2b.24

domenica 12 dicembre 2021 10:46

ESERCIZIO 2b.24

$$f(x,y) = \begin{cases} \frac{6}{5}(x^2+y) & \text{se} \quad x \in (0,1),\ y \in (0,1) \\ 0 & \text{altrimenti} \end{cases}$$

1) $f_X(x)$ e $f_Y(y)$. Si tratta di v.a indipendenti?
$cov(X,Y)$.
Trovare una diversa densità congiunta avente le stesse marginali

2) $P(2X<Y)$ e $P(X \le \frac{1}{2},\ Y \le \frac{1}{2})$

3) $Z = X \cdot Y$ e la densità di $W = \ln Z$

SVLG

1) $$f_X(x) = \int_0^1 \frac{6}{5}(x^2+y)\,dy = \frac{6}{5}\int (x^2+y)\,dy = \frac{6}{5}\left[x^2 y + \frac{y^2}{2}\right]_{y=0}^{y=1}$$

$$= \frac{6}{5}\left[x^2 + \frac{1}{2}\right]$$

2) $$f_Y(y) = \int_0^1 \frac{6}{5}(x^2+y)\,dx = \frac{6}{5}\left[\frac{x^3}{3} + yx\right]_{x=0}^{x=1} = \frac{6}{5}\left[\frac{1}{3}+y\right]$$

Verifichiamo se sono indipendenti:

$$\frac{6}{5}\left[x^2+\frac{1}{2}\right] \cdot \frac{6}{5}\left[\frac{1}{3}+y\right] = \left(\frac{6}{5}x^2 + \frac{3}{5}\right)\cdot\left(\frac{2}{5}+\frac{6}{5}y\right)$$

$$= \frac{12}{25}x^2 + \frac{36}{25}x^2 y + \frac{6}{25} + \frac{18}{25}y$$ NON SONO INDIPENDENTI!

calcoliamo la $cov(X,Y)$:

$$cov(X,Y) = E(XY) - E(X)\cdot E(Y)$$

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy \cdot x \cdot y$$

$$\int_0^1 \frac{6}{5}(x^2+y)\cdot y\cdot dy = \int_0^1\left(\frac{6}{5}x^2+\frac{6}{5}y\right)\cdot y\cdot dy = \int_0^1\left(\frac{6}{5}x^2y+\frac{6}{5}y^2\right)dy$$

$$= \frac{6}{5}x^2\frac{y^2}{2}+\frac{6}{5}\frac{y^3}{3} = \left[\frac{3}{5}x^2y^2+\frac{2}{5}y^3\right]_{y=0}^{y=1}$$

$$= \frac{3}{5}x^2+\frac{2}{5}$$

$$= \int_0^1\left(\frac{3}{5}x^2+\frac{2}{5}\right)\cdot x\cdot dx = \int_0^1 \frac{3}{5}x^3+\frac{2}{5}x = \frac{3}{5}\left[\frac{x^4}{4}\right]+\frac{2}{5}\left[\frac{x^2}{2}\right] = \frac{3}{20}\left[x^4\right]_0^1+\frac{1}{5}\left[x^2\right]_0^1$$

$$= \frac{3}{20}+\frac{1}{5} = \frac{3+4}{20} = \boxed{\frac{7}{20}}$$ ha sbagliato lui!

$$E(X) = \int_0^1 x\cdot\left(\frac{6}{5}x^2+\frac{3}{5}\right)dx = \int_0^1\left(\frac{6}{5}x^3+\frac{3}{5}x\right)dx = \frac{\cancel{6}^{3}}{5}\frac{x^4}{\cancel{4}_{2}}+\frac{3}{5}\frac{x^2}{2}$$

$$= \left[\frac{3}{10}x^4+\frac{3}{10}x^2\right]_0^1 = \frac{3}{10}+\frac{3}{10} = \frac{3+3}{10} = \frac{6}{10} = \boxed{\frac{3}{5}}$$

$$E(Y) = \int_0^1\left(\frac{6}{15}+\frac{6}{5}y\right)\cdot y\, dy = \int_0^1\left(\frac{6}{15}y+\frac{6}{5}y^2\right)dy = \left(\frac{6}{15}\frac{y^2}{2}+\frac{6}{5}\frac{y^3}{3}\right)$$

$$= \left[\frac{3}{15}y^2+\frac{2}{5}y^3\right]_0^1 = \frac{3}{15}+\frac{2}{5} = \frac{1}{5}+\frac{2}{5} = \boxed{\frac{3}{5}}$$

$$Cov(X,Y) = \frac{7}{20}-\left(\frac{9}{25}\right) = \frac{175-180}{500} = -\frac{5}{500} = \boxed{-\frac{1}{100}}$$

Una diversa densità congiunta avente le stesse marginali, si ottiene:

$$\boxed{g(x,y) = f(x)\cdot f(y)}\,!$$

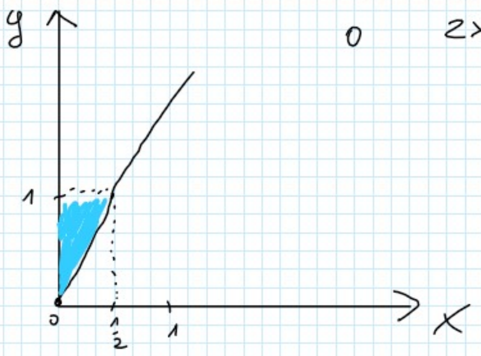
2) $P(2X<Y) = P(Y>2X) = \int_0^{\frac{1}{2}}\int_{2x}^{1}\frac{6}{5}(x^2+y)\,dx\,dy$

