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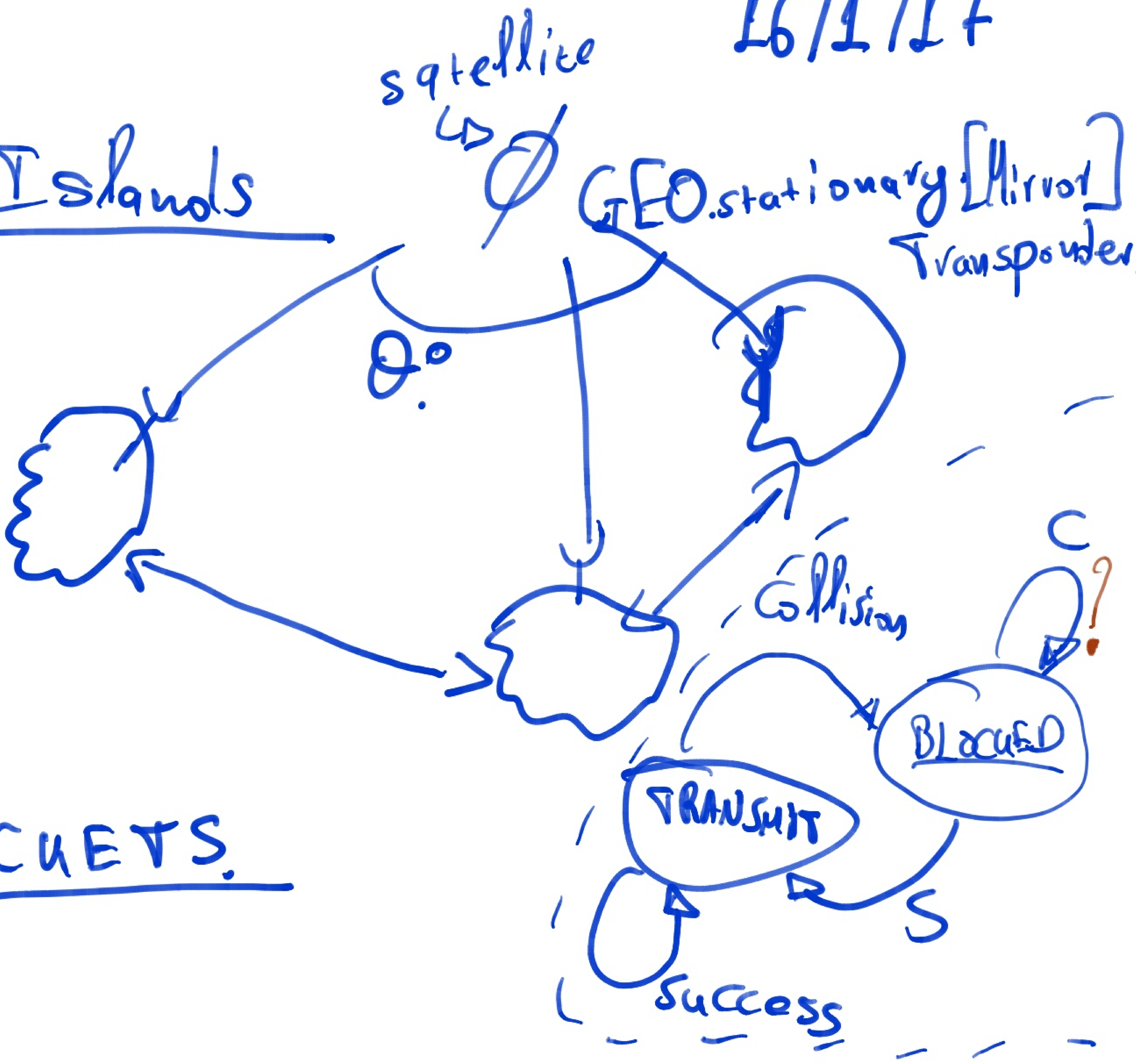
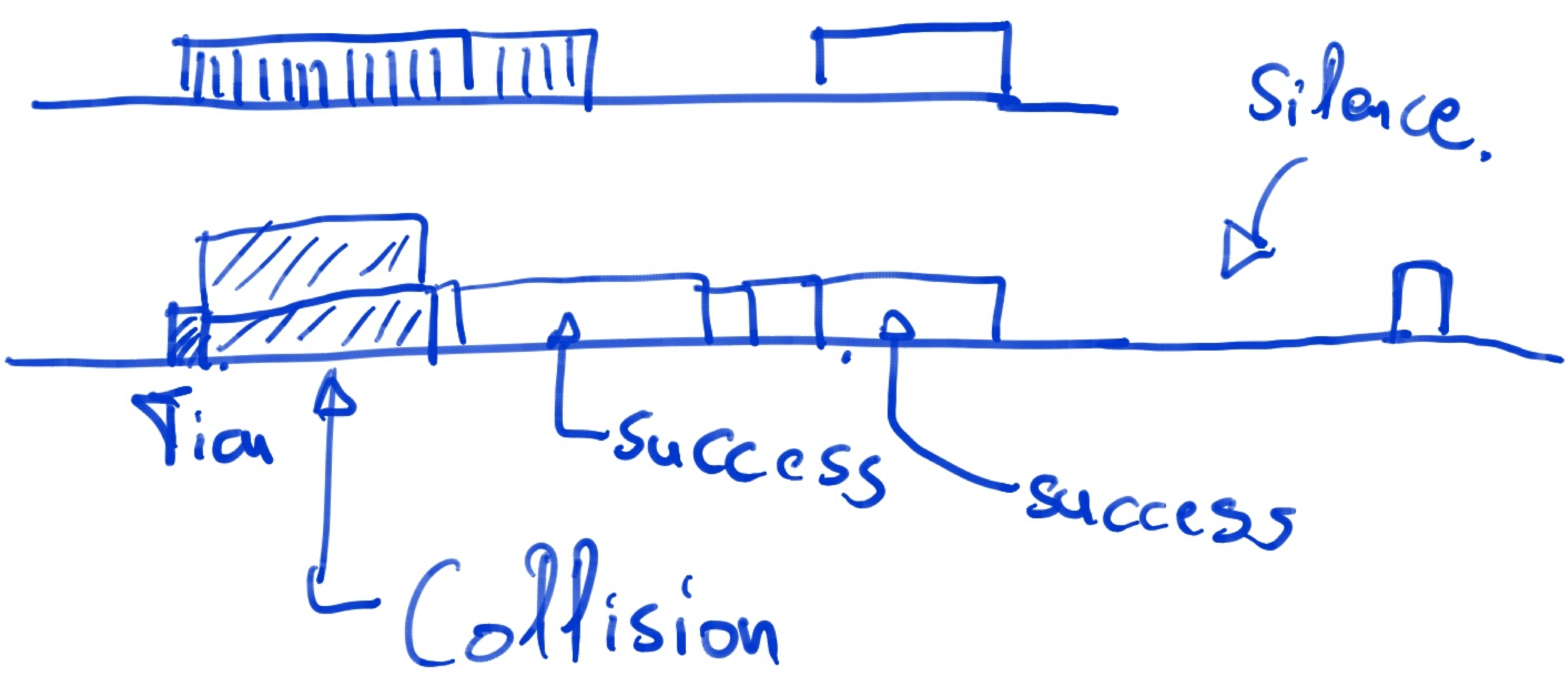
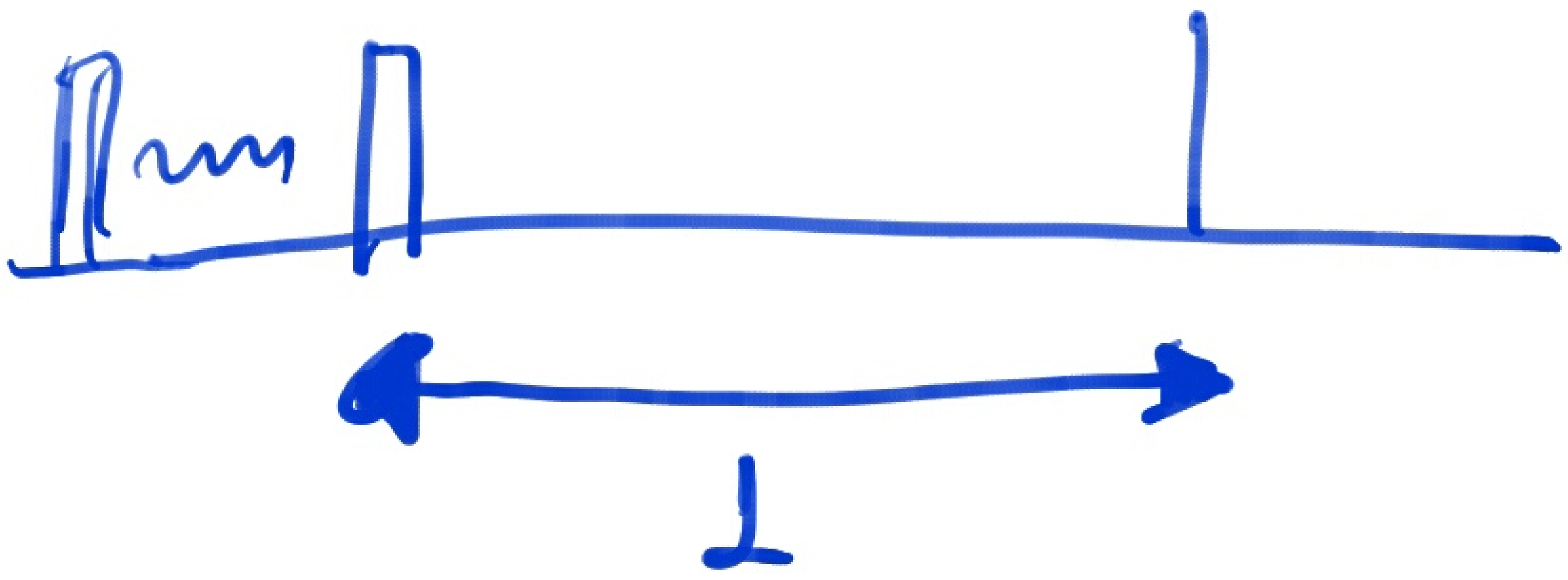
Islands



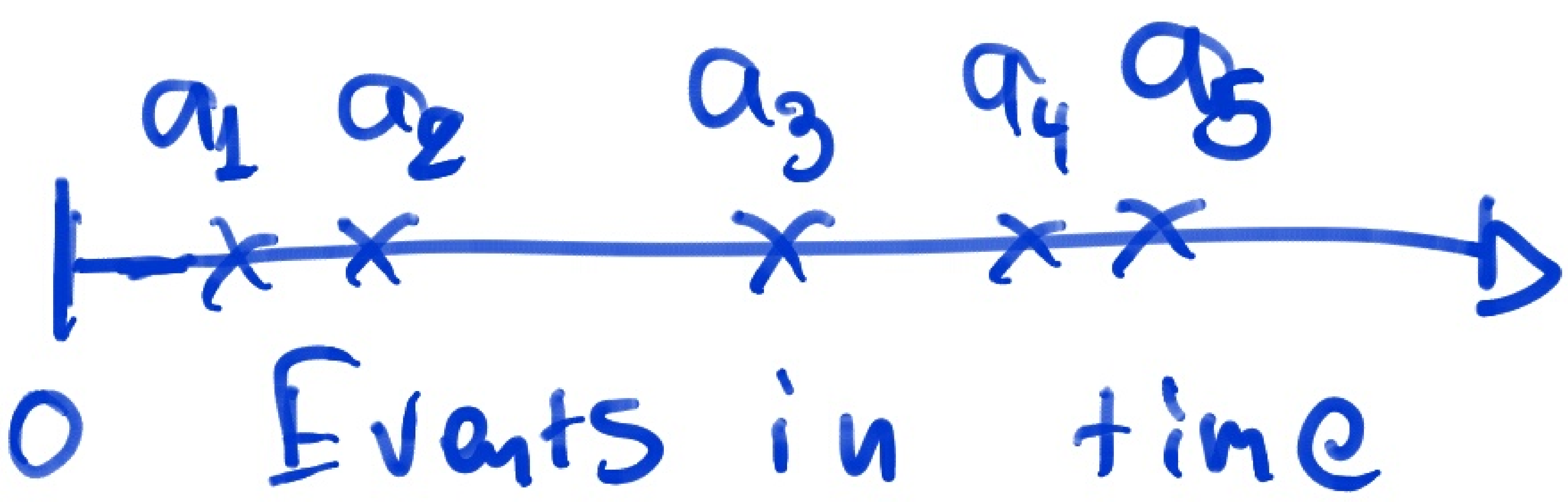
PACKETS





Transmissions come in a Poisson Process of rate Λ

$$\boxed{\text{Prob}[N \text{ transmissions in time } T] = e^{-\Lambda T}\frac{(\Lambda T)^N}{N!}}$$



Events in time

$$A_i = a_i - a_{i-1} \quad ; \quad a_0 = 0$$

+ $\{A_i\}$ are independent of each other

+ All distributed as follows:

$$\text{Prob}[A_i > x] = e^{-\Lambda x} \quad ; \quad \Lambda \text{ is the rate}$$

$\Lambda > 0$.

