

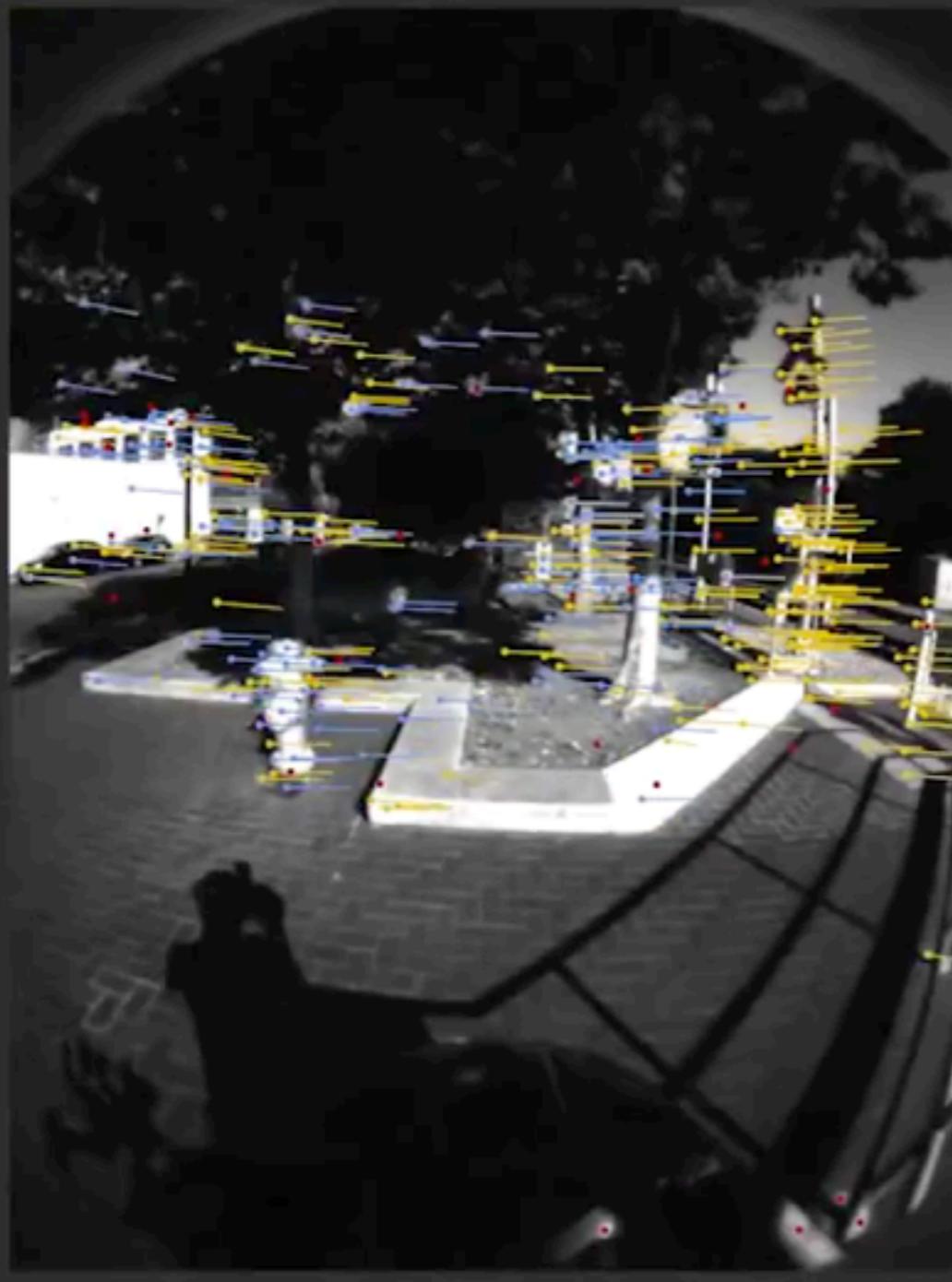


# Representing Robot Pose

The First-Order View

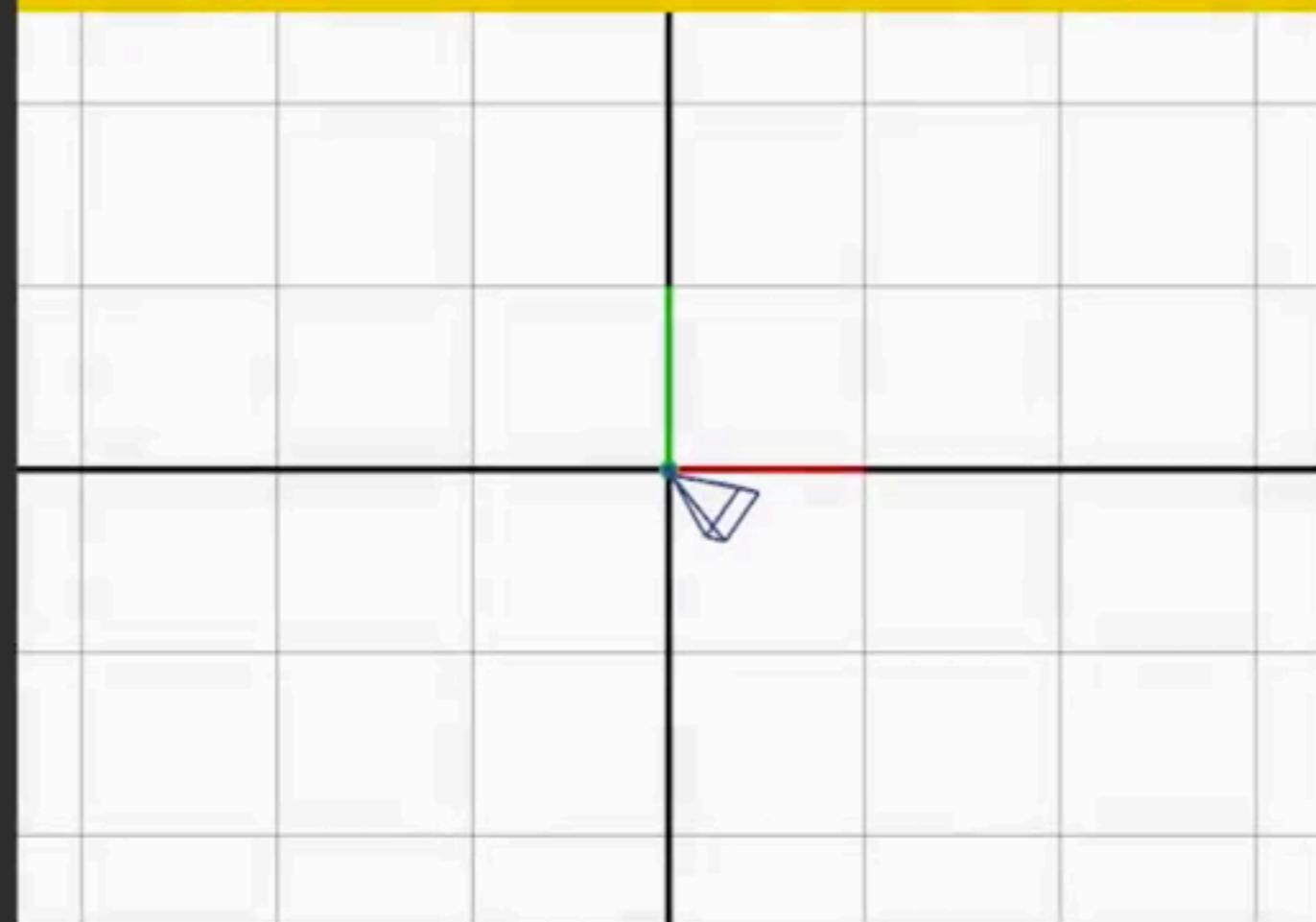
Paul Furgale, Hannes Sommer





Top-down graph plot

grid size: 1 m



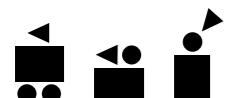
# **A Brief Sketch of Nonlinear Least Squares**

# Nonlinear Least Squares

Goal: estimate  $\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} (J(\mathbf{x}))$

Sum of  
Squared Errors:

$$J(\mathbf{x}) := \frac{1}{2} \sum_{n=1}^N \mathbf{e}_n(\mathbf{x})^T \mathbf{R}_n^{-1} \mathbf{e}_n(\mathbf{x})$$



# Nonlinear Least Squares

Goal: estimate  $\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} (J(\mathbf{x}))$

$$\mathbf{e}(\mathbf{x}) := \begin{bmatrix} \mathbf{e}_1(\mathbf{x}) \\ \vdots \\ \mathbf{e}_N(\mathbf{x}) \end{bmatrix}, \quad \mathbf{R} := \operatorname{diag}\{\mathbf{R}_1, \dots, \mathbf{R}_N\},$$

Block form:

$$J(\mathbf{x}) = \frac{1}{2} \mathbf{e}(\mathbf{x})^T \mathbf{R}^{-1} \mathbf{e}(\mathbf{x}).$$

# Nonlinear Least Squares

Goal: estimate  $\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} (J(\mathbf{x}))$

Current Guess

Perturbation

$$J(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x) = \frac{1}{2} \mathbf{e}(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x)^T \mathbf{R}^{-1} \mathbf{e}(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x)$$

Linearized:



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$$\approx \frac{1}{2} \left( \mathbf{e}(\bar{\mathbf{x}}) + \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}} \boldsymbol{\epsilon}_x \right)^T \mathbf{R}^{-1} \left( \mathbf{e}(\bar{\mathbf{x}}) + \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}} \boldsymbol{\epsilon}_x \right)$$

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Current Guess      Perturbation

$$\begin{aligned} J(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x) &= \frac{1}{2} \mathbf{e}(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x)^T \mathbf{R}^{-1} \mathbf{e}(\bar{\mathbf{x}} + \boldsymbol{\epsilon}_x) \\ &\approx \frac{1}{2} \left( \mathbf{e}(\bar{\mathbf{x}}) + \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}} \boldsymbol{\epsilon}_x \right)^T \mathbf{R}^{-1} \left( \mathbf{e}(\bar{\mathbf{x}}) + \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}} \boldsymbol{\epsilon}_x \right) \end{aligned}$$

Linearized:       $\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \mathbf{E} := \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}}, \quad \mathbf{e}(\boldsymbol{\epsilon}_x) := \bar{\mathbf{e}} + \mathbf{E}\boldsymbol{\epsilon}_x.$

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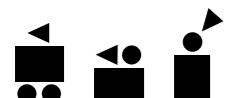
$$J(\boldsymbol{\epsilon}_x) = \frac{1}{2} \mathbf{e}(\boldsymbol{\epsilon}_x)^T \mathbf{R}^{-1} \mathbf{e}(\boldsymbol{\epsilon}_x),$$

# Nonlinear Least Squares

Goal: estimate  $\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} (J(\mathbf{x}))$

An intermediate goal:  $\boldsymbol{\epsilon}_x^* = \underset{\boldsymbol{\epsilon}_x}{\operatorname{argmin}} (J(\boldsymbol{\epsilon}_x))$

$$\frac{\partial J(\boldsymbol{\epsilon}_x)}{\partial \boldsymbol{\epsilon}_x}^T = \mathbf{E}^T \mathbf{R}^{-1} (\bar{\mathbf{e}} + \mathbf{E} \boldsymbol{\epsilon}_x^*) = 0$$

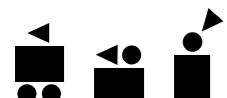


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Solve the  
linear system:

$$\boldsymbol{\epsilon}_x^* = -(\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1} \mathbf{E}^T \mathbf{R}^{-1} \bar{\mathbf{e}}$$

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Update the  
current guess:

$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \boldsymbol{\epsilon}_x^*$$

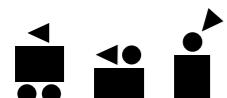


# Nonlinear Least Squares / Gauss Newton

Goal: estimate  $\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} J(\mathbf{x}) = \frac{1}{2}\mathbf{e}(\mathbf{x})^T \mathbf{R}^{-1} \mathbf{e}(\mathbf{x})$ .

- Start with an initial guess for  $\mathbf{x}$
- Iterate
  - Linearize

$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \mathbf{E} := \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}}, \quad \mathbf{e}(\epsilon_x) := \bar{\mathbf{e}} + \mathbf{E} \epsilon_x.$$



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- Solve the linear system
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Recover the covariance:

$$\mathbf{P} := (\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1},$$

$$\mathbf{x} = \mathbf{x}^* + \epsilon_x, \quad \epsilon_x \sim \mathcal{N}(\mathbf{0}, \mathbf{P}).$$

# Calculus

$$\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \Big|_{\bar{\mathbf{x}}}$$

# Linear Algebra

$$\epsilon_x^* = -(\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1} \mathbf{E}^T \mathbf{R}^{-1} \bar{\mathbf{e}}$$

# Probabilities

$$\mathbf{P} := (\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1},$$

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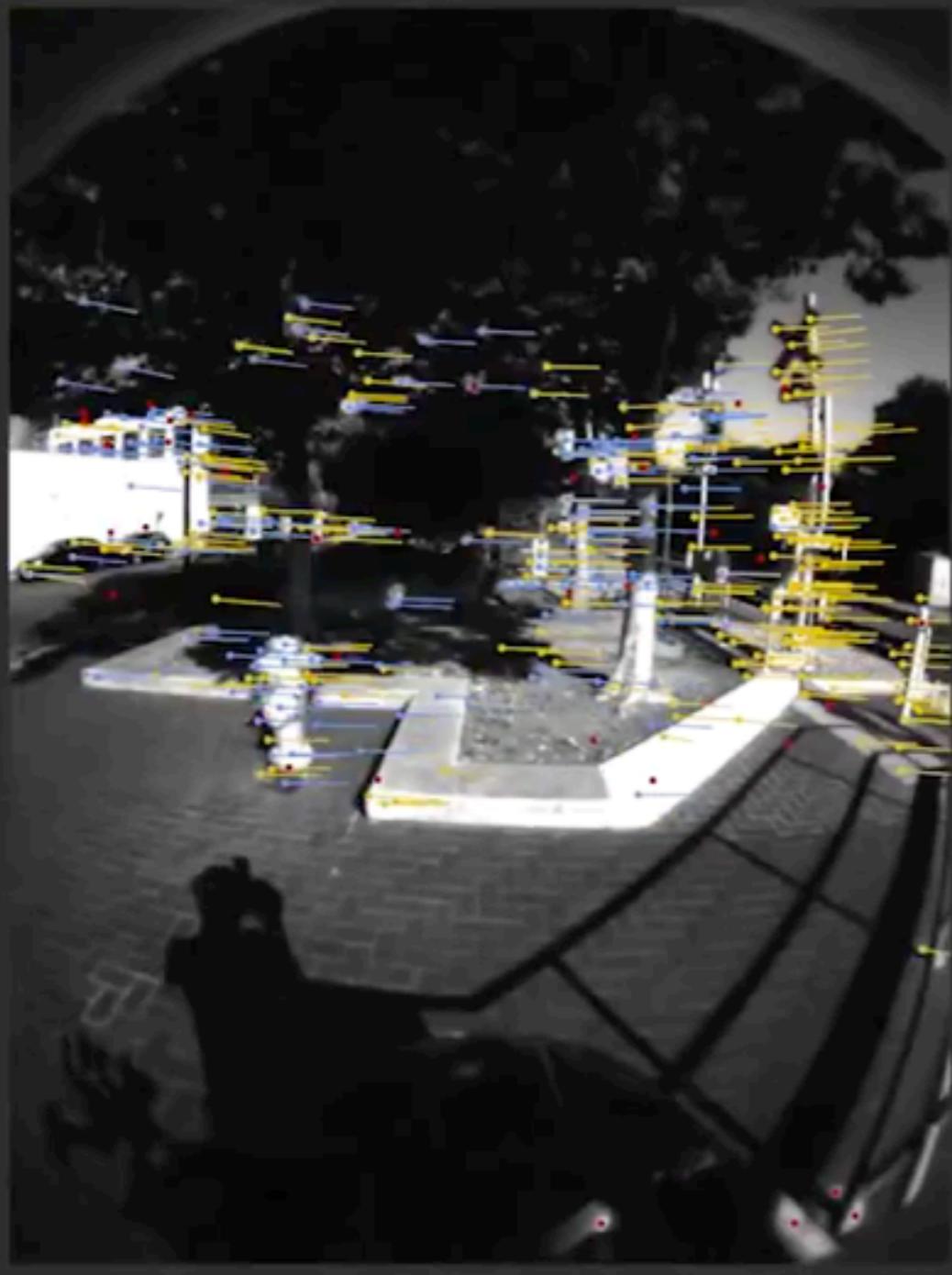
# Basic assumption

$$x \in \mathbb{R}^M$$

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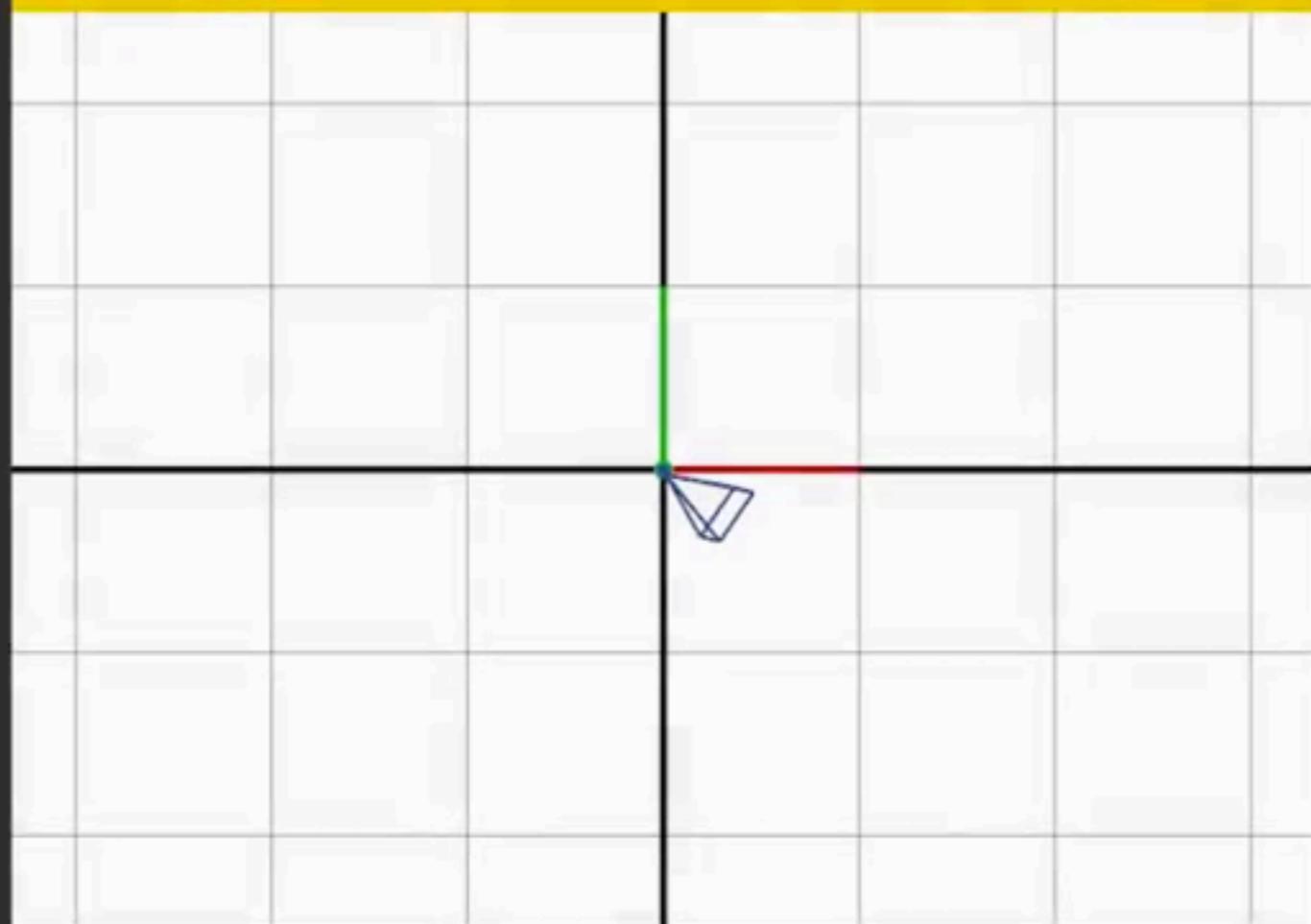
$$x \in \mathbb{R}^M$$

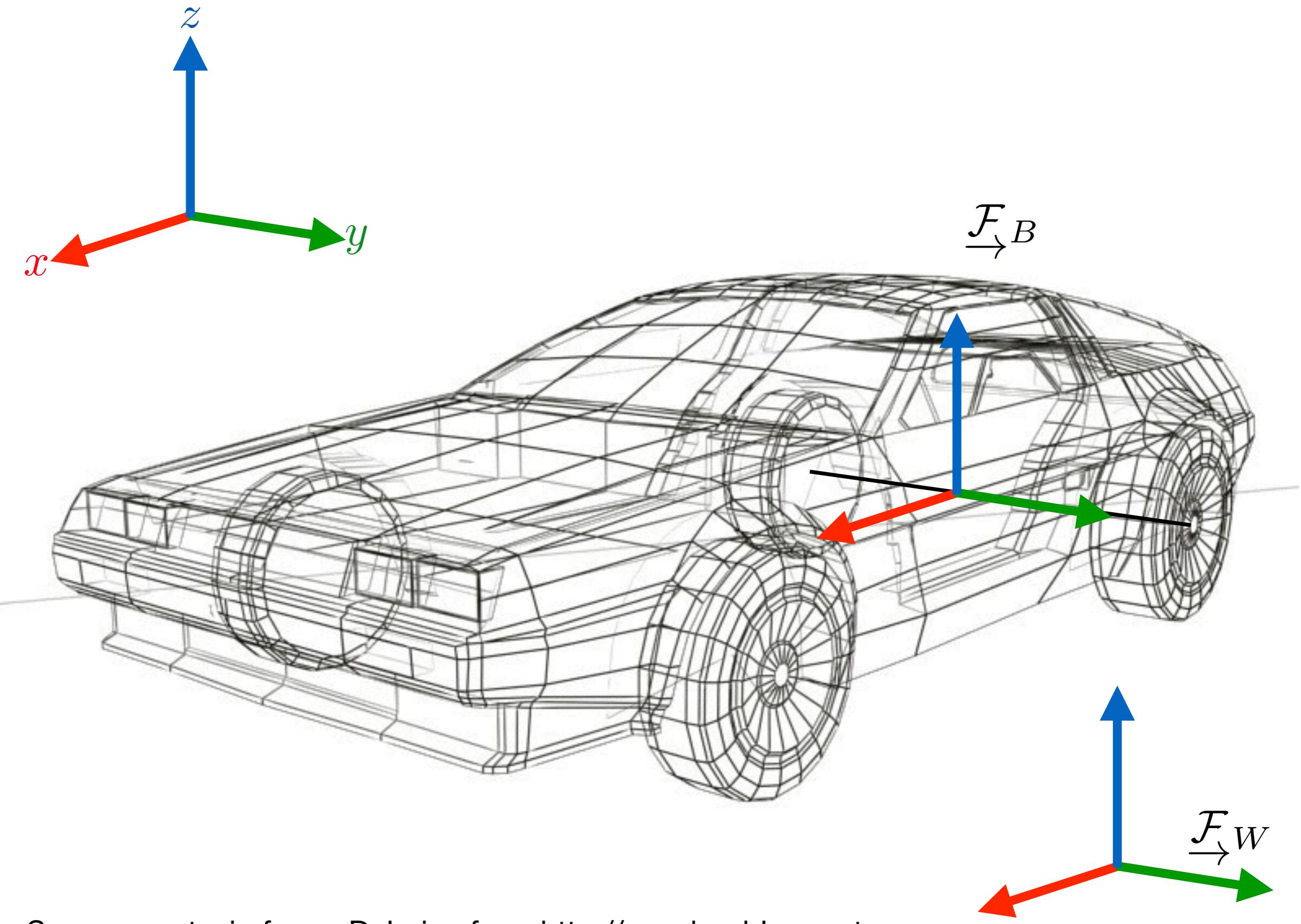
Not always true



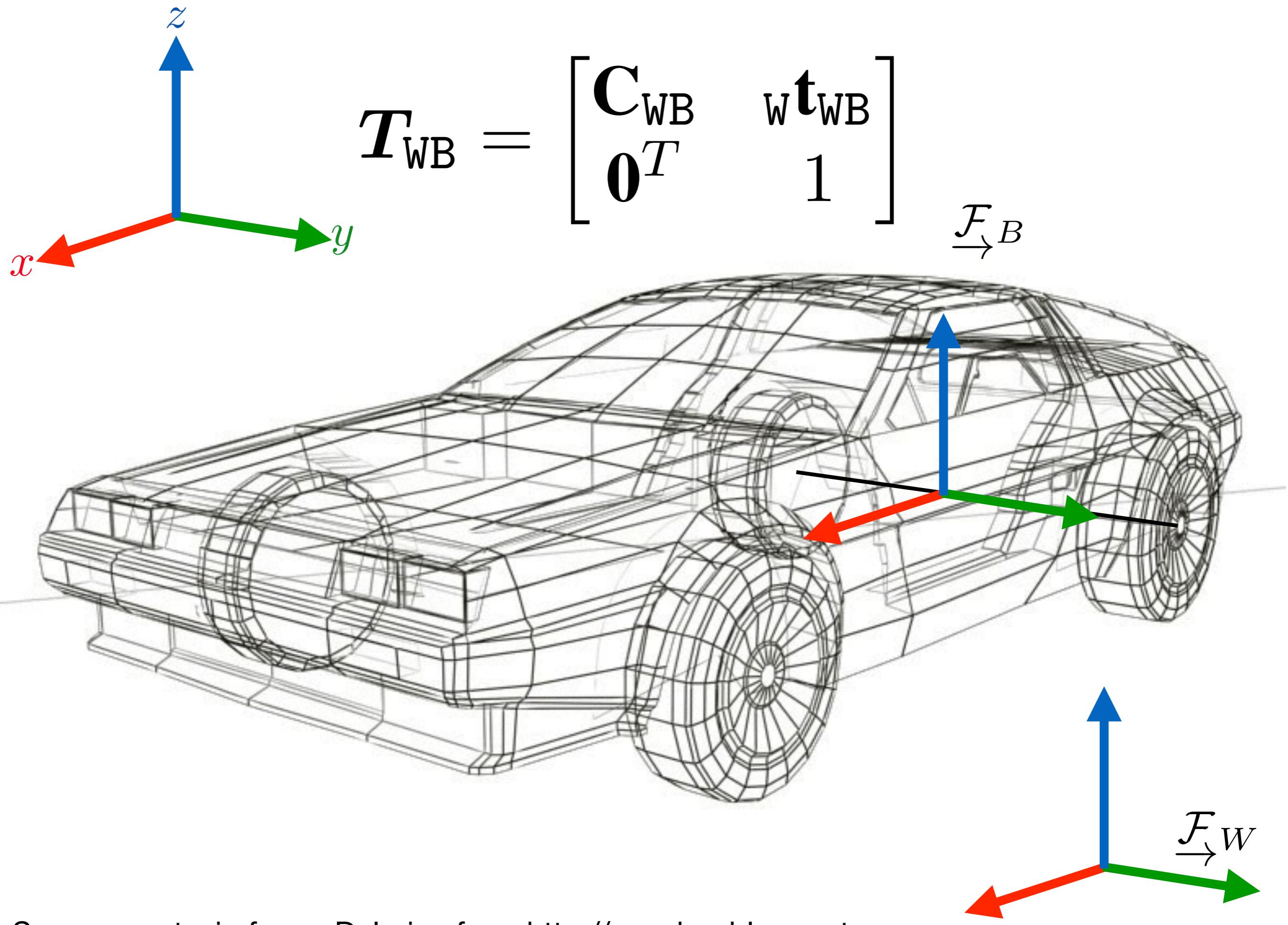
Top-down graph plot

grid size: 1 m

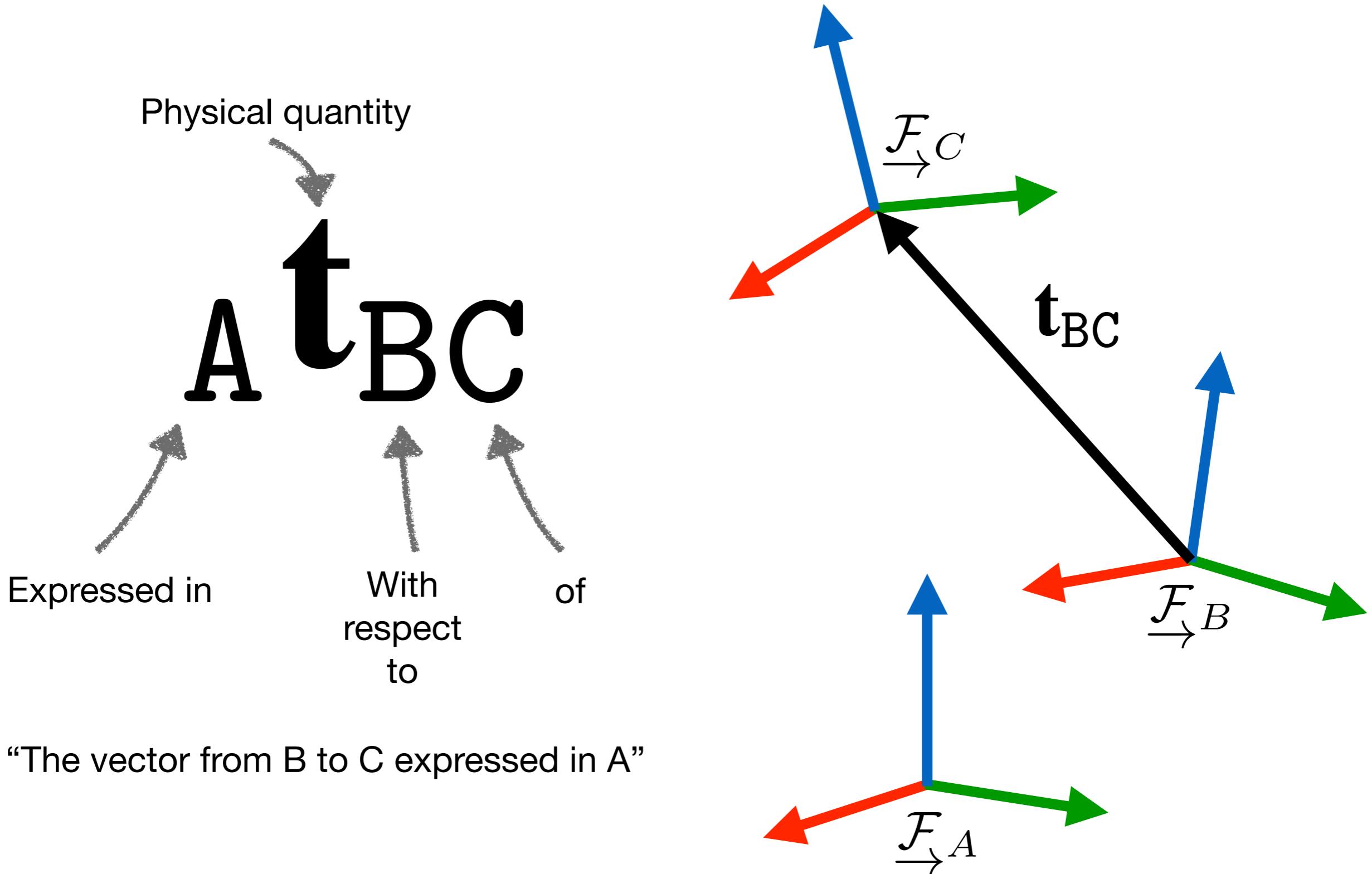




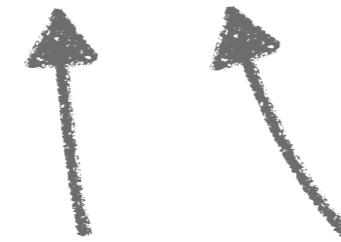
Super sweet wireframe DeLorian from <http://nanahra.blogspot.com>



