INSTRUCTOR'S SOLUTIONS MANUAL

DISCRETE MATHEMATICS

FIFTH EDITION

John A. Dossey
Illinois State University

Albert D. Otto
Illinois State University

Lawrence E. Spence
Illinois State University

Charles Vanden Eynden
Illinois State University

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Chapter 1

An Introduction to Combinatorial Problems and Techniques

1.1 THE TIME TO COMPLETE A PROJECT

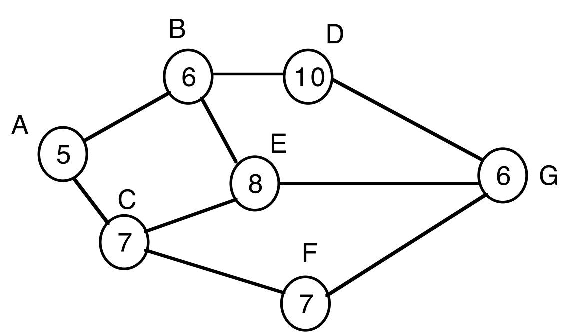
2. 31; A-B-E-G

4. 39; A-C-G-H

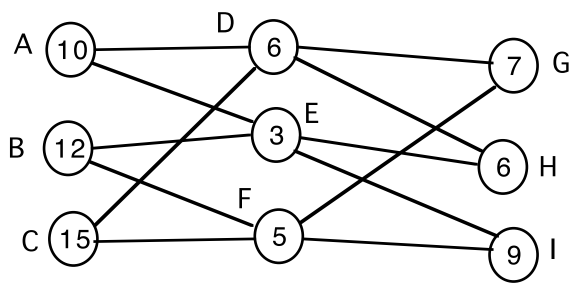
6. 16; B-D-F-H

8. 27; A-D-E-H

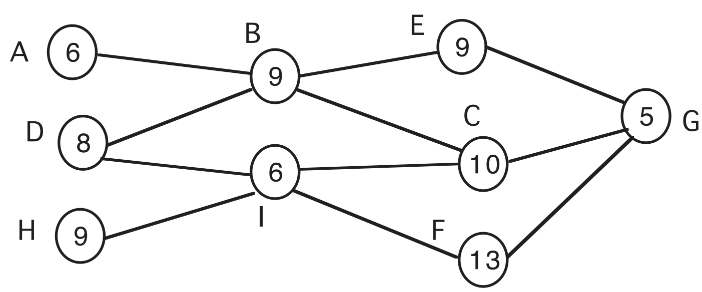
10. 27; A-B-D-G



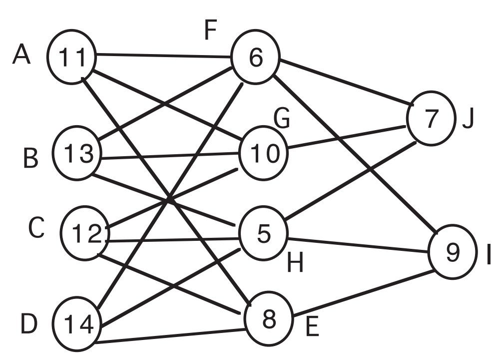
12. 29; C-F-I



14. 33; H-I-F-G



16. 31; D-E-I



18. 20 minutes

1.2 A MATCHING PROBLEM

2. 720 **4.** 210 **6.** 84 **8.** 1680

10. 19,958,400 **12.** $\frac{1}{8}$ **14.** 5040 **16.** 126

18. 210 **20.** 119 **22.** 1320 **24.** 5040

26. 240 **28.** 1200

1.3 A KNAPSACK PROBLEM

2. T **4.** F **6.** F **8.** F

10. T **12.** T **14.** T **16.** no

18. yes; 32

20. $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, $\{1,2,3,4\}$; 16

22. 128 **24.** 1024 **26.** 256 **28.** 26

30. $\{1,4,6,7,8,9,10,11,12\}$

1.4 ALGORITHMS AND THEIR EFFICIENCY

2. yes; 0 **4.** no

6. no **8.** -1, 9, 84; 3, 17, 84

10. -4, -4, 41, 95; 2, 11, 33, 95 **12.** 111000

14. 001010

16.

k	j	a_1	a_2	a_3
3		1	1	1
2		1	1	1
1		1	1	1
0		1	1	1

18.

k	j	a_1	a_2	a_3	a_4
4		1	1	1	0
		1	1	1	1

20. The circled numbers in the table below indicated the items being compared.

a_1	a_2	a_3	a_4	j	k
23	5	17	12	1	3
23	5	12	17		2
23	5	12	17		1
5	23	12	17	2	3
5	23	12	17		2
5	12	23	17	3	3
5	12	17	23		

22. The circled numbers in the table below indicated the items being compared.

a_1	a_2	a_3	a_4	a_5	j	k
88	2	75	10	48	1	4
88	2	75	10	48		3
88	2	10	75	48		2
88	2	10	75	48		1
2	88	10	75	48	2	4
2	88	10	48	75		3
2	88	10	48	75		2
2	10	88	48	75	3	4
2	10	88	48	75		3
2	10	48	88	75	4	4
2	10	48	75	88		

24. 6.5 years, 2.7 seconds

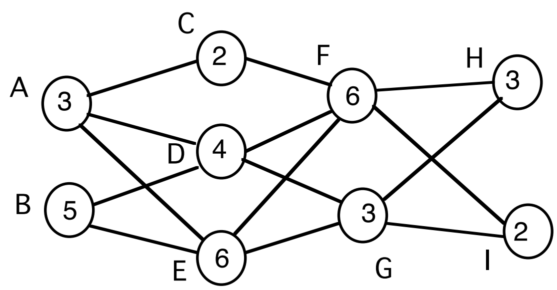
26. 2.3×10^{10} years, 12.5 seconds

28. $4n - 3$ **30.** $3n - 2$

32. -4, -4, 41, 95

SUPPLEMENTARY EXERCISES

2. 20; B-E-F-H



4. 336 **6.** 40 **8.** 14040

10. T **12.** F **14.** T

16. T **18.** 16 **20.** no

22. yes; 0 **24.** -5, 7, 7, 88

26. $\varnothing$, $\{4\}$, $\{3\}$, $\{3,4\}$, $\{2\}$, $\{2,4\}$, $\{2,3\}$, $\{2,3,4\}$, $\{1\}$, $\{1,4\}$, $\{1,3\}$, $\{1,3,4\}$, $\{1,2\}$, $\{1,2,4\}$, $\{1,2,3\}$, $\{1,2,3,4\}$

28. 4.92×10^8 years **30.** 4 **32.** $4r - 3$

Chapter 2

Sets, Relations, and Functions

2.1 SET OPERATIONS

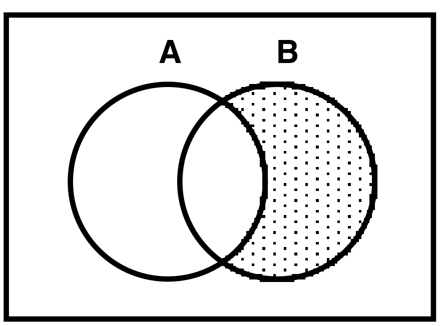
2. $A \cup B = \{1, 2, 4, 5, 6, 7, 9\}$, $A \cap B = \{1, 4, 6, 9\}$, $A - B = \varnothing$, $\overline{A} = \{2, 3, 5, 7, 8\}$, and $\overline{B} = \{3, 8\}$

4. $A \cup B = \{2, 3, 4, 5, 6, 7, 8, 9\}$, $A \cap B = \{7, 9\}$, $A - B = \{3, 4, 6, 8\}$, $\overline{A} = \{1, 2, 5\}$, and $\overline{B} = \{1, 3, 4, 6, 8\}$

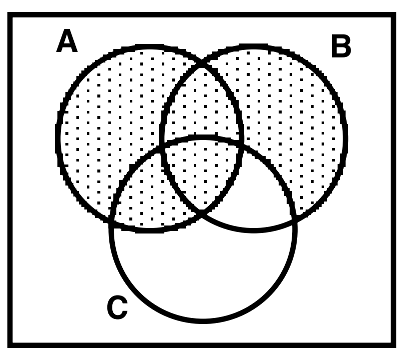
6. $\{(3,\ 1), (3,\ 2), (3,\ 3), (4,\ 1), (4,\ 2), (4,\ 3), (5,\ 1), (5,\ 2), (5,\ 3)\}$

8. $\{(p, a), (p, c), (p, e), (q, a), (q, c), (q, e), (r, a), (r, c), (r, e), (s, a), (s, c), (s, e)\}$

10.



12.



14. $A = \{1,\ 2\}$, $B = \{1,\ 3\}$, and $C = \{1\}$

16. $A = \{1,\ 2\}$, $B = \{1,\ 3\}$, and $C = \{1\}$

18. A

20. $\varnothing$

22. $A \cap B$

24. $A \cup \overline{B}$

26. The equality $A - B = B - A$ holds if and only if $A = B$.

28. The equality $A \cap B = A$ holds if and only if $A \subseteq B$.

2.2 EQUIVALENCE RELATIONS

2. reflexive and symmetric

4. reflexive, symmetric, and transitive

6. reflexive and symmetric

8. none

10. reflexive and symmetric

12. reflexive and transitive

14. The equivalence class of R containing ABCD consists of the string ABC and the strings of 4 letters having A as their first letter and C as their third letter. There are $26^2 = 676$ distinct equivalence classes of R.

16. The equivalence class of R containing $\{1,\ 2,\ 3\}$ is the set containing the following four elements of S: $\{1,\ 3\}$, $\{1,\ 2,\ 3\}$, $\{1,\ 2,\ 3,\ 4\}$, and $\{1,\ 3,\ 4\}$. There are 8 different equivalence classes of R, namely the sets consisting of the elements S, $S \cup \{2\}$, $S \cup \{4\}$, and $S \cup \{2, 4\}$ for every $S \subseteq \{1, 3, 5\}$.

18. The equivalence class of R containing $(5, 8)$ is the set

$$\{(x_1, x_2) \colon \text{each } x_i \text{ is an integer and } x_1 - x_2 = 5 - 8\}.$$

There are infinitely many distinct equivalence classes of R, namely, the sets of the form

$$\{(x_1, x_2) \colon \text{each } x_i \text{ is an integer and } x_1 - x_2 = k\},$$

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$.

20. $\{(1,1), (1,3), (1,6), (3,1), (3,3), (3,6), (6,1), (6,3), (6,6), (2,2), (2,5), (5,2), (5,5), (4,4)\}$

24. The equivalence classes have the form $E_1 \times E_2$, where E_i is an equivalence class of R_i.

28. There are 5 partitions of a set with three elements.

32. Let S be a nonempty set and R an equivalence relation on S. Then there is a function f with domain S such that $s_1\ R\ s_2$ if and only if $f(s_1) = f(s_2)$.

2.3 PARTIAL ORDERING RELATIONS

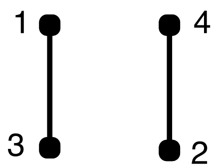
2. not antisymmetric

4. partial ordering

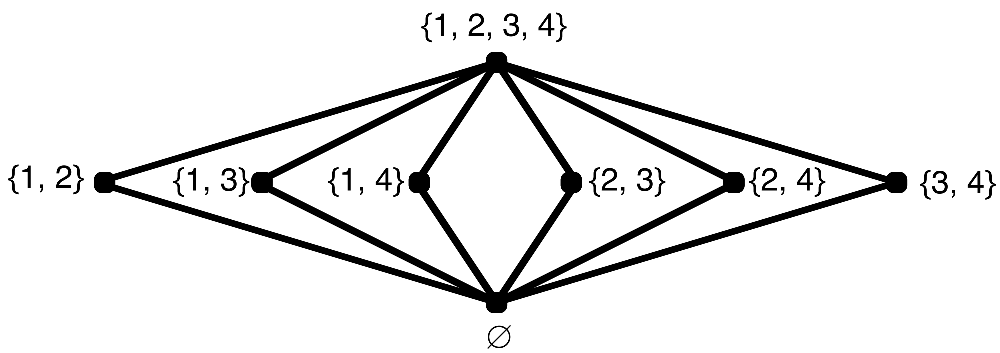
6. not antisymmetric

8. not antisymmetric

10.



12.



14. $R = \{(a, a), (b, b), (b, a), (c, c), (c, b), (c, a), (d, d), (d, a)\}$

16. $R = \{(1, 1), (2, 2), (2, 1), (2, 4), (4, 4), (8, 8), (8, 4)\}$

18. The maximal elements are {1}, {2}, and {3}; the only minimal element is {1, 2, 3}.

20. The only minimal element is 0; there are no maximal elements.

22. One possible sequence is 1, 3, 2, 4.

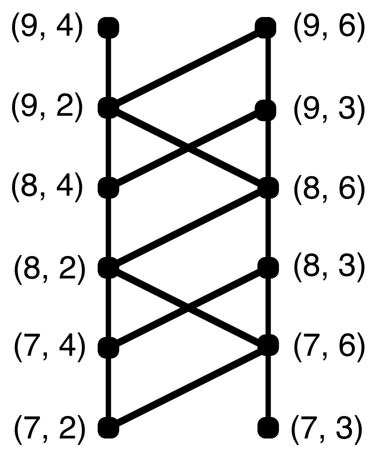
24. One possible sequence is 1, 3, 2, 6, 4, 12.

26. Let S denote the set of residents of Illinois and R be defined so that $x \; R \; y$ means that x is a sister of y.

28. Every prime integer is a minimal element; there are no maximal elements.

30. $(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)$

32.



36. **(a)** Suppose that y is element of S such that $x\ R\ y$ is false. If there is no element y_1 in S such that $y_1\ R\ y$, then y is a minimal element of S, contradicting that x is the unique minimal element of S. Thus there must be such an element y_1. If there is no element y_2 in S such that $y_2\ R\ y_1$, then y_1 is a minimal element of S, another contradiction. So there must be such an element y_2. Because S is finite, continuing in this manner must produce a minimal element y_k of S different from x. Because x is the unique minimal element of S, the assumption that there is an element y of S such that $x\ R\ y$ is false must be incorrect. Thus $x\ R\ s$ is true for every $s \in S$.

(b) Let Z denote the set of integers and $S = Z \cup \{\varnothing\}$. Let R be the relation defined on S by $x\ R\ y$ if and only if one of the following holds: (i) $x, y \in Z$ and $x \leq y$, or (ii) $x = y = \varnothing$. Then $\varnothing$ is the unique minimal element in S, but $\varnothing\ R\ z$ is false for every $z \in Z$.

40. 2^n

42. $2^n \cdot 3^{n(n-1)/2}$

2.4 FUNCTIONS

2. not a function with domain X

4. a function with domain X

6. a function with domain X

8. a function with domain X

10. a function with domain X

12. not a function with domain X

14. 4

16. 13

18. 8

20. 7

22. -1

24. 6

26. -2

28. 10

30. 0.78

32. 6.64

34. 3.22

36. -2.56

38. $gf(x) = g(f(x)) = \sqrt{x^2+1}$ and $fg(x) = f(g(x)) = x + 1$

40. $gf(x) = \dfrac{1}{3x}$ and $fg(x) = \dfrac{3}{x}$

42. $gf(x) = 5(2^x) - 2^{2x}$ and $fg(x) = 2^{5x-x^2}$

44. $gf(x) = \dfrac{4x-1}{2-x}$ and $fg(x) = \dfrac{3x-2}{3-x}$

46. one-to-one and onto

48. neither one-to-one nor onto

50. one-to-one but not onto

52. onto but not one-to-one

54. $f^{-1}(x) = \dfrac{x+2}{3}$

56. $f^{-1}(x)$ does not exist.

58. $f^{-1}(x)$ does not exist.

60. $f^{-1}(x) = \sqrt[3]{x+1}$

62. For $Y = \{x \in X\colon\ x \neq 0\}$, we have $g^{-1}(x) = \dfrac{-1}{x}$.

64. If $n < m$, there are no one-to-one functions; and if $m \leq n$, there are

$$P(n,m) = n(n-1)(n-2)\cdots(n-m+1)$$

one-to-one functions.

68. Let $X = \{1\}$, $Y = \{2,3\}$, and $Z = 4$. Define $f\colon\ X \to Y$ by$f(x) = 2$ for all $x \in X$ and $g\colon\ Y \to Z$ by $g(y) = 4$ for all $y \in Y$.

2.5 MATHEMATICAL INDUCTION

2. 7, 9, 13, 21, 37, 69

4. $x_n = \begin{cases} x & \text{if } n = 1 \\ x \cdot x^{n-1} & \text{if } n \geq 2 \end{cases}$

6. Let x_n denote the nth odd positive integer. Then $x_n = \begin{cases} 1 & \text{if } n = 1 \\ x_{n-1} + 2 & \text{if } n \geq 2. \end{cases}$

8. If $k = 1$, the sets $\{x_1, x_2, \ldots, x_k\}$ and $\{x_2, x_3, \ldots, x_{k+1}\}$ are disjoint.

10. If $k = 0$, the induction hypothesis cannot be applied to a^{k-1}.

28. $s_0 + s_1 + \cdots + s_n = (n+1)\left(s_0 + \frac{nd}{2}\right)$

2.6 APPLICATIONS

2. 56

4. 924

6. 120

8. 715

10. n

12. $r!$

14. 31

16. 4096

18. 128

20. 15

22. 36

24. 286

26. 126

28. 64

44. There are $\dfrac{n^2 + n + 2}{2}$ regions.

SUPPLEMENTARY EXERCISES

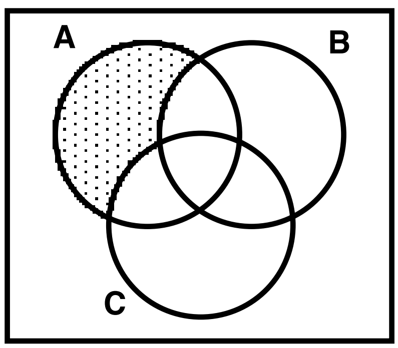
2. $\{1, 2, 3, 4, 5\}$

4. $\{1, 2, 4\}$.

6. $\{1, 2, 4, 5, 6\}$

8. $\{2, 3\}$

10.



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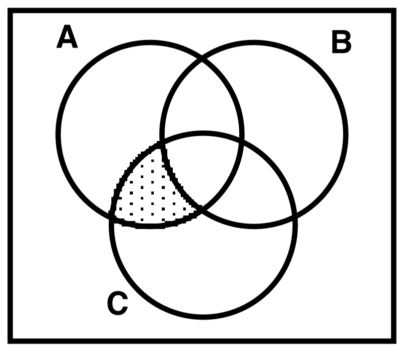


14. $gf(x) = 2(x^3 + 1) - 5 = 2x^3 - 3$ and $fg(x) = (2x - 5)^3 + 1 = 8x^3 - 60x^2 + 150x - 124$

16. not a function with domain X

18. not a function with domain X

20. neither one-to-one nor onto

22. one-to-one and onto

24. f^{-1} does not exist.

26. $f^{-1} = \sqrt[3]{x - 5}$

28. 84

30. 210

32. $\{1, 7\}$, $\{2, 6\}$, $\{3, 5\}$, $\{4\}$, $\{8\}$

34. the sets of positive and negative real numbers

36. 5

38. 6

40. **(b)** $a = \pm 1$

42. reflexive only

48. $x \vee x = x$ for all $x \in S$, $1 \vee y = y \vee 1 = y$ for all $y \in S$, $2 \vee 3 = 3 \vee 2 = 6$, $2 \vee 4 = 4 \vee 2 = 4$, $2 \vee 6 = 6 \vee 2 = 6$, $3 \vee 6 = 6 \vee 3 = 6$, $x \vee y$ is undefined in all other cases

52. Let S denote the set of all subsets of $\{1, 2, 3\}$ that contain at most two elements, and let R be the relation "is a subset of." If $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$, then $A \vee B = \{1, 2\}$, $B \vee C = \{2, 3\}$, and $A \vee C = \{1, 3\}$, but $(A \vee B) \vee C$ does not exist.

54. $[n] = \{-n, n\}$

56. $f(x) = \begin{cases} x & \text{if } x \equiv 0 \pmod{3} \\ x - 1 & \text{if } x \equiv 1 \pmod{3} \\ x - 2 & \text{if } x \equiv 2 \pmod{3} \end{cases}$

Chapter 3

Coding Theory

3.1 CONGRUENCE

2. The quotient q is 3, and the remainder r is 0.

4. $q = 12$ and $r = 7$

6. $q = -4$ and $r = 17$

8. $q = -6$ and $r = 9$

10. $p \not\equiv q \pmod{m}$

12. $p \equiv q \pmod{m}$

14. $p \equiv q \pmod{m}$

16. $p \not\equiv q \pmod{m}$

18. [5]

20. [4]

22. [2]

24. [8]

26. [3]

28. [5]

30. [3]

32. [6]

34. [1]

36. [4]

38. 11 A.M.

40. 5

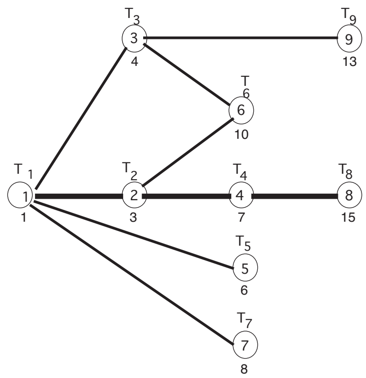
42. 3

44. February, March, and November

46. $[2] = [10] = [-6]$; $[7] = [39] = [-1] = [-17]$; $[45] = [-3]$

48. In $\mathcal{Z}_6$, $[2] \neq 0$ and $[3] \neq 0$, but $[2] \cdot [3] = [0]$.

50. **(a)**



(b) The whole project can be completed in 15 days.
(c) The critical path is $T_1 - T_2 - T_4 - T_8$.

52. **(b)** Take $a = \frac{1}{4}$, $b = 0$, $c = 1$, $x = 2$, $y = 0$, and $m = 2$.

3.2 THE EUCLIDEAN ALGORITHM

2. $-54, -27, -18, -9, -6, -3, -2, -1, 1, 2, 3, 6, 9, 18, 27, 54$

4. $-24, -12, -8, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 8, 12, 24$

6.

i	r_i
-1	341
0	217
1	124
2	93
3	31
4	0

[Print 31]

8.

i	r_i
-1	451
0	143
1	22
2	11
3	0

[Print 11]

10.

i	r_i
-1	89
0	55
1	34
2	21
3	13
4	8
5	5
6	3
7	2
8	1
9	0

[Print 1]

12. 30

14.

i	q_i	r_i	x_i	y_i
-1		2030	1	0
0		899	0	1
1	2	232	1	-2
2	3	203	-3	7
3	1	29	4	-9
4	7	0	-31	70

[Print 29, 4, -9]

16.

i	q_i	r_i	x_i	y_i
-1		231	1	0
0		182	0	1
1	1	49	1	-1
2	3	35	-3	4
3	1	14	4	-5
4	2	7	-11	14
5	2	0	26	-33

[Print 7, -11, 14]

18.

i	q_i	r_i	x_i	y_i
-1		555	1	0
0		2146	0	1
1	0	555	1	0
2	3	481	-3	1
3	1	74	4	-1
4	6	37	-27	7
5	2	0	58	-15

[Print 37, -27, 7]

20. **(a)** yes **(b)** no

22. **(a)** yes **(b)** yes

24. $x = 95$, $y = -133$

26. impossible

3.3 THE RSA METHOD

2. 10, 12, 14

4. 46, 7, 15

6.

Q	R	r_1	r_2	p	e
			1	30	29
14	1	33	30	33	14
7	0	18		18	7
3	1	18	30	18	3
1	1	18	30	18	1
0	1	18	30	18	0

[Print 30]

8.

Q	R	r_1	r_2	p	e
			1	7	53
26	1	49	7	49	26
13	0	64		64	13
6	1	37	79	37	6
3	0	16		16	3
1	1	10	34	10	1
0	1	100	94	100	0

[Print 94]

10.

Q	R	r_1	r_2	p	e
			1	12	101
50	1	144	12	144	50
25	0	523		523	25
12	1	59	331	1103	6
3	0	262		262	3
1	1	871	1114	871	1
0	1	59	70	59	0

[Print 70]

12. 120

14. 264

16. 5

18. 11

20. 21

22. 77

24. 8

30. $a = b = 2$, $x = 1$, $x' = 2$

3.4 ERROR-DETECTING AND ERROR-CORRECTING CODES

2. 0 **4.** 1 **6.** 0 **8.** 0

10. .9415 **12.** .0746 **14.** .0042 **16.** .9135

18. 1 **20.** 5 **22.** 2 **24.** 6

26. 1100 **28.** 10111 **30.** 3 **32.** 4

34. **(a)** 12 **(b)** 6 **36.** **(a)** 19 **(b)** 9

40. Let the message word x be

$$c_{11}c_{12}\ldots c_{1s}c_{21}c_{22}\ldots c_{2s}\ldots c_{s1}c_{s2}\ldots c_{ss}.$$

Choose $b_i = 0$ or $b_i = 1$ so that $c_{i1}c_{i2}\ldots c_{is} + b_i$ is even for $i = 1, 2, \ldots, s$, and choose $d_k = 0$ or $d_k = 1$ so that $c_{1k}c_{2k}\ldots c_{sk} + d_k$ is even for $k = 1, 2, \ldots, s$. Let the codeword corresponding to x be

$$c_{11}c_{12}\ldots c_{1s}c_{21}c_{22}\ldots c_{2s}\ldots c_{s1}c_{s2}\ldots c_{ss}b_1b_2\ldots b_sd_1d_2\ldots d_s.$$

This is an $(s^2, s^2 + 2s)$-block code in which the minimal distance between codewords is at least 3.

3.5 MATRIX CODES

2. 64 **4.** 1024

6. 00111111 **8.** 01111010

10. 11×7 **12.** 13×6

14. $\frac{7}{11}$ **16.** 00000, 01111, 10011, 11100

18. 000000, 001011, 010110, 011101, 100101, 101110, 110011, 111000

20. 0000000, 0011101, 0101011, 0110110, 1000111, 1011010, 1101100, 1110001

22.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

24.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

26.
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

28.
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

30.
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

32. yes

34. no, 1101

36. yes

38. no, 1011

40. **(a)** The codewords determined by z and w are [0 0 0 0 0 0 0 0] and [1 0 0 0 0 0 0 0]; so the distance between these codewords is 1.
(b) By Theorem 3.2(a), not all errors in a single digit can be detected.

42.
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

3.6 MATRIX CODES THAT CORRECT ALL SINGLE-DIGIT ERRORS

2. 1001, 10

4. 1100, ??

6. 0000, 10

8. 0110, 00

10. 0000, 111

12. 0111, ??

14. 1000, 011

16. 1101, 111

18. 0000, 101

20. 0010, 001

22. 1111, ??

24. 1010, 101

26. 0000, 010

28. 0110, 010

30. 10, 01, 10, ??

32. 14

34. 30

36. $k = 26$ and $n = 31$

38. $k = 1013$ and $n = 1023$

40. **(a)** $(1-p)^4$ **(b)** $(1-p)^7 + 7p(1-p)^6$

SUPPLEMENTARY EXERCISES

2. true

4. false

6. [8]

8. [3]

10. 11

12. 2 and 1

14. 4

16. 15

18. 47

20. $2 = 41a - 76b$

22. $47 = 6a + 11b$

24. 10, 6, 23, 21

26.

Q	R	r_1	r_2	p	e
			1	25	11
5	1	31	25	31	5
2	1	4	16	4	2
1	0	16		16	1
0	1	25	25	25	0

[Print 25]

28. 1680

30. 16

32. 0

34. .00000005572

36. 3

38. The minimal Hamming distance between codewords cannot exceed the number of 1s in the codeword containing the fewest 1s.

40. 2^k

42. $C(n, 0) + C(n, 1) + \cdots + C(n, s)$.

44. 001011111

46. 0000, 0111, 1000, 1111

48.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

50.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

52.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

54. 0111, 110

56. 0101, ??

58. 1101, ??

60. 1011, 010

62. **(a)** 57×63 **(b)** 63×6

64. 5

66. 1, 2, 4, 8, 16, 32, and 64 pounds

68. No; consider $a = 4$, $b = 2$, and $c = 8$.

Chapter 4

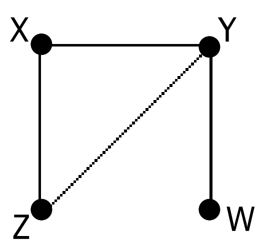
Graphs

4.1 GRAPHS AND THEIR REPRESENTATIONS

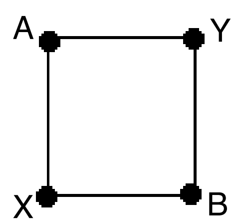
2. $\mathcal{V} = \{F, G, H\}$; $\mathcal{E} = \{\{F, G\}\}$

4. $\mathcal{V} = \{A, B, C, D\}$; $\mathcal{E} = \{\{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}\}$

6.



8.

