EE401: Advanced Communication Theory

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Introductory Concepts

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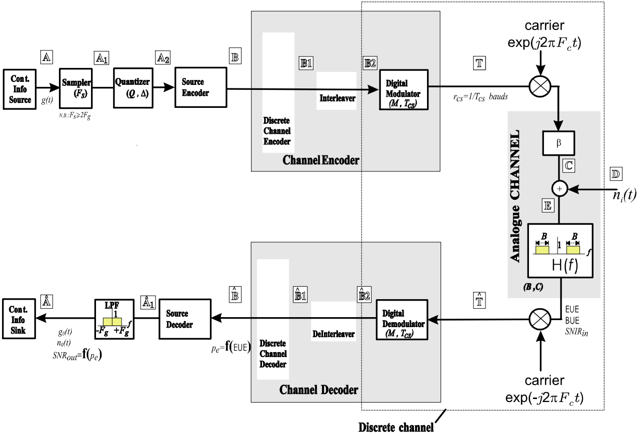
Introduction

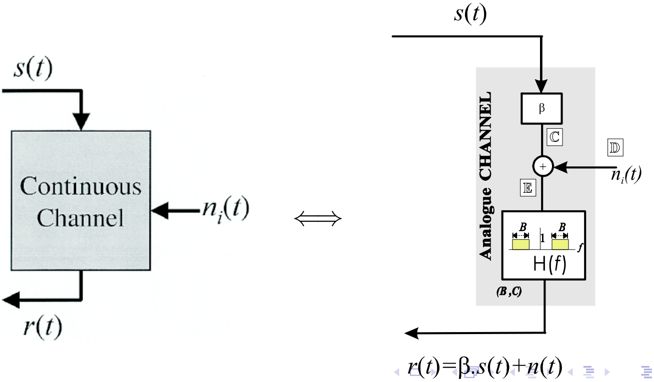
With reference to the following block structure of a Dig. Comm. System (DCS), this topic is concerned with the basics of both continuous and discrete communication channels.



- Just as with sources, communication channels are either
  - discrete channels, or
  - continuous channels
    1. wireless channels (in this case the whole DCS is known as a Wireless DCS)
    2. wireline channels (in this case the whole DCS is known as a Wireline DCS)
- Note that a continuous channel is converted into (becomes) a discrete channel when a **digital modulator** is used to feed the channel and a **digital demodulator** provides the channel output.
- Examples of channels - with reference to DCS shown in previous page,
  - discrete channels:
    - ★ input: $\mathrm{A2}$ - output: $\widehat{\mathrm{A}}2$ (alphabet: levels of quantiser - Volts)
    - ★ input: $\mathrm{B2}$ - output: $\widehat{\mathrm{B}}2$ (alphabet: binary digits or binary codewords)
  - continuous channels:
    - ★ input: $\mathrm{A1}$ - output: $\widehat{\mathrm{A}}1$, (Volts) - continuous channel (baseband)
    - ★ input: T, - output: $\widehat{\mathrm{T}}$ (Volts) - continuous channel (baseband),
    - ★ input: $\mathrm{T1}$ - output: $\widehat{\mathrm{T}}1$ (Volts) - continuous channel (bandpass).

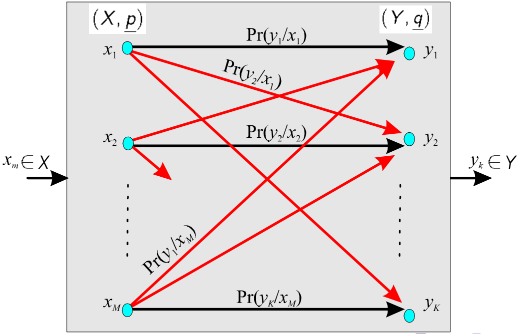
Continuous Channels

- A continuous communication channel (which can be regarded as an analogue channel) is described by
  - an input ensemble $(s(t), \text{pdf}_s(s))$ and $\text{PSD}_s(f)$
  - an output ensemble, $(r(t), \text{pdf}_r(r))$
  - the channel noise (AWGN) $n_i(t)$ and β,
  - the channel bandwidth B and channel capacity C.