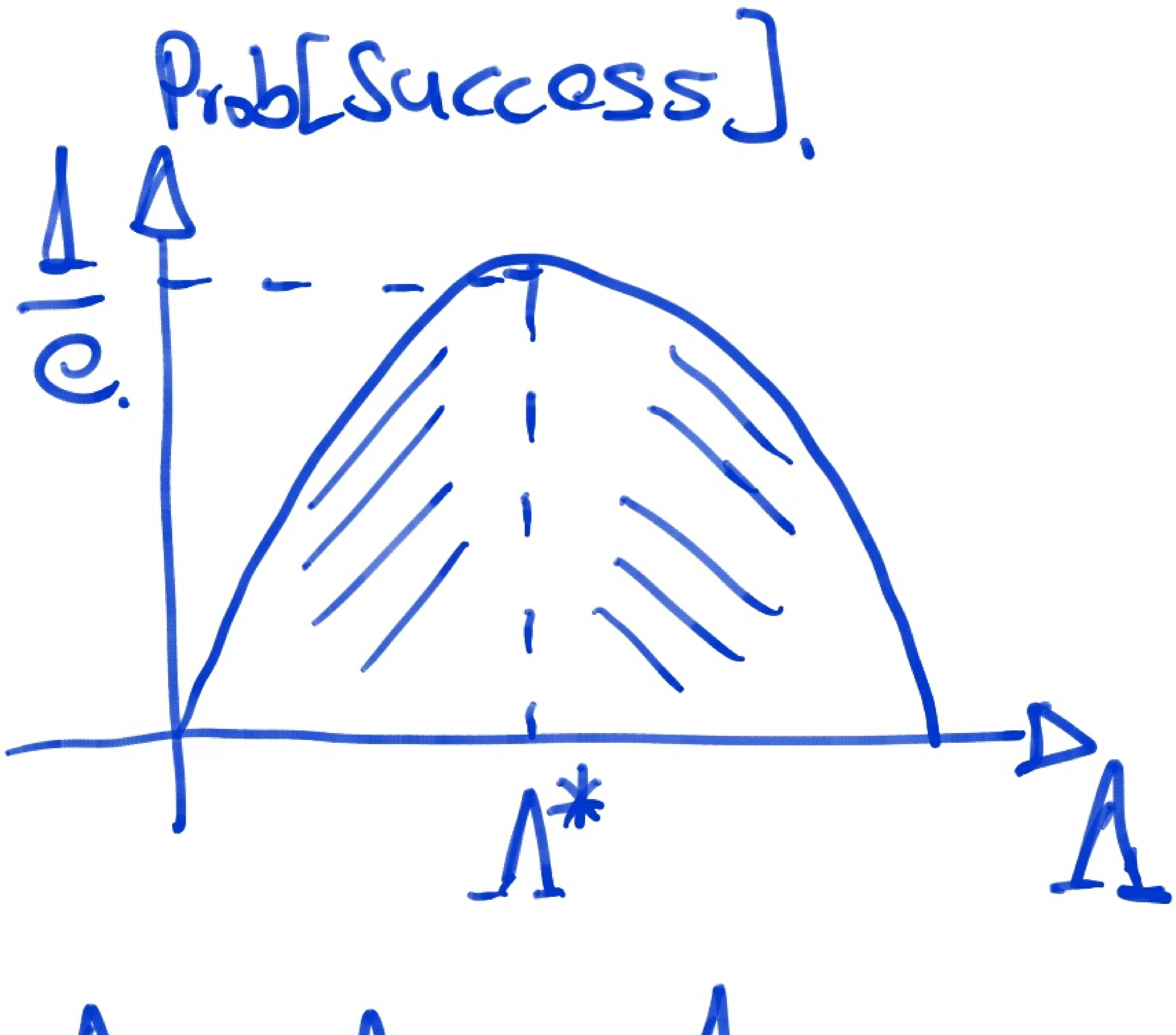
$$\text{Prob}[\text{Success}] = \Lambda e^{-\Lambda}$$

$$\text{Prob}[\text{COLLISION}] = 1 - \Lambda e^{-\Lambda} - e^{-\Lambda}$$

$$\frac{\partial \, \text{Prob}[\text{success}]}{\partial \Lambda} = e^{-\Lambda} - \Lambda e^{-\Lambda}$$

$$e^{-\Lambda}(1-\Lambda) = 0.$$

$$\boxed{\Lambda^* = 1}$$



$$\Lambda = \Lambda_B + \Lambda_F$$

blocked

fresh

$$\underbrace{\bar{\eta}_B \cdot b}_{\Lambda_B} + \Lambda_F = 1.$$

$$b^* = \frac{1 - \Lambda_F}{\bar{\eta}_B}$$

$\text{Prob}[\text{Silence}] = e^{-\Lambda} \sim \frac{1}{e}$ At optimum.

Gelenbe "Stability and optimal control ..."

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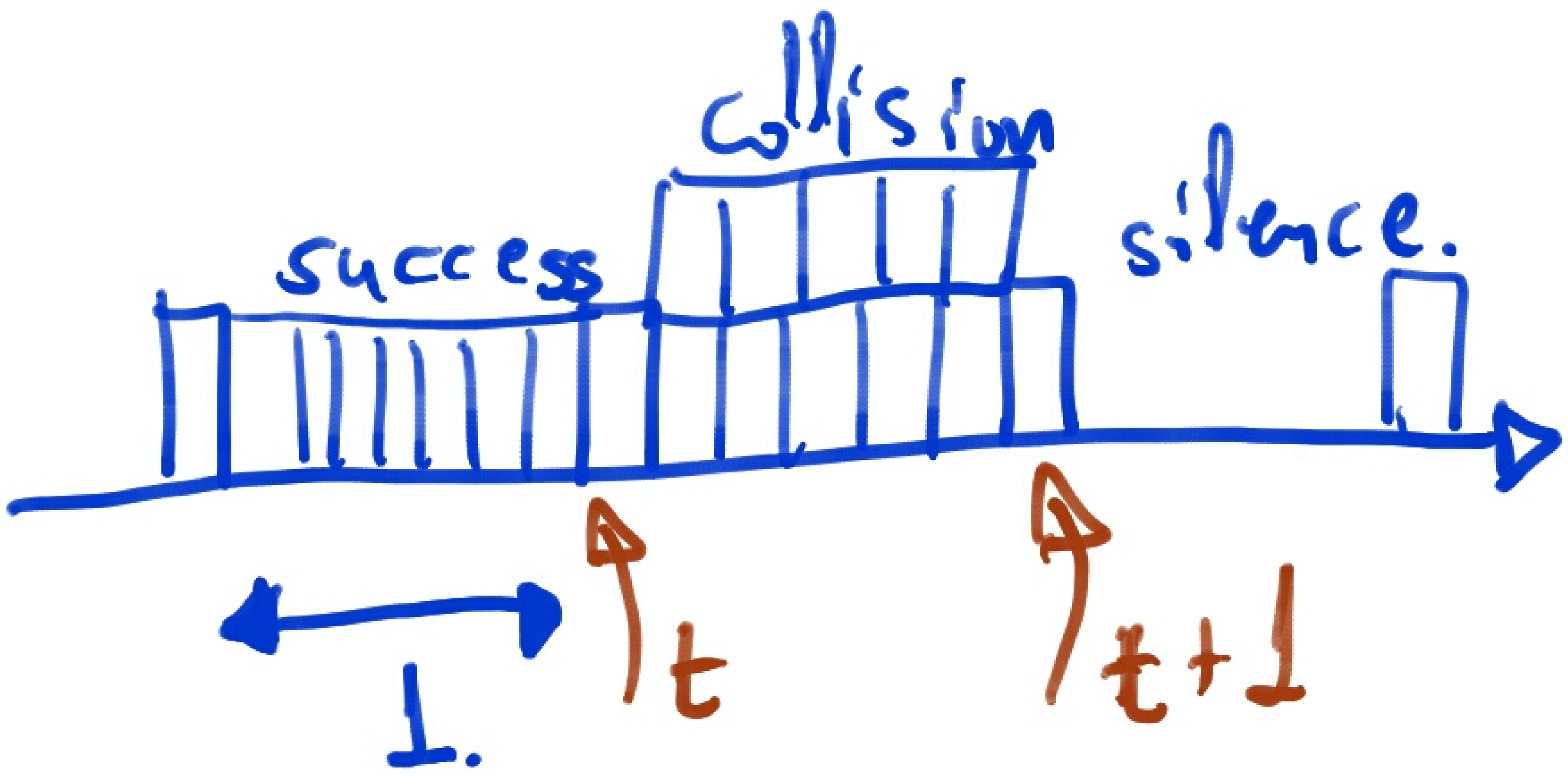


BLOCKED.

Number of blocked terminals:
$B(t)$

$t = 0, 1, 2, \dots$

$A(t)$: # of transmissions from Active terminals.

$N(t)$: # of transmissions from blocked terminals.

- $B(t+1) = B(t) - 1$ if $A(t) = 0$,
  $N(t) = 1$
  & $B(t) > 0$.

- $B(t+1) = B(t)$. if $A(t) = 0$,
  $N(t) \geq 2$
  & $B(t) > 0$.
  Or $A(t) = 0$ & $N(t) = 0$

  success. $\longrightarrow$ Or $A(t) = 1$ & $N(t) = 0$

- $B(t+1) = B(t) + A(t)$ if $A(t) = 1$, $N(t) \geq 1$.
  & $B(t) > 0$.
  Or $A(t) \geq 2$, any value of $N(t)$.

Assume that $B(t)=m$.

$$E[B(t+1)-B(t) \mid B(t)=m].$$

$$= -q_0 \cdot c_1(m) + 1 \cdot q_1 [1-c_0(m)].$$

$$+ \sum_{i=2}^{\infty} i \cdot q_i$$

average # of external arrivals.

$$= -\underbrace{[q_0 \cdot c_1 \cdot (m) + q_1 \cdot c_0(m)]}_{\text{Prob of success.}} + \boxed{\sum_{i=1}^{\infty} i \cdot q_i}$$

need to maximize this.

can't optimize here.

Call $q_i = \text{Prob}[A(t)=i]$.

$$c_i(m) = \text{Prob}[N(t)=i \mid B(t)=m].$$

$$c_i(0) = \begin{cases} 0 & \text{if } i \neq 0 \\ 1 & \text{if } i=0. \end{cases}$$

$$E[B(t+1)-B(t) \mid B(t)=m] =$$

$$= E[A(t)] - [\underbrace{q_0 C_1(m) + q_1 C_0(m)}_{\text{maximize}}]$$

as small as possible.

arbitrary.

cannot do anything.

at most equal to one

Heads → Transmit.

Probability b.

Tails → $1-b$.

$$C_0(m) = (1-b)^m$$

$$C_1(m) = \binom{m}{1} b (1-b)^{m-1} = m b (1-b)^{m-1}$$

$\Theta(m) \rightarrow$ (throughput)

MAXIMIZE: $a_1(1-b)^m + a_0 m b (1-b)^{m-1}$

by choosing b.

$$\frac{\partial \Theta(m)}{\partial b} = -a_1 \cdot m(1-b)^{m-1} + a_0 m (1-b)^{m-1} - a_0 m b (m-1)(1-b)^{m-2}$$

$$= m(1-b)^{m-2}\left[\underline{a_0(1-b) - a_0 b(m-1) - a_1(1-b)}\right].$$

Choose b from here.

$$a_0 - \cancel{a_0 b} - a_0 b m + \cancel{a_0 b} - a_1 + a_1 b =$$

$$a_0 - a_1 - b\left[a_0 m - a_1\right]$$

$$\Rightarrow \boxed{b^* = \frac{a_0 - a_1}{a_0 m - a_1}}$$

Ideally $E[B(t+1)-B(t) \mid B(t)=m] \leq 0$

Therefore, ideally $\sum_{i}^{\infty} i a_i \leq 1$

$\Rightarrow a_1 + 2a_2 + 3a_3 + \ldots \leq 1 \Rightarrow$

$1 - a_0 + [a_2 + 2a_3 + 3a_4 + \ldots] \leq 1 \Rightarrow$

$a_0 \geq a_2 + 2a_3 + 3a_4 + \ldots$

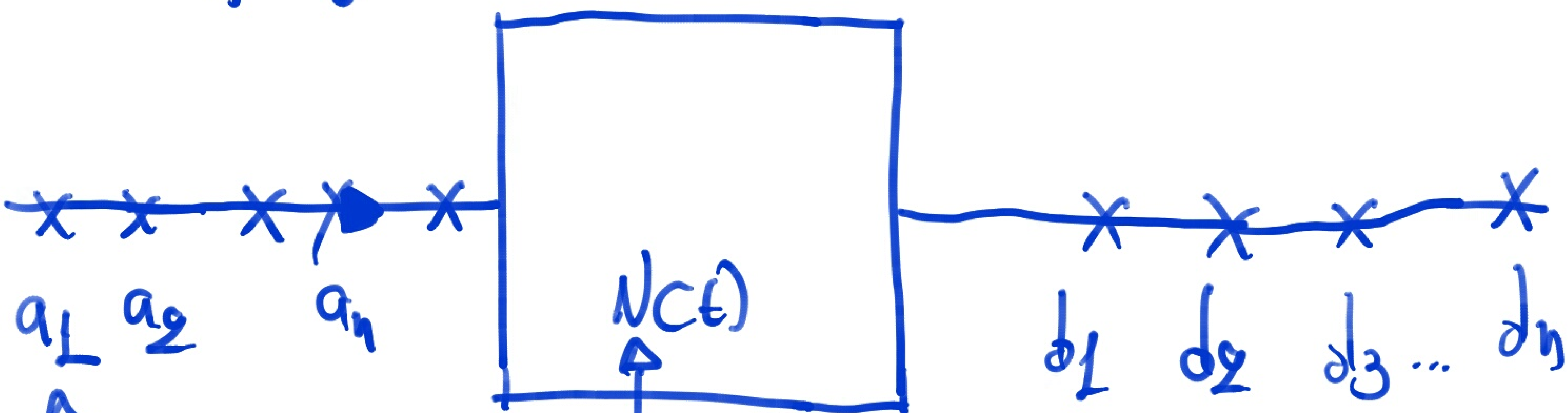
If external arrivals are Poisson with parameter λ

$a_0 = e^{-\lambda}, \; a_1 = \lambda \cdot e^{-\lambda}$

$e^{-\lambda} > \lambda \cdot e^{-\lambda} \Rightarrow \boxed{\lambda < 1}$

a_n^- just before

a_n^+ just after.

