Discrete Mathematics Quiz 1
2008.04.08

Name:________________ Student ID:________________

Instructions. This is a 2-hr close book quiz. Please manage your time well. No dictionary, calculator, PDA, or any other electronic device. Any dishonorable cheating behavior will give you a miserable future. (Totally 115 points)

1. [10%] Let $d_{\min}$ denote the minimum degree of a vertex in an undirected graph $G = (V, E)$, where V and E denote the set of vertices and edges. Show that the graph contains a simple path containing at least $d_{\min}$ edges. (hint: proof by contradiction, assume the longest simple path $v_0 - v_1 - ... - v_k$ contains $k < d_{\min}$ edges, and then argue why it can NOT be the longest path.)

   Ans: Suppose not, which means the longest simple path $v_0 - v_1 - ... - v_k$ contains $k < d_{\min}$ edges. Take vertex v_k for example, there exist at least $d_{\min}$ adjacent vertices to v_k, which means at least one vertex in $N - \{v_0, v_1, ..., v_{k-1}\}$ is adjacent to v_k, and thus it implies the simple path $v_0 - v_1 - ... - v_k$ can be at least one edge longer which contradicts the fact that it is already the longest.

2. [8%] In Z_7, which of the following congruence classes are equal?
   [2], [7], [10], [16], [39], [45], [-1], [-3], [-6], [-17], [-23]

   Ans:
   [2]=[16], [10]=[45], [39]=[-3]=[-17]

3. [62%] The following table lists a number of tasks that must be completed in order for a crew of workers to finish a project.

Task	A	B	C	D	E	F	G	H	I	J
Time (in days)	2	2	3	1	1	2	3	4	3	3

   Suppose the project starts at Apr. 9. Task A must be carried out before any other tasks can start. Task B must precede tasks E and F, and both E and F must be completed before H can begin. Tasks C and D must precede task G, which in turn must precede I. Task J must be carried out last. It is assumed that there are enough workers to carry out any number of tasks simultaneously.
   (a) [5%] Draw a PERT diagram showing the relations of these tasks.
   (b) [5%] Draw the critical path.
   (c) [5%] We say a "conflict of tasks" occurs when two tasks violate their precedent relations. Give a suitable sequence for conducting all the tasks so that no conflict of tasks occurs in the entire process.

(d) [5%] What is the fewest number of days needed to make this product?
(e) [5%] What is the earliest date for this project to be done?
(f) [5%] If we start to do task D in Apr.14, will it affect your answer in (e)? If yes, affect how much? If no, explain why not.
(g) [5%] What is the latest date for task H to begin, in order not to affect your answer in (e)?
(h) [9%] For any two tasks x and y, suppose we define $x\mathbf{R}y$ to be true if either task x equals to task y or task x can not be started until task y is completed. Explain why $\mathbf{R}$ is reflexive, antisymmetric, and transitive.
(i) [6%] Following (h), what are $B\mathbf{R}J$, $I\mathbf{R}C$, $H\mathbf{R}D$?(i.e. True or False?)
(j) [4%] Following (h), let S be set of all tasks and $\mathbf{R}$ be a relation on S. List all the minimal and maximal elements of S with respect to $\mathbf{R}$.
(k) [4%] Following (h), what are the infimum and supremum of E and F? what are the infimum and supremum of G and F?
(l) [4%] Following (h), is $(S, \mathbf{R})$ a lattice? Explain your answer.(an answer without explanation gets at most 1 point)