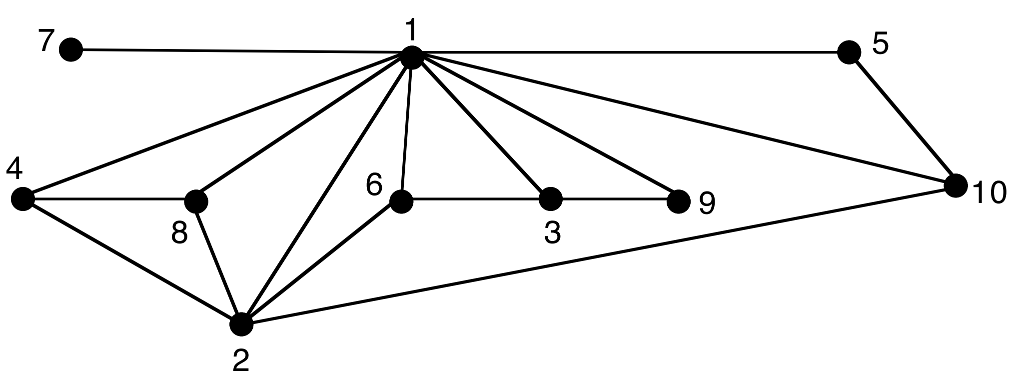


10. no

12. yes

14. no

18.



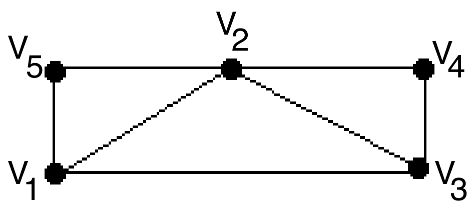
20. none, 0; C, 1

24. no

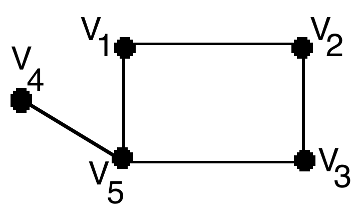
26. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $\begin{array}{ll} V_1: & V_2, V_3, V_4 \\ V_2: & V_1, V_3, V_4 \\ V_3: & V_1, V_2, V_4 \\ V_4: & V_1, V_2, V_3 \end{array}$

28. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ $\begin{array}{ll} V_1: & V_2, V_3, V_4 \\ V_2: & V_1, V_3 \\ V_3: & V_1, V_2, V_4 \\ V_4: & V_1, V_3 \end{array}$

30.



32.

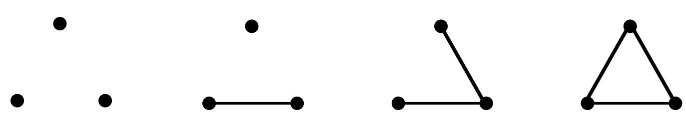


34. There are no edges.

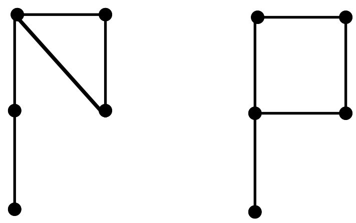
36. no

42. **(a)** yes **(b)** no **(c)** no

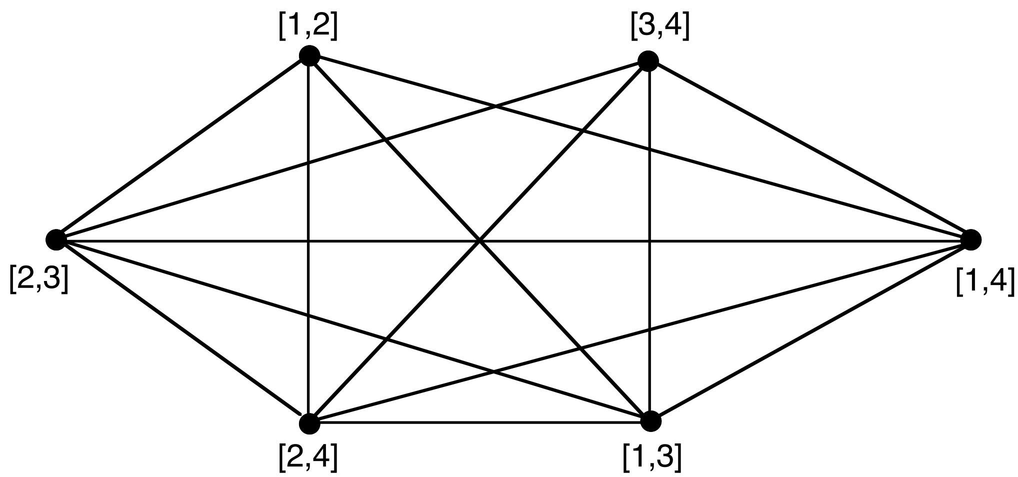
44.



46.



48.



50. n

52. the vertices corresponding to primes p such that $p > \dfrac{n}{2}$

4.2 PATHS AND CIRCUITS

2. not a graph

4. a graph

6. parallel edges: a, b, c; loops: f

8. parallel edges: none; loops: none

10. **(a)** e (1); b, c (2); a, f, b, a, d (5)
(b) e (1); a, d (2); b, c (2); a, f, c (3); b, f, d (3)
(c) e; b, c; a, d
(d) a, b, c, d (4); a, e, c, f (4); b, f, d, e (4); a, d, e (3); c, d, f (3); b, c, e (3); a, b, f (3)

12. no

14. yes

16. no

18. no

20. yes; $a, g, d, e, h, b, c, i, f$

22. no

24. yes, $a, b, h, i, m, k, e, d, f, j, g, c$

26. no

28. no

30. no

32. yes

34. yes

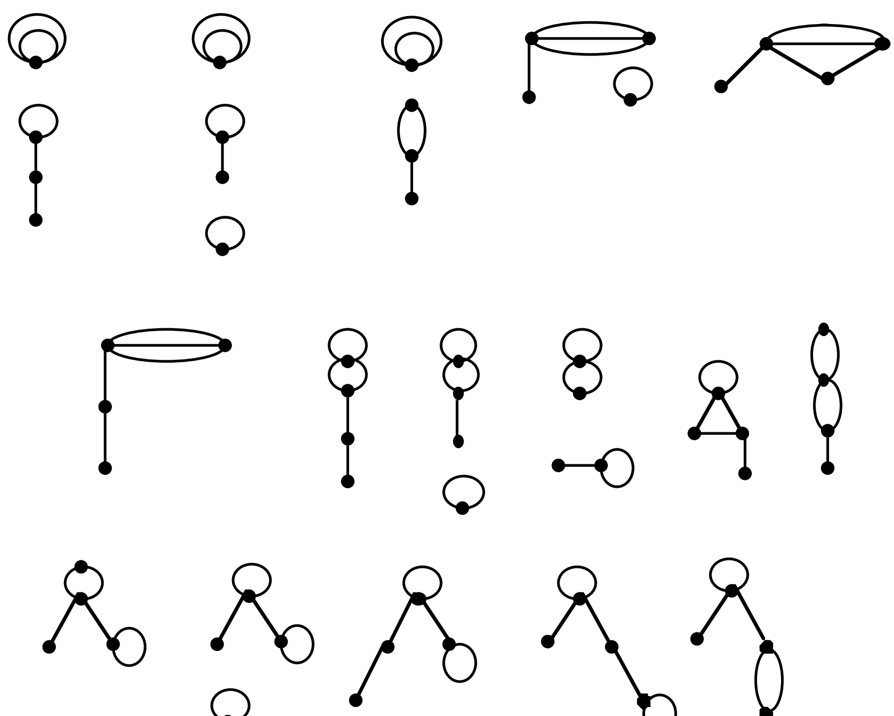
36. no

38. no

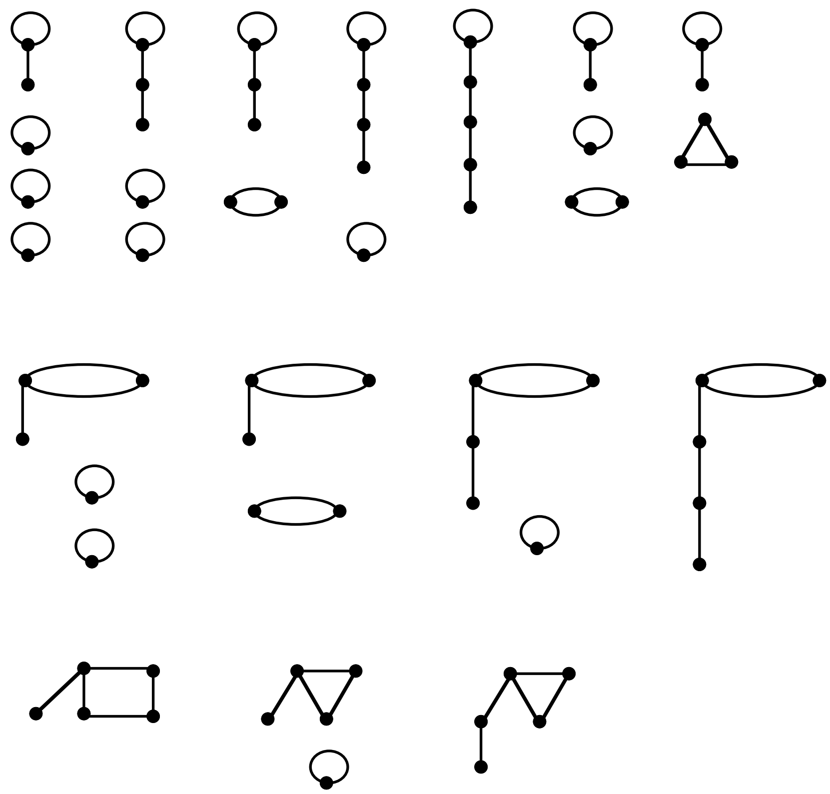
40. 1, 2, 3, 4, 5, 13, 12, 11, 10, 9, 8, 7, 6, 16, 17, 18, 19, 20, 14, 15, 1

42. 0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000

44.



46.



48. yes

50. no

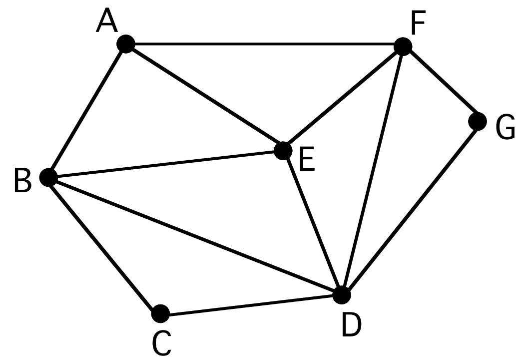
52. mn

54. $m = n > 1$

56. $n - 1$

58. **(a)** $\{A, D, E\}, \{B, C, F\}$ **(b)** $\{G\}, \{H\}, \{I\}$ **(c)** $\{J, O\}, \{N, K, L, M, P\}, \{Q\}$

60. **(a)**



(b) Euler path
(c) $A, F, G, D, F, E, A, B, E, D, B, C, D$

4.3 SHORTEST PATHS AND DISTANCE

2. 3; S, C, E, T

4. 11; $S, A, F, I, N, O, G, K, P, V, W, T$

6. C, 2; E, 3; F, 4; D, 7; G, 6; J, 5; H, 8; A, 11; S, C, D, H, A

8. C, 1; F, 2; J, 4; D, 4; G, 4; K, 5; E, 6; H, 5; L, 6; I, 8; A, 7; B, 8; S, C, D, H, E, A

10. S, B, C, A, G, D, T

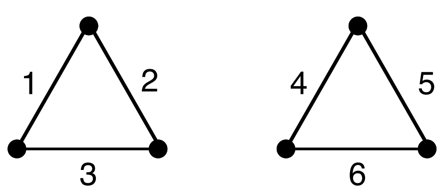
12. S, B, A, F, J, T

14. from V_1 to V_2: 1, 0, 4, 0; from V_1 to V_3: 1, 0, 4, 0

16. from V_1 to V_3: 1, 1, 6, 10; from V_2 to V_4: 0, 1, 2, 6

18. from V_1 to V_3: 1, 1, 6, 10; from V_2 to V_4: 0, 1, 2, 6

20.



4.4 COLORING A GRAPH

2. 4

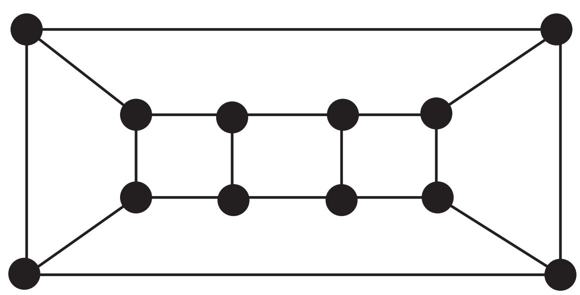
4. 2

6. 2

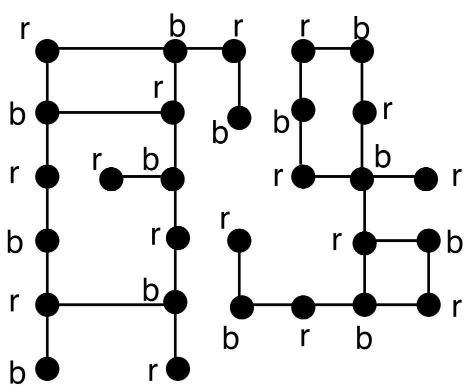
8. 3

10. 2, 2, 2

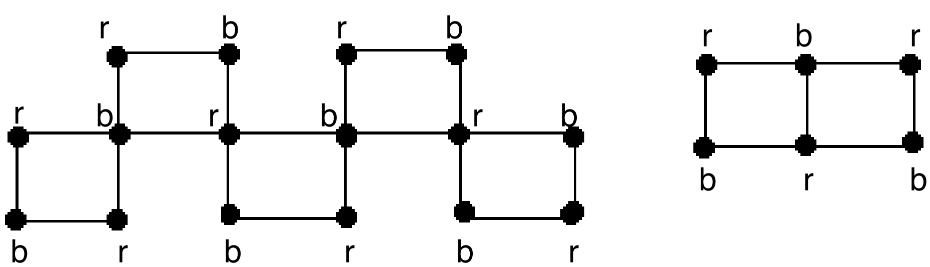
12.



16.



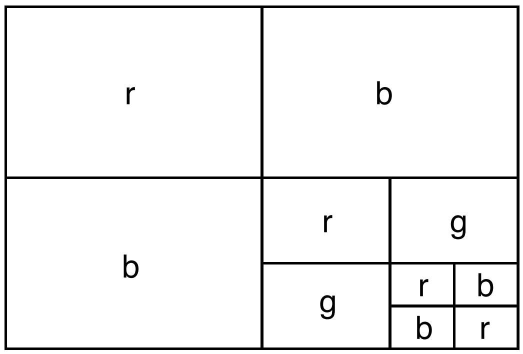
18.



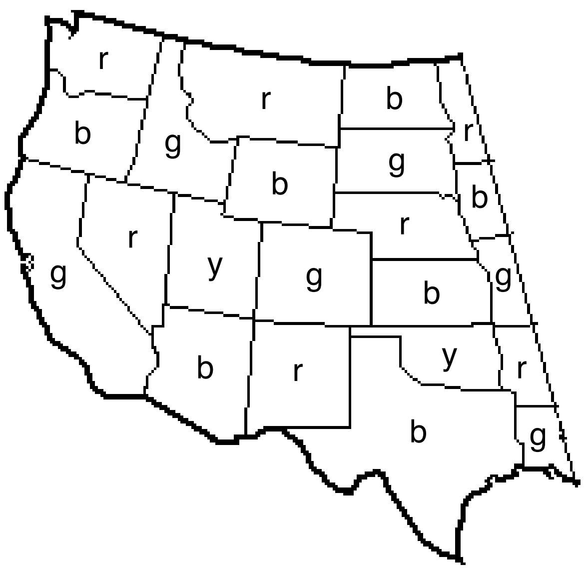
20. 27; 256

22. approximately 3.3 billion years, no

24.



26.



28. 3

30. yes

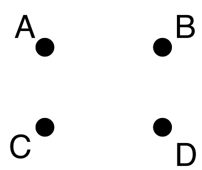
32. yes

4.5 DIRECTED GRAPHS AND MULTIGRAPHS

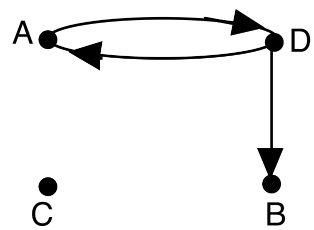
2. $\mathcal{V} = \{A, B\}$; $\mathcal{E} = \{\{A, B\}, \{B, A\}\}$

4. $\mathcal{V} = \{A, B, C, D, E\}$; $\mathcal{E} = \{\{B, A\}, \{C, B\}, \{B, D\}, \{D, E\}\}$

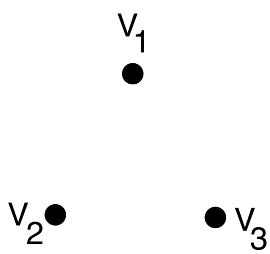
6.



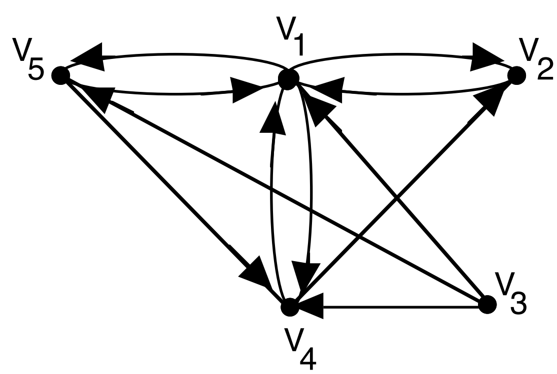
8.



10.



12.



14. none; none; 0; 0.

16. C; B, E; 1; 2

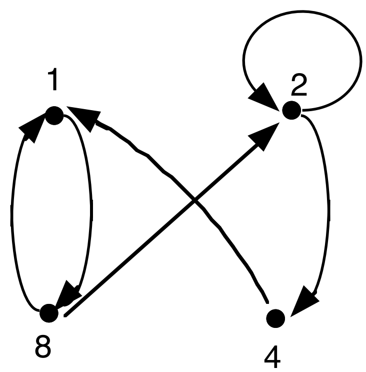
18. **(a)** A, B (1) **(b)** A, B, A (2); B, B (1)

20. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{array}{ll} A: & \text{none} \\ B: & C \\ C: & \text{none} \end{array}$

22. $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$ $\begin{array}{ll} A: & B, E \\ B: & C, D \\ C: & A, B \\ D: & \text{none} \\ E: & B \end{array}$

24. If a row contains only zeros, then the corresponding vertex has no directed edges leaving it. If a column contains only zeros, then the corresponding vertex has no directed edges coming into it.

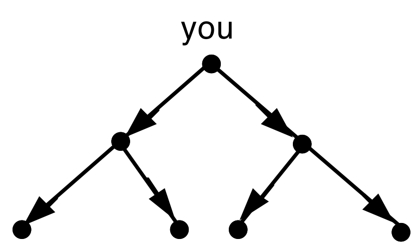
26.



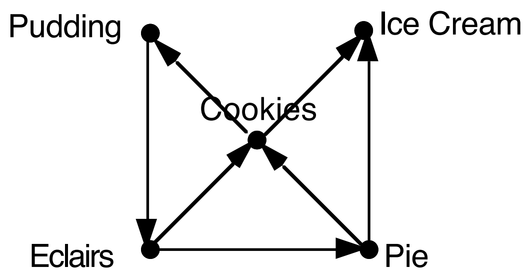
28. There will be a directed loop at each vertex. There will be parallel directed edges between every pair of vertices except for the pairs $\{1\}$ and $\{2\}$, $\{1\}$ and $\{3\}$, $\{2\}$ and $\{3\}$, $\{2\}$ and $\{1, 3\}$, $\{3\}$ and $\{1, 2\}$, $\{2, 3\}$ and $\{1, 2\}$, $\{2, 3\}$ and $\{1, 3\}$, $\{1, 3\}$ and $\{1, 2\}$.

30. A relation on a set is symmetric if $b\ R\ a$ is true whenever $a\ R\ b$ is true. In the case of a directed multigraph there is a directed edge from B to A whenever there is a directed edge from A to B.

32. The directed graph for the relation *is a child of* is shown below.



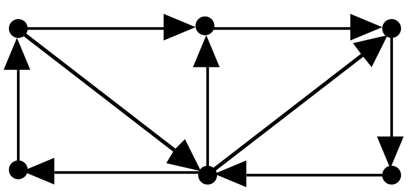
34. The directed graph for Susan's preferences is shown below.



36. yes

38. $n-1$

40.



42. no

44. Euler path: b, a, c, f, g, e, d

46. Euler path: $e, c, d, f, m, n, i, g, h, k, j, b, a$

48. none

50. 0011

54. b, d, e

56. $B; B, A; B, C; B, D$

58. no

60. This algorithm constructs a directed Euler circuit for a directed multigraph $\mathcal{D}$ (where the underlying multigraph is connected) where for each vertex the indegree is the same as the outdegree.

Step 1 (start directed path)
  (a) Set $\mathcal{E}$ to be the set of directed edges of $\mathcal{D}$.
  (b) Select a vertex, and set $\mathcal{C}$ to be the directed path consisting of this one vertex.
Step 2 (expand the directed path)
  while $\mathcal{E}$ is nonempty
    Step 2.1 (pick a starting vertex for expansion)
      (a) Set A to be a vertex in $\mathcal{C}$ that is incident with some directed

edge in $\mathcal{E}$.
(b) Set $\mathcal{P}$ to be the directed path consisting of just A.
Step 2.2 (expand $\mathcal{P}$ into a directed path from A to A)
(a) Set $B = A$.
(b) **while** *there is a directed edge e from B in $\mathcal{E}$*
(a) Remove e from $\mathcal{E}$.
(b) Replace B with the other vertex on e.
(c) Append the directed edge e and vertex B to the directed path $\mathcal{P}$.
endwhile
Step 2.3 (enlarge $\mathcal{C}$)
Replace any one occurrence of A in $\mathcal{C}$ with the directed path $\mathcal{P}$.
endwhile
Step 3 (output)
The directed path $\mathcal{C}$ is a directed Euler circuit.

62. 9; $S, E, F, L, G, B, C, H, M, T$

64. 7; S, A, B, F, G, J, O, T

66. Let $\mathcal{G}$ be a directed weighted graph in which there is more than one vertex and all weights are positive. This algorithm determines the distance and a shortest directed path from vertex S to every other vertex in $\mathcal{G}$. In the algorithm, $\mathcal{P}$ denotes the set of vertices with permanent labels. The predecessor of a vertex A is a vertex in $\mathcal{P}$ used to label A. The weight of the edge on vertices U and V is denoted by $W(U, V)$, and if there is no edge on U and V, we write $W(U, V) = \infty$.

Step 1 (label S)
(a) Assign S the label 0, and let S have no predecessor.
(b) Set $\mathcal{P} = \{S\}$.
Step 2 (label vertices)
Assign to each vertex V not in $\mathcal{P}$ the (perhaps temporary) label $W(S, V)$, and let V have the (perhaps temporary) predecessor S.
Step 3 (enlarge $\mathcal{P}$ and revise labels)
repeat
Step 3.1 (make another label permanent)
Include in $\mathcal{P}$ a vertex U having the smallest label of the vertices not in $\mathcal{P}$. (If there is more than one such vertex, arbitrarily choose any one of them.)
Step 3.2 (revise temporary labels)
For each vertex X not in $\mathcal{P}$ for which there is a directed edge from U to X, replace the label on X with the smaller of the old label on X and the sum of the label on U and $W(U, X)$. If the label on X was changed, let U be the new (perhaps

temporary) predecessor of X.
until $\mathcal{P}$ *contains every vertex of* $\mathcal{G}$
Step 4 (find distances and shortest directed paths)
The label on a vertex Y is its distance from S. If the label on Y is ∞, then there is no directed path, and hence no shortest directed path, from S to Y. Otherwise, a shortest directed path from S to Y is formed by using in reverse order the vertices Y, the predecessor of Y, the predecessor of the predecessor of Y, and so forth, until S is reached.

68. C, 2; F, 3; I, 3; D, 3; G, 5; J, 5; E, 5; H, 6; A, 9; B, 7; K, 6; S, C, D, E, A

70. C, 4; E, 2; G, 3; D, 6; F, 7; H, 3; I, 4; A, 6; B, 5; S, E, H, A

72. 1, 1, 1, 1; 1, 0, 0, 1

74. A directed graph $\mathcal{D}_1$ is isomorphic to a directed graph $\mathcal{D}_2$ when there is a one-to-one correspondence f between the vertices of $\mathcal{D}_1$ and $\mathcal{D}_2$ such that there is a directed edge in $\mathcal{D}_1$ from vertex U to vertex V if and only if there is a directed edge in $\mathcal{D}_2$ from vertex $f(U)$ to vertex $f(V)$.

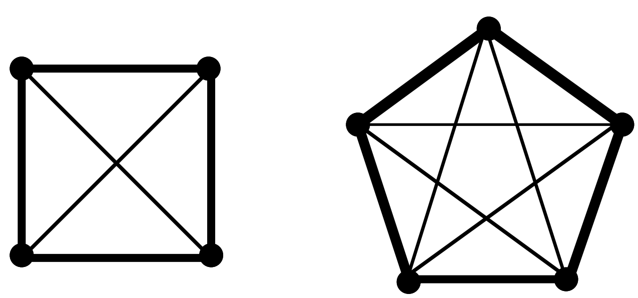
76. **(a)** no **(b)** yes

SUPPLEMENTARY EXERCISES

2. no, yes, no

6. Look at how the rows and columns of the first adjacency matrix are permuted.

10.



12. yes, $a, g, d, e, h, b, c, i, f$

14. yes

16. yes

18. C, 3; E, 4; H, 5; D, 5; F, 6; I, 6; A, 8; G, 7; B, 9. The shortest paths are S, C, E, F, G, A and S, C, E, F, G, B.

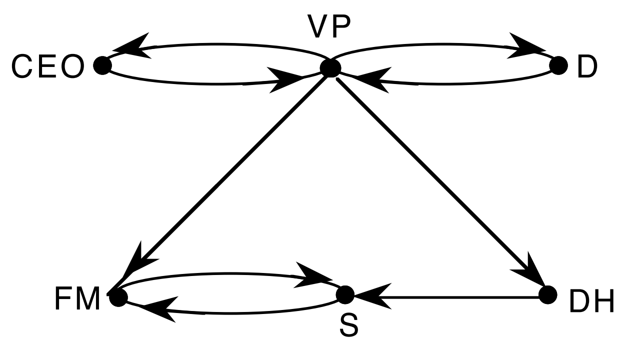
20. S, B, A, F, J, T

22. **(a)** 4 **(b)** 3

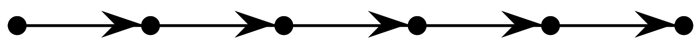
24. no

28. $P, N; P, H; P, K, O; P, K; P, L; P, K, M$

30.



32.



36. a, e, f; e, c, b; c, a, d

40. A, 5; B, 10; C, 4; D, 3; E, 5; F, 2; G, 4. The shortest paths are S, F, G, A and S, F, G, B.

Chapter 5

Trees

5.1 PROPERTIES OF TREES

2. yes

4. no

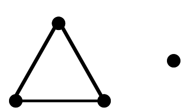
6. yes

8. no

10. 20

12. 33; no

14.

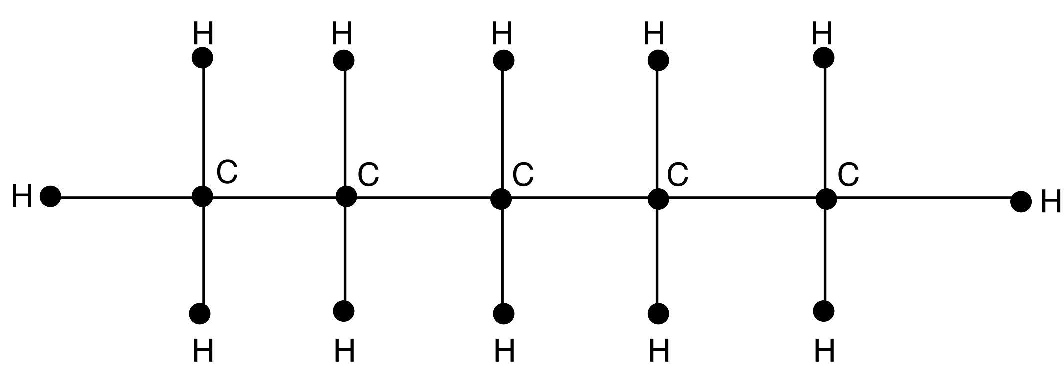


16. $n - 1$

18. $\frac{n(n-1)}{2}$

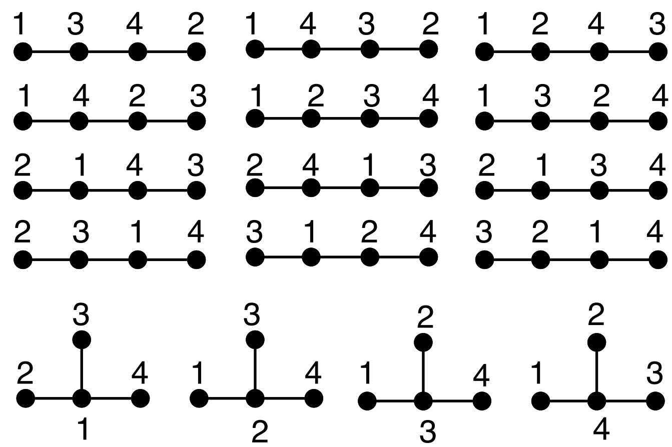
22. yes

24.



26. 9

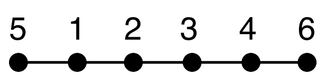
28.



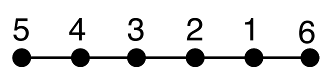
30. 3, 4; 4, 3; 2, 4; 4, 2; 2, 3; 3, 2; 1, 4; 4, 1; 1, 3; 3, 1; 1, 2; 2, 1; 1, 1; 2, 2; 3, 3; 4, 4

32. 1, 5, 6, 5, 4, 5, 9

34.

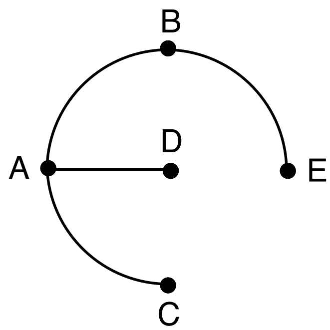


36.

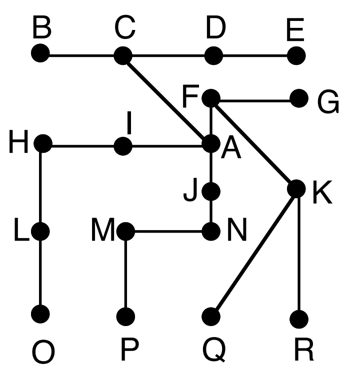


5.2 SPANNING TREES

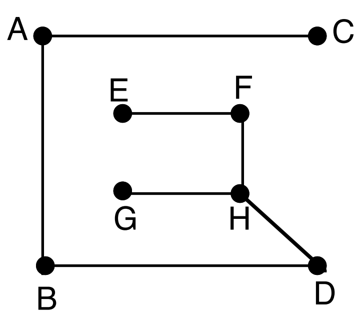
2.



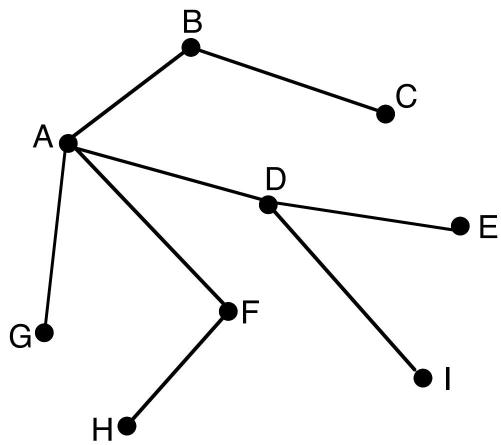
4.



6.

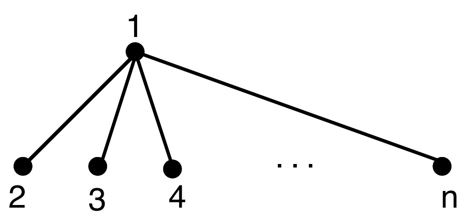


8.



10. n

12.



14. yes

16. no

18. $d, i, m, f, b, g, c, n, a$; 21

20. $e, f, b, c, j, g, d, k, q, i, o$; 36

22. $g, b, f, i, m, d, c, n, a$; 21

24. $g, c, j, b, f, e, d, k, q, i, o$; 36

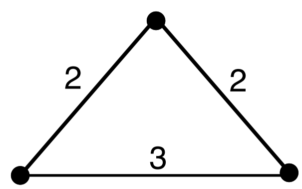
26. $a, e, h, k, p, j, n, f, b$; 36

28. $i, s, n, p, t, m, u, d, a, b, k$; 64

30. {Jones, Chen}, {Jones, Brown}, {Brown, Ritt}, {Ritt, Hill}

32. Modify Prim's algorithm so that in step 1 $\mathcal{T}$ is initialized to consist of the specified edge and $\mathcal{L}$ is to consist of the vertices of the specified edge; the rest of the algorithm continues in the same way. For the graph in Exercise 17 and the edge g, we obtain the edges g, f, c, and a.

34.



36. It does not consider the weights of the experiments.

38. g, f, a, c

40. k, e, f, i, c, j, a, d

42. Kruskal's algorithm is modified so that in step 1 $\mathcal{T}$ is initialized to consist of the specified edge and $\mathcal{S}$ is initialized to consist of all the edges of $\mathcal{G}$ except for the specified edge. The rest of the algorithm continues in the same way. For the graph in Exercise 17 and the edge d, we obtain the edges d, g, f, and a.

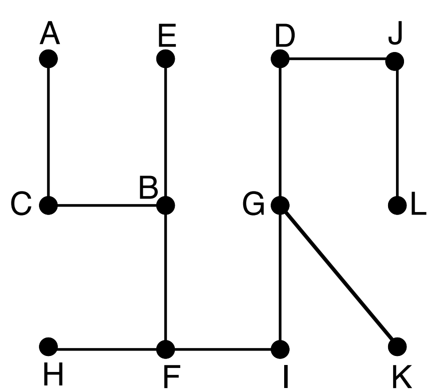
5.3 DEPTH-FIRST SEARCH

2. $A, C, B, E, F, H, I, G, D, J, L, K$

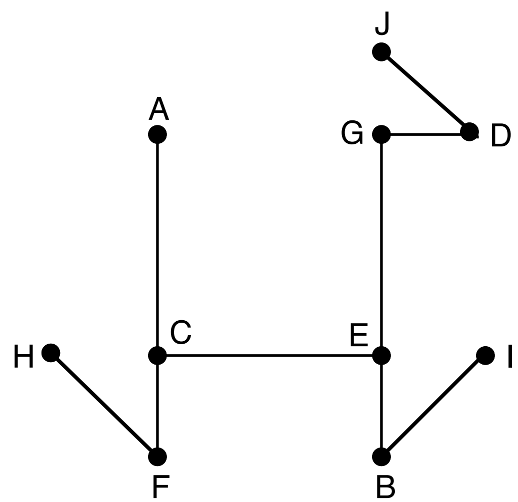
4. $A, C, E, B, I, G, D, J, F, H$

6. A, C, D, B, E, G, F, H

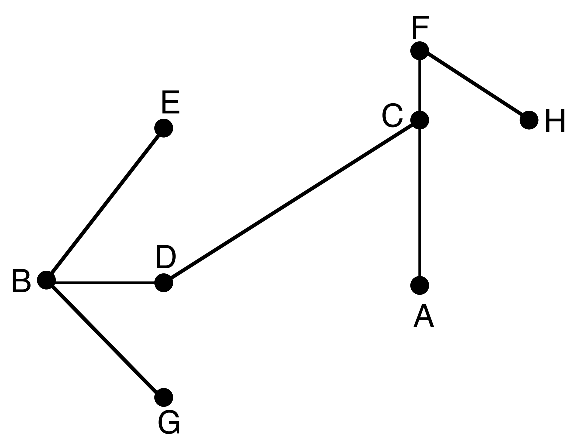
8.



10.



12.



14. $\{A, E\}$, $\{C, H\}$, $\{B, G\}$, $\{G, L\}$, $\{I, K\}$

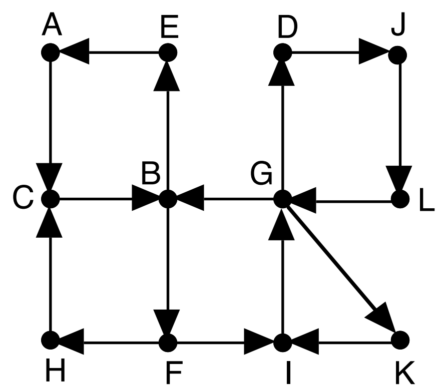
16. $\{A, G\}$, $\{C, H\}$, $\{E, I\}$, $\{J, G\}$

18. $\{D, A\}$, $\{C, H\}$, $\{C, E\}$, $\{E, D\}$, $\{D, G\}$

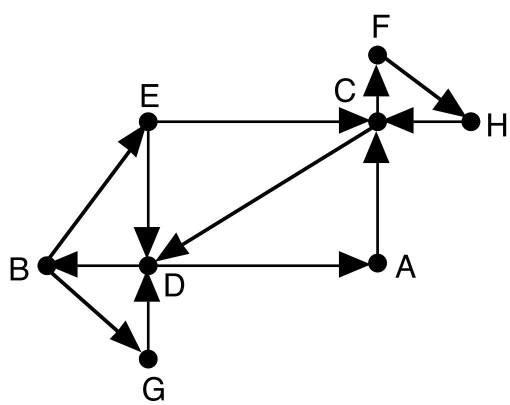
20. yes

22. no

24. yes



26. yes



28. yes

32. 6

34. $b \leq d$

5.4 ROOTED TREES

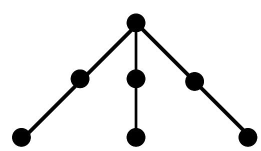
2. no

4. yes

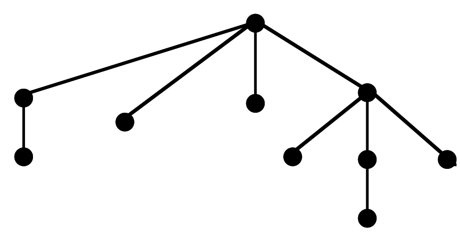
6. yes

8. yes

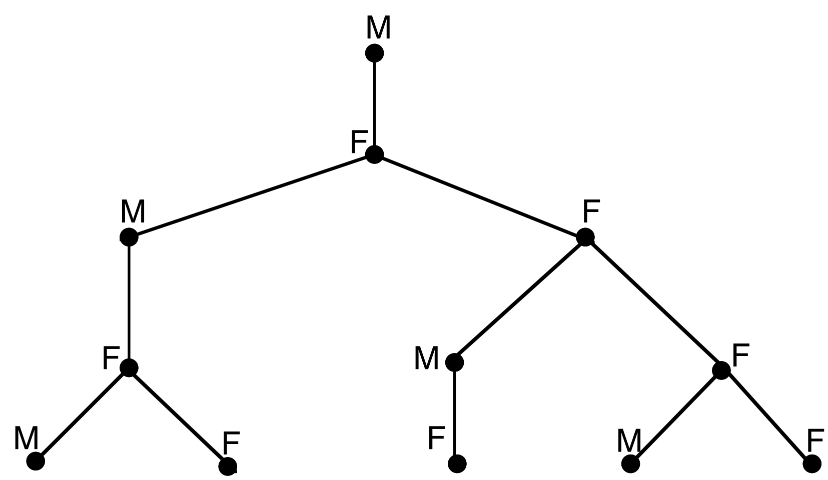
10.



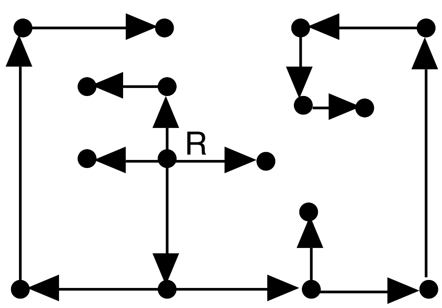
12.



16.



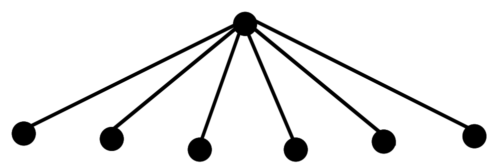
18.



20. **(a)** A **(b)** A, B, D, G **(c)** E, F, C, I, H **(d)** D
(e) E, F **(f)** G, H, I **(g)** A, D

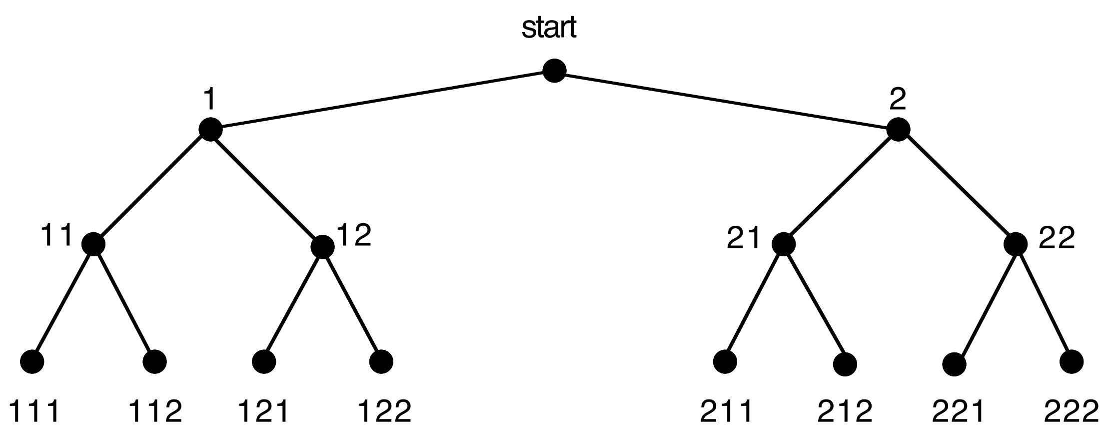
22. **(a)** C **(b)** C, G, A, B **(c)** D, E, H, F, I **(d)** C
(e) I **(f)** none **(g)** C

24.

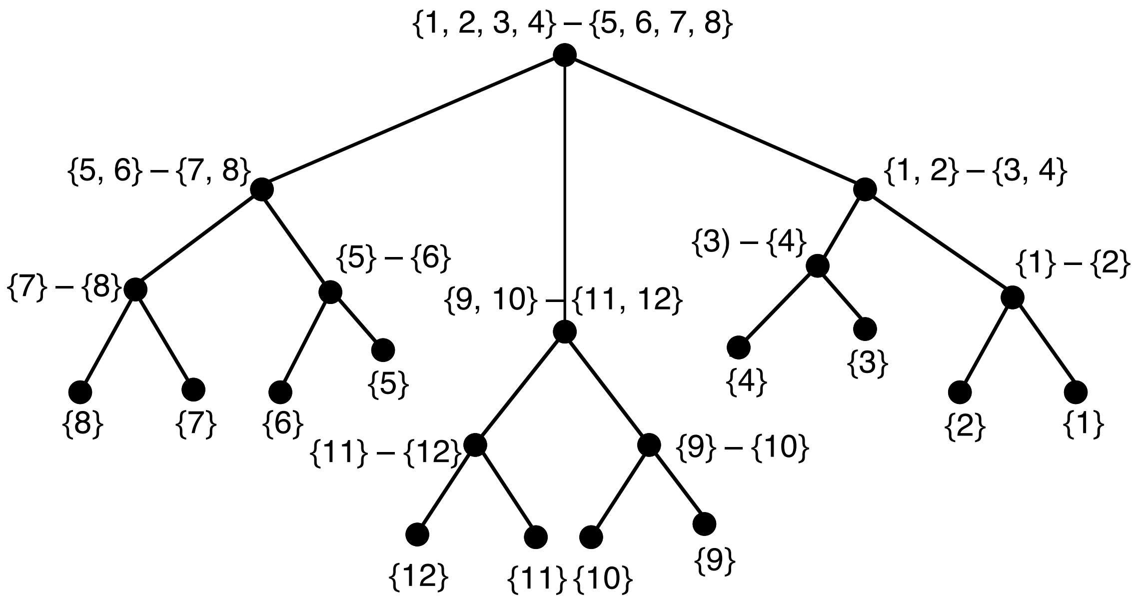


26. 52

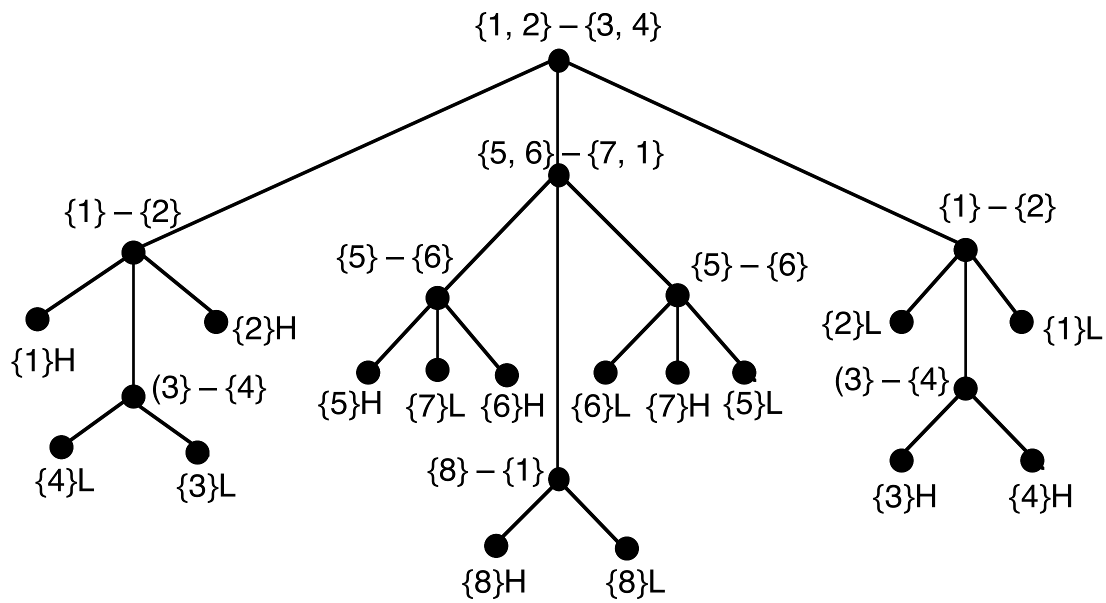
28.



30.


{1, 2, 3, 4} – {5, 6, 7, 8}
{5, 6} – {7, 8}
{1, 2} – {3, 4}
{3) – {4}
{1} – {2}
{5} – {6}
{7} – {8}
{9, 10} – {11, 12}
{4}
{3}
{5}
{8}
{7}
{6}
{2}
{1}
{11} – {12}
{9} – {10}
{12}
{11}
{10}
{9}

32.


{1, 2} – {3, 4}
{5, 6} – {7, 1}
{1} – {2}
{1} – {2}
{5} – {6}
{5} – {6}
{2}H
{1}H
{2}L
{1}L
(3} – {4}
{5}H
{7}L
{6}H
{6}L
{7}H
{5}L
(3} – {4}
{4}L
{3}L
{8} – {1}
{3}H
{4}H
{8}H
{8}L

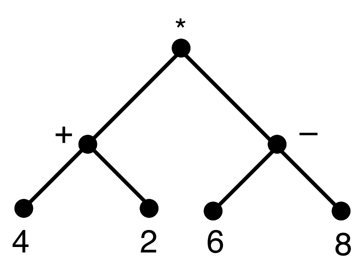
34. 1, 2, 4

36. 4

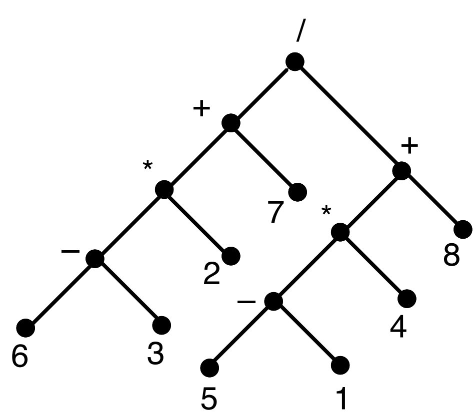
38. 2

5.5 BINARY TREES AND TRAVERSALS

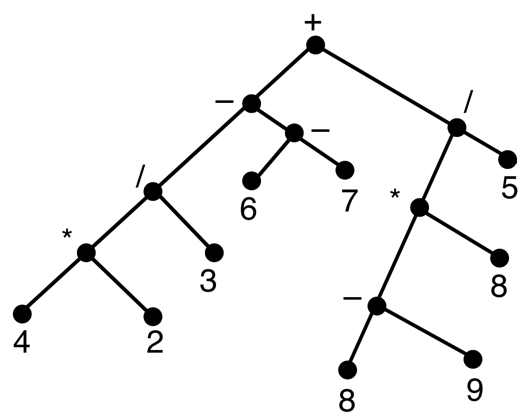
2.



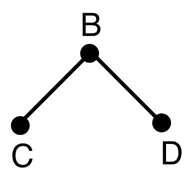
4.



6.

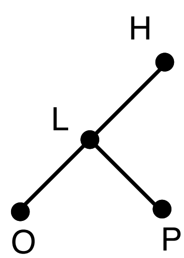


8.



10.

12.



14. A, B, C, D

16. $A, B, D, H, E, I, J, C, F, K, G, L$

18. $A, B, D, G, K, H, L, O, P, C, E, I, M, R, S, J, N, T, F$

20. C, D, B, A

22. $H, D, I, J, E, B, K, F, L, G, C, A$

24. $K, G, O, P, L, H, D, B, R, S, M, I, T, N, J, E, F, C, A$

26. A, C, B, D

28. $H, D, B, I, E, J, A, F, K, C, G, L$

30. $K, G, D, O, L, P, H, B, A, I, R, M, S, E, J, T, N, C, F$

32. $* \ + \ 4 \ 2 \ - \ 6 \ 8$

34. $/ \ + \ * \ - \ 6 \ 3 \ 2 \ 7 \ + \ * \ - \ 5 \ 1 \ 4 \ 8$

36. $+ \ - \ / \ * \ 4 \ 2 \ 3 \ - \ 6 \ 7 \ / \ * \ - \ 8 \ 9 \ 8 \ 5$

38. $4 \ 2 \ + \ 6 \ 8 \ - \ *$

40. $6 \ 3 \ - \ 2 \ * \ 7 \ + \ 5 \ 1 \ - \ 4 \ * \ 8 \ + \ /$

42. $4 \ 2 \ * \ 3 \ / \ 6 \ 7 \ - \ - \ 8 \ 9 \ - \ 8 \ * \ 5 \ / \ +$

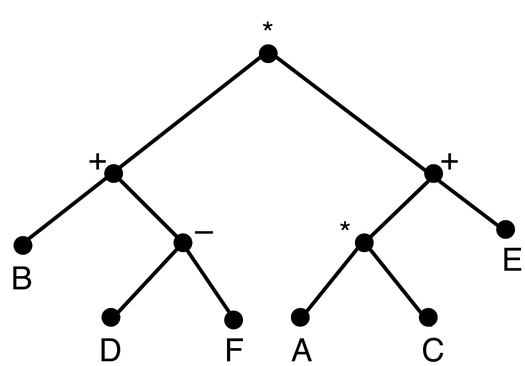
44. 24

46. -6

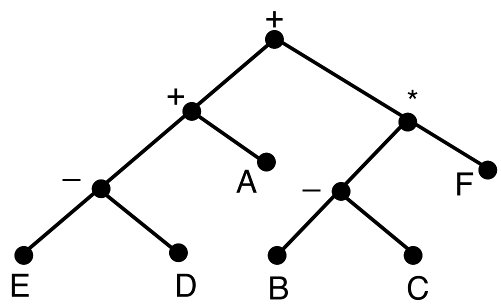
48. -25

50. -8

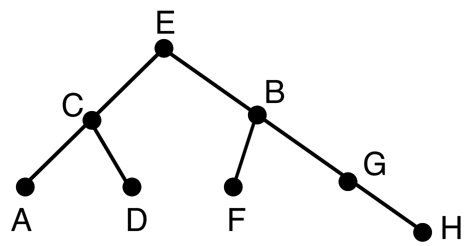
52.



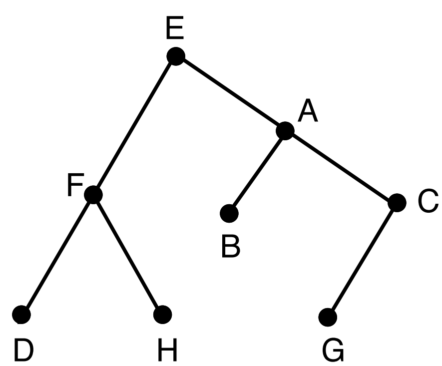
54.



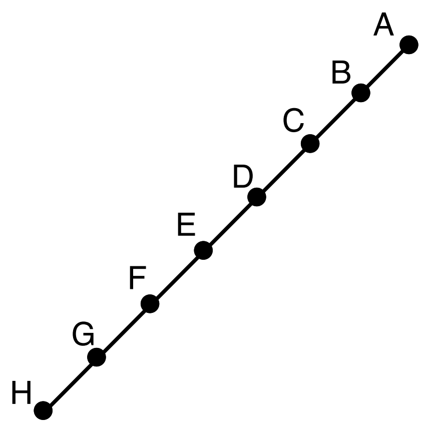
56.



58.



60.



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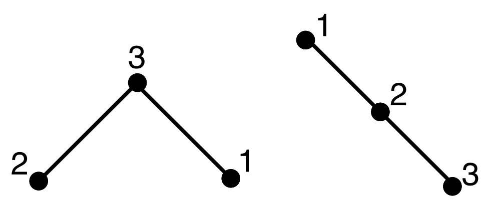
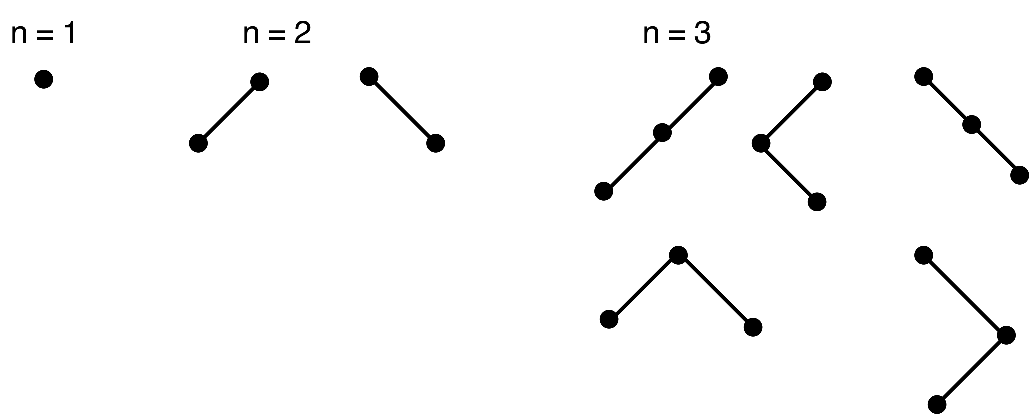


64.



5.6 OPTIMAL BINARY TREES AND BINARY SEARCH TREES

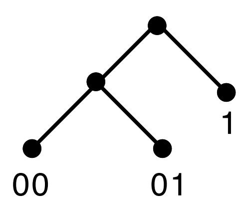
2. yes

4. yes

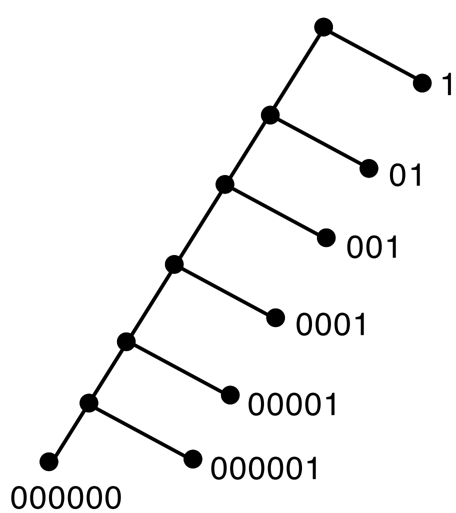
6. yes, {00, 10, 010, 011, 110, 111}

8. $a = 1$, $b = 1$, $c = 1$, $d = 1$

10.



12.



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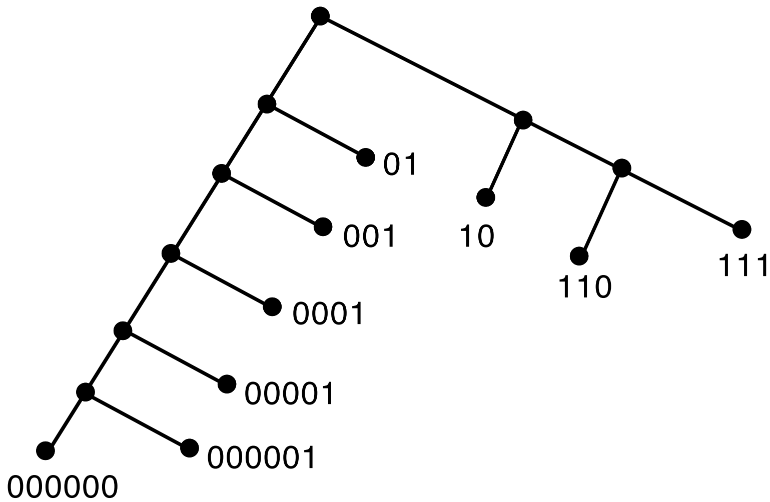
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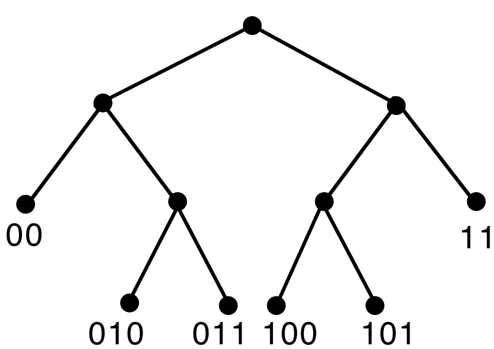
16.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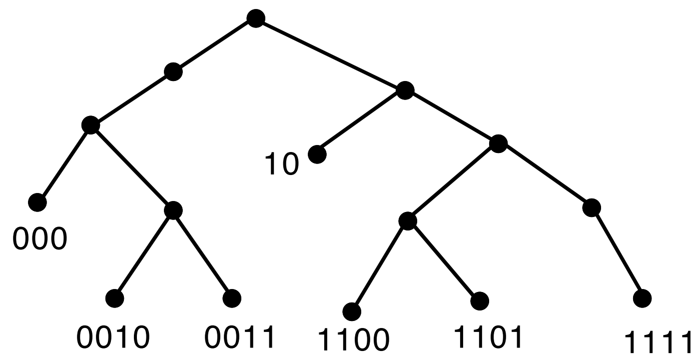
18.



