1 Integration

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $P = \{a = x_0 < x_1 < \cdots < x_n < b\}$ a partition of the interval $[a, b]$ and $\xi_k \in [x_k, x_{k+1}]$ points in each subinterval, then the sum

$$S(f, P, \xi) = \sum_{k=0}^{n-1} \left(\inf_{I_k} f \right) (x_{k+1} - x_k)$$

is called the Reimann sum to f and to

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$$U(f, P) := \sum_{k=0}^{n-1} (\inf_{I_k} f)(x_{k+1} - x_k)$$

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$$O(f, P) := \sum_{k=0}^{n-1} f(\sup_{I_k} f)(x_{k+1} - x_k)$$

(where $I_k = [x_k, x_{k+1}]$) are called the lower & upper Riemann sums. Similarly

$$\int_{\underline{a}}^{b} f dx = \sup \{U(f, P), P \in P\}$$

and

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$$\int_{a}^{\bar{b}} f dx = \inf \{O(f, P), P \in P\}$$

are called lower and upper Integrals of f. f is called Riemann integrable if

$$\int_{\underline{a}}^{b} f dx = \int_{a}^{\bar{b}} f dx$$

Fact

1. Each continuous function is Riemann integrable
2. Each monotone function is Riemann integrable

■

Properties

Let f, g Riemann integrable on I, $\alpha, \beta \in \mathbb{R}$ for

1. $$\int_a^b (\alpha f + \beta g)\, dx = \alpha \int_a^b f dx + \beta \int_a^b g dx$$

2. if $f(x) \leq g(x)$, $\forall x \in [a,b]$ then

$$\int_a^b f dx < \int_a^b g dx$$

3.

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| \, dx$$

4.

$$\left(\inf_I f \right) (b-a) \leq \int_a^b f(x) dx \leq \left(\sup_I f \right) (b-a)$$

5.

$$\int_a^b f dx = - \int_b^a f dx$$

6.

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx, \qquad \forall a,b,c \in \mathbb{R}$$

Fact (MWS der Integralrechnung)

$f : [a,b] \to \mathbb{R}$ continuous, then $\exists \rho \in (a,$

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$$\int_a^b f(x) dx = f(\rho)(b-a)$$

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Fact (Fundamental of calculus)

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1. Let $f : [a,b] \to \mathbb{R}$ continuous. Define

$$F(x) := \int_a^x f(t) \, dt \qquad \forall x \in [a,b]$$

   then F is differentiable and $F' = f$. F is called primitive (stammfunktion) of f.

2. If G is primitive of f, then $G = F + c$ for some constant

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3. Let F be primitive of f then

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$$\int_a^b f(x) dx = F(b) - F(a)$$

■

How do we calculate integrals?

1. **Partial Integration:** Follows from product rule for differentiation

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$
$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$$

2. **Substitution:** It follows from chain rule for differentiation

$$\int f(x)dx = \int f(\varphi(y))\,\varphi'(y)dy$$
$$\int_{\tilde{a}}^{\tilde{b}} f(x)dx = \int_a^b f(\varphi(y))\,\varphi'(y)dy$$

   where $\varphi(a) = \tilde{a}$, $\varphi(b) = \tilde{b}$.

3. **Partialfractions:** To integrate rational functions of the form $\frac{P(x)}{Q(x)}$ with P, Q polynomials.

   - If $\deg P < \deg Q$
     We expand the rational function in simpler rational functions by finding the roots of $Q(x)$. Say

     $Q(x) = (x - a_1) \dots (x - a_n) \cdot \left(A_1x^2 + B_1x + C_1\right) \dots \left(A_nx^2 + B_nx + C_n\right)$

     Then for each linear factor $(x - a_i)$ of multiplication r_i we need

$$\frac{\alpha_{i1}}{(x-a_i)} + \frac{\alpha_{i2}}{(x-a_i)^2} + \dots + \frac{\alpha_{ir}}{(x-a_i)^r}$$

     with constants $\alpha_{i1}, \dots, \alpha_{ir}$. For each quadratic factor $A_ix^2 + B_ix + C_i$ of multiply S we have

$$\frac{\beta_{i1}x + \gamma_{i1}}{A_ixr + B_ix + C_i} + \dots + \frac{\beta_{iS_i}x\gamma_{iS_i}}{A_ixr + B_ix + C_i}$$

     witch constants β_{ij}, γ_{ij}, $j = 1, \dots, S$.
   - If $\deg P \geq \deg Q$
     We first do long division to unite

$$\frac{P}{Q} \cdot R(x) + \frac{\tilde{P}(x)}{Q(x)}$$

     with polynomials $R(X)$ and $\tilde{P}(x)$ with $\deg \tilde{P} <$ deg. Proceed as in the first case