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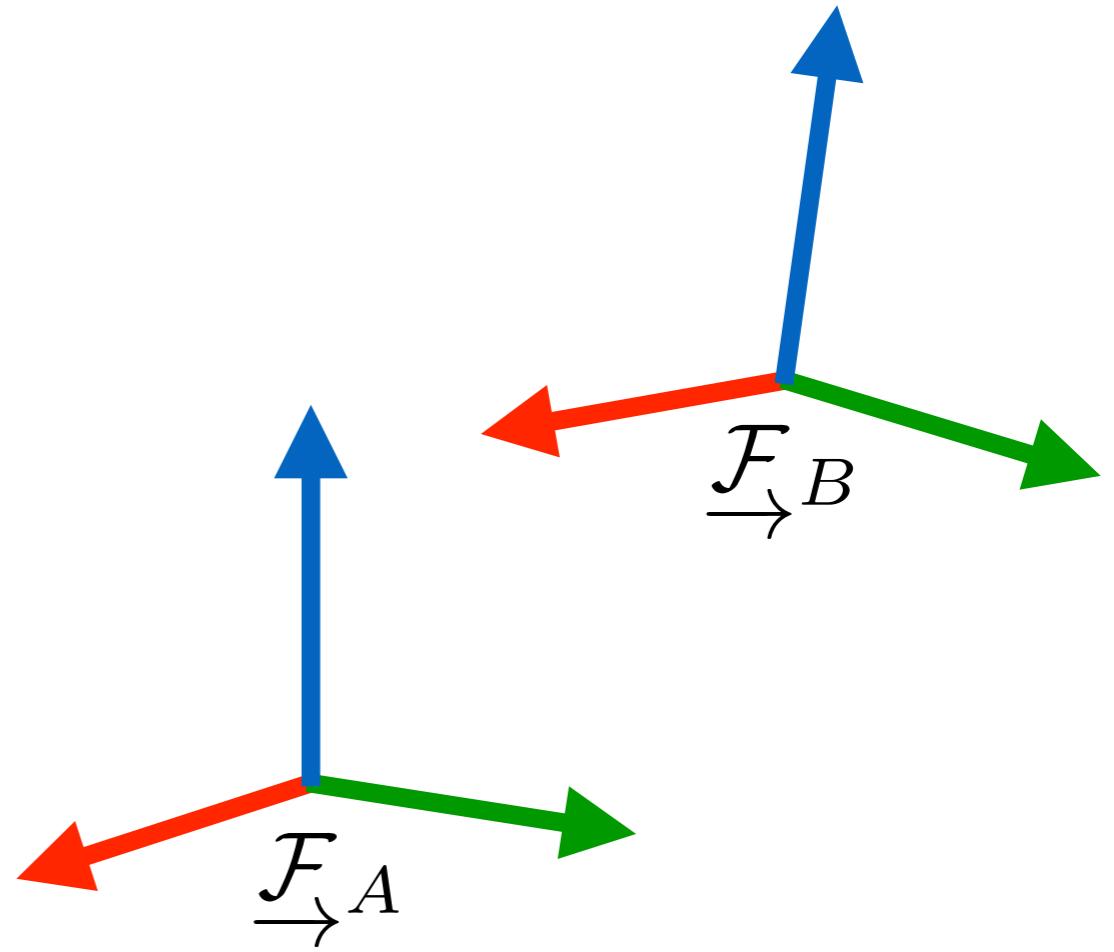


“The vector from B to C expressed in A”

$C_{AB}$ 

into      rotates  
vectors  
from

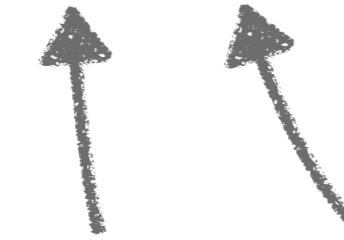
$${}_A V_{BC} = C_{ABB} V_{BC}$$



“The orientation/attitude of B with respect to A”

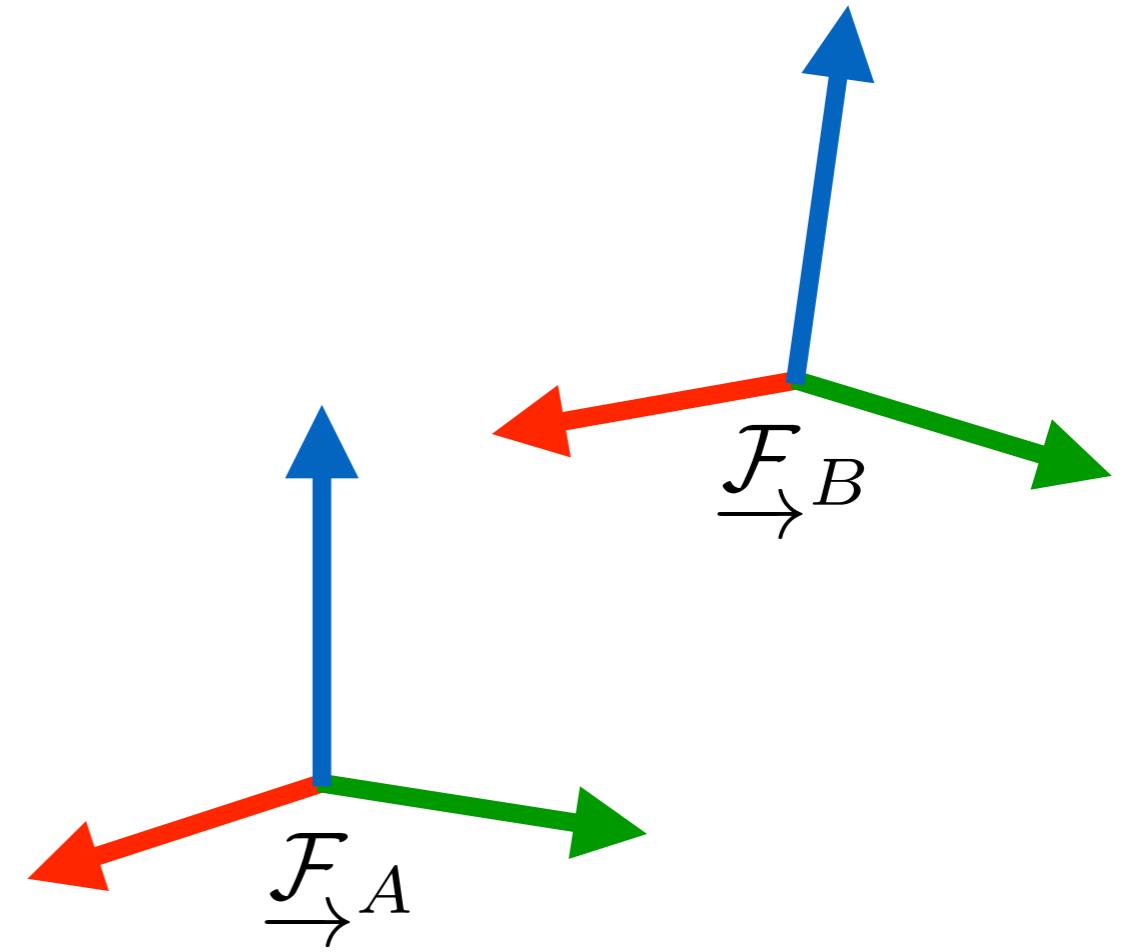
“Rotates vectors from frame B into frame A”

**C<sub>AB</sub>**



With  
respect  
to  
of

$$\mathbf{v}_B = \mathbf{C}_{AB} \mathbf{v}_A$$



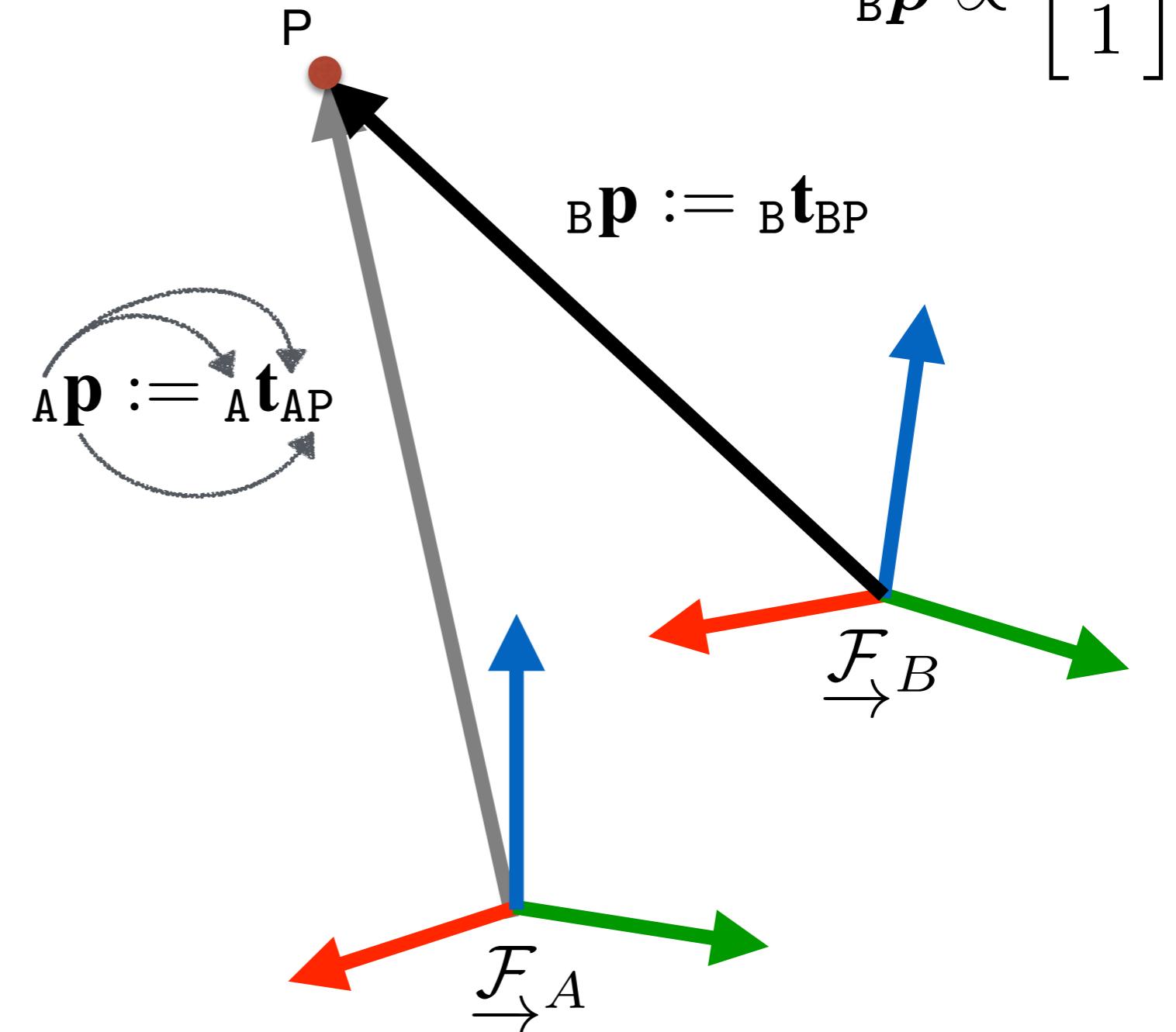
“The pose of B with respect to A”

“Transforms point from frame B into frame A”

$T_{AB}$

into  
transforms  
points  
from

$$A\mathbf{p} = T_{ABB}\mathbf{p}$$



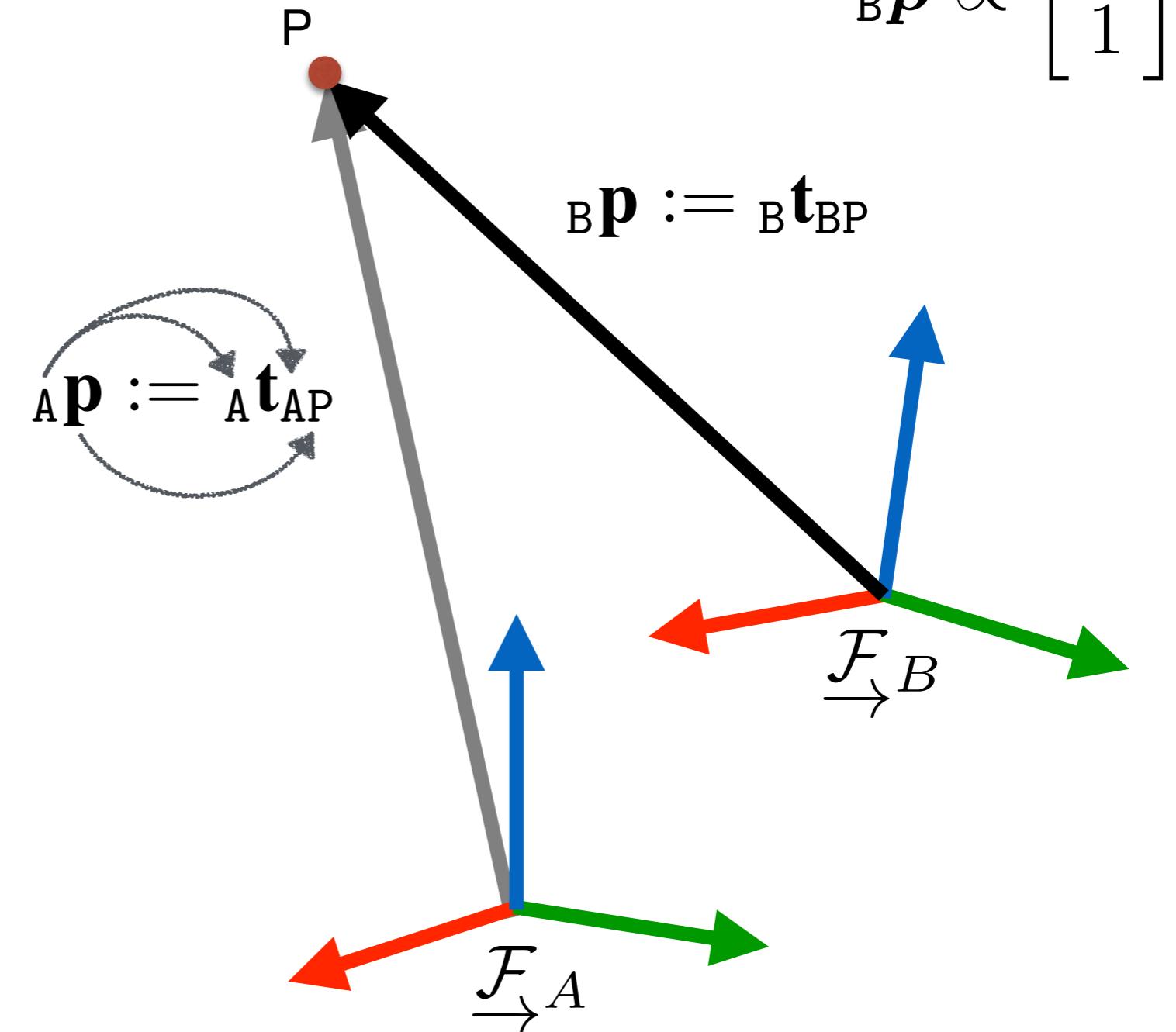
“The pose of B with respect to A”

“Transforms point from frame B into frame A”

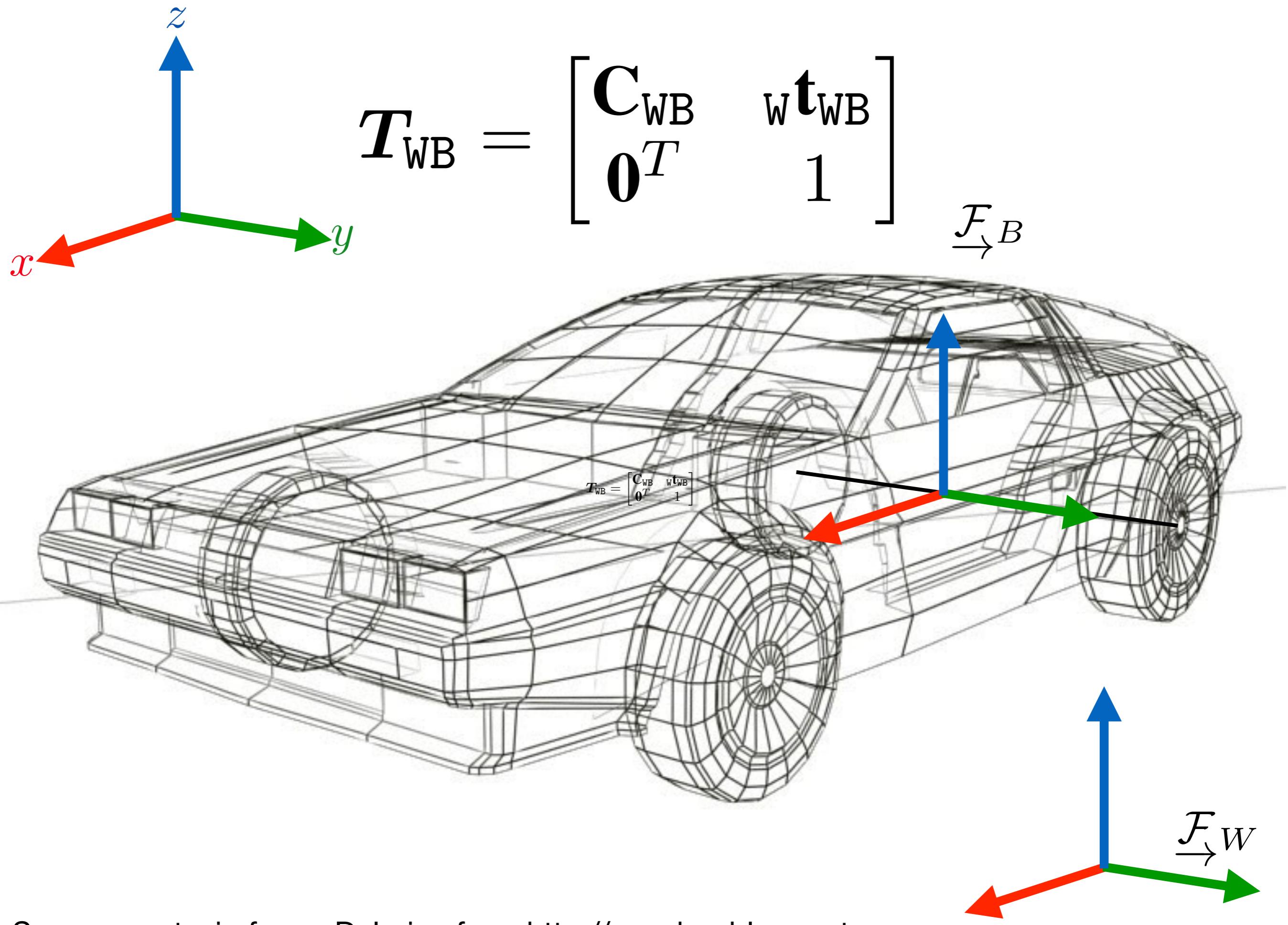
$T_{AB}$

With  
respect  
to  
of

$$A\mathbf{p} = T_{ABB}\mathbf{p}$$



$$B\mathbf{p} \propto \begin{bmatrix} {}^B\mathbf{p} \\ 1 \end{bmatrix}$$



Super sweet wireframe DeLorean from <http://nanahra.blogspot.com>

# Basic assumption

$$x \in \mathbb{R}^M$$

Does not hold for...

$c_{AB}, \Gamma_{AB}$

**These elements exist  
on manifolds**

**$C_{AB}$ ,  $T_{AB}$**

**$SO(3)$**

**$SE(3)$**

The Special Orthogonal Group

The Special Euclidean Group

# Nonlinear Least Squares / Gauss Newton

Goal: estimate  $\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} J(\mathbf{x}) = \frac{1}{2} \mathbf{e}(\mathbf{x})^T \mathbf{R}^{-1} \mathbf{e}(\mathbf{x})$ .

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$$\frac{\partial e(\mathbf{x})}{\partial \mathbf{x}} \Big|_{\bar{\mathbf{x}}}$$

# Linear Algebra

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**$C_{AB}$ ,  $T_{AB}$**

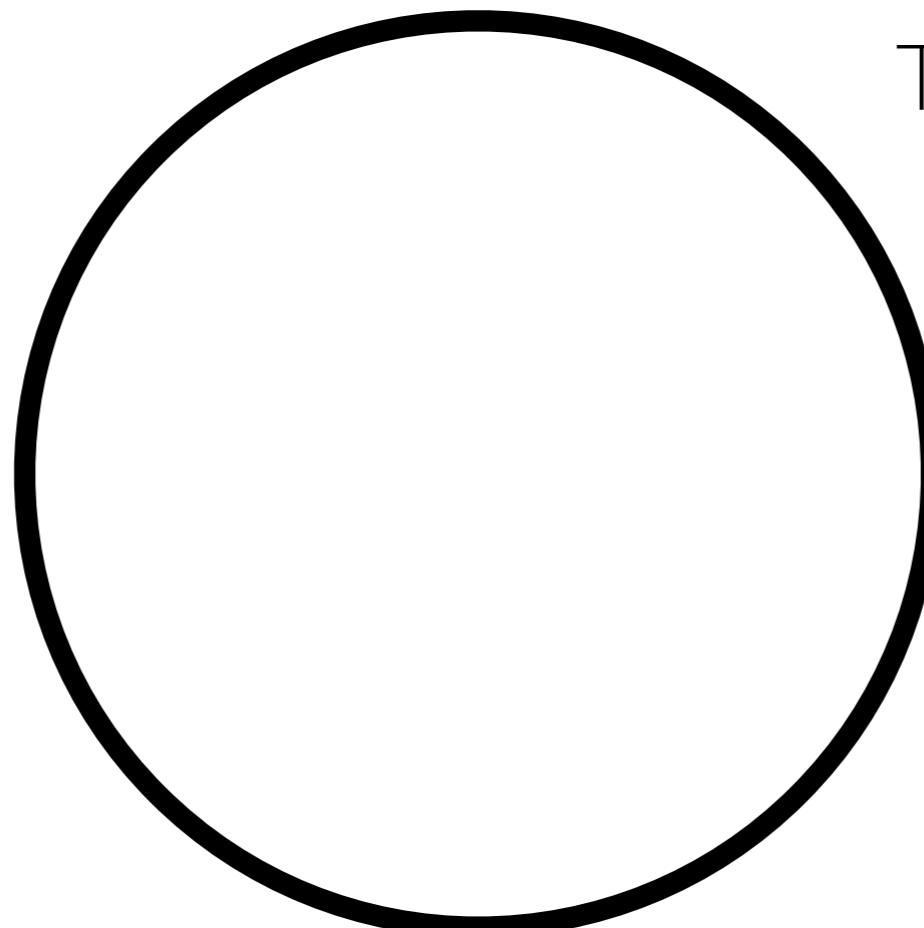
**$SO(3)$**

**$SE(3)$**

The Special Orthogonal Group

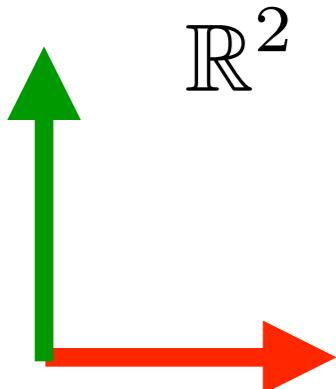
The Special Euclidean Group

# **A Brief Introduction to Manifolds**



The Manifold  
 $S^1$

The Ambient Space



# A Brief Introduction to Manifolds

- Topological Spaces
  - A set of points with the notion of a neighbourhood
  - This is enough to define continuous functions and curves

# A Brief Introduction to Manifolds

- Topological Spaces
- $N$ -Dimensional Topological Manifolds
  - A topological space that locally “looks like”  $\mathbb{R}^N$



# A Brief Introduction to Manifolds

- Topological Spaces
- $N$ -Dimensional Topological Manifolds
- $N$ -Dimensional Differentiable Manifolds
  - Curves have well-defined tangent vectors
  - Real-valued functions have well-defined directional derivatives

# A Brief Introduction to Manifolds

- Topological Spaces
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- $N$ -Dimensional Differentiable Manifolds
  - Curves have well-defined tangent vectors
  - Real-valued functions have well-defined directional derivatives

We can start to do calculus



# A Brief Introduction to Manifolds

- Topological Spaces
- $N$ -Dimensional Topological Manifolds
- $N$ -Dimensional Differentiable Manifolds (DM)



These are all restrictions

# A Brief Introduction to Manifolds

- **$N$ -Dimensional Differentiable Manifolds (DM)**

extension



restriction

- **Riemannian Manifolds**

- Introduce the notion of distance and angles
- We .... shortest connecting curves between points

