The Hanbury Brown and Twiss effect

Lecture 19

February 29th, 2008

1 Coherence of light

We have been considering the mode expansion of the vector potential

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, \alpha} c \sqrt{\frac{2\pi\hbar}{\omega_k}} \left[a_{\mathbf{k}, \alpha}(t) \boldsymbol{\varepsilon}_{\mathbf{k}, \alpha} e^{i\mathbf{k} \cdot \mathbf{r}} + a_{\mathbf{k}, \alpha}^\dagger(t) \boldsymbol{\varepsilon}_{\mathbf{k}, \alpha}^* e^{-i\mathbf{k} \cdot \mathbf{r}} \right] \tag{1}$$

with $\omega_{\mathbf{k}} = c|\mathbf{k}|$. The time dependence of the Fourier modes in the Heisenberg picture is $a_{\mathbf{k}, \alpha}(t) = a_{\mathbf{k}, \alpha}(0) e^{-i\omega_{\mathbf{k}} t}$.

Let's consider the intensity $\mathbf{E}^2$ of the electric field. Recalling that $\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$, we have

$$\mathbf{E}^2(0, t) = -\frac{2\pi}{V} \sum_{\mathbf{k}_1, \mathbf{k}_2} \hbar \sqrt{\omega_{\mathbf{k}_1} \omega_{\mathbf{k}_2}} \left(a_{\mathbf{k}_1}(t) - a_{\mathbf{k}_1}^\dagger(t) \right) \left(a_{\mathbf{k}_2}(t) - a_{\mathbf{k}_2}^\dagger(t) \right)$$

(we are going to drop the polarization label from now on as it creates unnecessary complications!)

If we consider a state with a fixed number of quanta in each mode

$$|\{N_{\mathbf{k}_i}\}\rangle = \prod_i \otimes |N_{\mathbf{k}_i}\rangle_{\mathbf{k}_i} \tag{2}$$

then the expectation value of the intensity is

$$\langle \{N_{\mathbf{k}_i}\} | \mathbf{E}^2(0, t) | \{N_{\mathbf{k}_i}\} \rangle = \frac{2\pi}{V} \sum_{\mathbf{k}} \hbar \omega_{\mathbf{k}} (2N_{\mathbf{k}} + 1) \tag{3}$$

If we had just $2N_{\mathbf{k}}$ in the sum, this would be proportional to the energy of each mode times the number of quanta in each mode, summed over all modes. In fact, we have a 'zero point' contribution even when $N_{\mathbf{k}} = 0$. It has exactly the same origin as the $\frac{1}{2}\hbar\omega$ ground state energy of the simple harmonic oscillator. In this case, the infinite number of modes means that the sum in Eq. (3) is not convergent.

Fortunately, this is not what a real experiment measures. To detect light when the number of photons is small we might use a device like a photomultiplier, which harnesses the photoelectric effect to convert a photon into a current pulse, destroying it in the process. Thus it is clear that we should have $2N_{\mathbf{k}}$ in the sum. We account for this property of the detector by arranging the mode operators inside any quantitiy that we measure in *normal order*. That

is, we put all annihilation operators to the right and all creation operators to the left. That way, we are guaranteed to get no signal in the vacuum. A normal ordered operator is usually sandwiched between two colons, so for the case of the electric field intensity we have

$$: \mathbf{E}^2(0,t) := \frac{2\pi}{V} \sum_{\mathbf{k}_1,\mathbf{k}_2} \hbar\sqrt{\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}} \left(a^\dagger_{\mathbf{k}_1}(t)a_{\mathbf{k}_2}(t) + a^\dagger_{\mathbf{k}_1}(t)a_{\mathbf{k}_2}(t) - a^\dagger_{\mathbf{k}_1}(t)a^\dagger_{\mathbf{k}_2}(t) - a_{\mathbf{k}_1}(t)a_{\mathbf{k}_2}(t) \right)$$

and

$$\langle\{N_{\mathbf{k}_i}\}| : \mathbf{E}^2(0,t) : |\{N_{\mathbf{k}_i}\}\rangle = \frac{4\pi}{V} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} N_{\mathbf{k}},$$

which is convergent if the high $\mathbf{k}$ states are empty.