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Beispiel

Berechne

1. $$\int_1^2 \frac{\sqrt{1+\ln(x)}}{x} dx$$

2. $$\int \cos(x)\cosh(x) dx$$

3. $$\int \frac{x^2-x+2}{x^3-x^2+x-1} dx$$

Solution

1. Note $\frac{1}{x}$ is the derivative of $(\ln(x)+1)$. Hence using substitution $u = (1+\ln(x)) \Rightarrow du = \frac{1}{x} dx$

$$\int_1^2 \frac{(1+\ln(x))^{\frac{1}{2}}}{x} dx = \int_1^{1+\ln(2)} u^{\frac{1}{2}} du = \left. \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^{1+\ln(2)} = \frac{2}{3}\left[(1+\ln(2))^{\frac{3}{2}} - 1\right]$$

2. This one is suitable for Integral by parts:

$$I = \int \underbrace{\cos(x)}_{u} \underbrace{\cosh(x)}_{v'} dx \qquad \boxed{\begin{array}{ll} u = \cos(x) & v' = \cosh(x) \\ u' = -\sin(x) & v = \sinh(x) \end{array}}$$

$$I = \cos(x)\sinh(x) + \int \underbrace{\sin(x)}_{u} \underbrace{\sinhh(x)}_{v'} dx \qquad \boxed{\begin{array}{ll} u = \sin(x) & v' = \sinh(x) \\ u' = \cos(x) & v = \cosh(x) \end{array}}$$

$$I = \cos(x)\sinh(x) + \left[\sin(x)\cosh(x) - \int \cos(x)\cosh(x) dx\right]$$

$$I = \cos(x)\sinh(x) + \sin(x)\cosh(x) - I$$

$$2I = \cos(x)\sinh(x) + \sin(x)\cosh(x)$$

$$x = \frac{1}{2}\left(\cos(x)\sinh(x) + \sin(x)\cosh(x)\right) + C$$

3. $$\int \frac{x^2-x+2}{x^3-x^2+x-1} dx$$

Note: $x^3 - x^2 + x - 1 = x^2(x-1) + (x-1) = (x^2+1)(x-1)$

$$\frac{x^2-x+2}{x^3-x^2+x-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{1}{x-1} - \frac{1}{x^2+1}$$

$$A(x^2+1) + (x-1)(Bx+C) = x^2 - x + 2$$

$$\begin{aligned}
x = 1 &\Rightarrow 2A = 2 \Rightarrow A = 1 \\
x = 2 &\Rightarrow 5 + 2B + C = 4 \Rightarrow 2B + C = -1 \\
x = 0 &\Rightarrow 1 - C = 2 \Rightarrow C = -1 \\
&\Rightarrow B = 0
\end{aligned}$$

$$\Rightarrow \int \frac{x^2 - x + 2}{x^3 - x^2 + x - 1} dx = \ln|x-1| - \tan^{-1}(x) + C$$

Improper Integrals

The improper integral of an integrable function f on (a, b) which is integrable on any subinterval $[a', b']$ we define the improper integral as

$$\int_a^b f(x)dx := \lim_{a' \searrow a} \lim_{b' \nearrow b} \int_{a'}^{b'} f(x)dx$$

Example (Basisprüfung Frühling 2010)

Untersuche ob das uneigentliche Integral $\int_1^\infty \frac{1}{x^2+x} dx$ konvergiert. Falls ja, berechne den Wert.

Note: since $\frac{1}{x^2+x} < \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2} dx$ is convergent, so is $\int \frac{1}{x^2+x}$

$$\begin{aligned}
\int \frac{1}{x^2 + x} &= \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \\
&= \ln|x| - \ln|x+1| = \ln\left|\frac{x}{x+1}\right|
\end{aligned}$$

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2 + x} &= \lim_{b \to \infty} \int_1^b \frac{1}{x^2 + x} dx \\
&= \lim_{b \to \infty} \ln\left|\frac{x}{x+1}\right|_1^b \\
&= \lim_{b \to \infty} \ln\left(\frac{b}{b+1}\right) - \ln\left(\frac{1}{2}\right) \\
&= \ln(1) + \ln(2) = \ln(2)
\end{aligned}$$

limit exists and is equal to $\ln(2)$

Fact

1.

$$\forall s \in \mathbb{R}, a > 0, \int_a^\infty \frac{dx}{x^s} = \begin{cases} \frac{a^{1-s}}{s-1} & s > 1 \\ \infty & s \leq 1 \end{cases}$$

2. If f is in $[a, \infty)$ continuous, and $\exists c$ and $s > 1$ so that $|f(x)| \leq \frac{c}{x^s}$, $\forall x \geq a$, then $\int\limits_a^\infty f(x)dx$ converges (Majorantenkriterium)

3. If f is in $[a, \infty)$ continuous, and $\exists c > 0$ $f(x) \geq \frac{c}{x}, \forall x \geq a$, then $\int\limits_a^\infty f(x)dx$ diverges to ∞.

