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AN INTRODUCTION TO MULTICOMPLEX SPACES AND FUNCTIONS

G. Baley Price

An Introduction to Multicomplex Spaces and Functions

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G. BALEY PRICE

University of Kansas
Lawrence, Kansas

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Foreword

This book arose out of a doctoral thesis in the early 1950s which was influenced by Professor Price. I expect it will influence new students now. It is a book for analysts and algebraists as well.

The theory of functions of one complex variable, whether treated in the manner of Gauss, Cauchy, Riemann, or Weierstrass, is one of the enduring staples of every mathematician's education. It was natural to attempt extension to the case of several complex variables, to see where the natural extensions occur and where they do not. This study was started by Weierstrass, followed by Poincaré, Cousin, and Picard, who solved and formulated basic problems.

In another direction there is the theory of functions of a quaternion variable, associated with Fueter in Switzerland. The author studies another extension. It was Segre in Italy who had the most definite influence. The book develops the subject as nearly parallel as was possible to the theory of functions of a complex variable. This is in spite of the fact that there exist divisors of zero. "They only add interest."

The mathematical community will be grateful to Professor Price for returning to the subject and writing *An Introduction to Multicomplex Spaces and Functions.*

Olga Taussky Todd
California Institute of Technology
Pasadena, California

Preface

This book treats two subjects: first, a class of Banach algebras known collectively as multicomplex spaces, and second, the theory of holomorphic functions defined on these multicomplex spaces. These spaces and functions are closely related to the space of complex numbers and to the theory of holomorphic functions of a complex variable. A brief history of the development of the ideas that have led to this study will be helpful.

Although Karl Friedrich Gauss (1777–1855) had very early discovered some of the properties of functions of a complex variable, he published little, and from 1814 on Augustin Louis Cauchy (1789–1857) became the effective founder of the theory of functions of a complex variable. The foundations of the subject rest on the concept of an algebra as well as on ideas about the theory of functions. Originally the only algebra known to mathematics was the algebra of real numbers, but the intrusion and discovery of the complex numbers initiated a broadening of the concept. George Peacock (1791–1858) in 1830 published his *Treatise on Algebra* in an effort to give algebra a logical structure similar to that in Euclid's *Elements*. In 1833 Sir William Rowan Hamilton (1805–1865) presented a paper to the Irish Academy in which he developed a formal algebra of real numbers which is precisely the algebra of the complex numbers as usually understood today. In 1843 Hamilton discovered quaternions, and in 1844 Hermann Günther Grassmann (1809–1877) published his *Ausdehnungslehre*. Benjamin Peirce (1809–1880) in 1864 presented his paper entitled *Linear Associative Algebra* to the American

Association for the Advancement of Science, but it was not published until 1881. There had been only one algebra at the beginning of the century, but Benjamin Peirce worked out multiplication tables for 162 linear associative algebras. Special algebras were discovered and investigated in detail; several have been described already. In addition, Arthur Cayley (1821–1895), Benjamin Peirce, and Charles S. Peirce (1839–1914) developed matrix algebras; William Kingdon Clifford (1845–1879) developed the Clifford algebras, of which octonians, or biquaternions, are special cases; and the vector analysis of Josiah Williard Gibbs (1839–1903) appeared in 1881 and 1884.

Finally, in 1892, in the search for and development of special algebras, Corrado Segre (1860–1924) published a paper [12] (see bibliography at the end of the book) in which he treated an infinite set of algebras whose elements he called bicomplex numbers, tricomplex numbers, ..., n-complex numbers, ... A bicomplex number is an element of the form $(x_1+i_1x_2)+i_2(x_3+i_1x_4)$, where $x_1, \ldots, x_4$ are real numbers, $i_1^2=i_2^2=-1$, and $i_1i_2=i_2i_1$. The bicomplex numbers can be embedded in euclidean space of 2^2 dimensions. Alternatively, a bicomplex number is an element $z_1+i_2z_2$, where z_1 and z_2 are complex numbers, and the rules of operation are formally the same as for complex numbers. Segre used two bicomplex numbers to form a tricomplex number $[(x_1+i_1x_2)+i_2(x_3+i_1x_4)]+i_3[(x_5+i_1x_6)+i_2(x_7+i_1x_8)]$. The units are 1, i_1, i_2, i_3, i_1i_2, i_1i_3, i_2i_3, $i_1i_2i_3$; all multiplications are commutative, and $i_1^2=i_2^2=i_3^2=-1$. The tricomplex numbers are embedded in euclidean space of 2^3 dimensions. Segre showed that his construction could be iterated indefinitely to form n-complex numbers which are embedded in euclidean space of 2^n dimensions. Segre showed that the bicomplex numbers contain divisors of zero, and he showed that every bicomplex number $z_1 + i_2z_2$ can be represented as the complex combination $(z_1-i_1z_2)[(1+i_1i_2)/2)] +(z_i + i_1z_2)[(1 - i_1i_2)/2]$ of the idempotent elements $(1 + i_1i_2)/2$ and $(1-i_1i_2)/2$. In this book, the elements of the algebras introduced by Segre are called bicomplex numbers and, collectively, multicomplex numbers. With the addition of the euclidean norm of the space in which Segre's algebras are embedded, they become the Banach algebras which are the bicomplex space and the multicomplex spaces of this book.

The theory of functions of a complex variable is based on the algebra of complex numbers, and the discovery of linear associative algebras has led to many efforts to develop similar theories of functions in other algebras. As early as 1894 Scheffers [11] investigated the generalization of functions of a complex variable, and by 1940 the literature on the subject was enormous: the bibliography of James A. Ward's paper [18] entitled *Theory of Analytic Functions in Linear Associative Algebras* contained 81 references (see also some sections in Hille's book [4]).

Michiji Futagawa [3] in 1928 and 1932 seems to have been the first to consider the theory of functions of a bicomplex variable. Although his quaternary variable is equivalent to the bicomplex variable, Futagawa did not treat it as such. The hypercomplex system of Friedrich Ringleb [9] is more general than the bicomplex algebra; he showed in 1933 that Futagawa's system is a special case of his own. Ringleb's results included one of the fundamental theorems of the subject; he showed that every holomorphic function of a bicomplex variable (a function that has a derivative) can be represented by two holomorphic functions of a complex variable. There was considerable activity in the field for several years: Scorza Dragoni wrote a paper [2] on holomorphic functions of a bicomplex variable in 1934; Ugo Morin (1901–?) investigated in 1935 the algebra of the bicomplex numbers [6]; Spampinato wrote three papers (see [13], [14], [15]) on functions of a bicomplex variable in 1935 and 1936; and Tsurusaburo Takasu (1890–?) published a paper [16] on a generalized bicomplex variable in 1943.

In 1953 James D. Riley published a paper [8] entitled *Contributions to the Theory of Functions of a Bicomplex Variable.* The first page of the paper contains the following footnote:

> This paper was written as a Ph.D. thesis at the University of Kansas under the supervision of Prof. V. Wolontis and many of the problems and numerous changes have been suggested by him. The author wishes to express his appreciation. The project was originally proposed by Prof. G. B. Price, and a preliminary investigation was made by him.

I no longer remember how or where I first learned about functions of a bicomplex variable, and the only outcome of my "preliminary investigation" was Riley's thesis, supervised by Wolontis. Other activities and subjects claimed my attention, and I had no opportunity to make any further study of the field. And there the matter stood until 1984, when Olga Taussky Todd expressed regret that I had published nothing on the subject. The remark led me to undertake the writing of this book.

I decided to make a fresh start, and I have developed the subject as nearly like the theory of functions of a complex variable as I could. I have used $\mathbb{C}_0$ to denote the real numbers $\mathbb{R}$; I have used $\mathbb{C}_1$ to denote the complex numbers $\mathbb{C}$, $\mathbb{C}_2$ to denote the bicomplex numbers. More generally, $\mathbb{C}_n$ denotes, for $n \geqslant 2$, the n-complex numbers of Segre. The addition of the norm in $\mathbb{R}^{2^n}$, in which $\mathbb{C}_n$ has a natural embedding, converts the n-complex numbers into a Banach algebra, which also is denoted by $\mathbb{C}_n$. The first four chapters of this book contain a detailed treatment of $\mathbb{C}_2$ and of the differentiable functions on $\mathbb{C}_2$.

The fifth chapter treats $\mathbb{C}_n$, $n \geqslant 3$, and its differentiable functions, and the emphasis is on large values of n. Insofar as my knowledge goes, this is the first treatment of this part of the subject. There is a wealth of homomorphic representations of the space $\mathbb{C}_n$ and of the holomorphic functions in it. There is a matrix algebra homomorphic to $\mathbb{C}_n$; the matrices in it are called Cauchy–Riemann matrices. The determinants of Cauchy–Riemann matrices can be factored into determinants of matrices of lower order. Elementary methods suffice for the proof for small values of n, but new methods are needed for large n because of the size and complexity of the mass of details. For example, the real Cauchy–Riemann matrix in $\mathbb{C}_{10}$ is a $2^{10} \times 2^{10}$ matrix with more than a million elements. A holomorphic function in $\mathbb{C}_n$ maps a set in $\mathbb{C}_n$ into a set in $\mathbb{C}_n$; properties of the Cauchy–Riemann matrices can be used to prove that the jacobian of the mapping is nonnegative at every point, and that it is positive at every point at which the derivative is not zero or a divisor of zero. A holomorphic function in $\mathbb{C}_n$ can be represented by 2^n functions of 2^n real variables; these functions satisfy a system of Cauchy–Riemann differential equations. A holomorphic function in $\mathbb{C}_n$ can be represented also by 2^{n-1} functions of 2^{n-1} complex variables (variables in $\mathbb{C}_1$); these functions satisfy another system of Cauchy–Riemann differential equations. These two representations are the first two in a sequence; in the final one, the holomorphic function can be represented by a pair of holomorphic functions of two variables in $\mathbb{C}_{n-1}$, and there is a corresponding pair of Cauchy–Riemann differential equations. The section headings in the Contents give a more complete indication of the topics treated in the entire book.

Among other results in $\mathbb{C}_n$, $n \geqslant 3$, Chapter 5 proves the fundamental theorem of the integral calculus, Cauchy's integral theorem, and Cauchy's integral formula. These theorems are proved in $\mathbb{C}_n$ without any appeal to, or use of, functions of a complex variable in $\mathbb{C}_1$. The theory of holomorphic functions in $\mathbb{C}_n$ seems to be as complete and detailed as the theory of holomorphic functions in $\mathbb{C}_1$, but it is more interesting because of the vastly richer structure. Some have stated that a theory of functions in $\mathbb{C}_n$, $n \geqslant 2$, is impossible because of the presence of divisors of zero, but the presence of these singular elements does not hinder the development of the theory – they only add interest.

The theory of holomorphic functions in $\mathbb{C}_n$ is a natural continuation of the theory of holomorphic functions of a complex variable. The results given above suggest some of the reasons the subject is interesting. I hope that this book provides an introduction which will display some of the beauty and interest of the field, and that it will make the subject accessible to others. The Epilogue (Chapter 6) describes areas that have not been examined thus far. The investigation of m-dimensional multicomplex spaces and of functions of several multicomplex variables seems to be a natural next step.

Anyone who has had an introduction to the theory of functions of a complex variable should be able to read this book without difficulty. With respect to multicomplex spaces and functions, it is self-contained and complete. The explanations are full and complete, and there are exercises to provide examples and to complete the exposition.

I take this opportunity to acknowledge my indebtedness to Olga Taussky Todd and to thank her for encouraging the exploration of the multicomplex spaces and their holomorphic functions and for writing the Foreword for this book. Also, it is a pleasure to acknowledge the assistance of Sharon Gumm and to thank her for her careful typing of the manuscript.

G. Baley Price

Contents

1

The Bicomplex Space

1. INTRODUCTION

This chapter contains an introduction to the Banach algebra of bicomplex numbers. This space of bicomplex numbers is the first in an infinite sequence of multicomplex spaces which are generalizations of the space of complex numbers.

For convenience, the real numbers $\mathbb{R}$ are usually denoted in this book by $\mathbb{C}_0$, and the complex numbers $\mathbb{C}$ are denoted by $\mathbb{C}_1$. An element in $\mathbb{C}_1$ is a number of the form

(1) $\qquad x_1 + i_1x_2, \qquad x_1, x_2 \text{ in } \mathbb{C}_0,\ i_1^2 = -1.$

An element in the space $\mathbb{C}_2$ of bicomplex numbers is a number of the form

(2) $$x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4,$$
$$x_1, \ldots, x_4 \text{ in } \mathbb{C}_0,\ i_1^2 = -1,\ i_2^2 = -1,\ i_1i_2 = i_2i_1.$$

Map the element (2) into the point $(x_1, \ldots, x_4)$ in $\mathbb{C}_0^4$; this mapping embeds $\mathbb{C}_2$ into $\mathbb{C}_0^4$. Addition of two elements in $\mathbb{C}_2$ and scalar multiplication of an element in $\mathbb{C}_2$ by a real number are defined in the usual way; $\mathbb{C}_2$ is a linear space with respect to addition and scalar multiplication. The norm of $x_1+i_1x_2+i_2x_3+i_1i_2x_4$ is defined to be the norm of $(x_1, \ldots, x_4)$ in $\mathbb{C}_0^4$; with this norm, $\mathbb{C}_2$ is a normed linear space. Since $\mathbb{C}_0^4$ is complete, $\mathbb{C}_2$ is complete and thus a Banach space. Two elements in $\mathbb{C}_2$ are multiplied as if they were polynomials; the assumptions in (2) show that $\mathbb{C}_2$ is closed under multiplication. Also, multiplication is associative and commutative, and $\mathbb{C}_2$ is a Banach algebra.

There is a second representation of elements in $\mathbb{C}_2$ which is important. From (2),

$$(3)\qquad x_1+i_1x_2+i_2x_3+i_1i_2x_4=(x_1+i_1x_2)+i_2(x_3+i_1x_4),$$

and (1) shows that $x_1+i_1x_2$ and $x_3+i_1x_4$ are complex numbers z_1 and z_2 in $\mathbb{C}_1$. Thus the element (2) can be represented as

$$(4)\qquad z_1+i_2z_2,\qquad z_1,\ z_2 \text{ in } \mathbb{C}_1,\ i_2^2=-1.$$

Map the element (4) into the point (z_1, z_2) in $\mathbb{C}_1^2$; this mapping embeds $\mathbb{C}_2$ into $\mathbb{C}_1^2$. Also, $\mathbb{C}_0$ is a subspace of $\mathbb{C}_1$, and $\mathbb{C}_1$ is a subspace of $\mathbb{C}_2$. The representation (4) is the reason why the elements (2) are called *bicomplex numbers.* In some cases the representation (4) is merely a matter of notational convenience; in others, it is essential to the development of the theory. Furthermore, some interesting problems arise from a comparison of results obtained from the two methods of representing elements in $\mathbb{C}_2$.

An element which is equal to its square is called an *idempotent* element. There are four idempotent elements in $\mathbb{C}_2$; they are

$$(5)\qquad 0,\ \ 1,\ \ \frac{1+i_1i_2}{2},\qquad \frac{1-i_1i_2}{2}.$$

Furthermore, for every $z_1+i_2z_2$ in $\mathbb{C}_2$,

$$(6)\qquad z_1+i_2z_2=(z_1-i_1z_2)\left(\frac{1+i_1i_2}{2}\right)+(z_1+i_1z_2)\left(\frac{1-i_1i_2}{2}\right).$$

Also, if $\|\ \|$ denotes the norm of elements in $\mathbb{C}_2$, then

$$(7)\qquad \|z_1+i_2z_2\|=(|z_1|^2+|z_2|^2)^{1/2}=(x_1^2+x_2^2+x_3^2+x_4^2)^{1/2},$$

$$(8)\qquad \|z_1+i_2z_2\|=\left(\frac{|z_1-i_1z_2|^2+|z_1+i_1z_2|^2}{2}\right)^{1/2}.$$

There is an important difference between $\mathbb{C}_1$ and $\mathbb{C}_2$: the complex numbers form a field, but the bicomplex numbers do not since they contain divisors of zero. Thus

$$(9)\qquad \left(\frac{1+i_1i_2}{2}\right)\left(\frac{1-i_1i_2}{2}\right)=0,\qquad \left\|\frac{1+i_1i_2}{2}\right\|=\left\|\frac{1-i_1i_2}{2}\right\|=\frac{\sqrt{2}}{2},$$

$$(10)\qquad \left(\frac{1+i_1i_2}{2}\right)^2=\frac{1+i_1i_2}{2},\qquad \left(\frac{1-i_1i_2}{2}\right)^2=\frac{1-i_1i_2}{2}.$$

The properties in (6), (9), and (10) show that the algebraic operations of addition, subtraction, multiplication, and division can be carried out on elements in $\mathbb{C}_2$ by performing the corresponding operations on the complex coefficients $z_1 - i_1 z_2$ and $z_1 + i_1 z_2$ in (6). The properties stated in (6)–(10), and the properties of the set of divisors of zero are the keys which unlock much of the theory. This chapter establishes the statements which have been made in this introduction and uses them to develop the fundamental properties of the bicomplex space $\mathbb{C}_2$.

