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Referee report on the PhD thesis

Ilias Ftouhi:
Diagrammes de Blaschke-Santaló
et autres problèmes en optimisation de forme

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The PhD thesis under report deals with some interesting problems in shape optimization theory, involving various shape functionals which depend on geometrical and spectral quantities. The main goal is to obtain efficient (or sharp in the best situations) inequalities related to the following quantities:

$d(\Omega)$ the diameter of a domain Ω;
$r(\Omega)$ the inradius of a domain Ω, defined as the radius of the largest ball included in Ω;
$|\Omega|$ the Lebesgue measure of a domain Ω;
$P(\Omega)$ the perimeter of a domain Ω, according to the De Giorgi definition

$$P(\Omega) = \sup\left\{\int_\Omega \operatorname{div}\phi\,dx \;:\; \phi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n),\ \|\phi\|_{L^\infty} \le 1\right\};$$

$h(\Omega)$ the Cheeger constant of a domain Ω, defined as

$$h(\Omega) = \inf\left\{\frac{P(E)}{|E|} \;:\; E \subset \Omega\right\};$$

$\lambda_1(\Omega)$ the first eigenvalue of the Laplace operator $-\Delta$ of a domain Ω with Dirichlet boundary conditions

$$\lambda_1(\Omega) = \min\left\{\Big[\int_\Omega |\nabla u|^2\,dx\Big]\Big[\int_\Omega u^2\,dx\Big]^{-1} \;:\; u \in H^1_0(\Omega),\ u \ne 0\right\}.$$

The introduction is very well presented and all the results in the thesis are clearly illustrated, together with some open problems and future research projects. In the subsequent sections the candidate studies:

- in Part I the Blaschke-Santaló diagrams for some choices of the quantities above and more precisely
  - volume, perimeter, Cheeger constant in section 2;
  - volume, perimeter, first Dirichlet eigenvalue in section 3;
  - some issues on Cheeger inequality for convex sets in section 4;
  - some numerical simulations related to Blaschke-Santaló diagrams in section 5;

- in Part II the question of the optimal location of a spherical obstacle in order to maximize the first Steklov eigenvalue. This is the content of section 6, where a domain $\Omega = B_1 \setminus \overline{B}_2$ is considered, with B_1 and B_2 balls of fixed radii, being $\overline{B}_2 \subset B_1$. It is shown that the concentric choice is the best one.

I liked a lot the thesis, which has been written very carefully, presenting all the problems in a very clear and complete way, from their initial formulation up to the numerical simulations, where new and unexpected difficulties arise. In particular, the numerical plot of Blaschke-Santaló diagrams is very well done, using random convex polygons. The part on Cheeger inequality is also very interesting and leads to new and sharper results with respect to what was known in the literature. Most of the results above are the object of some publications by the candidate: two of them are in a preprint form on `https://hal.archives-ouvertes.fr/hal` and three more are still in the form of working papers; I am confident that they will also be ready soon in a more definitive form.

Summarizing, my opinion on the thesis is strongly positive and I am therefore fully in favor of the thesis being admitted to the final discussion.

Pisa, December 28, 2020

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