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Example

$$\int\limits_1^\infty \frac{t^2+4}{(1+4t^2)^{\frac{3}{2}}} \text{ diverges}$$

$$\int\limits_1^\infty \frac{t^2+4}{(1+4t^2)^{\frac{5}{2}}} \text{ converges}$$

Because

$$\text{if } t \text{ is large enough, say } t > 2 \begin{cases} t^2 < t^2 + 4 < 2t^2 \\ 4t^2 < 1 + 4t^2 < 5t^2 \end{cases}$$

Hence

$$\frac{t^2}{(5t)^{2\alpha}} < \frac{t^2+4}{(1+4t^2)^\alpha} < \frac{2t^2}{(4t)^{2\alpha}}$$

$$\frac{C_1}{(t)^{2\alpha-2}} < \frac{t^2+4}{(1+4t^2)^\alpha} < \frac{C_2}{(4t)^{2\alpha-2}}$$

$\Rightarrow$ if $\alpha = \frac{3}{2}$

$$\frac{t^2+4}{(1+4t^2)^{\frac{3}{2}}} > \frac{C_1}{t}$$

And since $\int\limits_1^\infty \frac{dt}{t}$ diverges, so does

$$\int\limits_1^\infty \frac{t^2+4}{(1+4t^2)^{\frac{3}{2}}} dt$$

Similarly since for $\alpha = \frac{5}{3} : 2\alpha - 2 = \frac{10}{3} - 2 > 1$ and

$$\int \frac{C_2}{t^{\frac{4}{3}}} dt$$

converges, so does

$$\int \frac{t^2+4}{(1+4t^2)^{\frac{5}{3}}} dt < \int \frac{C_2}{t^{\frac{4}{3}}} dt$$

2 Differential Equations

2.1 Linear Differential equations with constant coefficients

To solve a linear differential equation of the form $Ly = b(x)$, where

$$L := \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1\frac{d}{dx} + a_0$$

$b(x)$ a function, and $a_i \in \mathbb{R}$

1. Find a homogeneous solution y_H, namely a solution of $Ly = 0$
2. Find a special solution y_S of $Ly = b(x)$ using the method “Ansatz von Typ der rechten Seite”
3. Then the general solution is given by

$$y = y_H + y_S$$

How to find the homogeneous solution y_H of $Ly = 0$

1. Find the characteristic polynom of L, namely

$$P_l(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

2. **Fact**

If $\lambda_1, \ldots, \lambda_{r \in \mathbb{C}}$ are the Pairwise distinct roots of $P(\lambda) = 0$ with associated multiplicates $m_1, \ldots, m_r$, then the functions

$$x \to x^k e^{\lambda_j x}, 1 \le k \le r, 0 \le k \le m_j$$

Form a system of fundamental solutions of the homogeneous equations $Ly = 0$

Variant

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If L has real coefficients, every pair of complex, non-real roots $\lambda_j = \mu_j \pm i v_j$ of multiply m_j give a fundamental solution

$$k^k e^{(\mu_j \pm i v_j)x} = x^k e^{\mu_j} \left(\cos(v_j x) \pm i \sin(v_j x)\right)$$

for $0 \le k < m_j$. So one can, as a basis, take

$$x^k e^{\mu_j x} \cos v_j x \text{ and } x^k e^{\mu_j x} \sin v_j x$$

instead of

$$x^k e^{(\mu_j + i v_j)x} \text{ and } x^k e^{(\mu_j - i v_j)x}$$

Then the general homogeneous solution is of the form

$$y_H(x) = \sum_{j=1}^{r} \sum_{k=0}^{m_j} c_{jk} x^k e^{\lambda_j x}$$

with constant c_{jk}

■

How to find the special solution of Inhomogeneous $Ly = b(x)$ using the method of "Ansatz"

Fact

1. Let $\lambda \in \mathbb{C}$. if λ is not a solution of $P_L(\lambda)$, then the inhomogeneous DGL $Ly = e^{\lambda x}$ has particular solutions $y = \frac{1}{P_L(x)} e^{\lambda x}$

2. Let $\lambda \in \mathbb{C}$, m its multiplicity as a solution of $P_L(\lambda) = 0$ (m can be zero, which means λ is not a solution of $P_L(\lambda) = 0$).

   Let $Q(x)$ a polynomial of degree k. Then a particular solution of $Ly(x) = Q(x)e^{\lambda x}$ is of the form
   $$y(x) = R(x)e^{\lambda x}$$
   for a polynomial $R(x)$ of degree $k + m$

3. Let l has a real coefficient. Let $\mu, v \in \mathbb{R}$, m the multiplicity of $\mu \pm iv$ as a solution of $P_L(\lambda) =$($m = 0$ means $\mu \pm iv$ is not a root of P_L). Let $Q(x)$ and $R(x)$ be polynomials of degree $\leq$. Then the particular solution of the inhomogeneous Differential equation
   $$Ly = Q(x)e^{\mu x}\cos vx + R(x)e^{\mu x}\sin x$$
   is of the form
   $$y(x) = S(x(e^{\mu x}\cos vx + T(x)e^{\mu x}\sin x$$
   for polynomials S, T of degree $\leq k + m$

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Boundary (Rand) or Initial value Problems (Anfangswert)

When we are given a differential equation $Ly = b(x)$ together with either

$$\begin{array}{rcl} \text{Boundary values} & \text{or} & \text{Initial values} \\ y(a_1) = A_1 & & y(a) = A_1 \\ y(a_2) = A_2 & & y'(a) = A_2 \\ \vdots & & \vdots \\ y(a_n) = A_n & & y^{(n-1)}(a) = A_n \end{array}$$

We first find the general solution $y = y_H + y_S$. Then we determine the constants $C_1, \ldots, c_n$ in the homogeneous solution using the given boundary or initial values.

