UNIVERSITAS INDONESIA

GRAVITATIONAL FIELD OF NONCANONICAL GLOBAL MONOPOLE: A STUDY OF THEIR BLACK HOLES AND COMPACTIFICATIONS

MASTER THESIS

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FACULTY OF MATHEMATICS AND NATURAL SCIENCES
GRADUATE PHYSICS (COURSE) PROGRAM
THEORETICAL AND APPLIED PHYSICS
DEPOK
2017

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MASTER THESIS
proposed in accordance to one of the requirements for the degree of
Master of Science

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PAGE OF ORIGINALITY STATEMENT

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ABSTRAK

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Judul	:	Medan Gravitasi dari Global Monopol Nonkanonik: Sebuah Studi Mengenai Lubang Hitam dan Kompaktifikasi

Dalam tesis ini, kami memberikan beberapa solusi medan gravitasi global monopol dan generalisasinya di dimensi yang lebih tinggi. Umumnya, kami mendiskusikan model matematika di *manifold* berdimensi tinggi M dengan $\dim(M) = p + D$. Kami mendiskusikan beberapa solusi lubang hitam, dimana kami berfokus pada horison-horisonnya. Kami juga mendiskusikan beberapa solusi kompaktifikasi (atau lebih akuratnya solusi metrik terfaktorisasi) dan diperlihatkan, dalam bentuk daftar, kompaktifikasi (faktorisasi) yang mungkin dari sebuah ruang berdimensi $(p + D)$ ke sebuah produk ruang $(p + 2) \times (D - 2)$ yang memiliki kelengkungan konstan.

Kata kunci: global monopol nonkanonik, lubang hitam, faktorisasi.

ABSTRACT

Name	:	Ilham Prasetyo
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In this thesis we present some gravitational field solutions of global monopoles and its generalizations in higher dimensions. In general, we discuss the mathematical model in a higher dimensional manifold M with $\dim(M) = p + D$. We discuss some blackhole solutions, whose horizons are what we focused on. We also discuss some compactification solutions (or more accurately factorized metric solutions) and list some possible compactification (factorization) channels from a $(p+D)$-dimensional space to a $(p+2)\times(D-2)$-space of constant curvature.

Keywords: noncanonical global monopole, blackhole, factorization.

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The topics in the introduction are quite many but I hope the language is easy enough for newcomer researchers so that this thesis will be useful for them.

Depok, 19 June 2017

Ilham Prasetyo

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CHAPTER 1
INTRODUCTION

1.1 Topological Defects

The motivation of studying monopole is basically because a phase transition in the early universe may have generated various kinds of topological defects, where monopole is one of them [1]. First of all, what are defects? Defects are some configuration that might form when a phase transition occurs. One may imagine bubbles of water vapor in boiled water. The air inside the bubble have different phase, that is gaseous, than outside the bubble, that is liquid. The bubble wall itself is a defect that separate them. Since the water as liquid and the vapor as a same field but differ in phase, the defect is a field configuration that separate two field with different phase. The configuration is called nontrivial due to the differential equations explaining it have nontrivial boundary conditions. These boundary conditions can be from topological considerations, hence the name *topological defects*. There are others, for example, that are not from topological arguments which are named *non-topological defects*.

Topological defects are a subset of a larger set of *solitons*, which is a subset of *solitary waves* [2]. Solitary waves are, in mathematical sense, localized non-singular solutions of any non-linear field differential equations. Intuitively, solitary waves have localized energy density that are undistorted with constant velocity, i.e., letting it run with a constant velocity will not differentiate its energy density compared to the waves that are stand still. In mathematical notations, it is

$$\epsilon(\vec{x}, t) = \epsilon(\vec{x} - \vec{u}t) \tag{1.1}$$

with $\vec{u}$ a velocity vector. Solitons, on the other hand, are restricted to solitary waves whose energy density at much later time are restored to their original shapes and velocities. Suppose its original energy density is $\epsilon_0(\vec{x} - \vec{u}t)$ and suppose there are other N wave solutions in the far past whose velocities and positions are arbitrary. These waves, whose velocity and position discrepancy of each is contained in $\vec{x}_i$, satisfy

$$\lim_{t \to -\infty} \epsilon(\vec{x}, t) = \sum_{i=1}^{N} \epsilon_0(\vec{x} - \vec{x}_i - \vec{u}t). \tag{1.2}$$

After a large time into the future, these waves also satisfy

$$\lim_{t\to+\infty} \epsilon(\vec{x},t) = \sum_{i=1}^{N} \epsilon_0(\vec{x}-\vec{x}_i-\vec{u}t+\vec{\delta}_i) \tag{1.3}$$

with $\vec{\delta}_i$ is a constant vectors.

There are many kinds of topological defects, such as domain walls, vortices, global strings, and monopoles. Domain walls and vortices are a common topic in condensed matter physics. These give rise to another area of research called topological phase transitions which investigate a change of matter properties that is related to topology. Global strings is another defect than is one of the hope for cosmologists to explain the problems of our cosmos, such as the cosmological constant problem. There are many type of monopoles, some of them are Dirac magnetic monopole and 't Hooft-Polyakov monopole. Next we discuss a *kink*, the most simple defect in a 2-dimensional spacetime. From this example we can see at least two reasons why a topological defect is very interesting: it has discrete charge that corresponds to its topology and it is a non-perturbative theory.

1.1.1 A (1+1)-dimensional Kink

Consider a Lagrangian density of a scalar field $\phi(x,t)$ in a $(1+1)$-dimensional flat spacetime, whose metric is $\eta_{\mu\nu} = \text{diag}(1,-1)$ with $\mu = 0,1 = t,x$, that has the following form

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - V(\phi), \tag{1.4}$$

with $V(\phi) = \frac{\lambda}{4}(\phi^2-\eta^2)^2$. This in turn give a non-linear differential equation is

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{\partial V}{\partial\phi} = 0 \tag{1.5}$$

and its energy functional is

$$E(\phi) = \int dx\, \eta_{00}\left(\left(\frac{\partial\phi}{\partial t}\right)^2 - \mathcal{L}\right) = \int dx \left(\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + V(\phi)\right). \tag{1.6}$$

This energy is minimized, i.e. $dE/d\phi = 0$, at $\phi = \pm\eta$. The localized solutions of this system must go to $\phi = \pm\eta$ at $t \to \pm\infty$.

Now we assume the model has at least a solution that satisfy the soliton's boundary conditions. These then allow us to consider a static solution $\phi(x)$ whose energy is

$$E(\phi) = \int dx \left(\frac{1}{2}\phi'^2 + V(\phi)\right), \tag{1.7}$$

where $\phi' = \frac{d\phi}{dx}$. This model is actually the *kink* model we refer before. Bogomolny [3] devices a method that reduces the Euler-Lagrange equation, which is a second order differential equation, to a first order one by making a squared term of the energy plus a term that correspond to lower bound of energy. This model turns out can use this method making its energy have the following form

$$E(\phi) = \int dx \left[\frac{1}{2}\left(\phi' \pm \sqrt{2V(\phi)}\right)^2 \mp \phi'\sqrt{2V(\phi)}\right], \tag{1.8}$$

which first order differential equation, or BPS equation, is from the squared term

$$\phi' \pm \sqrt{2V(\phi)} = 0 \tag{1.9}$$

and the energy's lower bound is

$$E \geq E_{\text{bound}}(\phi) = \mp \int d\phi \sqrt{2V(\phi)}. \tag{1.10}$$

This can be solved:

$$\int_{x_0}^{x} dx = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2V(\phi)}} = \pm\sqrt{\frac{2}{\lambda}} \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{(\phi^2 - \eta^2)}, \tag{1.11}$$

now suppose $\phi = \eta \tanh(\theta)$, $d\phi = \eta \operatorname{sech}^2(\theta) d\theta$,

$$\int_{x_0}^{x} dx = \pm\sqrt{\frac{2}{\lambda}} \int_{\theta_0}^{\theta} \frac{\eta \operatorname{sech}^2(\theta) d\theta}{\eta^2(\tanh^2(\theta) - 1)} = \mp\sqrt{\frac{2}{\lambda\eta^2}} \int_{\theta_0}^{\theta} d\theta, \tag{1.12}$$

thus we obtain the exact solution

$$\phi = \eta \tanh\left(\pm\sqrt{\frac{\lambda\eta^2}{2}} x\right). \tag{1.13}$$

This exact solution is the main advantage of Bogomolny method and we can see that with this method, we can define [4]

$$j^\mu = \epsilon^{\mu\nu}\partial_\nu \phi \tag{1.14}$$

so that it satisfy conservation of charge and current density, i.e. $\partial_\mu j^\mu = 0$ and we have the charge to be

$$Q = \int_{-\infty}^{\infty} dx j^0 = \int_{-\infty}^{\infty} \phi' dx = \phi(\infty) - \phi(-\infty) = 2\eta n, n \in \mathbb{Z}_2 = \{-1, 1\}. \tag{1.15}$$

($n = -1$ corresponds to *anti*-kink.) Here n actually corresponds to the so-called *topological invariant* of this field (explained later), j^μ and Q above turns out named as *topological current* and *topological charge*, respectively, that corresponds to the existence of *domain walls*. We also have the energy's lower bound turns out to be

$$E_{\text{bound}}(\phi) = \mp \int_{-\eta}^{\eta} d\phi \sqrt{\frac{\lambda}{2}} (\phi^2 - \eta^2) = \pm \frac{2\sqrt{2}}{3} \sqrt{\lambda} \eta^3 \simeq \frac{2\sqrt{2} m^3}{3\lambda}, \tag{1.16}$$

with mass scale $m \sim \sqrt{\lambda}\eta$ since, from the potential, we have $\phi \sim \eta$ and, from the equation of motion, the length scale is $\delta \sim (\lambda\eta^2)^{-1/2}$. As $\lambda \to 0$ we have $E \to \infty$, meaning that the defect becomes heavier as the coupling becomes weaker. This means that, not only the kink is stable by topologically, the properties of the kink will *not* be revealed by perturbation method.

1.1.2 Derrick's Theorem

Not only due to topology argument, the kink model is also stable by scaling argument according to Derrick's theorem [4, 5, 6]. Consider an energy functional at d-dimensional spacetime with the following form

$$E(\phi) = \int d^{d-1}x \left((\nabla\phi)^2 + V(\phi) \right). \tag{1.17}$$

Now rescale the length by $x^\mu \to \tilde{x}^\mu = \lambda x^\mu$. After rescaling we have $\phi \to \phi_\lambda = \phi(\lambda x)$, $d^{d-1}x \to \lambda^{-d+1}\, d^{d-1}\tilde{x}$ and $\nabla \to \lambda\tilde{\nabla}$ thus the energy functional becomes

$$E(\phi_\lambda) = \int d^{d-1}\tilde{x} \left(\lambda^{-d+3} (\tilde{\nabla}\phi_\lambda)^2 + \lambda^{-d+1} V(\phi_\lambda) \right). \tag{1.18}$$

The energy functional must be stable under such scaling, thus by variational principle it requires

$$\left. \frac{dE}{d\lambda} \right|_{\lambda=1} = \int d^{d-1}\tilde{x} \left((3-d)(\tilde{\nabla}\phi_\lambda)^2 + (1-d)V(\phi_\lambda) \right) = 0, \tag{1.19}$$

or $V(\phi_\lambda) = \frac{(3-d)}{(d-1)}(\tilde{\nabla}\phi_\lambda)^2$, and

$$\left.\frac{d^2E}{d\lambda^2}\right|_{\lambda=1} = \int d^{d-1}\tilde{x}\left((3-d)(2-d)(\tilde{\nabla}\phi_\lambda)^2 + d(d-1)V(\phi_\lambda)\right) > 0 \tag{1.20}$$

which by substituting $V(\phi_\lambda)$ we have

$$\left.\frac{d^2E}{d\lambda^2}\right|_{\lambda=1} = \int d^{d-1}\tilde{x}\ \ 2(3-d)(\tilde{\nabla}\phi_\lambda)^2 > 0 \tag{1.21}$$

meaning the model is stable at $d \leq 2$. At $d = 3$ we need more analysis.

Suppose we add a term $(\nabla\phi)^4$ in the Lagrangian density which make it no longer canonical

$$E(\phi) = \int d^{d-1}x\left((\nabla\phi)^2 + (\nabla\phi)^4 + V(\phi)\right). \tag{1.22}$$

After rescaling we have

$$E(\phi_\lambda) = \int d^{d-1}\tilde{x}\left(\lambda^{-d+3}(\tilde{\nabla}\phi_\lambda)^2 + \lambda^{-d+5}(\tilde{\nabla}\phi_\lambda)^4 + \lambda^{-d+1}V(\phi_\lambda)\right). \tag{1.23}$$

The first derivative implies

$$\left.\frac{dE}{d\lambda}\right|_{\lambda=1} = \int d^{d-1}\tilde{x}\left((3-d)(\tilde{\nabla}\phi_\lambda)^2 + (5-d)(\tilde{\nabla}\phi_\lambda)^4 + (1-d)V(\phi_\lambda)\right) = 0, \tag{1.24}$$

or $V(\phi_\lambda) = \frac{(3-d)}{(d-1)}(\tilde{\nabla}\phi_\lambda)^2 + \frac{(5-d)}{(d-1)}(\tilde{\nabla}\phi_\lambda)^4$ and the second derivative implies

$$\left.\frac{d^2E}{d\lambda^2}\right|_{\lambda=1} = \int d^{d-1}\tilde{x}\left(2(3-d)(\tilde{\nabla}\phi_\lambda)^2 + 4(5-d)(\tilde{\nabla}\phi_\lambda)^4\right) > 0, \tag{1.25}$$

thus the new model can be stable at $d \leq 2$. Adding more nonlinear kinetic term with more power, such as $(\tilde{\nabla}\phi_\lambda)^n$ with $n \geq 4$, does not change the upper bound of the spacetime's dimension.

Now we restrict the model with $V = 0$, in the non-canonical energy functional. An example of this is sigma model that we are going to use in this thesis. The energy functional after the scaling becomes

$$E(\phi_\lambda) = \int d^{d-1}\tilde{x}\left(\lambda^{-d+3}(\tilde{\nabla}\phi_\lambda)^2 + \lambda^{-d+5}(\tilde{\nabla}\phi_\lambda)^4\right). \tag{1.26}$$

The first derivative thus implies $(\tilde{\nabla}\phi_\lambda)^4 = \frac{(3-d)}{(d-5)}(\tilde{\nabla}\phi_\lambda)^2$ then the second derivative implies

$$\left.\frac{d^2E}{d\lambda^2}\right|_{\lambda=1} = \int d^{d-1}\tilde{x}\ \ 2(d-3)(\tilde{\nabla}\phi_\lambda)^2 \geq 0, \tag{1.27}$$

thus the new model is stable at $d \geq 4$ while at $d = 3$ we need more analysis. The upperbound vanishes and a lower bound now exists thus this gives us the freedom to apply arbitrary dimension.

1.2 A Few Concepts of Topology

The mathematics used to describe topology is quite technical thus to make this chapter short we briefly discuss a few of the main ideas that describes the topological point of view of gauge theories. Obviously, this description is incomplete and we encourage readers to see Ref. [7] for comprehensive explanations, where we heavily refer and Ref. [8] for more rigorous and explicit proofs.

Topological defects, as its name indicates, are dependent on the topology of the space where it lives, that is manifold $\mathcal{M}$. This manifold is locally Euclidean. As an example, one can locally zoom a surface of something in this three-dimensional world we live in, a soccer ball for instance, and see that its surface is two-dimensional flat space locally, $\mathbb{R}^2$. This manifold is a part of the tools called fibre bundles and the vacuum manifold is contained in it. This fibre bundle in turn have a correspondence to a certain homotopy group over the field in the fibre itself, which is its topological invariant.

1.2.1 Fibre Bundles

First, let us discuss fibre bundles. In a naive sense, one can think that the information regarding a topological defect is contained in a mathematical framework called fibre bundles, often symbolically summarised in $(E, \mathcal{P}, M, F, G)$, with M called basis manifold, E its entire bundle, F fibre attached in each points in M, G is a Lie group that is a structure in the fibre bundle, and $\mathcal{P}$ is projecting any point p in E to a point x in M, $\mathcal{P}(p) = x$. This is suitable for physicists since they almost always investigate something in a local area of M, denoted by an open set U_i in M whose sum is M itself, $\bigcup_{i \in I} U_i = M$. This U_i is called as an open cover of M, and it is common to think that M is a Hausdorff space, meaning that if we have two distinct points in M called p_i and p_j contained in each open cover U_i and U_j respectively, then one can have the two open cover to be disjoint, $U_i \cap U_j = \emptyset$. Each U_i, whose chart is $\psi_i(U_i) \subseteq \mathbb{R}^n$, has its own choice of coordinate basis in a vector space. This framework is equipped with a set of symmetry in mind that is denoted by a certain Lie group G, and this acts on F from left. One can construct a local trivialization from the entire bundle to a (U_i, Ψ_i) by a trivializing map $\Psi_i : U_i \times F \rightarrow E$ that is $\sigma^{-1}(p) = \Psi_i(p, f)$, where $\sigma : E \rightarrow M$ is a projection (surjective) map. If there is a two open cover that is connected and there is at least a point in it, $p \in U_i \cap U_j \neq \emptyset$, then there is a transition function $t_{ij} : U_i \cap U_j \to G$ such that $\Psi_j(p, f) = \Psi_i(p, t_{ij}(p)f)$. This makes an equivalence relation in E by $(p, f) \sim (p, t_{ij}(p)f)$. This equivalence relation is related to a homotopy class, or class of groups, of the fibre F, $\pi_n(F)$.

1.2.2 Homotopy Group

A first homotopy group $\pi_1(M)$, or usually called fundamental group, of a manifold M, that is path-connected, is what one can imagine as a equivalence relation between two path function in M, say a and b. Now for intuitive reason, which makes this discussion is quite naive, let us discuss the first homotopy group, or usually called fundamental group, $\pi_1(M)$. Let us imagine both paths map a same interval $[0, 1]$ to a different paths in M, $a, b : [0, 1] \to M$, by $a(0) = m_0 = b(0)$ its initial point and $a(1) = m_1 = b(1)$ its final point in M. Both paths are homotopic to each other relative to its initial and final points, $a \sim b$ rel $\{m_0, m_1\}$, if there exist a mediator map $c : [0, 1] \times [0, 1] \to M$ such that $c(t, 0) = a(t),\ c(t, 1) = b(t)$ with all t inside $[0, 1]$ and $c(0, s) = m_0,\ c(1, s) = m_1$ with s in $[0, 1]$. Then there is a equivalence relation between the two thus we can denote both a, b as something inside a set $[a]$ in $\pi_1(M)$. Usually one discuss loops in homotopy group whose loops have its initial and final point to be the same, $m_0 = m_1$, making $a \sim b$ rel $\{m_0\}$. The notion of loop a can be shrunk to a point means that b is just a constant map m_0, thus $a \sim m_0$ and a is inside $[m_0]$. If the fundamental group contains only $[m_0]$ then the fundamental group is said to be trivial $\pi_1(M) \cong \mathbb{I}$. In mathematics this means that M is simply-connected. One can construct more things with similar way of thinking to build the n-th homotopy group π_n with $n \geq 2$. The homotopy group is known as one of many kinds of topological invariants. In general calculating homotopy groups is a nontrivial task and it is an active research field in mathematics. In physics, the *boundary conditions* of the physical (scalar and gauge) fields determine the homotopy group. This conditions are usually determine the boundary of the spacetime and the inner space in the fields [2].

There is a correspondence between homotopy group and fibre bundles. There is a more specific type fibre bundle called principal G-bundle that differs to fibre bundle by taking its fibers to be identical to the structure group G. It also has a more structure, that is, G also act to F from the right. Principal G-bundle is actually related to homotopy group by the following theorem that we will not proof here (see Theorem 4.4.3 in Ref [8]): *Let G be a pathwise connected topological group. Then the set of equivalence classes of principal G-bundles over S^n, $n \geq 2$, is in one-to-*

one correspondence with the elements of $\pi_{n-1}(G)$. There is also a known result, whose proof is not easy and is not going to be discussed here, that $\pi_n(S^m) = 0$ for $n < m$ and $\pi_n(S^m) \cong \mathbb{Z}$ for $n = m$. This will be useful in our example later.

More specifically to topological solitons, they arise from considering models that use symmetry-breaking process. The vacuum manifold $\mathcal{M} = G/H$ is a quotient group of a larger group G and its subgroup H that contain identity element e that corresponds to the field itself. There exist a subgroup H in G whose elements are not affecting the field itself; H is the unbroken group from the broken G. For example, one can imagine $SU(2)$ broken while its subgroup $U(1)$ is unbroken. There are two theorems that are commonly used to calculate its homotopy class: *let G be path-connected and simply-connected with its subgroup H contain identity element e, then (1) the first fundamental theorem states that* $\pi_1(G/H) \cong \pi_0(H)$ *and (2) the second fundamental group states that* $\pi_2(G/H) \cong \pi_1(H)$. There is also a powerful tool that is quite frequently used called *exact homotopy sequences* that generalizes both theorems above. This exact sequence is a series of homomorphisms connecting homotopy groups:

$$\begin{aligned} ... \to \pi_n(G) \to \pi_n(G/H) \to \pi_{n-1}(H) \to \pi_{n-1}(G) \to ... \\ \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H). \end{aligned} \tag{1.28}$$

If there is an exact sequence $\mathcal{G}_1 \xrightarrow{f_1} \mathcal{G}_2 \xrightarrow{f_2} \mathcal{G}_3 \xrightarrow{f_3} \mathcal{G}_4$ with both $\mathcal{G}_1$ and $\mathcal{G}_4$ being trivial then f_1 is one-to-one (injective) since $f_1(\mathcal{G}_1) \subseteq \mathcal{G}_2$, f_3 is onto (surjective) since $f_3(\mathcal{G}_3) = e$ thus f_2 is an isomorphism.

Since the boundary conditions of the physical (scalar and gauge) fields determine the homotopy group [2] and the continuous group G is valid on all points in the inner space, a group H is only valid on the boundary. At the boundary, we then talk about G and H which is called as the broken and unbroken symmetry of the fields, respectively. We will see that, as an example in 't Hooft-Polyskove monopole model, both the scalar and gauge field map (one-to-one and onto) all points from the boundary of the physical space S^2 to the boundary of inner space $SO(3)/U(1) \cong S^2$ which give us its topological invariant $\pi_2(SO(3)/U(1)) \cong \pi_1(U(1)) \cong \mathbb{Z}$. Symmetry group $SO(3)$ and $U(1)$ is the broken and unbroken symmetry of the fields. The components of $\pi_1(U(1))$ turns out have a one-to-one correspondence to a set of equivalence classes of a principal $U(1)$-bundle over S^2, which is actually the framework of Dirac magnetic monopole [8].

