Student-generated examples in the learning of mathematics

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Stimulating Students To Construct Boundary Examples

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Abstract: We address three common difficulties encountered by students: not appreciating the necessity of conditions in a theorem before using it; using non-generic examples as if they were generic; and ignoring counter-examples as pathologies. We propose the conjecture that students would be more likely to remember and to appreciate the importance of conditions, if they were stimulated to construct examples for themselves which show why each of the conditions is necessary. Furthermore, constructing their own examples is likely to prompt them to explore the space of possibilities admitted by definitions, and hence to appreciate both the range of situations encompassed by the definition, and the force of both definition and theorem. We illustrate our conjectures with some particular but generic tasks for students, and we use these to consider what is involved in constructing examples, leading to ways to support students in learning how to construct them for themselves.

1. Method Our method of enquiry is to identify phenomena we wish to study, and to seek examples within our own experience. We then construct task-exercises to offer to others to see if they recognise what we find ourselves noticing. Through refinement and adjustment of task-exercises in the light of experience and of reading relevant literature, we both extend our own awarenesses, and offer others experiences which may highlight or even awaken sensitivities and awarenesses for them. These sensitivities and awarenesses may inform their future practice. As task-exercises are developed and shared, actions which exploit what is noticed become part of regular teaching for the benefit of students. Our method does not attempt to capture or cover the experience of readers. Rather it aims to make contact with that experience, perhaps challenging interpretations, perhaps pointing to features not previously noticed. The data of this method are the experiences generated, the sensitivities to notice which are enhanced. If you recognise at least something of what we are talking about as a result of having worked on these problems, you may be stimulated to look out for similar experiences in the future, and over time, begin to act upon what you notice. Validity in this method lies in you finding your actions being informed in the future, not in what we say.

The task-exercises which follow are intended to bring to the surface various features about the construction of examples to meet constraints, starting from the premise that if you simply ask students out of the blue to construct a mathematical object, they are likely to find it very difficult if not impossible. This may lead a tutor to lose confidence and to conclude that 'students can't do this sort of task'. The effect is a move from impoverished past experience (students are not aware of the construction of objects) to a continuation of impoverished experience (students are not called upon to construct examples, because 'it is too hard'), and so the cycle continues. The aim of this paper is to locate ways of breaking out of this cycle.

2. Task-Exercises

1 A Routine Problem: solve the differential equation $f''(x) + b f'(x) + c f(x) = 0$.

2 *A Mean Problem*: Observe that $\int_0^2 (1-x)\,dx = 0$. Generalise!

3 *A Divisory Problem*: Find all the positive integers which have an odd number of divisors.

4 *Another Square Problem*: Given two distinct straight lines L_1 and L_2 and a point E not on them, construct a square with one vertex at E, and one vertex on each of L_1 and L_2.

5 *Rolle Points*: Rolle's theorem tells us that any function differentiable on an interval has a point in the interior of that interval at which the slope of the function is the same as the slope of the chord between the points on the curve at the ends of that interval. Where on that interval would you expect to look for such a point? For example, are there any functions for which the Rolle point of every interval is the midpoint? A natural question to ask is whether there are any functions for which the Rolle point on any interval is, say, 2/3 of the way along the interval, or more generally ρ of the way along.

6 *Inflection points*: A common method for finding inflection points of a curve which is at least twice differentiable, is to differentiate twice and set equal to zero to find the abscissa. Sometimes this gives a correct answer for a correct reason, sometimes it gives a correct answer for a wrong reason, and sometimes it gives an incorrect answer. Construct examples which exemplify these three situations, and also a family of examples which include all three in each member, and thus might bring students up against these different possibilities. Must a function be twice differentiable to have an inflection point? What about members of the family $x^k\sin(1/x^2)$?

3. Specific Comments The first task (*A Routine Problem*) succumbs to a standard procedure, and indeed, few students are likely to solve it without having been shown a technique (both in particular and in general). But in being shown the technique, few probably ever think about what it is doing in terms of displaying a strategy: when in doubt try something you are familiar with and be prepared to adjust it, or better, express in symbols a general class of objects and then see if the constraints on parameters can be resolved. They see it simply as the solution to a problem, not a process of construction. Yet what it achieves is the construction of not just a single function but of a whole class of functions. The theorems of the topic show that this class includes all solutions. In other words, their structure characterises all solutions to the constraint.

For the second task (*A Mean Problem*), some students will think of varying the function but not the limits, while others will vary the limits but not the function. Some will try different specific linear or other functions, while others will write down a general (linear) function and then see what the integral condition imposes on the parameters. Some will remain with polynomials (a very fruitful as well as manageable constraint), while others will extend to familiar periodic functions. What they choose to vary and what they leave unvaried suggests where they are confident and where they are not. In any case, varying two things at once requires considerably more confidence and attention than varying just one.

Shifting attention to the fact that for a linear function, *any* interval with the zero point as the mid point will meet the constraint may raise an auxiliary question of whether there are any functions for which, for any interval with the zero point say 2/3 of the way along that interval, the integral will be zero, thus producing a sort of lopsided symmetry.

This problem nicely illustrates the situation in which incremental alterations can be made in the problem on the way to considerable generality. If the tutor's guidance keeps pushing the student to more and more variation, students will find themselves becoming aware of a wider and wider class of functions from which to choose when looking for examples meeting other constraints in the future. For example, if the tutor announces the existence of a function (held in a sealed envelope) known to be specified on [0, 1/2] and integrable there, can the student augment it to one on [0, 1] with the 0 integral property. This exposes students to a construction strategy for building functions by gluing them together at specified points.

The third task (*A Divisory Task*) is intended to highlight differences between leaping to the general but getting bogged down, and specialising to some simple examples to get a sense of what might be going on. For example, you can write down the general shape of a number (in a form which enables you to count factors) as the product of primes to various powers. You can then calculate the number of divisors, impose the condition that it must be odd, and look at what this says about your number. Alternatively you can locate some numbers which meet the conditions, and see what they have in common. Once you detect a pattern you can make a conjecture, and then try to justify that conjecture, perhaps using further work on examples as a source of insight. A related approach involves finding a way to count the number of divisors of a number from a list of its factors. In the process of investigation, students are likely to try various numbers, and perhaps even to begin seeing that writing down a number in base ten is not always the best way of exposing structure, enabling them in the future to construct numbers meeting constraints by choosing between different representations.

The fourth task (*Another Square Problem*) can be approached using a device which works extremely well in the context of dynamic geometry software. If a problem is too hard, remove a constraint, and then look for a locus. Here we require remove the requirement that one vertex lie on L_2, and we examine the loci of the other vertices of all squares meeting the remaining constraints. The fact that the loci are straight-lines informs us about what we need to prove, and also points the way to building a construction (Love 1996) The same strategy works in algebraic contexts. You remove or weaken a constraint, usually by expressing the more general using a parameter, and then look at the class of solutions this produces to see if you can meet the original constraint by suitable choice of parameter.

For the fifth task (*A Rolle Problem*), finding why there are no polynomials of degree larger than 1 for which ρ can be anything other than 1/2 is instructive because of the need to make use of the 'over every interval' condition. This is likely to call upon using the technique for forcing a polynomial to be constantly zero. Again one method is to try some simple functions (clearly a linear function works for any ρ, but might there be any others?), eventually stumbling on quadratics for the midpoint but failing with cubics and higher. Moving to exponential functions $e^{\lambda x}$ also fails, but by weakening the constraint, solutions to an adjusted problem can be found. For example, it *is* possible for intervals of a fixed width h to have a constant ρ (depending on h), for which every interval of width h has a Rolle Poin*t* ρ of the way along the interval. The possible values for ρ turn out to depend on λ, so that for any specified $\rho > 0$ there is a class of exponential functions which have the Rolle Point of the way along the interval for every interval of specified width (depending on ρ). This approach is analogous to the geometrical strategy in the previous problem: You weaken a constraint and try to solve that problem.

This approach to mathematical exploration is classic mathematical behaviour, but not one which is often drawn to students' attention. Consequently they are hampered if and when they are asked to construct an object meeting specified constraints for themselves. The method of trying a simple case (a pure quadratic, a pure linear), and then when one works generalising to a broader class (all quadratics) and when one fails, seeing why none of its class can work, is quintessential mathematical thinking.

Another approach exploits Taylor's theorem, but leads to the question of whether there are any solutions which do not have a Taylor approximation.

In the sixth task (*Inflection*), students are invited to exemplify the incorrect use of (part of) a technique, and to 'bury' a wrinkle or potential difficulty which is the subject of conditions surrounding the statement of the technique. Students whose method of finding inflection points of a curve is to differentiate twice and set equal to zero to find the abscissa get a correct answer for a correct reason on functions like $f(x) = x^3 - x$ where the slope at the inflection is *not* zero, though many expect an inflection point to have a zero slope. Students using the same method on $f(x) = x^3$ will get a correct answer for an incorrect reason (you have to reason about the change in $f'(x)$), and on $f(x) = x^4$ they will of course get an incorrect answer. The class of functions $x^k e^{\left(-1/x^2\right)}$ is not one many students are likely to come up with, but running into these functions in various contexts involving construction may alert them to their use (Michener 1978).

Trying to bury all three possibilities in one function could lead students to appreciate the underlying structure of what is going on (if the second derivative has repeated roots, the parity of the repetition determines whether there is an inflection or not). Such an example could then be used on other students to see if they are caught up in the mechanics of the technique and do not notice the wrinkles, or whether they correctly apply the technique and its conditions. Students might also wish to exemplify inappropriate use of the technique due to the function failing to have a first or perhaps a second derivative!

Caunt (1914 p137) gives the conditions of a point of inflection as $f''=0$ and changes sign in passing through it, or $f''=0$ and $f'''\neq 0$. He then draws up a table showing the effects of combinations of f' and f'' being positive, negative or zero, with illustrations. He also points out that an inflection point is a "point at which the tangent passes through three 'consecutive points' on the curve". SMP (1967 p212) distinguishes between stationary points which are also points of inflection, and oblique points of inflection, and suggests constructing a table showing the sign of the gradient as x increases. Neil & Shuard (1982 p54) first give an intrinsic definition (" a point at which the graph changes the direction in which it is bending"). They then give an intrinsic definition (f'' changes sign), and two warnings: that students often assume that the slope has to be 0 at an inflection point, and that "consideration of the second derivative should not be over emphasised".

4. General Comments Any technique which yields answers illustrates ways to construct mathematical objects. Solving a first order linear differential equation, finding an integral, disentangling eigenvalues and eigenvectors for a matrix, and changing basis are all examples of constructing an object with specified properties. Thus constructing mathematical objects is something students have seen and done a great deal of, yet for the most part they are unaware of it in these terms.

Curriculum topics can be seen as domains in which people have worked out techniques for resolving problems, usually through constructing objects meeting certain constraints. Where those techniques are deemed sufficiently simple, they are taught to students. But the teaching removes the creative aspects of problem resolution in favour of mechanical manipulations, and this means that students do not have to think about the class of objects from which they are trying to select a particular one. Since students are not used to thinking this way, they are often at sea when asked to construct an object for themselves having certain properties.

It follows that if students are going to be expected to construct other examples for themselves, they need to be aware of techniques as tools to call upon for this purpose. But there is a second level awareness which could also be offered to students. Where do techniques for solving specific classes of problems come from? How are they found?

Behind many of these techniques lies a very ancient and powerful principle, derived from the ancient roots of algebra: you describe what a general object looks like, then you impose constraints and use these to locate values of parameters. In arithmetic one proceeds from the known to the unknown through making calculations (synthesis); in algebra one proceeds from the unknown to the known (analysis): starting from the as-yet-unknown, expressing calculations as if one were checking a proposed solution, revealing relations and equations which are then resolved through algebraic techniques. Theon of Alexandria may have been the first to distinguish between *analysis* (beginning work with "the assumption of what is sought as though it were granted …" quoted in Klein 1932, p155) and *synthesis* (beginning work with "the assumption of what is granted …" quoted in Klein 1932, p155), but the idea has been taken up many times since. For example by Mary Boole (Tahta 1972 p55) who spoke of "acknowledging ignorance" by denoting what is not known by a label and then manipulating that label.

5. Constructing Boundary Examples Constructing boundary examples is a special case of constructing mathematical objects, which is why the bulk of the paper has focused on the more general problem. The problems discussed revealed among other things that the strategy of expressing a general object of particular form and then seeking constraints on the parameters in order to locate an object which meets the given constraints is powerful and pervasive. You start with maximum freedom (generality) and then you impose constraints. Another form of it is weakening a constraint to reveal a class of solutions, which sometimes enables a solution to be found to weaker constraints while the tighter constraint remains unsolved. This is what mathematicians consider to be 'progress'.

When students are offered examples to illustrate theorems, and even where these are boundary examples because they show why constraints are required, students have to make sense of what the examples are exemplifying. For example, MacHale (1980) pointed out that many students dismiss $f(x) = |x|$ as a pathology when shown that it is not differentiable at one point. They continue to act as if most functions are actually differentiable everywhere, and reasonably so because the particular example seems to them contrived. Whereas to the tutor it is patently obvious what is being exemplified and why the example is a boundary example (the same construction could be used to produce a wide range of functions differentiable at all sorts of different points), students do not usually have the same sense of generality as the tutor, and so may not appreciate the examples in the same way, at least without some help (eg. functions glued together continuously but not differentiably) at a range of points. In order to appreciate the particular, the students need to appreciate the general which it particularises, yet it is through the particular that they begin to appreciate the general!

If students are challenged to make us of the idea behind $f(x) = |x|$ to construct functions which are differentiable everywhere except at 1, 2, 3, … points, or at all points in $\{1/n: n \text{ an integer} \neq 0\}$, they may begin to appreciate the immense number and range of examples signified by and constructible from the one idea.

Furthermore, when an example is given, the students may alight on specific features which the tutor knows are irrelevant. Mariotti (1992) and Fischbein (1993) have studied this in detail in the context of geometry. But the same principle applies to the algebraic. Watson & Mason (1998) developed a technique which is specifically designed to alert students to the possibility that they have a narrow range of examples to call upon, and that they may be using overly special cases as their exemplars of properties. For example, here is a task-exercise which often serves this purpose with respect to continuous functions.

- Sketch the graph of a function on the interval [0, 1];
- Sketch the graph of a continuous function on the interval [0, 1];
- Sketch the graph of a differentiable function on the interval [0, 1];
- Sketch the graph of a continuous function on the interval [0, 1], with one of its extremal values at the left end of the interval [0, 1];
- Sketch the graph of a continuous function on the interval [0, 1], with both its extremal values at the end points of the interval [0, 1];
- Sketch the graph of a continuous function on the interval [0, 1], with its extremal values at the end points, and with a local maximum in the interior of the interval [0, 1];
- Sketch the graph of a continuous function on the interval [0, 1], with its extremal values at the end points, and with a local maximum and a local minimum in the interior of the interval [0, 1].

Now comes the interesting part! Work your way back through the examples, making sure that at each stage your example does *not* satisfy the constraints which follow! Thus your first example must be a function but must not be continuous; your last but one example must have a local maximum in the interior but not a local minimum. Finding that a set of constraints seem mutually incompatible is an excellent way to generate a conjecture leading to a little theorem. The structure of this kind of task forces students to become aware of a more general class of examples than they may have considered the first time.

Another device for developing awareness of the general class of objects which meet a given constraint is to ask students to construct a simple object, then a peculiar object (having something unusual about it; perhaps some feature that no-one else in the class is likely to think of), then to try to describe a general object of the class (Bills 1996). For example,

- Write down a number leaving a remainder of 1 on dividing by 7;
- Write down a number leaving a remainder of 1 on dividing by 7, but which you think no-one else (who is present, who is alive, …) will think of;
- Write down a description of all numbers leaving a remainder of 1 on dividing by 7.

Now we could go on and multiply two of these together and discover that it is of the same form, perhaps even exploring the notion of a prime in this restricted set of numbers closed under multiplication.

Another example related to *inflection* could be:

- Write down a cubic which passes through the origin and has an inflection point there.
- Write down a cubic which passes through the origin and has an inflection point there but which no-one else in the room is likely to write down.
- Write down all cubics which pass through the origin and have an inflection point there.

There is an opportunity to explore how much freedom there is in specifying the abscissae of inflection points for polynomials together with the slopes at those points.

In both these tasks, the effect of the second construction is to prompt students to think not just of one but of several, and then to become playful with some method of generating them. In the process they come into contact with the general.

A context for constructing boundary examples can also be provided by choosing several definitions, or candidates for definitions, of a single concept, and asking students to construct examples which distinguish between the definitions where possible. For example with inflection points, the definitions mentioned so far are but five among many others. A task can be constructed by listing the definitions and asking for examples which distinguish between the various definitions:

A point at which the graph changes the direction in which it is bending.

The slope of the tangent changes from increasing to decreasing or vice versa at the point.

A point at which the tangent passes through three 'consecutive points' on the curve.

A point at which the second derivative changes sign.

Either the second derivative is zero and changes sign in passing through it, or else $f'' = 0$ but $f''' \neq 0$.

Task: Construct examples where possible to show in what ways these definitions differ from each other, and examples which illustrate possible inflection points which are not covered by each definition. In what contexts might each be a sensible definition to choose?

6. Summary Using task-exercises selected to reveal or highlight aspects of example construction, we have suggested that any technique for solving a class of problems can be seen as a construction process, and that adopting this perspective would alert students to this way of thinking more generally. The mathematical strategy of writing down a general object and then imposing constraints to see if parameters can be chosen to meet those constraints mirrors the shift from synthesis to analysis identified by Theon in early Greek geometry, and lying at the heart of the development of algebra as a powerful problem solving device. Offering a particular example and then guiding students to use that idea to construct families of objects with similar properties encourages students to experience the significance of the 'single example'. If tutors become aware of construction techniques such as using glued functions to piece together a function with several specified properties, and if they also become aware that this can be generalised (meet some constraints with part of an object, and other constraints with another part and stick them together somehow), they can then use that awareness to stimulate students into using it, then draw students' attention to the use. Three structures were exemplified for tasks which encourage students to extend their access to paradigmatic and generic examples. Constructing classes of examples leads naturally to the problem of classifying or characterising all possible such examples.

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Mathematics: Art and Science

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Mathematics: Art and Science

A. Borel*

Editor's note: *Apart from some minor changes, the following article is a translation of the text of a lecture delivered, in German, at the Carl Friedrich von Siemens Stiftung, Munich, on May 7, 1981, and, in a slightly modified form, as the first of three "Pauli-Vorlesungen", on February 1, 1982, at the Federal School of Technology, Zurich.*

The Intelligencer *requested permission from the author to publish a translation of the text. We supplied the translation which the author checked and modified. We wish to thank him for his considerable help in improving the original translation.*

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Ladies and Gentlemen,

It is a great honor to be invited to address you here, but one which is fraught with difficulties. First, there is a rather natural reluctance for a practicing mathematician to philosophize about mathematics instead of just giving a mathematical talk. As an illustration, the English mathematician G. Hardy called it a "melancholy experience" to write *about* mathematics rather than just prove theorems! However, had I not surmounted that feeling, I wouldn't be here, so I need not dwell on it any more. More serious difficulties arise from the fact that there are mathematicians and nonmathematicians in the audience. Whether one should conclude from this that my talk is best suited for an empty audience is a question which every one of you will have answered within the next hour and therefore needs no further elaboration. The difficulty brought about by the presence of mathematicians here is that it makes me aware, almost painfully aware, that in fact everything about my topic has already been said, all arguments have already been presented and argued pro and con: Mathematics is only an art, or only a science, the queen of sciences, merely a servant of science, or even art and science combined. The very subject of my address, in Latin *Mathesis et Ars et Scientia Dicenda* appeared as the third topic in the defense of a dissertation in the year 1845. The opponent claimed it was only art, but not science [1]. It has occasionally been maintained that mathematics is rather trivial, almost tautological, and as such certainly unworthy of being regarded either as art or as science [2]. Most arguments can be supported by many references to outstanding mathematicians. It is even possible sometimes, by selective citation, to attribute widely different opinions to one and the same mathematician. So I would like to emphasize at the outset that the professional mathematicians assembled here are unlikely to hear anything new.

If I turn to the nonmathematicians, however, I encounter a much bigger, almost opposite problem: My task is to say something about the essence, the nature of mathematics. In so doing, however, I cannot assume that the object of my statements is common knowledge. Of course, I can presuppose a certain familiarity with Greek mathematics, Euclidean geometry, for example, perhaps the theory of conic sections, or even the rudiments of algebra or analytical geometry. But they have little to do with the object of present-day mathematical research: Starting from this more or less familiar ground, mathematicians have gone on to develop ever more abstract theories, which have less and less to do with everyday experience, even when they later find important applications in the natural sciences. The transition from one level of abstraction to the next was often very difficult even for the best mathematicians and represented at the time an extremely bold step. I couldn't possibly give a satisfactory survey of this accumulation of abstractions upon abstractions and of their applications in just a few minutes. Still I would feel quite uncomfortable simply to philosophize about mathematics without saying anything specific on its contents. I would also like to have a small supply of examples at hand to be able to illustrate general statements about mathematics or the position of mathematics with respect to art and natural sciences. I shall therefore attempt to describe, or at least to give an idea of, some such steps.

* Translated by Kevin M. Lenzen

In doing so I will not be able to define precisely all my terms and I don't expect full understanding by all. But that is not essential. What I want to communicate is really just a feeling for the nature of these transitions, perhaps even for their boldness and significance in the history of thought. And I promise not to spend any more than 20 minutes doing so.

A mathematician often aims for general solutions. He enjoys solving many special problems with a few general formulae. One can call this economy of thought or laziness. An age-old example is the solution to a second-degree equation, say

$$x^2 + 2bx + c = 0$$

Here b and c are given real numbers. We are looking for a real number x that will satisfy this equation. For centuries it has been known that x can be expressed in terms of b and c by the formula

$$x = -b \pm \sqrt{b^2 - c}$$

If $b^2 > c$, we can take the square root and get two solutions. If $b^2 = c$, then $x = -b$ is said to be a double solution. If $b^2 < c$, however, then we cannot take the square root and maintain, at least at the beginning secondary school level, that there is no solution.

In the sixteenth century similar formulas were devised for third- and even fourth-degree equations, such as the equation

$$x^3 + ax + b = 0$$

I won't write the formula out. It contains square roots and cube roots, so-called radicals. But an extremely interesting phenomenon was discovered that came to be called the *casus irreducibilis*. If this equation has three distinct real solutions and we apply the formula, which allows one in principle to compute them, then we meet square roots of negative numbers; at the outset these are meaningless. If we ignore the fact that they don't exist, however, and are not afraid to compute with them, then they cancel out and we get the solutions, provided we carefully follow certain formal rules. In short, starting from the given real numbers a,b, we arrive at the sought for ones by using "nonreal numbers". The square roots of negative numbers were called "imaginary numbers" to distinguish them from the real numbers, and controversies raged as to whether it was actually legitimate to use such nonreal numbers; Descartes, for example, did not want to have anything to do with them. Only around the year 1800 was a satisfactory solution—satisfactory for some at least—to this problem found. The real numbers are imbedded in a bigger system consisting of the points of the plane, i.e., pairs of real numbers, between which one defines certain operations which have the same formal properties as the four basic operations in arithmetic. The real numbers are identified with the points on the horizontal axis, and the square roots of negative numbers with those on the vertical axis. One then began to speak of complex (or imaginary) numbers. Formally we can use these mathematical objects almost as easily as the real numbers and can obtain solutions which are sometimes real, sometimes complex. For the second-degree equation mentioned earlier we can now say that there are two complex solutions if $b^2 < c$.

To a certain extent this is, of course, merely a convention, but it wasn't easy to grant to these complex numbers the same right to existence as to real numbers and not to regard them as a mere tool for arriving at real numbers. There was no strict definition of real numbers back then, but the close connection between mathematics and measurement or practical computation gave real numbers a certain reality in spite of the difficulties with irrational and negative numbers. It wasn't the same with complex numbers, however. That was a step in an entirely new direction, bringing a purely intellectual creation to the fore. As mathematicians became used to this new step, they began to realize that many operations performed with functions such as polynomials, trigonometric functions, etc., still made sense when complex values were accepted as arguments and as values. This marked the beginning of complex analysis or function theory. As early as 1811, the mathematician Gauss pointed out the necessity of devising such a theory for its own sake:

The point here is not practical utility, rather for me analysis is an independent science which would lose an extraordinary amount of beauty and roundness by discriminating against those fictitious quantities [3].

Apparently even he did not foresee the practical relevance complex analysis was later to achieve, as in the theory of electricity or in aerodynamics, for example.

But that is not the end of it. Allow me, if you will, to mention two further steps toward greater abstraction. Let us return to our second-degree equation. One can now say that it has, in general, two solutions which may be complex numbers. Similarly, an equation of the n-th degree has n solutions if one accepts complex numbers. From the Sixteenth Century on, people wondered whether there also was a general formula which would express the solutions of an equation of degree at least five from the coefficients by means of radicals. It was finally proved to be impossible. One proof (chronologically the third) was given by the French mathematician E. Galois within the framework of a more general theory which was not understood at the time and subsequently forgotten. Some 15 years later his work was rediscovered and understood only with great difficulty by a very few, so new was his viewpoint. Given an equation, Galois considered a certain set of permutations of the roots and showed that certain properties of this set of permutations are decisive. That was the beginning of an independent study of such sets of permutations which later came to be known as Galois groups. He showed that an equation is solvable by means of radicals only when the groups involved belong to a certain class; namely, the solvable groups, as they came to be called. The theorem mentioned earlier, regarding equations of degree at least five, is then a consequence of the fact that the group associated to a general equation of the n-th degree is solvable only when $n = 1, 2, 3, 4$ [4]. The important properties of such groups, for instance to be solvable, are actually independent of the nature of the objects to be permuted, and this led to the idea of an "abstract group" and to theorems of great significance, applicable in many areas of mathematics. But for many years this appeared to be nothing more than pure and very abstract mathematics. As a mathematician and a physicist were discussing the curriculum for physics at Princeton University around the year 1910, the physicist said they could no doubt leave out group theory, for it would never be applicable to physics [5]. Not 20 years later, three books on group theory and quantum mechanics appeared, and since then groups have been fundamental in physics as well.

The following will serve as a final example. I said earlier that we can consider complex numbers to be points in the plane. An Irish mathematician, N. R. Hamilton, wondered whether one could define an analogue of the four basic operations among the points of three-dimensional space, thus forming an even more comprehensive number system. It took him about 10 years to find the answer: It is not possible in three-dimensional space, but it is in four-dimensional space. We do not need to try to imagine just what four-dimensional space is here. It is simply a figure of speech for quadruples of real numbers instead of triples or pairs of real numbers. He called these new numbers quaternions. He did, however, have to do without one property of real or complex numbers which up until then had been taken for granted: commutativity in multiplication, i.e., $a \times b = b \times a$. He also showed that the calculus with quaternions had applications in the mathematical treatment of questions in physics and mechanics. Later, many other algebraic systems with a noncommutative product were defined, notably matrix algebras. This also appeared to be an entirely abstract form of mathematics, without connections to the outside world. In 1925, however, as Max Born was thinking about some new ideas of W. Heisenberg's, he discovered that the most appropriate formalism for expressing them was none other than matrix algebra, and this suggested that physical quantities be represented by means of algebraic objects which do not necessarily commute. This led to the uncertainty principle and was the beginning of matrix quantum mechanics, of the assignment of operators to physical quantities, which is at the basis of quantum mechanics [6].

With this last example I shall end my attempts to describe some mathematical topics. The examples are, of course, extremely incomplete and not at all representative of all areas of mathematics. They do have two properties in common, however, which I would like to emphasize since they are valid in a great many cases. First of all, these developments lead in the direction of ever greater abstraction, further and further away from nature. Second, abstract theories actually developed for their own sake have found important applications in the natural sciences. The suitability of mathematics to the needs of the natural sciences is in fact astonishingly great (one physicist spoke once of the "unreasonable effectiveness of mathematics" [7]) and is worthy of a far more detailed discussion than I can afford to enter into here.

The transition to ever greater abstraction is not to be taken for granted, as you may have gathered from Gauss' quotation. Mathematics was originally developed for practical purposes such as bookkeeping, measurements, and mechanics; even the great discoveries of the Seventeenth Century, such as infinitesimal and integral calculus, were at first primarily tools for solving problems in mechanics, astronomy, and physics. The mathematician Euler, who was active in all areas of mathematics and its applications—including shipbuilding—also wrote papers on pure number theory and more than once felt the need to explain that it was as justified and important as more practically oriented work [8]. Mathematics was from the very beginning, of course, a kind of idealization, but for a long time was not as far removed from reality or, more precisely, from our perception of reality, as in the examples mentioned earlier. As mathematicians went further in this direction, they became increasingly aware that a mathematical concept has a right to existence as soon as it has been defined in a logically consistent manner, without necessarily having a connection with the physical world; and that they had the right to study it even when there seemed to be no practical applications at hand. In short, this led more and more to "Pure Mathematics" or "Mathematics for Its Own Sake".

But if one leaves out the controlling function of practical applicability, the question immediately arises as to how one can make value judgments. Surely not all concepts and theorems are equal; as in G. Orwell's *Animal Farm*, some must be more so than others. Are there then internal criteria which can lead to a more or less objective hierarchy? You will notice that the same basic question can be asked about painting, music, or art in general: It thus becomes a question of aesthetics. Indeed, a usual answer is that mathematics is to a great extent an art, an art whose development has been derived from, guided by, and judged according to aesthetic criteria. For the lay person it is often surprising to learn that one can speak of aesthetic criteria in so grim a discipline as mathematics. But this feeling is very strong for the mathematician, even though it is difficult to explain: What are the rules of this aesthetic? Wherein lies the beauty of a theorem, of a theory? Of course there is no one answer that will satisfy all mathematicians, but there is a surprising degree of agreement, to a far greater extent, I think, than exists in music or painting.

Without wishing to maintain that I can explain this fully, I would like to attempt to say a bit more about it later. At the moment I shall content myself with the assertion that the analogy with art is one with which many mathematicians agree. For example, G. H. Hardy was of the opinion that, if mathematics has any right to exist at all, then it is only as art [9]. Our activity has much in common with that of an artist: A painter combines colors and forms, a musician tones, a poet words, and we combine ideas of a certain sort. The painter E. Degas wrote sonnets from time to time. Once, in a conversation with the poet S. Mallarmé, he complained that he found writing difficult even though he had many ideas, indeed an overabundance of ideas. Mallarmé answered that poems were made of words, not ideas [10]. We, on the other hand, work primarily with ideas.

This feeling of art becomes even stronger when one thinks of how a researcher works and progresses: One should not imagine that the mathematician operates entirely logically and systematically. He often gropes about in the dark, not knowing whether he should attempt to prove or disprove a certain proposition, and essential ideas often occur to him quite unexpectedly, without his even being able to see a clear and logical path leading to them from earlier considerations. Just as with composers and artists one should speak of inspiration [11].

Other mathematicians, however, are opposed to this view and maintain that an involvement with mathematics without being guided by the needs of the natural sciences is dangerous and almost certainly leads to theories which may be quite subtle and which may provide the mind with a peculiar pleasure, but which represent a kind of intellectual mirror that is completely worthless from the standpoint of science or knowledge. For example, the mathematician J. von Neumann wrote in 1947:

As a mathematical discipline travels far from its empirical sources, or still more, if it is second and third generation only indirectly inspired by ideas coming from "reality", it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely l'art pour l'art . . . *there is a great danger that the subject will develop along the line of least resistance . . . will separate into a multitude of insignificant branches. . . .*

In any event . . . the only remedy seems to me to be the rejuvenating return to the source: the reinjection of more or less directly empirical ideas [12].

Still others have taken a more intermediate stance: They fully recognize the importance of the aesthetic side of mathematics but feel that it is dangerous to push mathematics for its own sake too far. Poincaré, for example, had written earlier:

In addition to this it provides its disciples with pleasures similar to painting and music. They admire the delicate harmony of the numbers and the forms; they marvel when a new discovery opens up to them an unexpected vista; and does the joy that they feel not have an aesthetic character even if the senses are not involved at all? . . .

For this reason I do not hesitate to say that mathematics deserves to be cultivated for its own sake, and I mean the theories which cannot be applied to physics just as much as the others [13].

But a few pages further on he returns to this comparison and adds:

If I may be allowed to continue my comparison with the fine arts, then the pure mathematician who would forget the existence of the outside world could be likened to the painter who knew how to combine colors and forms harmoniously, but who lacked models. His creative power would soon be exhausted [14].

This denial of the possibility of abstract painting strikes me as especially noteworthy since we are in Munich, where, not much later, an artist would concern himself quite deeply with this question, namely, Wassily Kandinsky. It was sometime in the first decade of this century that he suddenly felt, after looking at one of his own canvases, that the subject can be detrimental to the painting in that it may be an obstacle to direct access to forms and colors; that is, to the actual artistic qualities of the work itself. But, as he wrote later [15], "a frightening gap" (*eine erschreckende Tiefe*) and a mass of questions confronted him, the most important of which was, "What should replace the missing subject?" Kandinsky was fully aware of the danger of ornamentation, of a purely decorative art, and wanted to avoid it at all costs. Contrary to Poincaré, however, he did not conclude that painting without a real subject had to be fruitless. In fact, he even developed a theory of the "inner necessity" and "intellectual content" of a painting. Since about 1910, as you know, he and other painters in increasing numbers have dedicated themselves to so-called abstract or pure painting which has little or nothing to do with nature.

If one does not want to admit an analogous possibility for mathematics, however, then one will be led to a conception of mathematics which I would like to summarize as follows: On the one hand, it is a science because its main goal is to serve the natural sciences and technology. This goal is actually at the origin of mathematics and is constantly a wellspring of problems. On the other hand it is an art because it is primarily a creation of the mind and progress is achieved by intellectual means, many of which issue from the depths of the human mind, and for which aesthetic criteria are the final arbiters. But this intellectual freedom to move in a world of pure thought must be governed to some extent by possible applications in the natural sciences.

However, this view is really too narrow, in particular the final clause is too limiting, and many mathematicians have insisted on complete freedom of activity. First of all, as was already pointed out, many areas of mathematics which have proved important for applications would not have been developed at all if one had insisted on applicability from the beginning. In spite of the above quotation, von Neumann himself pointed this out in a later lecture:

But still a large part of mathematics which became useful developed with absolutely no desire to be useful, and in a situation where nobody could possibly know in what area it would become useful: and there were no general indications that it even would be so. . . . This is true of all science. Successes were largely due to forgetting completely about what one ultimately wanted, or whether one wanted anything ultimately; in refusing to investigate things which profit, and in relying solely on guidance by criteria of intellectual elegance. . . .

And I think it extremely instructive to watch the role of science in everyday life, and to note how in this area the principle of laissez faire *has led to strange and wonderful results [16].*

Second, and for me more important, there are areas of pure mathematics which have found little or no application outside mathematics, but which one cannot help viewing as great achievements. I am thinking, for example, of the theory of algebraic numbers, class field theory, automorphic functions, transfinite numbers, etc.

Let us return to the comparison with painting once again and take as "subject" the problems which are drawn from the physical world. Then we see that we have painting drawn from nature as well as pure or abstract painting.

This comparison is, however, not yet entirely satisfactory, for such a description of mathematics would not encompass all its essential aspects, in particular its coherence and unity. Indeed, mathematics displays a coherence which I feel is much greater than in art. As a testimony to this, note that the same theorem is often proved independently by mathematicians living in widely separated locations, or that a considerable number of papers have two, sometimes more, authors. It can also happen that parts of mathematics which were developed completely independently of one another suddenly demonstrate deep-lying connections under the impact of new insights. Mathematics is, to a great extent, a collective undertaking. Simplifications and unifications maintain the balance with unending development and expansion; they display again and again a remarkable unity even though mathematics is far too large to be mastered by a single individual.

I think it would be difficult to account fully for this by appealing solely to the criteria mentioned earlier—namely, subjective ones like intellectual elegance and beauty, and consideration of the needs of natural sciences and technology. One is then led to ask whether there are criteria or guidelines other than those. In my opinion this is the case, and I would now like to complete the earlier description of mathematics by looking at it from a third standpoint and adding another essential element to it. In preparation for this I would like to digress, or at least apparently digress, and take up the question, Does mathematics have an existence of its own? Do we create mathematics or do we gradually discover theories which exist somewhere independently of us? If this is so, where is this mathematical reality located?

It is, of course, not absolutely clear that such a question is really meaningful. But this feeling—that mathematics somehow, somewhere, preexists—is widespread. It was expressed quite sharply, for example, by G. H. Hardy:

I believe that mathematical reality lies outside us, that our function is to discover or observe *it, and that the theorems which we prove, and which we describe grandiloquently as our "creations", are simply our notes of our observations. This view has been held, in one form or another by many philosophers of high reputation, from Plato onwards. . . . [17].*

If one is a believer, then one will see this preexistent mathematical reality in God. This was actually the belief of Hermite, who once said:

There exists, if I am not mistaken, an entire world which is the totality of mathematical truths, to which we have access only with our mind, just as a world of physical reality exists, the one like the other independent of ourselves, both of divine creation [18].

It wasn't too long ago that a colleague explained in an introductory lecture that the following question had occupied him for years: Why has God created the exceptional series?

But a reference to divine origin would hardly satisfy the nonbeliever. Many do, however, have a vague feeling that mathematics exists somewhere, even though, when they think about it, they cannot escape the conclusion that mathematics is exclusively a human creation.

Such questions can be asked of many other concepts such as state, moral values, religion, etc., and would probably be worthy of consideration all by themselves. But for want of time and competence, I shall have to content myself with a short and possibly oversimplified answer to this apparent dilemma by agreeing with the thesis that we tend to posit existence on all those things which belong to a civilization or culture in that we share them with other people and can exchange thoughts about them. Something becomes objective (as opposed to "subjective") as soon as we are convinced that it exists in the minds of others in the same form as it does in ours, and that we can think about it and discuss it together [19]. Because the language of mathematics is so precise, it is ideally suited to defining concepts for which such a consensus exists. In my opinion, that is sufficient to provide us with a *feeling* of an objective existence, of a reality of mathematics similar to that mentioned by Hardy and Hermite above, regardless of whether it has another origin, as Hardy and Hermite maintain. One could speculate forever on this last point, of course, but that is actually irrelevant to the continuation of this discussion.

