Quantum Entanglement and the Black Hole Information Paradox

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Abstract

The black hole (BH) information paradox first captured the attention of physicists through the pioneering work of Stephen Hawking in 1976. As quantum mechanics intertwined with the BHs, Hawking's insights shed light on a perplexing dilemma: the fate of information that falls beyond the event horizon. This paper discussed the fundamental principles of quantum mechanics and the black holes that give rise to this paradox. Focusing on the phenomenon of quantum entanglement and the black hole thermodynamics. Despite decades of work and theoretical exploration, the information paradox remains unresolved. Here, through a paper review, two notable proposals aimed at addressing this paradox are shown. These are black hole complementarity and firewalls.

Keywords: Black Holes, Quantum Entanglement, Information, Entropy

1 Introduction

Quantum mechanics is used to describe the behavior of matter and energy at small scales, typically at the level of atoms and subatomic particles. On the other hand, general relativity (GR) helps to understand the behavior of massive objects in space. Wheeler, Bekestein, and Hawkings were some of the pioneer physicists studying BHs from a quantum mechanics perspective. in this field. Hawkings demonstrated that quantum effects near the event horizon of a BH could lead to the spontaneous emission of particles, causing the BH to gradually lose mass and energy over time. This phenomenon implies that BHs are not completely black but instead emit radiation, eventually leading to their evaporation. His work on BH evaporation raised profound questions about the preservation of information in BH interactions. According to quantum mechanics, information cannot be destroyed, yet the process of BH evaporation seemed to suggest otherwise. This led to the formulation of the BH information paradox, which remains a topic of intense theoretical research in physics.

This paper focuses on the BH information paradox, which involves phenomena from both quantum mechanics and general relativity, especially quantum entanglement. Sections 2 and 3 cover important concepts of quantum mechanics and entanglement that will help to understand further sections. Sections 4 focus on BHs and their thermodynamics, while Section 5 focuses on the BH information paradox. As the paradox is actively researched, no definitive solution exists yet, but two proposed solutions, black hole complementarity, and firewalls, are reviewed in Section 6. Lastly, Section 7 provides a brief overview of the current status of the paradox and commentary on the debated solutions.

2 Concepts of Quantum Mechanics

Quantum mechanics describes the behavior of matter and energy at very small scales, such as atoms and subatomic particles. It provides a theoretical framework for understanding the fundamental principles governing the behavior of particles, including their wave-particle duality, probabilistic nature, and the quantization of physical properties. In this section, basic concepts of quantum mechanics [4] are shown, to later understand the information paradox and the BHs from a quantum mechanics perspective.

2.1 Operators, Unitarity and Complementarity

In quantum mechanics, some mathematical **operators** interact with wave functions in the Hilbert space. Ket and bra vectors can be written as column and row vectors respectively, and operators are commonly represented by matrices, depending on the choice of coordinate basis.

An operator acting on a state ψ is

$$\hat{O}|\psi\rangle = |\hat{O}\psi\rangle = \int \hat{O}(\mathbf{r})\psi(\mathbf{r})d\mathbf{r}$$

The dual of this ket-state $|\hat{O}\psi\rangle$ can be calculated by acting on the bra-state $\langle\psi|$ with the adjoint of the operator

$$\langle\psi|\hat{O}^\dagger = \langle\hat{O}^\dagger\psi| = \int \psi^*(\mathbf{r})\hat{O}^\dagger(\mathbf{r})d\mathbf{r}$$

The operator $\hat{O}$ has the eigenvalue α_n and the corresponding eigenstate $|\psi_n\rangle$, the same way, for the dual vector and adjoint operator the eigenvalue is the complex conjugate of α_n.

$$\hat{O}|\psi\rangle = \alpha_n|\psi\rangle \tag{1}$$

$$\langle\psi|\hat{O}^\dagger = \alpha_n^*\langle\psi| \tag{2}$$

Each observable[1] in physics corresponds to a Hermitian operator. A Hermitian operator is defined as being equal to its adjoint, $\hat{O} = \hat{O}^\dagger$, which gives $\alpha_n^* = \alpha_n$. The expectation value of an observable $\langle\hat{O}\rangle$ for the state $|\psi\rangle$ is $\langle\hat{O}\rangle = \langle\psi|\hat{O}|\psi\rangle$. Which is the average value of the measurement of the observable.