1.2.3 Dirac Magnetic Monopole

The concepts are quite difficult to grasp for unfamiliar readers hence it is very useful to discuss an example, which is Dirac magnetic monopole [9]. But before that, let us review a local gauge transformation in an abelian gauge theory. Locally transforming a Dirac field ψ with a generator $\alpha(x)$ from $U(1)$ to

$$\psi' = \exp(ie\alpha(x))\psi, \tag{1.29}$$

with ie a constant, in a Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu + m)\psi, \tag{1.30}$$

requires a coupling to an electromagnetic (EM) field by employing a minimal substitution, i.e, using the so-called covariant derivative

$$D_\mu \psi = \partial_\mu \psi + ieA_\mu \psi, \tag{1.31}$$

with A_μ the EM potential. To ensure the Lagrangian is invariant under this transformation $\mathcal{L}' = \mathcal{L}$, it requires A_μ to also transform with the form

$$A'_\mu = A_\mu - \partial_\mu \alpha. \tag{1.32}$$

The transformation of the potential field A_μ is also called as gauge transformation since

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.33}$$

is invariant under $U(1)$ transformation $F'_{\mu\nu} = F_{\mu\nu}$. The combined Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu + m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.34}$$

is thus invariant under $U(1)$ transformation.

The Dirac magnetic monopole may arise if the potential field has the following form

$$A_x^N = \frac{-gy}{r(r+z)},\ A_y^N = \frac{gx}{r(r+z)},\ A_z^N = 0, \tag{1.35}$$

or in spherical coordinates

$$\vec{A}^N(\vec{r}) = \frac{g(1-\cos\theta)}{r\sin\theta}\hat{e}_\phi, \tag{1.36}$$

where $\hat{e}_\phi = -\sin\phi\hat{e}_x + \cos\phi\hat{e}_y$. This satisfy the Coulomb version of magnetic field

$\vec{B} = \vec{\nabla} \times \vec{A}^N = g\hat{r}/r^2$, meaning that there is a monopole with strength g located at $r = 0$. This is also true for another type of potential field

$$A_x^S = \frac{gy}{r(r-z)},\ A_y^S = \frac{-gx}{r(r-z)},\ A_z^S = 0, \tag{1.37}$$

or in polar coordinates

$$\vec{A}^S(\vec{r}) = -\frac{g(1+\cos\theta)}{r\sin\theta}\hat{e}_\phi. \tag{1.38}$$

One can see that $\vec{A}^N$ ($\vec{A}^S$) is singular at $r = -z$ ($r = z$), or $\theta = \pi$ ($\theta = 0$), and the difference of both is

$$\vec{A}^N - \vec{A}^S = \frac{2g}{r\sin\theta}\hat{e}_\phi = \vec{\nabla}(2g\phi). \tag{1.39}$$

The potential field transform with $\alpha = 2g\phi$ and thus

$$\psi^N = \exp(2ieg\phi)\psi^S. \tag{1.40}$$

Let us imagine a 2-sphere S^2 with modulus one $|\vec{r}| = 1$ surrounds the monopole and set $\theta = \pi/2$ makes the singularities at $\theta = 0$ and $\theta = 2\pi$ do not show up. Its northern (southern) hemisphere U_N (U_S) receive magnetic flux from $\vec{A}^N$ ($\vec{A}^S$), denoted by Φ_N (Φ_S). ϕ is walking counter-clockwise thus for each hemisphere

$$\Phi_N = \int_N \vec{\nabla} \times \vec{A}^N \cdot d\vec{l} = \oint_{\text{equator}} \vec{A}^N \cdot d\vec{s}, \tag{1.41}$$

$$\Phi_S = \int_S \vec{\nabla} \times \vec{A}^S \cdot d\vec{l} = -\oint_{\text{equator}} \vec{A}^S \cdot d\vec{s}, \tag{1.42}$$

then the total flux is

$$\Phi = \Phi_N + \Phi_S = \oint_{\text{equator}} (\vec{A}^N - \vec{A}^S) \cdot d\vec{s} = \oint_{\text{equator}} \vec{\nabla}(2g\phi) \cdot d\vec{s} = 4\pi g. \tag{1.43}$$

As we go around the equator of the sphere n times from $\phi = 0$ to $\phi = 2\pi n$, we can see that $\exp(4ieg\pi) = 1 = \exp(2\pi n)$ means

$$g = \frac{n}{2e},\ n \in \mathbb{Z}, \tag{1.44}$$

which is known as the Dirac quantization condition for magnetic monopole with e electric charge. We can see that n is proportional the total flux Φ, thus Φ can be imagined as topological charge of Dirac monopole.

One can then see that the Dirac monopole can be seen in fibre bundle context, especially the theorem mentioned above that use principal G-bundle $(E, M, \mathcal{P}, G)$, that is, its fibre F is similar to its structure group G. The theorem from before indicate that Dirac magnetic monopole has its representation in principal $U(1)$-bundle over $M = S^2$. Since $U(1) \cong S^1$ and S^1 is pathwise connected, then $G = U(1)$ and thus $F = U(1)$. Then the equivalence classes in the bundle is in one-to-one correspondence to $\pi_n(F)$ and we know that $\pi_n(S^n) \cong \mathbb{Z}$, thus we have the topological invariant $\pi_1(U(1)/\mathbb{I})$ to be nontrivial, i.e., $\pi_1(U(1)) \cong \pi_1(S^1) \cong \mathbb{Z}$. Hence the quantized condition of Dirac magnetic monopole thus has one-to-one correspondence to this topological invariant. Furthermore, A_μ, times some constants, is also called as connection in the Hopf bundle $(S^3, \mathbb{CP}^1, \mathcal{P}, U(1))$, which is an example of principal $U(1)$-bundle. The base manifold $\mathbb{CP}^1$ is identified with S^2, whose section map into open covers U_N and U_S that cover S^3 gives, respectively, connections proportional to A^N and A^S (see Ref. [8] page 268-270 for more detailed discussions).

This is an example of a relation between the structure from the symmetry in a gauge field and a topological invariant through the use of fibre bundles. Although Dirac monopole has nontrivial first homotopy group, this not a topological soliton since its existence is postulated by symmetrizing the Maxwell equations $F^{\mu\nu} \to *F^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$, $*F^{\mu\nu} \to F^{\mu\nu}$ (or more explicitly $\vec{E} \to \vec{B}, \vec{B} \to -\vec{E}$); it does not arise from any symmetry-breaking process hence the fundamental theorems above cannot be used. There is another type monopole called 't Hooft-Polyakov monopole that arise from nonabelian gauge theory.

1.2.4 t' Hooft-Polyakov Monopole

Before continuing to the nonabelian gauge theory, let us some aspects of generalized global gauge transformations. Let G be the corresponding Lie group whose matrix representation $D(g)$ with $g \in G$ relate two different n-component scalar field ϕ_i and ϕ'_j with

$$\phi'_i = D_{ij}(g)\phi_j. \tag{1.45}$$

[It is a generalization of the previous global transformation $\psi \to \psi' = \exp(ie\alpha)\psi$ with α just a constant.] The $n \times n$ matrix representation D maps G to a vector space $\mathcal{V}$ spanned by ϕ_i preserve the group multiplication, $D(g_1 g_2) = D(g_1)D(g_2)$. Here g can be written as

$$g = \exp(i\omega_a L^a), \tag{1.46}$$

with $\omega_a \in \mathbb{R}$ and L^a is called as group generator. Basically $\exp : \mathcal{G} \to G$ maps Lie algebra $\mathcal{G}$ to its Lie group G and its Lie algebra admits the so-called commutation relation

$$[L^a, L^b] = if_{abc}L^c, \tag{1.47}$$

with f_{abc} is the structure constants of G. Now we for each L^a it has one $n \times n$ matrix representation T^a that also satisfy such commutation relation in $\mathcal{V}$. We claim without proof that $T^a = D(L^a)$. It is also common for physicist to use compact G and this property enable us to normalize the generators by

$$\text{tr}(T^a T^b) = \delta^{ab}. \tag{1.48}$$

[The construction above is a simplified version from what one can obtain in mathematics literatures thus readers are recommended to consult those, such as Ref. [8], for more precise and rigorous definitions.]

Consider Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - V(|\vec{\phi}|), \tag{1.49}$$

with the n-components of scalar field $\vec{\phi} = \{\phi^1, ..., \phi^n\}$ implies n-degrees of freedom. This is a generalized version of scalar field discussed before that gives the kink. The Lagrangian density is invariant under the global gauge transformation $\phi'_i = D_{ij}(g)\phi_j$, i.e., by shifting infinitesimally $\phi'_i = \phi_i + T^a_{ij}\phi_j$ the potential becomes

$$V(D_{ij}(g)\phi_j) = V(\phi_i) + \frac{\partial V}{\partial \phi_i} T^a_{ij}\phi_j + \frac{1}{2}\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}(T^a_{ik}\phi_k)(T^a_{jl}\phi_l) + ... \tag{1.50}$$

and since this need to be invariant $V(D_{ij}(g)\phi_j) = V(\phi_i)$, we need

$$\frac{\partial V}{\partial \phi_i} T^a_{ij}\phi_j = 0 \tag{1.51}$$

at the minima hence if the minima is located at $|\vec{\phi}| = 0$ then $T^a_{ij}\phi_j = 0$ is satisfied. This implies the Langrangian density is invariant for all $g \in G$ when the minima of the potential is located at $|\vec{\phi}| = 0$. If the minima is located at nonzero $|\vec{\phi}| = \phi_0$, the symmetry is spontaneously broken. Now we have H an unbroken group

$$H = \{g \in G | D(g)\phi_0 = \phi_0\}, \tag{1.52}$$

whose unbroken generator t^α of H satisfy

$$t^\alpha \phi_0 = 0. \tag{1.53}$$

We can choose t^α to be a subset of T^a with appropriate choice of T^a. Now if we perturb the scalar field with $\vec{\phi} = \vec{\phi}_0 + \vec{\phi}'$, we can get

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi'^a \partial^\mu \phi'^a - \frac{1}{2}\mu_{ij}^2 \phi'^i \phi'^j \tag{1.54}$$

with mass matrix $\mu_{ij}^2 = \frac{\partial^2 V}{\partial\phi_i \partial\phi_j}$.

Now let us consider a nonabelian gauge theory with the following Yang-Mills Lagrangian density

$$\mathcal{L} = \frac{1}{2}\mathcal{D}_\mu \phi^a \mathcal{D}^\mu \phi^a - V(|\phi|) - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu}, \tag{1.55}$$

with $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\epsilon^{abc}A_\mu^b A_\nu^c$ its Yang-Mills field strength tensor, $\mathcal{D}_\mu \phi^a = \partial_\mu \phi^a + e\epsilon^{abc}A_\mu^b \phi^c$ its *adjoint* covariant derivative where $T^a = iQ^a$ and Q^a is real antisymmetric matrix and its structure constant $f_{abc} = \epsilon_{abc}$. It is common to write the *adjoint* covariant derivative as $\mathcal{D}_\mu \phi = \mathcal{D}_\mu(\phi^a \sigma^a/2) = \partial_\mu \phi - ie[\mathcal{A}_\mu, \phi]$ where $\sigma^a/2$ are the Pauli matrices which are the matrix representation of generators of $SU(2)$, the field strength (in mathematics this is called as *curvature*) as $\mathcal{F}_{\mu\nu} = F^a{}_{\mu\nu}\sigma^a/2 = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - ie[\mathcal{A}_\mu, \mathcal{A}_\nu]$ with its vector potential (*connection*) as $\mathcal{A}_\mu = A^a{}_\mu L^a$. It is also invariant to local gauge transformation $\phi \to D(g)\phi$, $\mathcal{A}_\mu \to g\mathcal{A}_\mu g^{-1} + ie^{-1}g^{-1}\partial_\mu g$ making $\mathcal{D}_\mu \phi \to D(g)\mathcal{D}_\mu \phi$, $\mathcal{F}_{\mu\nu} \to g\mathcal{F}_{\mu\nu}g^{-1}$. Again with similar perturbation $\vec{\phi} = \vec{\phi}_0 + \vec{\phi}'$ we can get

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi'^a \partial^\mu \phi'^a - \frac{1}{2}\mu_{ij}^2 \phi'^i \phi'^j - \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2}M_{ab}^2 A_\mu^a A^{b\mu} + \mathcal{L}_{\text{int}}, \tag{1.56}$$

with vector field mass matrix $M_{ab}^2 = e^2(T^a T^b)_{ij}\phi_0^i \phi_0^j$.

Now we come into the 't Hooft-Polyakov monopole model, whose potential in the nonabelian gauge Lagrangian density is $V(|\phi|) = \frac{\lambda}{4}(\phi^a \phi^a - \eta^2)^2$. The triplet Higgs field ϕ^a with $a = 1, 2, 3$ has $SO(3)$ symmetry ($\cong SU(2)$). At the boundary of the system, $\phi^a \phi^a = \eta^2$, making the unbroken symmetry is $SO(2)$ ($\cong U(1)$). Hence the vacuum manifold is $\mathcal{M} = SO(3)/SO(2) \cong SU(2)/U(1) \cong S^2$. Also at the boundary, the scalar field $\{\phi^a\} : S^2 \to \mathcal{M}$ maps the boundary of the spacetime manifold, which is S^2, into the vacuum manifold $\mathcal{M}$, hence by exact homotopy sequences its homotopy group is $\pi_2(\mathcal{M}) \cong \pi_1(U(1)) \cong \mathbb{Z}$. The unbroken symmetry is the same as what Dirac monopole has, but with an additional advantage that is its vacuum manifold is noncontractible, or not simply-connected, due to its 2-loops are classified with a *winding number*. Using the normalized scalar field $\hat{\phi}^a = \phi^a/|\phi|$

and $a, i = 1, 2, 3$, the winding number is defined as [10]

$$N_\phi = \frac{1}{8\pi} \int dS^i \epsilon^{ijk} \epsilon_{abc} \hat{\phi}^a \partial_j \hat{\phi}^b \partial_k \hat{\phi}^c. \tag{1.57}$$

This winding number is integer valued $N_\phi \in \mathbb{Z}$. At the boundary, we know that

$$\int dS^i \epsilon^{ijk} \epsilon_{abc} \hat{\phi}^a \mathcal{D}_j \hat{\phi}^b \mathcal{D}_k \hat{\phi}^c = 0. \tag{1.58}$$

By multiplying with $1/8\pi$, using the definition of the adjoint covariant derivative and $\frac{1}{2}\epsilon^{ijk} F^a_{jk} = -B^a_i$, one can arrive at [10]

$$N_\phi = \frac{e}{4\pi} \int dS^i B^a_i \hat{\phi}^a. \tag{1.59}$$

This is equal to (1.43) if we identify by $g = (1/4\pi) \int dS^i B^a_i \hat{\phi}^a$ with g the magnetic strength. Hence $N_\phi = 1$ corresponds to $n = 2$ from Dirac quantization condition. Be careful that this identification is true only at the boundary. As an illustration, we consider an ansatz and we will see that the Lagrangian density contains a boundary term proportional to this winding number.

The trivial configuration is clearly just a constant $|\phi| = \eta$, but there is a nontrivial configuration proposed by 't Hooft [11] and Polyakov [12]

$$\phi^a = \eta f(r) \frac{x^a}{r}, \tag{1.60}$$

which is also called as *hedgehog ansatz* and has spherically symmetric property with x^a Cartesian coordinate x, y, z. (This is similar to an idea of Skyrme who thought that soliton models may be the solution to problems in nuclear physics [5]. He proposed the hedgehog ansatz that not only have the same form as above but there is also a nonzero time component of the scalar field. Later due to his brilliant idea, the soliton is named Skyrmion.) The scalar field can be seen as pointing at all directions like a hedgehog hence the name. The boundary condition is $f(r \to 0) = 0,\ f(r \to \infty) = 1$. Since this is static solution and due to the covariant derivative $\mathcal{D}_\mu$ contains A^a_μ, at spatial infinity the vector potential A^a_μ must also vanish. Thus it must satisfy

$$A^a_0 = 0,\ A^a_i = (1 - a(r)) \epsilon^{aij} \frac{x^j}{er^2}, \tag{1.61}$$

with boundary condition $a(r \to 0) = 1,\ a(r \to \infty) = 0$. Since $SO(3)$ symmetry imply $f_{abc} = \epsilon_{abc}$, we can see that at faraway $r \to \infty$

$$F^a_{0i} = 0,\ F^a_{ij} = \epsilon^{bik}(\delta^{ja}\delta^{lb} - \delta^{jb}\delta^{la})\frac{x^k x^l}{er^4} \simeq \epsilon^{ijk}\frac{x^a x^k}{er^4}, \tag{1.62}$$

thus F^a_{ij} asymptotically proportional to ϕ^a; its magnetic field is nonzero asymptotically. From the ansatz and using the fact that $\mathcal{A}_\mu$ is the component of one-form (so that one could change coordinate from Cartesian to spherical $(x, y, z) \to (r, \theta, \varphi)$ so the basis one-form changes with $dx^a = \sum_b (\partial x^a/\partial y^b) dy^b$ with $y^b = (r, \theta, \varphi)$), we obtain

$$\phi = \eta f(r)(\sin\theta\cos\varphi\sigma^1/2 + \sin\theta\sin\varphi\sigma^2/2 + \cos\theta\sigma^3/2), \tag{1.63}$$

$$\mathcal{A}_r = 0,\ \mathcal{A}_\theta = (1 - a(r))e^{-1}(\sin\varphi\sigma^1/2 - \cos\varphi\sigma^2/2), \tag{1.64}$$

$$\mathcal{A}_\phi = (1 - a(r))e^{-1}\sin\theta(\cos\theta(\cos\varphi\sigma^1/2 + \sin\varphi\sigma^2/2) - \sin\theta\sigma^3/2), \tag{1.65}$$

$$\mathcal{D}_r\phi = \eta f'(r)(\sin\theta\cos\varphi\sigma^1/2 + \sin\theta\sin\varphi\sigma^2/2 + \cos\theta\sigma^3/2), \tag{1.66}$$

$$\mathcal{D}_\theta\phi = \eta f(r)a(r)(\cos\theta\cos\varphi\sigma^1/2 + \cos\theta\sin\varphi\sigma^2/2 - \sin\theta\sigma^3/2), \tag{1.67}$$

$$\mathcal{D}_\varphi\phi = \eta f(r)a(r)\sin\theta(-\sin\varphi\sigma^1/2 + \cos\varphi\sigma^2/2), \tag{1.68}$$

$$\mathcal{F}_{r\theta} = a'(r)e^{-1}(-\sin\varphi\sigma^1/2 + \cos\varphi\sigma^2/2), \tag{1.69}$$

$$\mathcal{F}_{r\varphi} = a'(r)e^{-1}\sin\theta(-\cos\theta(\cos\varphi\sigma^1/2 + \sin\varphi\sigma^2/2) + \sin\theta\sigma^3/2), \tag{1.70}$$

$$\mathcal{F}_{\theta\varphi} = (1 - a(r)^2)e^{-1}\sin\theta(-\sin\theta(\cos\varphi\sigma^1/2 + \sin\varphi\sigma^2/2) - \cos\theta\sigma^3/2). \tag{1.71}$$

Hence, using $[\sigma^a, \sigma^b] = 2i\epsilon^{abc}\sigma^c$ and $\mathrm{tr}(\sigma^a\sigma^b) = 2\delta^{ab}$, for any diagonal metric $g_{\mu\nu}$ with signature $(+, -, -, -)$, we have

$$\begin{aligned}\mathcal{L} \;=\; & \frac{1}{2}g^{rr}(\eta f'(r))^2 + \frac{1}{2}(g^{\theta\theta} + g^{\varphi\varphi}\sin^2\theta)(\eta f(r)a(r))^2 - \frac{\lambda\eta^4}{4}(f(r)^2 - 1)^2 \\ & -\frac{1}{2}g^{rr}(g^{\theta\theta} + g^{\varphi\varphi}\sin^2\theta)\frac{a'(r)^2}{e^2} - \frac{1}{2}g^{\theta\theta}g^{\varphi\varphi}\sin^2\theta\frac{(1 - a(r)^2)^2}{e^2}.\end{aligned} \tag{1.72}$$

Notice that since this is a static configuration, all terms in the Lagrangian density must be semi-negative definite. Hence in literatures with metric sign $(-, +, +, +)$, the Lagrangian density is usually written by exchanging $\frac{1}{2}\mathcal{D}_\mu\phi^a\mathcal{D}^\mu\phi^a \to -\frac{1}{2}\mathcal{D}_\mu\phi^a\mathcal{D}^\mu\phi^a$ so that the terms with η^2 have their sign changed. In a flat space, we have the Lagrangian density as $(' \equiv \frac{d}{dr})$

$$\mathcal{L} = -\frac{\eta^2 f'^2}{2} - \frac{\eta^2 f^2 a^2}{r^2} - \frac{\lambda\eta^4}{4}(f^2 - 1)^2 - \frac{a'^2}{e^2r^2} - \frac{(a^2 - 1)^2}{2e^2r^2}, \tag{1.73}$$

and since it is a static system, then its energy is

$$E = 4\pi \int r^2 dr \left[\frac{\eta^2 f'^2}{2} + \frac{\eta^2 f^2 a^2}{r^2} + \frac{\lambda \eta^4}{4}(f^2-1)^2 + \frac{a'^2}{e^2 r^2} + \frac{(a^2-1)^2}{2e^2 r^4}, \right]. \tag{1.74}$$

The equation of motions are obviously second order derivative of f and a, but Bogomolny method with considering $V=0$ (or $\lambda = 0$ here) we have

$$E = 4\pi \int r^2 dr \left[\left\{ \frac{\eta f'}{\sqrt{2}} \mp \frac{(a^2-1)}{\sqrt{2} e r^2} \right\}^2 + \left\{ \frac{a'}{er} \mp \frac{\eta f a}{r} \right\}^2 \pm \frac{\eta [f(a^2-1)]'}{er^2} \right], \tag{1.75}$$

which tell us that the BPS equations are first order differential equations

$$f' = \pm \frac{(a^2-1)}{\eta e r^2}, \tag{1.76}$$

$$a' = \pm \eta e f a, \tag{1.77}$$

whose energy is

$$E \geq E_{\text{bound}} = \pm \frac{4\pi\eta}{e} f(a^2-1) \Big|_{r\to 0}^{r\to\infty} = \mp \frac{4\pi\eta}{e}. \tag{1.78}$$

This energy is called as the lowest energy and one can use it to determine the model's stability. At $r \to \infty$ we have $f = 1, f' = 0, a = 0, a' = 0$ and since $E = \int d^3x |\vec{B}|^2/2$ then

$$E = 4\pi \int r^2 dr \left[\frac{1}{2e^2 r^4} \right] = 4\pi \int r^2 dr \left[\frac{(\pm \vec{B} \cdot \hat{r})^2}{2} \right]. \tag{1.79}$$

We can see that if we have magnetic field at faraway is similar to Coulomb electric field $\vec{B} = (g/r^2)\hat{r}$ then its magnetic charge is $g = \mp 1/e$, which is similar to Dirac monopole with $n = \mp 2$.

