Problem 6.4.5

1.a)

2.a) $A = \{(x_1, x_2) \mid x_2 = 0\}$
$B = \{(x_1, x_2) \mid x_1 = x_2\}$

$x \in A \cap B = \{(0,0)\}$, as this is the only intersection between the two lines.

b) $A = \{(x_1, x_2) \mid x_2 = 0\}$
$B = \{(x_1, x_2) \mid x_1^2 + (x_2 - 1)^2 \leq 1\}$

Here $x_2 = 0$ is tangent to the ball at $(0,0) \Rightarrow x \in A \cap B = \{(0,0)\}$

c) $A = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$
$B = \{(x_1, x_2) \mid x_1 = x_2\}$

Here the line $x_1 = x_2$ intersects the two axes x_1, x_2 at $(0,0) \Rightarrow x \in A \cap B = \{(0,0)\}$

d) $A = \{(x_1, x_2) \mid x_2 = \sqrt{2}/2 = 1/\sqrt{2}\}$
$B = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$

Substituting $x_2 = 1/\sqrt{2} \Rightarrow x_1^2 + x_2^2 = 1 \Rightarrow x_1 = \pm \frac{1}{\sqrt{2}}$
$\Rightarrow x \in A \cap B = \{(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$

1.b)

2.a) $A = \{(x_1, x_2) \mid x_2 = 0\}$
$B = \{(x_1, x_2) \mid x_1 = x_2\}$

$x = (x_1, x_2)^T$, $a = (1,0)^T$, $\hat{a} = (1,0)$

$P_A = (x_1, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$b = (1,1)$, $\hat{b} = \frac{b}{\|b\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$P_B = (x \cdot \hat{b}) \cdot \hat{b} = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right) \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \left[\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}} \right] \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2}{2} \\ \frac{x_1 + x_2}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2} \right)$$

$R_A = 2 \cdot P_A - I = (2x_1, 0) - (x_1, x_2) = (x_1, -x_2)$
$R_B = 2 P_B - I = (x_1 + x_2, x_1 + x_2) - (x_1, x_2) = (x_2, x_1)$

2.b) $A = \{(x_1, x_2) \mid x_2 = 0\}$ $B = \{(x_1, x_2) \mid x_1^2 + (x_2 - 1)^2 \leq 1\}$

$P_A = (x_1, 0)$ $R_A = (x_1, -x_2)$

$$P_B = \frac{2x}{\|x\|} = \left(\frac{x_1}{\sqrt{x_1^2 + (x_2-1)^2}}, \frac{x_2 - 1}{\sqrt{x_1^2 + (x_2-1)^2}} \right) \quad R_B = 2 \cdot P_B - I = \left(\frac{2x_1}{\sqrt{x_1^2 + (x_2-1)^2}} - x_1, \frac{2(x_2 - 1)}{\sqrt{x_1^2 + (x_2-1)^2}} - x_2 \right)$$

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1.b)

2.c) $A = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \qquad B = \{(x_1, x_2) \mid x_1 = x_2\}$

$$P_A = \{(x_1, 0) \cup (0, x_2)\}$$
$$R_A = 2 \cdot P_A - I = \{(x_1, -x_2) \cup (-x_1, x_2)\}$$

$$P_B = \left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}\right), \quad R_B = 2 \cdot P_B - I = (x_2, x_1)$$

d) $A = \{(x_1, x_2) \mid x_2 = \sqrt{2}/2\}, \quad B = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$

$$P_A = (x_1, \sqrt{2}/2) \qquad R_A = (x_1, \sqrt{2} - x_2)$$

$$P_B = \frac{x}{\|x\|} = \left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}, \frac{x_2}{\sqrt{x_1^2+x_2^2}}\right) \qquad R_B = \left(\frac{2x_1}{\sqrt{x_1^2+x_2^2}} - x_1, \frac{2x_2}{\sqrt{x_1^2+x_2^2}} - x_2\right)$$

