Reverberation

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TSIA 206 - Speech and audio processing

Introduction

- Why study room acoustics?
  - To understand the quality criteria of a room
  - To design rooms that guarantee sound comfort
  - To understand and treat acoustic problems in premises (public places, theaters, homes)
- Impact on the recording and reproduction of sounds
  - Help in placing microphones and loudspeakers
  - Speech intelligibility, especially in sound reinforcement
- Digital processing of the audio signal
  - Apply reverberation to an anechoic signal:
    - Convolution with a measured Room Impulse Response (RIR)
    - Artificial reverberation
  - Cancel reverberation in a reverberated signal:
    - Dereverberation
    - Speech and music transcription
    - Source localization, separation

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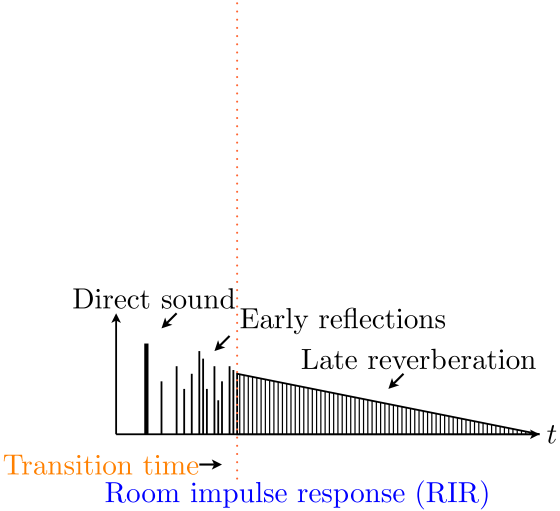
Outline

Part II

TF properties of reverberation

Time-frequency profile of reverberation



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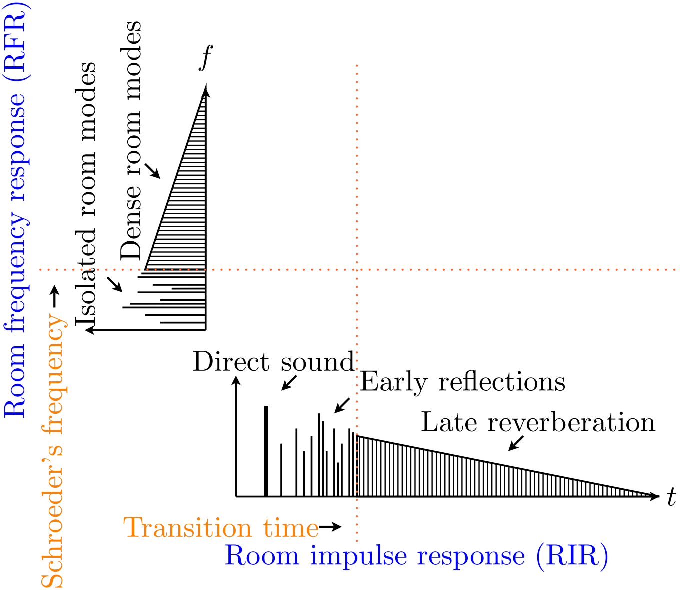

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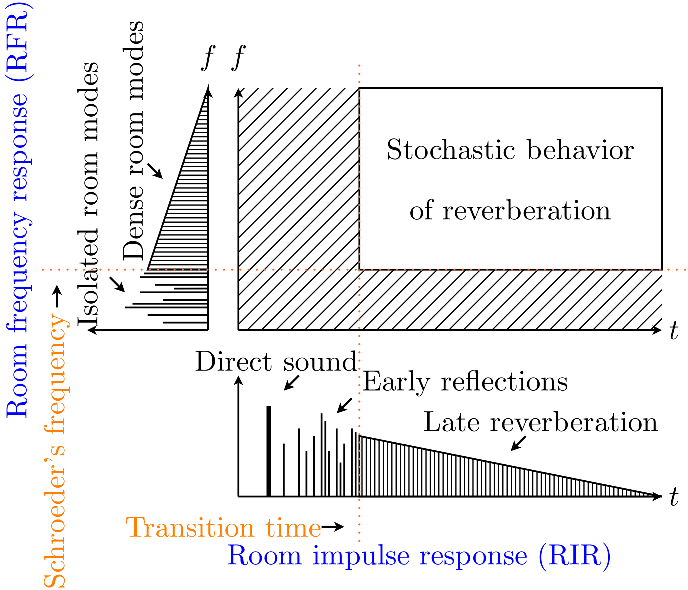

