

Graph Sparsification by Effective Resistances*

Daniel A. Spielman

Program in Applied Mathematics and
Department of Computer Science
Yale University

Nikhil Srivastava

Department of Computer Science
Yale University

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Abstract

We present a nearly-linear time algorithm that produces high-quality spectral sparsifiers of weighted graphs. Given as input a weighted graph $G = (V, E, w)$ and a parameter $\epsilon > 0$, we produce a weighted subgraph $H = (V, \tilde{E}, \tilde{w})$ of G such that $|\tilde{E}| = O(n \log n / \epsilon^2)$ and for all vectors $x \in \mathbb{R}^V$

$$(1 - \epsilon) \sum_{uv \in E} (x(u) - x(v))^2 w_{uv} \leq \sum_{uv \in \tilde{E}} (x(u) - x(v))^2 \tilde{w}_{uv} \leq (1 + \epsilon) \sum_{uv \in E} (x(u) - x(v))^2 w_{uv}. \quad (1)$$

This improves upon the spectral sparsifiers constructed by Spielman and Teng, which had $O(n \log^c n)$ edges for some large constant c , and upon the cut sparsifiers of Benczúr and Karger, which only satisfied (1) for $x \in \{0, 1\}^V$.

A key ingredient in our algorithm is a subroutine of independent interest: a nearly-linear time algorithm that builds a data structure from which we can query the approximate effective resistance between any two vertices in a graph in $O(\log n)$ time.

1 Introduction

The goal of sparsification is to approximate a given graph G by a sparse graph H on the same set of vertices. If H is close to G in some appropriate metric, then H can be used as a proxy for G in computations without introducing too much error. At the same time, since H has very few edges, computation with and storage of H should be cheaper.

We study the notion of spectral sparsification introduced by Spielman and Teng [25]. Spectral sparsification was inspired by the notion of cut sparsification introduced by Benczúr and Karger [5] to accelerate cut algorithms whose running time depends on the number of edges. They gave a nearly-linear time procedure which takes a graph G on n vertices with m edges and a parameter $\epsilon > 0$, and outputs a weighted subgraph H with $O(n \log n / \epsilon^2)$ edges such that the weight of every cut in H is within a factor of $(1 \pm \epsilon)$ of its weight in G . This was used to turn Goldberg and Tarjan's $\tilde{O}(mn)$ max-flow algorithm [16] into an $\tilde{O}(n^2)$ algorithm for approximate st -mincut, and

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appeared more recently as the first step of an $\tilde{O}(n^{3/2} + m)$ -time $O(\log^2 n)$ approximation algorithm for sparsest cut [19].

The cut-preserving guarantee of [5] is equivalent to satisfying (1) for all $x \in \{0, 1\}^n$, which are the characteristic vectors of cuts. Spielman and Teng [23, 25] devised stronger sparsifiers which extend (1) to all $x \in \mathbb{R}^n$, but have $O(n \log^c n)$ edges for some large constant c . They used these sparsifiers to construct preconditioners for symmetric diagonally-dominant matrices, which led to the first nearly-linear time solvers for such systems of equations.

In this work, we construct sparsifiers that achieve the same guarantee as Spielman and Teng's but with $O(n \log n / \epsilon^2)$ edges, thus improving on both [5] and [23]. Our sparsifiers are subgraphs of the original graph and can be computed in $\tilde{O}(m)$ time by random sampling, where the sampling probabilities are given by the effective resistances of the edges. While this is conceptually much simpler than the recursive partitioning approach of [23], we need to solve $O(\log n)$ linear systems to compute the effective resistances quickly, and we do this using Spielman and Teng's linear equation solver.

1.1 Our Results

Our main idea is to include each edge of G in the sparsifier H with probability proportional to its effective resistance. The effective resistance of an edge is known to be equal to the probability that the edge appears in a random spanning tree of G (see, e.g., [9] or [6]), and was proven in [7] to be proportional to the commute time between the endpoints of the edge. We show how to approximate the effective resistances of edges in G quickly and prove that sampling according to these approximate values yields a good sparsifier.

To define effective resistance, identify $G = (V, E, w)$ with an electrical network on n nodes in which each edge e corresponds to a link of conductance w_e (i.e., a resistor of resistance $1/w_e$). Then the effective resistance R_e across an edge e is the potential difference induced across it when a unit current is injected at one end of e and extracted at the other end of e . Our algorithm can now be stated as follows.

$H = \text{Sparsify}(G, q)$

Choose a random edge e of G with probability p_e proportional to $w_e R_e$, and add e to H with weight w_e / qp_e . Take q samples independently with replacement, summing weights if an edge is chosen more than once.

Recall that the *Laplacian* of a weighted graph is given by $L = D - A$ where A is the weighted adjacency matrix $(a_{ij}) = w_{ij}$ and D is the diagonal matrix $(d_{ii}) = \sum_{j \neq i} w_{ij}$ of weighted degrees. Notice that the quadratic form associated with L is just $x^T L x = \sum_{uv \in E} (x(u) - x(v))^2 w_{uv}$. Let L be the Laplacian of G and let \tilde{L} be the Laplacian of H . Our main theorem is that if q is sufficiently large, then the quadratic forms of L and \tilde{L} are close.

