Probability & Stochastic Processes

Coursework - 2017.

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2. a) $K = \{k^{th}$ card is the largest of the k cards drawn$\}$.

$M = \{k^{th}$ card is the largest of all m cards$\}$

$$P(M|K) = \frac{P(M \cap K)}{P(K)} \quad (1)$$

$P(M \cap K) = P(M)$ because if the k^{th} card is the largest in the second set it will be also the largest in the first one.

$$\text{Therefore, } P(M|K) = \frac{P(M)}{P(K)} = \frac{\frac{1}{m}}{\frac{1}{k}} = \frac{k}{m} \Rightarrow.$$

$$\boxed{P(M|K) = \frac{k}{m}}$$

b) i) $P(|X| \geq a) \leq \frac{E[|X|]}{a} = \sqrt{\frac{2}{\pi}} \cdot \frac{\cancel{\sigma}}{4\cancel{\sigma}} = \frac{1}{\sqrt{8\pi}} = \underline{0.2}$

ii) $P(|X-\mu| \geq a) = P(|X| \geq a) \leq \frac{\sigma^2}{a^2} = \frac{\cancel{\sigma^2}}{16\cancel{\sigma^2}} = \frac{1}{16} = \underline{0.0625}$

iii) $P(|X| > a) \leq 2 \cdot \min_{\lambda > 0} e^{-\lambda a} \phi_X(\lambda) = 2 \cdot \min_{\lambda > 0} e^{-a\lambda + \frac{\sigma^2 \lambda^2}{2}}$

Minimum when: $\lambda = \frac{a}{\sigma^2}$

So, $P(|X| > a) \leq 2 \cdot e^{-\frac{16\sigma^2}{2\sigma^2}} = 2 \cdot e^{-8} = \underline{0.000671}$

2. a) $f_{X_i}(x_i; c) = c e^{-c(x_i - x_0)}$

Since the samples are i.i.d. $\Rightarrow f_X(x_1, x_2, x_3, x_4, x_5; c) = c^5 \cdot e^{-c\sum_{i=1}^{5}(x_i - x_0)}$.

$= c^5 \cdot e^{-5c(\bar{x} - x_0)}$.

Therefore, $\frac{\partial}{\partial c} f(x_1, x_2, x_3, x_4, x_5; c) = \frac{\partial}{\partial c} c^5 \cdot e^{-5c(\bar{x} - x_0)} =$

$= 5c^4 e^{-5c(\bar{x} - x_0)} - c^5 5 \cdot (\bar{x} - x_0) e^{-5c(x - x_0)}$

Maximum when $\frac{\partial}{\partial c} f(x_1, x_2, x_3, x_4, x_5; c) = 0 \Rightarrow$

$c_{opt}^4 - c_{opt}^5 (\bar{x} - x_0) = 0 \Rightarrow c_{opt} = \frac{1}{\bar{x} - x_0}$

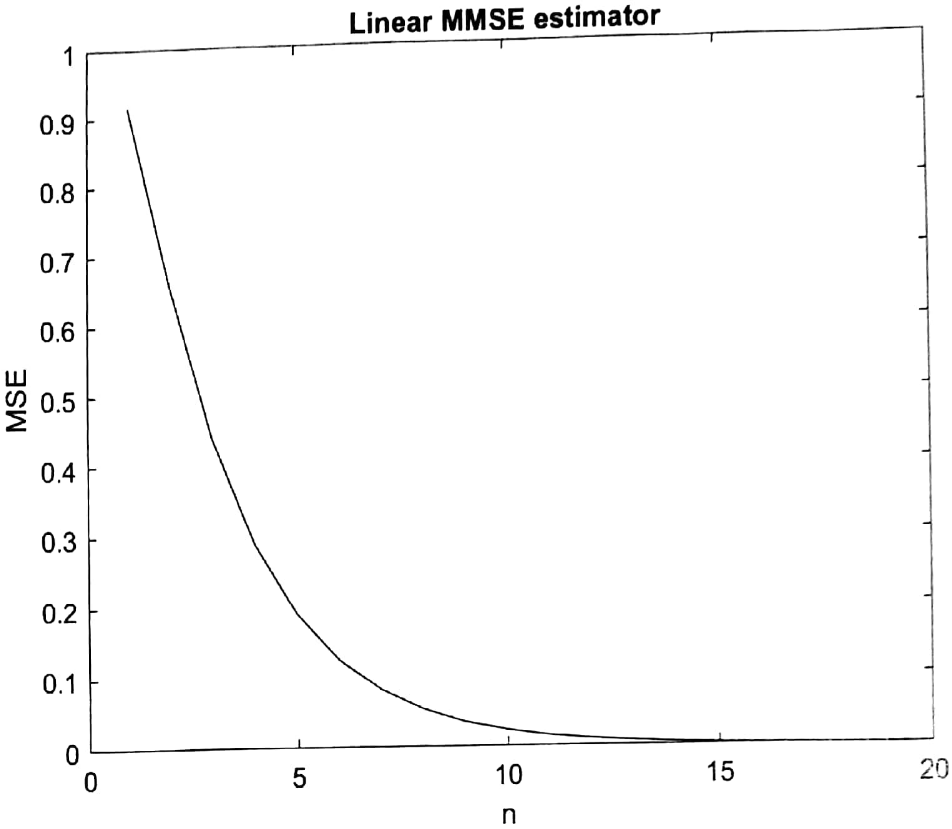
$\bar{x} = \frac{4.1 + 3.7 + 4.3 + 3.7 + 4.2}{5} = \frac{20}{5} = 4$