20. ROBIN

22. JBOND

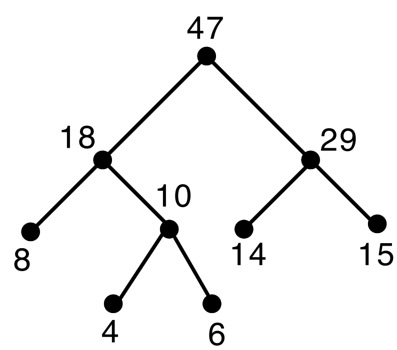
24. STARRAT

26. EATTHECAT

28. HOME

30. HELP

32.



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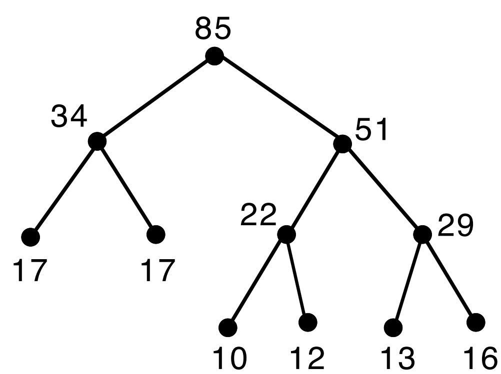


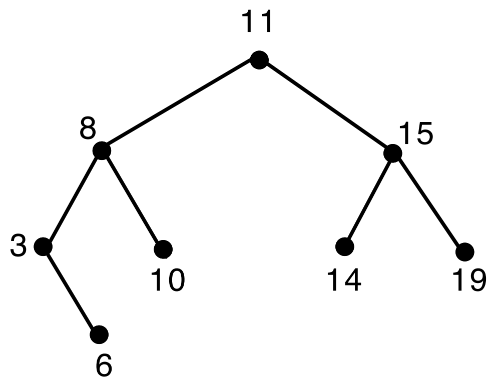
36. 366

38. 814

40. T: 11, A: 10, I: 01; L: 001, P: 0001, J: 0000

42. m: 1001, @: 011, c: 10001, s: 1011, po: 010, os: 1010, od: 10000, tid: 00, qod: 11

48.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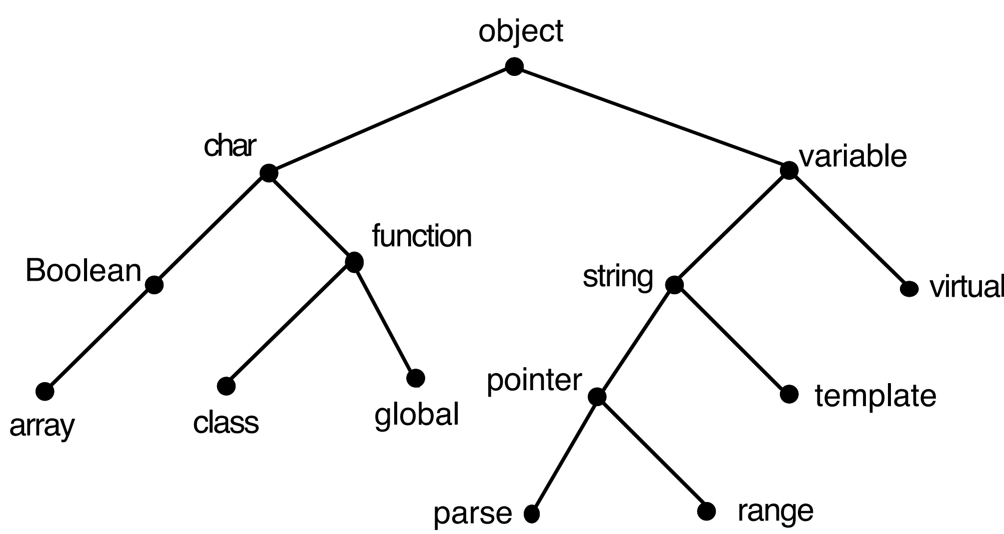
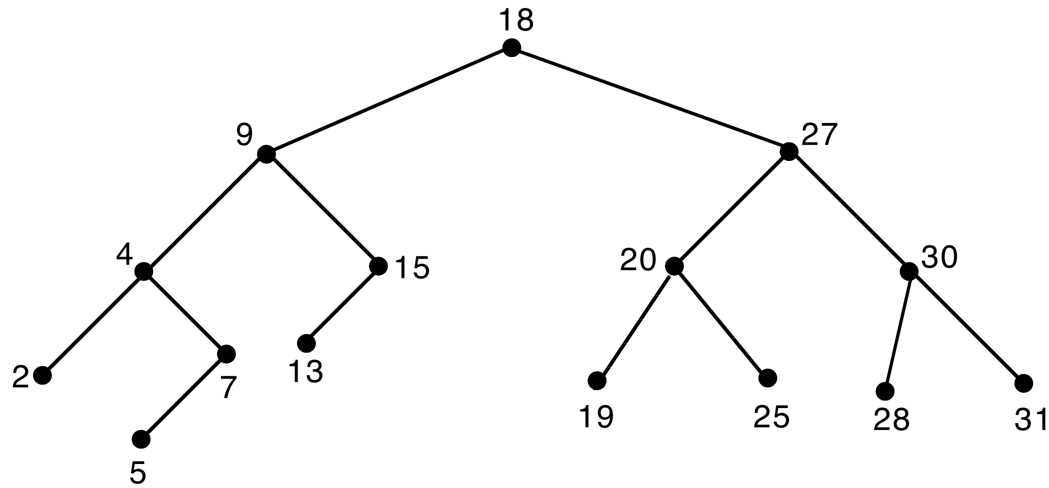
50.


object
char
variable
function
Boolean
string
virtual
pointer
template
array
class
global
parse
range

52.


18
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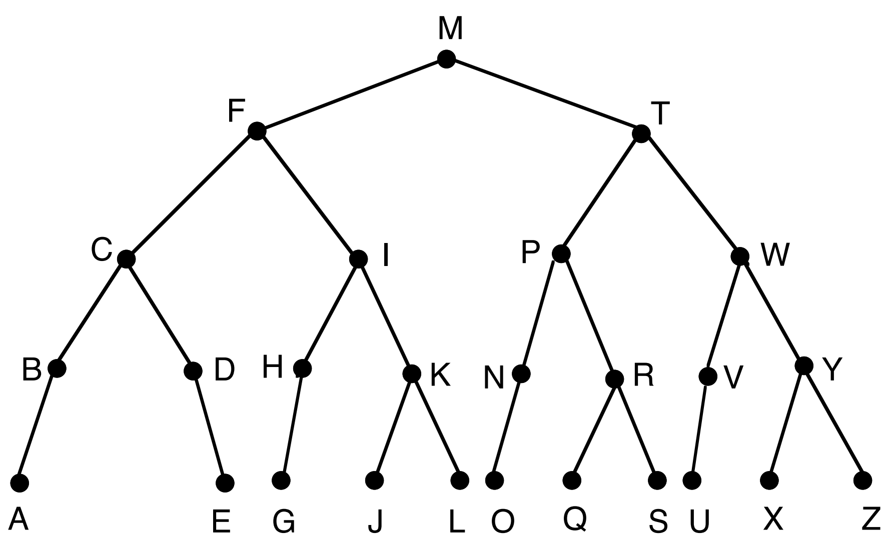
54.


M
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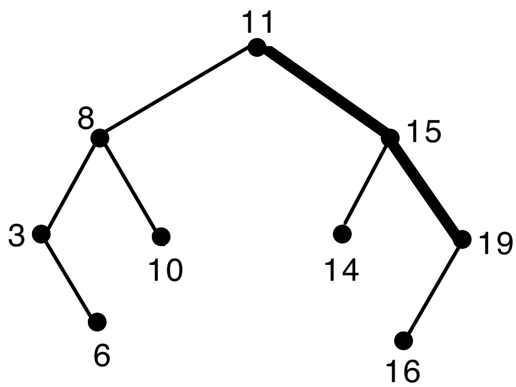
Z

56.


11
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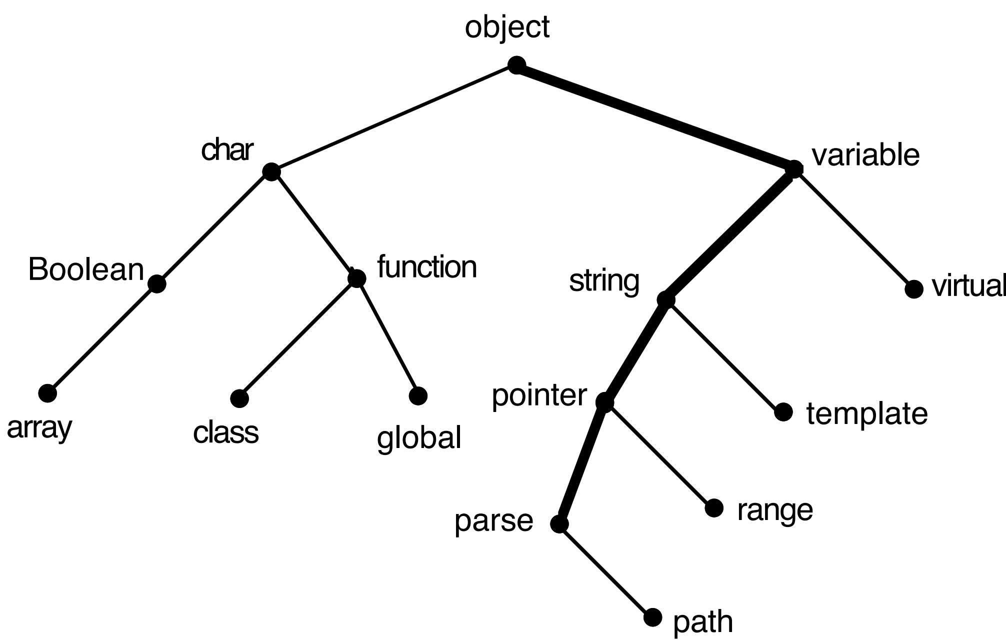
16

58.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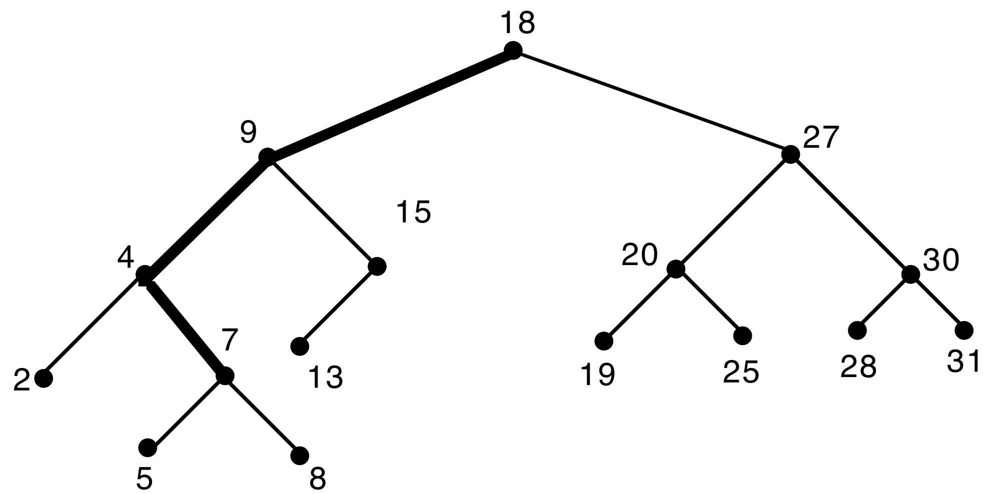
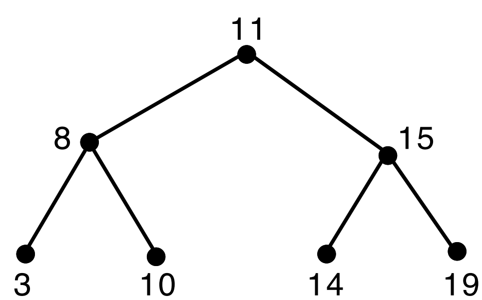
path

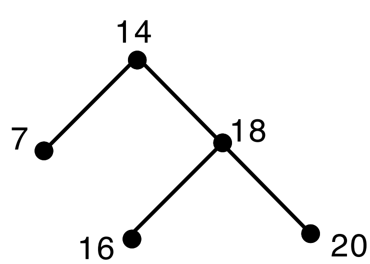
60.



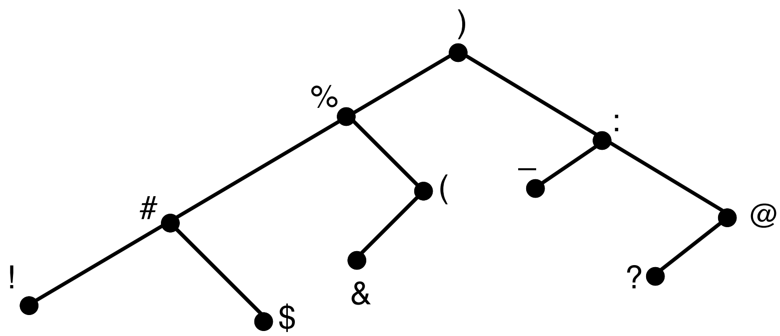
62.



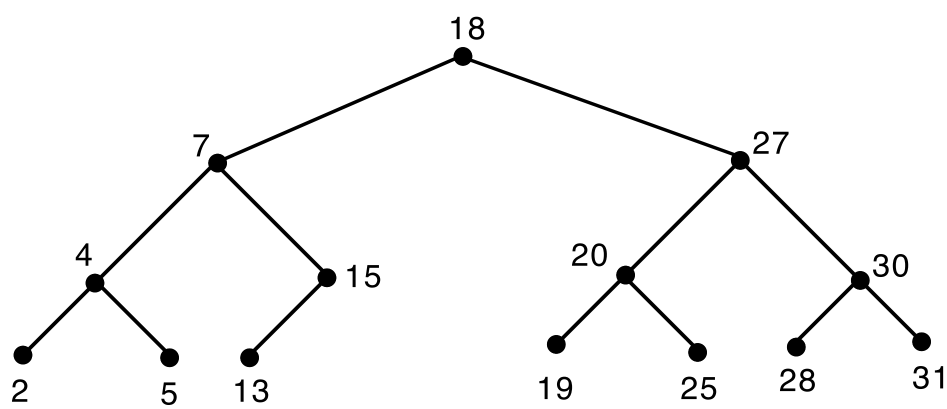
64.



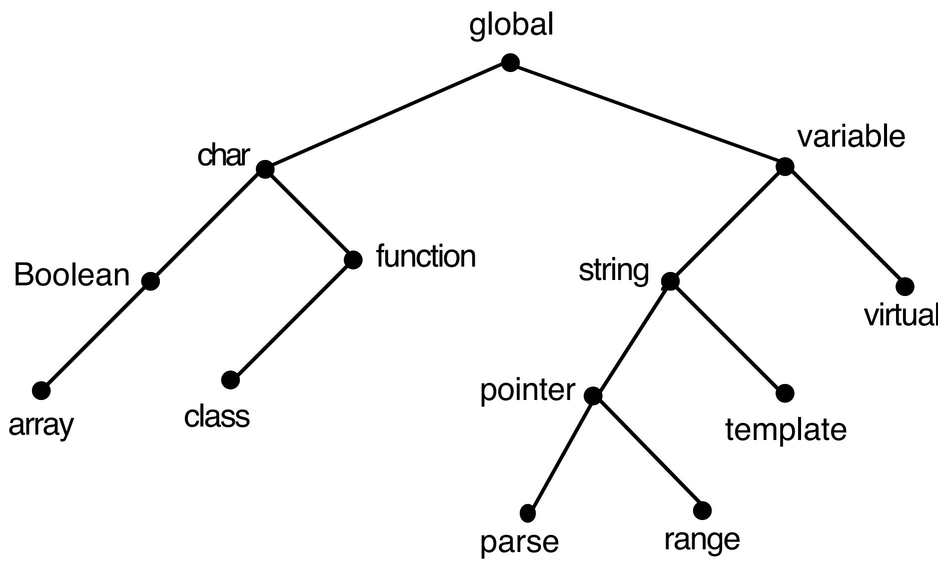
66.



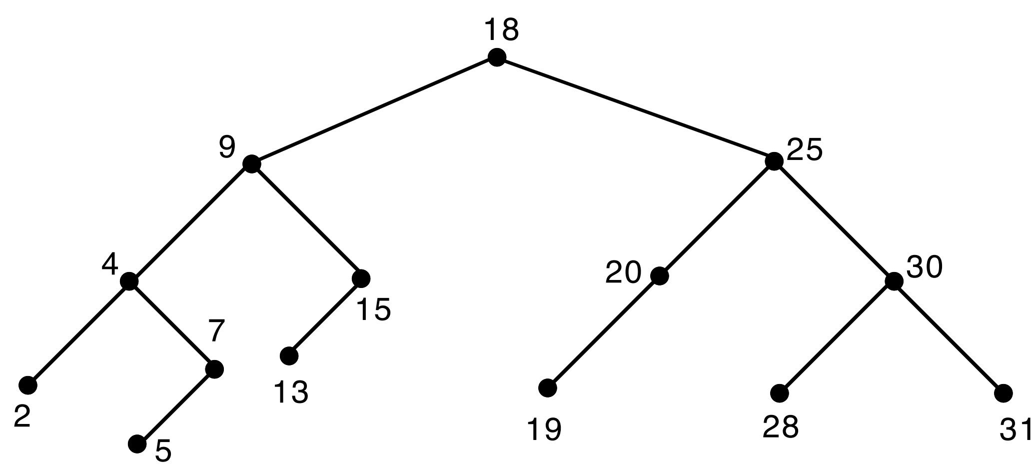
68.



70.


global
char
variable
Boolean
function
string
virtual
array
class
pointer
template
parse
range

72.


18
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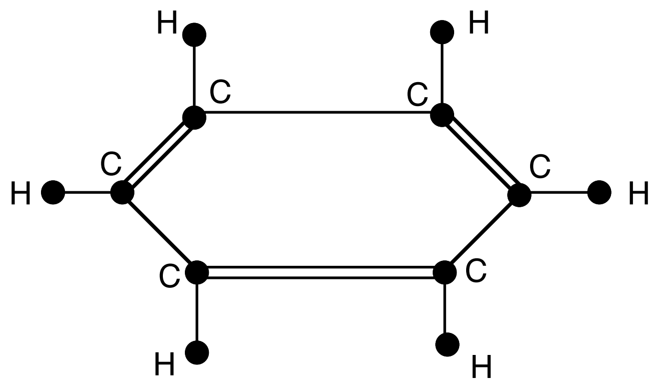
28
31

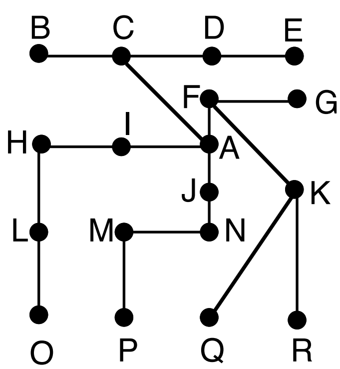
SUPPLEMENTARY EXERCISES

2. 2

4. no

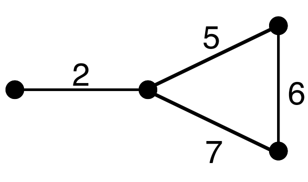


12.



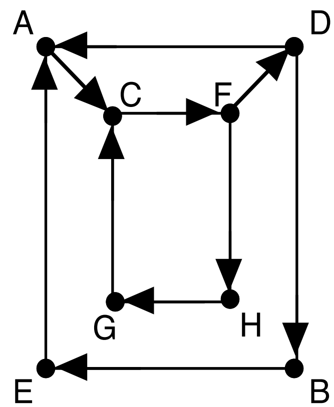
14. **(a)** k, e, f, i, j, d, c, a; 21 **(b)** $f, e, b, c, j, g, d, k, q, i, o$; 36

16.

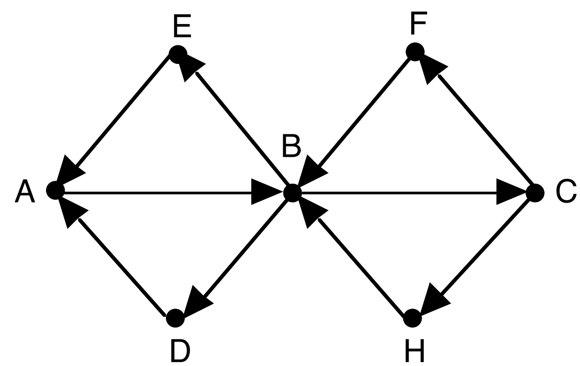


18. **(a)** A, C, F, D, B, E, H, G **(b)** A, B, C, F, H, D, E

20. **(a)**



(b)

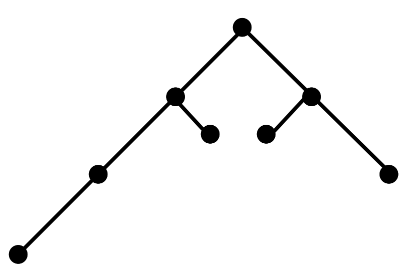


24. 1 2 1 3 1 2 3 1 3 2

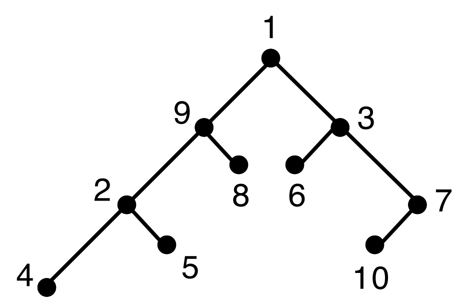
26. 8

28. $(q-1)p+1$

38.



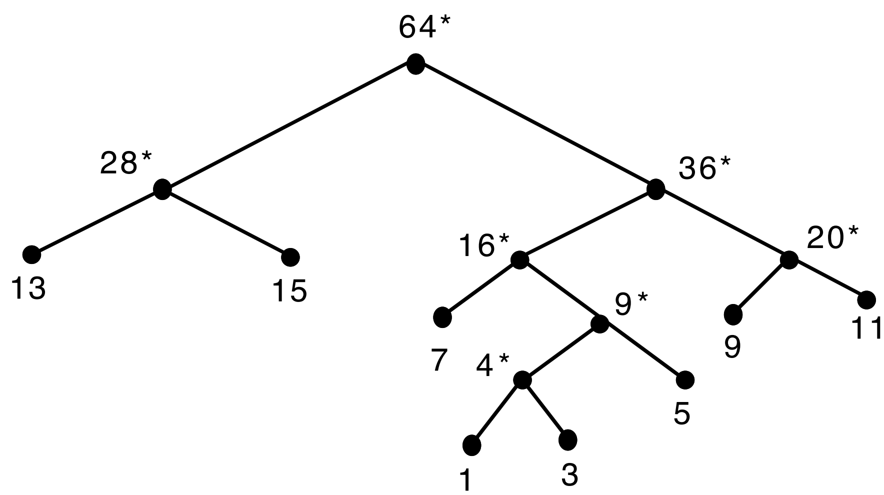
40.



42. $*\ a\ +\ b\ c\ =\ +\ *\ a\ b\ *\ a\ c$; $a\ b\ c\ +\ *\ =\ a\ b\ *\ a\ c\ *\ +$

44. yes, {11, 101, 0101, 00, 100, 011, 0100}

46.



Chapter 6

Matching

6.1 SYSTEMS OF DISTINCT REPRESENTATIVES

2. 1

4. 20

6. 12

8. no

10. yes

12. $\{1,2,4\}$

14. $\{3,5,6\}$

16. $\{1,2,4,5,6,7\}$

18. $k!/(k-n)!$

22. yes

28. 1

30. $k_1(k_2-1)(k_3-2)\cdots(k_n-n+1)$ if $k_i \geq i$ for all i; otherwise 0

6.2 MATCHINGS IN GRAPHS

2. no

4. yes; $\mathcal{V}_1 = \{1,3,7\}$, $\mathcal{V}_2 = \{2,4,5,6\}$

6. yes; $\mathcal{V}_1 = \{1,3,5,7,9,11,12\}, \mathcal{V}_2 = \{2,4,6,8,10,13\}$

8. $\{\{1,2\},\{3,4\},\{6,7\}\}$, $\{\{1,6\},\{3,4\},\{2,9\},\{5,8\},\{7,10\}\}$,
$\{\{1,4\},\{2,5\},\{3,6\},\{7,8\},\{9,10\},\{11,13\}\}$

10. $\{1,3,7\},\{1,2,3,7,8\},\{2,4,6,8,10,13\}$

12. The matrix of the graph is shown below.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

14. The matrix of the graph is shown below.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

16. The matrix of the graph is shown below.

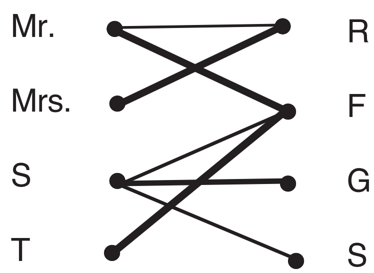
$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

18. The asterisks denote a maximum independent set of 1s.

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

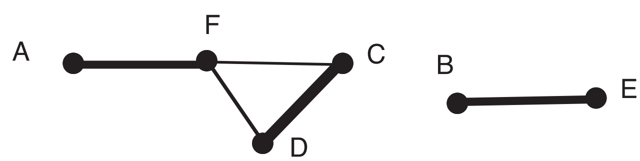
20. All rows; rows named 1, 5, 9 and columns named 6, 10; row named 5 and columns named 2, 8.

22. In the bipartite graph below, the maximum matching is indicated by the thicker edges.

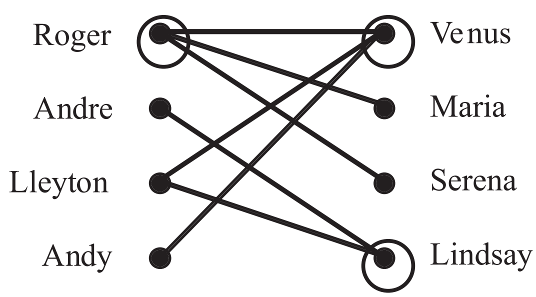


$$\begin{array}{c} \\ \text{Mr.} \\ \text{Mrs.} \\ \text{S} \\ \text{T} \end{array} \begin{array}{c} \begin{array}{cccc} \text{R} & \text{F} & \text{G} & \text{S} \end{array} \\ \begin{bmatrix} 1 & 1^* & 0 & 0 \\ 1^* & 0 & 0 & 0 \\ 0 & 1 & 1^* & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

24. In the graph below, the maximum matching is indicated by the thicker edges.

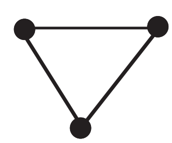


26. The circled vertices form a minimum covering in the graph below.



$$\begin{array}{c} \\ \text{Roger} \\ \text{Andre} \\ \text{Lleyton} \\ \text{Andy} \end{array} \begin{array}{c} \begin{array}{cccc} \text{V} & \text{M} & \text{S} & \text{L} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

28. In the graph below, a maximum matching has 1 edge, but a minimum covering has 2 vertices.



32. **(a)** 10 **(b)** 19

6.3 A MATCHING ALGORITHM

2. After applying step 2.2, the matrix is as follows.

$$\begin{array}{ccc}
 & \begin{array}{ccccc} A & B & C & D & E \end{array} & \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1^* & 0 & 1 & 0 & 1 \\ 0 & 1^* & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1^* & 0 \end{bmatrix} & \begin{array}{c} C\surd \\ E\surd \\ A \\ B \end{array} \\
 & \begin{array}{ccccc} 1\surd & 2\surd & \#\surd & & \#\surd \end{array} &
\end{array}$$

4. $4C, 1C, 1E$

6. $1D, 2C, 3A, 4B$

8. $1D, 2B, 3A, 4C$

10. $1D, 2C, 3B, 5A$

12. $\{1, C\}, \{2, D\}, \{3, B\}, \{4, A\}$

14. $\{1, C\}, \{2, A\}, \{3, B\}$

16. $\{1, D\}, \{2, E\}, \{3, A\}, \{4, B\}, \{5, C\}$

18. C, B, D, A

20. $3, 4, 2, 1$

22. $5, 9, 1, 6$

24. Barb, Erika, Deb, Andy

6.4 APPLICATIONS OF THE ALGORITHM

2. rows 2 and 4, column 1

4. row 5, columns 2 and 3

6. $\{3, B, C\}$

8. $\{2, 4, A, B\}$

10. $\{1, 3, 4, 5, 6\}$

12. $\{1, 2, 4, 5, 6, 7\}$

14. 4 minutes

6.5 THE HUNGARIAN METHOD

2. 10

4. 11

6. 9

8. 6

10. 22

12. 24

14. \$36,000

SUPPLEMENTARY EXERCISES

2. $|S_1 \bigcup S_2 \bigcup S_3 \bigcup S_5| = 3$

4. **(a)** $\{\{1,4\},\{2,8\},\{3,6\},\{5,11\},\{7,10\},\{9,12\}\}$

(b) $\{\{1,13\},\{2,14\},\{3,6\},\{4,5\},\{7,10\},\{8,9\},\{11,15\},\{12,16\}\}$

6. The matrix of the bipartite graph is shown below.

$$\begin{array}{c} \\ 1 \\ 3 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 4 & 5 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

8. all the rows

10. $\{\{1,A\},\{2,D\},\{3,C\},\{4,B\}\};\{1,2,3,4\}$

12. Dan with Kim, Ed with Mae, Fred with June, Gil with Lil, and Hal with Ivy

14. **(a)** $3+2+2+3=10$

(b) $3+4+5+2=14$ is minimal for a 4-element set

Chapter 7

Network Flows

7.1 FLOWS AND CUTS

2. a network with source E and sink A

4. not a network because no vertex has indegree 0

6. not a network because both A and E have outdegree 0

8. not a flow because the flow along arc (A, D) is negative

10. a flow with value 12

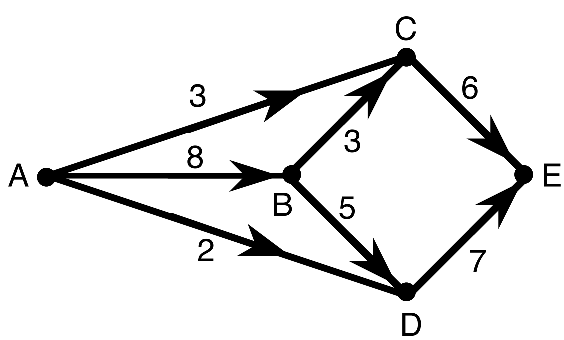
12. a flow with value 9

14. a cut with capacity 19

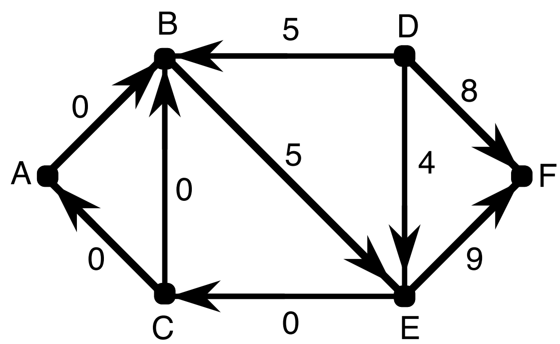
16. not a cut because $\mathcal{S}$ and $\mathcal{T}$ are not disjoint

18. not a cut because both the source and the sink are in $\mathcal{S}$

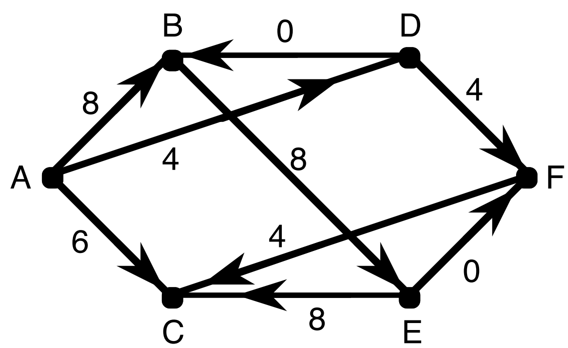
20.



