

## Esercizio 2b.33

$$f(x,y) = \begin{cases} x^a + y^a & \text{se } 0 < x < 1, \quad 0 < y < 1 \\ 0 & \text{altrimenti} \end{cases}$$

1) determinare  $a > 0$  affinché  $f(x,y)$  sia la densità congiunta.

per il valore di  $a$  trovato, trovare  $f_X(x)$  e  $f_Y(y)$  e dire se sono strettamente indipendenti.

2) media, varianza di  $X$  e  $Y$  e  $\text{cov}(X,Y)$

3) posto  $Z = X+Y$ , trovare la F.D.D. della densità di  $Z$ .

4)  $P(Y \leq \alpha X^2)$  e  $\beta > 0$  :  $P(Z \leq \beta) = \frac{1}{2\beta}$

SVLG

$$1) \int_0^1 \int_0^1 x^a + y^a \, dx \, dy = 1$$

$$\int x^a + y^a \, dy = \int x^a + \int y^a \, dy = x^a \cdot [y]_0^1 + \left[ \frac{y^{a+1}}{a+1} \right]_0^1 \\ = x^a + \frac{1}{a+1} \leq x^a + \frac{1}{a+1}$$

$$\int_0^1 \left( x^a + \frac{1}{a+1} \right) dx = \int x^a + \int \frac{1}{a+1} dx = \left[ \frac{x^{a+1}}{a+1} + \frac{x}{a+1} \right]_{x=0}^{x=1}$$

$$\frac{1}{a+1} + \frac{1}{a+1} = 1$$

$$\frac{2}{a+1} = 1$$

$$2 = a + 1$$

$$a = 1$$

$$f_X(x) = \int_0^1 (x+y) \, dy = x[y]_0^1 + \left[ \frac{y^2}{2} \right]_0^1 = \left[ x + \frac{1}{2} \right]$$

$$f_X(x) = \int_0^1 (x+y) dy = x \left[ y + \frac{y^2}{2} \right]_0^1 = \left[ x + \frac{1}{2} \right]$$

$$f_Y(y) = \int_0^1 (x+y) dx = \left[ \frac{x^2}{2} + xy \right]_0^1 = \left[ \frac{1}{2} + y \right]$$

Verifichiamo se sono indipendenti:

$$(x + \frac{1}{2})(\frac{1}{2} + y) = \frac{1}{2}x + xy + \frac{1}{4} + \frac{1}{2}y \quad \text{No, non sono indip.}$$

$$2) E(X) = \int_0^1 (x + \frac{1}{2}) \cdot x \cdot dx = \int_0^1 x^2 + \frac{1}{2}x = \left[ \frac{x^3}{3} + \frac{x^2}{6} \right]_0^1 = \frac{1}{3} + \frac{1}{6} = \left[ \frac{7}{12} \right]$$

$$E(Y) = \int_0^1 (y + \frac{1}{2}) \cdot y \cdot dy = \left[ \frac{y^2}{2} \right]_0^1 = \left[ \frac{7}{12} \right]$$

$$E(X^2) = \int_0^1 (x + \frac{1}{2}) \cdot x^2 \cdot dx = \int_0^1 x^3 + \frac{1}{2}x^2 = \left[ \frac{x^4}{4} + \frac{x^3}{6} \right]_0^1 = \frac{1}{4} + \frac{1}{6} = \frac{6+4}{24} = \frac{10}{24} = \left[ \frac{5}{12} \right]$$

$$E(Y^2) = \int_0^1 (y + \frac{1}{2}) \cdot y^2 = \left[ \frac{5}{12} \right]$$

$$\text{Var}(X) = E(X^2) - E^2(X) = \frac{5}{12} - \frac{49}{144} = \frac{60-49}{144} = \left[ \frac{11}{144} \right]$$

$$\text{Var}(Y) = \left[ \frac{11}{144} \right]$$

$$\text{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = \frac{1}{3} - \frac{49}{144} = \frac{48-49}{144} = \left[ -\frac{1}{144} \right]$$

$$\boxed{\rightarrow E(XY) = \int_0^1 \int_0^1 (x+y) \cdot x \cdot y \cdot dx \cdot dy}$$

$$\int_0^1 (x+y) y dy = \int x y + y^2 = \left[ \frac{xy^2}{2} + \frac{y^3}{3} \right]_0^1 \stackrel{y=1}{=} \left[ \frac{x}{2} + \frac{1}{3} \right]$$

$$\int_0^1 \left( \frac{x}{2} + \frac{1}{3} \right) \cdot x \cdot dx = \int \frac{x^2}{2} + \frac{x}{3} = \left[ \frac{x^3}{6} + \frac{x^2}{2} \right]_0^1 = \left[ \frac{1}{3} \right]$$

$$\int_0^1 \left( \frac{x}{2} + \frac{1}{3} \right) \cdot x \cdot dx = \int_0^1 \frac{x^2}{2} + \frac{x}{3} = \left[ \frac{x^3}{6} + \frac{x^2}{6} \right]_0^1 = \boxed{\frac{1}{3}}$$