Therefore, $c_{opt} = \frac{1}{4-1} \Rightarrow \boxed{c_{opt} = \frac{1}{3}}$

b).i) For $n=1$: $\underline{\underline{R}} \cdot \underline{c} = \underline{r} \Rightarrow R(0) \cdot c_1 = R(1) \Rightarrow c_1 = \frac{R(1)}{R(0)} = \frac{J_0(2\pi f_d \cdot 1)}{J_0(0)} \Rightarrow$

$$\boxed{c_1 = \frac{J_0(2\pi f_d)}{J_0(0)}}$$

ii) The plot of the mean-square error is presented in the following page:



It is clear that as the order of the MMSE estimator increases, the mean-square error is decreasing.

3. a) i) For a Poisson Process we have:

$$P[n(t_1,t_2)=k]=e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \text{ where } k=0,1,2,\dots \text{ and } t=t_2-t_1.$$

The probability of at most one failure in $[0,8)$ is:

$$P[n(0,8)=0]+P[n(0,8)=1]=e^{-0.25\cdot 8}\frac{(0.25\cdot 8)^0}{0!}+e^{-0.25\cdot 8}\frac{(0.25\cdot 8)^1}{1!}$$

$$=e^{-2}+2\cdot e^{-2}=\boxed{3\cdot e^{-2}}$$

The probability of at least two failures in $[8,16)$ is:

$$1-P[n(8,16)=0]-P[n(8,16)=1]=1-e^{-2}-2e^{-2}=\boxed{1-3\cdot e^{-2}}$$

The probability of at most 1 failure in $[16,24)$ is the same as the first probability calculated since the two time intervals are the same. Therefore:

$$P[n(16,24)=0]+P[n(16,24)=1]=3\cdot e^{-2}$$

The probability of having these failures in one experiment is:

$$(P[n(0,8)=0]+P[n(0,8)=1])\cdot(1-P[n(8,16)=0]-P[n(8,16)=1])$$
$$\cdot(P[n(16,24)=0]+P[n(16,24)=1])=3\cdot e^{-2}(1-3\cdot e^{-2})\cdot 3\cdot e^{-2}$$
$$=0.0979$$

We multiplied the three probabilities because if the intervals (t_1,t_2) and (t_3,t_4) are non-overlapping, then the random variables $n(t_1,t_2)$ and $n(t_3,t_4)$ are independent.

ii) If we want to calculate the probability that the third failure occurs after 8 hours, we have to calculate that exactly two failures occurred in the first 8 hours.

Therefore, $P(t_3 > 8) = P(n(0,8) = 2) = e^{-0.25 \cdot 8} \dfrac{(0.25 \cdot 8)^2}{2!} = e^{-2} \cdot \dfrac{4}{2}$

$\Rightarrow \boxed{P(t_3 > 8) = 0.27}$

b). $E[X(n)] = E[A\cos(n\lambda+\theta)] + E[B\sin(n\lambda+\theta)] = E[A]\cdot\cos(n\lambda+\theta) +$
$E[B]\sin(n\lambda+\theta) \Rightarrow \boxed{E[X(n)] = 0}$ (1)

• $R_{XX}(n_1, n_2) = E[X(n_1)X^*(n_2)] =$

$= E[(A\cos(n_1\lambda+\theta) + B\sin(n_1\lambda+\theta))\cdot(A\cos(n_2\lambda+\theta) + B\sin(n_2\lambda+\theta))] =$