Discrete Channels

- A discrete communication channel has a discrete input and a discrete output where
  - the symbols applied to the channel input for transmission are drawn from a finite alphabet, described by an input ensemble $(X, \underline{p})$ while
  - the symbols appearing at the channel output are also drawn from a finite alphabet, which is described by an output ensemble $(Y, \underline{q})$
  - the channel transition probability matrix $\mathbb{F}$.



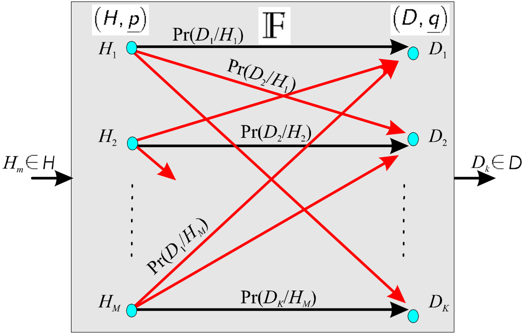
- In many situations the input and output alphabets X and Y are identical but in the general case these are different. Instead of using X and Y, it is common practice to use the symbols H and D and thus define the two alphabets and the associated probabilities as

$$\text{input:}\quad H = \{H_1, H_2, ..., H_M\} \quad \underline{p} = [\overbrace{\Pr(H_1)}^{\triangleq p_1}, \overbrace{\Pr(H_2)}^{\triangleq p_2}, ..., \overbrace{\Pr(H_M)}^{\triangleq p_M}]^T$$

$$\text{output:}\quad D = \{D_1, D_2, ..., D_K\} \quad \underline{q} = [\overbrace{\Pr(D_1)}^{\triangleq q_1}, \overbrace{\Pr(D_2)}^{\triangleq q_2}, ..., \overbrace{\Pr(D_K)}^{\triangleq q_K}]^T$$

where p_m abbreviates the probability $\Pr(H_m)$ that the symbol H_m may appear at the input while q_k abbreviates the probability $\Pr(D_k)$ that the symbol D_k may appear at the output of the channel.

- The probabilistic relationship between input symbols H and output symbols D is described by the so-called channel transition probability matrix $\mathbb{F}$, which is defined as follows:



$$\mathbb{F} = \begin{bmatrix} \Pr(D_1|H_1), & \Pr(D_1|H_2), & ..., & \Pr(D_1|H_M) \\ \Pr(D_2|H_1), & \Pr(D_2|H_2), & ..., & \Pr(D_2|H_M) \\ ..., & ..., & ..., & ... \\ \Pr(D_K|H_1), & \Pr(D_K|H_2), & ..., & \Pr(D_K|H_M) \end{bmatrix} \quad (1)$$

- $\Pr(D_k | H_m)$ denotes the probability that symbol $D_k \in D$ will appear at the channel output, given that $H_m \in H$ was applied to the input.
- The input ensemble $(H, \underline{p})$, the output ensemble $(D, \underline{q})$ and the matrix $\mathbb{F}$ fully describe the functional properties of the channel.
- The following expression describes the relationship between $\underline{q}$ and $\underline{p}$

$$\underline{q} = \mathbb{F}.\underline{p} \tag{2}$$

- Note that in a **noiseless channel**

$$\begin{aligned} D &= H \\ \underline{q} &= \underline{p} \end{aligned} \tag{3}$$

i.e the matrix $\mathbb{F}$ is an identity matrix

$$\mathbb{F} = \mathbb{I}_M \tag{4}$$

Joint transition Probability Matrix

- The joint probabilistic relationship between input channel symbols $H = \{H_1, H_2, ..., H_M\}$ and output channel symbols $D = \{D_1, D_2, ..., D_M\}$, is described by the so-called joint-probability matrix,

