ESERCIZIO 2b.29

si consideri:

$$f(x,y) = \begin{cases} 15x^2y & 0<x<y<1 \\ 0 & \text{ALTRIMENTI} \end{cases}$$

1) TROVARE la $f_X(x)$ e la $f_Y(y)$, calcolare la $\mathrm{cov}(x,y)$
2) $P(Y-4X<0)$
3) densità condizionale di X dato $Y=\frac{1}{2}$, $E(X|Y=\frac{1}{2})$

SVLG

1)
$$f_X(x) = \int_x^1 15x^2y\,dy = 15x^2\left[\frac{y^2}{2}\right]_x^1 = 15x^2\left[\frac{1}{2}-\frac{x^2}{2}\right]$$

$$= \frac{15}{2}x^2 - \frac{15}{2}x^4$$

$$f_Y(y) = \int_0^y 15x^2y\,dx = 15y\int_0^y x^2\,dx = 15y\left[\frac{x^3}{3}\right]_0^y = \left[\frac{15y\cdot y^3}{3}\right]$$

$$= \boxed{5y^4}$$

ora calcoliamo la $\mathrm{cov}(x,y)$.

$\mathrm{cov}(x,y) = E(xy) - E(x)\cdot E(y)$

$$E(xy) = \int_0^1\int_x^1 15x^2y\,dx\,dy \cdot x\cdot y = \int_x 15x^2y\,dy\cdot y = \int_x (15x^2y^2)\,dy = 15x^2\left[\frac{y^3}{3}\right]_x^1$$

(prendiamo gli estremi come valori da mettere nell'integrale)

$$= 15x^2\left[\frac{1}{3}-\frac{x^3}{3}\right] = \frac{15x^2}{3} - \frac{15x^5}{3} = \left[5x^2-5x^5\right]$$

$$\int_0^1 (5x^2-5x^5)\cdot x\,dx = \int_0^1 5x^3-5x^6\,dx = \left[\frac{5x^4}{4}-\frac{5x^7}{7}\right]_0^1 = \frac{5}{4}-\frac{5}{7} = \frac{35-20}{28} = \left[\frac{15}{28}\right]$$

$$\int_0 \dots dx = \int_0 5x - 5x\, dx = \left[\frac{5x}{4} - \frac{5x}{7}\right]_0 = \frac{5}{4} - \frac{5}{7} = \frac{35-20}{28} = \boxed{\frac{15}{28}}$$

ORA effettuiamo il calcolo delle medie:

$$E(x) = \int_0^1 x\left(\frac{15}{2}x^2 - \frac{15}{2}x^4\right)dx = \int_0^1 \frac{15x^3}{2} - \frac{15x^5}{2} = \left[\frac{15x^4}{8} - \frac{15x^6}{12}\right]_0^1 = \frac{15}{8} - \frac{15}{12} = \frac{45-30}{24} =$$

$$= \frac{15}{24} = \boxed{\frac{5}{8}}$$

$$E(y) = \int_0^1 y \cdot 5y^4\, dy = \int_0^1 5y^5\, dy = \left[\frac{5y^6}{6}\right]_0^1 = \boxed{\frac{5}{6}}$$

Ra posso calcolare la cov(x,y)!

$$\text{cov}(x,y) = E(xy) - E(x)\cdot E(y) = \frac{15}{28} - \left(\frac{5}{8}\cdot\frac{5}{6}\right) = \frac{15}{28} - \frac{25}{48} = \frac{720-700}{1344}$$

$$= \frac{20}{1344} = \frac{10}{672} = \boxed{\frac{5}{336}}$$

2) $P(y - 4x < 0) = P(y < 4x)$:

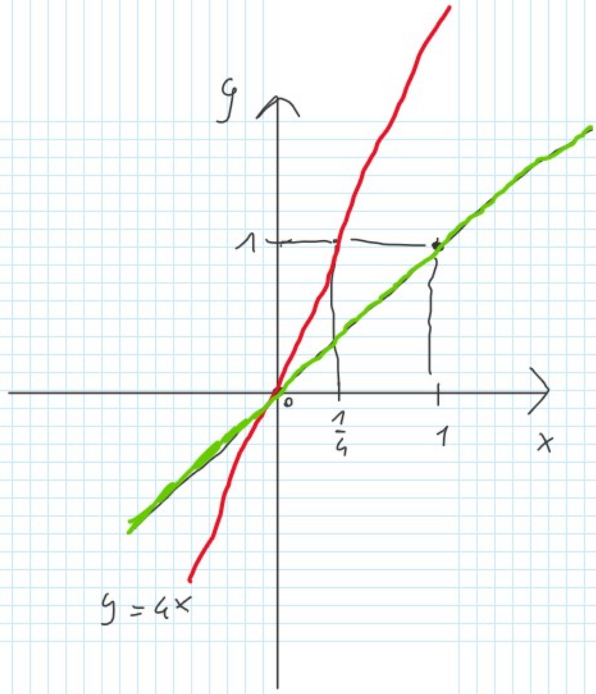
$y = 4x$

x	y
0	0
1	4
$\frac{1}{4}$	1

punto di rottura

$0 < x < y$

$x < y < 1$



$y = x$

x	y
0	0
1	1
$\frac{1}{4}$	$\frac{1}{4}$

$$= \boxed{\int_0^{\frac{1}{4}} \int_x^{4x} f(x,y)\,dx\,dy + \int_{\frac{1}{4}}^{1} \int_x^{1} f(x,y)\,dx\,dy}$$

quando viene assegnato un intervallo così complesso, bisogna effettuare l'intersezione con l'intervallo iniziale e la probabilità attuale!

- Risolvo la prima parte!

$$= \int_x^{4x} 15x^2 y \, dy = 15x^2 \left[\frac{y^2}{2}\right]_x^{4x} = 15x^2 \frac{16x^2}{2} = 15x^2 \cdot 8x^2 = \left[120x^4\right]$$

$$\int_0^{\frac{1}{4}} 120x^4 \, dx = \left[120 \frac{x^5}{5}\right] = 24\left[x^5\right]_0^{\frac{1}{4}} = 24 \cdot \frac{1}{1024} = \frac{\cancel{24}}{\cancel{1024}} = \boxed{\frac{3}{128}}$$

$12 \quad 6 \quad 3$

$512 \quad 256 \quad 128$

• Risolvo la seconda parte:

$$\int_x^1 (15x^2 y) \, dy = 15x^2 \left[\frac{y^2}{2}\right]_x^1 = 15x^2 \left[\frac{1}{2} - \frac{x^2}{2}\right] = \left[\frac{15x^2}{2} - \frac{15x^4}{2}\right]$$

$$\int_{\frac{1}{4}}^1 \frac{15x^2}{2} - \frac{15x^4}{2} \, dx = \frac{15x^3}{6} - \frac{\overset{3}{\cancel{15}}x^5}{\underset{2}{\cancel{10}}} = \left[\frac{15}{6}x^3 - \frac{3}{2}x^5\right]_{\frac{1}{4}}^1$$

$$= \left[\frac{15}{6} - \frac{3}{2} - \left(\frac{15}{384} - \frac{3}{2} \cdot \frac{1}{1024}\right)\right] = \left[\frac{15}{6} - \frac{3}{2} - \frac{15}{384} + \frac{3}{2048}\right]$$

$$= \frac{5}{2} - \frac{3}{2} - \frac{5}{128} + \frac{3}{2048} = \frac{5120 - 3072 - 80 + 3}{2048} = \left[\frac{1971}{2048}\right]$$

ORA sommiamo i valori trovati:

$$\frac{3}{128} + \frac{1971}{2048} = \frac{48 + 1971}{2048} = \frac{2019}{2048} = \left[\frac{63}{64}\right] \checkmark$$

3) $$f_{X|Y}\left(x \mid y = \tfrac{1}{2}\right) = \frac{f(x, \frac{1}{2})}{f_Y(\frac{1}{2})} = \frac{15x^2 \cdot \frac{1}{2}}{5\left(\frac{1}{2}\right)^4} = \frac{\frac{15}{2}x^2}{\frac{5}{16}} = \frac{\overset{3}{\cancel{15}}}{\underset{1}{\cancel{2}}} \cdot \frac{\overset{8}{\cancel{16}}}{\underset{1}{\cancel{5}}} = 24$$

(Porta sempre le funzioni indicatrici)

$$\cdot \, 1_{[0,1]}(x)$$

$$= \boxed{24x^2 \cdot 1_{\left[0,\frac{1}{2}\right]}(x)}$$

ora mi calcolo la media $E\left(x \mid y=\frac{1}{2}\right) = ?$

$$E\left(x \mid y=\frac{1}{2}\right) = \int_0^{\frac{1}{2}} x^2 24 \cdot x \, dx = \int_0^{\frac{1}{2}} 24x^3 = \frac{24x^4}{4} = \left[6x^4\right]_0^{\frac{1}{2}} = \frac{\cancel{6}^{3}}{\cancel{16}\,8} = \boxed{\frac{3}{8}}$$