$= E[A^2\cos(n_1\lambda+\theta)\cos(n_2\lambda+\theta) + AB\cos(n_1\lambda+\theta)\sin(n_2\lambda+\theta) + AB\sin(n_1\lambda+\theta)\cos(n_2\lambda+\theta)$
$+ B^2\sin(n_1\lambda+\theta)\sin(n_2\lambda+\theta)] =$

$\boxed{\text{uncorrelated}}$

$= E[A^2]\cos(n_1\lambda+\theta)\cos(n_2\lambda+\theta) + E[AB](\cos(n_1\lambda+\theta)\sin(n_2\lambda+\theta) +$
$\sin(n_1\lambda+\theta)\cos(n_2\lambda+\theta)) + E[B^2]\sin(n_1\lambda+\theta)\sin(n_2\lambda+\theta) =$

$= \frac{E[A^2]}{2}\left[\cos(n_1\lambda+\cancel{\theta}-n_2\lambda-\cancel{\theta}) + \cancel{\cos(n_1\lambda+\theta+n_2\lambda+\theta)}\right] +$

$\cancel{E[A]}^{0}\cdot\cancel{E[B]}^{0}[\cos(n_1\lambda+\theta)\sin(n_2\lambda+\theta) + \sin(n_1\lambda+\theta)\cos(n_2\lambda+\theta)] +$

$\frac{E[B^2]}{2}\left[\cos(n_1\lambda+\cancel{\theta}-n_2\lambda-\cancel{\theta}) - \cancel{\cos(n_1\lambda+\theta+n_2\lambda+\theta)}\right] =$

$= \frac{1}{2}\left[\cos(\lambda(n_1-n_2)) + \cos(\lambda(n_1-n_2))\right] = \cos(\lambda(n_1-n_2))$

Therefore, $\boxed{R_{XX}(n_1,n_2) = R_{XX}(n_1-n_2) = \cos[\lambda(n_1-n_2)]}$ (2).

From (1) and (2) $\Rightarrow X(n)$ is a wide-sense stationary random process.

c) For $Z(t)$ we have: $C_{ZZ}(\tau) = R_{ZZ}(\tau) - \mu_Z \cdot \mu_Z^*$

$$R_{ZZ}(\tau) = E[X(t_1)X(t_2)X(t_3)X(t_4)] = E[X(t_1)X(t_2)]E[X(t_3)X(t_4)]$$
$$+ E[X(t_1)X(t_3)]E[X(t_2)X(t_4)] + E[X(t_1)X(t_4)]E[X(t_2)X(t_3)] =$$
$$= 3R_{XX}^2(\tau). \quad (1)$$

Furthermore, $\mu_Z = E[Z(t)] = E[X^2(t)] = E[X(t_1)X^*(t_2)] = R_{XX}(\tau)$ (2)

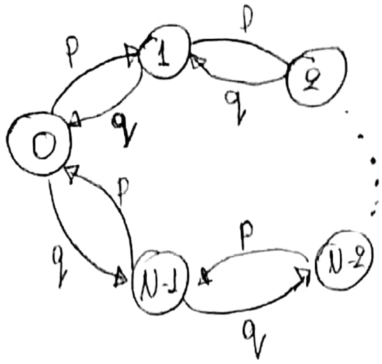
From (1) and (2) $\Rightarrow C_{ZZ}(\tau) = 3R_{XX}^2(\tau) - R_{XX}^2(\tau) \Rightarrow$

$$C_{ZZ}(\tau) = 2R_{XX}^2(\tau).$$

But, $R_{XX}(\tau) = C_{XX}(\tau)$ because $E[X(t)] = 0$

Therefore, $\boxed{C_{ZZ}(\tau) = 2C_{XX}^2(\tau)}$

4. a) The transition matrix is depicted into the following graph:



The limiting distribution is denoted as:

$$\Pi = (\pi_0, \pi_1, \dots, \pi_{N-1})$$

and in order to find it we solve the equations:

$$\Pi = \Pi \cdot P$$, where P is the given transition matrix.