- **Lie Groups**

- Adds the structure of a group: differentiable multiplication between points

**Both Differentiable Manifolds.**

**Both Lie Groups.**

**C<sub>AB</sub>, T<sub>AB</sub>**

**SO(3)**

The Special Orthogonal Group

**SE(3)**

The Special Euclidean Group

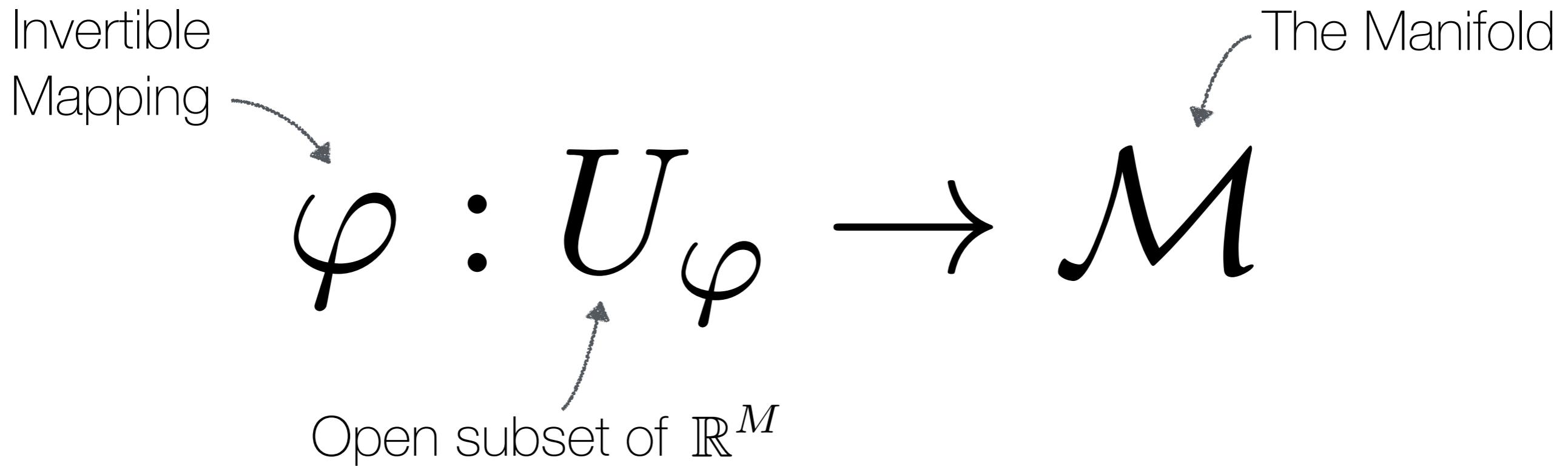
# Choosing an Atlas

- Both SO(3) and SE(3) are Differentiable Manifolds
- This implies that differentiation is possible. How?
- By choosing an *Atlas*

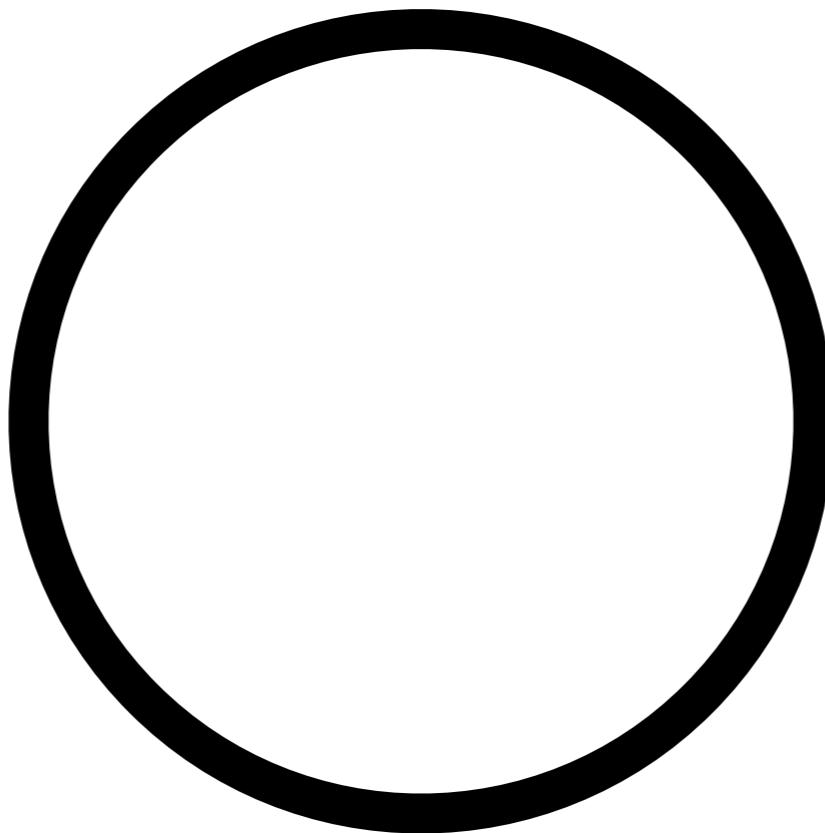
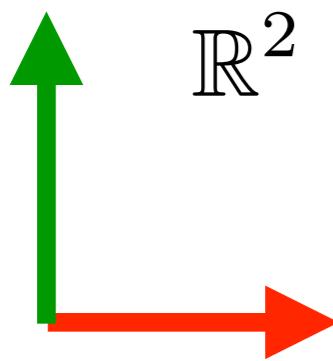


# Choosing an Atlas

- Charts:
  - For an  $M$ -Dimensional DM,  $\mathcal{M}$ , a *chart* is



The  
Ambient  
Space



1-Dimensional  
Manifold  
 $S^1$

Invertible  
Mapping

$\varphi$

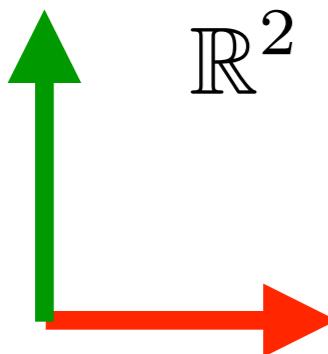
:  $U_\varphi$  →

$\mathcal{M}$

Open subset of  $\mathbb{R}^M$

The Manifold

The  
Ambient  
Space



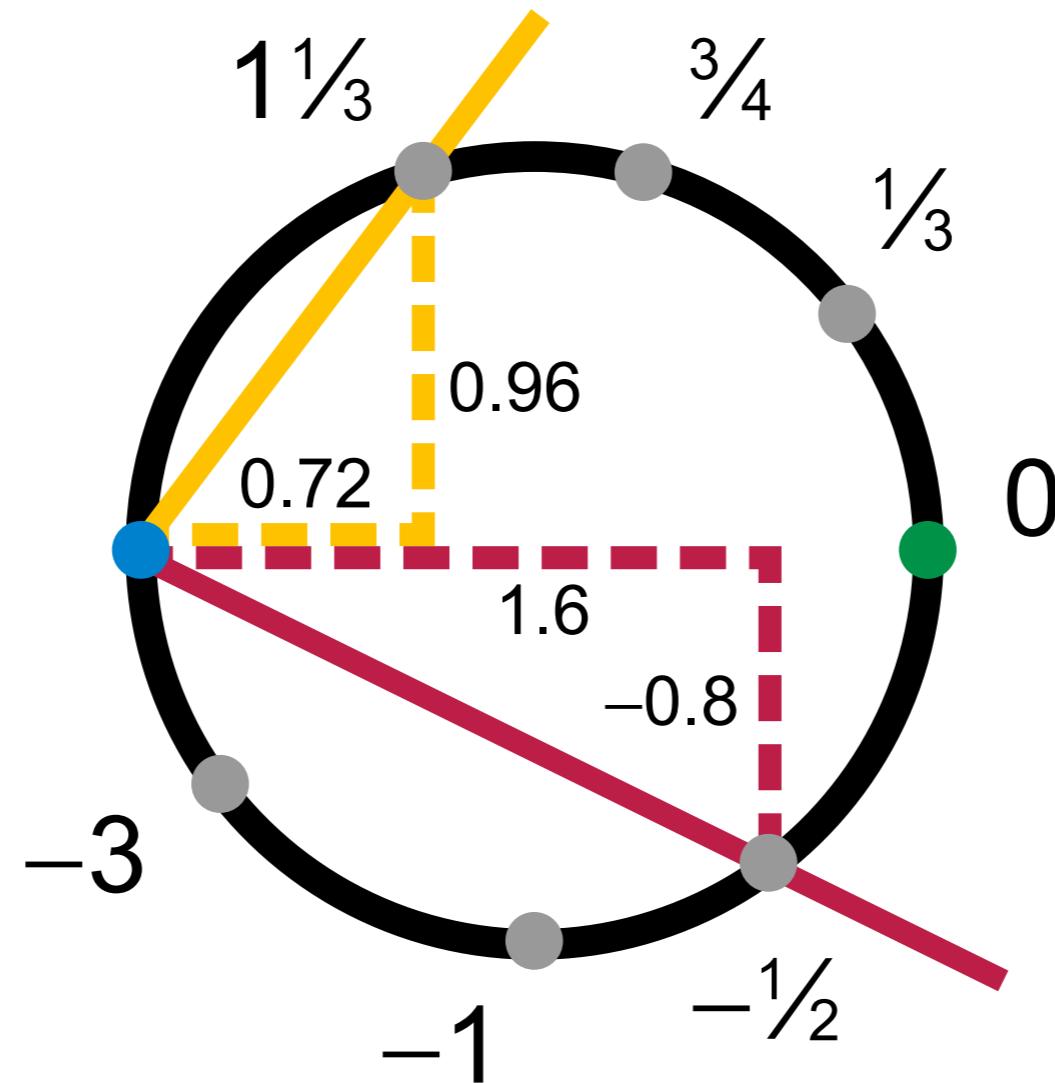
Invertible  
Mapping

$\varphi$

:  $U_\varphi$

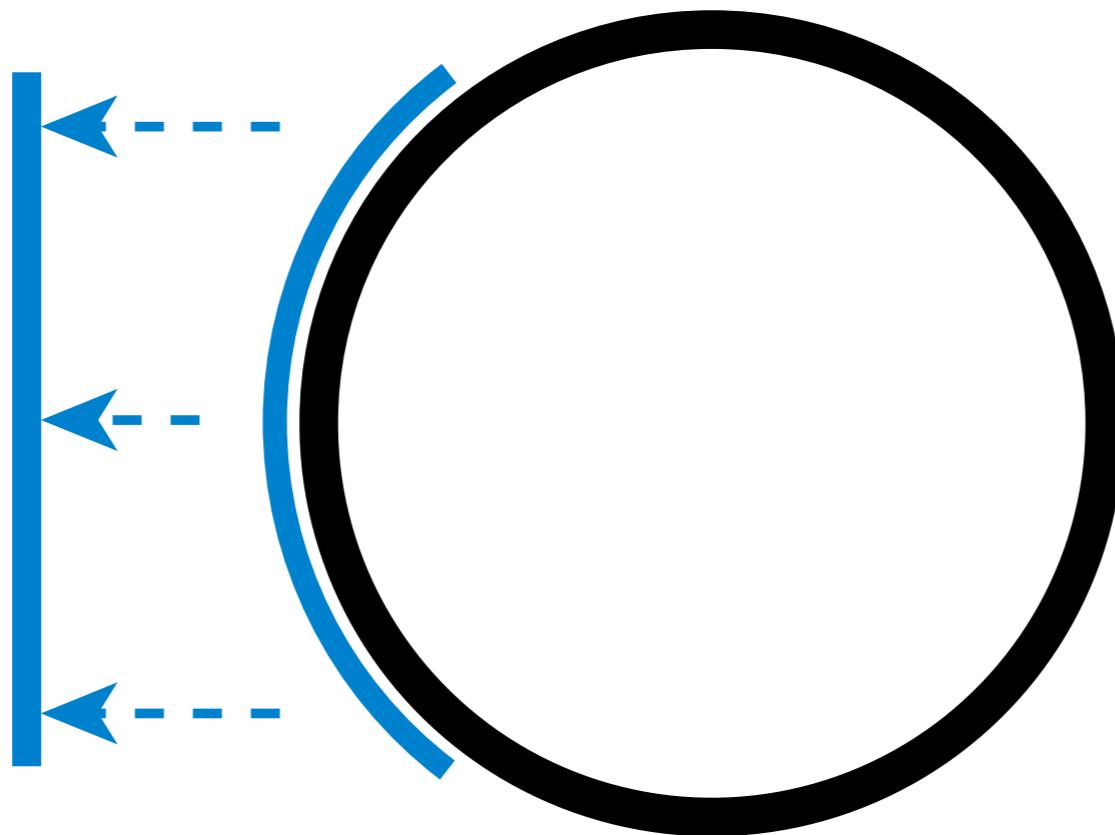
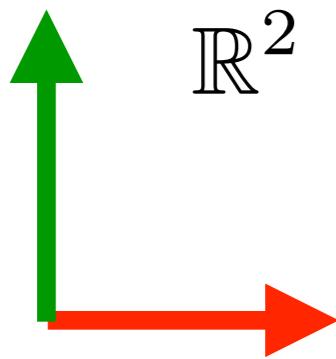
Open subset of  $\mathbb{R}^M$

1-Dimensional  
Manifold  
 $S^1$



The Manifold

The  
Ambient  
Space



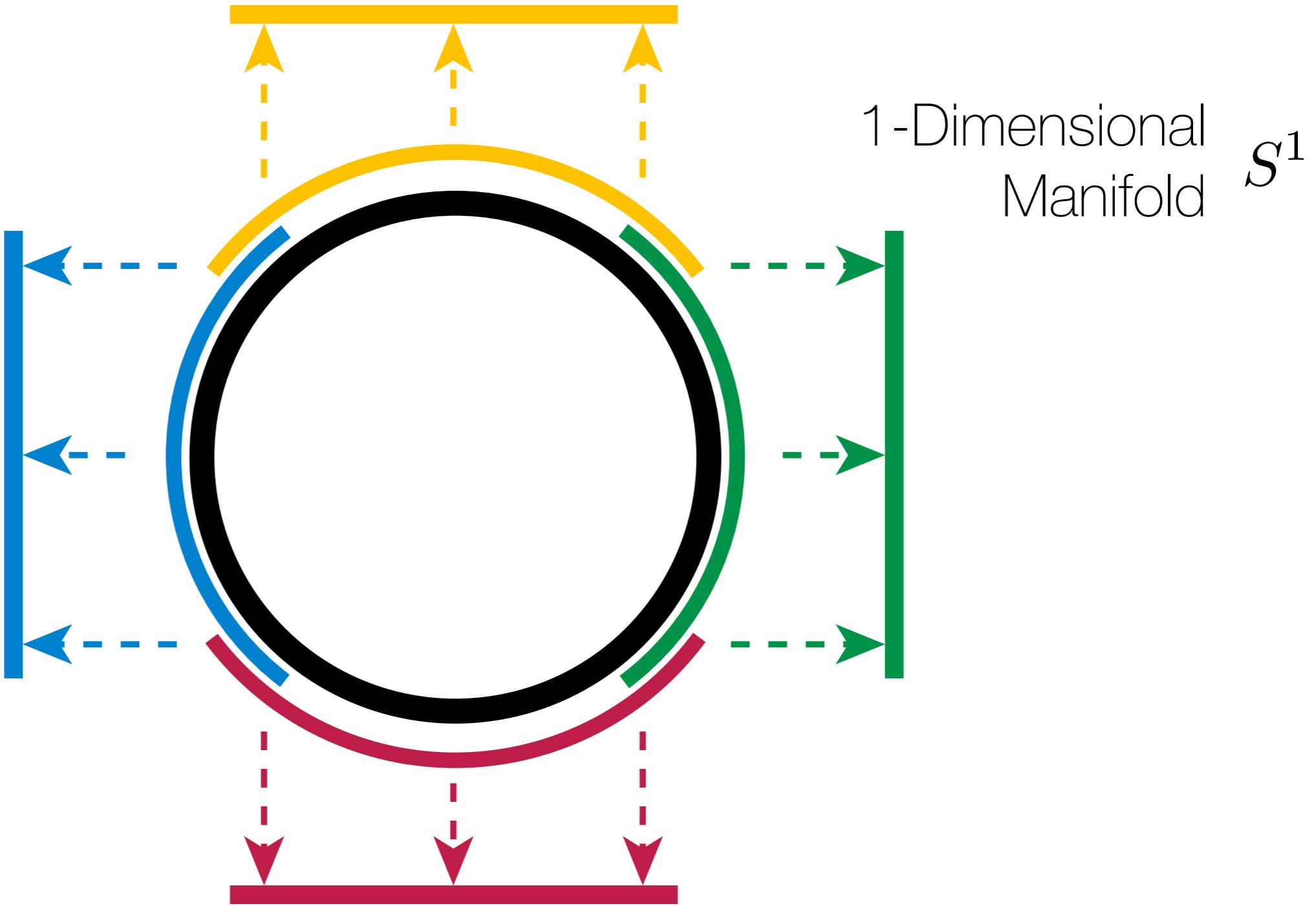
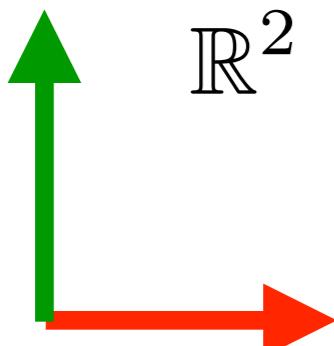
1-Dimensional  
Manifold  
 $S^1$

Invertible  
Mapping

$$\varphi : U_\varphi \rightarrow \mathcal{M}$$

An equation representing an invertible mapping  $\varphi$  from an open subset  $U_\varphi$  of  $\mathbb{R}^M$  to the manifold  $\mathcal{M}$ . The mapping is shown as a curved arrow pointing from  $U_\varphi$  to  $\mathcal{M}$ . The label "Open subset of  $\mathbb{R}^M$ " is positioned below  $U_\varphi$ , and "The Manifold" is positioned above  $\mathcal{M}$ .

The  
Ambient  
Space



- An *Atlas* is a collection of charts such that cover the manifold

# The Atlas Unlocks Calculus

$m \in \mathcal{M}, n \in \mathcal{N}$

$f : \mathcal{M} \rightarrow \mathcal{N}$

$M$ -dimensional       $N$ -dimensional

function

The diagram illustrates the concept of a function mapping between sets. At the top, two variables are shown:  $m \in \mathcal{M}$  and  $n \in \mathcal{N}$ . Above  $m$ , the text " $M$ -dimensional" is written with a curved arrow pointing down towards  $m$ . Above  $n$ , the text " $N$ -dimensional" is written with a curved arrow pointing down towards  $n$ . Below these, the function  $f : \mathcal{M} \rightarrow \mathcal{N}$  is defined, with a curved arrow pointing down towards it from the word "function".

# The Atlas Unlocks Calculus

$m \in \mathcal{M}, n \in \mathcal{N}$

$M$ -dimensional

$N$ -dimensional

function

The diagram illustrates the relationship between variables  $m$  and  $n$  through a function  $f$ . At the top, two arrows point downwards from the text labels " $M$ -dimensional" and " $N$ -dimensional" to the variables  $m$  and  $n$  respectively. Below these, another arrow points from the text label "function" to the equation  $n = f(m)$ .

# The Atlas Unlocks Calculus

$m \in \mathcal{M}, n \in \mathcal{N}$

$f : \mathcal{M} \rightarrow \mathcal{N}$

$M$ -dimensional       $N$ -dimensional

function

The diagram illustrates the concept of a function mapping between sets. At the top, two variables are shown:  $m \in \mathcal{M}$  and  $n \in \mathcal{N}$ . Above  $m$ , the text " $M$ -dimensional" is written with a curved arrow pointing down towards  $m$ . Above  $n$ , the text " $N$ -dimensional" is written with a curved arrow pointing down towards  $n$ . Below these, the function  $f : \mathcal{M} \rightarrow \mathcal{N}$  is defined, with a curved arrow pointing down towards it from the word "function".

# The Atlas Unlocks Calculus

$$f : \mathcal{M} \rightarrow \mathcal{N}$$

- The Jacobian encodes the first-order relationship that says “How do small changes in  $m$  become small changes in  $f(m)$ ? ”
- What should the “Jacobian” look like?

$$N \times M$$

# The Atlas Unlocks Calculus

$$f : \mathcal{M} \rightarrow \mathcal{N}$$

- Choose charts for  $m$  and  $f(m)$

$$\varphi_m : U_{\varphi_m} \rightarrow \mathcal{M}$$

$$\varphi_{f(m)} : U_{\varphi_{f(m)}} \rightarrow \mathcal{N}$$

- Define a mapping from  $\mathbb{R}^M$  to  $\mathbb{R}^N$  using the charts

$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$

# The Atlas Unlocks Calculus

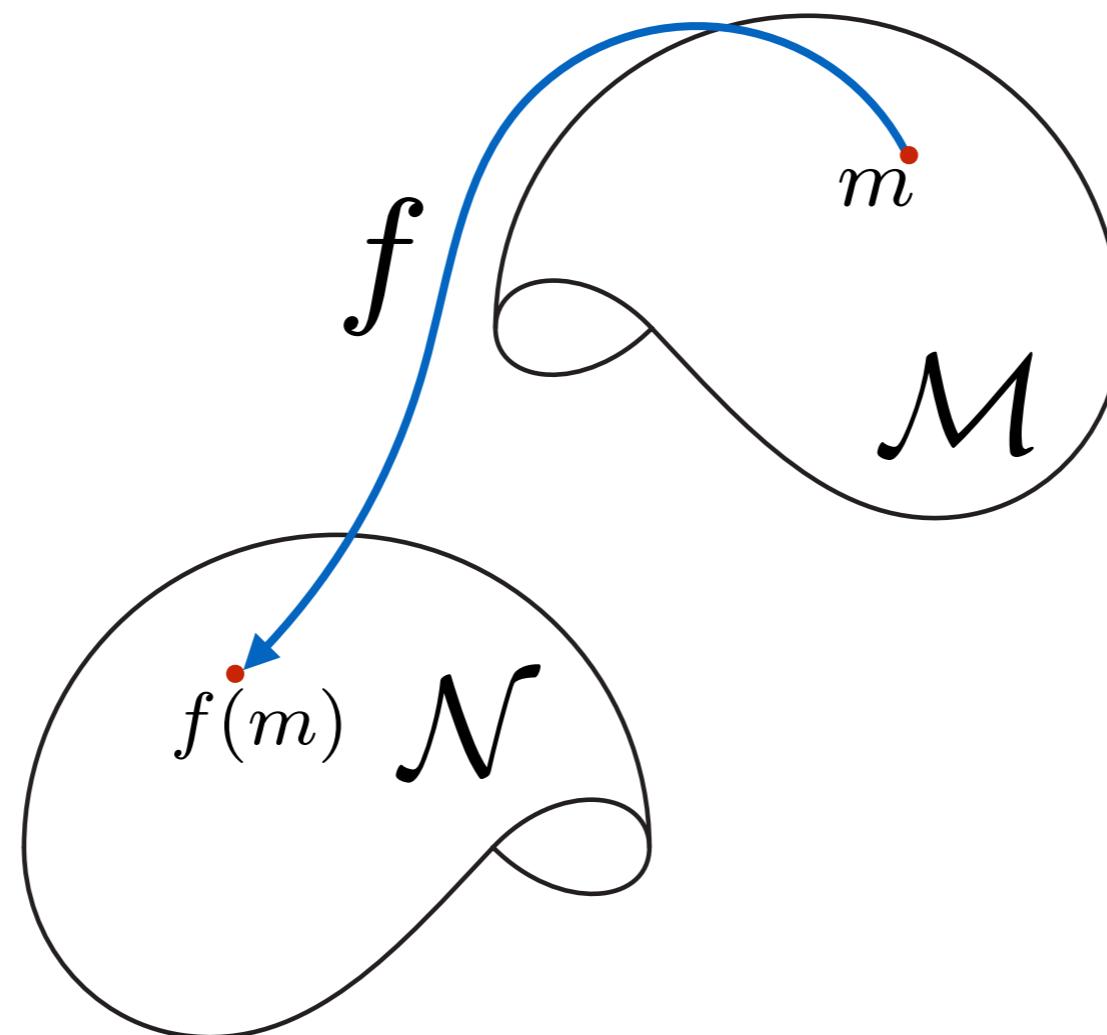
$$f : \mathcal{M} \rightarrow \mathcal{N}$$

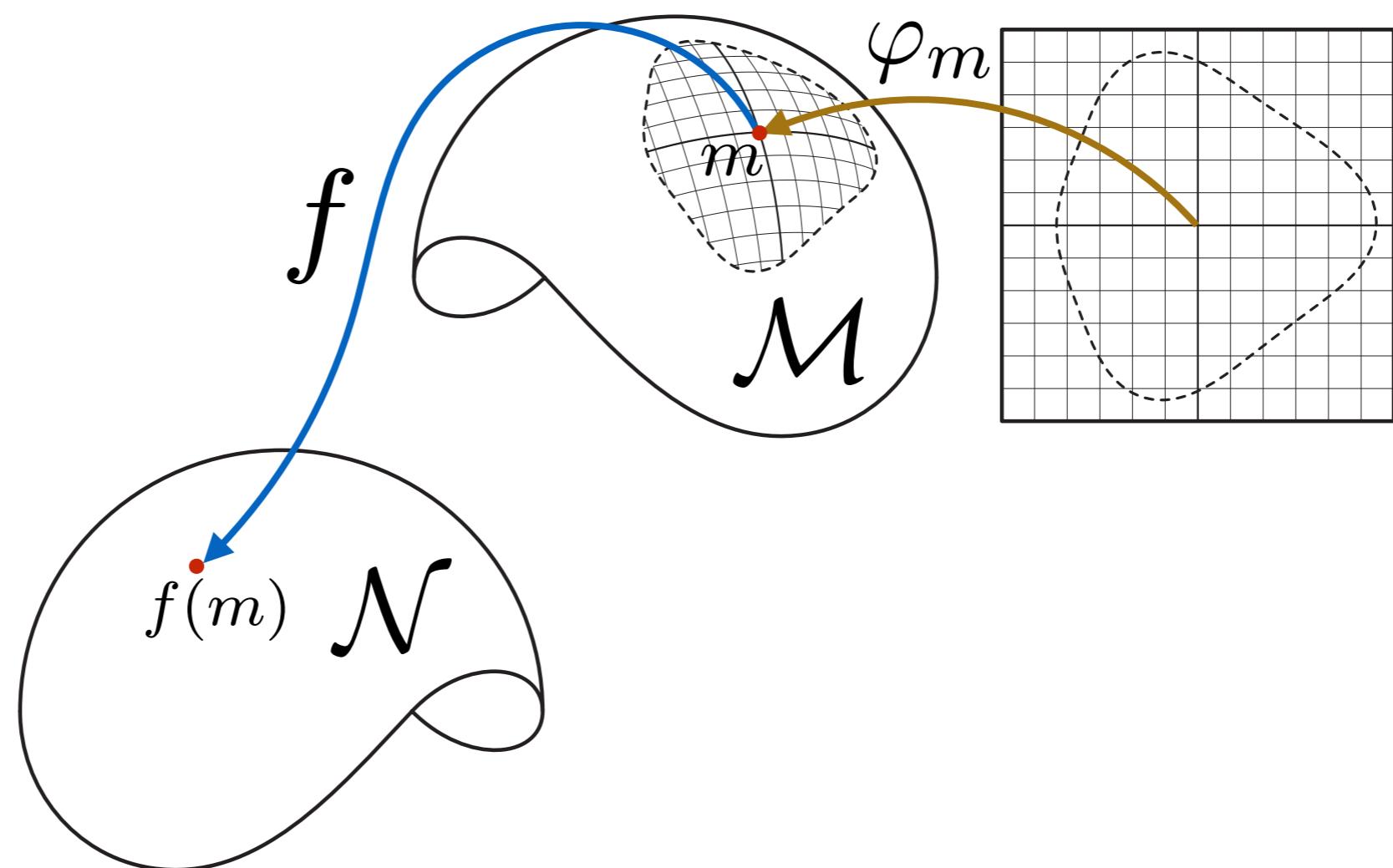
- Define a mapping from  $\mathbb{R}^M$  to  $\mathbb{R}^N$  using the charts

