ESERCIZIO 2b.23

$$f(x,y) = \begin{cases} 2(x+y) & \text{se } 0<x<y<1 \\ 0 & \text{altrimenti} \end{cases}$$

1) $f_X(x)$ e $f_Y(y)$, e calcolare la densità condizionata di X dato $y = \frac{1}{2}$
2) $cov(X,Y)$, $P(Y+X) \geq \frac{1}{2}$
3) $Z = \frac{X}{Y}$ ed il quantile di ordine $\frac{1}{2}$ di Z!

SVLG

1) $$f_X(x) = \int_x^1 2(x+y)\,dy = \int_x^1 2x+2y\,dy = \int 2x + \int 2y\,dy$$

$$= 2x[y]_x^1 + [y^2]_x^1 = 2x[1-x] + [1-x^2]$$

$$2x - 2x^2 + 1 - x^2 = \boxed{-3x^2+2x+1}$$

$$f_Y(y) = \int_0^y 2(x+y)\,dx = \int_0^y (2x+2y)\,dx = [x^2]_0^y + 2y[x]_0^y$$

$$= y^2 + 2y[y] = y^2 + 2y^2 = \boxed{3y^2}$$

like 2b.20

$$f_{X|Y}(x|y=\tfrac{1}{2}) = \frac{f(x,\frac{1}{2})}{f_Y(\frac{1}{2})} = \frac{2(x+\frac{1}{2})}{3\cdot\left(\frac{1}{2}\right)^2} \cdot 1_{[0,\frac{1}{2}]}(x)$$

$$= \frac{2x+1}{\frac{3}{4}} = \frac{4}{3}(2x+1) \cdot 1_{(0,\frac{1}{2})}(x)$$

2) verifichiamo se X e Y sono indipendenti:

$(-3x^2+2x+1)\cdot(3y^2) =$ NO, NON SONO INDIP.

$$cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \left(\frac{5}{12}\cdot\frac{3}{4}\right) = \frac{1}{3} - \frac{5}{16} = \frac{16-15}{48} = \boxed{\frac{1}{48}}$$

$$E(XY) = \int_0^1\int_x^1 f(x,y)\,x\cdot y\,dx\,dy$$

$$= \int_x^1 2(x+y)\, y\, dy = \int (2x+2y)\, y = \int 2xy + 2y^2 = \left[xy^2 + \frac{2}{3}y^3 \right]_x$$

$$= x + \frac{2}{3} - \left(x^3 + \frac{2}{3}x^3\right) = x + \frac{2}{3} - x^3 - \frac{2}{3}x^3$$

$$\int_0^1 \left(x + \frac{2}{3} - x^3 - \frac{2}{3}x^3\right) x\, dx = \int_0^1 x^2 + \frac{2}{3}x - x^4 - \frac{2}{3}x^4$$

$$= \frac{x^3}{3} + \frac{2}{3}\frac{x^2}{2} - \frac{x^5}{5} - \frac{2}{3}\frac{x^5}{5} = \left[\frac{x^3}{3} + \frac{x^2}{3} - \frac{x^5}{5} - \frac{2}{15}x^5\right]_0^1 = \frac{1}{3} + \frac{1}{3} - \frac{1}{5} - \frac{2}{15}$$

$$= \frac{5+5-3-2}{15} = \frac{5}{15} = \boxed{\frac{1}{3}}$$

$$E(x) = \int_0^1 \left(-3x^2 + 2x + 1\right) \cdot x \cdot dx =$$

$$= \int_0^1 \left(-3x^3 + 2x^2 + x\right) dx = \left[-\frac{3x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right]_0^1$$

