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Under-Sampled Phase Retrieval of Single Interference Fringe Based on Hilbert Transform

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ABSTRACT Phase retrieval from single interference fringe is important and effective method in obtaining the real phase distribution. The original phase can be retrieved by the line integral of its gradient expressed as sine and cosine components, which were gained by the Hilbert transform twice from a single interference fringe pattern. However, this method fails when the phase transformation of the interference fringe is too fast. In this paper, a novel method to recover the continuous phase of the whole field is proposed to solve the above problems. The shear interference technique is introduced into the phase retrieval method to build an exponential 2-D complex light field of natural base for the phase slope obtained by the Hilbert transform. Then, the expressions of phase slopes in x- and y-directions are constructed as a discrete Poisson equation. Therefore, the calculation of phase retrieval is equivalent to solve the discrete Poisson equation mathematically. Finally, the real phase is gotten by the weighted discrete cosine transform (WDCT) of the discrete Poisson equation. The simulation results verify the validity of this method and show that the proposed method can achieve the phase retrieval of the phase discontinuity in x- and y-directions, which leads to the under-sampled problem. It can restore the whole field phase distribution rapidly and accurately. Moreover, this method is applied to phase retrieval of interferometric synthetic aperture radar (InSAR) with the under-sampled problem in this paper. The experimental results show that this method can recover the phase of InSAR with the under-sampled problems caused by terrain abrupt change and so on. Compared with other commonly used methods, it achieved satisfactory results. This method provides a new idea for solving the under-sampled problem in the phase retrieval from a single-frame interference fringe.

INDEX TERMS Optical interferometry, phase retrieval, under-sampled, single interference fringe, Hilbert transform, shear interference.

I. INTRODUCTION

Optical interferometry can be used to measure the surface morphology, deformation, displacement, which was characterized by high accuracy, high sensitivity, non-contact and non-destructive [1]–[3]. However, in order to obtain the measured physical quantity we must restore the phase information from the interference fringes accurately, because the retrieved phase is wrapped within a range from $-\pi$ to π, i.e., modulo 2π. This process, which unwraps the wrapped phase to construct a continuous phase distribution, is called phase unwrapping [4], [5].

There are many phase unwrapping algorithms have been proposed. In terms of the working domains, these algorithms can be classified into two principal groups: temporal phase unwrapping and spatial phase unwrapping [5], [6].

Temporal phase unwrapping algorithms are robust and effective, but require multiframes of wrapped phase along the time axis, or multifrequency fringe patterns. On the other hand, spatial phase unwrapping algorithms have fewer restrictions, but these algorithms tend to fail in phase discontinuities and disjoint regions [6], [7].

Among various spatial phase unwrapping algorithms, they can be classified as phase shift method (requires multiple fringe patterns) [7]–[9] and single fringe retrieval methods [9]–[12].

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Phase shift method can remove the noise and has high measurement accuracy, but it requires at least three interference fringes to extract the phase. Comparing to the high externally conditional requirements and the large error influence of phase shift method, the single fringe phase retrieval methods, which requires only one interference fringe pattern to extract the phase and can also be used for transient measurements, is faster and more convenient.

At present, the proposed single fringe retrieval methods can be broadly divided into two categories. One is single non-carrier fringe pattern method, such as regularized phase tracking method, Regularization method, phase retrieval based on statistical filtering, polynomial parameter method based on genetic algorithm, two-dimensional nonlinear regression method, etc. These methods use iterative method to obtain the phase and require relatively high signal-to-noise ratio of the fringe pattern. If the noise is large, the final result cannot converge to the actual value. Also the image processing always takes a long time.

The second kind is the phase extraction method based on spatial carrier fringe pattern. These methods generally introduce carriers into the interference field, such as Fourier Transform Method, Wavelet Transform, Discrete Hilbert Transform, etc.

In 1974, J. Morlet firstly proposed Wavelet Analysis Theory [13], [14], a new transformation analysis method which has strong ability of resolving detail. However this method is essentially a window adjustable Fourier Transform. Its wavelet window of the signal must be smooth, otherwise the wrong solution will be produced in the frequency band after wavelet transform and the regional integration. It can't primarily get rid of the limitation of Fourier Transform.

In 1989, Takeda et al proposed a spatial carrier method based on Fourier Transform [15], [16]. Only one fringe pattern is used to finish phase information recovery, which is more suitable for dynamic measurement. However, when there are more mutations in the spatial distribution of the interferogram, its spectrum will be widened accordingly, which generates serious phase distortion of the filtered result.