't Hooft propose a different type of field strength

$$\mathcal{F}_{\mu\nu} = \frac{\phi^a}{|\phi|} F^a_{\mu\nu} - \frac{1}{e|\phi|^3} \epsilon^{abc} \phi^a \mathcal{D}_\mu \phi^b \mathcal{D}_\nu \phi^c, \tag{1.80}$$

whose scalar field points in the *same* direction $a = 3$ everywhere $\phi^a = \delta^{a3}|\phi|$, which makes square terms of A^a_μ vanishes at faraway distances

$$\mathcal{F}_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu. \tag{1.81}$$

Since at far distances $F_{ij}^3 = \epsilon^{ijk}\frac{r\cos\theta x^k}{er^4}\Big|_{\theta=0} = \epsilon^{ijk}\frac{x^k}{er^3}$ thus its electric field is zero $E^i = \mathcal{F}_{0i} = 0$ and its magnetic field is $B^k = \frac{1}{2}\epsilon^{ijk}\mathcal{F}_{ij} = \frac{x^k}{er^3}$. Hence the Coulomb-like magnetic field $\vec{B} = (g/r^3)\vec{r}$ has magnetic charge $g = 1/e$ and it is similar to Dirac monopole with $n = 2$.

There are other examples of topological defects that are classified by its homotopy class. With its topological invariant, we have some types of topological solitons: (1) domain walls have, e.g., $\pi_0(\mathbb{Z}_2/\mathbb{I}) \cong \mathbb{Z}_2$, (2) strings have, e.g., $\pi_1(U(1)/\mathbb{I}) \cong \mathbb{Z}$, (3) monopoles have, e.g., $\pi_2(SU(2)/U(1)) \cong \mathbb{Z}$, and (4) textures (also Skyrmions) have, e.g., $\pi_3(SU(2)/\mathbb{I}) \cong \mathbb{Z}$.

The important example that is going to be discussed later is, of course, global monopole by Barriola and Vilenkin [13]. This type of monopole arises from a global transformation group, unlike 't Hooft-Polykov monopole which is from a local one. This monopole lives in a vacuum manifold $\mathcal{M}$ has topological invariant $\pi_2(\mathcal{M}) \neq 0$. This means that the symmetry of gauge field has a mathematical structure that is preserved under a map to the second homotopy group of a certain vacuum manifold $\mathcal{M}$. Thus the non-trivial property of this group, that is the topological invariant $\pi_2(\mathcal{M}) \neq 0$, should also be contained in the field. This comes from an investigation by Kibble [14] that the existence of domain structures is dependent on the topology of the vacuum manifold, which then describe their classification in homotopy theory.

1.3 Barriola-Vilenkin Global Monopole

Now we briefly discuss Ref. [13], that is the starting point of this research. The model is firstly constructed by a Lagrangian density with the following form

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a - \frac{\lambda}{4}(\phi^a\phi^a - \eta^2)^2, \tag{1.82}$$

where ϕ^a is a triplet of scalar fields $a = 1, 2, 3$, and $g_{\mu\nu}$ is a metric of a manifold which has spherical symmetry, i.e., has the following form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = A(r)^2dt^2 - B(r)^2dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.83}$$

(For unfamiliar readers, repeated index means the index is summed over, i.e., $\phi^a\phi^a = \sum\limits_{a=1}^{3}\phi^a\phi^a$ and $g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a = \sum\limits_{a=1}^{3}\sum\limits_{\mu,\nu=0}^{3}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a$.) This scalar field has a global $O(3)$ symmetry that is broken spontaneously to $U(1)$. Since $U(1)$ is homomorphic to 2-sphere S^2, one can imagine the scalar fields to be constrained at the surface of S^2 if $\phi^a\phi^a = \eta^2$. The field configuration which describe monopole has the form of

$$\phi^a = \eta f(r)\frac{x^a}{r}, \tag{1.84}$$

with "Cartesian" coordinate x^a that has a relation with polar coordinate r, θ, φ as follows

$$x^1 = x = r\sin\theta\cos\varphi, \;\; x^2 = y = r\sin\theta\sin\varphi, \;\; x^3 = z = r\cos\theta. \tag{1.85}$$

What this configuration means is that the monopole is described by the solution $f(r)$ that depends only on r due to the spherical symmetry. The plot of potential term in the Lagrangian density with respect to $f(r)$ has a local maximum at $f = 0$ and global minimum at $f = \pm 1$. The equation of motion for the scalar field

$$\frac{1}{ABr^2}\left[\frac{Ar^2 f'}{B}\right]' - \frac{2f}{r^2} - \lambda\eta^2 f(f^2-1) = 0, \tag{1.86}$$

with $f' = df/dr$ and the components of energy momentum tensor are

$$T_0^0 = \frac{\eta^2}{2}\left(\frac{f'^2}{B^2} + 2\frac{f^2}{r^2}\right) + \frac{\lambda\eta^4}{4}(f^2-1)^2, \tag{1.87}$$

$$T_r^r = T_0^0 - \frac{\eta^2 f'^2}{B^2}, \tag{1.88}$$

$$T_\theta^\theta = T_\varphi^\varphi = T_0^0 - \frac{\eta^2 f^2}{r^2}. \tag{1.89}$$

These are obtained by the usual variational method. Numerically, f, A, and B is solved with the following boundary conditions

$$f(r\to 0) = 0, \; f(r\to\infty) = 1, A(r\to\infty) = B(r\to\infty) = 1. \tag{1.90}$$

This implies that the core of the monopole is located at $r = 0$ and has spherical shape with radius $r = \delta \gg 0$ ($f(\delta-\epsilon) \simeq 0, \; f(\delta+\epsilon) \simeq 1$ with ϵ a positive small number). And, $A = 1 = B$ means the spacetime is asymptotically flat.

We know that from the scalar field equation of motion, that the radius of the core δ has $f'(r \sim \delta \gg 0) = 0$ and $f''(r \sim \delta \gg 0) = 0$ thus we can ignore the most left term of Eq. (1.86) and we have $r^2 = 2/\lambda\eta^2(f^2-1)$ and f dimensionless. Hence, the units of radius is determined with

$$\delta \sim \lambda^{-1/2}\eta^{-1}. \tag{1.91}$$

Due to $T^0_0 \sim$ energy density, its energy density's unit at $f(r = 0) = 0, f'(r = 0) \ll 1$ is $\simeq \lambda\eta^4/4$ from the potential term at the most right term on Eq. (1.87). Since $\delta^3 \sim$ unit volume, the core's mass' units, which is equal to its energy since the system is considered to be not moving at all with respect to the choice of coordinate, is determined with

$$M_{\text{core}} \sim \lambda\eta^4\delta^3 \sim \lambda^{-1/2}\eta. \tag{1.92}$$

To ensure the gravity does not change significantly the interior structure of the core at small distances, M_{core} is required to be much smaller than the Plank mass m_P thus $\eta \ll m_P$.

We can consider the exterior solution where $f \simeq 1, f' = 0$ to simplify things. By this we can only consider the metric solutions $A(r)$ and $B(r)$. The components of the Ricci tensor from the metric are

$$\begin{aligned} R^0_0 &= \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB} + \frac{2A'}{rA}\right), \\ R^r_r &= \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB} - \frac{2B'}{rB}\right), \\ R^\theta_\theta = R^\varphi_\varphi &= \frac{1}{B^2}\left(\frac{1}{r^2} + \frac{A'}{rA} - \frac{B'}{rB}\right) - \frac{1}{r^2}. \end{aligned} \tag{1.93}$$

The Einstein equation $R^\mu_\nu = 8\pi G(T^\mu_\nu - \frac{1}{2}T\delta^\mu_\nu)$ and the energy-momentum tensor

$$T^0_0 = T^r_r = \frac{\eta^2}{r^2},\ \ T^\theta_\theta = T^\varphi_\varphi = 0, \tag{1.94}$$

then imply $A = B^{-1}$. Then using R^θ_θ we obtain the intended solution

$$B^{-2} = 1 - 8\pi G\eta^2 - \frac{2GM}{r}, \tag{1.95}$$

with M a constant of integration. This lead us to a Schwarszchild-like metric but with another term containing η with the form

$$ds^2 = \left(1 - 8\pi G\eta^2 - \frac{2GM}{r}\right)dt^2 - \frac{dr^2}{\left(1 - 8\pi G\eta^2 - \frac{2GM}{r}\right)} - r^2 d\Omega^2_{(2)}, \tag{1.96}$$

with $d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta d\varphi^2$ surface segment of S^2. This can be rescaled with $r = \tilde{r}(1 - 8\pi G\eta^2)^{1/2}, t = \tilde{t}/(1 - 8\pi G\eta^2)^{1/2}$ which makes

$$ds^2 = \left(1 - \frac{2G\tilde{M}}{\tilde{r}}\right)d\tilde{t}^2 - \frac{d\tilde{r}^2}{\left(1 - \frac{2G\tilde{M}}{\tilde{r}}\right)} - (1 - 8\pi G\eta^2)\tilde{r}^2 d\Omega^2_{(2)}, \tag{1.97}$$

with M also being rescaled $\tilde{M} = \frac{M}{(1-8\pi G\eta^2)^{3/2}}$. This hence allows us to interpret M as the mass of the monopole. We can see that if $M \gg \delta/G$ it describes a blackhole that contain a global monopole. This happens due to its Kretchmann scalar $K^2 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, i.e.

$$\begin{aligned} K^2 &= \frac{4G^2}{\tilde{r}^6\left(8\pi\eta^2 G - 1\right)^2}\Big(192\pi^2\eta^4 G^2\tilde{M}^2 - 48\pi\eta^2 G\tilde{M}^2 - 128\pi^2\eta^4 G\tilde{M}\tilde{r} \\ &\quad +3\tilde{M}^2 + 16\pi\eta^2\tilde{M}\tilde{r} + 64\pi^2\eta^4\tilde{r}^2\Big), \end{aligned} \tag{1.98}$$

implies absolute singularity at $r = 0$. By ignoring the term with M we obtain

$$ds^2 = d\tilde{t}^2 - d\tilde{r}^2 - (1-\Delta)\tilde{r}^2 d\Omega^2_{(2)}, \tag{1.99}$$

but with its area of the sphere now is $4\pi(1-\Delta)\tilde{r}^2$. One can imagine by dissecting some surface of the sphere, i.e., proportional to a deficit solid angle $\Delta = 8\pi G\eta^2$, and glue the leftover surface making its shape now like a rugby. Though the core exerts no gravitational pull at its surrounding, except at small distances near the core, at faraway it is not flat, i.e., the spacetime topology is not exactly Minkowskian. Since at $M = 0$ the Kretchmann scalar becomes

$$K^2 = \frac{256G^2\pi^2\eta^4}{\tilde{r}^4\left(8\pi\eta^2 G - 1\right)^2} \sim \frac{\lambda^2 M^4_{\text{core}}}{\tilde{r}^4\left(8\pi\eta^2 G - 1\right)^2}, \tag{1.100}$$

this implies singularity at $r = 0$ if the core mass $M_{\text{core}} \sim \eta/\lambda^{1/2}$ is nonzero. This singularity is smoothen out if there is no deficit angle, i.e., the core mass $M_{\text{core}} = 0$. At $\eta = 1/\sqrt{8\pi G}$ everywhere becomes singularity thus η must not be at this value and the metric does not allow $\eta > 1/\sqrt{8\pi G}$ hence it is necessary to satisfy

$$\eta < 1/\sqrt{8\pi G}. \tag{1.101}$$

The constant M can be estimated using $R^0_0 - \frac{1}{2}R\delta^0_0 = 8\pi G T^0_0$, i.e.,

$$\begin{aligned} B^{-2} &= 1 - \frac{8\pi G}{r}\int_0^r dr\, r^2 T^0_0 \\ &= 1 - \frac{8\pi G}{r}\int_0^r dr\, r^2\left(\frac{\eta^2}{2}\left(\frac{f'^2}{B^2} + 2\frac{f^2}{r^2}\right) + \frac{\lambda\eta^4}{4}(f^2-1)^2\right) \\ &= 1 - 8\pi G\eta^2 - \frac{8\pi G\eta^2}{r}\int_0^r dr\, r^2\left(\frac{f'^2}{2B^2} + \frac{f^2-1}{r^2} + \frac{\lambda\eta^2}{4}(f^2-1)^2\right). \end{aligned} \tag{1.102}$$

Comparing to the previous expression we obtain

$$M = 4\pi\eta^2 \int_0^r dr\, r^2 \left(\frac{f'^2}{2B^2} + \frac{f^2-1}{r^2} + \frac{\lambda\eta^2}{4}(f^2-1)^2 \right). \tag{1.103}$$

By estimation $f(\delta-\epsilon)=0, f(\delta+\epsilon)=1$ and $f'(\delta-\epsilon)=0=f'(\delta+\epsilon)$ with $\epsilon \ll \delta$, we can integrate the above expression to obtain

$$M = 4\pi\eta^2 \int_0^\delta dr\, r^2 \left(\frac{-1}{r^2} + \frac{\lambda\eta^2}{4} \right) + 4\pi\eta^2 \int_\delta^r dr\, r^2 0 = 4\pi\eta^2 \left(-\delta + \frac{\lambda\eta^2\delta^3}{12} \right). \tag{1.104}$$

It is physically reasonable to assume that the radius δ is unchanging, i.e., the mass of the monopole must be at the maximum,[1] i.e., $(dM/dr)_{r=\delta} \simeq 0$ which give us

$$\delta \simeq \frac{2}{\sqrt{\lambda}\eta}, \tag{1.105}$$

and thus negative mass

$$M \simeq -\frac{16\pi\eta}{3\sqrt{\lambda}}. \tag{1.106}$$

These properties of global monopole are what motivate us.

Olasagasti and Vilenkin in Ref. [15] had discussed this global defect at higher dimensional spacetimes with nonzero cosmological constant Λ. The authors use a spherically symmetric metric with $(p-1)$-dimensional brane with metric $\hat{g}_{\mu\nu}$ and solid angle of a $(n-1)$-sphere denoted by $d\Omega_{(n-1)}$ with basically the following form

$$ds^2 = A(r)^2 \hat{g}^{(p)}_{\mu\nu} dx^\mu dx^\nu - B(r)^2 dr^2 - C(r)^2 d\Omega^2_{(n-1)}. \tag{1.107}$$

They found that there is a deficit angle in flat higher dimensional spacetimes with $\Lambda = 0$. What is interesting are one class of solutions at $\Lambda > 0$ the manifold is an inflating spacetime with spherical symmetry, and one class of solutions at $\Lambda < 0$ has a spacetime with exponential warp factor with cigar-like geometry. This is similar to what Gregory obtained in Ref. [16] for $n = 2$ and Randall and Sundrum in Ref. [17] for $n = 1$.

This is verified by numerical calculation obtained by Cho and Vilenkin in Ref. [18] for $p = 4$ and $n = 3$. The solution is flat with constant solid angle for small η. At η sufficiently large enough $\Delta \sim 1$, the static solution gives singularity at a finite

[1] One might relate it to solving TOV equation numerically to obtain a mass of, say, a neutron star. One calculate it initially from any value of pressure at the center of the star to finally stop when the pressure is zero at the surface of the star. This boundary condition turns out give the mass curve to be increasing from zero at the center to a certain value at the surface. The plot of the mass over radius itself has nearly zero gradient at the surface.

distance from the core. This singularity can be removed if the solution is nonstatic allowing the defect worldsheet to inflate.

One can also arrive at the same results of the previous results mathematically using sigma model introduced in appendix A of Ref. [19]. The solutions obtained using sigma model has an advantage of being valid everywhere. This is what the previously discussed steps lacking by an approximation of $f \simeq 1$ at large r. This model has a slightly different Lagrangian density

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} h_{ab} \eta^2 \partial_\mu \phi^a \partial_\nu \phi^b - \frac{\lambda}{4} \eta^4 (h_{ab} \phi^a \phi^b - 1)^2, \tag{1.108}$$

with $h_{ab} = h_{ab}(\phi^c)$ is a metric of the 'inner' field of the scalar field with $a, b = 1, ..., n$ and $g_{\mu\nu}$ the same metric as Eq. (1.83). The scalar filed has n degrees of freedom. This model treats the scalar field to be constrained by

$$h_{ab} \phi^a \phi^b = 1, \tag{1.109}$$

hence the second term in the Lagrangian density is a term multiplied with a Lagrange multiplier λ and reducing one scalar field's degrees of freedom hence only $n - 1$ remains. The scalar field now must satisfy

$$\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} \eta^2 h_{ab} g^{\mu\nu} \partial_\nu \phi^b \right) = \frac{\eta^2}{2} g^{\alpha\beta} \partial_\alpha \phi^m \partial_\beta \phi^n \frac{\partial h_{mn}}{\partial \phi^a}, \tag{1.110}$$

with $|g| = |\det g_{\mu\nu}| = ABr^2 \sin\theta$. Now one can choose the simplest ansatz

$$\phi^a(\theta^a) = \theta^a, \tag{1.111}$$

with $\theta^a = \{\theta, \varphi\}$ the angle of 2-sphere of metric $g_{\mu\nu}$. This ansatz can satisfy Eq. (1.110) if

$$h_{ab}(\phi^c) = -\frac{1}{r^2} g_{ab}(r, \theta^c). \tag{1.112}$$

From these one obtain the components of the energy-momentum tensor to be

$$T_0^0 = T_r^r = \frac{\eta^2}{r^2}, T_\theta^\theta = 0, \tag{1.113}$$

and the Ricci tensor's components are the same as Eq. (1.93). These are the same as before and one will arrive at the same results as before.

The kinetic term in the Lagrangian density that we discussed before has canonical form. There are others that discussed the noncanonical form, e.g., power-law type in Refs. [20, 21] and DBI type in Ref. [22]. These form has the advantage of being stable at certain dimensions. Using Derrick's theorem one can see that the canonical form is stable only in a manifold $\mathcal{M}$ whose dimension is $\dim \mathcal{M} \leq 2$. By adding a non-linear term in the Lagrangian density, e.g., adding a squared term $-\beta^{-2}(\frac{1}{2}g^{\mu\nu}h_{ab}\eta^2\partial_\mu\phi^a\partial_\nu\phi^b)^2$ with β a coupling constant, stablility can be obtained at dimension $\dim(\mathcal{M}) \geq 3$. The nonlinearity of these forms give arise to interesting properties of, e.g., deficit angle and core's mass.

Now we are ready to discuss the outline of this master thesis. The following chapters will contains the results of our research.

1.4 The Outline

Now we know that a monopole is a non-contractible defect due to having its second fundametal group over 2-sphere is non-trivial $\pi_2(S^2) \cong \mathbb{Z}$ [14] thus it may exists at least theoretically. If coupled with gravity, the monopole gives a deficit solid angle in the spacetime around its core [13]. Due to the monopole is usually constructed to be living in a spontaneously symmetry breaking potential with its minimum located at η, the deficit angle gives an upper bound $\eta_c \equiv 1/\sqrt{8\pi G}$. If this is violated, the space-time becomes singular.

In [15] it is found that the space-time will have a cigar-like shape if $\eta > \eta_c$. In [18] and [23], the factorized solutions are shown numerically to be non-static. It also shown numerically that regular solutions may exists if $\eta \lesssim \sqrt{3/(8\pi G)}$ in [24], which singularity will emerge if this condition is unsatisfied. This is then interpreted as topological inflation solutions by [25, 26].

It is shown in [27] that solutions regarding deficit angle in both gravitational field and factorization due to the existence of global monopole can be seen. By modifying the metric, the Ricci tensor components then have additional terms. Those then are interpreted as contributions which makes the global defect in the space-time. Factorization by scalars can be done [28] and a sigma model can be used to investigate the factorization by global monopole [19].

Defect can arise without providing a symmetry-breaking potential term from non-linear kinetic terms in the Lagrangian density [29]. This is named in [20] as k-defect. In the case of topological defect from monopoles, it is called in [21] as k-monopole, with the properties of the gravitational field from Barriola-Vilenkin monopole in [13] still hold with slight differences. The mass can be positive or negative valued, which respectively make the gravitational field becomes attractive or repulsive.

This non-linearity can evade Derrick's theorem to enable a construction of a stable model in higher dimensions. By adding non-linear terms, stablility can be obtained at manifold $\mathcal{M}$ with dimension $\dim(\mathcal{M}) \geq 3$. The well-known Dirac-Born-Infeld theory [30, 22] is one such non-linearity. This advantage is a reason why it is used in other areas of physics, such as global strings [31], cosmic strings [32], chiral theory [33], solitons [6], and textures [34]. The power-law monopole is another example of k-defect.

In this thesis, we propose solutions from a non-linear sigma model in a finite higher-dimensional space-times. We use a $p + D$-dimensional space-time in investigating some solutions of a scalar field. Its metric is defined to have boost symmetry and spherical symmetry

$$ds^2 = g_{MN} dx^N dx^N = A^2(r)\hat{g}^{(p+1)}_{\mu\nu} dx^\mu dx^\nu - B^2(r)dr^2 - C^2(r)d\Omega^2_{(D-2)}, \quad (1.114)$$

with $M, N = 0, ..., p+D$. This metric consists of two spaces that are tied with each other: one has Lorentzian metric $\hat{g}_{\mu\nu}$ with signature $+--...$ with $(p+1)$-dimension denoted by greek indices $\mu, \nu = 0, ..., p$, the other is a $(D-2)$-sphere, S^{D-2}, denoted by $d\Omega^2_{D-2} = \tilde{g}_{ij} d\theta^i d\theta^j$ with lowercase latin indices $i, j = 1, ..., D-2$.

The model is as follows. Originally, the action is defined to be

$$\mathcal{S} = \int d^{p+D}x \sqrt{|g|} \left[\frac{R - 2\Lambda}{16\pi G} + \frac{1}{2}\partial_M \vec{\Phi} \partial^M \vec{\Phi} - \frac{\lambda}{4}\left(\vec{\Phi}^2 - \eta^2\right)^2 \right]. \quad (1.115)$$

(In Barriola-Vilenkin, $p = 0$ and $D = 4$.) In this model, the scalar field's degree of freedom is N, thus it is often denoted by a vector $\vec{\Phi}$ with N orthonormal basis. We choose the scalar field to be constrained on the minima of the potential $\vec{\Phi} \cdot \vec{\Phi} = \eta^2$, which makes them live only in the surface of S^{N-1}. The degrees of freedom is then reduced by one degree ($N - 1$).

With these constructions, we can write a non-linear sigma model as the following form

$$\mathcal{S} = \int d^{p+D}x \sqrt{|g|} \left[\frac{R - 2\Lambda}{16\pi G} - X - \frac{X^2}{\beta^2} \right], \quad (1.116)$$

with $X = -\frac{1}{2}\eta^2 h_{ij} \partial_M \phi^i \partial^M \phi^j$. The scalar field is defined to have a dependence on ϕ^i with $i, j = 1, \ldots, N-2$. $h_{ij} = h_{ij}(\phi^k) = \frac{\partial \vec{\Phi}}{\partial \phi^i} \cdot \frac{\partial \vec{\Phi}}{\partial \phi^j}$ is the inner space metric of the scalar field.