$$\mathbb{J} \triangleq \begin{bmatrix} \Pr(H_1, D_1), & \Pr(H_1, D_2), & ..., & \Pr(H_1, D_K) \\ \Pr(H_2, D_1), & \Pr(H_2, D_2), & ..., & \Pr(H_2, D_K) \\ ..., & ..., & ..., & ... \\ \Pr(H_M, D_1), & \Pr(H_M, D_2), & ..., & \Pr(H_M, D_K) \end{bmatrix}^T \quad (5)$$

- $\mathbb{J}$ is related to the forward transition probabilities of a channel with the following expression (compact form of Bayes' Theorem):

$$\mathbb{J} = \mathbb{F}.\underbrace{\begin{bmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_M \end{bmatrix}}_{\triangleq \text{diag}(\underline{p})} = \mathbb{F}.\text{diag}(\underline{p}) \quad (6)$$

- Note: This is equivalent to a new (joint) source having alphabet

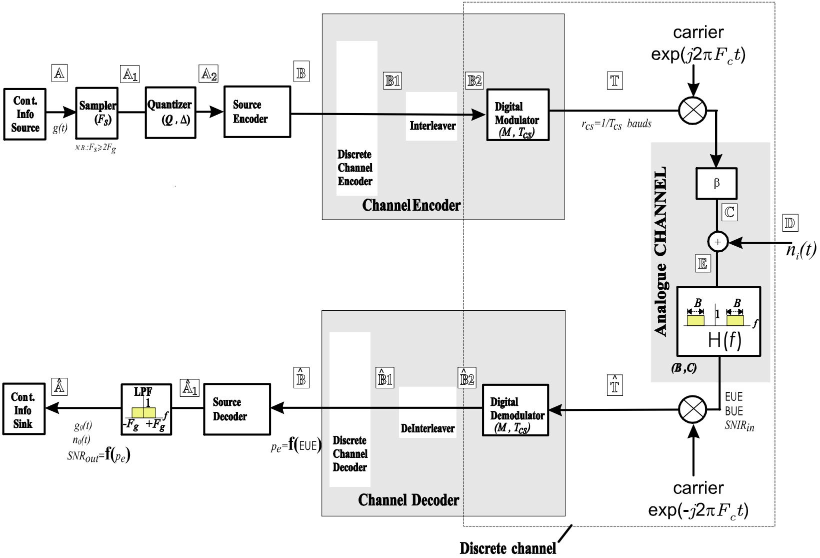
$$\{(H_1, D1), (H_1, D_2), ..., (H_M, D_K)\}$$

and ensemble (joint ensemble) defined as follows

$$(H \times D, \mathbb{J}) = \left\{ \begin{array}{c} (H_1, D_1), \Pr(H_1, D_1) \\ (H_1, D_2), \Pr(H_1, D_2) \\ ... \\ (H_m, D_k), \Pr(H_m, D_k) \\ ... \\ (H_M, D_K), \Pr(H_M, D_K) \end{array} \right\} \tag{7}$$

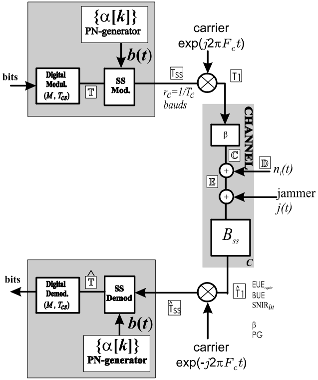
$$= \left\{ \left((H_m, D_k), \underbrace{\Pr(H_m, D_k)}_{=J_{km}} \right), \forall mk : 1 \leq m \leq M, 1 \leq k \leq K \right\}$$