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~~1.b)~~
2.b) $$\min \; (P_{x_1}-x_1)^2+(P_{x_2}-x_2)^2 \quad \text{s.t.} \quad x_1^2+(x_2-1)^2 \le 1$$

$$\Rightarrow \mathcal{L}(x_1,x_2,\lambda) = (P_{x_1}-x_1)^2+(P_{x_2}-x_2)^2+\lambda\left(x_1^2+(x_2-1)^2-1\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -2(P_{x_1}-x_1)+2\lambda x_1 = 0 \Rightarrow 2x_1+2\lambda x_1 = 2P_{x_1} \Rightarrow x_1 = P_{x_1}/1+\lambda$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -2(P_{x_2}-x_2)+2\lambda(x_2-1) = 0 \Rightarrow x_2-P_{x_2}+\lambda x_2-\lambda = 0 \Rightarrow x_2(1+\lambda) = P_{x_2}+\lambda$$
$$\Rightarrow x_2 = P_{x_2}+\lambda/1+\lambda$$

$$g(\lambda) = \left(P_{x_1}-\frac{P_{x_1}}{1+\lambda}\right)^2+\left(P_{x_2}-\frac{P_{x_2}+\lambda}{1+\lambda}\right)^2+\lambda\left(\frac{P_{x_1}^2}{(1+\lambda)^2}+\left(\frac{P_{x_2}+\lambda}{1+\lambda}-1\right)^2-1\right)$$

$$= \left(\frac{\lambda P_{x_1}}{1+\lambda}\right)^2+\left(\frac{\cancel{P_{x_2}}+\lambda P_{x_2}-\cancel{P_{x_2}}-\lambda}{1+\lambda}\right)^2+\lambda\left(\frac{P_{x_1}^2}{(1+\lambda)^2}+\left(\frac{P_{x_2}-1}{1+\lambda}\right)^2-1\right)$$

$$= \frac{\lambda^2 P_{x_1}^2}{(1+\lambda)^2}+\frac{\lambda^2(P_{x_2}-1)^2}{(1+\lambda)^2}+\frac{\lambda P_{x_1}^2}{(1+\lambda)^2}+\frac{\lambda(P_{x_2}-1)^2}{(1+\lambda)^2}-\lambda$$

$$= \frac{\lambda^2 P_{x_1}^2+\lambda P_{x_1}^2+\lambda^2(P_{x_2}-1)^2+\lambda(P_{x_2}-1)^2}{(1+\lambda)^2}-\lambda$$

$$\frac{\partial g}{\partial \lambda} = \frac{\left(2\lambda P_{x_1}^2+P_{x_1}^2+2\lambda(P_{x_2}-1)^2+(P_{x_2}-1)^2\right)(1+\lambda)^{\cancel{2}}-2(\cancel{1+\lambda})\left(\lambda^2 P_{x_1}^2+\lambda P_{x_1}^2+\lambda^2(P_{x_2}-1)^2+\lambda(P_{x_2}-1)^2\right)}{(1+\lambda)^{\cancel{4}\,3}}-1$$

$$= \frac{\left[P_{x_1}^2(2\lambda+1)+(P_{x_2}-1)^2(2\lambda+1)\right](1+\lambda)-2\left[\lambda P_{x_1}^2(\lambda+1)+\lambda(P_{x_2}-1)^2(\lambda+1)\right]}{(1+\lambda)^3}-1$$

$$= \frac{(2\lambda+1)(\cancel{1+\lambda})\left(P_{x_1}^2+(P_{x_2}-1)^2\right)-2(\cancel{1+\lambda})\left(\lambda P_{x_1}^2+\lambda(P_{x_2}-1)^2\right)}{(1+\lambda)^{\cancel{3}\,2}}-1$$

$$= \frac{(\cancel{2\lambda}+1)\left(P_{x_1}^2+(P_{x_2}-1)^2\right)-\cancel{2\lambda}\left(P_{x_1}^2+(P_{x_2}-1)^2\right)}{(1+\lambda)^2}-1 = 0$$

$$= \frac{P_{x_1}^2+(P_{x_2}-1)^2}{(1+\lambda)^2}-1 = 0 \Rightarrow P_{x_1}^2+(P_{x_2}-1)^2 = (1+\lambda)^2$$

$$\Rightarrow \lambda = \sqrt{P_{x_1}^2+(P_{x_2}-1)^2}-1$$

$$\therefore x_1 = \frac{P_{x_1}}{\sqrt{P_{x_1}^2+(P_{x_2}-1)^2}}, \quad x_2 = \frac{P_{x_2}+\sqrt{P_{x_1}^2+(P_{x_2}-1)^2}-1}{\sqrt{P_{x_1}^2+(P_{x_2}-1)^2}} = \frac{P_{x_2}-1}{\sqrt{P_{x_1}^2+(P_{x_2}-1)^2}}+1$$

$$P_B = \left(\frac{x_1}{\sqrt{x_1^2+(x_2-1)^2}},\ \frac{x_2-1}{\sqrt{x_1^2+(x_2-1)^2}}+1\right) \quad R_B = 2P_B-I = \left(\frac{2x_1}{\sqrt{x_1^2+(x_2-1)^2}}-x_1,\ \frac{2(x_2-1)}{\sqrt{x_1^2+(x_2-1)^2}}+2-x_2\right)$$

$$P_A = (x_1, 0) \quad R_A = (x_1, -x_2)$$

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1.d)