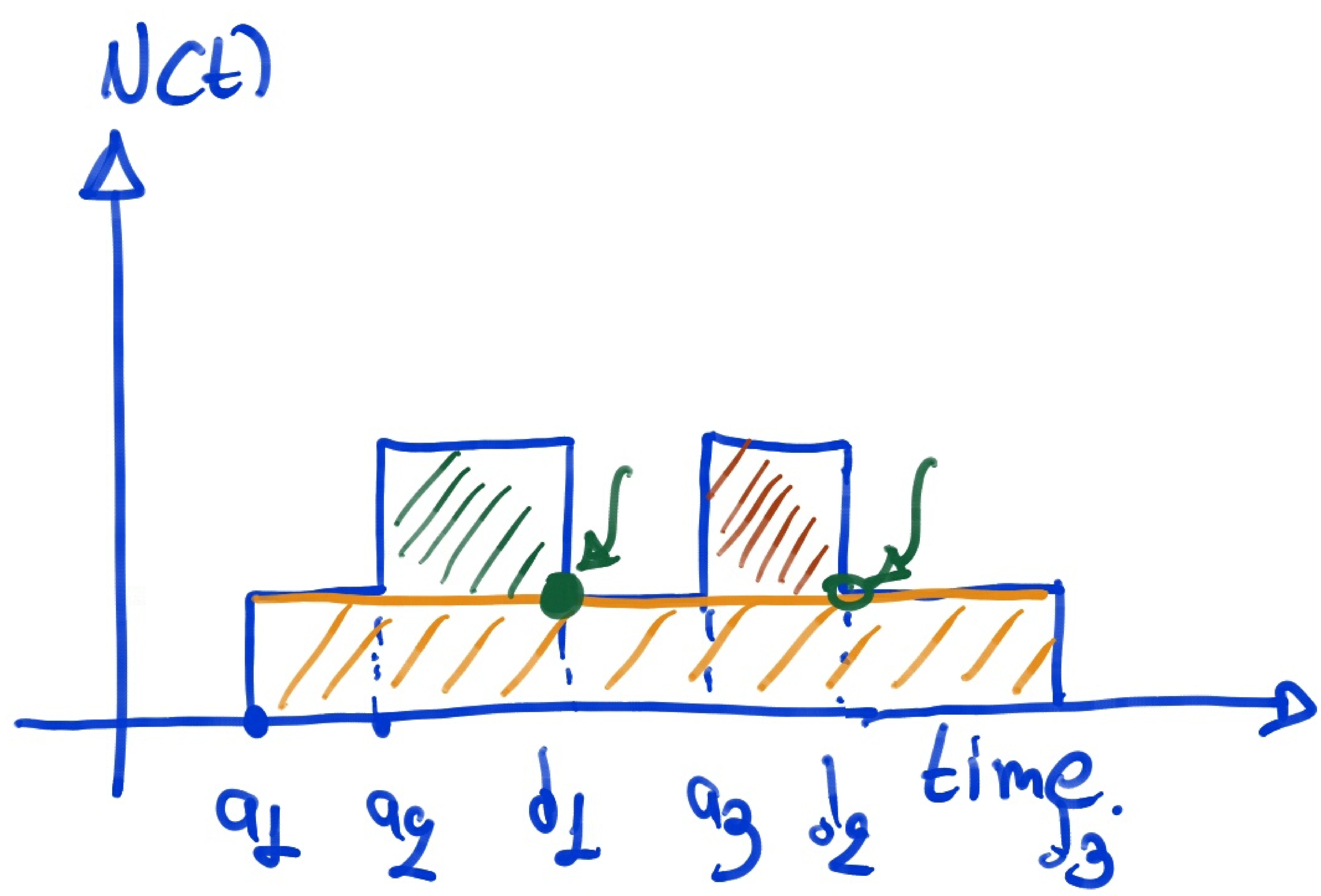


the time of the 1st arrival.

number of people in the box.

(packets in a router).

departure.



$$N(d_n^+) = N(d_n) - 1$$
$$N(a_n^+) = N(a_n) + 1$$

→ we assume continuity from the left.

$N(t) \rightarrow$ memory.

$a_i, d_i \rightarrow$ QoS.

Consider an object arriving at a_n. The object's departure time will be called $d_{D(n)} \neq d_n$.

Time spent in the system is

$$W_n = d_{D(n)} - a_n$$

$$\bar{N}(T) = \int_0^T N(t)\,dt$$ $\rightarrow$ Sum of rectangles according to the diagram.

$$S(n) = \inf\{i : d_i > a_n \text{ and } N(d_i^+) = N(a_n)\}.$$

In the diagram: $\delta_1 = 3. \Rightarrow$

$d_{\delta(1)} = d_3$

$\delta(2) = 1 \Rightarrow d_{\delta(2)} = d_1$

$d_{\delta(3)} = d_2.$

- Assume that $N(0) = 0$, $N(T) = 0$.

$$\int_0^T N(t)\,dt = \sum_{n=1}^{L(T)} [d_{\delta(n)} - a_n]$$

$$= \sum_{n=1}^{L(T)} [d_{D(n)} - a_n].$$

$$= \sum_{n=1}^{L(T)} w_n$$

Average Number in the system

$$\langle N \rangle = \frac{1}{T}\int_0^T N(t)\,dt \Rightarrow \int_0^T N(t)\,dt = T\langle N \rangle$$

Average Waiting time in the system

$$\langle W \rangle = \frac{1}{L(T)} \sum_{n=1}^{L(T)} W_n \Rightarrow$$

$$\sum_{n=1}^{L(T)} W_n = L(T)\langle W \rangle$$

So, $T \cdot \langle N \rangle = L(T) \cdot \langle W \rangle. \Rightarrow$

$$\boxed{\langle N \rangle = \frac{L(T)}{T}\langle W \rangle}$$

number of events per unit time (speed)

J.D.C. Little's Theorem

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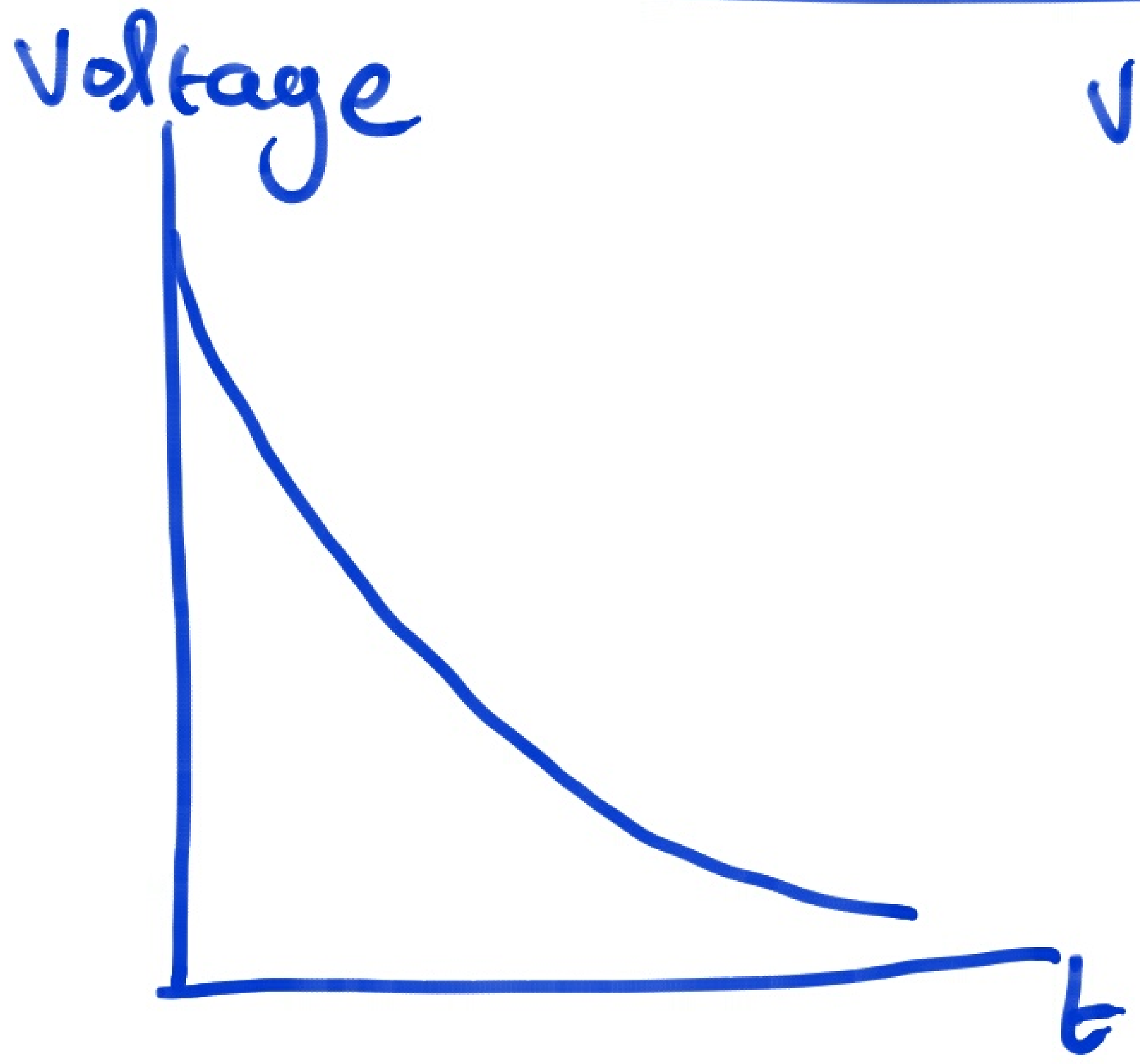
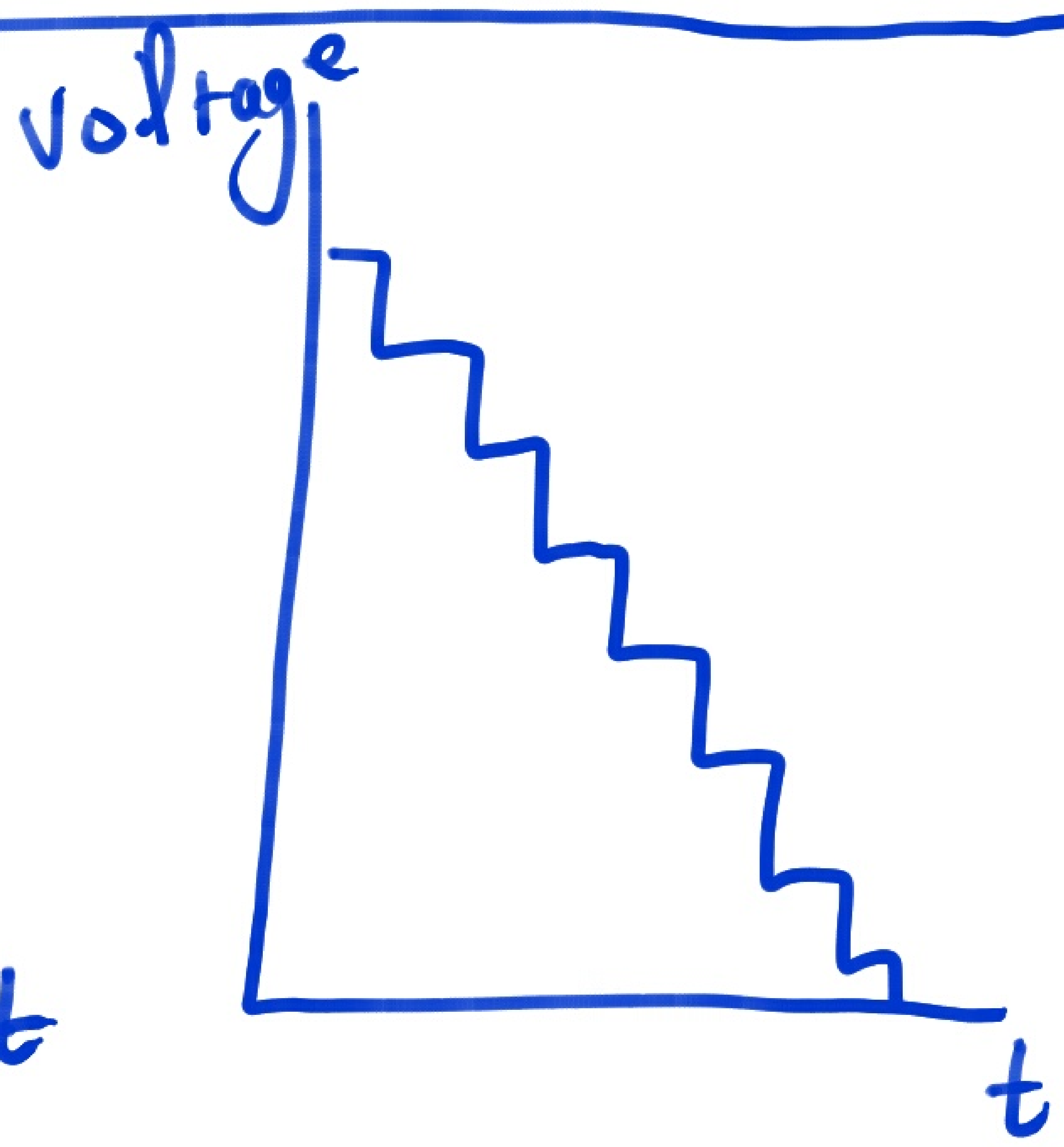
A BASIC INTRODUCTION TO ENERGY PACKET NETWORKS (EPN)





- What is the EPN?

