### 0.0.1 Differentiation in many variables

A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is (total) differentiable in $x_0$ if it exists a linear map $A : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = f(x_0) + A(x - x_0) + R(x_0, x)$$

Where $\lim_{x \to x_0} \frac{R(x_0,x)}{|x-x_0|} = 0$. In this case A is called the differential of $f$ at $x_0$ and it's denoted as $(df)(x_0)$.

Let $(A_1, A_2, ..., A_n)$ be a matrix representation of the linear map $A : \mathbb{R}^n \to \mathbb{R}$ (wrt to the standard Basis). Then $f$ differentiable at $x_0$ means:

$$f(x) = f(x_0) + A_1(x^1 - x_0^1) + A_2(x^2 - x_0^2) + ... + A_n(x^n - x_0^n) + R(x, x_0)$$

$P(x, x_0) = f(x_0) + \begin{bmatrix} A_1 & ... & A_n \end{bmatrix} \begin{bmatrix} x^1 - x_0^1 \\ ... \\ x^n - x_0^n \end{bmatrix}$ is the equation of the tangent plane at the point $f(x_0)$ on the surface formed by the graph of $f$.

**Fact** if $f : \Omega \to \mathbb{R}$ is differentiable in $x_0 \in \Omega$ then the partial derivative exist and the differential $df(x_0)$ has the matrix representation

$$\begin{pmatrix} \frac{\partial f}{\partial x}(x_0) & ... & \frac{\partial f}{\partial x^n}(x_0) \end{pmatrix} = \nabla f$$

the gradient of $f$.

**Fact** $f$ differentiable in $x_0 \Rightarrow f$ is continuous in $x_0$.

**Fact** If all partial derivatives of $f$ are continuous then $f$ is differentiable.

Using these last two facts and the definition of differentiability one can study if a given function is differentiable or not.

#### Differentiation rules

Let $f, g : \Omega \to \mathbb{R}$ be differentiable in $x_0$. Then:

1. $d(f \pm g)(x_0) = df(x_0) \pm dg(x_0)$
2. $d(fg)(x_0) = g(x_0)df(x_0) + f(x_0)dg(x_0)$
3. $d(f/g)(x_0) = \frac{g(x_0)df(x_0) - f(x_0)dg(x_0)}{(g(x_0))^2}$
4. Let $h : \mathbb{R} \to \mathbb{R}$ be differentiable in $g(x_0)$, then
$$d(hog)(x_0) = h'(g(x_0))dg(x_0)$$
5. Let $H : I \subset \mathbb{R} \to \Omega \subset \mathbb{R}^n$ be differentiable in $t_0 \in \mathbb{R}$ and $f$ differentiable in $H(t_0)$. Then
$$\frac{d}{dt}(foH)(t_0) = df(H(t_0))H'(t_0)$$
where $H(t) = (H_1(t), ..., H_n(t))$ and $H'(t) = (H_1'(t), ..., H_n'(t))$

#### Directional derivative

The directional derivative of $f$ in the direction of a unit vector $e \in \mathbb{R}^n - \{0\}$ is given by $d_e f(x_0) = \nabla f(x_0) \cdot e$.

**Particular higher derivatives**

One can similarly define higher derivatives order partial derivatives for functions $f \in C^m(\Omega)$.

**Fact (Schwarz)** if $f \in C^2(\Omega)$ then $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$ and in general for $f \in C^m(\Omega)$ all $n$ partial derivatives of $f$ of order $\leq m$ are independent of the order of differentiation. Using higher order derivatives one can analogous to the 1-dimensional case define a Taylor approximation of $f$.

