{-(\mu x + \mu y + \beta x y)}$$

$$= \frac{\mu^2}{z^2} \int_0^{+\infty} e^{-\mu x} \cdot e^{-\mu\left(\frac{1}{z} - 1\right)x} \, dx$$

$$= \frac{\mu^2}{z^2} \int_0^{+\infty} e^{-\mu x} \cdot e^{-\mu\left(\frac{1}{z}-1\right)x} \cdot x \, dx$$

$$= \frac{\mu^2}{z^2} \int_0^{+\infty} \cancel{e^{-\mu x}} \cdot e^{-\frac{\mu}{z}x} \cdot \cancel{e^{\mu x}} \cdot x \, dx$$

$$= \frac{\mu^2}{z^2} \int_0^{+\infty} x \cdot e^{-\frac{\mu}{z}x} \, dx$$

$$= \frac{\mu}{z} \int_0^{+\infty} \underbrace{\underbrace{x}\cdot \frac{\mu}{z} \cdot e^{-\frac{\mu}{z}x}}_{E(x) = \frac{1}{\lambda} = \exp\left(\frac{\mu}{z}\right)} dx$$

$$= \frac{\cancel{\mu}}{\cancel{z}} \cdot \frac{\cancel{z}}{\cancel{\mu}} = \boxed{1}$$

Cioè significa che $x \in [0,1]$ ed è uniformemente distribuita! in $[0,1]$.
e la sua f.d.d:

$$F_X(x) = \begin{cases} 0 & \text{se } z < 0 \\ z & \text{se } z \in (0,1) \\ 1 & \text{se } z > 1 \end{cases}$$

3) $E(z)$, $var(z)$!

- $E(z) = \int_0^1 z \cdot 1 \, dz = \left[\frac{z^2}{2}\right]_0^1 = \boxed{\frac{1}{2}}$

- $Var(z) = E(z^2) - E^2(z) = \frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \boxed{\frac{1}{12}}$

- $E(z^2) = \int_0^1 z^2 \cdot 1 = \left[\frac{z^3}{3}\right] = \boxed{\frac{1}{3}}$

4)

$$P\left(-2\sqrt{2} \leq Z < \frac{1}{7}\right) = P\left(0 \leq z < \frac{1}{7}\right) = \int^{\frac{1}{7}} 1 \, dz = \left[\frac{1}{7}\right]$$

$$P\left(-2\sqrt{2} \le z < \frac{1}{7}\right) = P\left(0 \le z < \frac{1}{7}\right) = \int_0 1\,dz = \boxed{\frac{1}{7}}$$

quando $z \in [0, 1]$