EE401: Advanced Communication Theory

Professor A. Manikas
Chair of Communications and Array Processing

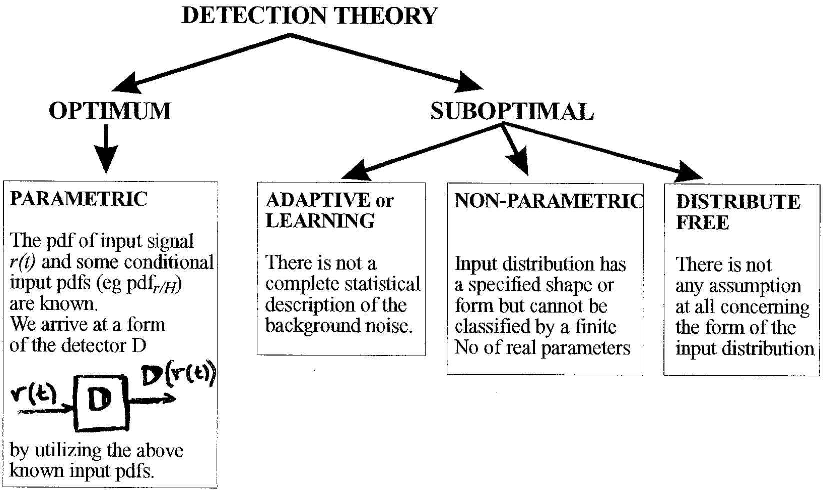
Imperial College London

Principles of Decision and Detection Theory

Table of Contents

Basic Decision Theory

- Detection Theory is concerned with determining the existence of a signal in the presence of noise - and can be classified as follows:

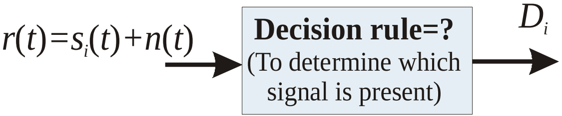


- N.B.:

1. **Adaptive or Learning** :
   - ★ system parameters change as a function of input;
   - ★ difficult to analyze;
   - ★ mathematically complex
2. **Non-Parametric** :
   - ★ fairly constant level of performance because these are based on general assumptions on the input pdf;
   - ★ easier to implement.
3. Consider a parametric detector which has been designed for Gaussian pdf input.
   - ★ **if input is actually Gaussian** : then parametric's performance is significantly better;
   - ★ **if input $\neq$Gaussian** but still symmetric: then non-parametric detector performance may be much better than the performance of the parametric detector

Hypothesis Testing in an M-ary Comm System

- A Hypothesis $\triangleq$ a statement of a possible condition
- In an M-ary Comm. System we have M hypotheses:



where $s_i(t)=$ one of M signal (channel symbols)

- hypotheses:

$$\begin{cases} H_1: & s_1(t) \text{ is } \overbrace{\text{present}}^{\text{is being sent}} & \text{with probability } \Pr(H_1) \\ H_2: & s_2(t) \text{ is present} & \text{with probability } \Pr(H_2) \\ \ldots & & \ldots \\ H_M: & s_M(t) \text{ is present} & \text{with probability } \Pr(H_M) \end{cases}$$

- The statistics of the observed signal

$$r(t) = s_i(t) + n(t) \tag{1}$$

  are affected by the presence of $s_1(t)$, or $s_2(t)$, ..., or $s_M(t)$

- If $s_i(t), \forall i$ are **known**

  then their distributions are **known**

  and the problem is **translated** to make a decision about one of the M distributions after having observed $r(t)$

  This is called 'Hypothesis Testing'

Terminology

- **A priori probabilities** :

$$\Pr(H_1), \Pr(H_2), ..., \Pr(H_M)$$

(these are calculated **BEFORE** the experiment is performed)

- **A posterior probabilities**

$$\Pr(H_1/r), \Pr(H_2/r), ..., \Pr(H_M/r)$$

That is, if r =observation variable
then we have M Conditional Probabilities

$$\Pr(H_i/r), \forall i \in [1, \ldots, M]$$

known as a POSTERIOR PROBABILITIES
(since these are calculated **AFTER** the experiment is performed).

- $\Pr(H_i / r)$ $\forall i$: difficult to find.
  A more natural approach is to find

$$\Pr(r / H_i), \forall i \tag{2}$$

  since in general $\text{pdf}_{r/H_i}, \forall i$
  - are known or
  - can be found

- **Likelihood Functions (LF)** :

$$\text{pdf}_{r/H_1}(r), \text{pdf}_{r/H_2}(r), \ldots, \text{pdf}_{r/H_M}(r) \tag{3}$$

  i.e. the M conditional probability density functions $\text{pdf}_{r/H_i}(r), \forall i$, are known as "Likelihood Functions"

- **Likelihood Ratio (LR)** : The ratio

$$\frac{\text{pdf}_{r/H_i}(r)}{\text{pdf}_{r/H_j}(r)} \text{ for } i \neq j \tag{4}$$

  is known as "Likelihood Ratio"

- **MAP Criterion**

  DECISION$-$choose hypothesis H_i i.e. D_i:

$$\text{iff } \Pr(H_i/r) > \Pr(H_j/r), \forall j : j \neq i \tag{5}$$

$$\begin{cases} D_1: & \text{iff } \Pr(H_1/r) > \Pr(H_j/r), \forall j \neq 1 \\ D_2: & \text{iff } \Pr(H_2/r) > \Pr(H_j/r), \forall j \neq 2 \\ \cdots & \cdots \\ D_M: & \text{iff } \Pr(H_M/r) > \Pr(H_j/r), \forall j \neq M \end{cases}$$

  This detection is known as max a posterior probability (MAP) criterion

- Equivalent MAP expressions:

$$
\begin{aligned}
D_i :& \quad \text{iff } \Pr(H_i) \times \text{pdf}_{r/H_i}(r) > \Pr(H_j) \times \text{pdf}_{r/H_j}(r); \forall j : j \neq i \\
&\Updownarrow \\
D_i =& \max_{j, \forall j} (\underbrace{\Pr(H_j) \times \text{pdf}_{r/H_j}(r)}_{\triangleq G_j})
\end{aligned}
$$

- Note:
  - The above can be easily proven using the Bayes rule

$$
\Pr(H_i/r) = \frac{\text{pdf}_{r/H_i}(r) . \Pr(H_i)}{\text{pdf}_r(r)} \tag{6}
$$

  - G_j is known as "decision variable"
  - In this topic the symbol G will be used to represent a "decision variable".

- Correlation between two **analogue energy signals** $r(t)$ and $s(t)$ of duration T_{cs}:

$$corr \triangleq \int_0^{T_{cs}} r(t).s(t).dt \tag{7}$$

- Correlation between the **discretised versions** $\underline{r}$ and $\underline{s}$ of the **signals** $r(t)$ and $s(t)$:

$$corr \triangleq \underline{r}^T \underline{s} \tag{8}$$

where

$$\underline{r} = [r_1, r_2, \ .... \ r_L]^T \tag{9}$$

and

$$\underline{s} = [s_1, s_2, \ .... \ s_L]^T \tag{10}$$

M-ary Decision Criteria

- Consider the sets of parameters $\mathcal{P}_1$ and $\mathcal{P}_2$ where:

  $\mathcal{P}_1$ denotes the set of parameters $\Pr(H_1)$,$\Pr(H_2)$, ..., $\Pr(H_M)$
  $\mathcal{P}_2$ represents the set of costs/weights C_{ij}, $\forall i, j$
  associated with the transition probabilities $\Pr(D_i|H_j)$
  (i.e. one weight for every element of the channel transition matrix $\mathbb{F}$ - see EE303, Topic on "Comm Channels")

- Estimate/identify the likelihood functions

$$\text{pdf}_{r|H_1}(r),\ \text{pdf}_{r|H_2}(r),\ \ldots,\ \text{pdf}_{r|H_M}(r)$$

- Decision: choose the hypothesis H_i (i.e. D_i) with the maximum $G_i(r)$ where $G_i(r)$ depends on the chosen criterion as follows:
  - ★ **BAYES** Criterion
  - ★ **Minimum Probability of Error** $(\min(\mathrm{p_e}))$ Criterion
  - ★ **MAP** Criterion
  - ★ **MINIMAX** Criterion
  - ★ Newman-Pearson (**N-P**) Criterion
  - ★ Maximum Likelihood (**ML**) Criterion

	$\mathcal{P}_1$	$\mathcal{P}_2$	choose Hypothesis with $\max(G_j(r))$
Bayes	**known**	**known**	$G_j(r) \triangleq \textbf{weight}_j \times \Pr(H_j) \times \mathrm{pdf}_{r\|H_j}$
$\min(p_e)$ or MAP	**known**	unknown	$G_j(r) \triangleq \Pr(H_j) \times \mathrm{pdf}_{r\|H_j}$
Minimax	unknown	**known**	$G_j(r) \triangleq \textbf{weight}_j \times \mathrm{pdf}_{r\|H_j}$
N-P	unknown	unknown	by solving a constraint optim. problem
ML	don't care	don't care	$G_j(r) \triangleq \mathrm{pdf}_{r\|H_j}$

- Notes:
  - Note-1:
    if an approximate/initial solution is required then any information about the sets of parameters $\mathcal{P}_1$ and/or $\mathcal{P}_2$ can be ignored. In this case the Maximum Likelihood (ML) Criterion should be used.
  - Note-2:

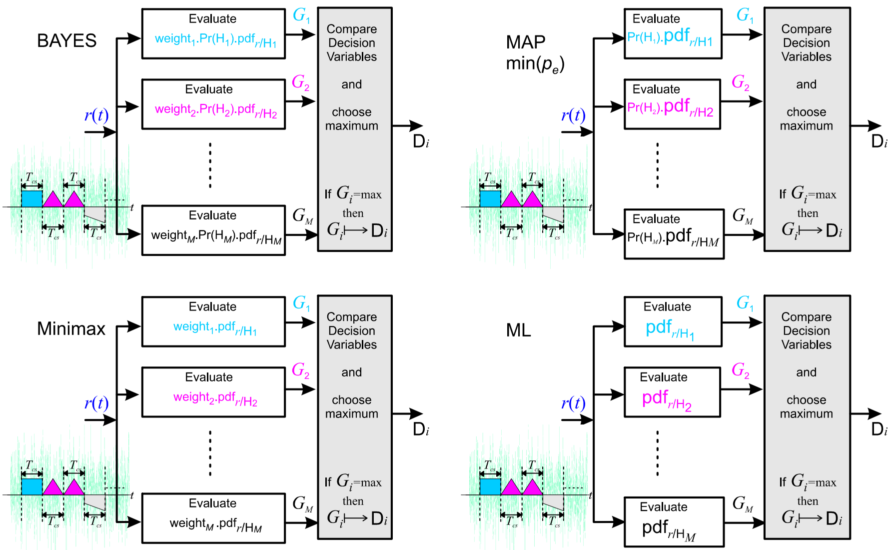
$$\textbf{weight}_{\ j} \triangleq \sum_{\substack{i=1 \\ i\neq j}}^{M} \left(C_{ij} - C_{jj}\right) \tag{11}$$

    or (since the term for $i = j$ is equal to zero) simply,

$$\textbf{weight}_{\ j} = \sum_{i=1}^{M} \left(C_{ij} - C_{jj}\right) \tag{12}$$

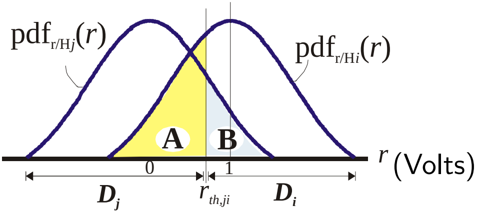
  - Note-3:
    Sometimes, for convenience, G_j will be used (i.e. $G_j \triangleq G_j(r)$) - i.e. the argument will be ignored.