In 2001, Wong extracted the phase information on the basis of once Hilbert transform [17]–[19]. After that, literature [20] put forward a method based on twice Hilbert Transform. In the method, the sinusoidal and cosine components of the interference fringes are first obtained from a single interference fringe, and then the phase slope can be gotten [21]–[23]. Along the horizontal direction (x-direction) and the vertical direction (y-direction) the phase slope is integrated and the original phase distribution can be finally obtained. Need not phase unwrapping, the method is convenient and can get rid of the phase error caused by the unpacking operation. However, this method can only deal with the continuous phase slope distribution. When the phase change is too fast, the phase and slope distributions in the horizontal and vertical directions are discontinuous. There will be obvious under-sampled problems [24]–[26], therefore the phase cannot be directly recovered by integration and the above method fails.

In order to solve the above problems of under-sampled with phase discontinuity generated from Hilbert transform in single interference fringe phase retrieval methods [27], a new method was put forward in this paper. The shear interference principle was used to build a two-dimensional compound optical field firstly. Then the phase slope expressions in x and y directions are constructed as discrete Poisson equation. Therefore the phase retrieval is equivalent to solve the discrete Poisson equation mathematically. Finally the weighted discrete cosine transform (WDCT) is used to retrieve the real phase.

Experiments show this method can avoid the under-sampled problem that caused by the discontinuity of phase slopes of x direction and y direction. The original phase can be recovered fast and accurately by this method.

In the practical application, the interference pattern generated by InSAR technology is similar to the situation [28]–[32] in this paper. Because of the sudden occurrence of clutter, surface subsidence and debris flow in the actual terrain, the under-sampled problems of local interference fringes often appear. The proposed method of this paper is applied to the phase retrieval method of InSAR interferogram with under-sampled problem and good results are obtained.

The structure of this paper is as follows. We present a comprehensive survey on phase unwrapping algorithms, especially on single fringe phase retrieval methods in section I. We start by the theoretical analysis in section II, which includes the principle of phase retrieval from single interference fringe pattern based on Hilbert Transform and the proposed method in this paper. This is followed by experiments and discussion in section III to verify and discuss the performance of the method proposed in this article. In section IV, a practical application with the proposed method in phase retrieval of InSAR interferogram was taken. Section V concludes the paper.

II. THEORETICAL ANALYSIS

A. THE PRINCIPLE OF PHASE RETRIEVAL FROM SINGLE INTERFERENCE FRINGE PATTERN BASED ON HILBERT TRANSFORM

By using Hilbert transform twice, the sine and cosine components of the interference fringes are obtained from a single interference fringe. The tangent of phase change between adjacent pixel points, which is also described as phase slope, can be gotten through the sine and cosine components from the arctangent transformation. When the starting point of the phase integral is selected, the phase slope can be integrated along the horizontal direction (x-direction) and the vertical direction (y-direction). Therefore the original phase distribution can be finally obtained.

Before further discussion, a brief definition and analysis of the fringe patterns are given, which will be used throughout the paper. A fringe pattern can be generally expressed as follows:

$$I(x, y) = a(x, y) + b(x, y)\cos[2\pi f_0 x + \phi(x, y)] \tag{1}$$

where $a(x, y)$ is background intensity, $b(x, y)$ and $\phi(x, y)$ are fringe amplitude and phase distribution respectively, and f_0 is the angular frequency of carrier fringe. Generally $a(x, y)$ and $b(x, y)$ can be regarded as constants because of changing slowly. The sine and cosine components are gained by Hilbert transform from a single carrier frequency interference fringe pattern. $I_1(x, y)$ is the sine component and can be obtained by Hilbert transform of $I(x, y)$:

$$I_1(x, y) = b(x, y)\sin[2\pi f_0 x + \phi(x, y)] \tag{2}$$

As we can see, the direct-current component of the interference pattern is filtered after Hilbert transform. Also, $I_2(x, y)$ is a cosine component, obtained by Hilbert transform of $I_1(x, y)$ shown as (3):

$$I_2(x, y) = -b(x, y)\cos[2\pi f_0 x + \phi(x, y)] \tag{3}$$

where $\theta(x, y) = 2\pi f_0 x + \phi(x, y)$ we can get the phase gradient in x direction:

$$\begin{aligned}\Delta\theta_x(x, y) &= \theta(x+1, y) - \theta(x, y) = \arctan\frac{\sin(\Delta\theta_x)}{\cos(\Delta\theta_x)}\\ &= \arctan\frac{I_2(x+1, y)I_1(x, y) - I_1(x+1, y)I_2(x, y)}{I_2(x+1, y)I_2(x, y) + I_1(x+1, y)I_1(x, y)}\end{aligned} \tag{4}$$

