Problem 4.1.9

1. $f(x) = |x|$, taking $\bar{x} = 0$, we have the following subgradient inequality

$$\langle \lambda, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) \Rightarrow \langle \lambda, x \rangle \leq f(x)$$

$\Rightarrow \lambda x \leq |x| \quad \forall x \in \mathbb{R} \Rightarrow \lambda \in [-1, 1] \Rightarrow \partial f(x) = [-1, 1], x = 0$

for $x < 0$ we have $\partial f(x) = \{-1\}$

for $x > 0$ we have $\partial f(x) = \{1\}$

2. $f(x) = \delta_{[0,+\infty)}$. Taking $C = [0, +\infty)$, $x_1, x_2 \in [0, +\infty)$ and $\lambda \in [0,1]$ where $x_1 \leq x_2$

$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \leq \lambda x_2 + (1-\lambda) x_2 = x_2$

$\lambda x_1 + (1-\lambda) x_2 \geq \lambda x_1 + (1-\lambda) x_1 = x_1$

$\Rightarrow [0, +\infty)$ is a closed, convex set in $\mathbb{R}$

$\Rightarrow \partial f(x) = \partial \delta_{[0,+\infty)}(x) = N_{[0,+\infty)}(x)$

3. $f(x) = \begin{cases} -\sqrt{x} & \text{if } x \in [0, +\infty) \\ +\infty & \text{if } x \notin [0, +\infty) \end{cases} = -\sqrt{x} + \delta_{[0,+\infty)}$

For $x \in (0, +\infty)$ and $g(x) = -\sqrt{x} \Rightarrow \partial g(x) = \{-\frac{1}{2\sqrt{x}}\}$.

For $\bar{x} = 0$, $g(x) = -\sqrt{x}$, we have

$$\langle \lambda, x - \bar{x} \rangle \leq g(x) - g(\bar{x}) \Rightarrow \lambda x \leq -\sqrt{x} \quad \forall x \in [0, +\infty)$$

$\Rightarrow \lambda \in (-\infty, 0)$

From 2. we have that $\partial \delta_{[0,+\infty)} = N_{[0,+\infty)}(x)$

: as we know $\partial (f+g)(x) = \partial f(x) + \partial g(x)$

$\Rightarrow$ we conclude that

$\partial f(x \neq 0) = \{-\frac{1}{2\sqrt{x}}, N_{[0,+\infty)}(x)\}$

$\partial f(x = 0) = \{(-\infty, 0), N_{[0,+\infty)}(x)\}$

Problem 4.1.14

1. $f(x) = |x|$

$f^*(y) = \sup_x \{xy - |x|\}$

if $y > 1$ $\sup_x \{xy - |x|\} = +\infty$

if $-1 \le y \le 1$ $\sup_x \{xy - |x|\} = 0$

if $y < -1$ $\sup_x \{xy - |x|\} = +\infty$

$$\Rightarrow f^*(y) = \begin{cases} 0, & |y| \le 1 \\ +\infty, & |y| > 1 \end{cases}$$

2. $f(x) = \delta_{[0,+\infty)} = \begin{cases} 0 & , x \in [0,+\infty) \\ +\infty & , x \notin [0,+\infty) \end{cases}$

we know from Problem 4.1.9 that the interval $[0,+\infty)$ is a closed, convex set. $\Rightarrow$ we understand that the conjugate of the support function of $[0,+\infty)$ is the indicator function $\delta_{[0,+\infty)}$. Given that $S(C,\cdot)(x)$ is lower-semi continuous, we have that $S^{**}(C,.)(x) = S(C,\cdot)(x)$ by the biconjugate theorem.

$\Rightarrow f^*(y) = S(C,\cdot)(y)$ where $C = [0,+\infty)$

Problem 4.1.14

3. $f(x) = \begin{cases} -\sqrt{x} & \text{if } x \in [0, +\infty) \\ +\infty & \text{if } x \notin [0, +\infty) \end{cases}$

let $g(x) = -\sqrt{x} \Rightarrow g^*(y) = \sup_{x \in [0, +\infty)} \{xy + \sqrt{x}\}$

for $y \geq 0$ we have $g^*(y) = +\infty$

Now looking at $\frac{d}{dx}\{xy + \sqrt{x}\} = y + \frac{1}{2\sqrt{x}} = 0 \Rightarrow x = \frac{1}{4y^2}$ $y < 0$

for $y < 0$ we have

$$g^*(y) = \frac{y}{4y^2} + \sqrt{\frac{1}{4y^2}} = \frac{1}{4y} - \frac{1}{2y} = -\frac{1}{4y}$$

$\therefore g^*(y) = \begin{cases} -\frac{1}{4y}, & y < 0 \\ +\infty, & y \geq 0 \end{cases}$

$\Rightarrow f^*(y) = \begin{cases} -\frac{1}{4y}, & y < 0 \\ +\infty, & y \geq 0 \end{cases}$

Problem 4.1.19

$q_A(x) = \frac{1}{2} x^T A x$ where A is real, positive definite square matrix

1. By lemma 4.1.8, we have that since $A \in P(n)$

$$\nabla^2 q_A(x) = A, \text{ where } A \text{ is a real, positive definite square matrix}$$

Given this fact about A, by Theorem 1.1.6 iii) as we have the hessian of $q_A(x)$ is positive definite for all $x \in X$, we have that q_A is strictly convex.

2. By lemma 4.1.18, we have that

$$\nabla q_A(x) = Ax, \text{ where } A \text{ is a real, positive definite square matrix}$$

$\Rightarrow q_A^*(y) = \sup_x \langle xy - \frac{1}{2} x^T A x \rangle$. Let $f : x \to xy - \frac{1}{2} x^T A x$

$\nabla f(x) = y - Ax = 0 \Rightarrow x = A^{-1} y$

Plug back into $f \Rightarrow y^T A^{-1} y - \frac{1}{2} (A^{-1} y)^T A (A^{-1} y)$

$$= y^T A^{-1} y - \frac{1}{2} y^T A^{-1} A A^{-1} y = y^T A^{-1} y - \frac{1}{2} y^T A^{-1} y = \frac{1}{2} y^T A^{-1} y$$

Problem 4.1.19

3. We have that $q_A(x) = \frac{1}{2} x^T A x$ is strictly convex, and that

$q_A^*(y) = \frac{1}{2} y^T A^{-1} y = q_{A^{-1}}(y)$

We know that if $A \succeq B \iff x^T A x \geq x^T B x \quad \forall x$

$\Rightarrow q_A(x) = \frac{1}{2} x^T A x \geq \frac{1}{2} x^T B x = q_B(x) \quad \forall x$

$\Rightarrow q_A^*(y) = \frac{1}{2} y^T A^{-1} y \leq \frac{1}{2} y^T B^{-1} y = q_B^*(y) \quad \forall y$ By lemma 4.1.12

$\Rightarrow A^{-1} \cancel{A} \preceq B^{-1}$

Now given $B^{-1} \succeq A^{-1} \iff y^T B^{-1} y \geq y^T A^{-1} y \quad \forall y$

$\Rightarrow q_{B^{-1}}(y) = \frac{1}{2} y^T B^{-1} y \geq \frac{1}{2} y^T A^{-1} y = q_{A^{-1}}(y) \quad \forall y$

$\Rightarrow q_B(x) = \frac{1}{2} x^T B x \leq \frac{1}{2} x^T A x = q_A(x) \quad \forall x$ By lemma 4.1.12

$\Rightarrow B \preceq A$

$\square$