Problem 6.3.5

1. We have $z = x - y \Rightarrow \|x-(y+\lambda e)\|^2 \geq \|z\|^2$

$\Rightarrow \|x-(y+\lambda e)\|^2 \geq \|x-y\|^2$

We know that $y$ is the unique solution to the strictly convex minimisation problem $\min_{\hat{y} \in M} \|\hat{y}-x\|$, and as $\lambda$ is a constant $>0$ and $y, e \in M$, we have $(y+\lambda e) \in M$ as well. We therefore know that since this is the case, and there is no value smaller for $\|x-y\|$ we have that

$$\|x-(y+\lambda e)\| \geq \|x-y\|$$

$$\Rightarrow \|x-(y+\lambda e)\|^2 \geq \|x-y\|^2$$

$$\|x-(y+\lambda e)\|^2 - \|z\|^2 \geq 0$$

$$\Rightarrow \|x\|^2 - 2\langle x, y+\lambda e\rangle + \|y+\lambda e\|^2 - \|x-y\|^2 \geq 0$$

$$\Rightarrow \cancel{\|x\|^2} - 2\langle x, y+\lambda e\rangle + \|y+\lambda e\|^2 - \cancel{\|x\|^2} + 2\langle x,y\rangle - \|y\|^2 \geq 0$$

$$\Rightarrow -2\cancel{\langle x,y\rangle} - 2\langle x,\lambda e\rangle + \cancel{\|y\|^2} + 2\langle y,\lambda e\rangle + \|\lambda e\|^2 + \cancel{2\langle x,y\rangle} - \cancel{\|y\|^2} \geq 0$$

$$\Rightarrow -2\langle x,\lambda e\rangle + 2\langle y,\lambda e\rangle + \|\lambda e\|^2 \geq 0$$

$$\Rightarrow -2\langle x-y,\lambda e\rangle + \|\lambda e\|^2 \geq 0$$

$$\Rightarrow -2\langle z,\lambda e\rangle + \|\lambda e\|^2 \geq 0 \Rightarrow \lambda^2\|e\|^2 - 2\lambda\langle z,e\rangle \geq 0$$

$$\Rightarrow \lambda[\lambda\|e\|^2 - 2\langle z,e\rangle] \geq 0$$

Problem 6.3.5

2. From 1. we have $\lambda[\lambda\|e\|^2 - 2\langle z,e\rangle] \geq 0$

$\Rightarrow \lambda\|e\|^2 - 2\langle z,e\rangle \geq 0$, taking the case $\lambda=0$ we have

$$-2\langle z,e\rangle \geq 0 \Rightarrow \langle z,e\rangle \leq 0$$

Knowing that since $\forall e\in M, \forall -e \in M$ aswell

$\Rightarrow \quad -2\langle z,-e\rangle \geq 0 \Rightarrow \langle z,e\rangle \geq 0$

$\Rightarrow \quad \langle z,e\rangle = 0$

we are

3. Given $x\in X$, $y\in M$, $z\in M^{\perp}$, and know that $y = \min_{\hat{y}\in M} \|\hat{y}-x\|$. Since the euclidean norm is strictly convex, we have that the minimiser is unique for each $x$. In addition, we have $z\in M^{\perp}$ and $x=y+z$. By Orthogonal Decomposition, we have that $y$, is the projection and hence is unique.

4. From 3. we have $\forall x\in X$, $\exists y\in M$, $z\in M^{\perp}$, and $x=y+z$

$\therefore \quad \|x\|^2 = \|y+z\|^2 = \|y\|^2 + \|z\|^2 + 2\langle y,z\rangle$.

Given $y\in M$, $z\in M^{\perp} := \{x \mid \langle x,m\rangle = 0 \ \forall m\in M\}$, we have that $\langle y,z\rangle = 0 \Rightarrow \|x\|^2 = \|y\|^2 + \|z\|^2$.

Again from 3. we have that $y$ is the projection of $x \Rightarrow y = Px$.