Example

1. Basisprüfung Frühling 2011
   (a) Bestimme alle Lösungen $y = y(x)$ der DGL $y^{(4)} - y = 0$ welche für $|x| \to \infty$ beschränkt bleiben
   (b) Bestimme eine Lösung $y = y(x)$ der DGL $y^{(4)} - y = e^{-x} + x$

Solution

(a) We first find the characteristic polynomial

$$x^4 - 1 = 0 \Rightarrow \left(x^2 - 1\right)\left(x^2 + 1\right) = 0$$
$$\lambda = \pm 1, \lambda = \pm i$$

The homogeneous solution is of the form

$$y_H = C_1 e^x + C_1 e^{-x} + C_3 \cos x + C_4 \sin x$$

Since

$$e^x \to \infty \text{ as } x \to \infty$$
$$e^{-x} \to \infty \text{ as } x \to -\infty$$

The solutions that satisfy are $y_H = C_3 \cos x + C_4 \sin x$ with $C_3, C_4 \in \mathbb{R}$

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(b) We use the superposition principle: Let $b_1(x) e^{-x}, b_2(x) = x$

i. y_{1S} is a solution of $Ly = b_1(x)$

ii. y_{2S} is a solution of $Ly = b_2(x)$

Then $y_{1S} + y_{2S}$ is a solution of $Ly = e^{-x} + x$. Since $\lambda = -1$ is a root of $P_L(\lambda) = 0$ of multiplicity, i.e. since e^{-x} is also a solution of $Ly = 0$

For y_{1S} we take as "ansatz" $y_{1S} = Cxe^{-x}$. For y_{2S} we take as "ansatz" $y_{2S} = Dx + E$ a polynomial of degree 1. So a special solution is of the form

$$y_S = Cxe^{-1} + Dx + E$$

To find the constants C, D, E, F we put this solution into the differential equation $y^{(4)} - y = e^{-x} +$

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$$\begin{aligned}
y'(x) &= C\left[e^{-x} - xe^{-x}\right] + D \\
y''(x) &= C\left[-e^{-x} - \left[e^{-x} - xe^{-x}\right]\right] = C\left[-2e^{-x} + xe^{-x}\right] \\
y^{(3)}(x) &= C\left[2e^{-x} + \left[e^{-x} - xe^{-x}\right]\right] = C\left[3e^{-x} - xe^{-x}\right] \\
y^{(4)}(x) &= C\left[-3e^{-x} + \left[e^{-x} - xe^{-x}\right]\right] = C\left[-4e^{-x} + xe^{-x}\right]
\end{aligned}$$

Then

$$\begin{aligned}
y_S^{(4)} - y_S(x) =& C \cdot 4e^{-x} + Cxe^{-x} - \left[Cxe^{-x} + Dx + E\right] \\
\Rightarrow& 4Ce^{-x} - Dx - E = e^{-x} + x \\
\Rightarrow& 4C = 1, D = -1
\end{aligned}$$

Hence a particular solution is

$$y_S = \frac{1}{4} xe^{-x} - x$$

General solution is

$$\begin{aligned}
y &= y_H + y_S \\
&= C_1 e^x + C_2 e^{-x} + C_3 \cos x + C_4 \sin x + \frac{1}{4} xe^{-x} - x
\end{aligned}$$

2. Basisprüfung Sommer 2013

   (a) Für welche Werte des Parameters $a \in \mathbb{R}$ strebt die allgemeine Lösung $y(x)$ der DGL $y'' + 2y' + ay = 0$ unabhängig von den Anfangsbedingungen gegen 0 für $x =$

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   (b) Finden Sie eine homogene DGL 2. ordnung mit konstanten Koeffizienten, deren allgemeine Lösung $y(x) = e^{-x} + 2xe^{-x}$ ist. Was sind dann die Anfangsbedingungen bey $x = 0$?

Solution

(a) The characteristic polynomial of $y'' + 2y' + ay = 0$ is $\lambda^2 + 2\lambda + a = 0$. The roots are

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 4a}}{2} = -1 \pm \sqrt{1 - a}$$

- If $1 - a < 0$ then there are 2 complex conjugate roots. Let $|1 - a| = b^2$ then the general solution is of the form

  $$y(x) = C_1 e^{-x} \cos(bx) + C_2 e^{-x} \sin(bx)$$

  and this go to zero of b, hence a since e^{-x} decays to zero.

