Discrete Mathematics Final Exam
2006.06.13

Name:________________________ Student ID:________________________

Instructions. This is a 2.9-hr close book quiz. Please manage your time well. No dictionary, calculator, PDA, or any other electronic device. Any dishonorable cheating behavior will give you a miserable future. (Totally 120 points)

1. [8%] Prove that $C_0^{2n+1} + C_1^{2n+1} + \cdots + C_n^{2n+1} = 2^{2n}$ for all nonnegative integers n.

   Ans:

   Since $C_r^{2n+1} = C_{2n+1-r}^{2n+1}$ and $C_0^n + C_1^n + \cdots + C_n^n = (1+1)^n$, we get
   $2 \cdot (C_0^{2n+1} + C_1^{2n+1} + \cdots + C_n^{2n+1}) = (C_0^{2n+1} + C_1^{2n+1} + \cdots + C_n^{2n+1}) + (C_{n+1}^{2n+1} + C_{n+2}^{2n+1} + \cdots + C_{2n+1}^{2n+1}) = (1+1)^{2n+1}$
   Thus $C_0^{2n+1} + C_1^{2n+1} + \cdots + C_n^{2n+1} = 2^{2n+1}/2 = 2^{2n}$

2. [6%] How many routes are there from the lower-left corner of an $n \times n$ squre grid to upper-right corner if we are restricted to traveling only to the right or upward? Explain your answer.

   Ans:

   Suppose a right and upward move is denoted by R and U, respectively. Then in such a square grid network we have to make n Rs and n Us. Any route will be a permutation of n Rs and n Us with C_n^{2n} possibilities.

3. [12%] Answer the following questions with explanation (answers with explanation give very few points):
[6%] (a) How many books must be chosen from among 24 mathematics books, 25 computer science books, 21 literature books, and 15 economic books in order to assure that there are at least 11 books on the same subject? Explain your answer.
[6%] (b) Given a list of 10 natural numbers $a_1, a_2, \cdots, a_{10}$, show that there exist indices i and j such that $1 \leq i < j \leq 10$ and $a_i + a_{i+1} + \cdots + a_{j-1} + a_j$ is divisible by 10.
Ans:

(a) There are four different kinds of books. The pigeonhole principle (i.e. some pigeonhole contains $11 = k+1$ pigeons, if the total number of pigeons is more than $k = 10$ times the number of pigeonholes which is 4) implies that a minimum of $4 \cdot (11-1)+1 = 41$ books must be chosen in order to assure that there are at least 12 books on the same subject.
(b) Consider the 10 numbers $s_i = \sum_{k=1}^{i} a_k$, then $\{s\} = \{a_1, a_1+a_2, a_1+a_2+a_3, \cdots, a_1 + a_2 + \cdots + a_{10}\}$. Select s_i and s_j where $1 \leq i < j \leq 10$, let their difference be $d_{ij} = s_j - s_i = a_i + a_{i+1} + \cdots + a_{j-1} + a_j$. If $s_i \equiv s_j (\bmod 10)$, we are done. Otherwise, distributing $\{s\}$ into the nine congruence classes $[1], [2], [3], \cdots, [9]$ in Z_{10}, there must be a class contains two numbers, say, s_k and s_l with $s_l > s_k$, by the pigeonhole principle. The difference $d_{kl} = a_k + a_{k+1} + \cdots + a_{l-1} + a_l$ will be divisible by 10, which is what we want.

4. [20%] A stack is an important data structure where the last item inserted into the stack must be the first item deleted (LIFO). Suppose we will insert all of the integers $1, 2, ..., n$ into a stack in sequence. Note that each integer from 1 through n enters and leaves the stack exactly once. We will denote that integer k enters the stack by writing k and denote that integer k leaves the stack by wrting $\overline{k}$. For example, if $n = 1$, there is only possible sequence: $1, \overline{1}$. For $n = 2$, there are two possibilities: $1, 2, \overline{2}, \overline{1}$ and $1, \overline{1}, 2, \overline{2}$. Likewise, for $n = 3$, there will be 5 possibilities for inserting the integers 1,2,3 into a stack and deleteing them from it.
   Let c_n be the number of possible sequences of $1, 2, ..., n$ that can result from the use of a stack in this manner. Let $c_0 = 1$. Answer the following questions.
   (a) [3%] What are c_1,c_2, and c_3?
   (b) [3%] Suppose 1 is the first integer deleted from the stack, how many possibilities will this happen? Explain your answer using c_k for some k.
   (c) [3%] Suppose 1 is the second integer deleted from the stack, how many possibilities will this happen? Explain your answer using c_k for some k.
   (d) [3%] Suppose 1 is the third integer deleted from the stack, how many possibilities will this happen? Explain your answer using c_k for some k.
   (e) [4%] Suppose 1 is the kth integer deleted from the stack, how many possibilities will this happen? Explain your answer using c_k for some k.
   (f) [4%] Explain how you compute for c_n using $c_0, ..., c_{n-1}$.

   Ans:

   (a) $c_1 = 1, c_2 = 2, c_3 = 5$
   (b) $1, \overline{1}$ is the only possibility, whereas the left $n - 1$ numbers have number of possbilities c_{n-1}, thus the total number of possibilities equals $c_0 c_{n-1}$
   (c) $1, 2, \overline{2}, \overline{1}$ is the only possibility where we only need to consider inserting/deleting 2 (i.e. only 1 number) which has c_1 posbility, whereas the left $n-2$ numbers have c_{n-2} possibilities, thus the total number of possibilities equals $c_1 c_{n-2}$.
   (d) Ignoring 1 and $\overline{1}$, we have to inserting/deleting integers 2 and 3 with c_2 possibilities, whereas the left $n - 3$ number have c_{n-3} possibilities, thus the total number of possibilities equals $c_2 c_{n-3}$.
   (e) In general, we have to inserting/deleting integers $2, 3, ..., k$ with c_{k-1} possibilities, whereas the left $n - k$ number have c_{n-k} possibilities, thus the total number of possibilities equals $c_{k-1} c_{n-k}$ by the multiplication principle.
   (f) Since 1 is possible to be the first, second, third,..., or the nth number to be deleted, by the addition principle, we will have the total number of possibilites $c_n = c_0 c_{n-1} + c_1 c_{n-2} + ... + c_{n-1} c_0$. for $n \geq 1$

5. [12%] Consider the sequences of n terms in which each term is -2, -1, 0 or 1. Let s_n denote the number of such sequences in which no term of -1 occurs before a term of 1 for $n \geq 1$
(a) [3%] What are s_1 and s_2? Answers without explanation get at most 1 point.
(b) [3%] Express s_n as a function of s_{n-1} and n
(c) [3%] Express s_n as a function of s_{n-1} and s_{n-2}
(d) [3%] Express s_n as a function of n

Ans:
(a) $s_1 = 4$, since any term can be the 1st term.
$s_2 = 3 + 4 + 4 + 4 = 15$ since
if 1 is the 2nd term, the first term can be -2,0,1;
if -2,0 or 1 is the 2nd term, the first term can be -2,-1,0 or 1.
(b) if the last term is 1, there are 3^{n-1} possibilities since the first $n-1$ terms are either -2, 0 or 1;
if the last term is 0, -1 or -2, there are there are s_{n-1} possibilities for each case.
By the principle of addition, $s_n = 3s_{n-1} + 3^{n-1}$
(c) $s_n - 3s_{n-1} = 3(s_{n-1} - 3s_{n-2})$, thus $s_n - 6s_{n-1} + 9s_{n-2} = 0$. Using $s_1 = 4$, $s_2 = 15$, and $s_n = (c_1 + nc_2)3^n$ we can get $s_n = 3^{n-1}(n+3)$

6. [16%] Given a recurrence relation $s_n = 4s_{n-1} - 3s_{n-2}$, $n \geq 2$, with $s_0 = 0$, $s_1 = 2$
   (a) [4%] Solve for s_n <u>without</u> using the characteristic polynomial or generating function
   (b) [4%] Solve for s_n using the characteristic polynomial
   (c) [4%] Show the power series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$
   (d) [4%] Solve for s_n using the generating function