The sigma model then gives us the equation of motion for the scalar field and the energy-momentum tensor

$$\frac{1}{\sqrt{|g|}}\partial_M\left(\sqrt{|g|}(1+2\beta^{-2}X)\eta^2 h_{ij}\partial^M\phi^j\right)=(1+2\beta^{-2}X)\frac{\eta^2}{2}\partial_M\phi^m\partial^M\phi^n\frac{\partial h_{mn}}{\partial\phi^i}, \tag{1.117}$$

$$T^M_N=\delta^M_N\left[\frac{\Lambda}{8\pi G}+X+\beta^{-2}X^2\right]+(1+2\beta^{-2}X)\eta^2 h_{ij}\partial^M\phi^i\partial_N\phi^j. \tag{1.118}$$

Choosing an ansatz that satisfies (1.117) now becomes crucial. Following [28, 19], we choose the simplest ansatz that respects spherical symmetry,

$$\phi^i(\theta^i)=\theta^i, \tag{1.119}$$

where now $i=1,...,D-2$. This ansatz satisfies if and only if we choose the internal metric that was dependent on the field ϕ^i now dependent on the metric that is dependent on the angles $\theta^1,...,\theta^{D-2}$

$$h_{ij}(\phi^k)=-C^{-2}(r)g_{ij}(r,\theta^k). \tag{1.120}$$

With this, $X=(D-2)\eta^2/2C^2$ which is independent to θ^i and thus (1.117) is satisfied.

Respectively, the metric and the Lagrangian density then give us the components for Einstein equations starting with p and D unspecified as follows:

$$R^0_0 = B^{-2}\left[\frac{A''}{A}-\frac{A'B'}{AB}+p\left(\frac{A'}{A}\right)^2+(D-2)\frac{A'C'}{AC}\right]+\frac{\hat{R}}{(p+1)A^2}, \tag{1.121}$$

$$R^r_r = B^{-2}\left[(p+1)\frac{A''}{A}+(D-2)\frac{C''}{C}-\frac{B'}{B}\left\{(p+1)\frac{A'}{A}+(D-2)\frac{C'}{C}\right\}\right], \tag{1.122}$$

$$R^\theta_\theta = B^{-2}\left[\frac{C''}{C}+\frac{C'}{C}\left\{(p+1)\frac{A'}{A}-\frac{B'}{B}+(D-3)\frac{C'}{C}\right\}\right]-\frac{k(D-3)}{C^2}. \tag{1.123}$$

and

$$T^0_0=T^r_r=\left[\frac{\Lambda}{8\pi G}+X+\beta^{-2}X^2\right], \tag{1.124}$$

$$T^\theta_\theta=\left[\frac{\Lambda}{8\pi G}+X+\beta^{-2}X^2\right]-(1+2\beta^{-2}X)\frac{\eta^2}{C^2}, \tag{1.125}$$

with $X=(D-2)\eta^2/2C^2$. [For readers interested on how to calculate these, we explain them in Appendix A in page 55.] Constants $\hat{R}$ and k comes from setting

$\hat{g}_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ to has constant curvature.[2] To simplify calculations from this point, we choose the brane to be flat $\hat{R} = 0$ by $\hat{g}_{\mu\nu} = \hat{\eta}_{\mu\nu}$ and, since $d\Omega_{D-2}$ denotes a surface segment of S^{D-2} with radius $= 1$, $k = 1$. We show them here since they will be used in the remaining chapters.

From here, we calculate the solutions by solving the above equations of motion. There are two class of solutions: *blackholes* and *compactification*. From here *compactification* more accurately means **factorized (metric) solutions** and we sometimes mention *compactification radius* that means **constant (metric) factor** implicitly. We discuss the model in separate chapters for different dimensions. Most of the analytic calculations are done by hand but if they are too complex we use *Mathematica* software. We write this thesis with the following key ideas.

Chapter I. **Introduction**
Here we describe this research background, the problems which we investigate, the boundaries of the problems, our goals in this research, and key ideas for the writing.

Chapter II. **Gravitation of Noncanonical Global Monopole in 4-dimensional Spacetimes**
Here we discuss gravitating global monopole in sigma model with noncanonical global monopole with nonlinear kinetic term in a 4-dimensional spacetime.

Chapter III. **Gravitation of Noncanonical Global Monopole in D-dimensional Spacetimes**
Here we focus on a D-dimensional space-time ($p = 0$) then we discuss the gravitational field. We discuss first when $D = 4$ then gradually adding the dimensions one by one $D = 5, 6, 7, ...$

Chapter IV. **Factorized Solutions in $(p + D)$-dimensional Spacetimes**
Here we focus only on obtaining factorization solutions at $p > 0, D > 4$.

Chapter V. **Conclusions**
Here we conclude with mentioning future projects from this.

[2] A space with constant curvature has the Riemann tensor with the form $R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc})$, with K Gaussian curvature of a d-dimensional space. Contracting a and c give us $R_{bd} = K(d-1)g_{bd}$ and contracting the rest give us $R = Kd(d-1)$. Three example of this space are flat space ($K = 0$), sphere with radius a ($K = 1/a^2$), and hyperboloid with "radius" a ($K = -1/a^2$). d-sphere and flat d-dimensional spacetime is used in metric $\tilde{g}_{ij}$ and $\hat{g}_{ij}$, respectively.

CHAPTER 2
GRAVITATION OF NONCANONICAL GLOBAL MONOPOLE IN 4-DIMENSIONAL SPACETIMES

Here we discuss the gravitational field and constant metric factorization solutions at $p = 0, D = 4$, which are the results we published in [35, 36]. The other spacetimes with $p = 0, D \geq 4$ and $p > 0, D \geq 4$ will be discussed in chapter 3 and 4, respectively.

2.1 Gravitational Field Solution

Now we discuss our results with $p = 0, D = 4$. First we choose ansatz $C(r) = r$. We can see that due to $T_0^0 = T_r^r$ we have

$$R_0^0 - R_r^r = 0 = B^{-2}\left[\frac{2A'}{Ar} + \frac{2B'}{Br}\right] \tag{2.1}$$

from setting $p = 0, D = 4$ in Eqs. (1.121)-(1.125), which after using boundary condition $A(r \to \infty) = 1 = B(r \to \infty)$ implies

$$A = B^{-1}. \tag{2.2}$$

This simplifies the rest of the Einstein equation that we are not use yet, which is

$$R_\theta^\theta = 8\pi G\left[T_\theta^\theta - \frac{T_0^0 + T_r^r + 2T_\theta^\theta}{2}\right] \tag{2.3}$$

or in explicit expression

$$B^{-2}\left[\frac{1}{r}\left\{-\frac{2B'}{B} + \frac{1}{r}\right\}\right] - \frac{1}{r^2} = -8\pi G\left[\frac{\Lambda}{8\pi G} + X + \beta^{-2}X^2\right] \tag{2.4}$$

with $X = \eta^2/r^2$, after a little arragement we have

$$\frac{1}{r^2}\left[\frac{r}{B^2}\right]' = \frac{1}{r^2} - 8\pi G\left[\frac{\Lambda}{8\pi G} + \frac{\eta^2}{r^2} + \beta^{-2}\frac{\eta^4}{r^4}\right], \tag{2.5}$$

and after integration with $2GM$ as constant of integration we obtain

$$B^{-2} = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3} - 8\pi G\eta^2 + 8\pi G\beta^{-2}\frac{\eta^4}{r^2}. \tag{2.6}$$

Inputting this to the metric we have

$$\begin{aligned} ds^2 &= \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3} - 8\pi G\eta^2 + 8\pi G\beta^{-2}\frac{\eta^4}{r^2}\right) dt^2 \\ &\quad - \frac{dr^2}{\left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3} - 8\pi G\eta^2 + 8\pi G\beta^{-2}\frac{\eta^4}{r^2}\right)} - r^2 d\Omega^2_{(2)}. \end{aligned} \tag{2.7}$$

This metric has absolute singularity at $r = 0$ due to its Kretchmann scalar $K^2 = R_{ABMN}R^{ABMN}$, whose explicit expression is

$$\begin{aligned} K^2 &= \frac{8}{B^4}\left[\left(\frac{A'}{Ar}\right)^2 + \left(\frac{B'}{Br}\right)^2\right] + \frac{4}{B^4}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)^2 + \frac{16}{r^4}\left(1 - \frac{1}{B^2}\right)^2 \\ &= \left[\frac{4GM}{r^3} - \frac{144\eta^4\pi G - 2\beta^2\Lambda r^4}{3\beta^2 r^4}\right]^2 \\ &\quad + \frac{16}{r^4}\left(\frac{\Lambda r^2}{3} + 8\pi G\eta^2 + \frac{2GM}{r} - \frac{8\eta^4\pi G}{\beta^2 r^2}\right)^2 \\ &\quad + \frac{4}{r^2}\left(-\frac{2\Lambda r}{3} + \frac{2GM}{r^2} - \frac{16\eta^4\pi G}{\beta^2 r^3}\right)^2, \end{aligned} \tag{2.8}$$

blows up at $r = 0$.

Now rescaling by $r = \tilde{r}(1 - 8\pi G\eta^2)^{1/2}, t = \tilde{t}(1 - 8\pi G\eta^2)^{-1/2}$ will give us

$$ds^2 = f_4(\tilde{r})d\tilde{t}^2 - \frac{d\tilde{r}^2}{f_4(\tilde{r})} - (1 - 8\pi G\eta^2)\tilde{r}^2 d\Omega^2_{(2)}, \tag{2.9}$$

with the rescaled metric solution

$$f_4(\tilde{r}) = 1 - \frac{2G\tilde{M}}{\tilde{r}} - \frac{\Lambda\tilde{r}^2}{3} + \frac{8\pi G\eta^4}{\tilde{\beta}^2\tilde{r}^2}, \tag{2.10}$$

and the rescaled constants $\tilde{M} = M(1 - 8\pi G\eta^2)^{-3/2}$ and $\tilde{\beta} = \beta(1 - 8\pi G\eta^2)$. After this, we suppress the tilde symbol above the rescaled r, t and constants in the rescaled metric solution, since this rescaling only alter the coordinate singularity and the absolute singularity is at $r = 0$. The coordinate singularity, commonly called as horizons, of this rescaled solution is what we discussed next. This rescaled metric solution turns out has the same feature as Reissner-Nördstrom-de Sitter but with “scalar charge” that comes from the nonlinear term and has deficit angle from

$$\eta^2 < 1/8\pi G. \tag{2.11}$$

If M is large enough, one can see this is a blackhole solution with three horizons, i.e., inner horizon r_-, outer horizon r_+, and cosmological horion r_c. Calculating each is not an easy task, thus we simplify this considering $\Lambda = 0$ and $\Lambda \neq 0$.

For the $\Lambda = 0$ case, one can see that the inner and outer horion can be calculated from $f_4(r) = 0$ to be having the following form

$$r_\pm = GM\left(1 \pm \sqrt{1 - \frac{8\pi\eta^4}{M^2G\beta^2}}\right). \tag{2.12}$$

The extremal case with both horizons coincide $r_- = r_+$ is also considered to be interesting due to some reasons, such as thermodynamic properties, which are not going to be discussed here. There is an upperbound of the nonlinearity constant

$$\beta^2 \leq \frac{8\pi\eta^4}{M^2G}, \tag{2.13}$$

whose equal sign correspond to the extremal case of $r_- = r_+$. This condition is not disturbing the existence of deficit angle. Thus here the horizons are dependent of β.

For the case of $\Lambda \neq 0$, we can consider three extremal cases, i.e., $r_- = r_+$, $r_+ = r_c$, and $r_- = r_+ = r_c$, by following Ref. [37]. We can re-express the solution by employing

$$f_4(r) = \left(1 - \frac{\rho}{r}\right)^2 \left(1 - \frac{\Lambda}{3}(r^2 + br + a)\right), \tag{2.14}$$

with ρ the coincide horizon. By comparing to (2.10) one obtain $b = 2\rho,\ a = 3\rho^2$ which implies

$$GM = \rho\left(1 - \frac{2\Lambda}{3}\rho^2\right), \tag{2.15}$$

$$\frac{8\pi G\eta^4}{\beta^2} = \rho^2\left(1 - \Lambda\rho^2\right). \tag{2.16}$$

From the ”electric charge” we can see that $\rho \leq 1/\sqrt{\Lambda}$, whose equal sign corresponds to Schwarzschild-de Sitter-like solution. From mass term M that must be positive definite we have $\rho^2 \leq 3/2\Lambda$ which is not violated. To obtain the

$r_- = r_+ = r_c$ case, one can see that

$$\left(1 - \frac{\Lambda}{3}(r^2 + 2\rho r + 3\rho^2)\right)_{r=\rho} = 0 \tag{2.17}$$

implies $\rho = 1/\sqrt{2\Lambda}$. Now let us substract this value with δ ($< 1/\sqrt{2\Lambda}$) and input this to the solution which give us

$$f_4(r)_{\rho=\frac{1}{\sqrt{2\Lambda}}-\delta} = \left(1 - \frac{(\frac{1}{\sqrt{2\Lambda}} - \delta)}{r}\right)^2 \left(\frac{1}{2} + \sqrt{2\Lambda}\delta - \Lambda\delta^2 - \frac{\Lambda}{3}r^2 + \frac{2\Lambda\delta - \sqrt{2\Lambda}}{3}r\right). \tag{2.18}$$

Then, the terms in the most right bracket has roots at

$$r = \rho_\pm = \left(\frac{1}{\sqrt{2\Lambda}} - \delta\right)\left(\pm\frac{2\sqrt{2}\sqrt{\Lambda\left(\delta^2(-\Lambda) + \sqrt{2}\delta\sqrt{\Lambda} + 1\right)}}{\sqrt{2}\sqrt{\Lambda} - 2\delta\Lambda} - 1\right). \tag{2.19}$$

When $\delta \to 0$, $\rho_+ = 1$ and $\rho_- = -3$. Suppose $\delta > 0$ then one of the roots will be larger than ρ,

$$\rho_+ \simeq \left(\frac{1}{\sqrt{2\Lambda}} - \delta\right)\left(1 + 3\sqrt{2}\delta\sqrt{\Lambda} + \frac{9\delta^2\Lambda}{2}\right). \tag{2.20}$$

This implies that at $0 < \rho < 1/\sqrt{2\Lambda}$ we have the case of $r_- = r_+$. Otherwise, we have ρ_+ less than ρ when $\delta < 0$, implies that at $1/\sqrt{2\Lambda} < \rho \leq 1/\sqrt{\Lambda}$ we have the case of $r_+ = r_c$. Hence we obtain the following

$$\underbrace{0 < \rho < 1/\sqrt{2\Lambda}}_{r_-=r_+}, \quad \overbrace{\rho = 1/\sqrt{2\Lambda}}^{r_-=r_+=r_c}, \quad \underbrace{1/\sqrt{2\Lambda} < \rho \leq 1/\sqrt{\Lambda}}_{r_+=r_c}. \tag{2.21}$$

One may verify this by plotting Eq. (2.10) with Eq. (2.15) and Eq. (2.16), which we show at Fig. 2.1.

2.2 Factorized Solution

Now we consider ansatz $C(r) = C = const.$ with the same spacetime dimension ($p = 0, D = 4$). Using the Einstein equations with the form $G^M_N = R^M_N - \delta^M_N R/2 =$

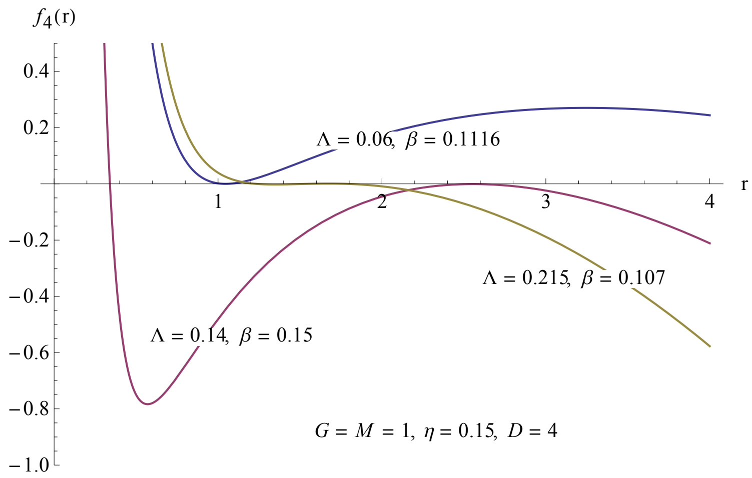

Figure 2.1: A plot of the three cases of extremal horizon in 4-dimensional spacetime.

$8\pi G T^M_N$ we have

$$\frac{1}{C^2} = \Lambda + \frac{8\pi G\eta^2}{C^2} + \frac{8\pi G\eta^4}{C^4\beta^2}, \tag{2.22}$$

$$-\frac{1}{B^2}\frac{A''}{A} + \frac{1}{B^2}\frac{A'B'}{AB} = \Lambda - \frac{8\pi G\eta^4}{C^4\beta^2}. \tag{2.23}$$

Now we calculate $A(r)$ and $B(r)$ first. Since C is just a constant, from (2.23) we have

$$B^{-2}\left(\frac{A'B'}{AB} - \frac{A''}{A}\right) = \pm\omega^2, \tag{2.24}$$

where we define a constant with real value ω

$$\pm\omega^2 \equiv \Lambda - \frac{8\pi G\eta^4}{C^4\beta^2} \tag{2.25}$$

Here we use plus-minus sign to ensure $\omega^2 \geq 0$. Now by choosing ansatz $B = 1$, Eq. (3.20) give us

$$ds^2 = \frac{1}{\omega^2}(\sin^2\chi\, dt^2 - d\chi^2) - C^2 d\Omega^2_{(2)}, \quad \text{for } +\omega^2, \tag{2.26}$$

$$ds^2 = dt^2 - dr^2 - C^2 d\Omega^2_{(2)}, \quad \text{for } \omega = 0, \tag{2.27}$$

$$ds^2 = \frac{1}{\omega^2}(\sinh^2\chi\, dt^2 - d\chi^2) - C^2 d\Omega^2_{(2)}, \quad \text{for } -\omega^2, \tag{2.28}$$

with $\chi \equiv \omega r$. Using another ansatz $B = A^{-1}$ will give us

$$ds^2 = (1 - \omega^2 r^2)\, dt^2 - \frac{dr^2}{(1 - \omega^2 r^2)} - C^2 d\Omega_{(2)}^2, \quad \text{for } +\omega^2, \tag{2.29}$$

$$ds^2 = dt^2 - dr^2 - C^2 d\Omega_{(2)}^2, \quad \text{for } \omega = 0, \tag{2.30}$$

$$ds^2 = (1 + \omega^2 r^2)\, dt^2 - \frac{dr^2}{(1 + \omega^2 r^2)} - C^2 d\Omega_{(2)}^2, \quad \text{for } -\omega^2. \tag{2.31}$$

The factorized space are $(dS_2 \;/\; M_2 \;/\; AdS_2) \times S^2$ when $B^{-2}\left(\frac{A'B'}{AB} - \frac{A''}{A}\right)\ (> / = / <)\ 0$ respectively.

Now we calculate the constant factor C. From (2.22) we get the constant factor to be

$$C^2 = \frac{8\pi G \eta^4}{(1 - 8\pi G \eta^2)\beta^2} \tag{2.32}$$

for $\Lambda = 0$ case and

$$C_\pm^2 = \frac{(1 - 8\pi G\eta^2) \pm \sqrt{(1 - 8\pi G\eta^2)^2 - 32\pi G\eta^4 \Lambda \beta^{-2}}}{2\Lambda} \tag{2.33}$$

for $\Lambda \neq 0$ case. At $\beta \to \infty$, $C^2 = C_-^2 = 0$ and $C_+^2 = (1 - 8\pi G\eta^2)/\Lambda$, thus $\Lambda = 0$ is forbidden.

Now for $\Lambda \neq 0$ we choose C_+^2. To guarantee $C^2 > 0$, the symmetry breaking scale must satisfy

$$\eta^2 < 1/8\pi G. \tag{2.34}$$

For $C_+^2 > 0$ it requires

$$\eta^2 \leq \frac{1}{8}\left(\frac{2\beta^2}{2\pi\beta^2 G - \Lambda} + \sqrt{\frac{2\beta^2\Lambda}{\pi G\left(\Lambda - 2\pi\beta^2 G\right)^2}}\right) \equiv \eta_+^2 \tag{2.35}$$

for $\Lambda > 2\pi\beta^2 G$,

$$\eta^2 \leq 1/16\pi G < 1/8\pi G \tag{2.36}$$

for $\Lambda = 2\pi\beta^2 G$, and

$$\eta^2 \leq \frac{1}{8}\left(\frac{2\beta^2}{2\pi\beta^2 G - \Lambda} - \sqrt{\frac{2\beta^2\Lambda}{\pi G\left(\Lambda - 2\pi\beta^2 G\right)^2}}\right) \equiv \eta_-^2 < 1/8\pi G \tag{2.37}$$

for $\Lambda < 2\pi\beta^2 G$. These satisfy

$$\eta_-^2 < \eta_+^2. \tag{2.38}$$

It is interesting that $1/8\pi G < \eta_+^2$ if $\Lambda < 2\pi\beta^2 G$ but this is a contradiction. This means that $\Lambda > 0$ gives an upperbound that is always less than $1/8\pi G$.

Table 2.1: Conditions for k-monopole with constant factorizable metric in 4-dimension.

	$B^{-2}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)>0$	$B^{-2}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)=0$	$B^{-2}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)<0$
$\Lambda > 2\pi\beta^2 G$	$\eta^2 < \eta_+^2$	$\eta^2 = \eta_+^2$	Cannot happen
$\Lambda = 2\pi\beta^2 G$	$\eta^2 < \eta_+^2$	$\eta^2 = 1/16\pi G$	Cannot happen
$\Lambda < 2\pi\beta^2 G$	$\eta^2 < \eta_-^2$	$\eta^2 = \eta_-^2$ or $\eta^2 = \eta_+^2$	Cannot happen
$\Lambda = 0$	Cannot happen	Cannot happen	$\eta^2 \neq \frac{1}{8\pi G}$ applies
$\Lambda < 0$		Cannot happen	

Substituting the conditions above ((2.34)-(2.37)) we compute numerically asking whether these conditions hold or not using the algebraic software we mentioned before. The results are shown in Table 2.1. This imply the following allowed factorization channels

$$dS_4 \longrightarrow \begin{cases} dS_2 \times S^2, \\ M_2 \times S^2, \end{cases} \tag{2.39}$$

$$M_4 \longrightarrow AdS_2 \times S^2. \tag{2.40}$$

These solutions are quite interesting but we do not obtain what is happening at $\eta^2 \geq 1/8\pi G$. Also, we are not certain whether these are stable or not. Due to the factorization process in these spacetimes leave us only 2-dimensional non-factorized spacetime, the gravity in this manifold will give us a divergent term in the metric and effective potential, which we will explain in section 5. This implies that we cannot guarantee the stability of these factorization solutions at $p = 0$.

