2b.28 (e(x) regole)) intervallo strano

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ESERCIZIO 2b.28

$$f(x,y) = \begin{cases} \dfrac{15}{x^3 y^4} & \text{se } 1 < x < y \\ 0 & \text{altrimenti} \end{cases}$$

1) $f_X(x)$ e $f_Y(y)$ e calcolare la densità condizionata di X dato $Y = 2$.

2) $Cov(X,Y)$ e $P(Y \leq X^2)$

3) Trovare la densità congiunta di $U = \log X$ e $V = \dfrac{X}{Y}$ e le relative densità marginali.

SVLG

1) $$\boxed{f_X(x)} = \int_x^{+\infty} \frac{15}{x^3 y^4}\, dy = \frac{15}{x^3} \int_x^{+\infty} y^{-4}\, dy = \frac{15}{x^3} \cdot \frac{y^{-3}}{-3} = -\frac{5}{x^3} \cdot \left[\frac{1}{y^3}\right]_x^{+\infty}$$

$$= -\frac{5}{x^3}\left[0 - \frac{1}{x^3}\right] = \boxed{+\frac{5}{x^6}}$$

$$\boxed{f_Y(y)} = \int_1^y \frac{15}{x^3 y^4}\, dx = \frac{15}{y^4} \int_1^y \frac{1}{x^3}\, dx = \frac{15}{y^4} \int_1^y x^{-3}\, dx = \frac{15}{y^4} \frac{x^{-2}}{-2} = -\frac{15}{2y^4} \cdot \left[\frac{1}{x^2}\right]_1^y$$

$$= -\frac{15}{2y^4} \cdot \left[\frac{1}{y^2} - 1\right] = \boxed{-\frac{15}{2y^6} + \frac{15}{2y^4}}$$

$$\boxed{f_{X|Y}(x \mid y=2)} = \frac{f(x,2)}{f_Y(2)} = \frac{\frac{15}{x^3\, 16}}{\frac{15}{32} - \frac{15}{128}} \cdot 1_{[1,2]}(x)$$

$1 < x < y$

$$= \frac{\frac{15}{16x^3}}{\frac{60-15}{128}} = \frac{\cancel{15}}{\cancel{16}x^3} \cdot \frac{\cancel{128}}{\cancel{45}} = \boxed{\left[\frac{8}{3x^3}\right] 1_{[1,2]}(x)}$$

2) $$Cov(X,Y) = E(XY) - E(X) \cdot E(Y)$$

$$E(XY) = \int_1^{+\infty} \int_x^{+\infty} f(x,y)\, x \cdot y\, dx\, dy$$

$+\infty$

Δ!

vedili smp come singoli:

$0 < x < y < 1$ e x

$E(XY) = \int_0^1 \int_x^1$

$E(X) = \int_0^1$

$E(Y) = \int_0^1$

$$= \int_x^{+\infty} \frac{15}{x^3 y^4}\cdot y\,dy = \int_x^{+\infty}\frac{15}{x^3y^3}\,dy = \frac{15}{x^3}\int_x y^{-3}\,dy = \frac{15}{x^3}\,\frac{y^{-2}}{-2} = -\frac{15}{2x^3}\left[\frac{1}{y^2}\right]_x^{+\infty}$$

$$= -\frac{15}{2x^3}\left[0-\frac{1}{x^2}\right] = \left[\frac{15}{2x^5}\right]$$

$$= \int_1^{+\infty}\frac{15}{2x^5}\,dx\cdot x = \int_1^{+\infty}\frac{15}{2x^4}\,dx = \frac{15}{2}\int_1^{+\infty} x^{-4}\,dx = \frac{15}{2}\,\frac{x^{-3}}{-3} = -\frac{5}{2}\cdot\left[\frac{1}{x^3}\right]_1^{+\infty}$$

$$= -\frac{5}{2}\left[0-1\right] = \boxed{\frac{5}{2}}$$

$$E(x) = \int_1^{+\infty} x\cdot\frac{5}{x^6}\cdot dx = \int_1^{+\infty}\frac{5}{x^5} = 5\int x^{-5}\,dx = 5\,\frac{x^{-4}}{-4} = -\frac{5}{4}\left[\frac{1}{x^4}\right]$$

$$= -\frac{5}{4}\left[0-1\right] = \frac{5}{4}$$

$1 < x < y$

$E(x) = \int_1^{+\infty} = E(y)$

$$E(y) = \int_1^{+\infty}\left(\frac{15}{2y^4} - \frac{15}{2y^6}\right)y\,dy = \int^{+\infty}\left(\frac{15}{2y^3} - \frac{15}{2y^5}\right)dy$$

$$= \int\frac{15}{2y^3} - \int\frac{15}{2y^5} = \frac{15}{2}\int y^{-3} - \frac{15}{2}\int y^{-5}\,dy$$

$$= \frac{15}{2}\,\frac{y^{-2}}{-2} - \frac{15}{2}\,\frac{y^{-4}}{-4} = -\frac{15}{4}\cdot\left[\frac{1}{y^2}\right]_1^{+\infty} + \frac{15}{8}\cdot\left[\frac{1}{y^4}\right]_1^{+\infty}$$

$$= -\frac{15}{4}\left[0-1\right] + \frac{15}{8}\left[0-1\right] = \frac{15}{4} - \frac{15}{8} = \frac{30-15}{8} = \left[\frac{15}{8}\right]$$

