EE303: Communication Systems

Professor A. Manikas
Chair of Communications and Array Processing

Imperial College London

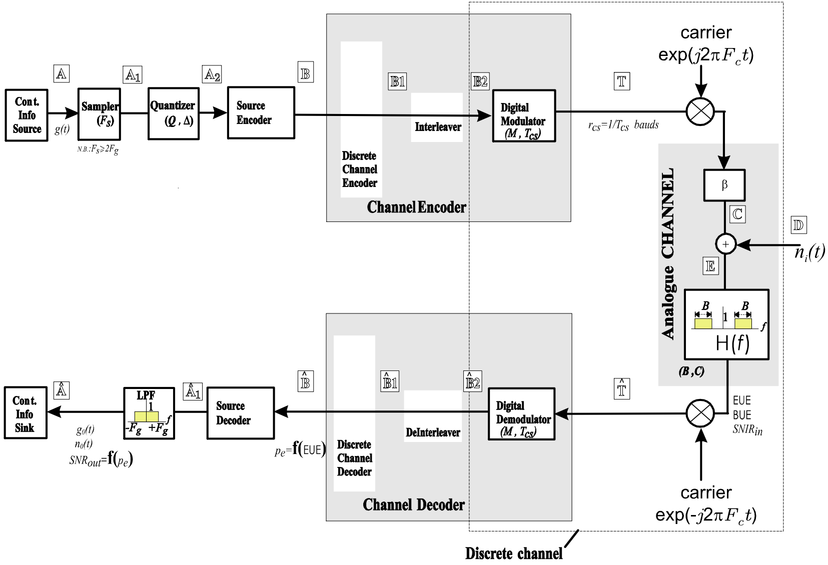
An Overview of Fundamentals: Information Sources

Table of Contents

Introduction

- **Wireline** and **fiber** communications as well as **wireless** communications are **fully digital**.
- The general block structure of a Digital Communication System is shown in the following page.
- it is common practice its **quality** to be expressed **in terms** of the **accuracy** with which the binary digits delivered at the output of the detector (point $\widehat{\mathrm{B}}2$) represent the binary digits that were fed into the digital modulator (point B2).
- It is generally taken that it is the fraction of the binary digits that are delivered back in error that is a measure of the quality of the communication system. This fraction, or rate, is referred to as the **bit error probability** , or, **Bit-Error-Rate BER** (point $\widehat{\mathrm{B}}2$).

Block Structure of a Digital Comm System



Classification of Information Sources

- Information sources (or communication sources), or simply sources can be classified as
  - **Discrete**
    - Discrete Memoryless Sources (MDS),
    - with Memory (e.g. Markov Sources)
  - **Continuous**
    - non-Gaussian
    - Gaussian
- Examples with reference to previous page's figure:
  - continuous: up to points $\boxed{A}$, $\boxed{A1}$, or $\boxed{T}$
  - discrete:
    - up to points $\boxed{A2}$ - levels of quantiser, or
    - up to points $\boxed{B}$, $\boxed{B1}$,or $\boxed{B2}$ - binary digits or binary codewords.
- The "**inverse**" of an information/communication source is an information/communication **sink** (discrete or continuous)
- N.B. - terminology: "continuous" = "analogue"

Discrete Memoryless Sources

- A source is called a **Discrete Source** if produces a sequence $\{X[n]\}$ of symbols one after another, with each symbol being drawn from a finite alphabet
$$X \triangleq \{x_1, x_2, ..., x_M\} \tag{1}$$
with a rate r_X symbols/sec,
and in which each symbol $x_m \in X$ is produced at the output of the source with some associated probability $\Pr(x_m)$ - abbreviated p_m, i.e.
$$\Pr(x_m) \triangleq p_m \tag{2}$$
- If successive outputs from a discrete source are **statistically independent**, or in other words, if at each instant of time the source chooses for transmission one symbol from the set $X = \{x_1, x_2, ..., x_M\}$ and its choices are independent from one time instant to the next, then the source is called a **Discrete Memoryless Source (DMS)**.

- If $\underline{p}$ represents the vector with elements the probabilities associated with the symbols of a source i.e.

$$\underline{p} \triangleq [p_1, p_2, ..., p_M]^T \tag{3}$$

then the set

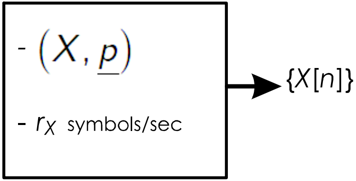
$$\left(X, \underline{p}\right) \triangleq \{(x_1, p_1), (x_2, p_2), ..., (x_M, p_M)\} \tag{4}$$

with

$$\sum_{m=1}^{M} p_m = 1 \tag{5}$$

is defined as the **Source Ensemble**

- A DMS source can be fully described by its **ensemble** $(X, \underline{p})$ and its **symbol rate** (symbols per second) r_X



- The point A2 may be considered as the input of a Digital Communication System where messages consist of sequences of "symbols" selected from an alphabet
  e.g. levels of a quantizer or telegraph letters, numbers and punctuations.