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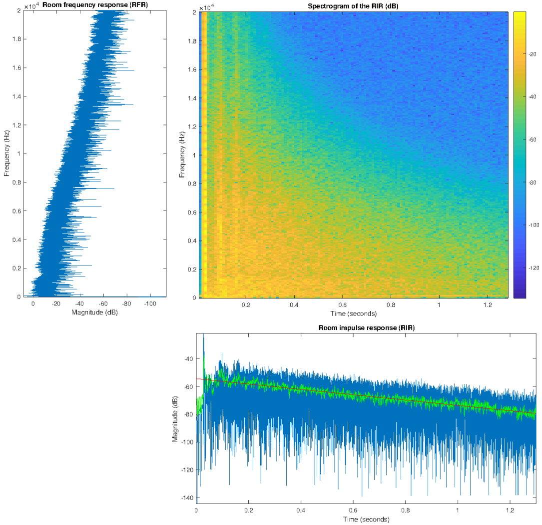
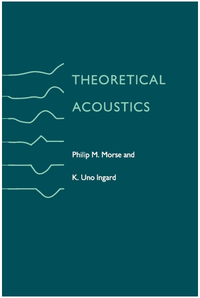


Part III

Fundamentals of room acoustics

References



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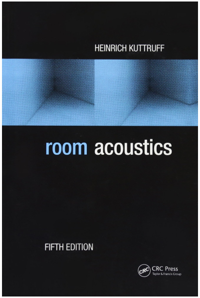
Princeton University Press, 1986



Room Acoustics, Heinrich Kuttruff, CRC Press, 2009

The wave equation

- **Wave equation**: $\forall \boldsymbol{x} \in \mathbb{R}^3, \forall t \in \mathbb{R}$,

$$\Delta p(\boldsymbol{x}, t) - \frac{1}{c^2}\frac{\partial^2 p(\boldsymbol{x}, t)}{\partial t^2} = 0 \qquad (1)$$

  - $p(\boldsymbol{x}, t)$ is the pressure wave at position $\boldsymbol{x}$ and time t
  - Δ is the Laplacian w.r.t. space, c is the speed of sound.
- **Classical solutions**:
  - **Plane wave** ($\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$)

$$p(\boldsymbol{x}, t) = e^{2\imath\pi\left(\boldsymbol{k}^\top \boldsymbol{x} + ckt\right)}$$

    were $\boldsymbol{k}$ is the *wave vector*, $k = \|\boldsymbol{k}\|_2$ is the *wave number*, $\lambda = \frac{1}{k}$ is the *wave length*, and $f = ck$ is the *frequency*
  - **Spherical wave**
    ($\Delta p = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial p}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 p}{\partial \varphi^2}$)

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Wave equation with initial conditions

- Initial conditions:

$$\forall \boldsymbol{x} \in \mathbb{R}^3, \begin{cases} p(\boldsymbol{x}, 0) &= c^2 g(\boldsymbol{x}) \\ \frac{\partial p}{\partial t}(\boldsymbol{x}, 0^+) &= 0 \end{cases}$$

- Plane wave representation of the solution: $\forall t \geq 0$,

$$p(\boldsymbol{x}, t) = c^2 H(t) Re \int_{\boldsymbol{k} \in \mathbb{R}^3} \widehat{g}(\boldsymbol{k}) e^{2\imath\pi\left(\boldsymbol{k}^\top \boldsymbol{x} + ckt\right)} d\boldsymbol{k}$$

where $H(t)$ denotes the Heaviside function, such that $H(t) = 1\ \forall t \geq 0$ and $H(t) = 0\ \forall t < 0$

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- Remark: this solution coincides on $t \in \mathbb{R}_+^*$ with the **causal** solution of the **inhomogeneous** wave equation: $\forall t \in \mathbb{R}$,

$$\Delta p(\boldsymbol{x}, t) - \frac{1}{c^2} \frac{\partial^2 p(\boldsymbol{x}, t)}{\partial t^2} = -g(\boldsymbol{x}) \dot{\delta}(t)$$

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Response to a point source in free space

- Presence of a point source at $\boldsymbol{x}_0 \in V$ emitting at $t = 0$.
- The wave equation (1) becomes inhomogeneous: $\forall t \in \mathbb{R}$,

$$\Delta p(\boldsymbol{x}, t) - \frac{1}{c^2}\frac{\partial^2 p(\boldsymbol{x}, t)}{\partial t^2} = -Q\rho_0 \delta(\boldsymbol{x} - \boldsymbol{x}_0)\dot{\delta}(t)$$