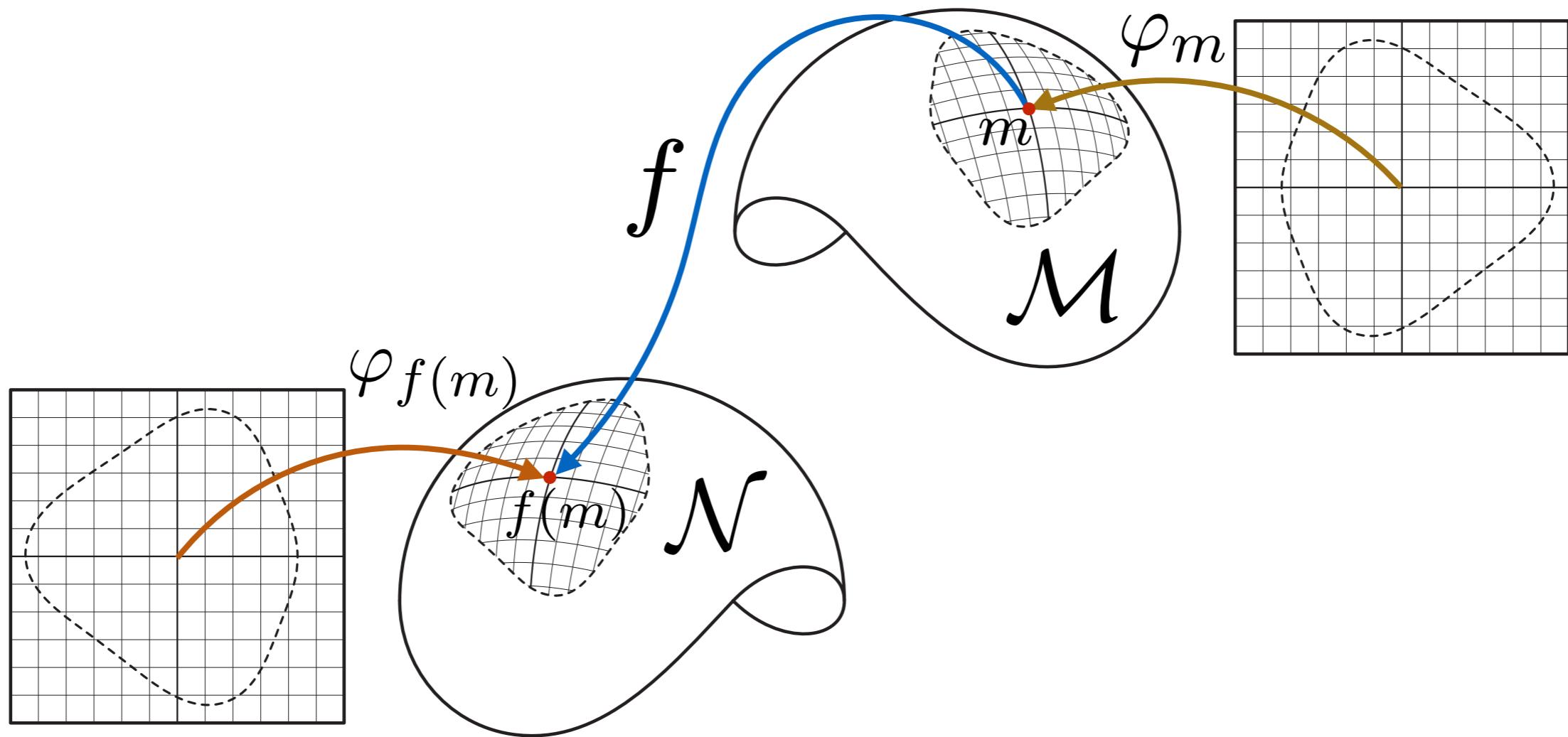
$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$

- Define the “Representing Matrix” of the differential of  $f$  at  $m$ , which depends on our choice of charts

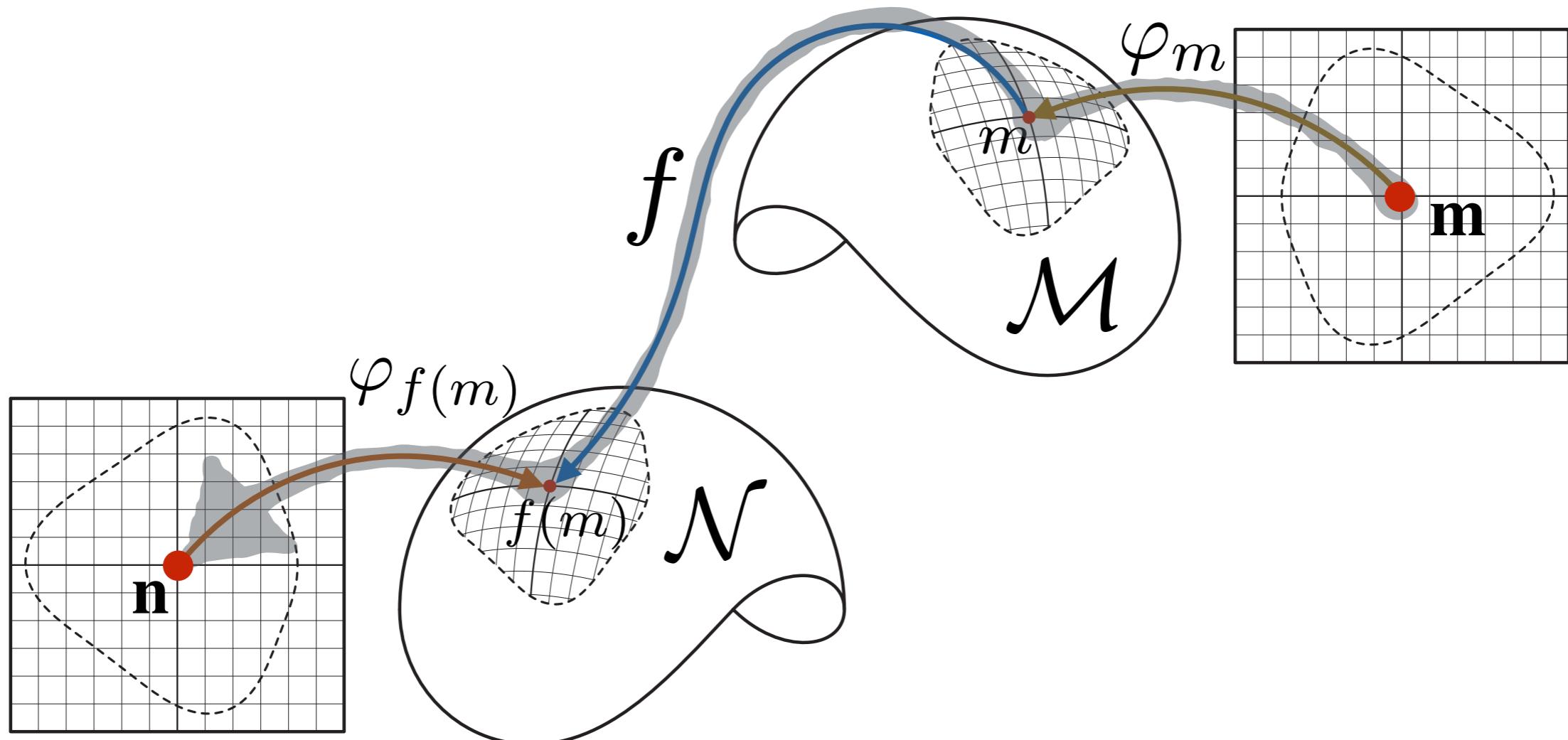
$$\mathbf{J}_m f := \left. \frac{\partial \mathbf{f}}{\partial \mathbf{m}} \right|_{\varphi_m^{-1}(m)} \in \mathbb{R}^{N \times M}$$



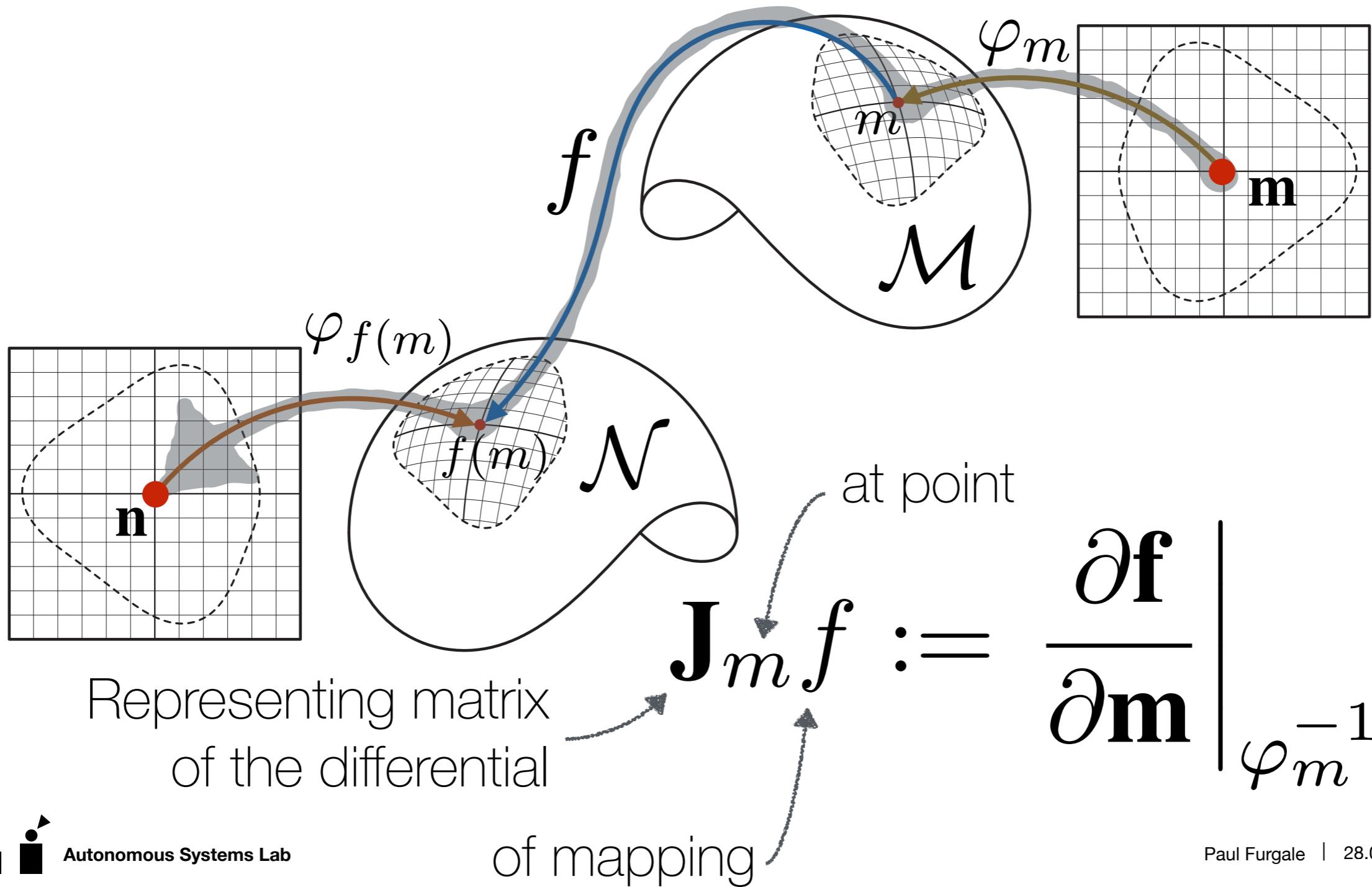




$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$



$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$



# Calculus

$$\frac{\partial e(\mathbf{x})}{\partial \mathbf{x}} \Big|_{\bar{\mathbf{x}}}$$



# Nonlinear Least Squares / Gauss Newton

Goal: estimate  $\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} J(\mathbf{x}) = \frac{1}{2} \mathbf{e}(\mathbf{x})^T \mathbf{R}^{-1} \mathbf{e}(\mathbf{x})$ .

- Start with an initial guess for  $\mathbf{x}$
- Iterate
  - Linearize
  - Solve the linear system
  - Update the parameters
  - Repeat until convergence

$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \mathbf{E} := \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}}, \quad \mathbf{e}(\epsilon_x) := \bar{\mathbf{e}} + \mathbf{E}\epsilon_x.$$

$$\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E} \epsilon_x^* = -\mathbf{E}^T \mathbf{R}^{-1} \bar{\mathbf{e}}$$

$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \epsilon_x^*$$

# Nonlinear Least Squares / Gauss Newton

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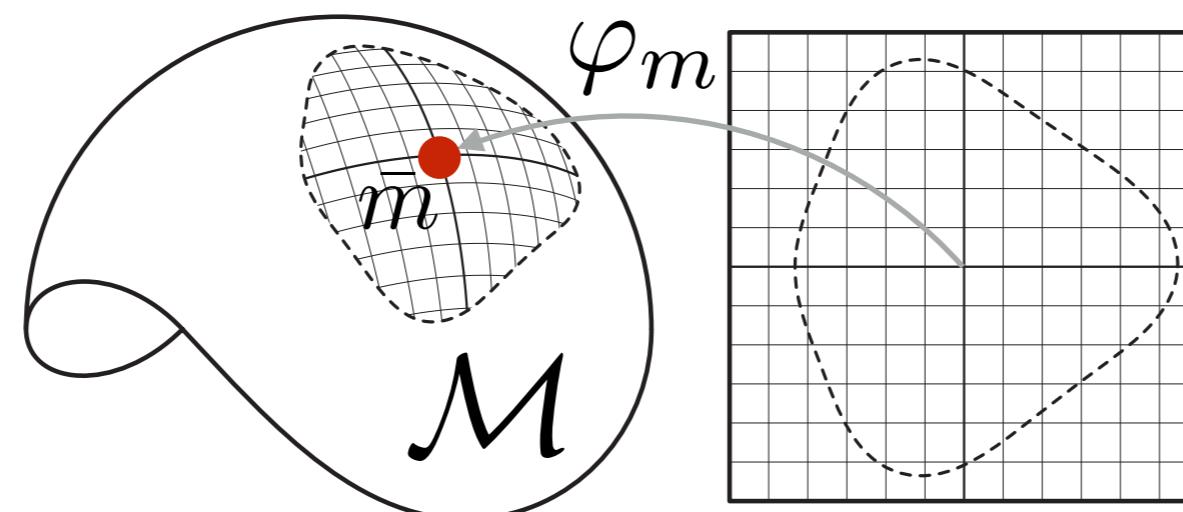
- Start with an initial guess for  $\mathbf{x}$
- Iterate
  - Linearize
  - Solve the linear system
  - Update the parameters
  - Repeat until convergence

Use the representing matrix and the chain rule

$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \boxed{\mathbf{E} := \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}}} \quad \mathbf{e}(\epsilon_x) := \bar{\mathbf{e}} + \mathbf{E}\epsilon_x.$$

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$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \epsilon_x^*$$



# Nonlinear Least Squares / Gauss Newton

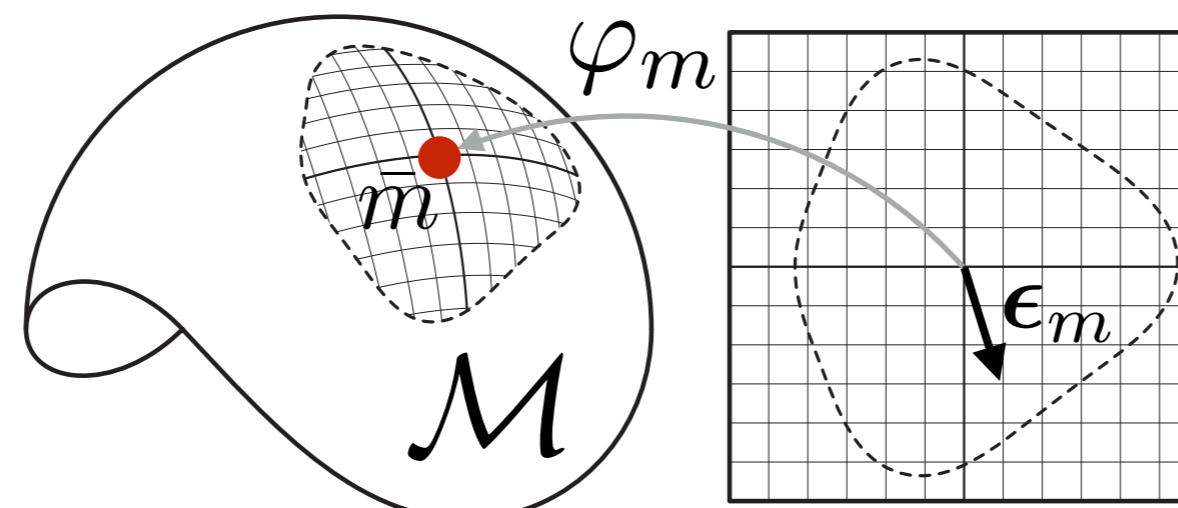
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- Start with an initial guess for  $\mathbf{x}$
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$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \epsilon_x^*$$



Solve for  
the update

# Nonlinear Least Squares / Gauss Newton

Goal: estimate  $\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} J(\mathbf{x}) = \frac{1}{2} \mathbf{e}(\mathbf{x})^T \mathbf{R}^{-1} \mathbf{e}(\mathbf{x})$ .

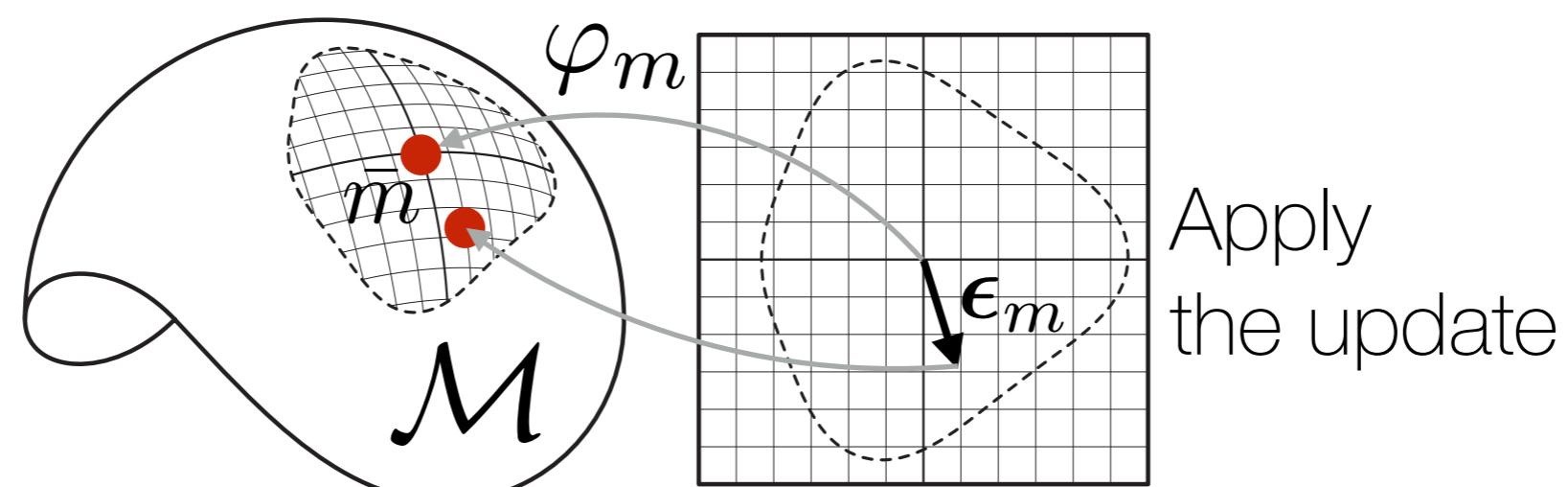
- Start with an initial guess for  $\mathbf{x}$
- Iterate
  - Linearize
  - Solve the linear system
  - Update the parameters
  - Repeat until convergence

$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \mathbf{E} := \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}}, \quad \mathbf{e}(\epsilon_x) := \bar{\mathbf{e}} + \mathbf{E}\epsilon_x.$$

$$\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E} \epsilon_x^* = -\mathbf{E}^T \mathbf{R}^{-1} \bar{\mathbf{e}}$$

$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \epsilon_x^*$$

$$\bar{m} \leftarrow \varphi_m(\varphi_m^{-1}(\bar{m}) + \epsilon_m)$$





# Linear Algebra

$$\epsilon_x^* = - (\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1} \mathbf{E}^T \mathbf{R}^{-1} \bar{\mathbf{e}}$$

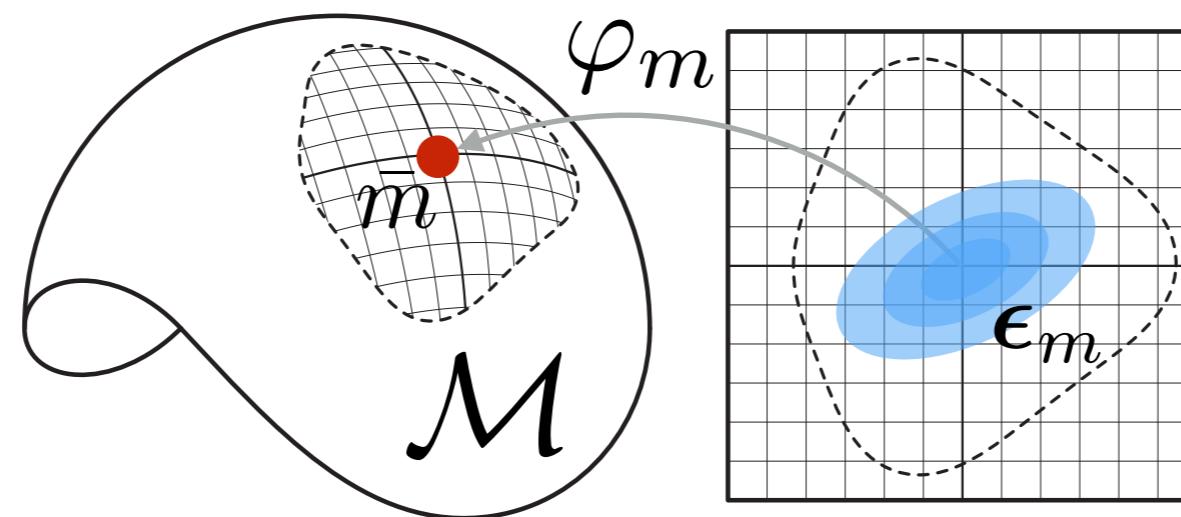
# Nonlinear Least Squares / Gauss Newton

Recover the covariance:

$$\mathbf{P} := (\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1},$$

$$\mathbf{x} = \mathbf{x}^* + \boldsymbol{\epsilon}_x, \quad \boldsymbol{\epsilon}_x \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$m = \varphi_m(\varphi_m^{-1}(\bar{m}) + \boldsymbol{\epsilon}_m)$$





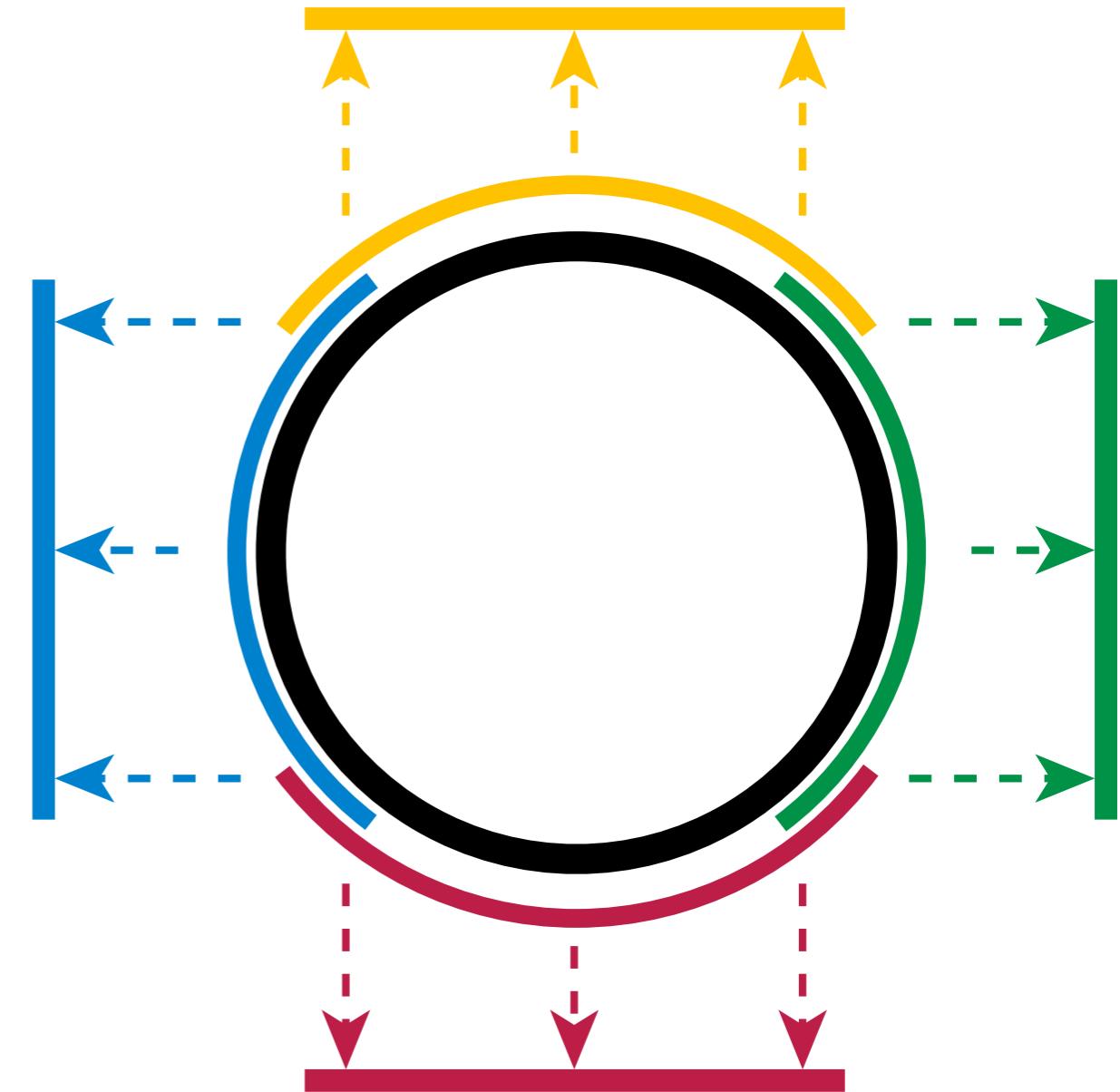
# Probabilities

$$\mathbf{P} := (\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1},$$

$$\mathbf{x} = \mathbf{x}^\star + \boldsymbol{\epsilon}_x, \quad \boldsymbol{\epsilon}_x \sim \mathcal{N}(0, \mathbf{P})$$

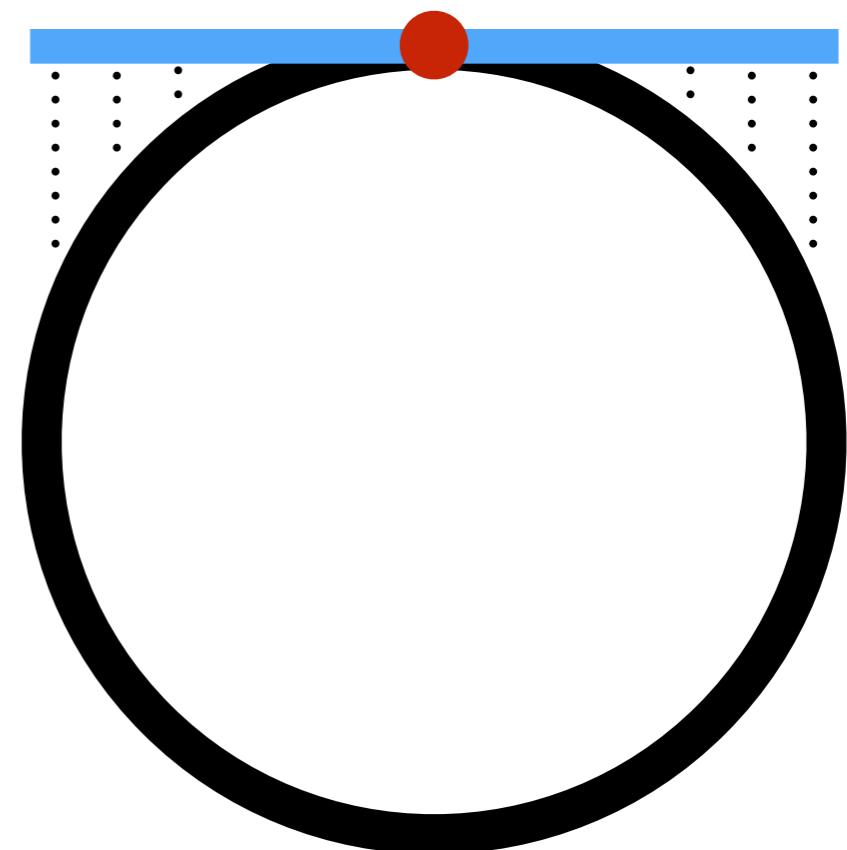
# Choosing an Atlas

- You can choose the Atlas you want
- Different choices will have benefits and drawbacks



