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The Ridge-Seeking Method for Obtaining the Skeleton of Digital Images

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Abstract—**A method for obtaining the skeleton of a segmented gray-scale image is introduced. The gray-scale information is used in this method to produce a connected and one-point wide skeleton positioned in the ridge areas.**

I. Introduction

The transformation of gray-scale images of elongated objects to their skeletons is a problem of great importance in pattern recognition. Skeletons retain the structural information and topological properties and can easily be used for feature extraction. The major requirements of a skeleton are connectivity, thinness, and its position [1].

There has been extensive research done in the area of skeletonization of elongated objects in black and white (binary) pictures [1]–[4]. The skeleton is obtained from the boundary of the binary picture. However, for many applications such as chromosomes, handwritten characters, and machine parts, the boundary is not well defined. The density is high in the ridge areas of the objects, and gradually diminishes outward, producing an image with a noisy boundary.

A few methods for the skeletonization of digital gray-scale pictures were proposed earlier. Levi and Montanari [5] obtain the gray weighted distance transform (GWDT) of the image by associating to each point the length of the minimal path from that point to the background. The skeleton points are those that do not belong to the minimal path of any other points to the background. This method requires the digital picture to be segmented into zeros (background) and nonzeros (objects). The skeleton resulting from this method does not guarantee that the connectivity, thinness, and position requirements will be satisfied.

Another approach called the min–max medial axis transformation (MMMAT) is based on the generalization of shrinking and expanding operations for gray-scale images [6]. Local minimum and local maximum operations in a gray-level picture correspond to shrinking and expanding operations of the binary picture, respectively [7]. The MMMAT algorithm does not require the picture to be segmented into objects and background; however, there is no assurance of obtaining a thin and connected skeleton. Also the number of iterations of the shrinking and expanding operations depends on the application.

Dyer and Rosenfeld [10] developed a thinning algorithm for gray-scale images that is a generalization of standard thinning alorithms for binary images. In their algorithm two points P and Q are connected if and only if the gray value of any element along the path that connects them is not less than both of them. This method defines the boundary point in a gray-scale image according to the gray-level values of its neighborhood and thins the image in a parallel mode, by substituting the boundary elements with its minimum neighbor if doing so does not disconnect its neighborhood according to the connectivity definition given above. In this technique the local neighborhood connectivity does not provide global connectivity for the skeleton and the connectedness of the skeleton is not guaranteed. Moreover, the skeleton does not lie along the high gray values, but rather it will be positioned in a central place determined by the boundary of the images.

In this correspondence, a ridge-seeking skeletonization method for gray-scale images is proposed. The input image is considered to be a set of connected nonzero elements as object areas and zero elements as background areas. Although this method requires that the elements of gray pictures with gray values below a certain threshold be set to zero, there is no commitment to a particular threshold value, i.e., the obtained skeleton does not depend on the boundary of the image. The gray value of each point in an image represents the degree of its membership. Using the gray value information, the method seeks the ridge areas to position the skeleton. The main characteristics of this algorithm are 1) the obtained skeleton of an object is positioned in the ridge (high gray value) areas and is independent of its undefined boundary; 2) the connectivity of the original image is preserved; and 3) the skeleton has a single element thickness.

N_4	N_3	N_2
N_5	N_0	N_1
N_6	N_7	N_8

Fig. 1. Point N_0 and its eight neighbors.

II. The Ridge Seeking Method

The ridge seeking method for obtaining the skeleton of gray-scale images consists of two phases: the contextual gray distance transformation (CGDT) phase and the thinning phase. The following definitions are used in the thinning phase.

Definition 1: A point N_0 in a 3×3 window, shown in Fig. 1, is a boundary point if its gray value is nonzero and at least one of its 4-neighbors has the value zero (note that in Fig. 1, points N_1, N_3, N_5, N_7 are called the 4-neighbors of point N_0 and points $N_1, N_2, N_3, \cdots, N_8$ are called the 8-neighbors of N_0).

