"Machine Learning and Computational Statistics"

9^{th} Homework

Exercise 1(*):

Suppose you are given a data set $Y=\{(y_i,\mathbf{x}_i), i=1,\dots,N\}$ where $y_i \in\{0,1\}$ is the class label for vector $\mathbf{x}_i \in R^l$. **Extract** the gradient descent logistic regression classifier for the two-class case (write in detail the algebraic manipulations using the hints in the relevant slides of its presentation).

Exercise 2:

Suppose you are given a data set $Y=\{(y_i,\mathbf{x}_i'), i=1,\dots,N\}$ where $y_i \in\{0,1\}$ is the class label for vector $\mathbf{x}_i' \in R^l$. Assume that y and $\mathbf{x}'$ are related via the following model: $y=f(\boldsymbol{\theta}'^{\mathrm{T}}\mathbf{x}'+\theta_0)$, where $\boldsymbol{\theta}$ and θ_0 are the model parameters and $f(z)=1/(1+\exp(-az))$.

(a) **Plot** the function $f(z)$ for various values of the parameter a.
(b) Propose a gradient descent scheme to **train** this model (that is, to estimate the values of the involved parameters), based on the **minimization** of the sum of error squares criterion, using Y.
(c) Can the model ever respond with a **"clear"** 1 or a **"clear"** 0, for a given $\mathbf{x}$?
(d) How can we interpret the response of the model for a given $\mathbf{x}$?
(e) Propose a way for leading the model responses very close to 1 (for class 1 vectors) or 0 (for class 0 vectors).

Hints:

(a) Use a more compact notation by setting $\mathbf{x}_i=[1\ \mathbf{x}_i']^{\mathrm{T}}$, $i=1,\dots,N$, and $\boldsymbol{\theta}=[\theta_0\ \boldsymbol{\theta}]^{\mathrm{T}}$. The model then becomes $y=f(\boldsymbol{\theta}^{\mathrm{T}}\mathbf{x})$.
(b) The sum of error squares criterion in this case is $J(\boldsymbol{\theta})=\sum_{n=1}^{N}\left(y_n-f(\boldsymbol{\theta}^T\boldsymbol{x}_n)\right)^2$.
(c) It is $f'(z)=\frac{df(z)}{dz}=af(z)\left(1-f(z)\right)$.

Exercise 3:

Consider a **two-class one-dimensional** classification problem and suppose you are given a data set

$$Y=\{(y_i,x_i),\ x_i\in R,\ \ y_i\in\{-1,+1\},\ i=1,\dots,N\}.$$

A classifier for this problem is of the form

"Assign a given x to class $+1$ (-1) if $x>T(<T)$"

Propose an (efficient) algorithm for determining the threshold T that borders the two classes, so that to minimize the classification error, **based on** the data set Y.

Hints:

(a) Exploit the fact that x_i's are **ordered** in the line of the real numbers.

(b) For your help and in order to get a visual inspection, consider a specific data set, e.g., $Y = \{\dots, (-1, -5), (-1, -3), (-1, 0), \ (+1, -1), \ (+1, 1), (+1, 3), \dots\}$ and mark on the line of the real numbers the data points, using different colors for points from different classes.

Exercise 4:

Consider the set-up of the example given in slide 43 of the 9th lecture. Verify that the node impurity decrease achieved by the rule "$x_1 \leq 3$" is equal to 0.42.

Exercise 5 (python code + text):

Consider a two-class, two-dimensional classification problem for which you can find attached two sets: one for training and one for testing (file *HW8.mat*). Each of these sets consists of pairs of the form $(y_i, \mathbf{x}_i)$, where y_i is the class label for vector $\mathbf{x}_i$. Let N_{train} and N_{test} denote the number of training and test sets, respectively. The data are given via the following arrays/matrices:

- ***train_x*** (a N_{train} x2 **matrix** that contains in its rows the training vectors $\mathbf{x}_i$)
- ***train_y*** (a N_{train}–dim. column **vector** containing the **class labels** (1 or 2) of the corresponding training vectors $\mathbf{x}_i$ included in ***train_x***).
- ***test_x*** (a N_{test}x2 **matrix** that contains in its rows the test vectors $\mathbf{x}_i$)
- ***test_y*** (a N_{test}–dim. column **vector** containing the **class labels** (1 or 2) of the corresponding test vectors $\mathbf{x}_i$ included in ***test_x***).

Assume that the two classes, ω_1 and ω_2 are modeled by normal distributions.

(a) Adopt the **Bayes classifier**.

i. Use the training set to **estimate** $P(\omega_1)$, $P(\omega_2)$, $p(\mathbf{x}|\omega_1)$, $p(\mathbf{x}|\omega_2)$ (Since $p(\mathbf{x}|\omega_j)$ is modeled a normal distribution, it is completely identified by $\boldsymbol{\mu}_j$ and Σ_j. Use the **ML estimates** for them as given in the lecture slides).

ii. **Classify** the points $\mathbf{x}_i$ of the test set, using the **Bayes classifier** (for each point apply the Bayes classification rule and keep the class labels, to an a N_{test}–dim. column **vector** , called ***Btest_y*** containing the **estimated class labels** (1 or 2) of the corresponding test vectors $\mathbf{x}_i$ included in ***test_x.***).

iii. Estimate the error classification probability ((1) **compare** ***test_y*** and ***Btest_y*** , (2) **count** the positions where both of them have the same class label and (3) **divide** with the total number of test vectors).

(b) Adopt the **naïve Bayes classifier**.

Recall that $\boldsymbol{x} = [x_1, x_2]^T$

i. Use the training set to estimate $P(\omega_1)$, $P(\omega_2)$, $p(x_1|\omega_1)$, $p(x_2|\omega_1)$, $p(x_1|\omega_2)$, $p(x_2|\omega_2)$ (Each $p(\boldsymbol{x}|\omega_j)$ is written as $p(\boldsymbol{x}|\omega_j) = p(x_1|\omega_j) * p(x_2|\omega_j)$. Use the **ML estimates** of the mean and variance for each one of the 1-dim. pdfs).

ii. Classify the points $\boldsymbol{x}_i = [x_{i1}, x_{i2}]^T$ of the test set, using the naïve Bayes classifier (Estimate $p(\boldsymbol{x}|\omega_j)$ with $p(x_{i1}|\omega_j) * p(x_{i2}|\omega_j)$ and then apply the Bayes rule. Keep the class labels, to an a N_{test}–dim. column **vector** , called ***NBtest_y*** containing the **estimated class labels** (1 or 2) of the corresponding test vectors $\boldsymbol{x}_i$ included in ***test_x***)

iii. Estimate the error classification probability (work as in the previous case).

(c) Adopt the **k-nearest neighbor classifier**, for $k = 5$ and estimate the classification error probability.

(d) Adopt the **logistic regression classifier** and (i) train it using the training set and then (ii) measure its performance on the test set.

(e) Depict graphically the training set, using different colors for points from different classes.

(f) Report the classification results obtained by the four classifiers and comment on them. Under what conditions, the first two classifiers would exhibit the same performance?

Hint: Use the attached Python code.