Zusammenfassung 19/12/2013

Every bounded from above/below set in R has a Supremum/Infimum.

Corollary 2.11

i. Falls E C F und F nach oben beschränkt ist, gilt

$$SUP(E) \leq SUP(F)$$

ii. Falls E C F und E nach unten beschränkt ist gilt

$$Inf(F) \leq Inf(E)$$

iii. Falls $\forall x \in E, \forall y \in F \qquad x \leq y$

Dann folgt $Sp(E) \leq Inf(F)$

Componentwise addition $R^n, +, -$

Scalar multiplication

$\langle R^n, +, x \rangle$ is a vectorspace

An important property is the triangular inequality

$$C \sim \left(R^2, \oplus, \odot \right)$$

Multiplication

$$(a,b)(c,d) = (ac - bd, ad + bc)$$

Sequences and series

Sequence $(a_n) \subset R$

$$a_n \to a \,(n \to \infty)$$

$$\forall \varepsilon > 0 \exists n(\varepsilon) > 0 : |a_n - a| < \varepsilon, n > n(\varepsilon)$$

Proprieties:

- if the sequence converges then the limit is unique
- if the sequence converges then it's bounded
- if the sequence is unbounded doesn't converge

Convergence criteria

I $\lim a_n b_n = ab$

...

Monotone convergence

a_n is bounded and monotone $\Rightarrow a_n$ converges

Example $a_n = \left(1 + \frac{1}{n}\right)^n \to e$

This theorem is handy for proving convergence of sequences defined by recursion.

Bolzano - Weierstrass

Every bounded sequence has a convergent subsequence.

Note: we can have infinite many convergent subsequence by removing the first element of the subsequence.

an is bounded, $a = \liminf(a_n)$

$$b = \limsup(a_n)$$

Then the following are equivalent

1 an converges to a

2 every subsequence converge to a

3 a=b

Cauchy criteria

an converges $\Leftrightarrow$ an is Cauchy

$$\forall \varepsilon > 0 \exists n(\varepsilon) : |a_n - a_m| < \varepsilon, m, n > n(\varepsilon)$$

Series

Sequence of the partial sum

$$S_n = a_1 + a_2 + \ldots . + a_n$$

$$\sum_{k=1}^{\infty} a_k \text{ converges } \Leftrightarrow \exists \lim S_n$$

Convergence criteria for series

- $\sum_{k=1}^{\infty} a_k$ converges $\Leftrightarrow \left| \sum_{k=n}^{\infty} a_k \right| < \varepsilon$
- $\sum_{k=1}^{\infty} a_k$ converges $\Rightarrow \lim(a_n) = 0$
- Comparison tests (Majoranten, Minoranten)

I. Find a k as starting point to compare the series

$$\exists k_0 : |a_k| \leq b_k, \forall k \geq k_0$$

Ii. $\sum b_k$ converges

$$\Rightarrow \sum a_k \quad \text{converges}$$

- Ratio test

$$limsup\left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum a_n \text{ converges}$$

$$liminf\left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \sum a_n \quad \text{diverges}$$

- root test

$$\text{limsup}^n\sqrt{a_n} < 1 \Rightarrow \quad \text{converges}$$

$$liminf\sqrt[n]{a_n} > 1 \Rightarrow \quad \text{diverges}$$

In both cases if lim=1 we don't have informations.

- absolute convergence

$$\sum |a_k| \text{converges} \Rightarrow \sum a_k \text{ converges}$$

The importance of absolute convergence is that we can reorder the terms of the sum as we want.

Continuity, limits of functions

f has a limit of a at x=x0

$\lim_{x \to x_0} f(x) = a$ if for every x_n with

$$\lim(x_n) = x_0, \lim f(x_n) = a$$

f is continuous in x0 if

$$f(\lim(x_n)) = \lim f(x_n)$$

Other definition

$$\forall \varepsilon > 0 \exists \delta > 0 : |x - x_0| < \delta \Rightarrow$$
$$|f(x) - f(x_0)| < \varepsilon$$

For f continuous on Ω

$$\forall x_0 \in \Omega, \forall \varepsilon > 0, \exists \delta > 0 : \forall x \in \Omega$$
$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Uniform (gleichmäßig) continuity

$$\forall \varepsilon > 0, \exists \delta > 0 : \forall x, x_0 \in \Omega$$
$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

If f is continuous on a compact set, then it is uniform continuous on this set.

Important properties of continuous functions

1. $f:[a,b]\to R$
   $\Rightarrow f([a,b])$ is bounded and there exist a Supremum and an Infimum (max and min)
2. Zwischenwertsatz

   Satz 4.19: Zwischenwertsatz

   Seien a <b in R und f [a, b]->R eine stetige Funktion mit f (a)<=f (b) dann gibt es zu jedem x in [a, b] mit f (x)=y

3 $f:[a,b]\to R$, continuous, strict monotone Bild (f)=[c, d]=[f (a),f (b)], f is bijektiv and the inverse is also continuous.

Punktweise und gleichmäßigeKonvergenz

Pointwise convergence

$$f_n \to f \quad \text{if} \quad \forall x \in \Omega \lim_{n\to\infty} f_n(x) = f(x)$$

In pointwise convergence one can have a sequence of continuous functions with the limit f not continuous.

This is cured by uniform convergence

A sequence of functions converges uniform to f if

$$\sup_{x\in\Omega} |f_k(x) - f(x)| \to 0 (k \to \infty)$$

If f_k converges to f uniformly and f_k are continuous, then also f is continuous.

Differentialrechnung

Definition: $f : I \to R$ ist in $x_0 \in I$ stetig wenn folgender Grenzwert existiert:

$$f\prime(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Mittelwertsatz der Differentialrechnung

Ist f auf [a,b] stetig und in]a,b[differenzierbar, so gibt es ein $c \in]a, b[$ mit

$$f\prime(c) = \frac{f(b) - f(a)}{b - a}$$

Umkehrsatz

Sei f differenzierbar, bijektiv, ... Dann

$$\left(f^{-1}\right)\prime(y) = \frac{1}{f\prime\left(f^{-1}(y)\right)}$$

Extremalstelle: trivial

Taylorreihe

Funktionen lassen sich in der Umgebung eines Punktes durch eine Potenzreihe beschreiben.
Die Taylorreihe von f um den Punkt a ist definiert durch

$$Tf(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!} (x-a)^{n}$$

Die linearisirung der Reihe mit Grad m entspricht das m-te Taylorpolynom.

$$T_m f(x) = \sum_{n=0}^{m} \frac{f^{n}(a)}{n!} (x-a)^{n}$$

Ist f (m+1) mal differenzierbar, so ist der Rest:

$$R_m f(x) = \frac{f^{m+1}(\xi)}{(m+1)!} (x-a)^{m+1}$$

Wobei $\xi \in (a, x)$

Auf jedem Fall muss gelten:

$$\lim_{x \to a} \frac{R_m f(x)}{(x-a)^m} = 0$$

Sonst wäre die Taylorreihe keine gute Approximation. Die Taylorpolynome sind sehr nützlich um Grenzwerte zu rechnen.