Information for students

This coursework is intended to be a sample exam paper. However, the level of difficulty may vary to some extent.

It accounts for 15% of the mark for this course.

Deadline: Friday, 5PM, December 15, 2017. Pease submit a hard (hand-written is fine) copy of your answers, as well as a PDF copy to Blackboard.

Do not submit the MATLAB codes.

The Questions

1. Random variables.

a) A pack contains m cards, labelled 1, 2, …, m. The cards are dealt out in a random order, one by one. Given that the kth card is the largest dealt in the first k cards dealt, what is the probability that it is also the largest in the pack? [10]

b) Let X be a Gaussian random variable with zero mean and variance σ^2. Estimate the tail probability $P(|X| > a)$ where $a = 4\sigma$ using

i) Markov inequality [5]

ii) Chebyshev inequality [5]

iii) Chernoff bound. [5]

Discuss your findings.

Hint: $E[|X|] = \sqrt{\frac{2}{\pi}}\sigma$ for a Gaussian random variable.

2. Estimation.

a) The random variable X has the truncated exponential density $f(x) = ce^{-c(x-x_0)}, \;\; x > x_0$. Let $x_0 = 1$. We observe the i.i.d. samples $x_i = 4.1$, 3.7, 4.3, 3.7, 4.2. Find the maximum-likelihood estimate of parameter c.

[10]

b) Consider the Rayleigh fading channel in wireless communications, where the channel coefficients $Y(n)$ has autocorrelation function

$$R_Y(n) = J_0(2\pi f_d n)$$

where J_0 denotes the zeroth-order Bessel function of the first kind (the function **besselj**(0,.) in MATLAB), and f_d represents the normalized Doppler frequency shift. Suppose we wish to predict $Y(n+1)$ from $Y(n)$, $Y(n-1)$, …, $Y(1)$. The coefficients of the linear MMSE estimator

$$Y(n+1) = \sum_{i=1}^{n} c_i Y(i)$$

are given by the Wiener-Hopf equation

$$\boldsymbol{Rc} = \boldsymbol{r}$$

where $\boldsymbol{c} = [c_1, c_2, \dots, c_n]^T$, $\boldsymbol{r} = [R_Y(n), R_Y(n-1), \dots, R_Y(1)]^T$, and $\boldsymbol{R}$ is a n-by-n matrix whose (i, j)th entry is $R_Y(i-j)$.

i) Give an expression for the coefficient of the first-order MMSE estimator, i.e., $n = 1$. [5]

ii) Let $f_d = 0.3$. Write a MATLAB program to compute the coefficients of the n-th order linear MMSE estimator and plot the mean-square error $\sigma_n^2 = r_0 - \boldsymbol{r}^* \boldsymbol{R}^{-1} \boldsymbol{r}$ as a function of n, for $1 \le n \le 20$. [10]

[As you may imagine, n cannot be greater than 2 for computation of this kind in an exam.]

3. Random processes.

a) The number of failures $N(t)$, which occur in a computer network over the time interval $[0, t)$, can be modelled by a Poisson process $\{N(t), t \geqslant 0\}$. On the average, there is a failure after every 4 hours, i.e. the intensity of the process is equal to $\lambda = 0.25$.

i) What is the probability of at most 1 failure in [0, 8), at least 2 failures in [8, 16), and at most 1 failure in [16, 24) ? (time unit: hour) [7]

ii) What is the probability that the third failure occurs after 8 hours? [4]

b) Consider the random process

$$X(n) = A\cos(n\lambda + \theta) + B\sin(n\lambda + \theta)$$

where A and B are uncorrelated random variables with zero means and unit variances, and θ is a fixed phase. Calculate the mean, autocorrelation function of $X(n)$ and determine whether it is wide-sense stationary or not. [6]

c) The random process $X(t)$ is Gaussian and wide-sense stationary with $E[X(t)] = 0$. Show that if $Z(t) = X^2(t)$, then autocovariance function $C_{ZZ}(\tau) = 2C_{XX}^2(\tau)$. [8]

Hint: For zero-mean Gaussian random variables X_k,

$$E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3]$$

4. Markov chains and martingales.

a) Cyclic random walk on a circle has the following transition matrix where $p + q = 1$:

$$P = \begin{pmatrix} 0 & p & 0 & & & & q \\ q & 0 & p & & & & \\ & q & 0 & p & & & \\ & & & . & . & . & \\ 0 & & & & q & 0 & p \\ p & & & & & q & 0 \end{pmatrix}$$

Find the limiting distribution.

[3]

b) Consider the gambler's ruin with state space $E = \{0,1,2,\ldots,N\}$ and transition matrix

$$P = \begin{pmatrix} 1 & & & & & & 0 \\ q & 0 & p & & & & \\ & q & 0 & p & & & \\ & & & . & . & . & \\ 0 & & & & q & 0 & p \\ & & & & & & 1 \end{pmatrix}$$

where $0 < p < 1$, $q = 1 - p$. This Markov chain models a gamble where the gambler wins with probability p and loses with probability q at each step. Reaching state 0 corresponds to the gambler's ruin.

i) Denote by S_n the gambler's capital at step n. Show that $Y_n = \left(\frac{q}{p}\right)^{S_n}$ is a martingale (DeMoivre's martingale). [4]

ii) Using the theory of stopping time, derive the ruin probability for initial capital i ($0 < i < N$). [4]

c) Let $N = 20$. Write a computer program to simulate the Markov chain in b). Starting from state i and run the Markov chain until reaching state 0. Repeat it for 100 times or more, and plot the ruin probabilities as a function of the gambler's initial capital i ($0 < i < N$), for

i) $p = 1/4$; [4]

ii) $p = 1/2$; [4]

iii) $p = 3/4$. [4]

Also plot the theoretic results of b) in the same figure as a benchmark.

[Obviously, such a question cannot be tested in this way in the exam!]