Before I elaborate on this, I would like to note that similar thoughts about our conception of physical reality have been expressed. For example, Poincaré wrote:

Our guarantee of the objectivity of the world in which we live is the fact that we share this world with other sentient beings. . . .
That is therefore the first requirement of objectivity: That which is objective must be common to more than one spirit and as a result be transmittable from one to the other . . . [20].

and Einstein:

By the aid of speech different individuals can, to a certain extent, compare their experiences. In this way it is shown that certain sense perceptions of different individuals correspond to each other, while for other sense perceptions no such correspondence can be established. We are accustomed to regard as real those sense perceptions which are common to different individuals, and which therefore are, in a measure, impersonal [21].

Now back to mathematics. Mathematicians share an intellectual reality, a gigantic number of mathematical ideas, objects whose properties are partly known and partly unknown, theories, theorems, solved and unsolved problems, which they study with mental tools. These problems and ideas are partially suggested by the physical world; primarily, however, they arise from purely mathematical considerations (such as groups or quaternions to go back to my earlier examples). This totality, although stemming from the human mind, appears to us to be a natural science in the normal sense, such as physics or biology, and is for us just as concrete. I would actually maintain that mathematics not only has a theoretical side, but also an experimental one. The former is clear: We strive for general theorems, principles, proofs and methods. That is the theory. But in the beginning one often has no idea of what to expect, and how to continue, and one gains understanding and intuition through experimentation, that is, through the study of special cases. First, one hopes to be led in this way to a sensible conjecture, and second, perhaps to stumble upon an idea that will lead to a general proof. It can also happen, of course, that certain special cases are of great interest in themselves. That is the experimental side. The fact that we operate with intellectual objects more than with real objects and laboratory equipment is actually not important. The feeling that mathematics is in this sense an experimental science is also not new. Hermite, for example, wrote to L. Königsberger around 1880:

The feeling expressed at that point in your letter where you say to me: "The more I think about all these things, the more I come to realize that mathematics is an experimental science like all other sciences," this feeling, I say, is also my feeling [22].

Traditionally, these experiments are carried out in one's head (or with pen and paper), and for this reason I have spoken of mental tools. I should add, however, that for about 20 years real apparatuses, namely, electronic computers, have been playing an increasing role. They have actually given this experimental side of mathematics a new dimension. This has advanced to the extent that one may already see important, reciprocal, and fascinating interactions between computer science and pure mathematics.

The word "science" in my title now takes a broader meaning: It refers not only to the natural sciences, as it did earlier, but also—and this to a much greater extent—to the conception of mathematics itself as an experimental and theoretical science, I would venture to say, as a *mental* natural science, as a natural science of the intellect, whose objects and modes of investigations are all creations of the mind.

This makes it somewhat easier for me to speak of motivation and aesthetics. If one does not want to take applications in the natural sciences as a yardstick, one is still not thrown back upon mere intellectual elegance. There still remain almost practical criteria; namely, applicability in mathematics itself. The consideration of this mathematical reality, the open problems, the structure, needs and connections among various areas, already indicates possibly fruitful, valuable directions and allows the mathematician to orient himself and attach relative values to problems as well as to theories. Often a test for the value of a new theory is whether it can solve old problems. *De facto,* this limits the freedom of a mathematician, in a way which is comparable to the constraints imposed on a physicist, who after all doesn't choose at random the phenomena for which he wants to construct a theory or to devise experiments. Many examples show that mathematicians have often been able to foresee how certain areas of mathematics will develop, which problems should be taken up and probably would be quickly solved. Rather often statements about the future of mathematics have proved true. Such predictions are not perfect, but they are successful enough to indicate a difference from art. Analogous relatively successful forecasts about the future of painting, for example, hardly exist at all.

I don't want to go too far in this, however, I suggested the concept of mathematics as a mental natural science as *one of three elements,* not as the whole. On the one hand, I don't want to overlook the importance of the interactions between mathematics and the natural sciences. First, it is a common saying that all disciplines in the natural sciences must strive for a mathematical formulation and treatment, indeed, that a discipline achieves the status of a science only when this has been carried out. Thus it is surely important that mathematicians try to help in this way. Second, it is doubtless a great achievement to formulate and treat complicated phenomena mathematically, and the new problems which are thereby introduced represent an enrichment for mathematics. One need only think of probability. I only mean that it is simply not necessary to put the idea of applicability in the foreground in order to do valuable mathematics. The history of mathematics shows that many outstanding achievements came from mathematicians who weren't thinking at all about external applications and who were led by purely mathematical considerations. And as was already mentioned and illustrated, these contributions often found important applications in the natural sciences or in engineering, often in completely unforeseen ways.

On the other hand, I don't want to say that one can foresee everything completely rationally. Actually, this isn't the case even in the natural sciences, especially since one often does not know in advance which experiments will prove interesting. Outstanding mathematicians have also been wrong and have sometimes, precisely in the name of applicability within mathematics, termed fruitless, idle, even dangerous, new ideas which later proved fundamental. The freedom not to consider practical applications, which von Neumann demanded for science as a whole, must also be demanded within mathematics.

One could object that this analogy between mathematics and natural sciences overlooks one essential difference: In the natural sciences or in technology one often encounters problems that one has to solve in order to advance at all. In the world of mathematical thought, one has still *de jure* the freedom to put aside apparently unsolvable, overly difficult problems and turn to other, more manageable ones; and maybe, in fact, follow the path of least resistance just as von Neumann had feared. Wouldn't that be a temptation for a mathematician who defines mathematics as "the art of finding problems that one can solve"? Interestingly enough, I heard this definition from a mathematician whose works are especially remarkable because they treated so many problems which seemed quite special at the time but which later proved fundamental and whose solutions opened up new paths, namely, Heinz Hopf.

It cannot be denied, however, that sometimes paths of least resistance are indeed followed, leading to trivial or meaningless work. It can also happen that a successful school later falls into a sterile period and then even, at worst, exerts a harmful influence. Remarkably enough, however, an antidote always comes along, a reaction which eliminates these mistaken paths and fruitless directions. Up until now mathematics has always been able to overcome such growth diseases, and I am convinced that it will always do so as long as there are so many talented mathematicians. It is very odd, however: Many of us have this feeling of a unity in mathematics, but it is dangerous to prescribe overly precise guidelines in the name of our conception of it. It is more important that freedom reign despite occasional misuse. Why this is so successful cannot be fully explained. If one thinks of Hopf, for example, one can, to a certain extent, see rational criteria in his choice of problems: They were for instance often the first special cases of a general problem for which known methods of proof were not applicable. He was of course aware of this. But that doesn't explain everything. He probably didn't always foresee how influential his work would become; and most likely did not worry about it. It is simply a part of the talent of a mathematician to be drawn to "good" problems, i.e., to problems which turn out to be significant later, even if it is not obvious at the time he takes them up. The mathematician is led to this partly by rational, scientific observations, partly by sheer curiosity, instinct, intuition, purely aesthetic considerations. Which brings me to my final subject, the aesthetic feeling in mathematics.

I already mentioned the idea of mathematics as an art, a poetry of ideas. With that as a starting point, one would conclude that, in order for one to appreciate mathematics, to enjoy it, one needs a unique feeling for intellectual elegance and beauty of ideas in a very special world of thought. It is not surprising that this can hardly be shared with nonmathematicians: Our poems are written in a highly specialized language, the mathematical language; although it is expressed in many of the more familiar languages, it is nevertheless unique and translatable into no other language; unfortunately, these poems can only be understood in the original. The resemblance to an art is clear. One must also have a certain education for the appreciation of music or painting, which is to say one must learn a certain language.

I have long agreed with such opinions and analogies. Without changing my fundamental position with regard to mathematics, I would nonetheless like to reformulate them somewhat in the direction of my previous statements. I believe that our aesthetics are not always so pure and esoteric but also include a few more earthly yardsticks such as meaning, consequences, applicability, usefulness—but within the mathematical science. Our judgment of a theorem, a theory, a proof is also influenced by this, but it is often simply equated to the aesthetic. I would like to try to explain this using Galois' theory, mentioned earlier. This theory is generally treasured as one of the most beautiful chapters in mathematics. Why? First, it solved a very old and, at that time, the most important question about equations. Second, it is an extremely comprehensive theory that goes far beyond the original question of solvability by radicals. Third, it is based on only a few principles of great elegance and simplicity which are formulated within a new framework with new concepts which demonstrate the greatest originality. Fourth, these new viewpoints and concepts, especially the concept of group, opened new paths and had a lasting influence on the whole of mathematics.

You will notice that of these four points only the third is a truly aesthetic judgment, and one about which one can have one's own opinion only when one understands the technical details of the theory. The others have a different character. One could make similar statements about theories in any natural science. They have a greater objective content, and a mathematician can have his own opinion about them even if he doesn't fully grasp the technical details of the theory. For the purpose of this discussion I have separated these four elements, but normally I would not always do so explicitly, and all four contribute to the impression of beauty. I do think that in this respect this example is fairly typical: What we describe as aesthetic is actually often a fusion of different views. For example, I would naturally find a method of proof more beautiful if it found new and unexpected applications, although the method itself hadn't changed. It may have become more important, but in and of itself not more beautiful. Since all this takes place within mathematics itself, it will hardly help the nonmathematician penetrate our aesthetic world. I hope, however, that it will help him find more plausible the fact that our so-called aesthetic judgments display a greater consensus than in art, a consensus that goes far beyond geographical and chronological limitations. In any case, I regard this as being a major factor. But once again, I must avoid taking this too far. It is a question of degree, not an absolute difference. An aesthetic judgment on the work of a composer or a painter also draws on external factors such as influence, predecessors, the position of the work with relation to other works, even if it is to a lesser extent. On the other hand, there are differences of opinion and fluctuations in time in the evaluation of mathematical works, though not to such a strong degree, I would add. All these nuances need a good deal of explanation which I cannot go into here for lack of time.

In the limited amount of time at my disposal, it would of course be easier to make only sweeping short statements about mathematics. But unfortunately, or fortunately, just as in other human undertakings to which many people contributed over many centuries, mathematics refuses to let itself be described by just a few simple formulas. Almost every general statement about mathematics has to be qualified somehow. One exception, perhaps the only one, might be this statement itself. I hope I have at least given the impression that mathematics is an extremely complex creation which displays so many essential traits in common with art and experimental and theoretical sciences that it has to be regarded as all three at the same time, and thus must be differentiated from all three as well.

I am aware that I have raised more questions than I have answered, treated too briefly those I have discussed and not even touched upon some important ones, such as the value of this creation. One can of course point to innumerable applications in the natural sciences and in engineering, many of which have a great influence on our daily life, thereby establishing a social right to existence for mathematics. But I must confess that, as a pure mathematician, I am more interested in an assessment of mathematics in itself. The contributions of the various mathematicians meld into an enormous intellectual construct which, in my opinion, represents an impressive testimony to the power of human thinking. The mathematician Jacobi once wrote that "the only purpose of science is to honor the human mind" [23]. I believe that this creation does indeed do the human mind great honor.

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Notes

1. The dissertation was by L. Kronecker, cf. *Werke*, 5 Vol., Teubner, Leipzig, 1895–1930, Vol. 1, p. 73. The opponent was G. Eisenstein.

   The source I am aware of for the name and the opinion of the opponent is a footnote by E. Lampe to a lecture by P. du Bois-Reymond, *Was will die Mathematik und was will der Mathematiker?*, published posthumously by E. Lampe in Jahresbericht der Deutschen Mathematiker-Vereinigung 19 (1910), 190–198.
2. For a discussion of a number of such opinions see A. Pringsheim, *Ueber den Wert und angeblichen Unwert der Mathematik*, Jahresbericht der Deutschen Mathematiker-Vereinigung 13 (1904), 357–382.
3. Letter to F. W. Bessel, 18 November 1811. See G. F. Auwers Verlag, *Briefwechsel zwischen Gauss und Bessel* Leipzig 1880, p. 156.
4. Actually the beginnings of group theory can already be traced to some earlier work, notably by Lagrange, which was in part familiar to Galois. The latter's standpoint was, however, so general and abstract and, in addition, so sketchily described that it was assimilated only slowly. For historical information on the theory of equations and the beginnings of group theory see, for example, N. Bourbaki, *Eléments d'histoire des mathématiques*, Hermann éd., Paris, 1969, third and fifth articles.
5. F. J. Dyson, *Mathematics in the physical sciences*, Scientific American 211 September (1964), 129–146.
6. See B. L. van der Waerden's historical introduction in "Sources in Quantum Mechanics", *Classics of Science*, Vol. 5, Dover Publications, New York, 1967, especially pp. 36–38. Cf. also Dirac's remarks on the introduction of non-commutativity in quantum mechanics in *loc cit.* [7].
7. E. P. Wigner, *The unreasonable effectiveness of mathematics*

in the natural sciences, Communications on Pure and Applied Mathematics 13 (1960), 1–14.

Among the many aspects of this interaction, the one that appears most remarkable to me is that the mathematical formalism sometimes leads to basic, new and purely physical ideas. One well-known example is the discovery of the positron: In 1928 P. A. M. Dirac set up quantum mechanic relativistic equations for the movement of the electron. These equations also allowed a solution with the same mass as the electron, but with the opposite electrical charge. All attempts to explain these solutions satisfactorily, or to eliminate them by some suitable modification of the equation, were unsuccessful. This led Dirac eventually to conjecture the existence of a particle with the necessary properties, which was later established by Anderson. For this see P. A. M. Dirac, "The development of quantum theory" (J. R. Oppenheimer Memorial Prize acceptance speech), Gordon and Breach, New York, 1971.

A newer and even more comprehensive example would be the use of irreducible representations of the special unitary group SU(3) in three complex variables, which led to the so-called "eightfold way". One of the first successes of this theory was quite striking, namely, the discovery of the particle Ω^-: Nine baryons were assigned, through consideration of two of their characteristic quantum numbers, to nine points of a very specific mathematical configuration consisting of 10 points in a plane [the 10 weights of an irreducible 10-dimensional representation of SU(3)]; this led M. Gell'man to conjecture that there should also be a particle corresponding to the tenth point, which would then possess certain well-defined properties. Such a particle was observed some two years later. A further development along these lines led to the theory of "quarks". For the beginnings of this theory see F. J. Dyson, *loc. cit.* [5] and M. Gell'man and Y. Ne'eman, *The Eightfold Way,* W. A. Benjamin, New York, 1964.

8. See a number of papers in L. Euler's *Opera Omnia,* especially I.2, 62–63, 285, 461, 576; I.3, 5.2. I want to thank A. Weil for pointing this out to me. Here is an example (translated from Latin by Weil), *loc. cit.* pp. 62–63, published in 1747:

"Nor is the author disturbed by the authority of the greatest mathematicians when they sometimes pronounce that number-theory is altogether useless and does not deserve investigation. In the first place, knowledge is always good in itself, even when it seems to be far removed from common use. Secondly, all the aspects of the truth which are accessible to our mind are so closely related to one another that we dare not reject any of them as being altogether useless. Moreover, even if the proof of some proposition does not appear to have any present use, it usually turns out that the method by which this problem has been solved opens the way to the discovery of more useful results.

"Consequently, the present author considers that he has by no means wasted his time and effort in attempting to prove various theorems concerning integers and their divisors. Actually, far from being useless, this theory is of no little use even in analysis. Moreover, there is little doubt that the method used here by the author will turn out to be of no small value in other investigations of greater import."

9. G. H. Hardy, *A Mathematician's Apology,* Cambridge University Press, 1940; new printing with a foreword by C. P. Snow, pp. 139–140.
10. P. Valéry, *Degas, danse, dessin,* A. Vollard éd., Paris, 1936; *Oeuvres II,* La Pléiade, Gallimard éd., Paris, 1966, pp. 1163–1240, especially pp. 1207–1209.
11. The following excerpt from a letter from C. F. Gauss to Olbers, written on 3 September 1805, shortly after Gauss had solved a problem (the "sign of the Gaussian Sums") he had been working on for years, can serve as an example:

"Finally, just a few days ago, success—but not as a result of my laborious search, but only by the grace of God I would say. Just as it is when lightning strikes, the puzzle was solved; I myself would not be able to show the threads which connect that which I knew before, that with which I had made my last attempt, and that by which it succeeded." See Gauss, *Gesammelte Werke,* Vol. 10_I, pp. 24–25. Here one must also mention H. Poincaré's description of some of his fundamental discoveries on automorphic functions. H. Poincaré, "L'invention mathématique" in *Science et Méthode,* E. Flammarion éd., Paris, 1908, Chap. III.

12. J. v. Neumann, "The mathematician" in Robert B. Heywood, *The Works of the Mind,* University of Chicago Press, 1947, pp. 180–187. *Collected Works,* 6 Vol., Pergamon, New York, 1961, Vol. I, pp. 1–9.
13. H. Poincaré, *La Valeur de la Science,* E. Flammarion, Paris, 1905, Chap. 5, p. 139. Actually, this chapter is the printed version of a lecture which Poincaré had delivered at the First International Congress of Mathematicians, Zurich, 1897.
14. *Loc. cit. [13]* p. 147.
15. W. Kandinsky, *Rückblick 1901–1913,* H. Walden ed., 1913. New printing by W. Klein Verlag, Baden-Baden, 1955, See pp. 20–21.
16. J. v. Neumann, "The role of mathematics in the science and in society", address to Princeton Graduate Alumni, June 1954. Cf. *Collected Works,* 6 Vol., Pergamon, New York, 1961, Vol. VI, pp. 477–490.
17. Cf. G. H. Hardy, *loc. cit.* [9] pp. 123–124.
18. G. Darboux, *La vie et l'Oeuvre de Charles Hermite,* Revue du mois, 10 January 1906, p. 46.
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20. H. Poincaré, *loc. cit.* [13] p. 262.
21. A. Einstein, *Vier Vorlesungen über Relativitätstheorie,* held in May 1921 at Princeton University, Fr. Vieweg und Sohn, Braunschweig, 1922, p. 1. English translation in: *The Meaning of Relativity,* Princeton University Press, Princeton, 1945.
22. Cf. L. Königsberger, *Die Mathematik eine Geistes-oder Naturwissenschaft?,* Jahresbericht der Deutschen Mathematiker-Vereinigung 23 (1914), 1–12.
23. In a letter of 2 July 1830, to A. M. Legendre, cf. C. G. J. Jacobi, *Gesammelte Werke,* G. Riemer, Berlin, 1881–1891, Vol. 1, pp. 453–455. Since this statement is sometimes misquoted, we prefer to give here its original context:

"Mais M. Poisson n'aurait pas dû reproduire dans son rapport une phrase peu adroite de feu M. Fourier, où ce dernier nous fait des reproches, à Abel et à moi, de ne pas nous être occupés de préférence du mouvement de la chaleur. Il est vrai que M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain et que sous ce titre une question de nombres vaut autant qu'une question du système du monde."

The Onto-Semiotic Approach to Research in Mathematics Education[1]

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Abstract: *In this paper we synthesise the theoretical model about mathematical cognition and instruction that we have been developing in the past years, which provides conceptual and methodological tools to pose and deal with research problems in mathematics education. Following Steiner's Theory of Mathematics Education Programme, this theoretical framework is based on elements taken from diverse disciplines such as anthropology, semiotics and ecology. We also assume complementary elements from different theoretical models used in mathematics education to develop a unified approach to didactic phenomena that takes into account their epistemological, cognitive, socio cultural and instructional dimensions.*

ZDM-Classification: D20, C30, C70

1. The need for a comprehensive approach to mathematics education

Mathematics Education is aimed to study the factors affecting the teaching and learning of mathematics and to develop programs to improve the teaching of mathematics. This goal was assumed by Steiner in his programme for the Theory of Mathematics Education, "*the development of a comprehensive view of mathematics education comprising research, development, and practice by means of a systemic approach"* (Steiner, 1985, p. 16).

In order to accomplish this aim Mathematics Education must consider the contributions of several disciplines: Psychology, Pedagogy, Sociology, Philosophy, etc. However, the use of these contributions in Mathematics Education should be based on an analysis of the nature of mathematics and mathematical concepts, and their personal and cultural development. Such epistemological analysis is essential in Mathematics Education, for it would be very difficult to suitably study the teaching and learning processes of undefined and vague objects.

Thus, research in Mathematics Education cannot ignore philosophical questions such as:

- What is the nature of mathematical objects?
- What roles human activity and socio-cultural processes play in the development of mathematical ideas?
- Is mathematics discovered or invented?
- Do formal definitions and statements cover the full meaning of concepts and propositions?
- What role is played, in the meaning of mathematical objects, by their relationships with other objects, the problems in which they are used and the different symbolic representations?

The relatively recent emergence of Mathematics Education as a scientific discipline explains the lack of a consolidated and dominant research paradigm. Diverse survey works (Ernest, 1994; Sierpinska and Lerman, 1996; Font, 2002) oriented to provide proposals for organizing the different research programmes in Mathematics Education have shown the diversity of theoretical approaches that are being developed at present. This variety is unavoidable, even enriching at some moments, but the progress of the discipline and the strengthening of its practical applications require a collective effort to identify the firm nucleus of concepts and methods that, in the long run, should crystallize in a true research program, using Lakatos's terminology (1983).

One main "meta-didactical" problem is the clarification of the theoretical notions that are used in the area of knowledge to analyse the cognitive phenomena, since we observe a variety of terms that have not been compared or clarified: knowledge, "savoir", competence, conception,

[1] In memoriam of Hans-Georg Steiner who encouraged our interest in the Theory of Mathematics Education and helped us recognise its complexity, and the need for adopting a holistic, interdisciplinary and systemic approach in our research.

concept, internal representation, conceptual image, invariant operative, meaning, etc.

Progress in the field requires contrasting these tools and possibly elaborating other new ideas that serve to more effectively carry out the intended work. It is also necessary to coherently articulate the diverse dimensions or facets implied, such as the ontological (types of objects and their nature), epistemological (access to knowledge), socio-cultural and instructional facets (teaching and learning in school institutions).

We believe it is necessary and possible to build a unified approach to mathematical knowledge and instruction that allows the overcoming of the dilemmas among the diverse competing paradigms: realism - pragmatism, individualism - institutional knowledge, constructivism – behaviourism, etc. To progress in this direction we should take into account some conceptual and methodological tools from holistic disciplines such as Semiotics, Anthropology and Ecology, coherently articulated with other disciplines such as Psychology and Pedagogy, which traditionally are immediate reference for Mathematics Education.

The Onto-Semiotic Approach to Mathematics Education that we present in this paper is a "home-grown model" that assumes a systemic and interdisciplinary perspective (Steiner, 1985) to deal with the complexity of Mathematics Education as a field of research, development and practice.

2. Towards a unified approach to mathematical knowledge and instruction

For the past 12 years we have been interested in the foundations for Mathematics Education Research (Batanero, Godino, Steiner and Wenzelburger, 1994) and we have developed diverse theoretical tools to deal with some of the mentioned questions. These tools were based on several theoretical antecedents, which are described and analysed in Godino (2003, chapter 2)[2]. These tools have been developed in three stages, in each of which the object of inquiry was progressively refined. Next we succinctly describe these three stages and the issues dealt with in each of them.

During the period 1993-98 (Godino and Batanero, 1994; Godino, 1996; Godino and Batanero, 1998), we progressively developed and refined the notions of "institutional and personal meaning[3] of a mathematical object" and their relationship to the notion of understanding. Starting from pragmatic assumptions, these ideas try to focus the interest of research on the institutional mathematical knowledge, without dismissing the individual subject towards which the educational effort is mainly addressed.

In a second stage (starting in 1998) we considered it necessary to elaborate a more specific ontological and semiotic model, since we realized that the epistemic and cognitive problems cannot be separated from the ontological reflection. For this reason we were interested in elaborating an ontology sufficiently rich to describe mathematical activity and the processes of communicating their "products." We tried to progress in developing a specific ontology and semiotic to study the processes of interpreting the systems of mathematical signs used in didactic interactions.

These questions are central in other disciplines (such as Semiotics, Epistemology and Psychology), although they have not provided a clear solution to these issues. The available answers are diverse, incompatible or difficult to articulate, as we can see, for example, in the dilemmas outlined by Peirce´s (1965), Saussure´s (1915) and Wittgenstein´s (1953) approaches. Moreover, the interest in using semiotic notions is also growing in mathematics education as it is shown in the monograph published by Anderson et al. (2003) and the special issue of *Educational Studies in Mathematics* (Sáenz-Ludlow and Presmeg, 2006).

We tried to provide a particular answer from the point of view of mathematics education, enlarging the investigations carried out so far on the institutional and personal meanings, and also adopting the idea of semiotic function and improving the associate mathematical ontology introduced in Godino and Recio (1998).

[2] The publications of Godino et al. can be downloaded from Internet: http://www.ugr.es/local/jgodino/

[3] Here, meaning is interpreted in terms of systems of practices related to the object.

In a third stage of our work we have been interested in theoretical models for mathematical instruction (Godino, Contreras and Font, 2006). We defined six dimensions in a mathematical instruction process, each of them modelled as a stochastic process with its respective state space and trajectory: epistemic (relating to institutional knowledge), educational (teachers' roles), student (students' roles), mediational (use of technological resources and time), cognitive (genesis of personal meanings) and emotional (students' attitudes, emotions, etc. when studying mathematics) trajectories.

The theoretical constructs elaborated during these three periods constitute the *onto-semiotic approach* that we synthesize in the next section.

3. Basic theoretical tools

The starting point for the *onto-semiotic approach* was an ontology of mathematical objects that takes into account the triple aspect of mathematics as a socially shared problem solving activity, a symbolic language and a logically organized conceptual system. Taking the problem-situation as the primitive notion, we defined the theoretical concepts of practice, (personal and institutional) object and meaning, with the purpose of making visible and operative, both the mentioned triple character of mathematics and the personal and institutional genesis of mathematical knowledge, as well as their mutual interdependence.

3. 1. Systems of operative and discursive practices linked to types of problems

We consider *mathematical practice* any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems (Godino and Batanero, 1998, p. 182). The practices can be idiosyncratic of a person or shared within an institution. An institution is constituted by the people involved in the same class of problem-situations, whose solution implies the carrying out of certain shared social practices and the common use of particular instruments and tools.

In the study of mathematics, more than a specific practice to solve a particular problem, we are interested in the systems of (operative and discursive) practices carried out by the people involved in certain types of problem-situations. For example, regarding the questions, What is the mathematical object[4] "arithmetic mean"?, What does it mean or does it represent the expression "arithmetic mean"?, we propose the following pragmatist answer: "The system of practices that a person carries out (personal meaning), or are shared within an institution (institutional meaning), to solve a type of problem-situations in which finding a representative of a set of data is required."

The socio-epistemic and cognitive relativity of meanings, when they are understood as systems of practices, and their use in the didactical analysis lead to introducing a basic typology of meanings. Regarding institutional meanings we distinguish the following types:

- *Implemented*: the system of practices that a teacher effectively implements in a specific teaching experience.
- *Assessed*: the system of practices that a teacher uses to assess his/her students' learning.
- *Intended*: the system of practices included in the planning of the study process.
- *Referential*: the system of practices used as reference to elaborate the intended meaning; for example that included in curricular documents. In a particular teaching experience the reference meaning will be part of a more global or *holistic meaning*, whose determination requires carrying out a historical and epistemological study to find the origin and evolution of the object

Regarding the personal meaning we introduce the following types:

- *Global*: set of personal practices that the subject is potentially able to carry out related to a specific mathematical object.
- *Declared*: the personal practices effectively shown in solving assessment tasks and questionnaires, independently if they are

[4] Initially we use the expression "mathematical object" as synonymous of "mathematical concept". Later we extend the use indicating any entity or thing to which we refer, or talk about it, be it real o imaginary and that intervenes in some way in mathematical activity.

correct or incorrect from the institutional point of view.

- *Achieved*: personal practices that fit the institutional meaning fixed by the teacher. The analysis of changes and evolution of personal meanings, as a result of the study process, will also serve to distinguish between *initial* and *final* personal achieved meanings.

In the *onto-semiotic* framework, teaching involves the participation of students in the community of practices sharing the institutional meaning, and learning is conceived as the students' appropriation of these meanings.

3.2. Objects involved and emerging from systems of practices

In mathematical practices ostensive (symbols, graphs etc.) and non-ostensive objects (brought to mind when doing mathematics), which are textually, orally, graphically or even gesturally represented intervene. New objects that come from the system of practices and explain their organization and structure (types of problems, procedures, definitions, properties, arguments), emerge[5]. If the system of practices is shared within an institution, the emerging objects are considered to be "institutional objects", whilst if these systems correspond to a person they are considered as "personal objects"[6]. The following types of primary mathematical objects are proposed:

- Language (terms, expressions, notations, graphics);
- Situations (problems, extra or intra-mathematical applications, exercises, etc.);
- Concepts, given by their definitions or descriptions (number, point, straight line, mean, function, etc.);
- Propositions, properties or attributes,
- Procedures (operations, algorithms, techniques);
- Arguments used to validate and explain the propositions and procedures (deductive, inductive, etc.).

These objects are organized in more complex entities, such as conceptual systems, theories, etc. The six types of primary objects postulated widen the traditional distinction between conceptual and procedural entities that is insufficient to describe the objects intervening and emerging from mathematical activity. The problem – situations promote and contextualise the activity; language (symbols, notations, graphics, …) represent the other entities and serve as tools for action; arguments justify the procedures and propositions that relate the concepts. These entities have to be considered as functional and relative to the language game (institutional frameworks and contexts of use) in which they participate; they have also a recursive character, in the sense that each object might be composed of other entities, depending on the analysis level, for example arguments might involve concepts, properties, operations, etc.

3.3. Relations between objects: Semiotic functions

Another component in the model is Hjelmslev´s (1943) notion of *function of sign*[7], that is the dependence between a text and its components and between these components themselves. In others words, the correspondences (relations of dependence or function) between an antecedent (expression, signifier) and a consequent (content, signified or meaning), established by a subject (person or institution) according to certain criteria or a corresponding code. These codes in mathematical activity can be rules (habits, agreements) that inform the subjects implied about the terms that should be put in correspondence in the fixed circumstances.
For us, the relations of dependence between expression and content can be representational (one object which is put in place of another for a certain purpose), instrumental (an object uses another as an instrument) and structural (two or more objects make up a system from which new objects emerge). In this way semiotic functions and the associated mathematical ontology, take into account the essentially relational nature of mathematics and radically generalize the notion of representation. The role of representation is not exclusively undertaken by language: in accordance with Peirce´s semiotic, we assume the different types of objects (problem-situations,

[5] "… mathematical discourse and its objects are *mutually constitutive*" (Sfard, 2000, p. 47)
[6] "Personal objects" include cognitive constructs such as conceptions, internal representations, conceptual images, etc.

[7] Named by Eco (1979) as semiotic function.

procedures, definitions, propositions and arguments) can also be expression or content of the semiotic functions.

3.4. Configuration of objects and mathematical processes

The notion of "system of practices" is useful for some types of macro-didactic analysis, particularly when comparing the particular form mathematical knowledge adopts in different institutional frameworks, contexts of use or language games. A finer description of mathematical activity requires the introduction of the six types of primary entities. These objects will form "configurations", that we define as the network of objects involved and emerging from the systems of practices and the relationships established between them. These configurations can be epistemic (networks of institutional objects) or cognitive (network of personal objects), and with the system of practices are the basic theoretical tools to describe mathematical knowledge, in its double personal and institutional facets.

These objects and relationships (configurations), emerge through time in both their personal and institutional facet, by means of mathematical processes, which we interpret as sequences of practices. The emergence of linguistic objects, problems, definitions, propositions, procedures and arguments take place throughout the respective primary mathematical processes of communication, problem posing, definition, enunciation, elaboration of procedures (algorithms, routines, ...) and argumentation. Problem solving and mathematical modelling should rather be considered as mathematical "hyper-processes", when involving complex configurations of primary mathematical processes (establishing connections between objects and generalizing techniques, rules and justifications). The effective realization of study processes also requires sequences of planning, monitoring and assessing that might be considered as meta-cognitive processes.

3.5. Contextual attributes

The notion of *language game* (Wittgenstein, 1953) plays an important role together with that of institution in our model. Here we refer to contextual factors to which the meanings of mathematical objects are relative and which attribute a functional nature to them. Mathematical objects intervening in mathematical practices or emerging from them, depend on the language game in which they take part, and can be considered from the following dual dimensions or facets (Godino, 2002):

Personal – institutional. Institutional objects, emerge from systems of practices shared within an institution, while personal objects emerge from specific practices from a person (Godino and Batanero, 1998, p. 185-6). "Personal cognition" is the result of individual thinking and activity when solving a given class of problems, while "institutional cognition" is the result of dialogue, agreement and regulation within the group of subjects belonging to a community of practices.

Ostensive – non ostensive. Mathematical objects (both at personal or institutional levels) are, in general, non perceptible. However, they are used in public practices through their associated *ostensives* (notations, symbols, graphs, etc.). The distinction between ostensive and non-ostensive is relative to the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation).

Extensive – intensive (example - type). An *extensive* object is used as a particular case (a specific example, i.e., the function $y = 2x+1$), of a more general class (i.e., the family of functions $y=mx+n$), which is an *intensive* object. The extensive / intensive duality is used to explain a basic feature of mathematical activity: the use of generic elements (Contreras and cols, 2005). This duality allows us to focus our attention on the dialectic between the particular and the general, which is a key issue in the construction and application of mathematical knowledge.

Unitary – systemic. In some circumstances mathematical objects are used as unitary entities (they are supposed to be previously known), while in other circumstances they are seen as systems that could be decomposed to be studied. For example, in teaching, addition and subtraction, algorithms, the decimal number system (tens, hundreds, …) is considered as something known, or as unitary entities. These same objects, in first grade, should be dealt with as systemic and complex objects to be learned.

Expression – content: they are the antecedent and consequent of semiotic functions. Mathematical activity is essentially relational, since the different objects described are not isolated, but they are related in mathematical language and activity by means of semiotic functions. Each type of object can play the role of antecedent or consequent (signifier or signified) in the semiotic functions established by a subject (person or institution).

These facets are grouped in pairs that are dually and dialectically complementary. They are considered as attributes applicable to the different primary and secondary objects, giving rise to different "versions" of the said objects. In Godino, Batanero and Roa (2005) the six types of primary entities and the five types of cognitive dualities are described using examples from a research in the field of combinatory reasoning.

In Figure 1 we represent the different theoretical notions that have been concisely described as an onto-semiotic model for mathematical knowledge. Here mathematical activity plays a central role and is modelled in term of systems of operative and discursive practices. From these practices the different types of mathematical objects, which are related among them building cognitive or epistemic configurations, emerge. Lastly, the objects that take part in mathematical practices and those emerging from these practices, depend on the language game in which they participate, and might be considered from the five facets of dual dimensions.

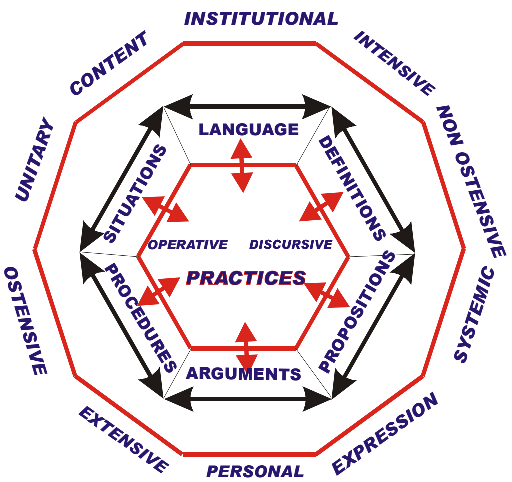

Figure 1: An onto-semiotics of mathematical knowledge

The types of objects described, summarised in figure 1 (systems of practices, emerging entities, configurations or onto-semiotic networks, the contextual attributes, together with the notion of semiotic function as the basic relational entity) make up an operative response to the ontological problem of representation and meaning of mathematical knowledge.

3.6. Understanding and knowing in the onto-semiotic approach

There are two basic ways to conceive "understanding": as a mental process or as a competence (Font 2001), which correspond to divergent or even conflicting epistemological conceptions. Cognitive approaches in Mathematics Education views understanding as a mental process, while the pragmatic position of the *onto-semiotic approach,* considers understanding as competence (a subject is said to understand a mathematical object when he/she uses it in a competent way in different practices).

However, considering the essential role played by the semiotic functions in the relational process carried out in mathematical activity (within a given language game) also lead to conceiving understanding in terms of semiotic functions (Godino, 2003). That is, we can interpret understanding of an object *O* by a subject X (person or institution) in terms of the semiotic functions that X can establish, in some fixed circumstances, in which *O* intervenes as expression or content. Each semiotic function implies a semiosis act by an interpretant agent and constitutes a knowledge. Speaking of knowledge is equivalent to speaking of the content of a (or many) semiotic function (s), and the variety of types of knowledge correspond to the diversity of semiotic functions that can be established among the diverse entities introduced in the theoretical model.

3.7. Didactical problems, practices, processes and objects

The theoretical model described for mathematical education can also be applied to other fields, particularly to pedagogical knowledge. In this case the problems from which knowledge emerge have a different nature, e.g.:

- What content should be taught in each context and circumstances?

- How should we allocate the diverse components and facets of contents through time?
- What model of the study process should be implemented in each circumstance?
- How should we plan, monitor and assess the teaching and learning processes?
- What factors condition the teaching and learning processes?

Here, the actions (didactical practices) implemented, their sequencing (didactical processes) and the emergent objects from these systems of practices (didactical objects) will be different from those arising in solving mathematical problems.

In the Theory of Didactical Configurations (Godino, Contreras y Font, 2006) that we are developing as a component of the onto-semiotic approach, we model the teaching and learning of a mathematical content as a multidimensional stochastic process composed of six sub-processes (epistemic, teacher's roles, students' roles, mediational, cognitive and emotional), and their respective trajectories and potential states. We introduce the *didactical configuration* as the primary unit for didactical analysis. This is constituted by the teacher – student interactions when studying a mathematical object or content and using some specific technological resources. Every instructional process is developed for a given time through a sequence of didactical configurations.