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- If $1 - a = 0$ then -1 is a double root. Then the solutions are of the form $C_1 e^{-x} + C_2 x e^{-x}$ and this will not decay to zero as $x \to \infty$

Similarly if $1 - a > 0$ then one of the roots will be positive if $\sqrt{1 - a} > 1$ and that root can lead to a growing solution.

Hence when $\sqrt{1 - a} > 1$, or $1 - a > 1$, or when $a < 0$, then $-1 + \sqrt{1 - a} > 0$, If on the other hang $0 < a < 1$ then again $1 \pm \sqrt{1 - a} < 0$ and both solutions are decaying. Hence if $0 < a < 1$ or $a > 1$ then the solutions is decaying of initial values, and if $a = 1$ or $a < 0$ then the solution can be growing.

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(b) Let $Ly = 0$ be of the form $y'' + a_1 y' + a_0 = 0$ then the characteristic polynomial is $\lambda^2 + a_1 \lambda + a_0$. Since we want e^{-x} and xe^{-x} to be the solutions, we should have $\lambda = -1$ with multiply 2, that means

$$P_L(\lambda) = \lambda^2 + 2\lambda + 1$$

and the differential equation is $y'' + 2y' + y = 0$. The general solution os this equation is

$$y_H = C_1 e^{-x} + C_2 x e^{-x}$$

Since the solution is $e^{-x} + 2xe^{-x}$, $C_1 = 1$, $C_2 = 2$ we must have $y(0) = e^{-0} + 2 \cdot 0 = 1$

$$y'(0) = -e^{-x} + 2 \left[e^{-x} - xe^{-x} \right] \Big|_{x=0} = -1 + 2 = 1$$

So we can choose $y(0) = 1$, $y'(0) = 1$

2.2 Solving Differential Equations by separation of variables

Example

1. (Frühling 2011) Bestimme die Lösung $y = y(x)$ der DGL $y' = e^{x-y}$ mit $y(0) = 0$

   Solution

$$\begin{aligned}
&\frac{dy}{dx} = e^x \cdot e^{-y} \\
&\Rightarrow e^y dy = e^x dx \\
&\Rightarrow \int e^y dx = \int e^x dx \\
&\Rightarrow e^y = e^x + C
\end{aligned}$$

   Since $y(0) = 0 \Rightarrow e^0 = e^0 + C \Rightarrow C = 0$ and $y = x$

2. (Herbst 2010) Bestimme eine Lösung der DGL

$$y + xy' = \frac{1}{y}\left(x + \frac{1}{x}\right), x > 0$$

   mit $y(1) = 2$, $y(x) > 0$, $\forall x$. (Hinweise: Sunstituire $u(x) = xy(x)$).

$$u'(x) = y(x) + xy'(x)$$

   Left hand side is

$$\begin{aligned}
u'(x) &= \frac{x}{y} + \frac{1}{xy} = \frac{x^2}{u} + \frac{1}{u} = \frac{x^2+1}{u} \\
\Rightarrow \frac{u'}{u} &= x^2 + 1 \Rightarrow \ln u = \frac{x^3}{3} + x + C \\
y(1) &= 2 \Rightarrow u(1) = 1 \cdot y(1) = 2 \\
\Rightarrow \ln 2 &= \frac{1}{3} + 1 + C \Rightarrow C = (\ln 2) - \frac{4}{3} \\
u &= Ce^{\frac{x^3}{3}} \cdot e^x
\end{aligned}$$

2.3 Differentiation in many variables

A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is (total) differentiable in x_0 if it exists a linear map $A : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = f(x_0) + A(x - x_0) + R(x_0, x)$$

Where $\lim_{x \to x_0} \frac{R(x_0,x)}{|x-x_0|} = 0$. In this case A is called the differential of f at x_0 and it's denoted as $(df)(x_0)$.

Let $(A_1, A_2, ..., A_n)$ be a matrix representation of the linear map $A : \mathbb{R}^n \to \mathbb{R}$ (wrt to the standard Basis). Then f differentiable at x_0 means:

$$f(x) = f(x_0) + A_1(x^1 - x_0^1) + A_2(x^2 - x_0^2) + ... + A_n(x^n - x_0^n) + R(x, x_0)$$

$$P(x, x_0) = f(x_0) + \begin{bmatrix} A_1 & ... & A_n \end{bmatrix} \begin{bmatrix} x^1 - x_0^1 \\ ... \\ x^n - x_0^n \end{bmatrix}$$

is the equation of the tangent plane at the point $f(x_0)$ on the surface formed by the graph of f.

Fact

if $f : \Omega \to \mathbb{R}$ is differentiable in $x_0 \in \Omega$ then the partial derivative exist and the differential $df(x_0)$ has the matrix representation

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0) & ... & \frac{\partial f}{\partial x^n}(x_0) \end{pmatrix} = \nabla f$$

the gradient of f.

■

Fact

f differentiable in $x_0 \Rightarrow f$ is continuous in x_0.

■

Fact

If all partial derivatives of f are continuous then f is differentiable

■

Using these last two facts and the definition of differentiability one can study if a given function is differentiable or not.

