2b.22
venerdì 10 dicembre 2021 09:53

ESERCIZIO 2b.22

$$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & \text{se} \quad 0 < x < y \\ 0 & \text{altrimenti} \end{cases}$$

gamma

1) $f_X(x)$, $f_Y(y)$
2) X e Y sono indipendenti?
3) $E(X+Y)$
4) densità di $X+Y$
5) $P(X \leq 3, Y \leq 2)$

SVLG

1) $$f_X(x) = \int_x^{+\infty} \lambda^2 e^{-\lambda y}\, dy = \lambda^2 \int_x^{+\infty} e^{-\lambda y}\, dy = -\lambda^2 \cdot \int_x^{+\infty} -\frac{\lambda}{\lambda} e^{-\lambda y}\, dy$$

$$= -\lambda \int -\lambda e^{-\lambda y}\, dy = -\lambda \left[e^{-\lambda y} \right]_x^{+\infty} = -\lambda \left[0 - e^{-\lambda x} \right]$$

$$= \boxed{\lambda e^{-\lambda x}}$$

$$f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y}\, dx = \lambda^2 e^{-\lambda y} \int 1 \cdot dx = \lambda^2 e^{-\lambda y} [x]_0^y$$

$$= \lambda^2 e^{-\lambda y} [y - 0] = \boxed{\lambda^2 y\, e^{-\lambda y}} \qquad \boxed{\Gamma(2, \lambda)}$$

2) $$\left[\lambda e^{-\lambda x} \cdot \lambda^2 y e^{-\lambda y}\right] = \lambda^3 y\, e^{-\lambda x - \lambda y}$$ NON sono indipendenti

3) $$E(X+Y) = E(X) + E(Y)$$

$$E(Y) = \int_0^{+\infty} \left(\lambda^2 y\, e^{-\lambda y}\right) \cdot y \cdot dy = \int_0^{+\infty} \lambda^2 y^2 e^{-\lambda y}\, dy$$

$$= \lambda^2 \int_0^{+\infty} y^2 e^{-\lambda y}\, dy =$$

$$\boxed{\Gamma(\underset{\alpha}{3}, \underset{\lambda}{\lambda}) = \frac{\lambda^3}{\Gamma(3)} \cdot y^2 \cdot e^{-\lambda y}}$$

$$\Gamma(3,\lambda) = \frac{\lambda}{\Gamma(3)} \cdot y \cdot e$$
$$\underset{\alpha}{3} \quad \underset{\lambda}{\lambda}$$

$$= \frac{\Gamma(3)}{\lambda^3} \cdot \lambda^2 \int_0^{+\infty} \underbrace{\frac{\lambda^3}{\Gamma(3)} \cdot y^2 \cdot e^{-\lambda y}}_{\text{gamma}(3,\lambda) = 1} = \frac{\Gamma(3)}{\lambda} = \boxed{\frac{2}{\lambda}}$$

$$E(x) + E(y) = \frac{1}{\lambda} + \frac{2}{\lambda} = \boxed{\frac{3}{\lambda}}$$

4) $Z = X + Y$

$$\phi = \begin{cases} X = X \\ Z = X + Y \end{cases} \qquad \phi^{-1} = \begin{cases} X = X \\ Y = Z - X \end{cases} \qquad J\phi^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad |\Delta| = 1$$

$$f_{X,Z}(x,z) = f_{X,Y}(x, z-x) \cdot 1$$

funz. indicatrici

$0 < x < y$ $\qquad$ $0 < x < z - x$

$x > 0$ $\qquad$ $x < z - x$

$x < y$ $\qquad$ $2x < z$

$x < \frac{z}{2}$

$$f_Z(z) = \int_0^{\frac{z}{2}} \lambda^2 e^{-\lambda(z-x)} dx = \int_0^{\frac{z}{2}} \lambda^2 e^{-\lambda z} \cdot e^{\lambda x} dx$$

$$= \lambda e^{-\lambda z} \int_0^{\frac{z}{2}} \lambda e^{\lambda x} dx$$

$$\frac{z}{2} - \frac{z}{1}$$

$$\frac{z - 2z}{2} =$$

$$= \lambda e^{-\lambda z} \cdot \left[e^{\lambda x}\right]_0^{\frac{z}{2}} = \lambda e^{-\lambda z}\left[e^{\lambda \frac{z}{2}} - 1\right]$$

$$= -\lambda e^{-\lambda z} + \lambda e^{-\frac{z}{2}\lambda} = \lambda\left(e^{-\frac{z}{2}\lambda} - e^{-\lambda z}\right)$$

5) $$P(X \leq 3, Y \leq 2) = \int_0^y \int_0^2 \lambda^2 e^{-\lambda y}\, dx\, dy$$

$0 < x < y$

$0 < y \leq 2$

$$= \int_0^2 \lambda^2 e^{-\lambda y}\, dy = -\lambda \int_0^2 -\lambda\, e^{-\lambda y}\, dy$$

$$-\lambda\left[e^{-\lambda y}\right]_0^2 = -\lambda\left[e^{-2\lambda} - 1\right] = -\lambda e^{-2\lambda} + \lambda$$

$$\int_0^y \left(-\lambda e^{-2\lambda} + \lambda\right) dx = \int \left(-\lambda e^{-2\lambda}\right) dx + \int \lambda\, dx$$

$$= -\lambda e^{-2\lambda}\,[x]_0^y + \lambda\,[x]_0^y$$

$$= -\lambda e^{-2\lambda} y + \lambda y$$

lui mette prima $\int_0^2 dy \int_0^y dx\; e^{-\lambda y}$??