Therefore:

$$\pi_0 = p\pi_1 + q\pi_{N-1}$$

$$\pi_1 = p\pi_2 + q\pi_0$$

$$\vdots$$

$$\pi_k = p\pi_{k+1} + q\pi_{k-1}.$$

$$\vdots$$

$$\pi_{N-1} = p\pi_0 + q\pi_{N-2}.$$

After hours of trying to find a closed form to solve the above equations, I did not manage to find something.
After searching on Papoulis' book I found that when N is an odd number the limiting distribution is:

$$\Pi = \left(\frac{1}{N}, \frac{1}{N}, \dots, \frac{1}{N}\right),$$

while when N is even there is no stable state, therefore a limiting distribution can not be found.

b) i) S_n denotes the gambler's capital at step n. Therefore, $S_{n+1} = S_n + Z_{n+1}$, where Z_{n+1} is the instantaneous gain or loss at step $n+1$.

Thus, $P\{Z_{n+1} = 1\} = p$ and $P\{Z_{n+1} = -1\} = q$.

So, $E[Y_{n+1} | Y_n, Y_{n-1}, \dots, Y_0] = E\left[\left(\frac{q}{p}\right)^{S_{n+1}} \middle| S_n, S_{n-1}, \dots, S_0\right] =$

$$= E\left[E\left(\frac{q}{p}\right)^{S_n + Z_{n+1}} \middle| S_n\right] = \left(\frac{q}{p}\right)^{S_n} \left(\frac{q}{p} \cdot p + \left(\frac{q}{p}\right)^{-1} q\right) = \left(\frac{q}{p}\right)^{S_n} = Y_n.$$

Therefore, since $E[Y_{n+1} | Y_n, Y_{n-1}, \dots, Y_0] = Y_n$, the sequence Y_n is a martingale.

ii) P_i is the probability of ruin for initial capital i, where $0 < i <$, and $(1-P_i)$ is the probability of gaining all wealth N.

T is a stopping time and hence from the theory of stopping time $E[Y_T] = E[Y_0] = \left(\frac{q}{p}\right)^i$ (1), because the gambler starts with initial capital equal to i.

Furthermore, $E[Y_T] = \left(\frac{q}{p}\right)^0 \cdot P_i + \left(\frac{q}{p}\right)^N (1 - P_i) = P_i + \left(\frac{q}{p}\right)^N (1 - P_i)$. (2)

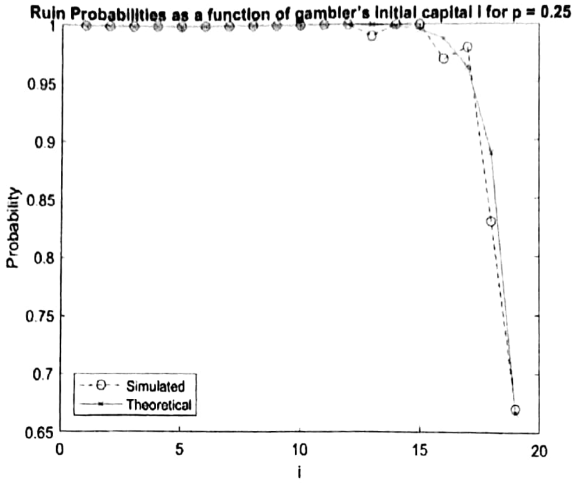
From (1) and (2) $\Rightarrow \left(\frac{q}{p}\right)^i = P_i + \left(\frac{q}{p}\right)^N (1 - P_i) \Rightarrow$

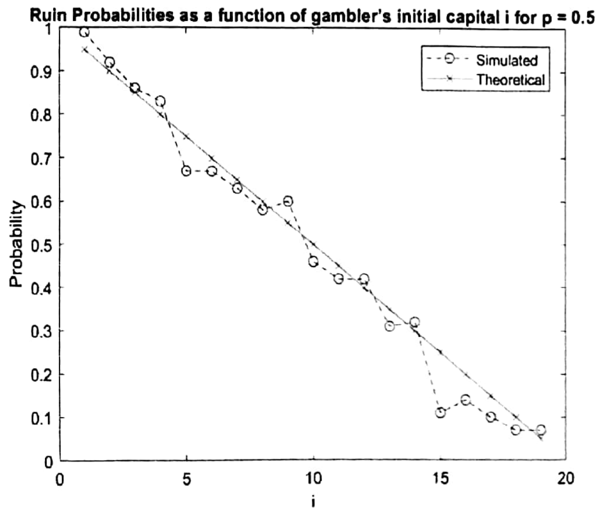
$$\left(\frac{q}{p}\right)^i - \left(\frac{q}{p}\right)^N = P_i \left(1 - \left(\frac{q}{p}\right)^N\right) \Rightarrow P_i = \frac{\left(\frac{q}{p}\right)^i - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N} =$$