This is still not quite the measurement we have in mind. We know from our studies of the photoelectric effect that for exciting atoms from their ground states only the positive frequency terms (i.e. the $a_{\mathbf{k}}(t)$ terms) in the vector potential contribute, so a photomultiplier really measures the square modulus of the *complex* electric field

$$\mathcal{E}(\mathbf{r},t) = \frac{i}{\sqrt{V}} \sum_{\mathbf{k}} \sqrt{2\pi\hbar\omega_k} a_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{r}}$$

Now let's consider the *fluctuations* in the intensity of the electric field. To quantify these, we will look at the correlation function of the intensity at two different times. Remembering to normal order, we get

$$\langle\{N_{\mathbf{k}_i}\}| : |\mathcal{E}(0,t_1)|^2|\mathcal{E}(0,t_2)|^2 : |\{N_{\mathbf{k}_i}\}\rangle$$
$$= \left(\frac{2\pi}{V}\right)^2 \sum_{\mathbf{k}_1,\mathbf{k}_2} \hbar^2\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2} N_{\mathbf{k}_1} N_{\mathbf{k}_2} \left[1 + \exp(i[\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2}][t_1 - t_2])\right] \qquad (4)$$

Normally we are interested in a statistical ensemble of states, so that there is an additional average the values of $\{N_{\mathbf{k}}\}$, with some probability distribution $P(\{N_{\mathbf{k}}\})$ (the Planck distribution, say, for a finite temperature system). We often denote this quantum *and* ensemble average with a double angular bracket

$$\langle\langle: |\mathcal{E}(0,t_1)|^2|\mathcal{E}(0,t_2)|^2 :\rangle\rangle = \sum_{\{N_{\mathbf{k}}\}} P(\{N_{\mathbf{k}}\})\langle\{N_{\mathbf{k}_i}\}| : |\mathcal{E}(0,t_1)|^2|\mathcal{E}(0,t_2)|^2 : |\{N_{\mathbf{k}_i}\}\rangle.$$

It's convenient to define a normalized correlation function

$$C(\tau) = \frac{\langle\langle: |\mathcal{E}(0,0)|^2|\mathcal{E}(0,\tau)|^2 :\rangle\rangle}{\langle\langle: |\mathcal{E}(0,0)|^2 :\rangle\rangle^2} = \frac{|\rho(0)|^2 + |\rho(\tau)|^2}{|\rho(0)|^2}$$

where $\rho(\tau)$ is just the Fourier transform of the spectral density

$$\rho(\tau) = \frac{2\pi}{V} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \langle N_{\mathbf{k}}\rangle e^{-i\omega_{\mathbf{k}}\tau}$$

For a thermal source $\langle N_{\mathbf{k}}\rangle = \left(e^{\beta\hbar\omega_{\mathbf{k}}} - 1\right)^{-1}$, with $\beta = 1/k_B T$

$$\begin{aligned}
\rho(\tau) &= 2\pi \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\hbar\omega_{\mathbf{k}}}{e^{\beta\hbar\omega_{\mathbf{k}}} - 1} e^{-i\omega_{\mathbf{k}}\tau} \\
&= \frac{1}{\pi}\frac{\hbar}{c^3} \int_0^\infty \omega^3 \frac{e^{-i\omega\tau}}{e^{\beta\hbar\omega_{\mathbf{k}}} - 1} d\omega = \frac{6}{\pi}\frac{(k_B T)^4}{c^3\hbar^3} \sum_{k=0}^\infty \frac{1}{(k+1+i\tau/\tau_{\text{coh}})^4} \\
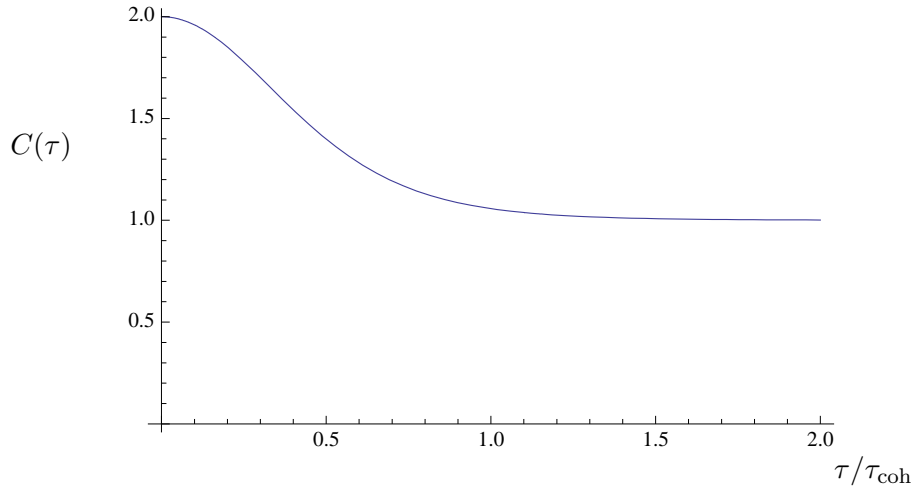&= \frac{6}{\pi}\frac{(k_B T)^4}{c^3\hbar^3} \zeta(4, 1+i\tau/\tau_{\text{coh}})
\end{aligned}$$

Figure 1: The function $C(\tau)$

(($\zeta(s, a)$ is the generalized zeta function) The important thing here is the timescale $\tau_{\mathrm{coh}} = \hbar/k_B T$ that sets the overall scale of $\rho(\tau)$: it is called the (thermal) *coherence time*. The corresponding function $C(\tau)$ is shown in Fig. 1. The important properties $C(0) = 2$ and $C(\tau \to \infty) = 1$, are independent of the precise form of $\rho(\tau)$ and rely on our choosing states of the form Eq. (2), rather than exactly how these states are occupied (although this choice does determine τ_{coh}: in general it is the inverse of the bandwidth of occupied states).

