

## ESERCIZIO 2b.33

$$f(x,y) = \begin{cases} x^a + y^a & \text{se } 0 < x < 1, \quad 0 < y < 1 \\ 0 & \text{altrimenti} \end{cases}$$

- 1) determinare  $a > 0$  affinché  $f(x,y)$  sia la densità congiunta.  
per il valore di  $a$  TROVATO, TROVARE  $f_X(x)$  e  $f_Y(y)$  e dire se sono stocasticamente indipendenti.
- 2) media, varianza di  $X$  e  $Y$  e  $\text{cov}(X,Y)$
- 3) Posto  $Z = X+Y$ , TROVARE la F.D.D e la densità di  $Z$ .
- 4)  $P(Y \leq X^2)$  e  $\beta > 0 : P(Z \leq \beta) = \frac{1}{24}$

SVLG

$$1) \int_0^1 \int_0^1 x^a + y^a \, dx \, dy = 1$$

$$\begin{aligned} \int x^a + y^a \, dy &= \int x^a + \int y^a \, dy = x^a \cdot [y]_0^1 + \left[ \frac{y^{a+1}}{a+1} \right]_0^1 \\ &= x^a + \frac{1^{a+1}}{a+1} = x^a + \frac{1}{a+1} \end{aligned}$$

$$\int_0^1 \left( x^a + \frac{1}{a+1} \right) dx = \int x^a + \int \frac{1}{a+1} \, dx = \left[ \frac{x^{a+1}}{a+1} + \frac{x}{a+1} \right]_{x=0}^{x=1}$$

$$\frac{1}{a+1} + \frac{1}{a+1} = 1$$

$$\frac{2}{a+1} = 1$$

$$2 = a+1$$

$$a = 1$$

$$f_X(x) = \int_0^1 (x+y) \, dy = x[y]_0^1 + \left[ \frac{y^2}{2} \right]_0^1 = \left[ x + \frac{1}{2} \right]$$

$$f_X(x) = \int_0^1 (x+y) dy = x \left[ y \right]_0^1 + \left[ \frac{y^2}{2} \right]_0^1 = \left[ x + \frac{1}{2} \right]$$

$$f_Y(y) = \int_0^1 (x+y) dx = \left[ \frac{x^2}{2} + yx \right]_0^1 = \left[ \frac{1}{2} + y \right]$$

Verifichiamo se sono indipendenti:

$$\left(x + \frac{1}{2}\right)\left(\frac{1}{2} + y\right) = \frac{1}{2}x + xy + \frac{1}{4} + \frac{1}{2}y \quad \text{NO, non sono indep.}$$

$$2) E(X) = \int_0^1 \left(x + \frac{1}{2}\right) \cdot x \cdot dx = \int_0^1 x^2 + \frac{1}{2}x = \left[ \frac{x^3}{3} + \frac{x^2}{4} \right]_0^1 = \frac{1}{3} + \frac{1}{4} = \left[ \frac{7}{12} \right]$$

$$E(Y) = \int_0^1 \left(y + \frac{1}{2}\right) \cdot y \cdot dy = \left[ \frac{7}{12} \right]$$

$$E(X^2) = \int_0^1 \left(x + \frac{1}{2}\right) \cdot x^2 \cdot dx = \int_0^1 x^3 + \frac{1}{2}x^2 = \left[ \frac{x^4}{4} + \frac{x^3}{6} \right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{6+4}{24} = \frac{10}{24} = \left[ \frac{5}{12} \right]$$

$$E(Y^2) = \int_0^1 \left(y + \frac{1}{2}\right) \cdot y^2 = \left[ \frac{5}{12} \right]$$

$$\text{Var}(X) = E(X^2) - E^2(X) = \frac{5}{12} - \frac{49}{144} = \frac{60-49}{144} = \left[ \frac{11}{144} \right]$$

$$\text{Var}(Y) = \left[ \frac{11}{144} \right]$$

$$\text{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = \frac{1}{3} - \frac{49}{144} = \frac{48-49}{144} = \left[ -\frac{1}{144} \right]$$

$$\rightarrow E(XY) = \int_0^1 \int_0^1 (x+y) \cdot x \cdot y \, dx \, dy$$

$$\int_0^1 (x+y) y \, dy = \int_0^1 xy + y^2 = \left[ \frac{xy^2}{2} + \frac{y^3}{3} \right]_{y=0}^{y=1} = \left[ \frac{x}{2} + \frac{1}{3} \right]$$

$$\int_0^1 \left(\frac{x}{2} + \frac{1}{3}\right) \cdot x \cdot dx = \int_0^1 \frac{x^2}{2} + \frac{x}{3} = \left[ \frac{x^3}{6} + \frac{x^2}{2} \right]_0^1 = \left[ \frac{1}{2} \right]$$

$$\int_0^1 \left( \frac{x}{2} + \frac{1}{3} \right) \cdot x \cdot dx = \int_0^1 \frac{x^2}{2} + \frac{x}{3} = \left[ \frac{x^3}{6} + \frac{x^2}{6} \right]_0^1 = \boxed{\frac{1}{3}}$$