$X \mid Y$ $\quad y=2x \quad$ $y\uparrow$

$$\int_{0} \int_{2x} \frac{6}{5}(x^2+y)\, dx\, dy$$

$y = 2x$

x	y
0	0
1	2
$\frac{1}{2}$	1



$$= \int_{2x}^{1} \frac{6}{5}(x^2+y)\, dy = \frac{6}{5}\left[x^2 y + \frac{y^2}{2}\right]_{y=2x}^{y=1}$$

$$= \frac{6}{5}\left[x^2 + \frac{1}{2} - \left(2x^3 + \frac{4x^2}{2}\right)\right] = \frac{6}{5}\left[x^2 + \frac{1}{2} - 2x^3 - 2x^2\right]$$

$$= \frac{6}{5}\left[-2x^3 - x^2 + \frac{1}{2}\right]$$

$$= \int_0^{\frac{1}{2}} \frac{6}{5}\left[-2x^3 - x^2 + \frac{1}{2}\right] dx = \frac{6}{5}\int_0^{\frac{1}{2}} \left(-2x^3 - x^2 + \frac{1}{2}\right) dx = \frac{6}{5}\left[-\frac{2x^4}{4} - \frac{x^3}{3} + \frac{1}{2}x\right]_0^{\frac{1}{2}}$$

$$= \frac{6}{5}\left[-\frac{x^4}{2} - \frac{x^3}{3} + \frac{x}{2}\right]_0^{\frac{1}{2}} = \frac{6}{5}\left[-\frac{1}{16}\cdot\frac{1}{2} - \frac{1}{8}\cdot\frac{1}{3} + \frac{1}{4}\right] = \frac{6}{5}\left[-\frac{1}{32} - \frac{1}{24} + \frac{1}{4}\right]$$

$$= \frac{6}{5}\left[\frac{-24-32+192}{768}\right] = \frac{136}{768}\cdot\frac{6}{5} = \boxed{\frac{17}{80}}$$

$$P\left(X \le \frac{1}{2}, Y \le \frac{1}{2}\right) = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} f(x,y)\, dx\, dy = \frac{1}{10}$$

$\downarrow$

come 26.22

3) $Z = X \cdot Y$

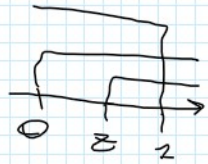
$$\phi = \begin{cases} X = X \\ Z = XY \end{cases} \quad \phi^{-1} = \begin{cases} X = X \\ Y = \frac{Z}{X} \end{cases} \qquad J\phi^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{z}{x^2} & \frac{1}{x} \end{bmatrix} \quad |\Delta| = \frac{1}{x}$$

$$\Phi = \begin{cases} \dots \\ z = xy \end{cases} \quad \Phi^{-1} = \begin{cases} \dots \\ y = \frac{z}{x} \end{cases} \qquad J_\Phi = \begin{bmatrix} \dots & \dots \\ -\frac{z}{x^2} & \frac{1}{x} \end{bmatrix} \quad |\Delta| = \frac{1}{x}$$

$$f_{x,z}(x,z) = f_{x,y}\left(x, \frac{z}{x}\right) \cdot \frac{1}{x}$$

studio delle funzioni indicatrici:

$$0 < x < 1 \qquad \begin{matrix} 0 < y < 1 \\ 0 < \frac{z}{x} < 1 \end{matrix} \rightarrow 0 < z < x \nearrow \begin{matrix} x > z \\ x > 0 \\ x < 1 \end{matrix}$$



$$f_z(z) = \int_z^1 \frac{6}{5}\left(x^2 + \frac{z}{x}\right) dx \cdot \frac{1}{x}$$

e $z = x \cdot y \in [0,1]$

ora risolvamo l'integrale:

$$\int_z^1 \left(\frac{6}{5}x^2 + \frac{6}{5}\frac{z}{x}\right)\frac{1}{x}\,dx = \int_z^1 \left(\frac{6x^{\cancel{2}}}{5\cancel{x}} + \frac{6z}{5x^2}\right)dx = \int_z^1 \frac{6}{5}x + \frac{6}{5}\frac{z}{x^2}\,dx$$

$$= \int \frac{6}{5}x + \int \frac{6}{5}\frac{z}{x^2}\,dx = \frac{6}{5}\left[\frac{x^2}{2}\right]_z^1 + \frac{6}{5}z\int x^{-2}\,dx$$

$$= \frac{6}{5}\cdot\frac{1}{2}\cdot\left[x^2\right]_z^1 + \frac{6}{5}z\,\frac{x^{-1}}{-1} = \frac{3}{5}\left[x^2\right]_z^1 - \left[\frac{6}{5}\frac{z}{x}\right]_{x=z}^{x=1}$$

$$= \frac{3}{5}\left[1 - z^2\right] - \frac{6}{5}z\cdot\left[\frac{1}{1} - \frac{1}{z}\right]$$

$$= \frac{3}{5} - \frac{3}{5}z^2 - \frac{6}{5}z + \frac{6\cancel{z}}{5\cancel{z}} = \frac{3}{5} + \frac{6}{5} - \frac{6}{5}z - \frac{3}{5}z^2 = \boxed{9 - 6z - 3z^2}$$

infine: $W = \ln z$

$$P(W \le w) = P(\ln z \le w) = P(z \le e^w) = P(z \le e^\tau)$$

$$f_z(e^\tau)\cdot f'(e^\tau) = \left(9 - 6e^\tau - 3e^{2\tau}\right)e^\tau$$

$$f_z(e^T) \cdot f'(e^T) = (9 - 6e^T - 3e^{2T}) \, e^T$$

dentro la densità di Z. $f_Z(e^T)$