The above expressions and in particular the origin of $C(0) = 2$ can be understood very simply within a classical picture in which the amplitudes $a_{\mathbf{k}}$ are complex Gaussian random variables, satisfying

$$\langle a_{\mathbf{k}_1}^* a_{\mathbf{k}_2} \rangle = \begin{cases} N_{\mathbf{k}_1} & \mathbf{k}_1 = \mathbf{k}_2 \\ 0 & \mathbf{k}_1 \neq \mathbf{k}_2 \end{cases}$$

Then $\mathcal{E}$ is a Gaussian complex variable and the above statistics imply $\langle |\mathcal{E}|^2 |\mathcal{E}|^2 \rangle = 2\langle |\mathcal{E}|^2 \rangle^2$ and so $C(0) = 2$. In other words, within classical wave theory $C(0) = 2$ is a simple property of incoherent light. By contrast for coherent light (where the amplitudes are not random) $\langle |\mathcal{E}|^2 |\mathcal{E}|^2 \rangle = \langle |\mathcal{E}|^2 \rangle^2$.

The quantum mechanical meaning of $C(0) = 2$ is very odd, however[1]. The arrival or photons at the detector is not uncorrelated, but rather there is a tendency for photons to arrive together within a time of order of the coherence time. That is, photons from an incoherent source (like the hot filament of a light bulb) tend to *bunch together*!

2   Intensity interferometry

[This discussion follows Baym, *Lectures on Quantum Mechanics*]
Things get even weirder when we consider two sources and two detectors (Fig. (2)). Let's first consider the classical picture, in which the amplitudes of the waves at detectors 1 and 2 are superpositions of waves arriving from sources A and B. The amplitudes at the detectors

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[1]There is a persistent historical tendency to say that light is behaving in a quantum way if it behaves like particles, whereas for matter things are the other way around!
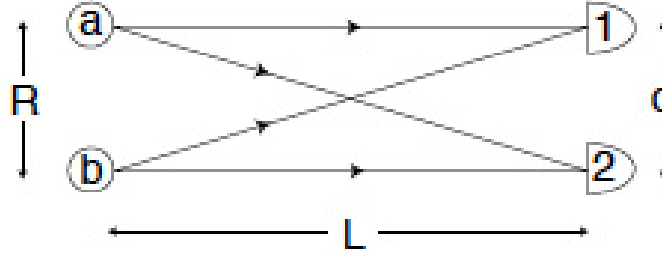
3

Figure 2: Two sources and two detectors. Figure from Ref. [1]

are then[2]

$$A_i = \alpha e^{ikr_{ai}+i\phi_a} + \beta e^{ikr_{bi}+i\phi_b}, \qquad i = 1, 2.$$

For incoherent light the amplitudes α β and the phases ϕ_a ϕ_b are random. The average intensity at the two detectors is then

$$\langle I_i \rangle = \langle |A_i|^2 \rangle = \langle |\alpha|^2 \rangle + \langle |\beta|^2 \rangle$$

with the cross terms have disappearing when we take the average over the phases. We get something surprising when we consider the correlation of the intensities at the two detectors

$$\begin{aligned} \langle I_1 I_2 \rangle &= \langle |\alpha|^2 \rangle + \langle |\beta|^2 \rangle + 2\langle |\alpha|^2 \rangle \langle |\beta|^2 \rangle \cos\left[k\left(r_{a1} - r_{a2} - r_{b1} + r_{b2}\right)\right] \\ &= \langle I_1 \rangle \langle I_2 \rangle + 2\langle |\alpha|^2 \rangle \langle |\beta|^2 \rangle \cos\left[k\left(r_{a1} - r_{a2} - r_{b1} + r_{b2}\right)\right] \end{aligned}$$

Defining a normalized correlation function in the same way as before

$$\begin{aligned} C_{12} = \frac{\langle I_1 I_2 \rangle}{\langle I_1 \rangle \langle I_2 \rangle} &= 1 + \frac{2\langle |\alpha|^2 \rangle \langle |\beta|^2 \rangle}{(\langle |\alpha|^2 \rangle + \langle |\beta|^2 \rangle)^2} \cos\left[k\left(r_{a1} - r_{a2} - r_{b1} + r_{b2}\right)\right] \\ &= 1 + \frac{2\langle |\alpha|^2 \rangle \langle |\beta|^2 \rangle}{(\langle |\alpha|^2 \rangle + \langle |\beta|^2 \rangle)^2} \cos(2\pi d\theta/\lambda) \end{aligned}$$

Where we have made a small angle approximation appropriate to the case when $L \gg R, d$, and where θ is the angular separation of the sources as viewed from the detectors. Thus we see that the intensity correlations display interference fringes as a function of detector separation – even though the sources are not coherent with each other!