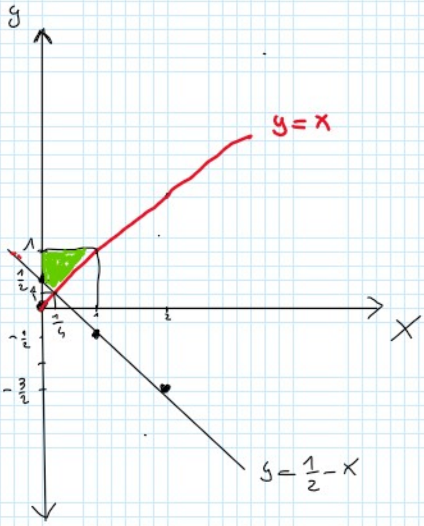
$$= -\frac{3}{4} + \frac{2}{3} + \frac{1}{2} = \frac{-9+8+6}{12} = \boxed{\frac{5}{12}}$$

$$E(y) = \int_0^1 (3y^2)\, y\, dy = \int_0^1 3y^3\, dy = \frac{3}{4}\left[y^4\right]_0^1 = \boxed{\frac{3}{4}}$$

- $P\left(x + y \geq \frac{1}{2}\right) = P\left(y \geq \frac{1}{2} - x\right)$

$y = \frac{1}{2} - x$

x	y
0	$\frac{1}{2}$ ✓
1	$-\frac{1}{2}$ ✗
$\frac{1}{2}$	0
$\frac{1}{4}$	$\frac{1}{4}$
2	$-\frac{3}{2}$



$P(y > x) \cap P\left(y \geq \frac{1}{2} - x\right)$

$$\int_0^{\frac{1}{4}} \int_{\frac{1}{2}-x}^{1} f(x,y)\, dx\, dy$$

$+$

$$\int_{\frac{1}{4}}^{1} \int_{x}^{1} f(x,y)\, dx\, dy$$

$$= \boxed{\int_0^{\frac14} \int_{\frac12 - x}^{1} f(x,y)\,dx\,dy + \int_{\frac14}^{1} \int_{x}^{1} f(x,y)\,dx\,dy}$$

RISOLVIAMO 1^a PARTE:

$$\int_{\frac12 - x}^{1} 2(x+y)\,dy = \int_{\frac12 - x}^{1} (2x+2y)\,dy = \left[2xy + y^2\right]_{y=\frac12 - x}^{y=1}$$

$$= \left[2x + 1 - \left(2x\left(\tfrac12 - x\right) + \left(\tfrac12 - x\right)^2\right)\right]$$

$$= \left[2x + 1 - \left(x - 2x^2 + \tfrac14 + x^2 - x\right)\right] = \left[2x + 1 - \cancel{x} + 2x^2 - \tfrac14 - x^2 \cancel{+x}\right]$$

$$= \left[2x + \tfrac34 + x^2\right] =$$

$$\int_0^{\frac14} \left(2x + \tfrac34 + x^2\right) dx = \left[x^2 + \tfrac34 x + \tfrac{x^3}{3}\right]_0^{\frac14} = \frac{1}{16} + \frac34 \cdot \frac14 + \frac{1}{64} \cdot \frac13$$

$$= \frac{1}{16} + \frac{3}{16} + \frac{1}{192} = \frac{12 + 36 + 1}{192} = \boxed{\frac{49}{192}}$$

sommiamo la seconda parte:

$$\int_{\frac14}^{1} \left(-3x^2 + 2x + 1\right) dx = \left[-x^3 + x^2 + x\right]_{\frac14}^{1} = \cancel{-1} + \cancel{1} + 1 - \left(-\tfrac{1}{64} + \tfrac{1}{16} + \tfrac14\right)$$

$$1 + \frac{1}{64} - \frac{1}{16} - \frac14 = \frac{64 + 1 - 4 - 16}{64} = \boxed{\frac{45}{64}}$$

$$\frac{45}{64} + \frac{49}{192} = \frac{135 + 49}{192} = \frac{\cancel{184}^{\cancel{92}\ \cancel{46}\ 23}}{\cancel{192}_{\cancel{96}\ \cancel{48}\ 24}} = \boxed{\frac{23}{24}}$$

3) $\boxed{Z = \frac{X}{Y}}$ calcolare la densità $z \in [0,1]$

$\boxed{\text{come ex 2016 TOPTI}}$; $\frac{y}{x} \le z$ $\quad y \le xz$ $\quad 0 < y < x < 2$

$$F_Z(z) = P\left(\frac{X}{Y} \le z\right) = P\left(Y \ge \frac{X}{Z}\right)$$

$$F_Z(z) = P\left(\frac{X}{Y} \le z\right) = P\left(Y \ge \frac{X}{Z}\right)$$

ORA, STUDIAMO le funzioni INDICATRICI.