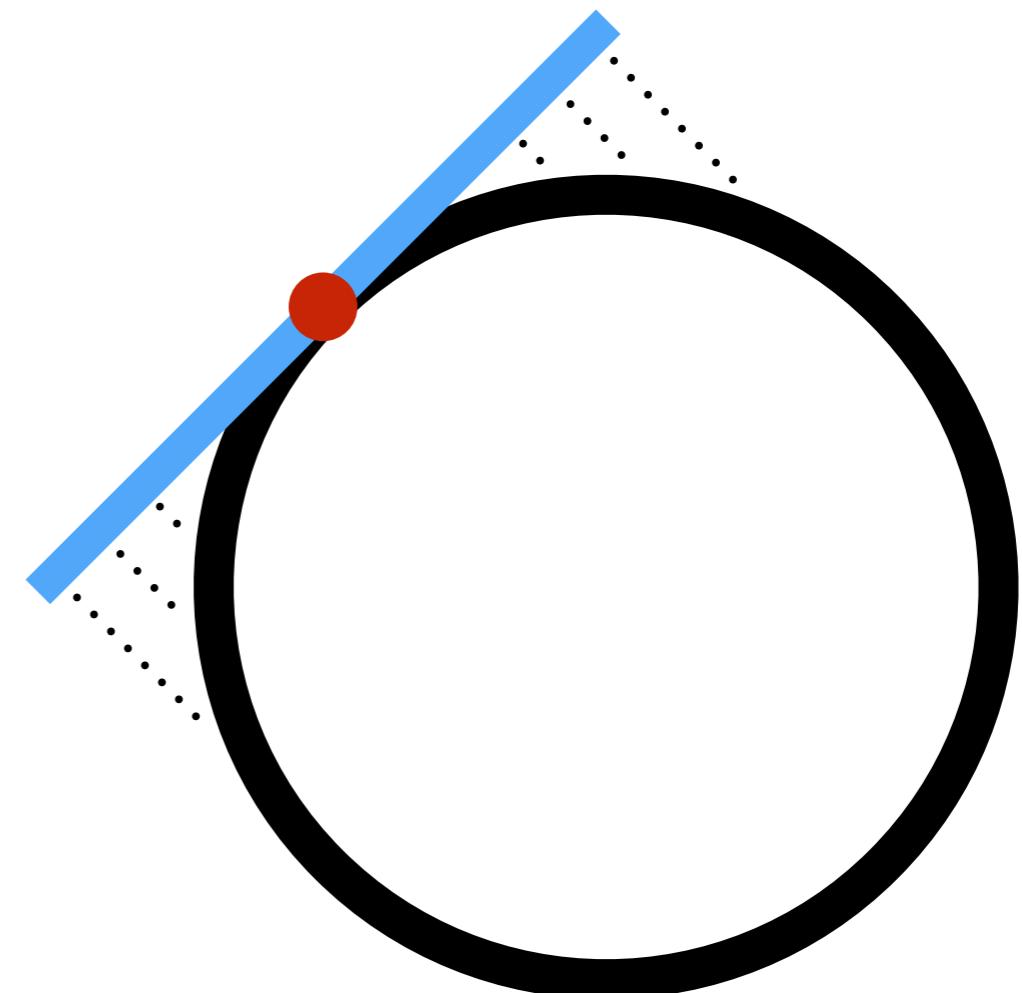
# Point-Wise Charts

- It is possible to define an atlas by specifying point-wise charts



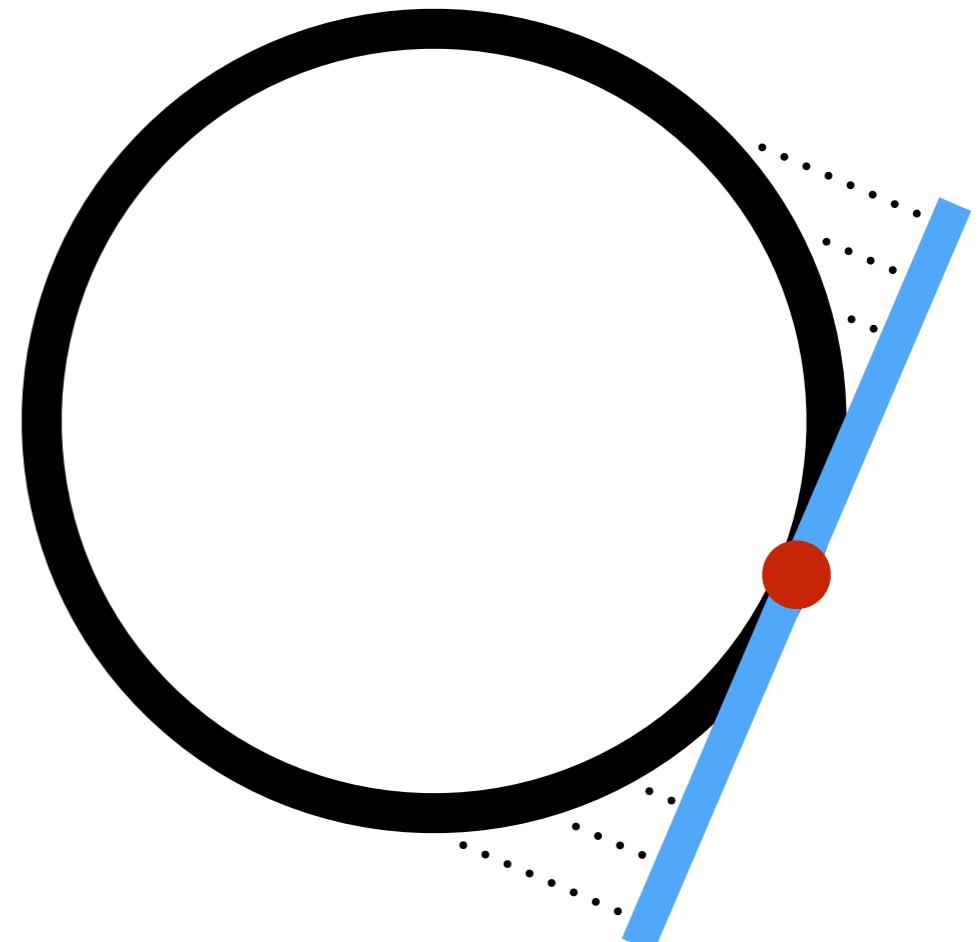
# Point-Wise Charts

- It is possible to define an atlas by specifying point-wise charts



# Point-Wise Charts

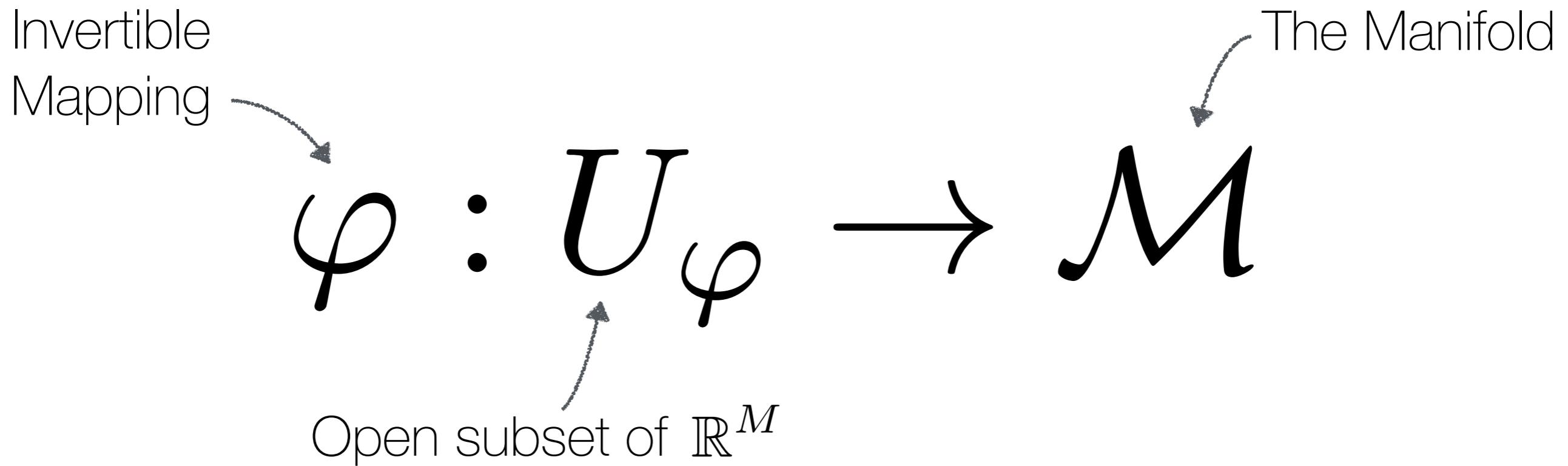
- It is possible to define an atlas by specifying point-wise charts
- Benefits:
  - Locally the best approximation to the manifold
  - Simplifies the computation of representing matrices



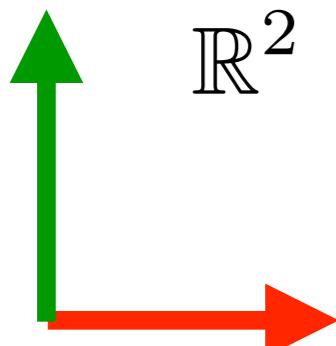
# **A Brief Introduction to Manifolds: Summary**

# Choosing an Atlas

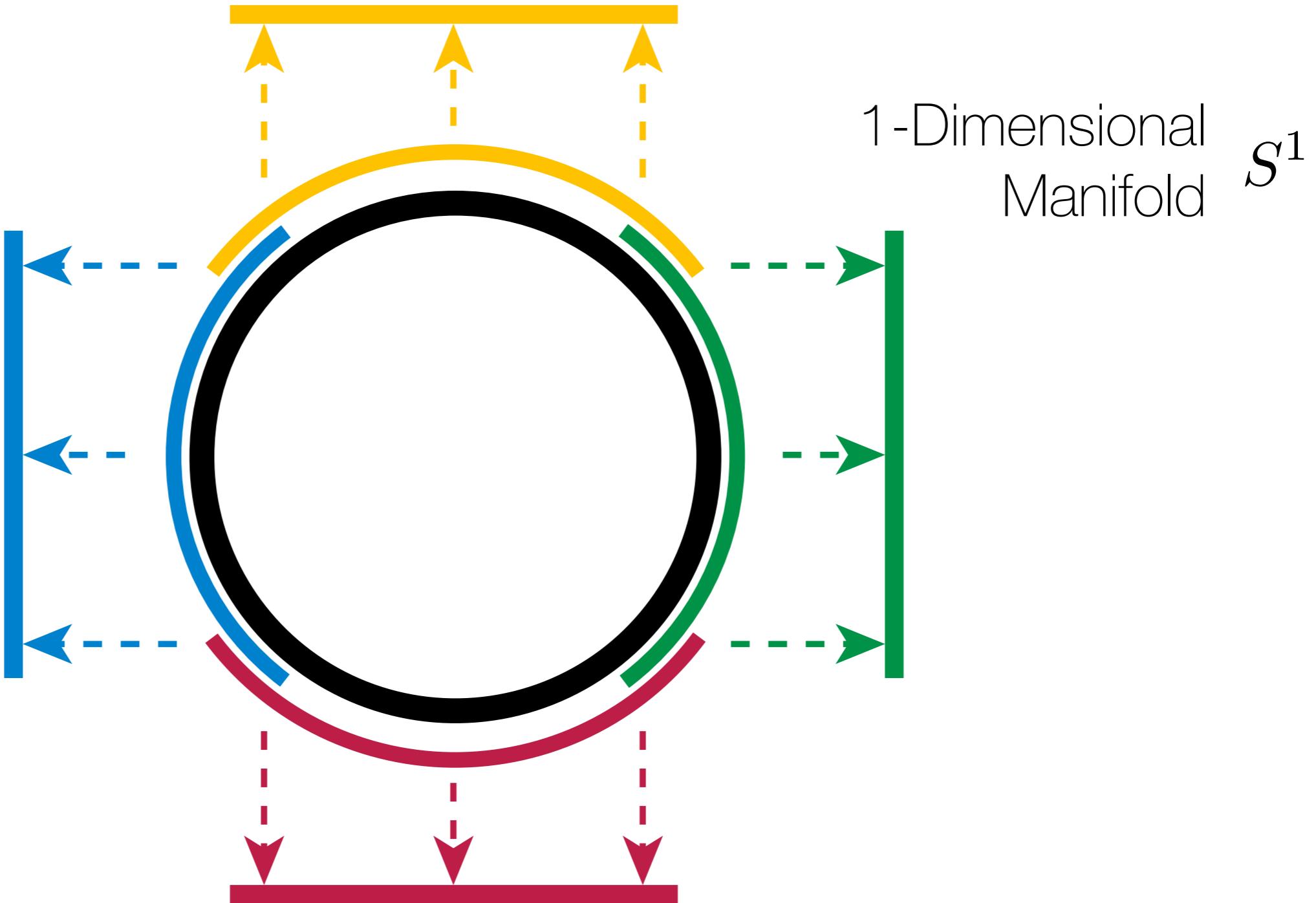
- Charts:
  - For an  $M$ -Dimensional DM,  $\mathcal{M}$ , a *chart* is



The  
Ambient  
Space

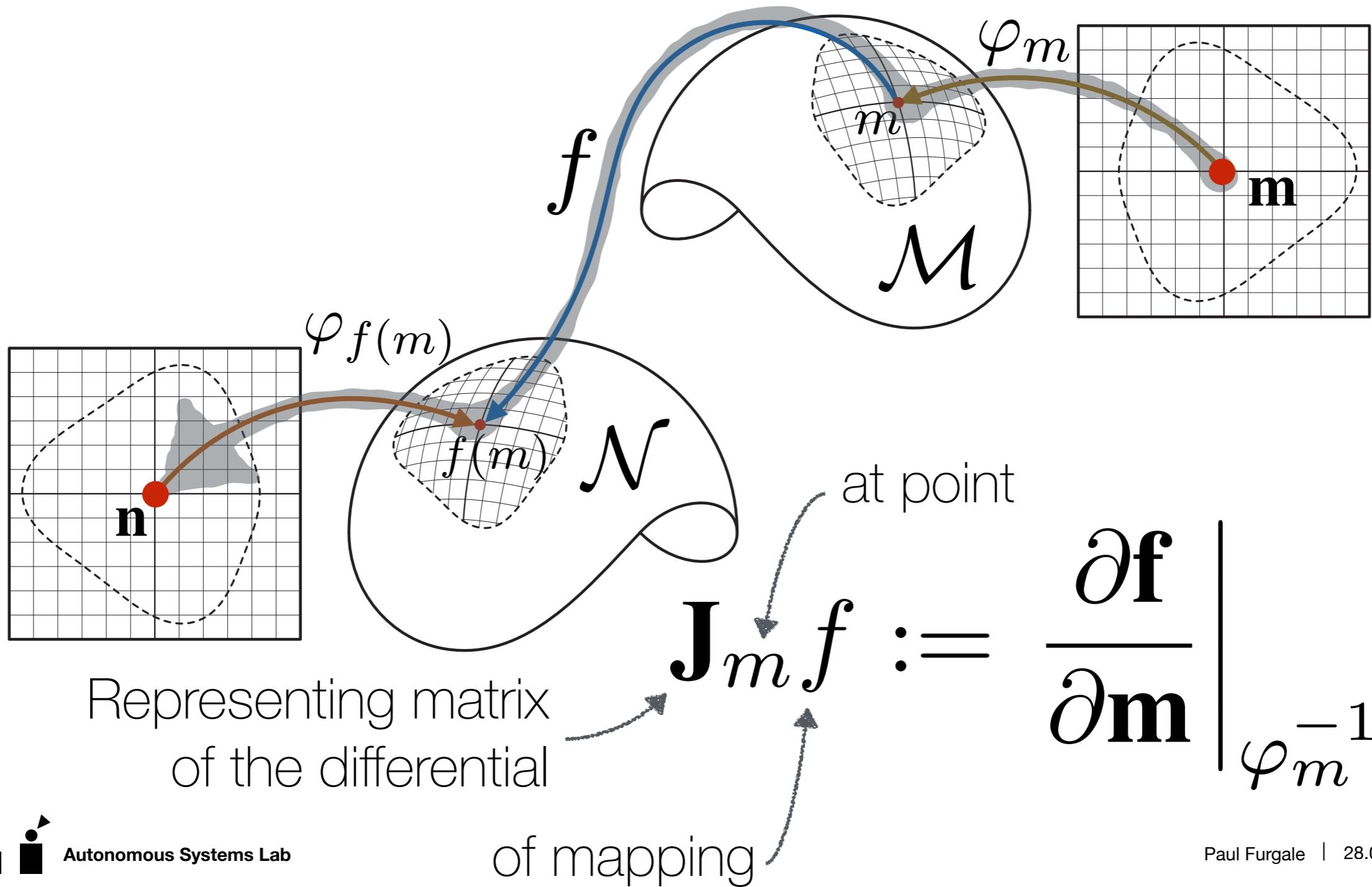


$\mathbb{R}^2$



- An *Atlas* is a collection of charts such that cover the manifold

$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$



**C<sub>AB</sub>**

**so(3)**

The Special Orthogonal Group

# The Special Orthogonal Group

$$SO(3) := \{ \mathbf{C} \in \mathbb{R}^{3 \times 3} \mid \mathbf{C}^T \mathbf{C} = \mathbf{I}, \det \mathbf{C} = 1 \}$$

Ambient Space

The Identity Matrix

$\mathbf{C}^T \equiv \mathbf{C}^{-1}$

- $SO(3)$  is a group, closed under standard matrix multiplication

# Atlases Covering SO(3)

- There are *many* atlases that cover SO(3)
  - Compositions of fundamental rotation matrices (Euler angles, Tait-Bryan angles)
  - Cayley rotational parameters
  - Axis/angle
  - Euler/Rodriguez parameters
- Each of these atlases has a singularity

# Atlases Covering SO(3)

- A singularity-free point-wise atlas:

$$\varphi_C(\phi) := \exp(\phi^\wedge) C$$

Skew-symmetric matrix  $\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}$



# The Exponential Map of $\text{SO}(3)$

$$\exp(\phi^\wedge)$$



# The Exponential Map of $SO(3)$

$$\exp(\phi^\wedge)$$

$$\exp(\phi^\wedge) \in SO(3)$$

# The Exponential Map of $SO(3)$

$$\exp(\phi^\wedge)$$

$$\exp(\phi^\wedge) \mathbf{C} \in SO(3)$$

# The Exponential Map of SO(3)

$$\exp(\phi^\wedge)$$

$$\exp(\phi^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!} (\phi^\wedge)^n$$



# The Exponential Map of SO(3)

$$\exp(\phi^\wedge)$$

$$\phi = \|\phi\|, \quad \mathbf{a} = \phi/\phi$$

$$\exp(\phi^\wedge) = \cos \phi \mathbf{1} + (1 - \cos \phi) \mathbf{a} \mathbf{a}^T + \sin \phi \mathbf{a}^\wedge$$



# The Exponential Map of SO(3)

$$\exp(\phi^\wedge)$$

$$\exp(\phi^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!} (\phi^\wedge)^n \approx 1 + \phi^\wedge$$



A first-order approximation

# The Exponential Map of SO(3)

$\log(C)$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}^\vee = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$



# The Exponential Map of SO(3)

$$\log(\mathbf{C})$$

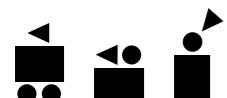
$$\theta = \arccos\left(\frac{\text{trace}(\mathbf{C}) - 1}{2}\right)$$

$$\log(\mathbf{C}) = \left( \frac{\theta}{2 \sin \theta} (\mathbf{C} - \mathbf{C}^T) \right)^\vee$$

# The Left-Exponential-Map Atlas for SO(3)

- A point-wise atlas:

$$\varphi_C(\phi) := \exp(\phi^\wedge)C$$

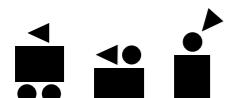


# The Left-Exponential-Map Atlas for SO(3)

- A point-wise atlas:

$$\varphi_C(\phi) := \exp(\phi^\wedge)C$$

$$\varphi_C^{-1}(R) := \log(RC^T)$$



# The Left-Exponential-Map Atlas for SO(3)

- A point-wise atlas:

$$\varphi_C(\phi) := \exp(\phi^\wedge)C$$

$$\varphi_C^{-1}(R) := \log(RC^T)$$

$$\varphi_C^{-1}(C) = 0$$

# The Left-Exponential-Map Atlas for SO(3)

- A point-wise atlas:

$$\begin{aligned}\varphi_C(\phi) &:= \exp(\phi^\wedge)C \\ &\approx (1 + \phi^\wedge)C\end{aligned}$$



# The Left-Exponential-Map Atlas for SO(3)

$$\begin{aligned}\varphi_C(\phi) &:= \exp(\phi^\wedge)C \\ &\approx (1 + \phi^\wedge)C\end{aligned}$$

- Benefits of the left-exponential-map atlas
  - Extremely simple linearized expression
  - Identities for manipulating expressions of this form
  - Known first order relationship with other rotation parameterizations

# Fundamental Identities (Cheat Sheet)

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$

$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$

$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

# Linearizing Expressions

$$\begin{aligned}\mathbf{a}^\wedge \mathbf{b} &= -\mathbf{b}^\wedge \mathbf{a} \\ \mathbf{C} \mathbf{a}^\wedge &= (\mathbf{C} \mathbf{a})^\wedge \mathbf{C} \\ \mathbf{a}^\wedge \mathbf{C} &= \mathbf{C} (\mathbf{C}^T \mathbf{a})^\wedge \\ (\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge\end{aligned}$$

- When someone gives you a rotation expression, you can linearize by (1) substituting in the linearized form (based on your chosen atlas)

$$\mathbf{C}_i = (1 + \phi_i^\wedge) \overline{\mathbf{C}}_i$$



The perturbation



The “large” rotation

# Linearizing Expressions

$$\begin{aligned}\mathbf{a}^\wedge \mathbf{b} &= -\mathbf{b}^\wedge \mathbf{a} \\ \mathbf{C} \mathbf{a}^\wedge &= (\mathbf{C} \mathbf{a})^\wedge \mathbf{C} \\ \mathbf{a}^\wedge \mathbf{C} &= \mathbf{C} (\mathbf{C}^T \mathbf{a})^\wedge \\ (\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge\end{aligned}$$

- Then you can rearrange the expression to bring it back into the form below. Use these two tools:
  1. Apply the fundamental identities
  2. Drop the products of “small” perturbations (this is consistent with the first order approximation).
- The result implicitly recovers the representing matrix

$$\mathbf{C}_i = (1 + \phi_i^\wedge) \overline{\mathbf{C}}_i$$

# Example: Transpose

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$

$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$

$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

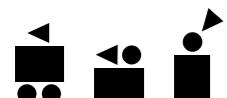
$$\mathbf{C}^T \approx ((1 + \phi^\wedge) \overline{\mathbf{C}})^T$$



# Example: Transpose

$$\begin{aligned}\mathbf{C}^T &\approx ((\mathbf{1} + \boldsymbol{\phi}^\wedge) \overline{\mathbf{C}})^T \\ &= \overline{\mathbf{C}}^T (\mathbf{1} + \boldsymbol{\phi}^\wedge)^T\end{aligned}$$

$$\begin{aligned}\mathbf{a}^\wedge \mathbf{b} &= -\mathbf{b}^\wedge \mathbf{a} \\ \mathbf{C} \mathbf{a}^\wedge &= (\mathbf{C} \mathbf{a})^\wedge \mathbf{C} \\ \mathbf{a}^\wedge \mathbf{C} &= \mathbf{C} (\mathbf{C}^T \mathbf{a})^\wedge \\ (\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge\end{aligned}$$



# Example: Transpose

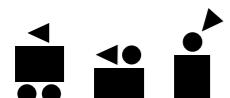
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$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\begin{aligned}\mathbf{C}^T &\approx ((\mathbf{1} + \boldsymbol{\phi}^\wedge) \overline{\mathbf{C}})^T \\&= \overline{\mathbf{C}}^T (\mathbf{1} + \boldsymbol{\phi}^\wedge)^T \\&= \overline{\mathbf{C}}^T (\mathbf{1}^T + (\boldsymbol{\phi}^\wedge)^T)\end{aligned}$$



# Example: Transpose

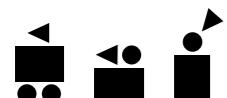
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$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\begin{aligned}\mathbf{C}^T &\approx ((\mathbf{1} + \boldsymbol{\phi}^\wedge) \overline{\mathbf{C}})^T \\&= \overline{\mathbf{C}}^T (\mathbf{1} + \boldsymbol{\phi}^\wedge)^T \\&= \overline{\mathbf{C}}^T (\mathbf{1}^T + (\boldsymbol{\phi}^\wedge)^T) \\&= \overline{\mathbf{C}}^T (\mathbf{1} - \boldsymbol{\phi}^\wedge)\end{aligned}$$



## Example: Transpose

$$\begin{aligned}\mathbf{a}^\wedge \mathbf{b} &= -\mathbf{b}^\wedge \mathbf{a} \\ \mathbf{C} \mathbf{a}^\wedge &= (\mathbf{C} \mathbf{a})^\wedge \mathbf{C} \\ \mathbf{a}^\wedge \mathbf{C} &= \mathbf{C} (\mathbf{C}^T \mathbf{a})^\wedge \\ (\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge\end{aligned}$$

$$\begin{aligned}\mathbf{C}^T &\approx ((1 + \phi^\wedge) \overline{\mathbf{C}})^T \\ &= \overline{\mathbf{C}}^T (1 + \phi^\wedge)^T \\ &= \overline{\mathbf{C}}^T (1^T + (\phi^\wedge)^T) \\ &= \boxed{\overline{\mathbf{C}}^T (1 - \phi^\wedge)}\end{aligned}$$



# Example: Transpose

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$

$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$

$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

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$$\begin{aligned}\mathbf{C}^T &\approx ((\mathbf{1} + \boldsymbol{\phi}^\wedge) \overline{\mathbf{C}})^T \\&= \overline{\mathbf{C}}^T (\mathbf{1} + \boldsymbol{\phi}^\wedge)^T \\&= \overline{\mathbf{C}}^T (\mathbf{1}^T + (\boldsymbol{\phi}^\wedge)^T) \\&= \overline{\mathbf{C}}^T (\mathbf{1} - \boldsymbol{\phi}^\wedge) \\&= \overline{\mathbf{C}}^T - \overline{\mathbf{C}}^T \boldsymbol{\phi}^\wedge\end{aligned}$$



# Example: Transpose

$$\begin{aligned}\mathbf{a}^\wedge \mathbf{b} &= -\mathbf{b}^\wedge \mathbf{a} \\ \mathbf{C} \mathbf{a}^\wedge &= (\mathbf{C} \mathbf{a})^\wedge \mathbf{C} \\ \mathbf{a}^\wedge \mathbf{C} &= \mathbf{C} (\mathbf{C}^T \mathbf{a})^\wedge \\ (\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge\end{aligned}$$

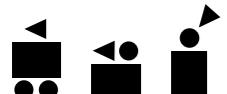
$$\begin{aligned}\mathbf{C}^T &\approx ((\mathbf{1} + \boldsymbol{\phi}^\wedge) \overline{\mathbf{C}})^T \\ &= \overline{\mathbf{C}}^T (\mathbf{1} + \boldsymbol{\phi}^\wedge)^T \\ &= \overline{\mathbf{C}}^T (\mathbf{1}^T + (\boldsymbol{\phi}^\wedge)^T) \\ &= \overline{\mathbf{C}}^T (\mathbf{1} - \boldsymbol{\phi}^\wedge) \\ &= \overline{\mathbf{C}}^T - \overline{\mathbf{C}}^T \boldsymbol{\phi}^\wedge \\ &= \overline{\mathbf{C}}^T - (\overline{\mathbf{C}}^T \boldsymbol{\phi})^\wedge \overline{\mathbf{C}}^T\end{aligned}$$