A didactical configuration includes an *epistemic configuration*, that is to say, a mathematical problem, the languages and actions required to solve it, rules (concepts, propositions and procedures), and argumentations, which are assumed by the teacher, students, or shared between them. There is also an *instructional configuration* made up by the teacher, students and the mediational objects (different resources) related to the mathematical content under study. The learning built throughout the process might be viewed as a set of *cognitive configurations*, that is the networks of objects emerging from or involved in the systems of personal practices that students carried out during the implementation of the epistemic configuration.

3.8. Didactical suitability criteria

We complement the theoretical notions described with the notion of *didactical suitability* of an instructional process, which is defined as the coherent and systemic articulation of the following six components (Godino, Wilhelmi and Bencomo, 2005; Godino, Contreras and Font, 2006), each of which is a matter of degree:

- *Epistemic suitability*, representativeness of institutional implemented (or intended) meaning as regards the reference meaning previously defined.
- *Cognitive suitability*, extent to which the institutional implemented (or intended) meaning is included in the students' "zone of proximal development" (Vygotski, 1934), and the closeness of personal meanings achieved to implemented (or intended) meaning.
- *Interactive suitability*, extent to which the didactical configurations and trajectories allow to identify and solve semiotic conflicts[8] that might happen during the instructional process.
- *Media/resources suitability*, availability and adequacy of material and temporal resources needed to develop the teaching and learning process.
- *Emotional suitability*, the students' involvement (interest, motivation, …) in the study process.
- *Ecological suitability*, extent to which the teaching and learning process fits the educational project, the school and society, and take into account the conditioning factors of the setting in which it is developed.

A higher suitability in one of these dimensions might not correspond to a high level of suitability in the other dimensions. Given preference to the different criteria will depend on the interactions among them; we then introduce *didactical suitability* as a systemic criteria of adequacy and appropriateness regarding the global educational project. This didactical suitability is relative to temporal, contextual and changing circumstances, which requires an inquiring and reflective attitude

[8] A *semiotic conflict* is any disparity or difference of interpretation between the meanings ascribed to an expression by two subjects (persons or institutions).

from the teacher and the people sharing the responsibility of an educational project.

The theoretical tools developed in the onto-semiotic approach to mathematical knowledge and instruction can be applied to analyse the teaching and learning process implemented in a particular teaching session, the planning and development of a didactical unit, or at a more global level, in the design and implementation of a course or curricular proposal. They also can be useful to analyse partial aspects of a study process, such as the didactical resources, textbooks, students' answers to specific tasks, etc.

4. Examples of application and comparison with other frameworks

Due to the space limitation it is not possible to describe a complete example of application of the framework described, which is being used as the theoretical background for several dissertations, articles and research reports. Some of these examples are described in the authors' web sites. To follow, we have prepared a short abstract of the application of the onto-semiotic approach in Godino, Batanero and Roa' research (2005), where they describe the mathematics activity carried out by a sample of university students when solving elementary combinatoric problems.

Our theoretical tools served to identify the variety of mathematical objects involved in combinatorial problem solving, beyond classical combinatorial formulae and showed examples for the cognitive dualities from which they can be considered, so as the semiotic functions that can be established among them. The students' errors and difficulties were explained by semiotic conflicts, i.e. as disparities between the student's interpretation and the meaning in the mathematics institution. As a result of this application, we provided original and relevant information to better understand the students' combinatorial thinking. The analysis also showed some "transparency illusions" in the teaching and assessing of combinatorics and suggested some ways to improve this teaching.

We believe the *onto-semiotic approach* might help compare the theoretical frameworks used in Mathematics Education and, to the same extent, to overcome some of their limitations for the analysis of mathematics cognition and instruction. The key role that we give to the notion of *mathematical practice*, and the features we attribute to it (any shared, situated, intentional action mediated by linguistic and material resources) might allow a coherent articulation with other theoretical frameworks, such as the social constructivism (Ernest, 1998), the socio-epistemology (Cantoral and Farfán, 2003), and the ethno-mathematical and socio-cultural approach to mathematical meaning and cognition (Radford, 2006).

In Godino, Font, Contreras and Wilhelmi (2006) we use the onto-semiotic approach to analyse and compare other theoretical frameworks, in particular, the theory of didactical situation (Brousseau, 1997), conceptual fields (Vergnaud, 1990), dialectic tool-object (Douady, 1986), anthropological theory of didactics (Chevallard, 1992), and semiotic registers (Duval, 1995).

5. Final reflections

The *onto-semiotic approach* is growing as a theoretical framework for Mathematics Education impelled by issues related to teaching and learning mathematics and the aspiration of achieving the articulation of the diverse dimensions and perspectives involved. In agreement with Steiner (1990), we are convinced that this work of articulation cannot be made through the superimposition of tools taken from different and heterogeneous theories. He conceived Mathematics Education as a scientific discipline in the centre of a complex, heterogeneous, social system – the System of Teaching Mathematics – and proposed beside Mathematics other referential sciences for our discipline, such as: Epistemology, Psychology Pedagogy Sociology and Linguistics. Each of these disciplines focuses its attention on partial aspects of the issues involved in teaching and learning mathematics, using their specific conceptual tools and methodologies. At a certain time this diversity of approaches might be inevitable, or even enriching, but we think that the progress in the discipline and the strengthening of its practical applications requires the emergence of a new global and unifying perspective.

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Developing authentic assessment: Case studies of secondary school mathematics teachers' experiences

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Developing Authentic Assessment: Case Studies of Secondary School Mathematics Teachers' Experiences

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Abstract: Authentic assessment techniques in mathematics raise issues that merit the attention of practitioners, educators, and researchers. Teacher training in assessment, the reliability and validity of authentic assessment, the variety of methods employed in such assessment, and the ways assessment is employed are all concerns that emerge as new assessment techniques other than the expected paper-and-pencil tests are implemented. At the secondary school level, new assessment techniques are emerging in mathematics classrooms and little is known about these experiences. This article specifically addresses the issues associated with authentic assessment by describing and exploring mathematics teachers' experiences as they implement these assessment techniques in their secondary school mathematics program. It summarizes the findings of a qualitative study of five secondary school mathematics teachers in Ontario, Canada, and offers suggestions as to how to support teachers through such a change project. How teachers use authentic assessment, the problems they encounter, and the theoretical and practical issues that emerge are all questions that require exploration and understanding.

Sommaire exécutif ; Les techniques d'évaluation authentiques en mathématiques soulèvent des questions qui méritent l'attention des enseignants, des didacticiens et des chercheurs. La formation des enseignants dans le domaine de l'évaluation, la fiabilité et la validité d'une évaluation authentique, la variété des méthodes employées dans cette évaluation et la façon dont ces méthodes sont appliquées sont autant de questions qui émergent à mesure que de nouvelles techniques d'évaluation, autres que les traditionnels tests sur papier, sont adoptées dans les salles de classe. Au niveau secondaire, de nouvelles techniques d'évaluation émergent dans les cours de mathématiques, mais les résultats de ces expériences sont à ce jour très peu connus.

Cet article fait le point sur les résultats d'une étude qualitative portant sur cinq enseignants de mathématiques à l'école secondaire, qui tentent d'appliquer des stratégies d'évaluation authentique dans leur pratique de l'enseignement. Cette étude a vu le jour après qu'une analyse de la recherche sur l'utilisation de l'évaluation authentique en mathématiques à l'école secondaire eut mis en évidence la quasi-totale absence de détails dans la littérature au sujet des convictions, des pratiques et des préoccupations des enseignants qui se proposent de changer leur méthodes d'évaluation de façon à se conformer aux pratiques courantes en enseignement des mathématiques. Or, il est nécessaire de connaître ces détails si l'on veut permettre aux enseignants de comprendre la valeur, les avantages et les difficultés liés à l'utilisation de l'évaluation authentique dans le cadre d'un programme de mathématiques à l'école secondaire. Grâce à cette compréhension accrue, les enseignants pourront être en mesure de participer au développement de connaissances dans le domaine de l'évaluation.

Une description détaillée des expériences des enseignants de mathématiques au secondaire qui appliquent des méthodes d'évaluation authentique en classe permet de mieux cerner le processus d'évaluation lui-même. L'article décrit les techniques d'évaluation dont se servent les enseignants, les raisons pour lesquelles ils s'en servent et les façons dont ils les utilisent. Cinq cas ont été étudiés au moyen d'une approche qualitative, combinant interviews, comptes-rendus, évaluations types et observations de classe, sur une période d'un an. Grâce à l'observation et à une série d'entrevues, les enseignants sont également amenés à réfléchir sur l'efficacité des techniques choisies, les sources qui soutiennent leur démarche, les dilemmes auxquels ils sont confrontés, leurs façons de faire face à ces dilemmes et enfin les aspects qui nécessitent un travail supplémentaire. Parmi les techniques d'évaluation authentique utilisées, mentionnons le journal de bord, l'évaluation de la performance, les rubriques, les listes de contrôle, l'évaluation par les pairs et l'autoévaluation.

Plusieurs aspects se sont avérés communs aux cinq participants. Les enseignants étaient tous fermement convaincus de l'importance de l'évaluation authentique. Parmi les raisons invoquées, citons la volonté d'insérer les mathématiques dans un contexte réaliste, l'importance de développer des compétences pour ce qui est des modes de pensée et de la résolution de problèmes, la nécessité pour les étudiants de comprendre les concepts mathématiques plutôt que d'imiter les processus algorithmiques, et enfin le fait que l'acquisition de connaissances métacognitives favorise l'apprentissage. Parmi les dilemmes les plus importants aux yeux des participants, mentionnons le difficile équilibre entre les attentes traditionnelles liées au curriculum et les activités d'évaluation authentiques, l'absence de critères de correspondance entre les nouvelles techniques d'évaluation et les méthodes traditionnelles, le sentiment d'isolement ou d'aliénation par rapport aux collègues, et le manque de temps pour explorer ou mettre au point de nouvelles ressources à intégrer dans les curriculums.

En résumé, les aspects clés qui préoccupent principalement les participants sont l'importance de lier entre eux le curriculum, l'enseignement et l'évaluation ; le rôle actif que jouent les administrateurs lorsqu'il s'agit de favoriser une réforme des pratiques dans le domaine de l'évaluation ; et enfin l'importance du soutien que pourrait fournir une certaine culture de collaboration chez les enseignants. En conséquence, cette étude a des implications significatives pour les enseignants et pour tous ceux qui ont à cœur la croissance professionnelle et le perfectionnement pédagogique des enseignants dans le domaine de l'évaluation.

Reform in mathematics education

In order to examine new techniques in mathematics assessment, it is first necessary to review the trends in mathematics education reform over the past decade and more. Mathematics education reform grew out of a perception that students were ill prepared mathematically to face the twenty-first century. *A Nation at Risk* (National Commission on Excellence in Education [NCEE], 1983) brought forward concerns about students' mathematical competence. In response to *A Nation at Risk*, two schools of thought emerged. One school urged a traditional approach that suggested a focus on the acquisition of mathematics 'skills' through rote memorization of algorithms that quickly brought students to an answer. The other school of thought encouraged understanding of mathematics through a constructivist approach with a focus on problem solving. *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 1989) was a seminal work that set guidelines for this problem-solving approach to mathematics. The new goals of mathematics education suggested

> that students should be exposed to numerous and varied interrelated experiences that encourage them to value the mathematical enterprise, to develop mathematical habits of mind, and to understand and appreciate the role of mathematics in human affairs; that they should be encouraged to explore, to guess, and even to make and correct errors so that they gain confidence in their ability to solve complex problems; that they should read, write, and discuss mathematics; and that they should conjecture, test, and build arguments about a conjecture's validity. (NCTM, 1989, p. 5)

This document and its more recent version, *Principles and Standards for School Mathematics* (NCTM, 2000), suggest that students should be able to build new mathematical knowledge through solving problems that arise in mathematics and other contexts. Thus, instructional and assessment tasks should emphasize connections within mathematics by embedding mathematics in relevant extended contexts that encourage students to explore and communicate mathematically (Mathematical Sciences Education Board and National Research Council [MSEB & NRC], 1993). Many have argued that students learn mathematics as they construct meanings for themselves through mathematical activity (Davis, Maher, & Noddings, 1990; Even & Tirosh, 2002).

The reform in mathematics education rests on the belief that understanding mathematics is more valuable than memorizing algorithms, and the use of problem solving puts the emphasis on students' 'understanding mathematics.' Studies support the claim that solving open-ended problems increases understanding and makes use of higher-order thinking skills (Romberg, Zarinnia, & Collis, 1990). Wood and Sellers (1997) indicate that students from problem-centred mathematics classrooms have a better understanding of mathematics and hold different beliefs about mathematics than students from other classrooms. Furthermore, metacognitive skills are both called upon and developed as students engage in mathematics problem solving (Brown & Baird, 1993; Schoenfeld, 1992).

A classroom environment where exploration is encouraged and perseverance is rewarded is likely to help students become good problem solvers. With encouragement and specific feedback from teachers, students develop an awareness of their own strategies and specific needs as they struggle with complex problems. The teacher can probe student thinking and encourage student suggestions while making it clear that solutions must ultimately meet certain mathematical standards (NCTM, 2000).

What is authentic assessment?

In a problem-centred curriculum, new assessment techniques that align with classroom activities and seek to assess students' understanding of mathematics concepts are required. Since traditional tests often focus only on the answer or the use of a suitable algorithm to reach the answer, authentic assessment techniques need to be employed to provide a broader range of measures. In this article, the term *authentic assessment* is used to describe assessment of this type: assessment that involves students in tasks that are worthwhile, significant, and meaningful and that resemble learning activities. Such assessment activities also encourage risk taking, allow for mathematical communication, and provide the opportunity to demonstrate the application of knowledge in unfamiliar settings. Other terms that are often used for authentic assessment include alternative, performance-based, or outcome-based assessment (Hart, 1994).

To capture the multi-faceted aspects of the problem-solving approach, a broad range of assessment techniques needs to be employed, such as structured interviews, concrete models, group problem solving, creative projects, and portfolio evaluations, as well as paper-and-pencil tests (Clarke, 1992; Goldin, 1992; Lampert, 2001). Authentic assessment includes the use of open-ended problems, scoring rubrics, student self-assessment, mathematics portfolios, and student journals. When a wide variety of these assessment techniques is used, teachers gain insight into the student's thinking and understanding of mathematics and students learn to describe their own problem-solving strategies (Maher, Davis, & Alston, 1992; Schoen, Cebulla, Finn, & Fi, 2003).

Design of the inquiry

The case studies reported in this article examine the beliefs, practices, and concerns of five teachers over a 1-year period and consider the support they require. In order to unearth their thoughts and experiences, multiple methods of data collection were used. A preliminary interview was held with each participant. Its purpose was to present the project and answer prospective participants' questions. These preliminary interviews were followed by four sets of individual interviews, one at the beginning of the project, two in the middle (following each of the observations), and one at the end. In most cases, the first interview elicited the extent to which authentic assessment was used in that teacher's classroom, the teacher's views on mathematics education, resources that aided the teacher with authentic assessment, and areas that created stumbling blocks to implementation. Two focus-group meetings were also held. Here teachers were able to discuss their assessment practices with one another. Furthermore, these served as support and facilitated an exchange of ideas. Each teacher's classroom was observed on two occasions when she or he was using authentic assessment to note how assessment practices played out. This provided a context through which to identify the practices that the teachers spoke of in their interviews. Each observation was followed by an interview, sometimes in person, sometimes by telephone due to hectic schedules, to obtain the teacher's perceptions of and response to the assessment activities. Throughout the study, samples of authentic assessment instruments that the teachers used, as well as school or department evaluation policies that would affect assessment practices, were collected. Each teacher maintained an assessment logbook, to supplement the interview and observation data, in which s/he frequently recorded the nature of their use of authentic assessment, including comments or reflections on the implementation of the particular assessment strategy. In most cases, the teachers included samples of the activity, samples of the assessment for that activity, and actual student responses, in order to fully 'round out' their brief description.

The participants

The five teachers selected for the study had already been incorporating some authentic assessment techniques into their secondary school mathematics classrooms. The teachers, all in the same school board, had been identified by the board mathematics coordinator as teachers who were attempting to use new assessment methods and they all agreed to be part of the study. The teachers came from four different schools, with two of the teachers at the same school. The teachers themselves were at very different stages in their careers and, consequently, issues of experience could emerge. As well, they varied in their expertise with authentic assessment, again allowing for differences to surface. An introduction to the participants and settings, identified by pseudonyms, follows.

Two of the participants, Gwen and Julia, taught at the same school. Gwen was completing her sixth year of teaching when the study began. Her teaching experience included a first year of teaching in an elementary school, followed by 5 years of teaching secondary school mathematics and computer science. She had an undergraduate degree in physical education and, at the time of the study, was taking courses towards a Master of Education degree. She had obtained her teaching qualifications in mathematics through additional qualification courses, following her first year of teaching. Julia was completing her fourth year of teaching at the start of the study and all of her teaching experience had been at the one school. Julia had wanted to be a mathematics teacher since she was in high school, where she participated in mathematics contests and volunteered as a peer tutor in mathematics classes. She then obtained her undergraduate degree in mathematics.

Dave was an experienced teacher who was in his first year as head of the mathematics department at his school. He was very involved in a local chapter of the provincial mathematics educators association. He frequently presented at conferences, both within the school board and at the provincial level. Dave had been teaching for 20 years, with a 5-year break in the middle of his teaching career when he was a financial planner. He has also had some experience teaching at the college level.

Miriam was an experienced teacher, who had taught for over 20 years and was the department head of mathematics at a large suburban high school. Miriam served on various committees in the school board that dealt with assessment and specifically with developing assessment scales for numeracy skills. She also taught a summer institute course to teachers within the board on using numeracy assessment scales.

Luke had been teaching for 11 years at both elementary and secondary levels. Luke had started his career teaching mathematics and working with students with specific learning needs in an elementary school for 2 years. He then moved into secondary school mathematics teaching and, at the beginning of the study, he was teaching mathematics at a large secondary school and was the assistant department head of mathematics. He had been at the school for about 6 years and, at the time, was teaching Grade 9, Grade 12, and calculus. In the middle of the study, after teaching secondary school mathematics for 8 years, Luke decided to make another change and moved from the secondary school to a senior elementary school to teach Grade 8 mathematics. Thus, during the second half of this study, Luke was teaching Grade 8 mathematics.

In summary, involved in the study there were two very experienced teachers who were also department heads, two teachers who were fairly new to the profession, and one teacher in the middle of his career. Two of the participants were male and three were female. They also had a variety of backgrounds and experiences, some with degrees in mathematics and others with degrees in other disciplines but with the required training to become mathematics teachers. Most of them were teaching at the secondary school level, with Luke moving to the Grade 8 level mid-way through the study. The common element was their choosing to include authentic assessment methods in their practice.

Findings

What the teachers were doing

Several areas were common to all five of the participants. The teachers had firm beliefs as to why it was important to use authentic assessment, including the beliefs that mathematics should be set in a realistic context; that developing thinking and problem-solving skills is critical; that understanding mathematics concepts rather than imitating algorithmic processes is essential; and that the development of metacognitive skills enhances students' learning.

All five participants had decided to include authentic assessment activities while working in a fairly traditional mathematics curriculum setting. Their teaching style, therefore, was dichotomous. At times, they used a traditional approach of giving examples at the chalkboard, after which students practised the skill. Other times, they engaged students in active problem solving that focused on developing mathematical understanding, presenting mathematics in context, and encouraging communication about mathematics. The goal, in the second instance, was for students to increase their understanding and to make connections among mathematics, other disciplines, and their world.

Each participant included group problem-solving activities as part of the classroom routine. Students were often presented with an open-ended problem, given in a realistic context. Students were encouraged to investigate the problem and to determine and present a solution. Problems often incorporated recently acquired skills or were an introduction to a new set of skills to be learned in a necessary circumstance. Such experiences agree with the definition of problem solving offered by Ginsberg et al. (1992) as being active, conjecturing, modelling, and applying skills. Frequently, students were given tools to help them solve the problem, such as manipulatives, chart or graph paper, graphing calculators, and/or computers. As Franke and Carey (1997) suggest, these concrete materials help bridge the gap between formal and informal thinking for students. Student solutions to the problems posed by their teachers were presented in a variety of ways, including written submissions with full justification for their answers and presentations of the solutions to the class, often using visuals or demonstrations. The teachers assessed these problem-solving situations through observation checklists while students worked, rubrics to determine levels of performance of a written submission, and teacher or peer assessment of classroom presentations.

All teachers demonstrated an intrinsic link between the types of activities they were doing and the assessment strategies they were using. As part of developing these classroom activities, the teachers also developed methods of assessing the activities, most of which could be called authentic assessment. Thus, another commonality was that all of the participants were striving to link their new instructional practices with new assessment practices. Similar results have been reported in several studies (Kulm, 1994; Lehman, 1995). New instructional practices necessitated new assessment practices. Teachers were choosing classroom activities to enhance the student's learning and the assessment grew out of the nature of the tasks that they were asking students to perform. 'You plan an activity and then you decide how it would be best to assess it' (Julia, informal discussion). The development of reflection and self-assessment in their students was also important to the participants. Assessment methods that were developed, therefore, encouraged the growth of metacognitive skills, such as self-reflection, responsibility for one's own learning, and self-confidence. This was exemplified by the use of portfolios by Gwen, strategic planning by Luke, learning logs by Miriam, and journals and questioning by Dave and Julia.

Why the teachers were using authentic assessment

None of the participants had been mandated to use authentic assessment in their classrooms. Rather, the choice to use authentic assessment activities was made by each individual teacher and came as a result of trying to link her/his assessment strategies to instructional practice. All of the teachers in the study had been introduced to new assessment strategies and problem-solving activities through their own professional development, reading, and voluntary attendance at workshops and conferences.

The reasons for using authentic assessment were interwoven through all of the interviews and focus-group meetings and a variety of reasons emerged. Ultimately, their choosing to use authentic assessment was a result of their choices of instructional activities. They chose such activities because they recognized the importance of applying mathematics in real contexts, creating connections between mathematics and the student's world, developing problem-solving skills, creating a deeper understanding of mathematics, developing metacognitive skills, and encouraging students' responsibility for their own learning. While these intentions may appear as separate ideas, they are firmly woven together. For instance, a teacher provides students with a problem-solving application of mathematics in a real context with several purposes in mind: to deepen a student's understanding of mathematics, to help the student make connections, and to help the student build confidence in himself or herself as a problem solver.

One reason for using authentic assessment activities cited by several participants was the belief that mathematics should be applied in realistic contexts to deepen students' understanding of mathematics and so that students recognize the value of mathematics. Students need to see the applications of mathematics within the classroom to be better prepared for applying mathematics outside of the classroom. Miriam suggested, 'It is the math that people are going to use in their lives' (focus-group interview). Luke insisted that you could not have authentic assessment unless you have authentic mathematics. He described his impression of authentic mathematics:

> For me, the notion of authenticity is 'Is the mathematics you're doing actually authentic mathematics?' Or is it just an isolated topic. And is the teacher as leader telling kids what to do? Or is it a chance to explore? Are they behaving like true problem solvers? Is the stuff that we are assessing even authentic, never mind the techniques that we are using? So can you have authentic assessment with a traditional type of curriculum? (Luke, focus-group interview)

Developing problem-solving expertise was also seen as an important reason for choosing authentic assessment activities. Problem solving often grew naturally out of the mathematics in context, an application of mathematics, or a mathematical dilemma or conjecture. For these teachers, it was not only important that students be able to perform mathematical algorithms such as solving quadratic equations or using matrices to solve linear systems. It was equally important that they be able to take a realistic problem and interpret it in such a way that they could build a mathematical model to solve the problem. This model could be algebraic, graphical, or geometric in nature. The leap between a problem in context and a mathematical model of the problem was often the most difficult one for students to make, especially if they were accustomed to mathematics in isolation. Yet, it is a significant one if they are going to be able to use mathematics outside of the classroom and see its relevance. This leap was also the crux of what these teachers called problem solving.

The teachers also believed that students reinforce their understanding of concepts through verbalization of those concepts. Gwen and Julia often used *think–pair–share* as a method of review. As students communicate with one another, the teacher circulates around the room and prompts where necessary. Gwen and Miriam discussed the benefits of students' working together and conferring about mathematics.

> **Gwen:** We really need to give them the opportunity to present things orally. Which they don't [normally do]. And they also need to write. I have kids do a lot of pair work where they have to explain to each other. 'Explain to your partner how to do this question.' If they [students] verbalize it, they understand it better.
>
> **Miriam:** Well, all kinds of research shows that if you tell someone else how to do it then you learn it better. (focus-group interview)

Developing students' confidence in doing mathematics was also important. Students require confidence to take necessary risks in problem solving, such as trying a different approach, brainstorming, or using trial and error. However, teachers also see that by engaging students in open-ended problem solving, where there is not necessarily only one correct answer and the teacher is open to alternative solutions, then the students develop confidence in themselves as problem solvers. Thus, problem solving both requires self-confidence and develops self-confidence. As Schoenfeld (1992) reports, students' beliefs and attitudes about learning mathematics affect their understanding. The development of metacognitive skills supports the development of problem-solving skills.

Teachers used authentic assessment and their linked activities so that students could develop a deeper understanding of mathematics. Teachers also wanted students to see the relevance of mathematics, be able to apply mathematics to solve problems in other contexts, and develop the confidence and skills necessary to make use of the mathematics that they learned.

Facing dilemmas

The greatest dilemmas faced by the participants included balancing traditional expectations in the curriculum with authentic assessment activities, matching new assessment techniques with traditional reporting methods, and dealing with feelings of isolation and alienation from colleagues. This particular section deals with these dilemmas. Other issues that arose involve the types of supports that teachers require, such as time to explore, develop, or find appropriate curriculum and assessment resources; administrative support; and sound professional development. These issues will be presented in the subsequent section on 'Supporting the Use of Authentic Assessment.'

Covering the content: Working within a traditional framework

All five teachers were trying to combine the traditional course content and traditional testing with other activities, which were generally problem-based or included self-reflection on the students' own learning and were assessed using authentic assessment strategies. These teachers had to cover prescribed content so that they could ensure that students were well prepared for the common course examination set by the teachers teaching the course. The teachers in the study frequently questioned the importance of the prescribed content and felt as though there were other, more significant mathematics concepts that could be experienced.

Covering the content is the term that teachers use to describe the pressure that they feel to 'pack in' many distinct (sometimes seemingly unconnected) topics that have been designated as part of the mathematics curriculum. The pressure to cram so many topics into so little time often forces teachers to teach mathematics as a series of memorized algorithms rather than to strive for student understanding of mathematics. The experiences of these participants appeared very similar to the experiences of the participants in a study by Rowley, Brew, and Ryan (1996), which reported that teachers felt pulled in two directions by the need to cover the content while, at the same time, doing activities that they believed would give students a deeper understanding of mathematics, help them make connections to mathematics, and allow them to take responsibility for their own learning. Lampert (2001) suggests that covering the curriculum assumes a linear approach to learning and that curriculum could be re-conceptualized around significant *conceptual fields*. She also suggests that such a re-conceptualized curriculum would prompt the need for authentic types of assessment.

The participants pointed to several ways of dealing with the dilemma of covering the content while also using problem-solving activities. At times, the teacher began the unit with a problem in context so that the students might see the purpose of pursuing a set of skills that would help them determine the solution to the problem. In other cases, the teacher would first teach the content and then use problem-solving activities as a follow-up. Julia mentioned that she saw a dichotomy between learning the rules and then applying them to an activity: 'If I feel that I'm just teaching the unit and the kids are just doing all the rules and I think that's kind of the way the unit is set up. Then I try to take the time to find an activity or something that will enhance that and make them think a little bit more. Because I think it helps them learn. It helps them become better students' (Julia, interview transcript). All noted, however, that mathematics education was in transition from a content-oriented curriculum toward a problems-based curriculum and expressed confidence that, in due time, the curriculum would better support the types of activities they felt were important. In fact, several of the participants felt that they could create such a curriculum, if given the time to create and implement it. Julia expressed her vision of what the curriculum could be:

> I want to start from scratch, sit down with somebody and completely change the way we teach Grade 9. I can picture what I would do. I want to do many more activities, much more discovery, much more group problem solving. I would love to do more projects as they are learning different skills as they work through the project, not so much a topic-by-topic approach. This means revamping the entire way we teach the course. (Julia, interview transcript)

In a focus-group discussion, Miriam confirmed that a new curriculum was needed and suggested that a problem-solving curriculum could lead in many different directions.

The participants felt that one stumbling block to introducing a problems-based curriculum was that the teacher needs to be very confident and knowledgeable about mathematics to resolve student questions and provide suitable prompts to students as they move through investigations and explorations. Luke suggested that teachers would need to be comfortable with a more open-ended approach: 'It means that you need a great knowledge of the course. Even an experienced math teacher, unless they know the course well and can see connections that can be made and can be drawn out would have a hard time. So you need to know the course and you need to know the mathematics' (Luke, focus-group interview). Dave was hopeful that the changes in the mathematics curriculum would help to alleviate the struggle that the participants felt. They viewed themselves in a state of transition. As with several other studies (Borko, 1997; Flexer, 1995; Ryan, 1994; Shepard, 1994) teachers reported making curricular changes such as re-sequencing curriculum, introducing new concepts, and emphasizing process rather than content.

As the teachers considered the pressure they felt to cover the content, the issue of examinations arose. They felt that exams were driving what was being taught and how it was being taught and that perhaps the format or purpose of exams should be reconsidered. Several of the participants felt compelled to cover the content so that their students would be prepared for the final examination. Julia believed that problem-solving and authentic-assessment activities were important but she did not want the time spent on those activities to be time taken away from examination content, since that might disadvantage her students. Gwen suggested that most teachers see the examinations as 'driving the curriculum.' Miriam suggested that this does not necessarily allow for the spontaneity that a problem-solving curriculum entails. This notion is supported by Neill and Medina (1989): 'As teaching becomes "coaching for the test" in too many schools, real learning and real thinking are crowded out. Among the instructional casualties are higher-order thinking skills' (p. 694).

Miriam suggested that a solution might be to have examinations disappear. Dave suggested that the final examination could take a completely different format, such as the in-class examination component, incorporating technology, that was used in his department. Several of the participants offered alternatives to traditional examinations, including performance assessment tasks as summative assessments.

Assessment and instruction are intrinsically linked. The teachers' need to use authentic assessment emerged as their instructional practices moved from a more traditional style to the use of such things as contextual problems, cooperative learning, and an emphasis on communication. In both assessment and instruction, they felt caught in the middle. They felt that they needed to teach a traditional curriculum to prepare students for a traditional examination. However, they also felt compelled to instruct, and thus assess, in a variety of ways.

The problem of isolation

The participants often spoke of feeling as though they were working in isolation, or worse, were seen as the odd person out. These participants saw themselves as doing things differently from their colleagues and they perceived that their colleagues saw them that way as well. The participants' feelings of isolation were very strong. They hesitated to share their ideas because they felt as though they needed to be on the defensive or be prepared for criticism. Julia mentioned that '[t]o a large degree I don't tell people what I'm doing or I'll wait until I've done it and see whether or not I really liked it before I tell them. But if it is something that I think went well and I think is worthwhile then I share it' (Julia, interview transcript). All of the participants expressed similar reservations. These teachers felt like outsiders in their own departments and often received negative feedback when they shared some of their ideas. In a focus-group meeting, Gwen reported that

> I said to one person in my department that I had done a self-evaluation at the end of a unit and they said 'Oh, I used to do that and try to relate it all to the report card but in the end I am going to give them the same check mark anyway so it doesn't really do me any good.' They seemed to miss the point that it was the kid who was going to get something out of the self-evaluation. (Gwen, focus-group interview)

Luke often expressed his frustration at feeling different from the rest of his department at the secondary school level. Although Luke reported that he had more time at the elementary level to try innovative ideas, he also reported that he still felt isolated. Other teachers were not attempting to implement and assess the curriculum in the same way that he did.

Even though these teachers felt isolated, they still continued to try new things in the classroom. They believed in what they were doing and they saw students developing an understanding of mathematics. They saw these experiences as part of their professionalism and often wondered or became frustrated that other teachers did not see it this way. The question arose, 'Why are the other colleagues so suspicious of these ideas?' Dave suggested that '[e]ven the young teachers have old thoughts' (interview transcript). Some suggested that these forms of assessment were difficult for many teachers because they might not feel comfortable enough with the mathematics to venture into open-ended problem solving: 'They are math teachers but they are not [math teachers], because they don't have a strong math background. I don't think that they have experienced math that way themselves and it is a pretty huge task' (Luke, interview transcript).

However, when the teachers shared their ideas with others in their department, they occasionally shifted other teachers' views of authentic assessment activities by giving them samples of activities that they had tried themselves. Miriam had an experience of teaching one of her colleagues a different way of teaching integers to Grade 9 students, using bingo chips:

> I know I had a teacher last year and I showed him how to teach integers using bingo chips and this guy finally tried it because I said that I am going to put a modelling question on integers using bingo chips on the exam so you will need to do it with his students. Well, there was a lot of grumbling and commotion and he tried it and then noticed that the kids were flying through the questions. He was sold on the idea. He fought for years and then tried it and was so excited and loved it. (Miriam, focus-group interview)

Luke suggested that teachers need to share ideas in order to develop new strategies. Yet, Luke also warned about being sensitive to other colleagues and not forcing these approaches as the 'right way' to teach. The participants felt that what they were doing had value; yet, they did not feel that they were 'authorities' who could prove that this was the best way to teach mathematics.

One problem with isolation is that it can lead to a lack of consistency in assessment among instructors. The reality of some teachers' incorporating new teaching and assessment strategies while others are not creates issues of incongruity, incoherence, and questionable accountability. Several of the participants recognized that there was a wide gap between colleagues about issues of assessment, and this concerned them. As supported by several studies (Morony & Olssen, 1994; Rowley, Leder, & Brew, 1994; Rowley et al., 1996) teachers need to be supported and provided with informal opportunities to discuss and debate issues. If teachers are going to increase their expertise with authentic assessment then teachers need to talk about their experiences. In fact, Miriam suggested that our first focus-group interview energized her. She believed that the discussion was very stimulating and commented that '[t]eachers just don't have enough time to talk to one another'(informal interview). Others confirmed this. Gwen and Julia often worked together in their department and developed and shared activities. They reported that this sharing helped to alleviate the feeling of isolation and helped them to develop and affirm new ideas through discussion. Collaborative work among teachers allows for the sharing of expertise in areas of assessment, and, as supported by several studies (Clarke, 1996; Morgan & Watson, 2002), greater consistency in assessment can be developed through discussion of assessment criteria and student work.

Authentic assessment and reporting

Several of the teachers discussed the difficulty of matching authentic assessment techniques with a more traditional method of reporting using percentage marks. The data gathered through authentic assessment frequently consists of levels on a rating scale or rubric or is anecdotal. This type of information is unsuited to being directly translated into a percentage mark. Wiggins (1994) confirms that translating rich assessment data into a mark or grade for a report card is a dilemma: 'Why, then, do we arbitrarily average grades and scores in school—where the dimensions of performance are even more complex and diverse—to arrive at a single grade per subject? Problem solving is not research, is not writing, is not discussing, is not accuracy, is not thoroughness, and is not mastery of the facts' (p. 35). The participants' experiences confirm this, as all of the participants found it difficult to convert authentic assessment to a mark at levels other than Grade 9. Gwen mentioned, 'I think the hardest thing right now is, except at the Grade 9 level, trying to relate authentic assessment to a mark. In the other levels, that's where I find I'm having a little bit of difficulty' (interview transcript). Julia also had difficulty using authentic assessment with her more senior grades, whereas, for Grade 9, she could report levels of student achievement. Julia used authentic assessment in her Grade 12 class and then converted that assessment data into a percentage mark. Using reporting methods that relate to levels definitely makes it easier to adapt authentic assessment to the evaluation of students. Luke used levels of achievement with his Grade 8 class, and Gwen and Julia reported achievement with check marks on a rating scale for their Grade 9 classes. These three participants believed that it was easier to link assessment and the reporting of progress when the emphasis was on levels of achievement rather than strictly on a percentage grade.

Gwen believed that students and parents received more information from a report card that discusses levels of achievement on several different outcomes rather than from an overall percentage mark. She was a firm advocate of this method of reporting and believed that parents and students benefited. Julia also believed that her school's Grade 9 report card was useful to students: 'I think just in terms of feedback to students, I think it's really helpful rather than just giving them a mark. I think it is good to continue to mark things in this way. Maybe I should be doing that more with my Grade 12 class. Maybe I should give them more anecdotal information about where their strengths are and where their weaknesses are' (interview transcript). Julia further remarked that the set-up of the Grade 9 report card actually encouraged her to use other forms of assessment rather than paper-and-pencil tests.

Supporting the use of authentic assessment

Several points arose from this study about enhancing the quality of authentic assessment practices, including the role a partnership between students and teachers plays in their being more aware of authentic assessment strategies, the importance of providing teacher resources about assessment and providing effective teacher development in assessment, and the value of administrative support.

Partnerships

Increasing colleagues', students', parents', and administrators' understanding of new assessment methods is essential to supporting teachers' use of authentic assessment practices. The participants considered collaboration with colleagues an important aspect in developing expertise and increasing consistency in assessment. It was also important to increase students' and parents' awareness of authentic assessment and to share with them the purpose of new instructional and assessment strategies. Lester and Kroll (1990) suggest that assessment methods communicate to students, parents, and administrators what is considered important. The participants explained that students became more comfortable with authentic assessment as their familiarity increased over time.

> Once you sort of go over it with them, later, then you can say this is sort of what I was looking for. Then they get better. They say 'Oh, that's what you mean, that's what you wanted.' ... That is sort of what happens when students get used to us giving them the right answer ... I mean I think it's not so much the students who need to change but it's the teachers too. (Miriam, focus-group interview)

Luke recognized that the students actually felt more relaxed with an authentic assessment activity because it alleviated the stress of a paper-and-pencil test, where resources or communication are prohibited. Several participants used exemplars (or samples of student work) that showed students the type of criteria that were used to reflect a particular level of achievement. Most of the participants discussed the assessment criteria with the students before initiating the assessment activity and often provided exemplars.