Operators with discrete spectrum[2] of eigenvalues offer a means to represent any wave function $|\psi\rangle$ within the Hilbert space as

$$|\psi\rangle = \sum_{n=1}^{\infty} c_n|\psi_n\rangle \tag{3}$$

Here, $|\psi_n\rangle$ represents the eigenstates of the observable. These eigenstates are orthonormal, denoted by $\langle\psi_n|\psi_m\rangle = \delta_{nm}$, and c_n are the probability amplitudes,

Unitarity says that time evolution operators must be unitary. For closed systems, time evolution operators are inherently unitary. An operator $\hat{U}$ is unitary if $\hat{U}\hat{U}^\dagger = \mathbb{I} = \hat{U}^\dagger\hat{U}$.

The time evolution operator $\hat{U}_t$ describes the future evolution of a quantum state according to

$$|\psi, t\rangle = \hat{U}_t|\psi, t_0\rangle$$

The unitarity of the time evolution operator ensures that the probabilities of all possible measurement outcomes sum up to one, regardless of when the measurement is made. When dealing with the BH information paradox, it's important to apply the principles of unitarity exclusively to closed systems. This ensures that information conservation holds as expected. Another aspect of quantum mechanics, that sets it apart from classical mechanics, is complementarity. **Complementarity** tells us that operators that do not commute cannot possess simultaneous eigenvalues, and thus cannot be in simultaneous eigenstates.

[1] Properties of the system that can be measured

[2] A discrete spectrum describes specific, separate values that certain physical properties, like energy levels in atoms or measurement outcomes, can have.

2.2 Pure and Mixed States

A system is considered **pure** if its description requires only one wave function, represented by a single ket vector $|\Psi\rangle$.

For instance, consider a fermion, which is a particle with spin $\frac{1}{2}$. This particle may spin up, spin down, or be in a superposition of the two. The pure basis states are $|\uparrow\rangle$ and $|\downarrow\rangle$, but even their superposition

$$|\Psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle$$

is still pure. c_1 and c_2 are the probability amplitudes.

A **mixed state**, in contrast, is a state that cannot be described by a single wave function. An ensemble can be in a pure or mixed state. If all the systems have the same quantum state then the ensemble is pure otherwise is mixed. A pure ensemble of particles that all have the same spin can be created by putting the particles through an apparatus that filters out the particles of one spin direction. Mixed ensembles are created all the time in processes where the states of the particles are determined randomly.

2.2.1 Density Matrix

Density matrices represent mixed states, whose coefficients correspond to probabilities of the system being in each pure state. Since mixed states are collections of pure states, pure states are a subset of mixed states. So, the density operator can be used to identify if a state is pure or mixed.

A fraction with relative population[3], W_i, is in the pure state $|\alpha^{(i)}\rangle$. They are expressed in terms of the basis vectors $|n\rangle$ of the Hilbert space

$$|\alpha^{(i)}\rangle = \sum_n c_n^{(i)} |n\rangle \tag{4}$$

where the coefficient can be calculated as $c_n^{(i)} = \langle n|\alpha^{(i)}\rangle \qquad c_{n'}^{(i)*} = \langle \alpha^{(i)}|n'\rangle$

The density matrix of a state will reflect the fraction of particles that are in each pure state in terms of the probabilities W_i. The definition of the density operator ρ is

$$\rho = \sum_{i=1}^{N} |\alpha^{(i)}\rangle W_i \langle \alpha^{(i)}| \tag{5}$$

2.2.2 Expectation Values

As previously mentioned, for a pure state, the expectation value of an operator can be calculated using its wave function.

For the pure state $|\alpha^{(i)}\rangle$ and operator A the expectation value is

$$\langle A\rangle_{\alpha^{(i)}} = \sum_{n'} \sum_n \langle n|\alpha^{(i)}\rangle \langle \alpha^{(i)}|n'\rangle \langle n'|A|n\rangle \tag{6}$$

For a mixed state is different since the probabilities of each wave function need to be taken into account. So the expectation value of A for a mixed state is

$$\langle A\rangle = \sum_n \langle n|\rho A|n\rangle \Longrightarrow \langle A\rangle = Tr(\rho A) \tag{7}$$

[3] How much each possible state contributes to the total population of states in a quantum system

Equation 7 is using the trace operator, $Tr()$. $Tr()$ of a square matrix is the sum of the diagonal elements, which also means that it is the sum of the eigenvalues λ_j of the matrix if the matrix is diagonalizable

$$Tr(\rho) = \sum_{i=1}^{N} \rho_{ii} = \sum_{j} \lambda_j \tag{8}$$

2.3 Quantum Entanglement

Entanglement involves a mutual dependence of properties among multiple particles, regardless of their spatial separation. Imagine a system with two or more particles. These particles are considered entangled if the entire system can only be described by a single wave function that cannot be separated into individual wave functions for each subsystem. For instance, a system of two particles can usually be described by combining two separate wave functions. However, in the case of entanglement, if particles A and B are non-entangled, the wave function $|\Psi\rangle$ of the entire system can be expressed as a tensor product of the two separate pure states $|\psi_A\rangle$ and $|\psi_B\rangle$.

The non-entangle state, also known as product or separate states, $|\Psi\rangle$ is

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$$

and the Hilbert space for the composite system is given by

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

where, $|\psi_A\rangle = \mathcal{H}_A$ and $|\psi_B\rangle = \mathcal{H}_B$

When particles A and B are entangled, the situation changes. Although a single wave function can still be employed to fully describe the system of entangled particles, it can't be split into separate parts for each particle. Any attempt to describe the particles individually would result in an incomplete picture because the properties of one particle depend on those of the other.

3 Concepts of General Relativity

Einstein's GR explains gravity as a consequence of the curvature of spacetime. While highly successful and supported by extensive observations, this theory is also intricate. In this section, some key aspects and tools of GR are essential for our discussion of BHs in Section 4.

Space is conceptualized as a three-dimensional set of points $\{x, y, z\}$, while spacetime encompasses a four-dimensional set $\{t, x, y, z\}$. The distinction lies in the fact that movement is restricted to one direction in time (forward), whereas in spatial dimensions, movement can occur in both directions. But how does this differ from Newtonian physics, which also incorporates time and space? The key distinction lies in their treatment: Newtonian physics treats space and time separately, while GR treats them as unified.

3.1 The Metric

The metric tensor serves as a valuable tool for describing the geometry of spacetime. It encapsulates the necessary information to characterize the curvature of the spacetime manifold.

In general relativity's curved spacetime, the metric tensor is represented by $g_{\mu\nu}$. Several properties and applications of the metric tensor in general relativity include determining the shortest distance between two points, and computing path length and proper time.

Examples of familiar metrics include those used in classical physics and special relativity. In classical physics, operating mostly in flat three-dimensional space, the metric is known as the Euclidean metric represented as $g_{\mu\nu}$. The metric tensor in special relativity is called the Minkowski metric, it describes four-dimensional flat spacetime and is represented as $\eta_{\mu\nu}$. These metrics are defined as:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

or equivalently in Cartesian and Spherical coordinates.

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

The metric can be represented in any coordinate system that covers the entire space under consideration, and the choice of coordinates depends on which system best describes the situation at hand.

It can be observed that the time component bears the opposite sign compared to the spatial components. Therefore, in the four-dimensional spacetime metrics utilized in special and general relativity, one component always possesses the opposite sign to the others. Conversely, in the Euclidean metric, which solely addresses spatial dimensions, all components share the same sign.

3.2 Einstein Equations

Below, the equations and each parameter of the Einstein Equations are introduced. These equations include a set of nonlinear second-order differential equations, yielding a metric tensor $g_{\mu\nu}$ as their solution.

The equations can be expressed in the following two forms

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{11}$$

where the energy-momentum tensor $T_{\mu\nu}$ is the tensor generalization of mass. It describes the distribution of mass/energy in the chosen region of spacetime. Meanwhile, $g_{\mu\nu}$ represents the metric tensor, which substitutes the gravitational potential in Newton's equation. The Ricci tensor[4] $R_{\mu\nu}$ describes the curvature of spacetime and involves the first and second derivatives of the metric.

In vacuum, $R_{\mu\nu} = 0$ as the energy-momentum tensor is zero.

3.2.1 The Schwarzschild Solution

The Schwarzschild metric is a fundamental solution to Einstein's equations of general relativity that describes the spacetime around a spherically symmetric metric. It provides a mathematical description of the geometry of spacetime around such a mass, including the curvature caused by gravity. It describes how spacetime is warped and stretched by the presence of mass.

A spherical symmetric solution to the vacuum, $R_{\mu\nu} = 0$, is the Schwarzschild metric, which is

[4] $R_{\mu\nu}$ and R (Ricci tensor and scalar) are related to the Riemann tensor, $R_{\rho\sigma\mu\nu}$, which is computed from the Christoffel symbols and their first derivatives. The Christoffel symbols are, constructed from the metric and its first derivatives.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{12}$$

in spherical coordinates, where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and G is Newton's gravitational constant.

The Schwarzschild metric describes the curvature of spacetime in the vacuum outside a spherically symmetric object of mass M. As $M \to 0$, it transforms into the Minkowski metric. Similarly, as $r \to \infty$, spacetime flattens.

One implication of the Schwarzschild metric is the existence of what it is now called a Schwarzschild Blac Hole. When the mass in question is compact enough, the curvature of spacetime becomes so severe that not even light can escape from its gravitational pull. This region of space is known as the event horizon, and it marks the boundary beyond which nothing can escape the BH's gravitational grip.

4 Black Holes and their Thermodynamics

BH is a collection of events that an outside observer at a distance $r > 2GM$, can never see. This implies that they are not simply very dense regions of space but rather regions of spacetime.

4.1 Event Horizon

The region in spacetime where $r = 2GM$ is called the **event horizon** of the BH. The definition of an event horizon is that it is a surface where anything that has fallen within it can never escape.

To see why this is the case for BHs, it can be observed what happens with (Equation 12) when $r < 2GM$. For $r > 2GM$ the first and second term in (Equation 12) has a negative/positive sign but for $r < 2GM$ the sign changes and

$$-\left(1 - \frac{2GM}{r}\right) dt^2 > 0$$

and

$$\left(1 - \frac{2GM}{r}\right)^{-1} dr^2 < 0$$

And the final term experienced no change of sign. $r^2 d\Omega^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) > 0$

In $r < 2GM$ the radial component is the component with the negative sign and thus corresponds to the time dimension. In a way the radial direction and the time direction change places. This means that within the event horizon moving forwards in time is the same as moving radially inwards towards the singularity at $r = 0$ However, in this region, it is more accurate to consider the singularity a time rather than a position in space [7]. Therefore, the reason nothing can escape the event horizon is not simply due to an immense pull of gravity but rather a result of the curvature of spacetime inside the BH.

4.2 Information in Black Holes

Now that the basic concepts in quantum mechanics and GR have been reviewed, BHs from a quantum mechanics perspective can be examined. Consider someone falling into a BH. This person is full of information, whether it be the color of his shoes, his position/velocity, or the spins of the protons in his body. However, once he passes the BH's event horizon, this person's information is classically inaccessible to an observer outside the BH. If this is the case, where is the information stored?

Einstein's theory implies that an outside observer will see his friend take infinite time to reach the event horizon, while it takes finite time for our friend to reach the event horizon in his frame! To an outside observer, this infinite approach makes his friend's information appear plastered on the surface of the BH. With the benefit of hindsight, this looks like an instance of the holographic principle. The holographic principle, a property of theories about quantum gravity, claims the information of a given volume is encoded entirely on its boundary region. In the case of BHs, BH's entropy depends only on the surface area of the event horizon. This aside is interesting; the fact that holography shows up here tells us what types of theories physicists use to analyze the BH information paradox.

By explicitly studying how BHs store information via entropy and how that information is radiated away over time, one will inevitably arrive at the BH information paradox.

4.3 Black Hole Entropy

Classically, one might say that a BH could simply 'swallow' infinite amounts of information. And classically, that information is stuck inside the BH forever, as nothing can travel faster than the speed of light. However, Bekenstein showed that BHs are thermal objects, having non-zero temperatures. Bekenstein calculated the entropy (a measure of the number of possible microstates in a system) of BHs. This entropy was a big first step toward the information paradox, as it led to the realization that BHs are indeed thermal objects, having the ability to radiate energy.

The derivation for the entropy of a BH can be done as follows [3]. Let's start with a BH mass M and radius given by

$$R = \frac{2MG}{c^2} \tag{13}$$

Now let's consider filling this BH with 'single bits' [5]. To represent adding only one bit of information to the BH, imaging adding photons of wavelength $\lambda \approx R$

To ensure that no extra information is encoded corresponding to where on the BH the photon was entered, these photons will impart the following energy into the BH.

$$\delta E - \frac{\hbar c}{\lambda} = \frac{\hbar c}{R}$$

Convert this into a small change in mass gained by the BH via $E = mc^2$

$$\delta M = \frac{\hbar}{cR}$$

Now, using Equation 13, the change in BH radius due to the addition of these photons can be expressed as:

$$\delta R = \frac{2G}{c^2}\frac{\hbar}{c^3 R} = \frac{2G\hbar}{c^3 R} \rightarrow R\delta R = \frac{2G\hbar}{c^3} \tag{14}$$

This allows an equation for the change in surface area ($\delta A = 2R\delta R$) due to the addition of one bit of information to the BH. Utilizing some statistical mechanics, the entropy of the BH can now be expressed as:

$$S_{BH} = \frac{k_B c^3 A}{4G\hbar} \tag{15}$$

This tells us that there is a finite measure of information (the Bekenstein-Hawking Entropy) that can fit on a BH of a surface are A.

The Bekenstein-Hawking Entropy [2] tells us how dense information is stored in BHs. Further, this entropy once again points us to the holographic principle: only the surface area of the event horizon affects how

[5]refers to the fundamental unit of information in quantum information theory

much information the BH stores. However, the most important implication for this paper is that BHs are themselves thermodynamic objects. BHs must have a temperature and thus be able to lose energy to their surroundings.

4.4 Hawking Radiation

Using the Bekenstein-Hawking Entropy (Equation 15) one can derive the temperature of a BH to be

$$T = \frac{\hbar c^3}{8\pi k_B G M} \tag{16}$$

This would imply that BHs can radiate energy to their surrounding environment and thus lose mass. In the classical limit, where BHs do not seem to have a temperature, this becomes apparent.

$$\hbar \to 0 \Longrightarrow T \to 0$$

This points to the fact that thermal radiation from black holes is a quantum effect. Hawking discovered the exact cause of this radiation, and, notably, it does not involve anything moving faster than light. Instead, Hawking showed that black hole radiation (named Hawking Radiation) is caused microscopically by particle-antiparticle pair production at the event horizon.

A particle and antiparticle can form in a vacuum in what are called vacuum fluctuations. These virtual particles usually shortly annihilate with each other according to the uncertainty principle $\Delta E \Delta t \geqslant \frac{\hbar}{2}$.

For example, an electron and positron pair that spawn from vacuum fluctuations will only survive approximately 10^{-22}s before annihilating.

Hawking realized that these vacuum fluctuations happen near the black hole event horizon. He found that there was some probability, dependent on the temperature of the black hole, that one of the virtual particles could escape the black hole horizon after creation. If the other virtual particle falls into the BH, then the two never have the chance to annihilate. The BH pays the price of this lost virtual particle's energy, and thus will slowly lose mass over time due to vacuum fluctuations at the event horizon.

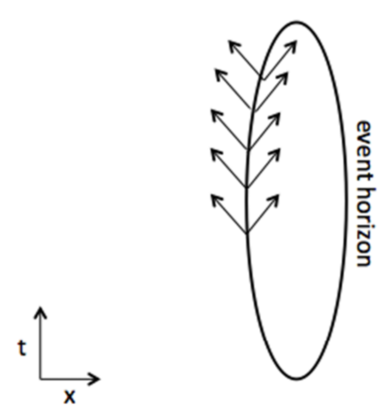

Figure 1: The pairs of arrows represent the particle-antiparticle pairs that are formed from vacuum fluctuations at the event horizon. One particle escapes while the other falls into the event horizon, causing the BH to lose mass. This process repeats over time, and the event horizon will shrink until the black hole has lost all its mass.

A particle-antiparticle pair that results in one instance of Hawking emission can be represented in the following way

$$a(T)|1\rangle_A|1\rangle_B + a(T)|0\rangle_A|0\rangle_B \tag{17}$$

Where $|1\rangle$ represents the presence of a virtual particle and $|0\rangle$ represents the absence of a virtual particle. The A bit is outside the black hole event horizon, while the B bit is inside. These bits are entangled because Alice, an observer outside the black hole, knows that there must be an antiparticle inside the black hole if she measures its corresponding particle on the outside. Although the coefficients $a(T)$ and $b(T)$ are written to suggest that this emission process is necessarily thermal, for our purposes, the state of one instance of Hawking Radiation can be written as:

$$|\psi_{HawkingRadiation}\rangle = \frac{|1,1\rangle + |0,0\rangle}{\sqrt{2}} \tag{18}$$

Now that there is an expression for the state of the Hawking emissions, it can be analyzed in terms of the quantum information tools built up in the previous sections.

5 The Information-Loss Problem

Hawking radiation is central to the BH information paradox, which underscores a conflict between quantum mechanics and GR. Initially proposed by Hawking [6], the concept later led to the assertion of information loss within black holes [5]. This paradox stems from uncertainty surrounding the fate of information entering a black hole, challenging both quantum mechanics and general relativity.

One approach to articulating the paradox involves examining the temporal evolution of entangled particles near a black hole's event horizon. Originating just outside the horizon, an entangled pair of particles arises from the vacuum, constituting a known pure state. One particle (A) falls into the black hole, while the other (B) escapes as Hawking radiation. Assuming information loss upon A's entry into the black hole, its destruction implies the loss of the pure entangled state. Consequently, only particle B remains, its state becoming mixed—a consequence to be demonstrated through subsequent calculations.

They entangled particles to be in the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\right)$$

which is a pure state.
The system of the entangled particles can be described by the reduced density matrix ρ_{AB}

$$\rho_{AB} = |\psi\rangle\langle\psi| \tag{19}$$

ρ_{AB} is then calculated

$$\rho_{AB} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Diagonalizing ρ_{AB} allows verification that it is a pure state. This can be achieved by calculating its eigenvalues λ_i. with the determinant $det(\rho_{AB} - \lambda\mathcal{I})$. The eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = 1$ giving the matrix ρ^D_{AB}

$$\rho^D_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

From ρ^D_{AB}, confirmation arises that the state is pure, as the only non-zero element appears as a one on the diagonal. This also means that the von Neumann entropy is zero, which it should be for a pure state.

Using the von Neumann entropy gives

$$S(\rho^D_{AB}) = -(ln(1)) = 0$$

Individually, A and B should be mixed states. This can be confirmed using $S(\rho_A)$ and $S(\rho_B) > 0$, or by diagonalizing the matrices ρ_A and ρ_B. To find ρ_A, the partial trace over B in the equation for ρ_{AB} is taken (using $Tr(|\psi\rangle\langle\phi|) = \langle\phi|\psi\rangle$).

$$\rho_A = Tr_B(\rho_{AB}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

ρ_A is diagonal. Calculating ρ_B reveals that $\rho_A = \rho_B$. Subsequently, $S(\rho_A) = S(\rho_B)$ is computed.

$$S(\rho_A) = -Tr(\rho_A ln(\rho_A)) = -\left(\frac{1}{2}ln\left(\frac{1}{2}\right) + \frac{1}{2}ln\left(\frac{1}{2}\right)\right) = ln(2) \approx 0.693 > 0$$

Since $S(\rho_A) = S(\rho_B) > 0$ both A and B are mixed states. Therefore B, if measured alone, is indeed a mixed state.

Over time the BH evaporates completely and all that is left is the Hawking radiation, which is an ensemble of particles in the mixed state B. The pure state of A and B has evolved into a mixed state. This contradicts the unitary time evolution of quantum mechanics. According to unitary evolution, a pure state will not evolve into a mixed state.

way of stating the information paradox is based on the assumption that information is forever lost if it enters the BH. Thus a pure state is turned into a mixed state when the BH disappears. The conclusion is that this violates the unitary time evolution of quantum mechanics.

The assumption is changed that information is lost inside a BH, instead assuming that infalling information is re-emitted in the Hawking radiation. A consequence of this change is a violation of GR. The central concept here is the equivalence principle. A consequence of the equivalence principle is that an observer in free fall crosses the event horizon, of a sufficiently large BH.