CHAPTER 3
GRAVITATION OF NONCANONICAL GLOBAL MONOPOLE IN D-DIMENSIONAL SPACETIMES

Here we discuss the other gravitational field and factorization solutions at $p = 0, D > 4$. These results are taken form a paper draft that is in peer-reviewing process by a journal.

The same as before, we apply $C(r) = r$. Now writing the Einstein equations in the form $R^A_B = 8\pi G(T^A_B - \delta^A_B \frac{T}{(D-2)})$ gives us

$$R^0_0 = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + (D-2)\frac{A'}{rA}\right] = -\frac{2}{D-2}\Lambda + \frac{(D-2)}{2}\frac{8\pi G\eta^4}{\beta^2 r^4}, \quad (3.1)$$

$$R^r_r = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} - (D-2)\frac{B'}{rB}\right] = -\frac{2}{D-2}\Lambda + \frac{(D-2)}{2}\frac{8\pi G\eta^4}{\beta^2 r^4}, \quad (3.2)$$

$$\begin{aligned} R^\theta_\theta &= \frac{1}{B^2}\left[\frac{A'}{rA} - \frac{B'}{rB} + \frac{D-3}{r^2}\right] - \frac{D-3}{r^2} \\ &= -\frac{2}{D-2}\Lambda - \frac{8\pi G\eta^2}{r^2} - \frac{(D-2)}{2}\frac{8\pi G\eta^4}{\beta^2 r^4}. \end{aligned} \quad (3.3)$$

Using these is more convenient since we can easily see that $R^0_0 = R^r_r$ will give us the same result as the previous discussion, that is $A = B^{-1}$. Also, from R^θ_θ we can see that we can obtain B since

$$R^\theta_\theta = \frac{1}{r^{D-2}}\left(\frac{r^{D-3}}{B^2}\right)' - \frac{D-3}{r^2}, \quad (3.4)$$

after we substitute A with B. With M a constant of integration, this in turn give us

$$B^{-2} = 1 - \frac{8\pi G\eta^2}{(D-3)} - \frac{2\Lambda r^2}{(D-2)(D-1)} - \frac{4(D-2)\pi G\eta^4}{(D-5)\beta^2 r^2} - \frac{2GM}{r^{(D-3)}}, \quad (3.5)$$

which is singular at $D = 5$. Again by rescaling the metric with $r \to r(1 - \frac{8\pi G\eta^2}{D-3})^{-1/2}, t \to t(1 - \frac{8\pi G\eta^2}{D-3})^{1/2}$ this give us

$$ds^2 = f_D(r)dt^2 - \frac{dr^2}{f_D(r)} - \left(1 - \frac{8\pi G\eta^2}{(D-3)}\right) r^2 d\Omega^2_{(D-2)}, \quad (3.6)$$

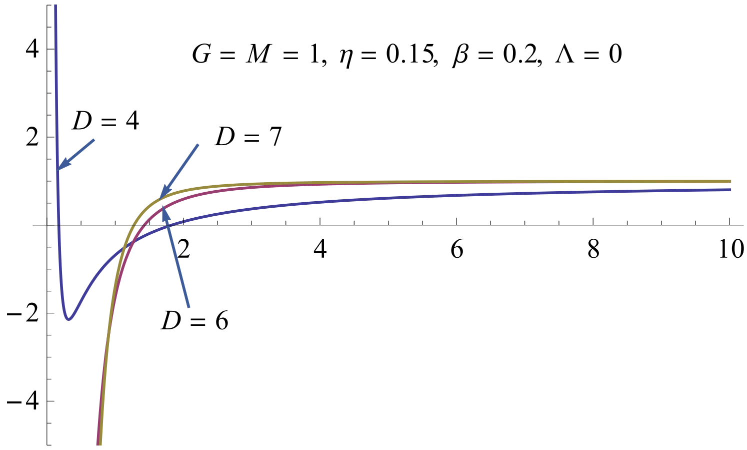

Figure 3.1: The case of different D with $\Lambda = 0$.

with the rescaled metric solution

$$f_D(r) = 1 - \frac{2\Lambda r^2}{(D-2)(D-1)} - \frac{4(D-2)\pi G\eta^4}{(D-5)\beta^2 r^2} - \frac{2GM}{r^{(D-3)}}, \tag{3.7}$$

and the rescaled constants $M \to M(1 - \frac{8\pi G\eta^2}{(D-3)})^{(1-D)/2}$ and $\beta \to \beta(1 - \frac{8\pi G\eta^2}{(D-3)})$.

If we plot (3.7), we will see that there is one horizon if $\Lambda = 0$ since the third term in (3.7) is not changing sign, unlike when $D = 4$. (The case of $\Lambda = 0$ with different D is shown in Fig. 3.1.) Finding the exact expression at the horizon $r = \rho$ when $\Lambda \leq 0$ is not easy but we can see that this is analogue to Tangherlini solutions, i.e. a higher dimensional Schwarszchild metric with $D \geq 4$

$$ds^2 = \left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right) dt^2 - \frac{dr^2}{(1 - (r_0/r)^{D-3})} - r^2 d\Omega^2_{(D-2)}, \tag{3.8}$$

but with deficit angle and additional complex or minus valued horizons, which are not evident in the plot. At $\Lambda < 0$, this is also true and it resemble Tangherlini-anti-de Sitter solutions

$$ds^2 = F_D dt^2 - \frac{dr^2}{F_D} - r^2 d\Omega^2_{(D-2)}, \tag{3.9}$$

with

$$F_D(r) = 1 - \left(\frac{r_0}{r}\right)^{D-3} + \frac{2|\Lambda| r^2}{(D-2)(D-1)}. \tag{3.10}$$

At the $\Lambda > 0$ case, we basically obtain another horizon called cosmological horizon ρ_c that is greater than its inner horizon r_+ resembling Tangherlini-de Sitter solutions

$$F_D(r) = 1 - \left(\frac{r_0}{r}\right)^{D-3} - \frac{2\Lambda r^2}{(D-2)(D-1)}, \tag{3.11}$$

but with negative or complex valued horizons which are unphysical. The blackhole

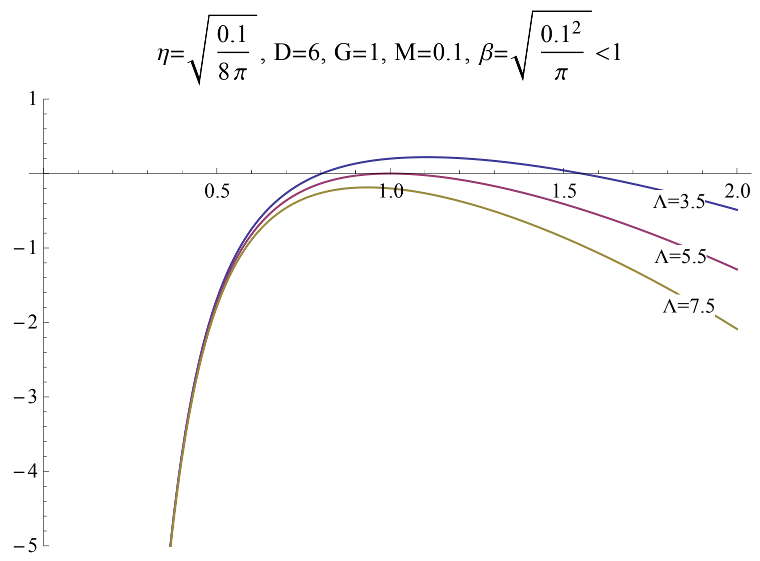

Figure 3.2: Transition of $6d$ black hole horizon from nonextremal, to extremal, and to naked singularity.

solution in 4-dimensional de Sitter spacetime can be decomposed into

$$F_4(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{\rho_c}{r}\right) g(r), \tag{3.12}$$

where r_+ its inner horizon and $g(r)$ contains the rest of unphysical horizons. These horizons can coincide making an extremal blackholes $r_+ = \rho_c = \rho$ which is related to acquiring thermodynamic properties of blackholes which we are not going to discuss here. Calculating extremal horizon ρ is not an easy task since F_4 is no longer a second order polynomial equation, even more so for F_D with $D > 4$. Fortunately, we can find ρ from (3.7) by employing the following function that factorize out the extremal horizon

$$f_D(r) = \left(1 - \frac{\rho}{r}\right)^2 \left[1 - \frac{\Lambda}{3}\left(r^2 + a + br + \frac{c_1}{r} + \frac{c_2}{r^2} + \cdots + \frac{c_{D-5}}{r^{D-5}}\right)\right], \tag{3.13}$$

where we determine the constants $a, b, c_1, \cdots, c_{D-5}$ by matching the above equation with Eq. (3.7). With induction, one leads to the following conditions to be satisfied

$$M = \frac{\rho^{D-3}}{(D-5)G}\left(\frac{4\Lambda\rho^2}{(D-1)(D-2)} - 1\right), \tag{3.14}$$

$$\frac{\eta^4}{\beta^2} = \frac{\rho^2}{(D-2)4\pi G}\left((D-3) - \frac{2\Lambda}{(D-2)}\rho^2\right). \tag{3.15}$$

The positivity of mass M and $\frac{\eta^4}{\beta^2}$ ratio implies $\sqrt{\frac{(D-1)(D-2)}{4\Lambda}} < \rho < \sqrt{\frac{(D-3)(D-2)}{2\Lambda}}$. In Figs. 3.2 and 3.3 we show how transition from nonextremal black hole can hap-

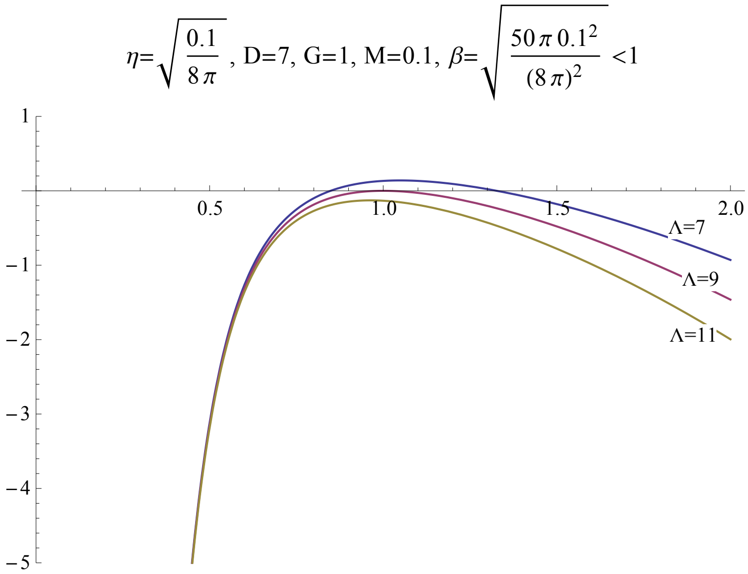

Figure 3.3: Transition of $7d$ black hole horizon from nonextremal, to extremal, and to naked singularity.

pen by varying Λ to extremal and to naked singularity in six and seven dimensions.

3.1 Factorized solutions

Another class of solutions to consider is when we set $C(r) = C = const$. This ansatz is similar to the case of spacetime factorization due to scalar field [28]. The Einstein's equations become

$$G_0^0 = \frac{(D-2)(D-3)}{2C^2} \tag{3.16}$$

$$G_\theta^\theta = -\frac{1}{B^2}\frac{A''}{A} + \frac{1}{B^2}\frac{A'B'}{AB} + \frac{(D-4)(D-3)}{2C^2} \tag{3.17}$$

with its matter part

$$G_0^0 = \Lambda + \frac{4(D-2)\pi G\eta^2}{C^2} + \frac{2(D-2)^2\pi G\eta^4}{C^4\beta^2}, \tag{3.18}$$

$$G_\theta^\theta = \Lambda + \frac{4(D-4)\pi G\eta^2}{C^2} + \frac{2(D-6)(D-2)\pi G\eta^4}{C^4\beta^2}. \tag{3.19}$$

Now we follow the previous section by first calculating the target spacetime after factoring its $D-2$-basis with a constant C. G_θ^θ can be rewritten as

$$B^{-2}\left(\frac{A'B'}{AB} - \frac{A''}{A}\right) = \omega^2, \tag{3.20}$$

where we define a constant ω

$$\omega^2 \equiv \Lambda - \frac{(D-4)(D-3-8\pi G\eta^2)}{2C^2} + \frac{2(D-6)(D-2)\pi G\eta^4}{C^4\beta^2}. \quad (3.21)$$

In general, ω^2 can be positive, zero, or negative. Now this can be solved, by taking an ansatz $B \equiv A^{-1}$, to give

$$ds^2 = \begin{cases} (1-\omega^2 r^2)\, dt^2 - \frac{dr^2}{(1-\omega^2 r^2)} - C^2 d\Omega^2_{D-2}, & \text{for } \omega^2 > 0, \\ dt^2 - dr^2 - C^2 d\Omega^2_{D-2}, & \text{for } \omega = 0, \\ (1+|\omega^2| r^2)\, dt^2 - \frac{dr^2}{(1+|\omega^2| r^2)} - C^2 d\Omega^2_{D-2}, & \text{for } \omega^2 < 0. \end{cases} \quad (3.22)$$

The resulting spacetimes (with its product of topological spaces) are Nariai ($dS_2 \times S^{D-2}$) [38], Plebanski-Hacyan ($M_2 \times S^{D-2}$) [39], and Bertotti-Robinson ($AdS_2 \times S^{D-2}$) [40, 41], respectively. We can see that ω^2 resembles the cosmological constant in the two-dimensional spacetimes with two factorized spacial dimensions.

We can solve for the radius from G^0_0. This give us

$$C^2 = \frac{4(D-2)\pi G\eta^4}{(D-3-8\pi G\eta^2)\beta^2} \quad (3.23)$$

for $\Lambda = 0$, and

$$C^2_\pm = \frac{(D-2)[(D-3-8\pi G\eta^2) \pm \sqrt{(D-3-8\pi G\eta^2)^2 - 32\pi G\eta^4\Lambda\beta^{-2}}]}{4\Lambda} \quad (3.24)$$

for $\Lambda \neq 0$. One can verify that as $\beta \to \infty$, $C^2_+ \to (D-2)(D-3-8\pi G\eta^2)/2\Lambda$, this approaches the Olasagasti-Vilenkin solution in Ref. [15]; when $\Lambda = 0$ the constant factor is not fixed by the theory. To ensure that C is real it requires

$$\eta < \frac{D-3}{8\pi G} = \eta_{crit} \quad (3.25)$$

when $\Lambda = 0$,

$$\eta \leq \frac{1}{\sqrt{8}} \sqrt{\frac{2\beta^2(D-3)}{2\pi\beta^2 G - \Lambda} - \sqrt{\frac{2\beta^2(D-3)^2\Lambda}{\pi G\left(\Lambda - 2\pi\beta^2 G\right)^2}}} \equiv \eta_{crit2} < \eta_{crit} \quad (3.26)$$

for $\Lambda < 2\pi G\beta^2$,

$$\eta \leq \frac{\eta_{crit}}{\sqrt{2}} \equiv \eta_{crit3} < \eta_{crit}, \quad (3.27)$$

Table 3.1: Conditions for k-monopole factorization in D-dimensions.

	$dS_2 \times S^{D-2}$	$M_2 \times S^{D-2}$	$AdS_2 \times S^{D-2}$
$\Lambda > 2\pi\beta^2 G$	$\eta^2 < \eta^2_{crit4}$	$\eta^2 = \eta^2_{crit4}$	cannot happen
$\Lambda = 2\pi\beta^2 G$	$\eta^2 < \eta^2_{crit3}$	$\eta^2 = \eta^2_{crit3}$	cannot happen
$\Lambda < 2\pi\beta^2 G$	$\eta^2 < \eta^2_{crit2}$	$\eta^2 = \eta^2_{crit2}$	cannot happen
$\Lambda = 0$	cannot happen	cannot happen	$\eta^2 < \eta^2_{crit}$
$\Lambda < 0$		cannot happen	

for $\Lambda = 2\pi G\beta^2$, and

$$\eta \leq \frac{1}{\sqrt{8}}\sqrt{\frac{2\beta^2(D-3)}{2\pi\beta^2 G - \Lambda} + \sqrt{\frac{2\beta^2(D-3)^2\Lambda}{\pi G\left(\Lambda - 2\pi\beta^2 G\right)^2}}} \equiv \eta_{crit4}, \tag{3.28}$$

for $\Lambda > 2\pi G\beta^2$. Again $\eta^2_{crit4} \geq \eta^2_{crit2}$ if $0 \leq \Lambda < 2\pi\beta^2 G$ but this is a contradiction, which means that $\eta^2_{crit4} < \eta^2_{crit2}$.

The factorized channel is denoted by $X_D \to Y_2 \times S^{D-2}$, where X and Y can each stands for the de Sitter, Minkowski, or Anti-de Sitter. To ensure which factorized channel between nine possible ones can take place, we check whether the condition of η that satisfies ω^2 simultaneously with $C^2 > 0$. This is done by checking whether conditions (3.25)-(3.28) satisfy the polynomial equations of $\omega^2 > 0$, $\omega^2 = 0$, or $\omega^2 < 0$. The results are shown in Table 3.1. Note that this is a list of classically-allowed conditions for factorization and that not all nine possibilities can happen. For instance, the case of $AdS_2 \times S^{D-2}$ factorization with $\Lambda < 2\pi\beta^2 G$. Solving the equation $\omega^2 < 0$ with this values in mind we can obtain $\eta^2 > \eta^2_{crit3}$. But this is contradicting (3.27), therefore we conclude that such factorization cannot happen. In short, below is the following possible channels

$$dS_D \longrightarrow \begin{cases} dS_2 \times S^{D-2}, \\ M_2 \times S^{D-2}, \end{cases} \tag{3.29}$$

$$M_D \longrightarrow AdS_2 \times S^{D-2}. \tag{3.30}$$

CHAPTER 4
FACTORIZED SOLUTIONS IN $(p+D)$-DIMENSIONAL SPACETIMES

Here we discuss the stability factorized solutions at $p > 0, D > 4$. We discuss the results which had been presented in Conference on Theoretical Physics and Nonlinear Phenomena 2016 and its proceeding paper will be published soon.

As stated before in the end of chapter 2, we apply $C(r) = Const.$ in metric (1.114). This in turn simplify eqs. (1.121)-(1.125), respectively, to

$$R^0_0 = B^{-2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + p\left(\frac{A'}{A}\right)^2\right], \tag{4.1}$$

$$R^r_r = B^{-2}\left[(p+1)\frac{A''}{A} - (p+1)\frac{B'}{B}\frac{A'}{A}\right], \tag{4.2}$$

$$R^\theta_\theta = -\frac{(D-3)}{C^2}. \tag{4.3}$$

and

$$T^0_0 = T^r_r = \left[\frac{\Lambda}{8\pi G} + X + \beta^{-2}X^2\right], \tag{4.4}$$

$$T^\theta_\theta = \left[\frac{\Lambda}{8\pi G} + X + \beta^{-2}X^2\right] - (1 + 2\beta^{-2}X)\frac{\eta^2}{C^2}, \tag{4.5}$$

with $X = (D-2)\eta^2/2C^2$. Now since we use the usual Einstein equation $G^M_N = R^M_N - \delta^M_N R/2 = 8\pi G T^M_N$ we show the components of the Einstein tensor

$$G^0_0 = pB^{-2}\left[-\frac{A''}{A} + \frac{A'B'}{AB} - \frac{(p-1)}{2}\left(\frac{A'}{A}\right)^2\right] + \frac{(D-2)(D-3)}{2C^2}, \tag{4.6}$$

$$G^r_r = -\frac{p(p+1)}{2B^2}\left(\frac{A'}{A}\right)^2 + \frac{(D-2)(D-3)}{2C^2} \tag{4.7}$$

$$G^\theta_\theta = (p+1)B^{-2}\left[-\frac{A''}{A} + \frac{A'B'}{AB} - \frac{p}{2}\left(\frac{A'}{A}\right)^2\right] - \frac{(D-4)(D-3)}{2C^2}. \tag{4.8}$$

Following the same algorithm in chapter 2 we will show that the spacetime after factorization is similar with Eqs. (2.28)-(2.31) but with different dimension. Using

$G^\theta_\theta - G^r_r$ which is

$$(p+1)B^{-2}\left[-\frac{A''}{A}+\frac{A'B'}{AB}\right]+\frac{(D-3)}{C^2}=-(1+2\beta^{-2}X)\frac{\kappa\eta^2}{C^2}$$

with $\kappa \equiv 8\pi G$, we can see that the most right term in the left hand side and the right hand side is just constants. Hence we define a real-valued constant ω with

$$\pm\omega^2=\frac{(D-3)}{(p+1)C^2}+(1+2\beta^{-2}X)\frac{\kappa\eta^2}{(p+1)C^2} \tag{4.9}$$

which makes

$$B^{-2}\left[\frac{A''}{A}-\frac{A'B'}{AB}\right]=\pm\omega^2 \tag{4.10}$$

solvable with ansatz $B=1$ or $B=A^{-1}$ as before. The former ansatz leads to

$$ds^2=\frac{1}{\omega^2}(\sin^2\chi\,\eta^{(p+1)}_{\mu\nu}dx^\mu dx^\nu-d\chi^2)-C^2d\Omega^2_{(D-2)},\quad \text{for } +\omega^2, \tag{4.11}$$

$$ds^2=\eta^{(p+1)}_{\mu\nu}dx^\mu dx^\nu-dr^2-C^2d\Omega^2_{(D-2)},\quad \text{for } \omega=0, \tag{4.12}$$

$$ds^2=\frac{1}{\omega^2}(\sinh^2\chi\,\eta^{(p+1)}_{\mu\nu}dx^\mu dx^\nu-d\chi^2)-C^2d\Omega^2_{(D-2)},\quad \text{for } -\omega^2, \tag{4.13}$$

with $\chi \equiv \omega r$, and the latter ansatz leads to

$$ds^2=(1-\omega^2r^2)\,\eta^{(p+1)}_{\mu\nu}dx^\mu dx^\nu-\frac{dr^2}{(1-\omega^2r^2)}-C^2d\Omega^2_{(D-2)},\quad \text{for } +\omega^2, \tag{4.14}$$

$$ds^2=\eta^{(p+1)}_{\mu\nu}dx^\mu dx^\nu-dr^2-C^2d\Omega^2_{(D-2)},\quad \text{for } \omega=0, \tag{4.15}$$

$$ds^2=(1+\omega^2r^2)\,\eta^{(p+1)}_{\mu\nu}dx^\mu dx^\nu-\frac{dr^2}{(1+\omega^2r^2)}-C^2d\Omega^2_{(D-2)},\quad \text{for } -\omega^2. \tag{4.16}$$

Both leads to same topologies classified by the value on $\pm\omega^2$. Thus the factorized space have the following topology: $(dS_{p+2}\ /\ M_{p+2}\ /\ AdS_{p+2})\times S^{D-2}$ when $B^{-2}\left(\frac{A'B'}{AB}-\frac{A''}{A}\right)\ (>\ /=/<)\ 0$.

Now there are actually two ways to calculate C: by using the remaining Einstein tensor or by dimensional reduction. We start first with the former method.