$\Rightarrow \|x\|^2 = \|y\|^2 + \|z\|^2 \Rightarrow \|x\|^2 = \|y\|^2 + \|x-y\|^2$

$\Rightarrow \|x\|^2 = \|y\|^2 + \cancel{\|x\|^2} - 2\langle x,y\rangle + \|y\|^2$

$\Rightarrow 2\langle x,y\rangle = 2\|y\|^2$

$\Rightarrow 2\langle x,Px\rangle = 2\|Px\|^2 \Rightarrow \|Px\|^2 = \langle Px,x\rangle$

Problem 6.3.6

1. We know that any $e \in C$ satisfies

$$\langle e - P_C u, u - P_C u \rangle \leq 0 \text{ for any } u$$

we first let $x = u$ and $e = P_C y$

$\Rightarrow \langle P_C y - P_C x, x - P_C x \rangle \leq 0 \Rightarrow \langle x - P_C x, P_C y - P_C x \rangle \leq 0$

now we let $y = u$ and $e = P_C x$

$\Rightarrow \langle P_C x - P_C y, y - P_C y \rangle \leq 0 \Rightarrow \langle y - P_C y, P_C x - P_C y \rangle \leq 0$

Adding the two, we get

$$\Rightarrow 0 \geq \langle x - P_C x - (y - P_C y), P_C y - P_C x \rangle$$
$$= \langle x - y - P_C x - P_C y, P_C y - P_C x \rangle$$
$$= -\langle x - y, P_C x - P_C y \rangle + \|P_C y - P_C x\|^2$$
$$\Rightarrow \langle x - y, P_C x - P_C y \rangle \geq \|P_C y - P_C x\|^2 = \|P_C x - P_C y\|^2$$

As $\|P_C y - P_C x\|^2$ is norm squared, ~~and~~ it must therefore be nonnegative

$$\Rightarrow \langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2 \geq 0$$

2.
$$\|\alpha x + (1-\alpha) y\|^2 + \alpha(1-\alpha)\|x - y\|^2$$
$$= \|\alpha x\|^2 + 2\langle \alpha x, (1-\alpha) y \rangle + \|(1-\alpha) y\|^2 + \alpha(1-\alpha)\left[\|x\|^2 - 2\langle x, y \rangle + \|y\|^2\right]$$
$$= \cancel{\|\alpha x\|^2} + 2\langle \alpha x, (1-\alpha) y \rangle + \|(1-\alpha) y\|^2 + \alpha\|x\|^2 - \cancel{\alpha^2\|x\|^2} - \alpha(1-\alpha) 2\langle x, y \rangle$$
$$+ \alpha(1-\alpha)\|y\|^2$$
$$= 2\langle \alpha x, (1-\alpha) y \rangle + \|y\|^2\left[(1-\alpha)^2 + \alpha(1-\alpha)\right] + \alpha\|x\|^2 - \alpha(1-\alpha) 2\langle x, y \rangle$$
$$= 2\langle \alpha x, (1-\alpha) y \rangle + \|y\|^2\left[1 - \cancel{2}\alpha + \cancel{\alpha^2} + \cancel{\alpha} - \cancel{\alpha^2}\right] + \alpha\|x\|^2 - \alpha(1-\alpha) 2\langle x, y \rangle$$
$$= \cancel{2\alpha(1-\alpha)\langle x, y \rangle} - \cancel{2\alpha(1-\alpha)\langle x, y \rangle} + \|y\|^2(1-\alpha) + \alpha\|x\|^2$$
$$= \alpha\|x\|^2 + (1-\alpha)\|y\|^2$$

Problem 6.3.6

3. From 2. we have

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2$$

if we let $\alpha = 2$, $x = Tx - Ty$, $y = x - y$, we have

$$\|2(\cancel{a}Tx - Ty) - (x-y)\|^2 - 2\|(Tx-Ty)-(x-y)\|^2 = 2\|Tx-Ty\|^2 - \|x-y\|^2$$

$$\Rightarrow \quad \|2(Tx-Ty) - (x-y)\|^2 = 2\|Tx-Ty\|^2 - \|x-y\|^2 + 2\|(Tx-Ty)-(x-y)\|^2$$

Using $R := 2T - I$

$$\|Rx - Ry\|^2 \Rightarrow \|2Tx - x - 2Ty + y\|^2 = \|(2T\cancel{y}x - 2Ty) - (x-y)\|^2$$

$$= \|2(T\cancel{y}x - Ty) - (x-y)\|^2 \Rightarrow 2\|Tx-Ty\|^2 - \|x-y\|^2 + 2\|(Tx-Ty)-(x-y)\|^2$$

by the expression above

$$\Rightarrow 2\|Tx-Ty\|^2 - \|x-y\|^2 + 2\|Tx - x - Ty + y\|^2$$

$$\Rightarrow 2\|Tx-Ty\|^2 - \|x-y\|^2 + 2\|(T-I)x - \cancel{y(I-T)}(T-I)y\|^2$$

$$= 2\|Tx-Ty\|^2 - \|x-y\|^2 + 2\|(I-T)x - (I-T)y\|^2$$

Problem 6.3.6

4 a $\Leftrightarrow$ b

we have $\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2$

$\Rightarrow 2\|Tx - Ty\|^2 + 2\|(I-T)x - (I-T)y\|^2 \leq 2\|x-y\|^2$

$\Rightarrow 2\|Tx - Ty\|^2 - \|x-y\|^2 + 2\|(I-T)x - (I-T)y \leq \|x-y\|^2$

From 3. LHS $= \|(2T-I)x - (2T-I)y\|^2$

$\Rightarrow \|(2T-I)x - (2T-I)y\|^2 \leq \|x-y\|^2$

$\Rightarrow \|(2T-I)x - (2T-I)y\| \leq \|x-y\|$

$\therefore$ we have $2T-I$ is non-expansive.

Problem 6.3.6

a. b$\Leftrightarrow$c we have $2T-I$ is non-expansive $\Rightarrow$ $T$ is firmly non-expansive

$$\Rightarrow \|Tx-Ty\|^2 + \|(I-T)x-(I-T)y\|^2 \le \|x-y\|^2$$

$$\Rightarrow \|Tx-Ty\|^2 \le \|x-y\|^2 - \|(I-T)x-(I-T)y\|^2$$

$$\Rightarrow \|Tx-Ty\|^2 \le \|x-y\|^2 - \|x-Tx\|^2 + 2\langle x-Tx, y-Ty\rangle - \|y-Ty\|^2$$

$$\Rightarrow \|Tx-Ty\|^2 \le \cancel{\|x\|^2} - 2\langle x,y\rangle + \cancel{\|y\|^2} - \cancel{\|x\|^2} + 2\langle x,Tx\rangle - \|Tx\|^2 + 2\langle x-Tx, y-Ty\rangle - \cancel{\|y\|^2} + 2\langle y,Ty\rangle - \|Ty\|^2$$

$$\Rightarrow \|Tx-Ty\|^2 \le 2\langle x,Tx\rangle - 2\langle x,y\rangle + 2\langle x-Tx, y-Ty\rangle + 2\langle y,Ty\rangle - \|Tx\|^2 - \|Ty\|^2$$

$$\Rightarrow \|Tx-Ty\|^2 \le 2\langle x,Tx\rangle - \cancel{2\langle x,y\rangle} + \cancel{2\langle x,y\rangle} - 2\langle x,Ty\rangle - 2\langle y,Tx\rangle + 2\langle Tx,Ty\rangle + 2\langle y,Ty\rangle - \|Tx\|^2 - \|Ty\|^2$$

$$\Rightarrow \|Tx-Ty\|^2 \le 2\langle x,Tx\rangle - 2\langle x,Ty\rangle - 2\langle y,Tx\rangle + 2\langle Tx,Ty\rangle + 2\langle y,Ty\rangle - \|Tx\|^2 - \|Ty\|^2$$

$$\Rightarrow \|Tx-Ty\|^2 \le 2\langle x,Tx\rangle - 2\langle x,Ty\rangle - 2\langle y,Tx\rangle + \cancel{2\langle Tx,Ty\rangle} + 2\langle y,Ty\rangle - \|Tx-Ty\|^2$$

$$\Rightarrow 2\|Tx-Ty\|^2 \le 2\langle x,Tx\rangle - 2\langle x,Ty\rangle - 2\langle y,Tx\rangle + 2\langle y,Ty\rangle$$

$$\Rightarrow \|Tx-Ty\|^2 \le \langle x,Tx\rangle - \langle y,Tx\rangle - \langle x,Ty\rangle + \langle y,Ty\rangle$$

$$\Rightarrow \|Tx-Ty\|^2 \le \langle Tx-Ty, x-y\rangle$$

As the norm squared is non-negative

$$\Rightarrow \langle Tx-Ty, x-y\rangle \ge \|Tx-Ty\|^2 \ge 0$$