Definition 2: A point N_0 is an end point if its gray value is nonzero and it has only one 4-neighbor with a nonzero value, moreover the only nonzero 4-neighbor of N_0 should have exactly two nonzero neighbors (an end point is the tip of a line).

Definition 3: A point N_0 is a local maximum point if its gray value is strictly greater than all of its 8-neighbors.

Definition 4: Two points P and Q of the object are 4-connected if and only if there exists a 4-path $(P_0, P_1, P_2, \cdots, P_n)$, with $P = P_0$ and $Q = P_n$, such that the gray value of all elements P_i, $0 \leqq i \leqq n$, are nonzero (a 4-path is a sequence of points $P_0, P_1, P_2, \cdots, P_n$, with P_i a 4-neighbor of P_{i-1}, $1 \leqq i \leqq n$). Note that this definition relates to our assumption of the set of nonzero elements as object areas and zero elements as background areas.

Definition 5: The strength of a 4-path $A = (P_0, P_1, P_2, \cdots, P_n)$, $S(A)$, is defined as the minimum gray value of all points P_i's, $1 \leqq i \leqq n-1$. A 4-path A is stronger than a 4-path B if $S(A) > S(B)$.

Definition 6: The strength of the 4-connectedness of two points is the strength of the strongest 4-path connecting them.

Note that for a point N_0 and its eight neighbors in Fig. 1, for any pair of N_1, N_3, N_5, N_7 there are exactly two 4-paths (alternate

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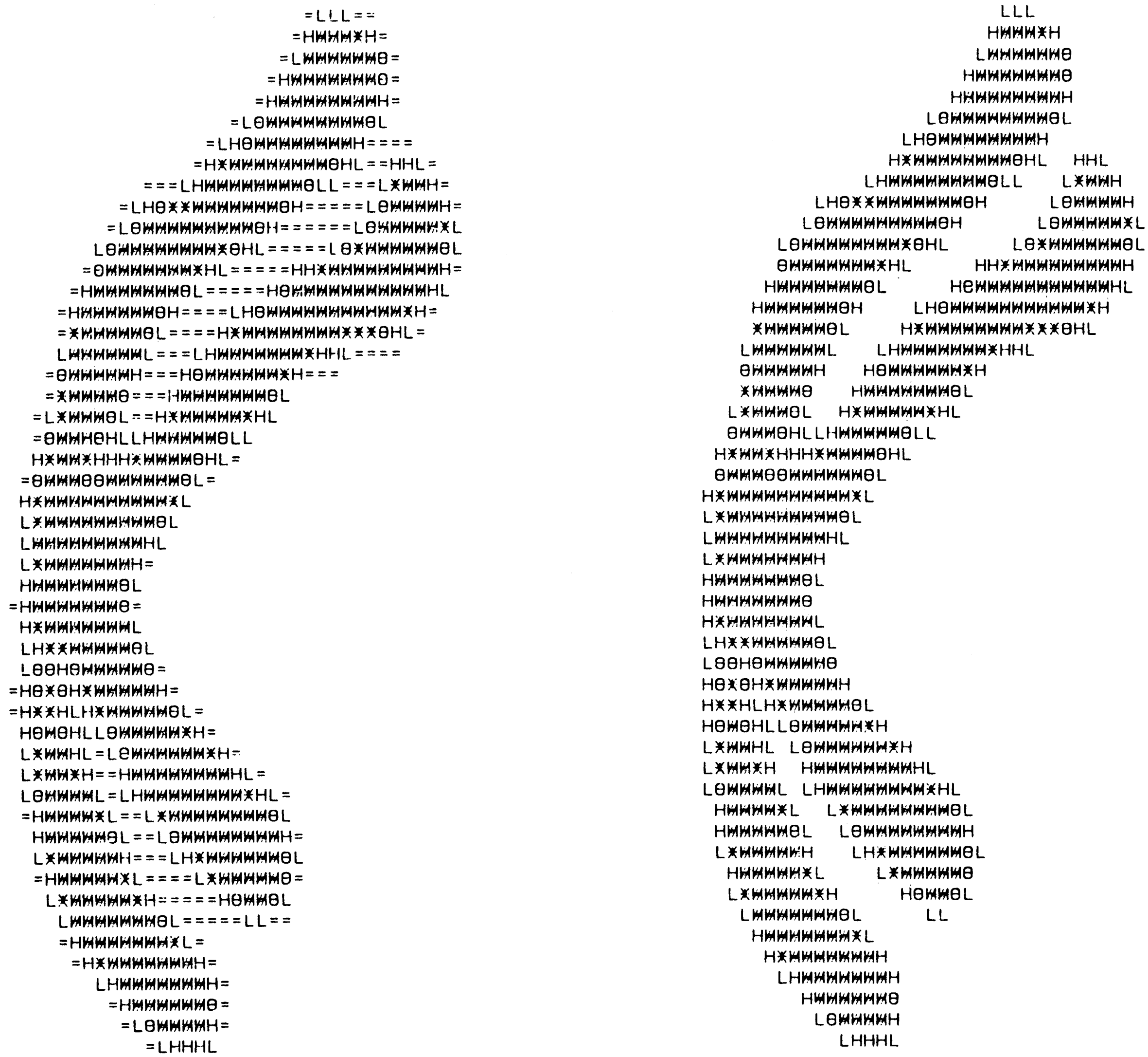


(a) (b)

Fig. 2. Chromosome image with no thresholding. (b) Chromosome image thresholded by 1. (c) The skeleton obtained from Fig. 2(a). (d) The skeleton obtained from Fig. 2(b).

paths) that connect them that do not include point N_0, for example for pair N_1N_3 the alternate paths are (N_1, N_2, N_3), $(N_1, N_8, N_7, N_6, N_5, N_4, N_3)$.

A. The Contextual Gray Distance Transform (CGDT)

In the first phase of this method we propose a CGDT of the original gray-scale image where the context as well as the position of a picture element play an important role in determining the distance of that element to the background. For the gray-scale image $f(i,j)$ defined in the domain $1 \leq i \leq M$, $1 \leq j \leq N$, assuming the frame of the picture consists of zero values, the CGDT (i,j) is defined as, $\mathrm{CGDT}(i,j) = \min[g_1(i,j), g_2(i,j)]$, where $g_1(i,j) = \min[g_1(i-1,j-1), g_1(i-1,j), g_1(i-1,j+1), g_1(i,j-1)]*(\mathrm{ave}/\mathrm{max})^2 + f(i,j)$ will be calculated in the forward raster sequence for $2 \leq i \leq M-1$, $2 \leq j \leq N-1$, and $g_2(i,j) = \min[g_2(i+1,j+1), g_2(i+1,j), g_2(i+1,j-1), g_2(i,j+1)]*(\mathrm{ave}/\mathrm{max})^2 + f(i,j)$ will be calculated in the reverse raster sequence for all $2 \leq i \leq M-1$, $2 \leq j \leq N-1$.

In the definition of functions g_1 and g_2, all elements in the first row and last row, as well as in the first column and last column assume the value zero, and these values will be used for the recursive definition of g_1 and g_2. Also, in the above functions, $\mathrm{ave} = [f(i,j-1) + f(i,j+1) + f(i-1,j) + f(i+1,j)]/4$, and max is the gray value that constitutes the bulk of the image, which can normally be determined from the classes of images to be processed. In a precise definition max is the gray value corresponding to the highest peak of the histogram of the digital image. However, the algorithm is not so sensitive to the choices of such a gray value, and in our implementation of algorithm, this value is known *a priori*, and no histogram construction is required.

The objectives of this transformation are first to construct a central ridge in the high gray value areas of the original gray-scale image. Secondly, since the value of the function CGDT at point (i,j) depends on the neighborhood of (i,j), then the transformation also does a smoothing operation.