$0 < \frac{x}{z} < 1$ $\quad$ $0 < x < y < 1$

$\downarrow$

$0 < x < z$ $\quad$ $\frac{x}{z} < y < 1$

$$\int_0^z dx \int_{\frac{x}{z}}^1 2(x+y)\, dy = \boxed{\frac{1}{3}z^2 + \frac{2}{3}z}$$

ora, derivando si trova la densità:

$\boxed{f_Z(z) = \frac{2}{3}(z+1)}$ se $x \in [0,1]$

MAT. IACOBIANA (STESSO PROCEDIMENTO)

(NON STUDIO I CASI PERCHÈ È UNA DIVISIONE)

$\boxed{z = \frac{x}{y}}$

$$\phi = \begin{cases} x = x \\ z = \frac{x}{y} \end{cases} \qquad \phi^{-1} = \begin{cases} x = x \\ y = \frac{x}{z} \end{cases} \qquad J\phi^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{1}{z} & -\frac{x}{z^2} \end{bmatrix} \quad |\Delta| = \frac{x}{z^2}$$

$$f_{X,Z}(x,z) = f_{X,Y}\left(x, \frac{x}{z}\right) \cdot \frac{x}{z^2}$$

$0 < x < y < 1$

$$\int_0^z 2\left(x + \frac{x}{z}\right) \cdot \frac{x}{z^2}\, dx$$

$\boxed{0 < x < 1}$

$$\int_0^z \left(2x + \frac{2x}{z}\right) \cdot \frac{x}{z^2}\, dx$$

$0 < y < 1$

$0 < \frac{x}{z} < 1$

$$\int_0^z \frac{2x^2}{z^2} + \frac{2x^2}{z^3}\, dx$$

$\boxed{0 < x < z}$

$$\int_0^z \frac{2x^2}{z^2} + \int \frac{2x^2}{z^3}\, dx$$

$$\frac{2}{z^2}\left[\frac{x^3}{3}\right]_0^z + \frac{2}{z^3}\left[\frac{x^3}{3}\right]_0^z = \frac{2z^{\cancel{3}}}{\cancel{z^2}3} + \frac{2\cancel{z^3}}{\cancel{z^3}3} = \boxed{\frac{2}{3}z + \frac{2}{3}} \text{ se } x \in (0,1)$$

ORA, TROVIAMO IL quantile di ORDINE $\frac{1}{2}$ di Z:

· PRENDO la densità e la derivo: (RISPETTO A Z):

$$\int \frac{2}{3}z + \frac{2}{3}\, dz = \boxed{\frac{z^2}{3} + \frac{2}{3}z}$$

$\hookrightarrow$ $\frac{z^2}{3} + \frac{2}{3}z = \frac{1}{2}$

$$\frac{z^2}{3} + \frac{2}{3}z = \frac{1}{2}$$

$$z^2 + 2z = \frac{3}{2}$$

$$2z^2 + 4z = 3$$

$$2z^2 + 4z - 3 = 0$$

$a = 2$

$b = 4$

$c = -3$

$$\Delta = 16 - 4(2)(-3) = 16 + 24$$

$$x_{1,2} = \frac{-4 \pm \sqrt{40}}{4}$$

$$\frac{-4 - \sqrt{40}}{4} = \frac{-4 - \sqrt{2^3 \cdot 5}}{4} = \frac{-4 - 2\sqrt{10}}{4} = \frac{-\cancel{2}(2 + \sqrt{10})}{\cancel{4}\,2} = \frac{-\sqrt{10} - 2}{2} \quad \times$$

$$\frac{-4 + \sqrt{40}}{4} = \frac{-4 + \sqrt{2^3 \cdot 5}}{4} = \frac{-4 + 2\sqrt{10}}{4} = \frac{\cancel{2}(-2 + \sqrt{10})}{\cancel{4}\,2} = \frac{\sqrt{10} - 2}{2} \quad ✓$$