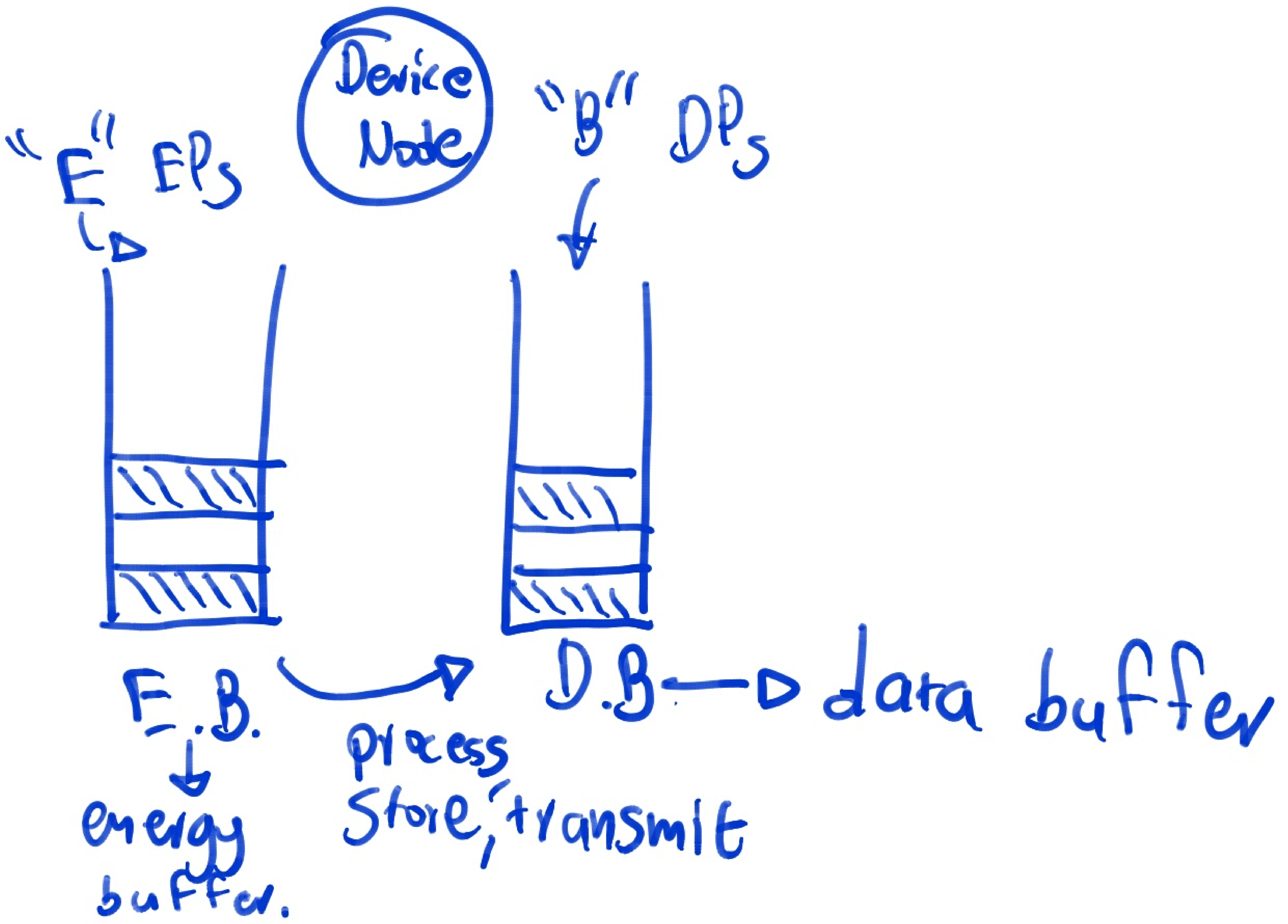
EPN is a paradigm where the energy to operate the system is consumed <u>intermittently</u> (or as discrete packet flows)

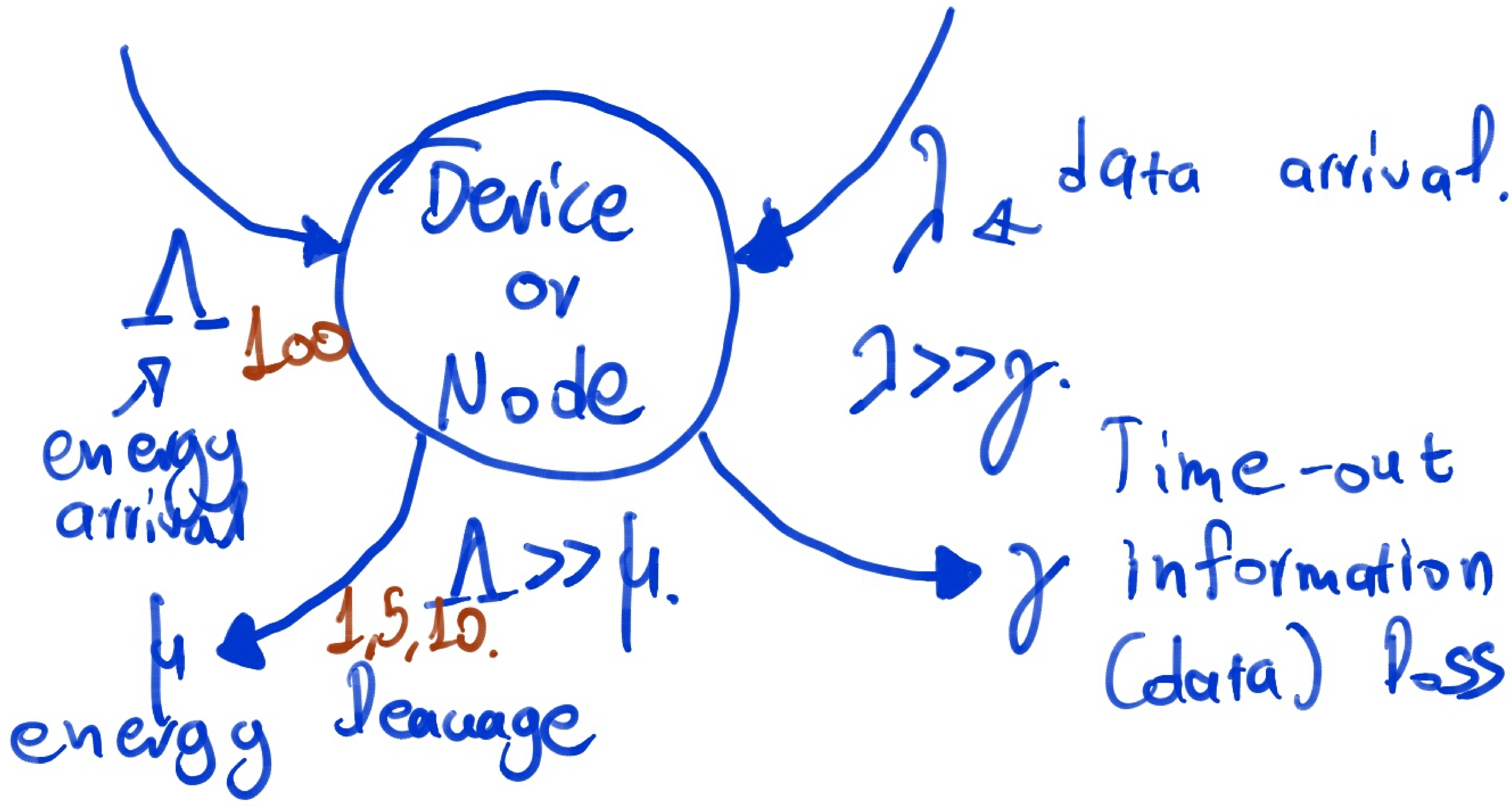
- Today, we consider the devices with EPN paradigm.

✓ Capture energy, from <u>intermittent</u> sources (vibrations, heat, light)

✓ Consume energy intermittently

✓ Both the arrival of data and energy can be modelled as random processes.





✓ Arrival of EPs (energy packets) and DP_s (data packets) are assumed to be Poisson process

✓ Device consumes 1 EP to sense, process, store and transmit 1 DP.

$$1\,EP \longleftrightarrow 1\,DP.$$

✓ The time that takes to create packets and transmit them is extremely fast compare to the rates Λ and λ so that the transmission (with all other processes) is instantaneous (takes zero time).

- What is the typical value of Λ?

We have 100 mW E.H. (energy harvesting) module.

We have 20 kbits packet size.

1 EP $\longleftrightarrow$ 1 DP

Assume 50 nJ (50 nW) consumes for transmitting 1 bit.

$$100 \cdot 10^{-3} = \Lambda \cdot 20 \cdot 10^{3} \cdot 50 \cdot 10^{-9}$$

$$\Rightarrow \boxed{\Lambda = 100}$$

$$\underset{\text{data}}{\downarrow}\qquad\underset{\text{energy}}{\downarrow}$$

at $t \geq 0$ $\quad (\underline{d}, \underline{e}) \to (D(t), E(t))$

State $\quad$ # of DPs at $t \geq 0$ $\quad$ # of EPs at $t \geq 0$

$(D(t) > 0, E(t) > 0)$ at t.

$$\downarrow$$

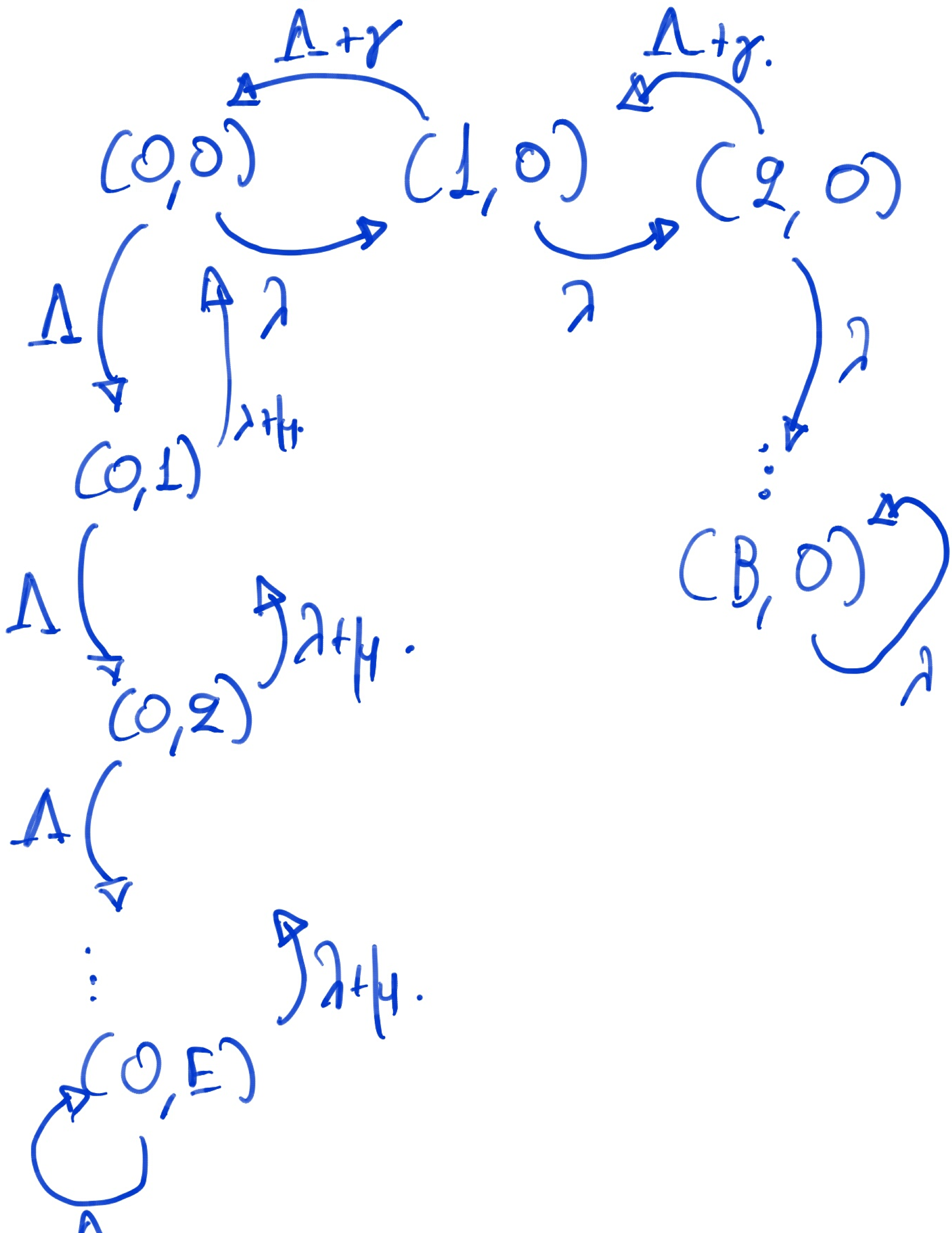
$(0, E(t) - D(t))$ if $E(t) > D(t)$

$(D(t) - E(t), 0)$ if $D(t) > E(t)$

There is not state that can have non-zero $E(t)$ and $D(t)$ at the same time t.

$$p(n,m,t) = P[D(t)=n,\ E(t)=m]$$

$$p(n,m) = \lim_{t\to\infty} P[D(t)=n,\ E(t)=m]$$



- State transition Diagram

$p(1,0) = ?$

$p(0,2) = ?$

$p(0,E) = ?$

$S = \{(0,0), (n,0), (0,m) : E \geq m > 0, B \geq n > 0\}$

↑ state space

- Global Balance Equations
(to find the stationary probability distributions of states)

Say stationary probability distribution of state i is π_i and transition rate from i to j is q_{ij}. The global balance equations:

$$\underbrace{\pi_i \sum_{j\in S-\{i\}} q_{ij}}_{\substack{\text{total flow}\\ \text{from out of}\\ \text{state } i \text{ into}\\ \text{other than } i}} = \underbrace{\sum_{j\in S-\{i\}} \pi_j q_{ji}}_{\substack{\text{total flow}\\ \text{out of all states}\\ j\neq i \text{ into state } i}}$$