Theorem 1. *Suppose G and $H = \text{Sparsify}(G, q)$ have Laplacians L and \tilde{L} respectively, and $1/\sqrt{n} < \epsilon \leq 1$. If $q = 9C^2 n \log n / \epsilon^2$, where C is the constant in Lemma 5 and if n is sufficiently large, then with probability at least $1/2$*

$$\forall x \in \mathbb{R}^n \quad (1 - \epsilon)x^T L x \leq x^T \tilde{L} x \leq (1 + \epsilon)x^T L x. \quad (2)$$

Sparsifiers that satisfy this condition preserve many properties of the graph. The Courant-Fischer Theorem tells us that

$$\lambda_i = \max_{S: \dim(S)=k} \min_{x \in S} \frac{x^T L x}{x^T x}.$$

Thus, if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of L and $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$ are the eigenvalues of \tilde{L} , then we have

$$(1 - \epsilon)\lambda_i \leq \tilde{\lambda}_i \leq (1 + \epsilon)\lambda_i,$$

and the eigenspaces spanned by corresponding eigenvalues are related. As the eigenvalues of the normalized Laplacian are given by

$$\lambda_i = \max_{S: \dim(S)=k} \min_{x \in S} \frac{x^T D^{-1/2} L D^{-1/2} x}{x^T x},$$

and are the same as the eigenvalues of the walk matrix $D^{-1}L$, we obtain the same relationship between the eigenvalues of the walk matrix of the original graph and its sparsifier. Many properties of graphs and random walks are known to be revealed by their spectra (see for example [6, 8, 15]). The existence of sparse subgraphs which retain these properties is interesting its own right; indeed, expander graphs can be viewed as constant degree sparsifiers for the complete graph.

We remark that the condition (2) also implies

$$\forall x \in \mathbb{R}^n \quad \frac{1}{1 + \epsilon} x^T L^+ x \leq x^T \tilde{L}^+ x \leq \frac{1}{1 - \epsilon} x^T L^+ x,$$

where L^+ is the pseudoinverse of L . Thus sparsifiers also approximately preserve the effective resistances between vertices, since for vertices u and v , the effective resistance between them is given by the formula $(\chi_u - \chi_v)^T L^+ (\chi_u - \chi_v)$, where χ_u is the elementary unit vector with a coordinate 1 in position u .

We prove Theorem 1 in Section 3. At the end of Section 3, we prove that the spectral guarantee (2) of Theorem 1 is not harmed too much if use approximate effective resistances for sampling instead of exact ones(Corollary 6).

In Section 4, we show how to compute approximate effective resistances in nearly-linear time, which is essentially optimal. The tools we use to do this are Spielman and Teng's nearly-linear time solver [23, 24] and the Johnson-Lindenstrauss Lemma [18, 1]. Specifically, we prove the following theorem, in which R_{uv} denotes the effective resistance between vertices u and v .

Theorem 2. *There is an $\tilde{O}(m(\log r)/\epsilon^2)$ time algorithm which on input $\epsilon > 0$ and $G = (V, E, w)$ with $r = w_{\max}/w_{\min}$ computes a $(24 \log n/\epsilon^2) \times n$ matrix \tilde{Z} such that with probability at least $1 - 1/n$*

$$(1 - \epsilon)R_{uv} \leq \|\tilde{Z}(\chi_u - \chi_v)\|^2 \leq (1 + \epsilon)R_{uv}$$

for every pair of vertices $u, v \in V$.

Since $\tilde{Z}(\chi_u - \chi_v)$ is simply the difference of the corresponding two columns of \tilde{Z} , we can query the approximate effective resistance between any pair of vertices (u, v) in time $O(\log n/\epsilon^2)$, and for all the edges in time $O(m \log n/\epsilon^2)$. By Corollary 6, this yields an $\tilde{O}(m(\log r)/\epsilon^2)$ time for sparsifying graphs, as advertised.

In Section 5, we show that H can be made close to G in some additional ways which make it more useful for preconditioning systems of linear equations.

1.2 Related Work

Batson, Spielman, and Srivastava [4] have given a deterministic algorithm that constructs sparsifiers of size $O(n/\epsilon^2)$ in $O(mn^3/\epsilon^2)$ time. While this is too slow to be useful in applications, it is optimal in terms of the tradeoff between sparsity and quality of approximation and can be viewed as generalizing expander graphs. Their construction parallels ours in that it reduces the task of spectral sparsification to approximating the matrix Π defined in Section 3; however, their method for selecting edges is iterative and more delicate than the random sampling described in this paper.

In addition to the graph sparsifiers of [5, 4, 23], there is a large body of work on sparse [3, 2] and low-rank [14, 2, 22, 10, 11] approximations for general matrices. The algorithms in this literature provide guarantees of the form $\|A - \tilde{A}\|_2 \leq \epsilon$, where A is the original matrix and \tilde{A} is obtained by entrywise or columnwise sampling of A . This is analogous to satisfying (1) only for vectors x in the span of the dominant eigenvectors of A ; thus, if we were to use these sparsifiers on graphs, they would only preserve the large cuts. Interestingly, our proof uses some of the same machinery as the low-rank approximation result of Rudelson and Vershynin [22] — the sampling of edges in our algorithm corresponds to picking $q = O(n \log n)$ columns at random from a certain rank $(n-1)$ matrix of dimension $m \times m$ (this is the matrix Π introduced in Section 3).

The use of effective resistance as a distance in graphs has recently gained attention as it is often more useful than the ordinary geodesic distance in a graph. For example, in small-world graphs, all vertices will be close to one another, but those with a smaller effective resistance distance are connected by more short paths. See, for instance [13, 12], which use effective resistance/commute time as a distance measure in social network graphs.