Similarly, the phase gradient in y direction is:

$$\begin{aligned}\Delta\theta_{\mathrm{y}}(x, y) &= \theta(x, y+1) - \theta(x, y) = \arctan\frac{\sin(\Delta\theta_y)}{\cos(\Delta\theta_y)}\\ &= \arctan\frac{I_2(x, y+1)I_1(x, y) - I_1(x, y+1)I_2(x, y)}{I_2(x, y+1)I_2(x, y) + I_1(x, y+1)I_1(x, y)}\end{aligned} \tag{5}$$

The primary phase distribution can be retrieved by the line integral of its gradients in x and y directions obtained by the (4) and (5) as follows:

$$\theta(x, y) = \int_{c(x_0, y_0)}^{c(x,y)} \Delta\theta_x(x, y)dx + \Delta\theta_y(x, y)dy + \theta(x_0, y_0) \tag{6}$$

Equation (6) expresses the line integral from point (x_0, y_0) to (x, y) along line C in any path. However, it must be ensured that $\Delta\theta_{\mathrm{x}}(x, y)$ and $\Delta\theta_{\mathrm{y}}(x, y)$ are continuous everywhere.

B. THE PRINCIPLE OF THE PROPOSED ALGORITHM

We can recover the original phase using the integral method by (6) when the gradients are continuous. However, when the phase gradient is discontinuous, the above method will fail and can't be applied to the phase retrieval.

In order to deal with this problem, we use shear interference technique to construct a discrete Poisson equation. And the phase recovery can be equivalent as calculating the solution of Poisson equation in Mathematics. From (4) and (5) it can be deduced that:

$$\begin{aligned}\phi(x, y+1) + \phi(x-1, y) + \phi(x, y-1)\\ + \phi(x+1, y) - 4\phi(x, y) = \rho(x, y)\end{aligned} \tag{7}$$

where,

$$\begin{aligned}\rho(x, y) = \Delta\theta_x(x, y) - \Delta\theta_x(x-1, y)\\ + \Delta\theta_y(x, y) - \Delta\theta_y(x, y-1)\end{aligned} \tag{8}$$

Clearly, (7) is a discrete Poisson Equation. However, if the phase changes too fast, $\Delta\theta_x(x, y) - \Delta\theta_x(x-1, y)$ is not continuous as calculated by $\Delta\theta_{\mathrm{x}}(x, y)$. For this reason, we built a two-dimensional light field in the complex domain based on the principle of shearing interference as follows:

$$V_1 = \exp[j\Delta\theta_x(x-1, y)] \tag{9}$$

where $j = \sqrt{-1}$. We make a move in x direction for 1 pixel and build a new optical field:

$$V_2 = \exp[j\Delta\theta_x(x, y)] \tag{10}$$

When V_2 is divided by V_1, we can get

$$K^x(x, y) = \frac{V_2}{V_1} = \frac{\exp[j\Delta\theta_x(x, y)]}{\exp[j\Delta\theta_x(x-1, y)]} \tag{11}$$

Therefore, we can obtain

$$\Delta\theta_x(x, y) - \Delta\theta_x(x-1, y) = \arctan[\frac{\mathrm{Im}(K^x(x, y))}{\mathrm{Re}(K^x(x, y))}] \tag{12}$$

In (10), the value of V_2 is identical regardless of whether $\Delta\theta_x(x, y)$ is continuous or not. According to Euler's theorem, we know the index, which is natural base e, can be turned into a complex forms of sine and cosine functions. So it eliminates the influence of its discontinuity by this way. Additionally, although the gradient of original phase changes rapidly, the change rate is low. Therefore, the twice discrete derivative of original phase $(\Delta\theta_x(x, y) - \Delta\theta_x(x-1, y))$ is continuous, whose value keeps between $-\pi$ and π. Similarly, the continuous distribution in the y direction is as follows:

$$\Delta\theta_y(x, y) - \Delta\theta_y(x, y-1) = \arctan[\frac{\mathrm{Im}(K^y(x, y))}{\mathrm{Re}(K^y(x, y))}] \tag{13}$$

Therefore, the (8) is amended as:

$$\rho(x, y) = \arctan\left[\frac{\mathrm{Im}(k^x(x, y))}{\mathrm{Re}(k^x(x, y))}\right] + \arctan\left[\frac{\mathrm{Im}(k^y(x, y))}{\mathrm{Re}(k^y(x, y))}\right] \tag{14}$$

We bring $\rho(x, y)$ into (7), then the phase retrieval is transformed into the process of calculating Poisson equation by discrete cosine transform.