Decision Criteria: Mathematical Architecture



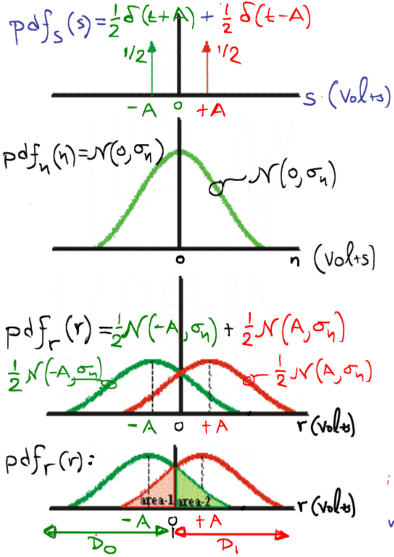
Examples

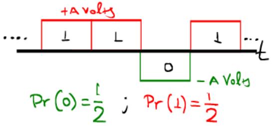
Example: Gaussian LFs

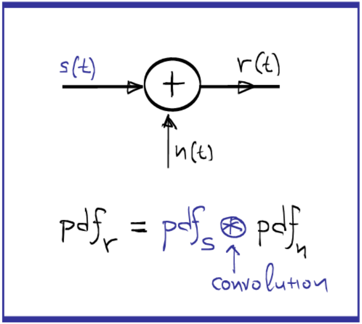


- By solving $G_j(r) = G_i(r)$ the decision threshold $r_{th,ji}$ can be estimated
- $\boxed{\text{area-A}} = \Pr(D_j|H_i)$ and $\boxed{\text{area-B}} = \Pr(D_i|H_j)$
- $p_{e,cs} = \sum_{j=1}^{M} \sum_{\substack{i=1 \\ i \neq j}}^{M} \Pr(D_j, H_i) =$ symbol error probability/rate (SER)

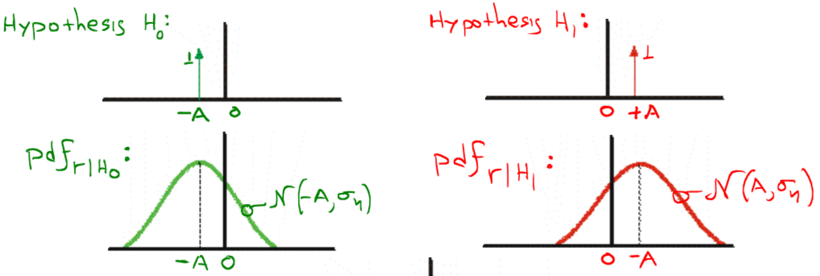
Example: Signal + Gaussian Noise



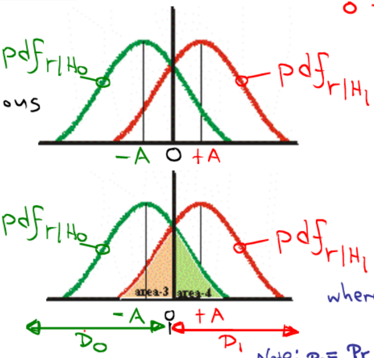




where $\begin{cases} \text{area-1} = Pr(D_0, H_1) \\ \text{area-2} = Pr(D_1, H_0) \end{cases}$



Likelihood functions put together



where $\begin{cases} \text{area-3} = \Pr(D_0|H_1) \\ \text{area-4} = \Pr(D_1|H_0) \end{cases}$

Note: $P_e = \underbrace{\Pr(D_0, H_1)}_{\Pr(D_0|H_1)\cdot\Pr(H_1)} + \underbrace{\Pr(D_1, H_0)}_{\Pr(D_1|H_0)\cdot\Pr(H_0)}$

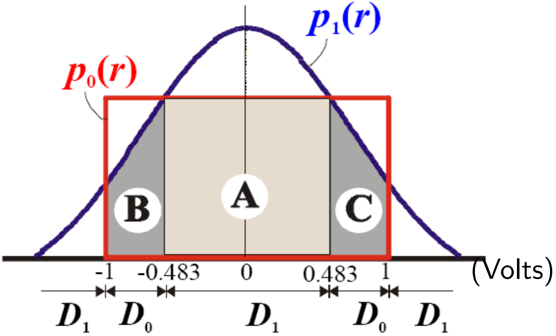
Example: Rectangular and Gaussian LFs

- A discrete channel employs two equally probable symbols and the likelihood functions are given by the following expressions

$$\mathrm{pdf}_{r|H_0} = \frac{1}{2}\mathrm{rect}\left\{\frac{r}{2}\right\}$$

$$\mathrm{pdf}_{r|H_1} = N\left(0, \sigma = \frac{1}{2}\right)$$

- If the detector employed has been designed in an optimum way, find the decision rule and model the above discrete channel.



Note:

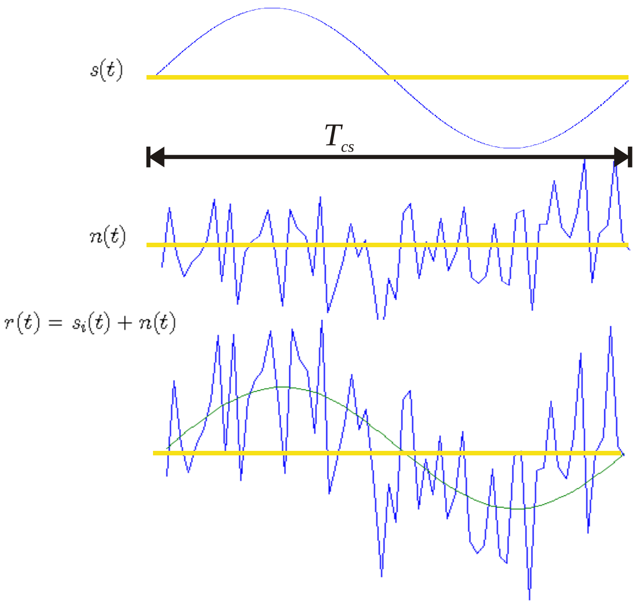
$$\boxed{\text{area B}} = \boxed{\text{area C}} = \mathbf{T}\left\{\frac{0.483}{1/2}\right\} - \mathbf{T}\left\{\frac{1}{1/2}\right\} = 0.1490:$$

$$\mathbf{Pr(D_0/H_1)} = \boxed{\text{area B} + \text{area C}} = 0.298$$

and $$\mathbf{Pr(D_1/H_0)} = \boxed{\text{area A}} = 0.483$$

Basic Concepts on Optimum Receivers

- Receiver = Detector + Decision Device
- Consider a M-ary communication model in which one of M signals $s_i(t)$, for $i = 1, 2, ..M$, is received in the time interval $(0, T_{\mathbf{cs}})$ in the presence of white noise
  i.e.
  $$r(t) = \left\{ \begin{array}{r} s_1(t) \\ or\ s_2(t) \\ \ldots \\ or\ s_M(t) \end{array} \right\} + n(t), \quad 0 \leq t \leq T_{cs} \tag{13}$$
  where $r(t)$ denotes the received **(observable)** signal.


$s(t)$
T_{cs}
$n(t)$
$r(t) = s_i(t) + n(t)$

The Concept of "Continuous Sampling"

Continuous Sampling helps to get a probabilistic description of a continuous signal:

1. assume that L amplitude samples of the received signal

$$r(t), \; 0 \leq t \leq T_{cs}$$

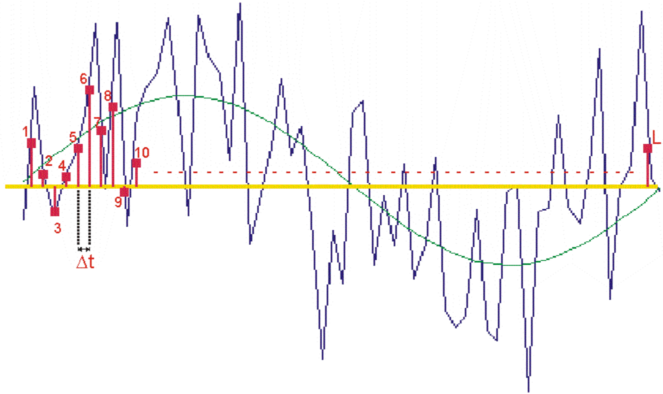
are available, i.e. $r_1, r_2, \ldots, r_L$,(with sampling frequency $\frac{1}{\triangle t}$)

where $r_k = \left\{ \begin{array}{r} s_{1k} \\ or \;\; s_{2k} \\ \ldots \\ or \;\; s_{Mk} \end{array} \right\} + n_k; k = 1, \ldots, L$

or, in more compact form,

$$\underline{r} = \left\{ \begin{array}{r} \underline{s}_1 \\ or \;\; \underline{s}_2 \\ \ldots \\ or \;\; \underline{s}_M \end{array} \right\} + \underline{n} \qquad \text{where} \left\{ \begin{array}{r} \underline{r} = \left[\; r_1, \quad r_2, \quad \ldots, \quad r_L \; \right]^T \\ \underline{n} = \left[\; n_1, \quad n_2, \quad \cdots, \quad n_L \; \right]^T \\ \underline{s}_i = \left[\; s_{i1}, \quad s_{i2}, \quad \cdots, \quad s_{iL} \; \right]^T \end{array} \right.$$

i.e. $\underline{r} = \underline{s}_i + \underline{n}$ (L samples)



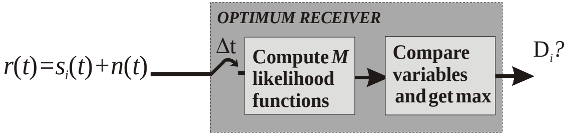
2. take the limit as $L \to \infty$, $\Delta t \to 0$, $L \times \Delta t \to T_{cs}$

