Problem 2.2.2

1. First show by induction that the matrix P that diagonalises A also diagonalises the matrices Y_n.

we have $Y_0 = 0 \Rightarrow$ ~~base~~ case $Y_1 = \frac{1}{2}(A + Y_0^2) = \frac{1}{2}A$

$\therefore$ we take the base case: $P^{-1}Y_1P = P^{-1}(\frac{1}{2}A)P = \frac{1}{2}D$

Assuming the P diagonalises Y_n, we look at Y_{n+1}

$$P^{-1}Y_{n+1}P = P^{-1}(\tfrac{1}{2}(A + Y_n^2))P = \tfrac{1}{2}P^{-1}AP + \tfrac{1}{2}P^{-1}Y_n^2P$$

$$= \tfrac{1}{2}D + \tfrac{1}{2}P^{-1}(Y_n \cdot Y_n)P = \tfrac{1}{2}D + \tfrac{1}{2}P^{-1}Y_nP \cdot (PP^{-1})Y_n$$

$$= \tfrac{1}{2}D + \tfrac{1}{2}D_n \cdot P^{-1}Y_nP = \tfrac{1}{2}D + \tfrac{1}{2}D_n^2$$

Applying this to the iteration $Y_{n+1} = \frac{1}{2}(A + Y_n^2)$, we get

$\cancel{D_{n+1} = \frac{1}{2}(A + \ldots)}$ $D_{n+1} = \frac{1}{2}D + \frac{1}{2}D_n^2$, which is the iteration on the reals. looking at $\{Y_{n+1} - Y_n\}$ we have

$\frac{1}{2}D + \frac{1}{2}D_n^2 - \frac{1}{2}D_n$ $\cancel{= \frac{1}{2}D + \frac{1}{2}D_n(D_n - 1)}$ as $I \succeq A \Rightarrow 1 \geq D$

since $A \in P(n)$ we have $D \geq 0 \Rightarrow \frac{1}{2}D + \frac{1}{2}D_n^2 \geq \frac{1}{2}D_n$

$\Rightarrow \{Y_n\}$ is non-decreasing. As $\{Y_n\}$ is non-decreasing and therefore bounded above, we take the limit of

$$D_{n+1} = \tfrac{1}{2}D + \tfrac{1}{2}D_n^2$$

$$\Rightarrow D_n = \tfrac{1}{2}D + \tfrac{1}{2}D_n^2$$

$$\Rightarrow \cancel{\tfrac{1}{2}D^2 - \tfrac{1}{2}D = 0} \quad \tfrac{1}{2}D_n^2 - D_n + \tfrac{1}{2}D = 0$$

which has roots $(1 - (1 - D)^{1/2})$ and $(1 + (1 - D)^{1/2})$

given $\{Y_n\}$ is non-decreasing, and $Y_0 = 0$, $n \geq 0$, after undiagonalising above, we get that $\{Y_n\}$ converges to $I + (I - A)^{1/2}$

Problem 3.1.8

We have the convex cone

$$\text{Cone co}\{(y_i, 1) \mid i = 1, \dots, m\}$$

and an element $b := (0, 1)$

We know that $b \in \text{Cone co}\{(y_i, 1) \mid i = 1, \dots, m\}$

if
$$\sum_{i=1}^{m} \lambda_i y_i = b, \quad \lambda_i \geq 0, \quad i = 1, \dots, m \quad \text{has a solution}$$

$$\Rightarrow \sum_{i=1}^{m} \lambda_i y_i = 0, \quad \lambda_i \geq 0, \quad i = 1, \dots, m \quad \text{has a solution}$$

By Farkas' lemma either the above has a solution or

$$\langle b, x \rangle > 0, \quad \langle y_i, x \rangle \leq 0 \quad \text{for } i = 1, \dots, m$$

has a solution

Given we have $b = (0, 1)$, taking $x = (x_1, x_2)$

$\langle b, x \rangle = x^T b > 0 \Rightarrow x > 0$

$\therefore \langle y_i, x \rangle \leq 0$ implies that since $x \geq 0$, $y_i \leq 0$

And given that if this system were to have a solution,
$b \notin \text{Cocone}\{(y_i, 1) \mid i = 1, \dots, m\} \Rightarrow (0, 1) \notin \text{Cocone}\{(y_i, 1) \mid i = 1, \dots, m\}$
$\Rightarrow y_i < 0$ which gives us

$$\langle y_i, x \rangle < 0 \quad \text{for } i = 1, \dots, m$$

And hence we have Gordon's Theorem.

Problem 3.1.9

we know $\bar{x}$ is stationary when

$$\max\{\langle \nabla f(\bar{x}), d\rangle, \langle \nabla g_i(\bar{x}), d\rangle \mid i \in I(\bar{x})\} \geq 0$$

where $I(\bar{x}) = \{i \mid g_i(\bar{x}) = 0\}$

$\Rightarrow$ $\bar{x}$ is not a stationary point when

$$\max\{\langle \nabla f(\bar{x}), d\rangle, \langle \nabla g_i(\bar{x}), d\rangle \mid i \in I(\bar{x})\} < 0 \quad (*)$$

By Gordon's Theorem, either $(*)$ has a solution, or the system

$$\lambda_0 \nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) = 0 \quad \text{has a solution} \quad (+)$$

$$\lambda_0, \lambda_i \geq 0, i = 1, \dots, m$$

$\Rightarrow$ we conclude that $(+)$ only has a ~~8to~~ solution iff $\bar{x}$ is a stationary point.

Hence we prove the Fritz-John optimality condition.