Block Structure of a Digital Comm System

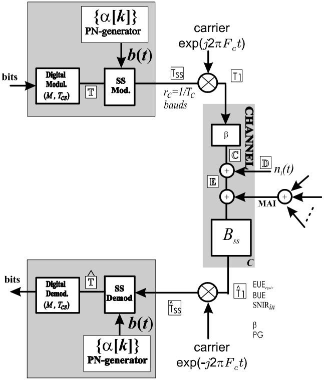


Block Structure of a Spread Spectrum Comm System

SSS:

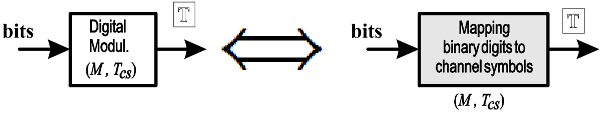


CDMA:

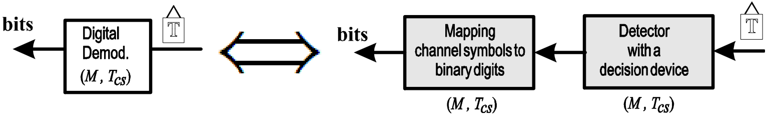


Digital Modulators/Demodulators

- A digital modulator is described by M **different channel symbols** . These channel symbols are **ENERGY SIGNALS** of duration T_{cs}.
- Digital Modulator:

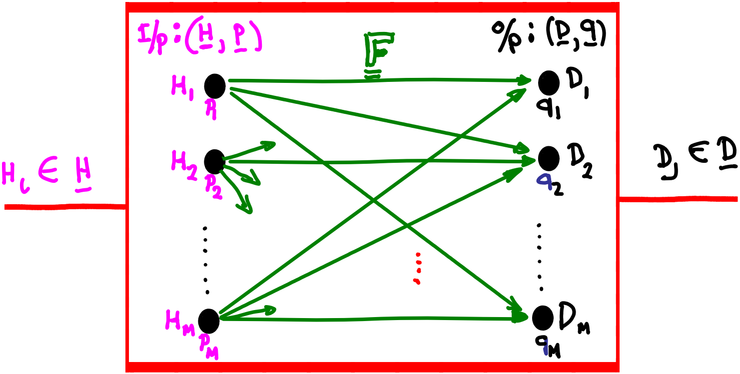


Digital Demodulator:



- Note:
  It is common practice to ignore the up/down conversion and to work in 'baseband'.

- If $M = 2$ $\Rightarrow$**Binary** Digital Modulator $\Rightarrow$**Binary** Comm. System
- If $M > 2$ $\Rightarrow$**M-ary** Digital Modulator $\Rightarrow$**M-ary** Comm. System



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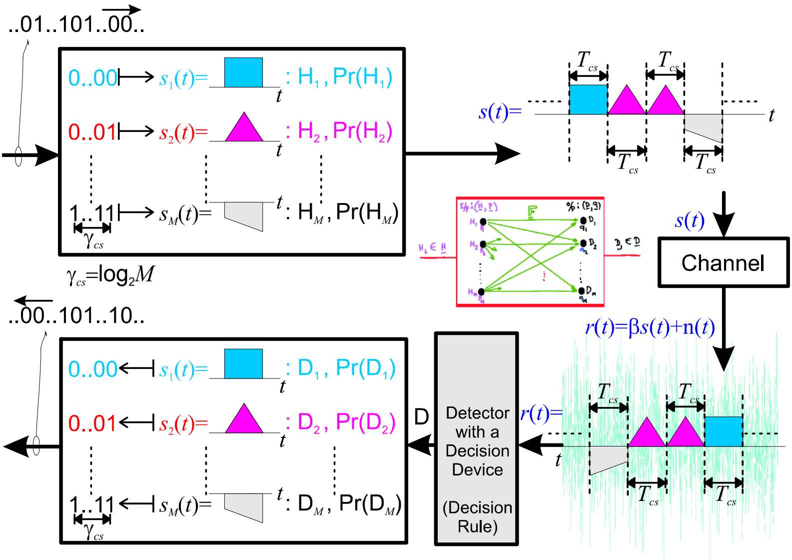
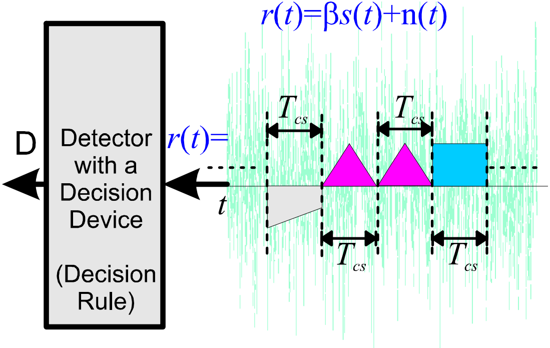


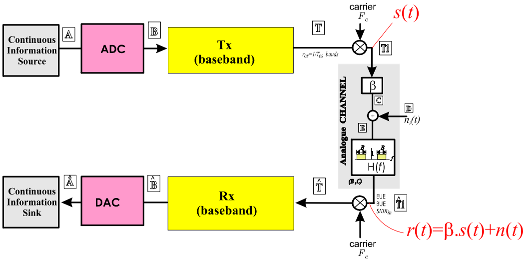
..01..101..00..
0..00 ⟼ $s_1(t)=$: H_1, Pr(H_1)
0..01 ⟼ $s_2(t)=$: H_2, Pr(H_2)
1..11 ⟼ $s_M(t)=$: H_M, Pr(H_M)
γ_{cs}
$\gamma_{cs}=\log_2 M$
$s(t)=$
T_{cs}
$s(t)$
Channel
$r(t)=\beta s(t)+n(t)$
$r(t)=$
Detector with a Decision Device
(Decision Rule)
D
..00..101..10..
0..00 ⟻ $s_1(t)=$: D_1, Pr(D_1)
0..01 ⟻ $s_2(t)=$: D_2, Pr(D_2)
1..11 ⟻ $s_M(t)=$: D_M, Pr(D_M)
t

- In this topic we will focus on the following block of a digital communication system ($\beta = 1$ is assumed):



This is the **'heart'** of a communication system and some elements of detection/decision theory will be employed for its investigation.

Appendices - A: Basic Performance Criteria

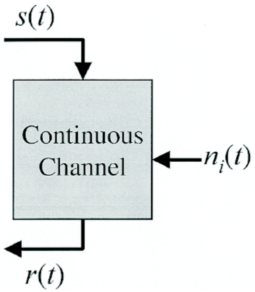


$$\mathrm{SNR}_{in} = \frac{\text{Power of signal at } \widehat{\mathrm{T}}}{\text{Power of noise at } \widehat{\mathrm{T}}} = \frac{\mathcal{E}\left\{(\beta s(t))^2\right\}}{\mathcal{E}\left\{n(t)^2\right\}} = \frac{\beta^2 P_s}{\underbrace{N_0 B}_{\triangleq P_n}} \tag{8}$$

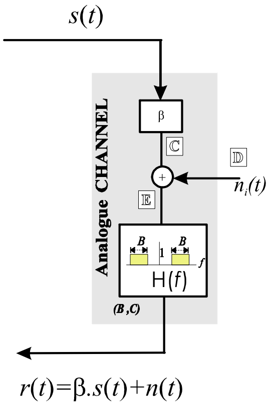
$$p_e = \text{BER at point } \widehat{\mathrm{B}} \tag{9}$$

$$\mathrm{SNR}_{out} = \frac{\text{Power of signal at } \widehat{\mathrm{A}}}{\text{Power of noise at } \widehat{\mathrm{A}}} = \underbrace{\mathrm{f}\{p_e\}}_{\text{denotes: a function of } p_e} \tag{10}$$