2 Preliminaries

2.1 The Incidence Matrix and the Laplacian

Let $G = (V, E, w)$ be a connected weighted undirected graph with n vertices and m edges and edge weights $w_e > 0$. If we orient the edges of G arbitrarily, we can write its Laplacian as $L = B^T W B$, where $B_{m \times n}$ is the *signed edge-vertex incidence matrix*, given by

$$B(e, v) = \begin{cases} 1 & \text{if } v \text{ is } e\text{'s head} \\ -1 & \text{if } v \text{ is } e\text{'s tail} \\ 0 & \text{otherwise} \end{cases}$$

and $W_{m \times m}$ is the diagonal matrix with $W(e, e) = w_e$. Denote the row vectors of B by $\{b_e\}_{e \in E}$ and the span of its columns by $\mathbb{B} = \text{im}(B) \subseteq \mathbb{R}^m$ (also called the *cut space* of G [15]). Note that $b_{(u,v)}^T = (\chi_v - \chi_u)$.

It is immediate that L is positive semidefinite since

$$x^T L x = x^T B^T W B x = \|W^{1/2} B x\|_2^2 \geq 0 \quad \text{for every } x \in \mathbb{R}^n.$$

We also have $\ker(L) = \ker(W^{1/2}B) = \text{span}(\mathbf{1})$, since

$$\begin{aligned} x^T Lx = 0 &\iff \|W^{1/2}Bx\|_2^2 = 0 \\ &\iff \sum_{uv \in E} w_{uv}(x(u) - x(v))^2 = 0 \\ &\iff x(u) - x(v) = 0 \quad \text{for all edges } (u, v) \\ &\iff x \text{ is constant, since } G \text{ is connected.} \end{aligned}$$

2.2 The Pseudoinverse

Since L is symmetric we can diagonalize it and write

$$L = \sum_{i=1}^{n-1} \lambda_i u_i u_i^T$$

where $\lambda_1, \dots, \lambda_{n-1}$ are the nonzero eigenvalues of L and u_1, \dots, u_{n-1} are a corresponding set of orthonormal eigenvectors. The *Moore-Penrose Pseudoinverse* of L is then defined as

$$L^+ = \sum_{i=1}^{n-1} \frac{1}{\lambda_i} u_i u_i^T.$$

Notice that $\ker(L) = \ker(L^+)$ and that

$$LL^+ = L^+L = \sum_{i=1}^{n-1} u_i u_i^T,$$

which is simply the projection onto the span of the nonzero eigenvectors of L (which are also the eigenvectors of L^+). Thus, $LL^+ = L^+L$ is the identity on $\text{im}(L) = \ker(L)^\perp = \text{span}(\mathbf{1})^\perp$. We will rely on this fact heavily in the proof of Theorem 1.

2.3 Electrical Flows

Begin by arbitrarily orienting the edges of G as in Section 2.1. We will use the same notation as [17] to describe electrical flows on graphs: for a vector $\mathbf{i}_{\text{ext}}(u)$ of currents injected at the vertices, let $\mathbf{i}(e)$ be the currents induced in the edges (in the direction of orientation) and $\mathbf{v}(u)$ the potentials induced at the vertices. By Kirchoff's current law, the sum of the currents entering a vertex is equal to the amount injected at the vertex:

$$B^T \mathbf{i} = \mathbf{i}_{\text{ext}}.$$

By Ohm's law, the current flow in an edge is equal to the potential difference across its ends times its conductance:

$$\mathbf{i} = WB\mathbf{v}.$$

Combining these two facts, we obtain

$$\mathbf{i}_{\text{ext}} = B^T (WB\mathbf{v}) = L\mathbf{v}.$$

If $\mathbf{i}_{\text{ext}} \perp \text{span}(\mathbf{1}) = \ker(L)$ — i.e., if the total amount of current injected is equal to the total amount extracted — then we can write

$$\mathbf{v} = L^+ \mathbf{i}_{\text{ext}}$$

by the definition of L^+ in Section 2.2.

Recall that the *effective resistance* between two vertices u and v is defined as the potential difference induced between them when a unit current is injected at one and extracted at the other. We will derive an algebraic expression for the effective resistance in terms of L^+ . To inject and extract a unit current across the endpoints of an edge $e = (u, v)$, we set $\mathbf{i}_{\text{ext}} = b_e^T = (\chi_v - \chi_u)$, which is clearly orthogonal to $\mathbf{1}$. The potentials induced by \mathbf{i}_{ext} at the vertices are given by $\mathbf{v} = L^+ b_e^T$; to measure the potential difference across $e = (u, v)$, we simply multiply by b_e on the left:

$$\mathbf{v}(v) - \mathbf{v}(u) = (\chi_v - \chi_u)^T \mathbf{v} = b_e L^+ b_e^T.$$

It follows that the effective resistance across e is given by $b_e L^+ b_e^T$ and that the matrix $BL^+ B^T$ has as its diagonal entries $BL^+ B^T(e, e) = R_e$.

3 The Main Result

We will prove Theorem 1. Consider the matrix $\Pi = W^{1/2} BL^+ B^T W^{1/2}$. Since we know $BL^+ B^T(e, e) = R_e$, the diagonal entries of Π are $\Pi(e, e) = \sqrt{W(e, e)} R_e \sqrt{W(e, e)} = w_e R_e$. Π has some notable properties.