  - ρ_0 is the static value of the gas density
  - Q is volume velocity of the point source
- Causal solution in free space:

$$\begin{aligned} p(\boldsymbol{x}, t) &= \frac{Q\rho_0}{4\pi}\frac{\dot{\delta}\left(t - \frac{\|\boldsymbol{x}-\boldsymbol{x}_0\|_2}{c}\right)}{\|\boldsymbol{x}-\boldsymbol{x}_0\|_2} \\ &= Q\rho_0 c^2 H(t) Re \int_{\boldsymbol{k}\in\mathbb{R}^3} e^{2\imath\pi\left(\boldsymbol{k}^\top(\boldsymbol{x}-\boldsymbol{x}_0)+c\|\boldsymbol{k}\|_2 t\right)} d\boldsymbol{k} \end{aligned}$$

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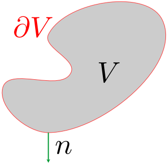
Boundary condition of the wave equation

- **Boundary condition** in an enclosure V: $\forall \boldsymbol{x} \in \partial V, \forall t \in \mathbb{R}$,



$$\frac{\partial}{\partial \boldsymbol{n}} p(\boldsymbol{x},t) + \beta(\boldsymbol{x},.) \overset{t}{*} \frac{1}{c} \frac{\partial p(\boldsymbol{x},t)}{\partial t} = 0 \tag{2}$$

  - $\widehat{\beta}(\boldsymbol{x},f) \in \mathbb{C}$, is the **specific admittance**, function of the position $\boldsymbol{x}$ and frequency f, such that $Re(\widehat{\beta}(\boldsymbol{x},f)) > 0$
  - $\frac{\partial}{\partial \boldsymbol{n}}$ denotes partial differentiation in the direction of the outward normal to the wall
- Reflection on a plane wall P with uniform admittance $\widehat{\beta}(f)$:
  - If the incident wave is a plane wave: $p_i(\boldsymbol{x},t) = e^{2\imath\pi\left(\boldsymbol{k}_i^\top \boldsymbol{x}+ft\right)}$
  - The reflected wave is: $p_r(\boldsymbol{x},t) = R(f)\, e^{2\imath\pi\left(\boldsymbol{k}_r^\top \boldsymbol{x}+ft\right)}$ (if $0 \in P$)
    - $R(f) \in \mathbb{C}$ is the *reflection factor*
    - $\boldsymbol{k}_r$ is the flipped wave vector w.r.t. reflection of plane P
  - Relation with the admittance: $R(f) = \frac{\cos(\theta)+\widehat{\beta}(f)}{\cos(\theta)-\widehat{\beta}(f)}$ where θ is the angle between $\boldsymbol{k}_i$ (or $\boldsymbol{k}_r$) and the normal to the wall

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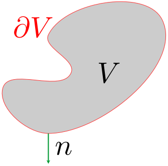

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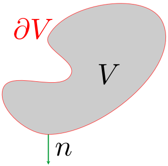
The Helmholtz equation: rigid walls case

- Rigid walls: $\widehat{\beta}(\boldsymbol{x}, f) = 0$ $(R(f) = 1)$
- **Helmholtz equation**: $\forall \boldsymbol{x} \in \mathbb{R}^3$,

$$\Delta\phi(\boldsymbol{x}) + 4\pi^2 k^2 \phi(\boldsymbol{x}) = 0 \tag{3}$$

  where $k \in \mathbb{C}$ is the **wave number**
- **Boundary condition** in an enclosure V: $\forall \boldsymbol{x} \in \partial V$,



$$\frac{\partial}{\partial \boldsymbol{n}}\phi(\boldsymbol{x}) = 0. \tag{4}$$

- The set of **wave numbers** k_n and unit **eigenfunctions** $\phi_n(\boldsymbol{x})$ solutions to (3) and (4) is discrete, indexed by an integer n
- Both the wave numbers k_n and eigenfunctions $\phi_n(\boldsymbol{x})$ are **real**
- The set $\{\phi_n(\boldsymbol{x})\}_{n\in\mathbb{N}}$ forms a **Hilbert** basis of $L^2(V)$

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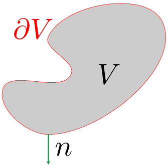

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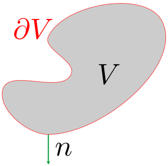

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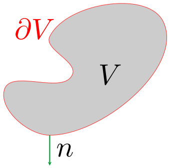

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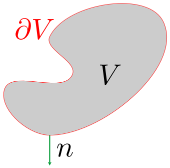

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- Both the wave numbers k_n and eigenfunctions $\phi_n(\boldsymbol{x})$ are **real**
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The Helmholtz equation: rigid walls case