B: Additive Noise

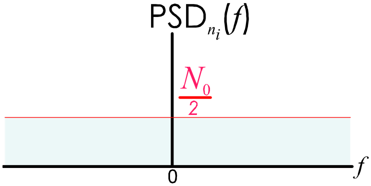


$\Longleftrightarrow$



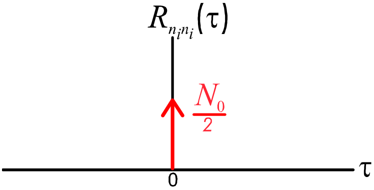
- types of channel signals
  - $s(t), r(t), n(t)$: bandpass
  - $n_i(t) = AWGN$: allpass

- $n_i(t)$
  - it is a random all-Pass signal
  - its Power Spectral Density is "White" i.e. "flat". That is,



$$\mathrm{PSD}_{n_i}(f) = \frac{N_0}{2} \quad (11)$$

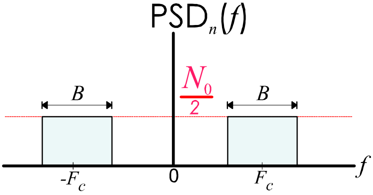
  - its amplitude probability density function is Gaussian
  - its Autocorrelation function $\left(\text{i.e. } \mathrm{FT}^{-1}\left\{\mathrm{PSD}(f)\right\}\right)$ is:



$$R_{n_i n_i}(\tau) = \frac{N_0}{2}\delta(\tau) \quad (12)$$

Bandlimited AWGN

- $n(t)$
  - it is a random Band-Pass signal of bandwidth B (equal to the channel bandwidth)
  - its Power Spectral Density is "bandmimited White". That is,



$$\mathrm{PSD}_{n_i}(f) = \frac{N_0}{2}\left(\mathrm{rect}\left\{\frac{f+F_c}{B}\right\} + \mathrm{rect}\left\{\frac{f-F_c}{B}\right\}\right) \tag{13}$$

  - Its power is:

$$\begin{aligned} P_n &= \sigma_n^2 = \int_{-\infty}^{\infty} PSD_{n_i}(f).df = \frac{N_0}{2} \times B \times 2 \\ &\Rightarrow P_n = N_0 B \end{aligned} \tag{14}$$

- more on $n(t)$:
  - its amplitude probability density function is Gaussian

$$\text{pdf}_n = \mathrm{N}(0, \sigma_n^2 = N_0 B) \tag{15}$$

  - It is also known as **bandlimited-AWGN**
  - It can be written as follows:

$$\begin{aligned} n(t) &= n_c(t)\cos(2\pi F_c t) - n_s(t)\sin(2\pi F_c t) && (16)\\ &= \underbrace{\sqrt{n_c^2(t) + n_s^2(t)}}_{\triangleq r_n(t)}\cos(2\pi F_c t + \phi_n(t)) && (17)\end{aligned}$$

  where

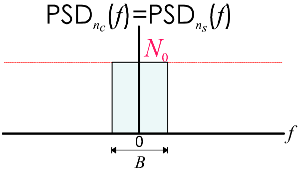
    - ★ $n_c(t)$ and $n_s(t)$ are random signals - with pdf=Gaussian distribution
    - ★ $r_n(t)$ is a random signal - with pdf=Rayleigh distribution
    - ★ $\phi_n(t)$ is a random signal - with pdf=uniform distribution: $[0, 2\pi])$

  N.B.: all the above are low pass signals & appear at Rx's o/p

  - Equ. 16 is known as **Quadrature Noise Representation** .