Lemma 3 (Projection Matrix). (i) Π is a projection matrix. (ii) $\text{im}(\Pi) = \text{im}(W^{1/2} B) = W^{1/2} \mathbb{B}$. (iii) The eigenvalues of Π are 1 with multiplicity $n - 1$ and 0 with multiplicity $m - n + 1$. (iv) $\Pi(e, e) = \|\Pi(\cdot, e)\|^2$.

Proof. To see (i), observe that

$$\begin{aligned} \Pi^2 &= (W^{1/2} BL^+ B^T W^{1/2})(W^{1/2} BL^+ B^T W^{1/2}) \\ &= W^{1/2} BL^+ (B^T WB) L^+ B^T W^{1/2} \\ &= W^{1/2} BL^+ LL^+ B^T W^{1/2} \quad \text{since } L = B^T WB \\ &= W^{1/2} BL^+ B^T W^{1/2} \\ &\qquad \text{since } L^+ L \text{ is the identity on } \text{im}(L^+) \\ &= \Pi. \end{aligned}$$

For (ii), we have

$$\text{im}(\Pi) = \text{im}(W^{1/2} BL^+ B^T W^{1/2}) \subseteq \text{im}(W^{1/2} B).$$

To see the other inclusion, assume $y \in \text{im}(W^{1/2} B)$. Then we can choose $x \perp \ker(W^{1/2} B) = \ker(L)$ such that $W^{1/2} Bx = y$. But now

$$\begin{aligned} \Pi y &= W^{1/2} BL^+ B^T W^{1/2} W^{1/2} Bx \\ &= W^{1/2} BL^+ Lx \quad \text{since } B^T WB = L \\ &= W^{1/2} Bx \quad \text{since } L^+ Lx = x \text{ for } x \perp \ker(L) \\ &= y. \end{aligned}$$

Thus $y \in \text{im}(\Pi)$, as desired.

For (iii), recall from Section 2.1 that $\dim(\ker(W^{1/2}B)) = 1$. Consequently, $\dim(\text{im}(\Pi)) = \dim(\text{im}(W^{1/2}B)) = n - 1$. But since $\Pi^2 = \Pi$, the eigenvalues of Π are all 0 or 1, and as Π projects onto a space of dimension $n - 1$, it must have exactly $n - 1$ nonzero eigenvalues.

(iv) follows from $\Pi^2(e, e) = \Pi(\cdot, e)^T \Pi(\cdot, e)$, since Π is symmetric. \square

To show that $H = (V, \tilde{E}, \tilde{w})$ is a good sparsifier for G , we need to show that the quadratic forms $x^T Lx$ and $x^T \tilde{L}x$ are close. We start by reducing the problem of preserving $x^T Lx$ to that of preserving $y^T \Pi y$. This will be much nicer since the eigenvalues of Π are all 0 or 1, so that any matrix $\tilde{\Pi}$ which approximates Π in the spectral norm (i.e., makes $\|\tilde{\Pi} - \Pi\|_2$ small) also preserves its quadratic form.

We may describe the outcome of $H = \mathbf{Sparsify}(G, q)$ by the following random matrix:

$$S(e, e) = \frac{\tilde{w}_e}{w_e} = \frac{(\# \text{ of times } e \text{ is sampled})}{qp_e}. \quad (3)$$

$S_{m \times m}$ is a nonnegative diagonal matrix and the random entry $S(e, e)$ specifies the ‘amount’ of edge e included in H by **Sparsify**. For example $S(e, e) = 1/qp_e$ if e is sampled once, $2/qp_e$ if it is sampled twice, and zero if it is not sampled at all. The weight of e in H is now given by $\tilde{w}_e = S(e, e)w_e$, and we can write the Laplacian of H as:

$$\tilde{L} = B^T \tilde{W} B = B^T W^{1/2} S W^{1/2} B$$

since $\tilde{W} = WS = W^{1/2} SW^{1/2}$. The scaling of weights by $1/qp_e$ in **Sparsify** implies that $\mathbb{E}\tilde{w}_e = w_e$ (since q independent samples are taken, each with probability p_e), and thus $\mathbb{E}S = I$ and $\mathbb{E}\tilde{L} = L$.

We can now prove the following lemma, which says that if S does not distort $y^T \Pi y$ too much then $x^T Lx$ and $x^T \tilde{L}x$ are close.

Lemma 4. *Suppose S is a nonnegative diagonal matrix such that*

$$\|\Pi S \Pi - \Pi \Pi\|_2 \leq \epsilon.$$

Then

$$\forall x \in \mathbb{R}^n \quad (1 - \epsilon)x^T Lx \leq x^T \tilde{L}x \leq (1 + \epsilon)x^T Lx,$$

where $L = B^T WB$ and $\tilde{L} = B^T W^{1/2} S W^{1/2} B$.

