Phase I

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Let $\mathbf{A} \in \mathbb{R}^{n \times p}$, $\mathbf{C} \in \mathbb{R}^{m \times p}$, $\mathbf{b} \in \mathbb{R}^{n}$, $\mathbf{d} \in \mathbb{R}^{m}$. We are interested in finding a vector $\mathbf{x}^{(0)}$ such that

$$\mathbf{x}^{(0)} \in \{\mathbf{C}\mathbf{x} \leq \mathbf{d}\} \cap \{\mathbf{A}\mathbf{x} = \mathbf{b}\},$$

where $\leq$ is taken component-wise. To do so is equivalent to solving the problem

$$\begin{aligned} &\text{minimize} && s \\ &\text{subject to} && \mathbf{C}\mathbf{x} \leq \mathbf{d} + s\mathbf{1}_m, \quad \mathbf{A}\mathbf{x} = \mathbf{b}. \end{aligned} \tag{0.1}$$

Define $\mathbf{u} = (\mathbf{x}^\intercal, s)^\intercal$, $\mathbf{e} = (\mathbf{0}_p^\intercal, 1)^\intercal$,

$$\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A} & \mathbf{0}_n \end{pmatrix}, \quad \tilde{\mathbf{C}} = \begin{pmatrix} \mathbf{C} & -\mathbf{1}_m \end{pmatrix}.$$

We can reformulate (0.1) as

$$\begin{aligned} &\text{minimize} && \mathbf{e}^\intercal \mathbf{u} \\ &\text{subject to} && \tilde{\mathbf{C}}\mathbf{u} \leq \mathbf{d}, \quad \tilde{\mathbf{A}}\mathbf{u} = \mathbf{b}. \end{aligned} \tag{0.2}$$

Define the logarithmic barrier function

$$B_t(\mathbf{u}) = t\mathbf{e}^\intercal \mathbf{u} - \sum_{i=1}^{m+1} \log(\tilde{d}_i - \tilde{\mathbf{c}}_i^\intercal \mathbf{u}).$$

Let $\mathbf{u}^*$ be a solution of (0.2), and $\mathbf{u}^*(t)$ be a solution of

$$\begin{aligned} \text{minimize} \quad & B_t(\mathbf{u}) \\ \text{subject to} \quad & \tilde{\mathbf{A}}\mathbf{u} = \mathbf{b}. \end{aligned} \tag{0.3}$$

As $t \to \infty$, $\mathbf{u}^*(t) \to \mathbf{u}^*$. We have

$$\begin{aligned} \nabla B_t(\mathbf{u}) &= t\mathbf{e} + \sum_{i=1}^{m+1} \frac{1}{\tilde{d}_i - \tilde{\mathbf{c}}_i^\intercal \mathbf{u}} \tilde{\mathbf{c}}_i, \\ \nabla^2 B_t(\mathbf{u}) &= \sum_{i=1}^{m+1} \frac{1}{(\tilde{d}_i - \tilde{\mathbf{c}}_i^\intercal \mathbf{u})^2} \tilde{\mathbf{c}}_i \tilde{\mathbf{c}}_i^\intercal. \end{aligned}$$

For each t, we will use Newton's method to solve (0.3). As an initial point for each inner iteration we choose $\mathbf{u}^{(0)}(t) = (\mathbf{x}^{(0)\intercal}, s^{(0)})^\intercal$ such that

$$\mathbf{A}\mathbf{x}^{(0)} = \mathbf{b} \quad \text{and} \quad s^{(0)} = \min\left\{\mathbf{C}\mathbf{x}^{(0)} - \mathbf{d}, 0\right\}.$$

Suppose we are at the t-th step of the outer iteration and at the ν-th step of the inner iteration. In the barrier method, the Newton step Δ, and the associated dual variable are given by the linear equations

$$\begin{pmatrix} \nabla^2 B_t(\mathbf{u}(t)^\nu) & \tilde{\mathbf{A}}^\intercal \\ \tilde{\mathbf{A}} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} -\nabla B_t(\mathbf{u}(t)^\nu) \\ \mathbf{0} \end{pmatrix}$$

We choose s with backtracking line search and define

$$\mathbf{u}(t)^{\nu+1} = \mathbf{u}(t)^\nu + s\Delta\mathbf{u}$$