> I tell them that I will be marking it and I tell them that they will be evaluating it as well. I talk to them about, or I usually write down on the overhead with them, 'What would really make a good assignment,' and some of the things that would tend to make a satisfactory kind of assignment and what would be something that would be considered to be not such a great assignment and we will write down together as a group some things to look for. (Julia, focus-group interview)

Some of the participants increased students' understanding of how authentic assessment applied to them by having the students help in creating rubrics for specific activities.

Resources

Several of the participants mentioned a lack of resources or the time for finding and adapting resources as a major stumbling block to their use of authentic assessment. Even when resources were located, they were not always suitable to a teacher's needs. Teachers need exemplars of problems, assessment tools, and samples of student work to assist them in determining levels of achievement. Miriam described the lack of time for finding and developing resources. Julia often found it difficult to create new activities to suit each unit and recognized that she often needed a resource to stimulate an idea that she could develop. Professional development and courses in authentic assessment are ways to help teachers locate, adapt, or develop resources. Dave suggested,

> What would be good would be a course of some kind in this stuff. It truly would be. It would take people who were excited and kind of lessen their load a bit. I mean it is hard to keep recreating everything by yourself. And for people who haven't tried it, it could be a relatively non-threatening introduction. I mean, if they could get some ready-made rubrics and activities that go with them then there might be a way to get them started. (Dave, interview transcript)

As a new department head, Dave recognized that teachers have very little time to investigate and implement new assessment strategies in their classrooms. Gwen also referred to the creation of activities' requiring a great deal of time and Luke reiterated the need for teachers to find time to share the resources and ideas that they have.

Attitude towards professional growth

Throughout this study, the participants' high level of commitment to professional growth was apparent. For instance, all of the participants had gone beyond the mandatory educational training to partake in or complete specialist courses, as in Dave's, Julia's, and Miriam's cases, or Master's degrees, as in Gwen's, Luke's, Dave's, and Miriam's cases. The participants were engaged not only in lifelong learning but also in seeking new resources, adapting their teaching, and generally enhancing their professional expertise. It has been suggested, 'Mathematics teachers need to continually explore mathematical concepts and ideas to be better prepared for different learning situations' (McDougall, 1997, p. 163). These findings are supported by the practice of the participants.

Examples of the teachers' commitment to professional growth occurred constantly. All of the participants attended workshops and conferences on their own time. Several of the participants stated that they started each new school year with a pledge to themselves to try one or two specific new ideas, such as introducing rubrics or portfolios.

> For instance, this year I am going to generate three more activities with rubrics, let's say. I do that. I commit myself to do that because I sometimes figure that if I don't do that then nothing happens. Because it is much easier to slide back into the traditional, or whatever you did last year (whether it was traditional or not). It is much easier to slide back. (Dave, interview transcript)

There was also significant evidence of these teachers as reflective practitioners:

> A lot of what I think about is what can I do differently? What didn't go well? What can I improve on? I often think okay, next year when I do this I'm going to do this, this way. Even sometimes I'll make notes to myself after I've done an activity that this needs to be worked on or I'll think of something like 'Gee a great way to start this section off would have been ...' (Julia, interview transcript)

The participants were willing to try new ideas and take risks. Gwen spoke of her willingness to try new things when she spoke of why she decided to use portfolios: 'Again, it is just more experimental. I wanted to try something different and see how it works, if it works and to see the amount of work it requires from students and from me and to see if it is worth it and if it is manageable' (Gwen, post-observation interview). This willingness to take risks requires that the teacher be confident of her/his own mathematical knowledge and insight. It would appear as though the participants solved the problems of creating a meaningful mathematics curriculum and displayed the characteristics they encouraged in their students: risk taking, confidence, collaborative work, and the use of appropriate resources. These characteristics were previously identified in several studies (Brown & Baird, 1993; Carpenter & Fennema, 1991) as being strong influences on teachers' use of problem-solving and authentic-assessment activities. Luke noted, 'You really have to believe [that this is the right thing to do]' (Luke, first interview).

Administrative support

School and board administrations can support new assessment practices in a variety of ways. They can help to provide time for teachers to collaborate, support professional development, communicate with parents and students about new assessment methods, and help to develop reporting methods that are aligned with assessment practices. Administrative support to facilitate finding time for teachers to work together was a strong influence on several of the participants. For instance, Miriam felt supported by a principal who found creative ways to build time into the schedule of the day for teachers to work together. Support for teachers' work is essential. 'Even if teachers are convinced of the benefits of using more innovative methods to evaluate their students, they are unlikely to succeed unless their supervisors, students, parents—and even their fellow teachers—understand and support their break with tradition' (Webb & Coxford, 1993, p. 13). Teachers need time to develop and find resources that are samples of good instructional and assessment practices and to organize and analyse samples of student work.

Reporting methods that align with new assessment practices are also useful. In Gwen and Julia's case, support through a compatible reporting system for Grade 9 encouraged a variety of assessment methods and provided parents with detailed information about student achievement. Administrative support is also required to educate parents and students as to new methods of assessment and reporting, thus increasing validity and accountability. Parents and students are very oriented towards percentage grades, especially in the senior levels. Parents, teachers, and students need more information about the value of authentic assessment techniques if these techniques are going to be credible. The creation and explanation of sample activities, sample assessments, and exemplars of different levels of work would be helpful to parents and students, as well as to teachers.

Conclusions

This study provides significant information about teachers' experiences in developing authentic assessment activities for their classrooms. How and why teachers use authentic assessment, the problems they encounter, and the ways that they can be supported are summarized in this concluding section.

The teachers in this study chose to use authentic assessment techniques because they were using innovative approaches to teaching mathematics. They believed that learning mathematics was more than practising algorithms and wanted to give students the opportunity to explore mathematical concepts, apply mathematics in a variety of situations, and become confident problem solvers. Their view of mathematics teaching and learning incorporated problem-solving challenges, communication of the process of mathematics, group work, and self-reflection on the process of learning mathematics, as well as developing traditional skills. They found that as they incorporated new instructional strategies in their practice, the sole use of paper-and-pencil tests was not comprehensive enough to show what students knew and could do. Thus, they developed and implemented other assessment instruments, such as portfolios, performance tasks, journals, and group projects.

The teachers faced several challenges. Their view of mathematics teaching and learning was often different from the views of their colleagues. In most cases, they were working on their own and felt a sense of isolation. They also were trying to implement innovative ideas that did not necessarily fit into a traditional mathematics curriculum and assessments that were more than a traditional examination. Often, they felt that they were met with skepticism from their colleagues. They also struggled with matching these assessment instruments with traditional reporting methods, for they felt as though the assessment evidence that they gathered could provide a great deal of information that could not fully be shown with a single numeric mark.

Participants suggested several ways that they could be supported. They felt that a problems-based curriculum would support their instructional and assessment strategies. They also recognized the importance of developing a collaborative teacher culture, particularly as teachers were struggling with new methods. They also valued the important role that administrators played in facilitating change in assessment practices.

The teachers in this study were constantly torn between delivering a curriculum that listed content topics and assumed a traditional teaching style and offering a problems-rich curriculum with a variety of teaching styles. If engaging instructional tasks and authentic assessment activities are to be implemented, then a curriculum that encourages and promotes such activities should be in place.

However, implementing a problems-rich curriculum may require much more than a new curriculum. This study examined the practice of teachers who chose to use authentic assessment in their classrooms. Their choices grew out of their view of mathematics, which in turn affected their instructional practice. They had a desire to increase student understanding through problem solving and incorporated this in their classroom instruction. They recognized a need to implement new assessment techniques and thus chose to grapple with issues of implementing these techniques. Thus, some would wonder whether educators are able to learn anything that would be applicable to teachers who are mandated to implement authentic assessment techniques. What can be said of teachers who may be expected to employ authentic assessment in their classrooms and do not 'believe,' as Luke suggests is necessary? The implementation of new assessment strategies must be coupled with or preceded by the implementation of instructional strategies that focus on problem solving and exploration. As well, teachers need to have an understanding of why they are implementing such strategies, so that classroom practices are purposeful. Teachers match their instructional and assessment style with how they think about mathematics and how they perceive students learn mathematics. Shepard (2000) suggests that implementing a reformed vision of curriculum and assessment can be overwhelming: 'Being able to ask the right questions at the right time, anticipate conceptual pitfalls, and have at the ready a repertoire of tasks that will help students take the next steps requires deep knowledge of subject matter. Teachers also need help in learning to use assessment in new ways' (Shepard, 2000, p. 12). New ideas about assessment and instruction are apt to be at odds with prevailing beliefs. This was shown by the sense of isolation of the participants. Teachers will need time to reflect on their own beliefs about mathematics and mathematics teaching and learning. They will also need to be exposed to new knowledge about how students learn and assimilate this knowledge before they will be able to think about changes in practice. New methods may need to be introduced gradually. Teachers who are new to the use of authentic assessment may be most comfortable with a course of study that gradually introduces relevant problem-solving activities and accompanying assessment methods.

A collaborative teacher culture needs to be supported and encouraged to serve as informal professional development that allows teachers to develop expertise and share resources. Sustained professional development and collective participation encourages professional communication and supports change in teaching practice (Garet, Porter, Desimone, Birman, & Yoon, 2001). Such collaboration also helps to develop the consistency in assessment practices that leads to greater accountability. Teachers also develop a system of support to counteract the feeling of isolation reported in several inquiries. Roulet (1998) examined the practice of exemplary mathematics teachers and his description of his participants, Randy and Jonathan, concurs with the experiences of isolation of the teachers in this study. The need for a collaborative work culture seems especially valid for teachers implementing new ideas that make them uneasy. Issues of increasing expertise, maintaining consistency, sharing resources, and providing mutual support will be best addressed through a coordinated effort.

Administrative support is required to give teachers time to learn new techniques and to share their knowledge, resources, frustrations, and successes. Administrators can also help to develop an understanding of the purpose of using authentic assessment. Increasing the validity and understanding of authentic assessment so that students and parents are more comfortable with incorporating authentic assessment in evaluation is critical.

The teachers in the five case studies chose to use authentic assessment and had been incorporating changes in curriculum and assessment practices as part of their own professional growth. Although they were not required to adopt authentic assessment practices, their stories have value to those who are expected to incorporate new assessment techniques into their practice. As well, the study's relevance is not restricted to mathematics educators but also speaks to other disciplines. Understanding teachers' experiences with authentic assessment helps the educational community recognize how to support teachers in assessment. Further, this article outlines the needs of these teachers and the conditions that support professional growth in the area of authentic assessment in mathematics.

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Computers in Early Childhood Mathematics [1]

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ABSTRACT Computers are increasingly a part of the lives of young children. This article reviews empirical studies that have investigated the implementation and use of computers in early childhood mathematics, from birth to grade 3. Major topics include general issues of children using computers, the use and efficacy of various types of computer programs for teaching and learning mathematics, and effective teaching strategies using computers.

Children Using Computers

Most schools have some computer technology, with the ratio of computers to students changing from 1:125 in 1984 and 1:22 in 1990 to 1:10 in 1997 (Clements & Nastasi, 1993; Coley et al, 1997). However, schools having computers does not mean children use computers. In one study, just 9% of fourth graders (they did not collect data on younger children) said they used a computer for schoolwork almost every day; 60% said they never used one. A study of preschool and kindergarten classrooms indicated low use by most teachers (Cuban, 2001). Nevertheless, there seems to be an increasing *potential* for children to use computers in early childhood settings. Is such use appropriate?

An old concern is that children must reach the stage of concrete operations before they are ready to work with computers. Research, however, has found that preschoolers are more competent than has been thought and can, under certain conditions, exhibit thinking traditionally considered 'concrete' (Gelman & Baillargeon, 1983). Furthermore, research shows that even young pre-operational children can use *appropriate* computer programs (Clements & Nastasi, 1992). A related concern is that computer use demands symbolic competence; that is, *computers* are not concrete. This ignores, however, that much of the activity in which young children engage *is* symbolic. They communicate with gestures and language, and they employ symbols in their play, song, and art (Sheingold, 1986).

Moreover, what is 'concrete' to the child may have more to do with what is meaningful and manipulable than with physical characteristics. One study compared a computer graphic felt board environment, in which children could freely construct 'bean stick pictures' by selecting and arranging beans, sticks, and number symbols, to a real bean stick environment (Char, 1989). The computer environment actually offered equal, and sometimes greater, control and flexibility to young children. Both environments were worthwhile, but one did not need to precede the other. Other studies show that computers enrich experience with regular manipulatives. Third-grade students who used both manipulatives and computer programs, or software, demonstrated a greater sophistication in classification and logical thinking, and showed more foresight and deliberation in classification, than did students who used only manipulatives (Olson, 1988).

Others argue that brain research indicates that children should not use computers (Healy, 1998). One could disagree with the interpretations of the research and its ramifications, but for our purposes, let it suffice to say that few neuroscientists believe that direct educational implications can be drawn from their field (Bruer, 1997; Cuban, 2001) – the implications are unwarranted and probably spurious. Finally, recent reports bring up the old issue of 'rushing' children. However, computers are no more dangerous than many of the other materials we use with young children, from pencils to books to tools; one can push a child to read or engage in other activities inappropriately early. They can all also be used to provide developmentally appropriate experiences. Furthermore, the construct of 'developmental appropriateness' continues to be refined. Following the National Association for the Education of Young Children (NAEYC), we define it as follows: developmentally appropriate means challenging but attainable for most children of a given age range, flexible enough to respond to inevitable individual variation, and, most important, consistent with children's ways of thinking and learning (Clements et al, in press). Therefore, the question is not if computers are 'concrete,' but whether they provide experiences that facilitate children's learning. Criticism (or proselytizing) not grounded in practice is unreliable. As just one initial example, critics have said, about children drawing shapes by giving Logo programming commands to a screen 'turtle', 'What does it mean to children to command a perfect square but still not be able to draw it by themselves?' (Cuffaro, 1984, p. 561). Research indicates, however, that Logo drawing experience allows some children to create pictures more elaborate than those that they can create by hand. Children modify their ideas and use these new ideas in all their artwork (Vaidya & McKeeby, 1984). Thus, what it means is that children can extend their experiences and their creative activities in learning to draw. Therefore, there seems to be no reason not to use computers *if* they can contribute to mathematical learning. Substantial evidence has also been generated addressing this question.

Computers, Mathematics, and Reasoning

Research has substantiated that computers can help young children learn mathematics. For example, one computer-based project showed positive and statistically significant improvement across grades and schools for three areas, reading, mathematics, and total battery scores (Kromhout & Butzin, 1993). Effects were largest for students in the program for more than one year, as well as those from minorities and free-lunch programs. In this section, I review research on computer-mediated practice, on-computer manipulatives, turtle geometry, and computer approaches to developing higher-order thinking skills.[2] For each of these, I describe some unique advantages of computers for educational practice.

Computer-mediated Practice

Children can use computer-assisted instruction (CAI) to practice arithmetic processes and to foster deeper conceptual thinking. Drill and practice software can help young children develop competence in such skills as counting and sorting (Clements & Nastasi, 1993). Indeed, the largest gains in the use of CAI have been in mathematics for *primary* grade children, especially in compensatory education (Ragosta et al, 1981; Lavin & Sanders, 1983; Niemiec & Walberg, 1984). Again, 10 minutes per day proved sufficient for significant gains; 20 minutes was even better. This CAI approach may be as or more cost-effective as other instructional interventions, such as peer tutoring and reducing class size (Niemiec & Walberg, 1987). Properly chosen, computer games may also be effective. Second graders with an average of one hour of interaction with a computer game over a two-week period responded correctly to twice as many items on an addition facts speed test as did students in a control group (Kraus, 1981).

How young can children be and still obtain such benefits? Three year-olds learned sorting from a computer task as easily as from a concrete doll task (Brinkley & Watson, 1987-88a). Reports of gains in such skills as counting have also been reported for kindergartners (Hungate, 1982). Similarly, kindergartners in a computer group scored higher on numeral recognition tasks than those taught by a teacher (McCollister et al, 1986). There was some indication, however, that instruction by a teacher was more effective for children just beginning to recognize numerals, but the opposite was true for more able children. Children might best work with such programs once they have understood the concepts; then, practice may be of real benefit. In addition, students with learning difficulties might be distracted by drill in a game format, which impairs their learning (Christensen & Gerber, 1990).

Unique capabilities of computers for providing practice include: the combination of visual displays, animated graphics and speech; the ability to provide feedback and keep a variety of records; the opportunity to explore a situation; and individualization. However, exclusive use of such drill software would do little to achieve the vision of the National Council of Teachers of Mathematics (2000) that children should be mathematically literate in a world where mathematics is rapidly growing and is extensively being applied in diverse fields. What other approaches help achieve that vision?

Turtle Geometry

Directing the movement of Logo's 'turtle' can also provide challenging learning experiences. In Logo, children give commands to direct an on-screen turtle to move through 'roads' or mazes or to draw shapes. Primary-grade children have shown greater explicit awareness of the properties of shapes and the meaning of measurements after working with Logo (Clements & Nastasi, 1993). For example, while drawing a face in Turtle Math™ (Clements & Meredith, 1994), Nina decided to draw her 'mouth with a smile' with exactly 200 turtle steps (approximately millimeters). Off-computer she wrote a procedure where the sides of the rectangle were 40 and 20 and the sides of each equilateral triangle were 10. She realized that the total perimeter of these figures was 20 short of 200 and changed just one side of each triangle to 20. Running these procedures on the computer, she remarked that changing the length of one side 'messed up' an equilateral triangle and consequently her 'smile'. She had to decide whether to compromise on the geometric shape or the total perimeter. Her final 'mouth' was a rectangle of 200 steps and her 'smile' was an equilateral triangle of 60 steps.

Logo programming is also a rich environment that elicits reflection on mathematics and one's own problem-solving. Students use certain mathematical notions in Logo programming, such as notions of inverse operation. First grader Ryan wanted to turn the turtle to point into his rectangle. He asked the teacher, 'What's half of 90?' After she responded, he typed RT 45. 'Oh, I went the wrong way.' He said nothing, eyes on the screen. 'Try LEFT 90,' he said at last. This inverse operation produced exactly the desired effect.

Other children may need teacher assistance to link their knowledge of mathematics to their computer work as well as Nina did. Teachers can ask children to reflect on their work, especially 'surprises,' when the computer does something other than what they want it to do. Such reflection can promote greater self-monitoring and may encourage them to find computer 'bugs' themselves (Clements et al, 1993).

Logo sometimes can be difficult for young children to comprehend. However, when the environment is gradually and systematically introduced to the children and when the micro-worlds are age-appropriate, they do not show signs of any problems (Clements, 1983-84; Brinkley & Watson, 1987-88b; Cohen & Geva, 1989; Watson et al, 1992; Howard et al, 1993; Allen et al, 1993). Thus, there is substantial evidence that young children can learn Logo and can transfer their knowledge to other areas, such as map-reading tasks and interpreting right and left rotation of objects.

Why should Logo be especially helpful in developing spatial concepts? From a Piagetian perspective, students construct initial spatial notions not from passive viewing, but from actions, both perceptual [3] and imagined, and from reflections on these actions (Piaget & Inhelder, 1967). These are critical foundations; however, unless they are mathematized [4], they remain only intuitions. Many experiences can help children reflect on and represent these actions; research indicates that Logo's turtle geometry is one potent type of experience. Logo environments are in fact action based. These actions are both perceptual – watching the turtle's movements – and physical – interpreting the turtle's movement as physical motion that could be performed oneself. By first having children form paths and shapes by walking, then using Logo, children can learn to think of the turtle's actions as ones that they can perform; that is, the turtle's actions become 'body syntonic.'

But why not just draw it without a computer? There are at least two reasons. First, drawing a geometric figure on paper, for example, is for most people a highly proceduralized and compiled process. Such a procedure is always run in its entirety. This is especially true for young children, who have not re-represented the sequential instructions that they implicitly follow. Then, they cannot alter the drawing procedure in any substantive manner (Karmiloff-Smith, 1990), much less consciously reflect on it. In creating a Logo procedure to draw the figure, however, students must analyze the visual aspects of the figure and their movements in drawing it, thus requiring them to reflect on how the components are put together. Writing a sequence of Logo commands, or a procedure, to draw a figure 'allows, or obliges, the student to externalize intuitive expectations. When the intuition is translated into a program it becomes more obtrusive and more accessible to reflection' (Papert, 1980, p. 145). That is, students must analyze the spatial aspects of the shape and reflect on how they can build it from components.

And they do. Primary-grade children have shown greater explicit awareness of the properties of shapes and the meaning of measurements after working with the turtle (Clements & Nastasi, 1993). They learn about measurement of length (Sarama, 1995; Campbell, 1987; Clements et al, 1997) and angle (du Boulay, 1986; Kieran, 1986; Olive et al, 1986; Frazier, 1987; Clements & Battista, 1989; Kieran & Hillel, 1990; Browning, 1991). One microgenetic study confirmed that students transform physical and mental action into concepts of turn and angle in combined off- and on-computer experiences (Clements & Burns, 2000). Students synthesized and integrated two schemes, turn as body movement and turn as number, as originally found (Clements et al, 1996). They used a process of psychological curtailment in which students gradually replace full rotations of their bodies with smaller rotations of an arm, hand, or finger, and eventually internalized these actions as mental imagery.

These effects are not limited to small studies. A major evaluation of a Logo-based geometry curriculum included 1624 students and their teachers and a wide assortment of research techniques, pre- and post-paper-and-pencil testing, interviews, classroom observations, and case studies (Clements et al, 2001). Across grades K–6, Logo students scored significantly higher than control students on a general geometry achievement test, making about double the gains of the control groups. These are especially significant because the test was paper-and-pencil, not allowing access to the computer environments in which the experimental group had learned and because the curriculum is a relatively short intervention, lasting only six weeks. Other assessments confirmed these results, and indicated that Logo was a particularly felicitous environment for learning mathematics, reasoning, and problem-solving.

As an example, consider a class of first graders, constructing rectangles with blocks, string, pencils and papers, pegboards, sticks, and computers (Clements et al, 2001). 'I wonder if I can tilt one,' mused a boy working with Logo. He turned the turtle, drew the first side, then was unsure about how much to turn at this strange new heading. He finally figured that it must be the same turn command as before. He hesitated again. 'How far now? Oh, it must be the same as its partner!' He easily completed his rectangle. The instructions he should give the turtle at this new orientation were initially not obvious. He analyzed the situation and reflected on the properties of a rectangle. Perhaps most important, he posed the problem for himself.

Students in another class had explored the notion that a square was a rectangle – a special type of rectangle. They then had created a parallelogram with Logo. One of the students came up to his teacher the next day and said that he was thinking about parallelograms at home. 'Is a rectangle a special parallelogram?' he asked. 'Why do you say so?' 'Because it's just like the rectangle procedure if it had 90° turns.' This conversation shows that the student had used his Logo experiences to extend his thinking about relationships between polygons (Clements et al, 2001).

These studies indicate that Logo, used thoughtfully, can provide an additional evocative context for young children's explorations of mathematical ideas. Such 'thoughtful use' includes structuring and guiding Logo work to help children form strong, valid mathematical ideas. Children do not appreciate the mathematics in Logo work unless teachers help them see the work mathematically. These teachers raise questions about 'surprises' or conflicts between children's intuitions and computer feedback to promote reflection. They pose challenges and tasks designed to make the mathematical ideas explicit for children. They help children build bridges between the Logo experience and their regular mathematics work (Clements, 1987; Watson & Brinkley, 1990/91). These suggestions are valid for most types of open-ended software and will be discussed in a later section.

Further, recent versions of Logo, such as Turtle Math, have built-in features that were designed based on research. As a small example, 'turn rays' (see Figure 1) help students differentiate between the 'turn angle' (exterior angle) and interior angle and help them conceptualize the measure of turns.

Research indicates that such features facilitate mathematical learning (see Clements et al, 2001, chapter 5).

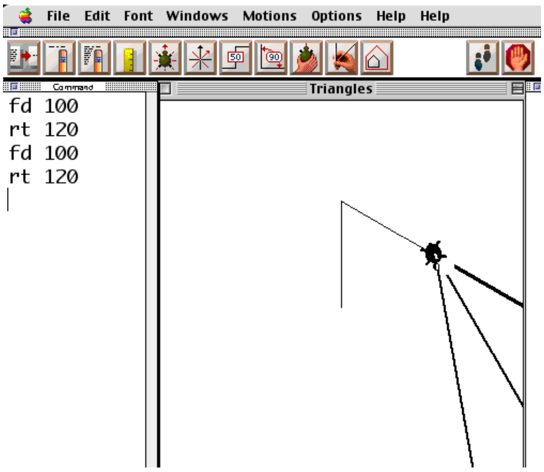

Figure 1. 'Turn Rays'.

In summary, Logo has some unique advantages (Clements & Battista, 1989, 1992) in that it links children's intuitive knowledge about moving and drawing to more explicit mathematical ideas, encourages the manipulation of specific shapes in ways that help students in viewing them as mathematical representatives of a class of shapes, facilitates students' development of autonomy in learning (rather than seeking authority) and positive beliefs about the creation of mathematical ideas, encourages wondering about and posing problems by providing an environment in which to test ideas and receive feedback about these ideas, helps connect visual shapes with abstract numbers, and fosters mathematical thinking (Clements, 1994).

Higher-order Thinking Skills

Computers can also help develop other higher-order thinking skills. Preschoolers who used computers scored higher on measures of metacognition (Fletcher-Flinn & Suddendorf, 1996). They were more able to keep in mind a number of different mental states simultaneously and had more sophisticated theories of mind than those who did not use computers. Several studies have reported that Logo experience significantly increases in both preschool and primary grade children's ability to monitor their comprehension and problem-solving processes; that is, to 'realize when you don't understand' (Clements & Gullo, 1984; Clements, 1986, 1990; Lehrer & Randle, 1986; Miller & Emihovich, 1986). This may reflect the prevalence of 'debugging' in Logo programming. Other abilities that may be positively affected include understanding the nature of a problem, representing that problem, and even 'learning to learn' (Lehrer & Randle, 1986; Clements, 1990). Along with the increase in metacognitive talk in writing and mathematics activities, there is a substantial argument that computers can foster young children's metacognition.

Problem-solving computer activities motivate children as young as kindergartners to make choices and decisions, alter their strategies, persist, and score higher on tests of critical thinking (Gélinas, 1986; Riding & Powell, 1987). Specially designed computer programs can improve analogical thinking of kindergartners (Klein & Gal, 1992); a variety of problem-solving CAI programs significantly increased first and second graders' ability to generalize and solve mathematics problems (Orabuchi, 1993). Several studies reveal that Logo is a particularly engaging activity to young children, fostering higher-order thinking in children from preschool through the primary grades, including special needs students (Degelman et al, 1986; Lehrer et al, 1986; Clements & Nastasi, 1988; Nastasi et al, 1990). Preschool and primary grade children develop the ability to understand the nature of problems and use representations such as drawings to solve them. When given opportunities to debug, or find and fix errors in Logo programs (Poulin-Dubois et al, 1989), they also increase their ability to monitor their thinking; that is, to realize when they are confused or need to change directions in solving a problem (Clements & Nastasi, 1992).

Unique advantages of computers for fostering higher-order thinking include: allowing children to create, change, save, and retrieve ideas, promoting reflection and engagement; connecting ideas from different areas, such as the mathematical and the artistic; providing situations with clear-cut variable means–end structure, some constraints, and feedback that students can interpret on their own; and so allowing children to interact, think, and play with ideas in significant ways, in some cases even with limited adult supervision (Clements, 1994).

Computer Manipulatives

In one approach, children explore shapes using general-purpose graphics programs or 'computer manipulatives.' Researchers observing such use observe that children learn to understand and apply concepts such as symmetry, patterns and spatial order. For example, Tammy overlaid two overlapping triangles on one square and colored select parts of this figure to create a third triangle which was not provided by the program. Not only did Tammy exhibit an awareness of how she had made this, but she also showed a higher-order awareness of the challenge it would be to others (Wright, 1994). As another example, young children used a graphics program to combine the three primary colors to create three secondary colors (Wright, 1994). Such complex combinatorial abilities are often thought out of reach of young children. In both these examples, the computer experience led the children to explorations that increased the boundaries of what they could do.

Computer manipulative programs extend general purpose graphics programs in allowing children to perform specific mathematical transformations on objects on the screen. For example, whereas physical base-ten blocks must be 'traded' (e.g. in subtracting, students may need to trade 1 ten for 10 ones), students can break a computer base-ten block into 10 ones. Such actions are more in line with the *mental actions* that we want students to learn. The computer also links the blocks to the symbols. For example, the number represented by the base-ten blocks is dynamically linked to the students' actions on the blocks, so that when the student changes the blocks the number displayed is automatically changed as well. This can help students make sense of their activity and the numbers.

Thus, computer manipulatives can provide unique advantages (Sarama et al, 1996; Clements & Sarama, 1998), including: saving and retrieving work, so children can work on projects over a long period (Ishigaki et al, 1996); offering a flexible and manageable manipulative, one that, for example, might 'snap' into position; providing an extensible manipulative, which you can resize or cut; linking the concrete and the symbolic with feedback, such as showing base-ten blocks dynamically linked to numerals; recording and replaying students' actions; and bringing mathematics to explicit awareness, for example, by asking children to consciously choose what mathematical operations (turn, flip, scale) to apply to them.

Integrated Approach

Of course, several approaches may be combined in one program. Julie Sarama and I designed our *Building Blocks* software to enable all young children to build solid content knowledge and develop higher-order, or critical, thinking. To achieve this, we needed to consider the audience, determine the basic approach to learning and teaching, and draw from theory and research in each phase of the design and development process. Based on theory and research on early childhood learning and teaching (Bowman et al, 2001; Clements, 2001), we determined that the basic approach of *Building Blocks* would be *finding the mathematics in, and developing mathematics from, children's activity*. The materials are designed to help children extend and mathematize their everyday activities, from building blocks to art to songs and stories to puzzles. Activities are designed based on children's experiences and interests, with an emphasis on supporting the development of *mathematical* activity. So, the materials do not rely on technology alone, but integrate three types of media: computers, manipulatives (and everyday objects), and print. Here I will briefly describe some of the computer activities.

Each activity has several levels, often containing quite different tasks. For example, in Double Trouble (see Figure 2), children decorate cookies, counting and adding to produce a given number of chips on each. At Level 1, children choose a twin cookie with the same number of chips as a given cookie. At Level 2, children make a twin cookie with the same number of chips as a given cookie. That is, they have to make a twin cookie with the same number of chips as a cookie Mrs Double shows them. At Level 3, children decorate cookies with a number given by Mrs Double, so they have to 'count out' the correct number twice, with no 'model.' At Level 4, children play a game in which they tell how many chips have been hidden under a napkin, when, for example, first two and then one more are shown to be placed under the napkin. At Level 5, children count on to 'fix' a cookie that has too few chips; for example, making a '4 cookie' into a '7 cookie.' In Free Explore, children make cookie problems for one another.

Figure 2. 'Double Trouble'.

As another example, in Party Time, children use one-to-one correspondence and counting to help set a table for a party. At Level 1, children get ready for a party by putting one of each item (such as plates, spoons, etc.) on each place mat. At Level 2, the character gets the items out, but asks the child to tell how many are needed. The child must count the place settings at the table. At Level 3, the character switches roles, telling the child how many place settings there are, and asking the child to get out that number of each item needed. In Free Explore, children create their own parties (see Figure 3).

Notice that free explore tasks – basically manipulatives in context – are critical to our design. As a final example, Shape Puzzles invites children to solve outline puzzles by putting together shapes. They use the tools to move the shapes into place. They move through research-based levels, from puzzles that are simple and 'obvious' (see Figure 4) to puzzles that are 'open' and challenging and require that children mentally combine shapes (Figure 5).

Again, there is a Free Explore activity here; children make their own shape puzzles for other children.

Figure 3. 'Free Explore'. Create your own parties.

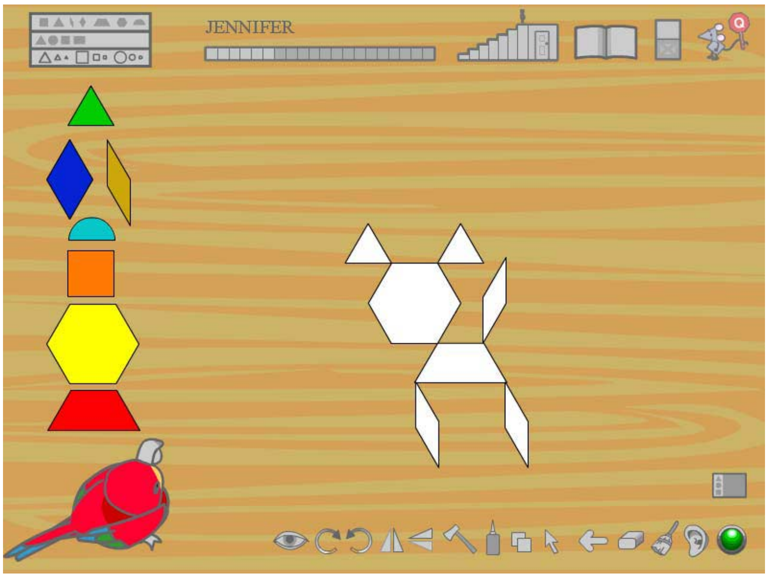

Figure 4. Shape puzzles – simple.

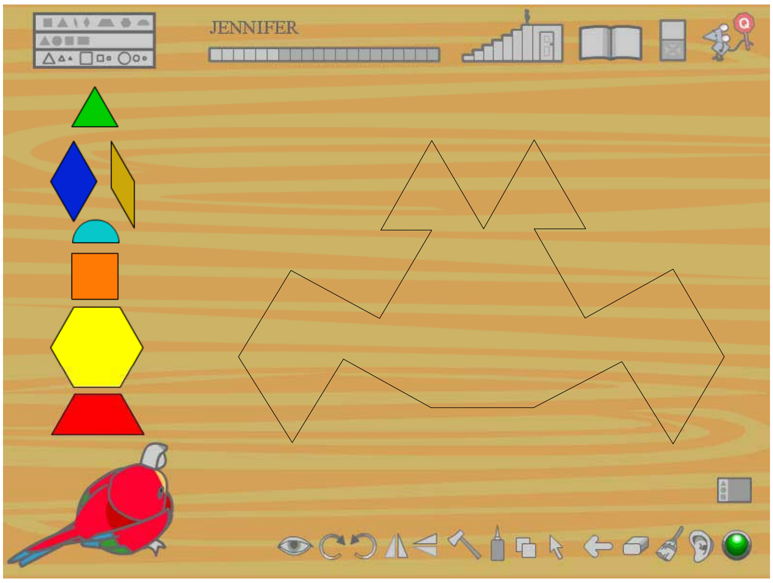

Figure 5. Shape puzzles – challenging.

Thus, children receive numerous opportunities for practice, within meaningful problem-solving contexts that require use of higher-order thinking skills. They also use higher-order and creative thinking in the free explore activities. Initial field testing of our first Building Blocks software product that embodies this integrated approach has been positive. In one study, preschoolers made substantial gains in both the areas of number and geometry (Sarama, in press).

The successful teachers in most of the studies in this entire section were consistently mediating children's interaction with the computer (Samaras, 1991). The importance of the teacher's role is the subject to which we now turn.

Teaching with Computers

Even in preschool, children can work cooperatively, with minimal instruction and supervision, if they have adult support initially (Rosengren et al, 1985; Shade et al, 1986). However, adults play a significant role in successful computer use. Children are more attentive, more interested, and less frustrated when an adult is present (Binder & Ledger, 1985). Thus, teachers may wish to make the computer one of many choices, placed where they can supervise and assist children.

Effective Strategies

Across the educational goals, we find that teachers whose children benefit significantly from using computers are always active. They closely guide children's learning of basic tasks, then encourage experimentation with open-ended problems. They are constantly encouraging, questioning, prompting, and demonstrating. Such scaffolding leads children to reflect on their own thinking behaviors and brings higher-order thinking processes to the fore. Such metacognitively oriented instruction includes strategies of identifying goals, active monitoring, modeling, questioning, reflecting, peer tutoring, discussion, and reasoning (Elliott & Hall, 1997). Whole group discussions that help children communicate about their solution strategies and reflect on what they have learned are also essential components of good teaching with computers (Galen & Buter, 2000).

Two studies show clearly that such scaffolding is critical. In the first (Yelland, 1994), children were only given instructions for specific tasks and then mostly left alone. These children rarely planned, were often off task, rarely cooperated, and displayed frustration and lack of confidence, and did not finish tasks. In the second study, using similar software and tasks (Yelland, 1998), the teacher scaffolded instruction by providing open-ended but structured tasks, holding group brainstorming sessions about problem-solving strategies, encouraging children to work collaboratively, asking them to think and discuss their plans before working at the computer, questioning them about their plans and strategies, and providing models of strategies as necessary. These children planned, worked on task collaboratively, were able to explain their strategies, were rarely frustrated, and completed tasks efficiently. They showed a high level of mathematical reasoning about geometric figures and motions, as well as number and measurement.

Such teaching is difficult. A balance of teacher guidance and children's self-directed exploration is necessary for children to learn to appropriate this new technology (Escobedo & Bhargava, 1991). In designing curriculum around open-ended software, research has shown that children work best when designated open-ended projects rather than asked merely to 'free explore' (Lemerise, 1993). They spend more time and actively search for diverse ways to solve the task. The group allowed to free explore grew disinterested quite soon. Models and sharing projects may also be helpful (Hall & Hooper, 1993). Effective teachers also integrate computers into the ongoing program. They balance and combine on-computer and off-computer activities and discuss computer activities in group sessions.

Although the more structured nature of typical CAI tasks allows research and recommendations regarding the time per day students should use the computer, teachers will have to use their knowledge of their classes and individual students in gauging time on more open-ended activities. Often, 10-20 minutes are not adequate; however, overuse should be monitored. There is some evidence that strict time limits for such activities can generate hostility and isolation instead of the usual positive effects of the computers on social communication (Hutinger et al, 1998). Care must be taken with very young children (i.e. less than four years of age) that they do not read a monitor for extended periods of time.

Arranging the Classroom Setting

The physical arrangement of the computers in the classroom can enhance their social use (Haugland & Shade, 1994; Shade, 1994), which also has positive effects on achievement (Clements & Nastasi, 1992). Computers in the classroom, rather than a laboratory, are more likely to facilitate positive social interactions and curriculum integration. Placing two seats in front of the computer and one at the side for the teacher can encourage positive social interaction. Placing computers close to each other can facilitate the sharing of ideas among children. Computers that are centrally located in the classroom invite other children to pause and participate in the computer activity. Such an arrangement also helps keep teacher participation at an optimum level. They are nearby to provide supervision and assistance as needed (Clements, 1991). Other factors, such as the ratio of computers to children, may also influence social behaviors. Less than a 10:1 ratio of children to computers might ideally encourage computer use, cooperation, and equal access to girls and boys (Lipinski et al, 1986; Yost, 1998). Cooperative use of computers raises achievement (Xin, 1999); a mixture of use in pairs and individual work may be ideal (Shade, 1994). It is critical to make sure special education children are accepted and supported. Only in these situations did they like to be included in regular classroom computer work (Xin, 1999).

In summary, we see that children can create complex simulations in second grade (Howland et al, 1997), direct the Logo turtle in preschool, and program in the primary grades, and create pictures and text at all age levels. Will teachers take the time to learn to support such challenging experiences?

Professional Development

If teachers are to take up that challenge, they need substantial professional development. Research has established that less than 10 hours of training can have a negative impact (Ryan, 1993). Further, only 15% reported receiving at least nine hours of training (Coley et al, 1997). Others have emphasized the importance of hands-on experience and warned against brief exposure to a variety of programs, rather than an in-depth knowledge of one (Wright, 1994).

Student teaching may have an adverse effect. Some pre-service teachers' cooperating teachers do not use technology and may actively impede the pre-service teachers' attempts at using technology in the practice of teaching (Bosch, 1993). Teachers at all levels need to be assisted in learning how to integrate computers into instruction (Coley et al, 1997), using models that have proven effective (Ainsa, 1992).

Final Words

The computer can offer unique opportunities for learning through exploration, creative problem-solving, and self-guided instruction. Realizing this potential demands a simultaneous focus on curriculum and technology innovations (Hohmann, 1994). Effectively integrating technology into the curriculum demands effort, time, commitment and sometimes even a change in one's beliefs. One teacher reflected, 'As you work into using the computer in the classroom, you start questioning everything you have done in the past and wonder how you can adapt it to the computer. Then, you start questioning the whole concept of what you originally did' (Dwyer et al, 1991).

Some criticize computer use, arguing that computers, by their nature, are mechanistic and algorithmic and support only uncreative thinking and production. However, adults increasingly view computers as valuable tools of creative production. Educational research indicates that there is no single 'effect' of the computer on mathematics achievement, higher-order thinking and creativity. Technology can support either drill or the highest-order thinking. Research also provides strong evidence that certain computer environments, such as word processing, art and design tools, computer manipulatives, and turtle graphics hold the potential for the computer's facilitation of these educational goals. There is equally strong evidence that the curriculum in which computer programs are embedded, and the teacher who chooses, uses, and infuses these programs, are essential elements in realizing the full potential of technology.

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Notes

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[2] There are other types of software, and many software titles – some intriguing – that are not discussed here. We report on those for which empirical evidence has been collected. This should not, of course, be taken as a commendation of

software discussed, especially compared to software which has not been studied. The length of our discussion of various types also reflects the size of the research corpus for each.

[3] Perceptual is used here, consistent with Piaget's original formulation, as meaning phenomena or experiences that depend on sensory input, in contrast to those that are represented mentally (and thus can be 're-presented' imagistically without sensory support). Thus, perceptual should not be confused with the notion that we, with Piaget, reject – that of 'immaculate perception' in which perceived objects are immediately registered in the brain.

[4] Mathematization emphasizes representing and elaborating mathematically – creating models of an everyday activity with mathematical objects, such as numbers and shapes; mathematical actions, such as counting or transforming shapes; and their structural relationships. Mathematizing involves reinventing, redescribing, reorganizing, quantifying, structuring, abstracting, and generalizing that which is first understood on an intuitive and informal level in the context of everyday activity.

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Cooperative study teams in mathematics classrooms

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This article describes a general instructional strategy designed to help students in the learning process from textbooks and to furnish opportunities for practice in critical reading. Students participate in cooperative learning by breaking the class up into small groups—the Study Teams—and providing them with worksheets and reading organizers, which organize the material into small items that reflect the major concepts in the reading material on which the study is focused. Some of the benefits that this type of instruction with Study Teams can produce are described.

1. Introduction

As mathematics teachers, we often voice complaints about our students' reluctance to read their mathematics textbooks, or their difficulties solving word problems due to poor reading skills. The typical mathematics student expects a teacher to explain what the book says. He may read a chapter in a history text and answer questions in the book without asking for any explanation; but in mathematics, after the student reads the material the typical response is: 'Fine, I read it. But what does it mean?'

Mathematics is a language that can neither be read nor understood without initiation. Students need to learn how to read mathematics, in the same way they learn how to read a novel or a poem, listen to music, or view a painting. Both a mathematics article and a novel are telling a story and developing complex ideas. The greatest difference is that a mathematical article does the job with a tiny fraction of the words and symbols used in a novel. In reading mathematics each word or symbol is important because there are many thoughts condensed into a few statements.

Mathematics is one area of the curriculum where traditionally, little reading occurs. For a variety of reasons, students do not know how to study mathematics [1]. Most of the time spent deliberately helping students learn to read focuses on literary and historical texts. Mathematical reading (and for that matter, mathematical writing) is rarely expected, much less considered to be an important skill, or one which can be increased by practice and training.

However, in a period of rapid technological change this situation is distressing and dangerous as well. The author believes that the school of the future will have to concentrate more on the relationship between people and knowledge than on knowledge itself. Increased use of information technology has led to an increase in the incidence with which quantitative information is presented in the printed

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media. There is a perceived need for citizens to be able to interpret such information. The pressure for this has been felt more and more in mathematics and science [2, 3].

In a time where there is a need for frequent adaptation of skills, it is generally agreed that teachers must do more than just teach students a certain body of facts. They are responsible for teaching students processes of thinking and learning so that they have the ability to become increasingly self-directed and not depend on the teacher [4]. In spite of advances in the electronic media the book still remains the best place to find and study a very large body of information. Current software can advance the printed media, but not replace it.

Therefore, the ability for critical reading may well be more valuable to students in the future than the actual topics that they study in the course. NCTM's Curriculum and Evaluation Standards for School Mathematics [5], emphasizes the importance of reading skills for all students and points out the fact that they will need specific instruction on how to read mathematical textbooks with understanding. 'Assignments that require students to read mathematics and respond both orally and in writing to questions based on their reading should be an integral part of the 9–12 mathematics program' (p. 142).

Cooperative learning has also been a subject of interest to researchers for the past three decades, and many research findings indicate that cooperative learning is an effective tool for improving academic achievement [6–9]. At the K-12 school levels, instruction using cooperative or collaborative learning techniques has gained in popularity, and there is a substantial body of literature supporting the idea that students can attain higher achievement, especially in mathematics, through working together in groups [10–14].

Until now, however, there has been no specific effort to focus on the importance of cooperative techniques designed to help students in reading mathematical textbooks with understanding. A goal that should be a conscious part of our teaching efforts is to teach our students to read and understand mathematics. This paper advocates this point of view. It presents a general instructional strategy designed to help students in the learning process from textbooks and to furnish opportunities for practice in critical reading. The author has used this technique during regular classroom instruction with first- and second-year algebra students, and all experiences described occurred in these teaching sections.

2. Instruction with cooperative Study Teams

The main idea of this strategy is that by participating in cooperative learning, students are able to learn concepts, processes and techniques presented in the textbook while working under the supervision of the classroom teacher. The following instructional sequence can be implemented:

1. Teacher introduces the unit to the class.

The unit may be a chapter section in an elementary mathematics textbook that can be taught during one class period: for instance, the quadratic equation and the nature of its roots. The teacher offers a unit overview in the whole class, which may include a general discussion about the unit objectives, a film or other media to generate excitement about the topic.

2. The whole class is divided into Study Teams.

The class is broken into groups of four or five students who constitute the Study Teams. The members of each group work together at one time following written instructions and are responsible for cooperating on learning tasks using the mathematics textbook.

Each Study Team must be a heterogeneous or mixed ability group. With a little planning, it is possible to have high, average and low-achieving students in each Study Team. In this way it is possible to pair a marginal student with an average or above average student for peer instruction and support, which will assist the marginal learners in improving their learning and self-confidence. Each student can be given a colour team badge to enhance the collaboration spirit of each team. Collaborative learning in the mathematics classrooms is a teaching strategy that holds increased promise for improving the mathematical skills and attitudes of students [15]. 'Small groups provide a forum in which students ask questions, discuss ideas, make mistakes, learn to listen to others' ideas, offer constructive criticism, and summarize their discoveries in writing' [5, p. 79]. Each team also selects a recorder-reporter who is responsible for reports to the whole class without mentioning any individual names in the report.

3. Teacher distributes Study Team worksheets.

Each worksheet contains a list of reading organizers that tells the learners what is important in each paragraph of the unit. The items on each sheet are questions based on the main ideas, concepts, processes and generalizations found in the unit. These items reflect the major concepts in the reading material on which the study must be focused. There is approximately one reading organizer for each paragraph. Appendix A presents a usable Study Team worksheet developed from an algebra textbook chapter on quadratic equations. If a chapter divides into five units, for instance, the teacher must prepare five worksheets and there will be five Study Team sessions.

4. Students learn concepts identified on the worksheet cooperatively.

In this session, students use the reading organizers to read paragraphs, point out portions, discuss concepts, keep notes on difficult areas, and agree on responses. Students read the textbook reflectively using pencil and paper. Rather than just keep on reading and waiting to see what the author's explanation is, they try to transform the content in such a way as to produce understanding. The advantage is that group therapy and peer learning is achieved. Students feel secure in the Study Team. Once the group has started its work then each member knows that the group will assist if he or she gets stuck or has been absent and needs information or did not understand something. Students can relate to one another better than to an instructor, at least when none of them understands a concept. It is fascinating to see the effects of this approach on students. Once when the author stopped to ask a member of a team if he needed help, he said: 'Maria is helping me!' giving me a funny look as if to say 'Go away'.

During the Study Team session the teacher moves among the groups organizing the class work, answering students' questions, explaining what the symbols mean, drawing out their ideas, giving suitable prompts and encouragement and facilitating their work. The instructor feels he is working 'with' students instead of 'on' them. The responsibility for working and learning becomes the students', not the teacher's. The students learn that a teacher helps to guide, but cannot learn for them.

5. Class discussion.

After the collaborative reading and learning guided by the worksheet the teacher allows time for the whole class to discuss those areas that need clarification involving students in a whole class discussion. The recorder from each team selects the difficult topics that the members have pointed out in their notebooks and asks the teacher to clarify them. The discussion session helps the class to elucidate key ideas and explain difficult points eliminating the possibility that an incorrect response has been given in the Study Teams. At this time, the teacher can correct errors, expand concepts and give further explanatory information or short demonstrations. At this session the teacher can also use supplementary materials that may include diagrams, graphs, or restudy on the unit topic (figure 1).

After this discussion each member of the Study Team completes its worksheet. A grade can be given to each student for completing the worksheet.

6. Teacher distributes summary cards.

If students have really understood something they are expected to explain it, and furthermore, requiring them to explain it helps them to understand the information. Verbalizing and writing also improves the ability to recall and organize information and serves as a powerful aid to learning [16].

Each member of the Study Team is given a 3×5 card and the students are asked to write on it, in their own words, a brief summary of the main points of the lesson. The students are encouraged to look at their textbooks and answer the open-ended request: 'What did I learn today?' (figure 2). This technique of writing on mathematics-related content helps students clarify and develop their understanding of a specific topic, as when putting down ideas on paper, they are faced with them and obliged to recognize their shortcomings. A grade can be given to each student for completing the summary card.

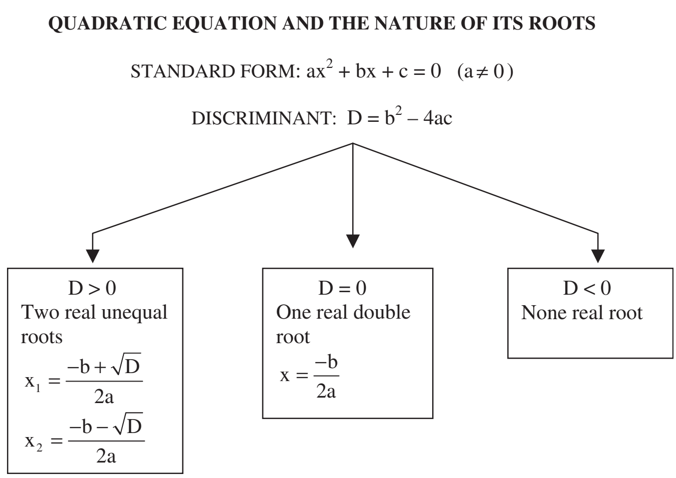

Figure 1. Conceptual diagram of the study unit.

Today I learn that:

The standard form of a quadratic equation is:
$ax^2 + bx + c = 0 \quad (a \neq 0)$.
The discriminant of a quadratic equation is: $D = b^2 - 4ac$.
The quadratic formula is $x = \dfrac{-b \pm \sqrt{D}}{2a}$.
The discrimination sign determines the nature of roots.

Figure 2. A student's summary card.

7. Students take individual tests.

Up to this point, the Study Team members have received identical grades for the group work on worksheets or the summary cards. The individual unit test is used to assess student mastery of the concepts. The test items should be based on the reading organizer worksheets and include all facts, concepts, and skills that are based on the unit as taught. This alignment between the unit as taught and the evaluation of students' outcome is important for slow learners and low ability students. Appendix B presents a list of focused study items and the corresponding evaluation items. Finally, each student receives a unit score based on a weighed combination of the collaborative group work and the individual test score.

3. Methodology

This article is a report on the experimental use of cooperative Study Teams in four secondary mathematics classrooms. The experiment has been running for three years (1999–2002), is integrated in an existing mathematics curriculum, and has included approximately 100 students in grade levels 8–10.

The experiment took place at three urban secondary schools in Patras, Greece. The student population was drawn from diverse socio-economic backgrounds and approximately half of each class were girls and half were boys. Cooperative learning skills such as positive interdependence, roles in a group, rules and procedures were all reviewed before the lessons began. The study was conducted two times a week during a 4-week period each time and covered a specific mathematics unit.

Finally, the participants were asked to reflect in writing, on the totality of their experiences at the conclusions of the teaching experiment. Students were asked to reflect on statements about mathematics and they were challenged to articulate their own perceptions of mathematics and their experiences with cooperative Study Teams.

These reflections, and also their reactions to the seven instruction sessions, were recorded in a class journal which circulated among the participants at the end of all study items and the corresponding evaluation items. The journal was returned to the author after two weeks. Since journal writing was new to the students, it was discussed with them, focusing on the purpose of keeping a journal and guiding them in the use of this activity [17].

In order to help them to express their feelings, it was proposed that they reflect on five items:

(1) How has cooperative Study Teams affected your learning of mathematics?
(2) How do you feel about instruction with cooperative Study Teams in math classes?
(3) What are the benefits of cooperative Study Teams in math classes?
(4) How could this instruction strategy be changed to be more effective?
(5) Would you like to continue this activity in mathematics?

Overall, the comments were very favourable and encouraging. Representative student comments included:

> 'It was great to get into mathematics a little. It broadened my horizons'.
>
> 'I learned to value cooperative work. It was fun and made very interesting by the instructor'.
>
> 'Working in a group can sometimes be a challenge because it helps in putting different ideas into perspective and allows verbalization of results, which is very helpful'.
>
> 'This has been one of the most beneficial courses I have taken'.
>
> 'I learned to use the text as the source of facts and not to depend on the teacher'.
>
> 'The more work that we do with quadratics and factorization, the more interesting it becomes. I never looked at math in such a way as we do now'.
>
> 'I liked today's class. I can't believe we came up with a theorem all on our own'.
>
> '... It's too bad the experiment period is almost over'.
>
> 'I felt... some sense of satisfactory and mastery. With help from another classmate I solved the problem at the end of the session'.
>
> 'You are broadened by new ideas, somebody else's viewpoint. You're able to see more'.
>
> 'For a moment we tried to solve the problem but we didn't and then we read. But our mistake was that we didn't take a pencil and a piece of paper and draw something'.

In addition, some weak students were interviewed before and after the sessions to determine their background and past experience with mathematics, and to allow them to express their feelings about mathematics.

4. Concluding remarks

The conclusions and comments expressed in the following come from reflecting on the author's teaching experience and in an analysis of the comments found in the journal writing of the many students who form the subject of the study. This evidence shows the possibility of engaging students in small-group mathematical activities with the purpose of helping them in the learning process from textbooks on mathematics-related content.

Certainly, we have not done a comparative analysis of these classes to see if this teaching strategy makes a difference in how much students learn, but we are convinced it helps. Without having provided conclusive evidence, we would like at least to suggest the hypothesis that involving students in different study experiences could contribute in a complementary way to their learning of mathematics.

Using this general instructional sequence, a single chapter can be taught in five to eight instructional periods. However, many extensive activities can be inserted between the several steps or some shortened or omitted. This depends on the ability level of the class and the background of the members in each group. We have found that the Study Team technique for learning from textbooks is easily adaptable to all secondary mathematics classes whatever the grade level but works best in mixed ability classes with a great number of low achievers and anxious students.

The use of this type of instruction with Study Teams has several advantages.

(1) It can help motivate students because everyone is involved in discussing and learning the material.
(2) Students are motivated to participate in meaningful reading and language experiences, to identify important concepts and to think about the meaning of these concepts.
(3) Study Team membership and peer tutoring give help to slow or disinterested students with poor mathematical backgrounds, bad experiences in mathematics, and bad attitudes who seldom receive classroom recognition. Students who know that they can depend on other group members to help and support do not feel the anxiety often experienced by those who do not understand the work. The classroom environment is less threatening, and anxiety is less likely to interfere with learning. Moreover, the student who helps others experiences gratification in giving.
(4) Students through Study Teams form new friendships and learn to appreciate differences in ability, differences in personal characteristics and differences in opinion. The cooperative-learning attitude offers a secure environment for everyone to make a contribution. Each student feels responsible for his/her own team performance and is rewarded for his/her contribution.
(5) Students are taught to read mathematics textbooks critically with systematic note-taking, outlining the most important areas, making connections between pictures, examples and diagrams, using pencil and paper and trying to interpret what the author is describing with symbols and words. This reading activity draws learners into the texts and encourages them to raise questions, make connections and, in general, actively work out meanings with the support of peers.
(6) Students also learn to use the text or their own summary as the source of facts and not to depend on the teacher for this information. They are expected to develop good study habits at their own level and prepared for future self-study. The mere fact that class time is devoted to reading

the textbook demonstrates to the students that the teacher values reading as a learning activity.

(7) Their teacher is no longer seen as the authority that dispenses knowledge to students, who merely absorb information. Instead, students become important resources for one another in the learning process.

During the last three years since the author has used this technique in the traditional classroom setting, most of the students have shown enjoyment in the explanatory nature of the learning process and appreciated the opportunity to work together rather than in silence on their own as in the traditional courses. Students' attitude, attendance, completion of assignments and willingness to participate in class has improved. In particular, the so-called 'poor' and unmotivated students have shown extraordinary creativity.

However, in the author's opinion, it is inappropriate to discuss or try to assess and compare different teaching methods or styles. Instead it should be more fruitful to ask whether the method chosen works or not with that particular instructor and that particular class and whether the method led to the goal set by the particular teaching sequence. With regard to our case, it seems that the Study Teams activities work well, that students learn their mathematics as well as developing good habits for future action, and that they like the method. Most of them are enthusiastic about trying something new and make an honest attempt to carry out the assignments.

Other colleagues who may want to use similar techniques based on the principles discussed in this article must have in mind that in order to operate in this way many students have to 'unlearn' habits built up over many years at school. Students may come from schools or classes with preconceptions about how they should behave in a mathematics class. The spirit of cooperation may be alien to students who have been schooled in an environment where the teacher carries the entire burden of the instruction and the students compete with one another to rise to the top. Therefore, it is recommended that these ideas should be continued for at least five courses before the students begin to accept them. This approach seems to have great promise and, with minor revisions each time, may challenge the near monopoly of the lecture as a principal means of mathematics teaching.

Appendix A Sample Study Team worksheet

READING GUIDE

DIRECTIONS: Meet with your team to read from your textbook about quadratic equations and the nature of their roots.
Use pencil and paper for note taking. Then discuss and write down the responses to the following items.

QUADRATIC EQUATIONS AND THE NATURE OF THEIR ROOTS
(Pages_______)

1. Is there a standard form for a quadratic equation? ____________________
   If so, what is it? ____

2. Describe the process of completing the square.

   ____

   What is its purpose? ______________________________

   ____

3. In terms of a, b and c, what is the discriminant, D?

   ____

   ____

   What is the significance of the discriminant? __________________

   ____

   ____

4. Write the quadratic formula: $x =$
   What is its use? ________________________________

5. Do all quadratic equations have two roots? __________________
   Explain why or why not. _____________________________

   ____

6. When are a quadratic equation's roots rational and when are they irrational?

   ____

   ____

7. Upon what does the nature of the roots depend? ________________

   ____

8. What is implied when $D > 0$? _________________________
   When $D = 0$? _________________________________
   When $D < 0$? _________________________________

Appendix B Focused study items and corresponding test items.

QUADRATIC EQUATIONS AND THE NATURE OF THEIR ROOTS

FOCUSED STUDY ITEMS	TEST ITEMS
The standard form of a quadratic equation is (page . . . .)	What are the a, b,c for the following quadratics: i) $3x^2 - x + 2$ ii) $x^2 - 7$ iii) $x^2 - 3x$ . . . .
The discriminant of a quadratic is (page . . .)	Compute the discriminant of the above quadratics . . . .

The quadratic formula
is (page . . . .)
. . . .

a) Find the roots of the equations

i) $2x^2 - x - 6 = 0$
. . . .
. . . .
ii) $t(3t - 10) = 25$
. . . .
. . . .

b) The formula $K = n(n - 3)/2$ yields the number of diagonals, K, in a polygon of n sides. Find the number of sides of a polygon having 54 diagonals.
. . . .
. . . .

The nature of roots
depends on (page . . .)
. . . .
. . . .

a) Find the value of the discriminant for each of the following quadratic equations and use it to tell how many real roots the equation has.

	D	N° of real roots
$x^2 - 3x + 2 = 0$	. . . .	. . . .
$x^2 - 5x + 4 = 0$	. . . .	. . . .
$x^2 - 7x + 6 = 0$	. . . .	. . . .
$x^2 - 9x + 8 = 0$	. . . .	. . . .

Can you find the pattern? . . . .
. . . .
Explain, by the means of D, why this form of quadratic equations have always real roots
. . . .
. . . .
Try to find another general form, which would give you quadratic equations which have always real roots . . . .
. . . .

b) Repeat the same work for the following equations:

	D	N° of real roots
$2x^2 - 2x + 1 = 0$	. . . .	. . . .
$5x^2 - 4x + 1 = 0$	. . . .	. . . .
$10x^2 - 6x + 1 = 0$	. . . .	. . . .
$17x^2 - 8x + 1 = 0$	. . . .	. . . .

Can you find the pattern? . . . .
. . . .

Explain, by the means of D, why this form of quadratic equations have always no real roots.
. . . .
. . . .

Try to find another general form, which would give you quadratic equations which have always no real roots . . . .
. . . .
. . . .

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Science and Language for English Language Learners in Relation to Next Generation Science Standards and with Implications for Common Core State Standards for English Language Arts and Mathematics

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Science and Language for English Language Learners in Relation to Next Generation Science Standards and with Implications for Common Core State Standards for English Language Arts and Mathematics

Okhee Lee[1], Helen Quinn[2], and Guadalupe Valdés[3]

The National Research Council (2011) released "A Framework for K–12 Science Education" that is guiding the development of the Next Generation Science Standards, which are expected to be finalized in early 2013. This article addresses language demands and opportunities that are embedded in the science and engineering practices delineated in the Framework. By examining intersections between learning of science and learning of language, the article identifies key features of the *language of the science classroom* as students engage in these language-intensive science and engineering practices. We propose that when students, especially English language learners, are adequately supported to "do" specific things with language, both science learning and language learning are promoted. We highlight implications for Common Core State Standards for English language arts and mathematics.

Keywords: bilingual/bicultural; language comprehension/development; policy analysis; science education

Common Core State Standards (CCSS) for English language arts and literacy and for mathematics (Common Core State Standards Initiative, 2010a, 2010b) have been adopted by most states and will affect instructional practice, curriculum, and assessment across the nation. The National Research Council (NRC, 2011) document "A Framework for K–12 Science Education: Practices, Crosscutting Concepts, and Core Ideas" (hereafter referred to as "the Framework") is the product of a committee of experts charged with developing a consensus view of what is important in K–12 science education grounded in an extensive review of the literature on science learning. Furthermore, this document is designed to guide the work of 26 lead states in developing Next Generation Science Standards (NGSS), a project coordinated by Achieve, Inc. The first draft of NGSS was released for public input in late spring 2012, the second draft close to the final form was released in January 2013, and NGSS are expected to be finalized in early 2013. We use the Framework as the base of our discussion in this article as NGSS faithfully follow the Framework as the foundation. Our purpose is not to reexamine the decisions made for the Framework or NGSS, but rather to explore and highlight their implications for English language learners (ELLs) in the science classroom.

This article discusses language learning challenges and opportunities that will emerge as ELLs engage with NGSS.[1] First, the article provides a brief overview of the Framework with a focus on language-intensive science and engineering practices (NRC, 2011). Second, it describes the literature on language in science learning and teaching. Third, it provides a perspective on second language acquisition and pedagogy. Fourth, by examining intersections between learning of science and learning of language, it identifies key features of the *language of the science classroom* as students engage in language-intensive science and engineering practices. Finally, it offers implications for research and policy.

The Framework refines what it means to promote learning science by moving away from prior approaches of detailed facts or loosely defined inquiry to a three-dimensional view of science and engineering practices, crosscutting concepts, and disciplinary core ideas. We argue for a parallel redefinition of what it means to support learning language in the science classroom by moving away from the traditional emphasis on language structure (phonology, morphology, vocabulary, and syntax) to an emphasis on language use for communication and learning. All students face language and literacy challenges and opportunities that are specific to science; such challenges and opportunities are

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Table 1
Three Dimensions of the Science Framework

Scientific and Engineering Practices	Crosscutting Concepts	Disciplinary Core Ideas
1. Asking questions (for science) and defining problems (for engineering) 2. Developing and using models 3. Planning and carrying out investigations 4. Analyzing and interpreting data 5. Using mathematics and computational thinking 6. Constructing explanations (for science) and designing solutions (for engineering) 7. Engaging in argument from evidence 8. Obtaining, evaluating, and communicating information	1. Patterns, similarity, and diversity 2. Cause and effect: Mechanism and explanation 3. Scale, proportion, and quantity 4. Systems and system models 5. Energy and matter: Flows, cycles, and conservation 6. Structure and function 7. Stability and change	Physical Sciences PS 1: Matter and its interactions PS 2: Motion and stability: Forces and interactions PS 3: Energy PS 4: Waves and their applications in technologies for information transfer Life Sciences LS 1: From molecules to organisms: Structures and processes LS 2: Ecosystems: Interactions, energy, and dynamics LS 3: Heredity: Inheritance and variation of traits LS 4: Biological evolution: Unity and diversity Earth and Space Sciences ESS 1: Earth's place in the universe ESS 2: Earth's systems ESS 3: Earth and human activity Engineering, Technology, and the Applications of Science ETS 1: Engineering design ETS 2: Links among engineering, technology, science, and society

amplified for ELLs and for other English speakers with limited standard English language and literacy development. We propose that when students, especially ELLs, are adequately supported to "do" specific things with language, both science learning and language learning are promoted. Our conceptualization of the *language of the science classroom* could inform the fields of science education, second language acquisition/English for speakers of other languages (ESOL)/English as a second language (ESL), and teacher preparation and professional development as the NGSS shape instructional practice, curriculum, and assessment in the coming years. Furthermore, this conceptualization could be applicable to other subjects, especially CCSS for English language arts and literacy and for mathematics. For example, the conceptual issues discussed in this article are used for the "Framework for English language proficiency development standards corresponding to the Common Core State Standards and the Next Generation Science Standards" (Council of Chief State School Officers [CCSSO], 2012).

Next Generation Science Standards with a Focus on Science and Engineering Practices

The Framework defines science learning as having three dimensions: (a) science and engineering practices, (b) crosscutting concepts, and (c) core ideas in each science discipline (see Table 1). The central content of the Framework document is a detailed explanation of what is intended in each dimension; how the three dimensions should be integrated in curriculum, instruction, and assessment; and how these dimensions progress in sophistication across K–12 grades. The meaning of the term *inquiry-based science* is refined and deepened by the explicit definition of the set of science and engineering practices. These practices are presented both as a representation of what scientists do as they engage in scientific inquiry and as a necessary part of what students must do both to learn science and to understand the nature of science. Furthermore, although both science learning and language learning demands increase as students progress across the grade levels, in this article we highlight general features and strategies that are common across grade levels. Much further work will be needed to develop grade-by-grade discrimination of how these general features and strategies are realized.

Engagement in any of the science and engineering practices involves both scientific sense-making and language use. The practices intertwine with one another in the sense-making process, which is a key endeavor that helps students to transition from their naïve conceptions of the world to more scientifically based conceptions. Engagement in these practices is also language intensive. In particular, we focus on four of the eight practices described in the Framework: (no. 2) developing and using models, (no. 6) constructing explanations (for science) and designing solutions (for engineering), (no. 7) engaging in argument from evidence, and (no. 8) obtaining, evaluating, and communicating information. These four practices are selected for the following reasons (see CCSSO, 2012 for all eight practices).

First, these practices are interrelated, in that each is used to support effective engagement in the others. For example, *argumentation from evidence* requires students to develop both mental and diagrammed *models* that clarify their thinking about the phenomenon or system under investigation and to construct *model-based explanations* using evidence (data and observations), logic, and verification. Argument is essential to support or critique a model or an explanation as well as its success or failure in explaining evidence about the phenomenon or system. Clearly, students must *obtain, evaluate and communicate information* as they engage in the process of constructing and critiquing explanations.

Second, these practices are language intensive and require students to engage in classroom science discourse (see literature review by G. Kelly, 2007). Students must read, write, view, and visually represent as they develop their models and explanations. They speak and listen as they present their ideas or engage in

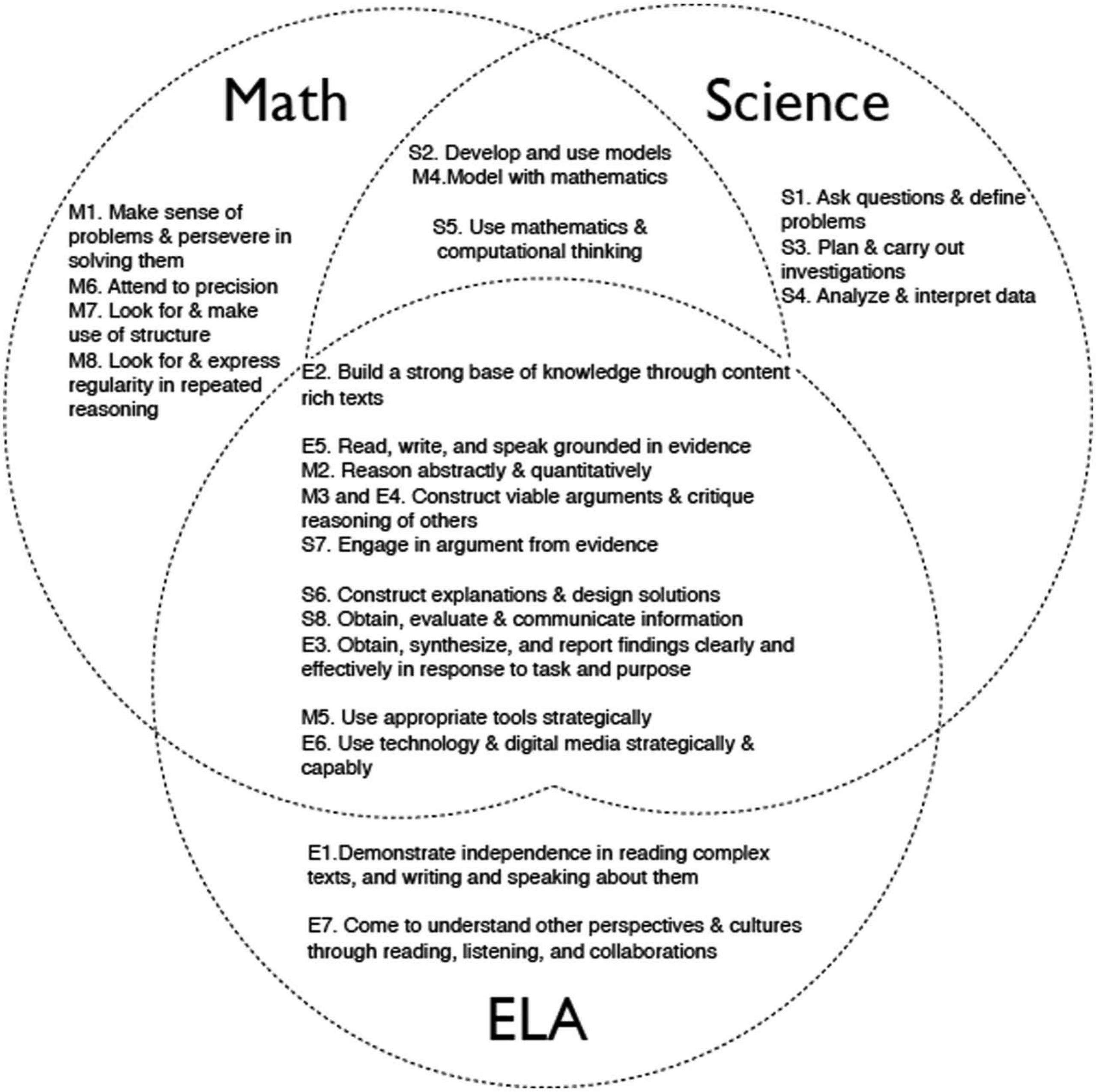

FIGURE 1. *Relationships and convergences found in the Common Core State Standards for Mathematics (practices), Common Core State Standards for English Language Arts and Literacy (student portraits), and the Science Framework (science and engineering practices)* The letter and number set preceding each phrase denotes the discipline and number designated by the content standards. The Science Framework is being used to guide the development of the Next Generation Science Standards.

reasoned argumentation with others to refine their ideas and reach shared conclusions. These practices offer rich opportunities and demands for language learning at the same time as they promote science learning.

Third, these practices are generally less familiar to many science teachers and require shifts for science teaching (Windschitl, Thompson, & Braaten, 2011). Teachers implementing these practices need both understanding of the practices and strategies to include all students regardless of English proficiency. The classroom culture of discourse must be developed and supported. Teachers need to ensure that all voices are respected, even as the process reveals limitations of a model or explanation, or "flawed" use of language.[2] For all students, the emphasis should be on making meaning, on hearing and understanding the contributions of others, and on communicating their own ideas in a common effort to build understanding of the phenomenon or to design solutions of the system being investigated and discussed.

Finally, the requirement for classroom discourse and the norms for this behavior are to a great extent common across all the science disciplines, and indeed across all the subject areas. The convergence of disciplinary practices across CCSS for mathematics and English language arts and literacy and NGSS are highlighted in Figure 1.

In short, as science classrooms incorporate the discourse-rich science and engineering practices described in the Framework, they will become richer language learning environments as well as richer science learning environments for all students. Engaging ELLs in these practices merits special attention, because such engagement can support both science and language learning.

Language in Science Learning and Teaching

To support students' engagement in the science and engineering practices highlighted in the previous section, teachers need a nuanced view of how language is used to construct and communicate meaning in science. Contemporary research on language in science learning and teaching highlights what students and teachers *do* with language as they engage in science inquiry and discourse practices (see literature review by Carlsen, 2007; G. Kelly, 2007).

Systemic Functional Linguistics Perspective

Systemic functional linguistics provides one perspective on how language is used in science learning. Halliday and Martin (1993) framed what they termed "the linguistic register"[3] of science classroom communication as a resource for meaning-making, not as a rigid set of conventions or a system of rules to be learned. Halliday (2002) argued that students must develop and understand the linguistic tools for meaning-making in science as comprising a unique linguistic register. This register provides tools for understanding what people are doing, what their relations are to each other, and how they are using language in the context of making scientific meaning.

Although the disciplinary language of science may seem quite different from everyday discourse, Halliday and Matthiessen (2004) argued that science and everyday discourse are dialogically complementary, interrelated, and synergistic and represent a fundamental continuity that provides different ways of depicting a common reality. Thus, the goal of learning the language of science is not to replace everyday language but to provide different tools for different communicative purposes. Teachers can support their students through classroom practices that make the features of the disciplinary language of science explicit, so that students build linguistic awareness and become comfortable using the disciplinary language for linguistically challenging tasks (Fang & Schleppegrell, 2008).

Discourse and Social Practice in the Science Classroom

Science education researchers have considered how discourse becomes part of the social practice of the science classroom (see literature review by G. Kelly, 2007). In perhaps the most widely cited work on how discourse is constructed in science classrooms, Lemke (1990) argued that discourse should be seen as "differentiated speech" that different groups of people and texts bring to science. For Lemke, the greatest challenge science teachers face is how to support students in building connections across differentiated speech forms, from everyday language to disciplinary discourse. In a similar vein, Gee (1990) proposed that to become competent users of the genre of classroom science discourse (which he distinguished from the genre of research science discourse), students must adopt certain communicative practices, such as those accepted for evaluating claims and representing scientific principles. These practices allow the speaker to be recognized as possessing scientific authority.

Using these conceptual ideas, a number of researchers have developed intervention studies to support teachers' and students' use of discourse and social practice in developing scientific understanding and engaging in scientific practices. For example, Engle and Conant (2002) identified four principles to foster productive disciplinary engagement in the classroom: problematizing content, giving students authority, holding students accountable to others and to disciplinary norms, and providing relevant resources. In a similar line of research, Cornelius and Herrenkohl (2004) mapped students' epistemological development in science onto the dynamic interactions of talk, text, other representations, and social interactions that took place in science classrooms. As another line of research, Windschitl et al. (2011) developed discourse tools and scaffolding to support novice science teachers in developing elements of expertlike teaching, with the greatest gains made in pressing their students for evidence-based scientific explanations through modeling and representations of challenging science concepts.

Language in Science Learning with ELLs

Although the research on discourse and social practice in the science classroom, as discussed above, has addressed the learning needs of all students, another area of research on language in science learning has focused on students who have traditionally been marginalized in science, including ELLs. This research has highlighted the importance of considering both the linguistic challenges and resources that ELLs and other traditionally marginalized students bring to the science classroom.

In perhaps the earliest line of research on science and language learning with ELLs, Warren, Rosebery, and colleagues (e.g., Rosebery, Warren, & Conant, 1992; Warren, Rosebery, & Conant, 1994) engaged students in scientific experimentation as a process of learning to appropriate scientific discourse and construct scientific meaning. The work focuses on the linguistic, cultural, conceptual, and imaginative resources that ELLs bring to the science classroom that can serve as intellectual resources for learning scientific knowledge and practices. In another longstanding line of research, Lee and colleagues (e.g., Lee, Buxton, Lewis, & LeRoy, 2006; Lee & Fradd, 1998) highlighted the importance of developing congruence between students' cultural and linguistic experiences and the specific demands of particular academic disciplines such as science. The work focuses on the link between the home and the academic discipline, especially when the two domains contain potentially discontinuous elements. In an emerging line of work, Brown (Brown, 2006; Brown & Ryoo, 2008) emphasized making the norms of scientific language explicit as a way to bridge the apparent cultural divide between learning to use the language of science and maintaining cultural identity. With ELLs, the work first focuses on discussion of scientific concepts in everyday English and then provides instructional scaffolds to help students convert the concepts into scientific language.

New Standards and Second Language Acquisition with ELLs

As we have pointed out above, the implementation of the CCSS and NGSS will affect instructional practice, curriculum, and assessment across the nation. Title III of the Elementary and Secondary Education Act (ESEA) will also require the states that have adopted these standards to develop English language proficiency development (ELPD) standards (see CCSSO, 2012, as one approach), ELPD assessments that are aligned with those standards, and annual measurable achievement objectives (AMAOs) for English language proficiency. State policies to meet existing federal requirements have many implications for the science education community. Important as the work on language and science has been to date, it is vital that, going forward, science educators and science education researchers understand the key assumptions made about second language acquisition (SLA) and second language pedagogy that inform the politics, policies, and practices surrounding the education of ELLs.

Here we present a brief discussion of first and second languages and their acquisition that emphasizes pragmatic, textual, and sociolinguistic competencies in SLA (Bachman, 1990; Canale & Swain, 1980). The view presented here is much broader than other familiar conceptualizations of language that are typically represented in English language development standards and assessments, both of which are primarily focused on grammatical competence (phonology, morphology, vocabulary, and syntax).

"Doing" Things With Language

Children naturally acquire both the ability to use language and implicit knowledge about not only the structure of that language (e.g., the sound patterns of words, the order of words) but also the conventions (e.g., when to speak or not speak and what to say to whom). By the time they arrive in school, children are skilled users of the variety of language functions used in their home and community. They *can do* many things with language. For example, they can argue with their siblings, complain, disagree, ask and answer questions, and make their needs and feelings known.

Children who are speakers of social or regional varieties of English that are generally referred to as "non-Standard English" may not have mastered the school-accepted ways of speaking to their teacher and classmates, asking and answering questions, or participating effectively in classroom interactions. However, these students can understand their teachers' explanations and instructions as well as what is said to them by their peers. They arrive at school ready, in varying degrees, to learn how *to do* many more things with their language and—from the perspective of systemic functional linguistics—to learn about the ways in which different aspects of language can be used skillfully to create textual meanings.

Students who are referred to as ELLs arrive at school with a well-established first language but at many different levels of English language development. Some may have little or no comprehension of English, whereas others may comprehend even subtle meanings but express themselves hesitantly, with simple and "flawed" language. However, when supported appropriately, most ELLs are capable of learning subjects such as science through their emerging language and of comprehending and carrying out sophisticated language functions (e.g., arguing from evidence, providing explanations) using less-than-perfect English. They *can do* a number of things using whatever level of English they have and can participate in science and engineering practices. By engaging in such practices, moreover, they grow in both their understanding of science and their language proficiency (i.e., capacity to do more with language).

Theories of Second Language Acquisition and Pedagogy

Second languages are acquired every day around the world in naturalistic contexts—outside the classroom—by individuals of different ages and backgrounds when interacting with speakers of that language with whom they need to communicate. According to Wong Fillmore (1992), two conditions are necessary to acquire a second language: (a) learners must have available to them speakers of the language who know the language well enough to provide both access to it and help in learning it, and (b) the social setting must bring learners and target language speakers into frequent enough contact to make language learning possible. In the case of children who arrive in the United States not speaking, or with limited, English, these conditions are seldom met. In many schools, ELLs have little contact with their English-speaking peers. Moreover, language is seen as an academic subject that has to be "taught" and "learned." Thus, "teaching" English to ELLs is seen primarily as the responsibility of specially trained language specialists, such as ESL and ESOL teachers.

Second language acquisition (SLA). Broadly, SLA theories divide into two perspectives (Zuengler & Miller, 2006). Traditional or "mainstream" SLA has viewed language learning as an individual cognitive task. Representatives of this perspective are what Johnson (2004) categorizes as information-processing approaches to SLA, including Gass and Selinker's model of SLA acquisition (2001) and Long's interaction hypothesis (1983, 1996). Progress in students' acquisition of the linguistic system is measured primarily by performance on discrete-point tests of grammatical forms.

The other, more socially oriented view of SLA, adopted in this article, is concerned with understanding how speakers of one language become users (speakers, listeners, writers, readers) of a second language. For researchers who ascribe to this view, the goal of second language learning is to use that language in order to function competently in a variety of contexts for a range of purposes. Such perspectives include Vygotskian sociocultural theory (Lantolf, 2000, 2006) and language socialization perspectives (Duff, 1995, 2002).

Second language pedagogy. Over 25 centuries of second language teaching (L. G. Kelly, 1969), there have been a series of pendulum shifts, debates, innovations, and controversies. In general, research on second language teaching (Ellis, 2005; Norris & Ortega, 2000, 2006) has been carried out primarily with adults in post-secondary settings.

Like the two perspectives on SLA, second language pedagogies can generally be classified as following one of two approaches: the structural and the experiential (Stern, 1990). Structural approaches based on traditional or mainstream SLA focus on "teaching" specific language elements (e.g., vocabulary, pronunciation, grammatical forms), assuming that these elements can be ordered and sequenced in a way that over time will lead to grammatical accuracy, greater complexity, and increased fluency. An example of such an approach is forms-focused instruction described by Long (1991) that is typical of most foreign language and ESL instruction and primarily follows a grammatical syllabus (e.g., modal verbs, present progressive, past tense, vocabulary lists), although also incorporating "communicative" activities using these forms.

Experiential approaches to teaching language, adopted in this article, are based on the socially oriented view of SLA. These approaches focus on supporting students' ability to do things with language, engaging them in purposeful activities, and providing them with opportunities for language use. An example of such an approach is task-based instruction that follows a curriculum of tasks (e.g., activities involving information gaps of various types in both written and oral language) around which students engage in actual communication (Pica, 2008).

Intersections between Learning of Science and Learning of Language

It is the perspective of this article that the opportunity for language development through use in context can be a language learning experience at the same time as it can be a science learning experience, provided that teachers understand how to include and support ELLs regardless of their levels of English proficiency. In this article, we identify key features of the *language of the science classroom* as students engage in language-intensive science and engineering practices. Before explaining our conceptualization, we examine one current approach to integrating the teaching of language and the teaching of science, *content-based language instruction.*

Content-Based Language Instruction

Content-based language instruction, at its best, integrates the teaching of language and the teaching of academic subjects (Scarcella, 2003; Schleppegrell, 2004; Snow, 2001). It was introduced to counter traditional "content-less" language instruction that focused primarily on forms and minimized the importance of meaningful and authentic use in the acquisition of language (Brinton, Snow, & Wesche, 1989). Originally taught by language specialists, including ESL and ESOL teachers, content-based language instruction was intended to provide students with increased motivation in subject matter as well as opportunities to experience larger discourse-level features and social interaction patterns essential to language use. However, ESL or ESOL teachers' inadequate content knowledge in multiple academic subjects has limited the success of this approach.

More recently, content-based language instruction has shifted to a "sheltered" model, in which content area classes for ELLs are taught by content area teachers with some training in language pedagogies, usually of the traditional type. Teachers are encouraged to focus on *both* content objectives *and* language objectives (Echevarria & Short, 2006; Echevarria & Vogt, 2008). Content-based language instruction is a valuable attempt to bring together subject matter instruction and second language instruction. Perhaps because content-based language instruction emerged within traditional language pedagogy, the attention of content area teachers is often directed at the study and practice of forms and language items such as vocabulary, phrases, or sentence frames.

Most science teachers today are likely to have learned about language primarily in these terms. When they are told that they must help students to acquire "academic language," there is much confusion about what this means. Indeed, this term is used variously by different scholars and professional development specialists.[4] Should they teach words from Coxhead's (1998) Academic Wordlist (e.g., assemble, prohibit, simulate) or should they limit themselves to the technical vocabulary of science? Should they be concerned with agreement of verbs and the correction of nonstandard forms (e.g., he done come) or should they model ways in which "real" scientists talk to each other?

In this article, we argue for two shifts: (a) a shift away from both content-based language instruction and the sheltered model to a focus on language-in-use environments and (b) a shift away from "teaching" discrete language skills to a focus on supporting language development by providing appropriate contexts and experiences. We envision science teachers who create carefully planned classrooms where students engage in science and engineering practices, such as evidence-based arguments and explanations of phenomena or systems. In such classrooms, ELLs are not left to sink or swim. They are supported in using multiple resources and strategies for learning science *and* developing English.

Language of the Science Classroom: Moving Toward Disciplinary Language of Science

The classroom language used to teach and learn a particular subject, such as English literature, history, or mathematics, draws from the disciplinary language and discourse conventions of the subject. Many features of classroom language and of written materials are common across subjects. There are, however, language and literacy challenges and opportunities that are specific to science. If science teachers are to engage ELLs in science and engineering practices, they must have a clear understanding of the ways that students and teachers use oral and written language to interact with each other and to obtain information from written materials. They must monitor individual students' language use to ensure that all students are comprehending the discourse and participating in it.

In this section, we describe and illustrate some of the ways that language is used in the teaching of science to provide teachers with a better understanding of what is currently being referred to as academic language and academic literacy. In order to be more specific, we introduce the term *language of the science classroom* that includes the registers (i.e., styles of talk) used in the science classroom by teachers and students as they participate in academic tasks and activities and demonstrate their knowledge in oral or written forms.[5] Language of the science classroom is grounded in colloquial or everyday language but moves toward the disciplinary language of science. For example, written materials used in science classrooms rarely represent the disciplinary language of professional scientists, but rather use styles and levels of language intended for science learners. As the grade level advances, written materials intended for learners tend to mirror disciplinary language more closely. Our intent is to be explicit about what science teachers and their students "do" with language in their classrooms.

As previously described, Table 1 summarizes (a) science and engineering practices, (b) crosscutting concepts, and (c) core ideas in each science discipline. Table 2 presents the four selected

Table 2
Science and Engineering Practices and Selected Language Functions

Practices	Scientific Sense-Making and Language Use	
Develop and use models	Analytical tasks	Develop and represent an explicit model of a phenomenon or system Use a model to support an explanation of a phenomenon or system Make revisions to a model based on either suggestions of others or conflicts between a model and observation
	Receptive language functions	Comprehend others' oral and written descriptions, discussions, and justifications of models of phenomena or systems Interpret the meaning of models presented in texts and diagrams
	Productive language functions	Communicate (orally and in writing) ideas, concepts, and information related to a model for a phenomenon or system • Label diagrams of a model and make lists of parts • Describe a model using oral and/or written language as well as illustrations • Describe how a model relates to a phenomenon or system • Discuss limitations of a model • Ask questions about others' models
Develop explanations (for science) and design solutions (for engineering)	Analytical tasks	Develop explanation or design Analyze the match between explanation or model and a phenomenon or system Revise explanation or design based on input of others or further observations Analyze how well a solution meets design criteria
	Receptive language functions	Comprehend questions and critiques Comprehend explanations offered by others Comprehend explanations offered by texts Coordinate texts and representations
	Productive language functions	Communicate (orally and in writing) ideas, concepts, and information related to an explanation of a phenomenon or system (natural or designed) • Provide information needed by listeners or readers • Respond to questions by amplifying explanation • Respond to critiques by countering with further explanation or by accepting as needing further thought • Critique or support explanations or designs offered by others
Engage in argument from evidence	Analytical tasks	Distinguish between a claim and supporting evidence or explanation Analyze whether evidence supports or contradicts a claim Analyze how well a model and evidence are aligned Construct an argument
	Receptive language functions	Comprehend arguments made by others orally Comprehend arguments made by others in writing
	Productive language functions	Communicate (orally and in writing) ideas, concepts, and information related to the formation, defense, and critique of arguments • Structure and order written or verbal arguments for a position • Select and present key evidence to support or refute claims • Question or critique arguments of others
Obtain, evaluate, and communicate scientific information	Analytical tasks	Coordinate written, verbal, and diagrammatic inputs Evaluate quality of an information source Evaluate agreement/disagreement of multiple sources Evaluate need for further information Summarize main points of a text or oral discussion
	Receptive language functions	Read or listen to obtain scientific information from diverse sources including lab or equipment manuals, oral and written presentations of other students, Internet materials, textbooks, science-oriented trade books, and science press articles Listen to and understand questions or ideas of others
	Productive language functions	Communicate (orally and in writing) ideas, concepts, and information related to scientific information • Present information, explanations, or arguments to others • Formulate clarification questions about scientific information • Provide summaries of appropriate information obtained for a specific purpose or audience • Discuss the quality of scientific information obtained from text sources based on investigating the scientific reputation of the source, and comparing information from multiple sources

Note. The analytical tasks, receptive language functions, and productive language functions included in this table are selective rather than exhaustive.

Table 3
Language of the Science Classroom

Features of Classroom Language	Teachers' Language Use and Tasks	Students' Language Use and Tasks		
	Oral and Written Receptive and Productive	Oral Receptive and Productive	Written: Receptive	Written: Productive
Modality	Explanations and presentations (one-to-many, many-to-many) Communication with small groups of students (one-to-group) Communication with individual students (one-to-one) Communication with parents (one-to-one)	Whole-classroom participation (one-to-many) Small group participation (one-to-group) Interaction with individual peers (one-to-one) Interaction with adults within school contexts (one-to-one)	Comprehension of written classroom and school-based formal and informal written communication	Production of written classroom and school-based formal and informal written communication • Written reports • Science journal entries
Registers	Colloquial + classroom registers + disciplinary language and terminology	Colloquial + classroom registers + disciplinary language and terminology	Science-learner written registers + disciplinary language and terminology + disciplinary discourse conventions	
Examples of Registers	Classroom registers: • Giving directions • Checking for understanding • Facilitating discussions Science discourse registers used for: • Describing models • Constructing arguments • Providing written or verbal explanation of a phenomenon or system	Classroom registers: • Comprehending oral directions • Asking for clarification • Participating in discussions Learner-appropriate science discourse registers and conventions used for: • Describing models • Constructing arguments • Providing oral explanations of a phenomenon or system	Classroom, school, and science-learner written registers: • Textbooks • Lab or equipment manuals • Writing by other students • Internet materials • Science-oriented trade books • Science press articles • Syllabi • School announcements • Formal documents (e.g., class assignment, quarterly grades, assessment results)	

science and engineering practices, types of analytical tasks that students engage in for each practice, and receptive (listening/reading) and productive (speaking/writing) language functions (see CCSSO, 2012, for all eight practices). These receptive and productive functions are what students "do" with language in order to accomplish analytical tasks. Table 2, then, unpacks the science and engineering practices presented in Table 1 and illustrates that as ELLs participate in each of these practices, they exercise certain analytical tasks to make sense of and construct scientific knowledge. Language serves as the vehicle to perform analytical tasks and ultimately to construct knowledge. Although analytical tasks and language functions are intrinsically interrelated, they deliberately appear separate in Table 2 to highlight the complexity of the language of the science classroom. To learn to perform analytical tasks and language functions over time, ELLs need access to a rich language environment in which frequent examples are part of everyday interactions.

Table 3 focuses on the language of the science classroom itself and, in column 1, highlights three key elements of classroom language use: *modality*, *registers*, and *examples of registers* in an attempt to move beyond simple definitions of "the language of science" as vocabulary or grammatical correctness. *Modality* refers to multiple aspects of the oral and written channels through which language is used. The table calls attention to the multiple features of students' and teachers' language use in the classroom while engaged in science and engineering practices. The table also makes evident that language used in the science classroom involves interactions between teacher and students, between students in small groups, by students with the entire class, and by students with various written materials. The rows labeled *Register* and *Examples of Register* highlight the various different registers used by both teachers and students to engage in interaction in the science classroom. These ways of using language range from the informal styles used by teachers to provide explanations, to the more formal, student-directed written styles used by classroom texts, and to the typical oral language used by students to interact with each other. In carrying out such practices, students grow in their ability to use appropriate registers.

Conclusions and Implications

NGSS will require major shifts in science education (NRC, 2011; see Table 1), comparable to major shifts due to CCSS for English language arts and literacy and for mathematics (Common Core State Standards Initiative, 2010a, 2010b). Across these three subject areas, the new standards share a common emphasis on disciplinary practices and classroom discourse (see Figure 1). As engagement in these practices is language intensive, it presents both language demands and opportunities for all students, especially ELLs.

Given the richness of science and engineering practices, NGSS will lead to science classrooms that are also rich language learning environments for ELLs. An important role of the science teacher is to encourage and support language use and development in the service of making sense of science. Participation in science and engineering practices should be expected of all students, and ELLs' contributions should be accepted and acknowledged for their value within the science discourse, rather than critiqued for their "flawed" use of language. This view is consistent with contemporary literature on language in science learning and teaching that highlights what students and teachers *do* with language as they engage in science inquiry and discourse practices (Carlsen, 2007; G. Kelly, 2007). This view is also consistent with current theories of SLA that emphasize what learners *can do* with language—the socially oriented (rather than individually oriented) view of SLA and the experiential (rather than structural) pedagogies.

Although content-based language instruction, sheltered instruction, and academic language instruction are valuable attempts to bring together subject matter instruction and second language instruction, their predominant emphases have been on the study and practice of language elements rather than on immersion in rich environments that use language for sense-making. In this article, we indicate how science and engineering practices involve a range of analytical tasks and language functions (see Table 2). We also stress the value of attention to the *language of the science classroom* that moves toward the disciplinary language of science (see Table 3).

NGSS will present both the need and opportunity to address a new set of research questions. What do science and engineering practices look like in the science classroom? How does student ability to engage in these practices progress over the grade levels? What are the supports needed for such engagement at a given grade level? What do science teachers and language specialists need to know about language demands and opportunities to support ELLs' engagement in these practices? What additional supports, both within and outside the science classroom, do ELLs need in order to most rapidly gain content-relevant language skills? What technologies and tools most effectively support this language learning?

Successful implementation of NGSS with ELLs will require political will, especially in the current accountability policy context, where these students tend to receive limited and inequitable science instruction because of the perceived urgency of developing basic literacy and numeracy. To the contrary, we argue that NGSS can provide a context where science learning and language learning can occur simultaneously. We also argue that ELLs' success in the science classroom will depend on shared responsibilities of teachers across subject areas, as learning of science and development of literacy and numeracy reinforce one another.

NOTES

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We would like to highlight that the article is based on collaboration among a science educator (Okhee Lee), a scientist (Helen Quinn), and a second language acquisition educator (Guadalupe Valdés), which represents the kinds of collaboration advocated in this article. We hope collaboration of this nature will become more common and frequent as CCSS and NGSS are implemented with all students, including ELLs.

1. In a companion paper that is posted on the project website ell.stanford.edu, we have offered effective classroom strategies to support science and language for ELLs in five domains: (a) literacy strategies for all students, (b) language support strategies with ELLs, (c) discourse strategies with ELLs, (d) home language support, and (e) home culture connections (Quinn, Lee, & Valdés, 2012).

2. We put the term "flawed" in quotations to stress that we do not consider incorrect grammar or nonnative-like speech to be a problem to be corrected in the moment. Rather, we stress that contributions should be valued for their role in the discourse regardless of the language level of the speaker.

3. The term *register* has been used in sociolinguistics to refer to a style of language defined by its context and different occasions of use. Ferguson (1994), one of the first scholars to use the term, defined register variation as ways of speaking in regularly occurring communication situations that, over time, will tend to develop similar vocabularies, characteristics of intonation and structure, and bits of syntax, that are different from the language of other communication situations. For a foundational discussion of register in sociolinguistics, the reader is referred to Biber and Finegan (1994). It is noted that Halliday and colleagues have a highly specialized definition of register that is not broadly used in general sociolinguistics. In this article, we use the term "register" following Ferguson and also "ways and styles of speaking" to refer to these regularly occurring uses of language.

4. For example, Solomon and Rhodes (1995) point out that academic language has generally been described in discrete linguistic terms focusing on lexis and syntax. It has also been described as "a compilation of unique language functions and structures that is difficult for minority students to master" (p. 2). Valdés (2004) lists various definitions of academic language used in the literature including "a set of intellectual practices," "language that follows stylistic conventions and is error free," "the language used in particular disciplines," and "the language needed to succeed academically in all content areas" (pp. 17–19).

5. Both students and teachers use "regular ways of speaking" for different purposes in the classroom. These different ways of speaking (e.g., talk between classmates, talk before the whole class, talk to present formal classroom reports) are acquired by individuals in the course of regularly interacting for particular purposes and in particular contexts. As pointed out in Footnote 3, a register is an accepted and shared "way of speaking" for specific purposes by individuals in interaction.

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On Proof and Progress in Mathematics [1]

WILLIAM P. THURSTON

This essay on the nature of progress in mathematics was stimulated by the article of Jaffe and Quinn, "Theoretical Mathematics: Toward a cultural synthesis of mathematics and theoretical physics" [2] Their article raises interesting issues that mathematicians should pay more attention to, but it also perpetuates some widely held beliefs and attitudes that need to be questioned and examined

The article had one paragraph portraying some of my work in a way that diverges from my experience, and it also diverges from the observations of people in the field whom I've discussed it with as a reality check

After some reflection, it seemed to me that what Jaffe and Quinn wrote was an example of the phenomenon that people see what they are tuned to see. Their portrayal of my work resulted from projecting the sociology of mathematics onto a one-dimensional scale (speculation versus rigor) that ignores many basic phenomena.

Responses to the Jaffe-Quinn article have been invited from a number of mathematicians, and I expect it to receive plenty of specific analysis and criticism from others Therefore, I will concentrate in this essay on the positive rather than on the contranegative. I will describe my view of the process of mathematics, referring only occasionally to Jaffe and Quinn by way of comparison

In attempting to peel back layers of assumptions, it is important to try to begin with the right questions:

1. What is it that mathematicians accomplish?

There are many issues buried in this question, which I have tried to phrase in a way that does not presuppose the nature of the answer

It would not be good to start, for example, with the question

> How do mathematicians prove theorems?

This question introduces an interesting topic, but to start with it would be to project two hidden assumptions:

(1) that there is uniform, objective and firmly established theory and practice of mathematical proof, and

(2) that progress made by mathematicians consists of proving theorems

It is worthwhile to examine these hypotheses, rather than to accept them as obvious and proceed from there.

The question is not even

> How do mathematicians make progress in mathematics?

Rather, as a more explicit (and leading) form of the question, I prefer

> How do mathematicians advance human understanding of mathematics?

This question brings to the fore something that is fundamental and pervasive: that what we are doing is finding ways for *people* to understand and think about mathematics.

The rapid advance of computers has helped dramatize this point, because computers and people are very different For instance, when Appel and Haken completed a proof of the 4-color map theorem using a massive automatic computation, it evoked much controversy. I interpret the controversy as having little to do with doubt people had as to the veracity of the theorem or the correctness of the proof. Rather, it reflected a continuing desire for *human understanding* of a proof, in addition to knowledge that the theorem is true.

On a more everyday level, it is common for people first starting to grapple with computers to make large-scale computations of things they might have done on a smaller scale by hand. They might print out a table of the first 10,000 primes, only to find that their printout isn't something they really wanted after all. They discover by this kind of experience that what they really want is usually not some collection of "answers"—what they want is *understanding*.

It may sound almost circular to say that what mathematicians are accomplishing is to advance human understanding of mathematics. I will not try to resolve this by discussing what mathematics is, because it would take us far afield Mathematicians generally feel that they know what mathematics is, but find it difficult to give a good direct definition. It is interesting to try. For me, "the theory of formal patterns" has come the closest, but to discuss this would be a whole essay in itself.

Could the difficulty in giving a good direct definition of mathematics be an essential one, indicating that mathematics is the smallest subject satisfying the following:

- Mathematics includes the natural numbers and plane and solid geometry.
- Mathematics is that which mathematicians study.
- Mathematicians are those humans who advance human understanding of mathematics.

In other words, as mathematics advances, we incorporate it into our thinking As our thinking becomes more sophisticated, we generate new mathematical concepts and new mathematical structures: the subject matter of mathematics changes to reflect how we think

If what we are doing is constructing better ways of thinking, then psychological and social dimensions are essential to a good model for mathematical progress These dimensions are absent from the popular model In caricature, the popular model holds that

D. mathematicians start from a few basic mathematical structures and a collection of axioms "given" about these structures, that

T there are various important questions to be answered about these structures that can be stated as formal mathematical propositions, and

P the task of the mathematician is to seek a deductive pathway from the axioms to the propositions or to their denials

We might call this the definition-theorem-proof (DTP) model of mathematics

A clear difficulty with the DTP model is that it doesn't explain the source of the questions Jaffe and Quinn discuss speculation (which they inappropriately label "theoretical mathematics") as an important additional ingredient Speculations consists of making conjectures, raising questions, and making intelligent guesses and heuristic arguments about what is probably true.

Jaffe and Quinn's DSTP model still fails to address some basic issues. We are not trying to meet some abstract production quota of definitions, theorems and proofs. The measure of our success is whether what we do enables *people* to understand and think more clearly and effectively about mathematics.

Therefore, we need to ask ourselves:

2. How do people understand mathematics?

This is a very hard question. Understanding is an individual and internal matter that is hard to be fully aware of, hard to understand and often hard to communicate. We can only touch on it lightly here.

People have very different ways of understanding particular pieces of mathematics. To illustrate this, it is best to take an example that practicing mathematicians understand in multiple ways, but that we see our students struggling with. The derivative of a function fits well. The derivative can be thought of as:

(1) Infinitesimal: the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function

(2) Symbolic: the derivative of x^n is nx^{n-1}, the derivative of $\sin(x)$ is $\cos(x)$, the derivative of $f\circ g$ is $f'\circ g*g'$, etc.

(3) Logical: $f'(x) = d$ if and only if for every ε there is a δ such that when $0 < |\Delta x| < \delta$,

$$\left|\frac{f(x+\Delta x)-f(x)}{\Delta x}-d\right| < \delta$$

(4) Geometric: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.

(5) Rate: the instantaneous speed of $f(t)$, when t is time

(6) Approximation: The derivative of a function is the best linear approximation to the function near a point

(7) Microscopic: The derivative of a function is the limit of what you get by looking at in under a microscope of higher and higher power.

This is a list of different ways of *thinking about* or *conceiving of* the derivative, rather than a list of different *logical definitions*. Unless great efforts are made to maintain the tone and flavor of the original human insights, the differences start to evaporate as soon as the mental concepts are translated into precise, formal and explicit definitions.

I can remember absorbing each of these concepts as something new and interesting, and spending a good deal of mental time and effort digesting and practicing with each, reconciling it with the others. I also remember coming back to revisit these different concepts later with added meaning and understanding.

The list continues; there is no reason for it ever to stop. A sample entry further down the list may help illustrate this. We may think we know all there is to say about a certain subject, but new insights are around the corner. Furthermore, one person's clear mental image is another person's intimidation:

37. The derivative of a real-valued function f in a domain D is the Lagrangian section of the cotangent bundle $T^*(D)$ that gives the connection form for the unique flat connection on the trivial $\mathbf{R}$-bundle $D \times \mathbf{R}$ for which the graph of f is parallel.

These differences are not just a curiosity. Human thinking and understanding do not work on a single track, like a computer with a single central processing unit Our brains and minds seem to be organized into a variety of separate, powerful facilities. These facilities work together loosely, "talking" to each other at high levels rather than at low levels of organization.

Here are some major divisions that are important for mathematical thinking:

(1) Human language. We have powerful special-purpose facilities for speaking and understanding human language, which also tie in to reading and writing Our linguistic facility is an important tool for thinking, not just for communication. A crude example is the quadratic formula which people may remember as a little chant, "ex equals minus bee plus or minus the square root of bee squared minus four ay see all over two ay " The mathematical language of symbols is closely tied to our human language facility The fragment of mathematical symbolese available to most calculus students has only one verb, "=". That's why students use it when they're in need of a verb. Almost anyone who has taught calculus in the U.S. has seen students instinctively write "$x^3 = 3x^2$" and the like

(2) Vision, spatial sense, kinesthetic (motion) sense People have very powerful facilities for taking in information visually or kinesthetically, and thinking with their spatial sense. On the other hand, they do not have a very good built-in facility for inverse vision, that is, turning an internal spatial understanding back into a two-dimensional image Con-

sequently, mathematicians usually have fewer and poorer figures in their papers and books than in their heads

An interesting phenomenon in spatial thinking is that scale makes a big difference. We can think about little objects in our hands, or we can think of bigger human-sized structures that we scan, or we can think of spatial structures that encompass us and that we move around in. We tend to think more effectively with spatial imagery on a larger scale: it's as if our brains take larger things more seriously and can devote more resources to them.

(3) Logic and deduction We have some built-in ways of reasoning and putting things together associated with how we make logical deductions: cause and effect (related to implication), contradiction or negation, *etc.*

Mathematicians apparently don't generally rely on the formal rules of deduction as they are thinking. Rather, they hold a fair bit of logical structure of a proof in their heads, breaking proofs into intermediate results so that they don't have to hold too much logic at once. In fact, it is common for excellent mathematicians not even to know the standard formal usage of quantifiers (for all and there exists), yet all mathematicians certainly perform the reasoning that they encode

It's interesting that although "or", "and" and "implies" have identical formal usage, we think of "or" and "and" as conjunctions and "implies" as a verb.

(4) Intuition, association, metaphor. People have amazing facilities for sensing something without knowing where it comes from (intuition); for sensing that some phenomenon or situation or object is like something else (association); and for building and testing connections and comparisons, holding two things in mind at the same time (metaphor). These facilities "listening" to my intuitions and associations, and building them into metaphors and connections. This involves a kind of simultaneous quieting and focusing of my mind. Words, logic, and detailed pictures rattling around can inhibit intuitions and associations

(5) Stimulus-response This is often emphasized in schools; for instance, if you see 3927×253, you write one number above the other and draw a line underneath, *etc.* This is also important for research mathematics: seeing a diagram of a knot, I might write down a presentation for the fundamental group of its complement by a procedure that is similar in feel to the multiplication algorithm.

(6) Process and time. We have a facility for thinking about processes or sequences of actions that can often be used to good effect in mathematical reasoning One way to think of a function is an action, a process, that takes the domain to the range. This is particularly valuable when composing functions. Another use of this facility is in remembering proofs: people often remember a proof as a process consisting of several steps. In topology, the notion of a homotopy is most often thought of as a process taking time. Mathematically, time is no different from one more spatial dimension, but since humans interact with it in a quite different way, it is psychologically very different.

3. How is mathematical understanding communicated?

The transfer of understanding from one person to another is not automatic. It is hard and tricky. Therefore, to analyze human understanding of mathematics, it is important to consider **who** understands **what**, and **when**.

Mathematicians have developed habits of communication that are often dysfunctional. Organizers of colloquium talks everywhere exhort speakers to explain things in elementary terms. Nonetheless, most of the audience at an average colloquium talk gets little of value from it. Perhaps they are lost within the first 5 minutes, yet sit silently through the remaining 55 minutes. Or perhaps they quickly lose interest because the speaker plunges into technical details without presenting any reason to investigate them At the end of the talk, the few mathematicians who are close to the field of the speaker ask a question or two to avoid embarrassment.

This pattern is similar to what often holds in classrooms, where we go through the motions of saying for the record what we think the students "ought" to learn, while the students are trying to grapple with the more fundamental issues of learning our language and guessing at our mental models Books compensate by giving samples of how to solve every type of homework problem Professors compensate by giving homework and tests that are much easier than the material "covered" in the course, and then grading the homework and tests on a scale that requires little understanding. We assume that the problem is with the students rather than with communication: that the students either just don't have what it takes, or else just don't care.

Outsiders are amazed at this phenomenon, but within the mathematical community, we dismiss it with shrugs

Much of the difficulty has to do with the language and culture of mathematics, which is divided into subfields. Basic concepts used every day within one subfield are often foreign to another subfield. Mathematicians give up on trying to understand the basic concepts even from neighboring subfields, unless they were clued in as graduate students.

In contrast, communication works very well within the subfields of mathematics Within a subfield, people develop a body of common knowledge and known techniques By informal contact, people learn to understand and copy each other's ways of thinking, so that ideas can be explained clearly and easily.

Mathematical knowledge can be transmitted amazingly fast within a subfield When a significant theorem is proved, it often (but not always) happens that the solution can be communicated in a matter of minutes from one person to another within the subfield. The same proof would be communicated and generally understood in an hour talk to members of the subfield It would be the subject of a 15- or 20-page paper, which could be read and understood in a few hours or perhaps days by members of the subfield

Why is there such a big expansion from the informal discussion to the talk to the paper? One-on-one, people use wide channels of communication that go far beyond formal mathematical language They use gestures, they draw pictures and diagrams, they make sound effects and use body language Communication is more likely to be two-way, so that people can concentrate on what needs the most attention With these channels of communication, they are in a much better position to convey what's going on, not just in their logical and linguistic facilities, but in their other mental facilities as well

In talks, people are more inhibited and more formal Mathematical audiences are often not very good at asking the questions that are on most people's minds, and speakers often have an unrealistic preset outline that inhibits them from addressing questions even when they are asked

In papers, people are still more formal Writers translate their ideas into symbols and logic, and readers try to translate back.

Why is there such a discrepancy between communication within a subfield and communication outside of subfields, not to mention communication outside mathematics?

Mathematics in some sense has a common language: a language of symbols, technical definitions, computations, and logic This language efficiently conveys some, but not all, modes of mathematical thinking Mathematicians learn to translate certain things almost unconsciously from one mental mode to the other, so that some statements quickly become clear Different mathematicians study papers in different ways, but when I read a mathematical paper in a field in which I'm conversant, I concentrate on the thoughts that are between the lines I might look over several paragraphs or strings of equations and think to myself "Oh yeah, they're putting in enough rigmarole to carry such-and-such idea" When the idea is clear, the formal setup is usually unnecessary and redundant—I often feel that I could write it out myself more easily than figuring out what the authors actually wrote. It's like a new toaster that comes with a 16-page manual If you already understand toasters and if the toaster looks like previous toasters you've encountered, you might just plug it in and see if it works, rather than first reading all the details in the manual

People familiar with ways of doing things in a subfield recognize various patterns of statements or formulas as idioms or circumlocution for certain concepts or mental images But to people not already familiar with what's going on the same patterns are not very illuminating; they are often even misleading The language is not alive except to those who use it

I'd like to make an important remark here: there are some mathematicians who are conversant with the ways of thinking in more than one subfield, sometimes in quite a number of subfields Some mathematicians learn the jargon of several subfields as graduate students, some people are just quick at picking up foreign mathematical language and culture, and some people are in mathematical centers where they are exposed to many subfields People who are comfortable in more than one subfield can often have a very positive influence, serving as bridges, and helping different groups of mathematicians learn from each other But people knowledgeable in multiple fields can also have a negative effect, by intimidating others, and by helping to validate and maintain the whole system of generally poor communication For example, one effect often takes place during colloquium talks, where one or two widely knowledgeable people sitting in the front row may serve as the speaker's mental guide to the audience

There is another effect caused by the big differences between how we think about mathematics and how we write it A group of mathematicians interacting with each other can keep a collection of mathematical ideas alive for a period of years, even though the recorded version of their mathematical work differs from their actual thinking, having much greater emphasis on language, symbols, logic and formalism But as new batches of mathematicians learn about the subject they tend to interpret what they read and hear more literally, so that the more easily recorded and communicated formalism and machinery tend to gradually take over from other modes of thinking

There are two counters to this trend, so that mathematics does not become entirely mired down in formalism First, younger generations of mathematicians are continually discovering and rediscovering insights on their own, thus reinjecting a diverse modes of human thought into mathematics

Second, mathematicians sometimes invent names and hit on unifying definitions that replace technical circumlocutions and give good handles for insights. Names like "group" to replace "a system of substitutions satisfying ", and "manifold" to replace

> We can't give coordinates to parametrize all the solutions to our equations simultaneously, but in the neighborhood of any particular solution we can introduce coordinates
>
> $(f_1(u_1, u_2, u_3), f_2(u_1, u_2, u_3), f_3(u_1, u_2, u_3), f_4(u_1, u_2, u_3), f_5(u_1, u_2, u_3))$
>
> where at least one of the ten determinants
>
> ...[ten 3×3 determinants of matrices of partial derivatives]...
>
> is not zero

may or may not have represented advances in insight among experts, but they greatly facilitate the *communication* of insights

We mathematicians need to put far greater effort into communicating mathematical *ideas* To accomplish this, we need to pay much more attention to communicating not just our definitions, theorems, and proofs, but also our ways of thinking We need to appreciate the value of different ways of thinking about the same mathematical structure

We need to focus far more energy on understanding and explaining the basic mental infrastructure of mathematics—with consequently less energy on the most recent results This entails developing mathematical language that is effective for the radical purpose of conveying ideas to people who don't already know them

Part of this communication is through proofs

4. What is a proof?

When I started as a graduate student at Berkeley, I had trouble imagining how I could "prove" a new and interesting mathematical theorem I didn't really understand what a "proof" was

By going to seminars, reading papers, and talking to other graduate students, I gradually began to catch on. Within any field, there are certain theorems and certain techniques that are generally known and generally accepted. When you write a paper, you refer to these without proof. You look at other papers in the field, and you see what facts they quote without proof, and what they cite in their bibliography You learn from other people some idea of the proofs. Then you're free to quote the same theorem and cite the same citations. You don't necessarily have to read the full papers or books that are in your bibliography. Many of the things that are generally known are things for which there may be no known written source. As long as people in the field are comfortable that the idea works, it doesn't need to have a formal written source

At first I was highly suspicious of this process. I would doubt whether a certain idea was really established But I found that I could ask people, and they could produce explanations and proofs, or else refer me to other people or to written sources that would give explanations and proofs. There were published theorems that were generally known to be false, or where the proofs were generally known to be incomplete. Mathematical knowledge and understanding were embedded in the minds and in the social fabric of the community of people thinking about a particular topic This knowledge was supported by written documents, but the written documents were not really primary

I think this pattern varies quite a bit from field to field I was interested in geometric areas of mathematics, where it is often pretty hard to have a document that reflects well the way people actually think. In more algebraic or symbolic fields, this is not necessarily so, and I have the impression that in some areas documents are much closer to carrying the life of the field. But in any field, there is a strong social standard of validity and truth Andrew Wiles's proof of Fermat's Last Theorem is a good illustration of this, in a field which is very algebraic. The experts quickly came to believe that his proof was basically correct on the basis of high-level ideas, long before details could be checked. This proof will receive a great deal of scrutiny and checking compared to most mathematical proofs; but no matter how the process of verification plays out, it helps illustrate how mathematics evolves by rather organic psychological and social processes.

When people are doing mathematics, the flow of ideas and the social standard of validity is much more reliable than formal documents People are usually not very good in checking *formal correctness* of proofs, but they are quite good at detecting potential weaknesses or flaws in proofs.

To avoid misinterpretation, I'd like to emphasize two things I am *not* saying. First, I am *not* advocating any weakening of our community standard of proof; I am trying to describe how the process really works Careful proofs that will stand up to scrutiny are very important I think the process of proof on the whole works pretty well in the mathematical community. The kind of change I would advocate is that mathematicians take more care with their proofs, making them really clear and as simple as possible so that if any weakness is present it will be easy to detect. Second, I am *not* criticizing the mathematical study of formal proofs, nor am I criticizing people who put energy into making mathematical arguments more explicit and more formal These are both useful activities that shed new insights on mathematics.

I have spent a fair amount of effort during periods of my career exploring mathematical questions by computer. In view of that experience, I was astonished to see the statement of Jaffe and Quinn that mathematics is extremely slow and arduous, and that it is arguably the most disciplined of all human activities. The standard of correctness and completeness necessary to get a computer program to work at all is a couple of orders of magnitude higher than the mathematical community's standard of valid proofs. Nonetheless, large computer programs, even when they have been very carefully written and very carefully tested, always seem to have bugs.

I think that mathematics is one of the most intellectually gratifying of human activities. Because we have a high standard for clear and convincing thinking and because we place a high value on listening to and trying to understand each other, we don't engage in interminable arguments and endless redoing of our mathematics. We are prepared to be convinced by others. Intellectually, mathematics moves very quickly Entire mathematical landscapes change and change again in amazing ways during a single career.

When one considers how hard it is to write a computer program even approaching the intellectual scope of a good mathematical paper, and how much greater time and effort have to be put into it to make it "almost" formally correct, it is preposterous to claim that mathematics as we practice it is anywhere near formally correct.

Mathematics as we practice it is much more formally complete and precise than other sciences, but it is much less formally complete and precise for its content than computer programs. The difference has to do not just with the amount of effort: the kind of effort is qualitatively different. In large computer programs, a tremendous proportion of effort must be spent on myriad compatibility issues: making sure that all definitions are consistent, developing "good" data structures that have useful but not cumbersome generality, deciding on the "right" generality for functions, *etc* The proportion of energy spent on the working part of a large program, as distinguished from the bookkeeping part, is surprisingly small. Because of compatibility issues that almost inevitably escalate out of hand because the "right" definitions change as generality and functionality are added, computer programs usually need to be rewritten frequently, often from scratch.

A very similar kind of effort would have to go into mathematics to make it formally correct and complete. It is not that formal correctness is prohibitively difficult on a small scale—it's that there are many possible choices of formalization on small scales that translate to huge numbers of interdependent choices in the large. It is quite hard to make these choices compatible; to do so would certainly entail going back and rewriting from scratch all old mathematical papers whose results we depend on. It is also quite hard to come up with good technical choices for formal definitions that will be valid in the variety of ways that mathematicians want to use them and that will anticipate future extensions of mathematics. If we were to continue to cooperate, much of our time would be spent with international standards commissions to establish uniform definitions and resolve huge controversies.

Mathematicians can and do fill in gaps, correct errors, and supply more detail and more careful scholarship when they are called on or motivated to do so. Our system is quite good at producing reliable theorems that can be solidly backed up. It's just that the reliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas.

On the most fundamental level, the foundations of mathematics are much shakier than the mathematics that we do. Most mathematicians adhere to foundational principles that are known to be polite fictions. For example, it is a theorem that there does not exist any way to ever actually construct or even define a well-ordering of the real numbers. There is considerable evidence (but no proof) that we can get away with these polite fictions without being caught out, but that doesn't make them right. Set theorists construct many alternate and mutually contradictory "mathematical universes" such that if one is consistent, the others are too. This leaves very little confidence that one or the other is the right choice or the natural choice. Gödel's incompleteness theorem implies that there can be no formal system that is consistent, yet powerful enough to serve as a basis for all of the mathematics that we do.

In contrast to humans, computers are good at performing formal processes. There are people working hard on the project of actually formalizing parts of mathematics by computer, with actual formally correct formal deductions. I think this is a very big but very worthwhile project, and I am confident that we will learn a lot from it. The process will help simplify and clarify mathematics. In not too many years, I expect that we will have interactive computer programs that can help people compile significant chunks of formally complete and correct mathematics (based on a few perhaps shaky but at least explicit assumptions), and that they will become part of the standard mathematician's working environment.

However, we should recognize that the humanly understandable and humanly checkable proofs that we actually do are what is most important to us, and that they are quite different from formal proofs. For the present, formal proofs are out of reach and mostly irrelevant: we have good human processes for checking mathematical validity.

5. What motivates people to do mathematics?

There is a real joy in doing mathematics, in learning ways of thinking that explain and organize and simplify. One can feel this joy discovering new mathematics, rediscovering old mathematics, learning a way of thinking from a person or text, or finding a new way to explain or to view an old mathematical structure.

This inner motivation might lead us to think that we do mathematics solely for its own sake. That's not true: the social setting is extremely important. We are inspired by other people, we seek appreciation by other people, and we like to help other people solve their mathematical problems. What we enjoy changes in response to other people. Social interaction occurs through face-to-face meetings. It also occurs through written and electronic correspondence, preprints, and journal articles. One effect of this highly social system of mathematics is the tendency of mathematicians to follow fads. For the purpose of producing new mathematical theorems this is probably not very efficient: we'd seem to be better off having mathematicians cover the intellectual field much more evenly. but most mathematicians don't like to be lonely, and they have trouble staying excited about a subject, even if they are personally making progress, unless they have colleagues who share their excitement.

In addition to our inner motivation and our informal social motivation for doing mathematics, we are driven by considerations of economics and status. Mathematicians, like other academics, do a lot of judging and being judged. Starting with grades, and continuing through letters of recommendation, hiring decisions, promotion decisions, referees reports, invitations to speak, prizes... we are involved in many ratings, in a fiercely competitive system.

Jaffe and Quinn analyze the motivation to do mathematics in terms of a common currency that many mathematicians believe in: credit for theorems.

I think that our strong communal emphasis on theorem-credits has a negative effect on mathematical progress. If what we are accomplishing is advancing human understanding of mathematics, then we would be much better off recognizing and valuing a far broader range of activity. The people who see the way to proving theorems are doing it in the context of a mathematical community; they are not doing it on their own. They depend on understanding of mathematics that they glean from other mathematicians. Once a theorem has been proven, the mathematical community depends on the social network to distribute the ideas to people who might use them further—the print medium is far too obscure and cumbersome.

Even if one takes the narrow view that what we are producing is theorems, the team is important. Soccer can serve as a metaphor. There might only be one or two goals during a soccer game, made by one or two persons. That does not mean that the efforts of all the others are wasted. We do not judge players on a soccer team only by whether they personally make a goal; we judge the team by its function as a team.

In mathematics, it often happens that a group of mathematicians advances with a certain collection of ideas. There are theorems in the path of these advances that will almost inevitably be proven by one person or another Sometimes the group of mathematicians can even anticipate what these theorems are likely to be. It is much harder to predict who will actually prove the theorem, although there are usually a few "point people" who are more likely to score. However, they are in a position to prove those theorems because of the collective efforts of the team The team has a further function, in absorbing and making use of the theorems once they are proven. Even if one person could prove all the theorems in the path single-handedly, they are wasted if nobody else learns them

There is an interesting phenomenon concerning the "point" people. It regularly happens that someone who was in the middle of a pack proves a theorem that receives wide recognition as being significant. Their status in the community—their pecking order—rises immediately and dramatically. When this happens, they usually become much more productive as a center of ideas and a source of theorems. Why? First, there is a large increase in self-esteem, and an accompanying increase in productivity. Second, when their status increases, people are more in the center of the network of ideas—others take them more seriously. Finally and perhaps most importantly, a mathematical breakthrough usually represents a new way of thinking, and effective ways of thinking can usually be applied in more than one situation

This phenomenon convinces me that the entire mathematical community would become much more productive if we open our eyes to the real values in what we are doing. Jaffe and Quinn propose a system of recognized roles divided into "speculation" and "proving" Such a division only perpetuates the myth that our progress is measured in units of standard theorems deduced This is a bit like the fallacy of the person who makes a printout of the first 10,000 primes. What we are producing is human understanding. We have many different ways to understand and many different processes that contribute to our understanding We will be more satisfied, more productive and happier if we recognize and focus on this.

6. Some personal experiences

Since this essay grew out of reflection on the misfit between my experiences and the description of Jaffe and Quinn's, I will discuss two personal experiences, including the one they alluded to

I feel some awkwardness in this, because I do have regrets about aspects of my career: if I were to do things over again with the benefit of my present insights about myself and about the process of mathematics, there is a lot that I would hope to do differently. I hope that by describing these experiences rather openly as I remember and understand them, I can help others understand the process better and learn in advance.

First I will discuss briefly the theory of foliations, which was my first subject, starting when I was a graduate student. (It doesn't matter here whether you know what foliations are.)

At that time, foliations had become a big center of attention among geometric topologists, dynamical systems people, and differential geometers. I fairly rapidly proved some dramatic theorems. I proved a classification theorem for foliations, giving a necessary and sufficient condition for a manifold to admit a foliation. I proved a number of other significant theorems. I wrote respectable papers and published at least the most important theorems. It was hard to find the time to write to keep up with what I could prove, and I built up a backlog.

An interesting phenomenon occurred. Within a couple of years, a dramatic evacuation of the field started to take place I heard from a number of mathematicians that they were giving or receiving advice not to go into foliations—they were saying that Thurston was cleaning it out. People told me (not as a complaint, but as a compliment) that I was killing the field. Graduate students stopped studying foliations, and fairly soon, I turned to other interests as well

I do not think that the evacuation occurred because the territory was intellectually exhausted—there were (and still are) many interesting questions that remain and that are probably approachable. Since those years, there have been interesting developments carried out by the few people who stayed in the field or who entered the field, and there have also been important developments in neighbouring areas that I think would have been much accelerated had mathematicians continued to pursue foliation theory vigorously.

Today, I think there are few mathematicians who understand anything approaching the state of the art foliations as it lived at that time, although there are some parts of the theory of foliations, including developments since that time, that are still thriving.

I believe that two ecological effects were much more important in putting a damper on the subject than any exhaustion of intellectual resources that occurred

First, the results I proved (as well as some important results of other people) were documented in a conventional, formidable mathematician's style. They depended heavily on readers who shared certain background and certain insights. The theory of foliations was a young, opportunistic subfield, and the background was not standardized. I did not hesitate to draw on any of the mathematics I had learned from others. The papers I wrote did not (and could not) spend much time explaining the background culture They documented top-level reasoning and conclusions that I often had achieved after much reflection and effort. I also threw out prize cryptic tidbits of insight, such as "the Godbillon-Vey invariant measures the helical wobble of a foliation", that remained mysterious to most mathematicians who read them This created a high entry barrier: I think many graduate students and mathematicians were discouraged that it was hard to learn and understand the proofs of key theorems.

Second is the issue of what is in it for other people in the subfield When I started working on foliations, I had the conception that what people wanted was to know the answers. I thought that what they sought was a collection of powerful proven theorems that might be applied to answer further mathematical questions But that's only one part of the story. More than the knowledge, people want *personal understanding*. And in our credit-driven system, they also want and need *theorem-credits*

I'll skip ahead a few years, to the subject that Jaffe and Quinn alluded to, when I began studying 3-dimensional manifolds and their relationship to hyperbolic geometry. (Again, it matters little if you know what this is about.) I gradually built up over a number of years a certain intuition for hyperbolic three-manifolds, with a repertoire of constructions, examples and proofs. (This process actually started when I was an undergraduate, and was strongly bolstered by applications of foliations.) After a while, I conjectured or speculated that all three-manifolds have a certain geometric structure; this conjecture eventually became known as the geometrization conjecture. About two or three years later, I proved the geometrization theorem for Haken manifolds. It was a hard theorem, and I spent a tremendous amount of effort thinking about it. When I completed the proof, I spent a lot more effort checking the proof, searching for difficulties and testing it against independent information.

I'd like to spell out more what I mean when I say I proved this theorem. It meant that I had a clear and complete flow of ideas, including details, that withstood a great deal of scrutiny by myself and by others. Mathematicians have many different styles of thought. My style is not one of making broad sweeping but careless generalities, which are merely hints of inspirations: I make clear mental models, and I think things through. My proofs have turned out to be quite reliable. I have not had trouble backing up claims or producing details for things I have proven. I am good in detecting flaws in my own reasoning as well as in the reasoning of others.

However, there is sometimes a huge expansion factor in translating from the encoding in my own thinking to something that can be conveyed to someone else. My mathematical education was rather independent and idiosyncratic, where for a number of years I learned things on my own, developing personal mental models for how to think about mathematics. This has often been a big advantage for me in thinking about mathematics, because it's easy to pick up later the standard mental models shared by groups of mathematicians. This means that some concepts that I use freely and naturally in my personal thinking are foreign to most mathematicians I talk to. My personal mental models and structures are similar in character to the kinds of models groups of mathematicians share—but they are often different models. At the time of the formulation of the geometrization conjecture, my understanding of hyperbolic geometry was a good example. A random continuing example is an understanding of finite topological spaces, an oddball topic that can lend good insight to a variety of questions but that is generally not worth developing in any one case because there are standard circumlocutions that avoid it.

Neither the geometrization conjecture nor its proof for Haken manifolds was in the path of any group of mathematicians at the time—it went against the trends in topology for the preceding 30 years, and it took people by surprise. To most topologists at the time, hyperbolic geometry was an arcane side branch of mathematics, although there were other groups of mathematicians such as differential geometers who did understand it from certain points of view. It took topologists a while just to understand what the geometrization conjecture meant, what it was good for, and why it was relevant.

At the same time, I started writing notes on the geometry and topology of 3-manifolds, in conjunction with the graduate course I was teaching. I distributed them to a few people, and before long many others from around the world were writing for copies. The mailing list grew to about 1200 people to whom I was sending notes every couple of months. I tried to communicate my real thoughts in these notes. People ran many seminars based on my notes, and I got lots of feedback. Overwhelmingly, the feedback ran something like "Your notes are really inspiring and beautiful, but I have to tell you that we spent 3 weeks in our seminar working out the details of §$n.n$ More explanation would sure help."

I also gave many presentations to groups of mathematicians about the ideas of studying 3-manifolds from the point of view of geometry, and about the proof of the geometrization conjecture for Haken manifolds. At the beginning, this subject was foreign to almost everyone. It was hard to communicate—the infrastructure was in my head, not in the mathematical community. There were several mathematical theories that fed into the cluster of ideas: three-manifold topology, Kleinian groups, dynamical systems, geometric topology, discrete subgroups of Lie groups, foliations, Teichmüller spaces, pseudo-Anosov diffeomorphisms, geometric group theory, as well as hyperbolic geometry.

We held an AMS summer workshop at Bowdoin in 1980, where many mathematicians in the subfields of low-dimensional topology, dynamical systems and Kleinian groups came.

It was an interesting experience exchanging cultures. It became dramatically clear how much proofs depend on the audience. We prove things in a social context and address them to a certain audience. Parts of this proof I could communicate in two minutes to the topologists, but the analysts would need an hour lecture before they would begin to understand it. Similarly, there were some things that could be said in two minutes to the analysts that would take an hour before the topologists would begin to get it. And there were many other parts of the proof which should take two minutes in the abstract, but that none of the audience at the time had the mental infrastructure to get in less than an hour.

At that time, there was practically no infrastructure and practically no context for this theorem, so the expansion from how an idea was keyed in my head to what I had to say to get it across, not to mention how much energy the audience had to devote to understand it, was very dramatic.

In reaction to my experience with foliations and in response to social pressures, I concentrated most of my attention on developing and presenting the infrastructure in what I wrote and in what I talked to people about. I explained the details to the few people who were "up" for it I wrote some papers giving the substantive parts of the proof of the geometrization theorem for Haken manifolds—for these papers, I got almost no feedback. Similarly, few people actually worked through the harder and deeper sections of my notes until much later.

The result has been that now quite a number of mathematicians have what was dramatically lacking in the beginning: a working understanding of the concepts and the infrastructure that are natural for this subject. There has been and there continues to be a great deal of thriving mathematical activity. By concentrating on building the infrastructure and explaining and publishing definitions and ways of thinking but being slow in stating or in publishing proofs of all the "theorems" I knew how to prove, I left room for many other people to pick up credit. There has been room for people to discover and publish other proofs of the geometrization theorem. These proofs helped develop mathematical concepts which are quite interesting in themselves, and lead to further mathematics.

What mathematicians most wanted and needed from me was to learn my ways of thinking, and not in fact to learn my proof of the general geometrization conjecture for Haken manifolds. It is unlikely that the proof of the general geometrization conjecture will consist of pushing the same proof further.

A further issue is that people sometimes need or want an accepted and validated result not in order to learn it, but so that they can quote it and rely on it.

Mathematicians were actually very quick to accept my proof, and to start quoting it and using it based on what documentation there was, based on their experience and belief in me, and based on acceptance by opinions of experts with whom I spent a lot of time communicating the proof. The theorem now is documented, through published sources authored by me and by others, so most people feel secure in quoting it; people in the field certainly have not challenged me about its validity, or expressed to me a need for details that are not available.

Not all proofs have an identical role in the logical scaffolding we are building for mathematics. This particular proof probably has only temporary logical value, although it has a high motivational value in helping support a certain vision for the structure of 3-manifolds. The full geometrization conjecture is still a conjecture. It has been proven for many cases, and is supported by a great deal of computer evidence as well, but it has not been proven in generality. I am convinced that the general proof will be discovered; I hope before too many more years. At that point, proofs of special cases are likely to become obsolete.

Meanwhile, people who want to use the geometric technology are better off to start off with the assumption "Let M^3 be a manifold that admits a geometric decomposition," since this is more general than "Let M^3 be a Haken manifold." People who don't want to use the technology or who are suspicious of it can avoid it. Even when a theorem about Haken manifolds can be proven using geometric techniques, there is a high value in finding purely topological techniques to prove it.

In this episode (which still continues) I think I have managed to avoid the two worst possible outcomes: either for me not to let on that I discovered what I discovered and proved what I proved, keeping it to myself (perhaps with the hope of proving the Poincaré conjecture), or for me to present an unassailable and hard-to-learn theory with no practitioners to keep it alive and to make it grow.

I can easily name regrets about my career. I have not published as much as I should. There are a number of mathematical projects in addition to the geometrization theorem for Haken manifolds that I have not delivered well or at all to the mathematical public. When I concentrated more on developing the infrastructure rather than the top-level theorems in the geometric theory of 3-manifolds, I became somewhat disengaged as the subject continued to evolve; and I have not actively or effectively promoted the field or the careers of the excellent people in it. (But some degree of disengagement seems to me an almost inevitable byproduct of the mentoring of graduate students and others: in order to really turn genuine research directions over to others, it's necessary to really let go and stop oneself from thinking about them very hard.)

On the other hand, I have been busy and productive, in many different activities. Our system does not create extra time for people like me to spend on writing and research; instead, it inundates us with many requests and opportunities for extra work, and my gut reaction has been to say 'yes' to many of these requests and opportunities. I have put a lot of effort into non-credit-producing activities that I value just as I value proving theorems: mathematical politics, revision of my notes into a book with a high standard of communication, exploration of computing in mathematics, mathematical education, development of new forms for communication of mathematics through the Geometry Center (such as our first experiment, the "Not Knot" video), directing MSRI, *etc.*

I think that what I have done has not maximized my "credits". I have been in a position not to feel a strong need to compete for more credits. Indeed, I began to feel strong challenges from other things besides proving new theorems.

I do think that my actions have done well in stimulating mathematics.

Notes

[1] Reprinted from "Proof and Progress in Mathematics", by William P. Thurston, *Bulletin of the American Mathematical Society* 30, Number 7, April 1994, pp. 161-177, by permission of the Author.

[2] Arthur Jaffee and Frank Quinn, "Theoretical Mathematics: Towards a Cultural Synthesis of Mathematics and Theoretical Physics", *Bulletin of the American Mathematical Society* 29, Number 1, July 1993.

Technological Pedagogical Content Knowledge in the Mathematics Classroom

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Abstract

Teacher knowledge has long been a focus of many educational researchers. Current conceptualizations of teacher knowledge are beginning to reflect the knowledge and skills teachers need to successfully navigate increasingly technologically-rich mathematical classrooms with the addition of knowledge domains such as technological pedagogical content knowledge (TPACK). This article situates TPACK in the mathematics classroom by developing four central components of knowledge necessary for technology-using mathematics teachers. This article concludes by presenting a portrait of effective TPACK in action and posing questions for technology-using teachers to consider as they embark on technology use in support of mathematics instruction. The intent of this article is not to offer a one-size-fits-all solution to the many issues surrounding technology use, but to provide the impetus for discussion and reflection among mathematics educators at all levels. (Keywords: TPACK, teacher knowledge, technology, mathematics)

Research on teachers' knowledge and beliefs has long remained a substantial area of inquiry in our explorations of the nature of teaching. How teachers' understanding of teaching, learning, students, and subject matter affects their everyday practice is an important aspect of our quest to understand the complex nature of teaching and the professional knowledge necessary for effective teaching. The influence of cognitive psychology on our understanding of learning has resulted in a number of ways to characterize the structure of knowledge in the individual (Borko & Putnam, 1996).

The development and analysis of teachers' formal knowledge by various researchers and educators is an effort to explain various areas of expertise related to teaching and to develop rationale theories on which to base practice (Friedson, 1986). Koehler and Mishra (2008) argue that teachers are autonomous agents with the power to significantly influence appropriate and inappropriate teaching. Thus, an understanding of the knowledge teachers must possess and access in various instructional settings has the potential to impact both teacher training and instructional practices.

With the emergence of technology's integral role in our daily lives and educational landscape at the beginning of the 21st century, researchers have begun to address the impact technology has on teacher knowledge. Although some researchers have begun by looking at the intersection of pedagogy and technology in the development of non-content-specific knowledge domains such as pedagogical technology knowledge (PTK) (Guerrero, 2005), others have examined the intersection of pedagogy and technology in the development of content-specific technological pedagogical content knowledge (TPACK) (Koehler & Mishra, 2008; Koehler, Mishra, & Yahya, 2007; Pierson, 2001).

TPACK

The TPACK (formerly TPCK) framework expands on Shulman's (1986a, 1986b, 1987) conceptualization of pedagogical content knowledge "…to describe how teachers' understanding of technologies and pedagogical content knowledge interact with one another to produce effective teaching with technology" (Koehler & Mishra, 2008, p. 12). In this model (see Figure 1), pedagogical knowledge, content knowledge, and technology knowledge intersect, interact, and influence one another to form and inform not only a teacher's understanding of content, pedagogy, and technology, but also combinations of these three knowledge domains. Together, these multiple knowledge domains intersect in the realm of TPACK to represent

> …an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones. (Koehler & Mishra, 2008, p. 17–18).

In short, TPACK is a rich understanding of how teaching and learning within a specific content area occur and change as a result of authentic, meaningful application of appropriate technologies. "A teacher capable of negotiating these relationships represents a form of expertise different from, and greater than, the knowledge of a disciplinary expert (say a mathematician or historian), a technology expert (a computer scientist), and a pedagogical expert (an experienced educator)" (Koehler, Mishra, & Polly, 2008, p. 1). Such an expert understands how technology influences decisions about content and pedagogy while also recognizing that content and pedagogy influence decisions about and uses of technology. As teachers think about teaching specific concepts, they must concurrently be thinking about how and if technology can be used to make the concept more accessible and understandable to their students. This type of knowledge domain requires deep content knowledge, fluid pedagogical knowledge, and knowledge not only of technology tools, but knowledge about how to teach with these tools.

TPACK and Mathematics

One area that has seen dramatic growth in the influence and applications of technology on the development of content and the evolution of instruction is mathematics. Math continues to evolve as a body of knowledge as technology[1] influences what we know, how we know it, what we teach, and how we teach it. Technology has had considerable impact on the development and expansion of new and existing mathematical concepts and applications in the past few decades. For example, technology has allowed us to apply computer-like algorithms to create, analyze, and recursively define fractals, fragmented geometric shapes, objects, or quantities that are reduced-size copies (or self-similar structures) of the whole. Fractals have emerged as especially useful applications in defining and measuring geographic and meteoric features and phenomenon. Similarly, technology has influenced content development and exploration in areas such as statistics, combinatorics, algebra, probability, geometry, and matrices by providing novice and expert mathematicians increased access, understanding, and application of advanced mathematical concepts through concrete modeling, iterative applications, and recursive functioning (Grandgenett, 2008).

Technology has also had considerable impact on how we think about teaching mathematics, especially at the K–12 level. The National Council of Teachers of Mathematics (NCTM), in its *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), states:

> Electronic technologies—calculators and computers—are essential tools for teaching, learning, and doing mathematics. They furnish visual images of mathematical ideas, they facilitate organizing and analyzing data, and they compute efficiently and accurately.… When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving. (p. 24).

The Association of Mathematics Teacher Educators (AMTE) (2006), in a recent position paper, agreed with the NCTM and further stated that "technology has become an essential tool for doing mathematics in today's world, and thus … it is essential for the teaching and learning of mathematics" (p. 1).

Properly implemented, technology changes how mathematics is taught by allowing teachers and students to focus on deep conceptual understanding over rote procedural skills through problem solving, reasoning, and decision making. There are many ways in which technology can be used to foster this type of mathematical thinking. For example, dynamic software environments, such as Geometer's Sketchpad, Cabri, Fathom, or Tinkerplots, make the exploration of core mathematical concepts tangible and interactive for students. These type of environments allow students to "…build and investigate mathematical models, objects, figures, diagrams, and graphs," (Key Curriculum Press, 2008, para. 1) in ways that bridge the gap between concrete and abstract. Handheld graphing devices allow students, through explorations and applications, to develop a deeper understanding of mathematical concepts and use higher-level approaches to solve mathematical problems. Handhelds also promote assimilation between mathematical concepts and their multiple representations (e.g., functions and their graphical, tabular, and symbolic representations). Wireless network technologies, such as the TI Navigator, promote improved student engagement, understanding, and performance by allowing for real-time tracking of student progress, collaborative lesson engagement, and instant feedback. Finally, virtual learning environments actively involve students in interactive mathematics instruction. Students are able to manipulate "physical" objects to visualize relationships and applications, form and test conjectures, and connect abstract concepts to concrete representations.

As researchers continue to pay attention to pedagogically appropriate uses of technology in professional development and classroom settings (e.g.,

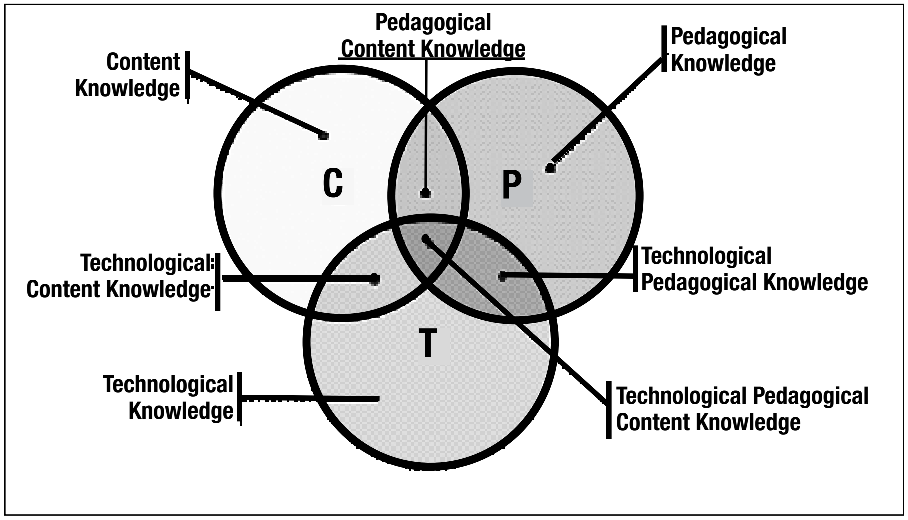

Figure 1. TPACK framework (source: www.tpck.org).

1 Here and throughout this discussion, the term technology refers to contemporary instructional and learning technologies, typically available in desktop or handheld form, that engage teachers and students in the teaching/learning process by promoting "…interactivity, multimodality, various new forms of communication, access to expertise, new varieties of resources, opportunities for stimulation, enhanced productivity, and so on" (Herman, 1994, p. 133).

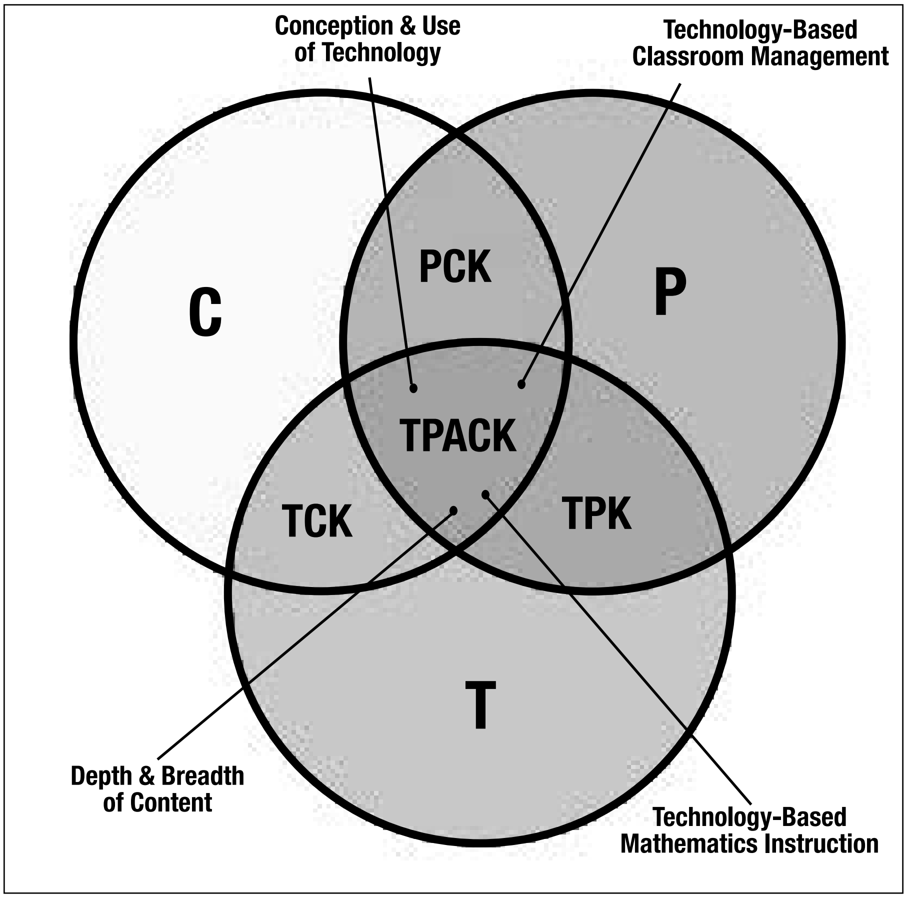

Figure 2: Four central components of mathematics-related TPACK and their derivation from pedagogical content knowledge (PCK), technological pedagogical knowledge (TPK), and technological content knowledge (TCK).

Means, 1994), theoretical development of content-specific technology-related knowledge domains continue to emerge. This article adds to the development of mathematical TPACK by identifying key components of technology use in the mathematics classroom, demonstrating robust TPACK through a classroom vignette, and addressing instructional implications of technology use through the development of questions to guide the technology-using mathematics teacher. The skills and knowledge related to technology use in the mathematics classroom are very similar to those referred to in other knowledge domains, and yet they are vastly different because of their reliance on technology as an instructional and educational tool. Technology has the unique characteristic of being an instructional tool that requires a specific set of operational skills while also necessitating a diverse array of instructional strategies and procedures.

Components of Mathematical TPACK

The knowledge needed to effectively employ technology as part of mathematics instruction includes technology-specific management, instructional, and pedagogical knowledge; increased mathematics subject-matter knowledge; and knowledge of when and how best to use technology to support mathematics instruction. TPACK in mathematics goes beyond knowledge of learning a technology tool and its operation, per se, to the dimension of how to operate a piece of technology to improve mathematics teaching and learning. Although this knowledge includes learning the basic operational skills, it embodies the aspects of technology most relevant to its capacity for use in instruction to improve teaching and learning. Nowhere are these intricacies of technology's effect on content and instruction more varied and applicable than in the mathematics classroom, where technology has the potential to change not only what we teach but how we teach it.

This article argues that TPACK in mathematics can be characterized by four central components (see Figure 2). The first component, a teacher's conception and use of technology, relates technology to pedagogical content knowledge by focusing on how the teacher can use technology to make the subject matter more comprehensible and accessible to students. The next two components of TPACK include elements associated with general pedagogical knowledge. These components encompass the general principles of instruction, organization, and classroom management specific to the application of technology in the mathematics classroom. The final component of TPACK relies on a teachers' subject-matter knowledge and deals with the increased responsibility teachers have to understand their content areas with both breadth and depth as a result of using technology as part of their instruction.

It should be noted that the number of TPACK components should not be thought of as a one-to-one correspondence with the subdomains of knowledge (i.e., PCK, TCK, and TPK) presented in the TPACK model. Rather, it is the result of careful analysis of seminal factors involved in technology-related pedagogy and subject matter specific to the content area under discussion. Consequently, this mathematical TPACK model posits one component related to each of PCK and TCK and two components related to TPK. This is not to say that TPK carries a greater weight or more importance in the development of TPACK in mathematics, but the issues related to instruction and management warranted their own consideration. Other content areas or conceptualizations of TPACK may include greater or fewer components, depending on the roles and interactions of technology, content, and pedagogy in the development of TPACK. As mentioned previously, the way that technology influences both pedagogy and content in mathematics is vastly different than the ways it may influence other content areas, so core components of TPACK may differ between mathematics and other content areas.

Conception and Use of Technology

The first component of TPACK includes a teacher's overarching conception of the use of technology in support of teaching and learning mathematics. It includes what the teacher believes about mathematics as a field, how he or she feels mathematics can best be addressed through the use of technology, and what is important for students to learn about mathematics through the use of technology. This component serves as the basis for the instructional and curricular decisions teachers make in rendering the subject matter more accessible to students.

The teacher must decide how to best use technology (if at all) to address the needs of the students, the content, and instruction and then decide which technology best accomplishes all these goals. Is exploratory data analysis (EDA) best taught through the use of spreadsheets, graphing calculators, or one of the many statistical programs available for all levels of students? There is no single correct answer to this question; the answer will depend largely on the skills of the teacher with various types of technology; the specific topic he or she is investigating within EDA; and the students' skills, needs, motivations, and prior understanding.

Most important, this component includes the knowledge of how to use technology in pedagogically appropriate ways that support instruction authentically rather than as a side-show tool. Although researchers are able to identify what is not "pedagogically appropriate," such as using technology for drill-and-skill or rote computation, they are less willing to specify exact elements of what appropriateness looks like. Because so much of appropriate technology use depends on the specific needs and nuances of each student, class, and teacher, pedagogically appropriate use of technology may vary from classroom to classroom. However, most researchers agree that, in general, pedagogically appropriate use implies a seamless integration of technology that promotes inquiry, reasoning, contextualized learning, and sense-making (Guerrero, 2005).

In mathematics, pedagogically appropriate uses of technology encompass the ways technology is used to represent and formulate mathematics so that it is understandable to students through the most useful representations, demonstrations, examples, and applications. For example, when teaching fractals, how should a teacher use technology to represent them and most effectively demonstrate their applications? Would an online virtual manipulative demonstrate their creation most efficiently? Would a technical computing environment make investigation into their iterative nature more accessible? Would dynamic geometry software demonstrate their applications most effectively? Of course, the answer depends on the exact nature of the content being covered, the instructional objectives, which tools are available to the teacher, and the tools with which he or she is most skilled.

As there is no single technological tool that is best for all instructional purposes, teachers must also be aware of the growing variety of tools available for their mathematics instruction. Within the mathematics classroom, this includes a thorough understanding of the uses and applications of, among other things, graphing calculators and their various programs and applications; data collection devices such as calculator-based rangers (CRBs), computer-based laboratories (CBLs), and other related scientific probes; spreadsheets; statistical software programs; dynamic geometry software programs; mathematical modeling and technical computing environments; online dynamic manipulatives; and the burgeoning use of networking tools such as TI Navigator. Though cost, time, and other factors limit teachers' instructional decisions, especially where technology is concerned, technology-using teachers must have a solid foundation for the decision of choosing one technology tool over another. Teachers must possess not only the knowledge of how to use the various features of each of these technologies, but also a thorough conceptualization of when and how to use them as instructional tools.

Technology-Based Mathematics Instruction

The second component of TPACK includes teachers' knowledge of and ability to maneuver through various instructional issues specifically related to the use of technology in support of mathematics teaching and learning. From this point of view, teachers need to understand that technology should be viewed as one instructional tool among many. It does not replace the teacher or any type of instruction, but should be included as part of a teacher's instructional repertoire.

Technology's success as a learning and instructional tool depends upon it being integrated into a meaningful curricular and instructional framework, and it should be used only when it is the most appropriate means of reaching an instructional goal (Sandholtz, Ringstaff, & Dwyer, 1997). Is an investigation of symmetry better suited to miras, geoboards, or dynamic geometry software? Then, once technology has been selected as the tool of choice, the teacher must decide which of various types of technology are best suited to the learning objectives and content of a given lesson. Is an investigation of linear functions better suited to graphing calculators, spreadsheets, virtual manipulatives, or dynamic geometry software?

Also included in this component is the teacher's ability to orchestrate the classroom environment in light of new demands and opportunities created by the use of technology. With the flourishing number of mobile and networking technologies, such as SmartBoards, interactive slates, and the TI Navigator system, teachers must have the ability to manage collaborative inquiry and share control of the technology with students and among students (Goss, Renshaw, Galbraith, & Geiger, 2000). As such, technology-using mathematics teachers need to be aware of and comfortable with a didactic shift in attention from them to the topic the class is exploring, and in their role as part ofteacher-directed instruction versus student-centered collaboration. They also need to recognize that technology may disrupt their instructional plans by uncovering insights into new and unexpected areas, and teachers should be comfortable with adapting to and making spontaneous changes in instruction. Now, more than ever, teachers need to be comfortable with and knowledgeable enough to go with that "teachable moment." By the same token, though, teachers should be aware of their responsibility to set boundaries on how far students can and should go in their investigations and individual work.

Finally, this instructional component of TPACK includes the ability to adjust the use of technology to serve the needs of a diverse array of students in terms of mathematical ability, affect, and interest. Just as students may lose interest in an assignment when it is too easy or too difficult, they may lose interest in using technology when its use does not take their personal needs into account (Sandholtz, Ringstaff, & Dwyer, 1997). Technology has long been used as a remediation tool, but some technology tools, such as the TI Navigator system, may actually make modified instruction to serve individual needs even easier by allowing the teacher to send, via a classroom network, different sets of students different sets of problems, instructions, or tasks.

Management

The third component of TPACK in mathematics covers management issues specifically related to teaching and learning with technology. The use of technology in instruction introduces a number of management variables and issues that teachers seldom encounter when their instruction does not use technology. Included here is a teacher's understanding of how to handle students' attitudes toward technology and their behavior as a result of using technology. How does one deal with issues such as students sending games and messages via graphing calculators, abusing data gathering probes, or using computers as physical shields to hide off-task behavior? On the other hand, teachers need to understand that some technologies may actually make management of instruction and behavior easier by providing constant access to student activity, progress, and understanding. The TI Navigator system allows teachers to grab screenshots of student work on the calculator at any point in time and provides teachers with instantaneous, formative assessment capabilities at any point in the lesson.

Management also encompasses teachers' understanding of how to deal with the physical environment (e.g., lighting, glare, setup of equipment, physical layout of room) and technical problems (e.g., broken hard drives, jammed printers, network problems, software restrictions, worn-out batteries) that arise as a result of using technology. Although ability to deal with such logistical elements of technology use improves over time, early on there is often a steep learning curve associated with managing all the physical and technical aspects of various technology tools in the mathematics classroom.

A final element of the management component of TPACK is a teacher's ability to maintain student engagement once the novelty effect has worn off. The use of technology has been shown to have positive effects on student attitudes, on-task behavior, initiative, engagement, and experimentation, but when used too often, too infrequently, or inappropriately, it can also result in student frustration, boredom, distraction, and unwillingness to transition to other activities (Sandholtz, Ringstaff, & Dwyer, 1997). As with any instructional tool, teachers need to know when and how to use technology to provide students with the most authentic learning environment possible. Using technology with every activity and for every instructional purpose is just as futile as using direct instruction for every topic and lesson.

Depth and Breadth of Mathematics Content

The fourth component of TPACK takes into account the increased responsibility teachers have to understand their mathematics in breadth and depth. Placing technology in the hands of students gives them the power to explore math to a depth that may be unfamiliar to the teacher (Goss et al., 2000). As a result, teachers need to be confident in their ability to handle students' investigations and inquiries. As with instructional flexibility, depth in content knowledge provides teachers with the ability and flexibility to explore, emphasize, or de-emphasize various mathematical topics that may arise in the course of instruction and investigation. When a student, using a graphing calculator, discovers an interesting fact about the slope of a tangent line while graphing quadratics, the teacher must decide if the findings are relevant and worth pursuing or tangential and best left alone. This requires content knowledge of not only functions and derivatives, but also a broader understanding of mathematics, the mathematics curricula, and where/how derivatives fit into the scope and sequence of both.

Along with having an extremely strong knowledge base in their subject matter, teachers must also possess a willingness to acknowledge their own subject-matter shortcomings. As a result of the depth and breadth of content that can be explored through technology, teachers need to understand that students may encounter topics and ideas that teachers may be unprepared to manage or address. In lieu of understanding every possible avenue a student's investigation and insight may take, the teacher needs to be able to acknowledge that they are unsure of a student's discovery, comments, or questions and must be willing to invest the time and energy to investigate these various content trails on their own.

A Portrait of TPACK in Action

Perhaps one of the best ways to grasp the complexity of TPACK in action is to examine each of the four components in the context of one teacher's technology use. Barbara, a secondary mathematics teacher at a rural high school in central California, has been teaching for 18 years and recently moved from the middle grades to her new position at her district's only high school. She has long been a proponent of technology in support of mathematics teaching and learning and spearheaded a controversial effort to require graphing calculators while at the middle school. She was an active participant in a long-term technology professional development program within the state and continues to be an ardent proponent of technology at all grade levels in her district. Barbara's technology use is, for lack of a better word, fluid. She demonstrates robust mathematical TPACK through her conceptions and use of technology, her technology-based math instruction and management skills, and her depth and breadth of mathematical content knowledge.

Conception and Use of Technology

Barbara is attracted to technology as an instructional tool because of its potential to improve her students' learning and depth of understanding. She uses technology because she feels it allows students to learn about "current math" and to learn math with more depth. She feels that technology can be applied in ways that make learning math more meaningful for the students and that technology allows her and her students to do things they would not be able to do otherwise. Technology enables her students to "see the math," make connections, and go more in depth with increased understanding. Barbara reflects her beliefs in the benefits of technology through the decisions she makes about using and implementing technology. In her own words, she sees the "big picture" and believes technology provides students with useful experiences that connect mathematics to their daily lives, and prepares them for a technologically savvy "real world."

Technology-Based Mathematics Instruction

Most of the decisions Barbara makes regarding technology use are aimed at making technology a natural part of the learning process rather than an object of study in and of itself. "I am just continually looking for how to make it seamless between teaching the math," she explains. Despite her obvious enthusiasm for and commitment to technology as an instructional tool, Barbara believes that students should be taught to think about technology as one of many tools in their mathematical repertoire. "It's a tool like anything else, like spell check or like pencils or using compasses," she says. For Barbara, technology should be used with, rather than blindly replace, other tools, such as mental math and manual computation with paper and pencil.

When she uses technology to teach a topic, it is clear that Barbara has carefully planned the lesson and knows where she wants it to end up. However, she is open to using student input to guide the nuances of how the lesson will get there. Barbara often uses questioning to guide students in a whole-class discovery-based discussion focusing on real-world theme problems that often span the course of several days. Barbara believes that students learn best when they see the connections within mathematics and the real world. For Barbara, technology is crucial to helping students make these connections by providing them with hands-on experiences and instant visuals, reinforcing concept links, and connecting math to real-world applications. In her attempts to make technology use seamless, Barbara plans her lessons so that students learn the technology as they learn the mathematics. As she introduces new material, she reinforces old technology-related skills and integrates new skills.

Depth and Breadth of Content

In a typical lesson, Barbara gives students a focus problem at the beginning of class, and, though the use of various questioning and discussion techniques, she gives students instructions for setting up their calculators and guides them through the exploration of a mathematical topic. These lessons are very interactive and involve a lot of give and take between Barbara and her students. These "theme problems" often run the course of several days. For example, students were investigating the rate of change in millimeters of the lean on the Tower of Pisa over the last 100 years. Although the problem started out as a theme problem that the class would pursue through two days of data analysis, it expanded into a problem that had students researching Tower of Pisa facts at home, arguing about an outlying point created through human interference versus natural causes, and wanting to explore center of gravity so they could determine when the tower would fall if its lean continued increasing at the current rate. This led to mini research projects, investigation of statistical outliers and their meaning, and construction of models to replicate "lean and fall" scenarios.

In follow-up interviews, Barbara mentioned that the problem had "morphed out of control" in several different directions than the one she intended, but she was going with what the students wanted to do because they were engaged, exploring some really rich content, and getting at the heart of some really teachable moments. Students were, of their own volition, asking about and investigating rich crosscurricular content areas that demonstrated not only true interest, but depth of understanding. She believed that engaging them through their own questions was well worth the risk of not necessarily knowing the outcome. Though she did not know the answer to every question the students asked, Barbara demonstrated flexibility in her willingness to explore content areas beyond her immediate grasp.

Management

Because of her extensive background with all types of technology, Barbara is very comfortable and proficient with technology, likes to try new things, and is willing to take the risk of experimenting in front of her students. She has become an expert at troubleshooting graphing calculators in a multi-platform setting. Although she now uses TI graphing calculators in her teaching, some of her students still own and use Casios that were required when they were in middle school. Barbara is able to move effortlessly from helping a TI student to helping a Casio student. Because she gives instructions only on the TI, Barbara often pairs her few Casio students with one another and frequently checks in with them to make

Table 1: Questions to Guide the Development and Use of Technology by Analyzing Each Component of TPACK

Component	Question
Instruction	Is technology the most appropriate instructional tool for teaching and learning this topic? How will technology affect the collaborative nature of my classroom? Will I be able to adapt my instruction based on student feedback, progress, and/or inquiry with technology? How will I adjust my instruction and the use of technology to meet individual student needs?
Management	How will I manage the physical logistics of technology? Where will we use technology? How many students per computer/calculator? Can I troubleshoot technical and/or application problems? How do I manage student progress and behavior? How do I encourage and maintain student engagement with technology-based lessons?
Depth and Breadth of Content	Do I have the mathematical knowledge to handle student inquiries that may take us beyond the intent of this lesson? Am I willing to acknowledge my own content-related shortcomings and invest the time and energy to investigate student-generated "content trails"?
Conception and Use of Technology	Is this topic best addressed through the use of technology? If so, how? What should students learn about this topic through the use of technology? How does technology improve teaching and learning of this topic? Is the use of technology in this lesson pedagogically appropriate? Do I have the skills to operate, navigate, and apply the various features of mathematics-related technology tools? Is technology fully integrated into this lesson or an add-on? Which technology will best support teaching and learning of a specific topic?

sure they are able to follow along. By her own account, part of her success at troubleshooting comes from her strict protocol for handling technology. She tells students not to touch the calculator screens, not to use anything other than their fingers to push the buttons, and to place their calculators on their desks at all times. "I don't know how much difference it makes," she admits. "It's just that I am more comfortable staying focused on math if I can walk around the room and see what they are doing, and I can troubleshoot faster."

Barbara exemplifies a teacher with comprehensive TPACK through her balanced use of technology as one tool within her instructional repertoire and her grounded beliefs that it is her responsibility to help prepare students for the tech-savvy real world. She firmly believes in the benefits of technology for making content accessible to her students but uses it within a meaningful curricular framework. Her teaching emphasizes collaborative inquiry through student-centered discussions and activities and often focuses on larger theme problems that run the course of several days. She has a solid background in mathematics and, though she clearly has instructional and content goals in mind, she is willing to explore unfamiliar areas if that is where student inquiries take a lesson.

Discussion

When choosing to use technology as part of their instructional repertoire, teachers must understand elements and implications of technology use related to instruction, management, content, pedagogy, and technology itself. Though Barbara's illustration provides an example of one teacher's use and conception of technology, the journey toward becoming such an authentic technology-using teacher takes time, energy, and commitment. Whether a novice technology user or a more experienced one such as Barbara, technology-using teachers are continually changing and growing in their conceptions and use of technology.

Questions, such as those provided in Table 1, provide a springboard for discussion and reflection centered on each of the four components of TPACK. These questions prompt theoretical and practical deliberation by both experienced and novice teachers, individuals and groups engaged in a technology-based change process, and teacher educators. Teachers will address various components of TPACK in different ways and must rely on their own expertise to begin thinking about some of the theoretical aspects of the application of technology in support of mathematics instruction.

Conclusion

Development and understanding of TPACK, especially as it relates to specific content areas, is imperative because of the importance of technology's appropriate use in educational settings. If technology is to influence teachers' practices in reform-oriented ways and improve students' learning by having a positive impact on engagement, achievement, and confidence, it must be successfully integrated into instruction in effective, authentic, and nonroutine ways. Ensuring technology's proper use in educational settings requires the development and understanding of the characteristics of teachers' technological pedagogical content knowledge base.

This article has attempted to address some of the central components of practical TPACK for the mathematics classroom. These include, but are not limited to, conception and use of technology; technology-based mathematics instruction and management; and depth and breadth of mathematics content. Although implicit in this knowledge base is facility with basic operational skills for various types of technology, TPACK most notably embodies the aspects of technology most relevant to its ability to be used as part of a teacher's instructional repertoire to improve teaching and learning. The working technology knowledge of a mathematics teacher using graphing calculators, computer software programs, and computer-based laboratories to deeply explore a mathematical topic is vastly different than that of an English teacher using the Internet and software programs to investigate and prepare literary documents. Each content area has specific instructional goals and needs that technology can address in a variety of ways. TPACK embodies a teacher's ability to distinguish between the types of technology that are most suited to their content area and make decisions regarding its appropriate application.

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ActiveMath: An Intelligent Tutoring System for Mathematics

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Abstract. ActiveMath is a web-based intelligent tutoring system for mathematics. This article presents the technical and pedagogical goals of ActiveMath, its principles of design and architecture, its knowledge representation, and its adaptive behavior. In particular, we concentrate on those features that rely on AI-techniques.

1 Introduction

Intelligent tutoring systems (ITSs) have been researched in AI now for several decades. With the enormous development and increasing availability of the Internet, the application of web-based learning systems becomes more likely and realistic and research for intelligent features receives more attention than before. As a result, a number of new ITS have been developed over the last five years, among them ActiveMath, a web-based, adaptive learning environment for mathematics.

These systems strive for improving long-distance learning, for complementing traditional classroom teaching, and for supporting individual and life-long learning. Web-based systems are available on central servers and allow a user to learn in her own environment and whenever it is appropriate for her.

Intelligent tutoring systems are a great field of application for AI-techniques. In a nutshell, our research for ActiveMath has used and further developed results in

- problem solving
- rule-based systems
- knowledge representation
- user modeling **?**
- adaptive systems and adaptive hyper-media
- diagnosis.

Learning environments have to meet realistic and complex needs rather than being a specific research system for specific demonstrations such as the famous blocksworld. Therefore, we point out important pedagogical and technical goals that our research for ActiveMath had to satisfy.

Pedagogical Goals

ActiveMath' design aims at supporting truly interactive, exploratory learning and assumes the student to be responsible for her learning to some extent. Therefore, a relative freedom for navigating through a course and for learning choices is given and by default, the student model is scrutable, i.e., inspectable and modifiable. Moreover, dependencies of learning objects can be inspected in a dictionary to help the student to learn the overall picture of a domain (e.g., analysis) and also the dependencies of concepts.

Several dimensions of adaptivity to the student and her learning context improve the learner's motivation and performance. Most previous intelligent tutor systems did not rely on an adaptive choice of content. A reason might be that the envisioned use was mostly in schools, where traditionally every student learns the same concepts for the same use. In colleges and universities, however, the same subject is already taught differently for different groups of users and in different contexts, e.g., statistics has to be taught differently for students of mathematics, for economics, or medicine. Therefore, the adaptive choice of content to be presented as well as examples and exercises is pivotal. In addition, an adaptation of examples and exercises to the student's capabilities is highly desirable in order to keep the learner in the zone of proximal development [13] rather than overtax or undertax her.

Moreover, web-based systems can be used in several learning contexts, e.g., long-distance learning, homework, and teacher-assisted learning. Personalization is required in all of them because even for teacher-assisted learning in a computer-free classroom with, say, 30 students and one teacher individualized learning is impossible. ActiveMath's current version provides adaptive content, adaptive presentation features, and adaptive appearance.

Technical Goals

Building quality hyper-media content is a time-consuming and costly process, hence the content should be *reusable* in different contexts. However, most of today's interactive textbooks consist of a collection of predefined

documents, typically canned `HTML` pages and multimedia animations. This situation makes a reuse in other contexts and a re-combination of the encoded knowledge impossible and inhibits a radical adaptation of course presentation and content to the user's needs.
ACTIVEMATH' knowledge representation contributes to re-usability and interoperability. In particular, it is compliant with the emerging knowledge representation and communication standards such as Dublin Core, `OpenMath`, `MathML`, and LOM[1]. Some of the buzzwords here are metadata, ontological XML, (`OMDoc` [8], and standardized content packaging. Such features of knowledge representations will ensure a long-term employment of the new technologies in browsers and other devices. In order to use the potential power of existing web-based technology e-Learning systems need an open architecture to integrate and connect to new components including student management systems such as WebCT, assessment tools, collaboration tools, and problem solving tools.

Organization of the Article This article provides an overview of the current ACTIVEMATH system. It describes some main features in more detail, in particular, the architecture and its components, the knowledge representation, the student model and the adaptation based on the information from the student model.

2 Architecture

The architecture of ACTIVEMATH, as sketched in Figure 1, strictly realizes the principle of separation of (declarative) knowledge and functionalities as well as the separation of different kinds of knowledge. For instance, pedagogical knowledge is stored in a pedagogical rule base, the educational content is stored in MBase, and the knowledge about the user is stored in the student model. This principle has proved valuable in many AI-applications and eases modifications as well as configurability and reuse of the system.

ACTIVEMATH has a client-server architecture whose client can be restricted to a browser. This architecture serves not only the openness but also the *platform independence*. On the client side, a browser – netscape higher than 6, Mozilla, or IE with MathPlayer – is sufficient to work with ACTIVEMATH. On the server-side components of ACTIVEMATH have been deliberately designed in a *modular* way in order to guarantee exchangeability and robustness.

[1] `http://ltsc.ieee.org/wg12/`

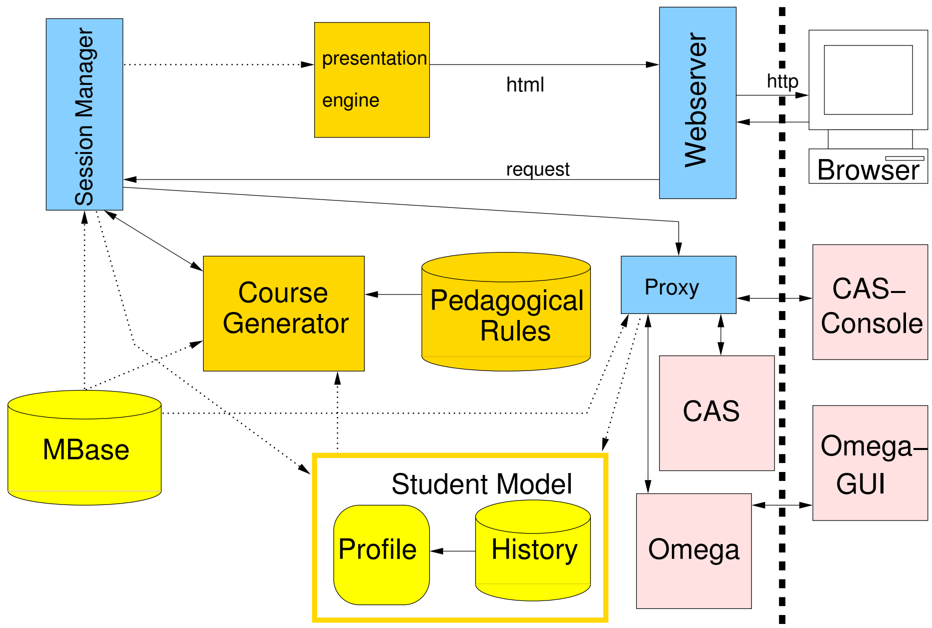

Fig. 1. Architecture of ActiveMath

When the user has chosen her goal concepts and learning scenario, the session manager sends a request to the *course generator*. The course generator is responsible for choosing and arranging the content to be learned. The course generator contacts the *mathematical knowledge base* in order to fetch the identifiers (IDs) of the mathematical concepts that are required for understanding the goal concepts, queries the student model in order to find out about the user's prior knowledge and preferences, and uses *pedagogical rules* to select, annotate, and arrange the content – including examples and exercises – in a way that is suitable for the learner. The resulting instructional graph, a list of IDs, is sent to the *presentation engine* that retrieves the actual mathematical content corresponding to the IDs and that transforms the `XML`-data to output-pages which are then presented via the user's browser.

The *course* generator and the suggestion mechanism [10] work with the rule-based system Jess [6] that evaluates the (pedagogical) rules in order to decide which particular adaptation and content to select and which actions to suggest. Jess uses the Rete algorithm [5] for optimization.

External systems such as the computer algebra systems Maple and MuPad and the proof planner Multi are integrated with ActiveMath.

They serve as cognitive tools [9] and support the learner in performing complex interactive exercises and they assist in producing feedback by evaluating the learner's input.Also, a diagnosis is passed to the student model in order to update the model.

In these exercises, ActiveMath does not necessarily guide the user strictly along a predefined expert solution. It may only evaluate whether the student's input is mathematically equivalent to an admissible subgoal, i.e., maybe irrelevant but not outside the solution space (see [3]). Moreover, the external systems can support the user by automated problem solving, i.e., they can take over certain parts in the problem solving process and thereby help the user to focus on certain learning tasks and to delegate routine tasks.

Actually, most diagnoses are known to be AI-hard problems. Most ITSs encode the possible problem solving steps and the most typical misconceptions into their solution space or into systems that execute them. From this encoding, the system diagnoses the misconception of a student. This is, however, (1) infeasible in realistic applications with large solution spaces and (2) it is in general impossible to represent all potential misconceptions of a student [17].

The *presentation engine* generates personalized web pages based on two frameworks: Maverick and Velocity. Maverick[2] is a minimalist MVC framework for web publishing using Java and J2EE, focusing solely on MVC logic. It provides a wiring between URLs, Java controller classes and view templates.

The presentation engine is a reusable component that takes a structure of `OMDocs` and transforms them into a presentation output that can be PDF (print format) or `HTML` with different maths-presentations – Unicode or `MathML` – (screen format) [7]. Basically, the presentation pipeline comprises two stages: stage 1 encompasses Fetching, Pre-Processing and Transformation, while stage 2 consists of Assembly, Personalization and optional Compilation. Stage 1 deals with individual content fragments or items, which are written in `OMDoc` and stored in a knowledge base. At this stage, content items do not depend on the user who is to view them. They have unique identifiers and can be handled separately. It is only in stage 2 that items are composed to user-specific pages.

[2] Maverick: `http://mav.sourceforge.net/`

3 Adaptivity

ActiveMath adapts its course generation (and presentation) to the student's

- technical equipment (customization)
- environment variables, e.g., curriculum, language, field of study (contextualization) and
- her cognitive and educational needs and preferences such as learning goals, and prerequisite knowledge (personalization).

As for personalization, individual preferences (such as the style of presentation), goal-competencies, and mastery-level are considered by the course generator. On the one hand, the goal-competencies are characterized by concepts that are to be learned and on the other hand, by the competency-level to be achieved: knowledge (k), comprehension (c), or application (a). The learner can initialize her student model by self-assessment of her mastery-level of concepts and choose her learning goals and learning scenario, for instance, the preparation for an exam or learning from scratch for k-competency level. The course generator processes this information and updates the student model and generates pages/sessions as depicted in the screenshots of Figure 2 and 3. These two screenshots differ in the underlying scenarios as the captions indicate.

Adaptation to the capabilities of the learner occurs in course generation as well as in the suggestion mechanism. The course generation checks whether the mastery-level of prerequisite concepts is sufficient for the goal competency. If not, it presents the missing concepts and/or explanations, examples, exercises for these concepts to the learner when a new session is requested. The suggestion mechanism acts dynamically in response to the student's activities. Essentially, this mechanism works with two blackboards, a diagnosis blackboard and a suggestion blackboard on which particular knowledge sources operate.

We also investigated special scenarios that support a student's metacognitive activities, such as those proposed in the seminal book 'How to Solve it' by Polya [15]. A Polya-scenario structures the problem solutions by introducing headlines such as "understand the problem", "make a plan", "execute the plan", and "look back at the solution". It augments and structures exercises with additional prompts similar to the above headlines [12].

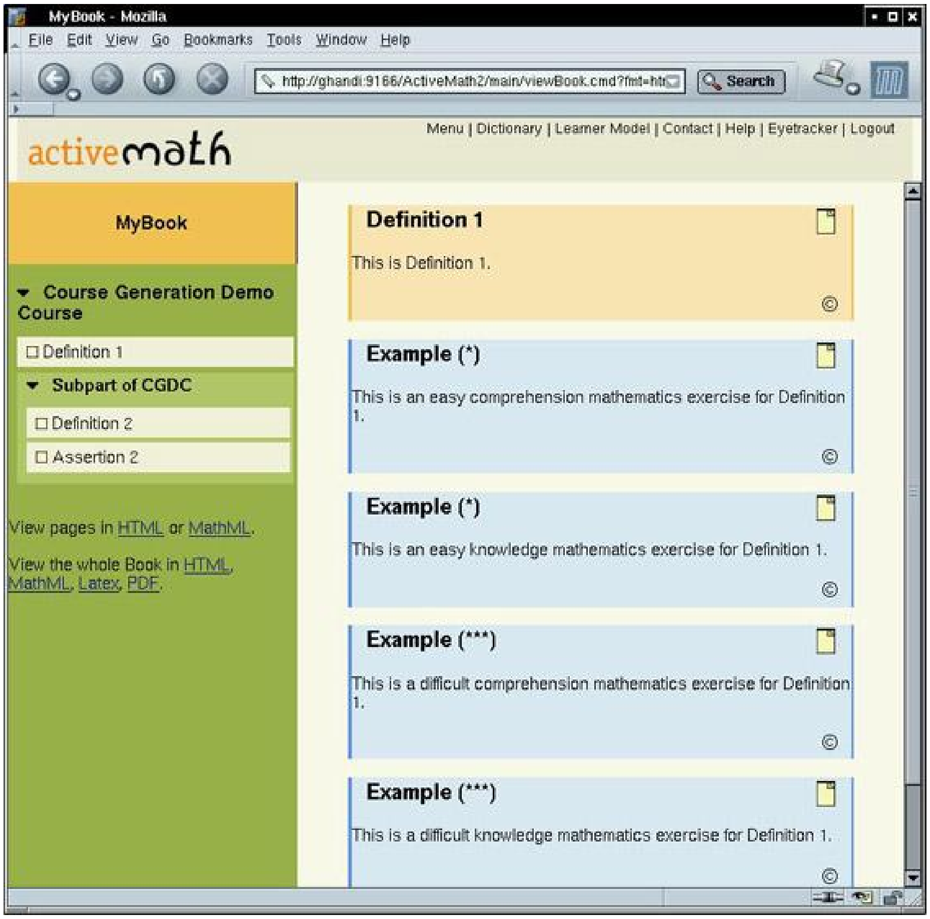

Fig. 2. A screen shot of an ActiveMath session for exam preparation

4 Student Modeling

User modeling has been a research area in AI for a long time. Actually, it started with early student modeling and still continues with the investigation of representational issues as well as diagnostic and updating techniques.

As ActiveMath' presentation is user-adaptive, it needs to incorporate persistent information about the user as well as a representation of the user's learning progress. Therefore, 'static' (wrt. the current session) properties such as field, scenario, goal concepts, and preferences as well as the 'dynamic' properties such as the mastery values for concepts and the student's actual behavior, are stored in the student model. These dif-

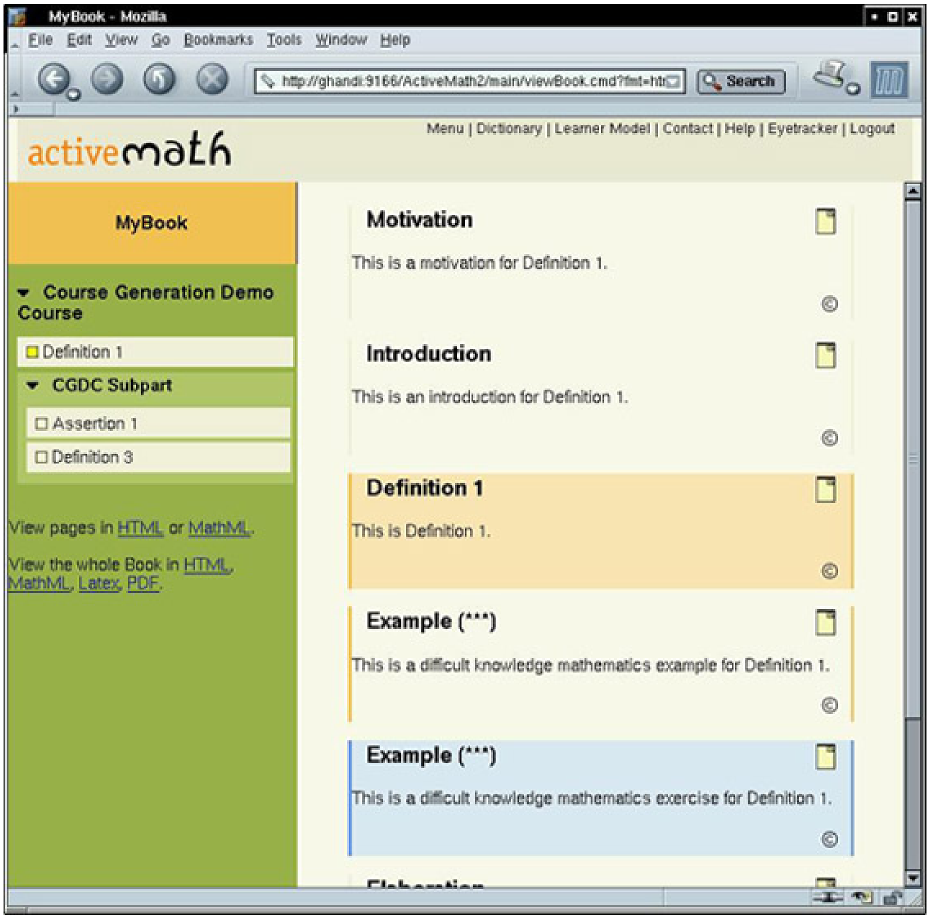

Fig. 3. k-level session of ActiveMath

ferent types of information are stored separately in the *history* and the static and dynamic *profiles*.

The profile is initialized with the learner's entries submitted to ActiveMath' registration page which describe the preferences (static), scenario, goals (static for the current session), and self-assessment values for knowledge, comprehension, and application of concepts (dynamic).

The history component stores the information about the learner's actions. Its elements contain information such as the IDs of the content of a read page or the ID of an exercise, the reading time, and the success rate of the exercise. Meanwhile, we developed a "poor man's eye-tracker" which allows to trace the student's attention and reading time in detail.

To represent the concept mastery assessment, the current (dynamic) profile contains values for a subset of the competences of Bloom's mastery taxonomy [2]:

- Knowledge
- Comprehension
- Application.

Finishing an exercise or going to another page triggers an updating process of the student model. Since different types of learner actions can exhibit different competencies, reading a concept mainly updates 'knowledge' values, following examples mainly updates 'comprehension', and solving exercises mainly updates 'application'. When the student model receives the notification that a student has finished reading a page, an evaluator fetches the list of its items and their types (concept, example, ...) and delivers an update of the values of those items. When the learner finishes an exercise, an appropriate evaluator delivers an update of the values of the involved concepts that depends on the difficulty and on the rating of how successful the solution was.

The student model is inspectable and modifiable by the student as shown in Figure 4. Our experience is that students tend to inspect their student model in order to plan what to learn next.

5 Knowledge Representation

As opposed to the purely syntactic representation formats for mathematical knowledge such as LaTex or `HTML`, the knowledge representation used by ActiveMath is the *semantic* `XML`-language `OMDoc` which is an extension of `OpenMath` [4]. `OpenMath` provides a collection of `OpenMath` objects together with a grammar for the representation of mathematical objects and sets of standardized symbols (the content-dictionaries). That is, `OpenMath` talks about objects rather than syntax.

Since `OpenMath` does not have any means to represent the content of a mathematical *document* nor its structure, `OMDoc` defines logical units such as “definition”, “theorem”, and “proof”. In addition, the purely mathematical `OMDoc` is augmented by educational metadata such as difficulty of a learning object or type of an exercise.

This representation has several advantages, among them

- it is human and machine understandable
- the presentation of mathematical objects can in principle be copied and pasted

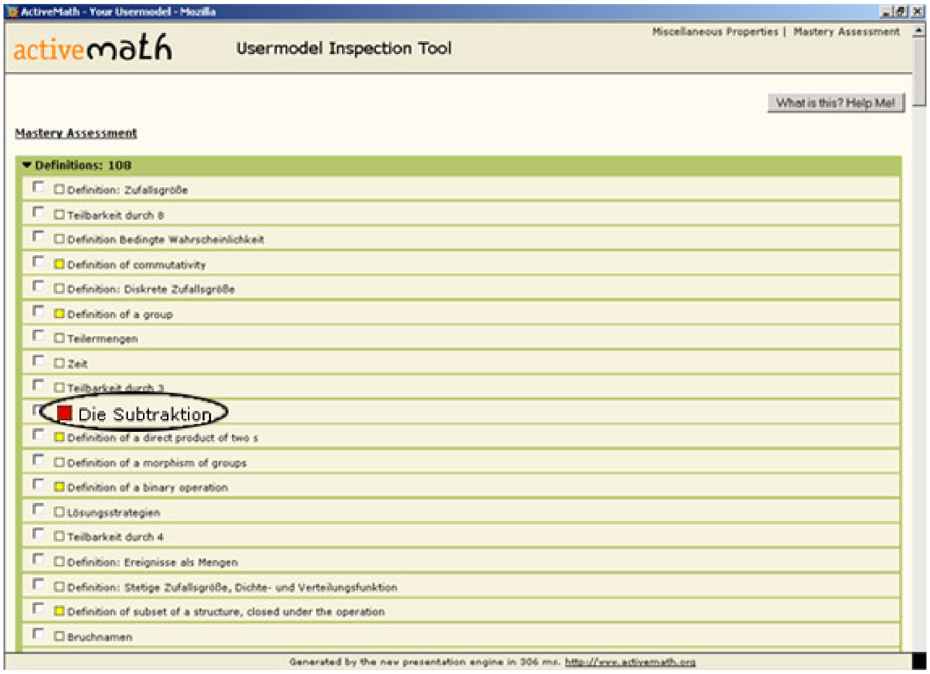

Fig. 4. Inspection of the student model (mastery-level)

– the presentations can automatically and dynamically be linked to concepts and learning objects and thus, found by ActiveMath's dictionary when clicking on a concept or formula in the course.
  For more details, see [11].

6 Conclusion, Related and Future Work

The intelligent tutoring systems group – at the DFKI Saarbrücken and at the University of Saarland – has been developing the web-based ITS ActiveMath now for several years. A demo (and demo guide) is available at `http://www.activemath.org`.

This system is configurable with pedagogical strategies, content, and presentational style sheets as well as with external problem solving systems. It employs a number of AI-techniques to realize adaptive course generation, student modeling, feedback, interactive exercises, and a knowledge representation that is expedient for the semantic Web.

Related Work Most web-based learning systems (particularly commercial ones) offer fixed multimedia web pages and facilities for user management and communication and most of them lack support for truly interactive problem solving and user-adaptivity. Moreover, they use proprietary knowledge representation formats rather than a standardized knowledge representation which is exchangeable between systems. Some user-adaptivity is offered by systems such as ELM-ART [18] and Metalink [14].

During the last decades research on pedagogy in the mathematics recognized that students learn mathematics more effectively, if the traditional rote learning of formulas and procedures is supplemented with the possibility to explore a broad range of problems and problem situations [16]. In particular, the international comparative study of mathematics teaching, TIMSS [1], has shown (1) that teaching with an orientation towards active problem solving yields better learning results in the sense that the acquired knowledge is more readily available and applicable especially in new contexts and (2) that a reflection about the problem solving activities and methods yields a deeper understanding and better performance.

Future Work We are now working on cognitively motivated extensions of ActiveMath by new types of examples and exercises that have shown their merit for learning. In particular, the student model will be enhanced by information about the learner's motivation such that the system can properly react to excitement, boredom and other motivational states.

Other extensions are being realized in the EU-project LeActiveMath that investigates natural language facilitiesfor a tutorial dialogue in interactive exercises and dialogues about the student model.

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