Proof. The assumption is equivalent to

$$\sup_{y \in \mathbb{R}^m, y \neq 0} \frac{|y^T \Pi(S - I)\Pi y|}{y^T y} \leq \epsilon$$

since $\|A\|_2 = \sup_{y \neq 0} |y^T A y| / y^T y$ for symmetric A . Restricting our attention to vectors in $\text{im}(W^{1/2}B)$, we have

$$\sup_{y \in \text{im}(W^{1/2}B), y \neq 0} \frac{|y^T \Pi(S - I)\Pi y|}{y^T y} \leq \epsilon.$$

But by Lemma 3.(ii), Π is the identity on $\text{im}(W^{1/2}B)$ so $\Pi y = y$ for all $y \in \text{im}(W^{1/2}B)$. Also, every such y can be written as $y = W^{1/2}Bx$ for $x \in \mathbb{R}^n$. Substituting this into the above expression we obtain:

$$\begin{aligned} & \sup_{y \in \text{im}(W^{1/2}B), y \neq 0} \frac{|y^T \Pi(S - I)\Pi y|}{y^T y} \\ &= \sup_{y \in \text{im}(W^{1/2}B), y \neq 0} \frac{|y^T (S - I)y|}{y^T y} \\ &= \sup_{x \in \mathbb{R}^n, W^{1/2}Bx \neq 0} \frac{|x^T B^T W^{1/2} S W^{1/2} B x - x^T B^T W B x|}{x^T B^T W B x} \\ &= \sup_{x \in \mathbb{R}^n, W^{1/2}Bx \neq 0} \frac{|x^T \tilde{L} x - x^T L x|}{x^T L x} \leq \epsilon. \end{aligned}$$

Rearranging yields the desired conclusion for all $x \notin \ker(W^{1/2}B)$. When $x \in \ker(W^{1/2}B)$ then $x^T L x = x^T \tilde{L} x = 0$ and the claim holds trivially. \square

To show that $\|\Pi S \Pi - \Pi \Pi\|_2$ is likely to be small we use the following concentration result, which is a sort of law of large numbers for symmetric rank 1 matrices. It was first proven by Rudelson in [21], but the version we state here appears in the more recent paper [22] by Rudelson and Vershynin.

Lemma 5 (Rudelson & Vershynin, [22] Thm. 3.1). *Let \mathbf{p} be a probability distribution over $\Omega \subseteq \mathbb{R}^d$ such that $\sup_{y \in \Omega} \|y\|_2 \leq M$ and $\|\mathbb{E}_{\mathbf{p}} yy^T\|_2 \leq 1$. Let $y_1 \dots y_q$ be independent samples drawn from \mathbf{p} . Then*

$$\mathbb{E} \left\| \frac{1}{q} \sum_{i=1}^q y_i y_i^T - \mathbb{E} yy^T \right\|_2 \leq \min \left(CM \sqrt{\frac{\log q}{q}}, 1 \right)$$

where C is an absolute constant.

We can now finish the proof of Theorem 1.

Proof of Theorem 1. **Sparsify** samples edges from G independently with replacement, with probabilities p_e proportional to $w_e R_e$. Since $\sum_e w_e R_e = \text{Tr}(\Pi) = n - 1$ by Lemma 3.(iii), the actual probability distribution over E is given by $p_e = \frac{w_e R_e}{n-1}$. Sampling q edges from G corresponds to sampling q columns from Π , so we can write

$$\begin{aligned} \Pi S \Pi &= \sum_e S(e, e) \Pi(\cdot, e) \Pi(\cdot, e)^T \\ &= \sum_e \frac{(\# \text{ of times } e \text{ is sampled})}{qp_e} \Pi(\cdot, e) \Pi(\cdot, e)^T \quad \text{by (3)} \\ &= \frac{1}{q} \sum_e (\# \text{ of times } e \text{ is sampled}) \frac{\Pi(\cdot, e)}{\sqrt{p_e}} \frac{\Pi(\cdot, e)^T}{\sqrt{p_e}} \\ &= \frac{1}{q} \sum_{i=1}^q y_i y_i^T \end{aligned}$$

for vectors y_1, \dots, y_q drawn independently with replacement from the distribution

$$y = \frac{1}{\sqrt{p_e}} \Pi(\cdot, e) \quad \text{with probability } p_e.$$

We can now apply Lemma 5. The expectation of yy^T is given by

$$\mathbb{E}yy^T = \sum_e p_e \frac{1}{p_e} \Pi(\cdot, e) \Pi(\cdot, e)^T = \Pi \Pi = \Pi,$$

so $\|\mathbb{E}yy^T\|_2 = \|\Pi\|_2 = 1$. We also have a bound on the norm of y :

$$\frac{1}{\sqrt{p_e}} \|\Pi(\cdot, e)\|_2 = \frac{1}{\sqrt{p_e}} \sqrt{\Pi(e, e)} = \sqrt{\frac{n-1}{R_e w_e}} \sqrt{R_e w_e} = \sqrt{n-1}.$$

Taking $q = 9C^2 n \log n / \epsilon^2$ gives:

$$\mathbb{E} \|\Pi S \Pi - \Pi \Pi\|_2 = \mathbb{E} \left\| \frac{1}{q} \sum_{i=1}^q y_i y_i^T - \mathbb{E}yy^T \right\|_2 \leq C \sqrt{\epsilon^2 \frac{\log(9C^2 n \log n / \epsilon^2)(n-1)}{9C^2 n \log n}} \leq \epsilon/2,$$

for n sufficiently large, as ϵ is assumed to be at least $1/\sqrt{n}$.

By Markov's inequality, we have

$$\|\Pi S \Pi - \Pi\|_2 \leq \epsilon$$

with probability at least 1/2. By Lemma 4, this completes the proof of the theorem. \square

We now show that using approximate resistances for sampling does not damage the sparsifier very much.

