A Book of Abstract Algebra
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Chapter Two Exercises
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A. Examples of Operations

1. $a * b = \sqrt{|ab|}$ on the set $\mathbb{Q}$.

Solution. This is not an operation on $\mathbb{Q}$. It violates $a * b$ being uniquely defined. For example $\sqrt{|4 \cdot 1|} = \sqrt{4} = \pm 2$.
It is also not closed. For example $\sqrt{|2 \cdot 1|} = \sqrt{2}$ which is not a rational number.

2. $a * b = a \ln b$ on the set $\{x \in \mathbb{R} : x > 0\}$.

Solution. This is a valid operation.

3. $a * b$ is a root of the equation $x^2 - a^2b^2 = 0$, on the set $\mathbb{R}$.

Solution. This is not an operation on $\mathbb{R}$ as it is not uniquely defined for any $a, b \in \mathbb{R}$ as we have roots $x = \pm ab$

4. Subtraction on the set $\mathbb{Z}$.

Solution. This is a valid operation.

5. Subtraction, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

Solution. This is not an operation on the given set as it is not closed. For example $4 - 5 = -1$ and -1 is not an element of the set.

6. $a * b = |a - b|$, on the set $\{n \in \mathbb{Z} : n \geq 0\}$.

Solution. This is a valid operation.

B. Properties of Operations

1. $x * y = x + 2y + 4$

Solution.
(i) This is not commutative.

$$x * y = x + 2y + 4; \;\; y * x = y + 2x + 4$$

(ii) This is not associative.

$$x * (y * z) = x * (y + 2z + 4) = x + 2(y + 2z + 4) + 4 = x + 2y + 4z + 12$$

$$(x * y) * z = (x + 2y + 4) * z = (x + 2y + 4) + 2z + 4 = x + 2y + 2z + 8$$

(iii) There is no identity element.

$$x * e = x : \;\; x + 2e + 4 = x \Rightarrow e = -2$$

$$x * (-2) = x + 2(-2) + 4 = x$$

$$(-2) * x = -2 + 2x + 4 \neq x$$

(iv) There is no identity, so there are no inverses.

2. $x * y = x + 2y - xy$

Solution.
(i) This is not commutative.

$$x * y = x + 2y - xy; \;\; y * x = y + 2x - xy$$

(ii) This is not associative.

$$\begin{aligned} x * (y * z) &= x * (y + 2z - yz) \\ &= x + 2(y + 2z - yz) - x(y + 2z - yz) \\ &= x + 2y + 4z - 2yz - xy - 2xz + xyz \end{aligned}$$

$$\begin{aligned} (x * y) * z &= (x + 2y - xy) * z \\ &= x + 2y - xy + 2z - (x + 2y - xy)z \\ &= x + 2y - xy + 2z - xz - 2yz + xyz \end{aligned}$$

(iii) There is no identity element.

$$x * e = x;\ x + 2e - xe = x \Rightarrow e = 0$$

$$x * 0 = x + 2(0) - x(0) = x$$

$$0 * x = 0 + 2x - (0)x = 2x$$

(iv) There is no identity, so there are no inverses.

3. $x * y = \|x + y\|$

Solution.
(i) The operation is commutative.

$$x * y = |x + y|$$

$$y * x = |y + x|$$

(ii) The operation is not associative.

$$x * (y * z) = x * |y + z| = |x + |y + z||$$

$$(x * y) * z = |x + y| * z = ||x + y| + z|$$

(iii) The operation has no identity element.

$$x * e = x \ \Rightarrow |x + e| = x;\ e = 0 \text{ and } e = -2x$$

(iv) There is no identity, so there are no inverses.

4.$x * y = \|x - y\|$

Solution.
(i) The operation is commutative.

$$x * y = |x - y|$$

$$y * x = |y - x|$$

(ii) The operation is not associative.

$$x * (y * z) = x * |y - z| = |x - |y - z||$$

$$*x * y) * z = |x - y| * z = ||x - y| - z|$$

(iii) The operation has no identity element.

$$x * e = x \ \Rightarrow |x - e| = x; e = 0 \text{ and } e = 2x$$

(iv) There is no identity, so there are no inverses.

5. $x * y = xy + 1$

Solution.
(i) The operation is commutative.

$$x * y = xy + 1$$

$$y * x = yx + 1$$

(ii) The operation is not associative.

$$x * (y * z) = x * (yz + 1) = x(yz + 1) + 1 = xyz + x + 1$$

$$(x * y) * z = (xy + 1) * z = (xy + 1)z + 1 = xyz + z + 1$$

(iii) The operation has no identity element.

$$x * e = x \ \Rightarrow xe + 1 = x; e = \frac{x - 1}{x}$$

(iv) There is no identity, so there are no inverses.

6. $x * y = \max(x, y)$

Solution.
(i) The operation is commutative.

$$x * y = \max(x, y)$$

$$y * x = \max(y, x)$$

(ii) The operation is associative.

$$x * (y * z) = x * \ \max(y, z) = \max(x, \max(y, z))$$

$$(x * y) * z = \max(x, y) * z = \max(\max(x, y), z)$$

(iii) The operation has no identity element.

$$x * e = x \ \Rightarrow \max(x, e) = x;\ e \leq x$$

(iv) There is no identity, so there are no inverses.