2. $\mathbb{C}_2$: A LINEAR SPACE

The purpose of this section is to define the set $\mathbb{C}_2$ of bicomplex numbers, the addition $\oplus$ of bicomplex numbers, the scalar multiplication $\odot$ of a bicomplex number by a real scalar a, and to prove that the system $(\mathbb{C}_2, \oplus, \odot)$ is a linear space.

2.1 DEFINITION The set $\mathbb{C}_2$ and $\ominus$ (equals) are defined by the following statements:

(1) $$\mathbb{C}_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 \colon x_1, \ldots, x_4 \text{ in } \mathbb{C}_0,\ i_1^2 = i_2^2 = -1,\ i_1i_2 = i_2i_1\};$$

(2) $$(x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) \ominus (y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4) \quad \text{if and only if } x_i = y_i,\ i = 1, \ldots, 4.$$

Addition is the operation on $\mathbb{C}_2$ defined by the function

(3) $$\oplus : \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{C}_2,\quad (x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4,\ y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4) \mapsto (x_1 + y_1) + i_1(x_2 + y_2) + i_2(x_3 + y_3) + i_1i_2(x_4 + y_4).$$

Scalar multiplication is the operation on $\mathbb{C}_2$ defined by the function

(4) $$\odot : \mathbb{C}_0 \times \mathbb{C}_2 \to \mathbb{C}_2,\quad (a,\ x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) \mapsto ax_1 + i_1ax_2 + i_2ax_3 + i_1i_2ax_4.$$

Observe that equality, addition, and scalar multiplication in $\mathbb{C}_2$ are defined in terms of equality, addition, and multiplication of real numbers. This fact provides the basis for the proof of the following theorem.

2.2 THEOREM The system $(\mathbb{C}_2, \oplus, \odot)$ is a linear space.

Proof. The set $\mathbb{C}_2$ with the operation addition is a commutative group. First, $\mathbb{C}_2$ is closed under addition, and addition is associative. The identity for addition is $0+i_1 0+i_2 0+i_1 i_2 0$, which will be denoted hereafter by 0 and called zero. The inverse of $x_1+i_1x_2+i_2x_3+i_1i_2x_4$ is $(-x_1)+i_1(-x_2)+i_2(-x_3)+i_1i_2(-x_4)$, and addition is commutative. All of these statements are true because the corresponding properties are true for addition in $\mathbb{C}_0$. If 1 is the unit in $\mathbb{C}_0$, then $1 \odot (x_1+i_1x_2+i_2x_3+i_1i_2x_4) = x_1+i_1x_2+i_2x_3+i_1i_2x_4$. Finally, scalar multiplication has the properties of closure, associativity, and distributivity required to complete the proof that $(\mathbb{C}_2, \oplus, \odot)$ is a linear space. □

In the future, equals $\ominus$, addition $\oplus$, and scalar multiplication $\odot$ will be denoted by $=$, $+$, and juxtaposition, respectively.

The space $\mathbb{C}_2$ of bicomplex numbers is embedded in $\mathbb{C}_0^4$ by mapping the bicomplex number $x_1+i_1x_2+i_2x_3+i_1i_2x_4$ into the point (x_1, x_2, x_3, x_4) in $\mathbb{C}_0^4$. The bicomplex numbers $x_1+i_1 0+i_2 0+i_1i_2 0$ are isomorphic to the real numbers $\mathbb{C}_0$, and for simplicity they are called real numbers; $0+i_1 0+i_2 0+i_1i_2 0$ and $1+i_1 0+i_2 0+i_1i_2 0$ are called zero and one and denoted by 0 and 1. The set $\{x_1+i_1x_2+i_2 0+i_1i_2 0\colon x_1, x_2 \in \mathbb{C}_0\}$ and $\mathbb{C}_1$ are isomorphic under corresponding operations.

Exercises

2.1 The results in this section have been given for the bicomplex space $\mathbb{C}_2$ with elements in the real form $x_1+i_1x_2+i_2x_3+i_1i_2x_4$. Repeat the entire section for the bicomplex space with elements in the complex form $z_1+i_2z_2$ [see (4) in Section 1].

2.2 There are two forms of the bicomplex space $\mathbb{C}_2$, one with real elements $x_1+i_1x_2+i_2x_3+i_1i_2x_4$ and the other with complex elements $z_1+i_2z_2$. In each form of $\mathbb{C}_2$ there is an operation called addition (see Definition 2.1 and Exercise 2.1). Establish an isomorphism between the two forms of $\mathbb{C}_2$ with respect to these operations of addition and thus prove that, from the abstract point of view, the two forms of $\mathbb{C}_2$ are identical.

3. $\mathbb{C}_2$: A BANACH SPACE

This section contains the definition of a norm $\|\ \|$ on the linear space $(\mathbb{C}_2, \oplus, \odot)$ and shows that the system $(\mathbb{C}_2, \oplus, \odot, \|\ \|)$ is a Banach space.

3.1 DEFINITION Define the function $\|\ \|: \mathbb{C}_2 \to \mathbb{R}_{\geqslant 0}$ as follows: for every $x_1+i_1x_2+i_2x_3+i_1i_2x_4$ in $\mathbb{C}_2$,

$$(1) \qquad \|x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4\| = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}.$$

3.2 THEOREM The function $\|\ \|: \mathbb{C}_2 \to \mathbb{R}_{\geqslant 0}$ is a norm on the linear space $(\mathbb{C}_2, \oplus, \odot)$.

Proof. The function $\| \ \|: \mathbb{C}_2 \to \mathbb{R}_{\geqslant 0}$ is known to have the following properties: for every $x_1+i_1x_2+i_2x_3+i_1i_2x_4$ and $y_1+i_1y_2+i_2y_3+i_1i_2y_4$ in $\mathbb{C}_2$ and a in $\mathbb{C}_0$,

(2) $\|x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4\| \geqslant 0,$

(3) $\|x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4\| = 0$ if and only if

$x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = 0,$

(4) $\|a(x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4)\| = |a| \, \|x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4\|,$

(5) $\|(x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) + (y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4)\|$

$\leqslant \|x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4\| + \|y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4\|.$

A function with these properties is called a norm; therefore $\| \ \|: \mathbb{C}_2 \to \mathbb{R}_{\geqslant 0}$ is a norm on $(\mathbb{C}_2, \oplus, \odot)$. □

The space $\mathbb{C}_0^4$ with the euclidean norm is known to be a complete space. Because $\mathbb{C}_2$ is embedded in $\mathbb{C}_0^4$ so that $x_1+i_1x_2+i_2x_3+i_1i_2x_4$ corresponds to (x_1, x_2, x_3, x_4), and because the norm on $\mathbb{C}_2$ is the same as the norm on $\mathbb{C}_0^4$, then the normed linear space $(\mathbb{C}_2, \oplus, \odot, \| \ \|)$ is a complete space. By definition, a space which is linear, normed, and complete is a Banach space. These statements prove the following theorem.

3.3 THEOREM The system $(\mathbb{C}_2, \oplus, \odot, \| \ \|)$ is a Banach space.

The norm $\| \ \|$ is defined in (1), but it has other representations. If $z_1 = x_1 + i_1x_2$ and $z_2 = x_3 + i_1x_4$, then

(6)
$$\begin{aligned} \|x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4\| &= (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2} \\ &= [|z_1|^2 + |z_2|^2]^{1/2} \\ &= \|z_1 + i_2z_2\|. \end{aligned}$$

Exercises

3.1 Prove the statements in equations (2)–(5). [*Hint.* To prove (5), assume Schwarz's inequality.]

3.2 If ζ, ζ_1, and ζ_2 denote elements in $\mathbb{C}_2$, prove that

(a) $\|-\zeta\| = \|\zeta\|$,

(b) $|\|\zeta_1\| - \|\zeta_2\|| \leqslant \|\zeta_1 + \zeta_2\| \leqslant \|\zeta_1\| + \|\zeta_2\|$,

(c) $|\|\zeta_1\| - \|\zeta_2\|| \leqslant \|\zeta_1 - \zeta_2\| \leqslant \|\zeta_1\| + \|\zeta_2\|$.

3.3 Let X be a set, and let $d: X \times X \to \mathbb{R}_{\geqslant 0}$, $(x, y) \mapsto d(x, y)$, be a function with the following properties: for every x, y, z in X,

(i) $d(x, y) \geqslant 0$, $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leqslant d(x, y) + d(y, z)$ (triangle inequality).

A function d with these properties is called a *distance function* on X, and $d(x, y)$ is called the *distance* from x to y. The system (X, d) is called a *metric space*.

Define a function $d: \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{R}_{\geqslant 0}$, $(\zeta_1, \zeta_2) \mapsto d(\zeta_1, \zeta_2)$, by setting $d(\zeta_1, \zeta_2) = \|\zeta_1 - \zeta_2\|$ for every ζ_1, ζ_2 in $\mathbb{C}_2$. Prove that this d is a distance function on $\mathbb{C}_2$ and thus that $(\mathbb{C}_2, d)$ is a metric space.

3.4 A Cauchy sequence in $\mathbb{C}_2$ is a function $s: \mathbb{N} \to \mathbb{C}_2$, $n \mapsto \zeta_n$, with the following property: to each $\varepsilon > 0$ there corresponds an n_0 in $\mathbb{N}$ for which $\|\zeta_n - \zeta_m\| < \varepsilon$ for all m, n such that $n \geqslant n_0$ and $m \geqslant n_0$.

(a) Prove that a Cauchy sequence in $\mathbb{C}_2$ is bounded.

(b) A normed linear space is said to be *complete* if every Cauchy sequence in the space has a limit in the space. Assume that the Bolzano–Weierstrass cluster point theorem is true in $\mathbb{C}_0^4$ and prove that $\mathbb{C}_2$ is a complete space and thus a Banach space.

(c) Assume that the Bolzano–Weierstrass cluster point theorem is true in $\mathbb{C}_0$ and use this fact to prove that $\mathbb{C}_2$ is complete.

3.5 If z, z_1, z_2 are in $\mathbb{C}_1$, show that $z_1 + i_2 z_2$ and $(zz_1) + i_2(zz_2)$ are in $\mathbb{C}_2$. Show also that $\|(zz_1) + i_2(zz_2)\| = |z|\ \|z_1 + i_2 z_2\|$.

4. MULTIPLICATION

This section defines multiplication $\otimes$ in $\mathbb{C}_2$; it establishes the properties of this operation; and it proves an inequality for the product of two elements in $\mathbb{C}_2$. Thus the system $(\mathbb{C}_2, \oplus, \odot, \|\ \|, \otimes)$ is shown to be a Banach algebra. In the future, multiplication $\otimes$ will usually be denoted by juxtaposition, and $\mathbb{C}_2$ will usually denote the system as well as the set of elements. A nonsingular element has a multiplicative inverse; a singular element does not have an inverse. The final theorem in the section identifies all of the singular elements in $\mathbb{C}_2$.

4.1 DEFINITION The product $(x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) \otimes (y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4)$ is the element in $\mathbb{C}_2$ obtained by multiplying $x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4$ and $y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4$ as if they were polynomials and then using the relations $i_1^2 = -1$, $i_2^2 = -1$, and $i_1i_2 = i_2i_1$ to simplify the result. The following display exhibits this product and the final result.

(1)

$$
\begin{array}{c|ccccccc}
 & y_1 & & i_1y_2 & & i_2y_3 & & i_1i_2y_4 \\
\hline
x_1 & x_1y_1 & + & i_1x_1y_2 & + & i_2x_1y_3 & + & i_1i_2x_1y_4 \\
i_1x_2 & +\ i_1x_2y_1 & - & x_2y_2 & + & i_1i_2x_2y_3 & - & i_2x_2y_4 \\
i_2x_3 & +\ i_2x_3y_1 & + & i_1i_2x_3y_2 & - & x_3y_3 & - & i_1x_3y_4 \\
i_1i_2x_4 & +\ i_1i_2x_4y_1 & - & i_2x_4y_2 & - & i_1x_4y_3 & + & x_4y_4
\end{array}
$$

(2) $$(x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4) \otimes (y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4)$$
$$= (x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4)$$
$$+ i_1(x_1y_2 + x_2y_1 - x_3y_4 - x_4y_3)$$
$$+ i_2(x_1y_3 - x_2y_4 + x_3y_1 - x_4y_2)$$
$$+ i_1i_2(x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1).$$

4.2 EXAMPLE Definition 4.1 shows that

(3) $$(2 + i_14 - i_23 + i_1i_25)(6 - i_18 + i_23 - i_1i_27)$$
$$= (18 - i_128 + i_256 + i_1i_252).$$

If z_1, z_2, w_1, w_2 are elements in $\mathbb{C}_1$, then

(4) $$(z_1 + i_2z_2)(w_1 + i_2w_2) = (z_1w_1 - z_2w_2) + i_2(z_1w_2 + z_2w_1).$$

The formula in (4) emphasizes once more the formal similarities of complex and bicomplex numbers.

4.3 THEOREM The following statements describe properties of multiplication in $\mathbb{C}_2$.

(5) $\mathbb{C}_2$ is closed under multiplication.

(6) Multiplication is associative.

(7) Multiplication is distributive with respect to addition.

(8) Multiplication is commutative.

(9) There is a unit element for multiplication; it is $(1+i_10+i_20+i_1i_20)$, which is usually denoted by 1.

Proof. The statements in the theorem can be verified by straightforward calculation which employs the definitions of the operations and the properties of $\mathbb{C}_2$. □

4.4 THEOREM If z is in $\mathbb{C}_1$ and $z_1+i_2z_2$ and $w_1+i_2w_2$ are in $\mathbb{C}_2$, then

(10) $$\|z(z_1 + i_2z_2)\| = |z| \; \|z_1 + i_2z_2\|,$$

(11) $$\|(z_1 + i_2z_2)(w_1 + i_2w_2)\| \leqslant \sqrt{2}\,\|z_1 + i_2z_2\| \; \|w_1 + i_2w_2\|.$$

The inequality in (11) is the best possible.