"I" and "Q" Noise Components

- $n_c(t)$ (i.e. "I") and $n_s(t)$ (i.e. "Q")
  - their Power Spectral Densities are:



$$\mathrm{PSD}_{n_c}(f) = \mathrm{PSD}_{n_s}(f) = N_0 \mathrm{rect}\left\{\frac{f}{B}\right\} \tag{18}$$

  - their power are:

$$\begin{aligned} P_{n_c} &= \sigma_{n_c}^2 = \int_{-\infty}^{\infty} PSD_{n_c}(f).df = N_0 \times B \\ \Rightarrow & \quad P_{n_c} = P_{n_s} = P_n = N_0 B \end{aligned} \tag{19}$$

  - Amplitude probability density functions: Gaussian,

$$\mathrm{pdf}_{n_c} = \mathrm{pdf}_{n_s} = \mathrm{N}(0, N_0 B) \tag{20}$$

  - are uncorrelated i.e. $\mathcal{E}\{n_c(t).n_s(t)\} = 0$

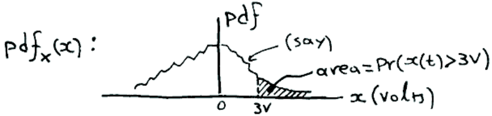
Tail function (or Q-function) for Gaussian Signals

Probablity and Probability-Density-Function (pdf)

- Consider a random signal $x(t)$ with a known amplitude probability density function $\mathrm{pdf}_x(\mathrm{x})$ - not necessarily Gaussian. Then the probability that the amplitude of $x(t)$ is greater than A Volts (say) is given as follows:

$$\Pr(x(t) > A) = \int_A^\infty \mathrm{pdf}_x(\mathrm{x}).d\mathrm{x} \tag{21}$$

- e.g.
  if $A = 3V \Rightarrow \Pr(x(t) > 3V) = \int_3^\infty \mathrm{pdf}_x(\mathrm{x}).d\mathrm{x} = \text{highlighted area}$



Gaussian pdf and Tail function

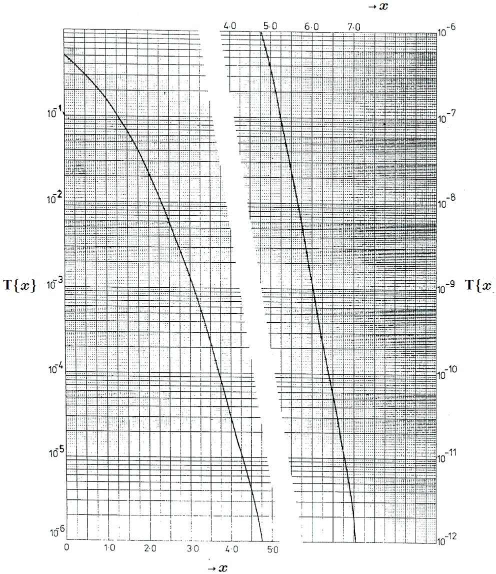
- If $\mathrm{pdf}_x(\mathrm{x}) =$ Gaussian of mean μ_x and standard deviation σ_x (notation used: $\mathrm{pdf}_x(\mathrm{x}) = \mathrm{N}(\mu_x, \sigma_x^2)$), then the above area is defined as the Tail-function (or Q-function)

$$\Pr(x(t) > A) = \int_A^\infty \mathrm{pdf}_x(\mathrm{x}).dx \triangleq \mathrm{T}\left\{\frac{|A-\mu_x|}{\sigma_x}\right\} \tag{22}$$

- e.g.
  - if $\mathrm{pdf}_x(\mathrm{x}) = \mathrm{N}(1,4)$ - i.e. $\mu_x = 0, \sigma_x = 2$ - and $A = 3V$ then $\Pr(x(t) > 3V) = \int_3^\infty \mathrm{pdf}_x(\mathrm{x}).dx \triangleq \mathrm{T}\left\{\frac{|3-1|}{2}\right\} = \mathrm{T}\{1\}$
  - if $\mathrm{pdf}_x(\mathrm{x}) = \mathrm{N}(0,1)$ and $A = 3V$ then $\Pr(x(t) > 3V) = \int_3^\infty \mathrm{pdf}_x(\mathrm{x}).dx \triangleq \mathrm{T}\{3\}$
- The Tail function graph is given in the next page