2.a) D-R: $x_{n+1} \in \frac{1}{2}(R_A R_B x_n + x_n) = \frac{1}{2}\left(R_A(x_2, x_1) + (x_1, x_2)\right) = \frac{1}{2}(x_1 + x_2, x_2 - x_1)$

$x^* = (0,0) \Rightarrow \lim_{k\to\infty} \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} = \frac{\frac{1}{2}\sqrt{(x_1+x_2)^2 + (x_2-x_1)^2}}{\sqrt{x_1^2 + x_2^2}} = \frac{1}{2}\frac{\sqrt{x_1^2 + 2x_1x_2 + x_2^2 + 2x_1x_2 + x_1^2}}{\sqrt{x_1^2+x_2^2}}$

$= \lim_{k\to\infty} \frac{1}{2}\frac{\sqrt{2x_1^2 + 2x_2^2}}{\sqrt{x_1^2+x_2^2}} = \lim_{k\to\infty} \frac{\sqrt{2}}{2}\frac{\sqrt{x_1^2+x_2^2}}{\sqrt{x_1^2+x_2^2}} = \frac{1}{\sqrt{2}} < 1 \Rightarrow$ R-linear

AP: $x_{n+1} \in P_A P_B x_n = \left(\frac{x_1+x_2}{2}, 0\right)$, $x^* = (0,0) \Rightarrow \lim_{k\to\infty} \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} = \lim_{k\to\infty} \frac{1}{2}\frac{x_1+x_2}{\sqrt{x_1^2+x_2^2}}$

$\approx \frac{1}{2} \Rightarrow$ R-linear

2.b) AP: $x_{n+1} \in P_A P_B x_n = \left(\frac{x_1}{\sqrt{x_1^2+(x_2-1)^2}}, 0\right)$, $x^* = (0,0) \Rightarrow \lim_{k\to\infty} \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}$

$= \lim_{k\to\infty} \frac{x_1}{\sqrt{x_1^2+x_2^2}\cdot\sqrt{x_1^2+(x_2-1)^2}} \approx \lim_{k\to\infty} \frac{x_1}{\sqrt{x_1^2 + 1}} \approx 1 \Rightarrow$ Sub-linear

D-R: $x_{n+1} \in \frac{1}{2}(R_A R_B x_n + x_n) = \frac{1}{2}\left(R_A\left(\frac{2x_1}{\sqrt{x_1^2+(x_2-1)^2}} - x_1, \frac{2(x_2-1)}{\sqrt{x_1^2+(x_2-1)^2}} + 2 - x_2\right) + (x_1, x_2)\right)$

$= \frac{1}{2}\left[\left(\frac{2x_1}{\sqrt{x_1^2+(x_2-1)^2}} - x_1, x_2 - 2 - \frac{2(x_2-1)}{\sqrt{x_1^2+(x_2-1)^2}}\right) + (x_1, x_2)\right] = \left(\frac{x_1}{\sqrt{x_1^2+(x_2-1)^2}}, x_2 - 1 - \frac{x_2-1}{\sqrt{x_1^2+(x_2-1)^2}}\right)$

$x^* = (0,0) \Rightarrow \lim_{k\to\infty} \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} = \lim_{k\to\infty} \frac{\sqrt{\frac{x_1^2}{x_2^2+(x_2-1)^2} + (x_2-1)^2 - \frac{2(x_2-1)^2}{\sqrt{x_1^2+(x_2-1)^2}} + \frac{(x_2-1)^2}{x_1^2+(x_2-1)^2}}}{\sqrt{x_1^2+x_2^2}}$

$= \lim_{k\to\infty} \frac{\sqrt{1 + (x_2-1)^2\left(1 - \frac{2}{\sqrt{x_1^2+(x_2-1)^2}}\right)}}{\sqrt{x_1^2+x_2^2}} = \lim_{k\to\infty} \frac{\sqrt{1 + (x_2-1)^2\left(\frac{\sqrt{x_1^2+(x_2-1)^2} - 2}{\sqrt{x_1^2+(x_2-1)^2}}\right)}}{\sqrt{x_1^2+(x_2-1)^2}}$