Measure of Information Generated by a DMS Source

Source Entropy

- Consider a discrete memoryless information source

$$(X, \underline{p}) = \begin{Bmatrix} (x_1, p_1) \\ (x_2, p_2) \\ \cdots \\ (x_M, p_M) \end{Bmatrix} \quad \text{with} \quad \sum_{m=1}^{M} p_m = 1 \tag{6}$$

Then, the average information per symbol generated by the source is given by the so-called **entropy** of the source i.e.

$$H_X \triangleq H_X(\underline{p}) = -\sum_{m=1}^{M} p_m \underbrace{\log_2(p_m)}_{\triangleq -\mathrm{I}(x_m)} \text{ bits/symbol} \tag{7}$$

$$= -\underline{p}^T \log_2(\underline{p}) \quad \text{bits/symbol} \tag{8}$$

Source Information Rate

- H_X is a measure of the **a priori uncertainty** associated with the source output or, equivalently, **a measure of the information obtained when the source output is observed**.
- The notion of entropy is not restricted to the case where the ensemble is finite
  i.e. M may be ∞.
- It can also be shown that H_X is the minimum number of binary digits bits needed to encode the source output.
- Based on entropy, the average **information bit-rate** at the output of the source is defined as follows:

$$r_{\inf} = \underset{\frac{symbols}{\text{sec}}}{\underset{\downarrow}{r_X}} \, . \, \underset{\frac{bits}{symbol}}{\underset{\downarrow}{H_X}} \quad \text{bits/sec} \tag{9}$$

- Thus, we have two types of 'bits' and, therefore, two types of 'rates'. That is,
  - **data rate** : r_b in $\frac{\textit{data bits}}{\text{sec}}$ or simply $\frac{\textit{bits}}{\text{sec}}$
  - **info rate** : r_{inf} in $\frac{\textit{information bits}}{\text{sec}}$ or simply $\frac{\textit{bits}}{\text{sec}}$
- NB:
  - In general:
$$r_{\text{inf}} \leq r_b \tag{10}$$
  - In ideal systems:
$$r_{\text{inf}} = r_b \tag{11}$$

Example 1

- If $X = \{0, 1\}$ with probablities

$$\underline{p} = \begin{bmatrix} \Pr(0) \\ \Pr(1) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \tag{12}$$

then

$$\begin{aligned}
\textbf{data rate} \quad &: \quad r_b = r_X = 10 \frac{bits}{\text{sec}} \\
\text{entropy} \quad &: \quad H_X = 1 \frac{bits}{\substack{\text{sec} \\ \text{symbol}}} = 1 \frac{info\ bit}{data\ bit} \\
\textbf{info rate} \quad &: \quad r_{\text{inf}} = r_X . H_X = 10 \frac{bits}{\text{sec}}
\end{aligned}$$

i.e.

$$r_b = r_{\text{inf}} = 10 \frac{bits}{\text{sec}} \tag{13}$$

Example 2

- If $X = \{0, 1\}$ with probablities

$$\underline{p} = \begin{bmatrix} \Pr(0) \\ \Pr(1) \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \tag{14}$$

then

$$\begin{aligned}
\textbf{data rate} \quad &: \quad r_b = r_X = 10 \frac{bits}{sec} \\
\text{entropy} \quad &: \quad H_X = 0.8813 \frac{bits}{\cancel{sec}\ symbol} = 0.8813 \frac{info\ bit}{data\ bit} \\
\textbf{info rate} \quad &: \quad r_{\inf} = r_X . H_X = 8.813 \frac{bits}{sec}
\end{aligned}$$

i.e.

$$r_b > r_{\inf} \tag{15}$$

A 'Joint' DMS Source

- ▶ Consider two information sources with the ensembles $(X, \underline{p})$ and $(Y, \underline{q})$, respectively, defined as follows

$$(X, \underline{p}) = \left\{ \begin{array}{c} (x_1, p_1) \\ (x_2, p_2) \\ \dots \\ (x_M, p_M) \end{array} \right\} \text{ with } \sum_{m=1}^{M} p_m = 1 \quad (16)$$

$$(Y, \underline{q}) = \left\{ \begin{array}{c} (y_1, q_1) \\ (y_2, q_2) \\ \dots \\ (y_K, q_K) \end{array} \right\} \text{ with } \sum_{k=1}^{K} p_k = 1 \quad (17)$$

where

$$\underline{p} = [\overbrace{\Pr(x_1)}^{p_1}, \overbrace{\Pr(x_2)}^{p_2}, \dots, \overbrace{\Pr(x_M)}^{p_M}]^T \quad (18)$$

$$\underline{q} = [\overbrace{\Pr(y_1)}^{q_1}, \overbrace{\Pr(y_2)}^{q_2}, \dots, \overbrace{\Pr(y_K)}^{q_K}]^T \quad (19)$$