Corollary 6. Suppose Z_e are numbers satisfying $Z_e \geq R_e/\alpha$ and $\sum_e w_e Z_e \leq \alpha \sum_e w_e R_e$ for some $\alpha \geq 1$. If we sample as in **Sparsify** but take each edge with probability $p'_e = \frac{w_e Z_e}{\sum_e w_e Z_e}$ instead of $p_e = \frac{w_e R_e}{\sum_e w_e R_e}$, then H satisfies:

$$(1 - \epsilon\alpha)x^T \tilde{L}x \leq x^T Lx \leq (1 + \epsilon\alpha)x^T \tilde{L}x \quad \forall x \in \mathbb{R}^n,$$

with probability at least 1/2.

Proof. We note that

$$p'_e = \frac{w_e S_e}{\sum_e w_e S_e} \geq \frac{w_e (R_e/\alpha)}{\alpha \sum_e w_e R_e} = \frac{p_e}{\alpha^2}$$

and proceed as in the proof of Theorem 1. The norm of the random vector y is now bounded by:

$$\frac{1}{\sqrt{p'_e}} \|\Pi(e, \cdot)\|_2 \leq \frac{\alpha}{\sqrt{p_e}} \sqrt{\Pi(e, e)} = \alpha \sqrt{n-1}$$

which introduces a factor of α into the final bound on the expectation, but changes nothing else. \square

4 Computing Approximate Resistances Quickly

It is not clear how to compute all the effective resistances $\{R_e\}$ exactly and efficiently. In this section, we show that one can compute constant factor approximations to all the R_e in time $\tilde{O}(m \log r)$. In fact, we do something stronger: we build a $O(\log n) \times n$ matrix \tilde{Z} from which the effective resistance between any two vertices (including vertices not connected by an edge) can be computed in $O(\log n)$ time.

Proof of Theorem 2. If u and v are vertices in G , then the effective resistance between u and v can be written as:

$$\begin{aligned} R_{uv} &= (\chi_u - \chi_v)^T L^+ (\chi_u - \chi_v) \\ &= (\chi_u - \chi_v)^T L^+ L L^+ (\chi_u - \chi_v) \\ &= ((\chi_u - \chi_v)^T L^+ B^T W^{1/2}) (W^{1/2} B L^+ (\chi_u - \chi_v)) \\ &= \|W^{1/2} B L^+ (\chi_u - \chi_v)\|_2^2. \end{aligned}$$

Thus effective resistances are just pairwise distances between vectors in $\{W^{1/2} B L^+ \chi_v\}_{v \in V}$. By the Johnson-Lindenstrauss Lemma, these distances are preserved if we project the vectors onto a subspace spanned by $O(\log n)$ random vectors. For concreteness, we use the following version of the Johnson-Lindenstrauss Lemma due to Achlioptas [1].

Lemma 7. *Given fixed vectors $v_1 \dots v_n \in \mathbb{R}^d$ and $\epsilon > 0$, let $Q_{k \times d}$ be a random $\pm 1/\sqrt{k}$ matrix (i.e., independent Bernoulli entries) with $k \geq 24 \log n / \epsilon^2$. Then with probability at least $1 - 1/n$*

$$(1 - \epsilon) \|v_i - v_j\|_2^2 \leq \|Qv_i - Qv_j\|_2^2 \leq (1 + \epsilon) \|v_i - v_j\|_2^2$$

for all pairs $i, j \leq n$.

Our goal is now to compute the projections $\{QW^{1/2} B L^+ \chi_v\}$. We will exploit the linear system solver of Spielman and Teng [23, 24], which we recall satisfies:

Theorem 8 (Spielman-Teng). *There is an algorithm $x = \text{STSolve}(L, y, \delta)$ which takes a Laplacian matrix L , a column vector y , and an error parameter $\delta > 0$, and returns a column vector x satisfying*

$$\|x - L^+ y\|_L \leq \epsilon \|L^+ y\|_L,$$

where $\|y\|_L = \sqrt{y^T L y}$. The algorithm runs in expected time $\tilde{O}(m \log(1/\delta))$, where m is the number of non-zero entries in L .

Let $Z = QW^{1/2} B L^+$. We will compute an approximation \tilde{Z} by using `STSolve` to approximately compute the rows of Z . Let the column vectors z_i and \tilde{z}_i denote the i th rows of Z and \tilde{Z} , respectively (so that z_i is the i th column of Z^T). Now we can construct the matrix \tilde{Z} in the following three steps.

1. Let Q be a random $\pm 1/\sqrt{k}$ matrix of dimension $k \times n$ where $k = 24 \log n / \epsilon^2$.
2. Compute $Y = QW^{1/2} B$. Note that this takes $2m \times 24 \log n / \epsilon^2 + m = \tilde{O}(m / \epsilon^2)$ time since B has $2m$ entries and $W^{1/2}$ is diagonal.
3. Let y_i , for $1 \leq i \leq k$, denote the rows of Y , and compute $\tilde{z}_i = \text{STSolve}(L, y_i, \delta)$ for each i .

We now prove that, for our purposes, it suffices to call **STSolve** with

$$\delta = \frac{\epsilon}{3} \sqrt{\frac{2(1-\epsilon)w_{min}}{(1+\epsilon)n^3 w_{max}}}.$$

Lemma 9. *Suppose*

$$(1-\epsilon)R_{uv} \leq \|Z(\chi_u - \chi_v)\|^2 \leq (1+\epsilon)R_{uv},$$

for every pair $u, v \in V$. If for all i ,

$$\|z_i - \tilde{z}_i\|_L \leq \delta \|z_i\|_L, \quad (4)$$

where

$$\delta \leq \frac{\epsilon}{3} \sqrt{\frac{2(1-\epsilon)w_{min}}{(1+\epsilon)n^3 w_{max}}} \quad (5)$$

then

$$(1-\epsilon)^2 R_{uv} \leq \|\tilde{Z}(\chi_u - \chi_v)\|^2 \leq (1+\epsilon)^2 R_{uv},$$

for every uv .