(è IN $E(xy)$ che nel secondo integrale modifico e valuto $y>x$, xchè in $E(x)$ e $E(y)$, dal primo all'ultimo.)

$$COV(X,Y) = \frac{5}{2} - \left[\frac{5}{4}\cdot\frac{15}{8}\right] = \frac{5}{2} - \left[\frac{75}{32}\right] = \frac{80-75}{32} = \left[\frac{5}{32}\right]$$

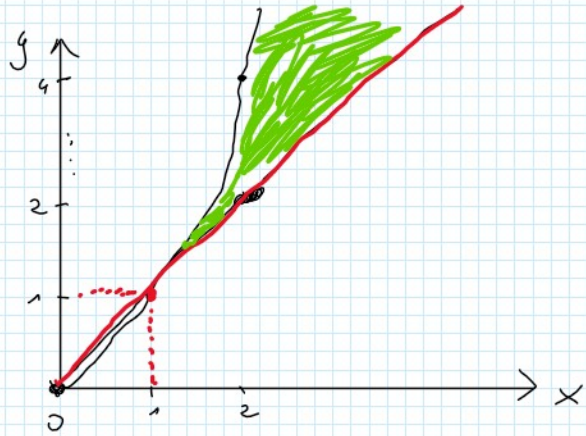
ORA:

$$P(y \le x^2) =$$

$$1 < x < y$$

$y = x^2$

x	y
1	1
2	4
3	9
$\frac{1}{2}$	$\frac{1}{4}$



$y = x$

x	y
0	0
1	1
2	2

$$\int_1^{+\infty} \int_x^{x^2} f(x,y)\, dx\, dy$$

$$= \int_x^{x^2} \frac{15}{x^3 y^4}\, dy = \frac{15}{x^3} \int y^{-4} dy = \frac{15}{x^3} \cdot \frac{y^{-3}}{-3} = \frac{-5}{x^3} \cdot \left[\frac{1}{y^3}\right]_x^{x^2}$$

$$= \frac{-5}{x^3}\left[\frac{1}{x^6} - \frac{1}{x^3}\right] = \left[-\frac{5}{x^9} + \frac{5}{x^6}\right]$$

$$= \int_1^{+\infty} \left(-\frac{5}{x^9} + \frac{5}{x^6}\right) dx = -5 \int x^{-9} + 5 \int x^{-6} dx = -5\, \frac{x^{-8}}{-8} + 5\, \frac{x^{-5}}{-5}$$

$$= \frac{5}{8} \cdot \left[\frac{1}{x^8}\right]_1^{+\infty} - \left[\frac{1}{x^5}\right]_1^{+\infty} = \frac{5}{8}[0 - 1] - [0 - 1]$$

$$-\frac{5}{8} + \frac{1}{1} = \frac{-5+8}{8} = \frac{3}{8}$$

3) $U = \log x \qquad V = \frac{x}{y}$

$$\phi = \begin{cases} U = \log x \\ V = \frac{x}{y} \end{cases} \qquad \phi^{-1} = \begin{cases} x = e^U \\ y = \frac{e^U}{V} \end{cases} \qquad J\phi^{-1} = \begin{bmatrix} e^U & 0 \\ \frac{e^U}{V} & -\frac{e^U}{V^2} \end{bmatrix}$$

$$|\Delta| = \frac{e^{2U}}{V^2}$$

$$f_{U,V}(u,v) = f_{x,y}\left(e^u, \frac{e^u}{v}\right) \cdot \frac{e^{2u}}{v^2}$$

INIZIO $\boxed{1 < x < y}$ $x \in [1, +\infty]$, $y \in [x, +\infty]$

ORA, studiamo le funzioni indicatrici:

$1 < x < +\infty$

$\downarrow$

$1 < e^u < +\infty$

$0 < U < +\infty$

$x < y < +\infty$

$\downarrow$

→ intervalli come nella cov(x,y)

$e^u < \frac{e^u}{v} < +\infty$

RACCOLGO e^u

$1 < \frac{1}{v} < +\infty$

INVERTO

$1 < V < 0$

$0 < V \leq 1$

P.S. = ho trovato U e V, non la U solo, xkè non devo fare l'integrale

$$f(u,v) = \underline{15}\ e^{-3u} \cdot e^{-4u} \cdot v^4 \cdot \frac{e^{2u}}{v^2} \cdot 1_{(0,+\infty)}(u) \cdot 1_{(0,1]}(V)$$

$$f(u,v) = 15\, e^{-7u} \cdot v^4 \cdot \frac{e^{2u}}{v^2}$$

$$f(u,v) = 15\, e^{-5u} \cdot v^2 \cdot 1_{(0,+\infty)}(u) \cdot 1_{(0,1]}(v)$$

$f_U(u) = 5e^{-5u} \cdot 1_{(0,+\infty)}(u)$

$f_V(v) = 3v^2 \cdot 1_{(0,1]}(v)$

U e V sono indipendenti!