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DETERMINING THE MAXIMAL FLOW IN A NETWORK BY THE METHOD OF PREFLOWS

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An algorithm is presented for determining the maximal flow in a network with an upper bound $O(n^3)$ on the number of operations, where n is the number of vertices of the network.

1. A flow network is an oriented graph $G = [V; U]$ with a set of vertices V, $|V| = n$, a set of arcs U, $|U| = p$, a real function, the "transmission capacity" of an arc, $c(u) > 0$, $u \in U$, a "source" $s \in V$, a "sink" $t \in V$. A function $f(u)$, $u \in U$, is called a flow if

1) $0 \leq f(u) \leq c(u) \quad \forall u \in U$,

2) $\operatorname{Div}(x) = \sum_{(x,y)\in U} f((x,y)) - \sum_{(z,x)\in U} f((z,x)) = 0 \qquad \forall x \in V/\{s, t\}$.

A flow with greatest output $v(f) = \operatorname{Div}(s) = -\operatorname{Div}(t)$ [1], is called maximal.

In [2] the problem is in effect reduced to the solution of not more than $n-1$ problems of the following type:

A network is given (a manual of shortest paths) $S_k = [V_k; U_k]$, $|V_k| \leq n$, $|U_k| \leq p$, $1 \leq k \leq n-1$, with "source" s, "sink" t, and "transmission capacity" $c_k(u)$, $u \in U_k$, such that any vertex, and also any arc, belong to some shortest (oriented) path from s to t (with a number k of arcs). Find a flow f_k such that for any (oriented) path ξ from s to t there is a saturated arc $u \in \xi$, i.e. $f_k(u) = c_k(u)$. f_k is called a *dead-end flow*.

An algorithm is presented for solving this problem in $O(n^2)$ operations.

2. In the manual $S = [V; U]$ (we shall omit the index k in what follows), by the lth layer, $l = 0, 1, \cdots, k$, is meant the set $O_l = \{x: x \in V, x$ being situated at distance l from $s\}$. In accordance with the definition $O_0 = \{s\}$, $O_k = \{t\}$.

Let $\rho(u)$, $u \in U$, be an admissible function, i.e. $0 \leq \rho(u) \leq c(u)$. Let ξ be called ρ-blocked if there is a saturated arc $u \in \xi$, i.e. $\rho(u) = c(u)$. An arc $u = (x, y)$ (respectively, vertex x') is called ρ-blocked if any path ξ of the form $x, u, y, \cdots, t$ (respectively, path ξ' of the form $x', \cdots, t$) is ρ-blocked. Thus a dead-end flow is a flow f for which the source s is f-blocked.

We shall employ the following notation: $\alpha(x)$, $x \in V$, is the set of arcs originating from x, and $\beta(x)$ is the set of incoming arcs at x;

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$$\overline{V} \doteqdot V/\{s, t\}; \quad \rho(U') \doteqdot \rho(u)|_{U'}, \quad |\rho(U'(x))| \doteqdot \sum_{u \in U} \rho(u), \quad U' \subseteq U$$

(the symbol $\doteqdot$ denotes equality by definition).

We call a function $g(u)$, $u \in U$, a *preflow* if the following properties are satisfied:

$P1$. $0 \leq g(u) \leq c(u)$.

$P2$. $|g(\beta(x))| \geq |g(\alpha(x))|$ $\forall x \in \overline{V}$.

$P3$. If $|g(\beta(x))| > |g(\alpha(x))|$, $x \in \overline{V}$ (x is called a *deficient vertex*), then the vertex x is g-blocked.

$P4$. The vertex s is g-blocked.

If the preflow g does not contain deficient vertices, then g is a dead-end flow.

3. Description of the algorithm. A dead-end flow in S is constructed iteratively. The ith iteration, $i = 1, 2, \cdots$, consists of two parts: a) completion to a preflow, b) balancing of the deficient vertices. The function obtained at the ith iteration as a result of completion we shall denote by g_i, and the function resulting from balancing by g_i^{bal}.

Each arc has a flow index: open or closed. If an arc becomes closed, then it remains so until the end of operation of the algorithm. Initially all arcs are open.

We shall consider that the set $M = \{m_1, m_2, \cdots, m_q\}$ is given in the form of a list if a certain ordering $m_{p_1}, m_{p_2}, \cdots, m_{p_q}$ is fixed by the designation of an initial element ($m_{\text{init}} = m_{p_1}$), a terminal element ($m_{\text{ter}} = m_{p_q}$), and for each element m_{p_j}, $j = 1, 2, \cdots, q$, a preceding element ($m_{p_{j-1}}$) and a succeeding element ($m_{p_{j+1}}$); we shall set $m_{p_0} = \varnothing$, $m_{p_{q+1}} = \varnothing$.

The following flow sets are given in the form of lists:

a) $\overline{\alpha}(x)$ $\forall x \in V$ is the set of open unsaturated arcs;

b) $\beta_\Delta(x)$ $\forall x \in \overline{V}$ is a set whose meaning can be made precise as follows. For each $z \in \beta_\Delta(x)$ a real number $\Delta(z) > 0$ is determined, called an *addition*. Initially, $\overline{\alpha}(x) = \alpha(x)$ $\forall x \in V$; $\beta_\Delta(x) = \varnothing$ $\forall x \in \overline{V}$.