- Rigid walls: $\widehat{\beta}(\boldsymbol{x}, f) = 0$ $(R(f) = 1)$
- **Helmholtz equation**: $\forall \boldsymbol{x} \in \mathbb{R}^3$,

$$\Delta\phi(\boldsymbol{x}) + 4\pi^2 k^2 \phi(\boldsymbol{x}) = 0 \qquad (3)$$

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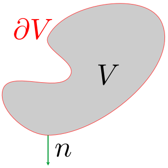

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Response to a point source in an enclosure

- Presence of a point source at $\boldsymbol{x}_0 \in V$ emitting at $t = 0$.
- The wave equation (1) becomes inhomogeneous: $\forall t \in \mathbb{R}$,

$$\Delta p(\boldsymbol{x}, t) - \frac{1}{c^2}\frac{\partial^2 p(\boldsymbol{x}, t)}{\partial t^2} = -Q\rho_0 \delta(\boldsymbol{x} - \boldsymbol{x}_0)\dot{\delta}(t)$$

  - ρ_0 is the static value of the gas density
  - Q is volume velocity of the point source
- Causal solution in an enclosure with rigid walls:

$$p(\boldsymbol{x}, t) = Q\rho_0 c^2 H(t) \sum_{n \in \mathbb{N}} \phi_n(\boldsymbol{x}_0)\phi_n(\boldsymbol{x}) \cos(2\pi c k_n t)$$

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Room impulse response (rigid walls)

- The room impulse response (RIR) h, between a punctual source position $\boldsymbol{x}_0 \in V$ and a punctual receiver position $\boldsymbol{x} \in V$, is defined as the Green's function of (1), which is the unique causal solution of the following inhomogeneous wave equation:

$$\forall t \in \mathbb{R}, \ \Delta h(\boldsymbol{x}, t) - \frac{1}{c^2} \frac{\partial^2 h(\boldsymbol{x}, t)}{\partial t^2} = -\delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t) \qquad (5)$$

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Example: eigenmodes of the shoebox room

- Shoebox room: $V = [0, L_x] \times [0, L_y] \times [0, L_z]$

$$\phi_{n_x,n_y,n_z}(\boldsymbol{x}) = \frac{2^{\delta_0(n_x)+\delta_0(n_y)+\delta_0(n_z)}}{8\sqrt{|V|}} \sum_{s_x,s_y,s_z=\pm 1} e^{2\imath\pi \boldsymbol{k}^\top_{s_x n_x, s_y n_y, s_z n_z} \boldsymbol{x}}$$

  - wave vectors $\boldsymbol{k}_{n_x,n_y,n_z} = \left[\frac{n_x}{2L_x}, \frac{n_y}{2L_y}, \frac{n_z}{2L_z}\right]$ for $(n_x, n_y, n_z) \in \mathbb{Z}^3$
  - wave numbers $k_{n_x,n_y,n_z} = \frac{1}{2}\sqrt{\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}}$

Eigenmodes in 1D

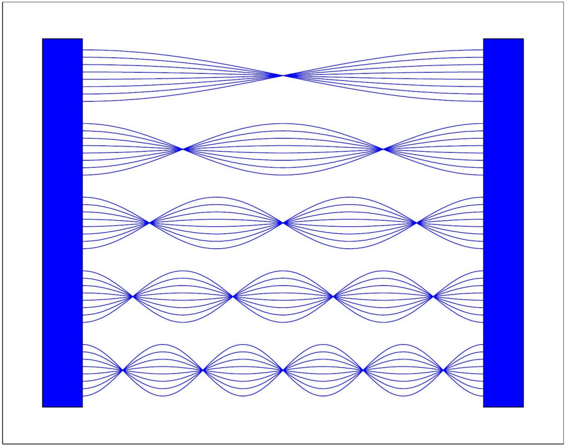

Figure: The first 5 modes of a tube closed at the ends

Eigenmodes in 2D

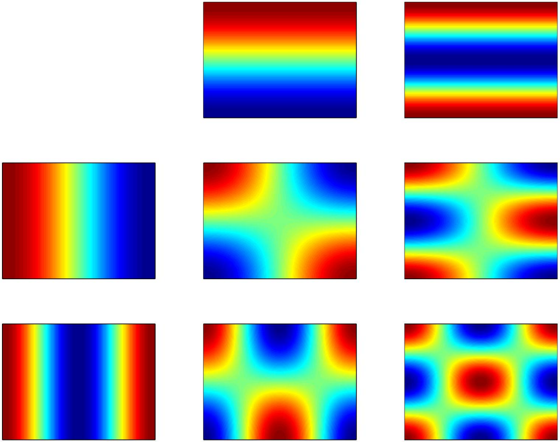

Figure: First modes of a rectangular room

The Helmholtz equation: non-rigid walls

- **Helmholtz equation**: $\forall \boldsymbol{x} \in \mathbb{R}^3$,

$$\Delta\varphi(\boldsymbol{x}, f) + 4\pi^2\kappa(f)^2\varphi(\boldsymbol{x}, f) = 0 \tag{6}$$

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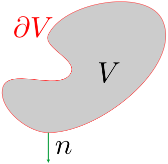

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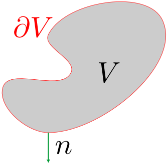

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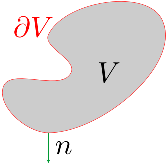

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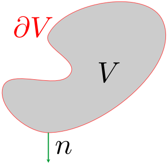

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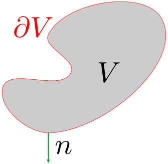

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Room impulse response (non-rigid walls)

- The room impulse response h is still defined as the unique causal solution to (5), with the boundary condition (2).
- Now this solution is obtained by inverse Fourier transform:

$$\forall \boldsymbol{x} \in V, \forall t \in \mathbb{R}, h(\boldsymbol{x}, t) = \int_{f \in \mathbb{R}} \left(\sum_{n \in \mathbb{N}} \frac{\varphi_n(\boldsymbol{x}_0, f)\, \varphi_n(\boldsymbol{x}, f)}{4\pi^2 (\frac{f^2}{c^2} - \kappa_n(f)^2)} \right) e^{2\imath\pi f t} df \tag{8}$$

- The integral can be calculated with the residue theorem:
  - Suppose that $\forall n \in \mathbb{N}$, a root of the equation $f = c\kappa_n(f)$ is $\nu_n = f_n + \imath\gamma_n$ with $f_n, \gamma_n > 0$.
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Example : eigenmodes of the shoebox room

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- Let $\varphi(\boldsymbol{x}, f) = \prod_{p=1}^{3} \varphi_p(x_p)$. Then $\forall p \in \{1, 2, 3\}$, we have

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Example : eigenmodes of the shoebox room

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Part IV

Geometrical acoustics

Ray-tracing in geometric acoustics

- Representing the reflection of waves on a plane surface as an optical reflection on a mirror.
- Holds only in the rigid walls case
- This leads to the concept of image source:

https://interactiveacoustics.info/html/GA_IS.html

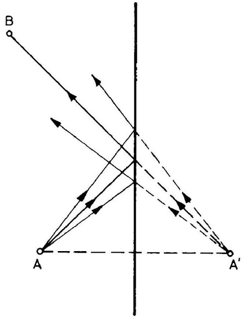


Figure: Construction of an image source

Diffuse reflections

- If the rays are scattered, the concept of mirror source no longer applies, it is necessary to take into account the angles of incidence in the ray tracing.

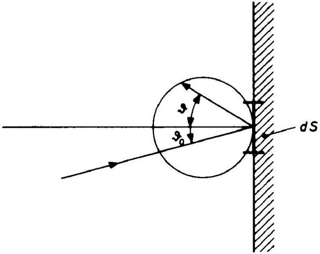
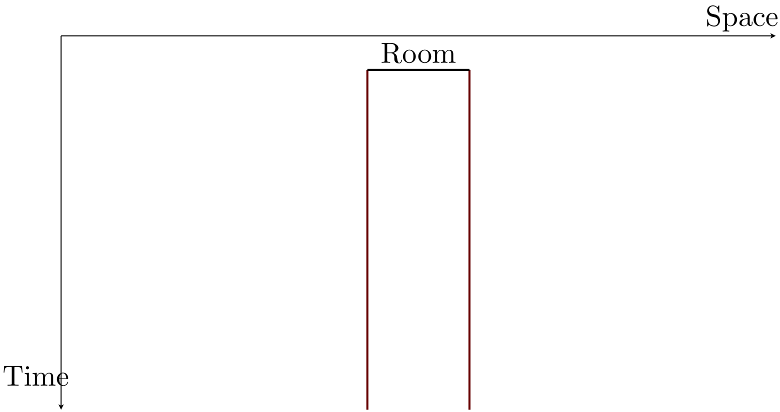