6 Possible Solutions: Complementarity or Firewalls

When dealing with BHs and potential solutions to the information paradox, one often encounters energies at the Planck scale or even higher. At these energy levels, the known laws of physics appear to break down. This is one reason why resolving the information paradox remains an active and somewhat speculative field of research.

Now, let's explore some of the proposed solutions to the information paradox. There are three main alternatives as to what happens to matter thrown into a BH. The first alternative is that the information is lost (destroyed at the singularity) which means that some do not even consider the information paradox a paradox and simply accept that information seems to be lost inside a black hole [10]. The second and third alternatives are that the information escapes with the Hawinkg radiation or the information is stored in remnants.

In the following sections, the option two will be discussed. The first solution that will be discussed is called black hole complementarity [9]. After that, firewalls will be discussed, as the idea of a firewall came from a critique against black hole complementarity [1].

6.1 Complementarity

Black hole complementarity (BHC) solves the information paradox by suggesting that BHs have a surface membrane, a so-called stretched horizon [9], that absorbs and re-emits the infalling information. This membrane only exists from the perspective of an outside observer far away from the black hole. BHC uses a complementary approach to explaining the contradictory experiences of different observers. To explain the viewpoint of BHC, the following discussion is based on the postulates used in [9].

Postulate 1 implies that information is not lost in the BH but instead emitted as Hawking radiation. Postulate 2, also known as effective field theory, states that there is nothing unexpected about the physics outside the stretched horizon of the black hole. The outside can be described by semiclassical gravity. The third postulate emphasizes that to a distant observer, a black hole appears to be a normal quantum mechanical system with a finite number of states. Apart from these postulates, BHC includes a crucial assumption, which is that an observer in free fall crossing the event horizon does not experience anything strange, according to the equivalence principle stated in Section 6.2. The key point of BHC is that it claims that even though the perspectives of several observers may differ when looking at one perspective at a time no physical laws are violated. The concept is usually explained by studying the perspectives of two observers. Bob, one of the observers, is far away from the BH and he is observing Alice, who is in free fall and about to enter the black hole (See Figure 2).

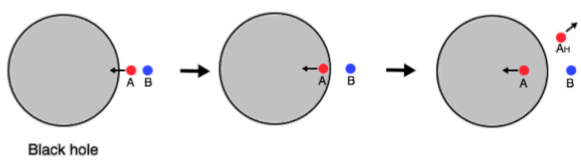

Figure 2: Particles A and B are entangled. A enters the black hole and B stays outside. Hawking radiation A_H is after some time sent out due to A entering the black hole. B is entangled to both A and A_H. Alice on the side the black hole measures A and B about the side the black hole measures both B and A_H. Bob then jumps to get the data from Alice, thus collecting data on A, B, and A_H which leads to him discovering in a violation of quantum mechanics.

Alice, crossing a large black hole's event horizon, experiences no noticeable effects due to the equivalence principle, being in free fall and facing negligible tidal forces. From her perspective, no radiation or information can escape the black hole without violating causality. On the contrary, Bob, observing from outside, sees Alice slow down and redshift as she approaches the horizon. He interprets this as Alice being absorbed by a membrane of infalling matter with high temperature, giving them vastly different experiences of the event horizon: Alice passes it unnoticed, while Bob sees her vaporized by the surrounding membrane.

Following postulate 1, proponents of black hole complementarity (BHC) believe that information falling into a black hole is later emitted as thermal radiation. This resolves the unitarity problem but introduces challenges such as the violation of the no-cloning theorem, as Alice and her Hawking radiation copy both exist. BHC suggests that both Alice's and Bob's perspectives are valid and complementary: Alice's information is conserved within the black hole, while Bob observes the emitted radiation. Bob, unable to observe beyond the event horizon, cannot verify Alice's survival or access any information she sends due to redshift. Alice and Bob could observe entangled particles, with Alice measuring one entering the black hole and Bob measuring the emitted Hawking radiation. If Bob enters the black hole after measuring the radiation, he could receive data from Alice, resulting in two copies of the same information, violating the no-cloning theorem.

Bob should also find a violation of the monogamy of entanglement since both Alice's particle and the Hawking radiation should be entangled with Bob's particle outside the black hole. However, as explained below, according to [8] this experiment would only work if, again, energies of the Planck scale or beyond were used to transmit the data.