Optimum M-ary Receivers

- **Objective** : to design a receiver which operates on $r(t)$ and chooses one of the following M hypotheses:

$$\left\{\begin{array}{lll} \boxed{H_1} & : & r(t) = s_1(t) + n(t) \\ \boxed{H_2} & : & r(t) = s_2(t) + n(t) \\ \cdots & & \cdots \\ \boxed{H_M} & : & r(t) = s_M(t) + n(t) \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{lll} \boxed{H_1} & : & \underline{r} = \underline{s}_1 + \underline{n} \\ \boxed{H_2} & : & \underline{r} = \underline{s}_2 + \underline{n} \\ \cdots & & \cdots \\ \boxed{H_M} & : & \underline{r} = \underline{s}_M + \underline{n} \end{array}\right.$$

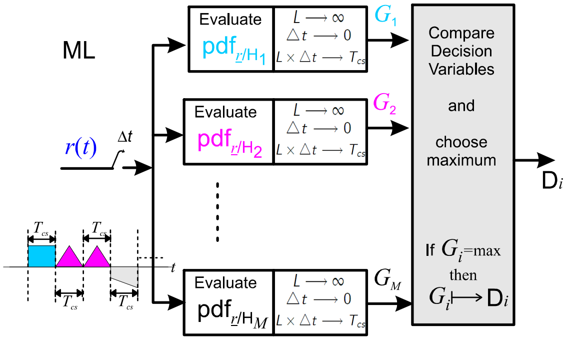
- **Optimum Decision Rule**:
  depends on the likelihood functions $\text{pdf}_{r|H_{\mathbf{i}}|}(r)$
- **Optimum Receiver - Initial Conceptual Structure:**



- **Assumptions** :
  - Noise: $\text{PSD}_{n_i}(f) = \frac{N_0}{2} \Rightarrow \text{PSD}_n(f) = \frac{N_0}{2}\text{rect}\left\{\frac{f}{2B}\right\}$
    Therefore, the received signal is sampled at intervals

$$\Delta t = \frac{1}{2B} \tag{14}$$

  - **Sampling**: the samples are uncorrelated and statistically independent
- **Example - ML Receiver with "Cont. Sampling"** :



Gaussian Multivariable Distribution

- Consider the Gaussian random vector: $\underline{r} = [r_1, r_2, ..., r_L]^T$

Then

$$\mathrm{pdf}(\underline{r}) = \frac{1}{\sqrt{(2\pi)^L \det(\mathbb{R}_{rr})}} \exp\left\{-\frac{(\underline{r}-\underline{\mu}_r)^T \mathbb{R}_{rr}^{-1}(\underline{r}-\underline{\mu}_r)}{2}\right\} \quad (15)$$

where

$$\underline{mean} = \underline{\mu}_r = \mathcal{E}\{\underline{r}\} = [\mathcal{E}\{r_1\}, \mathcal{E}\{r_2\}, ..., \mathcal{E}\{r_L\}]^T \quad (16)$$

$$\mathrm{cov}(\underline{r}) = \mathbb{R}_{rr} = \mathcal{E}\left\{\left(\underline{r}-\underline{\mu}_r\right)\left(\underline{r}-\underline{\mu}_r\right)\right\}^T \quad (17)$$

- If $\underline{r} = \underline{s} + \underline{n}$ then the mean $\underline{\mu}_r$ and covariance $\mathbb{R}_{rr}$ are given as follows:

$$\underline{\mu}_r = \underline{s} \text{ and } \mathbb{R}_{rr} = \mathbb{R}_{nn} \tag{18}$$

proof :

$$\begin{aligned}
\mathcal{E}\{\underline{r}\} = \underline{\mu}_r &\triangleq \mathcal{E}\{\underline{s}+\underline{n}\} = \mathcal{E}\{\underline{s}\} + \mathcal{E}\{\underline{n}\} = \mathcal{E}\{\underline{s}\} = \underline{s} \qquad (19)\\
\text{cov}\{\underline{r}\} &\triangleq \mathbb{R}_{rr} = \mathcal{E}\left\{(\underline{r} - \mathcal{E}\{\underline{r}\})(\underline{r} - \mathcal{E}\{\underline{r}\})^T\right\}\\
&= \mathcal{E}\left\{(\underline{r} - \underline{s})(\underline{r} - \underline{s})^T\right\}\\
&= \varepsilon\{\underline{nn}^T\}\\
\Rightarrow \text{cov}\{\underline{r}\} &= \mathbb{R}_{nn} = \sigma_n^2 \mathbb{I}_L = N_0 B \mathbb{I}_L \qquad (20)
\end{aligned}$$

- Note that:

$$\text{If } \mathbb{R}_{nn} = \sigma_n^2 \mathbb{I}_L \text{ then } \det(\mathbb{R}_{nn}) = \left(\sigma_n^2\right)^L \tag{21}$$

Likelihood Functions (Continuous Sampling)

- $\text{pdf}_{r/H_i}(r)$: Based on the above (for AWGN) and for "Continuous Sampling" we have

$$\text{pdf}_{r/H_i}(\underline{r}) = \left(\frac{1}{\sqrt{2\pi\sigma_n^2}}\right)^L \cdot \exp\left\{-\frac{(\underline{r}-\underline{s}_i)^T(\underline{r}-\underline{s}_i)}{2\sigma_n^2}\right\} \qquad (22)$$

- However $\left\{\begin{array}{c} \sigma_n^2 = N_0 B \\ \triangle t = \frac{1}{2B} \end{array}\right\} \Longrightarrow \sigma_n^2 = \frac{N_0}{2\cdot\triangle t}$
- Therefore

$$\text{pdf}_{r/H_i}(\underline{r}) = \left(\sqrt{\frac{\Delta t}{\pi N_0}}\right)^L \cdot \exp\left\{-\frac{(\underline{r}-\underline{s}_i)^T(\underline{r}-\underline{s}_i)\cdot\Delta t}{N_0}\right\}$$

  and for "Continuous Sampling", i.e. $L \longrightarrow \infty, \triangle t \longrightarrow 0, L \times \triangle t \longrightarrow T_{cs}$ we have

$$\text{pdf}_{r/H_i}(r(t)) = const \cdot \exp\left\{-\frac{1}{N_0}.\int_0^{T_{cs}} (r(t) - s_i(t))^2 dt\right\} \qquad (23)$$

- **ML APPROACH** using "Continuous Sampling":
  Rule : choose the hypothesis $H_{\mathbf{i}}$ with the maximum likelihood $\text{pdf}_{r/H_{\mathbf{i}}}(r(t))$ where

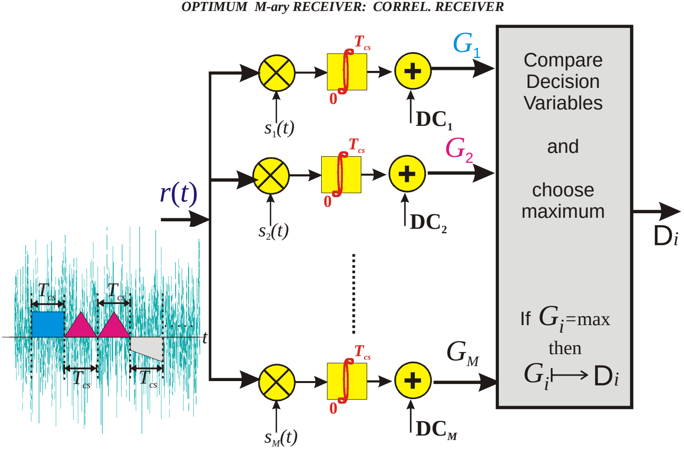
$$\begin{aligned}\text{pdf}_{r/H_{\mathbf{i}}}(r(t)) &= const \cdot \exp\left\{-\frac{1}{N_0}.\int_0^{T_{cs}}(r(t)-s_i(t))^2dt\right\} \quad (24)\\ \text{for } i &= 1,\ldots,M\end{aligned}$$

- **MAP APPROACH** using "Continuous Sampling":
  This is a more general approach than the Maximum Likelihood
  Rule : choose the hypothesis H_i with the maximum $\Pr(H_i)\times\ \text{pdf}_{r/H_i}(r(t))$ where

$$\begin{aligned}\text{pdf}_{r/H_i}(r(t)) &= const.\exp\left\{-\frac{1}{N_0}.\int_0^{T_{cs}}(r(t)-s_i(t))^2dt\right\} \quad (25)\\ \text{for } i &= 1,\ldots,M\end{aligned}$$

$$\begin{aligned}\text{i.e.}\qquad & \max\left\{\Pr(H_i)\times\text{pdf}_{r/H_i}(r(t))\right\}\\ =\ & \max\left\{\int_0^{T_{cs}}r(t)s_i^*(t)dt+\frac{N_0}{2}\ln\left(\Pr\left(H_i\right)\right)-\frac{1}{2}E_i\right\} \quad (26)\end{aligned}$$

- Equation-26 (for proof see Appendix 2) suggests that an M-ary receiver will have the following form:



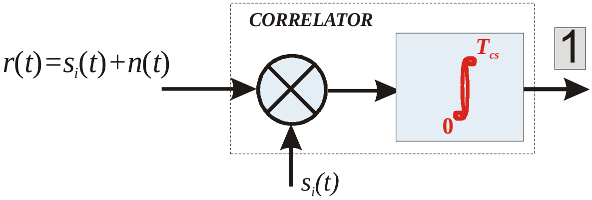
where $\mathrm{DC}_i \equiv \frac{N_0}{2}\ln(\Pr(H_i)) - \frac{1}{2}E_i$ with $i = 1, 2, \ldots, M$

- Note that if the a priori probabilities are equal,
  i.e.
  $$\Pr(H_1) = \Pr(H_2) = \cdots = \Pr(H_i) \tag{27}$$
  and the signals have the same energy,
  i.e.
  $$E_1 = E_2 = \cdots = E_M$$
  then Equation-26 is simplified as follows:
  $$\begin{aligned} \max\left\{\Pr(H_i) \times \mathrm{pdf}_{r/H_i}(r(t))\right\} &= \max\{\mathrm{pdf}_{r/H_i}(r(t))\} \\ &= \max\left\{\int_0^{T_{cs}} r(t)s_i^*(t) \cdot dt\right\} \end{aligned} \tag{28}$$
- N.B. - ML receiver: similar structure to MAP with $\mathrm{DC}_i = \frac{1}{2}E_i$

Optimum M-ary Receivers using Matched Filters

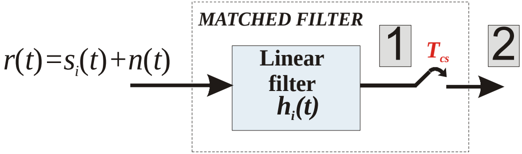
Known Signals in White Noise

- Consider the output of one branch of the correlation receiver:



$$\boxed{\text{at point-1}} = output = \int_0^{T_{cs}} r(t) \cdot s_i(t) \cdot dt$$

- In this section we will try to replace the **correlator** of a correlation receiver with a **linear filter** (**known as Matched Filter**).



$$\boxed{\text{at point-1}} \quad = \quad \int_0^t r(u) \cdot h_i(t-u) \cdot du$$

$$\boxed{\text{at point-2}} \quad = \quad output = \int_0^{T_{cs}} r(u) \cdot h_i(T_{cs}-u) \cdot du$$

- N.B.:
  compare correlator's o/p (point-1) with matched filter's o/p (point-2).