Figure: Ideal scattering model during a reflection

Geometrical acoustics in 1D



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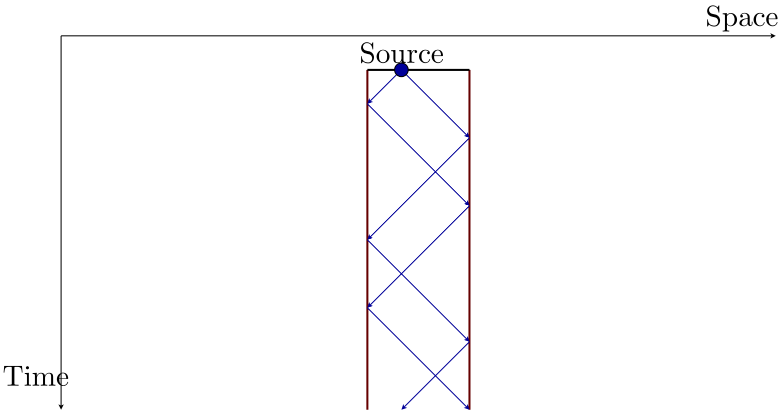

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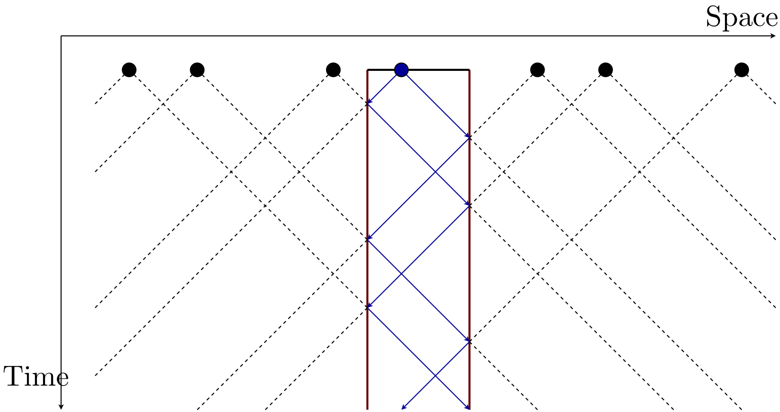

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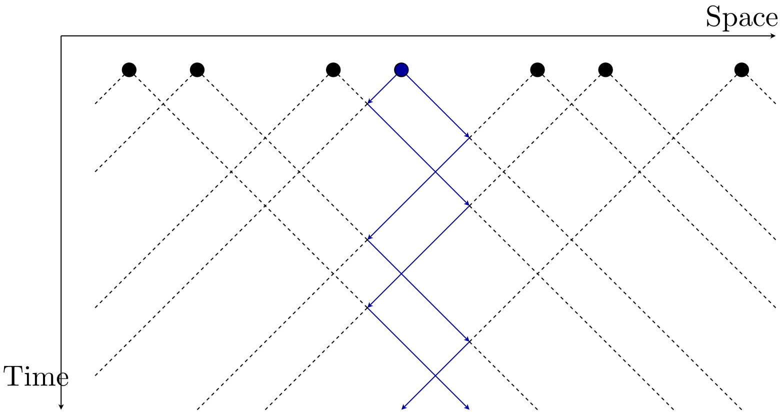

Geometrical acoustics in 2D and 3D

- The concept of image sources still applies when angles between edges or faces are of the form $\theta = \frac{\pi}{n}$ with $n \in \mathbb{N}^*$
- Cristallographic restriction theorem (in 2D and 3D):
  - Admissible angles for a polygon or polyhedron are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$
- 2D solutions: admissible polygons are the:
  - Rectangle (angles of $\frac{\pi}{2}$ only)
  - Isosceles right triangle (angles of $\frac{\pi}{2}$ and $\frac{\pi}{4}$)
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- 3D solutions: admissible polyhedra are right prisms with bases among 2D solutions
- For all other room shapes, geometrical acoustics is a high-frequency approximation, and involves *occlusion* of the image sources in parts of the room

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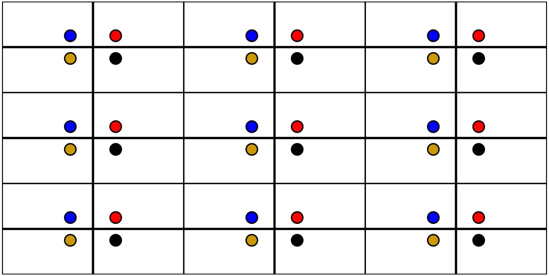
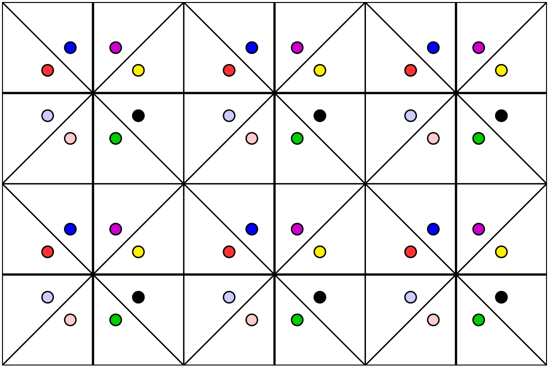
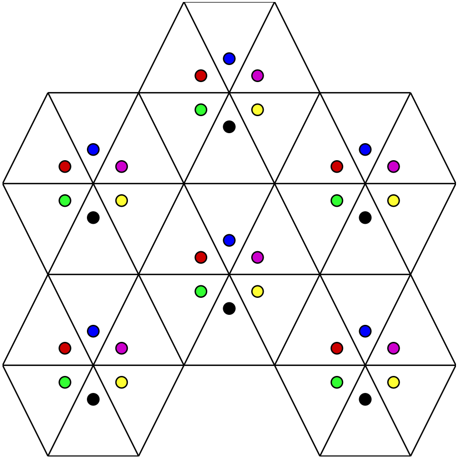
Rectangular room