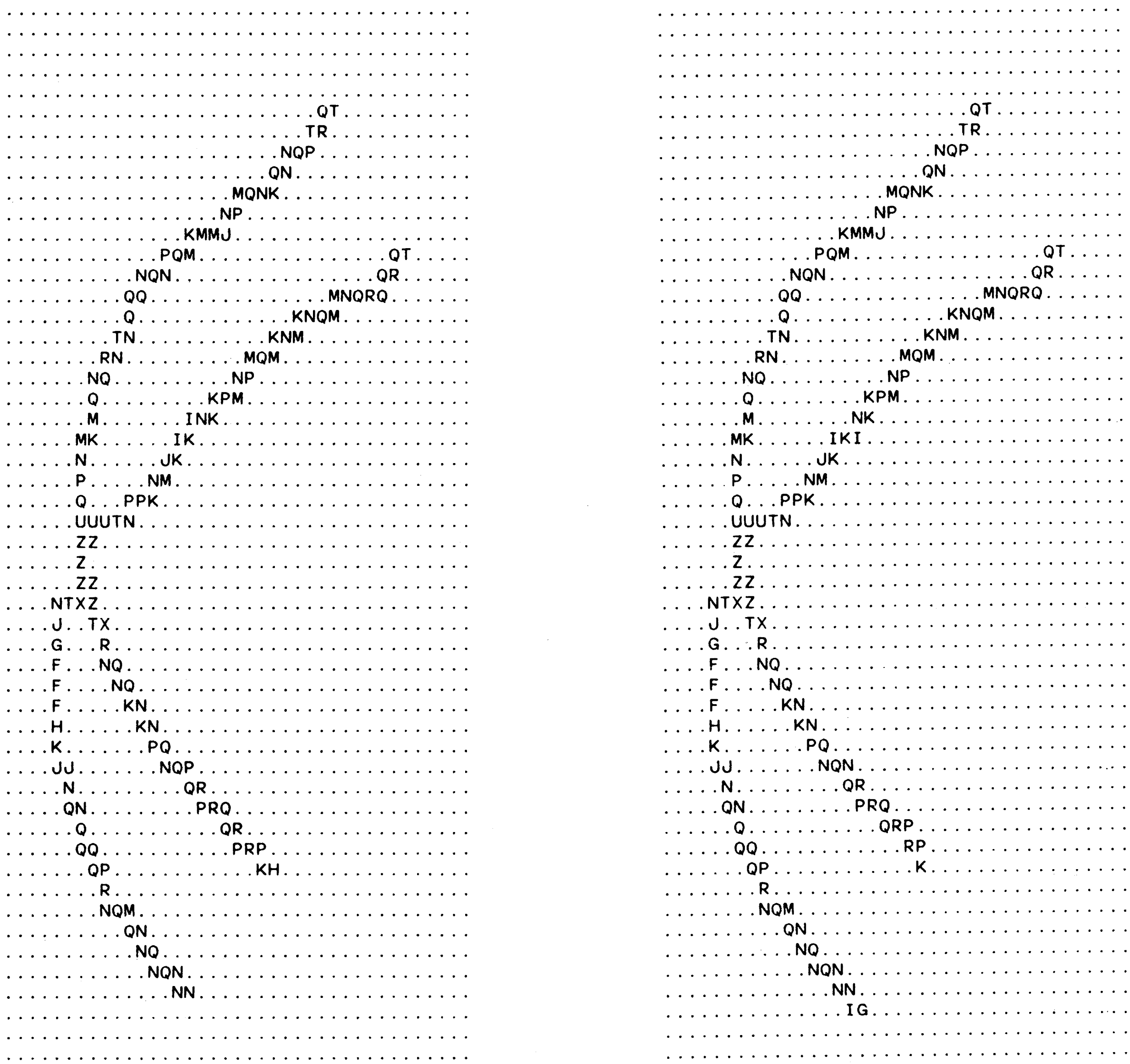
B. Thinning

The thinning process requires several passes as we scan the CGDT image in the forward raster sequence. During each pass, the point N_0 will be removed (changed to zero) if the following conditions are satisfied:

1) N_0 is a boundary point.

2) N_0 is not a local maximum point. A local maximum point in which its gray value is strictly greater than all of its neighbors values, is expected to be a skeleton point. Such a consideration will help to speed up the process of obtaining the skeleton.

3) N_0 is not an end point.



(c) (d)

Fig. 2. (*Continued*).