- Let us form a joined source where its symbols are taken to be pairs of symbols drawn from the two original sources X and Y.
  The new source with alphabet

$$\{(x_1, y_1), (x_1, y_2), \ldots, (x_2, y_1), \ldots, (x_m, y_k), \ldots, (x_M, y_K)\}$$

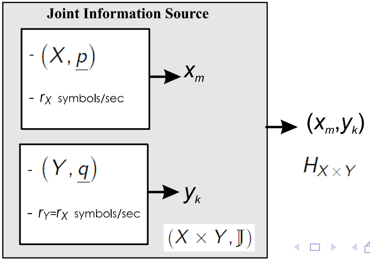
  is known as **joint source** and its ensemble as **joint ensemble** defined as follows:

$$(X \times Y, \mathbb{J}) = \left\{ \begin{array}{c} (x_1, y_1), \Pr(x_1, y_1) \\ (x_1, y_2), \Pr(x_1, y_2) \\ \ldots \\ (x_m, y_k), \Pr(x_m, y_k) \\ \ldots \\ (x_M, y_K), \Pr(x_M, y_K) \end{array} \right\} \tag{20}$$

$$= \left\{ \left((x_m, y_k), \underbrace{\Pr(x_m, y_k)}_{=J_{km}} \right), \forall mk : 1 \leq m \leq M, 1 \leq k \leq K \right\}$$

- Note that the joint probabilistic relationship between the symbols of X and Y, described by the so-called **joint-probability matrix** ,

$$\mathbb{J} = \begin{bmatrix} \overbrace{\Pr(x_1, y_1)}^{\triangleq J_{11}}, & \overbrace{\Pr(x_1, y_2)}^{\triangleq J_{21}}, & \ldots, & \overbrace{\Pr(x_1, y_K)}^{\triangleq J_{K1}} \\ \overbrace{\Pr(x_2, y_1)}^{\triangleq J_{12}}, & \overbrace{\Pr(x_2, y_2)}^{\triangleq J_{22}}, & \ldots, & \overbrace{\Pr(x_2, y_K)}^{\triangleq J_{K,2}} \\ \ldots, & \ldots, & \ldots, & \ldots \\ \overbrace{\Pr(x_M, y_1)}^{\triangleq J_{1M}}, & \overbrace{\Pr(x_M, y_2)}^{\triangleq J_{2M}}, & \ldots, & \overbrace{\Pr(x_M, y_K)}^{\triangleq J_{KM}} \end{bmatrix}^T$$



$$H_X \triangleq -\sum_{m=1}^{M} p_m \underbrace{\log_2(p_m)}_{\triangleq -\mathrm{I}(x_m)} = -\underline{p}^T \log_2(\underline{p}) \qquad \frac{bits}{symbol} \tag{21}$$

$$H_Y \triangleq -\sum_{k=1}^{K} q_k \underbrace{\log_2(q_k)}_{\triangleq -\mathrm{I}(x_m)} = -\underline{q}^T \log_2(\underline{q}) \qquad \frac{bits}{symbol} \tag{22}$$

$$H_{X \times Y} \triangleq -\sum_{m=1}^{M} \sum_{k=1}^{K} J_{km} \underbrace{\log_2(J_{km})}_{\triangleq -\mathrm{I}(x_m, y_k)} \tag{23}$$

$$= -\underline{1}_K^T \left(\underbrace{\mathbb{J} \odot \log_2(\mathbb{J})}_{K \times M \quad matrix} \right) \underline{1}_M \; \frac{bits}{symbol} \tag{24}$$

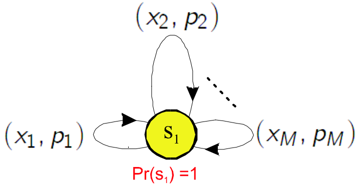
where

$$\underline{1}_M = \text{a column vector of } M \text{ ones}$$

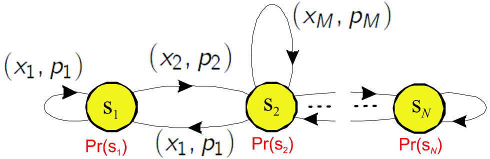
Markov Discrete Information Sources

Entropy

- A DMS can be modelled as with a single-state transition diagram



- Markov sources are modelled by Markov state transition diagrams of N states ($N > 1$), e.g.



