0.0.1 Differentiation in many variables

A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is (total) differentiable in x_0 if it exists a linear map $A : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = f(x_0) + A(x - x_0) + R(x_0, x)$$

Where $\lim_{x \to x_0} \frac{R(x_0,x)}{|x-x_0|} = 0$. In this case A is called the differential of f at x_0 and it's denoted as $(df)(x_0)$.

Let $(A_1, A_2, ..., A_n)$ be a matrix representation of the linear map $A : \mathbb{R}^n \to \mathbb{R}$ (wrt to the standard Basis). Then f differentiable at x_0 means:

$$f(x) = f(x_0) + A_1(x^1 - x_0^1) + A_2(x^2 - x_0^2) + ... + A_n(x^n - x_0^n) + R(x, x_0)$$

$P(x, x_0) = f(x_0) + \begin{bmatrix} A_1 & ... & A_n \end{bmatrix} \begin{bmatrix} x^1 - x_0^1 \\ ... \\ x^n - x_0^n \end{bmatrix}$ is the equation of the tangent plane at the point $f(x_0)$ on the surface formed by the graph of f.

Fact if $f : \Omega \to \mathbb{R}$ is differentiable in $x_0 \in \Omega$ then the partial derivative exist and the differential $df(x_0)$ has the matrix representation

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0) & ... & \frac{\partial f}{\partial x^n}(x_0) \end{pmatrix} = \nabla f$$

the gradient of f.

Fact f differentiable in $x_0 \Rightarrow f$ is continuous in x_0.

Fact If all partial derivatives of f are continuous then f is differentiable.

Using these last two facts and the definition of differentiability one can study if a given function is differentiable or not.

Differentiation rules

Let $f, g : \Omega \to \mathbb{R}$ be differentiable in x_0. Then:

1. $d(f \pm g)(x_0) = df(x_0) \pm dg(x_0)$
2. $d(fg)(x_0) = g(x_0)df(x_0) + f(x_0)dg(x_0)$
3. $d(f/g)(x_0) = \frac{g(x_0)df(x_0) - f(x_0)dg(x_0)}{(g(x_0))^2}$
4. Let $h : \mathbb{R} \to \mathbb{R}$ be differentiable in $g(x_0)$, then

$$d(hog)(x_0) = h'(g(x_0))dg(x_0)$$

5. Let $H : I \subset \mathbb{R} \to \Omega \subset \mathbb{R}^n$ be differentiable in $t_0 \in \mathbb{R}$ and f differentiable in $H(t_0)$. Then

$$\frac{d}{dt}(foH)(t_0) = df(H(t_0))H'(t_0)$$

where $H(t) = (H_1(t), ..., H_n(t))$ and $H'(t) = (H_1'(t), ..., H_n'(t))$

Directional derivative

The directional derivative of f in the direction of a unit vector $e \in \mathbb{R}^n - \{0\}$ is given by $d_e f(x_0) = \nabla f(x_0) \cdot e$.

Particular higher derivatives

One can similarly define higher derivatives order partial derivatives for functions $f \in C^m(\Omega)$.

Fact (Schwarz) if $f \in C^2(\Omega)$ then $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$ and in general for $f \in C^m(\Omega)$ all n partial derivatives of f of order $\leq m$ are independent of the order of differentiation. Using higher order derivatives one can analogous to the 1-dimensional case define a Taylor approximation of f.

Fact Let $f \in C^m(\Omega), f : \Omega \to \mathbb{R}, \Omega \in \mathbb{R}$ and $x_1, x_0 \in \Omega$. Then

$$f(x_1) = f(x_0) + \nabla f(x_0)(x_1 - x_0) + \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)(x_i^i - x_0^i)(x_1^j - x_0^j) + R_3(f, x_1, x_0)$$

Where $\lim_{x_1 \to x_0} \frac{R(f,x_1,x_0)}{||x_1 - x_0||} = 0$

The analogue of the second derivative is given by the matrix of partial derivatives of order 2. This matrix is called the Hesse-matrix of f.

$$\mathrm{Hess}(f) = \nabla^2 f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_{i,j=1,\dots,n}$$

The extrema of a function $f : \Omega \to \mathbb{R}$

Definition A point $x \in \Omega$ is called a critical point if $\nabla f(x) = 0$.

Fact f differentiable, x_0 is called local extrema of f then x_0 is a critical point.

Fact Let x_0 be a critical point of f. Then we have three different cases:

1. x_0 is a local minima if $\nabla^2 f(x_0)$ is positive definite.
2. x_0 is a local maxima if $\nabla^2 f(x_0)$ is negative definite.
3. Otherwise it is a saddle point (Sattelpunkt).

Fact Let $f : \Omega \to \mathbb{R}$ be continuous and differentiable on an open set $\Omega \subset \mathbb{R}^n$. Let $\partial\Omega$ be the boundary of Ω. Then every global extrema of f is either a critical point of f in Ω or a global extremal point of $f|_{\partial\Omega}$ (f restricted to the boundary).

Example (FS 2011) Sei $f(x,y) = 4x^2y^2 - x^2 - 4y^2 + 1$. Bestimme die globalen Extrema von f auf dem Gebiet $\Omega = \{(x,y) = \frac{x^2}{4} + y^2 \leq 1, y \geq 0\}$.

Solution We first find the critical points:

$$\nabla f = \begin{pmatrix} 8xy^2 - 2x \\ 8x^2y - 8y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 2x(4y^2 - 1) = 0 \Rightarrow & x = 0 \text{ or } y = \pm\frac{1}{2} \\ 8y(x^2 - 1) = 0 \Rightarrow & x = \pm 1 \text{ or } y = 0 \end{aligned}$$

$(0,0), (\pm 1, \pm\frac{1}{2})$ are the critical points of f. Since $y \geq 0$ we only take $P_1 = (0,0), P_{2,3} = (\pm 1, \frac{1}{2})$. Then we need to compute Hes(f).

$$\mathrm{Hess}(f) = \begin{pmatrix} 8y^2 - 2 & 16xy \\ 16xy & 8x^2 - 8 \end{pmatrix}$$

$$\text{Hess}(f)(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -8 \end{pmatrix} \Rightarrow \text{ negative definite } \Rightarrow \text{ local maxima}$$

$$\text{Hess}(f)(1, \frac{1}{2}) = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix} \Rightarrow \text{ indefinite}$$

$$\text{Hess}(f)(-1, \frac{1}{2}) = \begin{pmatrix} 0 & -8 \\ -8 & 0 \end{pmatrix} \Rightarrow \text{ indefinite}$$

To find global extrema we need to look at f on the boundary of Ω, which is the curve $\frac{x^2}{4} + y^2 = 1$ and the line $y = 0$. First on the line $y = 0$ let $g = f|_{y=0} = -x^2 + 1$.

$$g'(x) = -2x \Rightarrow x = 0, P_1(0,0) \text{ is a point we need to check}$$

We also need to check the corners $P_{4,5} = (\pm 2, 0)$. On the ellipse: let $h = f|_{\frac{x^2}{4}+y^2=1} = -x^4 + 4x^2 - 3$.

$$h'(x) = -4x^3 + 8x = 0 \Rightarrow P_6 = (0,1), P_{7,8} = (\pm\sqrt{2}, \frac{1}{\sqrt{2}})$$

Now we look at the values of f at these points. $f(P_1) = f(0,0) = 1, f(P_{2,3}) = 0, f(P_{4,5}) = -3, f(P_6) = -3, f(P_{7,8}) = 1$. f has also a minima at $(\pm 2, 0), (0,1)$ and a maxima at $(0,0), (\pm\sqrt{2}, \frac{1}{\sqrt{2}})$.

Example (FS 2010)

1. Bestimme das Taylorpolynom erster Ordnung der Funktion $f(x,y) = e^{x^2}(x+y)$ um dem Punkt $(1,1)$.

2. Bestimme $c \in \mathbb{R}$ so dass der Vektor $\begin{pmatrix} 1 \\ -1 \\ c \end{pmatrix}$ tangential an den Graphen $g(f) = \{(x, y, f(x,y)) : (x,y) \in \mathbb{R}^2$ im Punkt $(1,1,z)$ liegt.

Solution

1. $\nabla f = \begin{pmatrix} 2xe^{x^2}(x+y) + e^{x^2} \\ e^{x^2} \end{pmatrix}, \nabla f(1,1) = \begin{pmatrix} 5e \\ e \end{pmatrix}, f(1,1) = 2e$

   $$f(x,y) = 2e + \begin{pmatrix} 5e \\ e \end{pmatrix}(x-1, y-1) + r_2(x,y) = 2e + 5e(x-1) + e(y-1) + (x-1)(\nabla^2 f)(t)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$$

   The Taylor polynomial of order 1 is $z = 2e + 5e(x-1) + e(y-1)$

2. The vector $(1, -1, c)^T$ must be perpendicular to the normal vector of the plane, that is (from part 1) $n = (5e, e, -1)^T$. Hence

   $$(1, -1, c) \cdot (5e, e, -1) = 0 \Rightarrow 4e - c = 0 \Rightarrow c = 4e$$

0.0.2 Line(Weg) integral

Let $v : \Omega \to \mathbb{R}^n$ be a vector field and γ a curve with parametrization $\gamma : [a,b] \to \Omega, t \to \gamma(t)$. Then the line integral of v along γ is defined as

$$\int_\gamma v ds = \int_a^b < v(\gamma(t)), \gamma'(t) > dt$$

Facts

1. $\int_\gamma vds$ is independent of the parametrization of the path.
2. $\int_{\gamma_1+\gamma_2} vds = \int_{\gamma_1} vds + \int_{\gamma_2} vds$
3. $\int_\gamma vds = -\int_{-\gamma} vds$
4. If v is the gradient vector field associated to a function f i.e. $v = df$ then $\int_\gamma vds = f(\gamma(b)) - f(\gamma(a))$, where $\gamma : [a, b] \to \Omega$.