22.



24.

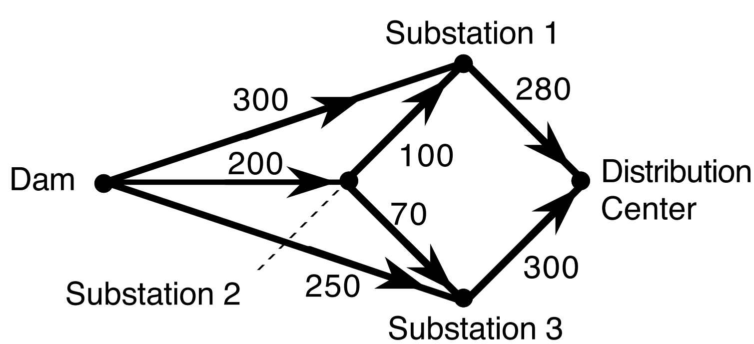


26. $\{A, B, D\}$, $\{C, E\}$

28. $\{D\}$, $\{A, B, C, E, F\}$

30. $\{A, B\}$, $\{C, D, E, F\}$

32.



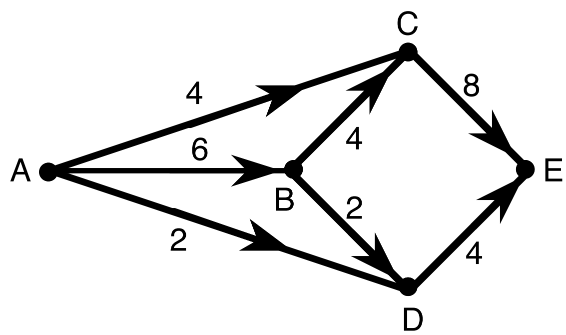
34. $f(\mathcal{U},\mathcal{V}) = 0$ and $f(\mathcal{V},\mathcal{U}) = 9$

7.2 A FLOW AUGMENTATION ALGORITHM

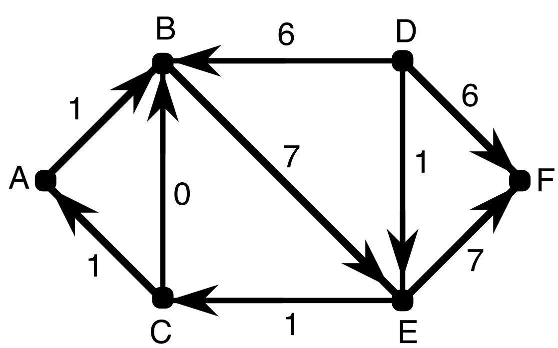
2. 1.9

4. 3

6.



8.



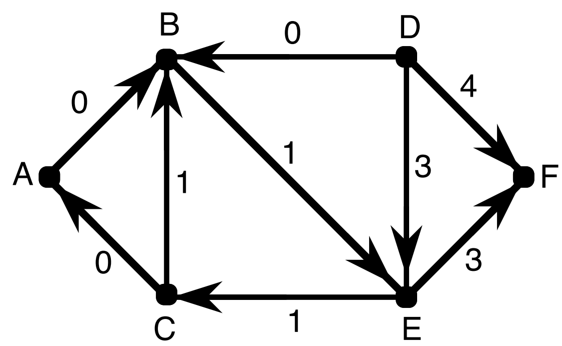
10. The given flow is maximal.

12. Increase the flow by 3 along D, B, E, F.

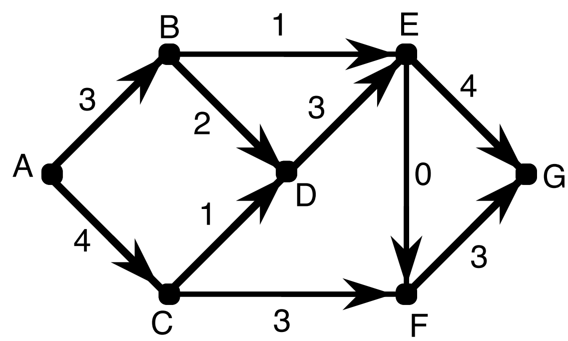
14. Increase the flow by 2 along A, B, D, F, E, C.

16. Increase the flow by 4 along A, B, D, C, F, G.

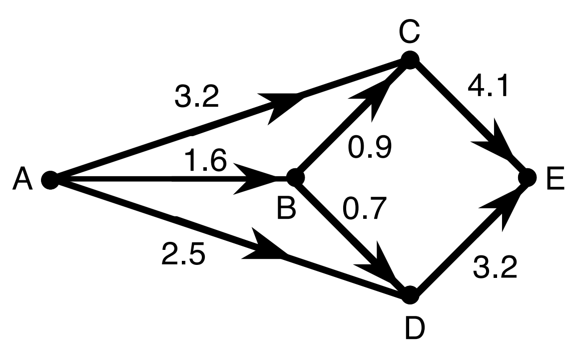
18.


B
0
D
0
4
1
3
A
1
F
0
3
C
1
E

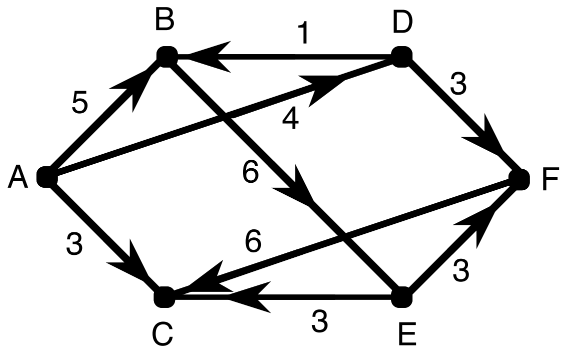
20.


B
1
E
3
2
3
4
A
D
0
G
1
4
3
C
3
F

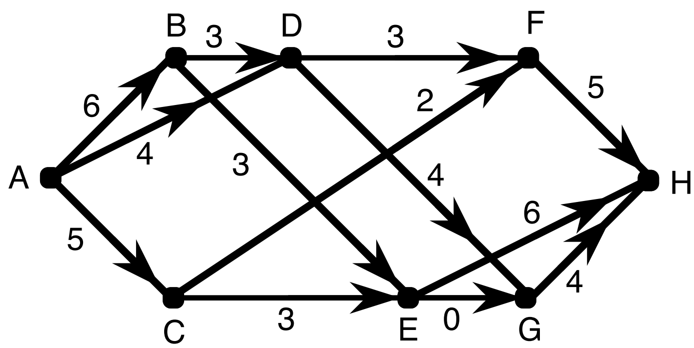
22.


C
3.2
4.1
1.6
0.9
A
B
0.7
E
2.5
3.2
D

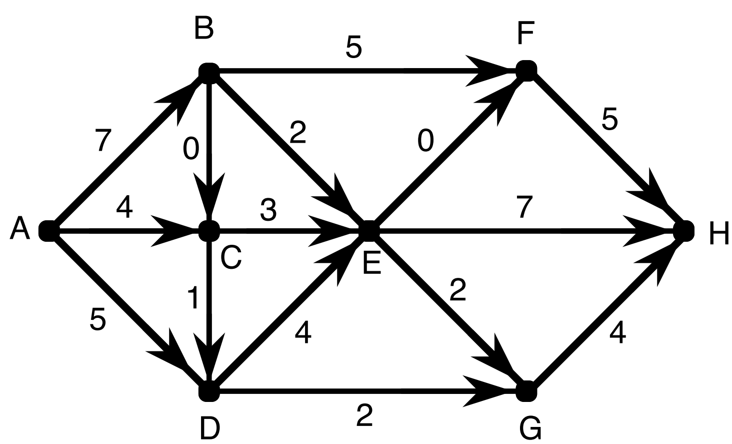
24.


B
1
D
5
4
3
A
6
F
3
6
3
3
C
E

26.


B
3
D
3
F
6
2
5
4
3
A
4
H
5
6
4
C
3
E
0
G

28.


B
5
F
7
0
2
0
5
A
4
3
7
C
E
H
1
2
5
4
4
D
2
G

7.3 THE MAX-FLOW MIN-CUT THEOREM

2. 25

4. 32

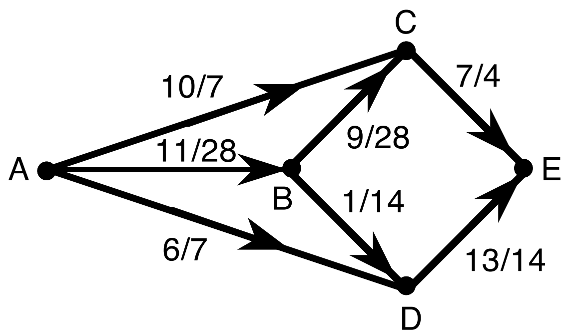
6. $\{D, B\}$, $\{A, C, E, F\}$

8. $\{A, B, D, E, F\}$, $\{C\}$

10. $\{A, C\}$, $\{B, D, E\}$

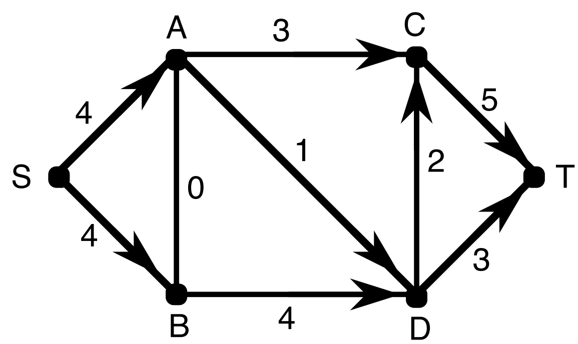
12. $\{A, B\}$, $\{C, D, E, F, G\}$

14.



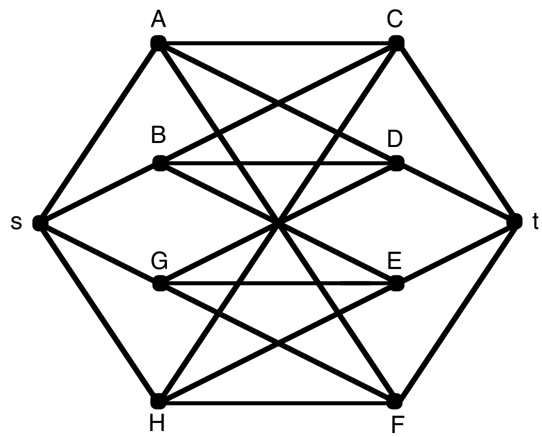
20. (S, A), (S, B), and (S, D).

22.

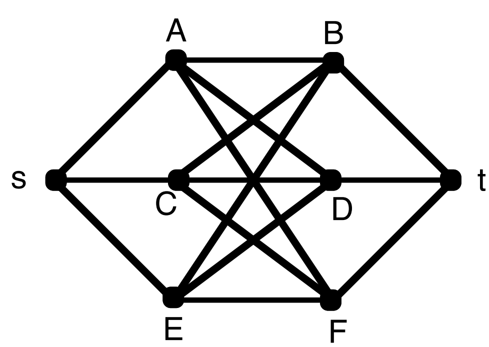


7.4 FLOWS AND MATCHINGS

2. bipartite; $\mathcal{V}_1 = \{A, B, G, H\}$ and $\mathcal{V}_2 = \{C, D, E, F\}$
In the network below, all arcs have capacity 1 and are directed from left to right.



4. bipartite; $\mathcal{V}_1 = \{A, C, E\}$ and $\mathcal{V}_2 = \{B, D, F\}$
In the network below, all arcs have capacity 1 and are directed from left to right.



6. not bipartite

8. The given matching is a maximum.

10. $\{(A, X), (B, Z), (C, W), (D, Y)\}$

12. $\{(A, a), (B, b), (C, c), (E, d)\}$

14. $\{(1, C), (2, A), (5, D)\}$

16. A matching of actresses to accents is: Sally–English, Tess–German, Ursula–French, Vickie–Danish, and Winona–Chinese.

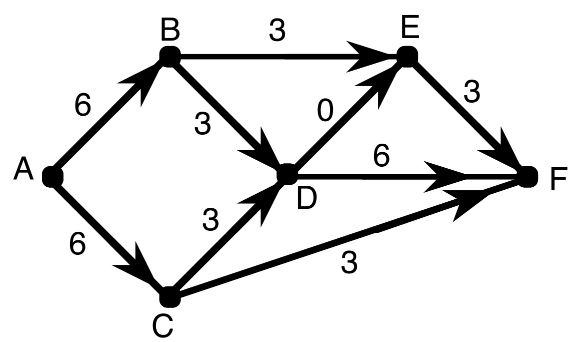
18. Only Professors Abel, Crittenden, and Forcade can teach courses 1, 3, 4, and 6. Hence it is impossible to schedule these four courses without assigning more than one course to some professor.

20. 3, 1, 2, 5, 4

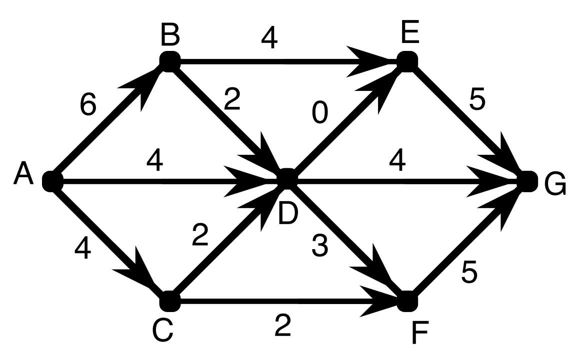
22. One possible matching is: Ann and Harry, Betty and Ian, Carol and Jim, Diane and Frank, and Ellen and Gregory.

SUPPLEMENTARY EXERCISES

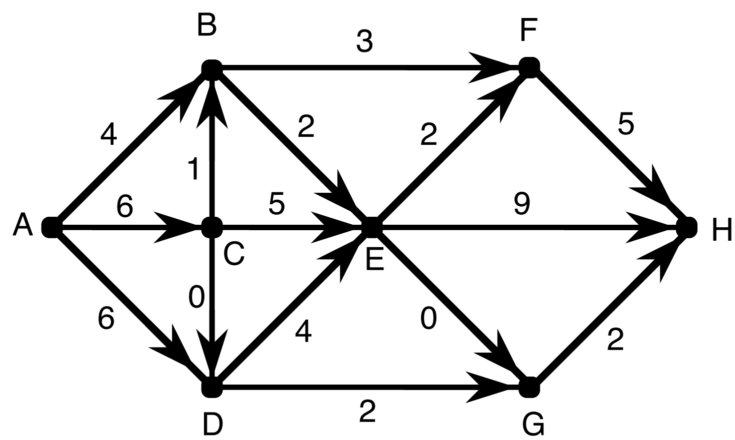
2. A minimal cut is $\{A, B, C, D, E\}$, $\{F\}$. A maximal flow is shown below.



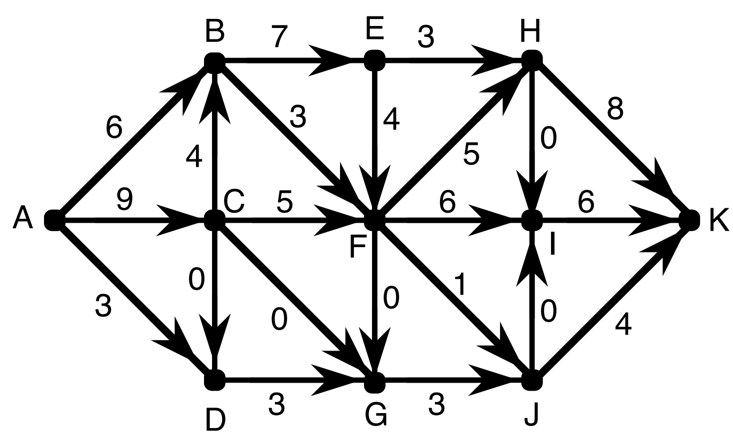
4. A minimal cut is $\{A, B, C, D, E\}$, $\{F, G\}$. A maximal flow is shown below.



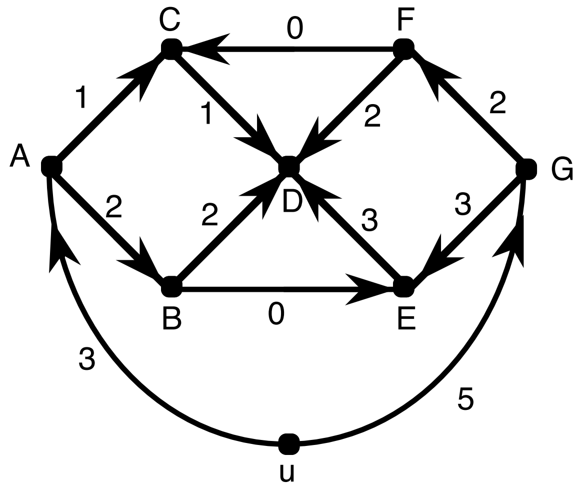
6. A minimal cut is $\{A, D, G\}$, $\{B, C, E, F, H\}$. A maximal flow is shown below.



8. A minimal cut is $\{A, D, G\}$, $\{B, C, E, F, H, I, J, K\}$. A maximal flow is shown below.

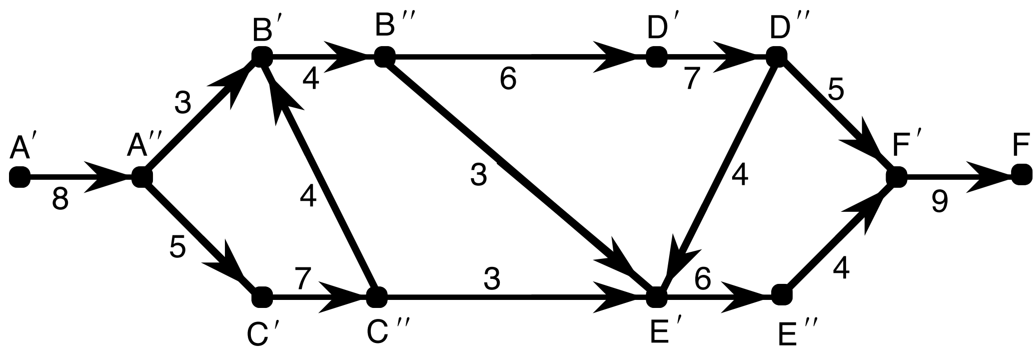


12.



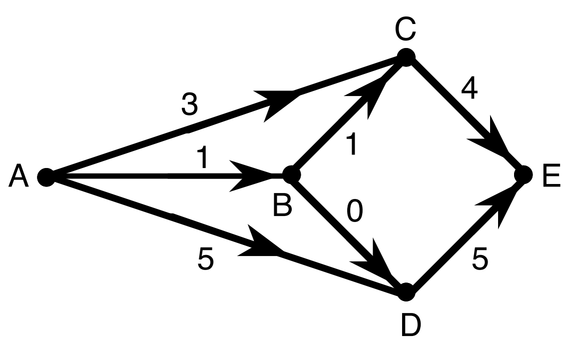
C
0
F
1
1
2
2
A
D
G
2
2
3
3
B
0
E
3
5
u

16.



B′
B″
D′
D″
4
6
7
5
3
A′
A″
F′
F″
8
4
3
4
9
5
7
3
6
4
C′
C″
E′
E″

18.



C
3
4
1
1
A
B
E
0
5
5
D

Chapter 8

Counting Techniques

8.1 PASCAL'S TRIANGLE AND THE BINOMIAL THEOREM

2. 21 **4.** 792 **6.** 6

8. 126 **10.** 57,915 **12.** 4608

14. The $n = 7$ row of Pascal's triangle is: 1, 7, 21, 35, 35, 21, 7, 1.

16. $(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

18. $(x - 2y)^5 = x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$

20. 495 **22.** 126 **24.** 56

8.2 THREE FUNDAMENTAL PRINCIPLES

2. 8 **4.** 27

6. 34 **8.** 57

10. 128 **12.** 1,048,576

14. 24 **16.** **(a)** 216 **(b)** 120

18. **(a)** 5040 **(b)** 144 **(c)** 720 **(d)** 1440

20. 279,936

22. 8

24. 24

26. 24,336,000

28. **(a)** 1320 **(b)** 60 **(c)** 312 **(d)** 360

30. 88

32. **(a)** 1680 **(b)** 108

36. **(a)** 4096 **(b)** 4083

8.3 PERMUTATIONS AND COMBINATIONS

2. 35

4. 70

6. 120

8. 1320

10. 336

12. $n(n-1)$

14. 720

16. 2520

18. 1820

20. 495

22. 15

24. **(a)** 220 **(b)** 1320

26. 254,016,000

28. 19,958,400

30. **(a)** 1001 **(b)** 120 **(c)** 48 **(d)** 154

36. $C(k,k) + C(k+1,k) + \cdots + C(n,k) = C(n+1,k+1)$ for $0 \le k \le n$

8.4 ARRANGEMENTS AND SELECTIONS WITH REPETITIONS

2. 10,080

4. 1260

6. 28

8. 286

10. 630,630

12. 90

14. 1001

16. 15

18. $\dfrac{52!}{(13!)^4} \approx 5.364 \times 10^{28}$

20. 369,600

22. 34,650

24. 120

26. 1050

28. 3360

30. 28

32. 88

34. $C(m-1, m-n) + C(m-1, n-1)$

8.5 PROBABILITY

2. $\dfrac{1}{3}$

4. $\dfrac{1}{216}$

6. $\dfrac{1}{8}$

8. $\dfrac{35}{128}$

10. $\dfrac{6}{625}$

12. $\dfrac{14}{969}$

14. $\dfrac{150}{169}$

16. $\dfrac{1024}{59{,}049}$

18. $\dfrac{60}{143}$

20. $\dfrac{25}{28}$

22. $\dfrac{2}{143}$

24. $\dfrac{315}{8192}$

26. $\dfrac{14{,}175}{4^8}$

28. $\dfrac{63}{125}$

30. **(a)** $\dfrac{14}{323}$ **(b)** $\dfrac{160}{323}$ **(c)** $\dfrac{135}{323}$

32. $\dfrac{198}{4165}$

34. $\dfrac{9}{2548}$

36. $\dfrac{17{,}296}{527{,}085}$

8.6 THE PRINCIPLE OF INCLUSION-EXCLUSION

2. 102

4. 25

6. 294

8. 154

10. 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321

12. 14,670

14. $\dfrac{4}{15}$

16. 74,670

18. $\dfrac{1561}{4096}$

20. 864

22. 1560

24. $\dfrac{11}{30}$

28. $(-1)^{n+1}$

30. $S(n, k)$

36. $m! \cdot S(n, m)$

38. 67,132,800

8.7 GENERATING PERMUTATIONS AND r-COMBINATIONS

2. $p > q$ **4.** $p < q$ **6.** $p > q$

8. (3, 6, 4, 2, 5, 1) **10.** (4, 1, 2, 3, 5, 6) **12.** (5, 2, 1, 3, 4, 6)

14. (1, 2, 4, 3, 5, 6) **16.** (4, 6, 1, 2, 3, 5) **18.** (2, 3, 4, 1, 5, 6)

20. {1, 2, 4, 5, 7} **22.** {1, 3, 7, 8, 9} **24.** {2, 4, 6, 8, 9}

26. {3, 5, 6, 7, 8} **28.** {4, 5, 6, 7, 9} **30.** {5, 6, 7, 8, 9}

32. {1, 2, 3}; {1, 2, 4}; {1, 2, 5}; {1, 2, 6}; {1, 3, 4}; {1, 3, 5}; {1, 3, 6}; {1, 4, 5}; {1, 4, 6}; {1, 5, 6}; {2, 3, 4}; {2, 3, 5}; {2, 3, 6}; {2, 4, 5}; {2, 4, 6}; {2, 5, 6}; {3, 4, 5}; {3, 4, 6}; {3, 5, 6}; {4, 5, 6}

SUPPLEMENTARY EXERCISES

2. 6720 **4.** 210

6. $x^7 + 14x^6y + 84x^5y^2 + 280x^4y^3 + 560x^3y^4 + 672x^2y^5 + 448xy^6 + 128y^7$

8. $625x^4 - 1000x^3y + 600x^2y^2 - 160xy^3 + 16y^4$

10. {1, 4, 5, 6, 7} **12.** $-198,016$

14. 13 **16.** 720

18. 3304 **20.** 792

22. $\dfrac{13}{66}$ **24.** 136

26. $\dfrac{1}{26}$ **28.** 504

30. 28

32. $\dfrac{5}{3888}$

34. 330

36. 20

38. 118,755

40. 68

42. $\dfrac{49{,}875}{13^5}$

44. 630

46. 455

50. 119

52. $\displaystyle\sum_{k=0}^{n}(-1)^k C(n,k)\cdot 2^k\cdot(2n-k)!$

56. $C(n+k-1,n)$

58. **(a)** Each term in s_n lies one row higher and one term to the right of its predecessor. Consequently the terms in s_n lie along a diagonal line extending upward and to the right from the left edge of the triangle.