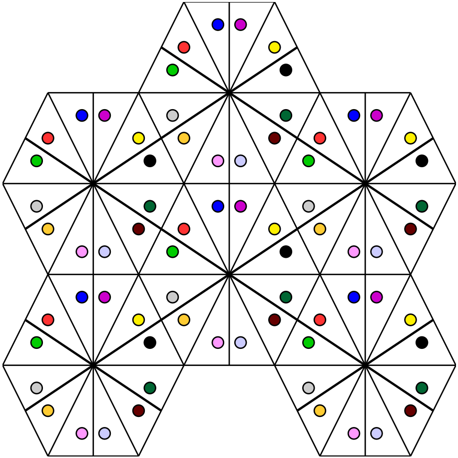
Isosceles right triangle



Equilateral triangle



Hemi-equilateral triangle



Part V

Measurements

Measurement tools

- Direct measurement of reverberation time
  - Excitation by pink noise
  - Analysis by octave or third of octave
- Measurement of the room response by a Dirac pulse
  - Use of a gun firing blanks
- Measurements of the response by pseudo random excitation
  - MLS sequences
  - Golay sequences
- Measurements with modulated sinusoidal excitation
  - Equivalent to pulse synthesis (radar)
  - Logarithmic modulation, nonlinear kernels

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Room response (1)

Note the exponential decrease of the response and its duration

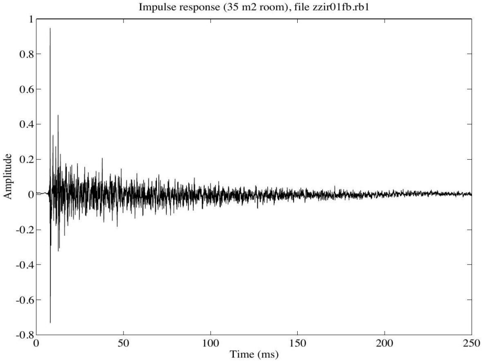

Figure: Measured response in a room of 35 m^2

Room response (2)

Zoom on the early part of the same response

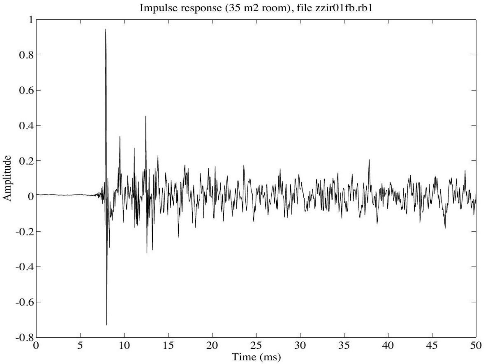

Figure: First milliseconds of the same response (room of 35 m^2)

Energy decay curve

- EDC = Energy Decay Curve
- Excitation of a loudspeaker by white noise until $t = 0$:

$$e(t) \sim \mathcal{N}\left(0, \sigma^2\right) \text{ for } t \leq 0 \text{ and } e(t) = 0 \text{ for } t > 0$$

- Calculation of the energy of the signal captured on a microphone:

$$C(t) = \mathbb{E}\left[\left|\int_{\tau=0}^{\infty} h(\tau)e(t-\tau)d\tau\right|^2\right] = \sigma^2 \int_{\tau=t}^{\infty} h(\tau)^2 d\tau$$

- Reverberation time: measures the time T60 that it takes for the reverberation to decrease by 60dB:

$$-60 = 10\log\left(e^{-2\bar{\gamma}T_{60}\ \mathrm{dB}}\right) \qquad (\bar{\gamma} = \text{average damping})$$

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Characterization of the response

From the energy decay curve:

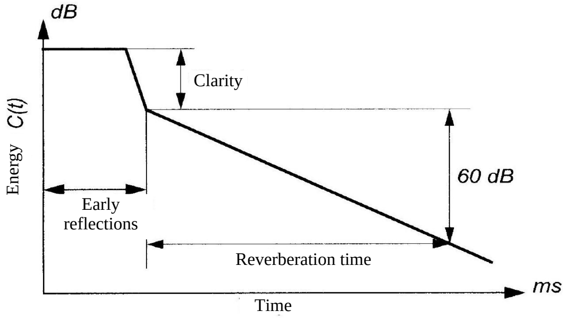

Figure: Parameters extracted from the energy decay curve (EDC)

Part VI

Stochastic reverberation models

Time domain

Schroeder (1962) and Moorer (1979): the RIR is modeled as

$$h(t) = b(t)e^{-at}H(t)$$

- $b(t)$ is a centered white Gaussian process
- $a > 0$ is related to the reverberation time: $\mathrm{RT}_{60} = \frac{3\ln(10)}{a}$
- $H(t)$ is the Heaviside function: $H(t)=1\ \forall t \geq 0$, $H(t)=0\ \forall t<0$

Manfred R. Schroeder. Frequency-correlation functions of frequency responses in rooms.

The Journal of the Acoustical Society of America, 34(12):1819–1823, 1962

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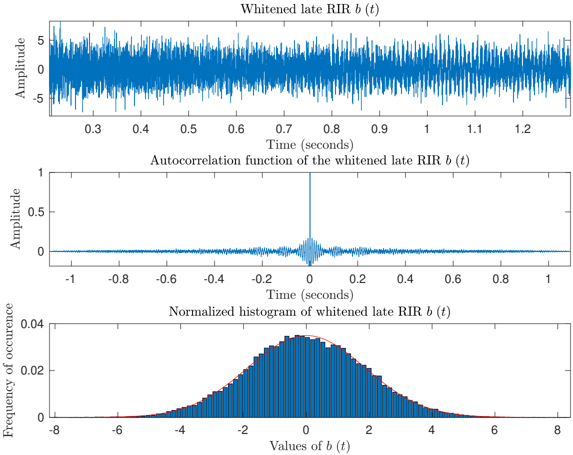
Validation of time model (classroom)



Frequency domain

- Moorer (1979): the RIR is modeled as

$$h(t) = b(t)e^{-at}H(t)$$

- The RFR is the Fourier transform of the RIR:

$$\widehat{h}(f) = \int_{t\in\mathbb{R}} h(t)e^{-2\imath\pi ft}dt$$

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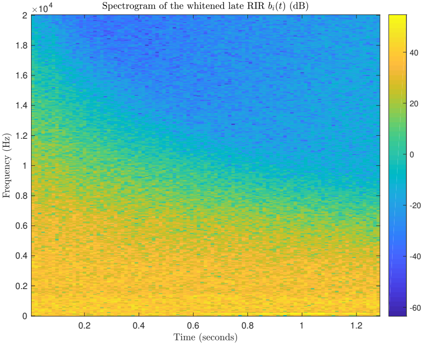

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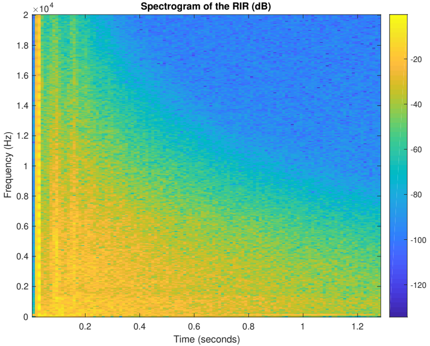

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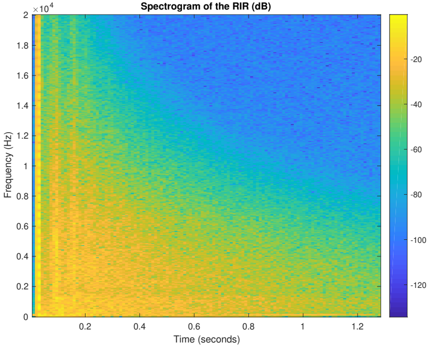

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- Correlation at freq. f between sensors (Cook et al., 1955):

$$\operatorname{corr}\left[\widehat{h}(\boldsymbol{x}_1, f), \widehat{h}(\boldsymbol{x}_2, f)\right] = \operatorname{sinc}\left(\frac{2\pi f D}{c}\right)$$

  - $D = \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2$ is the distance between sensors
  - c is the speed of sound in the air (≈ 343 m/s)
- Assumption: diffuse acoustic field
  - Sum of plane waves
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Part VII

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  - Energy decay curve, clarity
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