The existence of noise and shadow in some areas of the interferogram may cause phase errors, which will lead to the phase retrieved inaccuracy or even completely wrong. Taking into account the effects of noise, the $\rho(x, y)$ can be corrected by the weighting factor to prevent the spread of errors and compensate its smoothing effect. The weighted is defined as:

$$\begin{cases} p(x, y) = \Delta\theta_x(x, y) - \Delta\theta_x(x-1, y)\\ q(x, y) = \Delta\theta_y(x, y) - \Delta\theta_y(x, y-1)\\ u(x, y) = unit(filt[p(x, y)^2 + q(x, y)^2]) \end{cases} \tag{15}$$

In (15), $unit(.)$ is normalization processing, $filt[.]$ represents mean filter, $u(x, y)$ is weight, we can obtain the $\rho^*(x, y)$ after the weighting treatment:

$$\rho^*(x, y) = [1 + u(x, y)]\rho(x, y) \tag{16}$$

Then, by calculating Poisson equation we can obtain the true phase.

III. EXPERIMENTS AND DISCUSSIONS

In order to verify the performance of the method proposed in this article, we carry out a series experiments to testify its effectiveness.

A. RESULTS OF THE PHASE RETRIEVAL AND THE COMPARISON

The intensity of the interference fringe pattern can be generated by MATLAB expressed as (17):

$$I(x, y) = 0.5 + 0.5\cos[2\pi f_0 x + \phi(x, y)] \tag{17}$$

where $\phi(x, y) = 2{*}peaks(400)$ and f_0 is the angular frequency of the carrier fringe, whose values is 0.125. The results and the comparison of phase retrieval from single interference fringe pattern by the line integral of its gradients and the proposed method were showed in Fig. 1.

Fig. 1(a) represents the original phase and its interference fringe pattern is Fig. 1(b). Fig. 1(c) and (d) are the gradients of the original phase in horizontal and vertical directions respectively. We can find the gradients are continuous because of slowly changing of the phase. The phase can be retrieved by the integral of phase gradient shown in Fig. 1(e). Fig. 1(f) is the result of retrieved phase by the proposed method of this paper. The phase distribution in a single column (y = 200 pixel) obtained by the integral algorithm and the proposed method represented by line segment and point line respectively are showed in Fig. 1(g). It can be seen the retrieved phase by the two methods are both consistent with the original phase. They can recover the phase well when the phase changes slowly.

When the phase changes too fast, the phase gradients are discontinuous. This will bring some difficulties to the phase retrieval by the original algorithm. Therefore, this paper introduces the shearing interference technology and proposes a novel method. We obtain a carrier frequency interference fringe pattern by Matlab, with the intensity expressed by (17), and $\phi(x, y) = 100 * peaks(400)$ to verify its validity experimentally. The results and a comparison of the two methods are shown in Fig. 2.

Fig. 2(a) is the original phase which changes quickly. Its interference fringe pattern is Fig. 2(b). The phase gradients in horizontal and vertical directions are $\Delta\theta_x(x, y)$ and $\Delta\theta_y(x, y)$ shown in Figs. 2(c) and (d). It can be seen the original phase cannot be obtained by the integral method because of the discontinuous gradients. The result of phase retrieval by the original method is failed as shown in Fig. 2(g). However, by the proposed algorithm, which introduced the shearing interference, $\Delta\theta_x(x, y)$ and $\Delta\theta_y(x, y)$ are transformed into