For state $(0,0)$:

$$\boxed{p(0,0)(\Lambda+\lambda) = p(1,0)(\lambda+\mu) + p(1,0)(\Lambda+\gamma).} \quad 1$$

$$p(1,0)(\Lambda+\lambda+\gamma) = p(0,0)(\lambda) + p(2,0)(\Lambda+\gamma)$$

$$p(2,0)(\Lambda+\lambda+\gamma) = p(1,0)\lambda + p(3,0)(\Lambda+\gamma) \quad 4$$

$$\vdots$$

$$\boxed{p(n,0)(\Lambda+\lambda+\gamma) = p(n-1,0)\lambda + p(n+1,0)(\Lambda+\gamma) \quad B>n>0}$$

$$\boxed{p(B,0)(\Lambda+\gamma)=p(B-1,0)(\lambda)} \quad (5)$$

$$\boxed{\begin{aligned}p(0,m)(\Lambda+\lambda+\mu)&=p(0,m-1)\Lambda\\&+p(0,m+1)(\lambda+\mu). \quad E>m>0\end{aligned}} \quad (2)$$

$$\boxed{p(0,E)(\lambda+\mu)=\Lambda p(0,E-1)} \quad (3)$$

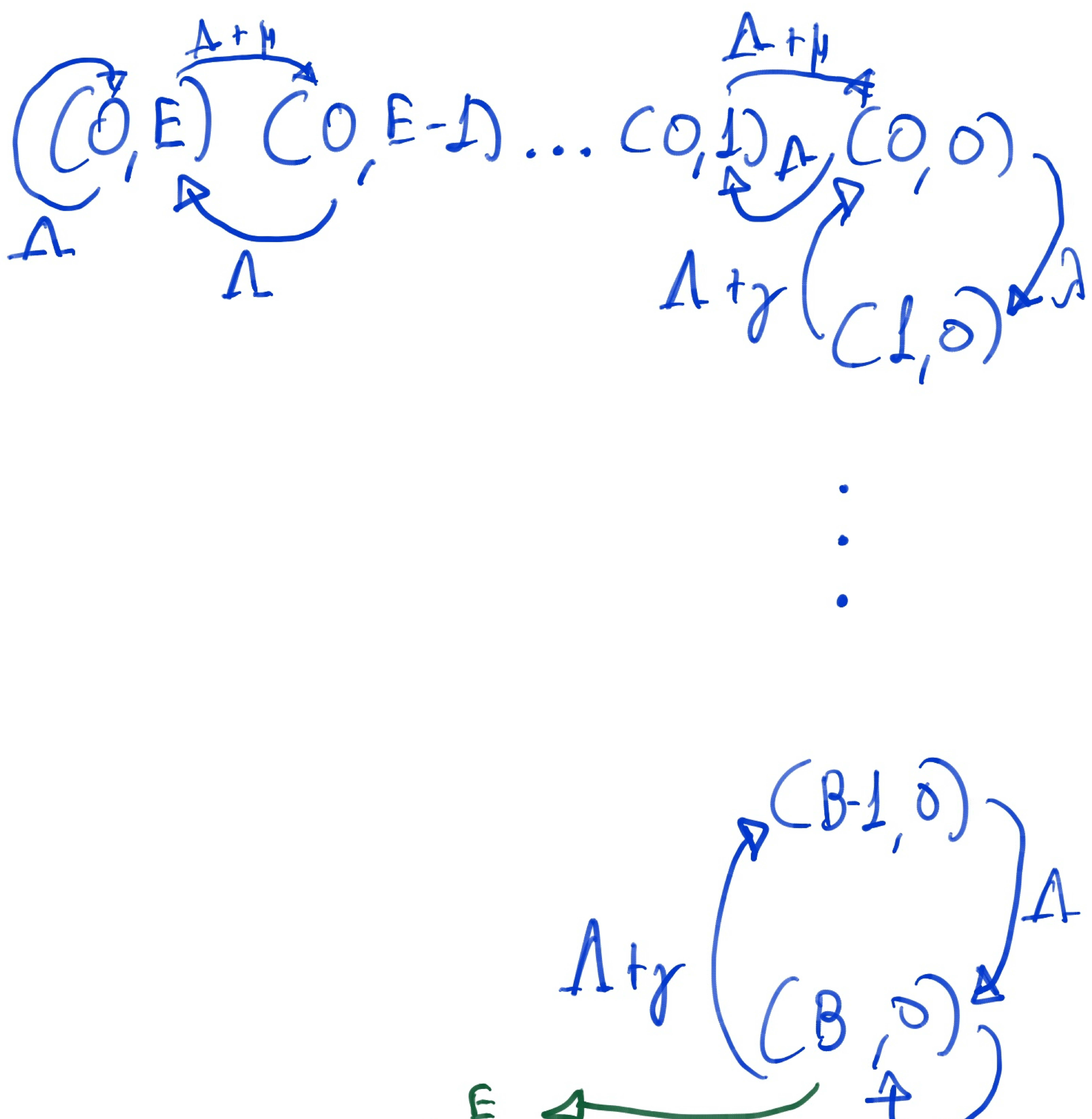
Assume Eq. 2 has a solution of the form:

$$p(0,m)=Q^m\cdot c_1, \quad Q=\frac{\Lambda}{\lambda+\mu} \quad \text{and}$$

c_1 is a constant.

Eq. 4:

$$p(n,0)=a^n\cdot c_2, \quad a=\frac{\lambda}{\Lambda+\gamma} \quad \text{and } c_2 \text{ is a const.}$$



$$L_d = \lambda \sum_{m=0}^{E} p(B,m)$$
$$= \lambda p(B,0)$$

because if we have B we cannot have an energy packet

$$S = \{(0,0),(n,0),(0,m) : B > n > 0, E > m > 0\}$$

$$\cancel{c_1}\,\cancel{\alpha^{m}}(\lambda+\Lambda+\mu)=\cancel{c_1}\,\cancel{\alpha^{m-1}}(\Lambda)+\cancel{c_1}\,\underset{\alpha^2}{\cancel{\alpha^{m+1}}}(\lambda+\mu)$$

$$\Rightarrow \frac{\Lambda}{\lambda+\mu}(\Lambda+\lambda+\mu)=\Lambda+\frac{\Lambda^2}{\lambda+\mu}.$$

$$=\frac{(\lambda+\mu+\Lambda)\Lambda}{\lambda+\mu}.$$

Therefore, the solution works.

Eq. 1:

$$p(0,0)(\lambda+\mu)=\frac{\Lambda}{\cancel{\lambda+\mu}}c_1(\lambda+\mu)+\frac{\lambda}{\cancel{\Lambda+\gamma}}c_2\cancel{(\Lambda+\gamma)}$$

$$\Rightarrow 0=\Lambda(c_1-p(0,0))+\lambda(c_2-p(0,0))$$

$$\Rightarrow c_1=c_2=p(0,0).$$

How can we guarantee that this solution is unique?

Aperiodicity: the period of a state i is

returns empty set

$$p(i) = \gcd\{n : p_{ii}^{(n)} > 0\}$$ i.e. $p(i) = 1$, state i is aperiodic

↳ greater common divider.

Irreducibility: If it is possible to get to any state from any state then it is called irreducible.

* Theorem:

A finite irreducible, aperiodic Markov Chain has a unique stationary probability distribution.

Solution

$$p(0,m) = \left(\frac{\Lambda}{\lambda+\mu}\right)^m p(0,0) \quad E \geq m > 0$$

$$p(n,0) = \left(\frac{\lambda}{\Lambda+\gamma}\right)^n p(0,0) \quad B \geq n > 0.$$

$$1 = p(0,0) + \sum_{n=1}^{B} p(n,0) + \sum_{m=1}^{E} p(0,m)$$

$$= p(0,0)\left[1 + \sum_{n=1}^{B} a^n + \sum_{m=1}^{E} Q^m\right] =$$

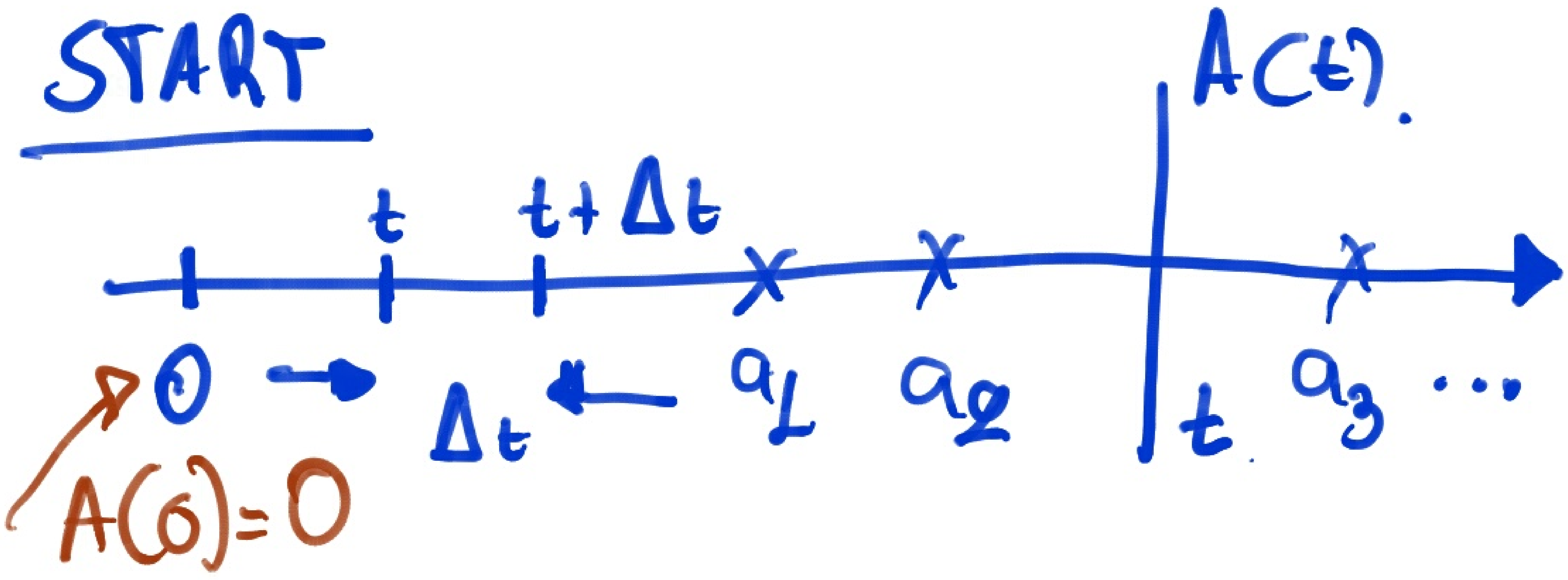
$$p(0,0)\left[1+\frac{a(a^{B}-1)}{(a-1)}+\frac{Q(Q^{E}-1)}{Q-1}\right]$$

$$\Rightarrow \boxed{p(0,0)=\frac{(1-a)(1-Q)}{a^{B+1}(Q-1)+Q^{E+1}(a-1)+1-aQ}}$$

6/2/18

10^9 Nodes $\sim 10^6$ Nodes

Operator.



Arrivals (event happens)

t+Δt not included

Prob[1 arrival in [t, t+Δt[ ] =

t is included.

$$\lambda \Delta t + O(\Delta t), \quad \lambda > 0 \text{ real number.}$$

$O(\Delta t)$ is ANY function of Δt

such that $\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0.$

Prob[0 arrival in $[t, t+\Delta t[$]

$= 1 - \lambda \Delta t + O(\Delta t)$

Prob[$k > 1$ arrivals in $[t, t+\Delta t[$]

$= O(\Delta t) \rightarrow$ negligible.

$A(t)$: total number of arrivals by time t

$$\text{Prob}[A(t) = n] \equiv p_n(t)$$

$$p_0(t+\Delta t) = p_0(t) \cdot \overbrace{\left(1 - \lambda \Delta t + O(\Delta t)\right)}^{\text{No one came.}}$$

$$\lim_{\Delta t \to 0} \frac{p_0(t+\Delta t) - p_0(t)}{\Delta t} = -\lambda \cancel{\Delta t}\, p_0(t) + p_0(t) \cancel{\frac{O(\Delta t)}{\Delta t}} \to 0$$

$$\boxed{\frac{d}{dt} p_0(t) = -\lambda p_0(t)}$$

$$p_0(0) = 1.$$

$$\boxed{p_0(t) = e^{-\lambda t}}$$

exponential distribution.

if $\{a_i\}$ are the arrival instants $p_0(t) = \text{Prob}[a_1 > t] = e^{-\lambda t}$

$$n > 0: p_n(t+\Delta t) = p_n(t)(1 - \lambda \Delta t + O(\Delta t))$$

$$+ p_{n-1}(t)(\lambda \cdot \Delta t + O(\Delta t))$$

$$+ \sum_{l=0}^{n-2} p_l(t) O(\Delta t)$$

$$\frac{p_n(t+\Delta t)-p_n(t)}{\Delta t} = p_n(t)\,\cancel{\frac{O(\Delta t)}{\Delta t}}\to 0$$

$$+p_{n-1}(t)\,\cancel{\frac{O(\Delta t)}{\Delta t}}\to 0$$

$$+\sum_{l=0}^{n-2} p_l(t)\,\cancel{\frac{O(\Delta t)}{\Delta t}}\to 0$$

After taking $\lim \Delta t \to 0$:

$$\sum_{n=0}^{\infty} x^n \left| \begin{array}{l} \frac{d}{dt} p_n(t) = -\lambda p_n(t) + \lambda p_{n-1}(t), \ n \geq 1 \\ 1 \ \frac{d}{dt} p_0(t) = -\lambda p_0(t) \end{array} \right.$$

$$|x| \leq 1: G(x,t) = \sum_{n=0}^{\infty} p_n(t)\, x^n$$

$$\frac{d}{dt} G(x,t) = -\lambda G(x,t) + \lambda x \sum_{n=0}^{\infty} p_n(t) x^n$$

$$= -\lambda(1-x) G(x,t).$$

$$\boxed{G(x,0) = 1.}$$

$$G(x,t) = e^{-\lambda(1-x)t} = e^{-\lambda t} \cdot e^{\lambda x t}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} x^n$$

$$\boxed{p_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}}$$

POISSON DISTRIBUTION.

$$E[A(t)] = \sum_{n=0}^{\infty} n \cdot p_n(t) = \sum_{n=1}^{\infty} n e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