**Fact** Let $f \in C^m(\Omega), f : \Omega \to \mathbb{R}, \Omega \in \mathbb{R}$ and $x_1, x_0 \in \Omega$. Then

$$f(x_1) = f(x_0) + \nabla f(x_0)(x_1 - x_0) + \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)(x_i^i - x_0^i)(x_1^j - x_0^j) + R_3(f, x_1, x_0)$$

Where $\lim_{x_1 \to x_0} \frac{R(f,x_1,x_0)}{||x_1 - x_0||} = 0$

The analogue of the second derivative is given by the matrix of partial derivatives of order 2. This matrix is called the Hesse-matrix of $f$.

$$\text{Hess}(f) = \nabla^2 f = \left( \frac{\partial^2 f}{\partial x^i \partial x^j} \right)_{i,j=1,\dots,n}$$

**The extrema of a function $f : \Omega \to \mathbb{R}$**

**Definition** A point $x \in \Omega$ is called a critical point if $\nabla f(x) = 0$.

**Fact** $f$ differentiable, $x_0$ is called local extrema of $f$ then $x_0$ is a critical point.

**Fact** Let $x_0$ be a critical point of $f$. Then we have three different cases:

1. $x_0$ is a local minima if $\nabla^2 f(x_0)$ is positive definite.
2. $x_0$ is a local maxima if $\nabla^2 f(x_0)$ is negative definite.
3. Otherwise it is a saddle point (Sattelpunkt).

**Fact** Let $f : \Omega \to \mathbb{R}$ be continuous and differentiable on an open set $\Omega \subset \mathbb{R}^n$. Let $\partial\Omega$ be the boundary of $\Omega$. Then every global extrema of $f$ is either a critical point of $f$ in $\Omega$ or a global extremal point of $f|_{\partial\Omega}$ (f restricted to the boundary).

**Example (FS 2011)** Sei $f(x, y) = 4x^2y^2 - x^2 - 4y^2 + 1$. Bestimme die globalen Extrema von $f$ auf dem Gebiet $\Omega = \{(x, y) = \frac{x^2}{4} + y^2 \leq 1, y \geq 0\}$.

**Solution** We first find the critical points:

$$\nabla f = \begin{pmatrix} 8xy^2 - 2x \\ 8x^2y - 8y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 2x(4y^2 - 1) = 0 \Rightarrow & x = 0 \text{ or } y = \pm\frac{1}{2} \\ 8y(x^2 - 1) = 0 \Rightarrow & x = \pm 1 \text{ or } y = 0 \end{aligned}$$

$(0, 0), (\pm 1, \pm\frac{1}{2})$ are the critical points of $f$. Since $y \geq 0$ we only take $P_1 = (0, 0), P_{2,3} = (\pm 1, \frac{1}{2})$. Then we need to compute Hes(f).

$$\text{Hess}(f) = \begin{pmatrix} 8y^2 - 2 & 16xy \\ 16xy & 8x^2 - 8 \end{pmatrix}$$

$$\text{Hess}(f)(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -8 \end{pmatrix} \Rightarrow \text{ negative definite } \Rightarrow \text{ local maxima}$$

$$\text{Hess}(f)(1, \frac{1}{2}) = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix} \Rightarrow \text{ indefinite}$$

$$\text{Hess}(f)(-1, \frac{1}{2}) = \begin{pmatrix} 0 & -8 \\ -8 & 0 \end{pmatrix} \Rightarrow \text{ indefinite}$$

To find global extrema we need to look at $f$ on the boundary of $\Omega$, which is the curve $\frac{x^2}{4} + y^2 = 1$ and the line $y = 0$. First on the line $y = 0$ let $g = f|_{y=0} = -x^2 + 1$.

$$g'(x) = -2x \Rightarrow x = 0, P_1(0,0) \text{ is a point we need to check}$$

We also need to check the corners $P_{4,5} = (\pm 2, 0)$. On the ellipse: let $h = f|_{\frac{x^2}{4}+y^2=1} = -x^4 + 4x^2 - 3$.

$$h'(x) = -4x^3 + 8x = 0 \Rightarrow P_6 = (0,1), P_{7,8} = (\pm\sqrt{2}, \frac{1}{\sqrt{2}})$$

Now we look at the values of $f$ at these points. $f(P_1) = f(0,0) = 1, f(P_{2,3}) = 0, f(P_{4,5}) = -3, f(P_6) = -3, f(P_{7,8}) = 1$. $f$ has also a minima at $(\pm 2, 0), (0,1)$ and a maxima at $(0,0), (\pm\sqrt{2}, \frac{1}{\sqrt{2}})$.