4) N_0's removal does not change the connectedness of its neighbors.

5) N_0's removal does not weaken the strength of the 4-connectedness of its neighbors. This is called the "ridge seeking criterion," and satisfaction of this criterion requires that N_0 not be on the strongest path connecting any pair of N_1, N_3, N_5, N_7.

This process will continue until there is no point removed during an entire pass.

The above conditions 1) through 3) are trivial, however, conditions 4) and 5) need some elaboration. Designate the pairs $N_1N_3, N_3N_5, N_5N_7, N_1N_7$ as corner pairs and N_1N_5, N_3N_7 as cross pairs, and also assume $V(N_I)$ represents the gray value at point N_I. We will show that conditions 4) and 5) are satisfied if the logical equation (1) holds.

$$C(1) \cdot \text{and} \cdot C(2) \cdot \text{and} \cdot C(3) \cdot \text{and} \cdot C(4) \cdot \text{and} \cdot L(1) \cdot \text{and} \cdot L(2) = \cdot \text{true} \cdot . \quad (1)$$

where

$$\underset{I \in \{1,2,3,4\}}{C(I)} = \begin{cases} \cdot\text{true}\cdot & \text{if either } \min\left[V(N_{2I-1}), \quad V(N_{2I+1}|\text{mod } 8)\right] = 0 \\ & \text{or } \min\left[V(N_{2I-1}), \quad V(N_{2I+1}|\text{mod } 8)\right] > 0 \\ & \text{and } V(N_{2I}) \geqq V(N_0) \\ \cdot\text{false}\cdot & \text{otherwise} \end{cases}$$

and

$$\underset{K \in \{1,2\}}{L(K)} = \begin{cases} \cdot\text{false}\cdot & \text{if for the path } (N_{2K-1}, N_0, N_{2K+3}), \quad V(N_{2K+1}) \\ & \text{and } V(N_{2K+5}|\text{mod } 8) \text{ are both less than } V(N_0) \\ & \text{when } \min\left[V(N_{2K-1}), \quad V(N_{2K+3})\right] > 0 \\ \cdot\text{true}\cdot & \text{otherwise} \end{cases}$$