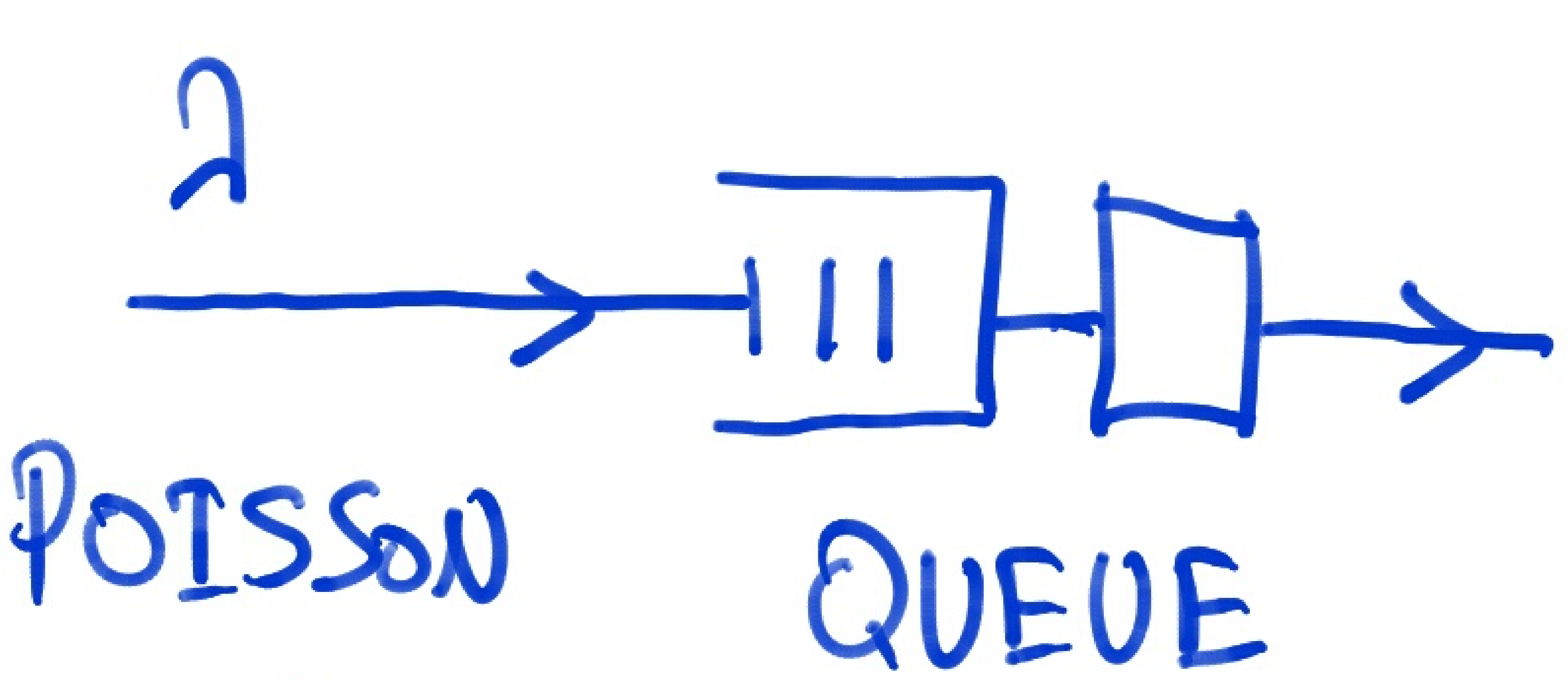
$$= e^{-\lambda t}\lambda t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$= e^{-\lambda t}\lambda t \underbrace{\sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!}}_{e^{\lambda t}}$$

$$= \lambda t.$$

Exp. dist

$$P_0(t) \equiv \text{Prob}[a_1 > t] = e^{-\lambda t}$$



Assume FIFO

exponential service times $E[S_i] = \frac{1}{\mu}$

Service

Assume

Arrival times $\{a_i\}$; Service Times $\{S_i\}$.

independent and identically dist.

$$\text{Prob}[s_1, s_2, \ldots, s_i] = \prod_{j=1}^{i} \text{Prob}[s_i]$$

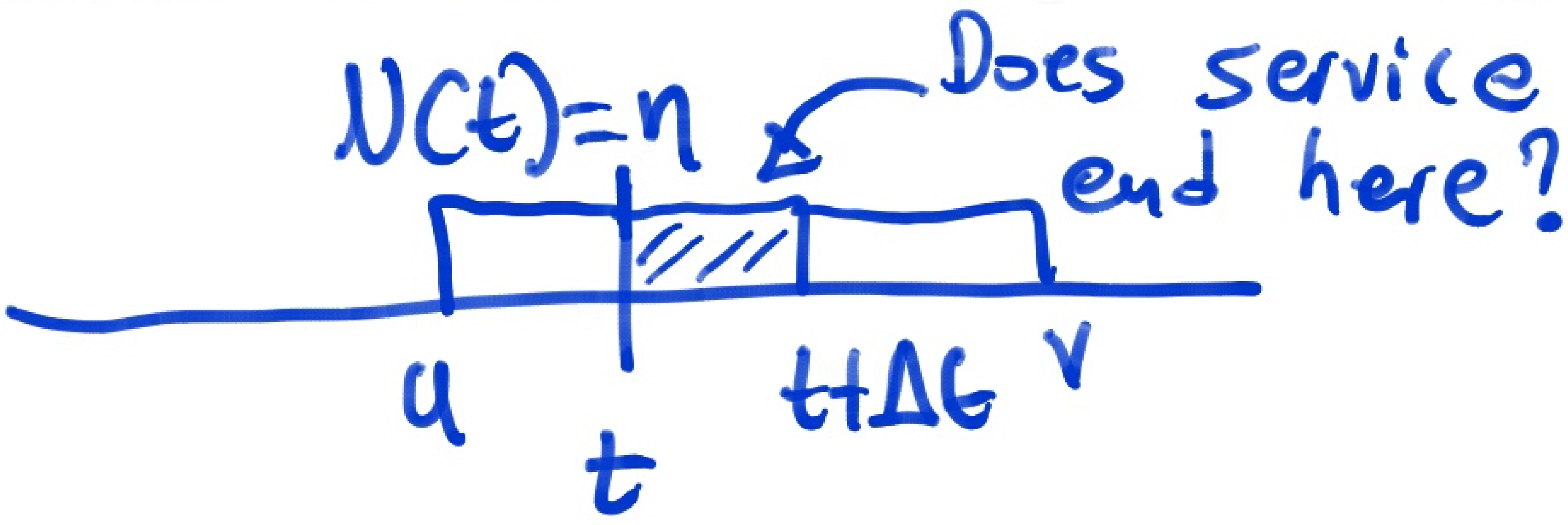
$$\text{Prob}[s_i] = \text{Prob}[s_j] \quad j \neq i$$

We are interested in the queue length.

$$\text{Prob}[N(t)=n] \equiv p_n(t)$$

$$n \geq 1: p_n(t+\Delta t) = p_n(t)\left[1-\lambda\Delta t+O(\Delta t)\right] \cdot \left[1-\mu\Delta t+O(\Delta t)\right]$$

Prob. of zero departures.



v is the end time

$$\mathrm{Prob}\big[\underbrace{v-u}_{s_i} > \underbrace{t+\Delta t-u}\big] \Longleftrightarrow$$

the departure does not occur in time $[t, t+\Delta t[$.

- $$\mathrm{Prob}[v-u > t+\Delta t-u \mid v-u > t-u] =$$

$$= \frac{P[v-u>t+\Delta t-u \ \cancel{\cap\ v-u>t}]}{P[v-u>t-u]}$$ redundant

$$= \frac{e^{-\mu(t+\Delta t-u)}}{e^{-\mu(t-u)}} = e^{-\mu\Delta t} \cong 1-\mu\Delta t+O(\Delta t)$$

- $$\mathrm{Prob}[v-u< t+\Delta t-u \mid v-u>t-u] =$$

$$\frac{P[t-u<v-u<t+\Delta t-u]}{P[v-u>t-u]} =$$

$$\frac{1-e^{-t+\Delta t-u}-(1-e^{-(t-u)\mu})}{e^{-\mu(t-u}} =$$

$$\frac{e^{-\mu(t-\mu)} - e^{-\mu(t+\Delta t-u)}}{e^{-\mu(t-u}} =$$

$$= 1 - e^{-\mu\Delta t} \cong \mu\Delta t + O(\Delta t).$$

$$p_n(t+\Delta t) = p_n(t)\left[1-\lambda\Delta t+O(\Delta t)\right]$$
$$\cdot\left[1-\mu\Delta t+O(\Delta t)\right] +$$
$$p_{n-1}(t)\left[\lambda\Delta t+(O\Delta t)\left[1-\mu\Delta t \;\; O(\Delta t)\right]\right.$$
$$+p_{n+1}(t)\left[1-\lambda\Delta t+O(\Delta t)\right]$$
$$\left[\mu\Delta t+O(\Delta t)\right]$$

$$p_n(t+\Delta t)-p_n(t) = -\lambda\Delta t\, p_n(t) - \mu\Delta t\, p_n(t)$$
$$+\lambda\Delta t\, p_{n-1}(t)$$
$$+\mu\Delta t\, p_{n+1}(t) + O(\Delta t)$$

$n \geq 1$: $\frac{d}{dt} p_n(t) = -p_n(t)(\lambda + \mu_n) + \lambda p_{n-1}(t) + \mu_n p_{n-1}(t)$

$$\frac{d}{dt} p_0(t) = -\lambda p_0(t) + \mu p_1(t)$$

Steady-State or Stationary solution

$$\frac{d}{dt} p_n(t) = 0 \quad \forall n$$

$$p_0 \lambda = \mu p_1 \Rightarrow p_1 = \frac{\lambda}{\mu} p_0.$$

$$G(x) = \sum_{n=0}^{\infty} p_n x^n$$

$$\lambda \cdot G(x) + \mu \sum_{n=1}^{\infty} p_n x^n =$$

$$\lambda \cdot G(x) + \mu [G(x) - p_0] = \frac{\mu}{x} [G(x) - p_0]$$
$$+ \lambda x G(x).$$

$$G(x)\lambda(1-x)+\mu G(x)\left[1-\frac{1}{x}\right]$$

$$=\mu p_0\left[1-\frac{1}{x}\right]$$

$$G(x)=\frac{\mu p_0\left[1-\frac{1}{x}\right]}{\lambda(1-x)+\mu\left[1-\frac{1}{x}\right]}$$

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$$p_n = \lim_{t \to \infty} \text{Prob}[N(t)=n]$$

$$\underbrace{\frac{d}{dt} p_n(t)}_{0} = \ldots$$

$$p_0 \cdot \lambda = \mu \cdot p_1$$

$n \geq 1 \quad p_n(\lambda+\mu) = \lambda p_{n-1} + \mu p_{n+1}$

$$\boxed{\sum_{n=0}^{\infty} p_n = 1.}$$

$$p_1 = \frac{\lambda}{\mu} p_0 \quad \begin{cases} p_n \overset{?}{=} \left(\frac{\lambda}{\mu}\right)^n p_0 \\ \end{cases}$$

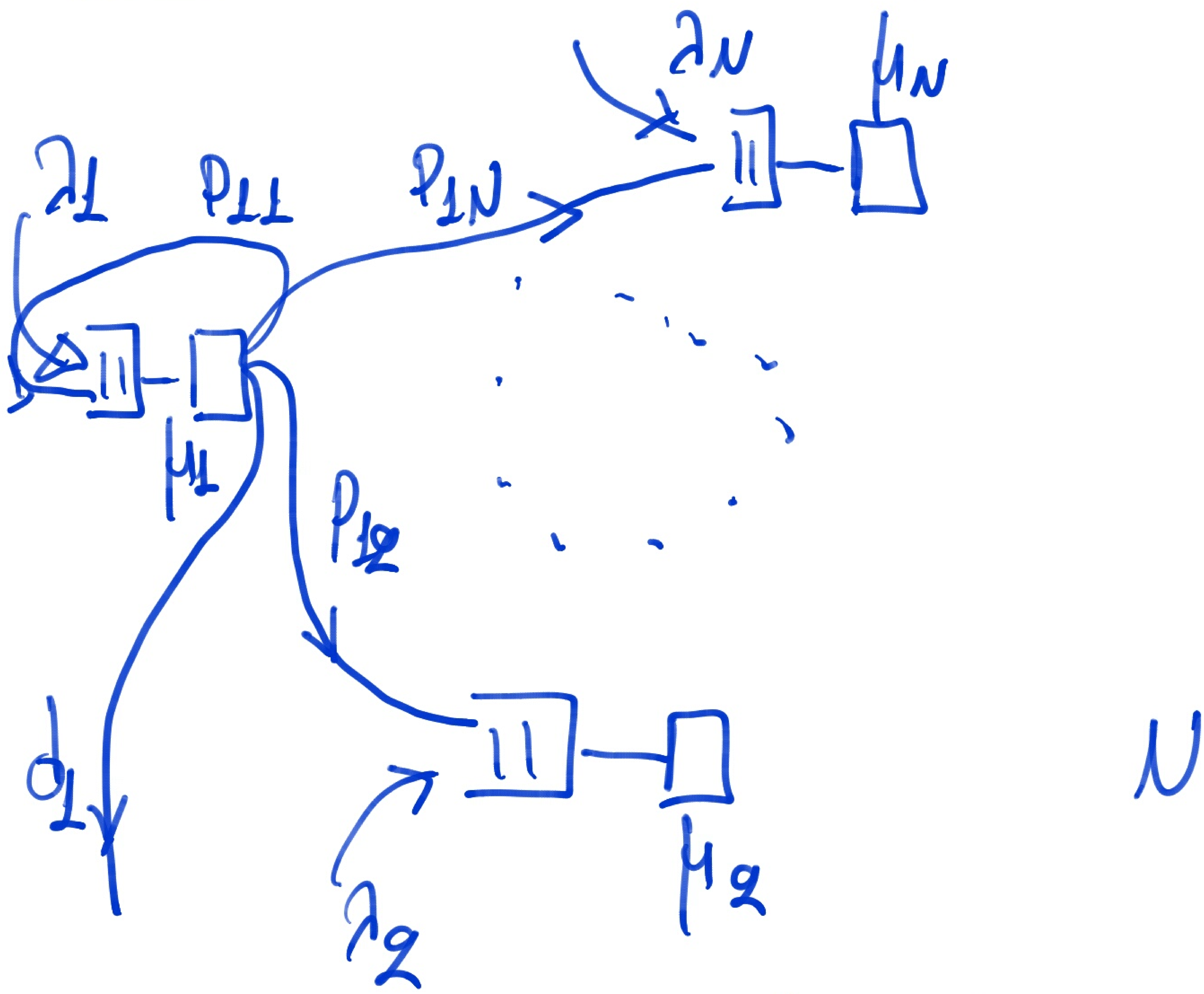
$$\left(\frac{\lambda}{\mu}\right)^n \cdot \cancel{p_0}(\lambda+\mu) = \left(\frac{\lambda}{\mu}\right)^{n-1} \cancel{p_0}\,\lambda + \left(\frac{\lambda}{\mu}\right)^{n+1} \cancel{p_0} \cdot \mu \quad \checkmark$$

$$p_0 \sum_{n=0}^{\infty} \underbrace{\left(\frac{\lambda}{\mu}\right)}_{<1}^{n}$$

If $\lambda < \mu$: then $\dfrac{p_0}{1-\frac{\lambda}{\mu}} = 1$

$$\Rightarrow \boxed{p_0 = 1 - \frac{\lambda}{\mu}}$$

Jackson's Theorem



$$d_i = 1 - \sum_{j=1}^{N} p_{ij}$$

$$\begin{pmatrix} \lambda_i, \mu_i \\ p_{ij} \end{pmatrix}$$

Assume

Poisson external arrivals

Exponential service times.

$$p(k,t) = \mathrm{Prob}[K_1(t)=k_1, \dots, K_N(t)=k_N]$$

$k_i \geq 0$ integers.