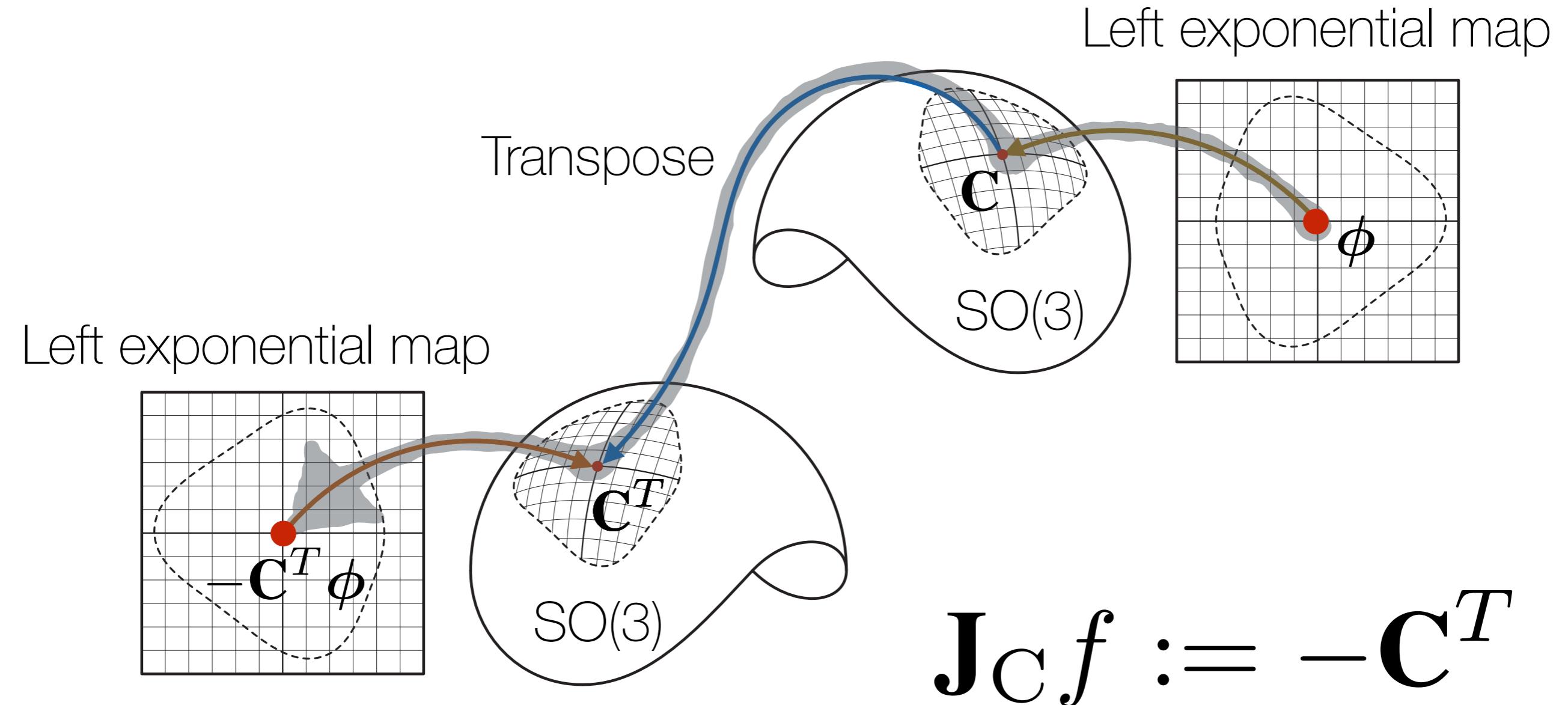


# Example: Transpose

$$\begin{aligned}\mathbf{a}^\wedge \mathbf{b} &= -\mathbf{b}^\wedge \mathbf{a} \\ \mathbf{C} \mathbf{a}^\wedge &= (\mathbf{C} \mathbf{a})^\wedge \mathbf{C} \\ \mathbf{a}^\wedge \mathbf{C} &= \mathbf{C} (\mathbf{C}^T \mathbf{a})^\wedge \\ (\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge\end{aligned}$$

$$\begin{aligned}\mathbf{C}^T &\approx ((\mathbf{1} + \boldsymbol{\phi}^\wedge) \overline{\mathbf{C}})^T \\ &= \overline{\mathbf{C}}^T (\mathbf{1} + \boldsymbol{\phi}^\wedge)^T \\ &= \overline{\mathbf{C}}^T (\mathbf{1}^T + (\boldsymbol{\phi}^\wedge)^T) \\ &= \overline{\mathbf{C}}^T (\mathbf{1} - \boldsymbol{\phi}^\wedge) \\ &= \overline{\mathbf{C}}^T - \overline{\mathbf{C}}^T \boldsymbol{\phi}^\wedge \\ &= \overline{\mathbf{C}}^T - (\overline{\mathbf{C}}^T \boldsymbol{\phi})^\wedge \overline{\mathbf{C}}^T \\ &= (\mathbf{1} + (-\overline{\mathbf{C}}^T \boldsymbol{\phi})^\wedge) \overline{\mathbf{C}}^T\end{aligned}$$





# Example: Multiplication

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$

$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$

$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\mathbf{C}_1 \mathbf{C}_2 \approx (\mathbf{1} + \phi_1^\wedge) \overline{\mathbf{C}}_1 (\mathbf{1} + \phi_2^\wedge) \overline{\mathbf{C}}_2$$



# Example: Multiplication

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$

$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$

$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

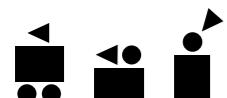
$$\begin{aligned}\mathbf{C}_1 \mathbf{C}_2 &\approx (\mathbf{1} + \phi_1^\wedge) \overline{\mathbf{C}}_1 (\mathbf{1} + \phi_2^\wedge) \overline{\mathbf{C}}_2 \\ &= \overline{\mathbf{C}}_1 \overline{\mathbf{C}}_2 + \phi_1^\wedge \overline{\mathbf{C}}_1 \overline{\mathbf{C}}_2 + \phi_1^\wedge \overline{\mathbf{C}}_1 \phi_2^\wedge \overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_1 \phi_2^\wedge \overline{\mathbf{C}}_2\end{aligned}$$



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$$\begin{aligned}\mathbf{a}^\wedge \mathbf{b} &= -\mathbf{b}^\wedge \mathbf{a} \\ \mathbf{C} \mathbf{a}^\wedge &= (\mathbf{C} \mathbf{a})^\wedge \mathbf{C} \\ \mathbf{a}^\wedge \mathbf{C} &= \mathbf{C} (\mathbf{C}^T \mathbf{a})^\wedge \\ (\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge\end{aligned}$$

$$\begin{aligned}\mathbf{C}_1 \mathbf{C}_2 &\approx (\mathbf{1} + \phi_1^\wedge) \bar{\mathbf{C}}_1 (\mathbf{1} + \phi_2^\wedge) \bar{\mathbf{C}}_2 \\ &= \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 + \phi_1^\wedge \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 + \phi_1^\wedge \bar{\mathbf{C}}_1 \phi_2^\wedge \bar{\mathbf{C}}_2 + \bar{\mathbf{C}}_1 \phi_2^\wedge \bar{\mathbf{C}}_2 \\ &\approx \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 + \phi_1^\wedge \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 + \bar{\mathbf{C}}_1 \phi_2^\wedge \bar{\mathbf{C}}_2\end{aligned}$$



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$$\begin{aligned}\mathbf{C}_1 \mathbf{C}_2 &\approx (\mathbf{1} + \phi_1^\wedge) \overline{\mathbf{C}}_1 (\mathbf{1} + \phi_2^\wedge) \overline{\mathbf{C}}_2 \\ &= \overline{\mathbf{C}}_1 \overline{\mathbf{C}}_2 + \phi_1^\wedge \overline{\mathbf{C}}_1 \overline{\mathbf{C}}_2 + \phi_1^\wedge \overline{\mathbf{C}}_1 \phi_2^\wedge \overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_1 \phi_2^\wedge \overline{\mathbf{C}}_2 \\ &\approx \overline{\mathbf{C}}_1 \overline{\mathbf{C}}_2 + \phi_1^\wedge \overline{\mathbf{C}}_1 \overline{\mathbf{C}}_2 + \overline{\mathbf{C}}_1 \phi_2^\wedge \overline{\mathbf{C}}_2 \\ &= \overline{\mathbf{C}}_1 \overline{\mathbf{C}}_2 + \phi_1^\wedge \overline{\mathbf{C}}_1 \overline{\mathbf{C}}_2 + (\overline{\mathbf{C}}_1 \phi_2)^\wedge \overline{\mathbf{C}}_1 \overline{\mathbf{C}}_2\end{aligned}$$



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$$\begin{aligned}\mathbf{C}_1 \mathbf{C}_2 &\approx (\mathbf{1} + \phi_1^\wedge) \bar{\mathbf{C}}_1 (\mathbf{1} + \phi_2^\wedge) \bar{\mathbf{C}}_2 \\ &= \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 + \phi_1^\wedge \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 + \phi_1^\wedge \bar{\mathbf{C}}_1 \phi_2^\wedge \bar{\mathbf{C}}_2 + \bar{\mathbf{C}}_1 \phi_2^\wedge \bar{\mathbf{C}}_2 \\ &\approx \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 + \phi_1^\wedge \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 + \bar{\mathbf{C}}_1 \phi_2^\wedge \bar{\mathbf{C}}_2 \\ &= \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 + \phi_1^\wedge \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 + (\bar{\mathbf{C}}_1 \phi_2)^\wedge \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2 \\ &= (\mathbf{1} + (\phi_1 + \bar{\mathbf{C}}_1 \phi_2)^\wedge) \bar{\mathbf{C}}_1 \bar{\mathbf{C}}_2\end{aligned}$$

# Example: Multiplication with a Vector

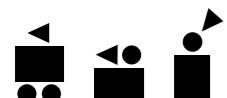
$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$

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$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

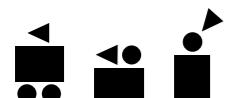
$$\mathbf{C}\mathbf{v} \approx (1 + \phi^\wedge) \overline{\mathbf{C}}(\bar{\mathbf{v}} + \boldsymbol{\epsilon}_v)$$



# Example: Multiplication with a Vector

$$\begin{aligned}\mathbf{a}^\wedge \mathbf{b} &= -\mathbf{b}^\wedge \mathbf{a} \\ \mathbf{C} \mathbf{a}^\wedge &= (\mathbf{C} \mathbf{a})^\wedge \mathbf{C} \\ \mathbf{a}^\wedge \mathbf{C} &= \mathbf{C} (\mathbf{C}^T \mathbf{a})^\wedge \\ (\mathbf{a}^\wedge)^T &= -\mathbf{a}^\wedge\end{aligned}$$

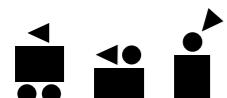
$$\begin{aligned}\mathbf{Cv} &\approx (1 + \phi^\wedge) \bar{\mathbf{C}} (\bar{\mathbf{v}} + \boldsymbol{\epsilon}_v) \\ &= \bar{\mathbf{C}} \bar{\mathbf{v}} + \phi^\wedge \bar{\mathbf{C}} \bar{\mathbf{v}} + \phi^\wedge \bar{\mathbf{C}} \boldsymbol{\epsilon}_v + \bar{\mathbf{C}} \boldsymbol{\epsilon}_v\end{aligned}$$



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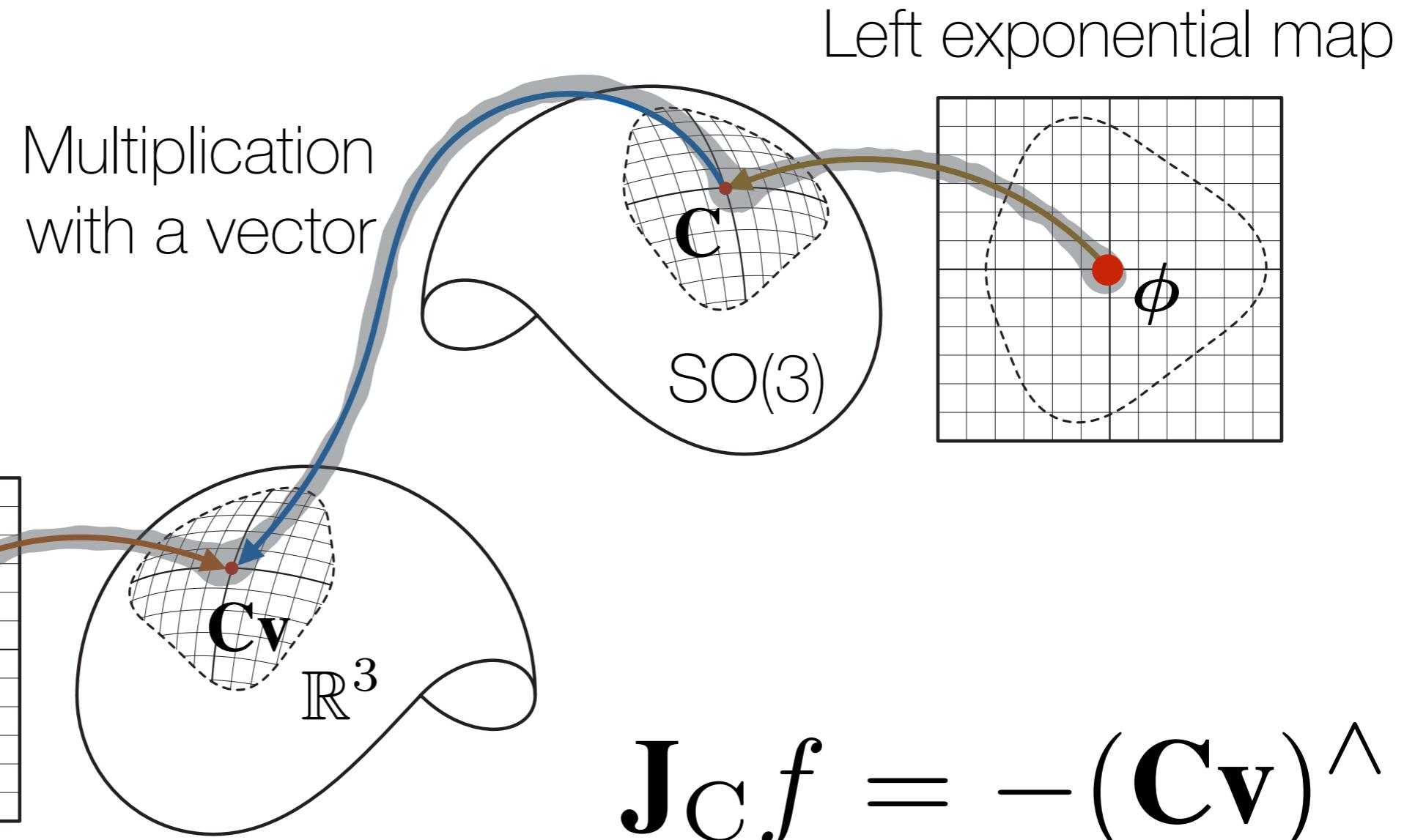


# Example: Multiplication with a Vector

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$$\begin{aligned}\mathbf{C}\mathbf{v} &\approx (1 + \phi^\wedge) \overline{\mathbf{C}} (\bar{\mathbf{v}} + \epsilon_v) \\ &= \overline{\mathbf{C}} \bar{\mathbf{v}} + \phi^\wedge \overline{\mathbf{C}} \bar{\mathbf{v}} + \phi^\wedge \overline{\mathbf{C}} \epsilon_v + \overline{\mathbf{C}} \epsilon_v \\ &\approx \overline{\mathbf{C}} \bar{\mathbf{v}} + \phi^\wedge \overline{\mathbf{C}} \bar{\mathbf{v}} + \overline{\mathbf{C}} \epsilon_v \\ &= \overline{\mathbf{C}} \bar{\mathbf{v}} + (-\overline{\mathbf{C}} \bar{\mathbf{v}})^\wedge \phi + \overline{\mathbf{C}} \epsilon_v\end{aligned}$$





# Checking your answers

- When developing code for optimization packages, always always always check your answers using finite differences
- General strategy:
  - Write down the exact charts that you are using
  - Implement the charts and the nonlinear function
  - Evaluate the Jacobian with respect to the charts
  - Estimate derivatives with:
    - <http://www.mathworks.com/matlabcentral/fileexchange/13490-adaptive-robust-numerical-differentiation>
    - Or, write unit tests for your functions

# Checking your answers: Example

## jacobianest

gradest: estimate of the Jacobian matrix of a vector valued function of n variables  
usage: [jac,err] = jacobianest(fun,x0)

arguments: (input)

fun - (vector valued) analytical function to differentiate.  
fun must be a function of the vector or array x0.

x0 - vector location at which to differentiate fun

If x0 is an nxm array, then fun is assumed to be  
a function of n\*m variables.

arguments: (output)

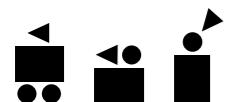
jac - array of first partial derivatives of fun.  
Assuming that x0 is a vector of length p  
and fun returns a vector of length n, then  
jac will be an array of size (n,p)

err - vector of error estimates corresponding to  
each partial derivative in jac.



# Checking your answers: Example

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*c)'*c ), zeros(3,1) )
```



# Checking your answers: Example

```
>> jacobianest(@(phi) so3log( (so3exp(phi)*c)'*c ), zeros(3,1) )
```

Matlab anonymous function of phi

# Checking your answers: Example

$$\varphi_C(\phi) := \exp(\phi^\wedge)C$$

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*c)'*c ), zeros(3,1) )
```

Chart on the input



# Checking your answers: Example

$$\varphi_C^{-1}(C) = 0$$

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*C)'*C ), zeros(3,1) )
```

The linearization point of the chart on the input



# Checking your answers: Example

$$\mathbf{C}^T$$

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*c)'*c ), zeros(3,1) )
```

The function (transpose on the result)



# Checking your answers: Example

$$\varphi_C^{-1}(\mathbf{R}) := \log(\mathbf{RC}^T)$$

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*C)'*C ), zeros(3,1) )
```

The inverse chart of the result

# Checking your answers: Example

```
>> jacobianest( @(phi) so3log( (so3exp(phi)*c)'*c ), zeros(3,1) )  
ans =  
  
-0.6315    -0.4226    0.6501  
-0.2276    -0.7005    -0.6764  
-0.7413    0.5751    -0.3462  
  
>> -c'  
  
ans =  
  
-0.6315    -0.4226    0.6501  
-0.2276    -0.7005    -0.6764  
-0.7413    0.5751    -0.3462
```



**C<sub>AB</sub>**

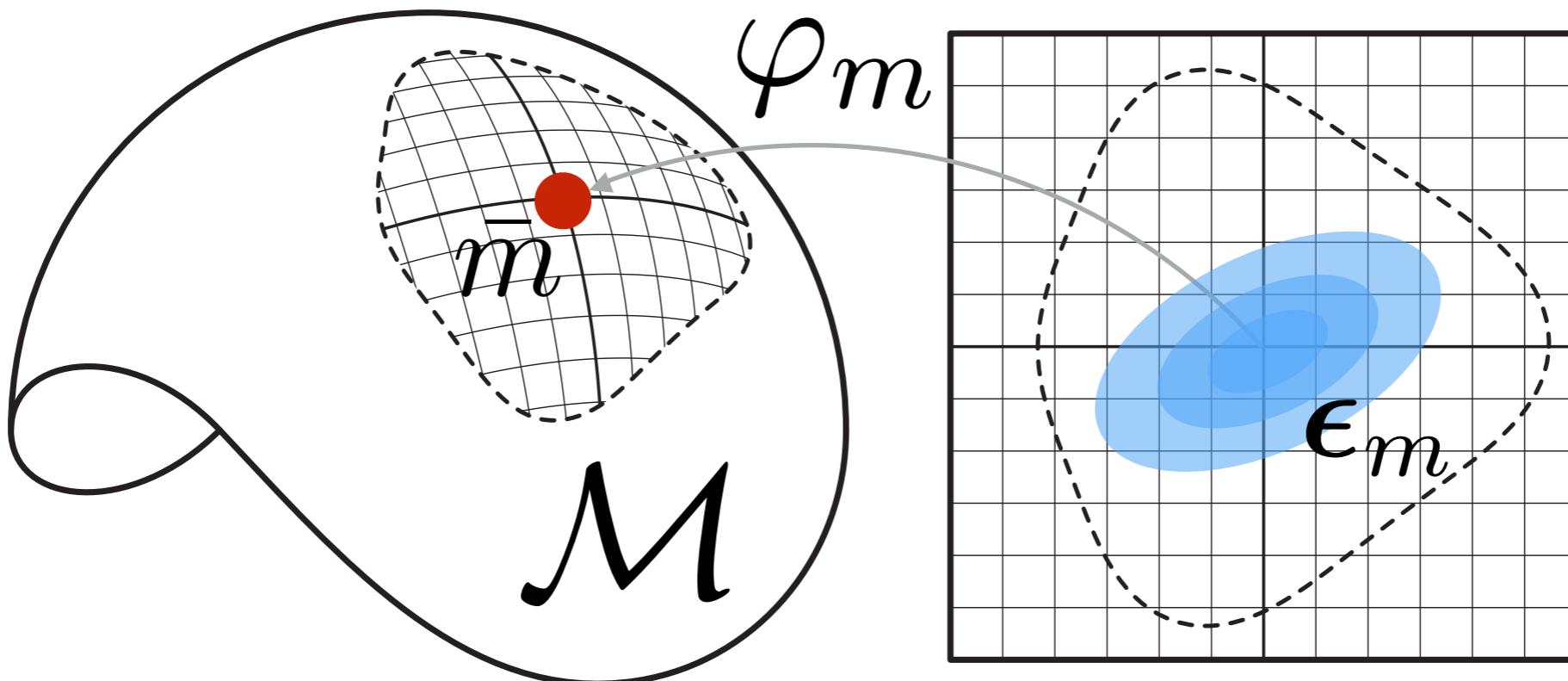
**so(3)**

Uncertainties on The Special Orthogonal Group

# SO(3) – Uncertainty

- Uncertainty is often represented in the chart
- Whatever chart you use for optimization, that is where the uncertainty lives

$$\epsilon_m \sim \mathcal{N}(0, P)$$



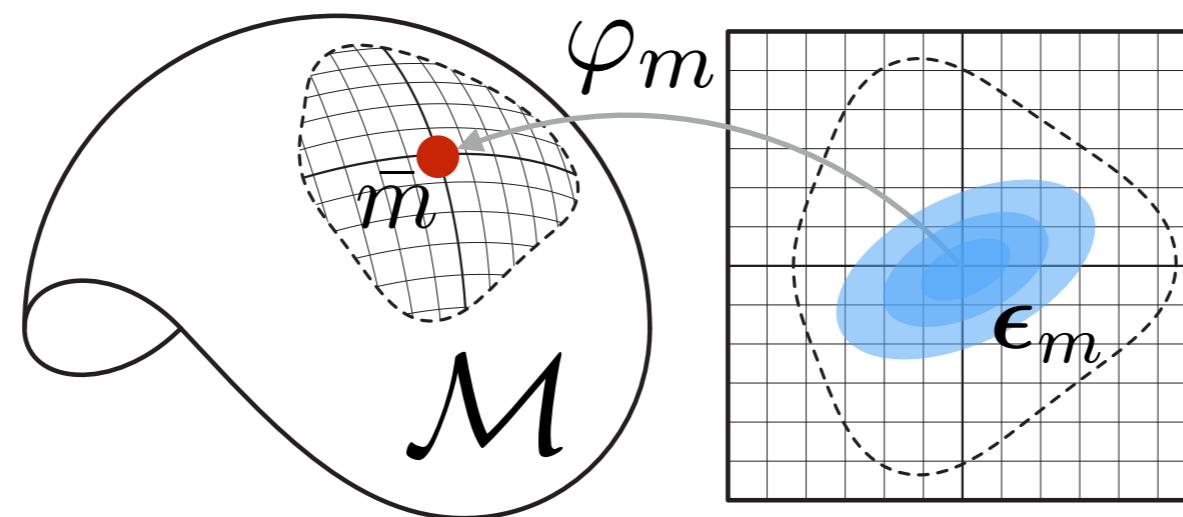
# SO(3) – Uncertainty

Recover the covariance:

$$\mathbf{P} := (\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1},$$

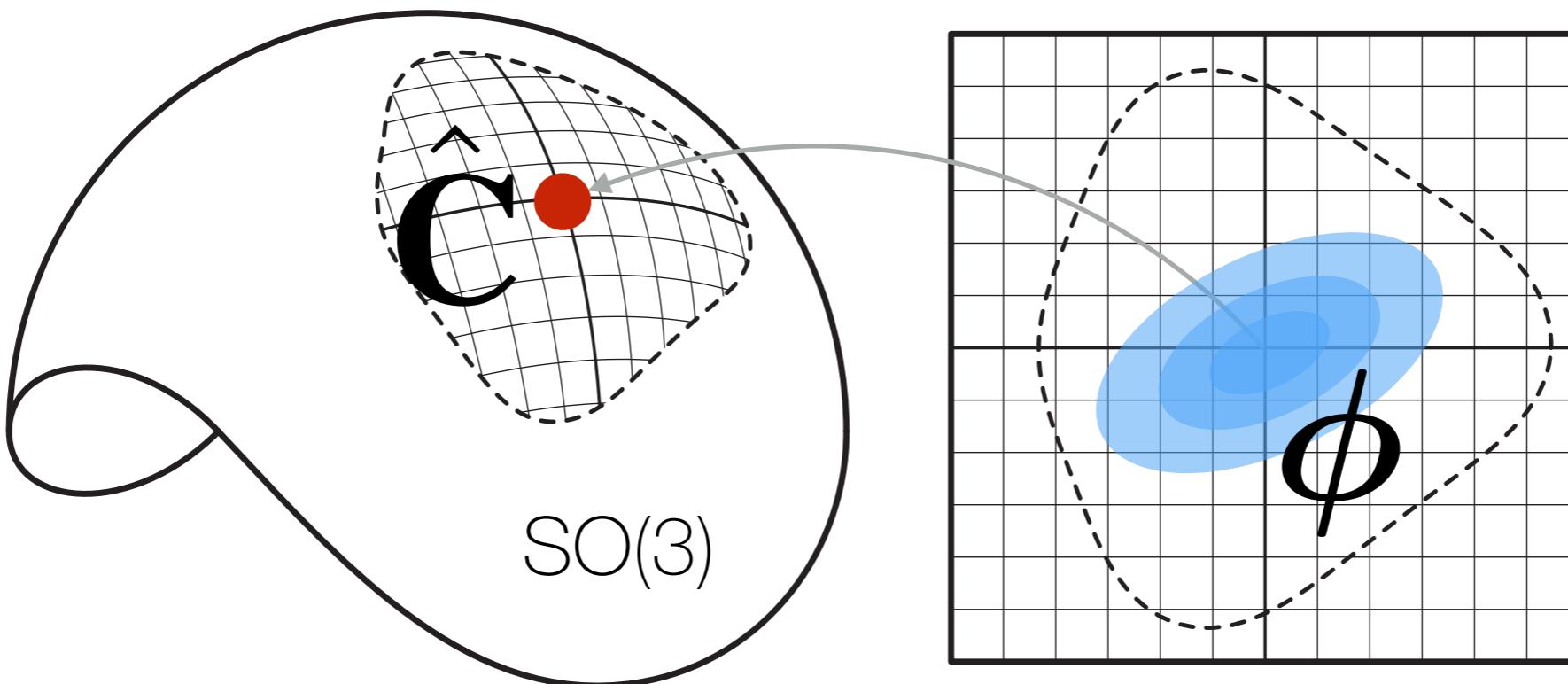
$$\mathbf{x} = \mathbf{x}^* + \boldsymbol{\epsilon}_x, \quad \boldsymbol{\epsilon}_x \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$

$$m = \varphi_m(\varphi_m^{-1}(\bar{m}) + \boldsymbol{\epsilon}_m) \quad \boldsymbol{\epsilon}_m \sim \mathcal{N}(\mathbf{0}, \mathbf{P})$$



# SO(3) – Uncertainty

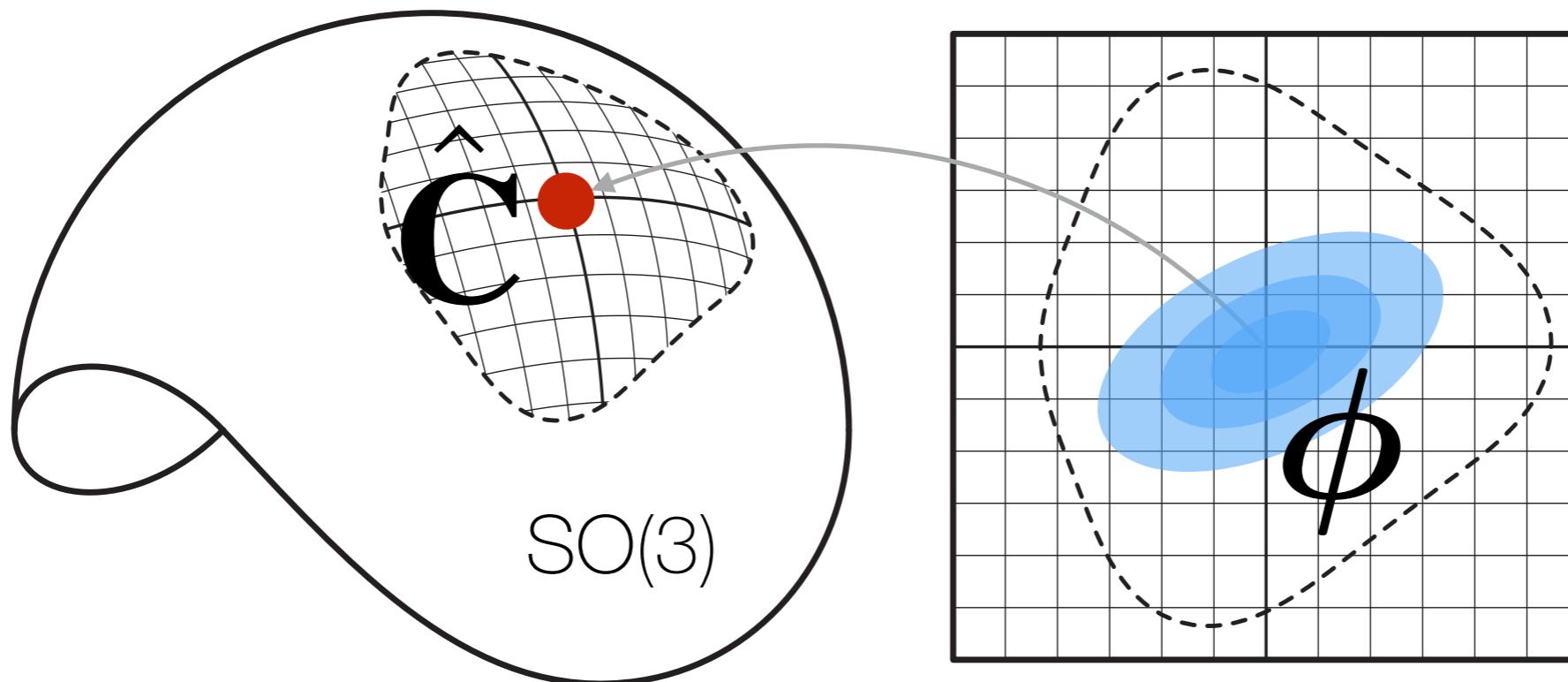
$$\mathbf{C} = \exp(\phi)\hat{\mathbf{C}}, \quad \phi \sim \mathcal{N}(0, \mathbf{P})$$



This is called an *injection* of noise onto SO(3)

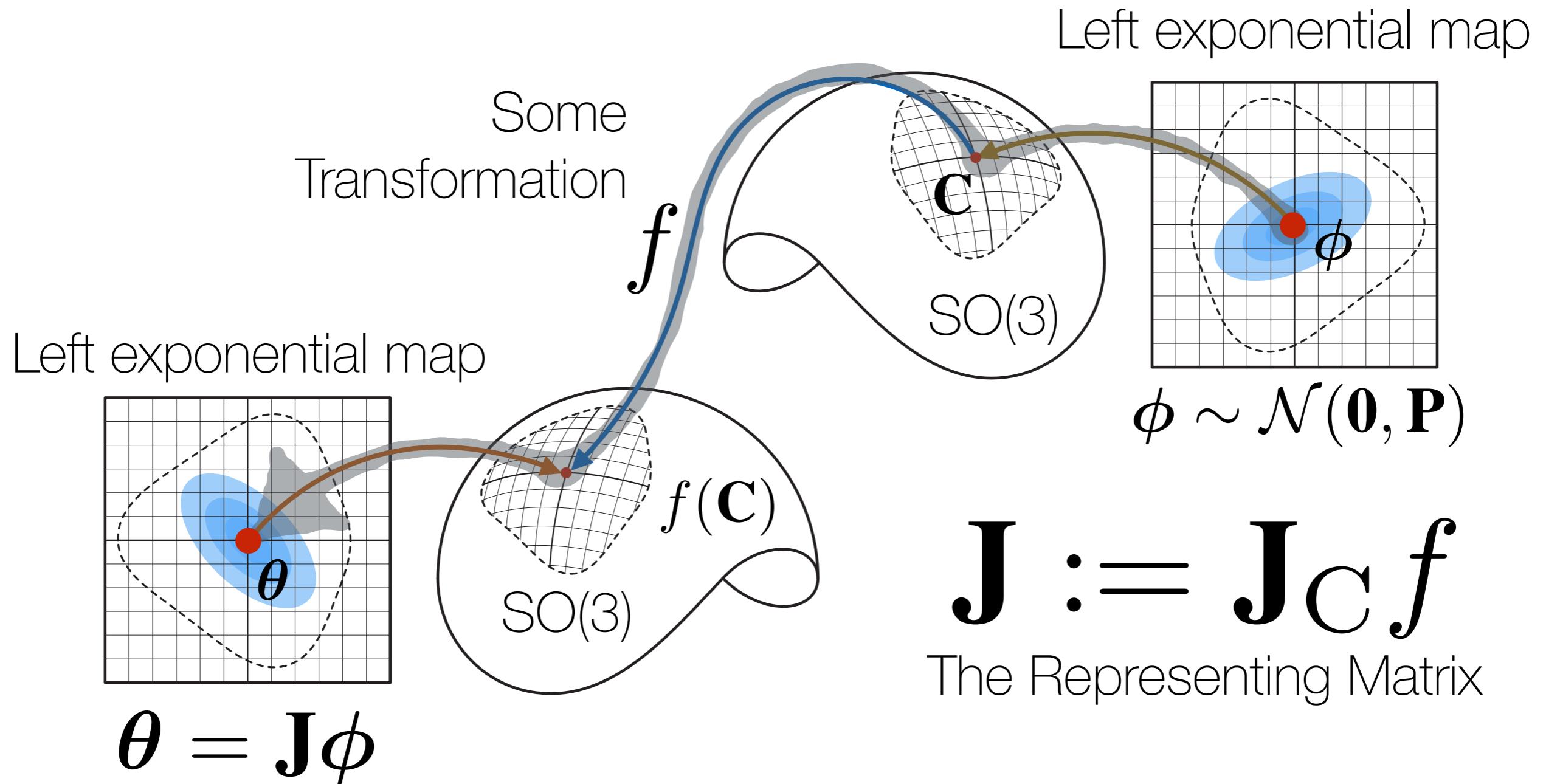
# SO(3) – Uncertainty

$$\mathbf{C} \sim \mathcal{N}(\hat{\mathbf{C}}, \mathbf{P})$$

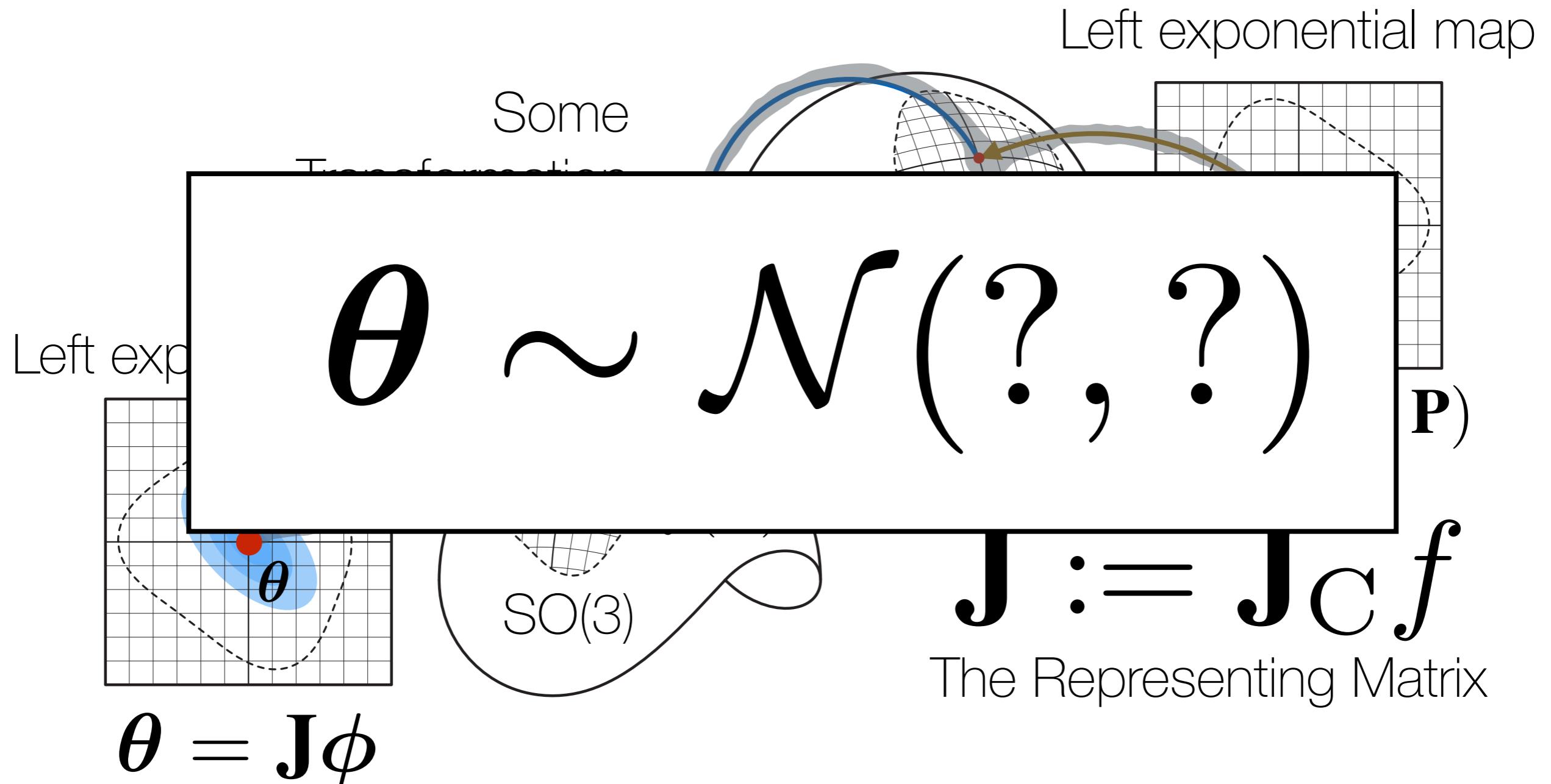


This is called an *injection* of noise onto  $\text{SO}(3)$

# SO(3) – Transforming Uncertainties (to first order)



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# SO(3) – Transforming Uncertainties (to first order)

$$\theta = \mathbf{J}\phi, \quad \phi \sim \mathcal{N}(0, \mathbf{P})$$

$$E[\theta]$$



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$$\begin{aligned} E[\theta] \\ = E[\mathbf{J}\phi] \end{aligned}$$



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$$\begin{aligned} E[\theta] \\ = E[\mathbf{J}\phi] \\ = \mathbf{J}E[\phi] \end{aligned}$$



# SO(3) – Transforming Uncertainties (to first order)

$$\theta = \mathbf{J}\phi, \quad \phi \sim \mathcal{N}(0, \mathbf{P})$$

$$\begin{aligned} E[\theta] \\ = & E[\mathbf{J}\phi] \\ = & \mathbf{J}E[\phi] \\ = & 0 \end{aligned}$$



# SO(3) – Transforming Uncertainties (to first order)

$$\theta = \mathbf{J}\phi, \quad \phi \sim \mathcal{N}(0, \mathbf{P})$$

$$E[\theta]$$

$$\theta \sim \mathcal{N}(\text{?}, \text{?})$$

$$= 0$$



# SO(3) – Transforming Uncertainties (to first order)

$$\theta = \mathbf{J}\phi, \quad \phi \sim \mathcal{N}(0, \mathbf{P})$$

$$E[\theta]$$

$$\theta \sim \mathcal{N}(0, ?)$$

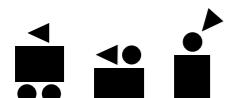
$$= 0$$



# SO(3) – Transforming Uncertainties (to first order)

$$\theta = \mathbf{J}\phi, \quad \phi \sim \mathcal{N}(0, \mathbf{P})$$

$$E [\theta\theta^T]$$



# SO(3) – Transforming Uncertainties (to first order)

$$\theta = \mathbf{J}\phi, \quad \phi \sim \mathcal{N}(0, \mathbf{P})$$

$$\begin{aligned} & E [\theta\theta^T] \\ &= E [\mathbf{J}\phi\phi^T\mathbf{J}^T] \end{aligned}$$



# SO(3) – Transforming Uncertainties (to first order)

$$\theta = \mathbf{J}\phi, \quad \phi \sim \mathcal{N}(0, \mathbf{P})$$

$$\begin{aligned} & E[\theta\theta^T] \\ &= E[\mathbf{J}\phi\phi^T\mathbf{J}^T] \\ &= \mathbf{J}E[\phi\phi^T]\mathbf{J}^T \end{aligned}$$



# SO(3) – Transforming Uncertainties (to first order)

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$$\begin{aligned} & E[\theta\theta^T] \\ &= E[\mathbf{J}\phi\phi^T\mathbf{J}^T] \\ &= \mathbf{J}E[\phi\phi^T]\mathbf{J}^T \\ &= \mathbf{JPJ}^T \end{aligned}$$



# SO(3) – Transforming Uncertainties (to first order)

$$\theta = \mathbf{J}\phi, \quad \phi \sim \mathcal{N}(0, \mathbf{P})$$

$$E [\theta\theta^T]$$

$$\theta \sim \mathcal{N}(0, ?)$$

$$= \mathbf{J} \mathbf{P} \mathbf{J}^T$$



# SO(3) – Transforming Uncertainties (to first order)

$$\theta = \mathbf{J}\phi, \quad \phi \sim \mathcal{N}(0, \mathbf{P})$$

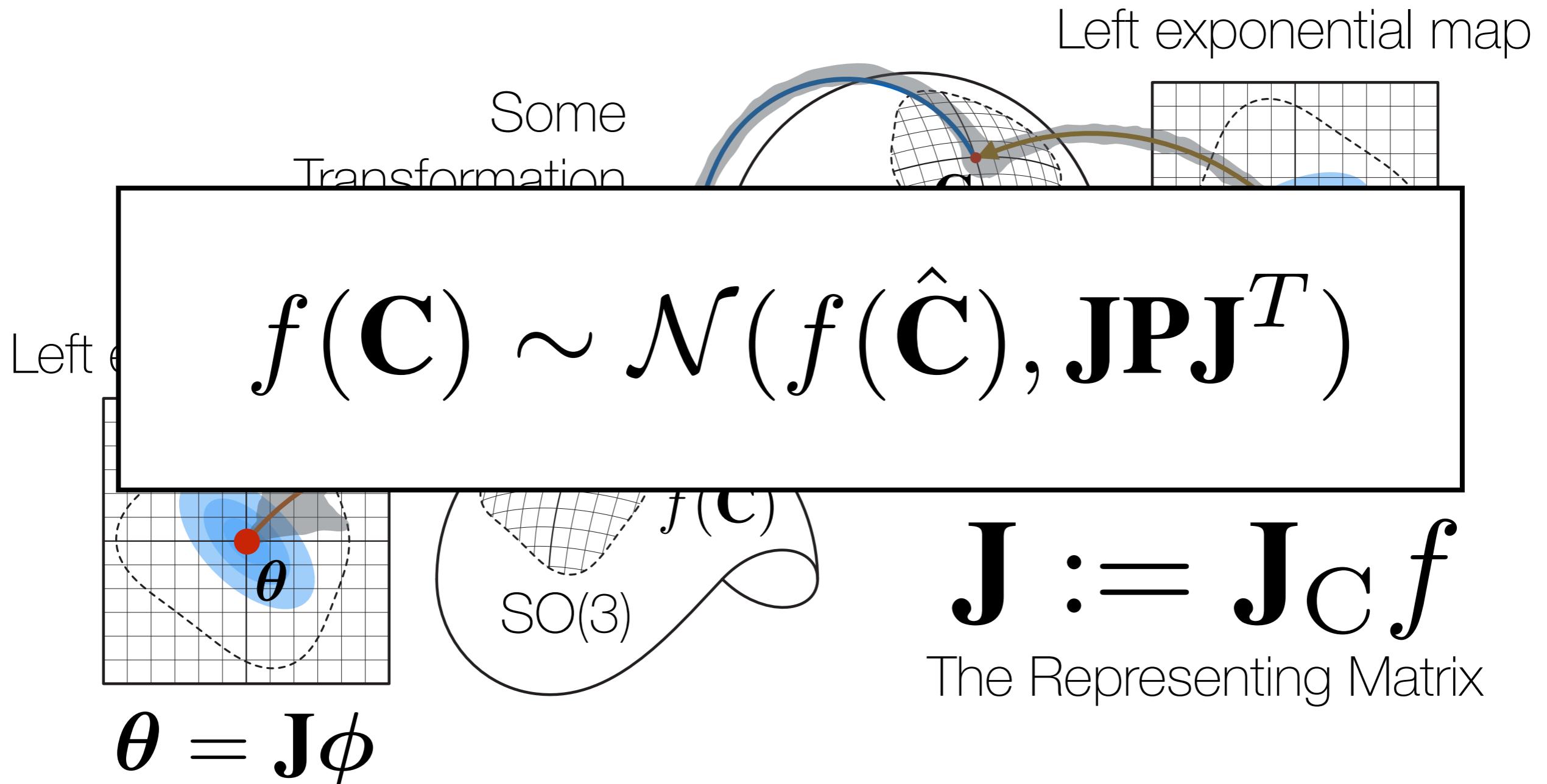
$$E [\theta\theta^T]$$

$$\theta \sim \mathcal{N}(0, \mathbf{JPJ}^T)$$

$$= \mathbf{JPJ}^T$$



# SO(3) – Transforming Uncertainties (to first order)

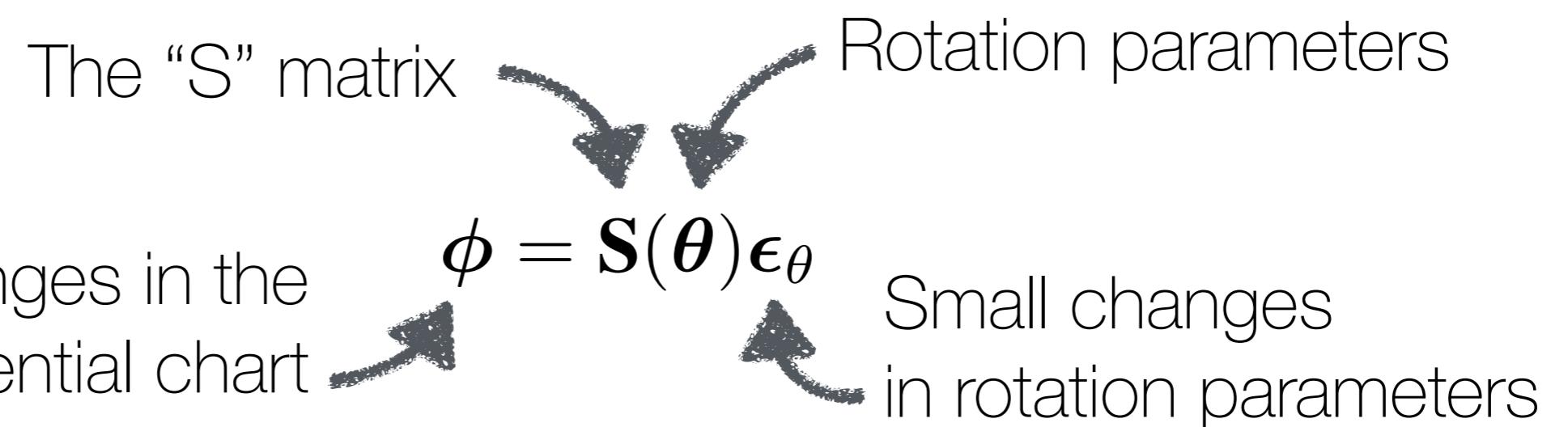


# Relating Different Parameterizations

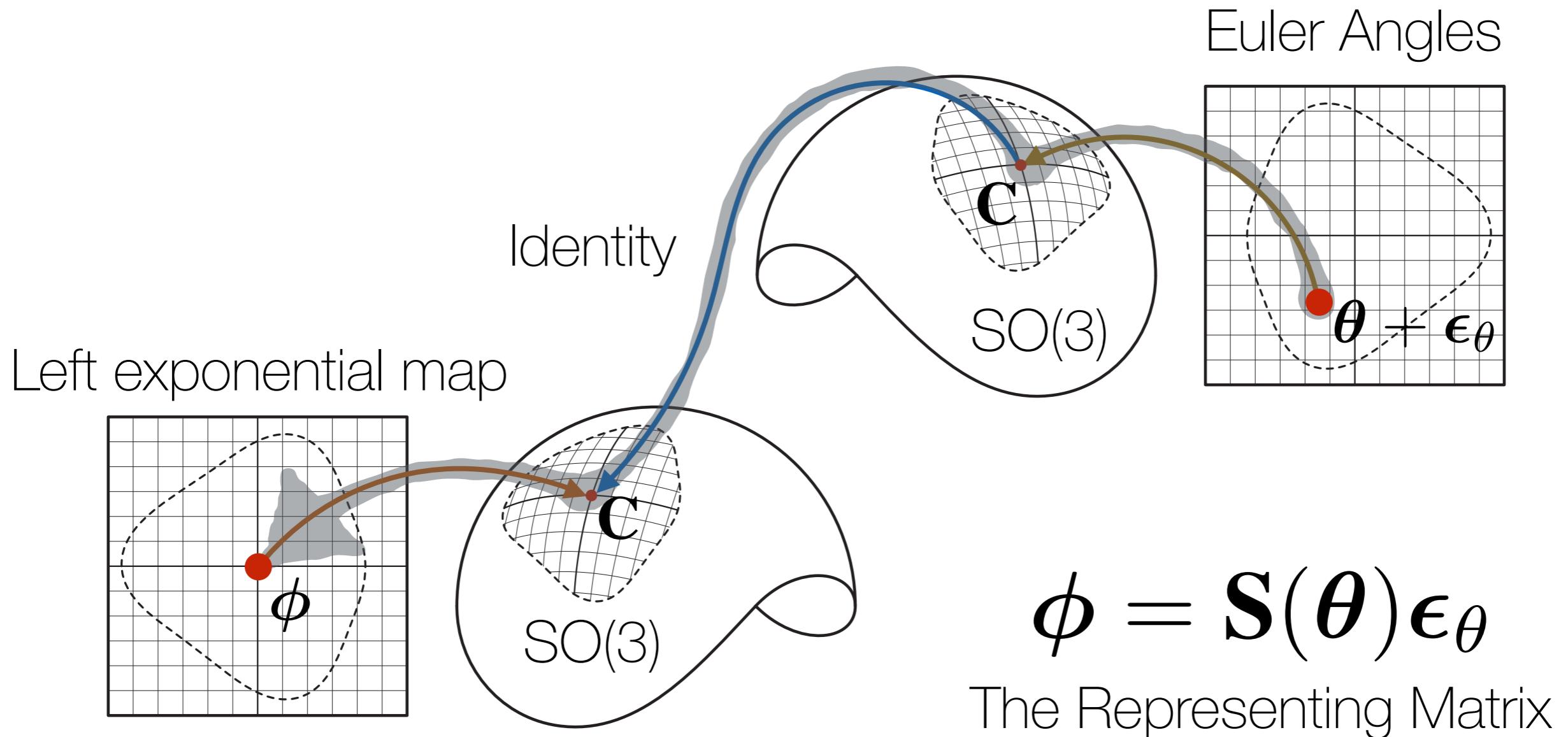
- We know the mappings between the left-exponential chart and other rotation parameterizations
- Amazingly, this is the same relationship between angular velocity and parameter rates
- The formulas are available in standard textbooks on spacecraft attitude dynamics

# Relating Different Parameterizations

- We know the mappings between the left-exponential chart and other rotation parameterizations
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# Relating Different Parameterizations



## 30 ROTATIONAL KINEMATICS

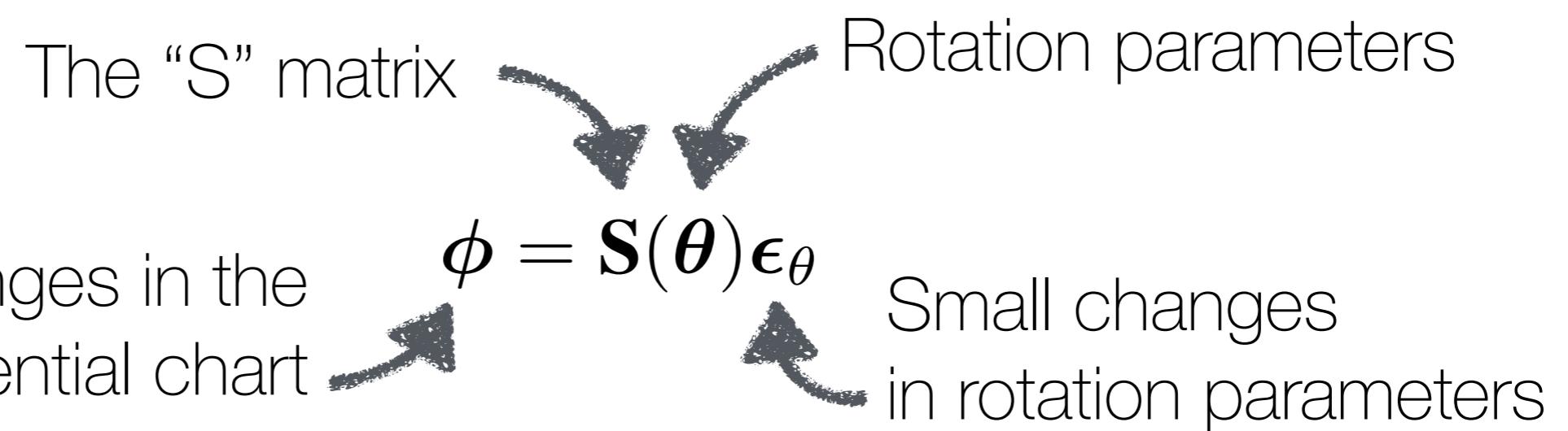
**TABLE 2.3**  
**Comparison of Parameter Alternatives**

Name	<i>n</i>	Parameters, $\mathbf{v}$	Definition	$\mathbf{C}(\mathbf{v})$
Direction cosines	9	$c_{ij}$	(2.1, 6)	$\mathbf{C} = \{c_{ij}\}$
Direction cosines	6	$c_{ij}$	(2.1, 6)	$\mathbf{C} = \{c_{ij}\}$
Axis / angle variables	4	$a_1, a_2, a_3, \phi$	Euler's theorem	$\mathbf{C} = 1 \cos \phi + (1 - \cos \phi)aa^T - \sin \phi \mathbf{a}^\times$
Axis / angle variables	3	$\phi_1, \phi_2, \phi_3$	$\phi = \phi \mathbf{a}$	$\mathbf{C} = 1 + b_1(\phi)\mathbf{a}^\times + b_2(\phi)\mathbf{a}^\times\mathbf{a}^\times$ $b_1 \triangleq -\phi^{-1} \sin \phi;$ $b_2 \triangleq 2\phi^{-2} \sin^2 \frac{1}{2}\phi$
Euler – Rodriguez (see Probs. 2.12, 19)	3	$p_1, p_2, p_3$	$\mathbf{p} = \mathbf{a} \tan \frac{1}{2}\phi$	$\mathbf{C} = 1 + \frac{2}{1 + \mathbf{p}^T \mathbf{p}} (\mathbf{p}^\times \mathbf{p}^\times - \mathbf{p}^\times)$
Euler parameters	4	$\varepsilon_1, \varepsilon_2, \varepsilon_3, \eta$	$\varepsilon = \mathbf{a} \sin \frac{1}{2}\phi$ $\eta = \cos \frac{1}{2}\phi$	$\mathbf{C} = (\eta^2 - \varepsilon^T \varepsilon)1 + 2\varepsilon \varepsilon^T - 2\eta \varepsilon^\times$
Euler parameters	3	$\varepsilon_1, \varepsilon_2, \varepsilon_3$	$\varepsilon = \mathbf{a} \sin \frac{1}{2}\phi$	$\mathbf{C} = 1 + 2\varepsilon^\times \varepsilon^\times - 2(1 - \varepsilon^T \varepsilon)^{1/2} \varepsilon^\times$
