Problem 10.2.2

1. $\langle x^*, x\rangle = |\sum_{i=1}^{m} x_i \cdot x_i^*| \leq \sum_{i=1}^{m} |x_i \cdot x_i^*| \leq \|x_i^*\|_\infty \sum_{i=1}^{m} |x_i| = \|x^*\|_\infty \|x\|_1$

Where equality holds when ~~x* is a constant vector~~ $x^* = \text{Sign}(x)$

2. $h(x) = \|x\|_1 \Rightarrow h^*(x^*) = \sup_x \{\langle x, x^*\rangle - \|x\|_1\}$

$$= \sup_{\|x\|_1 = k,\, k\in\mathbb{R}} \{\langle x, x^*\rangle - \|x\|_1\} = \sup_{k\in\mathbb{R}} \sup_{\|x\|_1 = k} \{\langle x, x^*\rangle - \|x\|_1\}$$

$$= \sup_{k\in\mathbb{R}} \{k\|x^*\|_\infty - k\} = \sup_{k\in\mathbb{R}} k(\|x^*\|_\infty - 1) = \begin{cases} +\infty & \text{if } \|x^*\|_\infty > 1 \\ 0 & \text{if } \|x^*\|_\infty \leq 1 \end{cases}$$

$$= \delta_{B_1^\infty(0)}(x)$$

3. From 1. we have $\langle x^*, x\rangle \leq \|x^*\|_\infty \|x\|_1$

We also have $h(x) = \|x\|_1$, and $h^*(x^*) = \delta_{B_1^\infty(0)}(x^*)$

$$\Rightarrow h(x) + h^*(x^*) = \begin{cases} +\infty, & \text{when } \|x^*\|_\infty > 1 \\ \|x\|_1, & \text{when } \|x^*\|_\infty \leq 1 \end{cases}$$

if $x^* \in \partial h(x)$, by the subgradient equality we have $\langle x^*, x\rangle \geq h(x) + h^*(x^*) = \begin{cases} +\infty, & \text{if } \|x^*\|_\infty > 1 \\ \|x\|_1, & \text{if } \|x^*\|_\infty \leq 1 \end{cases}$

As $\langle x^*, x\rangle \geq +\infty$ is infeasible $\Rightarrow \langle x^*, x\rangle \geq \|x\|_1$ and $\|x^*\|_\infty \leq 1$

From 1. we have $\langle x^*, x\rangle \leq \|x\|_1 \cdot \|x^*\|_\infty$. Given $\|x^*\|_\infty \leq 1$

$\Rightarrow \langle x^*, x\rangle \leq \|x\|_1 \cdot \|x^*\|_\infty \leq \|x\|_1$

$\Rightarrow$ we must have $\langle x^*, x\rangle = \|x\|_1 \cdot \|x^*\|_\infty = \|x\|_1$

By Fenchel's equality, given $x^* \in \partial h(x)$, ~~we have~~ $\cancel{\langle x^*, x\rangle = g(x) + g^*(x^*) = \|x\|_1 = \|x\|_1 \cdot \|x^*\|_\infty \Rightarrow \|x^*\|_\infty = 1 \text{ when } x \neq 0}$

we have $\langle x^*, x\rangle = h(x) + h^*(x^*)$

$$= \|x\|_1 = \|x\|_1 \cdot \|x^*\|_\infty$$

$$\Rightarrow \|x^*\|_\infty = 1 \text{ when } x \neq 0$$

3. Continued

Now given $\|x\|_1 \|x^*\|_\infty = \langle x, x^* \rangle$ and $\|x^*\|_\infty \le 1$

we know that $h(x) + h^*(x^*) = \|x\|_1$ as previously shown
we also know that when $x \neq 0$, $\|x^*\|_\infty = 1 \Rightarrow \langle x, x^* \rangle = \|x\|_1$

$\Rightarrow$ we have $h(x) + h^*(x^*) = \langle x^*, x \rangle$ which by fenchel's equality means $x^* \in \partial h(x)$

4. $f(x^*) = \epsilon \|x^*\|_\infty \Rightarrow f^*(x) = \sup_{x^*} \{ \langle x^*, x \rangle - \epsilon \|x^*\|_\infty \} = \sup_{\epsilon \|x^*\|_\infty = k, k \in \mathbb{R}} \{ \langle x^*, x \rangle - \epsilon \|x^*\|_\infty \}$

$= \sup_{k \in \mathbb{R}, \epsilon \|x^*\|_\infty = k} \{ \langle x^*, x \rangle - \epsilon \|x^*\|_\infty \} = \sup_{k \in \mathbb{R}} \{ \frac{k}{\epsilon} \|x\|_1 - k \}$

$= \sup_{k \in \mathbb{R}} k \{ \frac{1}{\epsilon} \|x\|_1 - 1 \} = \begin{cases} +\infty & \text{if } \frac{1}{\epsilon} \|x\|_1 > 1 \\ 0 & \text{if } \frac{1}{\epsilon} \|x\|_1 \le 1 \end{cases}$

$= \begin{cases} +\infty & \text{if } \|x\|_1 > \epsilon \\ 0 & \text{if } \|x\|_1 \le \epsilon \end{cases} = \delta_{B_\epsilon^1(0)}(x)$

By the biconjugate theorem, we have $f(x^*) = f^{**}(x^*)$. Given that $g(x) = f^*(x)$
$\Rightarrow f^{**}(x^*) = f(x^*) = g^*(x^*) = \epsilon \|x^*\|_\infty$

5. Again from 1. we have $\langle x^*, x \rangle \le \|x\|_1 \|x^*\|_\infty$
and from 4. we have $g(x) = \delta_{B_\epsilon^1(0)}(x)$ and $g^*(x^*) = \epsilon \|x^*\|_\infty$
$\Rightarrow g(x) + g^*(x^*) = \begin{cases} +\infty, & \|x\|_1 > \epsilon \\ \epsilon \|x^*\|_\infty, & \|x\|_1 \le \epsilon \end{cases}$

Given $x^* \in \partial g(x) \Rightarrow$ By the subgradient inequality we have $\langle x^*, x \rangle \ge g(x) + g^*(x^*)$

Since $\langle x^*, x \rangle \ge +\infty$ is infeasible, we must have $\|x\|_1 \le \epsilon$
$\Rightarrow \langle x^*, x \rangle \ge \epsilon \|x^*\|_\infty$
From 1. however, we have $\langle x^*, x \rangle \le \|x\|_1 \cdot \|x^*\|_\infty \le \epsilon \|x^*\|_\infty$
$\Rightarrow$ we must have $\langle x^*, x \rangle = \|x\|_1 \cdot \|x^*\|_\infty = \epsilon \|x^*\|_\infty$

By fenchel's equality, we have $\langle x^*, x \rangle = g(x) + g^*(x^*) = \epsilon \|x^*\|_\infty = \|x\|_1 \cdot \|x^*\|_\infty$
$\Rightarrow$ when $x \neq 0$, $\|x\|_1 = \epsilon$

5. cont

if we have $\langle x^*, x\rangle = \|x\|_1 \cdot \|x^*\|_\infty$ and $\|x\|_1 \leq t$

we have $g(x) + g^*(x^*) = t\|x^*\|_\infty$, we also have that since $\|x\|_1 = t$

$\langle x^*, x\rangle = t\|x^*\|_\infty \Rightarrow g(x) + g^*(x^*) = \langle x^*, x\rangle$

$\Rightarrow$ By Fenchel's equality, we then have $x^* \in \partial g(x)$

6.

$0 \in \partial_x\{\frac{1}{2}\|y - Ax\|_2^2 + \delta_{B_t^1(0)}(x)\} \Rightarrow -A^T(y - Ax) + \partial_x \delta_{B_t^1(0)}(x) = 0$

$\Rightarrow \partial_x \delta_{B_t^1(0)}(x) = A^T(y - Ax) \Rightarrow x^* \in \partial_x B_t^1(0)(x) \Rightarrow x^* \in N_{B_t^1(0)}(x)$

$0 \in \partial_x\{\frac{1}{2}\|y - Ax\|_2^2 + r\|x\|_1\} \Rightarrow -A^T(y - Ax) + \partial_x r\|x\|_1 = 0$

$\Rightarrow \partial_x r\|x\|_1 = A^T(y - Ax) \Rightarrow x^* \in \partial_x \|x\|_1 \Rightarrow x^* \in S_r(x)$

$$x^* \in \begin{cases} r\,\mathrm{sign}(x_i) & \text{if } x_i \neq 0 \\ |x\,\mathrm{sign}(x_i)| \leq r & \text{if } x_i = 0 \end{cases}$$