Discrete Mathematics Final Exam

2006.06.13

Name:_____ Student ID:_____

Instructions. This is a 2.9-hr close book quiz. Please manage your time well. No dictionary, calculator, PDA, or any other electronic device. Any dishonorable cheating behavior will give you a miserable future. (Totally 120 points)

1. [8%] Prove that $C_0^{2n+1} + C_1^{2n+1} + \cdots + C_n^{2n+1} = 2^{2n}$ for all nonnegative integers n.

   Ans:

2. [6%] How many routes are there from the lower-left corner of an $n \times n$ squre grid to upper-right corner if we are restricted to traveling only to the right or upward? Explain your answer.

   Ans:

3. [12%] Answer the following questions with explanation (answers with explanation give very few points):
   [6%] (a) How many books must be chosen from among 24 mathematics books, 25 computer science books, 21 literature books, and 15 economic books in order to assure that there are at least 11 books on the same subject? Explain your answer.
   [6%] (b) Given a list of 10 natural numbers $a_1, a_2, \cdots, a_{10}$, show that there exist indices i and j such that $1 \leq i < j \leq 10$ and $a_i + a_{i+1} + \cdots + a_{j-1} + a_j$ is divisible by 10.
   Ans:

4. [20%] A stack is an important data structure where the last item inserted into the stack must be the first item deleted (LIFO). Suppose we will insert all of the integers $1, 2, ..., n$ into a stack in sequence. Note that each integer from 1 through n enters and leaves the stack exactly once. We will denote that integer k enters the stack by writing k and denote that integer k leaves the stack by wrting $\overline{k}$. For example, if $n = 1$, there is only possible sequence: $1, \overline{1}$. For $n = 2$, there are two possibilities: $1, 2, \overline{2}, \overline{1}$ and $1, \overline{1}, 2, \overline{2}$. Likewise, for $n = 3$, there will be 5 possibilities for inserting the integers 1,2,3 into a stack and deleteing them from it.
Let c_n be the number of possible sequences of $1, 2, ..., n$ that can result from the use of a stack in this manner. Let $c_0 = 1$. Answer the following questions.
(a) [3%] What are c_1,c_2, and c_3?
(b) [3%] Suppose 1 is the first integer deleted from the stack, how many possibilities will this happen? Explain your answer using c_k for some k.
(c) [3%] Suppose 1 is the second integer deleted from the stack, how many possibilities will this happen? Explain your answer using c_k for some k.
(d) [3%] Suppose 1 is the third integer deleted from the stack, how many possibilities will this happen? Explain your answer using c_k for some k.
(e) [4%] Suppose 1 is the kth integer deleted from the stack, how many possibilities will this happen? Explain your answer using c_k for some k.
(f) [4%] Explain how you compute for c_n using $c_0, ..., c_{n-1}$.

Ans:

5. [12%] Consider the sequences of n terms in which each term is -2, -1, 0 or 1. Let s_n denote the number of such sequences in which no term of -1 occurs before a term of 1 for $n \geq 1$
   (a) [3%] What are s_1 and s_2? Answers without explanation get at most 1 point.
   (b) [3%] Express s_n as a function of s_{n-1} and n
   (c) [3%] Express s_n as a function of s_{n-1} and s_{n-2}
   (d) [3%] Express s_n as a function of n

   Ans:

6. [16%] Given a recurrence relation $s_n = 4s_{n-1} - 3s_{n-2}$, $n \geq 2$, with $s_0 = 0$, $s_1 = 2$
   (a) [4%] Solve for s_n without using the characteristic polynomial or generating function
   (b) [4%] Solve for s_n using the characteristic polynomial
   (c) [4%] Show the power series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$
   (d) [4%] Solve for s_n using the generating function

   Ans:

7. [8%] A code for $\{a, b, c, d, e\}$ is given by $a : 00$, $b : 01$, $c : 101$, $d : x10$, $e : yz1$, where $x, y, z \in \{0, 1\}$. Determine x, y and z so that the given code is a prefix code

   Ans:

8. [8%] The National Security Agency is helping American diplomats in foreign countries send coded messages back to the State Department in Washington, D.C. These messages are to be sent using the characters R, I, H, V with an expected usage rate of 40, 35, 20, 5, respectively, per 100 characters. Find an assignment of codewords that minimizes the number of bits needed to send a message. (i.e. using sequences of 0 or 1 to express R, I, H, V with minimal number of bits)

   Ans:

9. [20%] Given an undirected graph $G = (V, E)$ where V is the vertex set and E is the edge set.
(a) [5%] Give a method to check whether G is connected. Briefly explain your method and discuss its complexity
(b) [5%] Give a method to check whether G contains an odd cycle. Briefly explain your method and discuss its complexity
(c) [5%] Give a method to detect bridges on G (if a bridge exists, your method should find it. Otherwise, your method should know there exists no bridge). Briefly explain your method and discuss its complexity
(d) [5%] Find a strongly connected orientation for the graph in Fig 1.

Ans:

10. [10%] The following questions are about traversals on binary trees.
    (a) [5%] A binary tree has preorder: $TSRFDIHEZGMLJNQ$ and
    postorder: $FIHDRZGESJNLQMT$, draw the binary tree
    (b) [5%] Given postorder: $DEBHGFCA$ and inorder: $DBEACHGF$, draw the binary tree

    Ans: