Quadratic Program solved with Interior Point method

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Suppose we have the problem

$$\underset{\mathbf{x}\in\mathbb{R}^p}{\operatorname{argmin}} \quad \frac{1}{2}\mathbf{x}^\intercal\mathbf{H}\mathbf{x} + \mathbf{f}^\intercal\mathbf{x},$$
$$\text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b},$$

where $\mathbf{f} \in \mathbb{R}^p$, $\mathbf{H} \in \mathbb{R}^{p\times p}$, $\mathbf{H} \succ \mathbf{0}$, $\mathbf{A} \in \mathbb{R}^{n\times p}$, $\mathbf{b} \in \mathbb{R}^n$.

Let $\mathbf{b} = (b_1, \ldots, b_n)^\intercal$, $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)^\intercal$, where $\mathbf{a}_i \in \mathbb{R}^p$. Define the logarithmic barrier function

$$B(\mathbf{x}, \mu) = \frac{1}{2}\mathbf{x}^\intercal\mathbf{H}\mathbf{x} + \mathbf{f}^\intercal\mathbf{x} - \mu\sum_{i=1}^{n}\log(b_i - \mathbf{a}_i^\intercal\mathbf{x}).$$

The gradient of B w.r.t. $\mathbf{x}$ is

$$\mathbf{F}(\mathbf{x}) = \mathbf{H}\mathbf{x} + \mathbf{f} + \mu\sum_{i=1}^{n}\frac{\mathbf{a}_i}{b_i - \mathbf{a}_i^\intercal\mathbf{x}}$$

Define the variable $\lambda_i = \mu/(b_i - \mathbf{a}_i^\intercal\mathbf{x})$. Replacing λ_i into the gradient we get

$$\mathbf{F}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{H}\mathbf{x} + \mathbf{f} + \sum_{i=1}^{n}\lambda_i\mathbf{a}_i$$

If we define $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^\intercal$ the gradient becomes

$$\mathbf{F}(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{H}\mathbf{x} + \mathbf{f} + \mathbf{A}^\intercal \boldsymbol{\lambda}.$$

Define the matrices $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $\mathbf{C} = \mathrm{diag}(b_1 - \mathbf{a}_1^\intercal \mathbf{x}, \ldots, b_n - \mathbf{a}_n^\intercal \mathbf{x})$, $\mathbf{c} = (b_1 - \mathbf{a}_1^\intercal \mathbf{x}, \ldots, b_n - \mathbf{a}_n^\intercal \mathbf{x})^\intercal$ and

$$\mathbf{G}(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} \mathbf{H}\mathbf{x} + \mathbf{f} + \mathbf{A}^\intercal \boldsymbol{\lambda} \\ \boldsymbol{\Lambda}\mathbf{c} - \mu \mathbf{1}_n \end{pmatrix}.$$

The jacobian of $\mathbf{G}$ is

$$\mathbf{J}_{\mathbf{G}}(\mathbf{x}, \boldsymbol{\lambda}) = \begin{pmatrix} \mathbf{H} & \mathbf{A}^\intercal \\ -\boldsymbol{\Lambda}\mathbf{A} & \mathbf{C} \end{pmatrix}.$$

To find $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ such that $\mathbf{G}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = \mathbf{0}$ we use Newton's method. We start at $(\mathbf{x}^{(0)}, \boldsymbol{\lambda}^{(0)})$, which can be the unconstrained solution of the original problem, and then at the t-th iteration we get

$$\begin{pmatrix} \mathbf{x}^{(t+1)} \\ \boldsymbol{\lambda}^{(t+1)} \end{pmatrix} = \begin{pmatrix} \mathbf{x}^{(t)} \\ \boldsymbol{\lambda}^{(t)} \end{pmatrix} - [\mathbf{J}_{\mathbf{G}}(\mathbf{x}^{(t)}, \boldsymbol{\lambda}^{(t)})]^{-1} \mathbf{G}(\mathbf{x}^{(t)}, \boldsymbol{\lambda}^{(t)})$$