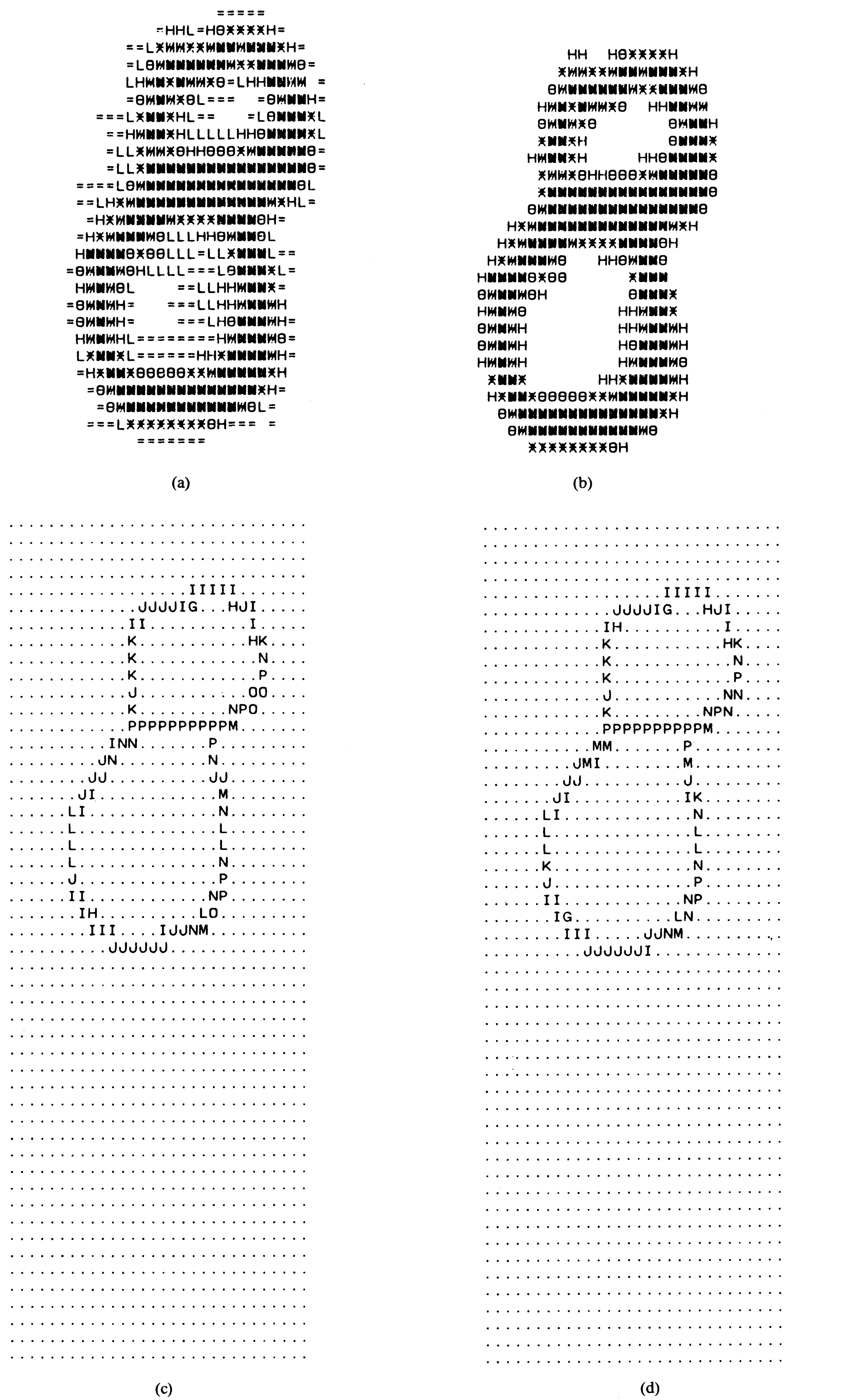


Fig. 3. (a) A handwritten character 8 with fuzzy boundary. (b) Character 8 thresholded by 2. (c) The skeleton obtained from Fig. 3(a). (d) The skeleton obtained from Fig. 3(b).

The logical variables $C(1)$, $C(2)$, $C(3)$, $C(4)$ and $L(1)$, $L(2)$ correspond to the connectivity of the four corner pairs $N_1N_3, N_3N_5, N_5N_7, N_7N_1$ and the connectivity of two cross pairs N_1N_5 and N_3N_7, respectively. Note that the constraint for $C(I)$ being true plays an important role and it leads to having a strong connection between elements that are not yet removed, otherwise $\min[V(N_{2I-1}), V(N_{2I+1}|\mathrm{mod}\, 8)] = 0$ implies that one of the end points of path $(N_{2I-1}, N_{2I}, N_{2I+1})$ are equal to zero (i.e., it belongs to the background) and we are unconcerned about a strong connection between the background and object.

Lemma 1: $C(I) = \cdot\mathrm{true}\cdot$ implies that the removal of boundary point N_0 does not affect the connectivity of the Ith corner and satisfies the ridge seeking criterion.

Proof: Let us consider corner pair N_1N_3 and $C(1) = \cdot\mathrm{true}\cdot$ as a condition for the connectedness of N_1 and N_3. The proof for the connectedness of other corner pairs is similar. $\mathrm{Min}[V(N_1), V(N_3)] = 0$ implies there is no connected path between N_1 and N_3, hence the removal of N_0 does not affect the connectedness. However, when $\min[V(N_1), V(N_3)] > 0$, then removal of N_0 requires that at least one of the two alternate paths (N_1, N_2, N_3) or $(N_3, N_4, N_5, N_6, N_7, N_8, N_1)$ be connected. For $\min[V(N_1), V(N_3)] > 0$, either $V(N_5)$ or $V(N_7)$ must be zero (because we consider only boundary points for removal), which in either case implies that the alternate path $(N_3, N_4, N_5, N_6, N_7, N_8, N_1)$ is disconnected, and thus $V(N_2) \neq 0$ is sufficient for existence of a connected path between N_1 and N_3 after removal of N_0. Note that, in this case, $C(1) = \cdot\mathrm{true}\cdot$ implies $V(N_2) \geqq V(N_0) \neq 0$, which satisfies the ridge seeking criterion, condition 5).