$= \lim_{k\to\infty} \frac{\sqrt{\sqrt{x_1^2+(x_2-1)^2} + (x_2-1)^2\left(\sqrt{x_1^2+(x_2-1)^2} - 2\right)}}{(x_1^2+(x_2-1)^2)^{1/4}(x_1^2+x_2^2)^{1/2}} \approx \frac{\left(\sqrt{x_1^2+(x_2-1)^2} + (x_2-1)^2\left(\sqrt{x_1^2+(x_2-1)^2} - 2\right)\right)^{1/2}}{(x_1^2+x_2^2)^{3/4}}$

$\approx 1 \Rightarrow$ Sublinear
(confirmed in code)

1.d)

2.c) AP: $x_{n+1} \in P_A P_B x_n = \left(\frac{x_1+x_2}{2}, 0\right)$, $x^* = (0,0) \Rightarrow \lim_{k\to\infty} \frac{\|x_{n+1}-x^*\|}{\|x_n - x^*\|}$

$\Rightarrow \lim_{k\to\infty} \frac{1}{2}\frac{(x_1+x_2)}{\sqrt{x_1^2+x_2^2}} \simeq \frac{1}{2} < 1 \Rightarrow$ R-linear

D-R: $x_{n+1} \in \frac{1}{2}\left(R_A R_B x_n + x_n\right) = \frac{1}{2}\left((-x_2, x_1)+(x_1,x_2)\right) = \left(\frac{x_1-x_2}{2}, \frac{x_1+x_2}{2}\right)$

$x^*=(0,0) \Rightarrow \lim_{k\to\infty} \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|} = \lim_{k\to\infty} \frac{\sqrt{x_1^2-2x_1x_2+x_2^2+x_2^2+2x_1x_2+x_1^2}}{\sqrt{x_1^2+x_2^2}} = \lim_{k\to\infty} \frac{\sqrt{2}}{2}\frac{\sqrt{x_1^2+x_2^2}}{\sqrt{x_1^2+x_2^2}}$

$= \frac{1}{\sqrt{2}} < 1 \Rightarrow$ R-linear

2.d) ~~AP~~ AP: $x_{n+1} \in P_A P_B x_n = \left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}, \frac{\sqrt{2}}{2}\right)$, $x^* = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \therefore \lim_{k\to\infty} \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}$

$= \lim_{k\to\infty} \frac{\sqrt{\left(\frac{x_1}{\sqrt{x_1^2+x_2^2}} - 1/\sqrt{2}\right)^2}}{\sqrt{(x_1-1/\sqrt{2})^2+(x_2-1/\sqrt{2})^2}} = \lim_{k\to\infty} \frac{\sqrt{2}x_1 - \sqrt{x_1^2+x_2^2}}{\sqrt{2}\sqrt{x_1^2+x_2^2}\cdot\sqrt{(x_1-1/\sqrt{2})^2+(x_2-1/\sqrt{2})^2}}$ ~~$\simeq 1 \Rightarrow$ Sublinear~~

$= \lim_{k\to\infty} \frac{\sqrt{2}x_1 - \sqrt{x_1^2+1/2}}{\sqrt{2}\sqrt{x_1^2+1/2}\cdot(x_1-1/\sqrt{2})^2} \simeq 1 \Rightarrow$ Sublinear (confirmed in code)

~~D-R~~

D-R: $x_{n+1} \in \frac{1}{2}\left(R_A R_B x_n + x_n\right) = \frac{1}{2}\left(R_A R_B x_n + x_n\right) = \frac{1}{2}\left[\left(\frac{2x_1}{\sqrt{x_1^2+x_2^2}} - x_1, \sqrt{2} + x_2 - \frac{2x_2}{\sqrt{x_1^2+x_2^2}}\right) + (x_1, x_2)\right]$

$= \left(\frac{x_1}{\sqrt{x_1^2+x_2^2}}, \frac{1}{\sqrt{2}} + x_2 - \frac{x_2}{\sqrt{x_1^2+x_2^2}}\right)$, $x^* = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \Rightarrow \lim_{k\to\infty} \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}$

$= \lim_{k\to\infty} \frac{\sqrt{\left(\frac{x_1}{\sqrt{x_1^2+x_2^2}} - \frac{1}{\sqrt{2}}\right)^2 + \left(x_2 - \frac{x_2}{\sqrt{x_1^2+x_2^2}}\right)^2}}{\sqrt{x_1^2+x_2^2}\sqrt{(x_1-\frac{1}{\sqrt{2}})^2+(x_2-1/\sqrt{2})^2}} \simeq 1$ (confirmed in code) $\Rightarrow$ Sublinear