2.3.1 Differentiation rules

Let $f, g : \Omega \to \mathbb{R}$ be differentiable in x_0. Then:

1. $d(f \pm g)(x_0) = df(x_0) \pm dg(x_0)$
2. $d(fg)(x_0) = g(x_0)df(x_0) + f(x_0)dg(x_0)$
3. $d(f/g)(x_0) = \frac{g(x_0)df(x_0) - f(x_0)dg(x_0)}{(g(x_0))^2}$
4. Let $h : \mathbb{R} \to \mathbb{R}$ be differentiable in $g(x_0)$, then
$$d(hog)(x_0) = h'(g(x_0))dg(x_0)$$
5. Let $H : I \subset \mathbb{R} \to \Omega \subset \mathbb{R}^n$ be differentiable in $t_0 \in \mathbb{R}$ and f differentiable in $H(t_0)$. Then
$$\frac{d}{dt}(foH)(t_0) = df(H(t_0))H'(t_0)$$
where $H(t) = (H_1(t), ..., H_n(t))$ and $H'(t) = (H_1'(t), ..., H_n'(t))$

2.3.2 Directional derivative

The directional derivative of f in the direction of a unit vector $e \in \mathbb{R}^n - \{0\}$ is given by $d_e f(x_0) = \nabla f(x_0) \cdot e$.

2.3.3 Particular higher derivatives

One can similarly define higher derivatives order partial derivatives for functions $f \in C^m(\Omega)$.

Fact (Schwarz)

if $f \in C^2(\Omega)$ then $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$ and in general for $f \in C^m(\Omega)$ all n partial derivatives of f of order $\leq m$ are independent of the order of differentiation. Using higher order derivatives one can analogous to the 1-dimensional case define a Taylor approximation of f.

■

Fact

Let $f \in C^m(\Omega), f : \Omega \to \mathbb{R}, \Omega \in \mathbb{R}$ and $x_1, x_0 \in \Omega$. Then

$$f(x_1) = f(x_0) + \nabla f(x_0)(x_1 - x_0) + \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)(x_i^i - x_0^i)(x_1^j - x_0^j) + R_3(f, x_1, x_0)$$

Where $\lim_{x_1 \to x_0} \frac{R(f,x_1,x_0)}{||x_1 - x_0||} = 0$

■

The analogue of the second derivative is given by the matrix of partial derivatives of order 2. This matrix is called the Hesse-matrix of f.

$$\text{Hess}(f) = \nabla^2 f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right)_{i,j=1,\ldots,n}$$

2.3.4 The extrema of a function $f : \Omega \to \mathbb{R}$

Definition

A point $x \in \Omega$ is called a critical point if $\nabla f(x) = 0$.

Fact

f differentiable, x_0 is called local extrema of f then x_0 is a critical point

■

Fact

Let x_0 be a critical point of f. Then we have three different cases:

1. x_0 is a local minima if $\nabla^2 f(x_0)$ is positive definite.
2. x_0 is a local maxima if $\nabla^2 f(x_0)$ is negative definite.
3. Otherwise it is a saddle point (Sattelpunkt).

■

Fact

Let $f : \Omega \to \mathbb{R}$ be continuous and differentiable on an open set $\Omega \subset \mathbb{R}^n$. Let $\partial\Omega$ be the boundary of Ω. Then every global extrema of f is either a critical point of f in Ω or a global extremal point of $f|_{\partial\Omega}$ (f restricted to the boundary)

■

Example (FS 2011)

Sei $f(x, y) = 4x^2y^2 - x^2 - 4y^2 + 1$. Bestimme die globalen Extrema von f auf dem Gebiet $\Omega = \{(x, y) = \frac{x^2}{4} + y^2 \leq 1, y \geq 0\}$.

Solution

We first find the critical points:

$$\nabla f = \begin{pmatrix} 8xy^2 - 2x \\ 8x^2y - 8y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x(4y^2 - 1) = 0 \Rightarrow x = 0 \text{ or } y = \pm\frac{1}{2}$$
$$8y(x^2 - 1) = 0 \Rightarrow x = \pm 1 \text{ or } y = 0$$

$(0, 0), (\pm 1, \pm\frac{1}{2})$ are the critical points of f. Since $y \geq 0$ we only take $P_1 = (0, 0), P_{2,3} = (\pm 1, \frac{1}{2})$. Then we need to compute Hess(f).

$$\text{Hess}(f) = \begin{pmatrix} 8y^2 - 2 & 16xy \\ 16xy & 8x^2 - 8 \end{pmatrix}$$

$$\text{Hess}(f)(0, 0) = \begin{pmatrix} -2 & 0 \\ 0 & -8 \end{pmatrix} \Rightarrow \text{ negative definite } \Rightarrow \text{ local maxima}$$

$$\text{Hess}(f)(1, \frac{1}{2}) = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix} \Rightarrow \text{ indefinite}$$

$$\text{Hess}(f)(-1, \frac{1}{2}) = \begin{pmatrix} 0 & -8 \\ -8 & 0 \end{pmatrix} \Rightarrow \text{ indefinite}$$