(b) $s_n = F_{n+1}$, the $(n+1)st$ Fibonacci number

60. The number of distributions in each of the cases is given in the table below.

Case	$m>n$	$m=n$	$m<n$
1.	m^n	m^n	m^n
2.	0	$m!\cdot S(n,m)$	$m!\cdot S(n,m)$
3.	$P(m,n)$	$P(m,n)$	0
4.	0	$n!$	0
5.	$C(m+n-1,n)$	$C(m+n-1,n)$	$C(m+n-1,n)$
6.	0	$C(n-1,m-1)$	$C(n-1,m-1)$
7.	$C(m,n)$	$C(m,n)$	0
8.	0	1	0

Chapter 9

Recurrence Relations and Generating Functions

9.1 RECURRENCE RELATIONS

2. 12

4. 1063

6. -533

8. -120

10. 4

12. -10

14. Since the growth from one year to the next is 2 dollars, the next year's fee is the previous year's fee plus 2 dollars. The initial value is 50 dollars. Thus the recurrence relation and initial conditions are:

$$m_n = m_{n-1} + 2 \text{ for } n \geq 1 \qquad \text{and} \qquad m_0 = 50.$$

16. Since the change due to interest is 6 percent of the value per year, the next year's value is the previous year's value plus a 6 percent increase, that is 1.06 times the previous year's value. The initial value is 800 dollars. Thus the recurrence relation and initial conditions are:

$$b_n = 1.06b_{n-1} \text{ for } n \geq 1 \qquad \text{and} \qquad b_0 = 800.$$

18. Tom's salary changes in two ways from one year to the next. First, he receives a 5 percent raise on his previous salary. This portion of the next year's salary is found by multiplying the previous salary by 1.06. After the cost-of-living percent increase has been added, the additional 1000 dollar increase is added. The initial value is 24,000 dollars. Thus the recurrence relation and initial conditions are:

$$s_n = 1.05s_{n-1} + 1000 \text{ for } n \geq 1 \qquad \text{and} \qquad s_0 = 24{,}000.$$

20. The amount of change from one year to the next can be viewed as a 73 percent survival rate. To this number, we add the new births (5 birds) to get the present year's total. Since there are 975 birds in existence at the beginning, this is the initial value. Thus the recurrence relation and initial conditions are:

$$j_n = 0.73j_{n-1} + 5 \text{ for } n \geq 1 \qquad \text{and} \qquad j_0 = 975.$$

22. The number of ways stamps can be affixed to the envelope depends on the number of stamps already there. In a manner similar to Example 9.3, we consider the type of stamp that completes the total postage of n cents. This can either be a 2-, 3-, or 5-cent stamp. In each case, the previous stamps are worth $n-5$ cents, $n-3$ cents, or $n-2$ cents, respectively. Thus the recurrence relation and initial conditions are:

$$s_n = s_{n-2} + s_{n-3} + s_{n-5} \text{ for } n > 5 \qquad \text{and} \qquad s_1 = 0, s_2 = 1, s_3 = 1, s_4 = 1, s_5 = 3.$$

24. By the analysis used in proving Theorem 1.3 (see page 88 of the textbook), we see that the number of subsets of a set with n elements is 2 times the number of subsets of a set with $n-1$ elements. Thus the recurrence relation and initial conditions are:

$$s_n = 2s_{n-1} \text{ for } n \geq 1 \qquad \text{and} \qquad s_0 = 1.$$

26. Consider an acceptable n-bit string. If it ends in 1, then its first $n-1$ bits form an acceptable $(n-1)$-bit string; so there are s_{n-1} of these strings. If an acceptable n-bit string ends in 0, then bit $n-1$ must be a 1, and the first $n-2$ bits form an acceptable string of length $n-2$; so there are s_{n-2} of these strings. Thus the recurrence relation and initial conditions are:

$$s_n = s_{n-1} + s_{n-2} \text{ for } n \geq 3 \qquad \text{and} \qquad s_1 = 2, s_2 = 3.$$

By iterating the recurrence relation, we find that

$$\begin{aligned} s_3 &= s_2 + s_1 = 3 + 2 = 5, \\ s_4 &= s_3 + s_2 = 5 + 3 = 8, \\ s_5 &= s_4 + s_3 = 8 + 5 = 13, \\ s_6 &= s_5 + s_4 = 13 + 8 = 21. \end{aligned}$$

28. Note that two L-shaped pieces can be assembled to form a 2×3 rectangle. A 6×2 board can be covered by two of these 2×3 rectangles (and hence by four L-shaped pieces). In general, changing the board from size $6 \times (n-1)$ to size $6 \times n$ increases the area of the board by 6 units, and so the larger board requires 1 additional rectangle (two additional L-shaped pieces) to cover it. Thus the recurrence relation and initial conditions are:

$$p_n = p_{n-1} + 2 \text{ for } n \geq 3 \qquad \text{and} \qquad p_2 = 4.$$

30. Consider a permutation of n digits in which each digit is either in its natural position or adjacent to it. If n is in its natural position (last), then the preceding digits form an acceptable permutation of $n-1$ digits. Hence there are p_{n-1} permutations in this case. On the other hand, if n is not last, it must be next-to-last, so that the last two digits are n and $n-1$, respectively. In this case, the preceding digits form an acceptable permutation of $n-2$ digits. Thus there are p_{n-2} permutations in this case. Therefore the recurrence relation and initial conditions are:

$$p_n = p_{n-1} + p_{n-2} \text{ for } n \geq 3 \qquad \text{and} \qquad p_1 = 1, p_2 = 2.$$

32. For $n = 1$, there are 5 stacks of chips, all of which are acceptable. For $n = 2$, there are $5^2 - 1 = 24$ acceptable stacks of chips. (The only unacceptable stack contains two red chips.) In general, consider an acceptable stack of n chips. If the top chip is not red, then it can be any one of 4 colors, and the first $n-1$ chips can be any acceptable stack of $n-1$ chips. Hence there are $4s_{n-1}$ stacks in which the top chip is not red. In addition, there are stacks where the top chip is red. In these, the chip that is next to the top must be any of the four colors other than red, and the first $n-2$ chips can be any acceptable stack of $n-2$ chips. Thus there are $4s_{n-2}$ stacks in which the top chip is red. Therefore the recurrence relation and initial conditions are:

$$s_n = 4s_{n-1} + 4s_{n-2} \text{ for } n \geq 3 \qquad \text{and} \qquad s_1 = 5, s_2 = 24.$$

34. Without loss of generality, we can consider the elements of the set to be $1, 2, \ldots, n$ and can list the elements of the subsets in increasing order. Clearly, there is one three-element subset for $n = 3$. Now consider a 3-element subset of a set of n elements, where $n \geq 4$. If the last element is not n, then the subset is a 3-element subset of $\{1, 2, \ldots, n-1\}$. There are s_{n-1} subsets of this type. In addition, if the last element of the set is n, so that the set has the form $\{—, —, e_n\}$, the first two elements of the set can be selected in $C(n-1, 2) = \frac{1}{2}(n-1)(n-2)$ ways. Thus the recurrence relation and initial conditions are:

$$s_n = s_{n-1} + \frac{1}{2}(n-1)(n-2) \text{ for } n \geq 4 \qquad \text{and} \qquad s_3 = 1.$$

36. Clearly $s_1 = 1$. For a 2×2 checkerboard, there are four 1×1 squares and one 2×2 square, so that $s_2 = 5$. In general, when compared to an $(n-1) \times (n-1)$ checkerboard, an $n \times n$ checkerboard contains one additional row and column. In addition to the s_{n-1} squares contained in an $(n-1) \times (n-1)$ checkerboard, this permits an additional $2n-1$ squares of dimension 1×1, $2n-3$ squares of dimension 2×2, ..., and 1 square of dimension $n \times n$. The number of additional squares is therefore

$$(2n-1) + (2n-3) + \cdots + 1 = n^2,$$

and so the recurrence relation and initial conditions are:

$$s_n = s_{n-1} + n^2 \text{ for } n \geq 2 \qquad \text{and} \qquad s_1 = 1.$$

38. Let s_n denote the number of n-bit strings that contain neither the pattern 1000 nor the pattern 0011, and let u_n and v_n denote the number of such strings that end in 0 and end in 1, respectively. Thus, for every positive integer n, we have

$$s_n = u_n + v_n.$$

An acceptable n-bit string must end in 10, 00, 01, or 11. If it ends in 10, then its first $n-1$ bits form an acceptable string of length $n-1$ that ends in 1. Hence there are v_{n-1} strings of length n ending in 10.

Now consider an acceptable n-bit string ending in 00. Its first $n-1$ bits can be any acceptable string of length $n-1$ that ends in 0, except for one ending in 100. Because the number of strings of length $n-1$ ending in 100 is v_{n-3}, there are $u_{n-1} - v_{n-3}$ acceptable strings of length n that end in 00.

In an acceptable n-bit string ending in 01, the first $n-1$ bits form an acceptable string of length $n-1$ that ends in 0, and so there are u_{n-1} acceptable n-bit strings ending in 01.

Finally, if an acceptable n-bit string ends in 11, then its first $n-1$ bits form an acceptable string of length $n-1$ that ends in 1, but not 001. The number of acceptable $(n-1)$-bit strings ending in 1 is v_{n-1}. Of these, the number that end in 001 is $u_{n-3} - v_{n-5}$, since u_{n-3} is the number of strings that end in 001 and we must subtract the v_{n-5} strings that complete the pattern 1000 on bit $n-2$. Therefore the number of acceptable n-bit strings ending in 11 is $v_{n-1} - (u_{n-3} - v_{n-5})$.

Combining the results of the four preceding paragraphs, we see that

$$\begin{aligned} s_n &= v_{n-1} + (u_{n-1} - v_{n-3}) + u_{n-1} + [v_{n-1} - (u_{n-3} - v_{n-5})] \\ &= 2(u_{n-1} + v_{n-1}) - (u_{n-3} + v_{n-3}) + v_{n-5} \\ &= 2s_{n-1} - s_{n-3} + v_{n-5}. \end{aligned}$$

Solving the preceding equation for v_{n-5} yields

$$s_n - 2s_{n-1} + s_{n-3} = v_{n-5}.$$

We claim that

$$s_n = 2s_{n-1} - u_{n-2} \qquad \text{for } n \geq 4.$$

To see why, suppose that $n \geq 4$ and we put a 0 or 1 on the end of an acceptable $(n-1)$-bit string and obtain an unacceptable n-bit string. Since the last four bits of this unacceptable n-bit string must be 1000 or 0011, there must be a 0 in bit $n-2$. Moreover, bit $n-3$ can be either 1 or 0. Thus any acceptable $(n-1)$-bit string that becomes unacceptable by adding bit n must be an acceptable $(n-2)$-bit string ending in 0, and conversely any acceptable $(n-2)$-bit string ending in 0 gives rise to exactly one acceptable $(n-1)$-bit string that becomes unacceptable when bit n is included. It follows that the number of acceptable n-bit strings equals the number of ways to put a 0 or 1 on the end of an acceptable $(n-1)$-bit string, minus u_{n-2}. This justifies our claim.

Substituting $n-3$ for n in the preceding displayed equation gives $s_{n-3} = 2s_{n-4} - u_{n-5}$ for $n \geq 7$, which can be rewritten as

$$u_{n-5} = 2s_{n-4} - s_{n-3} \qquad \text{for } n \geq 7.$$

Combining this equation with the previous one involving v_{n-5} yields

$$\begin{aligned} s_{n-5} &= u_{n-5} + v_{n-5} \\ &= (2s_{n-4} - s_{n-3}) + (s_n - 2s_{n-1} + s_{n-3}) \\ &= s_n - 2s_{n-1} + 2s_{n-4} \end{aligned}$$

for $n \geq 7$. Solving this equation for s_n, we get

$$2s_{n-1} - 2s_{n-4} + s_{n-5} = s_n \qquad \text{for } n \geq 7,$$

which is the desired recurrence relation.

9.2 THE METHOD OF ITERATION

2. The expression is correct for $n = 0$: $s_0 = 4(2^0) + 3 = 7$. Suppose that it is correct for $n = k \geq 0$, that is, suppose that $s_k = 4(2^k) + 3$. By recurrence,

$$s_{k+1} = 2[4(2^k) + 3] - 3 = 4(2^{k+1}) + 6 - 3 = 4(2^{k+1}) + 3.$$

Hence the statement holds for $n = k+1$ and, by the principle of mathematical induction, for each nonnegative integer n.

4. The expression is correct for $n = 0$: $s_0 = 3^0(3+0) = 3$. Suppose that it is correct for $n = k \geq 0$, that is, suppose that $s_k = 3^k(3+k)$. By recurrence,

$$s_{k+1} = 3[3^k(3+k)] + 3^{k+1} = 3^{k+1}(3) + 3^{k+1}(3) + 3^{k+1} = 3^{k+1}[3 + (k+1)].$$

Hence the statement holds for $n = k+1$ and, by the principle of mathematical induction, for each nonnegative integer n.

6. The expression is correct for $n = 1$: $s_0 = 1 \cdot C(0,0) = 1$. Suppose that it is correct for $n = k \geq 1$, that is, suppose that $s_k = \frac{1}{k}C(2k-2, k-1)$. By recurrence,

$$\begin{aligned} s_{k+1} &= \frac{4(k+1)-6}{k+1} \cdot \left[\frac{1}{k} \cdot \frac{(2k-2)!}{(k-1)!\,(k-1)!}\right] \\ &= \frac{2(2k-1)}{(k+1)k} \cdot \frac{(2k-2)(2k-1)\cdots(k!)}{k!} \\ &= \frac{1}{k+1} \cdot \frac{(2k)!}{k!\,k!}. \end{aligned}$$

Hence, the statement holds for $n = k+1$ and, by the principle of mathematical induction, for each positive integer n.

8. The expression is correct for $n = 1$: $C(2n+1,3) = C(3,3) = 1$. Suppose that it is correct for $n = k \geq 1$, that is, suppose that $C(2k+1,3) = \frac{(2k+1)!}{(2k-2)!(3!)}$ is a solution to the given recurrence relation. By recurrence,

$$\begin{aligned} s_{k+1} &= s_k + (2(k+1)-1)^2 \\ &= s_k + 4k^2 + 4k + 1 \\ &= \frac{(2k+1)!}{2k!\,3!} \cdot \frac{[(2k+1)6 + (2k)(2k-1)]}{4k^2 + 1k + 6} \\ &= C(2k+3,3) \\ &= C(2(k+1)+1,3). \end{aligned}$$

Hence the statement holds for $n = k+1$ and, by the principle of mathematical induction, for each positive integer n.

10. The solution is $C(2n+1,3) = \frac{(2n+1)(2n)(2n-1)}{2}$. This can be proved by mathematical induction by an argument like that in Exercise 8.

12. We have

$$s_0 = 3, \quad s_1 = -2(3), \quad s_2 = -2[-2(3)] = (-2)^2(3), \quad s_3 = -2[(-2)^2(3)] = (-2)^3(3).$$

Conjecture that $(-2)^n(3)$ is a solution to the recurrence relation for $n \geq 0$. This can be proved by mathematical induction.

14. We have

$$s_0 = 7, \quad s_1 = 7 + (-2), \quad s_2 = 7 + 2(-2), \quad s_3 = 7 + 3(-2).$$

Conjecture that $7 - 2n$ is a solution to the recurrence relation for $n \geq 0$. This can be proved by mathematical induction.

16. We have

$$s_0 = -4, \quad s_1 = 4 + 10 = 14 \quad s_2 = -14 + 10 = -4, \quad s_3 = 4 + 10 = 14.$$

Conjecture that $9(-1)^{n+1} + 5$ is a solution to the recurrence relation for $n \geq 0$. This can be proved by mathematical induction.

18. Conjecture that $s_n = \frac{3}{4}(-3)^n + \frac{5}{4}$ is a solution to the recurrence relation for $n \geq 0$. This can be proved by mathematical induction.

20. Conjecture that $s_n = (-1)^n(n+6)$ is a solution to the recurrence relation for $n \geq 0$. This can be proved by mathematical induction.

22. Conjecture that $s_n = 5 + 5n + n^2$ is a solution to the recurrence relation for $n \geq 0$. This can be proved by mathematical induction.

24. Conjecture that $s_n = \frac{1}{3}(2^{n+2} - 1)$ is a solution to the recurrence relation for $n \geq 0$. This can be proved by mathematical induction.

26. **(a)** Since the volume grows at the rate of 0.2 percent per day, the amount on a given day is found by multiplying the preceding day's volume by 1.002. The initial value is 10 cubic feet. Thus the recurrence relation and initial condition are $m_n = 1.002m_{n-1}$ for $n \geq 1$ and $m_0 = 10$.

(b) Conjecture a solution of $m_n = 10(1.002)^n$ for $n \geq 0$. This can be verified by mathematical induction.

28. **(a)** In the first month, there is just one pair of rabbits. At the end of one month, there is still this one pair of rabbits. Thus the initial conditions are $r_0 = 1$ and $r_1 = 1$. After the second month of life, each pair of rabbits breed 2 other pairs of rabbits. Hence, at the end of each month, the rabbit pairs from the previous month remain, as well as 2 additional pairs for any pairs of rabbits that were born at least two months before. Thus the recurrence relation is: $r_n = r_{n-1} + 2r_{n-2}$ for $n \geq 2$.

(b) Conjecture a solution of $r_n = \frac{1}{3}[2^{n+1} - (-1)^{n+1}]$ for $n \geq 0$. This can be verified by mathematical induction.

30. We see that $s_1 = 2$, $s_2 = 5$, and $s_3 = 14$. In general, if an acceptable sequence of length n ends in -1 or 0, its first $n-1$ terms form an acceptable sequence of length $n-1$. So there are $2s_{n-1}$ such sequences. In a sequence of length n that ends with a 1, the first $n-1$ terms must form a sequence of length $n-1$ that contains an odd number of 1s. So there are $3^{n-1} - s_{n-1}$ such sequences. Thus the recurrence relation is

$$s_n = 2s_{n-1} + (3^{n-1} - s_{n-1}) = s_{n-1} + 3^{n-1}$$

for $n \geq 2$. Conjecture a solution of $s_n = \frac{3^n+1}{2}$ for $n \geq 0$. This can be verified by mathematical induction.

32. In the cofactor method of evaluating the determinant of an $n \times n$ matrix, the determinant is written as a sum of n terms, each of which involves the evaluation of a determinant of an $(n-1) \times (n-1)$ matrix. Thus $a_n = (n-1) + na_{n-1}$ for $n \geq 2$ with an initial condition of $a_2 = 1$. Conjecture a solution of $a_n = n! - 1$ for $n \geq 1$. This can be verified by mathematical induction.

34. Consider the minimal number of moves m_n necessary to move n disks from the leftmost spoke to the rightmost spoke or vice versa, subject to the condition that a disk may not be placed on a smaller disk. For $n = 1$, we may move the single disk to the middle spoke and then to the rightmost spoke; thus $m_1 = 2$. In general, we must proceed as follows.

(1) Move the smallest $n-1$ disks from the leftmost spoke to the rightmost spoke.

(2) Move the largest disk to the middle spoke.
(3) Move the smallest $n-1$ disks from the rightmost spoke to the leftmost spoke.
(4) Move the largest disk from the middle spoke to the rightmost spoke.
(5) Move the smallest $n-1$ disks from the leftmost spoke to the rightmost spoke.

Steps (2) and (4) require only 1 move, and steps (1), (3), and (5) require m_{n-1} moves. Thus we have the recurrence relation and initial comdition

$$m_n = 3m_{n-1} + 2 \text{ for } n \geq 1 \qquad \text{and} \qquad m_1 = 2.$$

By using the method of iteration as in Example 9.2, we obtain the formula

$$m_n = 3^n - 1 \text{ for } n \geq 1.$$

9.3 LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

2. Applying Theorem 8.1, we find that $s_n = 1$ for $n \geq 0$.

4. Applying Theorem 8.1, we find that $s_n = 2(1.5)^n + 2$ for $n \geq 0$.

6. Applying Theorem 8.1, we find that $s_n = 32 - 10n$ for $n \geq 0$.

8. Applying Theorem 8.1, we find that $s_n = -5(-2)^n$ for $n \geq 0$.

10. Applying Theorem 8.1, we find that $s_n = (-2.5)(-1)^n + 3.5$ for $n \geq 0$.

12. Applying Theorem 8.1, we find that $s_n = (-3)(10)^n + 5$ for $n \geq 0$.

14. Applying Theorem 8.2, we find that $s_n = (-1)^n(3 - 4n)$ for $n \geq 0$.

16. Applying Theorem 8.2, we find that $s_n = 3(-2)^n - 4(2)^n$ for $n \geq 0$.

18. Applying Theorem 8.2, we find that $s_n = (1 + 2n)3^n$ for $n \geq 0$.

20. Applying Theorem 8.2, we find that $s_n = 6(-3)^n - 4(-5)^n$ for $n \geq 0$.

22. Applying Theorem 8.2, we find that $s_n = 3(4)^n - 2(6)^n$ for $n \geq 0$.

24. Applying Theorem 8.2, we find that $s_n = (-3 + 5n)(2)^n$ for $n \geq 0$.

26. **(a)** The recurrence relation and initial condition are $s_n = 0.9s_{n-1} + 1200$ for $n \geq 1$ and $s_0 = 15,000$.
(b) Because 2009 corresponds to $n = 9$, the town would issue 13,162 licenses.
(c) In the long run, 12,000 licenses will be issued per year, the limit of the expression $s_n = (0.90)^n(3000) + 12,000$ as n increases without bound.

28. Letting v_n represent the value of the account after n years, we have the recurrence relation $v_n = 1.08v_{n-1} + 2000$ and the initial condition $v_0 = 0$. Applying Theorem 9.1, we obtain the formula

$$v_n = 25{,}000(1.08)^n - 25{,}000.$$

To obtain the value immediately after the last deposit, this expression must be evaluated for $n = 35$, giving \$344,633.61.

30. A recurrence relation for the loan is $v_n = v_{n-1}(1.01075) - 175$, with an initial value v_0. Use Theorem 9.1 to find a formula for v_n. Then evaluate this formula for $n = 60$, and set the resulting expression equal to the total payments of \$10,500. Solving this equation for v_0 gives $v_0 = \$7708.60$, the price of the car.

32. Let v_n denote the value of the account after n years. Then v_n satisfies the recurrence relation

$$v_n = 1.06(v_{n-1} - v_{n-2}) + 1.0816v_{n-2} = 1.06v_{n-1} + 0.0216v_{n-2} \text{ for } n \geq 2$$

and the initial conditions $v_0 = 1100$ and $v_1 = 1166$. This is a second-order homogeneous linear difference equation with constant coefficients, whose auxiliary equation is $x^2 = 1.06x + 0.0216$. Now

$$x^2 - 1.06x - 0.0216 = (x - 1.08)(x + 0.02),$$

and so $v_n = c_1(1.08)^n + c_2(-0.02)^n$ for some constants c_1 and c_2 that satisfy the system of equations

$$\begin{aligned} c_1 + \qquad c_2 &= 1100 \\ 1.08c_1 - 0.02c_2 &= 1166. \end{aligned}$$

Solving this system, we obtain $c_1 = 1080$ and $c_2 = 20$. Hence

$$v_n = 1080(1.08)^n + 20(-0.02)^n \text{ for } n \geq 0.$$

34. If $s_n = s_0 + s_0 r + s_0 r^2 + \cdots + s_0 r^n$, then s_n satisfies $s_n = rs_{n-1} + s_0$. Thus, by Theorem 9.1,

$$\begin{aligned} s_n &= r^n \left(s_0 + \frac{s_0}{r-1} \right) - \frac{s_0}{r-1} \\ &= s_0 \frac{r^{n+1} - 1}{r - 1}. \end{aligned}$$

36. Suppose $a = 1$. Then $t_0 = s_0 = 1\left(s_0 + \frac{0b}{2}\right) = s_0$. Thus the formula is correct for $n = 0$. Assume that it is correct for some nonnegative integer k, and consider the case

$n = k + 1$. We have

$$\begin{aligned}
t_{k+1} &= s_0 + s_1 + s_2 + \cdots + s_k + s_{k+1} \\
&= t_k + s_{k+1} \\
&= t_k + (as_k + b) \\
&= t_k + [(s_0 + (k+1)b] \\
&= (k+1)\left[s_0 + \frac{kb}{2}\right] + [(s_0 + (k+1)b] \\
&= (k+2)s_0 + \frac{(k+2)(k+1)b}{2} \\
&= (k+2)\left[s_0 + \frac{(k+1)b}{2}\right].
\end{aligned}$$

Thus the formula is correct for $n = k + 1$ and so, by the principle of mathematical induction, for all nonnegative integers n. This verifies the formula when $a = 1$.

Now suppose $a \neq 1$. For $n = 0$, we have

$$t_0 = s_0 = \left[\frac{a^1 - 1}{a - 1}\right] s_0 + b\left(\frac{a - a + 0}{(a-1)^2}\right) = s_0.$$

Thus, the the formula is correct for $n = 0$. Assume that it is correct for some nonnegative integer k, and consider the case $n = k + 1$.

$$\begin{aligned}
t_{k+1} &= s_0 + s_1 + s_2 + ... + s_k + s_{k+1} \\
&= t_k + s_{k+1} = t_k + as_k + b \\
&= \left(\frac{a^{k+1} - 1}{a - 1}\right) s_0 + b\left[\frac{a^{k+1}(k+1)a + k}{(a-1)^2}\right] + \left[a^{k+1}\left(s_0 + \frac{1}{a-1}\right) - \frac{1}{a-1}\right] \\
&= \left[\frac{(a^{k+2} - 1)s_0}{a - 1}\right] + b\left[\frac{a^{k+2} - a(k+2) + (k+1)}{(a-1)^2}\right].
\end{aligned}$$

Thus the formula is correct for $n = k + 1$ and so, by the principle of mathematical induction, for all nonnegative integers n. This verifies the formula when $a \neq 1$.

38. If $n = 0$, then

$$s_n = \left[s_0 + \frac{0(s_1 - s_0 r)}{r}\right] r^0 = s_0.$$

If $n = 1$, then

$$s_n = \left[s_0 + \frac{1(s_1 - s_0 r)}{r}\right] r = s_1.$$

9.4 ANALYZING THE EFFICIENCY OF ALGORITHMS WITH RECURRENCE RELATIONS

2. -35

4. 28

6. -2487

8. -14

10. The binary search algorithm proceeds as in the table below.

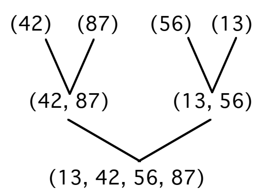
b	e	m	a_m	Is $a_m = 177$?
1	125	$\left\lfloor \frac{1}{2}(1+125) \right\rfloor = 63$	63	no; greater
1	62	$\left\lfloor \frac{1}{2}(1+62) \right\rfloor = 31$	31	no; greater
1	30	$\left\lfloor \frac{1}{2}(1+30) \right\rfloor = 15$	15	no; less
16	30	$\left\lfloor \frac{1}{2}(16+30) \right\rfloor = 23$	23	no; greater
16	22	$\left\lfloor \frac{1}{2}(16+22) \right\rfloor = 19$	19	no; greater
16	18	$\left\lfloor \frac{1}{2}(16+18) \right\rfloor = 17$	17	yes

12. The binary search algorithm proceeds as in the table below.

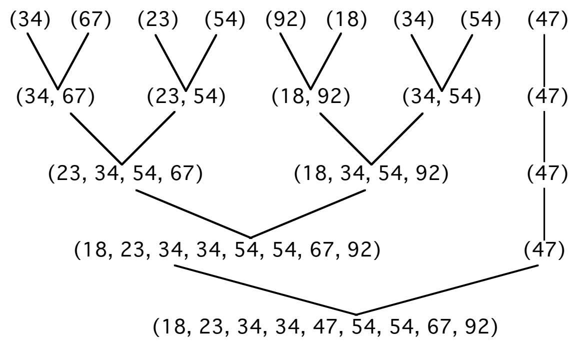
b	e	m	a_m	Is $a_m = 305$?
1	100	$\lfloor \frac{1}{2}(1 + 100) \rfloor = 50$	200	no; less
51	100	$\lfloor \frac{1}{2}(51 + 100) \rfloor = 75$	250	no; less
76	100	$\lfloor \frac{1}{2}(76 + 100) \rfloor = 88$	276	no; less
89	100	$\lfloor \frac{1}{2}(89 + 100) \rfloor = 94$	288	no; less
95	100	$\lfloor \frac{1}{2}(95 + 100) \rfloor = 97$	294	no; less
98	100	$\lfloor \frac{1}{2}(98 + 100) \rfloor = 99$	298	no; less
100	100	100	300	no; less
101	100			

Since $b > e$, the target t is not in the list.

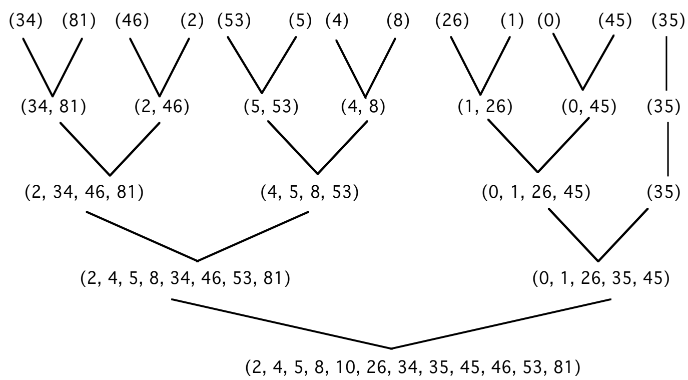
14. The diagram below shows the iterations of the merge sort algorithm.



16. The diagram below shows the iterations of the merge sort algorithm.



18. The diagram below shows the iterations of the merge sort algorithm.



20. When choosing between equal items for inclusion in list C, the merging algorithm selects items from list B before those in list A.

22. We have $\lfloor -3.1 \rfloor + \lfloor -6.9 \rfloor = (-4) + (-7) = -11 \neq -10 = \lfloor -3.1 - 6.9 \rfloor$.

24. Let $\lfloor x \rfloor = m$, where m is an integer. Then, $m \leq x < m + 1$, and so

$$m + n \leq x + n < m + n + 1.$$

Thus $\lfloor x + n \rfloor = m + n = \lfloor x \rfloor + n$.

26. When $n = 0$, the only elementary operation is the comparison performed in the **while** statement in step 2. Thus the initial condition is $e_0 = 1$. In general, when step 2 is performed for a value of k such that $k \leq n$, there is one comparison performed, one addition and k multiplications in step 2(a), and one addition in step 2(b). The total number of elementary operations in this case is $k+3$. Because step 2 is completed one additional time (with $k = n$) for a polynomial of degree n than it is for a polynomial of degree $n - 1$, we see that the recurrence relation is $e_n = e_{n-1} + (n + 3)$ for $n \geq 1$.

28. The number of comparisons c_n for 2^n items is found by doubling the number for 2^{n-1} items and adding 1 for the final comparison. Thus $c_n = 2c_{n-1} + 1$ for $n \geq 2$ and $c_1 = 0$.

30. Applying the method of iteration, we see that

$$\begin{aligned}
e_0 &= 1 \\
e_1 &= e_0 + 4 = 1 + 4 \\
e_2 &= e_1 + 5 = 1 + 4 + 5 \\
e_3 &= e_2 + 6 = 1 + 4 + 5 + 6 \\
e_4 &= e_3 + 7 = 1 + 4 + 5 + 6 + 7.
\end{aligned}$$

Thus we conjecture that

$$\begin{aligned}
e_n &= 1 + 4 + 5 + \cdots (n+3) \\
&= [1 + 2 + 3 + 4 + 5 + \cdots (n+3)] - (2+3) \\
&= \frac{(n+4)(n+3)}{2} - 5 \\
&= \frac{1}{2}(n^2 + 7n + 2).
\end{aligned}$$

This conjecture can be verified by mathematical induction.

32. Applying Theorem 9.1, we have $c_n = 2^n - 1$ for $n \geq 1$.

34. We will use the inequality $\frac{k!}{2^k} \geq 1$ for $k \geq 4$, which is easily established by mathematical induction.

Let $n = 2m$ be an even positive integer. To prove that $(2m)! \geq (2m)^m$, we will prove the equivalent inequality

$$\frac{(2m)!}{(2m)^m} \geq 1. \tag{1}$$

This inequality is easily checked for $m = 1$, 2, 3, and 4. Assume that it holds for a nonnegative integer $k \geq 4$. For $m = k+1$, we have

$$\begin{aligned}
\frac{[2(k+1)]!}{[2(k+1)]^{k+1}} &= \frac{(2k+2)(2k+1)(2k)\cdots(k+1)(k!)}{(2k+2)(2k+2)(2k+2)\cdots(2k+2)(2k+2)} \\
&= \frac{(2k+2)}{(2k+2)}\left[\frac{(2k+1)}{(2k+2)}\cdots\frac{(k+2)}{(2k+2)}\right]\frac{(k+1)k!}{1} \\
&= \left[\frac{(2k+1)}{(2k+2)}\cdots\frac{(k+2)}{(2k+2)}\right]\frac{(k+1)k!}{1} \\
&= \frac{1}{2^k}\left[\frac{(2k+1)}{(k+1)}\cdots\frac{(k+2)}{(k+1)}\right]\frac{(k+1)!}{1} \\
&> \frac{k!}{2^k} \\
&\geq 1.
\end{aligned}$$

Thus (1) holds for $m = k+1$ and so, by the principle of mathematical induction, it holds for all positive integers m.

36. Let $n = 2m+1$, where m is a nonnegative integer. As in Exercise 34, we will prove that

$$\frac{(2m+3)!}{\left[(2m+3)^{m+1}(2m+3)^{1/2}\right]} \geq 1.$$

This inequality is easily checked for $m = 1$, 2, 3, and 4. Assume that it holds for a nonnegative integer $k \geq 4$. For $m = k+1$, we have

$$\begin{aligned}
\frac{(2k+3)!}{(2k+3)^{k+1}(2k+3)^{1/2}} &= \frac{2k+3}{2k+3}\cdot\frac{(2k+2)(2k+1)\cdots(k+3)}{2^k\left(k+\frac{3}{2}\right)\left(k+\frac{3}{2}\right)\cdots\left(k+\frac{3}{2}\right)}\cdot\frac{(k+2)(k+1)}{(2k+3)^{1/2}}\cdot\frac{k!}{1} \\
&= \frac{(2k+2)(2k+1)\cdots(k+3)}{\left(k+\frac{3}{2}\right)\left(k+\frac{3}{2}\right)\cdots\left(k+\frac{3}{2}\right)}\cdot\frac{(k+2)(k+1)}{(2k+3)^{1/2}}\cdot\frac{k!}{2^k} \\
&\geq \frac{k!}{2^k}.
\end{aligned}$$

As in Exercise 34, the final fraction is greater than or equal to 1 for $k \geq 4$, and thus the desired inequality is true for all odd integers by the principle of mathematical induction.

38. Note that if r is odd, then

$$\left\lfloor\frac{r+1}{2}\right\rfloor = \frac{r+1}{2},$$

and so the desired equality is clear. Assume, therefore, that r is even, and so

$$\left\lfloor\frac{r+1}{2}\right\rfloor = \frac{r+1}{2} - \frac{1}{2} = \frac{r}{2} < \frac{r+1}{2}.$$

Thus $\log_2 \lfloor \frac{r+1}{2} \rfloor < \log_2 \left(\frac{r+1}{2}\right)$, and so $\left\lfloor \log_2 \lfloor \frac{r+1}{2} \rfloor \right\rfloor \le \left\lfloor \log_2 \left(\frac{r+1}{2}\right) \right\rfloor$. Assume by way of contradiction that

$$\left\lfloor \log_2 \left\lfloor \frac{r+1}{2} \right\rfloor \right\rfloor < \left\lfloor \log_2 \frac{r+1}{2} \right\rfloor .$$

Let $\lfloor \log_2 \left(\frac{r+1}{2}\right) \rfloor = n$, which is at least 1 since $r > 0$. Then

$$\log_2 \left(\frac{r}{2}\right) = \log_2 \left\lfloor \frac{r+1}{2} \right\rfloor < n \le \log_2 \left(\frac{r+1}{2}\right),$$

and hence

$$\log_2 r - \log_2 2 < n \le \log_2 (r+1) - \log_2 2.$$

Adding 1 to each member of the inequality gives

$$\log_2 r < n+1 \le \log_2 (r+1),$$

and hence (taking 2 to the power of the three terms) $r < 2^{n+1} \le r+1$. Since 2^{n+1} and r are integers, we must have $2^{n+1} = r+1$. However, this is impossible because $r+1$ is odd.

9.5 COUNTING WITH GENERATING FUNCTIONS

2. $B + C = 2 + 2x - x^2 + 5x^4 + x^5$

4. $AC = 1 + x - x^3 + x^5 + x^6$

6. $C + F = 2 - x - x^3 + 2x^4 - x^5 + x^6 - x^7 + \cdots$

8. $CF = 1 - x + x^4 - x^5 + x^6 - x^7 + \ldots$

10. $DC = 1 + x + x^4 + x^5 + x^6 + x^7 + \cdots$

12. $FD = 1 + x^2 + x^4 + x^6 + \cdots$

14. $(1+x)(1+x)(1+x)(1+x) = 1 + 4^x + 6x^2 + 4x^3 + x^4$

16. The desired generating function is

$$\begin{aligned}&(1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4)(1 + x^2 + x^4 + x^6)\\ &\quad = 1 + 2x + 4x^2 + 6x^3 + 8x^4 + 9x^5 + 10x^6 + \cdots .\end{aligned}$$

18. $(1 + x + x^2 + x^3 + x^4)(1 + x^2 + x^4 + x^6) = 1 + x + 2x^2 + 2x^3 + 3x^4 + 2x^5 + 3x^6 + \cdots$

20. $(1 + x^3 + x^6 + x^9 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots) = 1 + x^2 + x^3 + x^4 + x^5 + 2x^6 + \cdots$

22. $(1+x)^3(1+x+x^2+x^3+\cdots) = 1+4x+7x^2+8x^3+8x^4+8x^5+8x^6+\cdots$

24. $(1+x^7)(1+x^7)(1+x^7)(1+x^9+x^{18}+x^{27}+x^{36}+\cdots) = 1+3x^7+x^9+\cdots$

26. $(x+x^2+x^4+x^8+\cdots)^2 = x^2+2x^3+x^4+2x^5+\cdots$

28. $a_r = 1$ for $r=0$ and $a_r = 0$ for all $r \geq 1$, as $F = (1+x+x^2+x^3+\cdots)(1-x) = 1$

30. $a_r = 0$ for r odd and $a_r = 1$ for r even

32. $a_r = \dfrac{(r+1)(r+2)}{2}$

34. The desired generating function is

$$\begin{aligned} F &= (x+x^2+x^4+x^8+\cdots)(x^2+x^3+x^5+x^7+\cdots) \\ &= x^3+2x^4+x^5+2x^6+2x^7+x^8+2x^9+x^{10}+\cdots. \end{aligned}$$

The smallest $r > 2$ such that $a_r = 0$ is $r = 16$.

9.6 THE ALGEBRA OF GENERATING FUNCTIONS

2. $(1-5x)^{-1} = 1+5x+25x^2+125x^3+\ldots$

4. $(1-3x+9x^2-27x^3+\ldots)^{-1} = 1+3x$

6. $(1+2x^3)^{-1} = 1-2x^3+4x^6-8x^9+16x^{12}-\cdots$

8. $(1+x+x^3)^{-1} = 1-x+x^2-2x^3+3x^4-4x^5+\cdots$

10. $(\frac{1}{3}+x^4)^{-1} = 3-9x^4+27x^8-81x^{12}+243x^{16}-\cdots$

12. $S = \dfrac{3-x}{1-x^2}$

14. $S = \dfrac{2-x}{1-x+3x^2}$

16. $S = \dfrac{2x}{x^2+3x-1}$

18. $S = \dfrac{-3+17x-15x^2}{(1-x)(1-4x+5x^2)}$

20. $S = \dfrac{1-x+2x^2}{1-2x-x^2+x^3}$

22. $a=-1$, $b=3$

24. $a = 0.6$, $b = 0.4$

26. $a = 1$, $b = 2$

28. $3 - (-3)^n$

30. $3 - (-2)^{n+1}$

32. 2 if n is odd, 3 if $n \equiv 0 \pmod 4$, and 1 if $n \equiv 2 \pmod 4$

34. If S is a generating function for $\{s_n\}$ with s_0 given, then

$$\begin{aligned} S &= s_0 + s_1x + s_2x^2 + s_3x^3 + \cdots \\ &= s_0 + (as_0 + b)x + (a^2s_0 + ab + b)x^2 + (a^3s_0 + a^2b + ab + b)x^3 + \cdots \\ &= s_0 + ax(s_0 + as_0x + b + a^2s_0x^2 + ab + bx^2 + \dots) + (bx + bx^2 + bx^3 + \cdots) \\ &= s_0 + axS + bx(1-x)^{-1}. \end{aligned}$$

36. Checking $n = 0$, we have $s_n = (s_0 + \frac{b}{a-1})a^0 - \frac{b}{a-1}$. Suppose that the formula holds for all nonnegative integers less than or equal to k, and consider s_{k+1}:

$$\begin{aligned} s_{k+1} &= a\left[\left(s_0 + \frac{b}{a-1}\right)a^k - \frac{b}{a-1}\right] + b \\ &= a^{k+1}s_0 + \frac{a^{k+1}b}{a-1} - \frac{b}{a-1} \\ &= \left(s_0 + \frac{b}{a-1}\right)a^{k+1} - \frac{b}{a-1}. \end{aligned}$$

Thus the formula holds for $n = k+1$ when it holds for $n = k$. Hence, by the strong principle of mathematical induction, it holds for all nonnegative integers.

38. We have

$$\begin{aligned} S &= s_0 + s_1x + s_2x^2 + s_3x^3 + \cdots \\ &= s_0 + s_1x + (as_1x^2 + bs_0x^2) + (a^2s_1x^5 + abs_0x^5) + \cdots \\ &= s_0 + s_1x + ax(s_1x + s_2x^2 + s_3x^3 + \cdots) + bx(s_0 + s_1x + s_2x^2 + \cdots) \\ &= s_0 + s_1x + ax(S - s_0) + bx^2S. \end{aligned}$$

40. For $n \geq 0$, we have

$$\begin{aligned}
S &= \frac{c_1}{(1-r_1x)} + \frac{c_2}{(1-r_2x)} \\
&= c_1\left(\frac{1}{1-r_1x}\right) + c_2\left(\frac{1}{1-r_2x}\right) \\
&= c_1(1 + r_1x + r_1x^2 + r_1x^3 + \cdots) + c_2(1 + r_2x + r_2x^2 + r_2x^3 + \cdots) \\
&= (c_1 + c_2) + (c_1r_1 + c_2r_2)x + (c_1r_2 + c_2r_2)x^2 + \cdots \\
&= c_1(r_1)^n + c_2(r_2)^n.
\end{aligned}$$

42. We have

$$\begin{aligned}
S &= \frac{k_1}{(1-rx)} + \frac{k_2}{(1-rx)^2} \\
&= k_1(1 + rx + rx^2 + rx^3 + \cdots) + k_2(1 + 2r + 3r^2 + 4r^4 + \cdots) \\
&= k_1r^n + k_2(n+1)r^n.
\end{aligned}$$

SUPPLEMENTARY EXERCISES

2. $s_5 = 0$

4. $s_5 = 477$

6. $v_n = 1.02v_{n-1} - 200$ for $n \geq 1$ and $v_0 = 20{,}000$

8. $v_n = 0.88v_{n-1}$ for $n \geq 1$ and $v_0 = 18{,}000$

10. $c_n = 2.5c_{n-1}$ for $n \geq 1$ and $c_0 = 500$

12. \$15,500

14. Evaluating the formula for $n = 1$, we have $2^0 + 2 = 3 = s_1$. Thus the formula is correct for $n = 1$. Suppose that it is correct for an integer $k \geq 1$, and consider the formula for $k + 1$. Because

$$s_{k+1} = 2(2^k + 2) - 2 = 2^{k+1} + 2,$$

the formula is correct for $k + 1$ when it holds for k. Thus, by the principle of mathematical induction, the formula is correct for all positive integers.

16. For $n = 0$, we have $2^0 + 2(3^0) + 0 - 7 = -4$; thus the formula is correct for $n = 0$. Suppose that the formula is correct for all nonnegative integers $n \leq k$. Then

$$\begin{aligned} s_{k+1} &= 5[2^k + 2(3^k) + k - 7] - 6[2^{k-1} + 2(3^{k-1}) + (k-1) - 7] + 2(k+1) - 21 \\ &= 2(2^k) - 6(3^k) + (k+1) - 7. \end{aligned}$$

Hence the formula is correct for $n = k + 1$ and so, by the principle of mathematical induction, for all nonnegative integers.

18. $s_n = 2 + 7^n$

20. $s_n = 5^n - 3(2^n)$

22. The ratios for the values in Exercise 21 are 1.333, 1.750, 1.571, 1.636, 1.611, 1.621, 1.617, 1.618, and 1.618. In addition, $\frac{1+\sqrt{5}}{2} \approx 1.618$.

24. $s_n = 9(2^n) - 5(-1)^n$ and $t_n = 6(2^n) - 5(-1)^n$

26. If u_n satisfies (9.7), then

$$bu_n = a_1 bu_{n-1} + a_2 bu_{n-2} + \cdots + a_n bu_0.$$

If v_n satisifies (9.7), then

$$cv_n = a_1 cv_{n-1} + a_2 cv_{n-2} + \cdots + a_n cv_0.$$

Therefore

$$\begin{aligned} bu_n + cv_n &= (a_1 bu_{n-1} + a_2 bu_{n-2} + \cdots + a_n bu_0) + (a_1 cv_{n-1} + a_2 cv_{n-2} + \cdots + a_n cv_0) \\ &= a_1(bu_{n-1} + cv_{n-1}) + a_2(bu_{n-2} + cv_{n-2}) + \cdots + a_n(bu_0 + cv_0). \end{aligned}$$

Thus $bu_n + cv_n$ satisfies (9.7).

28. $s_n = (5 + n - 3n^2)2^n$ for $n \geq 0$

30. Let u_n be a particular solution of (9.8) for $n \geq k$. Then

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k} + f(n)$$

for $n \geq k$. Suppose that w_n is a solution to (9.8) for $n \geq k$. Then

$$w_n = a_1 w_{n-1} + a_2 w_{n-2} + \cdots + a_k w_{n-k} + f(n)$$

for $n \geq k$. Consider $v_n = w_n - u_n$ for $n \geq k$. For $n \geq k$, we have

$$v_n = a_1 v_n + a_2 v_{n-2} + \cdots + a_k v_{n-k},$$

so that v_n satisfies (9.7) for $n \geq k$. Hence $w_n = u_n + v_n$ for $n \geq k$.

32. $s_n = 4(-1)^n + 6^n - n + 2$

34. The binary search algorithm proceeds as follows.

b	e	m	a_m	Is $a_m = 7$?
1	4	$\lfloor \frac{1}{2}(1+4) \rfloor = 2$	4	no; less
3	4	$\lfloor \frac{1}{2}(3+4) \rfloor = 3$	6	no; less
4	4	$\lfloor \frac{1}{2}(4+4) \rfloor = 4$	8	no; greater
5	4			

Since $b > e$ in the last line of the table, the target number (7) is not in the given list.

36. 3

38. 4

40. (7) and (3, 4, 5)

42. $S = \dfrac{1+x}{(1-x)^2}$

44. $S = x(1-x^2)^{-1}$

46. $s_n = -4(-6)^n$

48. $s_n = 2^n + 5$

50. a_r is the coefficient of x^r in $(1 + x + x^2 + \cdots)^3$

52. a_r is the coefficient of x^r in $(1 + x + x^2 + x^3)^8$

54. a_r is the coefficient of x^r in $(x^3 + x^4 + \cdots)(1 + x + x^2)$

Chapter 10

Combinatorial Circuits and Finite State Machines

10.1 LOGICAL GATES

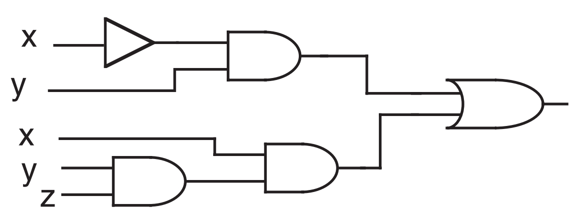
2. $(x \vee y) \wedge (x \wedge y)$

4. $(x \wedge y)' \wedge (x' \wedge y)$

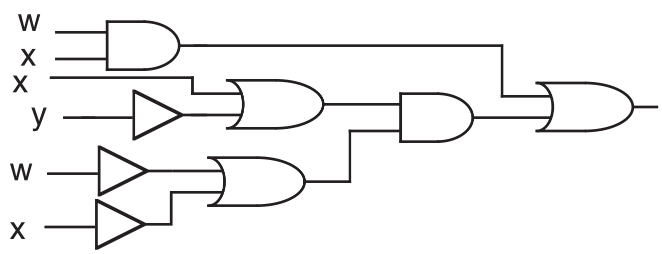
6. $(x \wedge y') \wedge (x' \wedge y)$

8. $(x' \wedge y') \wedge (x \vee y)$

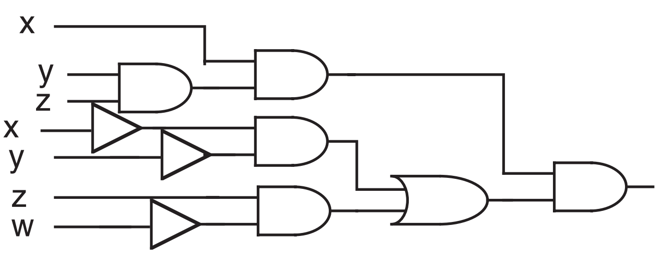
10. The circuit representing the given Boolean expression is shown below.



12. The circuit representing the given Boolean expression is shown below.



14. The circuit representing the given Boolean expression is shown below.



16. 0

18. 0

20.

x	y	*Output*
0	0	1
0	1	1
1	0	1
1	1	0

22.

x	y	z	*Output*
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

24.

x	y	*Output*
0	0	0
0	1	1
1	0	1
1	1	1

26.

x	y	*Output*
0	0	0
0	1	1
1	0	1
1	1	1

28.

x	y	z	output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

30. not equivalent

32. not equivalent

34. not equivalent

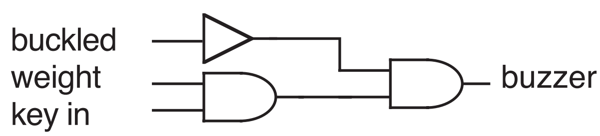
36. not equivalent

38. not equivalent

40. equivalent

42. equivalent

44. The circuit for the seatbelt buzzer is shown below.



46. We make a circuit a directed graph, where the input variables, output, and logical gates are vertices. There is a directed edge from U to V in case there is a line in the circuit from U to an input of V, where V is a logical gate, or from U to V, where U is a logical gate and V is the output vertex. The condition is that this directed graph have no directed circuits.

10.2 CREATING COMBINATORIAL CIRCUITS

10. (c), (f), (a), (g)

12. (c), (c), (f), (f), (g), (g), (f)

14. (a), (a), (a), (a), (c), (c), (f), (f), (g), (g), (f)

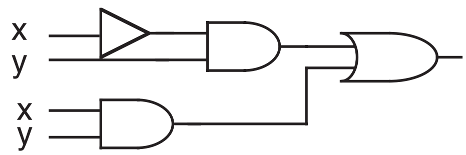
16. (j), (j), (a)

18. (j), (b), (a), (d)

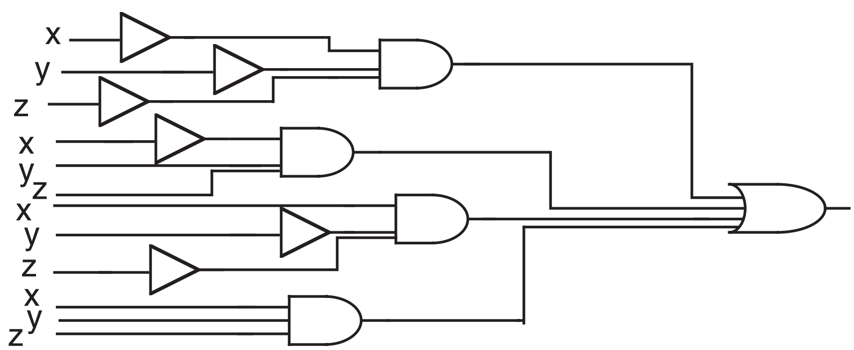
20. When $x = 0$ and $y = 1$, the first is 1 and the second 0.

22. When $x = 0$, the first is 1 and the second 0.

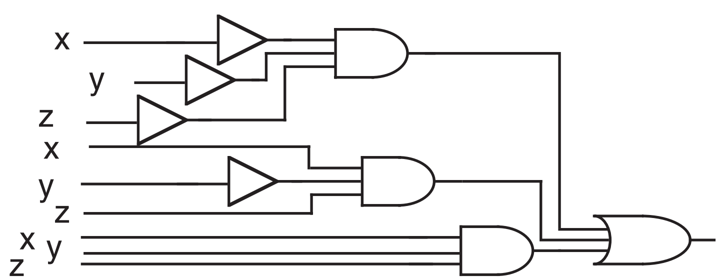
24. $(x' \wedge y) \vee (x \wedge y)$



26. $(x' \wedge y' \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z)$



28. $(x' \wedge y' \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z)$



30. 9

32. 12

34. 13

36. $(G_1 \wedge (G_2 \vee G_3)) \vee (G_1' \wedge G_2' \wedge G_3)$

38. (a), (b), (g)

10.3 KARNAUGH MAPS

2. $(x' \wedge y' \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$

4. $(x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z')$

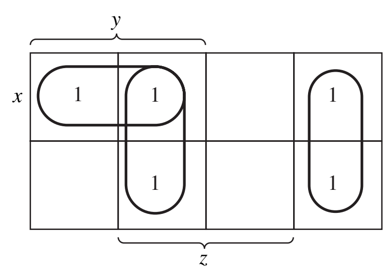
6. $(w' \wedge x' \wedge y' \wedge z') \vee (w' \wedge x \wedge y' \wedge z) \vee (w' \wedge x \wedge y \wedge z) \vee (w \wedge x' \wedge y' \wedge z') \vee (w \wedge x \wedge y' \wedge z) \vee (w \wedge x \wedge y \wedge z)$

8. $(x \wedge y) \vee z$

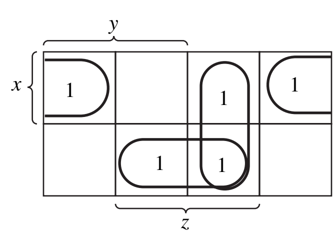
10. $(x \wedge z) \vee (x' \wedge z') \vee (y' \wedge z)$

12. $(w' \wedge x) \vee (y' \wedge z) \vee (w \wedge x' \wedge z) \vee (x \wedge y \wedge z')$

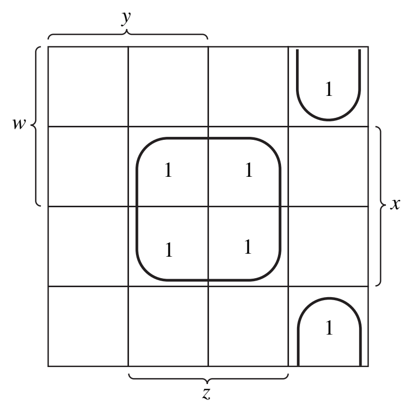
14. The Karnaugh map for the given Boolean expression is shown below.



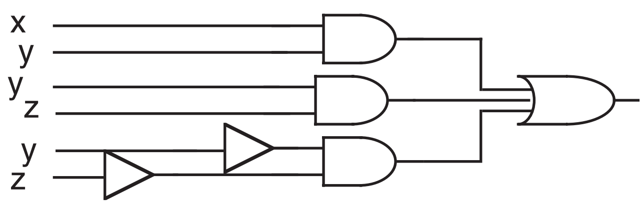
16. The Karnaugh map for the given Boolean expression is shown below.



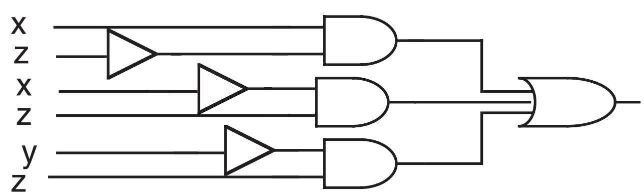
18. The Karnaugh map for the given Boolean expression is shown below.



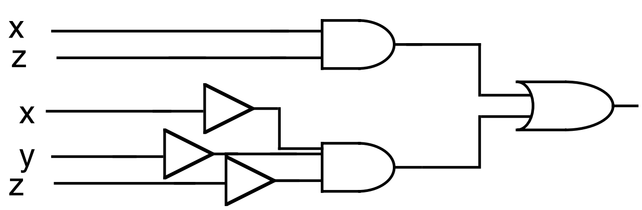
20. $(x \wedge y) \vee (y \wedge z) \vee (y' \wedge z')$



22. $(x \wedge z') \vee (x' \wedge z) \vee (y' \wedge z)$



24. $(x \wedge z) \vee (x' \wedge y' \wedge z')$



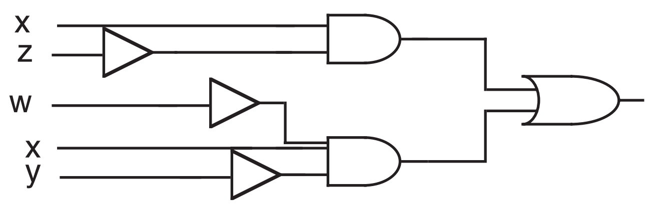
26. $(x \wedge y \wedge z) \vee (x' \wedge y') \vee (y' \wedge z')$

28. $y \vee (x' \wedge z')$

30. $(x \wedge z) \vee (y' \wedge z)$

32. $(x \wedge y \wedge z') \vee (w \wedge y \wedge z) \vee (w \wedge y' \wedge z') \vee (w' \wedge y' \wedge z) \vee (w' \wedge x' \wedge y')$

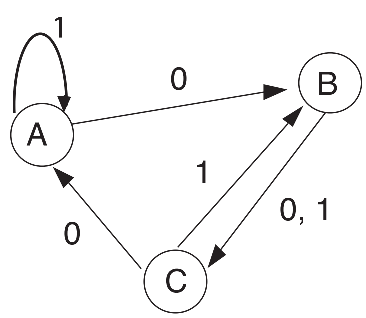
34. The simplified circuit is shown below.



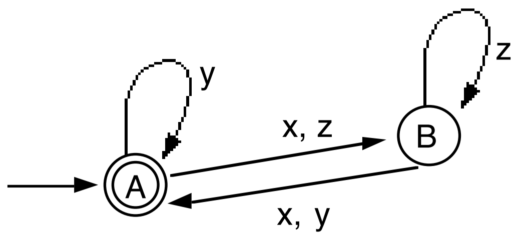
36. 9

10.4 FINITE STATE MACHINES

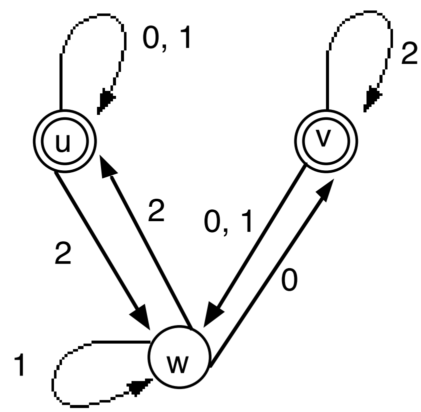
2. The transition diagram is shown below.



4. The transition diagram is shown below.



6. The transition diagram is shown below.



8.