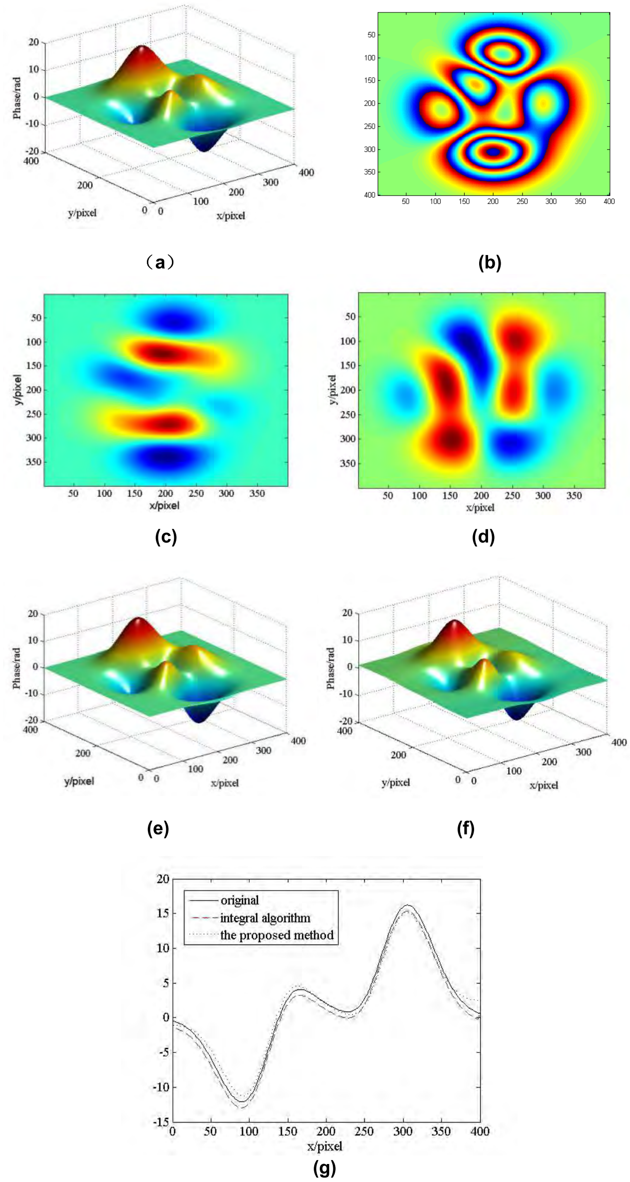

FIGURE 1. Comparison of phase retrieval results between the integral method and the proposed method when phase changes slowly. (a) Original phase. (b) Interference fringe pattern. (c) $\Delta\theta_x(x, y)$. (d) $\Delta\theta_y(x, y)$. (e) Phase retrieval result by the integral method. (f) Phase retrieval result by the proposed method.(g) The Comparison in x axis direction.

sine and cosine functions through building 2-D complex optical field. Therefore the discontinuous points are eliminated according to Euler's theorem, $\Delta\theta_x(x, y) - \Delta\theta_x(x - 1, y)$ and $\Delta\theta_y(x, y) - \Delta\theta_y(x, y - 1)$ are also continuous, as shown in Figs. 2(e) and (f). We can obtain the reconstructed phase by calculating the Poisson equation using weight discrete cosine transform shown in Fig. 2(h). The distribution of the retrieved phase in a single column (y = 200 pixel) by the integral algorithm and by the improved algorithm are showed in Fig. 2(i). Clearly the original method fails completely, while the point line represents the retrieved phase by the

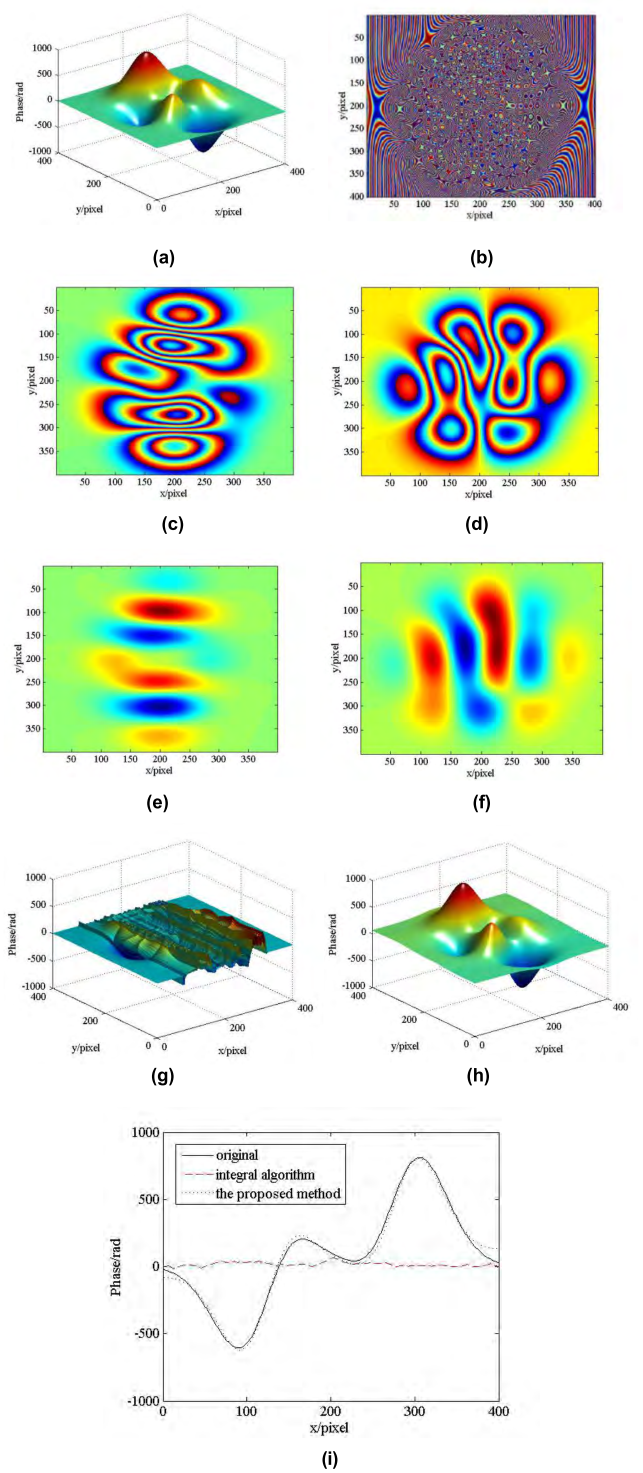

FIGURE 2. Comparison of the phase retrieval results between the two methods as phase changes quickly. (a) Original phase. (b) Interference fringe pattern. (c) $\Delta\theta_x(x, y)$. (d) $\Delta\theta_y(x, y)$. (e) $\Delta\theta_x(x, y) - \Delta\theta_x(x-1, y)$. (f) $\Delta\theta_y(x, y) - \Delta\theta_y(x, y-1)$. (g) Recovered phase by the integral method. (h) Recovered phase by the proposed method.(i) The Comparison in x axis direction between the two methods.

proposed method agrees well with the original. The experiments indicate that the under-sampled phase retrieval from single interference fringe pattern based on Hilbert transform have different results by the original method and the proposed method. The original method based on Hilbert transform, which retrieved the original phase by the integral of phase gradient of x and y direction, cannot get the real phase because of the discontinuous gradients. However the proposed algorithm can solve the above under-sampled problem caused by the discontinuity of phase gradient. It is more suitable for phase retrieval when the phase changes rapidly.

B. EXPERIMENTS OF THE PHASE RETRIEVAL WITH NOISE

In order to take into account the effects of noise, we carried out some experiments for an interference fringe pattern with noise. The result of phase retrieval by the proposed algorithm is shown in Fig. 3.

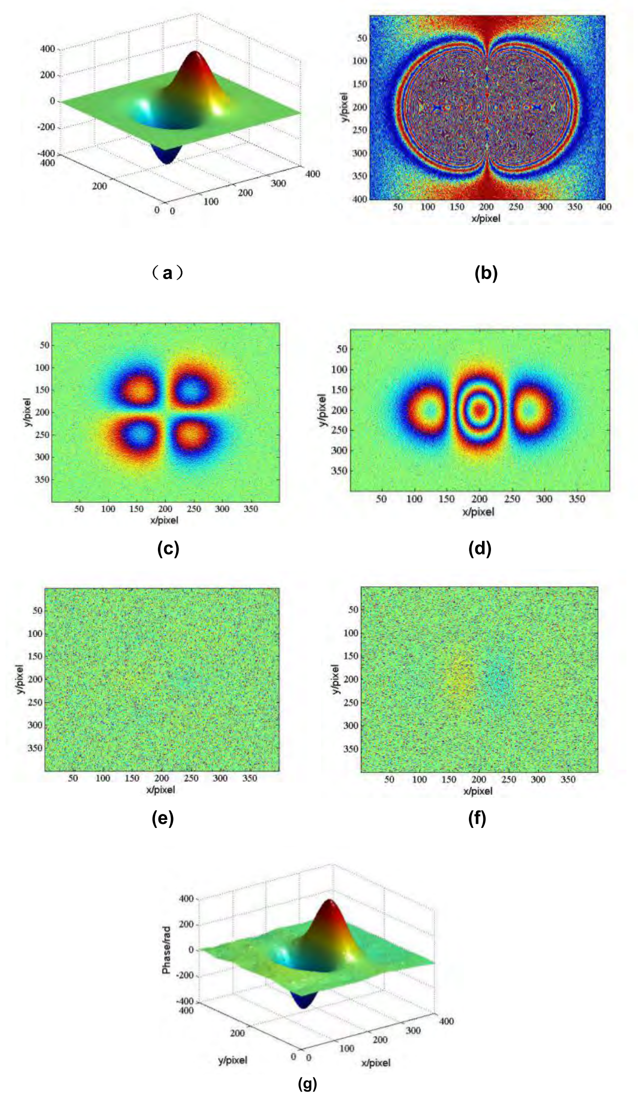

FIGURE 3. Phase retrieved results with noise by the proposed method. (a) Original phase. (b) Interference fringe pattern with noise. (c) $\Delta\theta_x(x, y)$. (d) $\Delta\theta_y(x, y)$. (e) $\Delta\theta_x(x, y) - \Delta\theta_x(x-1, y)$. (f) $\Delta\theta_y(x, y) - \Delta\theta_y(x, y-1)$. (g) Phase retrieval by the proposed method.

Fig. 3(a) is the original phase. Its interference fringe pattern with noise is showed in Fig. 3(b). The phase gradients which are discontinuous in horizontal and vertical directions are represented in Figs. 3(c) and (d). Obviously, there are many discontinuous breakpoints and the original integral method cannot retrieve the original phase. After dealing with the proposed method of this paper, the breakpoints are eliminated and the continuous discrete derivative of phase gradient can be obtained as shown in the Fig. 3(e) and Fig. 3(f). Finally, 3D phase retrieval result of the proposed method is shown in Fig. 3(g). It can be seen the retrieved phase is agree well with the origin phase by this method.

IV. PHASE RETRIEVAL RESULTS OF THE INSAR INTERFEROGRAM

Terrain mutations are frequently encountered in InSAR measurement techniques. Therefore the under-sampled problems of local interference fringes often appear and the phase retrieval methods of InSAR interferogram is similar to the situation studied in this paper. In order to verify the reliability of this algorithm further, the actual InSAR data of 400∗400 generated by the antenna signal of a mountain which was collected by ESA (European Space Agency) were analyzed and the results of the experiments were shown in Fig. 4. We proceeded the experiments between our method and some commonly used methods, including branch-cut phase unwrapping method [33], quality map phase unwrapping method[34], least square phase unwrapping method [35], and three times integral method (proposed by us in [36]). The comparison and analysis were given after the experiments.

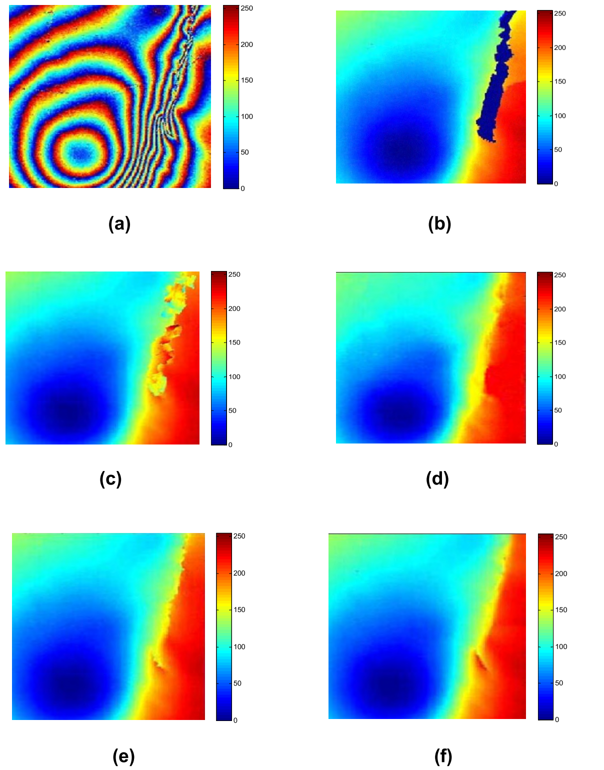

FIGURE 4. The results of some classic algorithms and the proposed method. (a) Original wrapped phase. (b) The result of the branch-cut phase unwrapping method. (c) The result of the quality map phase unwrapping method. (d) The result of the least square phase unwrapping method. (e) The result of three times integral phase retrieval method. (f) The result of the proposed method in this paper.

Fig. 4(a) is the original wrapped phase, it can be seen the interference fringe is clear overall. However, there is a under-sampled area in the upper right part of the interferogram due to the terrain mutation, which makes the phase unwrapping difficult and results in the failure of the branch-cut phase unwrapping algorithm and quality map phase unwrapping algorithm, as shown in Fig. 4(b) and (c). Because there are no suitable way to deal with the under-sample area in the above two method. In general, the unwrapping results are relatively smooth of the least squares phase unwrapping method by using gradient values defining the weight. However, the local phase information is lost and the phase unwrapping results at the dense fringes are less reliable as shown in Fig. 4(d). By using the shear technique into the phase unwrapping algorithm in this paper, the problem that the phase gradient is too large to lead the under-sampled is solved. It can be seen in Fig. 4(e) and (f), the quality of phase unwrapping has been greatly improved. In Fig. 4(e) we used the three times integral of partial derivative of the slope, which was proposed by us before in [36]. A detailed analysis of the operation efficiency and the quality of these phase unwrapping algorithms will be presented in the next part.

It is difficult to distinguish the results are good or bad from the unwrapped images merely. We use the relevant data to further demonstrate the effectiveness of the proposed method. The operation efficiency and the unwrapping quality of the different phase unwrapping algorithms are often analyzed from the aspects of value ε, running time, discontinuous point number, and the unwrapping phase root mean square σ.

The definition of the value ε is as (18):

$$\varepsilon = \frac{1}{MN}\sum_{i=0}^{M-2}\sum_{j=0}^{N-1}\omega_{i,j}^{x}\left|\Phi_{i+1,j}-\Phi_{i,j}-\Delta_{i,j}^{x}\right|^{P} + \frac{1}{MN}\sum_{i=0}^{M-1}\sum_{j=0}^{N-2}\omega_{i,j}^{y}\left|\Phi_{i,j+1}-\Phi_{i,j}-\Delta_{i,j}^{y}\right|^{P} \tag{18}$$

where M and N are the line number and the column number, $\Phi_{i,j}$ is the unwrapping phase in the point (i, j) and $\omega_{i,j}^{x}$, $\omega_{i,j}^{y}$ are the weights of the wrapped phase gradient $\Delta_{i,j}^{x}$, $\Delta_{i,j}^{y}$.

The definition of unwrapping phase root mean square σ (rad) is as (19):

$$\sigma = \sqrt{\frac{\sum_{m=0}^{i-1}\sum_{n=0}^{j-1}(\psi_{m,n}-\Phi_{m,n}^{w})^{2}}{i\times j}} \tag{19}$$

where i and j are the line number and the column number of the phase, $\psi_{m,n}$ is the original wrapped phase, and $\Phi_{m,n}^{w}$ is the unwrapped phase.

The value of ε reflects the fluctuation of the unwrapped phase difference and is inversely related to the phase unwrapping quality. Under the premise of ensuring the quality of the unwrapping, the shorter the run time is, the better the phase unwrapping algorithm is.

The anti-distortion ability of the phase unwrapping algorithm is determined by the proportion of the discontinuous points. For the better anti-distortion algorithm, the non-continuous points of the algorithm are also relatively smaller. The unwrapping phase root mean square analyzes the effect of the unwrapping algorithm in suppressing the error transfer. The results of the detailed analysis of the five methods are shown in Table 1.

TABLE 1. Analysis for phase unwrapping results.

phase unwrapping algorithm	ε	run time /s	discontinuous points	unwrapping phase root mean square σ(rad)
branch-cut phase unwrapping	0.1972	3.70	1.48%	5.0297
quality map phase unwrapping	0.1034	30.39	2.04%	3.1776
Least square phase unwrapping	0.0346	4.25	0.03%	0.9821
three times integral method	0.0009	3.75	0.25%	0.3797
the proposed method	0.0103	6.21	0.13%	0.2851

The value of ε reflects the fluctuation of the unwrapped phase difference. By analysis of Table 1, the ε of the branch-cut phase unwrapping algorithm and the quality map phase unwrapping algorithm is larger. This is because the failure of the branch tangents pass through the under-sampled area to lead to the local phase is not processed and the phase unwrapping result of quality map phase unwrapping algorithm is poor, which leads to greater volatility compared with the original wrapped phase. In terms of operation time, the integral time of branch cutting method is the shortest. But it is the result of the omission of processing partial wrapped phase. The quality map phase unwrapping algorithm takes the longest time. Because of introducing the shearing interference technology, the operation time of the proposed method is a little longer than the least square phase unwrapping method.

From the point of discontinuity, the phase unwrapping results are bad of the branch-cut phase unwrapping algorithm and the quality map phase unwrapping algorithm, because the pixel resolution is difficult to satisfy the large phase change rate of the local phase fringe density region. Although the discontinuity points in the recovered phase are almost zero, the region with fast phase change is completely smoothed out because of the too strong smoothness of the Least squares phase method. By studying the unwrapping phase root mean square σ, it can be seen the proposed method has the best ability of suppressing the error transfer.

In summary, the first three methods are failed in under-sampled phase retrieval, while the last two have the better results in under-sampled phase retrieval compared with the common used methods. The proposed one is the best in under-sampled phase retrieval, especially in terms of discontinuous points and unwrapping phase root mean square. Although the running time is a little longer than the three times integral method, it is still fast enough to retrieve the real phase in practical applications.

V. CONCLUSIONS

In this paper, an under-sampled phase retrieval method of single interference fringe is proposed based on Hilbert transformation. The two-dimensional complex light field is constructed according to the principle of shear interference. The phase slope expression of x direction and y direction is constructed as a discrete Poisson equation. The phase retrieval is equivalent to mathematically solve the discrete Poisson equation. This method solves the problem of under-sampled when the phase slope is discontinuous in the phase retrieval of the single interference fringes by using Hilbert transform and avoids the influence of rapid phase slope transformation in the horizontal direction and the vertical direction. It has been verified by experiments that this method can quickly and accurately restore full phase distribution. Also compared with commonly used methods in practical applications, the validity and practicability of our method are verified.

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