To find global extrema we need to look at f on the boundary of Ω, which is the curve $\frac{x^2}{4} + y^2 = 1$ and the line $y = 0$. First on the line $y = 0$ let $g = f|_{y=0} = -x^2 + 1$.

$$g'(x) = -2x \Rightarrow x = 0, P_1(0, 0) \text{ is a point we need to check}$$

We also need to check the corners $P_{4,5} = (\pm 2, 0)$. On the ellipse: let $h = f|_{\frac{x^2}{4}+y^2=1} = -x^4 + 4x^2 - 3$.

$$h'(x) = -4x^3 + 8x = 0 \Rightarrow P_6 = (0,1), P_{7,8} = (\pm\sqrt{2}, \frac{1}{\sqrt{2}})$$

Now we look at the values of f at these points. $f(P_1) = f(0,0) = 1, f(P_{2,3}) = 0, f(P_{4,5}) = -3, f(P_6) = -3, f(P_{7,8}) = 1$. f has also a minima at $(\pm 2, 0), (0,1)$ and a maxima at $(0,0), (\pm\sqrt{2}, \frac{1}{\sqrt{2}})$.

Example (FS 2010)

1. Bestimme das Taylorpolynom erster Ordnung der Funktion $f(x,y) = e^{x^2}(x+y)$ um dem Punkt $(1,1)$.
2. Bestimme $c \in \mathbb{R}$ so dass der Vektor $\begin{pmatrix} 1 \\ -1 \\ c \end{pmatrix}$ tangential an den Graphen $g(f) = \{(x, y, f(x,y)) : (x,y) \in \mathbb{R}^2$ im Punkt $(1,1,z)$ liegt.

Solution

1. $$\nabla f = \begin{pmatrix} 2xe^{x^2}(x+y) + e^{x^2} \\ e^{x^2} \end{pmatrix}, \nabla f(1,1) = \begin{pmatrix} 5e \\ e \end{pmatrix}, f(1,1) = 2e$$

$$\begin{aligned} f(x,y) &= 2e + \begin{pmatrix} 5e \\ e \end{pmatrix} (x-1, y-1) + r_2(x,y) \\ &= 2e + 5e(x-1) + e(y-1) + (x-1)(\nabla^2 f)(t) \begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \end{aligned}$$

   The Taylor polynomial of order 1 is $z = 2e + 5e(x-1) + e(y-1)$

2. The vector $(1, -1, c)^T$ must be perpendicular to the normal vector of the plane, that is (from part 1) $n = (5e, e, -1)^T$. Hence

$$(1, -1, c) \cdot (5e, e, -1) = 0 \Rightarrow 4e - c = 0 \Rightarrow c = 4e$$

2.4 Line (Weg) integral

Let $v : \Omega \to \mathbb{R}^n$ be a vector field and γ a curve with parametrization $\gamma : [a,b] \to \Omega, t \to \gamma(t)$. Then the line integral of v along γ is defined as

$$\int_\gamma v ds = \int_a^b < v(\gamma(t)), \gamma'(t) > dt$$

Facts

1. $\int_\gamma vds$ is independent of the parametrization of the path.
2. $\int_{\gamma_1+\gamma_2} vds = \int_{\gamma_1} vds + \int_{\gamma_2} vds$
3. $\int_\gamma vds = -\int_{-\gamma} vds$
4. If v is the gradient vector field associated to a function f i.e. $v = df$ then $\int_\gamma vds = f(\gamma(b)) - f(\gamma(a))$, where $\gamma : [a,b] \to \Omega$.

Equivalent one can change everything in terms of 1-forms $\lambda = \lambda_1 dx^1 + \lambda_2 dx^2 + ... + \lambda_n dx^n$. Then

$$\int_\gamma \lambda = \int_a^b \lambda(\gamma(t))\gamma'(t)dt$$

Fact

$\lambda : \Omega \to L(\mathbb{R}^\ltimes, \mathbb{R})$ a constant 1-form, then the following are equivalent:

1. $\exists f \in C'(\Omega) : df = \lambda$
2. For every 2 continuous C'-paths γ_1, γ_2 with $\gamma_i : [a_i, b_i] \to \Omega$ with the same beginning and ending points:
$$\int_{\gamma_1} \lambda = \int_{\gamma_2} \lambda$$
3. For every closed curve γ, $\int_\gamma = 0$

■

Definition

A vector field $V : \Omega \to \mathbb{R}^n$ is called conservative if $\int_\gamma V ds = 0$ for every closed curve γ.