7. $x * y = \frac{xy}{x+y+1}$ on $\mathbb{R}^+$

Solution.
(i) The operation is commutative.

$$x * y = \frac{xy}{x+y+1}$$

$$y * x = \frac{yx}{y+x+1}$$

(ii) The operation is associative.

$$x * (y * z) = x * \frac{yz}{y+z+1} = \frac{x\frac{yz}{y+z+1}}{x + \frac{yz}{y+z+1} + 1} = \frac{xyz}{xy + xz + yz + x + y + z + 1}$$

$$(x * y) * z = \frac{xy}{x+y+1} * z = \frac{\frac{xy}{x+y+1}z}{\frac{xy}{x+y+z} + z + 1} = \frac{xyz}{xy + xz + yz + x + y + z + 1}$$

(iii) The operation has no identity element.

$$x * e = x \ \Rightarrow \frac{xe}{x+e+1} = x; \ \Rightarrow 0 = x^2 + x$$

(iv) There is no identity, so there are no inverses.

C. Operations on a Two -Element Set

Let A be the two-element set $A = \{a, b\}$.

1. Write the tables of all 16 operations on A.

Solution.

O_1

(x, y)	$x * y$
(a, a)	a
(a, b)	a
(b, a)	a
(b, b)	a

O_2

(x, y)	$x * y$
(a, a)	a
(a, b)	a
(b, a)	a
(b, b)	b

O_3

(x, y)	$x * y$
(a, a)	a
(a, b)	a
(b, a)	b
(b, b)	a

O_4

(x, y)	$x * y$
(a, a)	a
(a, b)	a
(b, a)	b
(b, b)	b

O_5

(x, y)	$x * y$
(a, a)	a
(a, b)	b
(b, a)	a
(b, b)	a

O_6

(x, y)	$x * y$
(a, a)	a
(a, b)	b
(b, a)	a
(b, b)	b

O_7

(x, y)	$x * y$
(a, a)	a
(a, b)	b
(b, a)	b
(b, b)	a

O_8

(x, y)	$x * y$
(a, a)	a
(a, b)	b
(b, a)	b
(b, b)	b

O_9

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	a
(b,b)	a

O_{10}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	a
(b,b)	b

O_{11}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	b
(b,b)	a

O_{12}

(x,y)	$x*y$
(a,a)	b
(a,b)	a
(b,a)	b
(b,b)	b

O_{13}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	a
(b,b)	a

O_{14}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	a
(b,b)	b

O_{15}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	b
(b,b)	a

O_{16}

(x,y)	$x*y$
(a,a)	b
(a,b)	b
(b,a)	b
(b,b)	b

2. Identify which of the operations 0_1 to O_{16} are commutative.

Solution. O_1, O_2, O_7, O_8, O_9, O_{10}, O_{15}, O_{16} are all commutative as $a * b = b * a$.

3. Identify which of the operations, among 0_1 to O_{16}, are associative.

Solution.
O_1: is associative.
O_3: $b*(a*b) = b*a = b \neq (b*a)*b = b*b = a$
O_4: is associative.
O_5: $b*(a*b) = b*b = a \neq (b*a)*b = a*b = b$
O_6: is associative.
O_7: is associative.
O_8: is associative.
O_9: $a*(a*b) = a*a = b \neq (a*a)*b = b*b = a$.
O_{10}: is associative.
O_{11}: $a*(a*b) = a*a = b \neq (a*a)*b = b*b = a$.
O_{12}: $a*(b*a) = a*b = a \neq (a*b)*a = a*a = b$.
O_{13}: $a*(b*a) = a*a = b \neq (a*b)*a = b*a = a$.
O_{14}: $a*(b*a) = a*a = b \neq (a*b)*a = b*a = a$.
O_{15}: $a*(a*b) = a*b = b \neq (a*a)*b = b*b = a$.
$O_(16)$: is associative.

4. For which of the operations 0_1 to O_{16} is there an identity element.

Solution. O_2, O_7, O_8, O_{10}

5. For which of the operations 0_1 to O_{16} does every element have an inverse.

Solution. We only have to look at the operations that have inverses. For O_2 a does not have an inverse, and for O_8 b does not have an inverse. Each element of O_7 and O_{10} is its own inverse.

D. Automata: The Algebra of Input/Output Sequences

1. Prove the operation defined above is associative.

Proof: Let $a, b, c \in A^*$ where $a = a_1a_2 \ldots a_i$, $b = b_1b_2 \ldots b_j$ and $c = c_1c_2 \ldots c_k$.
Then $(ab)c = (a_1a_2 \ldots a_ib_1b_2 \ldots b_j)c = a_1a_2 \ldots a_ib_1b_2 \ldots b_jc_1c_2 \ldots c_k$
And $a(bc) = a(b_1b_2 \ldots b_jc_1c_2 \ldots c_k) = a_1a_2 \ldots a_ib_1b_2 \ldots b_jc_1c_2 \ldots c_k$.
$\therefore a(bc) = (ab)c$ and concatenation is associative. ■

2. Explain why the operation is not commutative.

Solution. The operation is not commutative as it can be shown that $ab \neq ba$. For example let $a = 111$ and $b = 000$ be elements of $A = \{0, 1\}$ then $ab = 111000$ and $ba = 000111$.

3. Prove that there is an identity element for this operation.

Proof: Let $a \in A^*$ Then

$$a\lambda = a \text{ and } \lambda a = a$$

Therefore λ is an identity element by definition of identity. ■