- If we choose the impulse response of the linear filter as

$$\boxed{h_i(t) = s_i(T_{cs} - t) \quad 0 \leq t \leq T_{cs}} \tag{29}$$

then that linear filter, defined by the above equation, is called Matched Filter.

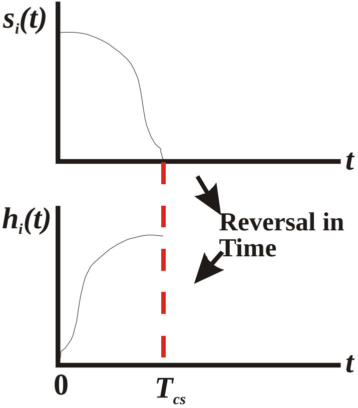
$$\boxed{\text{at point-2}} = \int_0^{T_{cs}} r(u) \cdot s_i(u) \cdot du$$

$$\Longrightarrow \quad \boxed{\text{at point-2}} = \int_0^{T_{cs}} r(t) \cdot s_i(t) \cdot dt$$

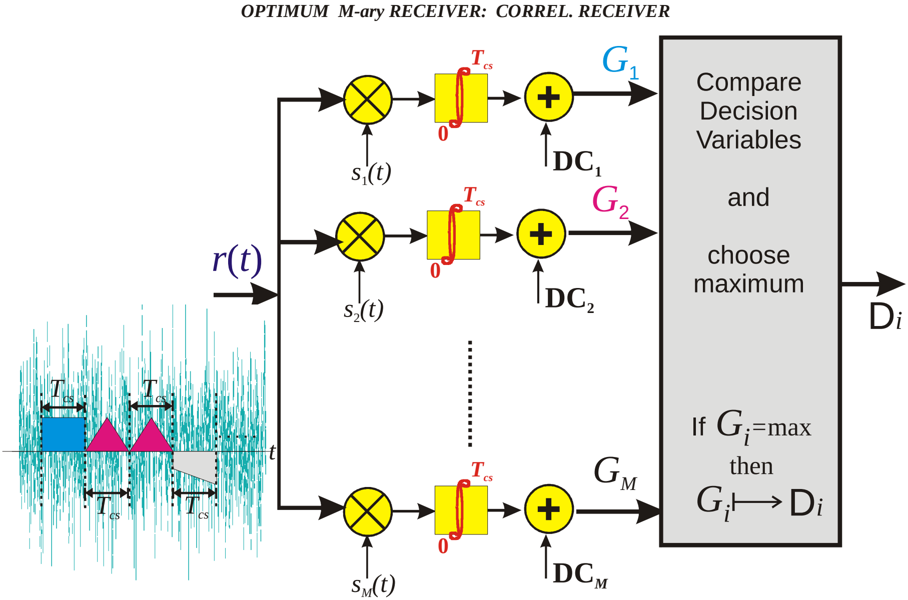
- N.B.:

$$\boxed{\text{Output Of Correlator}} \underset{\text{only at time } t=T_{cs}}{\underset{\uparrow}{=}} \boxed{\text{Output Of Matched Filter}}$$

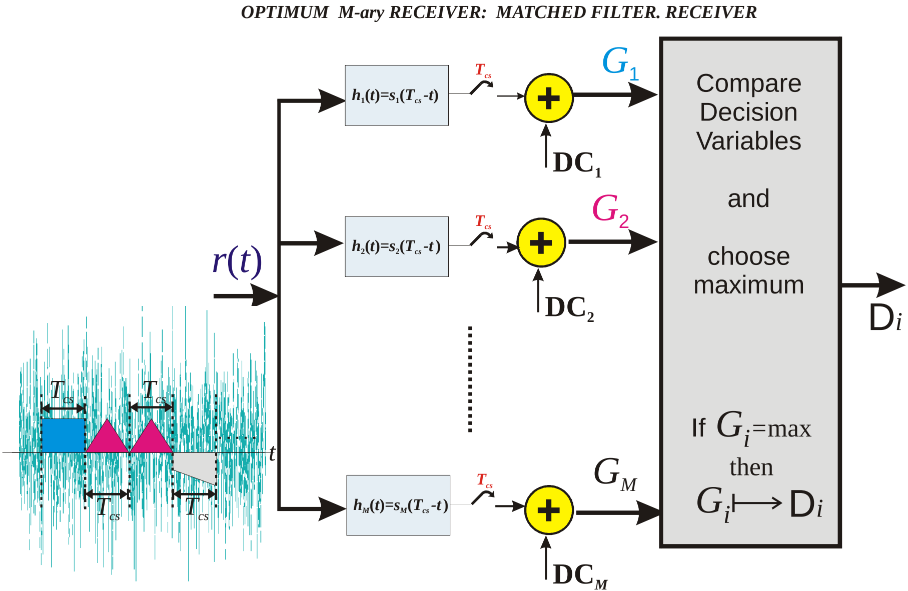
- example:



CORRELATION RECEIVER:



or, MATCHED FILTER RECEIVER:



Signals and Matched Filters in Freq. Domain

- Let $h(t) = \begin{cases} s(T_{cs} - t) & 0 \leq t \leq T_{cs} \\ 0 & \text{elsewhere} \end{cases}$
- $\boxed{\text{Time Domain}} \overset{FT}{\longmapsto} \boxed{\text{Freq. Domain}}$

$$\begin{aligned} s(t) \overset{FT}{\longmapsto} S(f) \quad &= \int_0^{T_{cs}} s(t) e^{-j2\pi ft} dt \\ h(t) \overset{FT}{\longmapsto} H(f) \quad &= \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt \\ &= \int_o^{T_{cs}} s(T_{cs} - t) e^{-j2\pi ft} dt \\ &= \int_0^{T_{cs}} s(u) e^{-j2\pi f(T_{cs}-u)} du \\ &= e^{-j2\pi fT_{cs}} \int_0^{T_{cs}} s(u) e^{+j2\pi fu} du \\ &= e^{-j2\pi fT_{cs}} \cdot S^*(f) \end{aligned}$$

therefore

$$\boxed{H(f) = e^{-j2\pi fT_{cs}} \cdot S^*(f)} \tag{30}$$

Known Signals in Non-White Noise

- Now let us assume that the noise $n(t)$ is described as follows:

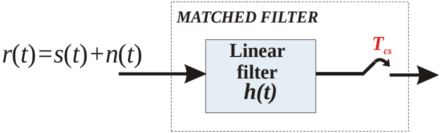
$$\boxed{n(t) = \begin{cases} \text{zero mean} \\ R_{nn}(\tau) = \text{not necessarily white or Gaussian} \end{cases}} \tag{31}$$

- Let

$$r(t) = \underbrace{s(t)}_{\text{completely known}} + n(t) \tag{32}$$

- objective: design a linear filter $h(t)$ such that

$$\underset{\text{at } t=T_{cs}}{\text{SNR}_{out}} = \max \tag{33}$$



$$\begin{aligned} output &= \int_0^{T_{cs}} h(\tau)\cdot \overbrace{r(T_{cs}-\tau)}^{\substack{s(T_{cs}-\tau)+n(T_{cs}=\tau)\\ \Downarrow}} \cdot dt \\ &= \underbrace{\int_0^{T_{cs}} h(\tau)\cdot s(T_{cs}-\tau)dt}_{=\widetilde{s}(T_{cs})} + \underbrace{\int_0^{T_{cs}} h(\tau)\cdot n(T_{cs}-\tau)d\tau}_{\widetilde{n}(T_{cs})} \end{aligned}$$

$$\therefore \text{SNR}_{out} = \frac{\mathcal{E}\{\widetilde{s}(T_{cs})^2\}}{\mathcal{E}\{\widetilde{n}(T_{cs})^2\}} = \frac{\widetilde{s}(T_{cs})^2}{\mathcal{E}\{\widetilde{n}(T_{cs})^2\}} \tag{34}$$

$$\therefore \quad \max_{h(t)}(\text{SNR}_{out}) = \left\{\min_{h(t)} \mathcal{E}\{\widetilde{n}(T_{cs})^2\} \text{ with } \widetilde{s}(T_{cs}) = \text{constant}\right\} \tag{35}$$

$$\therefore \quad \min_{h(t)} \xi \text{ where } \xi = \mathcal{E}\{\widetilde{n}(T_{cs})^2\} - \lambda\cdot\widetilde{s}(T_{cs}) \tag{36}$$

- i.e

$$\text{optimum impulse response} \triangleq h_{opt} = \arg\min_{h(t)} \xi \tag{37}$$

$$\text{where } \xi = \mathcal{E}\{\widetilde{n}(T_{cs})^2\} - \lambda\cdot\widetilde{s}(T_{cs})$$

- The solution of Equation 37 is the solution of the following

  FREDHOLM INTEGRAL EQUATION of 1st KIND:

$$\boxed{\int_0^{T_{cs}} h_{opt}(z).R_{nn}(\tau - z).dz = s(T_{cs} - \tau);\ 0 \leq \tau \leq T_{cs}} \quad (38)$$

GENERAL EXPRESSION FOR MATCHED FILTERS

- Proof of the above result (Equation 38) is not important. What is really important is that the proof does not involve any assumption about the pdf or PSD of the noise (i.e. that the noise is white Gaussian).