Completion to a preflow. Let $i - 1$ iterations, $i \geq 2$, already have been effected, the function g_{i-1}^{bal} determined, and the assertions verified (for $r = i - 1$):

1°. For g_r^{bal} properties $P1$, $P2$, $P4$ are satisfied, and a layer $O_{s(r)}$ is determined such that $\forall x \in \bigcup_{i=1}^{s(r)-1} O_l$ $P3$ is satisfied and in the layers $O_{s(r)+1}, O_{s(r)+2}, \cdots, O_{k-1}$ there are no deficient vertices.

2°. All closed arcs of g_r^{bal} are blocked.

3°. For each arc $u \in \overline{\alpha}(x)$ $\forall x \in V$, except perhaps for the first, $g_r^{\text{bal}}(u) = 0$.

Completion at the ith iteration consists in the construction of a preflow g_i from g_{i-1}^{bal}. We shall say that on the arc $u \in U$ an *active assignment* is produced, if $g_i(u) > g_{i-1}^{\text{bal}}(u)$. We shall say that in all the remaining arcs a *passive assignment* is produced.

For all the arcs incident to vertices of the layers $O_0, O_1, \cdots, O_{s(i-1)-1}$, we assume $g_i = g_{i-1}^{\text{bal}}$ (passive assignment).

A *layered circuit* of the vertices of the layers $O_{s(i-1)}, O_{s(i-1)+1}, \cdots, O_{k-1}$ is produced: first the vertices of the layer $O_{s(i-1)}$ are sorted out, then those of

$O_{s(i-1)+1}$, and so forth up to O_{k-1}, inclusively. Suppose the vertices $x_1, x_2, \cdots, x_N$ have already been traversed in a layered circuit, and let $x_{N+1} \in O_j$ be the next vertex. $g_i(\beta(x))$ is already determined for any vertex $x \in O_j$, and

$$|g_i(\beta(x_{N+1}))| \geqslant |g_{i-1}^{\mathrm{bal}}(\alpha(x_{N+1}))|.$$

a) If $|g_i(\beta(x_{N+1}))| = |g_{i-1}^{\mathrm{bal}}(\alpha(x_{N+1}))|$, then we set $g_i(\alpha(x_{N+1})) = g_{i-1}^{\mathrm{bal}}(\alpha(x_{N+1}))$ (passive assignment).

b) If $|g_i(\beta(x_{N+1}))| < |g_{i-1}^{\mathrm{bal}}(\alpha(x_{N+1}))|$ and $\bar{\alpha}(x_{N+1}) = \varnothing$ then we set $g_i(\alpha(x_{N+1})) = g_{i-1}^{\mathrm{bal}}(\alpha(x_{N+1}))$ (passive assignment).

c) Let $|g_i(\beta(x_{N+1}))| > |g_{i-1}^{\mathrm{bal}}(\alpha(x_{N+1}))|$ and $\bar{\alpha}(x_{N+1}) \neq \varnothing$. Let $\bar{\alpha}(x) = \{u_1, u_2, \cdots \cdots, u_q\}$.

We set $g_i(\alpha(x_{N+1})/\bar{\alpha}(x_{N+1})) = g_{i-1}^{\mathrm{bal}}(\alpha(x_{N+1})/\bar{\alpha}(x_{N+1}))$ (passive assignment). For the arcs $u_0 = \varnothing, u_1, u_2, \cdots, u_l$, $0 \leq 1 < q$, suppose g_i has already been determined and $|g_i(\hat{\alpha}(x_{N+1}))| < |g_i(\beta(x_{N+1}))|$, where $\hat{\alpha}(x_{N+1}) = \alpha(x_{N-1})/\bar{\alpha}(x_{N+1}) \cup \{u_0, u_1, \cdots, u_l\}$ is the set of arcs with already given g_i. For the next arc u_{l+1} of the list $\bar{\alpha}(x_{N+1})$ we set

$$g_i(u_{l+1}) = \min\{c(u_{l+1}), |g_i(\beta(x_{N+1}))| - |g_i(\hat{\alpha}(x_{N+1}))|\}$$

(active assignment). We make active assignments up to the arc u_m (inclusive) for which either: 1) $|g_i(\alpha(x_{N+1})/\bar{\alpha}(x_{N+1}) \cup \{u_1, u_2, \cdots, u_m\})| = |g_i(\beta(x_{N+1}))|$, or 2) $m = q$. For the arcs $u_{m+1}, u_{m+2}, \cdots, u_q$ (in case 1) we set $g_i = g_{i-1}^{\mathrm{bal}} = 0$ (passive assignment). The new list $\bar{\alpha}(x_{N+1})$ takes the form $\{u_m, u_{m+1}, \cdots, u_q\}$ if $g_i(u_m) < c(u_m)$, and takes the form $\{u_{m+1}, u_{m+2}, \cdots, u_q\}$ if $m < q$ and $g_i(u_m) = c(u_m)$.