   This is known as the *Hanbury Brown and Twiss* effect, after its discoverers, the radio astronomers Robert Hanbury Brown and Richard Twiss. It is the basis of the technique of *intensity interferometry* that they pioneered and applied to the measurement of the angular size of stars, starting in 1950. Measuring a small angle requires a large distance between detectors. Conventional interferometry involves comparing *amplitudes* at two detectors, which becomes very hard for large separations. By contrast, intensity interferometry does not even require a physical connection between detectors – the correlations may be extracted later. Although conventional interferometry has now replaced intensity interferometry in radio astronomy, the latter has become a valuable tool in collision experiments in nuclear and particle physics. You can read more about these applications (as well as the history of the discovery of the Hanbury Brown and Twiss effect) in Ref. [1]

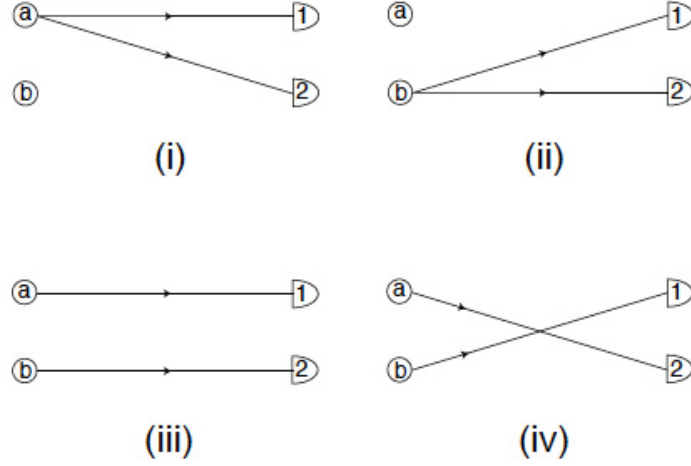[2]We ignore the small effect of one source being closer to one of the detectors.

Figure 3: The four processes contributing to the two-photon probability. Figure from Ref. [1].

3 Quantum mechanics of HBT

After using intensity interferometry to measure the angular size of the sun using two radio telescopes, Hanbury Brown and Twiss went on to demonstrate intensity correlations from an incoherent light source [2]. Their experiment correlated photon counts from two photomultipliers, so was firmly in the quantum regime. While the classical wave explanation of the previous section seemed reasonable for radio waves, their observation of correlations between photon counts was more controversial, with some claiming that their existence would require a modification of quantum theory.

To think about the problem quantum mechanically, notice that the normal ordered correlation function of the electric fields at two different points at equal times is really measuring

$$\langle\langle: |\mathcal{E}(\mathbf{r}_1, 0)|^2 |\mathcal{E}(\mathbf{r}_2, 0)|^2 :\rangle\rangle \sim \langle\langle a^\dagger(\mathbf{r}_1, 0) a^\dagger(\mathbf{r}_2, 0) a(\mathbf{r}_2, 0) a(\mathbf{r}_1, 0)\rangle\rangle.$$

This is the squared amplitude of a state with a photon removed at each of detector 1 and detector 2, averaged over all initial states. Thus we should consider the processes by which one photon can end up at each of the two detectors. Evidently there are four possibilities, illustrated in Fig. 3. Processes (i) and (ii), where both photons originate in one of the sources, describe different quantum states. Thus each contributes its amplitude squared to overall probability, with no possibility for interference. Interference can arise from the second two processes, but only if we add the amplitudes together before squaring, to give a cross term. How can we justify this? The important point is that photons are *indistinguishable* particles, so that processes (iii) and (iv) are different components of the same quantum state, giving an amplitude for the process in question of

$$\psi(1, 2) = \frac{1}{\sqrt{2}} \left[\phi_a(1) \phi_b(2) + \phi_a(2) \phi_b(1) \right] \tag{5}$$

Thus if we want our particle description to produce the same physics as our wave description, we see that it is necessary to introduce the idea of indistinguishability (and also to

specify that photons are bosons, corresponding to the plus sign in Eq. (5)). This is a two-way street however, and it leads to a powerful idea. Any system of identical bosons can be described in the same language that we used for photons, that is, as a system of oscillators. The development of this realization leads to the technique of *second quantization*, to which we will turn next.

References

[1] G. Baym, Acta Physica Polonica B **29**, 1839 (1998).

[2] R. Brown and R. Twiss, Nature **177**, 27 (1956).