$$e_i = (0, \dots, \underset{\text{i-th position.}}{1}, \dots, 0) \qquad k_i^+ = k + e_i$$

$$k_i^- = k - e_i \quad \underbrace{\text{if } k_i \geq 1}_{\mathbb{1}[k_i > 0]} \qquad \mathbb{1}[k_i>0] = \begin{cases} 0 & \text{if } k_i \leq 0 \\ 1 & \text{if } k_i \geq 1 \end{cases}$$

$$k_{ij}^{+-} = k + e_i - e_j \quad \text{if } k_j > 0.$$

$$\overbrace{\frac{d}{dt} p(k,t)}^{\to 0} = \sum_{i=1}^{N} \lambda_i \, p(k_i^-, \cancel{t}) \, \mathbb{1}[k_i > 0]$$

$$+ \sum_{i=1}^{N} \mu_i \, p(k_i^+, \cancel{t}) \cdot d_i$$

$$+ \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \mu_i P_{ij} \, p(k_{ij}^{+-}, \cancel{t}) \, \mathbb{1}[k_j > 0]$$

$$-\left[\sum_{i=1}^{N} \lambda_i p(k,\cancel{t}) + \sum_{i=1}^{N} \mu_i (1-P_{ii}) p(k,\cancel{t}) \mathbb{1}[k_i>0]\right]$$

We are going to deal with the steady state.

$$p(k) = \lim_{t \to \infty} p(k,t) \; ; \; \frac{d}{dt} p(k,t) = 0$$

Consider

$$\Lambda_i = \lambda_i + \sum \Lambda_j P_{ji}$$

Theorem (JACKSON) ~1965.

$$p(k) = \prod_{i=1}^{N} \left(\frac{\Lambda_i}{\mu_i}\right)^{k_i} \left(1 - \frac{\Lambda_i}{\mu}\right) \quad \text{If } \Lambda_i < \mu_i$$

check

$$\Lambda = \lambda + \Lambda P \Rightarrow \Lambda(1-P) = \lambda$$

$$\Rightarrow \boxed{\Lambda = \lambda[1-P]^{-1}}$$

Divide by $P(k)$

$$0 = \sum_{i=1}^{N} \lambda_i \frac{\mu_i}{\Lambda_i} 1[k_i > 0]$$

$$+ \sum_{i=1}^{N} \cancel{\mu_i} \frac{\Lambda_i}{\cancel{\mu_i}} \lambda_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i P_{ij} \frac{\Lambda_i}{\mu_i} \frac{\mu_j}{\Lambda_j} 1[k_j > 0]$$

$$- \left[\sum_{i=1}^{N} \lambda_i + \sum_{i=1}^{N} \mu_i (1 - p_{ii}) 1[k_i > 0] \right]$$

$$\sum_{i}^{N} \frac{\lambda_i \mu_i}{\Lambda_i} 1[k_i > 0] + \sum_{i}^{N} \cancel{\Lambda_i} - \sum_{j}^{N} \cancel{(\Lambda_j - \lambda_j)}$$

$$+ \sum_{i} \sum_{\substack{j \\ i \neq j}} \frac{\mu_j^*}{\cancel{\Lambda_j}} 1[k_j > 0] \Lambda_i P_{ij} \rightarrow \cancel{\Lambda_j^*} - \lambda_j$$

$$= \sum_i \lambda_i + \sum_i \mu_i (1-p_{ii}^{*})\, 1[k_i>0] . \quad \checkmark$$

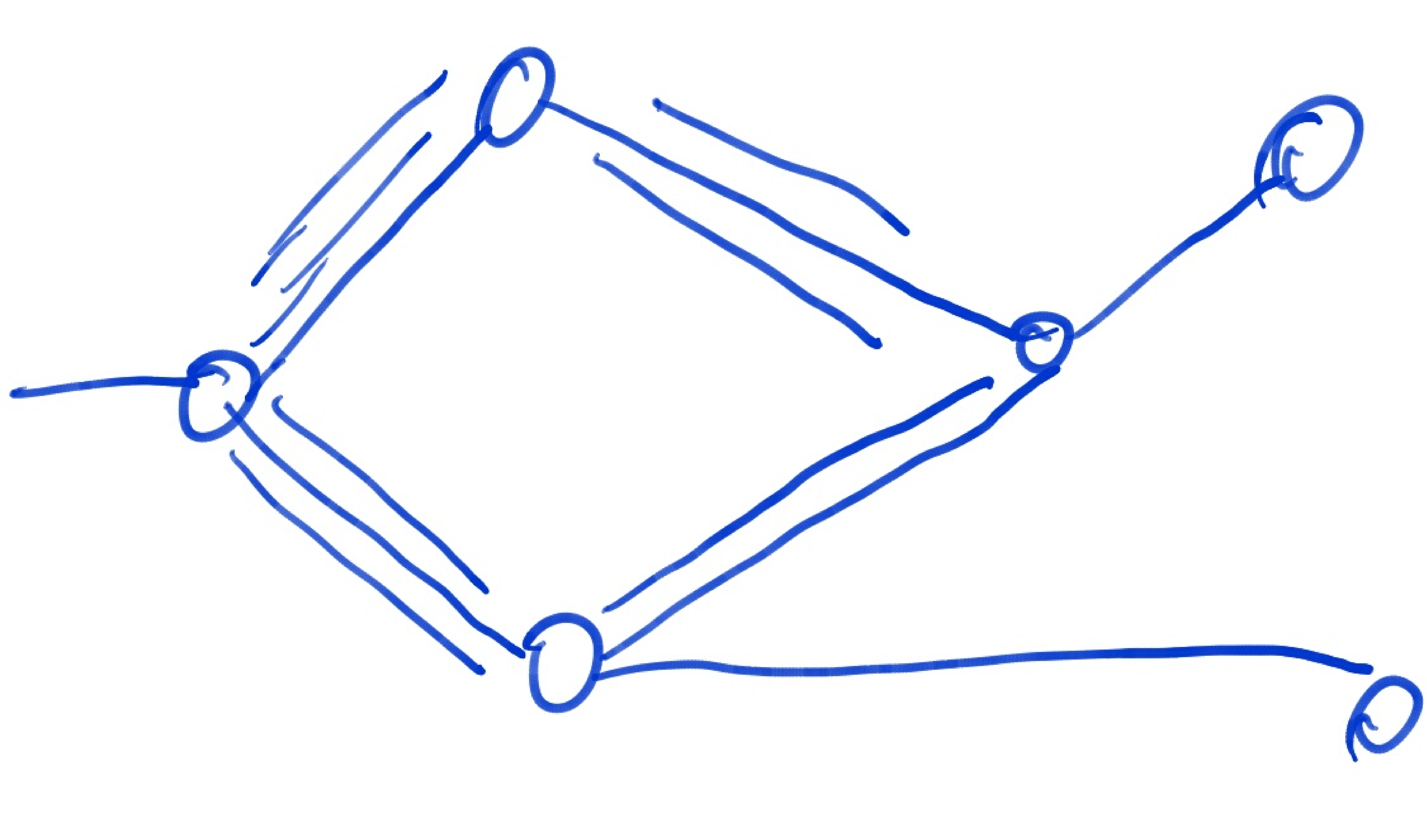
$$p(k) = \prod_{i=1}^{N} \left(\frac{\Lambda_i}{\mu_i}\right)^{k_i} \left(1-\frac{\Lambda_i}{\mu_i}\right)$$

*Packet loss.

$$\Lambda_i = \lambda_i + \sum_j \Lambda_j p_{ji} (1-L_j)$$

loss prob. in node j

Paths are deterministic



Π_i Set of paths starting at node i

$= \Pi_{i1}, \Pi_{i2}, \ldots$

$\Pi_{il} = (i, i_2(\pi_{il}), i_3(\pi_{il}), \ldots)$

how do we define a path?

T_{il} traffic in packets/second for Π_{il}.

$\lambda_i = \sum_{\text{over all } l} T_{il}$

$$P_{ij} = \frac{\sum_{\text{paths}} T(\text{prefix}, i, j, \text{suffix})}{\sum_{\text{all}} \sum_{\text{paths}} (\text{prefix}, i, m, \text{suffix})}$$

$$\begin{cases} \sum_{i=1}^{N} k_i(t) = K. \end{cases}$$

$$p(k) \sum_{i=1}^{N} \mu_i \mathbb{1}[k_i > 0] = \sum \Big[\mu_i p_{ij} p(k_{ij}^{+-}) \, \mathbb{1}[k_j > 0]$$

$$\sum_{i=1}^{N} k_i = K.$$

Single server queue

First-In-First-Out Service

Arrival instants are $a_1, a_2, \ldots, a_n, \ldots$

Departure instants are $d_1, d_2, \ldots, d_n, \ldots$

Service times are $S_1, S_2, \ldots, S_n, \ldots$

$$d_{n+1} = \begin{cases} d_n + S_{n+1} & \text{if } a_{n+1} < d_n. \\ a_{n+1} + S_{n+1} & \text{if } a_{n+1} \geq d_n. \end{cases}$$

$$\underbrace{W_{n+1}}_{\text{waiting time}} = \begin{cases} 0 & \text{if } a_{n+1} \geq d_n. \\ \underbrace{d_n - a_{n+1}}_{a_n + W_n + S_n - a_{n+1}.} & \text{if } a_{n+1} < d_n. \end{cases}$$

$$W_{n+1} = \begin{cases} W_n + S_n - (a_{n+1} - a_n) & \text{if} \\ 0 & \text{if } \underline{a_{n+1} \geq a_n + W_n + S_n} \end{cases}$$

$a_{n+1} - a_n \geq W_n + S_n$

$0 \geq (W_n + S_n - (a_{n+1} - a_n))$

$$w_{n+1} = [w_n + S_n - (a_{n+1} - a_n)]^+ \leftarrow \text{Lindley's Equation}$$

$$[x]^+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

$$E[a_{n+1} - a_n] = \frac{1}{\lambda} \leftarrow \text{interarrival time.}$$

$$E[S_n] = \frac{1}{\mu}$$

If $E[S_n] > E[a_{n+1} - a_n] \Rightarrow$ instability

$\Rightarrow w_n$ will increase

20/2/18.

Closed network

N nodes

K customers.

$p(\underline{k})$; $\underline{k} = (k_1, \dots, k_N)$

$$\sum_{i=1}^{N} k_i = K, \quad k_i \geq 0$$

Q: How many vectors $\underline{k}$?

Answers

1) N

2) $N!$

3) $\dfrac{(K+N-1)!}{(N-1)!\,K!}$

isomorphic structure

$$(k_1, \dots, k_N) \to (\underbrace{1, \dots, 1}_{k_1}, 0, \underbrace{1, \dots, 1}_{k_2}, 0, \dots, 0, \underbrace{1, \dots, 1}_{k_N})$$

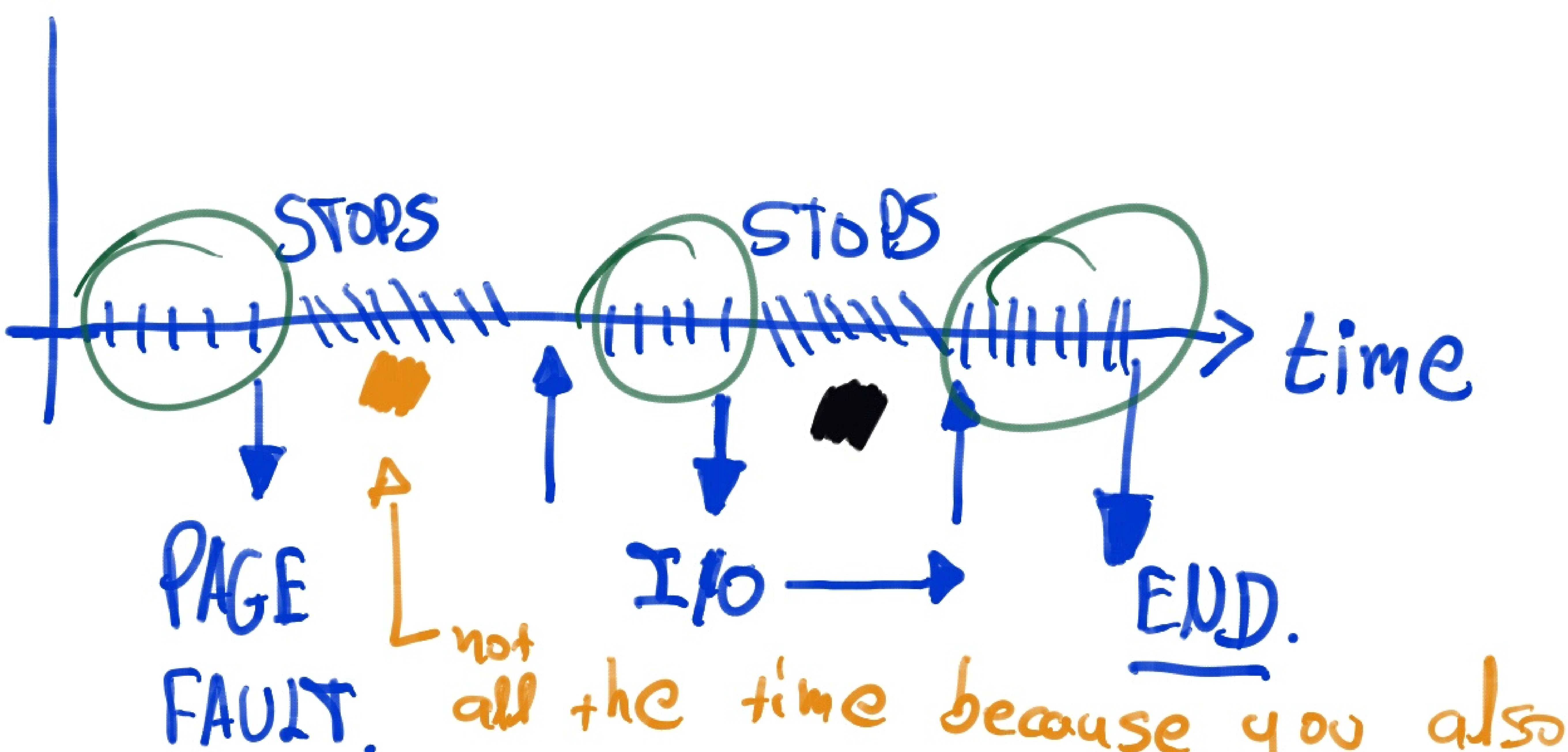
$K+N-1$

$$M=1000,\ N=10: \quad \frac{(1009)!}{(1000)!\ 9!} = \frac{1001 \times 1002 \times \ldots \times 1009}{8!}$$