Proof. Since z can be considered to be the bicomplex number $z+i_20$, the product $z(z_1+i_2z_2)$ is defined; equation (4) shows that

(12) $$z(z_1 + i_2z_2) = (zz_1) + i_2(zz_2).$$

Thus multiplication by z has the character of scalar multiplication. By (6) in Section 3,

$$\begin{aligned}(13)\qquad \|z(z_1 + i_2 z_2)\| &= (|zz_1|^2 + |zz_2|^2)^{1/2} = (|z|^2|z_1|^2 + |z|^2|z_2|^2)^{1/2}\\ &= |z|(|z_1|^2 + |z_2|^2)^{1/2} = |z|\ \|z_1 + i_2 z_2\|.\end{aligned}$$

Thus (10) is true. Next, since multiplication is distributive with respect to addition,

$$(14)\qquad (z_1 + i_2 z_2)(w_1 + i_2 w_2) = z_1(w_1 + i_2 w_2) + i_2 w_2(w_1 + i_2 w_2).$$

Since $\|z_1(w_1 + i_2 w_2)\| = |z_1|\ \|w_1 + i_2 w_2\|$ and $\|i_2 z_2(w_1 + i_2 w_2)\| = |z_2|\ \|w_1 + i_2 w_2\|$, the triangle inequality for the norm shows that

$$\begin{aligned}(15)\qquad \|(z_1 + i_2 z_2)(w_1 + i_2 w_2)\| &\leqslant |z_1|\ \|w_1 + i_2 w_2\| + |z_2|\ \|w_1 + i_2 w_2\|\\ &\leqslant (|z_1| + |z_2|)\|w_1 + i_2 w_2\|.\end{aligned}$$

Now Schwarz's inequality shows that

$$(16)\qquad (|z_1| + |z_2|) \leqslant \sqrt{2}(|z_1|^2 + |z_2|^2)^{1/2} = \sqrt{2}\,\|z_1 + i_2 z_2\|,$$

and (15) and (16) establish (11). To prove that the inequality in (11) is the best possible, observe that

$$\begin{aligned}(17)\qquad &\left\|\frac{1 + i_1 i_2}{2}\right\| = \frac{\sqrt{2}}{2}, \qquad \left\|\frac{1 - i_1 i_2}{2}\right\| = \frac{\sqrt{2}}{2},\\ &\left\|\frac{1 + i_1 i_2}{2}\frac{1 + i_1 i_2}{2}\right\| = \left\|\frac{1 + i_1 i_2}{2}\right\| = \frac{\sqrt{2}}{2} = \sqrt{2}\left\|\frac{1 + i_1 i_2}{2}\right\|\ \left\|\frac{1 + i_1 i_2}{2}\right\|,\\ &\left\|\frac{1 - i_1 i_2}{2}\frac{1 - i_1 i_2}{2}\right\| = \left\|\frac{1 - i_1 i_2}{2}\right\| = \frac{\sqrt{2}}{2} = \sqrt{2}\left\|\frac{1 - i_1 i_2}{2}\right\|\ \left\|\frac{1 - i_1 i_2}{2}\right\|.\end{aligned}$$

The proof of Theorem 4.4 is complete. □

4.5 EXAMPLE Although (17) shows that there are pairs of numbers $z_1 + i_2 z_2$ and $w_1 + i_2 w_2$ in $\mathbb{C}_2$ for which the equality holds in (11), there are many pairs for which the inequality holds. The extreme case of an inequality is the following:

$$(18)\qquad \left\|\frac{1 + i_1 i_2}{2}\frac{1 - i_1 i_2}{2}\right\| = \|0\| = 0 < \sqrt{2}\left\|\frac{1 + i_1 i_2}{2}\right\|\ \left\|\frac{1 - i_1 i_2}{2}\right\|.$$

In most cases, the norm of a product lies between the extremes shown in (17) and (18). For example,

(19) $$\|2 + i_1 4 - i_2 3 + i_1 i_2 5\| = \sqrt{54},$$
$$\|6 - i_1 8 + i_2 3 - i_1 i_2 7\| = \sqrt{158},$$

and the norm of the product is, by (3),

(20) $$\|18 - i_1 28 + i_2 56 + i_1 i_2 52\| = \sqrt{6948}.$$

Thus

(21) $$\|(2 + i_1 4 - i_2 3 + i_1 i_2 5)(6 - i_1 8 + i_2 3 - i_1 i_2 7)\|$$
$$= \frac{\sqrt{6948}}{\sqrt{54}\sqrt{158}} \|2 + i_1 4 - i_2 3 + i_1 i_2 5\| \; \|6 - i_1 8 + i_2 3 - i_1 i_2 7\|.$$

Since

(22) $$0 < \frac{\sqrt{6948}}{\sqrt{54}\sqrt{158}} < 1,$$

the two elements in $\mathbb{C}_2$ in (19) satisfy the inequality in (11), but this case is quite different from the extreme situations shown in (17) and (18).

4.6 THEOREM The system $(\mathbb{C}_2, \oplus, \odot, \| \, \|, \otimes)$ is a Banach algebra.

This theorem follows from the definition of a Banach algebra and the properties of the system which have been established in preceding theorems. In the usual definition of a Banach algebra, the norm of the product of two elements is required to be equal to or less than the product of the norms of these elements. Thus, strictly speaking, $\mathbb{C}_2$ is a modified Banach algebra.

4.7 DEFINITION If $\zeta_1 : (z_1 + i_2 z_2)$ and $\zeta_2 : (w_1 + i_2 w_2)$ are two elements in $\mathbb{C}_2$ and $\zeta_1 \zeta_2 = 1$, then each of the elements ζ_1 and ζ_2 is said to be the (multiplicative) *inverse* of the other. An element which has an inverse is said to be *nonsingular*, and an element which does not have an inverse is said to be *singular*.

4.8 THEOREM An element $\zeta_1 : (z_1 + i_2 z_2)$ is nonsingular if and only if

(23) $$|z_1^2 + z_2^2| \neq 0,$$

and it is singular if and only if

(24) $$|z_1^2 + z_2^2| = 0.$$

Proof. By definition, ζ_1 is nonsingular if and only if there is a $\zeta_2:(w_1+i_2w_2)$ such that $\zeta_1\zeta_2=1$. By (4) this equation is equivalent to the following system:

$$z_1w_1 - z_2w_2 = 1, \qquad z_2w_1 + z_1w_2 = 0. \tag{25}$$

This system has a (unique) solution for (w_1, w_2) if

$$\det\begin{bmatrix} z_1 & -z_2 \\ z_2 & z_1 \end{bmatrix} = z_1^2 + z_2^2 \neq 0. \tag{26}$$

Thus ζ_1 is nonsingular if (23) is true. If (24) is true, the following considerations show that (25) has no solution. There are two possibilities: (a) $z_1=z_2=0$, and (*b*) $z_2=\pm i_1z_1\neq 0$. In case (a) the two equations in (25) are inconsistent, and the system has no solution. If $z_2=i_1z_1$, then (25) becomes

$$z_1(w_1 - i_1w_2) = 1, \qquad z_1(w_1 - i_1w_2) = 0; \tag{27}$$

if $z_2 = -i_1z_1$, then (25) can be simplified to

$$z_1(w_1 + i_1w_2) = 1, \qquad z_1(w_1 + i_1w_2) = 0. \tag{28}$$

The systems in (27) and (28) are inconsistent, and thus (25) has no solution if (24) is true. Thus ζ_1 is nonsingular if and only if (23) is true, and it is singular if and only if (24) is true. □

4.9 COROLLARY The element $z_1+i_2z_2$ is nonsingular if and only if

$$|z_1 - i_1z_2|\,|z_1 + i_1z_2| \neq 0, \tag{29}$$

and it is singular if and only if

$$|z_1 - i_1z_2|\,|z_1 + i_1z_2| = 0. \tag{30}$$

4.10 THEOREM Let $\zeta_1:(z_1+i_2z_2)$ and $\zeta_2:(w_1+i_2w_2)$ be two elements in $\mathbb{C}_2$. Then $\zeta_1\zeta_2$ is singular if and only if at least one of the elements ζ_1 and ζ_2 is singular.

Proof. By (4),

$$(z_1 + i_2z_2)(w_1 + i_2w_2) = (z_1w_1 - z_2w_2) + i_2(z_1w_2 + z_2w_1). \tag{31}$$

Also

$$\begin{aligned} &(z_1w_1 - z_2w_2)^2 + (z_1w_2 + z_2w_1)^2 = (z_1^2 + z_2^2)(w_1^2 + w_2^2), \\ &|(z_1w_1 - z_2w_2)^2 + (z_1w_2 + z_2w_1)^2| = |z_1^2 + z_2^2|\,|w_1^2 + w_2^2|. \end{aligned} \tag{32}$$

Then Theorem 4.8 shows that $\zeta_1\zeta_2$ is singular if and only if at least one of the elements ζ_1 and ζ_2 is singular. □

4.11 EXAMPLE The elements

(33) $$\zeta_1 : \left(\frac{1 + i_1 i_2}{2}\right), \qquad \zeta_2 : \left(\frac{1 - i_1 i_2}{2}\right),$$

are nonzero elements, but since

(34) $$\left(\frac{1}{2}\right)^2 + \left(\frac{i_1}{2}\right)^2 = 0, \qquad \left(\frac{1}{2}\right)^2 + \left(\frac{-i_1}{2}\right)^2 = 0,$$

each of them is singular by Theorem 4.8. Then Theorem 4.10 shows that, for every element $z_1 + i_2 z_2$ in $\mathbb{C}_2$, the elements $\zeta_1(z_1 + i_2 z_2)$ and $\zeta_2(z_1 + i_2 z_2)$ are singular. Theorem 4.8 provides a second proof; since

(35) $$\left(\frac{1 + i_1 i_2}{2}\right)(z_1 + i_2 z_2) = \left(\frac{z_1}{2} - \frac{i_1 z_2}{2}\right) + i_2\left[i_1\left(\frac{z_1}{2} - \frac{i_1 z_2}{2}\right)\right],$$

then

(36) $$\left(\frac{z_1}{2} - \frac{i_1 z_2}{2}\right)^2 + \left[i_1\left(\frac{z_1}{2} - \frac{i_1 z_2}{2}\right)\right]^2 = 0,$$

and $\zeta_1(z_1 + i_2 z_2)$ is singular by (24) in Theorem 4.8. Similar arguments show that $\zeta_2(z_1 + i_2 z_2)$ is singular.

4.12 DEFINITION Define $V: \mathbb{C}_2 \to \mathbb{R}_{\geqslant 0}$ to be the function such that, if $\zeta : (x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4)$ is in $\mathbb{C}_2$, then

(37) $$V(\zeta) = |(x_1 + i_1 x_2)^2 + (x_3 + i_1 x_4)^2|^2.$$

4.13 THEOREM If $\zeta_1 : (x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4)$ and $\zeta_2 : (y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4)$ are elements in $\mathbb{C}_2$, then

(38) $$V(\zeta_1) \geqslant 0,$$

(39) $$V(\zeta_1) = (x_1^2 - x_2^2 + x_3^2 - x_4^2)^2 + (2x_1 x_2 + 2x_3 x_4)^2,$$

(40) $$V(\zeta_1 \zeta_2) = V(\zeta_1) V(\zeta_2).$$

Proof. The inequality (38) follows from the definition of $V(\zeta)$ in (37). To establish (39), calculate $V(\zeta_1)$ from the definition in (37). To prove (40), represent ζ_1 and ζ_2 as $z_1 + i_2 z_2$ and $w_1 + i_2 w_2$, respectively. Then (31) and (37) show that

(41) $$V(\zeta_1 \zeta_2) = |(z_1 w_1 - z_2 w_2)^2 + (z_1 w_2 + z_2 w_1)^2|^2.$$

Then since $V(\zeta_1) = |z_1^2 + z_2^2|^2$ and $V(\zeta_2) = |w_1^2 + w_2^2|^2$ by (37), equations (41) and (32) show that

(42) $$V(\zeta_1 \zeta_2) = |z_1^2 + z_2^2|^2 \, |w_1^2 + w_2^2|^2 = V(\zeta_1) V(\zeta_2). \qquad \square$$

4.14 COROLLARY The element $\zeta : (z_1 + i_2 z_2)$ is nonsingular if and only if $V(\zeta) > 0$, and is singular if and only if $V(\zeta) = 0$.

Definition 4.12 and inequality (38) show that this corollary is merely a restatement of Theorem 4.8.

There is only one element in $\mathbb{C}_0$ which does not have a multiplicative inverse; it is the element 0. Denote the set whose only element is 0 by $\mathcal{O}_0$. There is only one element in $\mathbb{C}_1$ which does not have an inverse; it is $0 + i_1 0$. Denote the set whose only element is $0 + i_1 0$ by $\mathcal{O}_1$. Example 4.11 shows that there are many elements in $\mathbb{C}_2$ which do not have inverses; denote the set of these elements by $\mathcal{O}_2$. Since $\mathbb{C}_0$ is isomorphic to a subset of $\mathbb{C}_1$ and $\mathbb{C}_1$ is isomorphic to a subset of $\mathbb{C}_2$, it is customary to say simply that $\mathbb{C}_0$ *is* a subset of $\mathbb{C}_1$ and $\mathbb{C}_1$ *is* a subset of $\mathbb{C}_2$. Then $0 + i_1 0 + i_2 0 + i_1 i_2 0$ is the single element in $\mathcal{O}_0$ and $\mathcal{O}_1$, and it belongs also to $\mathcal{O}_2$. Thus

$$\mathcal{O}_0 = \mathcal{O}_1 \subsetneqq \mathcal{O}_2. \tag{43}$$

Exercises

4.1 Verify each of the following statements:

(a) $$\left(\frac{1+i_1 i_2}{2}\right)\left(\frac{1+i_1 i_2}{2}\right) = \frac{1+i_1 i_2}{2}, \qquad \left(\frac{1-i_1 i_2}{2}\right)\left(\frac{1-i_1 i_2}{2}\right) = \frac{1-i_1 i_2}{2}, \qquad \left(\frac{1+i_1 i_2}{2}\right)\left(\frac{1-i_1 i_2}{2}\right) = 0;$$

(b) $$\left\|\frac{1+i_1 i_2}{2}\right\| = \frac{\sqrt{2}}{2}, \qquad \left\|\frac{1-i_1 i_2}{2}\right\| = \frac{\sqrt{2}}{2};$$

(c) $$V\left(\frac{1+i_1 i_2}{2}\right) = 0, \qquad V\left(\frac{1-i_1 i_2}{2}\right) = 0;$$

(d) $(1 + i_1 i_2)/2$ and $(1 - i_1 i_2)/2$ are singular elements.

(e) If $z_1 + i_2 z_2$ is an element in $\mathbb{C}_2$, then $(z_1 + i_2 z_2)[(1 + i_1 i_2)/2]$ and $(z_1 + i_2 z_2)[(1 - i_1 i_2)/2]$ are singular elements.

4.2 Prove the following theorem: if $z_1 + i_2 z_2$ and $w_1 + i_2 w_2$ are elements in $\mathbb{C}_2$, then

(a) $$z_1 + i_2 z_2 = (z_1 - i_1 z_2)\left(\frac{1+i_1 i_2}{2}\right) + (z_1 + i_1 z_2)\left(\frac{1-i_1 i_2}{2}\right),$$

$$w_1 + i_2 w_2 = (w_1 - i_1 w_2)\left(\frac{1+i_1 i_2}{2}\right) + (w_1 + i_1 w_2)\left(\frac{1-i_1 i_2}{2}\right);$$

(b) $(z_1+i_2z_2)+(w_1+i_2w_2)$

$$=[(z_1-i_1z_2)+(w_1-i_1w_2)]\left(\frac{1+i_1i_2}{2}\right)+[(z_1+i_1z_2)$$

$$+(w_1+i_1w_2)]\left(\frac{1-i_1i_2}{2}\right);$$

(c) $(z_1+i_2z_2)(w_1+i_2w_2)$

$$=(z_1-i_1z_2)(w_1-i_1w_2)\left(\frac{1+i_1i_2}{2}\right)$$

$$+(z_1+i_1z_2)(w_1+i_1w_2)\left(\frac{1-i_1i_2}{2}\right).$$

4.3 If $z_1+i_2z_2$ is an element in $\mathbb{C}_2$, show that

$$(z_1+i_2z_2)\left(\frac{1+i_1i_2}{2}\right)=(z_1-i_1z_2)\left(\frac{1+i_1i_2}{2}\right),$$

$$(z_1+i_2z_2)\left(\frac{1-i_1i_2}{2}\right)=(z_1+i_1z_2)\left(\frac{1-i_1i_2}{2}\right).$$

4.4 If $z_1+i_2z_2$ is a singular element, show that at least one of the following statements is true:

$$z_1+i_2z_2=(z_1-i_1z_2)\left(\frac{1+i_1i_2}{2}\right),$$

$$z_1+i_2z_2=(z_1+i_1z_2)\left(\frac{1-i_1i_2}{2}\right).$$

4.5 Prove the following statements:

(a) If c_1 and c_2 are in $\mathbb{C}_1$ and

$$c_1\left(\frac{1+i_1i_2}{2}\right)+c_2\left(\frac{1-i_1i_2}{2}\right)=0,$$

then $c_1=0$ and $c_2=0$.

(b) If both of the equations in Exercise 4.4 are true, then $z_1+i_2z_2=0$.

5. FRACTIONS AND QUOTIENTS

This section contains the definition of the quotient of two elements in $\mathbb{C}_2$, and it shows that a quotient (fraction) exists if and only if the divisor (denominator) is nonsingular. Next, the section establishes the standard cancellation laws and some necessary inequalities for quotients.

5.1 DEFINITION Let $\zeta_1:(z_1+i_2z_2)$ and $\zeta_2:(w_1+i_2w_2)$ be elements in $\mathbb{C}_2$. If there exists a unique element $\eta:(u_1+i_2u_2)$ in $\mathbb{C}_2$ such that $\zeta_1=\zeta_2\eta$, then the quotient (or fraction) ζ_1/ζ_2 exists and

$$\frac{\zeta_1}{\zeta_2}=\eta. \tag{1}$$

5.2 THEOREM The fraction ζ_1/ζ_2 is defined if and only if ζ_2 is nonsingular. If ζ_2 is nonsingular and ζ_2^{-1} denotes the inverse of ζ_2, then

$$\frac{\zeta_1}{\zeta_2}=\zeta_1\zeta_2^{-1}. \tag{2}$$

Proof. By (1), ζ_1/ζ_2 is a bicomplex number $\eta:(u_1+i_2u_2)$ such that $\zeta_1=\zeta_2\eta$. This equation is equivalent to the following system of equations in the unknowns u_1, u_2:

$$w_1u_1 - w_2u_2 = z_1, \qquad w_2u_1 + w_1u_2 = z_2. \tag{3}$$

This system of equations has a unique solution if and only if $w_1^2+w_2^2\neq 0$, that is, if and only if ζ_2 is nonsingular (see Theorem 4.8). Thus if ζ_2 is nonsingular, there is an element η in $\mathbb{C}_2$ such that $\zeta_1=\zeta_2\eta$. Since ζ_2 is nonsingular, it has an inverse ζ_2^{-1}, and $\zeta_1\zeta_2^{-1}=\zeta_2\eta\zeta_2^{-1}=\zeta_2\zeta_2^{-1}\eta=\eta$. Therefore, $\eta=\zeta_1\zeta_2^{-1}$, and the proof of (2) and of Theorem 5.2 is complete. □

5.3 COROLLARY If $\zeta_1:(z_1+i_2z_2)$ and $\zeta_2:(w_1+i_2w_2)$ are elements in $\mathbb{C}_2$ and $w_1^2+w_2^2\neq 0$, then the fraction ζ_1/ζ_2 is defined and

$$\frac{\zeta_1}{\zeta_2}=\frac{z_1w_1+z_2w_2}{w_1^2+w_2^2}+i_2\left(\frac{z_2w_1-z_1w_2}{w_1^2+w_2^2}\right) \tag{4}$$
$$=\frac{(z_1+i_2z_2)(w_1-i_2w_2}{w_1^2+w_2^2}.$$

Proof. The value of ζ_1/ζ_2 is $u_1+i_2u_2$, where u_1 and u_2 satisfy the system of equations in (3). The solution of these equations gives the result in (4). Formally, the value of the quotient is obtained as in the complex case:

$$\frac{z_1+i_2z_2}{w_1+i_2w_2}=\frac{(z_1+i_2z_2)(w_1-i_2w_2)}{(w_1+i_2w_2)(w_1-i_2w_2)} \tag{5}$$
$$=\frac{(z_1+i_2z_2)(w_1-i_2w_2)}{w_1^2+w_2^2}.$$
□

5.4 THEOREM (Cancellation Laws)
Let $\zeta_1, \zeta_2, \zeta_3$ be elements in $\mathbb{C}_2$.

(6) If ζ_3 is nonsingular and $\zeta_1\zeta_3=\zeta_2\zeta_3$, then $\zeta_1=\zeta_2$.

(7) If ζ_2 and ζ_3 are nonsingular, then $\dfrac{\zeta_1\zeta_3}{\zeta_2\zeta_3}=\dfrac{\zeta_1}{\zeta_2}$.

Proof. Since ζ_3 is nonsingular, it has an inverse ζ_3^{-1} by Definition 4.7. Since $\zeta_1\zeta_3=\zeta_2\zeta_3$ by hypothesis, then $(\zeta_1\zeta_3)\zeta_3^{-1}=(\zeta_2\zeta_3)\zeta_3^{-1}$. Then since multiplication is associative, $\zeta_1(\zeta_3\zeta_3^{-1})=\zeta_2(\zeta_3\zeta_3^{-1})$ or $\zeta_1=\zeta_2$. Thus (6) is true. Consider (7). Since ζ_2 and ζ_3 are nonsingular by hypothesis, then $\zeta_2\zeta_3$ is nonsingular by Theorem 4.10. Thus both of the fractions in (7) exist by Theorem 5.2. Let η denote the value of $\zeta_1\zeta_3/\zeta_2\zeta_3$. Then

$$\zeta_1\zeta_3=(\zeta_2\zeta_3)\eta \tag{8}$$

and

$$\zeta_1=\zeta_2\eta \tag{9}$$

by (6). Thus by (8) and (9),

$$\frac{\zeta_1\zeta_3}{\zeta_2\zeta_3}=\eta=\frac{\zeta_1}{\zeta_2}, \tag{10}$$

and (7) is true. The proof of Theorem 5.4 is complete. □

5.5 THEOREM If $w_1+i_2w_2$ is a nonsingular element in $\mathbb{C}_2$, then

$$\begin{aligned}\frac{\sqrt{2}}{2}\frac{\|z_1+i_2z_2\|}{\|w_1+i_2w_2\|}&\leqslant\left\|\frac{z_1+i_2z_2}{w_1+i_2w_2}\right\|\\&=\frac{\|(z_1+i_2z_2)(w_1-i_2w_2)\|}{|w_1^2+w_2^2|}\leqslant\frac{\sqrt{2}\|z_1+i_2z_2\|\,\|w_1+i_2w_2\|}{|w_1^2+w_2^2|}.\end{aligned} \tag{11}$$

Proof. Since $w_1+i_2w_2$ is nonsingular, then $w_1^2+w_2^2\neq 0$ and the fraction $(z_1+i_2z_2)/(w_1+i_2w_2)$ exists by Corollary 5.3; its value is an element $u_1+i_2u_2$ in $\mathbb{C}_2$ such that

$$z_1+i_2z_2=(w_1+i_2w_2)(u_1+i_2u_2). \tag{12}$$

Then Theorem 4.4 (11) shows that

$$\begin{gathered}\|z_1+i_2z_2\|\leqslant\sqrt{2}\|w_1+i_2w_2\|\,\|u_1+i_2u_2\|,\\ \frac{\sqrt{2}}{2}\frac{\|z_1+i_2z_2\|}{\|w_1+i_2w_2\|}\leqslant\left\|\frac{z_1+i_2z_2}{w_1+i_2w_2}\right\|,\end{gathered} \tag{13}$$

and the inequality on the left in (11) is true. By (4) and Theorem 4.4,

$$\begin{aligned}\left\|\frac{z_1+i_2z_2}{w_1+i_2w_2}\right\|&=\left\|\frac{(z_1+i_2z_2)(w_1-i_2w_2)}{w_1^2+w_2^2}\right\|\\&=\frac{\|(z_1+i_2z_2)(w_1-i_2w_2)\|}{|w_1^2+w_2^2|}\\&\leqslant\frac{\sqrt{2}\|z_1+i_2z_2\|\,\|w_1+i_2w_2\|}{|w_1^2+w_2^2|}.\end{aligned} \tag{14}$$

□

5.6 COROLLARY If $w_1 + i_2 w_2$ is nonsingular, then

$$(15)\qquad \frac{\sqrt{2}}{2}\frac{\|z_1 + i_2 z_2\|}{\|w_1 + i_2 w_2\|} \leqslant \left\|\frac{z_1 + i_2 z_2}{w_1 + i_2 w_2}\right\| \leqslant \frac{\sqrt{2}\,\|z_1 + i_2 z_2\|\,\|w_1 + i_2 w_2\|}{[V(w_1 + i_2 w_2)]^{1/2}}.$$

Proof. The inequalities in (15) follow from (11) and the definition of $V(w_1 + i_2 w_2)$ in Definition 4.12. □

5.7 EXAMPLE In both $\mathbb{C}_1$ and $\mathbb{C}_2$, the norm of a fraction is large when the denominator is close to a singular element. At first glance, the situation in the two spaces seems different, but the apparent difference results from the fact that there is only one singular element (namely zero) in $\mathbb{C}_1$ but many singular elements in $\mathbb{C}_2$. In (15), $V(w_1 + i_2 w_2)$ can be considered to be a measure of how close $w_1 + i_2 w_2$ is to a singular element. If $w_2 = i_1 r w_1$, where $r \in \mathbb{C}_0$, then

$$(16)\qquad [V(w_1 + i_2 w_2)]^{1/2} = |w_1^2|\,|1 - r^2|,$$
$$\|w_1 + i_2 w_2\| = |w_1|(1 + r^2)^{1/2},$$

$$(17)\qquad \left\|\frac{z_1 + i_2 z_2}{w_1 + i_2 w_2}\right\| \leqslant \frac{\sqrt{2}\,\|z_1 + i_2 z_2\|(1 + r^2)^{1/2}}{|w_1|\,|1 - r^2|}.$$

If $|1 - r^2|$ is close to zero, then $V(w_1 + i_2 w_2)$ is close to zero although $\|w_1 + i_2 w_2\|$ is not small. Thus the bound for $(z_1 + i_2 z_2)/(w_1 + i_2 w_2)$ on the right in (15) may be very large [see (17)] in spite of the fact that $\|w_1 + i_2 w_2\|$ is also large. The bound shown in (15) is necessarily large because the norm of the fraction $(z_1 + i_2 z_2)/(w_1 + i_2 w_2)$ is large in situations such as the one described in (16) and (17) [compare (11)].

Exercises

5.1 If $w_1^2 + w_2^2 \neq 0$, show that

$$\frac{z_1 + i_2 z_2}{w_1 + i_2 w_2} = (z_1 + i_2 z_2)\left(\frac{1}{w_1 + i_2 w_2}\right).$$

5.2 If $w_1^2 + w_2^2 \neq 0$, show that

$$\frac{z_1 + i_2 z_2}{w_1 + i_2 w_2} = \frac{(z_1 + i_2 z_2)(w_1 - i_2 w_2)}{(w_1 + i_2 w_2)(w_1 - i_2 w_2)},$$

and explain how this equation provides an easy way to calculate the value of the bicomplex fraction on the left.

5.3 (a) If $c \in \mathbb{C}_1$, the bicomplex numbers of the form c and $i_2 c$ can be called complex numbers and pure imaginary bicomplex numbers, respectively. If $c \neq 0$, show that division by c and $i_2 c$ is always possible.
(b) Assume that $c \neq 0$, and express each of the following fractions as a bicomplex number:

$$\frac{z_1 + i_2 z_2}{c}, \qquad \frac{z_1 + i_2 z_2}{i_2 c}.$$

5.4 Show that division by the following bicomplex numbers is impossible:

(a) $\dfrac{1 + i_1 i_2}{2}, \qquad \dfrac{1 - i_1 i_2}{2};$

(b) $(z_1 + i_2 z_2)\left(\dfrac{1 + i_1 i_2}{2}\right), \qquad (z_1 + i_2 z_2)\left(\dfrac{1 - i_1 i_2}{2}\right),$

$(z_1 + i_2 z_2) \in \mathbb{C}_2$.

5.5 Let $\zeta_1, \ldots, \zeta_4$ be elements in $\mathbb{C}_2$, and assume that ζ_2 and ζ_4 are nonsingular. Show that each of the fractions in the following equations is defined and that the equations hold.

$$\frac{\zeta_1}{\zeta_2}\frac{\zeta_3}{\zeta_4} = \frac{\zeta_1\zeta_3}{\zeta_2\zeta_4}, \qquad \frac{\zeta_1}{\zeta_2} + \frac{\zeta_3}{\zeta_2} = \frac{\zeta_1 + \zeta_3}{\zeta_2}, \qquad \frac{\zeta_1}{\zeta_2} + \frac{\zeta_3}{\zeta_4} = \frac{\zeta_1\zeta_4 + \zeta_2\zeta_3}{\zeta_2\zeta_4}.$$

5.6 (a) Let A be a 2 by 2 matrix $[a_{ij}]$ whose elements a_{ij} are bicomplex numbers in $\mathbb{C}_2$. Show that the definition of the determinant of a complex matrix can be applied to define a determinant of A which has all of the elementary properties of the determinant of a complex matrix.
(b) Consider the system of equations which, in matrix form, is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \qquad c_1, c_2 \text{ in } \mathbb{C}_2.$$

If $\det A$ is a nonsingular element in $\mathbb{C}_2$, show that the system of equations has a unique solution which is

$$\zeta_1 = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\det A}, \qquad \zeta_2 = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{\det A}.$$

(c) Does Cramer's rule apply in this case? Is the solution ζ_1, ζ_2 obtained in (b) a pair of numbers in $\mathbb{C}_1$ or a pair of numbers in $\mathbb{C}_2$? If $c_1 = c_2 = 0$, what is the solution of the system of equations?
(d) Solve the same problem for a similar system of 3 equations in 3 unknowns, and for n equations in n unknowns.

6. THE IDEMPOTENT REPRESENTATION

This section defines idempotent elements and shows that there are four idempotent elements in $\mathbb{C}_2$. Two of these idempotent elements, namely $(1+i_1i_2)/2$ and $(1-i_1i_2)/2$, play an important role since every element in $\mathbb{C}_2$ has a unique representation as a linear combination of them. This section presents the properties of these idempotent elements, and it describes the properties of $\mathbb{C}_2$ and of its operations in terms of the idempotent representation of its elements.

6.1 DEFINITION Let ζ_1 and ζ_2 be elements in $\mathbb{C}_2$. If $\zeta_1^2=\zeta_1$, then ζ_1 is called an *idempotent* element. If $\zeta_1\neq 0$, $\zeta_2\neq 0$, and $\zeta_1\zeta_2=0$, then ζ_1 and ζ_2 are called *divisors of zero.*

6.2 THEOREM There are four and only four idempotent elements in $\mathbb{C}_2$, and they are

(1) $$0, \qquad 1, \qquad \frac{1+i_1i_2}{2}, \qquad \frac{1-i_1i_2}{2}.$$

Proof. Let $z_1+i_2z_2$ be an element in $\mathbb{C}_2$. Then $z_1+i_2z_2$ is an idempotent element if and only if $(z_1+i_2z_2)^2=z_1+i_2z_2$. This equation is equivalent to the following two equations:

(2) $$z_1^2-z_2^2=z_1, \qquad 2z_1z_2=z_2.$$

The second of these equations is satisfied if $z_2=0$ or $z_1=\frac{1}{2}$. If $z_2=0$, the first equation is satisfied by $z_1=0$ or $z_1=1$. If $z_1=\frac{1}{2}$, the first equation is satisfied by $z_2=\pm i_1/2$. Thus the two equations in (2) have the four solutions

(3) $$\begin{array}{llll} z_1=0, & z_1=1, & z_1=1/2, & z_1=1/2, \\ z_2=0, & z_2=0, & z_2=i_1/2, & z_2=-i_2/2. \end{array}$$

The corresponding elements $z_1+i_2z_2$ are those in (1), and the proof of Theorem 6.2 is complete. □

The following notation will be used for the third and fourth idempotent elements in (1):

(4) $$e_1=\frac{1+i_1i_2}{2}, \qquad e_2=\frac{1-i_1i_2}{2}.$$

6.3 THEOREM The idempotent elements e_1, e_2 have the following properties:

(5) $$e_1^2=e_1, \qquad e_2^2=e_2, \qquad e_1e_2=0;$$

(6) $$\|e_1\|=\sqrt{2}/2, \qquad \|e_2\|=\sqrt{2}/2;$$

(7) $V(e_1)=0$, $V(e_2)=0$, and e_1, e_2 are singular elements in $\mathbb{C}_2$;

(8) e_1, e_2 are linearly independent with respect to complex constants of combination; that is, if c_1, c_2 are in $\mathbb{C}_1$ and $c_1e_1+c_2e_2=0$, then $c_1=c_2=0$.

Proof. The statements in (5) can be verified easily from the definitions in (4), and (6) follows from (4) and Definition 3.1. Statement (7) follows from (4), Definition 4.12, and Corollary 4.14. To prove (8), let c_1, c_2 be complex numbers in $\mathbb{C}_1$. Then the equation $c_1e_1+c_2e_2=0$ is

$$c_1\left(\frac{1+i_1i_2}{2}\right)+c_2\left(\frac{1-i_1i_2}{2}\right)=0. \tag{9}$$

This equation is equivalent to the system

$$c_1+c_2=0, \qquad i_1c_1-i_1c_2=0. \tag{10}$$

Since the determinant of the matrix of coefficients in this system is $-2i_1$, then $c_1=0$ and $c_2=0$. The proof of all parts of Theorem 6.3 is complete. □

6.4 THEOREM (Idempotent Representation) Every element $\zeta:(z_1+i_2z_2)$ in $\mathbb{C}_2$ has the following unique representation:

$$\zeta=(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2. \tag{11}$$

Proof. Let c_1, c_2 be complex members in $\mathbb{C}_2$ such that $c_1e_1+c_2e_2=\zeta$. Then

$$c_1\left(\frac{1+i_1i_2}{2}\right)+c_2\left(\frac{1-i_1i_2}{2}\right)=z_1+i_2z_2, \tag{12}$$

and this equation is equivalent to the following system of equations:

$$\frac{c_1}{2}+\frac{c_2}{2}=z_1, \qquad \frac{i_1c_1}{2}-\frac{i_1c_2}{2}=z_2. \tag{13}$$

These equations have the following unique solution:

$$c_1=z_1-i_1z_2, \qquad c_2=z_1+i_1z_2. \tag{14}$$

Thus ζ has the unique representation shown in (11), and the proof of Theorem 6.4 is complete. □

6.5 DEFINITION Equation (11) is the *idempotent representation* of the element $\zeta:(z_1+i_2z_2)$ in $\mathbb{C}_2$. Also, $z_1-i_1z_2$ and $z_1+i_1z_2$ are the *idempotent components* of $z_1+i_2z_2$.

6.6 THEOREM Let $z_1+i_2z_2$ and $w_1+i_2w_2$ be elements in $\mathbb{C}_2$. Then

$$\begin{aligned}(15)\qquad &(z_1+i_2z_2)+(w_1+i_2w_2)\\ &\quad=[(z_1-i_1z_2)+(w_1-i_1w_2)]e_1+[(z_1+i_1z_2)+(w_1+i_1w_2)]e_2,\end{aligned}$$

$$(16)\qquad (z_1 + i_2z_2)(w_1 + i_2w_2) \\ = [(z_1 - i_1z_2)(w_1 - i_1w_2)]e_1 + [(z_1 + i_1z_2)(w_1 + i_1w_2)]e_2,$$

$$(17)\qquad (z_1 + i_2z_2)^n = (z_1 - i_1z_2)^n e_1 + (z_1 + i_1z_2)^n e_2,\ n = 0, 1, \ldots.$$

If $(w_1 - i_1w_2) \neq 0$ and $(w_1 + i_1w_2) \neq 0$, then

$$(18)\qquad \frac{z_1 + i_2z_2}{w_1 + i_2w_2} = \left(\frac{z_1 - i_1z_2}{w_1 - i_1w_2}\right)e_1 + \left(\frac{z_1 + i_1z_2}{w_1 + i_1w_2}\right)e_2.$$

Proof. Equation (15) follows from the idempotent representation of elements in $\mathbb{C}_2$ and from properties of addition and multiplication. To prove (16), multiply the idempotent representations of $z_1+i_2z_2$ and $w_1+i_2w_2$ and then use the properties of e_1, e_2 in (5) and the properties of multiplication to simplify the result. Induction and (16) can be used to prove (17). Consider (18). Since $(w_1-i_1w_2)\neq 0$ and $(w_1+i_1w_2)\neq 0$ by hypothesis, $w_1+i_2w_2$ is nonsingular, and by Theorem 5.2 the fraction $(z_1+i_2w_2)/(w_1+i_2w_2)$ is defined; let $u_1+i_2u_2$ denote its value. Then by Definition 5.1,

$$(19)\qquad z_1 + i_2z_2 = (w_1 + i_2w_2)(u_1 + i_2u_2).$$

Since

$$(20)\qquad z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2, \\ (w_1 + i_2w_2)(u_1 + i_2u_2) \\ = [(w_1 - i_1w_2)(u_1 - i_1u_2)]e_1 + [(w_1 + i_1w_2)(u_1 + i_1u_2)]e_2,$$

equation (19) and the fact that the idempotent representation is unique show that

$$(21)\qquad z_1 - i_1z_2 = (w_1 - i_1w_2)(u_1 - i_1u_2), \\ z_1 + i_1z_2 = (w_1 + i_1w_2)(u_1 + i_1u_2).$$

These equations show that, since $(w_1-i_1w_2)(w_1+i_1w_2)\neq 0$,

$$(22)\qquad u_1 - i_1u_2 = \frac{z_1 - i_1z_2}{w_1 - i_1w_2}, \qquad u_1 + i_1u_2 = \frac{z_1 + i_1z_2}{w_1 + i_1w_2}.$$

Thus

$$(23)\qquad \frac{z_1 + i_2z_2}{w_1 + i_2w_2} = u_1 + i_2u_2 \\ = (u_1 - i_1u_2)e_1 + (u_1 + i_1u_2)e_2, \\ = \left(\frac{z_1 - i_1z_2}{w_1 - i_1w_2}\right)e_1 + \left(\frac{z_1 + i_1z_2}{w_1 + i_1w_2}\right)e_2.$$

Therefore, (18) is true, and the proof of Theorem 6.6 is complete. □

6.7 COROLLARY If $z_1+i_2z_2$ and $w_1+i_2w_2$ are elements in $\mathbb{C}_2$ such that $z_1-i_1z_2=0$ and $w_1+i_1w_2=0$ (or $z_1+i_1z_2=0$ and $w_1-i_1w_2=0$), then $(z_1+i_2w_2)(w_1+i_2w_2)=0$. If $z_1+i_2z_2\neq 0$ and $w_1+i_2w_2\neq 0$, then these elements are divisors of zero.

Proof. If $z_1-i_1z_2=0$ and $w_1+i_1w_2=0$ (or $z_1+i_1z_2=0$ and $w_1-i_1w_2=0$), then Theorem 6.6 (16) shows that

$$(z_1+i_2z_2)(w_1+i_2w_2)=0e_1+0e_2=0. \tag{24}$$

If, in addition, $z_1+i_2z_2\neq 0$ and $w_1+i_2w_2\neq 0$, then these elements are divisors of zero by Definition 6.1. □

6.8 THEOREM If $z_1+i_2z_2$ is an element in $\mathbb{C}_2$, then

$$\left(\frac{|z_1-i_1z_2|^2+|z_1+i_1z_2|^2}{2}\right)^{1/2}=(|z_1|^2+|z_2|^2)^{1/2}=\|z_1+i_2z_2\|. \tag{25}$$

Proof. Let $z_1=x_1+i_1y_1$ and $z_2=x_2+i_1y_2$. Then

$$\begin{aligned} z_1-i_1z_2&=(x_1+i_1y_1)-i_1(x_2+i_1y_2)=(x_1+y_2)+i_1(y_1-x_2),\\ z_1+i_1z_2&=(x_1+i_1y_1)+i_1(x_2+i_1y_2)=(x_1-y_2)+i_1(y_1+x_2). \end{aligned} \tag{26}$$

Then

$$\begin{aligned} |z_1-i_1z_2|^2+|z_1+i_1z_2|^2&=(x_1+y_2)^2+(y_1-x_2)^2\\ &\quad+(x_1-x_2)^2+(y_1+x_2)^2\\ &=2(x_1^2+y_1^2+x_2^2+y_2^2)\\ &=2(|z_1|^2+|z_2|^2)\\ &=2\|z_1+i_2z_2\|^2. \end{aligned} \tag{27}$$

The formula in (25) follows from these equations. □

Exercises

6.1 Show that an element $z_1+i_2z_2$ in $\mathbb{C}_2$ is zero if and only if both of its idempotent components are zero.

6.2 Define the i_2-conjugate bicomplex number of $z_1+i_2z_2$ to be $z_1-i_2z_2$. Show that $(z_1+i_2z_2)(z_1-i_2z_2)$ is a complex number in $\mathbb{C}_1$. Compare this result with the corresponding property of conjugate complex numbers in $\mathbb{C}_1$.

6.3 Define the i_1i_2-conjugate bicomplex number of $z_1+i_2z_2$ to be $\bar{z}_1-i_2\bar{z}_2$.
(a) Show that

$$(z_1+i_2z_2)(\bar{z}_1-i_2\bar{z}_2)=|z_1-i_1z_2|^2e_1+|z_1+i_1z_2|^2e_2.$$

(b) Show that $z_1+i_2z_2=0$ if and only if $(z_1+i_2z_2)(\bar{z}_1-i_2\bar{z}_2)=0$. Compare this result with a property of complex numbers in $\mathbb{C}_1$.

6.4 Find the bicomplex numbers which are nth roots of unity. [*Solution.* The problem is to find the solutions in $\mathbb{C}_2$ of the equation $(z_1+i_2z_2)^n=1$. Use the idempotent representation to show that this equation is

$$(z_1 - i_1z_2)^n e_1 + (z_1 + i_1z_2)^n = 1e_1 + 1e_2.$$

Show that this equation is satisfied if and only if

$$(z_1 - i_1z_2)^n = 1, \qquad (z_1 + i_1z_2)^n = 1.$$

Since these are polynomial equations in $\mathbb{C}_1$, each equation has n roots; they are the n nth roots of unity. Denote them by $\omega_1, \ldots, \omega_n$. Show that the equation $(z_1+i_2z_2)^n=1$ has exactly n^2 bicomplex roots, and that they are $\omega_ie_1+\omega_je_2$ for $i, j=1, \ldots, n$.]

6.5 Show that

$$\sum_{k=0}^{n} (a_k + i_2b_k)(z_1 + i_2z_2)^k = 0, \qquad a_k + i_2b_k \in \mathbb{C}_2,\ a_n^2 + b_n^2 \neq 0,$$

has n^2 roots (if roots are counted with their multiplicities), and explain how to find these roots. [*Hint.* Set $z_1+i_2z_2=(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2$, and use Theorem 6.6 to show that the given polynomial equation is

$$\left[\sum_{k=0}^{n} (a_k - i_1b_k)(z_1 - i_1z_2)^k\right]e_1 + \left[\sum_{k=0}^{n} (a_k + i_1b_k)(z_1 + i_1z_2)^k\right]e_2 = 0e_1 + 0e_2.$$

Show that the roots of the given equation can be found by solving the two equations

$$\sum_{k=0}^{n} (a_k - i_1b_k)(z_1 - i_1z_2)^k = 0, \qquad a_n - i_1b_n \neq 0,$$

$$\sum_{k=0}^{n} (a_k + i_1b_k)(z_1 + i_1z_2)^k = 0, \qquad a_n + i_1b_n \neq 0.]$$

6.6 Consider again the polynomial equation in Exercise 6.5. Assume that $a_n-i_1b_n\neq 0$, $a_n+i_1b_n=0$, and $a_{n-1}+i_1b_{n-1}\neq 0$. Show that the equation has $n(n-1)$ roots if the roots are counted with their multiplicities.

6.7 Let $z_1+i_2z_2$ and $w_1+i_2w_2$ be two elements in $\mathbb{C}_2$. Show that $z_1+i_2z_2=w_1+i_2w_2$ if and only if $z_1-i_1z_2=w_1-i_1w_2$ and $z_1+i_1z_2=w_1+i_1w_2$.

6.8 Let $[a_{ij}]$, $i, j = 1, \dots, n$, be a matrix with elements a_{ij} in $\mathbb{C}_2$. As a matter of notation, set $a_{ij} = \alpha_{ij} e_1 + \beta_{ij} e_2$.

(a) Use Theorem 6.6 and the definition of the determinant to show that

$$\det[a_{ij}] = \det[\alpha_{ij}]e_1 + \det[\beta_{ij}]e_2.$$

(b) Show that $\det[a_{ij}]$ is nonsingular if and only if $\det[\alpha_{ij}] \neq 0$ and $\det[\beta_{ij}] \neq 0$.

6.9 Consider the following system of equations (compare Exercise 5.6):

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Here $[a_{ij}]$ is the matrix in Exercise 6.8. Also, $c_1, \dots, c_n$ are given numbers in $\mathbb{C}_2$ and $\zeta_1, \dots, \zeta_n$ are unknown numbers in $\mathbb{C}_2$. As a matter of notation, set

$$\zeta_i = z_i e_1 + w_i e_2, \qquad c_i = \gamma_i e_1 + \delta_i e_2, \qquad i = 1, \dots, n.$$

(a) Show that the solution of the given system can be found by solving two systems

$$\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix},$$

$$\begin{bmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & & \vdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}.$$

(b) Assume that $\det[a_{ij}]$ is a nonsingular element in $\mathbb{C}_2$ and show that the given system of equations has a unique solution.

(c) Assume again that $\det[a_{ij}]$ is nonsingular. Use a second method to show that the given system of equations has a unique solution and to find this solution.

6.10 Consider again the system of equations in Exercise 6.9. Investigate the solution of the system under the assumption that

(a) $\det[\alpha_{ij}] \neq 0$, $\det[\beta_{ij}] = 0$,

(b) $\det[\alpha_{ij}] = 0$, $\det[\beta_{ij}] = 0$.

6.11 Use the formula in Theorem 6.8 (25) to prove the following properties of the norm $\| \ \|$: for every $z_1 + i_2 z_2$ and $w_1 + i_2 w_2$ in $\mathbb{C}_2$ and z in $\mathbb{C}_1$:

(a) $\|z_1 + i_2 z_2\| \geqslant 0$, $\|z_1 + i_2 z_2\| = 0$ if and only if $z_1 + i_2 z_2 = 0$;

(b) $\|z(z_1 + i_2 z_2)\| = |z| \, \|z_1 + i_2 z_2\|$;

(c) $\|(z_1+i_2z_2)+(w_1+i_2w_2)\| \leqslant \|z_1+i_2z_2\| + \|w_1+i_2w_2\|$;
(d) $\|(z_1+i_2z_2)(w_1+i_2w_2)\| \leqslant \sqrt{2}\,\|z_1+i_2z_2\|\,\|w_1+i_2w_2\|$.
Compare this exercise with Theorems 3.2 and 4.4.

6.12 If $z_1+i_2z_2$ is in $\mathbb{C}_2$, show that

$$\|(z_1+i_2z_2)^n\| \leqslant 2^{(n-1)/2}\|z_1+i_2z_2\|^n, \qquad n=1,2,\ldots.$$

6.13 (a) If

$$z_1+i_2z_2=(2+3i_1)+i_2(4+5i_1),$$
$$w_1+i_2w_2=(1-2i_1)+i_2(3-2i_1),$$

show that

$$\|z_1+i_2z_2\|=(54)^{1/2}, \qquad \|w_1+i_2w_2\|=(18)^{1/2},$$
$$\|(z_1+i_2z_2)(w_1+i_2w_2)\|=(940)^{1/2},$$

and thus that

$$0<\|(z_1+i_2z_2)(w_1+i_2w_2)\|<\|z_1+i_2z_2\|\,\|w_1+i_2w_2\|.$$

(b) Prove that

$$|z_1-i_1z_2|^2|w_1-i_1w_2|^2+|z_1+i_1z_2|^2|w_1+i_1w_2|^2$$
$$\leqslant |z_1-i_1z_2|^2|w_1+i_1w_2|^2+|z_1+i_1z_2|^2|w_1-i_1w_2|^2$$

is a necessary and sufficient condition that $z_1+i_2z_2$ and $w_1+i_2w_2$ satisfy the inequality

$$\|(z_1+i_2z_2)(w_1+i_2w_2)\| \leqslant \|z_1+i_2z_2\|\,\|w_1+i_2w_2\|.$$

(c) Verify that the numbers $z_1+i_2z_2$ and $w_1+i_2w_2$ in (a) satisfy the inequality in (a) by showing that they satisfy a sufficient condition obtained by the methods explained in (b).

6.14 Theorems 3.2 and 4.4 have shown that

$$0 \leqslant \|(z_1+i_2z_2)(w_1+i_2w_2)\| \leqslant \sqrt{2}\,\|z_1+i_2z_2\|\,\|w_1+i_2w_2\|.$$

Verify that there exist many pairs of nonzero numbers $z_1+i_2z_2$ and $w_1+i_2w_2$ in $\mathbb{C}_2$ for which the equality holds on the left, and many other pairs of nonzero numbers for which the equality holds on the right. (*Hint.* Theorem 6.3.)

6.15 Let z, z_1, z_2 and w_1, w_2 denote complex numbers in $\mathbb{C}_1$. Define a system $(\mathbb{A}, +, \cdot, \times, N)$ by the following statements:

Elements in $\mathbb{A}$: $(z_1-i_1z_2,\ z_1+i_1z_2)$
Equals $(=)$: $(z_1-i_1z_2,\ z_1+i_1z_2)=(w_1-i_1w_2,\ w_1+i_1w_2)$ if and only if $z_1-i_1z_2=w_1-i_1w_2$ and $z_1+i_1z_2=w_1+i_1w_2$.

Addition $(+)$: $(z_1-i_1z_2,\ z_1+i_1z_2)+(w_1-i_1w_2,\ w_1+i_1w_2)$
$=[(z_1-i_1z_2)+(w_1-i_1w_2),\ (z_1+i_1z_2)+(w_1+i_1w_2)]$.
Scalar multiplication $(\cdot)$: $z\cdot(z_1-i_1z_2,\ z_1+i_1z_2)=[z(z_1-i_1z_2),\ z(z_1+i_1z_2)]$.
Multiplication $(\times)$: $(z_1-i_1z_2,\ z_1+i_1z_2)\times(w_1-i_1w_2,\ w_1+i_1w_2)$
$=[(z_1-i_1z_2)(w_1-i_1w_2),\ (z_1+i_1z_2)(w_1+i_1w_2)]$.
Norm N: $N(z_1-i_1z_2,\ z_1+i_1z_2)=\left(\dfrac{|z_1-i_1z_2|^2+|z_1+i_1z_2|^2}{2}\right)^{1/2}$.

(a) Show that the system $(\mathbb{A}, +, \cdot, \times, N)$ is a Banach algebra.
(b) Show that, under the correspondence

$$z_1+i_2z_2 \leftrightarrow (z_1-i_1z_2,\ z_1+i_1z_2),$$

the Banach algebra $(\mathbb{C}_2, \oplus, \odot, \otimes, \|\ \|)$ and $(\mathbb{A}, +, \cdot, \times, N)$ are isomorphic and that $\|z_1+i_2z_2\|=N(z_1-i_1z_2,\ z_1+i_1z_2)$.

7. TWO PRINCIPAL IDEALS

The purpose of this section is to define a principal ideal in an algebra and then to describe the principal ideals in $\mathbb{C}_2$ determined by the idempotent elements e_1 and e_2. The section investigates the properties of these ideals and uses them to solve polynomial equations in $\mathbb{C}_2$.

7.1 DEFINITION An *ideal* I in an algebra A is a nonempty subset of A with the following properties:

(1) If α_1 and α_2 are in I, then $\alpha_1-\alpha_2$ is in I;

(2) If α is in I and a is in A, then $a\alpha$ is in I.

The ideal determined by an element β in A is $\{a\beta : a\in A\}$, and it is called a *principal ideal*. The principal ideals in $\mathbb{C}_2$ determined by e_1 and e_2 are denoted by I_1 and I_2 respectively; thus

(3) $I_1=\{(z_1+i_2z_2)e_1 : (z_1+i_2z_2)\in\mathbb{C}_2\}$,

(4) $I_2=\{(z_1+i_2z_2)e_2 : (z_1+i_2z_2)\in\mathbb{C}_2\}$.

Since $z_1+i_2z_2=(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2$, then

(5) $(z_1+i_2z_2)e_1=(z_1-i_1z_2)e_1, \qquad (z_1+i_2z_2)e_2=(z_1+i_1z_2)e_2,$

and the ideals I_1, I_2 in $\mathbb{C}_2$ have in addition the following descriptions:

(6) $I_1=\{(z_1-i_1z_2)e_1 : z_1, z_2\in\mathbb{C}_2\}$,

(7) $I_2=\{(z_1+i_1z_2)e_2 : z_1, z_2\in\mathbb{C}_2\}$.

7.2 THEOREM The ideals I_1, I_2 are linear subspaces in $\mathbb{C}_2$ which have the single element 0 in common:

(8) $\quad I_1 \cap I_2 = \{0\}.$

Proof. If all elements in $\mathbb{C}_2$ are multiplied by a fixed element ζ_0 in $\mathbb{C}_2$, the set $\mathbb{C}_2$ is transformed into a subset of $\mathbb{C}_2$. This transformation is a linear transformation since

(9) $\quad \zeta_0(z_1\zeta_1 + z_2\zeta_2) = z_1(\zeta_0\zeta_1) + z_2(\zeta_0\zeta_2).$

Then by the definitions of I_1 and I_2 in (3) and (4), these ideals are linear subspaces of $\mathbb{C}_2$. To prove (8), let $(z_1 - i_1z_2)e_1$ in I_1 equal $(w_1 + i_1w_2)e_2$ in I_2 [see (6) and (7)]. Thus

(10) $\quad (z_1 - i_1z_2)e_1 = (w_1 + i_1w_2)e_2.$

Since, by Theorem 6.3 (8), e_1 and e_2 are linearly independent with respect to complex constants of combinations, then $z_1 - i_1z_2 = 0$ and $w_1 + i_1w_2 = 0$. Therefore, 0 is the one and only element in $I_1 \cap I_2$, and the proof of Theorem 7.2 is complete. □

7.3 THEOREM The product of a number in I_1 and a number in I_2 is zero. Two elements in $\mathbb{C}_2$ are divisors of zero if and only if one is in $I_1 - \{0\}$ and the other is in $I_2 - \{0\}$.

Proof. Let $(z_1 - i_1z_2)e_1$ and $(w_1 + i_1w_2)e_2$ be elements in I_1 and I_2 respectively [see (6) and (7)]. Then

(11) $\quad (z_1 - i_1z_2)e_1(w_1 + i_1w_2)e_2 = (z_1 - i_1z_2)(w_1 + i_1w_2)e_1e_2 = 0,$

and the first statement in the theorem is true. If $(z_1 - i_1z_2)e_1 \neq 0$ and $(w_1 + i_1w_2)e_2 \neq 0$, then these two numbers are divisors of zero. To show that there are no other divisors of zero, let $z_1 + i_2z_2$ and $w_1 + i_2w_2$ be two elements in $\mathbb{C}_2$ such that

(12) $\quad z_1 + i_2z_2 \neq 0, \qquad w_1 + i_2w_2 \neq 0, \qquad (z_1 + i_2z_2)(w_1 + i_2w_2) = 0.$

By Theorem 6.6 (16),

(13) $$(z_1 + i_2z_2)(w_1 + i_2w_2) = (z_1 - i_1z_2)(w_1 - i_1w_2)e_1 + (z_1 + i_1z_2)(w_1 + i_1w_2)e_2.$$

Since $(z_1 + i_2z_2)(w_1 + i_2w_2) = 0$, then

(14) $\quad (z_1 - i_1z_2)(w_1 - i_1w_2) = 0, \qquad (z_1 + i_1z_2)(w_1 + i_1w_2) = 0.$

Equations (14) and (12) show that one of the following two cases holds:

(15) $\quad (z_1 - i_1z_2) \neq 0, \qquad (z_1 + i_1z_2) = 0, \qquad (w_1 - i_1w_2) = 0,$
$(w_1 + i_1w_2) \neq 0;$

(16) $(z_1 - i_1z_2) = 0, \qquad (z_1 + i_1z_2) \neq 0, \qquad (w_1 - i_1w_2) \neq 0,$
$(w_1 + i_1w_2) = 0.$

If (15) holds, then $z_1+i_2z_2$ is in $I_1 - \{0\}$ by (6) since $z_1+i_2z_2=(z_1-i_1z_2)e_1$, and $w_1+i_2w_2$ is in $I_2-\{0\}$ by (7) since $w_1+i_2w_2=(w_1+i_1w_2)e_2$. If (16) holds, similar considerations show that $z_1+i_2z_2$ is in $I_2-\{0\}$ and $w_1+i_2w_2$ is in $I_1-\{0\}$. In both cases, $z_1+i_2z_2$ and $w_1+i_2w_2$ are divisors of zero. The proof of Theorem 7.3 is complete. □

7.4 THEOREM An element $z_1+i_2z_2$ in $\mathbb{C}_2$ is singular if and only if $z_1+i_2z_2 \in I_1 \cup I_2$; it is nonsingular if and only if $z_1+i_2z_2 \notin I_1 \cup I_2$.

Proof. By Corollary 4.9, $z_1+i_2z_2$ is singular if and only if $|z_1-i_1z_2|\,|z_1+i_1z_2|=0$. Since $z_1+i_2z_2=(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2$, equations (6) and (7) show that $z_1+i_2z_2$ is singular if and only if $z_1+i_2z_2 \in I_1 \cup I_2$. Similarly, $z_1+i_2z_2$ is nonsingular if and only if $z_1+i_2z_2 \notin I_1 \cup I_2$. □

7.5 COROLLARY If $\mathcal{O}_2$ is the set of singular elements in $\mathbb{C}_2$ as defined at the end of Section 4, then

(17) $\mathcal{O}_2 = I_1 \cup I_2.$

An element in the complement of $\mathcal{O}_2$, that is, a nonsingular element, is often called a regular element in $\mathbb{C}_2$.

7.6 THEOREM If $\zeta:(x_1+i_1x_2+i_2x_3+i_1i_2x_4)$ denotes an element in $\mathbb{C}_2$, then

(18) $I_1 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4\colon x_1 - x_4 = 0 \text{ and } x_2 + x_3 = 0\},$

(19) $I_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4\colon x_1 + x_4 = 0 \text{ and } x_2 - x_3 = 0\}.$

Proof. By the idempotent representation,

(20)
$$\begin{aligned}(x_1 + i_1x_2) &+ i_2(x_3 + i_1x_4)\\ &= [(x_1 + i_1x_2) - i_1(x_3 + i_1x_4)]e_1 + [(x_1 + i_1x_2)\\ &\quad + i_1(x_3 + i_1x_4)]e_2.\end{aligned}$$

Now (16) shows that ζ is in I_1 if and only if $(x_1+i_1x_2)+i_1(x_3+i_1x_4)=0$, that is, if and only if $x_1-x_4=0$ and $x_2+x_3=0$. Thus (18) is true. Similarly, (7) shows that ζ is in I_2 if and only if $(x_1+i_1x_2)-i_1(x_3+i_1x_4)=0$, that is, if and only if $x_1+x_4=0$ and $x_2-x_3=0$. Thus (19) is true, and the proof is complete. □

Theorem 7.6 provides an easy proof that $I_1 \cap I_2 = \{0\}$, for if ζ is in I_1 and in I_2, then

(21) $$x_1 - x_4 = 0, \quad x_2 + x_3 = 0, \qquad x_1 + x_4 = 0, \qquad x_2 - x_3 = 0.$$

The only solution of these four equations is $x_i = 0$, $i = 1, \ldots, 4$. Thus $\zeta = 0$, and (8) is true.

7.7 THEOREM The ideals I_1, I_2 are closed sets in $\mathbb{C}_2$; the set $\mathcal{O}_2$ of singular elements in $\mathbb{C}_2$ is closed in $\mathbb{C}_2$; and the set of regular elements (the complement of $\mathcal{O}_2$) is an open set in $\mathbb{C}_2$. Every point in $\mathcal{O}_2$ is a limit point of the set of regular elements.

Proof. Each of the planes $x_1 - x_4 = 0$ and $x_2 + x_3 = 0$ is a closed set in $\mathbb{C}_0^4$. Since the intersection of two closed sets is a closed set, (18) shows that I_1 is closed. Similar arguments show that I_2 is closed. Since the union of two closed sets is closed, $I_1 \cup I_2$ is closed, and $\mathcal{O}_2$ is closed by (17). Then the set of regular elements, the complement of $\mathcal{O}_2$, is an open set. Finally, if ζ is in $I_1 \cup I_2$, then every neighborhood of ζ contains points $x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$ which are not in $I_1 \cup I_2$ by (18) and (19); that is, every neighborhood of ζ contains regular elements, and ζ is a limit point of the set of regular elements in $\mathbb{C}_2$. The proof of Theorem 7.7 is complete. □

The existence of divisors of zero in $\mathbb{C}_2$ but not in $\mathbb{C}_1$ is one of the significant differences between the algebras $\mathbb{C}_1$ and $\mathbb{C}_2$, and the solution of polynomial equations emphasizes this difference. Exercise 6.5 outlines a proof that the equation

(22) $$\sum_{k=0}^{n} (a_k + i_2 b_k)(z_1 + i_2 z_2)^k = 0, \qquad a_n^2 + b_n^2 \neq 0,$$

has n^2 solutions. A further examination of this equation is instructive. Assume that (22) has the following n roots.

(23) $$r_k + i_2 s_k, \qquad k = 1, \ldots, n.$$

Then the remainder theorem and the factor theorem show that (22) can be given the form

(24) $$(a_n + i_2 b_n) \prod_{k=1}^{n} [(z_1 + i_2 z_2) - (r_k + i_2 s_k)] = 0.$$

Since $a_n^2 + b_n^2 \neq 0$ by (22), then $a_n + i_2 b_n$ is nonsingular, and (24) is equivalent to

(25) $$\prod_{k=1}^{n} [(z_1 + i_2 z_2) - (r_k + i_2 s_k)] = 0.$$

This form of the equation displays n of its roots, and it is not immediately apparent how the remaining n^2-n roots are to be found. Theorem 7.3 supplies the answer. There are exactly two ways in which a value for $z_1+i_2z_2$ causes the polynomial in (25) to vanish; they are the following: (a) the value of $z_1+i_2z_2$ makes one of the factors equal to zero; and (b) the value of $z_1+i_2z_2$ makes one factor equal to an element in I_1 and another factor equal to an element in I_2. These facts will now be used to find the n^2 roots of the polynomial equation $P(z_1+i_2z_2)=0$ in (25). Assume that this equation has n distinct roots as follows:

(26) $\quad r_k + i_2s_k = (r_k - i_1s_k)e_1 + (r_k + i_1s_k)e_2;$

(27) $\quad r_p - i_1s_p \neq r_q - i_1s_q, \qquad r_p + i_1s_p \neq r_q + i_1s_q,$

$\quad p \neq q,\ p, q = 1, \ldots, n.$

7.8 THEOREM If the equation $P(z_1+i_2z_2)=0$ in (25) has n distinct roots which satisfy (27), then it has n^2 roots; they are

(28) $\quad (r_p - i_1s_p)e_1 + (r_q + i_1s_q)e_2, \qquad p, q = 1, \ldots, n,$

and these n^2 roots are distinct.

Begin the proof of this theorem by first proving the following lemma.

7.9 LEMMA If $P(z_1+i_2z_2)=0$ in (25) has two roots

(29) $\quad r_p + i_2s_p, \qquad r_q + i_2s_q,$

which satisfy (27), then

(30) $\quad (r_p - i_1s_p)e_1 + (r_q + i_1s_q)e_2,$

(31) $\quad (r_q - i_1s_q)e_1 + (r_p + i_1s_q)e_2,$

are also two roots of $P(z_1+i_2z_2)=0$; they are distinct and distinct from the roots in (29).

Proof. To prove the lemma, show that there is a unique value of $z_1+i_2z_2$ such that

(32) $\quad [(z_1 + i_2z_2) - (r_p + i_2s_p)] \in I_1 - \{0\},$

$\quad [(z_1 + i_2z_2) - (r_q + i_2s_q)] \in I_2 - \{0\},$

and also a unique value for $z_1+i_2z_2$ such that

(33) $\quad (z_1 + i_2z_2) - (r_p + i_2s_p) \in I_2 - \{0\},$

$\quad (z_1 + i_2z_2) - (r_q + i_2s_q) \in I_1 - \{0\}.$

By (6) and (7), $z_1+i_2z_2$ satisfies (32) if and only if there exist numbers w_p and w_q in $\mathbb{C}_1$ such that

(34) $\quad (z_1 + i_2z_2) - (r_p + i_2s_p) = w_pe_1, \qquad w_p \neq 0,$

(35) $\quad (z_1 + i_2z_2) - (r_q + i_2s_q) = w_qe_2, \qquad w_q \neq 0.$

The unknown quantities in these equations are z_1, z_2, w_p, and w_q; equations (34) and (35) are equivalent to the following equations:

(36)
$$z_1 - r_p = \frac{w_p}{2}, \qquad z_2 - s_p = \frac{i_1w_p}{2},$$
$$z_1 - r_q = \frac{w_q}{2}, \qquad z_2 - s_q = -\frac{i_1w_q}{2}.$$

These equations are linear in z_1, z_2, w_p, and w_q. Since the determinant of their matrix of coefficients is $i_1/2$, they have the following unique solution:

(37)
$$z_1 = \frac{(r_p + r_q) + i_1(s_p - s_q)}{2}, \qquad z_2 = \frac{-i_1(r_p - r_q) + (s_p + s_q)}{2},$$
$$w_p = -(r_p - r_q) + i_1(s_p - s_q), \qquad w_q = (r_p - r_q) + i_1(s_p - s_q).$$

Because $r_p+i_2s_p$ and $r_q+i_2s_q$ are roots of (25) which satisfy (27), the last two equations in (37) show that $w_p \neq 0$ and $w_q \neq 0$. Thus there exists a unique value for $z_1 + i_2z_2$ which satisfies (34) and (35); this value is a root of the polynomial equation (25) by Theorem 7.3. Since by (37), (34), and (35),

(38) $\quad z_1 + i_2z_2 = (r_q - i_1s_q)e_1 + (r_p + i_1s_p)e_2,$

this root is different from the two roots in (29). In the same way, (6) and (7) show that $z_1+i_2z_2$ satisfies (33) if and only if there exist elements w_p and w_q in $\mathbb{C}_1$ such that

(39) $\quad (z_1 + i_2z_2) - (r_p + i_2s_p) = w_pe_2, \qquad w_p \neq 0,$

(40) $\quad (z_1 + i_2z_2) - (r_q + i_2s_q) = w_qe_1, \qquad w_q \neq 0.$

As before, the unknown quantities in these equations are z_1, z_2, w_p, and w_q; equations (39) and (40) are equivalent to the following equations:

(41)
$$z_1 - r_p = \frac{w_p}{2}, \qquad z_2 - s_p = -\frac{i_1w_p}{2},$$
$$z_1 - r_q = \frac{w_q}{2}, \qquad z_2 - s_q = \frac{i_1w_q}{2}.$$

These equations are linear in z_1, z_2, w_p, and w_q; the determinant of their matrix of coefficients is $-i_1/2$; and their unique solution is

(42)
$$z_1 = \frac{(r_p + r_q) - i_1(s_p - s_q)}{2}, \qquad z_2 = \frac{i_1(r_p - r_q) + (s_p + s_q)}{2},$$

$$w_p = (r_p - r_q) - i_1(s_p - s_q), \qquad w_q = -(r_p - r_q) - i_1(s_p - s_q).$$

Because $r_p + i_2 s_p$ and $r_q + i_2 s_q$ are roots of (25) which satisfy (27), the last two equations in (42) show that $w_p \neq 0$ and $w_q \neq 0$. Thus there exists a unique value for $z_1 + i_2 z_2$ which satisfies (39) and (40) and hence (33); this value is a root of the polynomial equation (25) by Theorem 7.3. Since by (42),

$$(43) \qquad z_1 + i_2 z_2 = (r_p - i_1 s_p)e_1 + (r_q + i_1 s_q)e_2,$$

and since the roots in (29) satisfy (27) by hypothesis, the root in (43) is distinct from the one in (38) and from those in (29). Thus the polynomial equation $P(z_1 + i_2 z_2) = 0$ has the two roots in (30) and (31) [see (38) and (43)] in addition to the two roots in (29), and the four roots are distinct. The proof of Lemma 7.9 is complete. □

Proof of Theorem 7.8. The equation $P(z_1 + i_2 z_2) = 0$ has, by hypothesis, the following n roots,

$$(44) \qquad (r_k - i_1 s_k)e_1 + (r_k + i_1 s_k)e_2, \qquad k = 1, \ldots, n;$$

they satisfy (27). Lemma 7.9 shows that, corresponding to each distinct pair $(r_p + i_2 s_p)$, $(r_q + i_2 s_q)$ of these roots, the equation $P(z_1 + i_2 z_2) = 0$ has two additional roots as follows:

$$(45) \qquad (r_p - i_1 s_p)e_1 + (r_q + i_1 s_q)e_2,$$

$$(46) \qquad (r_q - i_1 s_q)e_1 + (r_p + i_1 s_p)e_2, \qquad p \neq q.$$

Since a pair of roots can be selected from the n roots in (44) in $n(n-1)/2$ ways, and since two roots can be constructed from each pair, there are $n(n-1)$ roots of the form shown in (45) and (46). Thus the total number of roots is $n + n(n-1)$ or n^2. The roots in (44), (45), and (46) can be described as follows:

$$(47) \qquad (r_p - i_1 s_p)e_1 + (r_q + i_1 s_q)e_2, \qquad p, q = 1, \ldots, n.$$

Since the n roots in (26) satisfy (27) by hypothesis, the n^2 roots in (47) are distinct. The proof of Theorem 7.8 is complete. □

Exercises

7.1 Establish each of the following statements in two ways:

(a) $3 + 7i_1 - 7i_2 + 3i_1 i_2$ and $(a - i_1 b) + i_2(b + i_1 a)$, $a - i_1 b \neq 0$, are in $I_1 - \{0\}$.

(b) $7 + 4i_1 + 4i_2 - 7i_1 i_2$ and $(a + i_1 b) + i_2(b - i_1 a)$, $a + i_1 b \neq 0$, are in $I_2 - \{0\}$.

(c) The product of an element in (a) and an element in (b) is zero.

(d) $\|3 + 7i_1 - 7i_2 + 3i_1 i_2\| = 116^{1/2}$;
$\|(a + i_1 b) + i_2(b - i_1 a)\| = [2(a^2 + b^2)]^{1/2}$.

7.2 Show that the following equation is satisfied by every $z_1+i_2z_2$ in I_2:

$$\sum_{k=1}^{n} (a_k + i_2b_k)(z_1 + i_2z_2)^k = 0,$$

$$(a_k + i_2b_k)\in I_1 \text{ for } k = 1,\ldots,n.$$

7.3 Prove the following fundamental theorem of algebra: Every polynomial equation

$$\sum_{k=0}^{n} (a_k + i_2b_k)(z_1 + i_2z_2)^k = 0, \qquad a_n^2 + b_n^2 \neq 0, \quad n \geqslant 1,$$

has at least one root in $\mathbb{C}_2$. (*Hint*. Exercise 6.5.)

7.4 Prove the following Remainder Theorem: Let $P(z_1+i_2z_2)$ denote the polynomial

$$\sum_{k=0}^{n} (a_k + i_2b_k)(z_1 + i_2z_2)^k = 0, \qquad a_n^2 + b_n^2 \neq 0, n \geqslant 1,$$

and let $Q(z_1+i_2z_2)$ and R be the quotient and constant remainder obtained by dividing $P(z_1+i_2z_2)$ by $(z_1+i_2z_2)-(r_1+i_2r_2)$. Then

(a) $P(z_1+i_2z_2)=[(z_1+i_2z_2)-(r_1+i_2r_2)]\ Q(z_1+i_2z_2)+R$;

(b) $Q(z_1+i_2z_2)$ is a polynomial of degree $n-1$ whose leading coefficient is $a_n+i_2b_n$;

(c) $R=P(r_1+i_2r_2)$.

7.5 Prove the following factor theorem: If $P(z_1+i_2z_2)$ is the polynomial in Exercise 7.4, and if $P(r_1+i_2r_2)=0$, then $[(z_1+i_2z_2)-(r_1+i_2r_2)]$ is a factor of $P(z_1+i_2z_2)$. Thus if $P(r_1+i_2r_2)=0$, then

$$P(z_1 + i_2z_2) = [(z_1 + i_2z_2) - (r_1 + i_2r_2)]\ Q(z_1 + i_2z_2).$$

7.6 Let $P(z_1 + i_2z_2)$ be the polynomial in Exercise 7.4. Prove that $P(z_1+i_2z_2)$ can be factored as follows:

$$P(z_1+i_2z_2)=(a_n+i_2b_n)\prod_{k=1}^{n} [(z_1+i_2z_2)-(r_k+i_2s_k)].$$

7.7 Let $P(z_1+i_2z_2)$ be the polynomial $(z_1+i_2z_2)^2-5(z_1+i_2z_2)+6$.

(a) Show that

$$2, \quad 3, \quad \frac{5-i_1i_2}{2}, \quad \frac{5+i_1i_2}{2},$$

are four roots of $P(z_1+i_2z_2)=0$.

(b) Use the factor theorem in Exercise 7.5 to show that $P(z_1+i_2z_2)$ can be factored into linear factors in two essentially different ways as follows:

$$P(z_1+i_2z_2)=[(z_1+i_2z_2)-2][(z_1+i_2z_2)-3];$$

$$P(z_1+i_2z_2)=\left[(z_1+i_2z_2)-\left(\frac{5-i_1i_2}{2}\right)\right]\left[(z_1+i_2z_2)-\left(\frac{5+i_1i_2}{2}\right)\right].$$

7.8 Let $P(z_1+i_2z_2)$ be the polynomial

$$\sum_{k=0}^{n}(a_k+i_2b_k)(z_1+i_2z_2)^k=0, \qquad a_n^2+b_n^2\neq 0,\ n\geq 1.$$

(a) Use the fundamental theorem of algebra, the factor theorem, and the methods used to prove Theorem 7.8 to show that the equation $P(z_1+i_2z_2)=0$ has n^2 roots (which may not all be distinct).

(b) Use the factor theorem to prove that $P(z_1+i_2z_2)$ can be factored into linear factors as follows:

$$P(z_1+i_2z_2)=(a_n+i_2b_n)\prod_{k=1}^{n}[(z_1+i_2z_2)-(r_k+i_2s_k)].$$

(c) Assume that no two of the roots of $P(z_1+i_2z_2)=0$ are equal. Prove that $P(z_1+i_2z_2)$ can be factored into linear factors in $n!$ essentially different ways. [*Hint*. The equation $P(z_1+i_2z_2)=0$ has a root $r_1+i_2s_1$ by the fundamental theorem of algebra. Then

$$P(z_1+i_2z_2)=[(z_1+i_2z_2)-(r_1+i_2s_1)]\,Q_1(z_1+i_2z_2).$$

The equation $Q_1(z_1+i_2z_2)=0$ has a root $r_2+i_2s_2$, and

$$\begin{aligned}&P(z_1+i_2z_2)\\&\quad=[(z_1+i_2z_2)-(r_1+i_2s_1)][(z_1+i_2z_2)\\&\qquad-(r_2+i_2s_2)]\,Q_2(z_1+i_2z_2).\end{aligned}$$

A continuation of this process shows that $P(z_1+i_2z_2)=0$ has n roots, and then the methods used in proving Theorem 7.8 show that it has n^2 roots. To factor $P(z_1+i_2z_2)$, use any one of the $(n-1)^2$ roots of $Q_1(z_1+i_2z_2)=0$ for the second factor, and so on. This process constructs $n^2(n-1)^2\cdots 2^2 1^2$, or $(n!)^2$, strings of factors. Since each set of n factors can be arranged in $n!$ different orders, there are $(n!)^2/n!$, or $n!$, essentially different ways to factor $P(z_1+i_2z_2)$. Compare Exercise 7.7.]

7.9 (a) Show that the following equation [a special case of (22)] has n roots in $\mathbb{C}_1$:

$$\sum_{k=0}^{n}a_k(z_1+i_2z_2)^k=0, \qquad a_k\in\mathbb{C}_1,\ k=0, 1,\ldots, n,\ a_n\neq 0.$$

(b) Assume that no two of the complex roots in (a) are equal. Prove that the equation has $n(n-1)$ distinct bicomplex roots which are in $\mathbb{C}_2$ but not in $\mathbb{C}_1$.

(c) Prove that the n^2 distinct roots of the equation in (a) occur in i_2-conjugate bicomplex pairs. Compare Exercise 7.7. (*Hint.* Exercise 6.2. The i_2-conjugate bicomplex number of a number in $\mathbb{C}_1$ is the number itself.)

7.10 (a) Show that the following equation [a special case of (22)] has n roots in $\mathbb{C}_1$:

$$\sum_{k=0}^{n} a_k(z_1+i_2z_2)^k=0, \qquad a_k \text{ in } \mathbb{C}_0,\ k=0,1,\ldots,n,\ a_n\neq 0.$$

(b) Show that, in (a), the complex roots in $\mathbb{C}_1$ occur in conjugate complex pairs. (*Hint.* The conjugate complex number of a number in $\mathbb{C}_0$ is the number itself.)

(c) Assume that the n complex roots of the equation in (a) are distinct numbers in $\mathbb{C}_1$. Prove that the equation has $n(n-1)$ distinct bicomplex roots which are not in $\mathbb{C}_1$.

(d) Prove that the roots of the equation in (a) occur in i_1i_2-conjugate bicomplex pairs. (*Hint.* Exercise 6.3.)

8. THE AUXILIARY COMPLEX SPACES

Define the complex spaces A_1, A_2 as follows:

$$(1)\qquad \begin{aligned} A_1 &= \{z_1 - i_1z_2\colon z_1 \text{ and } z_2 \text{ in } \mathbb{C}_1\},\\ A_2 &= \{z_1 + i_1z_2\colon z_1 \text{ and } z_2 \text{ in } \mathbb{C}_1\}. \end{aligned}$$

Since each element in $\mathbb{C}_1$ can be represented in the form $z_1-i_1z_2$ and $z_1+i_1z_2$ (and in many ways), the elements in A_1 and A_2 are the same as the elements in $\mathbb{C}_1$. Nevertheless, because of the special representations $z_1-i_1z_2$ and $z_1+i_1z_2$, the special notation A_1 and A_2 is convenient. The idempotent representation $(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2$ associates with each point $z_1+i_2z_2$ in $\mathbb{C}_2$ the points $z_1-i_1z_2$ and $z_1+i_1z_2$ in A_1 and A_2 respectively, and to each pair of points $(z_1-i_1z_2, z_1+i_1z_2)$ in $A_1\times A_2$ there corresponds a unique point in $\mathbb{C}_2$. Define functions $h_1:\mathbb{C}_2\to A_1$, $h_2:\mathbb{C}_2\to A_2$, and $H:A_1\times A_2\to\mathbb{C}_2$ as follows:

$$(2)\qquad \begin{aligned} &h_1(z_1+i_2z_2)=z_1-i_1z_2, && z_1+i_2z_2 \text{ in } \mathbb{C}_2,\ z_1-i_1z_2 \text{ in } A_1;\\ &h_2(z_1+i_2z_2)=z_1+i_1z_2, && z_1+i_2z_2 \text{ in } \mathbb{C}_2,\ z_1+i_1z_2 \text{ in } A_2;\\ &H(z_1-i_1z_2, z_1+i_1z_2) &&\\ &\quad=(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2, && (z_1-i_1z_2, z_1+i_1z_2)\in A_1\times A_2. \end{aligned}$$

The purpose of this section is to establish the properties of these functions or mappings.

The functions h_1, h_2, restricted to a set X in $\mathbb{C}_2$, map X into sets X_1, X_2 in A_1, A_2 respectively; the function H, restricted to a set A in $A_1 \times A_2$, maps A into a set Y in $\mathbb{C}_2$. Thus

$$(3)\qquad \begin{aligned} &h_1(X) = X_1, &&X \subset \mathbb{C}_2,\ X_1 \subset A_1;\\ &h_2(X) = X_2, &&X \subset \mathbb{C}_2,\ X_2 \subset A_2;\\ &H(A) = Y, &&A \subset A_1 \times A_2,\ Y \subset \mathbb{C}_2. \end{aligned}$$

8.1 DEFINITION Let R and S be rings. A function $h: R \to S$, $u \mapsto h(u)$, is called a *homomorphism* if and only if

$$(4)\qquad h(u+v) = h(u) + h(v), \qquad u, v \text{ in } R,$$

$$(5)\qquad h(uv) = h(u)h(v).$$

8.2 THEOREM The mapping $h_1: \mathbb{C}_2 \to A_1$ is a homomorphism which maps I_2 into $\{0\}$ in A_1, and $h_2: \mathbb{C}_2 \to A_2$ is a homomorphism which maps I_1 into $\{0\}$ in A_2.

Proof. By Theorem 6.6,

$$(6)\qquad \begin{aligned} &(z_1 + i_2 z_2) + (w_1 + i_2 w_2)\\ &\quad = [(z_1 - i_1 z_2) + (w_1 - i_1 w_2)]e_1\\ &\qquad + [(z_1 + i_1 z_2) + (w_1 + i_1 w_2)]e_2, \end{aligned}$$

$$(7)\qquad \begin{aligned} &(z_1 + i_2 z_2)(w_1 + i_2 w_2)\\ &\quad = [(z_1 - i_1 z_2)(w_1 - i_1 w_2)]e_1\\ &\qquad + [(z_1 + i_1 z_2)(w_1 + i_1 w_2)]e_2. \end{aligned}$$

Then by (2),

$$(8)\qquad \begin{aligned} h_1[(z_1 + i_2 z_2) + (w_1 + i_2 w_2)] &= (z_1 - i_1 z_2) + (w_1 - i_1 w_2)\\ &= h_1(z_1 + i_2 z_2) + h_1(w_1 + i_2 w_2),\\ h_1[(z_1 + i_2 z_2)(w_1 + i_2 w_2)] &= (z_1 - i_1 z_2)(w_1 - i_1 w_2)\\ &= h_1(z_1 + i_2 z_2)h_1(w_1 + i_2 w_2). \end{aligned}$$

Thus h_1 is a homomorphism by Definition 8.1. In the same way, (6) and (7) and the definition of h_2 in (2) show that h_2 is a homomorphism. The proof will be completed by showing that

$$(9)\qquad h_1(I_2) = \{0\},$$

$$(10)\qquad h_2(I_1) = \{0\}.$$

If $z_1+i_2z_2$ is in I_2, then (7) in Section 7 shows that $z_1-i_1z_2=0$; then (2) shows that $h_1(z_1+i_2z_2)=0$, and (9) is true. In the same way, if $z_1+i_2z_2$ is in I_1, then $z_1+i_1z_2=0$ by (6) in Section 7, and $h_2(z_1+i_2z_2)=0$ by (2). Thus (10) is true, and the proof of Theorem 8.2 is complete. □

Theorem 8.2 emphasizes that the mappings $h_1:\mathbb{C}_2\to A_1$ and $h_2:\mathbb{C}_2\to A_2$ are many-to-one mappings. Nevertheless, h_1 and h_2 map each element $z_1+i_2z_2$ into a unique pair of elements $(z_1-i_1z_2,\ z_1+i_1z_2)$ in $A_1\times A_2$. Thus given a point $z_1+i_2z_2$ in $\mathbb{C}_2$, the equations

$$(11)\qquad z_1-i_1z_2=w_1,\qquad z_1+i_1z_2=w_2,$$

define a unique element (w_1,w_2) in $A_1\times A_2$ which corresponds to $z_1+i_2z_2$ in $\mathbb{C}_2$. Furthermore, given (w_1,w_2) in $A_1\times A_2$, the equations (11) have the unique solution

$$(12)\qquad z_1=\frac{w_1+w_2}{2},\qquad z_2=\frac{i_1(w_1-w_2)}{2},$$

which defines the unique point $z_1+i_2z_2$ in $\mathbb{C}_2$ which corresponds to the element (w_1,w_2) in $A_1\times A_2$.