Lemma 2: $L(K) = \cdot\mathrm{true}\cdot$ and each $C(I) = \cdot\mathrm{true}\cdot$ implies that removal of N_0 does not affect the connectivity of the Kth cross pair elements and satisfies the ridge seeking criterion.

Proof: For any cross pair, say N_3N_7, the removal of N_0 does not disconnect the pair N_3N_7 if at least one of the alternate paths i.e., $(N_3, N_4, N_5, N_6, N_7)$ or $(N_3, N_2, N_1, N_8, N_7)$ is connected. Since N_0 is a boundary point and $V(N_3), V(N_7)$ are both nonzero, then at least one of the two values $V(N_1)$ or $V(N_5)$ is equal to zero, consequently one of the alternate paths is disconnected. Assuming $V(N_5) = 0$, then $V(N_1) \neq 0$ is necessary for the existence of the alternate path $(N_3, N_2, N_1, N_8, N_7)$. Now to prove that $V(N_1) \neq 0$ is sufficient, it is clear that for $V(N_3)$, $V(N_1)$, and $V(N_7)$ all being nonzero, $V(N_2)$ and $V(N_8)$ are required to be greater than or equal to $V(N_0)$ for $C(1)$ and $C(4)$ to be true, as a result, the path $(N_3, N_2, N_1, N_8, N_7)$ exists. Moreover, in this case $L(2) = \cdot\mathrm{true}\cdot$ implies $V(N_1) \geqq V(N_0) \neq 0$, which also satisfies the ridge seeking criterion—condition 5).

Theorem: The resulting skeleton of a connected object is also connected.

Proof: Lemmas 1 and 2 provide local connectivity. As a result, the connectivity of the obtained skeleton is preserved.

III. Discussion

The ridge seeking method for obtaining the skeleton operates in a sequential mode. To obtain a single point wide and connected skeleton, during each pass of the thinning phase as soon as the removal of a point is certain (all conditions of the thinning phase are satisfied for the point under consideration), the evaluation of $C(I)$ and $L(K)$ of the neighboring points to be sequentially analyzed must take this new value zero into consideration.

Fig. 2 shows the chromosome images and their skeletons. In Fig. 2(a) the arms of the chromosome are touching, and the resulting skeletons from Fig. 2(a) (with no thresholding) and Fig. 2(b) (thresholded by 1) are shown in Figs. 2(c) and 2(d), respectively. The obtained skeletons are almost precisely the same. The imprecision is due to the CGDT, where it produces almost precisely the same transformed image (CGDT image) for both unthresholded and thresholded images. It should be noted that sometimes an extra point may appear in the branch point area of the skeleton, simply because this extra point was a local maximum and was not removed. These extra points can easily be removed by one more pass. Figs 3(a) and 3(b) show a handwritten character 8 with a fuzzy boundary and the same character after fuzz removal. Application of the algorithm to Figs. 3(a) and 3(b) produce corresponding skeletons shown in Figs. 3(c) and 3(d).

Note also, the resulting skeletons are 4-connected. The 8-connectivity case was considered, however, the process is computationally unattractive for the case of gray-scale images. An 8-connected skeleton can easily be obtained from a 4-connected one.

The technique developed in this correspondence deals with obtaining a connected skeleton of a one-point wide width and located in the ridge areas from a segmented digital image. Although it is required that the input image be segmented into zeros and nonzeros, the obtained skeleton does not depend on the choice of a particular threshold value, and it will be positioned in the ridge areas of the image.

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A Learning Algorithm for the Finite-Time Two-Armed Bandit Problem

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***Abstract*—A simple algorithm for the finite-time two-armed bandit problem is proposed. In this algorithm, the whole process is divided into the first estimating process and the next controlling process. Efficient methods by using approximation for computing the optimal length of the estimating process are provided.**

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