Euler angles	3	$\theta_1, \theta_2, \theta_3$	Fig. 2.2	$\mathbf{C} = \mathbf{C}_i(\theta_1) \mathbf{C}_j(\theta_2) \mathbf{C}_k(\theta_3);$ $i \neq j; j \neq k$ (see Table 2.1)

Name	$\omega = \mathbf{S}(\mathbf{v})\mathbf{v}$	$\mathbf{v} = \mathbf{S}^{-1}(\mathbf{v})\omega$
Direction cosines	$\omega^* = -\dot{\mathbf{C}}\mathbf{C}^T \equiv \mathbf{C}\dot{\mathbf{C}}^T$	$[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3] \triangleq \mathbf{C}$ $\dot{\mathbf{c}}_1 = \mathbf{c}_1^* \omega; \dot{\mathbf{c}}_2 = \mathbf{c}_2^* \omega; \dot{\mathbf{c}}_3 = \mathbf{c}_3^* \omega$ $\mathbf{c}_1 = \mathbf{c}_1^* \omega; \mathbf{c}_2 = \mathbf{c}_2^* \omega; \mathbf{c}_3 = \mathbf{c}_1^* \mathbf{c}_2$
Direction cosines	—	
Axis / angle variables	$\omega = \dot{\phi} \mathbf{a} - (1 - \cos \phi) \mathbf{a}^* \mathbf{a} + \mathbf{a} \sin \phi$	$\mathbf{a} = \frac{1}{2}(\mathbf{a}^* - \cot \frac{1}{2}\phi \mathbf{a}^* \mathbf{a}^*)\omega$ $\dot{\phi} = \mathbf{a}^T \omega$
Axis / angle variables	$\omega = (1 + c_1(\phi)\phi^* + c_2(\phi)\phi^*\phi^*)\dot{\phi}$ $c_1 \triangleq -2\phi^{-2} \sin^2 \frac{1}{2}\phi;$ $c_2 \triangleq \phi^{-3}(\phi - \sin \phi)$	$\dot{\phi} = (1 + \frac{1}{2}\phi^* + a_1(\phi)\phi^*\phi^*)\omega$ $\phi \triangleq (\phi^T \phi)^{1/2}; a_1 \triangleq \phi^{-2}(1 - \frac{1}{2}\phi \cot \frac{1}{2}\phi)$
Euler – Rodriguez (see Probs. 2.12, 19)	$\omega = \frac{2}{1 + \mathbf{p}^T \mathbf{p}}(1 - \mathbf{p}^*)\mathbf{p}$	$\mathbf{p} = \frac{1}{2}(\mathbf{p}\mathbf{p}^T + 1 + \mathbf{p}^*)\omega$
Euler parameters	$\omega = 2(\eta \dot{\epsilon} - \dot{\eta} \epsilon) - 2\epsilon^* \dot{\epsilon}$	$\dot{\epsilon} = \frac{1}{2}(\epsilon^* + \eta \mathbf{1})\omega$ $\dot{\eta} = -\frac{1}{2}\epsilon^T \omega$
Euler parameters	$\omega = 2 \left[ \frac{1 + \epsilon^* \epsilon^*}{(1 - \epsilon^T \epsilon)^{1/2}} - \epsilon^* \right] \dot{\epsilon}$	$\dot{\epsilon} = \frac{1}{2}\{\epsilon^* + (1 - \epsilon^T \epsilon)^{1/2} \mathbf{1}\}\omega$
Euler angles	$\omega = [1_i \quad \mathbf{C}_i \mathbf{1}_j \quad \mathbf{C}_i \mathbf{C}_j \mathbf{1}_k] \dot{\theta}$	$\dot{\theta} = \mathbf{S}^{-1}(\theta_1, \theta_2)\omega$ (see Table 2.2)

# Relating Different Parameterizations

- We know the mappings between the left-exponential chart and other rotation parameterizations
- Once you know the S matrix for each parameterization, you can transform uncertainties between parameterizations



# Relating Different Rotation Parameterizations

$$\phi = S(\theta)\epsilon_\theta$$

Hughes, P. C. (1986). *Spacecraft Attitude Dynamics*. John Wiley & Sons, New York.



**T<sub>AB</sub>**

**SE(3)**

The Special Euclidean Group

# SE(3) - The Special Euclidean Group

- Things work exactly the same
  - Tim Barfoot and Paul Furgale (2014). Associating Uncertainty with Three-Dimensional Poses for use in Estimation Problems. *Robotics, IEEE Transactions on*, 30(3), pp. 679-693.

# **Summary**

# Nonlinear Least Squares / Gauss Newton

Goal: estimate  $\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} J(\mathbf{x}) = \frac{1}{2} \mathbf{e}(\mathbf{x})^T \mathbf{R}^{-1} \mathbf{e}(\mathbf{x})$ .

- Start with an initial guess for  $\mathbf{x}$
- Iterate
  - Linearize
  - Solve the linear system
  - Update the parameters
  - Repeat until convergence

$$\bar{\mathbf{e}} := \mathbf{e}(\bar{\mathbf{x}}), \quad \mathbf{E} := \left. \frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \right|_{\bar{\mathbf{x}}}, \quad \mathbf{e}(\epsilon_x) := \bar{\mathbf{e}} + \mathbf{E} \epsilon_x.$$

$$\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E} \epsilon_x^* = -\mathbf{E}^T \mathbf{R}^{-1} \bar{\mathbf{e}}$$

$$\bar{\mathbf{x}} \leftarrow \bar{\mathbf{x}} + \epsilon_x^*$$

Recover the covariance:

$$\mathbf{P} := (\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1},$$

$$\mathbf{x} = \mathbf{x}^* + \epsilon_x, \quad \epsilon_x \sim \mathcal{N}(\mathbf{0}, \mathbf{P}).$$

# Calculus

$$\frac{\partial \mathbf{e}(\mathbf{x})}{\partial \mathbf{x}} \Big|_{\bar{\mathbf{x}}}$$

# Linear Algebra

$$\epsilon_x^* = -(\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1} \mathbf{E}^T \mathbf{R}^{-1} \bar{\mathbf{e}}$$

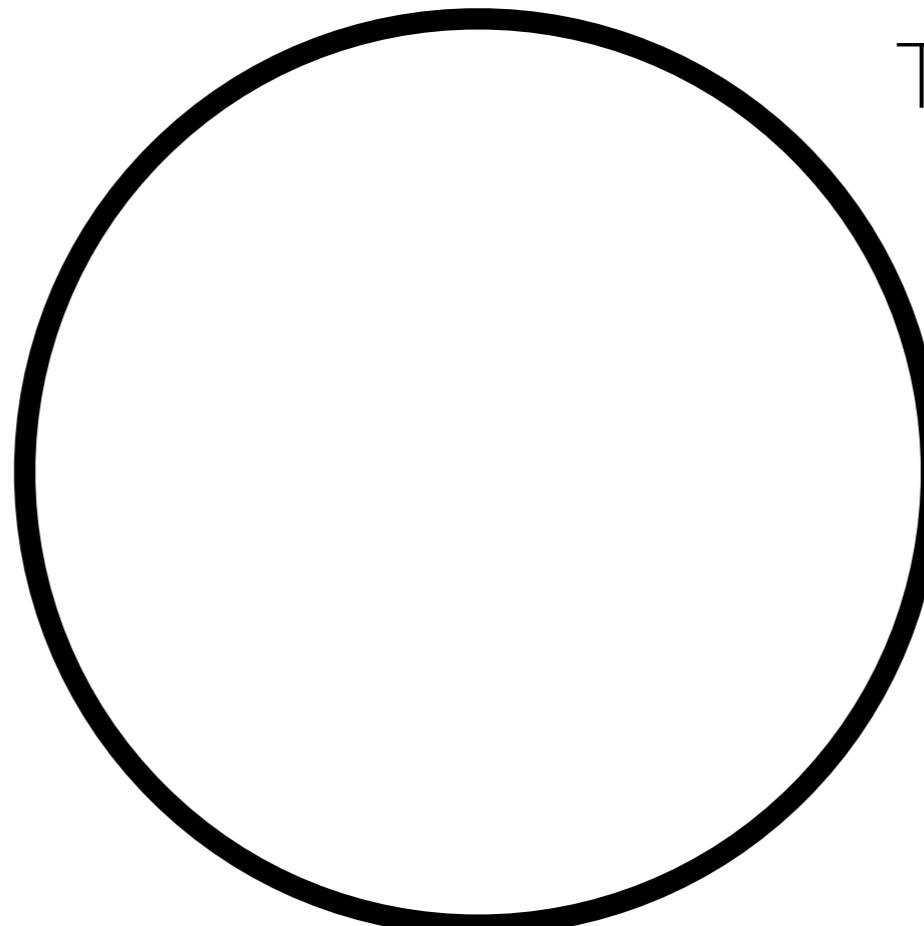
# Probabilities

$$\mathbf{P} := (\mathbf{E}^T \mathbf{R}^{-1} \mathbf{E})^{-1},$$

$$\mathbf{x} = \mathbf{x}^\star + \boldsymbol{\epsilon}_x, \quad \boldsymbol{\epsilon}_x \sim \mathcal{N}(0, \mathbf{P})$$

# Basic assumption

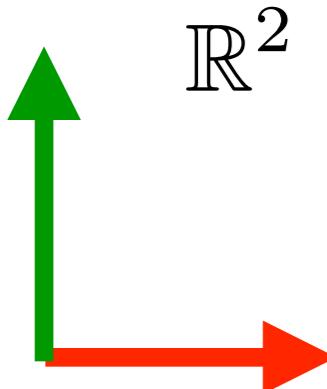
$$x \in \mathbb{R}^M$$



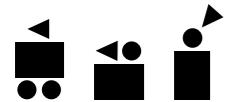
The Manifold

$$S^1$$

The Ambient Space

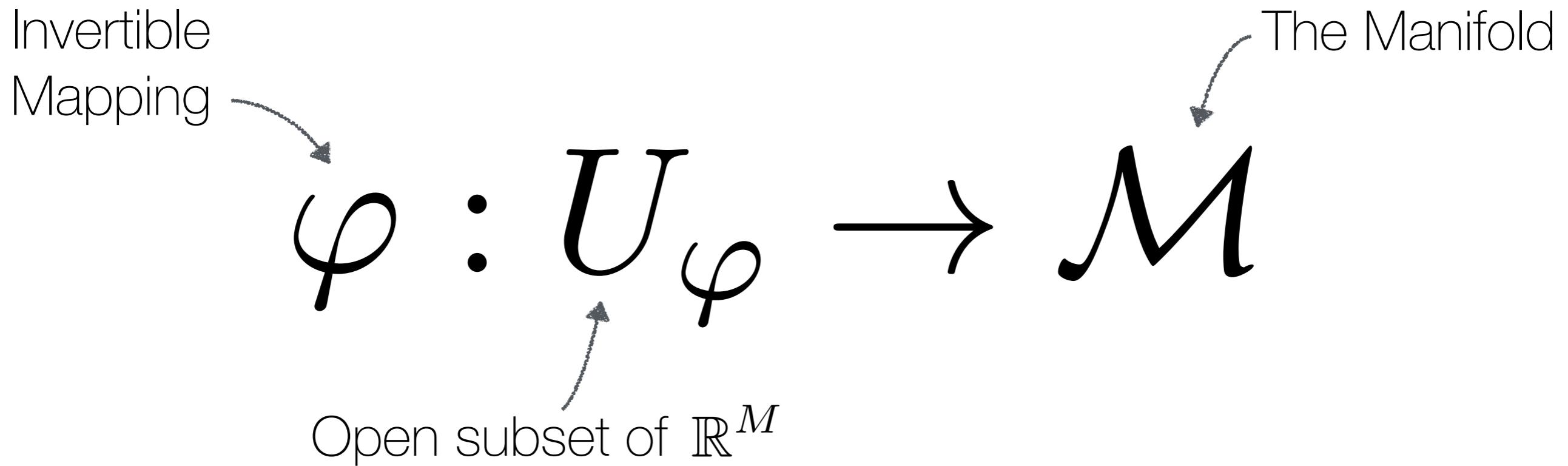


$$\mathbb{R}^2$$

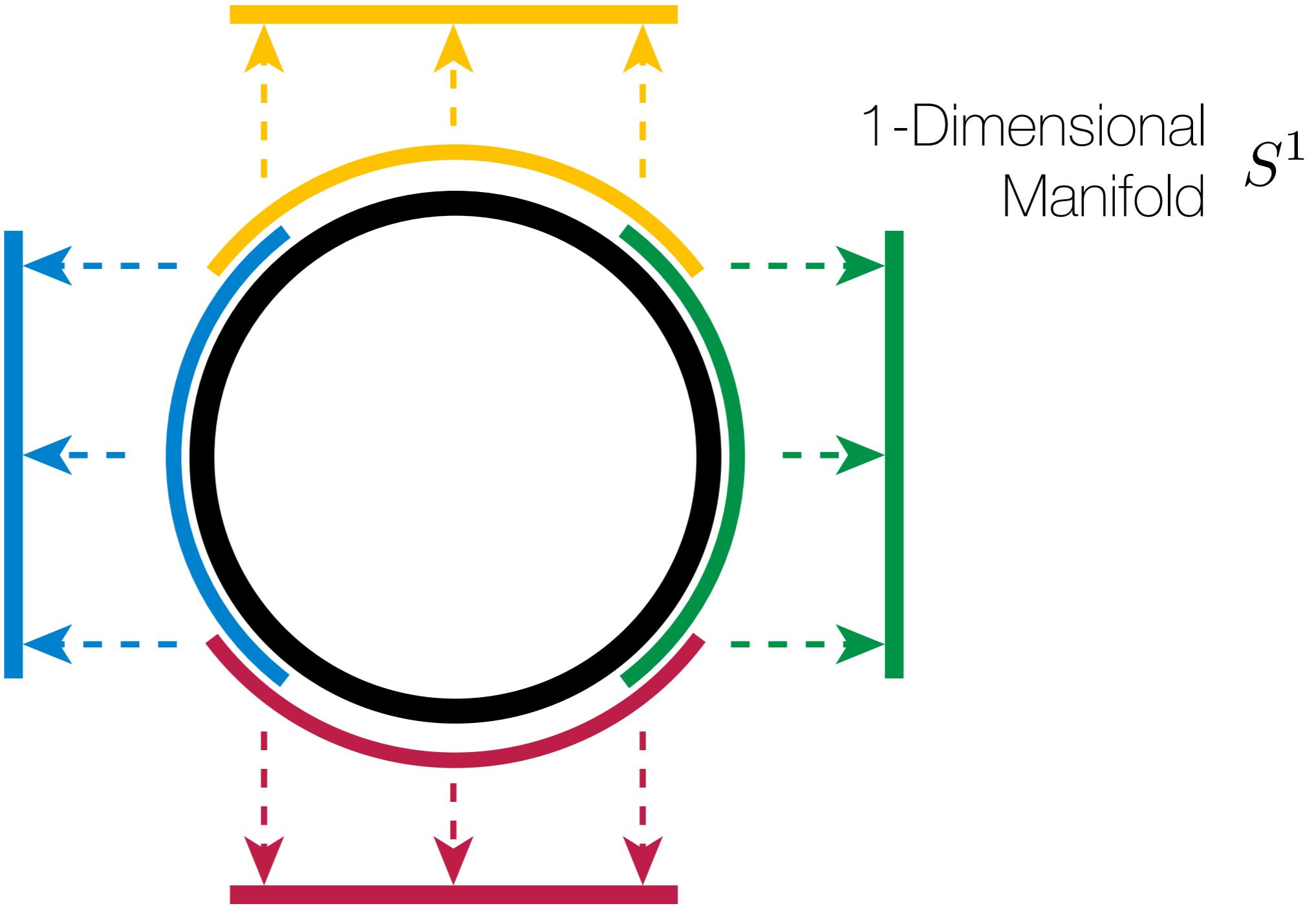
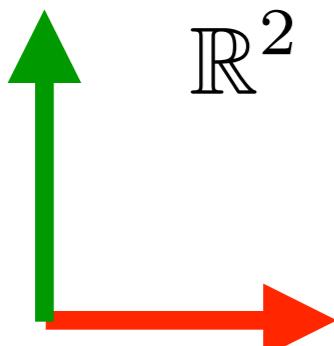


# Choosing an Atlas

- Charts:
  - For an  $M$ -Dimensional DM,  $\mathcal{M}$ , a *chart* is

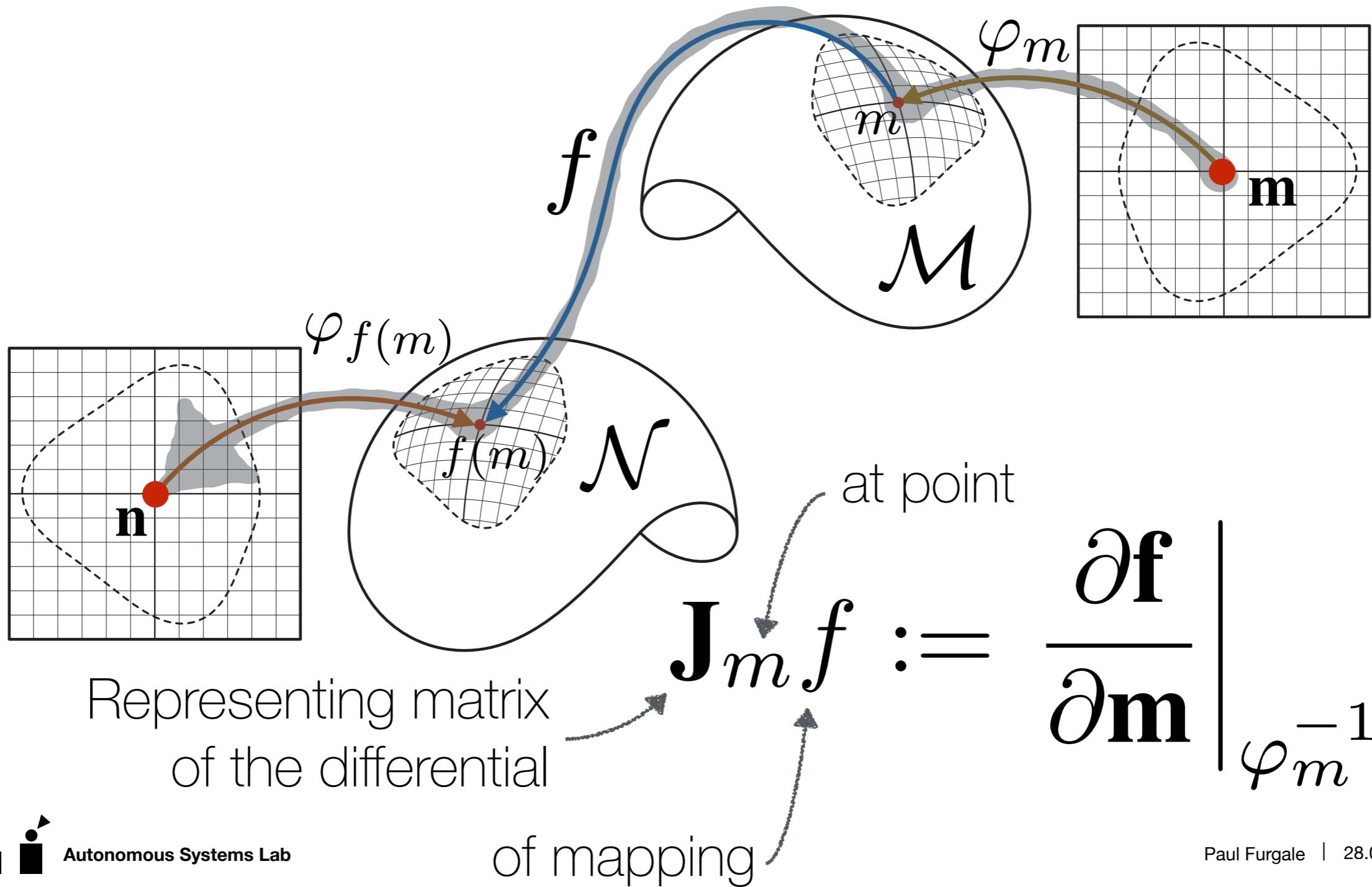


The  
Ambient  
Space



- An *Atlas* is a collection of charts such that cover the manifold

$$\mathbf{f} := \varphi_{f(m)}^{-1} \circ f \circ \varphi_m$$



**C<sub>AB</sub>**

**so(3)**

The Special Orthogonal Group

# Fundamental Identities (Cheat Sheet)

$$\mathbf{a}^\wedge \mathbf{b} = -\mathbf{b}^\wedge \mathbf{a}$$

$$\mathbf{C}\mathbf{a}^\wedge = (\mathbf{C}\mathbf{a})^\wedge \mathbf{C}$$

$$\mathbf{a}^\wedge \mathbf{C} = \mathbf{C}(\mathbf{C}^T \mathbf{a})^\wedge$$

$$(\mathbf{a}^\wedge)^T = -\mathbf{a}^\wedge$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}$$

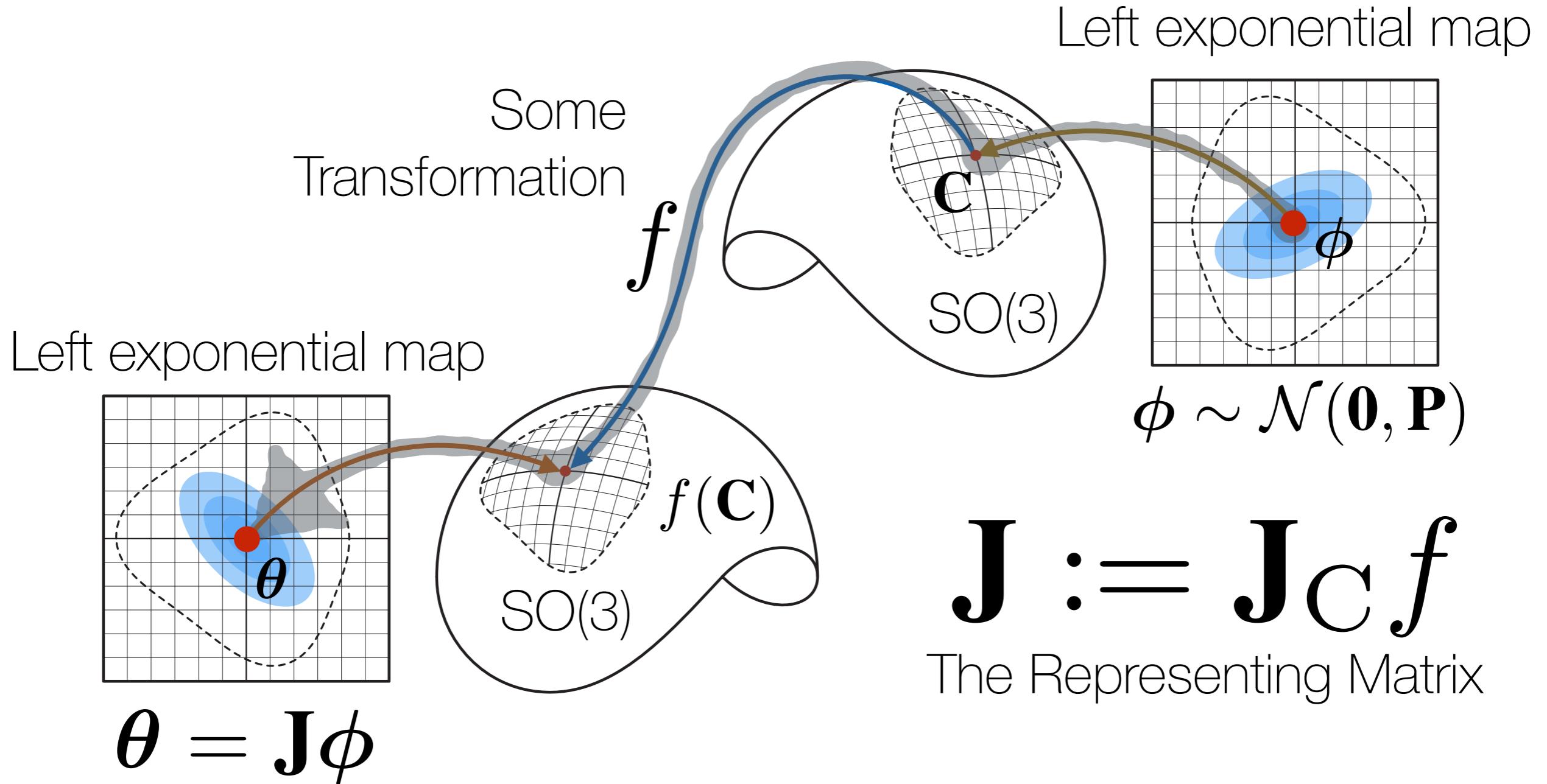


# Checking your answers

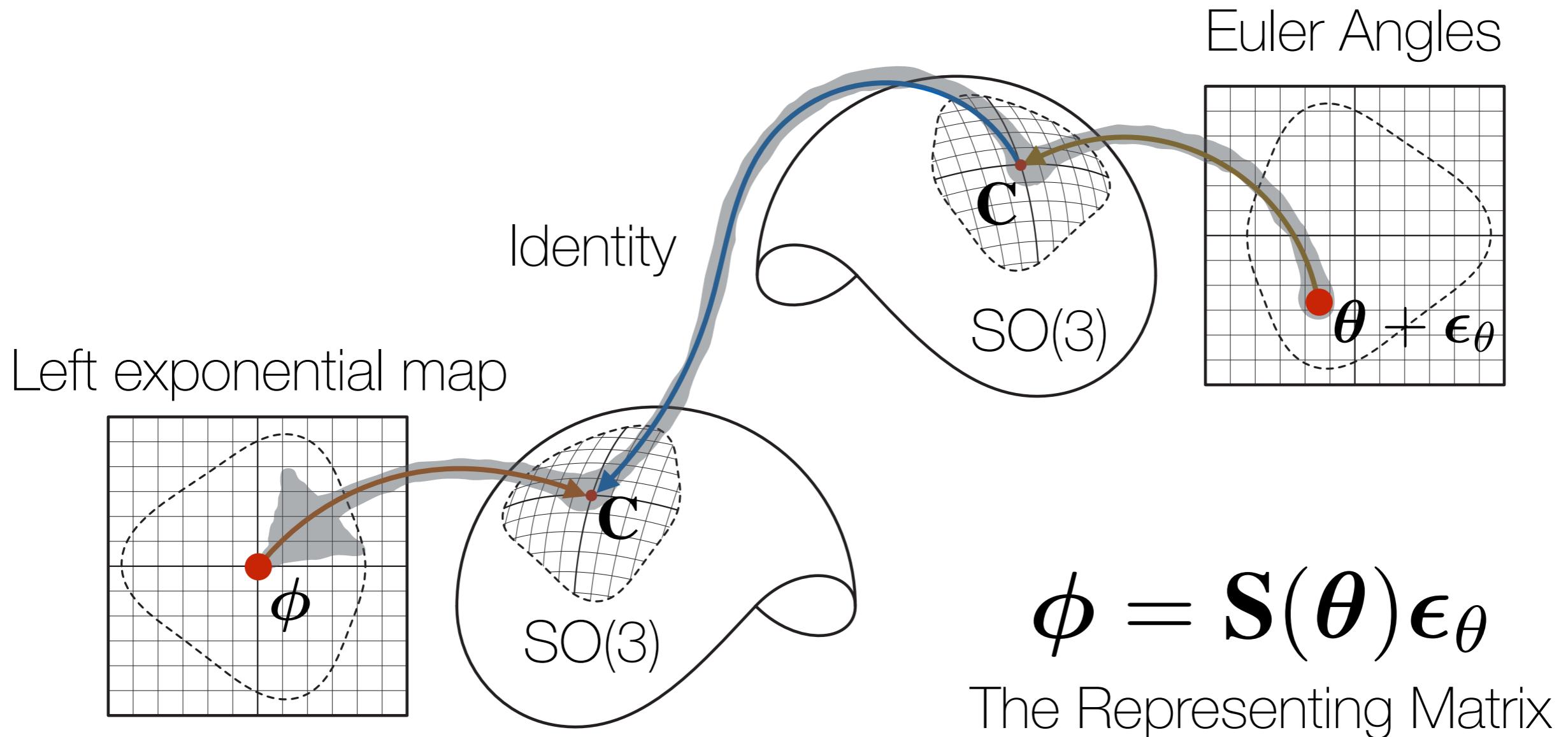
```
>> jacobianest( @(phi) so3log( (so3exp(phi)*c)'*c ), zeros(3,1) )
```



# SO(3) – Transforming Uncertainties (to first order)



# Relating Different Parameterizations



# Reading List

- Steven Lavalle, Planning Algorithms, Cambridge University Press 2006 (<http://planning.cs.uiuc.edu/>)
- Murray, Richard M., et al. A mathematical introduction to robotic manipulation. CRC press, 1994. ([http://nyx-www.informatik.uni-bremen.de/444/1/li\\_book\\_06.pdf](http://nyx-www.informatik.uni-bremen.de/444/1/li_book_06.pdf))
- Hughes, Peter C. Spacecraft attitude dynamics. Courier Dover Publications, 2012.