Combining above equations by $pG^\theta_\theta-(p+1)G^0_0-G^r_r$ we have

$$-2\Lambda-\frac{(p+D-2)\kappa\eta^2}{C^2}-\frac{(2p+D-2)(D-2)\kappa\eta^4}{2C^4\beta^2}=-\frac{(p+D-2)(D-3)}{C^2}, \tag{4.17}$$

which is actually just $(p+D-2)R^\theta_\theta$ thus we are in the right track. Solving this is classified into two class: (1) $\Lambda = 0$ will give us

$$C^2 = \frac{(2p+D-2)(D-2)\kappa\eta^4}{2\beta^2(p+D-2)(D-3-\kappa\eta^2)}, \tag{4.18}$$

and (2) $\Lambda \neq 0$ give us

$$\begin{aligned} C_{\pm}^2 &= \frac{(p+D-2)(D-3-\kappa\eta^2)}{4\Lambda} \\ &\pm \frac{\sqrt{(p+D-2)^2(D-3-\kappa\eta^2)^2 - 4\Lambda(2p+D-2)(D-2)\kappa\eta^4\beta^{-2}}}{4\Lambda}. \end{aligned} \tag{4.19}$$

At weak coupling $\beta \to \infty$, only C_+^2 that is not vanished ($C_-^2 \to 0 \leftarrow C^2$ and $C_+^2 \to (p+D-2)(D-3-\kappa\eta^2)/2\Lambda$) and it is in agreement with Ref. [15]. Notice that since $C^2 > 0$, Eq. (4.18) requires

$$\eta^2 < \eta_{crit}^2 \equiv (D-3)/8\pi G \tag{4.20}$$

when $\Lambda = 0$. For $C_+^2 > 0$, it turns out that it must satisfy $\Lambda > 0$. We obtain three classes of solution depending to the value of the cosmological constant Λ: (1) when

$$\Lambda = \frac{2\pi D^2 G + 4\pi DGp - 8\pi DG + 2\pi Gp^2 - 8\pi Gp + 8\pi G}{4\beta^2 + \beta^2 D^2 - 4\beta^2 D + 2\beta^2 Dp - 4\beta^2 p} \equiv \Lambda_c,$$

it requires

$$\eta^2 \leq \frac{D-3}{16\pi G} < \frac{(D-3)}{8\pi G}, \tag{4.21}$$

(2) when $\Lambda < \Lambda_c$, it requires

$$\eta^2 \leq \eta_{-(p,D)}^2 < (D-3)/8\pi G, \tag{4.22}$$

and (3) when $\Lambda > \Lambda_c$, it requires

$$\eta^2 \leq \eta_{+(p,D)}^2. \tag{4.23}$$

Explicit expression of both $\eta_{-(p,D)}^2$ and $\eta_{+(p,D)}^2$ are too long to be shown here, thus we do not show them but they satisfy

$$\eta_{-(p,D)}^2 < \eta_{+(p,D)}^2. \tag{4.24}$$

It is interesting that $1/8\pi G < \eta_{+(p,D)}^2$ if $\Lambda < \Lambda_c$ but this is a contradiction. This means that $\Lambda > 0$ gives an upperbound that is always less than $(D-3)/8\pi G$. Notice that $\Lambda < 0$ is not allowed since it makes $C_+^2 < 0$.

4.1 Checking Stability

Now we move to the second method: introducing an effective potential, that is dependent on a conformal factor in its metric, by dimensional reduction method. This actually a way to verify stability, yet the result of C^2 from this method are the same as from solving the Einstein equations directly. The main idea is simple: we introduce a conformal factor b, also called as *radion*, in the metric and perform a *Weyl (conformal) transformation* in the action with the form of *Einstein frame*. By this we reduce the dimension of integral in the action. After this, the action has the form of *Jordan frame*. This can be transformed back into *Einstein frame* also by using another *Weyl transformation*, which again reduce the dimension. The result is an additional terms in the Lagrangian density that are called effective potential of the radion b, $V(b)$. The resulting action also has less integration since the dimension is reduced, hence the name dimensional reduction. [To see what Weyl transformation, Einstein frame, and Jordan frame are, reader can consult Appendix B in page 59.]

Let us consider a metric with signature $(+ - - \ldots)$

$$ds^2 = G_{MN}^{(p+D)} dx^A dx^B, \tag{4.25}$$

with $G_{MN}^{(p+D)}$ metric on a $(p+D)$-dimensional Einstein frame action

$$\mathcal{S} = \int d^{p+D}x \sqrt{|G|} \left[\frac{\mathcal{R}^{(p+D)}}{2\kappa} + \mathcal{L}_m \right], \tag{4.26}$$

with $\kappa^{-1/2}$ denoting a Plank mass in the $(p+D)$-dimensional spacetime. Now we perform Weyl transformation by defining the metric as

$$ds^2 = g_{\mu\nu}^{(p+2)}(x) dx^\mu dx^\nu - b^2(x) \gamma_{ij}^{(D-2)}(y) dy^i dy^j, \tag{4.27}$$

with $\gamma_{ij}^{(D-2)}$ a metric of a space with constant curvature whose radius is R_0. The action now becomes

$$\mathcal{S} = \frac{V_{(D-2)}}{2\kappa} \int d^{p+2}x \sqrt{|g|} \left[b^{D-2} \mathcal{R}^{(p+D)} + 2\kappa b^{D-2} \mathcal{L}_m \right], \tag{4.28}$$

which has Jordan frame form. We again transform it back into Einstein frame action by another Weyl transformation

$$g_{\mu\nu}^{(p+2)}(x) = b^{2a}(x)\, \tilde{g}_{\mu\nu}^{(p+2)}(x), \tag{4.29}$$

with a a constant to be determined. After a lengthy algebra, $a = -(D-2)/p$ makes the action has Einstein frame form

$$\mathcal{S} = \frac{V_{(D-2)}}{2\kappa}\int d^{p+2}x\sqrt{|\tilde{g}|}\left[\tilde{\mathcal{R}}^{(p+2)} + \frac{(D-2)(D-3)}{b^{2(p+D-2)/p}R_0^2} + \frac{2(D-2)}{p}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu(\tilde{\partial}_\nu \ln b)\right.$$
$$\left. - \frac{(D-2)(p+D-2)}{p}\tilde{g}^{\mu\nu}(\tilde{\partial}_\mu \ln b)(\tilde{\partial}_\nu \ln b) + 2\kappa b^{-(D-2)/p}\tilde{\mathcal{L}}_m\right]. \tag{4.30}$$

By argument of Gauss law

$$\int d^{p+2}x\sqrt{|\tilde{g}|}\tilde{\nabla}^\nu(\tilde{\partial}_\nu \ln b) = \int d^{p+2}x\tilde{\nabla}^\nu(\sqrt{|\tilde{g}|}\tilde{\partial}_\nu \ln b) = \left[\sqrt{|\tilde{g}|}\tilde{\partial}_\nu \ln b\right]_{x=-\infty}^{x=\infty}, \tag{4.31}$$

thus third term is just a boundary term hence we can remove it and the Euler-Lagrange equation will not be affected. To simplify the expressions, we define the radion b as

$$b \equiv \exp\left[\sqrt{\frac{p}{(D-2)(p+D-2)}}\frac{\psi}{M_P}\right], \tag{4.32}$$

with $M_P \equiv \sqrt{V_{(D-2)}/\kappa}$ a $(p+2)$-dimensional Planck mass. The final result is an action in Einstein frame

$$\mathcal{S} = \int d^{p+2}x\sqrt{|\tilde{g}|}\left[\frac{M_P^2\tilde{\mathcal{R}}^{(p+2)}}{2} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\psi\tilde{\partial}_\nu\psi - V(\psi)\right], \tag{4.33}$$

where $V(\psi)$ an effective potential. This contains

$$V(\psi) = -e^{\sigma\psi/M_P}\frac{M_P^2(D-2)(D-3)}{2R_0^2} - e^{\chi\psi/M_P}M_P^2\kappa\tilde{\mathcal{L}}_m, \tag{4.34}$$

with $\sigma \equiv -2\sqrt{(p+D-2)/p(D-2)}$ and $\chi \equiv -2\sqrt{(D-2)/p(p+D-2)}$ whose $p \neq 0$. After performing two Weyl transformations, the metric has the following final form

$$ds^2 = e^{\chi\psi/M_P}\tilde{g}_{\mu\nu}^{(p+2)}dx^\mu dx^\nu - e^{\xi\psi/M_P}\gamma_{ij}^{(D-2)}dy^i dy^j, \tag{4.35}$$

with $\xi \equiv 2\sqrt{p/(D-2)(p+D-2)}$. Here we can see that $p=0$ leads to $\sigma=\infty$ and $\chi=\infty$ and $\xi=0$, thus we must restrict p with $p\geq 1$. Since we are interested in $D\geq 4$, then $p+D\geq 5$.

Now we can use the above results. We set the metric (4.35) to be

$$ds^2 = e^{\chi\psi/M_P}\left[A^2(r)\hat{\eta}_{\mu\nu}^{(p+1)}dx^\mu dx^\nu - B^2(r)dr^2\right] - e^{\xi\psi/M_P}\left[C^2 d\Omega_{(D-2)}^2\right], \tag{4.36}$$

which become Eq. (1.114) as $\psi=0$. This makes $R_0 = C$ and we get the Lagrangian density

$$\tilde{\mathcal{L}}_m = -\frac{\Lambda}{\kappa} + K(X), \tag{4.37}$$

with $K(X) = -X - \beta^{-2}X^2$ and $X = \frac{(D-2)\eta^2}{2C^2}e^{-\xi\psi/M_P}$. ($X$ has this form since X is dependent on the metric and the Weyl transformation transform the metric.) Then the effective potential is

$$\begin{aligned} V(\psi) &= -e^{\sigma\psi/M_P}\frac{M_P^2(D-2)(D-3)}{2C^2} + e^{\chi\psi/M_P}M_P^2\Lambda \\ &\quad + M_P^2\kappa\left[\frac{(D-2)\eta^2}{2C^2}e^{(\chi-\xi)\psi/M_P} + \frac{(D-2)^2\eta^4}{4\beta^2C^4}e^{(\chi-2\xi)\psi/M_P}\right]. \end{aligned} \tag{4.38}$$

One can see that the potential's minimum can be at $\psi = c$ where c is anywhere. But since the metric (4.35) goes to (1.114) at $\psi = 0$, we set the minimum at $\psi = 0$. By setting the minima/maxima to be at $\psi = 0$, we can calculate C that satisfy both $V'(\psi = 0) = 0$ and $V''(\psi = 0) > 0$, whose explicit expressions are, respectively,

$$\chi\Lambda + \frac{(D-2)}{2}\left[(\chi-\xi)\kappa\eta^2 - \sigma(D-3)\right]\frac{1}{C^2} + \left[\frac{(\chi-2\xi)(D-2)^2\kappa\eta^4}{4\beta^2}\right]\frac{1}{C^4} = 0, \tag{4.39}$$

and

$$\frac{4(D-2)\Lambda}{p(p+D-2)} + \frac{2(\kappa\eta^2 - D + 3)(p+D-2)}{pC^2} + \frac{(2p+D-2)^2(D-2)\kappa\eta^4}{p(p+D-2)\beta^2C^4} > 0. \tag{4.40}$$

First we investigate what is the radius C that makes $V'(\psi = 0) = 0$. By solving Eq. (4.39) we obtain

$$C^2 = \frac{(2p+D-2)(D-2)\kappa\eta^4}{2\beta^2(p+D-2)(D-3-\kappa\eta^2)}, \tag{4.41}$$

and

$$C^2 = \frac{(p+D-2)(D-3-\kappa\eta^2)}{4\Lambda}\left(1 \pm \sqrt{1 - \frac{(2p+D-2)(D-2)\kappa\eta^4\beta^{-2}}{4\Lambda(p+D-2)^2(D-3-\kappa\eta^2)^2}}\right), \tag{4.42}$$

when $\Lambda = 0$ and $\Lambda \neq 0$, respectively. These are *actually* (4.18) and (4.19) which we find using Einstein's equations, but now we can check its stability. But these expressions are not easy to use to be substituted to $V''(\psi = 0) \geq 0$.

Fortunately, from solving (4.39) with roots $1/C^2$ we can obtain an inverse form of both (4.41) and (4.42) which is

$$\frac{1}{C_{\pm}^2}=\frac{(D-3-\kappa\eta^2)(p+D-2)\beta^2}{(2p+D-2)(D-2)\kappa\eta^4}\left(1\pm\sqrt{1-\frac{4\Lambda(2p+D-2)(D-2)\kappa\eta^4}{(p+D-2)^2(D-3-\kappa\eta^2)^2\beta^2}}\right). \tag{4.43}$$

We can see that $1/C_{-}^2 = 2\Lambda/(p+D-2)(D-3-\kappa\eta^2)$ as $\beta\to\infty$. This may implies $1/C_{-}^2$ ($1/C_{+}^2$) corresponds to the inverse of Eq. (4.42) with plus (minus) sign. The term inside square root of (4.43) needs to be positive thus we need $\beta^2 > 0$ when $\Lambda \le 0$ or

$$\beta^2 \ge \beta_{crit}^2 \equiv \frac{4\Lambda(2p+D-2)(D-2)\kappa\eta^4}{(p+D-2)^2(D-3-\kappa\eta^2)^2}, \tag{4.44}$$

when $\Lambda > 0$. The conditions on η, that are (4.20)-(4.23) must be satisfied, thus the condition for $\Lambda < 0$ is eliminated and Eq. (4.44) still hold.

Now we set that the minimum of the potential at $\psi = 0$ positioned at zero, $V(\psi = 0) = 0$, happens when $\Lambda = \Lambda_*$, and this must satisfy both (4.39) and (4.40).[1] By substituting (4.43) when $\Lambda = \Lambda_*$ in (4.38) after setting $\psi = 0$ and since $V(\psi = 0, \Lambda = \Lambda_*) = 0$, we have

$$\begin{aligned}0 = \Lambda_* &- \frac{(D-3-\kappa\eta^2)^2(p+D-2)\beta^2}{2(2p+D-2)\kappa\eta^4}\left(1\pm\sqrt{1-\frac{4\Lambda_*(2p+D-2)(D-2)\kappa\eta^4}{(p+D-2)^2(D-3-\kappa\eta^2)^2\beta^2}}\right)\\ &+\frac{(D-3-\kappa\eta^2)^2(p+D-2)^2\beta^2}{4(2p+D-2)^2\kappa\eta^4}\left(1\pm\sqrt{1-\frac{4\Lambda_*(2p+D-2)(D-2)\kappa\eta^4}{(p+D-2)^2(D-3-\kappa\eta^2)^2\beta^2}}\right)^2,\end{aligned} \tag{4.45}$$

with the plus and minus sign correspond to $1/C_{+}^2$ and $1/C_{-}^2$ respectively. By squaring the last term on the right hand side

$$\begin{aligned}0 = \Lambda_* &- \frac{(D-3-\kappa\eta^2)^2 p(p+D-2)\beta^2}{2(2p+D-2)^2\kappa\eta^4}\left(1+\frac{2\Lambda_*(2p+D-2)(D-2)\kappa\eta^4}{p(p+D-2)(D-3-\kappa\eta^2)^2\beta^2}\right)\\ &\mp\frac{(D-3-\kappa\eta^2)^2 p(p+D-2)\beta^2}{2(2p+D-2)^2\kappa\eta^4}\sqrt{1-\frac{4\Lambda_*(2p+D-2)(D-2)\kappa\eta^4}{(p+D-2)^2(D-3-\kappa\eta^2)^2\beta^2}}\end{aligned} \tag{4.46}$$

and by transferring the most right term on the left hand side to the left hand side we

[1] This treatment follows the method used in [19].

have

$$\pm\sqrt{1-\frac{4\Lambda_*(2p+D-2)(D-2)\kappa\eta^4}{(p+D-2)^2(D-3-\kappa\eta^2)^2\beta^2}}=\frac{2\Lambda_*(2p+D-2)(2p)\kappa\eta^4}{p(p+D-2)(D-3-\kappa\eta^2)^2\beta^2}-1. \tag{4.47}$$

Now by squaring both sides we have a formula resembling $a\Lambda_*^2+b\Lambda_*+c=0$ thus $\Lambda_*=0$ or

$$\Lambda_*=\frac{(D-3-\kappa\eta^2)^2\beta^2}{4\kappa\eta^4}\geq 0. \tag{4.48}$$

Now Λ_* contains η,β, and κ, thus the above expressions can be simplified: the expression of (4.43) become

$$\frac{1}{C_\pm^2}=\frac{4(p+D-2)\Lambda_*}{(2p+D-2)(D-2)(D-3-\kappa\eta^2)}\left(1\pm\sqrt{1-\frac{\Lambda(2p+D-2)(D-2)}{\Lambda_*(p+D-2)^2}}\right), \tag{4.49}$$

and substitute this into (4.38) that become

$$\begin{aligned}\frac{V(\psi=0)}{M_P^2} &= \frac{2p\Lambda}{(2p+D-2)}-\frac{2p(p+D-2)\Lambda_*}{(2p+D-2)^2}\\ &\times\left(1\pm\sqrt{1-\frac{\Lambda(2p+D-2)(D-2)}{(p+D-2)^2\Lambda_*}}\right).\end{aligned} \tag{4.50}$$

and into (4.40) that become

$$\begin{aligned}V''(\psi=0) &= -\frac{8\Lambda}{(p+D-2)}+\frac{8(p+D-2)\Lambda_*}{(2p+D-2)(D-2)}\\ &\times\left(1\pm\sqrt{1-\frac{\Lambda(2p+D-2)(D-2)}{(p+D-2)^2\Lambda_*}}\right),\end{aligned} \tag{4.51}$$

with the plus and minus sign correspond to $1/C_+^2$ and $1/C_-^2$ respectively. Now we have the potential and its second derivative as a function of Λ.

Now we can determine the factorized space and its stability in each class of solutions separated by the value of Λ. By this we can obtain tunneling channels $Y_{p+D}\longrightarrow Z_{p+2}\times S^{D-2}$ (with Λ and $V(\psi=0)$ denoting Y_{p+D} and Z_{p+2} respectively). Since $1/C_+^2=2(D-3-\kappa\eta^2)(p+D-2)\beta^2/(2p+D-2)(D-2)\kappa\eta^4$, we choose $1/C_+^2$ for $\Lambda=0$. On the other hand, we choose $1/C_-^2$ for $\Lambda>0$ (the requirements of η do not allow $\Lambda<0$) since $1/C_-^2=2\Lambda/(p+D-2)(D-3-\kappa\eta^2)$

as $\beta \to \infty$. In the case of $\Lambda = 0$,

$$\frac{V(\psi=0,C_+,\Lambda=0)}{M_P^2} = -\frac{4p(p+D-2)\Lambda_*}{(2p+D-2)^2} < 0, \tag{4.52}$$

$$V''(\psi=0,C_+,\Lambda=0) = \frac{16(p+D-2)\Lambda_*}{(2p+D-2)(D-2)} > 0, \tag{4.53}$$

since $\Lambda_* > 0$. In the case of $\Lambda > 0$,

$$\frac{V(\psi=0,C_-)}{M_P^2} = \frac{2p\Lambda}{(2p+D-2)} - \frac{2p(p+D-2)\Lambda_*}{(2p+D-2)^2} \times \left(1-\sqrt{1-\frac{\Lambda(2p+D-2)(D-2)}{(p+D-2)^2\Lambda_*}}\right), \tag{4.54}$$

$$V''(\psi=0,C_-) = -\frac{8\Lambda}{(p+D-2)} + \frac{8(p+D-2)\Lambda_*}{(2p+D-2)(D-2)} \times \left(1-\sqrt{1-\frac{\Lambda(2p+D-2)(D-2)}{(p+D-2)^2\Lambda_*}}\right). \tag{4.55}$$

By considering Λ to be positive but very near to zero

$$\frac{V(\psi=0,C_-)}{M_P^2} \simeq \frac{2p\Lambda}{(2p+D-2)} - \frac{\Lambda(D-2)p}{(2p+D-2)(p+D-2)} = \frac{2p\Lambda}{(p+D-2)} > 0, \tag{4.56}$$

$$V''(\psi=0,C_-) \simeq -\frac{8\Lambda}{(p+D-2)} + \frac{8\Lambda}{(p+D-2)} = 0, \tag{4.57}$$

and at $\Lambda = (p+D-2)^2\Lambda_*/(2p+D-2)(D-2)$

$$\frac{V(\psi=0,C_-)}{M_P^2} = \frac{2p(p+D-2)^2\Lambda_*}{(2p+D-2)^2(D-2)} - \frac{2p(p+D-2)\Lambda_*}{(2p+D-2)^2} = \frac{2p^2(p+D-2)\Lambda_*}{(2p+D-2)^2(D-2)} > 0, \tag{4.58}$$

$$V''(\psi=0,C_-) = -\frac{8(p+D-2)\Lambda_*}{(2p+D-2)(D-2)} + \frac{8(p+D-2)\Lambda_*}{(2p+D-2)(D-2)} = 0. \tag{4.59}$$

Both cases need the third derivative of V, which are

$$V'''(\psi=0,C_-) \simeq 8\sqrt{\frac{p}{(D-2)(p+D-2)}}\frac{\Lambda_*}{M_P}\frac{[(p+D-2)^3-(D-2)^2]}{p^2(p+D-2)} > 0 \tag{4.60}$$

for $\Lambda \ll \Lambda_*$ case, and

$$V'''(\psi = 0, C_-) = \frac{[(p+D-2)^3 - (D-2)^2]}{(D-2)p^2}\frac{\Lambda_*}{M_P}\sqrt{\frac{p}{(D-2)(p+D-2)}} > 0 \tag{4.61}$$

for $\Lambda = (p+D-2)^2\Lambda_*/(2p+D-2)(D-2)$ case. These imply that the compactification channels $dS_{p+D} \to (dS, M)_{p+2} \times S^{D-2}$ are unstable since the potential at $\psi = 0$ is a strictly increasing point of inflection. These imply that $dS_{p+D} \to dS_{p+2} \times S^{D-2}$ is unstable. We also point out that at $\Lambda = \Lambda_*$ we have $V = 0$ and since we know that $0 < \Lambda_* < (p+D-2)^2\Lambda_*/(2p+D-2)(D-2)$ then it may also be an increasing point of inflection hence we also have $dS_{p+D} \to dS_{p+2} \times S^{D-2}$ that is also unstable. In summary, the obtained compactification channels are

$$dS_{p+D} \longrightarrow \begin{cases} dS_{p+2} \times S^{D-2}, \\ M_{p+2} \times S^{D-2}, \end{cases} , \tag{4.62}$$

$$M_{p+D} \longrightarrow AdS_{p+2} \times S^{D-2}, \tag{4.63}$$

which is identical the solutions we obtained in the previous chapters at $p = 0$ but here we know that (4.62) is unstable while (4.63) is stable for $p > 0$.

CHAPTER 5
CONCLUSIONS

Here we conclude this thesis with the following list.