A Special Case

- In the case of white noise the above Equation (Equation-38) can be solved easily as follows:

$$\int_0^{T_{cs}} h_{opt}(z) \cdot \underbrace{\frac{N_0}{2} \cdot \delta(\tau - z)}_{=R_{nn}(\tau - z)} \cdot dz = s(T_{cs} - \tau) \qquad (39)$$

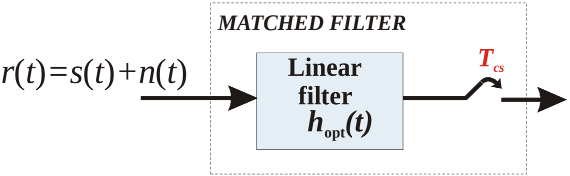
$$\Longrightarrow \quad h_{opt}(\tau) \cdot \frac{N_0}{2} = s(T_{cs} - \tau) \Longrightarrow$$

$$\boxed{h_{opt} = \frac{2}{N_0} \cdot s(T_{cs} - \tau)} \qquad (40)$$

- Be careful - SNR is not influenced by the factor $\frac{2}{N_0}$

Output SNR

- Consider next you have got the optimum impulse response $h_0(t)$ (by using Equation-38). This impulse response provides the maximum output-SNR which can be estimated as follows:



- Important Question: $\quad \text{SNR}_{\text{out}} = ?$
  we can answer this question as follows: $\searrow$

$$\text{o/p} = \int_0^{T_{cs}} h_{opt}(\tau).\underbrace{\underline{r(T_{cs}-\tau)}}_{\substack{\uparrow \\ s(T_{cs}-\tau)+n(T_{cs}-\tau)}} d\tau$$

$$= \underbrace{\int_0^{T_{cs}} h_{opt}(\tau).s(T_{cs}-\tau)d\tau}_{=\widetilde{s}(T_{cs})} + \underbrace{\int_0^{T_{cs}} h_{opt}(\tau).n(T_{cs}-\tau)d\tau}_{=\widetilde{n}(T_{cs})} \quad (41)$$

therefore,

$$SNR_{out_{\max}} = \frac{\mathcal{E}\{\widetilde{s}(T_{cs})^2\}}{\mathcal{E}\{\widetilde{n}(T_{cs})^2\}} = \frac{\widetilde{s}(T_{cs})^2}{\mathcal{E}\{\widetilde{n}(T_{cs})^2\}} = \quad (42)$$

$$= \cdots\cdots$$

...(for you)...

. . . (for you). . .

i.e. $$\boxed{\mathrm{SNR}_{\substack{out\\ \max}} = \frac{\widetilde{s}(T_{cs})^2}{\mathcal{E}\{\widetilde{n}(T_{cs})^2\}} = \int_0^{T_{cs}} h_{opt}(z) \cdot s(T_{cs} - z)dz} \qquad (43)$$

A Special Case: Output SNR (Signal plus white Noise)

- For white noise we have seen that: $\boxed{h_{opt}(z) = \frac{2}{N_0}.s(T_{cs} - z)}$
- Then Equation-43 becomes:

$$\begin{aligned}
\mathrm{SNR}_{\substack{out\\ \max}} &= \int_0^{T_{cs}} \underbrace{\frac{2}{N_0} \cdot s(T_{cs} - z)}_{=h_{opt}} \cdot s(T_{cs} - z) \cdot dz \\
&= \frac{2}{N_0} \int_0^{T_{cs}} s^2(T_{cs} - z) \cdot dz = \\
\Longrightarrow \mathrm{SNR}_{\substack{out\\ \max}} &= 2\frac{E}{N_0} \text{ for white noise} \qquad (44)
\end{aligned}$$

- Provided that the filter is matched to the signal, it is obvious from Equation-44 that

$$\begin{aligned}
\mathrm{SNIR}_{\substack{out\\ \max}} &\neq \mathrm{f}\{\text{signal waveform}\} \\
&\neq \mathrm{f}\{\text{signal bandwidth}\} \\
&\neq \mathrm{f}\{\text{peak power}\} \\
&\neq \mathrm{f}\{\text{time duration}\}
\end{aligned}$$

Approximation To Matched Filter Solution

- We have seen that the matched filter can be found by solving the following integral equation:

$$\boxed{\int_0^{T_{cs}} h_{opt}(z) \cdot R_{nn}(\tau - z) \cdot dz = s(T_{cs} - \tau);\ 0 \leq \tau \leq T_{cs}}$$

MATCHED FILTER GENERAL EQUATION
(a Fredholm Integral of the 1st kind.)

- However, it is very difficult to solve the above equation in a general case.

- N.B.: solution=easy when noise=white

- In order to find an approximation to matched filter solution, i.e.

$$w_{\mathbf{0}}(z) \simeq h_{opt}(z)$$

we need to relax the condition $0 \leq \tau \leq T_{cs}$ to $-\infty \leq \tau \leq \infty$

$$0 \leq \boldsymbol{\tau} \leq T_{\mathbf{cs}} \ \mathit{to} \ -\infty \leq \tau \leq \infty$$

.

Then

$$\int_{-\infty}^{\infty} \underline{w_0(z)} R_{nn}(\tau - z) \cdot dz = s(\underline{T_0} \quad - \tau) \tag{45}$$

Maximizes SNR at some time

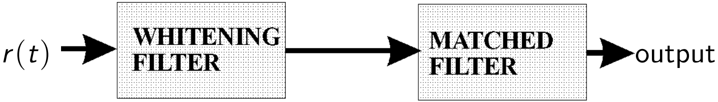
However the above integral is a convolution integral

$$\therefore \quad W_0(f) \cdot \mathrm{PSD}_n(f) = S^*(f).e^{-j2\pi f T_0} \tag{46}$$

$$\Longrightarrow \quad W_0(f) = \frac{S^*(f).e^{-j2\pi f T_0}}{\mathrm{PSD}_n(f)} \tag{47}$$

Whitening Filter

- $r(t) = \underbrace{s(t)}_{\substack{\uparrow \\ known}} + \overbrace{n(t)}^{\substack{non\text{-}white \\ \downarrow}} : i/p \rightarrow$ WHITENING FILTER $\longrightarrow$ MATCHED FILTER $\rightarrow o/p$: $s(t) + \underbrace{n_i(t)}_{\substack{\uparrow \\ white}}$
- whitening filter:
  - attenuates⟶regions in Frequ Domain in which <u>NOISE=LARGE</u>
  - accentuates⟶regions in Frequ Domain in which <u>NOISE=LOW</u>
- combined Whitening Filter-Matched Filter transfer function:



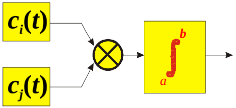
$$\text{combined transfer function} \equiv \mathbb{L}(f) = \frac{\mathbb{S}^*(f).e^{-j2\pi f T_0}}{\text{PSD}_n(f)}$$

Optimum M-ary Rx based on Signal Constellation

Introduction - Orthogonal Signals

- Two signals $c_i(t)$ and $c_j(t)$ are called orthogonal signals (i.e. $c_i(t) \perp c_j(t)$) in the interval $[a, b]$ iff:

$$\int_a^b c_i(t) \cdot c_j^*(t) dt = \begin{cases} E_{c_i} & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{48}$$



$$= \begin{cases} E_{c_i} & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

- Note:
  - if $E_{c_i} = 1$ then the signals are called orthonormal
  - $\int_a^b c_i(t).c_i^*(t)\, dt = E_{c_i} =$ the energy of $c_i(t)$ in the interval $[a, b]$.

The “Approximation Theorem” of an Energy Signal

- **Theorem** : Consider an energy signal $s(t)$. Furthermore consider a set of D orthogonal signals

$$\{c_1(t),\ c_2(t),\ ...,\ c_D(t)\}$$

Then

- The signal $s(t)$ can be approximated by a linear combination of the orthogonal signals $c_i(t)$ as follows:

$$\widehat{s}(t) = \sum_{i=1}^{D} w_i^* c_i(t) = \underline{w}_s^H \underline{c}(t) \tag{49}$$

where $\begin{cases} \underline{w}_s = [w_1, w_2, \ldots, w_D]^T \\ \underline{c}(t) = [c_1(t), c_2(t), \ldots, c_D(t)]^T \end{cases}$

- $\underline{w}_s$ is a vector of coefficients or weights

- The best approximation is the one which uses the following coefficients

$$\underline{w}_s = \frac{1}{\underline{E}_c} \odot \int_a^b s(t) \cdot \underline{c}^*(t)dt = \text{optimum weights} \tag{50}$$

where

$$\underline{E}_c = [E_{c_1}, E_{c_2}, \ldots, E_{c_D}]^T \tag{51}$$

and $\odot$ denotes Hadamard multiplication

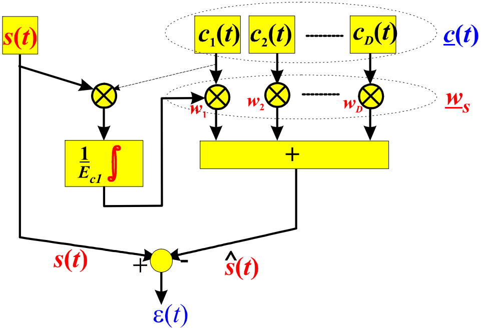
- The above coefficients are optimum in the sense that they minimize the energy E_ϵ of the error signal

$$\epsilon(t) = s(t) - \widehat{s}(t) \tag{52}$$

■

Comments on the "Approximation" Theorem

- The above theorem states that any energy signal can be approximated by a linear combination of a set of orthogonal signals and can be equivalent described by the following diagram:



- The energy of the error signal $\epsilon(t)$ is minimum when the set of signals

$$\{\epsilon(t), c_1(t), c_2(t), ..., c_D(t)\} = \text{is an orthogonal set}$$

$$\text{i.e} \quad \text{if} \quad \left\{ \begin{array}{c} \epsilon(t) \perp c_i(t), \forall i \\ \text{or} \\ \int_a^b \left(s(t) - \underline{w}_s^H \underline{c}(t)\right) .\underline{c}^*(t).dt = \underline{0} \end{array} \right\} \text{ then } \quad E_\epsilon = \min \tag{53}$$

- In general

$$s(t) \approx \widetilde{s}(t) \tag{54}$$

- However, if

$$s(t) = \widehat{s}(t) \tag{55}$$

then $\{c_1(t), c_2(t), ..., c_D(t)\} \triangleq$ a **complete set** of orthog. signals.

Gram-Schmidt Orthogonalization

- Suppose that we have a set of energy signals

$$\{s_1(t),\, s_2(t),\, ...,\, s_M(t)\} = \textbf{non-orthogonal}$$

and we wish to construct a set of orthogonal signals

$$\{c_1(t),\, c_2(t),\, ...,\, c_D(t)\} = \textbf{orthogonal}$$

- One popular approach to construct this orthogonal set is the **GRAM-SCHMIDT orthogonalisation** , described as follows:

1st orthogonal signal: $c_1(t) = \dfrac{s_1(t)}{\sqrt{\text{energy of numerator } E_{1,num}}}$

2nd orthogonal signal: $c_2(t) = \dfrac{s_2(t) - a_{21}\cdot c_1(t)}{\sqrt{\text{energy of numerator } E_{2,num}}}$

$a_{21} = \displaystyle\int_{-\infty}^{+\infty} s_2(t)\, c_1^*(t)\, dt$

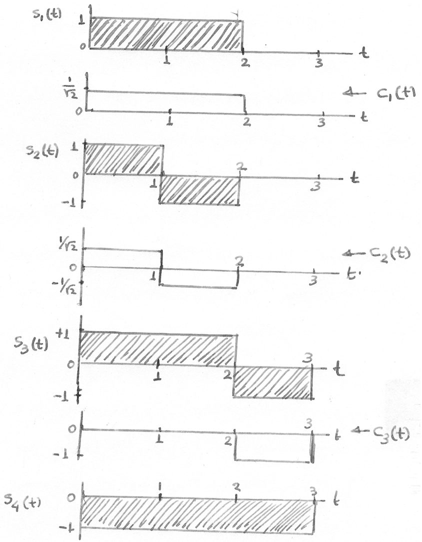
⋮

or, in general,

kth orthogonal signal: $c_k(t) = \dfrac{s_k(t) - \sum_{i=1}^{k-1} a_{ki}\cdot c_i(t)}{\sqrt{\text{energy of numerator } E_{k,num}}}$

$a_{ki} = \displaystyle\int_{-\infty}^{+\infty} s_k(t)\cdot c_i^*(t)\, dt$

Example



$c_4(t) = 0$

M-ary Signals: Energy and Cross-Correlation

- **Binary Com. Systems** : use 2 possible waveforms $\{s_0(t),\ s_1(t)\}$ or $\{s_1(t), s_2(t)\}$
- **M-ary Com. Systems** : use M possible waveforms $\{\, s_1(t), ..., s_M(t)\}$
  Note: these waveforms are energy signals of duration T_{cs}
- The M signals (or channel symbols) are characterized by their energy E_i

$$E_i = \int_0^{T_{cs}} s_i^2(t)dt \quad \forall i \tag{56}$$

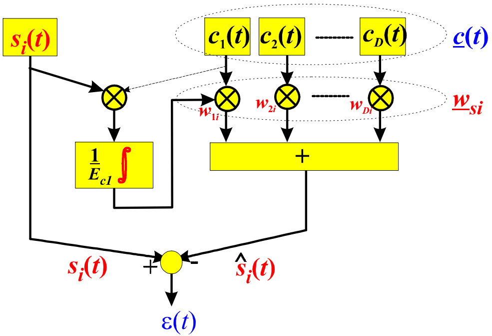
- Furthermore their similarity (or dissimilarity) is characterized by their cross-correlation

$$\rho_{ij} = \frac{1}{\sqrt{E_i E_j}} \int_0^{T_{cs}} s_i(t) \cdot s_j^*(t)dt \tag{57}$$

- The M signals may be also expressed, by using the Approx. Theorem described in the previous section, as a linear combination of a set of D orthogonal signals $\{c_k(t)\}$

$$s_i(t) = \underline{w}_{s_i}^H \underline{c}(t); \forall i \tag{58}$$

$$\text{where } \underline{c}(t) = [c_1(t), c_2(t), \ldots, c_D(t)]^T \tag{59}$$



- Let us define the following $(D \times D)$ matrix

$$
\begin{aligned}
\mathbb{R}_{cc} &= \int_0^{T_{cs}} \underline{c}(t).\underline{c}(t)^H.dt \\
&= \begin{bmatrix}
\int_0^{T_{cs}} c_1^2(t)dt, & \int_0^{T_{cs}} c_1(t)c_2^*(t)dt, & ..., & \int_0^{T_{cs}} c_1(t)c_D^*(t).dt \\
\int_0^{T_{cs}} c_2(t).c_1^*(t).dt, & \int_0^{T_{cs}} c_2^2(t).dt & ..., & \int_0^{T_{cs}} c_2(t)c_D^*(t).dt \\
..., & ..., & ..., & ... \\
\int_0^{T_{cs}} c_D(t).c_1^*(t).dt, & \int_0^{T_{cs}} c_D(t).c_2^*(t).dt, & ..., & \int_0^{T_{cs}} c_D^2(t).dt
\end{bmatrix}
\end{aligned}
$$

i.e.

$$
\mathbb{R}_{cc} = \int_0^{T_{cs}} \underline{c}(t).\underline{c}(t)^H.dt = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix} = \mathbb{I}_D \tag{60}
$$

- Then, by using Equation-56 in conjunction with Equation-58, the energy E_i of the signal $s_i(t), \forall i$, can be expressed as follows:

$$
\begin{aligned}
E_i &= \int_0^{T_{cs}} \underbrace{\underline{w}_{s_i}^H \underline{c}(t)}_{=s_i(t)} \cdot \underbrace{\underline{c}(t)^H \underline{w}_{s_i}}_{=s_i^*(t)} \cdot dt \\
&= \underline{w}_{s_i}^H \left(\int_0^{T_{cs}} \underline{c}(t)\underline{c}(t)^H dt \right) \underline{w}_{s_i} \\
&= \underline{w}_{s_i}^H \cdot \mathbb{R}_{cc} \cdot \underline{w}_{s_i} \\
&= \underline{w}_{s_i}^H \cdot \mathbb{I}_D \cdot \underline{w}_{s_i} \\
&= \underline{w}_{s_i}^H \underline{w}_{s_i} \\
&= \left\| \underline{w}_{s_i} \right\|^2
\end{aligned}
\tag{61}
$$

- Similarly, by using Equation-57 in conjunction with Equation-58 the cross-correlation coef. $\rho_{ij} \quad \forall ij$ becomes

$$
\begin{aligned}
\rho_{ij} &= \frac{1}{\sqrt{E_i E_j}} \int_0^{T_{cs}} \underline{w}_{s_i}^H \cdot \underline{c}(t) \cdot \underline{c}(t)^H \cdot \underline{w}_{s_j} \cdot dt \\
&= \frac{1}{\sqrt{E_i E_j}} \cdot \underline{w}_{s_i}^T \cdot \left(\int_0^{T_{cs}} \underline{c}(t) \cdot \underline{c}(t)^H \cdot dt \right) \cdot \underline{w}_{s_j} \\
&= \frac{1}{\sqrt{E_i E_j}} \underline{w}_{s_i}^H \cdot \mathbb{R}_{cc} \cdot \underline{w}_{s_j} \\
&= \frac{1}{\sqrt{E_i E_j}} \underline{w}_{s_i}^H \cdot \mathbb{I}_D \cdot \underline{w}_{s_j} \\
&= \frac{1}{\sqrt{E_i E_j}} \underline{w}_{s_i}^H \underline{w}_{s_j} = \frac{\underline{w}_{s_i}^H \underline{w}_{s_j}}{\left\| \underline{w}_{s_i} \right\| \left\| \underline{w}_{s_j} \right\|}
\end{aligned}
$$

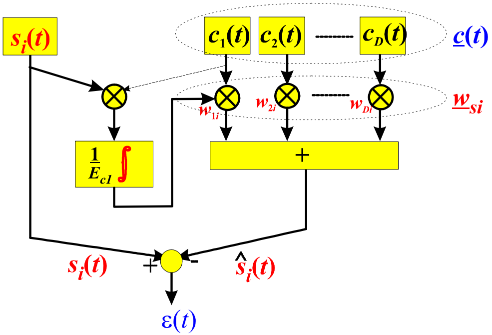
i.e.

$$E_i = \underline{w}_{s_i}^H \underline{w}_{s_i} = \left\| \underline{w}_{s_i} \right\|^2 \tag{62}$$

$$\rho_{ij} = \frac{1}{\sqrt{E_i E_j}} \underline{w}_{s_i}^H \cdot \underline{w}_{s_j} = \frac{\underline{w}_{s_i}^H \underline{w}_{s_j}}{\left\| \underline{w}_{s_i} \right\| \left\| \underline{w}_{s_j} \right\|} \tag{63}$$

Signal Constellation

- With reference to the figure below



- If the error signal $\epsilon(t) = 0$

then the knowledge of the vector $\underline{w}_s$ is as good as knowing the transmitted signal $s(t)$

or, in the case of M signals (M-ary system) $\{\text{knowledge of } s_i(t)\} = \{\text{knowledge of } \underline{w}_{s_i}\}, \forall i$

- Therefore, we may represent the signal $s_i(t)$ by a <u>point</u> $(\underline{w}_{s_i})$ in a D-dimensional Euclidean space with

$$D \leq M \tag{64}$$

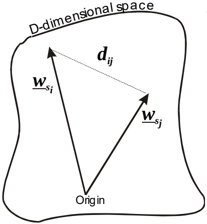
- The **set of points (vectors)** specified by the columns of the matrix

$$\mathbb{W} = \left[\underline{w}_{s_1}, \quad \underline{w}_{s_2}, \quad \ldots, \quad \underline{w}_{s_M}\right] \tag{65}$$

is known as **"signal constellation"** .

Distance between two M-ary signals

- The distance between two signals $s_i(t)$ and $s_j(t)$ is the Euclidean distance between their associate vectors $\underline{w}_{s_i}$ and $\underline{w}_{s_j}$



i.e.
$$\begin{aligned} d_{ij} &= \left\| \underline{w}_{s_i} - \underline{w}_{s_j} \right\| \\ &= \sqrt{\left(\underline{w}_{s_i} - \underline{w}_{s_j}\right)^H \left(\underline{w}_{s_i} - \underline{w}_{s_j}\right)} \\ &= \sqrt{\underline{w}_{s_i}^H \underline{w}_{s_i} + \underline{w}_{s_j}^H \underline{w}_{s_j} - 2\,\mathrm{Re}\left(\underline{w}_{s_i}^H \underline{w}_{s_j}\right)} \\ &= \sqrt{E_i + E_j - 2\rho_{ij}\sqrt{E_i E_j}} \end{aligned}$$

$$\boxed{d_{ij}^2 = E_i + E_j - 2\rho_{ij}\sqrt{E_i E_j}} \tag{66}$$

It is clear from the above that the Euclidean distance d_{ij} of two signals indicates, like the cross-correlation coefficient, the similarity or dissimilarity of the signals.

- An **Important Bound** involving d_{ij} :
  - If $p_{e,cs(s_i(t))}$ denotes the prob. of error associated with the channel symbol $s_i(t)$, it can be found that (see S. Haykin p.498)

$$p_{e,cs(s_i(t))} \leq \sum_{\substack{j=1 \\ j \neq i}}^{M} \mathrm{T}\left\{\frac{d_{ij}}{\sqrt{2N_0}}\right\} \tag{67}$$

Then, the symbol error probability $p_{e,sc}$ is bounded as follows:

$$\boxed{p_{e,sc} \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{j=1 \\ j \neq i}}^{M} \mathrm{T}\left\{\frac{d_{ij}}{\sqrt{2N_0}}\right\}} \tag{68}$$

  - N.B.: the following expression was used to go from Equ.67 to Equ.68

$$p_{e,sc} = \frac{1}{M} \sum_{i=1}^{M} p_{e,cs(s_i(t))} \text{ (see S.Haykin p.498)} \tag{69}$$

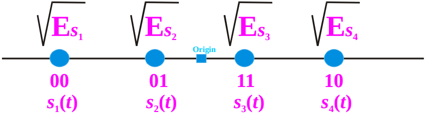
- "minimum distance": $d_{\min} \triangleq \min_{\forall i,j} \{d_{ij}\}$

Examples of Signal Constellation Diagram

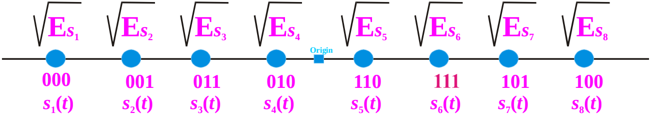
Consider an M-ary System having the following signals

$$\{s_1(t), s_2(t), \ldots, s_M(t)\} \text{ with } 0 \leq t \leq T_{cs}$$

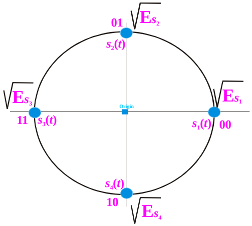
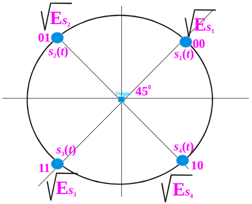
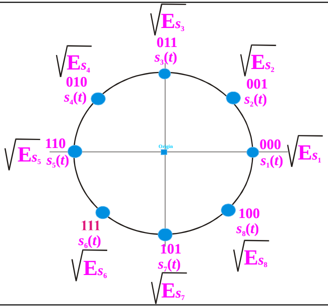
- M-ary ASK
  - channel symbols: $s_i(t) = A_i \cdot \cos(2\pi F_c t)$ where A_i =given= $2i - 1 - M$ (say)
  - dimensionality of signal space $= D = 1$
  - if $M = 4$ then



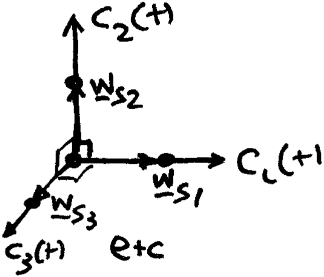
  - if $M = 8$ then



- M-ary PSK
  - channel symbols: $s_i(t) = A.\cos\left(2\pi F_c t + \frac{2\pi}{M}.(i-1) + \phi\right)$
    for $i = 1, 2, ..,$
  - dimensionality of signal-space $= D = 2$

$M = 4$ & $\phi = 0^\circ$	$M = 4$ & $\phi = 45^\circ$	$M = 8$ & $\phi = 0^\circ$
		

- M-ary FSK:
  very difficult to be represented using constellation diagram

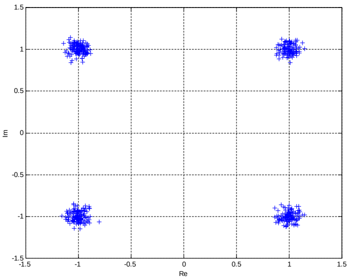


$$f_i - f_j = n \frac{1}{2T_{cs}}$$

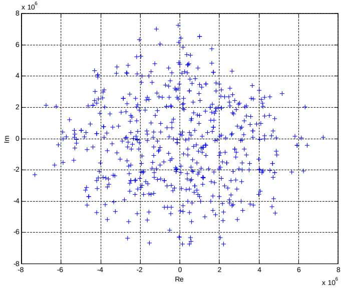
$$f_i + f_j = m \frac{1}{2T_{cs}}$$

$$n, m = \text{integers}$$

Examples of Plots of Decision Variables (QPSK- Receiver)



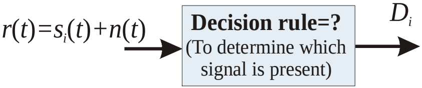
"good"



"bad"

Optimum M-ary Receiver using the Signal-Vectors

- Let us consider again the general M-ary problem:



where $s_i(t)$ = one of M signals (channel symbols)

- Hypotheses:

$$\begin{cases} H_1: & s_1(t) \text{ is present with probability } \Pr(H_1) \\ H_2: & s_2(t) \text{ is present with probability } \Pr(H_2) \\ \cdots & \cdots\cdots\cdots \\ H_M: & s_M(t) \text{ is present with probability } \Pr(H_M) \end{cases}$$

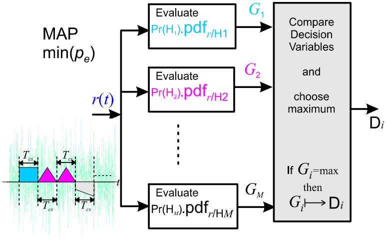
- we have seen that the MAP criterion, using the "CONTINUOUS SAMPLING" concept, is described by the following rule:
  Rule: choose the hypothesis H_i with the maximum

$$\Pr(H_i) \times \text{pdf}_{r/H_i}(r(t))$$

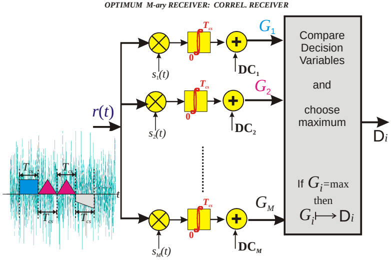
where

$$\begin{aligned} & \max\left\{\Pr(H_i) \times \text{pdf}_{r/H_i}(r(t))\right\} \\ = \;& \max\left\{\int_0^{T_{cs}} r(t)s_i^*(t)dt + \frac{N_0}{2}\ln(\Pr(H_i)) - \frac{1}{2}E_i\right\} \end{aligned} \tag{70}$$

- Remember that this rule can also be described by the following two equivalent receivers:



$\Longleftrightarrow$



where

$$\mathrm{DC}_i = \frac{N_0}{2}\ln(\Pr(H_i)) - \frac{1}{2}E_i \tag{71}$$

- Now let us use the Approx. Theorem of energy signals to represent the received signals $r(t)$ as well as the M-ary signal $s_i(t)$, $\forall i$,

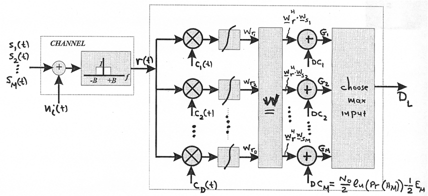
$$\text{i.e.} \quad \begin{cases} r(t) = \underline{w}_r^H \underline{c}(t) \\ s_i(t) = \underline{w}_{s_i}^H \underline{c}(t) \end{cases} \Longrightarrow \begin{aligned} s_i^*(t) &= (\underline{w}_{s_i}^H \underline{c}(t))^* \\ &= \underline{c}(t)^H \underline{w}_{s_i} \; \forall i \end{aligned}$$

In this case the MAP receiver (Equ-70) becomes

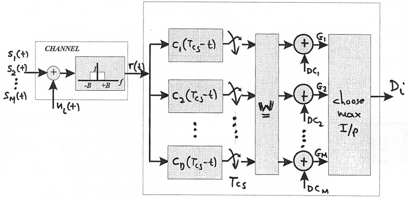
$$\begin{aligned}
& \max \left\{ \Pr\left(H_{\mathbf{i}}\right) \times \text{pdf}_{r/H_{\mathbf{i}}}(r(t)) \right\} \\
= \; & \max \left\{ \int_0^{T_{cs}} r(t) s_i^*(t) dt + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\} \\
= \; & \max \left\{ \int_0^{T_{cs}} \underline{w}_r^H \underline{c}(t) \cdot \underline{c}(t)^H \underline{w}_{s_i} dt + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\} \\
= \; & \max \left\{ \underline{w}_r^H \left(\int_0^{T_{cs}} \underline{c}(t) \cdot \underline{c}(t)^H dt \right) \underline{w}_{s_i} + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\} \\
= \; & \max \left\{ \underline{w}_r^H \underline{w}_{s_i} + \frac{N_0}{2} \ln(\Pr(H_i)) - \frac{1}{2} E_i \right\}
\end{aligned}$$

$$\text{i.e.}\quad \max\left\{\Pr(H_i)\times \text{pdf}_{r/H_i}(r(t))\right\} = \max\left\{\underline{w}_r^H\underline{w}_{s_i} + \frac{N_0}{2}\ln(\Pr(H_i)) - \frac{1}{2}E_i\right\} \tag{72}$$

The above equation may be implemented as follows:



or

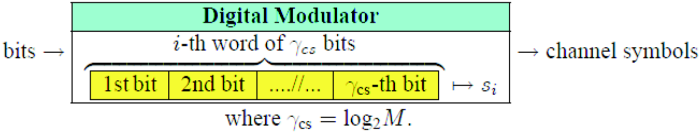


Encoding M-ary Signals

- The performance of M-ary systems is evaluated by means of the average *probability of symbol error* $p_{e,cs}$, which, for $M > 2$, is different than the average *probability of bit error* (or Bit-Error-Rate BER), p_e .

$$\text{That is } \begin{cases} p_{e,cs} \neq p_e & \text{for } M > 2 \\ p_{e,cs} = p_e & \text{for } M = 2 \end{cases} \tag{73}$$

  However, because we transmit binary data, the probability of bit error p_e is a more natural parameter for performance evaluation than $p_{e,cs}$.
- Although, these two probabilities are related, i.e. $p_e = \text{f}\{p_{e,cs}\}$ their relationship depends on the encoding approach which is employed by the digital modulator for mapping binary digits to M-ary signals (channel symbols)



Important Relationships Between BER and SER

- If the encoder provides a mapping where adjacent symbols differ in one binary digit then it can be proved (see Haykin p500) that

$$\frac{1}{\gamma_{cs}} \cdot p_{e,cs} \leq p_e \leq p_{e,cs} \tag{74}$$

e.g. M-ary PSK (for Gray code, otherwise difficult):

$$BER = p_e = \frac{1}{\gamma_{cs}} \cdot p_{e,cs} \tag{75}$$

  - N.B.: Note that Gray encoder provides a mapping where adjacent symbols differ in one binary digit. This property is very important because the most likely errors (due to channel noise effects) are related with an erroneous selection (wrong decision) of an adjacent symbol.

- If all M-ary signals are equally likely, then it can be proved (see Haykin p.500) that

$$p_e = \frac{2^{\gamma_{cs}-1}}{2^{\gamma_{cs}}-1} \cdot p_{e,cs} \tag{76}$$

which, for large γ_{cs} is simplified to

$$p_e = \frac{1}{2} \cdot p_{e,cs} \tag{77}$$

- Example:

$$M\text{-ary FSK:} \qquad BER = p_e = \frac{2^{\gamma_{cs}-1}}{2^{\gamma_{cs}}-1} \cdot p_{e,cs} \tag{78}$$

Appendix-1: Proof of Equation-26

The above rule can be rewritten as follows:

$$
\begin{aligned}
& \max\left\{\Pr(H_i) \times \mathrm{pdf}_{r/H_\mathbf{i}}(r(t))\right\} \\
= \; & \max\left\{\ln\left(\Pr(H_i)\right) + \ln\left(const.\right) - \frac{1}{N_0}\int_0^{T_{cs}} (r(t) - s_i(t))^2 dt\right\} \\
= \; & \max\left\{\ln\left(\Pr(H_i)\right) - \frac{1}{N_0}\int_0^{T_{cs}} (r(t) - s_i(t))^2 dt\right\} \\
= \; & \max\left\{\ln\left(\Pr(H_i)\right) - \frac{1}{N_0}\int_0^{T_{cs}} r^2(t)dt - \frac{1}{N_0}\int_0^{T_{cs}} s_i^2(t)dt + \frac{2}{N_0}\int_0^{T_{cs}} r(t)s_i^*(t)dt\right\} \\
= \; & \max\left\{\ln\left(\Pr(H_i)\right) - \frac{1}{N_0}E_r - \frac{1}{N_0}E_i + \frac{2}{N_0}\int_0^{T_{cs}} r(t)s_i^*(t)dt\right\} \\
= \; & \max\left\{\frac{N_0}{2}\ln\left(\Pr(H_i)\right) - \frac{1}{2}E_i + \int_0^{T_{cs}} r(t)s_i^*(t)dt\right\} \\
= \; & \max\left\{\int_0^{T_{cs}} r(t)s_i^*(t)dt + \frac{N_0}{2}\ln\left(\Pr\left(H_i\right)\right) - \frac{1}{2}E_i\right\}
\end{aligned}
$$

Appendix-2: Walsh-Hadamard Orthogonal Sets and Signals

- $\mathbb{H}_1 = [1]$

  $$\mathbb{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (2 \times 2) \text{ matrix}$$

  $$\mathbb{H}_{64} = \underbrace{\mathbb{H}_2 \otimes \mathbb{H}_2 \otimes \mathbb{H}_2 \otimes \mathbb{H}_2 \otimes \mathbb{H}_2 \otimes \mathbb{H}_2}_{6-\text{times}}$$

  where $\otimes$ denotes the Kronecker product of two matrices

  e.g. for two matrices $\mathbb{A}$ and $\mathbb{B}$

  $$\text{if } \mathbb{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ then } \mathbb{A} \otimes \mathbb{B} = \begin{bmatrix} A_{11}\mathbb{B} & A_{12}\mathbb{B} \\ A_{21}\mathbb{B} & A_{22}\mathbb{B} \end{bmatrix}$$

- Properties:
  - $\mathbb{H}_{64}^T \, \mathbb{H}_{64} = 64 \, \mathbb{I}_{64}$
  - Let $\underline{h}_i$ denote the i-th column of $\mathbb{H}_{64}$

    i.e. $\mathbb{H}_{64} = \left[\underline{h}_1, \quad \underline{h}_2, \quad \ldots, \quad \underline{h}_{64} \right]$

- The following example illustrates an application of "Walsh Hadamard" in a mobile communications where (each user in the same cell and same frequency channel is assigned one column of Walsh matrix,

$$\underbrace{....\ m_1, m_2, m_3,\ ....,\ m_k,\ ....}_{\text{input binary sequence}} \rightarrow \boxed{\begin{array}{c}\textbf{Transmitter}\\ \text{Walsh}\\ \text{2nd column}\\ \underline{\mathbf{h}}_2\end{array}} \rightarrow \underbrace{....\ m_1\underline{\mathbf{h}}_2^T, m_2\underline{\mathbf{h}}_2^T, m_3\underline{\mathbf{h}}_2^T,\ ....,\ m_k.\underline{\mathbf{h}}_2^T\ \ ....}_{\text{output binary sequence}}$$

$$\underbrace{....\ m_1\underline{\mathbf{h}}_2^T, m_2\underline{\mathbf{h}}_2^T, m_3\underline{\mathbf{h}}_2^T,\ ....,\ m_k\underline{\mathbf{h}}_2^T\ \ ....}_{\text{input binary sequence}} \rightarrow \left\{\begin{array}{l} \rightarrow \boxed{\begin{array}{c}\textbf{Receiver-A}\\ \text{Walsh}\\ \text{2nd column}\\ \underline{\mathbf{h}}_2/64\end{array}} \rightarrow \underbrace{....\ m_1, m_2, m_3,\ ....,\ m_k,\ ....}_{\text{output binary sequence}} \\ \rightarrow \boxed{\begin{array}{c}\textbf{Receiver-B}\\ \text{Walsh}\\ \text{3rd column}\\ \underline{\mathbf{h}}_3/64\end{array}} \rightarrow \underbrace{....\ 0,\ 0,\ \ 0,\ ....,\ 0,\ ....}_{\text{output binary sequence}} \end{array}\right.$$

Appendix-3: IS95 - Walsh Function of order-64

Walsh functions of order 64, as indexed in IS-95 (W_i is the Walsh notation, and H_i is the Hadamard notation)

W_0	H_0	0000000000000000	0000000000000000	0000000000000000	0000000000000000
W_{63}	H_1	0101010101010101	0101010101010101	0101010101010101	0101010101010101
W_{31}	H_2	0011001100110011	0011001100110011	0011001100110011	0011001100110011
W_{32}	H_3	0110011001100110	0110011001100110	0110011001100110	0110011001100110
W_{15}	H_4	0000111100001111	0000111100001111	0000111100001111	0000111100001111
W_{48}	H_5	0101101001011010	0101101001011010	0101101001011010	0101101001011010
W_{16}	H_6	0011110000111100	0011110000111100	0011110000111100	0011110000111100
W_{47}	H_7	0110100101101001	0110100101101001	0110100101101001	0110100101101001
W_7	H_8	0000000011111111	0000000011111111	0000000011111111	0000000011111111
W_{56}	H_9	0101010110101010	0101010110101010	0101010110101010	0101010110101010
W_{24}	H_{10}	0011001111001100	0011001111001100	0011001111001100	0011001111001100
W_{39}	H_{11}	0110011010011001	0110011010011001	0110011010011001	0110011010011001
W_8	H_{12}	0000111111110000	0000111111110000	0000111111110000	0000111111110000
W_{55}	H_{13}	0101101010100101	0101101010100101	0101101010100101	0101101010100101
W_{23}	H_{14}	0011110011000011	0011110011000011	0011110011000011	0011110011000011
W_{40}	H_{15}	0110100110010110	0110100110010110	0110100110010110	0110100110010110
W_3	H_{16}	0000000000000000	1111111111111111	0000000000000000	1111111111111111
W_{60}	H_{17}	0101010101010101	1010101010101010	0101010101010101	1010101010101010
W_{28}	H_{18}	0011001100110011	1100110011001100	0011001100110011	1100110011001100
W_{35}	H_{19}	0110011001100110	1001100110011001	0110011001100110	1001100110011001
W_{12}	H_{20}	0000111100001111	1111000011110000	0000111100001111	1111000011110000
W_{51}	H_{21}	0101101001011010	1010010110100101	0101101001011010	1010010110100101
W_{19}	H_{22}	0011110000111100	1100001111000011	0011110000111100	1100001111000011
W_{44}	H_{23}	0110100101101001	1001011010010110	0110100101101001	1001011010010110
W_4	H_{24}	0000000011111111	1111111100000000	0000000011111111	1111111100000000
W_{59}	H_{25}	0101010110101010	1010101001010101	0101010110101010	1010101001010101
W_{27}	H_{26}	0011001111001100	1100110000110011	0011001111001100	1100110000110011
W_{36}	H_{27}	0110011010011001	1001100101100110	0110011010011001	1001100101100110
W_{11}	H_{28}	0000111111110000	1111000000001111	0000111111110000	1111000000001111
W_{52}	H_{29}	0101101010100101	1010010101011010	0101101010100101	1010010101011010
W_{20}	H_{30}	0011110011000011	1100001100111100	0011110011000011	1100001100111100
W_{43}	H_{31}	0110100110010110	1001011001101001	0110100110010110	1001011001101001
W_1	H_{32}	0000000000000000	0000000000000000	1111111111111111	1111111111111111
W_{62}	H_{33}	0101010101010101	0101010101010101	1010101010101010	1010101010101010
W_{30}	H_{34}	0011001100110011	0011001100110011	1100110011001100	1100110011001100
W_{33}	H_{35}	0110011001100110	0110011001100110	1001100110011001	1001100110011001
W_{14}	H_{36}	0000111100001111	0000111100001111	1111000011110000	1111000011110000
W_{49}	H_{37}	0101101001011010	0101101001011010	1010010110100101	1010010110100101
W_{17}	H_{38}	0011110000111100	0011110000111100	1100001111000011	1100001111000011
W_{46}	H_{39}	0110100101101001	0110100101101001	1001011010010110	1001011010010110
W_6	H_{40}	0000000011111111	0000000011111111	1111111100000000	1111111100000000
W_{57}	H_{41}	0101010110101010	0101010110101010	1010101001010101	1010101001010101
W_{25}	H_{42}	0011001111001100	0011001111001100	1100110000110011	1100110000110011
W_{38}	H_{43}	0110011010011001	0110011010011001	1001100101100110	1001100101100110
W_9	H_{44}	0000111111110000	0000111111110000	1111000000001111	1111000000001111
W_{54}	H_{45}	0101101010100101	0101101010100101	1010010101011010	1010010101011010
W_{22}	H_{46}	0011110011000011	0011110011000011	1100001100111100	1100001100111100
W_{41}	H_{47}	0110100110010110	0110100110010110	1001011001101001	1001011001101001
W_2	H_{48}	0000000000000000	1111111111111111	1111111111111111	0000000000000000
W_{61}	H_{49}	0101010101010101	1010101010101010	1010101010101010	0101010101010101
W_{29}	H_{50}	0011001100110011	1100110011001100	1100110011001100	0011001100110011
W_{34}	H_{51}	0110011001100110	1001100110011001	1001100110011001	0110011001100110
W_{13}	H_{52}	0000111100001111	1111000011110000	1111000011110000	0000111100001111
W_{50}	H_{53}	0101101001011010	1010010110100101	1010010110100101	0101101001011010
W_{18}	H_{54}	0011110000111100	1100001111000011	1100001111000011	0011110000111100
W_{45}	H_{55}	0110100101101001	1001011010010110	1001011010010110	0110100101101001
W_5	H_{56}	0000000011111111	1111111100000000	1111111100000000	0000000011111111
W_{58}	H_{57}	0101010110101010	1010101001010101	1010101001010101	0101010110101010
W_{26}	H_{58}	0011001111001100	1100110000110011	1100110000110011	0011001111001100
W_{37}	H_{59}	0110011010011001	1001100101100110	1001100101100110	0110011010011001
W_{10}	H_{60}	0000111111110000	1111000000001111	1111000000001111	0000111111110000
W_{53}	H_{61}	0101101010100101	1010010101011010	1010010101011010	0101101010100101
W_{21}	H_{62}	0011110011000011	1100001100111100	1100001100111100	0011110011000011
W_{12}	H_{63}	0110100110010110	1001011001101001	1001011001101001	0110100110010110