Proof. Consider an arbitrary pair of vertices u, v . It suffices to show that

$$\left| \|Z(\chi_u - \chi_v)\| - \|\tilde{Z}(\chi_u - \chi_v)\| \right| \leq \frac{\epsilon}{3} \|Z(\chi_u - \chi_v)\| \quad (6)$$

since this will imply

$$\begin{aligned} \left| \|Z(\chi_u - \chi_v)\|^2 - \|\tilde{Z}(\chi_u - \chi_v)\|^2 \right| &= \left| \|Z(\chi_u - \chi_v)\| - \|\tilde{Z}(\chi_u - \chi_v)\| \right| \cdot \left| \|Z(\chi_u - \chi_v)\| + \|\tilde{Z}(\chi_u - \chi_v)\| \right| \\ &\leq \frac{\epsilon}{3} \cdot \left(2 + \frac{\epsilon}{3} \right) \|Z(\chi_u - \chi_v)\|^2. \end{aligned}$$

As G is connected, there is a simple path P connecting u to v . Applying the triangle inequality twice, we obtain

$$\begin{aligned} \left| \|Z(\chi_u - \chi_v)\| - \|\tilde{Z}(\chi_u - \chi_v)\| \right| &\leq \left\| (Z - \tilde{Z})(\chi_u - \chi_v) \right\| \\ &\leq \sum_{ab \in P} \left\| (Z - \tilde{Z})(\chi_a - \chi_b) \right\|. \end{aligned}$$

We will upper bound this later term by considering its square:

$$\begin{aligned}
\left(\sum_{ab \in P} \| (Z - \tilde{Z})(\chi_a - \chi_b) \|^2 \right)^2 &\leq n \sum_{ab \in P} \| (Z - \tilde{Z})(\chi_a - \chi_b) \|^2 && \text{by Cauchy-Schwarz} \\
&\leq n \sum_{ab \in E} \| (Z - \tilde{Z})(\chi_a - \chi_b) \|^2 \\
&= n \| (Z - \tilde{Z})B^T \|^2_F && \text{writing this as a Frobenius norm} \\
&= n \| B(Z - \tilde{Z})^T \|^2_F \\
&\leq \frac{n}{w_{min}} \| W^{1/2} B(Z - \tilde{Z})^T \|^2_F && \text{since } \| W^{-1/2} \|_2 \leq 1/\sqrt{w_{min}} \\
&\leq \delta^2 \frac{n}{w_{min}} \| W^{1/2} B Z^T \|^2_F \\
&&&\text{since } \| W^{1/2} B(z_i - \tilde{z}_i) \|^2 \leq \delta^2 \| W^{1/2} B z_i \|^2 \text{ by (4)} \\
&= \delta^2 \frac{n}{w_{min}} \sum_{ab \in E} w_{ab} \| Z(\chi_a - \chi_b) \|^2 \\
&\leq \delta^2 \frac{n}{w_{min}} \sum_{ab \in E} w_{ab} (1 + \epsilon) R_{ab} \\
&\leq \delta^2 \frac{n(1 + \epsilon)}{w_{min}} (n - 1) && \text{by Lemma 3.(iii).}
\end{aligned}$$

On the other hand,

$$\| Z(\chi_u - \chi_v) \|^2 \geq (1 - \epsilon) R_{uv} \geq \frac{2(1 - \epsilon)}{nw_{max}},$$

by Proposition 10. Combining these bounds, we have

$$\begin{aligned}
\left| \frac{\| Z(\chi_u - \chi_v) \| - \| \tilde{Z}(\chi_u - \chi_v) \|}{\| Z(\chi_u - \chi_v) \|} \right| &\leq \delta \left(\frac{n(1 + \epsilon)}{w_{min}} (n - 1) \right)^{1/2} \cdot \left(\frac{nw_{max}}{2(1 - \epsilon)} \right)^{1/2} \\
&\leq \frac{\epsilon}{3} && \text{by (5),}
\end{aligned}$$

as desired. \square

Proposition 10. *If $G = (V, E, w)$ is a connected graph, then for all $u, v \in V$,*

$$R_{uv} \geq \frac{2}{nw_{max}}.$$

Proof. By Rayleigh's monotonicity law (see [6]), each resistance R_{uv} in G is at least the corresponding resistance R'_{uv} in $G' = w_{max} \times K_n$ (the complete graph with all edge weights w_{max}) since G' is obtained by increasing weights (i.e., conductances) of edges in G . But by symmetry each resistance R'_{uv} in G' is exactly

$$\frac{\sum_{uv} R'_{uv}}{\binom{n}{2}} = \frac{(n-1)/w_{max}}{n(n-1)/2} = \frac{2}{nw_{max}}.$$

Thus $R_{uv} \geq \frac{2}{nw_{max}}$ for all $u, v \in V$. \square

Thus the construction of \tilde{Z} takes $\tilde{O}(m \log(1/\delta)/\epsilon^2) = \tilde{O}(m \log r/\epsilon^2)$ time. We can then find the approximate resistance $\|\tilde{Z}(\chi_u - \chi_v)\|^2 \approx R_{uv}$ for any $u, v \in V$ in $O(\log n/\epsilon^2)$ time simply by subtracting two columns of \tilde{Z} and computing the norm of their difference. \square

Using the above procedure, we can compute arbitrarily good approximations to the effective resistances $\{R_e\}$ which we need for sampling in nearly-linear time. By Corollary 6, any constant factor approximation yields a sparsifier, so we are done.

5 An Additional Property

Corollary 6 suggests that **Sparsify** is quite robust with respect to changes in the sampling probabilities p_e , and that we may be able to prove additional guarantees on H by tweaking them. In this section, we prove one such claim.

The following property is desirable for using H to solve linear systems (specifically, for the construction of *ultrasparsifiers* [23, 24], which we will not define here):

$$\text{For every vertex } v \in V, \quad \sum_{e \ni v} \frac{\tilde{w}_e}{w_e} \leq 2 \deg(v). \quad (7)$$

This says, roughly, that not too many of the edges incident to any given vertex get blown up too much by sampling and rescaling. We show how to incorporate this property into our sparsifiers.

Lemma 11. *Suppose we sample $q > 4n \log n / \beta$ edges of G as in **Sparsify** with probabilities that satisfy*

$$p_{(u,v)} \geq \frac{\beta}{n \min(\deg(u), \deg(v))}$$

for some constant $0 < \beta < 1$. Then with probability at least $1 - 1/n$,

$$\sum_{e \ni v} \frac{\tilde{w}_e}{w_e} \leq 2 \deg(v) \quad \text{for all } v \in V.$$

Proof. For a vertex v , define i.i.d. random variables X_1, \dots, X_q by:

$$X_i = \begin{cases} \frac{1}{p_e} & \text{if } e \ni v \text{ is the } i\text{th edge chosen} \\ 0 & \text{otherwise} \end{cases}$$

so that X_i is set to $1/p_e$ with probability p_e for each edge e attached to v . Let

$$D_v = \sum_{e \ni v} \frac{\tilde{w}_e}{w_e} = \sum_{e \ni v} \frac{(\# \text{ of times } e \text{ is sampled})}{qp_e} = \frac{1}{q} \sum_{i=1}^q X_i.$$

We want to show that with high probability, $D_v \leq 2 \deg(v)$ for all vertices v . We begin by bounding

the expectation and variance of each X_i :

$$\mathbb{E}X_i = \sum_{e \ni v} p_e \frac{1}{p_e} = \deg(v)$$

$$\begin{aligned}\mathbf{Var}(X_i) &= \sum_{e \ni v} p_e \left(\frac{1}{p_e^2} - \frac{1}{p_e} \right) \\ &\leq \sum_{e \ni v} \frac{1}{p_e} \\ &\leq \sum_{(u,v) \ni v} \frac{n \min(\deg(u), \deg(v))}{\beta} \quad \text{by assumption} \\ &\leq \sum_{(u,v) \ni v} \frac{n \deg(v)}{\beta} \\ &= \frac{n \deg(v)^2}{\beta}\end{aligned}$$

Since the X_i are independent, the variance of D_v is just

$$\mathbf{Var}(D_v) = \frac{1}{q^2} \sum_{i=1}^q \mathbf{Var}(X_i) \leq \frac{n \deg(v)^2}{\beta q}.$$

We now apply Bennett's inequality for sums of i.i.d. variables (see, e.g., [20]), which says

$$\mathbb{P}[|D_v - \mathbb{E}D_v| > \mathbb{E}D_v] \leq \exp \left(\frac{-(\mathbb{E}D_v)^2}{\mathbf{Var}(D_v)(1 + \frac{\mathbb{E}D_v}{q})} \right)$$

We know that $\mathbb{E}D_v = \mathbb{E}X_i = \deg(v)$. Substituting our estimate for $\mathbf{Var}(D_v)$ and setting $q \geq 4n \log n / \beta$ gives:

$$\begin{aligned}\mathbb{P}[D_v > 2 \deg(v)] &\leq \exp \left(\frac{-\deg(v)^2}{\frac{n \deg(v)^2}{\beta q} \left(1 + \frac{\deg(v)}{q} \right)} \right) \\ &\leq \exp \left(\frac{-\beta q}{2n} \right) \quad \text{since } 1 + \frac{\deg(v)}{q} \leq 2 \\ &\leq \exp(-2 \log n) = 1/n^2.\end{aligned}$$

Taking a union bound over all v gives the desired result. \square

Sampling with probabilities

$$p'_e = p'_{(u,v)} = \frac{1}{2} \left(\frac{\|Zb_e^T\|^2 w_e}{\sum_e \|Zb_e^T\|^2 w_e} + \frac{1}{n \min(\deg(u), \deg(v))} \right)$$

satisfies the requirements of both Corollary 6 (with $\alpha = 2$) and Lemma 11 (with $\beta = 1/2$) and yields a sparsifier with the desired property.

Theorem 12. *There is an $\tilde{O}(m/\epsilon^2)$ time algorithm which on input $G = (V, E, w), \epsilon > 0$ produces a weighted subgraph $H = (V, \tilde{E}, \tilde{w})$ of G with $O(n \log n/\epsilon^2)$ edges which, with probability at least $1/2$, satisfies both (2) and (7).*

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