	I	II	III	IV
0	I	I	I	I
1	II	III	IV	III

initial state I

10.

	a	b	c	d
0	b	c	c	c
1	d	c	a	b

initial state d, accepting states a, c

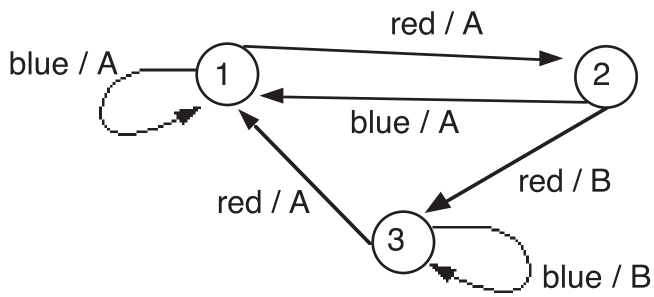
12. A

14. III

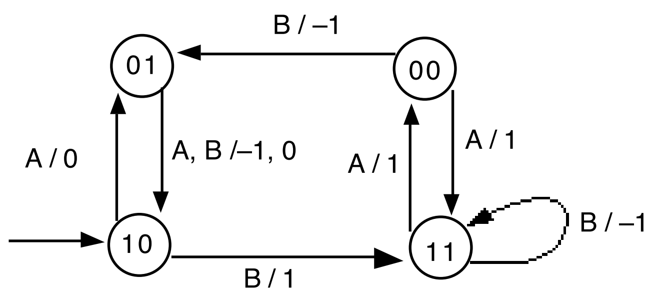
16. yes

18. no

20. The transition diagram is shown below.



22. The transition diagram is shown below.



24.

	A	B	C	D	A	B	C	D
0	A	A	B	C	0	0	1	2
1	B	C	D	D	1	2	3	4

26.

	apple	pear	plum	apple	pear	plum
0	pear	plum	apple	t	t	s
1	plum	plum	plum	s	t	t

initial state pear

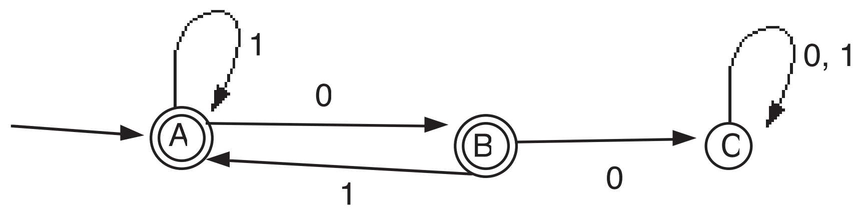
28. $1, 1, 1, -1, -1, -1$

30. $t\ s\ s\ s\ t\ t\ t\ s$

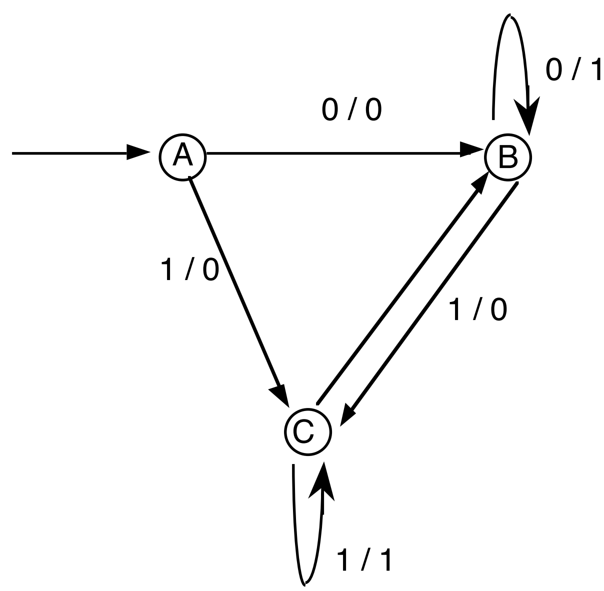
32. All strings ending in 0.

34. All strings with at least two groups of 1s separated by one or more 0s.

36. The finite state machine shown below accepts a string if and only if it does not contain consecutive 0s.



38. The finite state machine shown below produces an output string that contains as many 1s as there are pairs of consecutive 0s or 1s in the input string.

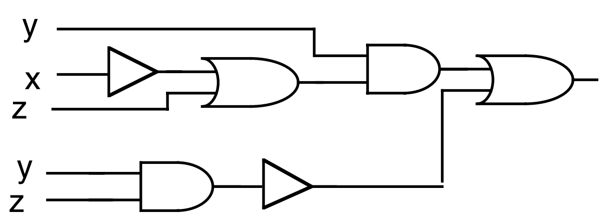


SUPPLEMENTARY EXERCISES

2. The truth table is shown below.

x	y	z	*Output*
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

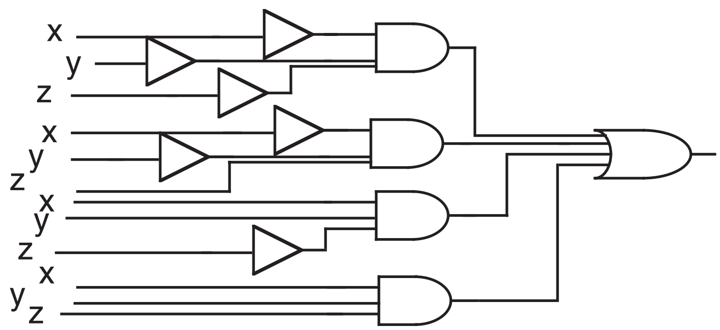
The corresponding circuit is as follows.



4.

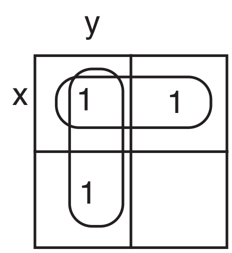
x	y	z	*Lights*
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

6. $(x' \wedge y' \wedge z') \vee (x' \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y \wedge z)$

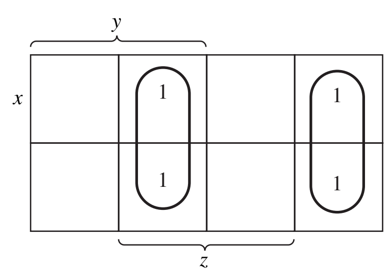


17 gates

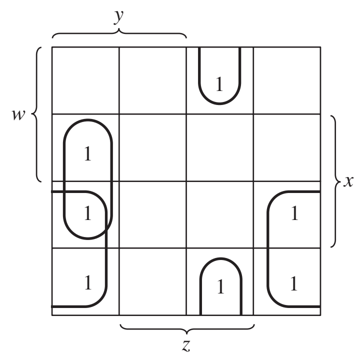
8. **(a)** The Karnaugh map is shown below.



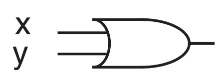
(b) The Karnaugh map is shown below.



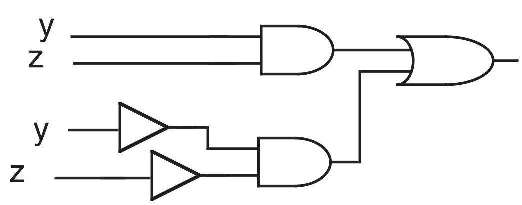
(c) The Karnaugh map is shown below.



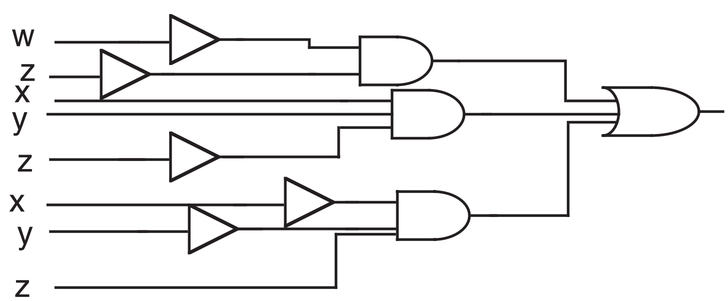
10. **(a)** $x \vee y$



(b) $(y \wedge z) \vee (y' \wedge z')$

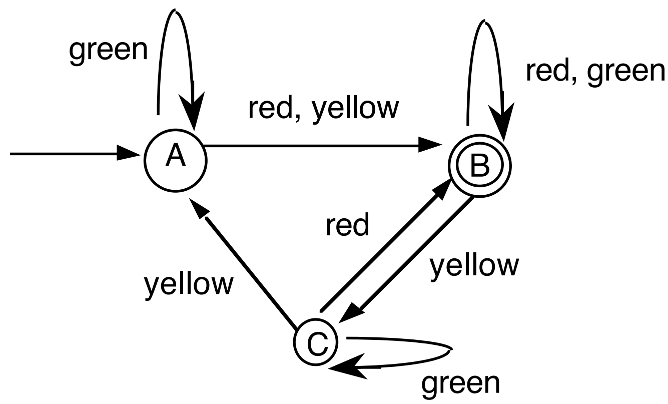


(c) $(w' \wedge z') \vee (x \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$



12. **(a)** $(x \wedge y' \wedge z) \vee (x' \wedge z')$ **(b)** $(w' \wedge y') \vee (w' \wedge x \wedge z')(x' \wedge y' \wedge z)$

14. The transition diagram is shown below.

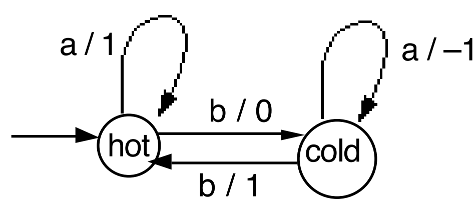


16.

	x	y	z
3	y	y	x
5	y	z	y

initial state z; accepting states x, z

18. The transition diagram is shown below.

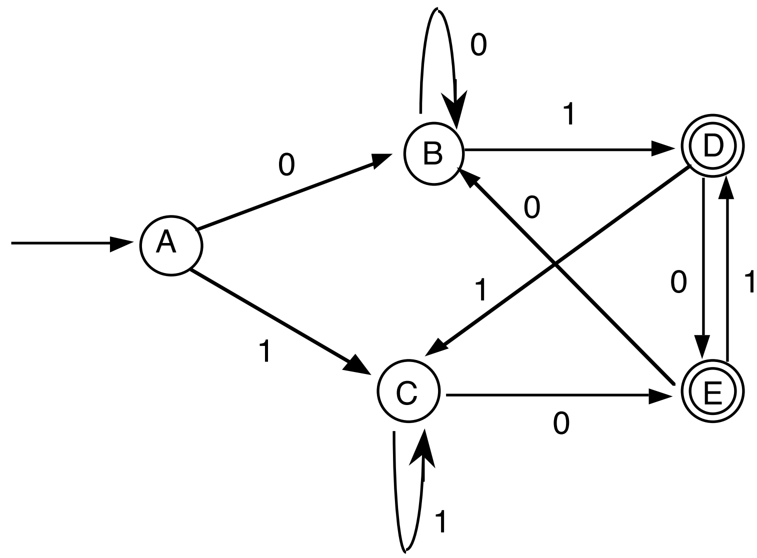


20.

	1	x	cider	1	x	cider
0	x	x	x	a	foam	b
1	cider	1	x	b	b	c

initial state 1

22. The finite state machine is shown below.



Appendix A

An Introduction to Logic and Proof

A.1 STATEMENTS AND CONNECTIVES

2. not a statement

4. false statement

6. true statement

8. true statement

10. false statement

12. not a statement

14. Christmas is not celebrated on December 25.

16. It has snowed in Chicago.

18. No person is rich.

20. There exists a millionaire that does not pay taxes.

22. There exists a resident of Chicago who does not love the Cubs.

24. There is a farmer in South Dakota.

26. **(a)** The conjunction is: Oregon borders Canada and Egypt is in Asia. The conjunction is false because at least one (in this case, both) of the simple statements from which it is formed is false.
(b) The disjunction is: Oregon borders Canada or Egypt is in Asia. The disjunction is false, because both of the simple statements from which it is formed are false.

28. **(a)** The conjunction is: Cardinals are red and robins are blue. The conjunction is false because at least one (in this case, the second) of the simple statements from which it is formed is false.

(b) The disjunction is: Cardinals are red or robins are blue. The disjunction is true because at least one (in this case, the first) of the simple statements from which it is formed is true.

30. **(a)** The conjunction is: Oranges are fruit, and potatoes are vegetables. The conjunction is true because both of the simple statements from which it is formed are true.

(b) The disjunction is: Oranges are fruit, or potatoes are vegetables. The disjunction is true because at least one (in this case, both) of the simple statements from which it is formed is true.

32. **(a)** The conjunction is: Algebra is an English course, and accounting is a business course. The conjunction is false.

(b) The disjunction is: Algebra is an English course, or accounting is a business course. The conjunction is true.

34. **(a)** The converse is: If I take a break, then I have completed this assignment.

(b) The inverse is: If I don't complete this assignment, then I won't take a break.

(c) The contrapositive is: If I don't take a break, then I won't complete this assignment.

36. **(a)** The converse is: If I get a B for the course, then I got an A on the final exam.

(b) The inverse is: If I don't get an A on the final exam, then I won't get a B for the course.

(c) The contrapositive is: If I don't get a B for the course, then I didn't get an A on the final exam.

A.2 LOGICAL EQUIVALENCE

2. The truth table is as follows.

p	q	$\sim p$	$\sim q$	$(\sim p) \vee q$	$(\sim q) \wedge p$	$(\sim p \vee q) \wedge (\sim q \wedge p)$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

4. The truth table is as follows.

p	q	$\sim p$	$\sim q$	$(\sim p) \wedge q$	$(\sim q) \vee p$	$(\sim p \wedge q) \to (\sim q \vee p)$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	F	T	T

6. The truth table is as follows.

p	q	r	$\sim q$	$(\sim q) \vee r$	$p \to (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

8. The truth table is as follows.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$\sim[p \wedge (q \vee r)]$
T	T	T	T	T	F
T	T	F	T	T	F
T	F	T	T	T	F
T	F	F	F	F	T
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	F	F	T

10. The truth table is as follows.

p	q	r	$\sim q$	$r \wedge \sim q$	$q \vee p$	$(r \wedge \sim q) \leftrightarrow (q \vee p)$
T	T	T	F	F	T	F
T	T	F	F	F	T	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	T	F	F	F	T	F
F	F	T	T	T	F	F
F	F	F	T	F	F	T

12. If a truth table for $(p \to q) \vee (\sim q \wedge p)$ is constructed, the entries in the column corresponding to the statement $(p \to q) \vee (\sim q \wedge p)$ are all T. Hence this statement is a tautology.

14. If a truth table for $\sim(\sim p \wedge q) \to (\sim q \vee p)$ is constructed, the entries in the column corresponding to the statement $\sim(\sim p \wedge q) \to (\sim q \vee p)$ are all T. Hence this statement is a tautology.

16. If a truth table for $[(p \wedge q) \to r] \to [\sim r \to (\sim p \vee \sim q)]$ is constructed, the entries in the column corresponding to the statement $[(p \wedge q) \to r] \to [\sim r \to (\sim p \vee \sim q)]$ are all T. Hence this statement is a tautology.

18. We proceed as in Example A.8 by making a truth table containing columns for each of the given statements. If the 4 rows of this table are arranged as in Exercise 2, then the entries in the column corresponding to the statements p and $p \vee (p \wedge q)$ are T, T, F, F. Because the entries in the columns corresponding to p and $p \vee (p \wedge q)$ are the same, these statements are logically equivalent.

20. We proceed as in Example A.8 by making a truth table containing columns for each of the given statements. If the 4 rows of this table are arranged as in Exercise 2, then the entries in the column corresponding to the statements $p \leftrightarrow q$ and $(\sim p \vee q) \wedge (\sim q \vee p)$ are T, F, F, T. Because the entries in the columns corresponding to $p \leftrightarrow q$ and $(\sim p \vee q) \wedge (\sim q \vee p)$ are the same, these statements are logically equivalent.

22. We proceed as in Example A.8 by making a truth table containing columns for each of the given statements. If the 8 rows of this table are arranged as in Exercise 6, then the entries in the column corresponding to the statements $(p \to q) \to r$ and $(p \vee r) \wedge (q \to r)$ are T, F, T, T, T, F, T, F. Because the entries in the columns corresponding to $(p \to q) \to r$ and $(p \vee r) \wedge (q \to r)$ are the same, these statements are logically equivalent.

24. We proceed as in Example A.8 by making a truth table containing columns for each of the given statements. If the 8 rows of this table are arranged as in Exercise 6, then the entries in the column corresponding to the statements $p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$ are T, T, T, F, T, T, T, T. Because the entries in the columns corresponding to $p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$ are the same, these statements are logically equivalent.

26. **(c)** The table containing the verification of statement (c) is arranged with 8 rows as in Exercise 6. The entries in the column corresponding to each of the given statements are T, F, F, F, F, F, F, F. Thus the two statements are logically equivalent.

(d) The table containing the verification of statement (d) is arranged with 8 rows as in Exercise 6. The entries in the column corresponding to each of the given statements are T, T, T, T, T, T, T, F. Thus the two statements are logically equivalent.

28. **(h)** The table containing the verification of statement (h) is arranged with 4 rows as in Exercise 2. The entries in the column corresponding to each of the given statements are F, T, T, T. Thus the two statements are logically equivalent.

30. Prepare a truth table containing the statement $[(p \to q) \wedge \sim q] \to \sim p$ with its 4 rows arranged as in Exercise 2. The entries in the column corresponding to the statement are T, T, T, T. Because these entries are all T, the statement is a tautology.

32. Prepare a truth table containing the statement $[(p \vee q) \wedge \sim p] \to q$ with its 4 rows arranged as in Exercise 2. The entries in the column corresponding to the statement are T, T, T, T. Because these entries are all T, the statement is a tautology.

34. **(a)** The truth table for $p \,|\, p$ and $\sim p$ is as follows.

p	$p \,\|\, p$	$\sim p$
T	F	F
F	T	T

Because the last two columns are identical, $p \,|\, p$ is logically equivalent to $\sim p$.

(b) The truth table for $(p \,|\, p) \,|\, (q \,|\, q)$ and $p \vee q$ is as follows.

p	q	$p \,\|\, p$	$q \,\|\, q$	$(p \,\|\, p) \,\|\, (q \,\|\, q)$	$p \vee q$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

Because the last two columns are identical, $(p \,|\, p) \,|\, (q \,|\, q)$ is logically equivalent to $p \vee q$.

(c) The truth table for $(p \,|\, q) \,|\, (p \,|\, q)$ and $p \wedge q$ is as follows.

p	q	$p \,\|\, q$	$p \,\|\, q$	$(p \,\|\, q) \,\|\, (p \,\|\, q)$	$p \wedge q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	T	T	F	F
F	F	T	T	F	F

Because the last two columns are identical, $(p \,|\, q) \,|\, (p \,|\, q)$ is logically equivalent to $p \wedge q$.

(d) The truth table for $p \,|\, (q \,|\, q)$ and $p \to q$ is as follows.

p	q	$q \,\|\, q$	$p \,\|\, (q \,\|\, q)$	$p \to q$
T	T	F	T	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Because the last two columns are identical, $p\,|\,(q\,|\,q)$ is logically equivalent to $p \to q$.

A.3 METHODS OF PROOF

2. The truth table is as follows.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

Since the last column, which corresponds to the law of the syllogism, contains only T's, the law is a tautology.

4. Suppose that there exists a rational number r such that $r^2 = 3$. Then $r = m/n$ for some integers m and n that have no common factors. Squaring both sides of this equation, we obtain $m^2 = 3n^2$. Thus the integer m^2 is divisible by 3, and so m is divisible by 3. That is, $m = 3p$ for some integer p. Substituting this expression for m in the previous equation, we have $(3p)^2 = 3n^2$. Hence $9p^2 = 3n^2$, which simplifies to $3p^2 = n^2$. Therefore n^2 is divisible by 3, and so n is divisible by 3. But then both m and n have a common factor of 3, contradicting our assumption. It follows that there does not exist a rational number r such that $r^2 = 3$.

6. Suppose that ac divides bc. Then $acm = bc$ for some positive integer m. Dividing by the positive integer c gives $am = b$. Thus a divides b.

8. Suppose that a divides b. Then there exists a positive integer c such that $b = ac$. Since c is a positive integer, we have $b = ac \geq a \cdot 1 = a$.

10. Suppose that a divides both b and $b + 2$. Then there exist positive integers c and d such that $ac = b$ and $ad = b+2$. Hence $a(d-c) = ad-ac = (b+2)-b = 2$. Therefore a divides 2, whose only positive divisors are 1 and 2. Thus $a = 1$ or $a = 2$.

12. We prove the statement by mathematical induction. The statement is true for $n = 11$ because $12 \cdot 9 = 108 \leq 110 = (11)^2 - 11$. Suppose that the statement is true for some integer $k \geq 11$. Then $12(k-2) < k^2 - k$, that is, $12k - 24 < k^2 - k$. Now

$$\begin{aligned}(k+1)^2 - (k+1) &= (k^2 + 2k + 1) - (k+1) \\ &= k^2 + k = (k^2 - k) + 2k \\ &\geq (12k - 24) + 2k = 12k + (2k - 24) \\ &\geq 12k + (22 - 24) = 12k - 2 \\ &\geq 12k - 12 = 12(k-1).\end{aligned}$$

This proves the given statement true for $n = k+1$, and so it is true for all integers greater than 10 by the principle of mathematical induction.

14. Suppose that both x and y are odd. Then there exist integers m and n such that $x = 2m+1$ and $y = 2n+1$. Hence

$$xy = (2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1.$$

Because $2(2mn + m + n)$ is even, it follows that xy is odd.

16. We may assume that x is a nonnegative integer, because if 3 divides xy, then 3 also divides $(-x)y$. Suppose that 3 divides xy but that 3 does not divide x. Then $xy = 3k$ for some integer k. Furthermore, by the division algorithm, we may write $x = 3q + r$, where $0 \leq r < 3$. Note that $r \neq 0$ since 3 does not divide x, and so $r = 1$ or $r = 2$. Multiplying the equation $x = 3q + r$ by y yields $xy = 3qy + ry$. Hence $ry = xy - 3qy = 3k - 3qy = 3(k - qy)$. This proves that y is divisible by 3 if $r = 1$. When $r = 2$, we have $2y = 3(k - qy)$, which implies that $3y = 3(k - qy) + y$. Hence $y = 3y - 3(k - qy) = 3(y - k + qy)$ is divisible by 3. Thus, no matter whether $r = 1$ or $r = 2$, we see that y is divisible by 3. So if 3 divides xy but 3 does not divide x, then 3 divides y; that is, if 3 divides xy, then 3 must divide x or y.

18. Suppose that 3 divides both x and y. Then there exist integers m and n such that $x = 3m$ and $y = 3n$. Therefore $ax + by = a(3m) + b(3n) = 3(am + bn)$, and so 3 divides $ax + by$.

20. Suppose that a and b are both odd. Then there exist integers m and n such that $a = 2m+1$ and $b = 2n+1$. Hence

$$\begin{aligned}a^2 + b^2 &= (2m+1)^2 + (2n+1)^2 \\ &= (4m^2 + 4m + 1) + (4n^2 + 4n + 1) \\ &= 4(m^2 + m + n^2 + n) + 2.\end{aligned}$$

Thus if $a^2 + b^2$ is divided by 4, the remainder is 2. It follows that $a^2 + b^2$ is not divisible by 4.

22. If both x and y are odd integers, then xy is odd by Exercise 14. Thus we need only to establish that if xy is odd, then both x and y are odd. We will prove the contrapositive:

If not both x and y are odd, then xy is not odd. This statement is equivalent to the statement: If at least one of x and y is even, then xy is even. Assume that at least one of x and y is even, say, x. Then there exists an integer m such that $x = 2m$, and so $xy = (2m)y = 2(my)$, which is even. This proves that if xy is odd, then both x and y are odd, completing the proof of the given biconditional statement.

24. Three consecutive positive odd integers can be represented as $2k+1$, $2k+3$, and $2k+5$ for some positive integer k. Note that k can be written in the form $k = 3n$, $k = 3n+1$, or $k = 3n+2$ for some positive integer n. If $k = 3n$, then $2k+3 = 2(3n)+3 = 3(2n+1)$ is divisible by 3. If $k = 3n+1$, then $2k+1 = 2(3n+1)+1 = 6n+3 = 3(2n+1)$ is divisible by 3. Finally, for $k = 3n+2$, we have $2k+5 = 2(3n+2)+5 = 6n+9 = 3(2n+3)$ is divisible by 3. Thus, no matter which of the three forms k has, at least one of the integers $2k+1$, $2k+3$, and $2k+5$ is divisible by 3 and hence is not prime.

26. For any positive integer n, we have

$$n^4 - n^2 = n^2(n^2 - 1) = n^2(n+1)(n-1).$$

Because $n-1$, n, and $n+1$ are consecutive integers, either one of them is divisible by 6, or one of them is divisible by 2 and a different one of them is divisible by 3. Hence their product is divisible by 6, and so $n^2(n+1)(n-1) = n^4 - n^2$ is also divisible by 6.

28. Suppose, by way of contradiction, that there are only a finite number of primes, say, $p_1, p_2, \ldots, p_k$. Consider the positive integer $n = p_1p_2\cdots p_k + 1$, the product of all of the primes plus 1. Dividing n by any of the primes p_i leaves a remainder of 1, and so n is not divisible by any prime. This means that n is prime, contradicting that $p_1, p_2, \ldots, p_k$ is the entire collection of primes. Hence our assumption must been false, that is, there are an infinite number of primes.

SUPPLEMENTARY EXERCISES

2. false statement

4. false statement

6. true statement

8. false statement

10. The negation is: There exists an isosceles triangle that is not equilateral. (true)

12. The negation is: Red is not a primary color, or blue is a primary color. (true)

14. The negation is: For every integer x, $x^2 \geq 1$. (false)

16. The negation is: There are not three solutions to $x^3 - x = 0$. (false)

18. **(a)** $2^3 = 8$ and $3^2 = 8$. (false)
(b) $2^3 = 8$ or $3^2 = 8$. (true)

20. **(a)** Roses are animals, and tigers are plants. (false)
(b) Roses are animals, or tigers are plants. (false)

22. **(a)** If $(3+3)^2 = 3^2 + 3^2$, then $(3+3)^2 = 18$. (true)
(b) If $(3+3)^2 \neq 18$, then $(3+3)^2 \neq 3^2 + 3^2$. (true)
(c) If $(3+3)^2 \neq 3^2 + 3^2$, then $(3+3)^2 \neq 18$. (true)

24. **(a)** If $3^2 + 2 \times 3 \times 3 + 3^2 = 6^2$, then $(3+3)^2 = 6^2$. (true)
(b) If $(3+3)^2 \neq 6^2$, then $3^2 + 2 \times 3 \times 3 + 3^2 \neq 6^2$. (true)
(c) If $3^2 + 2 \times 3 \times 3 + 3^2 \neq 6^2$, then $(3+3)^2 \neq 6^2$. (true)

26. The truth table is as follows.

p	q	$\sim[(p \vee q) \wedge p] \leftrightarrow (\sim p \wedge q)$
T	T	T
T	F	T
F	T	T
F	F	F

28. The truth table is as follows.

p	q	r	$[(p \to r) \wedge (p \to q)] \to [p \to (r \wedge q)]$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

30. The statement is a tautology.

32. The statement is not a tautology. (It is false if p, q, and r are all false.)

34. No, the statements are not logically equivalent. (They have different values if p and r are false and q is true.)

36. If the truth table is arranged as in Exercise 28, then the column corresponding to each given statement is T, T, T, T, T, T, F, F. So the statements are logically equivalent.

38. We will prove the contrapositive: If x is divisible by 5, then x is not even or x has zero as a unit's digit. We must show that if x is divisible by 5, then either x is odd or x has zero as a unit's digit. Suppose then that x is divisible by 5 and that x is not odd. Because x is divisible by 5, there exists an integer m such that $x = 5m$. Because x is also even, m is even. But then the units digit of $5m$ must be zero, because 5 times any even integer has a unit's digit of zero. Thus the contrapositive of the given statement is true, and hence the given statement is true.

40. The statement is false for $n = 24$. In this case $6n - 1 = 6(24) - 1 = 143$ and $6n + 1 = 6(24) + 1 = 145$. Both these integers are composite because the first is divisible by 11 and the second is divisible by 5.

42. For any positive integer n, we have $s = n + n^2 = n(n + 1)$. Because n and $n + 1$ are consecutive positive integers, one of these must be even, and hence their product s is even.

44. Let $k = c + c^3$, where c is a positive integer. If c is even, then c^3 is even, and hence $k = c + c^3$ is even. If c is odd, then c^3 is odd, and thus $k = c + c^3$ is even. Therefore, in either case, k is even, and the given statement is true.

Appendix B

Matrices

2. not defined

4.
$$\begin{bmatrix} 0 & 2 & 5 \\ 3 & 0 & 6 \end{bmatrix}$$

6.
$$\begin{bmatrix} -2 & -4 & -6 \\ 1 & -1 & 9 \\ 4 & 7 & 14 \end{bmatrix}$$

8. not defined

10.
$$\begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix}$$

12.
$$\begin{bmatrix} 11 & 16 \\ 8 & 27 \end{bmatrix}$$

14.
$$\begin{bmatrix} 5 & 12 \\ -12 & 5 \end{bmatrix}$$

16.
$$\begin{bmatrix} 49 & 102 \\ 51 & 151 \end{bmatrix}$$

18.
$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

20.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

22.
$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

24.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$