Ans:
(a) see Fig.1
(b) A-C-G-I-J
(c) ABCDEFGHIJ (there are many other permutations, as long as it follows topological ordering)
(d) 14 days
(e) Apr.22, since Apr.9+14-1=Apr.22
(f) Yes, the earliest date for G to start was supposed to be Apr.14, determined by C. If D starts at Apr. 14, the earliest date for G to start becomes Apr.15 which will make the tight schedule 1 day longer.
(g) Apr.16. Since the earliest time for J to start is Apr.20 (Apr.22-3+1), which means the latest date for H to finish is Apr.19, and thus Apr.19-4+1=Apr.16 is the latest date for H to start, without affecting the tightest schedule in (e).
(h) $\mathbf{R}$ is obviously reflexive since a task x equals to itself.
Suppose $x \neq y$. If $x\mathbf{R}y$ is true, task x is not started until y is completed, and thus y must have been started before x is completed, which means $y\mathbf{R}x$ is false. This makes the only chance to have both $x\mathbf{R}y$ and $y\mathbf{R}x$ be true is to let $x = y$, which is true since $\mathbf{R}$ is reflexive. This shows $\mathbf{R}$ is antisymmetric.
Suppose $x \neq y$. If both $x\mathbf{R}y$ and $y\mathbf{R}z$ are true, which means y is completed before x is started, and z is completed before y is started, thus z is completed before x is started and $x\mathbf{R}z$ becomes true. So $\mathbf{R}$ is transitive.
(i) $B\mathbf{R}J = False$, $I\mathbf{R}C = True$, $H\mathbf{R}D = False$
(j) minimal elements: J; maximal element: A
(k) infimum of E,F: H, supremum of E,F: B; infimum of G,F: J, supremum of G,F: A
(l) yes, since all pairs of tasks uniquely meet or join each other.

4. [16%] Let X denote the set of real numbers. Suppose we have a function $f : X \to X$, $f(x) = \frac{1}{|x|+1}$
(a) [4%] Is f well-defined? why?
(b) [2%] $rng\ f$ =?
(c) [6%] Is f injective? why? Is f surjective? why?
(d) [4%] Derive f^{-1}, if one exists; or explain why it does not exist.

Ans:
(a) yes. since for each $x \in X$, $f(x)$ is uniquely defined in X (i.e. for a given $x \in X$, there will be only one $f(x)$)
(b) Let $y = \frac{1}{|x|+1}$.
If $x \geq 0$, $y = \frac{1}{x+1}$, $x = \frac{1-y}{y} = f^{-1}(y)$ with $y \in X - \{0\}$
If $x < 0$, $y = \frac{1}{-x+1}$, $x = \frac{y-1}{y} = f^{-1}(y)$ with $y \in X - \{0\}$
It is easy to show $y \in (0, 1]$, since $|x| \geq 0$, $1 + |x| \geq 1$, which means $\frac{1}{|x|+1} \leq 1$; also, $\lim_{x\to\infty} \frac{1}{|x|+1} = 0$.
Thus $rng\ f = (0, 1]$.
(c) f is NOT injective since if $x_1 = -x_2 \neq 0$, $f(x_1) = f(x_2)$
f is NOT surjective since $rng\ f = (0, 1] \neq X = R$.
(d) by (c), we know f^{-1} does not exist, since it is not one-to-one correspondence.

5. [19%] If X has m elements and Y has n elements,
(a) [2%] Is it possible that $m > n$, if there is a one-to-one function with domain X and codomain Y? Explain your answer.
(b) [5%] How many one-to-one functions are there with domain X and codomain Y?
(c) [2%] Is it possible that $m > n$, if there is an onto function with domain X and codomain Y? Explain your answer.
(d) [10%] How many onto functions are there with domain X and codomain Y?

Ans:
(a) No, since such a function will make some element in X not mappable to Y.
(b) When $m \leq n$, totally there are $P^n_m = n \cdot (n-1) \cdot \ldots \cdot (n-m+1)$ possible mapping relations (i.e. functions).
(c) Yes, as long as each element in X is mapped to an element in Y, and each element in $\cancel{Y}$ is mapped.
(d) It is not possible that $m < n$, if there is an onto function with domain X and codomain Y, since it will force some element in X maps to more than one element in Y, contradicts the definition of functions. Thus for an onto function with domain X and codomain Y, $m \geq n$. In this case, the total possibilities can be calculated by the total number of functions - $\sum_{i=1}^{n-1}$(the total number of functions without mapping i elements in Y) which equals to $n^m - \sum_{i=1}^{n-1}(-1)^{i+1} C^n_i (n-i)^m$.