1. We are able to find blackhole solutions in four and higher dimensional spacetimes. The solution in four dimension ($p = 0, D = 4$) resembles Reissner-Nördtrom-de Sitter solution but with deficit solid angle present. This makes η must always be smaller than a certain upper bound η_{crit}. In higher dimension with $p = 0, D > 5$, the solution resemble Tangherlini-de Sitter solution also with its η having an upper bound. We do not obtain blackhole solutions at $p > 0$.

2. The factorized (compactified for $p > 0$) solutions can be obtained for $p \geq 0, D \geq 4$. The stability of these can be verified for $p > 0$ case. When η is above its critical value, we do not obtain any factorized solutions.

The proposed model in this thesis is noncanonical by adding just one higher order kinetic term in the Lagrangian density. This actually is incomplete since in effective field theory the noncanonical terms are infinitely many. Dirac-Born-Infeld (DBI) term, for example, are more suitable to use for more physically reasonable models up to some coupling constants. We are currently investigating noncanonical global defects with DBI term and we hope that in the near future we can discuss the thermodynamic properties of its blackhole solutions.

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APPENDICES

APPENDIX A: FINDING RICCI TENSOR USING TETRAD METHOD

To calculate Ricci tensor is known to be quite tedious. From a certain metric g_{ab}, one then calculate the nonzero components of connection coefficient

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{bc} \right),$$

which are quite a lot then calculate the Ricci tensor

$$R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{cd} \Gamma^d_{ab} - \Gamma^c_{bd} \Gamma^d_{ac},$$

which are more prone error in analytic calculations due to lots of terms caused by the summation. Here we present a simpler one known as tetrad method which is less prone to error even though quite hard to grasp at first glance. This method heavily uses the fact that we use the torsionless property of a manifold and this simplify the Cartan structure equations which is used to calculate Riemann tensor. But before using this, one can ”flatten” a metric by using orthonormal basis known as vielbein or tetrad. One can find detailed explanation of these in appendix J of Ref. [42] while here we only demonstrate the calculations.

Suppose we use metric ansatz with the following form

$$ds^2 = g_{AB} dx^A dx^B = A^2(r) dt^2 - F^2(r) \hat{g}_{\mu\nu} dx^\mu dx^\nu - B^2(r) dr^2 - C^2(r) \tilde{g}_{ab} dx^a dx^b, \tag{1}$$

with metric with hat ($\mu, \nu = 1, \cdots, p$) that represents p-brane with constant curvature and metric with tilde ($i, j = \theta_1, \theta_2, \cdots, \theta_{(D-2)}$) on which represents $(D-2)$-dimensional space also with constant curvature. Here we set the latter metric $\tilde{g}_{ab}$ with specific feature of its Ricci scalar having the form of $\tilde{R} = k(D-2)(D-3)$, with $k = 1$ denoting $(D-2)$-sphere with radius $= 1$, $k = -1$ denoting $(D-2)$-dimensional hyperbolic space also with radius $= 1$, and $k = 0$ denoting $(D-2)$-dimensional flat space. We calculate the Riemann by ”flatten” the metric using vierbein with hatted indices

$$g = g_{AB} dx^A dx^B = \eta_{\hat{A}\hat{B}} \omega^{\hat{A}} \otimes \omega^{\hat{B}}$$

and $\eta_{\hat{A}\hat{B}} = \mathrm{diag}(1, -1, -1, ...)$, which will give us

$$\omega^{\hat{t}} = Adt,\ \omega^{\hat{\mu}} = F\hat{\omega}^{\hat{\mu}},\ \omega^{\hat{r}} = Bdr,\ \omega^{\hat{a}} = C\tilde{\omega}^{\hat{a}}. \tag{2}$$

Using Cartan 1st equation

$$T^{\hat{A}} = d\omega^{\hat{A}} + \Theta^{\hat{A}}_{\ \hat{B}} \wedge \omega^{\hat{A}},$$

and $T^{\hat{A}} = 0$ denoting vanishing torsion, we obtain four contracted connection coefficients (with $A' = \frac{dA}{dr}$)

$$\Theta^{\hat{t}}_{\ \hat{r}} = \frac{A'}{AB}\omega^{\hat{t}} = \Theta^{\hat{r}}_{\ \hat{t}}, \tag{3}$$
$$\Theta^{\hat{\mu}}_{\ \hat{\nu}} = \hat{\Theta}^{\hat{\mu}}_{\ \hat{\nu}}, \tag{4}$$
$$\Theta^{\hat{a}}_{\ \hat{b}} = \tilde{\Theta}^{\hat{a}}_{\ \hat{b}}, \tag{5}$$
$$\Theta^{\hat{\mu}}_{\ \hat{r}} = \frac{F'}{BF}\omega^{\hat{\mu}} = -\Theta^{\hat{r}}_{\ \hat{\mu}}, \tag{6}$$
$$\Theta^{\hat{a}}_{\ \hat{r}} = \frac{C'}{BC}\omega^{\hat{a}} = -\Theta^{\hat{r}}_{\ \hat{a}}. \tag{7}$$

Using Cartan 2nd equation

$$\frac{1}{2!}R^{\hat{A}}_{\ \hat{B}\hat{C}\hat{D}}\omega^{\hat{C}} \wedge \omega^{\hat{D}} = d\Theta^{\hat{A}}_{\ \hat{B}} + \Theta^{\hat{A}}_{\ \hat{C}} \wedge \Theta^{\hat{C}}_{\ \hat{B}},$$

and changing from vierbein to the original one by $R^{A}_{\ BCD} = R^{\hat{A}}_{\ \hat{B}\hat{C}\hat{D}}\omega_{\hat{A}}^{\ A}\omega^{\hat{B}}_{\ B}\omega^{\hat{C}}_{\ C}\omega^{\hat{D}}_{\ D}$ using

$$\omega^{\hat{t}}_{\ t} = A = \frac{1}{\omega_{\hat{t}}^{\ t}}, \tag{8}$$
$$\omega^{\hat{\mu}}_{\ \mu} = F = \frac{1}{\omega_{\hat{\mu}}^{\ \mu}}, \tag{9}$$
$$\omega^{\hat{r}}_{\ r} = B = \frac{1}{\omega_{\hat{r}}^{\ r}}, \tag{10}$$
$$\omega^{\hat{a}}_{\ a} = C = \frac{1}{\omega_{\hat{a}}^{\ a}}, \tag{11}$$

we obtain non-zero components of Riemann tensor as follows

$$R_{\mu t\nu t} = \frac{1}{B^2}\frac{A'F'}{AF}g_{\mu\nu}g_{tt}, \tag{12}$$
$$R_{atbt} = \frac{1}{B^2}\frac{A'C'}{AC}g_{ab}g_{tt}, \tag{13}$$
$$R_{\mu a\nu b} = \frac{1}{B^2}\frac{C'F'}{CF}g_{\mu\nu}g_{ab}, \tag{14}$$
$$R_{rtrt} = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)g_{rr}g_{tt}, \tag{15}$$
$$R_{r\mu r\nu} = \frac{1}{B^2}\left(\frac{F''}{F} - \frac{B'F'}{BF}\right)g_{rr}g_{\mu\nu}, \tag{16}$$
$$R_{rarb} = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right)g_{rr}g_{ab}, \tag{17}$$
$$R_{\mu\nu\alpha\beta} = \hat{R}_{\mu\nu\alpha\beta} + \frac{1}{B^2}\frac{F'^2}{F^2}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}), \tag{18}$$
$$R_{abcd} = \tilde{R}_{abcd} + \frac{1}{B^2}\frac{C'^2}{C^2}(g_{ac}g_{bd} - g_{ad}g_{bc}). \tag{19}$$

From these we can find the Ricci tensor

$$R_{tt} = g_{tt}\frac{1}{B^2}\left[p\frac{A'F'}{AF} + \frac{A''}{A} - \frac{A'B'}{AB} + (D-2)\frac{A'C'}{AC}\right], \tag{20}$$
$$R_{\mu\nu} = g_{\mu\nu}\frac{1}{B^2}\left[\frac{A'F'}{AF} + \frac{F''}{F} - \frac{B'F'}{BF} + (D-2)\frac{C'F'}{CF} + (p-1)\frac{F'^2}{F^2}\right] + \hat{R}_{\mu\nu}, \tag{21}$$
$$R_{rr} = g_{rr}\frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + p\frac{F''}{F} - p\frac{B'F'}{BF} + (D-2)\frac{C''}{C} - (D-2)\frac{B'C'}{BC}\right], \tag{22}$$
$$R_{ab} = g_{ab}\frac{1}{B^2}\left[\frac{C''}{C} - \frac{B'C'}{BC} + \frac{A'C'}{AC} + p\frac{C'F'}{CF} + (D-3)\frac{C'^2}{C^2}\right] + \tilde{R}_{ab}, \tag{23}$$
$$\hat{R}_{\mu\nu} = \frac{1}{-pF^2}\hat{R}g_{\mu\nu}, \tag{24}$$
$$\tilde{R}_{ab} = \frac{1}{-(D-2)C^2}k(D-2)(D-3)g_{ab}. \tag{25}$$

Setting $F = iA$ and including dt^2 into $\hat{g}_{\mu\nu}dx^\mu dx^\nu$ we get the Ricci tensor as follows

$$R^t_t = B^{-2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + p\left(\frac{A'}{A}\right)^2 + (D-2)\frac{A'C'}{AC}\right] + \frac{\hat{R}}{(p+1)A^2}, \tag{26}$$
$$R^r_r = B^{-2}\left[(p+1)\frac{A''}{A} + (D-2)\frac{C''}{C} - \frac{B'}{B}\left\{(p+1)\frac{A'}{A} + (D-2)\frac{C'}{C}\right\}\right], \tag{27}$$
$$R^\theta_\theta = B^{-2}\left[\frac{C''}{C} + \frac{C'}{C}\left\{(p+1)\frac{A'}{A} - \frac{B'}{B} + (D-3)\frac{C'}{C}\right\}\right] - \frac{k(D-3)}{C^2}. \tag{28}$$

This is used in an alternative of the Einstein tensor

$$R_B^A = \kappa \left(T_B^A - \frac{1}{p+D-2} T \delta_B^A \right) \tag{29}$$

with the right hand side calculated from

$$T_B^A = g^{AC} T_{BC} = -g^{AC} \frac{2}{\sqrt{|g|}} \frac{\delta(\sqrt{|g|}\mathcal{L}_m)}{\delta g^{BC}}, \tag{30}$$

with the Lagrangian density of the matter $\mathcal{L}_m$ is provided. Note that this definition must be checked by seeing the sign of T_0^0. One can find literatures, e.g. [42], discussing perfect fluid that is homogeneous and isotropic in a flat spacetime implies

$$T_{00} = \epsilon, T_{ij} = p_{(i)} \delta_{ij},$$

i.e., T_{00} is energy density and T_{ii} is pressure along i-th axis. (The other T_{0i} corresponds to momentum density along i-th axis.) Energy density and pressure can be thought as matter field moving along time axis and coordinate axis, respectively. It is clear that energy density of a physically acceptable model must be positive definite. Be aware that this is often discussed using metric with $(+--...)$ signature, but it turns out to be true also for metric with $(+--\ldots)$ signature. However this form can be quite inconvenient since for a metric with zero off-diagonal terms the zero-zero component of the energy-momentum component is often with the following form

$$T_{00} = g_{00} \times (\text{matter terms}) \, .$$

This is the reason why it is more convenient to check using T_0^0 whether the calculation of energy-momentum tensor is true or not since it is independent from g_{00}. Thus the condition for a physically acceptable model is

$$T_0^0 \begin{cases} \geq 0, & \text{for } (+--\ldots), \\ \leq 0, & \text{for } (-++\ldots). \end{cases} \tag{31}$$

APPENDIX B: CONFORMAL (WEYL) TRANSFORMATION

Conformal transformation begins with defining the metric of an n-dimensional space into another metric with a conformal factor ω

$$\tilde{g}_{mn} = \omega^2 g_{mn}. \tag{32}$$

This makes the connection transformed by

$$\tilde{\Gamma}^a{}_{bc} = \frac{1}{2}\tilde{g}^{ad}(\tilde{g}_{bd,c}+\tilde{g}_{cd,b}-\tilde{g}_{bc,d}) = \Gamma^a{}_{bc}+[\delta^a_b(\ln\omega)_{,c}+\delta^a_c(\ln\omega)_{,b}-g^{af}g_{bc}(\ln\omega)_{,f}] = \Gamma^a{}_{bc}+C^a{}_{bc}. \tag{33}$$

This makes the covariant derivative of, e.g., a vector transform as

$$\tilde{\nabla}_b V^a = \tilde{\partial}_b V^a + \tilde{\Gamma}^a_{bc}V^c \to \nabla_b V^a = \partial_b V^a + \Gamma^a_{bc}V^c + C^a_{bc}V^c \tag{34}$$

The Riemann tensor require lengthy calculation:

$$\tilde{R}^a_{bcd} = \tilde{\Gamma}^a{}_{bd,c} - \tilde{\Gamma}^a{}_{bc,d} + \tilde{\Gamma}^a{}_{ec}\tilde{\Gamma}^e{}_{bd} - \tilde{\Gamma}^a{}_{ed}\tilde{\Gamma}^e{}_{bc}$$

$$\tilde{\Gamma}^a{}_{bd,c} = \Gamma^a{}_{bd,c} + C^a_{bd,c} = \Gamma^a{}_{bd,c} + [\delta^a_b(\ln\omega)_{,d} + \delta^a_d(\ln\omega)_{,b} - g^{af}g_{bd}(\ln\omega)_{,f}]_{,c}$$

$$= +\Gamma^a{}_{bd,c}+[\delta^a_b(\ln\omega)_{,cd}+\delta^a_d(\ln\omega)_{,bc}-g^{af}_{,c}g_{bd}(\ln\omega)_{,f}-g^{af}g_{bd,c}(\ln\omega)_{,f}-g^{af}g_{bd}(\ln\omega)_{,fc}]$$

$$-\tilde{\Gamma}^a{}_{bc,d} = -\Gamma^a{}_{bc,d} - C^a_{bc,d} = -\Gamma^a{}_{bc,d} - [\delta^a_b(\ln\omega)_{,c} + \delta^a_c(\ln\omega)_{,b} - g^{af}g_{bc}(\ln\omega)_{,f}]_{,d}$$

$$= -\Gamma^a{}_{bc,d}+[-\delta^a_b(\ln\omega)_{,cd}-\delta^a_c(\ln\omega)_{,bd}+g^{af}_{,d}g_{bc}(\ln\omega)_{,f}+g^{af}g_{bc,d}(\ln\omega)_{,f}+g^{af}g_{bc}(\ln\omega)_{,fd}]$$

$$\tilde{\Gamma}^a{}_{ec}\tilde{\Gamma}^e{}_{bd} = \Gamma^a{}_{ec}\Gamma^e{}_{bd} + C^e{}_{bd}\Gamma^a{}_{ce} + C^a{}_{ec}\Gamma^e{}_{db} + C^a{}_{ec}C^e{}_{db}$$

$$-\tilde{\Gamma}^a{}_{ed}\tilde{\Gamma}^e{}_{bc} = -\Gamma^a{}_{ed}\Gamma^e{}_{bc} - C^e{}_{bc}\Gamma^a{}_{de} - C^a{}_{ed}\Gamma^e{}_{cb} - C^a{}_{ed}C^e{}_{cb}$$

$$C^e{}_{bd}\Gamma^a{}_{ce} = [\Gamma^a{}_{cb}(\ln\omega)_{,d} + \Gamma^a{}_{cd}(\ln\omega)_{,b} - \Gamma^a{}_{ce}g^{ef}g_{bd}(\ln\omega)_{,f}]$$

$$-C^e{}_{bc}\Gamma^a{}_{de} = [-\Gamma^a{}_{db}(\ln\omega)_{,c} - \Gamma^a{}_{dc}(\ln\omega)_{,b} + \Gamma^a{}_{de}g^{ef}g_{bc}(\ln\omega)_{,f}]$$

$$C^e{}_{bd}\Gamma^a{}_{ce}-C^e{}_{bc}\Gamma^a{}_{de} = \Gamma^a{}_{cb}(\ln\omega)_{,d}-\Gamma^a{}_{db}(\ln\omega)_{,c}-\Gamma^a{}_{ce}g^{ef}g_{bd}(\ln\omega)_{,f}+\Gamma^a{}_{de}g^{ef}g_{bc}(\ln\omega)_{,f}$$

$$C^a{}_{ec}\Gamma^e{}_{db} = [\Gamma^a{}_{db}(\ln\omega)_{,c} + \Gamma^e{}_{db}\delta^a_c(\ln\omega)_{,e} - \Gamma^e{}_{db}g^{af}g_{ec}(\ln\omega)_{,f}]$$

$$-C^a{}_{ed}\Gamma^e{}_{cb} = [-\Gamma^a{}_{cb}(\ln\omega)_{,d} - \Gamma^e{}_{cb}\delta^a_d(\ln\omega)_{,e} + \Gamma^e{}_{cb}g^{af}g_{ed}(\ln\omega)_{,f}]$$

$$C^a{}_{ec}C^e{}_{db} = \delta^a_b(\ln\omega)_{,c}(\ln\omega)_{,d} + \delta^a_c(\ln\omega)_{,b}(\ln\omega)_{,d} - g^{af}g_{bc}(\ln\omega)_{,f}(\ln\omega)_{,d}$$

$$+\delta^a_d(\ln\omega)_{,c}(\ln\omega)_{,b}+\delta^a_c(\ln\omega)_{,d}(\ln\omega)_{,b}-g^{af}g_{dc}(\ln\omega)_{,f}(\ln\omega)_{,b}$$

$$-(\ln\omega)_{,c}g^{af}g_{bd}(\ln\omega)_{,f}-\delta^a_c(\ln\omega)_{,e}g^{ef}g_{bd}(\ln\omega)_{,f}+g^{af}(\ln\omega)_{,f}g_{bd}(\ln\omega)_{,c}$$

$$-C^a{}_{ed}C^e{}_{cb}=-\delta^a_b(\ln\omega)_{,d}(\ln\omega)_{,c}-\delta^a_d(\ln\omega)_{,b}(\ln\omega)_{,c}+g^{af}g_{bd}(\ln\omega)_{,f}(\ln\omega)_{,c}$$

$$-\delta^a_c(\ln\omega)_{,d}(\ln\omega)_{,b}-\delta^a_d(\ln\omega)_{,c}(\ln\omega)_{,b}+g^{af}g_{cd}(\ln\omega)_{,f}(\ln\omega)_{,b}$$

$$+(\ln\omega)_{,d}g^{af}g_{bc}(\ln\omega)_{,f}+\delta^a_d(\ln\omega)_{,e}g^{ef}g_{bc}(\ln\omega)_{,f}-g^{af}(\ln\omega)_{,f}g_{bc}(\ln\omega)_{,d}$$

The red colored texts are vanishing. Thus we obtain Riemann tensor as follows

$$\tilde{R}^a{}_{bcd}=R^a{}_{bcd}+\delta^a_d(\ln\omega)_{,b;c}-g^{af}_{,c}g_{bd}(\ln\omega)_{,f}-g^{af}g_{bd,c}(\ln\omega)_{,f}-g^{af}g_{bd}(\ln\omega)_{,f;c}$$

$$-\delta^a_c(\ln\omega)_{,b;d}+g^{af}_{,d}g_{bc}(\ln\omega)_{,f}+g^{af}g_{bc,d}(\ln\omega)_{,f}+g^{af}g_{bc}(\ln\omega)_{,f;d}$$

$$-\Gamma^a{}_{ce}g^{ef}g_{bd}(\ln\omega)_{,f}+\Gamma^a{}_{de}g^{ef}g_{bc}(\ln\omega)_{,f}+\Gamma^e{}_{db}\delta^a_c(\ln\omega)_{,e}$$

$$-\Gamma^e{}_{db}g^{af}g_{ec}(\ln\omega)_{,f}-\Gamma^e{}_{cb}\delta^a_d(\ln\omega)_{,e}+\Gamma^e{}_{cb}g^{af}g_{ed}(\ln\omega)_{,f}$$

$$-g^{af}g_{bc}(\ln\omega)_{,f}(\ln\omega)_{,d}+\delta^a_c(\ln\omega)_{,d}(\ln\omega)_{,b}-\delta^a_c(\ln\omega)_{,e}g^{ef}g_{bd}(\ln\omega)_{,f}$$

$$-\delta^a_d(\ln\omega)_{,b}(\ln\omega)_{,c}+g^{af}g_{bd}(\ln\omega)_{,f}(\ln\omega)_{,c}+\delta^a_d(\ln\omega)_{,e}g^{ef}g_{bc}(\ln\omega)_{,f}.$$

or after eliminating some terms

$$\begin{aligned}\tilde{R}^a{}_{bcd} &= R^a{}_{bcd}+\delta^a_d(\ln\omega)_{,b;c}-\delta^a_c(\ln\omega)_{,b;d}-g^{af}g_{bd}(\ln\omega)_{,f;c}+g^{af}g_{bc}(\ln\omega)_{,f;d}\\ &\quad -g^{af}g_{bc}(\ln\omega)_{,f}(\ln\omega)_{,d}+\delta^a_c(\ln\omega)_{,d}(\ln\omega)_{,b}-\delta^a_c(\ln\omega)_{,e}g^{ef}g_{bd}(\ln\omega)_{,f}\\ &\quad -\delta^a_d(\ln\omega)_{,b}(\ln\omega)_{,c}+g^{af}g_{bd}(\ln\omega)_{,f}(\ln\omega)_{,c}+\delta^a_d(\ln\omega)_{,e}g^{ef}g_{bc}(\ln\omega)_{,f},\end{aligned} \tag{35}$$

with $V^a_{;b}=\nabla_b V^a$ and $V^a_{,b}=\partial_b V^a$. Using δ^c_a we can obtain Ricci tensor

$$\tilde{R}_{bd}=R_{bd}-g^{af}g_{bd}(\ln\omega)_{,f;a}+(n-2)[-(\ln\omega)_{,b;d}+(\ln\omega)_{,d}(\ln\omega)_{,b}-g_{bd}g^{ef}(\ln\omega)_{,e}(\ln\omega)_{,f}]. \tag{36}$$

After multiply it by $\tilde{g}^{bd}=\omega^{-2}g^{bd}$ its Ricci scalar is

$$\tilde{R}=\omega^{-2}[R-2(n-1)g^{ab}(\ln\omega)_{,a;b}-(n-1)(n-2)g^{ab}(\ln\omega)_{,a}(\ln\omega)_{,b}]. \tag{37}$$

B.1: The Uniqueness of Weyl Tensor under Conformal Transformation

Weyl tensor captures information from Riemann tensor that Ricci tensor can not obtain. Ricci tensor contains only the trace part of Riemann tensor, thus the non-trace term are missing. These term are contained in the so-called Weyl tensor, which is defined as

$$C_{abcd} = R_{abcd} + \frac{R}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}) - \frac{1}{n-2}(g_{ac}R_{bd} + R_{ac}g_{bd} - g_{ad}R_{bc} - R_{ad}g_{bc}). \tag{38}$$

