Problem 8.2.4

$$\min_{x} \|x\|_1$$
$$\text{Subj. to } Ax=b$$

$\Rightarrow \quad \min_{(x,z)} \{ f(x) + \|z\|_1 \mid x - z = 0 \} \qquad f(x) = \delta_{C = \{x \mid Ax - b = 0\}}$

$x^{k+1} \in P_C(z^k - \nu^k) \; - \; \min \frac{1}{2}\|x^k - (z^k - \nu^k)\|^2$
$\text{Subj. to } Ax = b$

$= \mathcal{L} = \frac{1}{2}\|x - (z-\nu)\|^2 + \lambda(Ax - b)$

$\nabla_x \mathcal{L} = x - (z - \nu) + A^T\lambda = 0 \Rightarrow x = (z - \nu) - A^T\lambda$

$\Rightarrow \mathcal{L} = \frac{1}{2}\|(z-\nu) - (z-\nu) - A^T\lambda\|^2 + \lambda(A(z-\nu) - AA^T\lambda - b)$

$= \frac{1}{2}\|-A^T\lambda\|^2 + \lambda(A(z-\nu) - AA^T\lambda - b)$

$\nabla_\lambda \mathcal{L} = AA^T\lambda + A(z-\nu) - 2AA^T\lambda - b = 0$

$\Rightarrow -AA^T\lambda = b - A(z-\nu)$

$\Rightarrow \lambda = (-AA^T)^{-1}b + (AA^T)^{-1}A(z-\nu) = (AA^T)^{-1}[A(z-\nu) - b]$

$\Rightarrow x^{k+1} = (z^k - \nu^k) - A^T(AA^T)^{-1}[A(z^k - \nu^k) - b] = (z^k - \frac{\lambda^k}{\rho^k}) - A^T(AA^T)^{-1}[A(z^k - \frac{\lambda^k}{\rho}) - b]$

$z^{k+1} = \operatorname{argmin}_z \mathcal{L}_{\rho^k}(x^{k+1}, z, \lambda^k) = \operatorname{argmin}_z \|z\|_1 + \lambda^T(x^{k+1} - z) + \frac{\rho}{2}\|x^{k+1} - z\|$

$\Rightarrow \nabla_z \sum_{i=1}^{m} |z_i| + \lambda^{k^T}(x - z) + \frac{\rho}{2}\|x - z\|^2$

$= \partial\left[\sum_{i=1}^{m} |\cdot|\right](z^{k+1}) - \lambda^k - \rho(x^{k+1} - z^{k+1}) \ni 0$

$\Rightarrow 0 \in \frac{1}{\rho}\partial|\cdot|(z_i^{k+1}) + z_i^{k+1} - (x^{k+1} + \frac{\lambda^k}{\rho})_i$

$\Rightarrow z^{k+1} = S_{\frac{1}{\rho}}(x^{k+1} + \frac{\lambda^k}{\rho}) = S_{\frac{1}{\rho}}(x^{k+1} + \nu^k)$

$\lambda^{k+1} = \lambda^k + \rho(x^{k+1} - z^{k+1})$

Problem 8.2.5

$$\min_{x} \frac{1}{2}\|Ax-b\|^2 + \gamma\|x\|_1$$

$$\Rightarrow \min_{(x,z)} \left\{ \frac{1}{2}\|Ax-b\|^2 + \gamma\|z\|_1 \mid x - z = 0 \right\}$$

$x^{k+1} \in \operatorname{argmin}_x \mathcal{L}_{\rho^k}(x, z^k, \lambda^k) = \frac{\partial}{\partial x} \frac{1}{2}\|Ax-b\|^2 + \gamma\|z^k\|_1 + \lambda^k(x - z^k) + \frac{\rho^k}{2}\|x - z^k\|^2$

$\Rightarrow A^T(Ax-b) + \lambda^k + \rho^k(x - z^k) = 0 \Rightarrow A^TAx - A^Tb + \lambda^k + \rho^k x - \rho^k z^k = 0$

$\Rightarrow (A^TA + \rho^k)x = A^Tb + \rho^k z^k - \lambda^k$

$\Rightarrow x^{k+1} = (A^TA + \rho^k)^{-1}[A^Tb + \rho^k z^k - \lambda^k]$

$z^{k+1} \in \operatorname{argmin}_z \mathcal{L}_{\rho^k}(x^{k+1}, z, \lambda^k) = \frac{\partial}{\partial z}\frac{1}{2}\|Ax^{k+1}-b\|^2 + \gamma\|z\|_1 + \lambda^k(x^{k+1} - z) + \frac{\rho^k}{2}\|x^{k+1} - z\|^2$

$= \frac{\partial}{\partial z} \frac{1}{2}\|Ax^{k+1}-b\|^2 + \gamma \sum_{i=1}^{m} |z_i| + \lambda^k(x^{k+1} - z) + \frac{\rho^k}{2}\|x^{k+1} - z\|^2$

$\Rightarrow \gamma \cdot \left[\frac{\partial}{\partial z}\left[\sum_{i=1}^{m} |\cdot|\right](z^{k+1}) - \lambda^k - \rho^k(x^{k+1} - z^{k+1})\right] \ni 0$

$\Rightarrow x^{k+1} - \frac{\lambda^k}{\rho^k} \in \left[\frac{\gamma}{\rho^k}\frac{\partial}{\partial z}\left[\sum_{i=1}^{m} |\cdot|\right](z^{k+1}) + z^{k+1}\right] \Rightarrow z^{k+1} = S_{\gamma/\rho^k}(x^{k+1} - \mu^k)$

where $\mu^k = \frac{\lambda^k}{\rho^k}$

$\mu^{k+1} = \mu^k + (x^{k+1} - z^{k+1}) \Rightarrow \lambda^{k+1} = \lambda^k + \rho(x^{k+1} - z^{k+1})$