$$3) \quad \boxed{Z = X + Y} \quad \begin{matrix} x \in [0, 1] \\ y \in [0, 1] \end{matrix}$$

$$\phi = \begin{cases} X = x \\ Z = x + y \end{cases}$$

$$\phi^{-1} = \begin{cases} X = x \\ y = z - x \end{cases}$$

$$J\phi^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad |J\phi^{-1}| = 1$$

$$f_{X,Z}(x,z) = f_{X,Y}(x, z-x) \cdot 1$$

$$f_Z(z) = \int_{\mathbb{R}} \cancel{x} + z - \cancel{x} \cdot 1 = \int_{\mathbb{R}} z \, dx$$

$$\boxed{0 < X < 1}$$

$$0 < Y < 1$$

$$0 < Z - X < 1$$

$$\begin{aligned} Z - X &> 0 \\ -X &> -Z \end{aligned}$$

$$X < Z$$

$$\begin{aligned} Z - X &< 1 \\ -X &< 1 - Z \end{aligned}$$

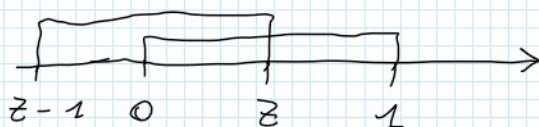
$$X > Z - 1$$

$$\boxed{X \in (Z-1, Z)}$$

$$\bullet \quad \underline{X \in (0, 1) \cap X \in (Z-1, Z) \quad \text{and} \quad Z \in (0, 2)}$$

$$\bullet \text{ se } Z < 0 = 0$$

$$\bullet \text{ se } Z \in (0, 1):$$



$$\int_0^Z z \, dx = [zx]_0^Z = z^2$$

• se  $z \in (1, 2)$ :



$$\int_{z-1}^1 z dx = z \left[ x \right]_{z-1}^1 = z(-z+2) = -z^2 + 2z = z(2-z)$$

• se  $z > 2 = 0$

**Ricapitolando:**

$$f_z(z) = \begin{cases} 0 & \text{se } z < 0 \\ z^2 & \text{se } z \in (0, 1) \\ z(2-z) & \text{se } z \in (1, 2) \\ 0 & \text{se } z > 2 \end{cases}$$

ora serve la f.d.d.:

Bisogna integrare da  $-\infty$  a  $z$   $\int_{-\infty}^z f_z(z) dz = \int_0^z f_z(z) dz$

$$\int_0^z z^2 dz = \left[ \frac{z^3}{3} \right]$$

$$\int_0^z (2z - z^2) dz = \left[ z^2 - \frac{z^3}{3} \right]$$

$$F_z(z) = \begin{cases} 0 & \text{se } z < 0 \\ \frac{z^3}{3} & \text{se } z \in (0, 1) \\ -\frac{z^3}{3} + z^2 & \text{se } z \in (1, 2) \\ 0 & \text{se } z > 2 \end{cases}$$

$$\int \int f(x, y) dx dy$$

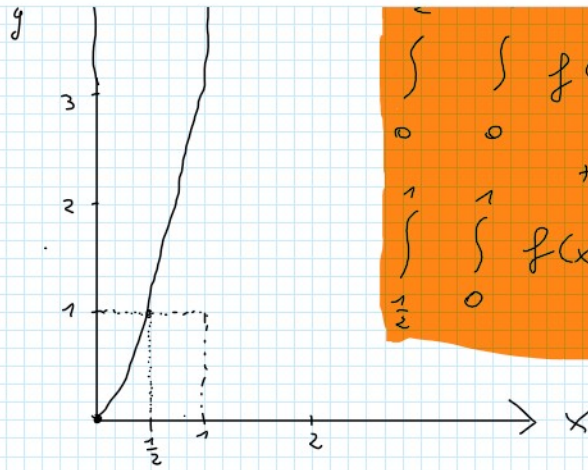
$$4) P(y \leq 4x^2) =$$

$$x \in (0, 1)$$

$$y \in (0, 1)$$

$$y = 4x^2$$

x	y
0	0
1	4
$\frac{1}{2}$	1



$$\int_0^1 \int_0^1 f(x, y) dx dy + \int_{\frac{1}{2}}^1 \int_0^1 f(x, y) dx dy$$

$$\int_0^{\frac{1}{2}} \int_0^{4x^2} (x+y) dx dy + \int_{\frac{1}{2}}^1 \int_0^1 (x+y) dx dy = \left[ \frac{59}{80} \right]$$

$$\text{ORA } P(Z \leq \beta) = \frac{1}{24}$$

$$\frac{Z^3}{3} = \frac{1}{24}$$

$$Z^3 = \frac{3}{24} = \frac{1}{8}$$

$$\rightarrow Z^3 = \frac{1}{8}$$

$$Z = \frac{1}{2}$$