(We apologize if this cause a confusion, but this is different from $C^a_{\ bc}$, that is the connection coefficient transformation term.) Here we investigate its (anti)symmetry. First let us see what happens when we contract it with g^{ac}

$$g^{ac}C_{abcd} - R_{bd} = \frac{R}{(n-2)(n-1)}(ng_{bd} - g_{bd}) - \frac{1}{n-2}(nR_{bd} + Rg_{bd} - R_{bd} - R_{bd})$$

$$= \frac{R}{(n-2)(n-1)}(n-1)g_{bd} - (R_{bd} + \frac{R}{n-2}g_{bd}) = -R_{bd},$$

which is why Weyl tensor is trace-free. Now exchanging $c \leftrightarrow d$ we have

$$C_{abdc} = -R_{abcd} - \frac{R}{(n-2)(n-1)}(-g_{ad}g_{bc} + g_{ac}g_{bd}) + \frac{1}{n-2}(-g_{ad}R_{bc} - R_{ad}g_{bc} + g_{ac}R_{bd} + R_{ac}g_{bd}).$$

which makes $C_{abcd} = -C_{abdc}$. Now exchange $a \leftrightarrow c$

$$C_{cbad} = R_{cbad} + \frac{R}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{cd}g_{ab}) - \frac{1}{n-2}(g_{ac}R_{bd} + R_{ac}g_{bd} - g_{cd}R_{ab} - R_{cd}g_{ab}),$$

then $b \leftrightarrow d$

$$C_{cdab} = R_{abcd} + \frac{R}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{bc}g_{ad}) - \frac{1}{n-2}(g_{ac}R_{bd} + R_{ac}g_{bd} - g_{bc}R_{ad} - R_{bc}g_{ad}),$$

which is $C_{cdab} = C_{abcd}$. Now verifying $C_{abcd} + C_{acdb} + C_{adbc} = 0$,

$$C_{abcd} = R_{abcd} + \frac{R}{(n-2)(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}) - \frac{1}{n-2}(g_{ac}R_{bd} + R_{ac}g_{bd} - g_{ad}R_{bc} - R_{ad}g_{bc})$$

$$C_{acdb} = R_{acdb} + \frac{R}{(n-2)(n-1)}(g_{ad}g_{cb} - g_{ab}g_{cd}) - \frac{1}{n-2}(g_{ad}R_{cb} + R_{ad}g_{cb} - g_{ab}R_{cd} - R_{ab}g_{cd})$$

$$C_{adbc} = R_{adbc} + \frac{R}{(n-2)(n-1)}(g_{ab}g_{dc} - g_{ac}g_{db}) - \frac{1}{n-2}(g_{ab}R_{dc} + R_{ab}g_{dc} - g_{ac}R_{db} - R_{ac}g_{db}).$$

Adding all up will diminish all terms and give us zero. This tool is quite interesting since it is invariant to conformal transformation thus the name Weyl transformation. Now let us see what is happening to the Weyl tensor after conformal transformation

$$\tilde{C}^a{}_{bcd} = \tilde{R}^a{}_{bcd} + \frac{\tilde{R}}{(n-2)(n-1)}(\delta^a_c \tilde{g}_{bd} - \delta^a_d \tilde{g}_{bc}) - \frac{1}{n-2}(\delta^a_c \tilde{R}_{bd} + \tilde{R}^a_c \tilde{g}_{bd} - \delta^a_d \tilde{R}_{bc} - \tilde{R}^a_d \tilde{g}_{bc})$$

$$= R^a{}_{bcd} + g^{af} g_{bc} (\ln\omega)_{,f;d} - g^{af} g_{bc} (\ln\omega)_{,f} (\ln\omega)_{,d} - g^{af} g_{bd} (\ln\omega)_{,f;c} + g^{af} g_{bd} (\ln\omega)_{,f} (\ln\omega)_{,c}$$
$$+\delta^a_d (\ln\omega)_{,b;c} - \delta^a_d (\ln\omega)_{,b} (\ln\omega)_{,c} + \delta^a_d (\ln\omega)_{,e} g^{ef} g_{bc} (\ln\omega)_{,f}$$
$$+\delta^a_c (\ln\omega)_{,d} (\ln\omega)_{,b} - \delta^a_c (\ln\omega)_{,e} g^{ef} g_{bd} (\ln\omega)_{,f} - \delta^a_c (\ln\omega)_{,b;d}$$
$$+\frac{(\delta^a_c g_{bd} - \delta^a_d g_{bc})}{(n-2)(n-1)}[R - 2(n-1) g^{fg} (\ln\omega)_{,f;g} - (n-1)(n-2) g^{fg} (\ln\omega)_{,f} (\ln\omega)_{,g}]$$
$$-\frac{\delta^a_c \left\{R_{bd} - g^{fg} g_{bd} (\ln\omega)_{,f;g} + (n-2)[(\ln\omega)_{,d} (\ln\omega)_{,b} - g_{bd} g^{fg} (\ln\omega)_{,f} (\ln\omega)_{,g} - (\ln\omega)_{,b;d}]\right\}}{(n-2)}$$
$$-\frac{g_{bd} \left\{R^a_c - \delta^a_c g^{fg} (\ln\omega)_{,f;g} + (n-2)[(\ln\omega)_{,c} (\ln\omega)_{,e} g^{ae} - \delta^a_c g^{fg} (\ln\omega)_{,f} (\ln\omega)_{,g} - (\ln\omega)_{,e;c} g^{ae}]\right\}}{n-2}$$
$$+\frac{\delta^a_d \left\{R_{bc} - g^{fg} g_{bc} (\ln\omega)_{,f;g} + (n-2)[(\ln\omega)_{,c} (\ln\omega)_{,b} - g_{bc} g^{fg} (\ln\omega)_{,f} (\ln\omega)_{,g} - (\ln\omega)_{,b;c}]\right\}}{n-2}$$
$$+\frac{g_{bc} \left\{R^a_c - \delta^a_d g^{fg} (\ln\omega)_{,f;g} + (n-2)[(\ln\omega)_{,d} (\ln\omega)_{,e} g^{ae} - \delta^a_d g^{fg} (\ln\omega)_{,f} (\ln\omega)_{,g} - (\ln\omega)_{,e;d} g^{ae}]\right\}}{n-2}$$
$$= R^a{}_{bcd} + \frac{R(\delta^a_c g_{bd} - \delta^a_d g_{bc})}{(n-2)(n-1)} - \frac{1}{n-2}(\delta^a_c R_{bd} + R^a_c g_{bd} - \delta^a_d R_{bc} - R^a_d g_{bc}) = C^a{}_{bcd}.$$

It is proven that Weyl tensor transform in a simple way: $\tilde{C}^a{}_{bcd} = C^a{}_{bcd}$ or $\tilde{C}_{abcd} = \omega^2 C_{abcd}$.

B.2: Transforming Jordan frame to Einstein frame

In string theory people usually start from an action in Jordan frame to formulate a specific model in scalar-tensor theory introduced in Ref. [42]. Let us consider an action of scalar-tensor model in n-dimensional manifold

$$\mathcal{S} = \int d^n x \sqrt{|g|} \left[\frac{f(\lambda)\mathcal{R}}{16\pi G} - \frac{h(\lambda)}{2} g^{mn} \nabla_m \lambda \nabla_n \lambda - U(\lambda) + \mathcal{L}_m(g_{ab}, \psi_i)\right], \quad (39)$$

where on the right hand side the first term implies coupling of a scalar field with the curvature, the second and the third term imply the scalar field Lagrangian density, and the last term imply Lagrangian density of other fields. We can perform Weyl rescaling by

$$g_{mn} = f^{2a}\tilde{g}_{mn} \tag{40}$$

with a a constant to be determined which give us $\sqrt{|g|} = \sqrt{|\tilde{g}|}f^{na}$, $(-h/2)g^{mn}\nabla_m\lambda\nabla_n\lambda = (-h/2)f^{-2a}\tilde{g}^{mn}\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda$, and

$$\mathcal{R} = f^{-2a}\left[\tilde{\mathcal{R}} - 2(n-1)\tilde{g}^{mn}\tilde{\nabla}_m\tilde{\nabla}_n\mathrm{ln}f^a + (n-2)(n-1)\tilde{g}^{mn}\tilde{\nabla}_m\mathrm{ln}f^a\tilde{\nabla}_n\mathrm{ln}f^a\right]. \tag{41}$$

Performing some algebra to change from $\nabla_m\mathrm{ln}f^a$ to $\nabla_a\lambda$ as follows

$$\begin{aligned}
\mathcal{R} &= f^{-2a}\tilde{\mathcal{R}} - 2(n-1)af^{-2a-1}\tilde{g}^{mn}\tilde{\nabla}_m\tilde{\nabla}_n f + [(n-2)a+2](n-1)af^{-2a-2}\tilde{g}^{mn}\tilde{\nabla}_m f\tilde{\nabla}_n f. \\
&= f^{-2a}\tilde{\mathcal{R}} - 2(n-1)af^{-2a-1}\tilde{g}^{mn}\tilde{\nabla}_m\left[f'(\lambda)\tilde{\nabla}_n\lambda\right] \\
&\quad +[(n-2)a+2](n-1)af^{-2a-2}\tilde{g}^{mn}[f'(\lambda)]^2\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda \\
&= f^{-2a}\tilde{\mathcal{R}} - 2(n-1)af^{-2a-1}\tilde{g}^{mn}\left[f''(\lambda)\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda + f'(\lambda)\tilde{\nabla}_m\tilde{\nabla}_n\lambda\right] \\
&\quad +[(n-2)a+2](n-1)af^{-2a-2}\tilde{g}^{mn}[f'(\lambda)]^2\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda \\
&= f^{-2a}\tilde{\mathcal{R}} - 2(n-1)af^{-2a-1}f'(\lambda)\tilde{g}^{mn}\tilde{\nabla}_m\tilde{\nabla}_n\lambda \\
&\quad +\left\{[(n-2)a+2](n-1)af^{-2a-2}[f'(\lambda)]^2 - 2(n-1)af^{-2a-1}f''(\lambda)\right\}\tilde{g}^{mn}\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda.
\end{aligned}$$

Let us define an effective potential

$$V_{\text{eff}}(\lambda, f^{2a}\tilde{g}_{ab}, \psi_i) = \frac{U(\lambda) - \mathcal{L}_m(f^{2a}\tilde{g}_{ab}, \psi_i)}{f^{na}} \tag{42}$$

thus we have the terms outside the potential

$$\begin{aligned}
\frac{f\mathcal{R}}{16\pi G} &- \frac{h}{2}f^{-2a}\tilde{g}^{mn}\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda = f^{1-2a}\frac{\tilde{\mathcal{R}}}{16\pi G} - \frac{2(n-1)a}{16\pi G}f^{-2a}f'(\lambda)\tilde{g}^{mn}\tilde{\nabla}_m\tilde{\nabla}_n\lambda \\
&\quad +\left\{[(n-2)a+2][f'(\lambda)]^2 - 2f''(\lambda)f\right\}(n-1)a\frac{f^{-2a}}{16\pi Gf}\tilde{g}^{mn}\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda \\
&\quad -\frac{h}{2}f^{-2a}\tilde{g}^{mn}\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda \\
&= f^{1-2a}\frac{\tilde{\mathcal{R}}}{16\pi G} - \frac{2(n-1)a}{16\pi G}f^{-2a}f'(\lambda)\tilde{g}^{mn}\tilde{\nabla}_m\tilde{\nabla}_n\lambda \\
&\quad +\left[\left\{[(n-2)a+2][f'(\lambda)]^2 - 2f''(\lambda)f\right\}\frac{(n-1)a}{16\pi Gf} - \frac{h}{2}\right]f^{-2a}\tilde{g}^{mn}\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda,
\end{aligned}$$

which after substituting into the action

$$\begin{aligned}\mathcal{S} &= \int d^n x \sqrt{|\tilde{g}|}\left[f^{1+(n-2)a}\frac{\tilde{\mathcal{R}}}{16\pi G} - \frac{2(n-1)a}{16\pi G} f^{(n-2)a} f'(\lambda)\tilde{g}^{mn}\tilde{\nabla}_m\tilde{\nabla}_n\lambda\right. \\ &+ \left[\left\{[(n-2)a+2][f'(\lambda)]^2 - 2f''(\lambda)f\right\}\frac{(n-1)a}{16\pi G f} - \frac{h}{2}\right] f^{(n-2)a}\tilde{g}^{mn}\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda \\ &\left. - V_{\text{eff}}(\lambda, f^{2a}\tilde{g}_{ab}, \psi_i)\right], \end{aligned} \tag{43}$$

implies

$$a = 1/(2-n). \tag{44}$$

We thus have the intended Einstein frame

$$\begin{aligned}\mathcal{S} &= \int d^n x \sqrt{|\tilde{g}|}\left[\frac{\tilde{\mathcal{R}}}{16\pi G} - \frac{K^2(\lambda)}{2}\tilde{g}^{mn}\tilde{\nabla}_m\lambda\tilde{\nabla}_n\lambda - V_{\text{eff}}(\lambda, f^{2/(2-n)}\tilde{g}_{ab}, \psi_i)\right. \\ &\left. + \frac{(n-1)}{(n-2)8\pi G}\frac{f'(\lambda)}{f}\tilde{g}^{mn}\tilde{\nabla}_m\tilde{\nabla}_n\lambda\right], \end{aligned} \tag{45}$$

with the last term as the surface term and

$$K^2(\lambda) = \frac{h}{f} - \frac{(n-1)}{(n-2)8\pi G}\left(\frac{f'(\lambda)}{f}\right)^2 - \frac{2(n-1)}{(n-2)8\pi G}\left[\frac{f'(\lambda)}{f}\right]'. \tag{46}$$

By ignoring the surface term and using

$$\Phi = \int K(\lambda)d\lambda, \tag{47}$$

we have a simplified form

$$\mathcal{S} = \int d^n x \sqrt{|\tilde{g}|}\left[\frac{\tilde{\mathcal{R}}}{16\pi G} - \frac{1}{2}\tilde{g}^{mn}\tilde{\nabla}_m\Phi\tilde{\nabla}_n\Phi - V_{\text{eff}}\left\{\Phi, f^{2/(2-n)}(\Phi)\tilde{g}_{ab}, \psi_i\right\}\right]. \tag{48}$$

B.3: Dimensional Reduction Method

Now we perform a dimensional reduction following Ref. [4] to obtain a different type of effective potential. (This is what we use in Section 5.) This is basically a process that translate an Einstein frame to a Jordan frame and then translate again to Einstein frame. Consider a metric with signature $(+--\ldots)$

$$ds^2 = G^{(p+D)}_{MN} dx^A dx^B, \tag{49}$$

with $G_{MN}^{(p+D)}$ metric on $(p+D)$ dimensional action

$$\mathcal{S}=\int d^{p+D}x\sqrt{|G|}\left[\frac{\mathcal{R}^{(p+D)}}{2\kappa}+\mathcal{L}_m\right]. \tag{50}$$

By defining inside the metric with radion b

$$ds^2=g_{\mu\nu}^{(p+2)}(x)dx^\mu dx^\nu-b^2(x)\gamma_{ij}^{(D-2)}(y)dy^i dy^j, \tag{51}$$

we can perform dimensional reduction. With Weyl transformation we obtain the scalar curvature

$$\mathcal{R}^{(p+D)}=\mathcal{R}^{(p+2)}+\frac{\mathcal{R}^{(D-2)}}{b^2}-2(D-2)g^{\mu\nu}\nabla_\mu(\partial_\nu\ln b)-(D-2)(D-1)g^{\mu\nu}(\partial_\mu\ln b)(\partial_\nu\ln b), \tag{52}$$

and we define $(D-2)$ dimension space has constant curvature that has radius of curvature R_0 thus its scalar curvature is

$$\mathcal{R}^{(D-2)}=\frac{(D-2)(D-3)}{R_0^2}, \tag{53}$$

and its surface area is just a constant

$$\int d^{D-2}y\sqrt{\gamma}=V_{(D-2)} \tag{54}$$

But due to the integral becomes $\int d^{p+D}x\sqrt{|G|}=\int d^{p+2}xV_{(D-2)}\sqrt{|g|}b^{D-2}$, the action become

$$\mathcal{S}=\frac{V_{(D-2)}}{2\kappa}\int d^{p+2}x\sqrt{|g|}\left[b^{D-2}\mathcal{R}^{(p+D)}+2\kappa b^{D-2}\mathcal{L}_m\right], \tag{55}$$

which is known as Jordan's frame action. To change it into Einstein frame action, we perform conformal (Weyl) transformation by defining a new metric

$$g_{\mu\nu}^{(p+2)}(x)=\omega^2(x)\tilde{g}_{\mu\nu}^{(p+2)}(x). \tag{56}$$

This lead us to a new connection coefficient

$$\Gamma^\rho{}_{\mu\nu}=\tilde{\Gamma}^\rho{}_{\mu\nu}+\delta^\rho_\nu\tilde{\partial}_\mu\ln\omega+\delta^\rho_\mu\tilde{\partial}_\nu\ln\omega-\tilde{g}^{\rho\lambda}\tilde{g}_{\mu\nu}\tilde{\partial}_\lambda\ln\omega, \tag{57}$$

and after some long algebraic calculation we obtain its scalar curvature become

$$\mathcal{R}^{(p+2)} = \omega^{-2}\left[\tilde{\mathcal{R}}^{(p+2)} - 2(p+1)\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu(\tilde{\partial}_\nu \ln\omega) - p(p+1)\tilde{g}^{\mu\nu}(\tilde{\partial}_\mu \ln\omega)(\tilde{\partial}_\nu \ln\omega)\right]. \tag{58}$$

Note that $\sqrt{|g|} = \sqrt{|\tilde{g}|}\omega^{p+2}$ and it makes $\sqrt{|g|}b^{D-2}\mathcal{R}^{(p+2)} = \sqrt{|\tilde{g}|}\omega^{p+2}b^{D-2}\mathcal{R}^{(p+2)}$ thus we define the conformal factor and b to vanish with

$$\omega(x) = b^a(x), \tag{59}$$

with a a constant to be determined. This makes

$$\mathcal{R}^{(p+2)} = b^{-2a}\left[\tilde{\mathcal{R}}^{(p+2)} - 2a(p+1)\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu(\tilde{\partial}_\nu \ln b) - a^2p(p+1)\tilde{g}^{\mu\nu}(\tilde{\partial}_\mu \ln b)(\tilde{\partial}_\nu \ln b)\right], \tag{60}$$

and

$$\begin{aligned}
\mathcal{R}^{(p+D)} &= \mathcal{R}^{(p+2)} + \frac{(D-2)(D-3)}{b^2R_0^2} \qquad (61)\\
&\quad + b^{-2a}\left[-2(D-2)\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu(\tilde{\partial}_\nu \ln b) - (D-2)[2ap+D-1]\tilde{g}^{\mu\nu}(\tilde{\partial}_\mu \ln b)(\tilde{\partial}_\nu \ln b)\right],\\
&= b^{-2a}\left\{\tilde{\mathcal{R}}^{(p+2)} + \frac{(D-2)(D-3)}{b^{2-2a}R_0^2}\right. \qquad (62)\\
&\quad - [a^2p(p+1) + (D-2)(2ap+D-1)]\tilde{g}^{\mu\nu}(\tilde{\partial}_\mu \ln b)(\tilde{\partial}_\nu \ln b)\\
&\quad \left. -[2a(p+1) + 2(D-2)]\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu(\tilde{\partial}_\nu \ln b)\right\}.
\end{aligned}$$

Notice that $\sqrt{|\tilde{g}|}\omega^{p+2}b^{D-2}\mathcal{R}^{(p+2)} = \sqrt{|\tilde{g}|}b^{a(p+2)}b^{D-2}b^{-2a}\{\dots\} = \sqrt{|\tilde{g}|}\{\dots\}$ is required to obtain Einstein's frame thus

$$a = -(D-2)/p. \tag{63}$$

This makes

$$\begin{aligned}
\{\dots\} &= \tilde{\mathcal{R}}^{(p+2)} + \frac{(D-2)(D-3)}{b^{2(p+D-2)/p}R_0^2} - \frac{(D-2)(p+D-2)}{p}\tilde{g}^{\mu\nu}(\tilde{\partial}_\mu \ln b)(\tilde{\partial}_\nu \ln b)\\
&\quad + \frac{2(D-2)}{p}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu(\tilde{\partial}_\nu \ln b) \qquad (64)
\end{aligned}$$

and with $\tilde{\mathcal{L}}_m$ is $\mathcal{L}_m$ after conformal transformation (57), we obtain

$$\begin{aligned}
\mathcal{S} &= \frac{V_{(D-2)}}{2\kappa}\int d^{p+2}x\sqrt{|\tilde{g}|}\left[\tilde{\mathcal{R}}^{(p+2)} + \frac{(D-2)(D-3)}{b^{2(p+D-2)/p}R_0^2} + \frac{2(D-2)}{p}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu(\tilde{\partial}_\nu \ln b)\right.\\
&\quad \left. -\frac{(D-2)(p+D-2)}{p}\tilde{g}^{\mu\nu}(\tilde{\partial}_\mu \ln b)(\tilde{\partial}_\nu \ln b) + 2\kappa b^{-(D-2)/p}\tilde{\mathcal{L}}_m\right]. \qquad (65)
\end{aligned}$$

The third term in the integral is just a total derivative thus by Gauss' theorem we can choose $\tilde{\partial}_\nu \ln b = 0$ at the boundary. Define

$$b = \exp\left[\sqrt{\frac{p}{(D-2)(p+D-2)}}\frac{\psi}{M_P}\right], \tag{66}$$

with $M_P \equiv \sqrt{V_{(D-2)}/\kappa}$ as a $(p+2)$-dimensional Planck mass we obtain our intended action in Einstein's frame

$$\mathcal{S} = \int d^{p+2}x\sqrt{|\tilde{g}|}\left[\frac{M_P^2\tilde{\mathcal{R}}^{(p+2)}}{2} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\psi\tilde{\partial}_\nu\psi - V(\psi)\right], \tag{67}$$

which has effective potential

$$V(\psi) = -e^{\sigma\psi/M_P}\frac{M_P^2(D-2)(D-3)}{2R_0^2} - e^{\chi\psi/M_P}M_P^2\kappa\tilde{\mathcal{L}}_m, \tag{68}$$

and the metric become

$$ds^2 = e^{\chi\psi/M_P}\tilde{g}_{\mu\nu}^{(p+2)}dx^\mu dx^\nu - e^{\xi\psi/M_P}\gamma_{ij}^{(D-2)}dy^i dy^j, \tag{69}$$

with

$$\chi = -2\sqrt{\frac{(D-2)}{p(p+D-2)}}, \tag{70}$$

$$\sigma = -2\sqrt{\frac{(p+D-2)}{p(D-2)}}, \tag{71}$$

$$\xi = 2\sqrt{\frac{p}{(D-2)(p+D-2)}}. \tag{72}$$