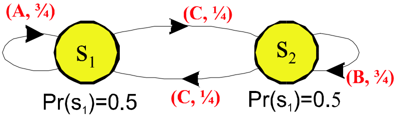
- The entropy of a Markov Source is given as follows:

$$H = \sum_{i=1}^{N} \Pr(s_i).H_i \quad \frac{bits}{symbol} \tag{25}$$

where

$$\begin{aligned} N &= \text{number of states} \\ \Pr(s_i) &= \text{Probability to be on the } i^{\text{th}} \text{state } s_i \\ H_i &= \text{entropy of the } i^{\text{th}} \text{ state (considered as a DMS)} \end{aligned}$$

- Example: Consider a two-state Markov source which gives the symbols A,B,C according to the following model:

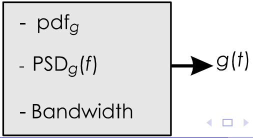


Then (for you)

- $H_1 =?$
- $H_2 =?$
- $H = ?$ bits/symbol

Continuous Sources/Signals

- A source whose output is a continuous signal in both amplitude and time is called a Continuous Source.
- However, this definition may be relaxed to include also signals that are continuous in amplitude but discrete in time, e.g. point $\boxed{\text{A2}}$. That is the condition is relaxed to only "continuous in amplitude".
- A continuous source which relates directly to analogue signal waveforms is described by its **ensemble** $\left(g(t), \text{pdf}_g(g)\right)$ where $\text{pdf}_g(g)$ denotes the amplitude probability density function of $g(t)$.
- However, in order to describe fully a continuous source $g(t)$, in **addition to the source ensemble** , the following parameters of $g(t)$ should be specified/identified
  - the **power spectral density** $\text{PSD}_g(f)$,
  - the **bandwidth** .



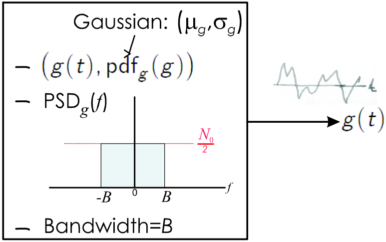
Measure of Information Generated by a Continuous Source

Continuous Source: Differential Entropy

- Entropy of an analoque source $= \infty$
- Differential entropy:

$$H_{\mathrm{g}} \triangleq -\int_{-\infty}^{\infty} \mathrm{pdf}_g(\mathrm{g}).\log_2(\mathrm{pdf}_g(\mathrm{g})).dg \tag{26}$$

- The term "entropy" in this course (for analogue signals) will be used to refer to "differential entropy"
- Entropy of a Gaussian Source



$$H_g = \log_2 \sqrt{2\pi e \sigma_g^2} = \max \tag{27}$$

Important Relationships

- At the output of an information source $g(t)$ the following are very important:

$$\begin{aligned}
\text{Entropy} &: H_g & (28)\\
\max(Entropy) &: \text{Gaussian Entropy} = \log_2 \sqrt{2\pi e \sigma_g^2} & (29)\\
\text{Entropy Power} &: N_g \triangleq \frac{1}{2\pi e} 2^{2H_g} & (30)\\
\text{Average Power} &: P_g \triangleq \mathcal{E}\left\{g(t)^2\right\} & (31)\\
\text{In general} &: P_g \geqq N_g & (32)
\end{aligned}$$

$$\begin{aligned}
\text{if } \mathrm{pdf}_g &= \text{Gaussian} \Rightarrow P_g = N_g & (33)\\
\text{if } \mathrm{pdf}_g &\neq \text{Gaussian} \Rightarrow P_g > N_g & (34)
\end{aligned}$$

A Note on Information Sinks

- A communication sink is the destination of the symbols produced by a source and transmitted from the input to the output of a channel.
- It has one input and no output and can be seen as the inverse of a source. That is, matching its associated source, a sink is either
  - continuous, or
  - discrete
- Examples (with reference to the block structure of a comm system):
  - continuous: from ponts $\boxed{\widehat{\mathrm{A}}}$, $\boxed{\widehat{\mathrm{A1}}}$, or $\boxed{\widehat{\mathrm{T}}}$
  - discrete: form points $\boxed{\widehat{\mathrm{B}}}$, $\boxed{\widehat{\mathrm{B1}}}$ or $\boxed{\widehat{\mathrm{B2}}}$,

- A sink is described by
  1. a performance (fidelity) criterion $\rho(X, Y)$
     - ★ e.g. criterion for a discrete sink: BER
     - ★ e.g. criterion for a continuous sink: SNR
  2. a maximum allowed distortion $D_{\max}$ (e.g. $D_{\max} = 10^{-3}$ i.e. $\mathrm{BER} < 10^{-3}$)