**Example (FS 2010)**

1. Bestimme das Taylorpolynom erster Ordnung der Funktion $f(x,y) = e^{x^2}(x+y)$ um dem Punkt $(1,1)$.

2. Bestimme $c \in \mathbb{R}$ so dass der Vektor $\begin{pmatrix} 1 \\ -1 \\ c \end{pmatrix}$ tangential an den Graphen $g(f) = \{(x,y,f(x,y)) : (x,y) \in \mathbb{R}^2$ im Punkt $(1,1,z)$ liegt.

**Solution**

1. $\nabla f = \begin{pmatrix} 2xe^{x^2}(x+y) + e^{x^2} \\ e^{x^2} \end{pmatrix}, \nabla f(1,1) = \begin{pmatrix} 5e \\ e \end{pmatrix}, f(1,1) = 2e$

   $$f(x,y) = 2e + \begin{pmatrix} 5e \\ e \end{pmatrix} (x-1, y-1) + r_2(x,y) = 2e + 5e(x-1) + e(y-1) + (x-1)(\nabla^2 f)(t) \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$$

   The Taylor polynomial of order 1 is $z = 2e + 5e(x-1) + e(y-1)$

2. The vector $(1, -1, c)^T$ must be perpendicular to the normal vector of the plane, that is (from part 1) $n = (5e, e, -1)^T$. Hence

   $$(1, -1, c) \cdot (5e, e, -1) = 0 \Rightarrow 4e - c = 0 \Rightarrow c = 4e$$

### 0.0.2 Line(Weg) integral

Let $v : \Omega \to \mathbb{R}^n$ be a vector field and $\gamma$ a curve with parametrization $\gamma : [a,b] \to \Omega, t \to \gamma(t)$. Then the line integral of $v$ along $\gamma$ is defined as

$$\int_\gamma v ds = \int_a^b < v(\gamma(t)), \gamma'(t) > dt$$

**Facts**

1. $\int_\gamma vds$ is independent of the parametrization of the path.
2. $\int_{\gamma_1+\gamma_2} vds = \int_{\gamma_1} vds + \int_{\gamma_2} vds$
3. $\int_\gamma vds = -\int_{-\gamma} vds$
4. If v is the gradient vector field associated to a function $f$ i.e. $v = df$ then $\int_\gamma vds = f(\gamma(b)) - f(\gamma(a))$, where $\gamma : [a, b] \to \Omega$.

Equivalent one can change everything in terms of 1-forms $\lambda = \lambda_1 dx^1 + \lambda_2 dx^2 + ... + \lambda_n dx^n$. Then

$$\int_\gamma \lambda = \int_a^b \lambda(\gamma(t))\gamma'(t)dt$$

**Fact** $\lambda : \Omega \to L(\mathbb{R}^\ltimes, \mathbb{R})$ a constant 1-form, then the following are equivalent:

1. $\exists f \in C'(\Omega) : df = \lambda$
2. For every 2 continuous C'-paths $\gamma_1, \gamma_2$ with $\gamma_i : [a_i, b_i] \to \Omega$ with the same beginning and ending points:
$$\int_{\gamma_1} \lambda = \int_{\gamma_2} \lambda$$
3. For every closed curve $\gamma$, $\int_\gamma = 0$

**Definition** A vector field $V : \Omega \to \mathbb{R}^n$ is called conservative if $\int_\gamma V ds = 0$ for every closed curve $\gamma$.

**Fact** For a simply connected region $\Omega$, we have

$$\text{V is conservative} \iff v = \nabla f \text{ for some function } f$$