If an active assignment is made on the arc $u = (x_{N+1}, y)$, then u is added to the end of the list $\beta_\Delta(y)$ (even if the arc u is already contained in $\beta_\Delta(y)$) and we define $\Delta(u)$ to be equal to $g_i(u) - g_{i-1}^{\mathrm{bal}}(u)$.

In the construction of g_i we set $g_0^{\mathrm{bal}} \equiv 0$, $s(0) = 0$, $g_i(\alpha(s)) = c(\alpha(s))$ (active assignment) and proceed further in accordance with the completion algorithm.

Balancing. Let $d(i)$ be the *maximal index* of a layer in which g_i has a deficient vertex (x). At the vertex x an operation of balancing is effected, i.e. determination of $g_i^{\mathrm{bal}}(\beta(x))$ so that $|g_i^{\mathrm{bal}}(\beta(x))| = |g_i(\alpha(x))|$ and $g_i^{\mathrm{bal}}(u) \leq g_i(u)$ $\forall u \in \beta(x)$. We effect balancing at the vertex x in accordance with the list $\beta_\Delta(x)$, starting from its terminus and decreasing g_i at the expense of "additions". Let u_l be the next element in $\beta_\Delta(x)$ and suppose $|g_i^{\mathrm{bal}}(\hat{\beta}(x))| + |g_i(\beta(x)/\hat{\beta}(x))| > |g_i(\alpha(x))|$, where $\hat{\beta}(x) \subset \beta(x)$ is the set of arcs for which g_i^{bal} is defined. We set

$$g_i^{\mathrm{bal}}(u_l) = \max\{g_i(u_l) - \Delta(u_l), |g_i(\alpha(x)) - g_i^{\mathrm{bal}}(\hat{\beta}(x))| - |g_i(\beta(x)/\hat{\beta}(x)/u_l)|\}$$

(active assignment) and proceed to u_{l-1}, and so on until $|g_i^{\mathrm{bal}}(\beta(x))| + |g_i(\beta(x)/\hat{\beta}(x))| = |g_i(\alpha(x))|$, where $\hat{\beta}(x)$ is the (new) set with already determined g_i^{bal}. For all $u \in \beta(x)/\hat{\beta}(x)$ we put $g_i^{\mathrm{bal}}(u) = g_i(u)$ (passive assignment). Each arc $u = (y, x) \in \beta(x)$ is "closed" and is stricken from the list $\alpha(y)$ (if it is present in it).

If there are deficient vertices in the layer $O_{d(i)}$, then we effect balancing independently for each one. For all arcs with not already designated g_i^{bal} we set $g_i^{\text{bal}} = g_i$ (passive assignment). We set $s(i) = d(i) - 1$.

Lemma 1. 1) *g_i is a preflow.* 2) *assertions* 1°–3° *are valid for $r = i$.* 3) *If a vertex $x \in V$ is g_i^{bal}-blocked, then it is g_i-blocked and g_i^{bal}-blocked.*

The lemma is proved by induction on i (for $i = 1$ the proof is immediate).

To each "addition" Δ we associate the index $e(\Delta)$ of that iteration at which it occurred as a result of completion. Let $k_{i,j}$ be the maximal index of the additions relative to all the arcs incoming to vertices of the layer O_j before commencement of balancing in the ith iteration.

Lemma 2. *In the balancing at the ith iteration an active assignment can be made only for those arcs $u \in \beta_\Delta(x)$, $x \in O_{d(i)}$, for which $e(\Delta(u)) = k_{i,d(i)} g_i^{\text{bal}}(u) \geq g_i(u) - \Delta(u)$.*

With the aid of Lemmas 1 and 2 one can prove the

Theorem. *At each vertex $x \in \overline{V}$ balancing is carried out not more than once.*

It follows from the theorem that the number of iterations is not greater than $n - 2$. The final function is a flow, and since s is g_i-blocked, it follows from Lemma 1 that this flow is dead-end.

4. Estimate of the number of operations. In a passive assignment no operations are effected. Owing to the utilization of lists, the number of operations in the construction of a dead-end flow is $\eta = O(n^2 + \overline{\eta})$, where η is the number of active assignments. From the Theorem it follows that $\overline{\eta} = O(p + \eta_{\text{a}})$, where η_{a} is the number of active assignments in the completions. We say that as a result of an active assignment in completion on an arc u the event A occurs if the arc u is saturated, and the event B occurs if it is not saturated.

Lemma 3. 1) *The total number of events A is not greater than p.*

2) *In the course of one iteration in $\alpha(x)$, $\forall x \in V$ not more than one event B occurs.*

From Lemma 3 it follows that $\eta_{\text{a}} = p + \eta'$ and $\eta' = O(n^2)$, where η' is the total number of events B, whence it follows that $\eta = O(n^2 - \eta') = O(n^2)$.

There is a modification of the method presented for finding a dead-end flow for which $\eta = O(p + \eta')$. One can construct extremal examples of networks supporting the precision of the estimate $O(n^3)$ for finding the maximal flow by means of the given and modified methods.

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