3)

$$Z = X + Y$$

$x \in [0, 1]$   
 $y \in [0, 1]$

$$\Phi = \begin{cases} X = x \\ Z = x + y \end{cases} \quad \bar{\Phi} = \begin{cases} X = x \\ y = z - x \end{cases} \quad J\bar{\Phi} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad |D| = 1$$

$$f_{X, Z}(x, z) = f_{x, y}(x, z - x) \cdot 1$$

$$f_Z(z) = \int_{\mathbb{R}} \cancel{X + Z - X} \cdot 1 = \int_{\mathbb{R}} z dx$$

$$0 < x < 1$$

$$0 < y < 1$$

$$0 < z - x < 1$$

$$z - x > 0$$

$$-x > -z$$

$$x < z$$

$$z - x < 1$$

$$-x < 1 - z$$

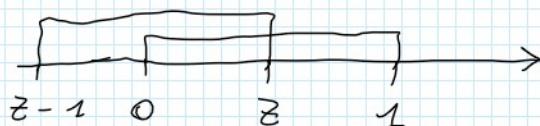
$$x > z - 1$$

$$x \in (z - 1, z)$$

- $x \in (0, 1) \cap x \in (z - 1, z) \text{ con } z \in (0, 2)$

- se  $z < 0 = 0$

- se  $z \in (0, 1)$ :



$$\int_0^z z dx = [zx]_0^z = z^2$$

• se  $z \in (1, 2)$ :



$$\int_{z-1}^1 z dx = z \left[ x \right]_{z-1}^1 = z(-z+2) = -z^2 + 2z = z(z-2)$$

• se  $z > 2 = 0$

Ricapitolando:

$$f_z(z) = \begin{cases} 0 & \text{se } z < 0 \\ z^2 & \text{se } z \in (0, 1) \\ z(2-z) & \text{se } z \in (1, 2) \\ 0 & \text{se } z > 2 \end{cases}$$

Ora serve la f.d.d.

Bisogna integrare da  $\int_{-\infty}^z f_z(z) dz = \int_0^z f_z(z) dz$

$$\int_0^z z^2 dz = \left[ \frac{z^3}{3} \right]$$

$$\int_0^z (2z - z^2) dz = \left[ z^2 - \frac{z^3}{3} \right]$$

$$F_z(z) = \begin{cases} 0 & \text{se } z < 0 \\ \frac{z^3}{3} & \text{se } z \in (0, 1) \\ -\frac{z^3}{3} + z^2 & \text{se } z \in (1, 2) \\ 0 & \text{se } z > 2 \end{cases}$$

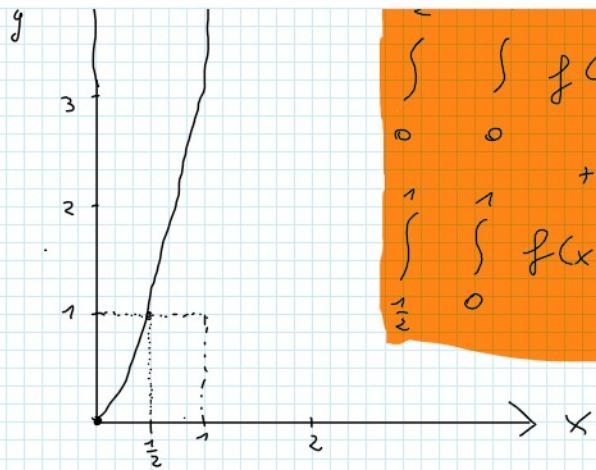
$$\frac{1}{2} q \times z \int \int f(x, y) dx dy$$

$$4) P(G \leq 4x^2) =$$

$$x \in (0,1) \quad | \quad g = 4x^2$$

$$y \in (0,1)$$

$$\begin{array}{c|c} x & g \\ \hline 0 & 0 \\ 1 & 4 \\ \frac{1}{2} & 1 \end{array}$$



$$\int_0^1 \int_0^{4x^2} f(x,y) dx dy + \int_{\frac{1}{2}}^1 \int_0^1 f(x,y) dx dy$$

$$\int_0^{\frac{1}{2}} \int_0^{4x^2} (x+y) dx dy + \int_{\frac{1}{2}}^1 \int_0^1 (x+y) dx dy = \boxed{\frac{59}{80}}$$

$$\text{OBA } P(Z \leq \beta) = \frac{1}{24}$$

$$\frac{z^3}{3} = \frac{1}{24}$$

$$z^3 = \frac{3}{24} 8$$

$$\boxed{z = \frac{1}{2}}$$