Let X be a set in $\mathbb{C}_2$. The restrictions $h_1|_X:X\to A_1$ and $h_2|_X:X\to A_2$ map X into sets X_1 and X_2 [see (3)] as follows:

$$(13)\qquad X_1=\{w_1\in A_1:w_1=h_1(z_1+i_2z_2),\ z_1+i_2z_2\in X\},$$

$$(14)\qquad X_2=\{w_2\in A_2\colon w_2=h_2(z_1+i_2z_2),\ z_1+i_2z_2\in X\}.$$

An understanding of the relation between X and the pair X_1,X_2 is important for later work, and several examples will provide an introduction to the study of these sets.

8.3 EXAMPLE Let X be the set $\{(z_1^k+i_2z_2^k)$ in $\mathbb{C}_2$: $k=1,\ldots,n\}$ such that

$$(15)\qquad z_1^k+i_2z_2^k=(z_1^k-i_1z_2^k)e_1+(z_1^k+i_1z_2^k)e_2,$$

$$(16)\qquad z_1^p-i_1z_2^p\neq z_1^q-i_1z_2^q,\qquad z_1^p+i_1z_2^p\neq z_1^q+i_1z_2^q,\qquad p\neq q,\ p,q=1,\ldots,n.$$

Then

$$(17)\quad h_1(z_1^k+i_2z_2^k)=z_1^k-i_1z_2^k,\qquad h_2(z_1^k+i_2z_2^k)=z_1^k+i_1z_2^k,\qquad k=1,\ldots,n,$$

$$(18)\quad X_1=\{(z_1^k-i_1z_2^k)\in A_1\colon k=1,\ldots,n\},\quad X_2=\{(z_1^k+i_1z_2^k)\in A_2\colon k=1,\ldots,n\}.$$

In this case $h_1|_X$ and $h_2|_X$ are one-to-one mappings of X into A_1 and A_2 respectively. The cartesian product of X_1 and X_2 is

$$(19)\qquad \{(z_1^p-i_1z_2^p,\ z_1^q+i_1z_2^q)\colon p,q=1,\ldots,n\}.$$

Now H maps $X_1 \times X_2$ into a set in $\mathbb{C}_2$, but this set is not X since

(20) $$X = \{H(z_1^k - i_1 z_2^k, z_1^k + i_1 z_2^k) \text{ in } \mathbb{C}_2\colon k = 1, \ldots, n\}.$$

It is necessary to know how the points in X_1 are paired with the points in X_2 in order to construct X. As (20) shows, X is the image under H of a proper subset of $X_1 \times X_2$.

8.4 EXAMPLE Let X be the following set of elements in $\mathbb{C}_2$:

(21) $$(z_1^p - i_1 z_2^p)e_1 + (z_1^q + i_1 z_2^q)e_2, \qquad p, q = 1, \ldots, n,$$

(22) $$(z_1^p - i_1 z_2^p) \neq (z_1^q - i_1 z_2^q), \qquad (z_1^p + i_1 z_2^p) \neq (z_1^q + i_1 z_2^q), \qquad p, q = 1, \ldots, n.$$

Then

(23) $$h_1[(z_1^p - i_1 z_2^p)e_1 + (z_1^q + i_1 z_2^q)e_2] = z_1^p - i_1 z_2^p, \qquad p, q = 1, \ldots, n.$$

(24) $$h_2[(z_1^p - i_1 z_2^p)e_1 + (z_1^q + i_1 z_2^q)e_2] = z_1^q + i_1 z_2^q, \qquad p, q = 1, \ldots, n.$$

In this case, $h_1|_X$ and $h_2|_X$ are n-to-1 mappings of X into A_1 and A_2, respectively. Because of the special nature of X, it can be reconstructed easily from X_1 and X_2. Since

(25) $$\begin{aligned} X &= \{(z_1^p - i_1 z_2^p)e_1 + (z_1^q + i_1 z_2^q)e_2\colon p, q = 1, \ldots, n\} \\ &= \{H(z_1^p - i_1 z_2^p, z_1^q + i_1 z_2^q)\colon p, q = 1, \ldots, n\}, \end{aligned}$$

then X is the image under H of the cartesian product $X_1 \times X_2$.

8.5 EXAMPLE Let X_1, X_2 be given sets of elements w_1, w_2 in A_1, A_2, respectively. Set

(26) $$X = \{z_1 + i_2 z_2 \text{ in } \mathbb{C}_2\colon z_1 + i_2 z_2 = w_1 e_1 + w_2 e_2,\ w_1 \in X_1,\ w_2 \in X_2\}.$$

In this case, X is the image under H [see (2)] of the set

(27) $$\{(w_1, w_2)\colon (w_1, w_2) \in X_1 \times X_2\}.$$

Thus X is the image under H of the cartesian set $X_1 \times X_2$.

8.6 THEOREM Let X_1, X_2 be sets in A_1, A_2 which have more than one point each, and let X be the set in $\mathbb{C}_2$ such that $X = H(X_1, X_2)$. Then

(28) each of the mappings $h_1|_X : X \to X_1$ and $h_2|_X : X \to X_2$ is a many-to-one mapping;

and

(29) there is a one-to-one correspondence between points $z_1 + i_2 z_2$ in X and pairs of points (w_1, w_2) in the cartesian product $X_1 \times X_2$.

Proof. To prove (28), let $a-i_1b$ be a fixed point in X_1, and let w_2 be a variable point in X_2. Set

(30) $$z_1 + i_2z_2 = (a - i_1b)e_1 + w_2e_2.$$

Then $z_1+i_2z_2$ is in X, and $h_1(z_1+i_2z_2)=a-i_1b$. Thus corresponding to each point w_2 in X_2 there is a point $z_1+i_2z_2$ in X such that $h_1(z_1+i_2z_2)=a-i_1b$. Since X_2 has more than one point by hypothesis, $h_1|_X$ is a many-to-one mapping. Similar arguments show that $h_2|_X$ is a many-to-one mapping, and (28) is true. To prove (29), observe first that to each point $z_1+i_2z_2$ in X there corresponds a unique pair (w_1, w_2) in $X_1 \times X_2$ by (2). To complete the proof, show as follows that $(h_1|_X, h_2|_X): X \to X_1 \times X_2$ has an inverse. If $(w_1, w_2) \in X_1 \times X_2$, then

(31) $$w_1 = z_1 - i_1z_2, \qquad w_2 = z_1 + i_1z_2.$$

Since these are the linear equations (11) which have the unique solution (12), there is a unique point $z_1+i_2z_2$ which corresponds to (w_1, w_2) in $X_1 \times X_2$; hence, (29) is true, and the proof is complete. □

8.7 THEOREM Let X be a set in $\mathbb{C}_2$, and let h_1 and h_2 map X into X_1 in A_1 and X_2 in A_2, respectively.

(32) If X is an open set in $\mathbb{C}_2$, then X_1 and X_2 are open sets in A_1 and A_2.

(33) If X is a convex set in $\mathbb{C}_2$, then X_1 and X_2 are convex sets in A_1 and A_2.

(34) If X is star-shaped with respect to $a+i_2b$ in $\mathbb{C}_2$, then X_1 and X_2 are star-shaped with respect to $a-i_1b$ and $a+i_1b$, respectively.

Proof. To prove (32), show that each point in X_1 has a neighborhood in X_1 and that each point in X_2 has a neighborhood in X_2. Let w_1^0 be a point in X_1; then there is some point $a+i_2b$ in X such that $w_1^0=h_1(a+i_2b)=a-i_1b$. Also, $h_2(a+i_2b)$ is a point $w_2^0=a+i_1b$ in X_2. Since X is open, there is a neighborhood $N(a+i_2b, \varepsilon)$ which is contained in X. We shall show that $N(a-i_1b, \varepsilon)\subset X_1$ and $N(a+i_1b, \varepsilon)\subset X_2$. Choose arbitrary points w_1 and w_2 such that

(35) $$w_1 \in N(a - i_1b, \varepsilon), \qquad w_2 \in N(a + i_1b, \varepsilon).$$

Proof is still required to show that $w_1 \in X_1$ and $w_2 \in X_2$. Because of (35),

(36) $$|w_1 - (a - i_1b)| < \varepsilon, \qquad |w_2 - (a + i_1b)| < \varepsilon.$$

Let $z_1+i_2z_2$ be the unique point in $\mathbb{C}_2$ which corresponds to (w_1, w_2) by (12); then

(37) $$w_1=h_1(z_1+i_2z_2)=z_1-i_1z_2, \qquad w_2=h_2(z_1+i_2z_2)=z_1+i_1z_2.$$

The proof will show that $z_1+i_2z_2$ is not only in $\mathbb{C}_2$ but also in X. By (36), (37), and Theorem 6.8,

$$\begin{aligned} (38)\qquad & \|(z_1+i_2z_2)-(a+i_2b)\| \\ &= \left[\frac{|(z_1-i_1z_2)-(a-i_1b)|^2+|(z_1+i_1z_2)-(a+i_1b)|^2}{2}\right]^{1/2} \\ &< \left(\frac{\varepsilon^2+\varepsilon^2}{2}\right)^{1/2} = \varepsilon. \end{aligned}$$

Then $(z_1+i_2z_2)\in N(a+i_2b,\varepsilon)$, and therefore $w_1\in X_1$ and $w_2\in X_2$ by (37). To summarize, $a-i_1b$ is an arbitrary point in X_1; and w_1, which was chosen as an arbitrary point in $N(a-i_1b,\varepsilon)$, is in X_1. Therefore, $N(a-i_1b,\varepsilon)\subset X_1$ and X_1 is an open set in A_1. A similar proof can be given to show that a neighborhood $N(c+i_1d,\varepsilon)$ of an arbitrary point $c+i_1d$ in X_2 is in X_2. Thus the proof is complete that X_1 and X_2 are open sets in A_1 and A_2, respectively.

To prove (33), let w_1^1 and w_1^2 be two points in X_1. Then there are points $a+i_2b$ and $c+i_2d$, not necessarily unique, such that

$$(39)\qquad w_1^1=h_1(a+i_2b)=a-i_1b,\qquad w_1^2=h_1(c+i_2d)=c-i_1d.$$

Also, there are points w_2^1 and w_2^2 in X_2 such that

$$(40)\qquad w_2^1=h_2(a+i_2b)=a+i_1b,\qquad w_2^2=h_2(c+i_2d)=c+i_1d.$$

Then $a-i_1b$ and $c-i_1d$ are in X_1, and $a+i_1b$ and $c+i_1d$ are in X_2. Since X is convex by hypothesis, then $t(a+i_2b)+(1-t)(c+i_2d)$, $0\leqslant t\leqslant 1$, is in X and therefore

$$\begin{aligned} (41)\qquad & t(a-i_1b)+(1-t)(c-i_1d)\in X_1, \\ & t(a+i_1b)+(1-t)(c+i_1d)\in X_2,\qquad 0\leqslant t\leqslant 1. \end{aligned}$$

The first of these equations shows that X_1 is convex. To show that X_2 is convex, choose two arbitrary points in X_2 and use a similar argument to show that the segment connecting them is in X_2. The proof of (33) is complete.

To prove (34), assume that X is star-shaped with respect to $a+i_2b$. Then for every $z_1+i_2z_2$ in X the points

$$(42)\qquad t(z_1+i_2z_2)+(1-t)(a+i_2b),\qquad 0\leqslant t\leqslant 1,$$

are also in X, and

$$\begin{aligned} (43)\qquad & t(z_1-i_1z_2)+(1-t)(a-i_1b)\in X_1, \\ & t(z_1+i_1z_2)+(1-t)(a+i_1b)\in X_2,\qquad 0\leqslant t\leqslant 1. \end{aligned}$$

If w_1^0 is an arbitrary point in X_1, then there is at least one point $z_1+i_2z_2$ in X such that $w_1^0=h_1(z_1+i_2z_2)=z_1-i_1z_2$. For this point, $z_1+i_1z_2$ is in X_2. Also,

References

G. Birkhoff and S. Mac Lane , A Survey of Modern Algebra, Revised Edition, Macmillan, New York, 1953, xi + 472pp.

G. Scorza Dragoni , Sulle funzioni olomorfe di una variabile bicomplessa, Reale Accad. d'Italia, Mem. Classe Sci. Fis., Mat. Nat. 5 (1934), 597–665.

Michiji Futagawa , On the theory of functions of a quaternary variable, Tôhoku Math. J. 29 (1928), 175–222; 35 (1932), 69–120.

E. Hille , Functional Analysis and Semi-Groups, Vol. 31, American Mathematical Society Colloquium Publications, New York, 1948.

E. Hille , Analytic Function Theory, Chelsea Publishing, New York, Vol. I, 1982, xi + 308 pp.; Vol. II, 1977, xii + 496pp.

U. Morin , Richerche sull'algebra bicomplessa, Mem. Acad. Ital. 6 (1935), 1241–1265.

G. Baley Price , Multivariable Analysis, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1984, xiv + 655 pp.

James D. Riley , Contributions to the theory of functions of a bicomplex variable, Tôhoku Math. J. 2nd series 5 (1953), 132–165.

Friedrich Ringleb , Beiträge zur Funktionentheorie in hyperkomplexen Systemen, I, Rend. Circ. Mat. Palermo 57 (1933), 311–340.

G. Battista Rizza , Teoremi e formule integrali nelle algebre ipercom-plesse, Atti IV Cong. Un. mat. Ital. 2 (1953), 204–209.

G. Scheffers , Verallgemeinerung der Grundlagen der gewöhnlich complexen Funktionen, I, II, Berichte über die Verhandlungen der Sächsis- chen Akademie der Wissenschaften zu Leipzig, Math.-Phys. Klasse 45 (1893), 828–848; 46 (1894), 120–134.

C. Segre , Le Rappresentazioni Reali delle Forme Complesse e Gli Enti Iperalgebrici, Math. Ann. 40 (1892), 413–467.

N. Spampinato , Estensione nel Campo Bicomplesso di Due Teoremi, del Levi-Civita e del Severi, per le Funzione Olomorfe di Due Variabili Complesse, I, II, Atti Reale Accad. Naz. Lincei, Rend. (6) 22 (1935), 38–43, 96–102.

N. Spampinato , Sulla Rappresentazione delle Funzioni di Variabile Bicomplessa Totalmente Derivabili, Ann. Mat. Pura Appl. (4) 14 (1936), 305–325.

N. Spampinato , Sulle Funzione di Variabili bicomplessa o Biduale, Scritte Matematici Offerto a Luigi Berzolari, published by Zanichelli, 1936, pp. 595–611.

Tsurusaburo Takasu , Theorie der Funktionen einer allgemeinen bikomplexen Veranderlichen, I, Tôhoku Sci. Rep. 32 (1943), 1.

V. A. Tret'jakov , On the properties of some elementary functions that are defined on the algebra of bicomplex numbers (Russian), pp. 99–106 in Mathematical Analysis and the Theory of Functions (Russian), edited by I. I. Bavrin , Moskov. Oblast. Ped. Inst., Moscow, 1980, 164 pp.

J. A. Ward , Theory of analytic functions in linear associative algebras, Duke Math. J. 7 (1940), 233–248.

M. Spivak , Calculus on Manifolds, W. A. Benjamin , New York and Amsterdam, 1965, xii +146 pp.

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The manuscript, written and supplemented from lecture notes by Erwin Bareiss, was approved by Rudolf Fueter (1879–1950), and typed by Wilfred Bauert. It summarizes Professor Fueter's research in hyper complex function theory, and points out where more research would lead to new insight in number theory, function theory and its applications.