Fact

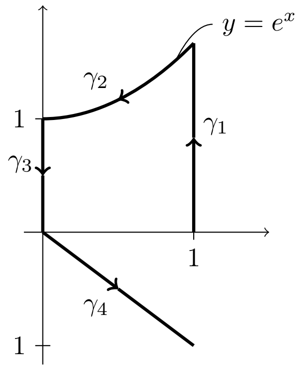
For a simply connected region Ω, we have

$$\text{V is conservative} \iff v = \nabla f \text{ for some function } f$$

■

Example (FS 2010)

Berechne das Wegintegral $\int_\gamma vds$ entlang das eingezeichneten Weg für das Vektorfeld $v(x,y) = \begin{pmatrix} xy^2 \\ -y \end{pmatrix}$



$$\int_\gamma vds = \int\limits_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} vds$$

Solution

We first parametrize the curves:

1. $\gamma_1 = (1, t), t \in [0, e] \Rightarrow \gamma_1' = (0, 1)$
2. $-\gamma_2 = (t, e^t), t \in [0, 1] \Rightarrow -\gamma_2' = (1, e^t)$
3. $\gamma_3 = (0, 1 - t), t \in [0, 1] \Rightarrow \gamma_3' = (0, -1)$
4. $\gamma_4 = (t, -t)t \in [0, 1] \Rightarrow \gamma_4' = (1, -1)$

$$\int_\gamma vds = \int_0^e (t^2, -t) \cdot (0, 1)dt + \int_1^0 (te^{2t}, -e^t) \cdot (1, e^t)dt + ...$$

See the last example of Green's theorem for the rest.

2.5 Integration in $\mathbb{R}^n$

The Riemann integral in $\mathbb{R}^n$ is constructed in analogue way to the case in n=1. With Riemann sums over subintervals is replaced with sums over ßub-rectangles". dx is replaced with an n-dimensional volume element $dvol_n$ which we denote also by $d\mu(x)$.

Fact

For a normal-region

$$D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \phi(x) < y < \psi(x)\}$$

Where ϕ, ψ are continuous functions: $\mathbb{R} \to \mathbb{R}$ we have

$$\int_D fd\mu(x) = \int_a^b \int_{\phi(x)}^{\psi(x)} f(x, y)dydx$$

■

Fact

For a rectangle $Q = [a, b] \times [c, d] \in \mathbb{R}^2$

$$\int_Q f d\mu = \int_a^b \int_c^d f dy dx = \int_c^d \int_a^b f dx dy$$

■

2.5.1 Substitution in $\mathbb{R}^n$

Let $u, v \in \mathbb{R}^n$ open intervals, $\Phi : u \to v$ bijective continuous differentiable with $\det(d\Phi(y)) \neq 0 \forall y \in u$. Then for $f : v \to \mathbb{R}$ continuous we have

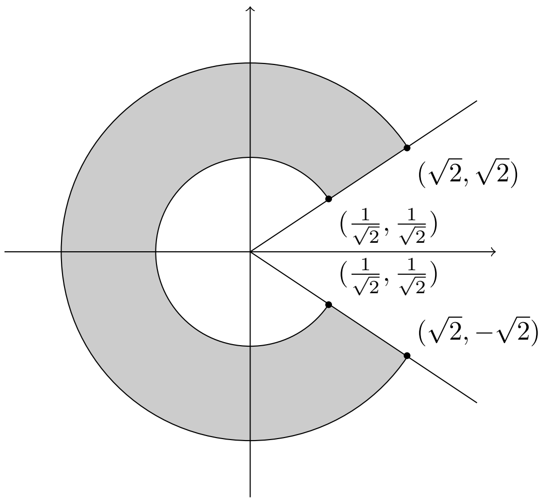
$$\int_v f(x) d\mu(x) = \int_u f(\Phi(y)) |\det(d\Phi(y))| d\mu(y)$$

Example (FS 2010)

Bestimme das Integral

$$\int_\Omega x\sqrt{x^2 + y^2} \log\left(\sqrt{x^2 + y^2}\right) d\mu(x, y)$$

wobei



Solution

Given the symmetry of the region it's better to use polar coordinates. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\sigma) \\ r\sin(\sigma) \end{pmatrix} \Rightarrow dxdy = rdrd\sigma$$

The region in (r,σ) domain is simply $1 \leq r \leq 2$, $\frac{\pi}{4} \leq \sigma \leq \frac{7\pi}{4}$.

$$\begin{aligned}
\int_\Omega x\sqrt{x^2+y^2}\log\sqrt{x^2+y^2}d\mu(x,y) &= \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}}\int_1^2 r\cos(\sigma)r\log(r)rdrd\sigma \\
&= \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}}\cos(\sigma)d\sigma\int_1^2 r^3\log(r)dr \\
&= \left[-\sqrt{2}+\frac{1}{4}x^4\log(x)-\frac{x^4}{16}\right]_1^2 \\
&= \log(16)-\frac{15}{16}
\end{aligned}$$

2.6 Green's theorem

Let $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ has a C' parametrization. Let $U \subset \Omega$ and $f = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ where $P, Q \in C'(U)$. Then

$$\int\int_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\mu = \int_{\partial\Omega} Pdx + Qdy$$

Let $V = (P, Q)$ be a vectorfield then

$$\int_{\partial\Omega} V ds = \int\int_\Omega (rot(V)d\mu$$

Where $\text{rot}(V) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ and the line integral is taken around the boundary of Ω in positive sense.

add last example (it's either in my notes nor in her)