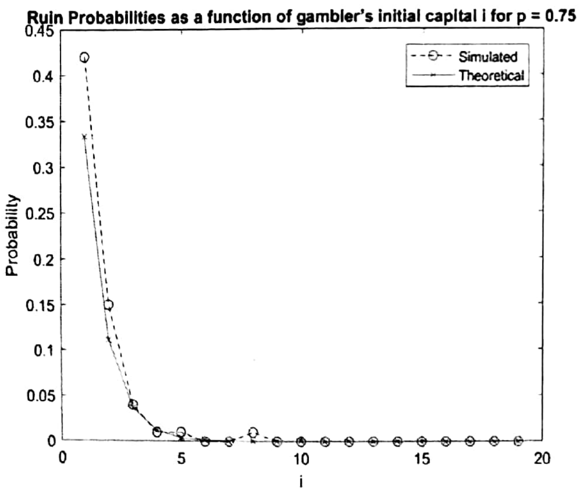
$$= \frac{\dfrac{\left(\frac{q}{p}\right)^i - \left(\frac{q}{p}\right)^N}{\left(\frac{q}{p}\right)^N}}{\dfrac{1 - \left(\frac{q}{p}\right)^N}{\left(\frac{q}{p}\right)^N}} = \frac{\left(\frac{q}{p}\right)^{i-N} - 1}{\left(\frac{p}{q}\right)^N - 1} \Rightarrow \boxed{P_i = \frac{1 - \left(\frac{p}{q}\right)^{N-i}}{1 - \left(\frac{p}{q}\right)^N}}$$

c)

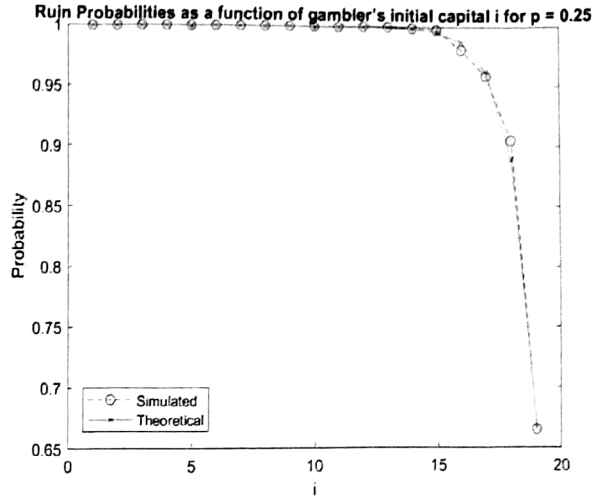
100 iterations

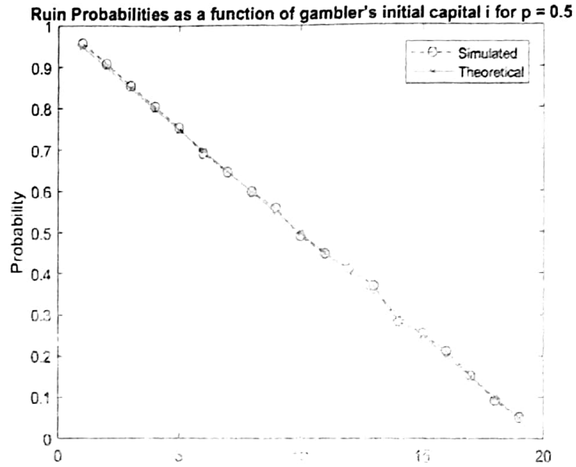


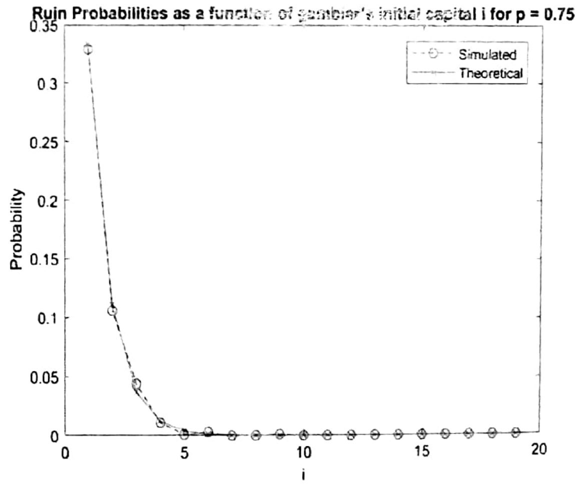




1000 iterations.







It is obvious that as the number of iterations increases, the simulated line tends to become identical to the theoretical one.