Equivalent one can change everything in terms of 1-forms $\lambda = \lambda_1 dx^1 + \lambda_2 dx^2 + ... + \lambda_n dx^n$. Then

$$\int_\gamma \lambda = \int_a^b \lambda(\gamma(t))\gamma'(t)dt$$

Fact $\lambda : \Omega \to L(\mathbb{R}^{\ltimes}, \mathbb{R})$ a constant 1-form, then the following are equivalent:

1. $\exists f \in C'(\Omega) : df = \lambda$
2. For every 2 continuous C'-paths γ_1, γ_2 with $\gamma_i : [a_i, b_i] \to \Omega$ with the same beginning and ending points:
$$\int_{\gamma_1} \lambda = \int_{\gamma_2} \lambda$$
3. For every closed curve γ, $\int_\gamma = 0$

Definition A vector field $V : \Omega \to \mathbb{R}^n$ is called conservative if $\int_\gamma V ds = 0$ for every closed curve γ.

Fact For a simply connected region Ω, we have

$$\text{V is conservative} \iff v = \nabla f \text{ for some function } f$$

Example (FS 2010) Berechne das Wegintegral $\int_\gamma vds$ entlang das eingezeichneten Weg fÃ¼r das Vektorfeld $v(x, y) = \begin{pmatrix} xy^2 \\ -y \end{pmatrix}$

Solution $int_\gamma vds = \int_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} vds$. We first parametrize the curves:

1. $\gamma_1 = (1, t), t \in [0, e] \Rightarrow \gamma_1' = (0, 1)$
2. $-\gamma_2 = (t, e^t), t \in [0, 1] \Rightarrow -\gamma_2' = (1, e^t)$
3. $\gamma_3 = (0, 1 - t), t \in [0, 1] \Rightarrow \gamma_3' = (0, -1)$
4. $\gamma_4 = (t, -t)t \in [0, 1] \Rightarrow \gamma_4' = (1, -1)$

$$\int_\gamma vds = \int_0^e (t^2, -t) \cdot (0, 1)dt + \int_1^0 (te^{2t}, -e^t) \cdot (1, e^t)dt + ...$$

See the last example of Green's theorem for the rest.

0.0.3 Integration in $\mathbb{R}^n$

The Riemann integral in $\mathbb{R}^n$ is constructed in analogue way to the case in n=1. With Riemann sums over subintervals is replaced with sums over "sub-rectangles". dx is replaced with an n-dimensional volume element $dvol_n$ which we denote also by $d\mu(x)$.

Fact For a normal-region

$$D = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, \phi(x) < y < \psi(x)\}$$

Where ϕ, ψ are continuous functions: $\mathbb{R} \to \mathbb{R}$ we have

$$\int_D f d\mu(x) = \int_a^b \int_{\phi(x)}^{\psi(x)} f(x,y) dy dx$$

Fact For a rectangle $Q = [a,b] \times [c,d] \in \mathbb{R}^2$

$$\int_Q f d\mu = \int_a^b \int_c^d f dy dx = \int_c^d \int_a^b f dx dy$$

Substitution in $\mathbb{R}^n$

Let $u, v \in \mathbb{R}^n$ open intervals, $\Phi : u \to v$ bijective continuous differentiable with $\det(d\Phi(y)) \neq 0 \forall y \in u$. Then for $f : v \to \mathbb{R}$ continuous we have

$$\int_v f(x) d\mu(x) = \int_u f(\Phi(y)) |\det(d\Phi(y))| d\mu(y)$$

Example (FS 2010) Bestimme das Integral $\int_\Omega x\sqrt{x^2+y^2}\log(\sqrt{x^2+y^2}) d\mu(x,y)$ wobei

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Solution Given the symmetry of the region it's better to use polar coordinates. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\sigma) \\ r\sin(\sigma) \end{pmatrix} \Rightarrow dxdy = r dr d\sigma$$

The region in (r,σ) domain is simply $1 \leq r \leq 2, \frac{\pi}{4} \leq \sigma \leq \frac{7\pi}{4}$.

$$\int_\Omega x\sqrt{x^2+y^2}\log\sqrt{x^2+y^2} d\mu(x,y) = \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} \int_1^2 r\cos(\sigma) r \log(r) r dr d\sigma =$$

$$= \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} \cos(\sigma) d\sigma \int_1^2 r^3 \log(r) dr = \left[-\sqrt{2} + \frac{1}{4}x^4\log(x) - \frac{x^4}{16}\right]_1^2 = \log(16) - \frac{15}{16}$$

0.0.4 Green's theorem

Let $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ has a C' parametrization. Let $U \subset \Omega$ and $f = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ where $P, Q \in C'(U)$. Then

$$\int\int_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\mu = \int_{\partial\Omega} P dx + Q dy$$

Let $V = (P,Q)$ be a vectorfield then

$$\int_{\partial\Omega} V ds = \int\int_\Omega (rot(V)) d\mu$$

Where $rot(V) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ and the line integral is taken around the boundary of Ω in positive sense.

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