Bob would have to wait a long time outside the black hole to collect data from the infalling particle. Postulate 1 states that a black hole's entire life can be described using the unitary time evolution operator, meaning the system of the black hole and emitted radiation always remains in a pure state, despite individual Hawking radiation particles being in mixed states due to entanglement with partner particles in the black hole. As the black hole emits radiation, it becomes entangled with earlier emitted radiation, starting at the Page time when half its initial entropy is emitted. For an observer to gather information thrown into the black hole, they must detect both early and later emitted radiation, requiring collecting the complete system of entangled particles. However, due to the long lifetime of a black hole, Bob doesn't have enough time to collect data from both outside and inside the black hole, unless Alice sends it using super-Planckian frequencies. Alice's data also has a limited lifetime as she and her data approach the singularity where they will be destroyed. Describing events inside and outside the black hole without using Planck-scale physics is physically impossible, similar to the impossibility of simultaneously knowing the position and momentum of a particle. The data from measurements inside and outside the black hole can be seen as complementary, with different observers finding the information at different events in spacetime. This introduces more uncertainty into physics, sacrificing the independence of when and where events occur from observers' perspectives, akin to the uncertainty introduced by the wave-particle duality of matter and the Heisenberg uncertainty principle.

6.2 Firewalls

A relatively recent contribution to the discussion of the information paradox is the concept of firewalls. In an article published in 2012, Almheiri, Marolf, Polchinski, and Sully (AMPS) present the firewall as a potential solution to the information paradox, as they claim to have found inconsistencies in BHC. The firewall solution suggests that around a black hole, there is a surface of very energetic quanta which will destroy anything that tries to get past the event horizon. AMPS argues for the existence of this firewall mainly based on their claim that BHC is incomplete, which they show via a thought experiment. Here, the argument in [1] is going to be examined.

Assuming that postulate 2 and the equivalence principle are still valid, a contradiction is found that BHC cannot explain. Consider a BH past its Page time and a pair of entangled Hawking particles A and B (See Figure 3). As per usual, A enters the black hole while B escapes to infinity. As B is part of the later radiation, it is entangled with particle C from the early radiation, see Figure 6. However, B cannot be entangled with both A and C according to the monogamy of entanglement.

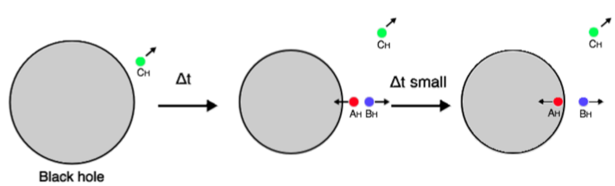

Figure 3: Particle C_H is part of the early radiation emitted from a black hole. Later, after the Page time, the entangled particles A_H and $B_H$are created. A_H enters the black hole while B_H escapes. As B_H is part of the late radiation it is entangled to C_H.

AMPS argues that a single observer can measure all three particles, violating the monogamy of entanglement. To avoid this, they suggest that an infalling observer who finds two particles entangled cannot find the third particle entangled. Instead, they propose the existence of a firewall consisting of high-energy particles behind the event horizon, breaking entanglement between particles. The exact nature and formation of the firewall are unclear, but it's proposed to form right behind the event horizon, independent of the observer. This contrasts with the stretched horizon of BHC, which depends on the observer. While controversial due to its violation of the equivalence principle, AMPS sees it as the least radical solution, preserving unitarity in quantum mechanics.

7 Conclusion

The information paradox involves key concepts from both quantum mechanics and general relativity, such as entanglement, unitarity, black holes, the equivalence principle, and Hawking radiation. Despite being proposed over 40 years ago, it remains unsolved and has led to various hypotheses and ideas. Hawking initially suggested information loss in black hole evaporation, but later favored unitarity. Others, like Susskind, propose solutions involving concepts like wormholes. Both these solutions have faced criticism and controversy. BHC claims to preserve unitarity but introduces new uncertainty by challenging the independence of when and where events occur. Firewalls propose a violation of the equivalence principle but lack a precise formation process. There are other proposed solutions, but consensus remains elusive. Physicists anticipate that resolving the paradox will require new insights into quantum gravity and possibly novel physics.

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