The graph below shows the Tail function $\mathbf{T}\{x\}$ which represents the area from x to ∞ of the Gaussian probability density function N(0,1), i.e.

$$\mathbf{T}\{x\} = \int_x^\infty \frac{1}{\sqrt{2\pi}}.\exp\left(-\frac{y^2}{2}\right)dy$$



Note that if $x > 6.5$ then $\mathbf{T}\{x\}$ may be approximated by $\mathbf{T}\{x\} \approx \frac{1}{\sqrt{2\pi}.x}.\exp\left\{-\frac{x^2}{2}\right\}$

Fourier Transform Tables

	Description	Function	Transformation
1	Definition	$g(t)$	$G(f)=\int_{-\infty}^{\infty} g(t).e^{-j2\pi ft}dt$
2	Scaling	$g\left(\frac{t}{T}\right)$	$\lvert T\rvert.G(fT)$
3	Time shift	$g(t-T)$	$G(f).e^{-j2\pi fT}$
4	Frequency shift	$g(t).e^{j2\pi Ft}$	$G(f-F)$
5	Complex conjugate	$g^*(t)$	$G^*(-f)$
6	Temporal derivative	$\frac{d^n}{dt^n}g(t)$	$(j2\pi f)^n.G(f)$
7	Spectral derivative	$(-j2\pi t)^n.g(t)$	$\frac{d^n}{df^n}G(f)$
8	Reciprocity	$G(t)$	$g(-f)$
9	Linearity	$A.g(t)+B.h(t)$	$A.G(f)+B.H(f)$
10	Multiplication	$g(t).h(t)$	$G(f)*H(f)$
11	Convolution	$g(t)*h(t)$	$G(f).H(f)$
12	Delta function	$\delta(t)$	1
13	Constant	1	$\delta(f)$
14	Rectangular function	$\mathbf{rect}\{t\} \triangleq \begin{cases} 1 & \text{if } \lvert t\rvert<\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$	$\mathbf{sinc}\{f\} \triangleq \frac{\sin(\pi f)}{\pi f}$
15	Sinc function	$\mathbf{sinc}(t)$	$\mathbf{rect}\{f\}$
16	Unit step function	$u(t) \triangleq \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$	$\frac{1}{2}\delta(f)-\frac{j}{2\pi f}$
17	Signum function	$\mathrm{sgn}(t) \triangleq \begin{cases} 1 & t>0 \\ -1 & t<0 \end{cases}$	$-\frac{j}{\pi f}$
18	decaying exp (two-sided)	$e^{-\lvert t\rvert}$	$\frac{2}{1+(2\pi f)^2}$
19	decaying exp (one-sided)	$e^{-\lvert t\rvert}.u(t)$	$\frac{1-j2\pi f}{1+(2\pi f)^2}$
20	Gaussian function	$e^{-\pi t^2}$	$e^{-\pi f^2}$
21	Lambda function	$\Lambda\{t\} \triangleq \begin{cases} 1-t & \text{if } \quad 0\le t\le 1 \\ 1+t & \text{if } -1\le t\le 0 \end{cases}$	$\mathbf{sinc}^2\{f\}$
22	Repeated function	$\mathbf{rep}_T\{g(t)\}=g(t)*\mathbf{rep}_T\{\delta(t)\}$	$\left\lvert\frac{1}{T}\right\rvert \mathbf{comb}_{\frac{1}{T}}\{G(f)\}$
23	Sampled function	$\mathbf{comb}_T\{g(t)\}=g(t).\mathbf{rep}_T\{\delta(t)\}$	$\left\lvert\frac{1}{T}\right\rvert \mathbf{rep}_{\frac{1}{T}}\{G(f)\}$