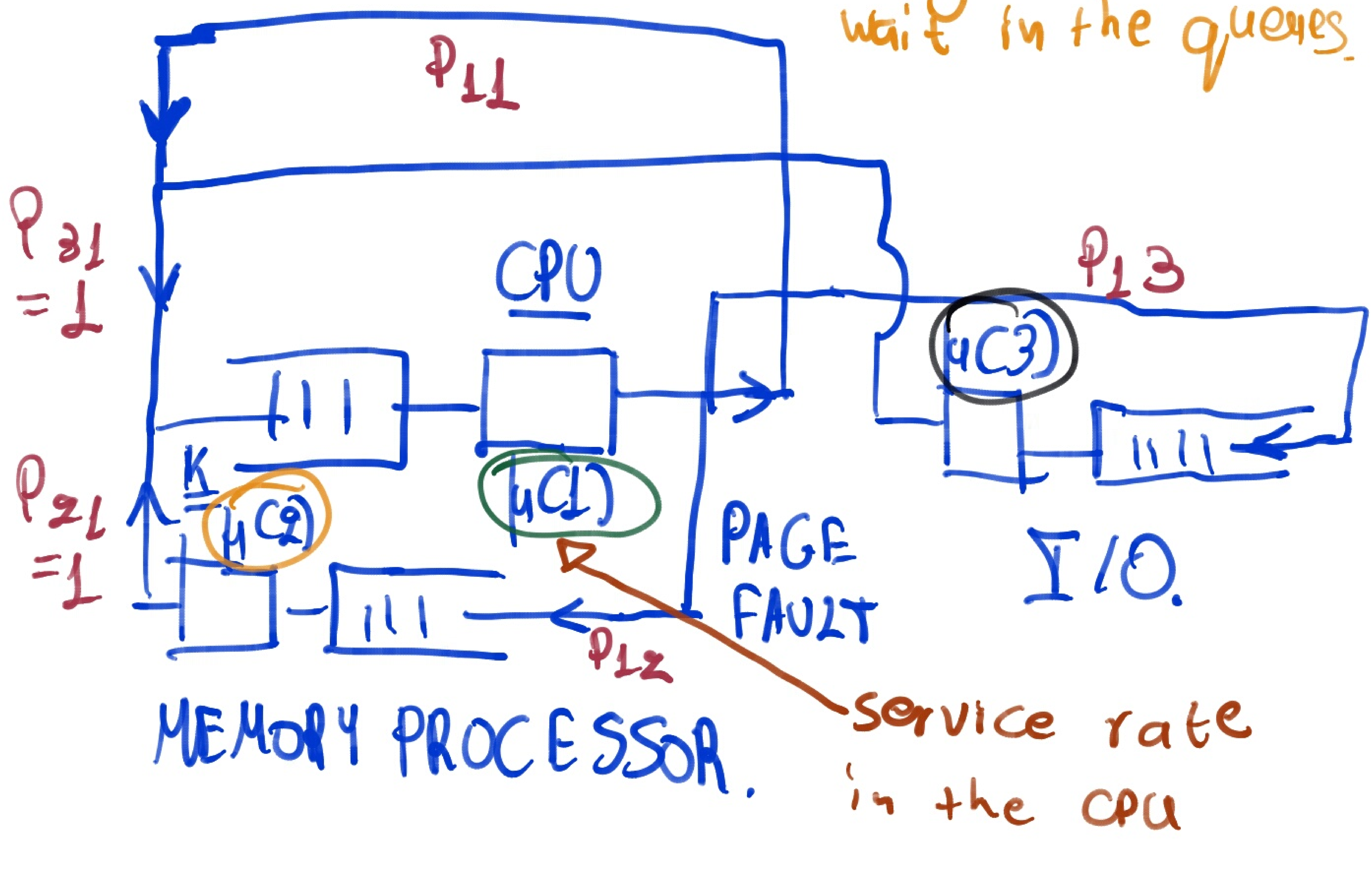
$\Uparrow$

$\boxed{\text{Computational Complexity.}}$

COMPUTER PROGRAM.



all the time because you also wait in the queues.



$\frac{1}{\mu(1)}$ = avg duration of uninterrupted execution at CPU.

$\frac{1}{\mu(2)}$ = avg time to fetch a page

$\frac{1}{\mu(3)}$ = avg time for I/O.

PAGE FAULTS → MEMORY LIMITATIONS.

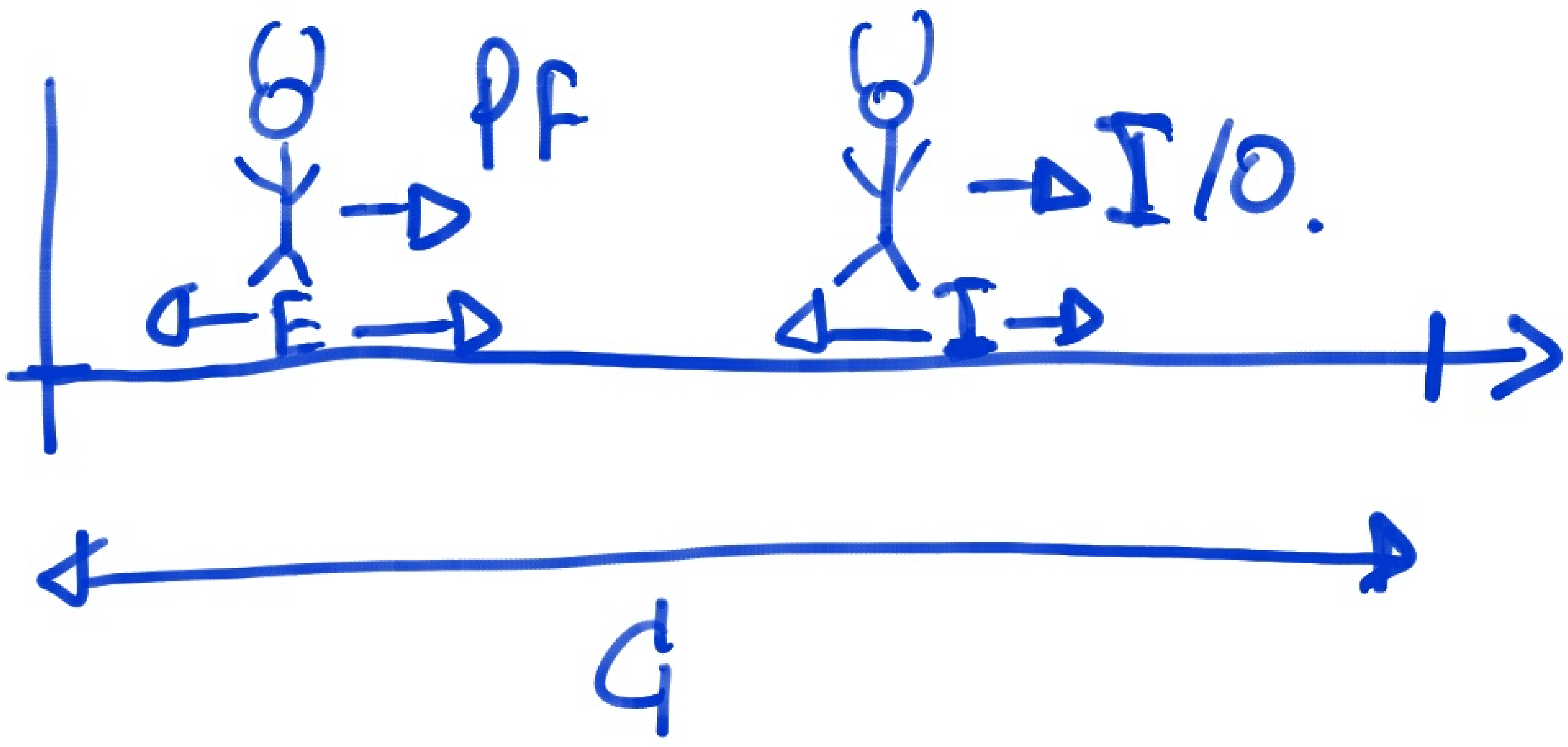
P_{11} → END OF EXECUTION OF THE PROGRAM.

P_{12} → PAGE FAULTS (PF).

P_{13} → I/O.

PF → MEMORY SIZE S

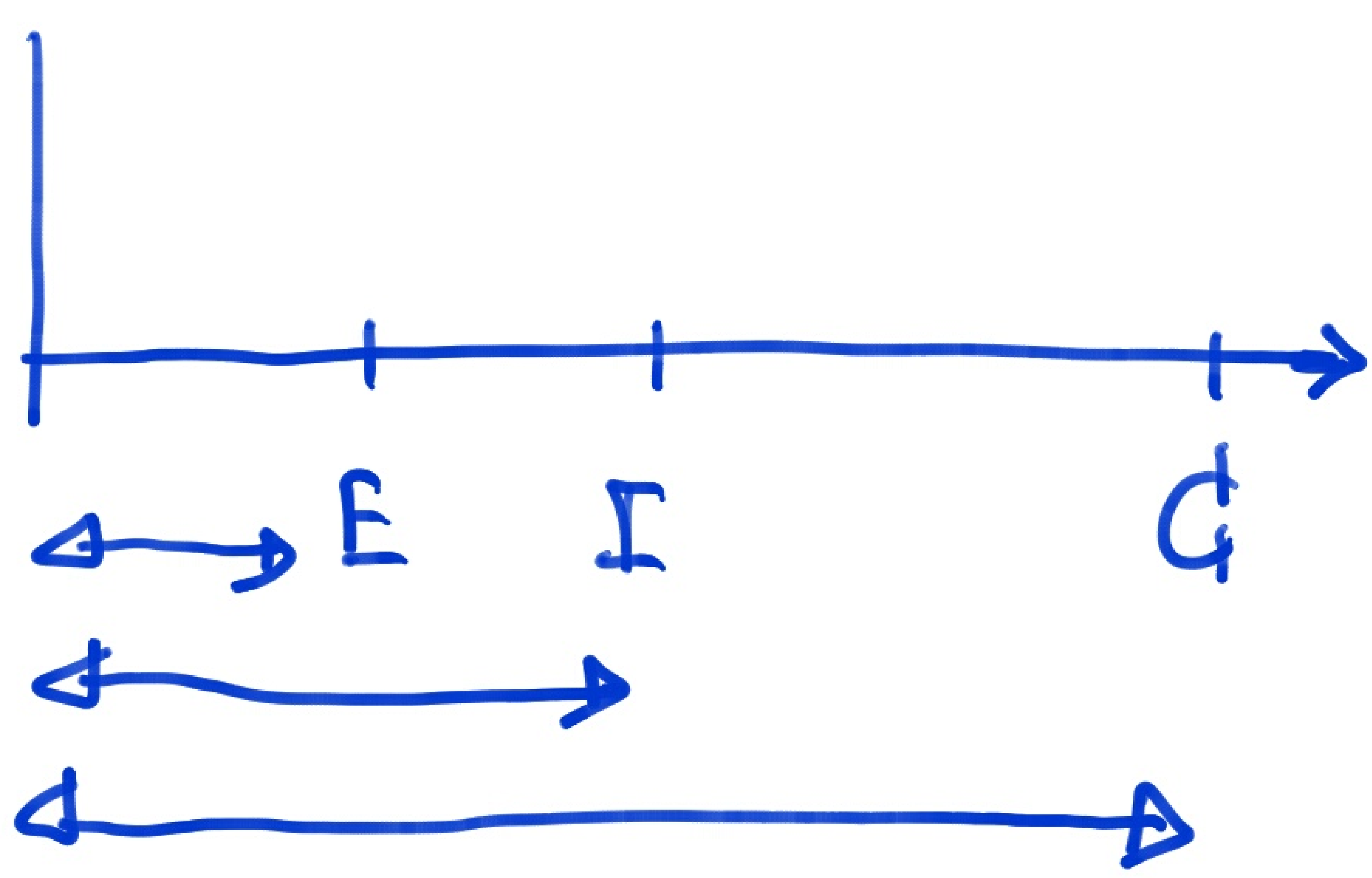
S is the amount of memory given to the program



uninterrupted program
exec time

$C = ||||| + |||||| + ||||||$

We assume that C, E, I are the
average values of three independent
exponential random variables.



$$\frac{1}{\mu(C)} = E[\underbrace{\min(e, i, c)}]$$

$E[c] = C$

$E[i] = I$

$E[e] = E.$

$$\mathrm{Prob}[\min(e, \iota, c) > x]$$

$$= \mathrm{Prob}[e > x, \iota > x, c > x]$$

$$= e^{-\frac{1}{E}x} \cdot e^{-\frac{1}{I}x} \cdot e^{-\frac{1}{C}x} = e^{-x\left[\frac{1}{E}+\frac{1}{I}+\frac{1}{C}\right]}$$

$$\boxed{(uCI) = \frac{1}{C} + \frac{1}{E} + \frac{1}{I}}$$

$$p_{11} = \mathrm{Prob}[c < e, c < \iota]$$

$$p_{12} = \mathrm{Prob}[e < c, e < \iota]$$

$$p_{13} = \mathrm{Prob}[\iota < c, \iota < e].$$

$$p_{11} = \int_0^\infty dx \underbrace{\frac{1}{C} e^{-\frac{1}{C}x}}_{\mathrm{Prob}\ x \le c < x+dx.} \cdot e^{-\frac{1}{I}x} \cdot e^{-\frac{1}{E}x} = \frac{\frac{1}{C}}{\frac{1}{C}+\frac{1}{E}+\frac{1}{I}}$$

$$P_{12} = \frac{1/E}{\frac{1}{C} + \frac{1}{I} + \frac{1}{E}}$$

$$P_{13} = \frac{1/I}{\frac{1}{C} + \frac{1}{I} + \frac{1}{E}}.$$

$$p(k) = G(K, N) \prod_{i=1}^{N} \left(\frac{v_i}{\mu(i)} \right)^{k_i}$$

$$v_i = \sum_{j=1}^{N} v_j P_{ji}$$

set $v_1 = 1$: average number of visits to queue 1.

$$v_2 = P_{12}$$

$$v_3 = P_{13}$$