   Ans:
   (a) Let $a_n = s_n - 3s_{n-1}$, we get $a_n = a_{n-1} = ... = a_1 = 2$. So, $s_n + 1 = 3(s_{n-1} + 1)$. Let $b_n = s_n + 1$, we get $b_n = 3b_{n-1} = ... = 3^{n-1}b_1 = 3^n b_0 = 3^n$, thus $s_n = 3^n - 1$
   (b) characteristic polynomial: $x^2 - 4x + 3$ with root=1 or 3
   $s_n = c_1 + c_2 \cdot 3^n$, using $s_0 = 0$, $s_1 = 2$, we shall get $c_1 = -1, c_2 = 1$
   (c) Let $F = 1 + x + x^2 + x^3 + ...$, then $F(1-x) = 1$, $F = 1/(1-x)$
   (d) generating function $F = s_0 + s_1 x + s_2 x + ...$
   $= s_0 + s_1 x + (4s_1 - 3s_0)x^2 + (4s_2 - 3s_1)x^3 + ...$
   $= s_0 + s_1 x + 4x[s_1 x + s_2 x^2 + s_3 x^3 + ...] - 3x^2[s_0 + s_1 x + s_2 x^2 + s_3 x^3 + ...]$
   $= 2x + 4x[F - 0] - 3x^2 F$,
   we get $F(1 - 4x + 3x^2) = 2x$, $F = \frac{1}{1-3x} - \frac{1}{1-x} = \sum_{n=0}^{\infty}(3^n - 1)x^n$, thus $s_n = 3^n - 1$

7. [8%] A code for $\{a, b, c, d, e\}$ is given by a : 00, b : 01, c : 101, d : $x10$, e : $yz1$, where $x, y, z \in \{0, 1\}$. Determine x, y and z so that the given code is a prefix code

   Ans:

   If $x = 0$, b becomes internal node; so $x = 1$; Likewise, $y = z = 1$ or a, b, c become internal nodes.

8. [8%] The National Security Agency is helping American diplomats in foreign countries send coded messages back to the State Department in Washington, D.C. These messages are to be sent using the characters R, I, H, V with an expected usage rate of 40, 35, 20, 5, respectively, per 100 characters. Find an assignment of codewords that minimizes the number of bits needed to send a message. (i.e. using sequences of 0 or 1 to express R, I, H, V with minimal number of bits)

   Ans:

   There are many solutions. The Huffman's optimal binary tree algorithm will first merges V and H, and then merges I, and finally merges with R. After constructing the optimal binary tree, assign 0 on the left branch and 1 on the right branch.

9. [20%] Given an undirected graph $G = (V, E)$ where V is the vertex set and E is the edge set.
(a) [5%] Give a method to check whether G is connected. Briefly explain your method and discuss its complexity
(b) [5%] Give a method to check whether G contains an odd cycle. Briefly explain your method and discuss its complexity
(c) [5%] Give a method to detect bridges on G (if a bridge exists, your method should find it. Otherwise, your method should know there exists no bridge). Briefly explain your method and discuss its complexity
(d) [5%] Find a strongly connected orientation for the graph in Fig 1.

Ans:

(a) Apply any search algorithm, e.g. BFS, DFS, starting from a vertex in G. If there exists vertex that has not been labeled, we know G is disconnected. Complexity for DFS/BFS is $O(|E|)$
(b) By Theorem 4.8, A graph G is 2-colorable iff it contains no odd cycle
Step 1. Apply BFS algorithm,
Step 2. coloar all vertices of odd (even) distance label to be red (black)
Step 3. check each edge $(i, j) \in E$, if $color[i] = color[j]$, it is NOT 2-colorable (i.e. it contains an odd cycle), STOP
Step 4. otherwise, it is 2-colorable (i.e. it contains no odd cycles)

Step 1 takes $O(|E|)$ time; Step 2 takes $O(|V|)$ time; Step 3 takes $O(|E|)$. Overall, it takes $O(|E|)$ since usually $O(|V|) < O(|E|)$
(c)
for each edge $e = (i, j) \in E$ **do**
    remove e from G
    apply DFS/BFS starting from i (or j)
    if there exists an unlabeled vertex
        e is a bridge, **STOP**
end for
G has no bridge

Complexity: each BFS/DFS takes $O(|E|)$ time, totally $O(|E|)$ iterations, thus $O(|E|^2)$ time.
(d) apply DFS, record order of visiting. assign orientation from smaller order to larger order for DFS tree edges; assign orientation from larger order to smaller order for DFS tree edgesfor non-DFS tree edges

10. [10%] The following questions are about traversals on binary trees.
(a) [5%] A binary tree has preorder: $TSRFDIHEZGMLJNQ$ and
postorder: $FIHDRZGESJNLQMT$, draw the binary tree
(b) [5%] Given postorder: $DEBHGFCA$ and inorder: $DBEACHGF$, draw the binary tree

Ans: