Minimal Number of Monochromatic Edges in Bicolored Graphs

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Abstract

The max-cut problem as part of extremal combinatorics was formulated by Paul Erdős in the 1960s ([6]). Multiple papers have been written on the topic regarding its role in mathematics as an extremal combinatorics problem and from an algorithmic perspective. This problem has been previously analyzed for various types of graphs([4], [8], [9]), such as complete and H-free graphs for specific graphs H([3], [7]).

In this project, we provide several bounds on the minimal number of edges in the maximum cut of various graphs. Sharp bounds are derived for the vertex-neighboring graph of a three-dimensional grid using a computer-aided double counting argument, and based on our results, we formulate a conjecture for the optimal coloring of these graphs. A general bound in terms of the chromatic number of the graph is obtained using the probabilistic method. This result is used to prove two bounds on the size of the maximal cut for planar graphs in terms of the number of its vertices and the number of its edges, respectively.

1 Introduction

Some graphs can be split into two parts (two colors), so that the ends of each edge are in different colors. Such graphs are called bipartite. However, not all graphs are bipartite. Consequently, in the field of graph theory, the function $\chi(G)$ is defined as the minimal number of colors in which we need to color the graph G, such that all edges are between differently colored vertices. A natural question that arises is what is the minimal number of monochromatic edges in bicolored graphs. This is the famous maximum cut problem which is known to be NP - complete. However, sharp bounds can be found for some specific graphs(tabular, planar). A graph whose vertices are the cells of an n-dimensional grid and has edges between vertex-neighboring cells is called tabular throughout this project.

Similarly to Erdős et al. [1] we define $N(G, r)$ to be the maximal number of complete monochromatic graphs K_r that are pairwise disjoint. In this project, we analyze bounds on $N(G, 2)$ for tabular and non-tabular graphs.

Notations:

1. Big $\mathcal{O}$ notation: A function $f(x) \in \mathcal{O}(g(x))$ if there exist $c, x_0 > 0$, such that $f(x) \leq c \cdot g(x)$ for all $x \geq x_0$.

2. Big Ω notation: A function $f(x) \in \Omega(g(x))$ if there exist $c, x_0 > 0$, such that $f(x) \geq c \cdot g(x)$ for all $x \geq x_0$.

3. Θ notation: A function $f(x) \in \Theta(g(x))$ if there exist $c_1, c_2, x_0 > 0$, such that $c_1 \cdot g(x) \geq f(x) \geq c_2 \cdot g(x)$ for all $x \geq x_0$.

2 Tabular graphs

We consider the problem for a specific family of graphs, which is obtained by taking the unit cells of a $k \times n \times m$ lattice ($k \leq n \leq m$) as vertices and having edges between cells with a common vertex. We call them tabular graphs. For this construction, we define a striped coloring to be a partition of the grid into $1 \times n \times m$ planes colored by alternating the two colors. The two-dimensional case of the problem ($k = 1$) was proposed by Alexander Ivanov to a Bulgarian olympiad. The proof can be found here [2].

Lemma 1. *For a $2 \times 2 \times m$ grid the striped coloring is optimal. Furthermore,*

$$N(G_{2\times2\times m}, 2) = 6m.$$

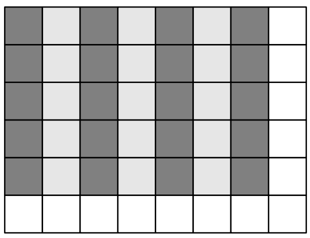
Proof. We will use induction on m. For $m = 2$ all cells have a common vertex, if there are k white and ℓ black cells then the number of edges between monochromatic cells is $\binom{k}{2} + \binom{\ell}{2} \geq 2\binom{4}{2} = 12$. The result follows from Jensen's inequality. Suppose that the inductive hypothesis is true for a given $m \geq 2$. For a $2 \times 2 \times (m+1)$ lattice we know that in the first m $2 \times 2 \times 1$ layers the minimum number of edges between monochromatic vertices is achieved with the striped coloring. It is $6m$. It follows from the case $m = 2$ that in the last two $2 \times 2 \times 1$ layers there are at least 12 edges between monochromatic vertices, and those in the second to last layer are counted twice. Let their number be t i.e. the number of edges between monochromatic vertices is at least $6m + 12 - t$, but we have $t \leq \binom{4}{2} = 6$, which implies $N(G_{2\times2\times(m+1)}, 2) \geq 6(m+1)$. Equality is achieved when the first m layers are colored with the striped coloring, and the last two layers have 4 unit cells of each color, i.e. when the whole lattice is striped. $\square$

Lemma 2. *For a $2 \times n \times m$ grid the striped coloring is optimal. Furthermore,*

$$N(G_{2\times n\times m}, 2) = 5nm - 4m.$$

Proof. We proceed by induction with an inductive step from $2 \times n \times m$ to $2 \times (n+1) \times (m+1)$. See Figure 1.

Figure 1: Inductive step.



Consider the lattice as an $n \times m$ table of $1 \times 1 \times 2$ parallelepipeds, which we call dominoes. There are three types of dominoes - containing two white cubes, two black cubes, and one white and one black cube, which we call white, black, and mixed dominoes, respectively. The white and black dominoes contribute one edge each to the total. Additionally, if two dominoes have a vertex in common, we can measure their mutual contribution to the total number of edges by the following rule:

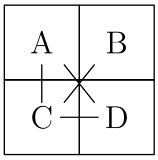
	white	black	mixed
white	4	0	2
black	0	4	2
mixed	2	2	2

By the induction hypothesis, the upper left $n \times m$ rectangle has at least $5nm - 4m$ edges. This is achieved when we color with stripes.

Remark 1. *In the bottom right 2×2 square of dominoes there are at least 12 edges from Lemma 1, and at most one of them is already counted in the bottom left $n \times m$ rectangle, i.e. we have at least 11 edges that we have not yet counted.*

In order to bound the remaining edges, we prove the following sublemma:

Lemma 2.1. *Suppose that we have a 2×2 square of dominoes, which we will denote by A, B, C, D, as shown below. Then the number of edges that come from the adjacencies of the pairs (A, C); (B, C); (C, D); (A, D), and from the domino C itself (unless it is mixed) is at least 5.*



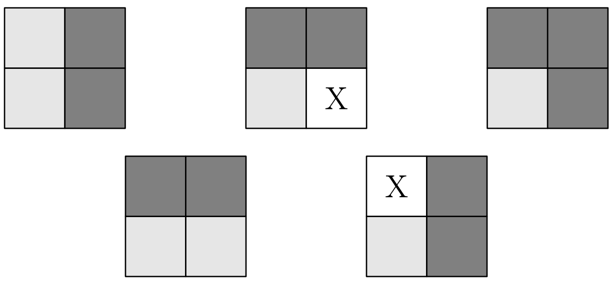
Proof. If C is a mixed domino, we have at least 6 edges that come from the adjacency of dominoes (C, A); (C, B); (C, D). Otherwise, we can assume without loss of generality that C is white.

If either A or D is also white, we have at least 5 edges - 4 from its adjacency with C and one from C itself. We have no more edges if and only if the other two dominoes are black.

If either A or D is mixed, then we again have at least 5 edges - 2 between (A, D), 2 between the mixed one and C, and one from C itself. In this case, equality is achieved only when the other two dominoes are black.

If both A and D are black, we again have at least 5 edges - 4 between (A, D) and one from C itself. B must be black, or otherwise, it would contribute with more edges.

Remark: Our lemma is proven, and the equality cases are the following:



We use light grey, dark grey and X to illustrate white, black, and mixed dominoes, respectively. Note that if C is black, we have to switch the light grey and dark grey colors in the equality cases above. $\square$

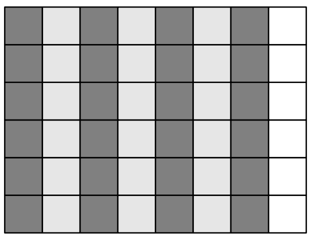
Let us introduce a coordinate system with $(1,1)$ related to its bottom-left-most domino and $(m+1, n+1)$ related to its top-right-most domino. From Lemma 2.1 for the 2×2 squares of dominoes with bottom-left corner $(m-1,1)$; $(m-2,1)$; $\ldots$; $(1,1)$, there are at least $5(m-1)$ new edges, and if we use the Lemma 2.1 for the $90°$ rotations of the 2×2 domino squares whose top-right corners are $(m+1,3)$; $(m+1,4)$; $(m+1,5), \ldots, (m+1, n+1)$, respectively, we get that there are at least $5(m-1)$ additional edges. Thus the total number of edges between monochromatic vertices is at least:

$$\begin{aligned} M &= 5nm - 4m + 5(m-1+n-1) + 11 \\ &= 5nm + 5m + 5n + 5 - 4(m+1) = 5(n+1)(m+1) - 4(m+1). \end{aligned}$$

It remains to check if equality occurs iff the entire board is striped. Let us analyze it. First, the top-left $n \times m$ rectangle must be striped due to the induction hypothesis.

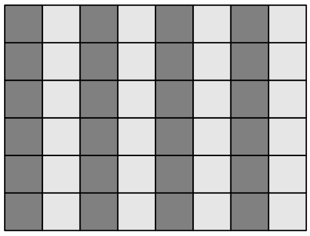
Let's consider the $(1,1)$ domino. Equality has to be achieved in Lemma 2.1 and from its equality cases we can conclude that only the first one is applicable, since $(1,2)$ and $(2,2)$ are differently colored. Hence $(1,1)$ must be colored as $(1,2)$, and $(2,1)$ must be colored as $(2,2)$. The same thing holds for $(2,1)$; $(3,1)$; $\ldots (m-1,1)$ and so any $(i,1)$ for $i = 1,2,\ldots m$ must be colored as $(i,2)$. See Figure 2.

Figure 2: The first n columns are striped.



From the fact that $(m,1)$ and $(m,2)$ must be monochromatic it follows that $(m+1,1)$ and $(m+1,2)$ are in the opposite color, as equality is achieved in Remark 1. We already know the coloring of $(m+1,2)$; $(m,2)$; $(m,3)$, thus to achieve equality in Lemma 2.1 $(m+1,3)$ must be colored as $(m+1,2)$. The same holds true for $(m+1,4)$; $(m+1,5)$; $(m+1,1)$; $\ldots$; $(m+1,n+1)$, i.e., the board must be colored in stripes. See Figure 3. $\square$

Figure 3: Striped coloring.



Lemma 3. *For an $n \times n \times n$ grid the following bound is achieved:*

$$N(G_{n\times n\times n}, 2) = 4n^3 + \Theta(n^2).$$

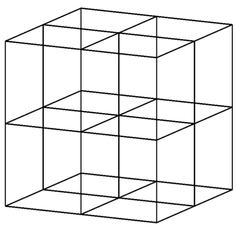
Proof. Note that if we color the lattice in stripes, we get n $1\times n\times n$ layers. Each layer contains $(n-2)^2$ cells with eight neighbors each, $4(n-2)$ cells with five neighbors each, and four cells with 3 neighbors each. This gives rise to a construction with

$$\frac{n\Big(8(n-2)^2 + 5\cdot 4(n-2) + 3\cdot 4\Big)}{2} = 4n^3 - 6n^2 + 2n$$

edges between monochromatic vertices, i.e., $N(G_{n\times n\times n}, 2) = 4n^3 + \Omega(n^2)$.

Throughout, vertices of unit cells (not vertices of the graph) that are not part of the grid surface will be called interior points. For each interior point i, we call the $2\times 2\times 2$ cube of cells that have that interior point as a vertex its neighborhood, and we denote it by $\mathcal{N}_i$. See Figure 4.

Figure 4: A neighborhood $\mathcal{N}_i$ of an interior point i.



For each interior point i and its neighborhood - $\mathcal{N}_i$, we will associate a counter c_i that starts at 0 and is incremented according to the following rule: for every two monochromatic cells adjacent by a vertex, we add $\frac{1}{s}$ to the counter of each of their common interior points, provided they have s interior points in common. Thus, $\sum_{\text{cells } i} c_i =$ monochromatic edges.

Note that the counter c_i of each interior i point depends only on the color of the cells in $\mathcal{N}_i$. This allows us to consider a finite number of cases for the colors of the cells in $\mathcal{N}_i$, and to find the one where c_i is minimal. To this end, we will consider four cases for the interior point, namely:

1. a corner point; where its neighborhood touches three of the cube's $n \times n \times n$ faces;
2. an edge point, where its neighborhood touches exactly two of the cube's faces;
3. a face point, where its neighborhood touches exactly one of the cube's faces;
4. a center point, where its neighborhood does not touch the cube's faces.

Since the cases are $2^8 = 256$, we use a computer program to check them (Note that they could be reduced, but again they would be more than 30).

For every two cells in the neighborhood - a, b we define a weight $f(a,b) = \frac{1}{s_1}$, where s_1 is be the number of interior points common to a and b. In all cases, we can determine $f(a,b)$, and then the counter will be equal to the sum of the weights of the edges between the monochromatic cells. Computing these weights is quite technical and lengthy, but for the sake of completeness, it can be found in Appendix A.

The computer program uses the following algorithm:

Algorithm 1 Finding the smallest value of the counter

```
f(i, j) := weights of the edges
minimal_counter := INF
for all colorings β of the neighborhood do
    counter := 0
    for all pairs of cells (i, j) do
        if β(i) = β(j) then
            counter += f(i, j)
        end if
    end for
    minimal_counter = min(minimal_counter, counter)
end for
```

The result of the program is the following:

	Corner point	Edge point	Face point	Central point
Minimal value of the counter	7.5	5	4.5	4

From this and the fact that there are in total 8 corner points, $12(n-3)$ edge points, $6(n-3)^2$ face points, and $(n-3)^3$ central points, it follows that the number of monochromatic edges is at least:

$$N(G_{n\times n\times n}, 2) \leq 8 \times 7.5 + 12(n-3) \times 5 + 6(n-3)^2 \times 4.5 + (n-3)^3 \times 4$$
$$= 4n^3 - 9n^2 + 6n + 15.$$

This gives us the upper bound: $N(G_{n\times n\times n}, 2) = 4n^3 + \mathcal{O}(n^2)$. □

Remark 2. *The above two bounds coincide for $n = 3$, which means that, in this case, the striped coloring is optimal.*

Remark 3. *When the neighborhood is colored in stripes, equality holds in all local bounds for the counters.*

From the above results, we can formulate the following conjecture:

Conjecture 1. *The optimal coloring for three-dimensional grids is the striped coloring.*

3 Non-tabular graphs

A natural continuation of the aforementioned problem is the generalization for all graphs and other specific cases. In this section, we develop two bounds for $N(G, 2)$ for planar graphs and a general bound for graphs (possibly non-planar) in terms of their chromatic number.

Lemma 4. *For a graph G with chromatic number $\chi(G) = n$ the following inequality holds:*

$$N(G,2) \leq C|E| \text{ , where } C = \begin{cases} \dfrac{n-2}{2(n-1)} & \text{for even } n, \\ \\ \dfrac{n-1}{2n} & \text{for odd } n. \end{cases}$$

Proof. Since $\chi(G) = n$, the graph can be colored in n colors with no monochromatic edges. We want to merge approximately half of these colors into one color, and merge the remaining colors into another color. For this purpose, we use the probabilistic method. First, we take at random $\lfloor \frac{n}{2} \rfloor$ colors to be recolored white and the remaining colors to be recolored black. Then we calculate the probability that an edge ends up between monochromatic vertices. Finally, knowing this, we can calculate the expected value of the number of edges between monochromatic vertices.

Case 1. n is even.
We take an arbitrary edge $e = (u, v)$ where $u, v \in V$. Let us denote by $c : V \to \{0, 1\}$ the two-coloring. To ensure $c(u) = c(v)$, we first have two ways of choosing their color. If they are white, we want to choose $\frac{n}{2}$ from the rest to be black, and conversely, if they are black, we want to choose $\frac{n}{2}$ from the rest to be white. This implies:

$$\begin{aligned} C := \mathbb{P}\Big(c(u) = c(v)\Big) &= \frac{2 \cdot \binom{n-2}{\frac{n}{2}}}{\binom{n}{\frac{n}{2}}} = \frac{\frac{2\cdot(n-2)!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}-2\right)!}}{\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}} \\ &= \frac{2 \cdot \left(\frac{n}{2}\right)\left(\frac{n}{2} - 1\right)}{n(n-1)} = \frac{n-2}{2(n-1)}. \end{aligned}$$

Case 2. n is odd.
Again we want to calculate $\mathbb{P}\Big(c(u) = c(v)\Big)$, but this time we will have to split the colors into two unequal parts. We want to choose $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ colors to recolor white and $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ colors to recolor black. Therefore:

$$\begin{aligned} C := \mathbb{P}\Big(c(u) = c(v)\Big) &= \frac{\binom{n-2}{\lfloor \frac{n}{2} \rfloor}}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} + \frac{\binom{n-2}{\lceil \frac{n}{2} \rceil}}{\binom{n}{\lceil \frac{n}{2} \rceil}} = \frac{\binom{n-2}{\lfloor \frac{n}{2} \rfloor} + \binom{n-2}{\lceil \frac{n}{2} \rceil}}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \qquad = \frac{\binom{n-1}{\lceil \frac{n}{2} \rceil}}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \\ &= \frac{(n-1)!\,\lfloor \frac{n}{2} \rfloor!\,\lceil \frac{n}{2} \rceil!}{n!\,\lceil \frac{n}{2} \rceil!\,\big(\lfloor \frac{n}{2} \rfloor - 1\big)\,!} = \frac{\lfloor \frac{n}{2} \rfloor}{n} = \frac{n-1}{2n}. \end{aligned}$$

Let $f : E \to \{0, 1\}$ be such that for all $e \in E$, $f(e) = \begin{cases} 1, \text{ if } c(u) = c(v), \\ 0, \text{ otherwise.} \end{cases}$
From here the number of edges between monochromatic vertices is equal to $X = \sum\limits_{e \in E} f(e)$. Then the expected value of X is:

$$\mathbb{E}[X] = \sum_{(u,v) \in E} \mathbb{P}\Big(c(u) = c(v)\Big) = C|E|,$$

thus there is a coloring, such that $X \le C|E|$. $\square$

Lemma 5. *For a planar graph $G_P(E, V)$ the following bound holds*

$$N(G_P, 2) \leq \frac{|E|}{3}.$$

Proof. This follows directly from Lemma 4 and the four-color theorem which states that $\chi(G_P) \leq 4$ for a planar graph G_P [5]. $\square$

Remark 4. *From Euler's formula* $|V|-|E|+|F|= 2$ *and the fact that every face in a planar graph is surrounded by at least 3 edges, therefore* $2|E|\geq 3|F|$ *and* $|E|\leq 3|V|-6$. *From this fact and Lemma 5 we have:*

$$N(G_P, 2) \leq |V|-2.$$

4 Conclusion and future development

For the tabular graphs, we proved that the striped coloring is optimal for a $2 \times n \times m$ lattice and obtained bounds close to each other for the general case $k \times n \times m$, which is presumably the most significant result in the project. The proof of these bounds uses a promising method that could be applied to other graphs - splitting the number of monochromatic edges between counters and finding a local bound for each counter. For non-tabular graphs, we showed a relation between the size of the max-cut and the chromatic number of graphs, and derived a bound using the number of vertices for a planar graph.

In the future, we could prove or disprove Conjecture 1, and prove or disprove that the striped coloring is optimal for n-dimensional grids. For a wide range of graphs other bounds can be analyzed, such as non-orthogonal tabular graphs.

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A Determining the weights of the adjacencies in a neighbourhood

In this appendix, we show how the weights of the edges are determined between two monochromatic cells in the neighborhood of an interior point. The weights are different for the four types of interior points.

The following diagrams show the weights of all adjacencies. The black dots and squares in the vertices of the cube show the eight cells in the neighborhood. The weights of the edges between any two of them are written next to the segment that connects these two vertices. The weights of the edges between opposite vertices of the cube are not shown, but all of them are equal to one.

Figure 5: Corner interior cell. The cell that coincides with the vertex of the $n \times n \times n$ grid is marked by a square.

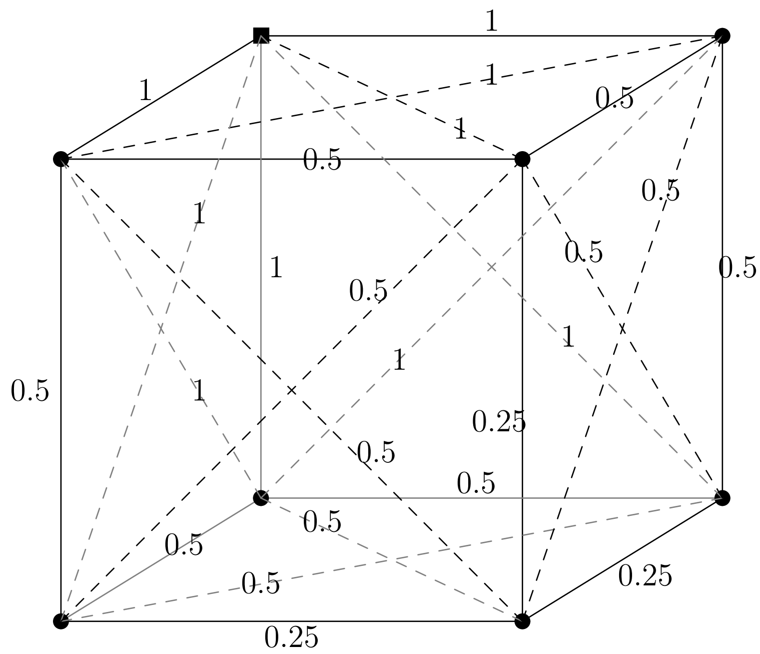

Figure 6: Edge interior cell. The edge that coincides with the edge of the $n \times n \times n$ grid is marked by two squares.

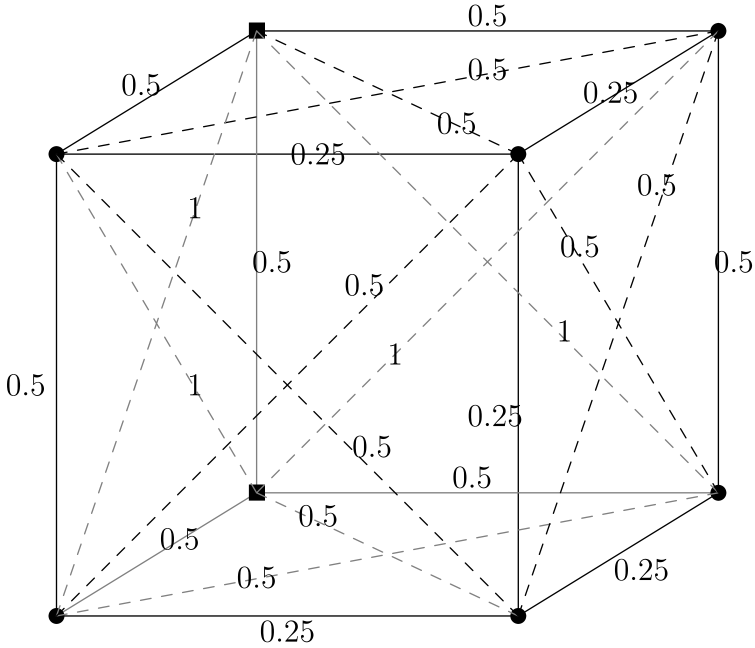

Figure 7: Face interior point. The face of the $2 \times 2 \times 2$ neighborhood that lies on a face of the $n \times n \times n$ grid is marked by squares.

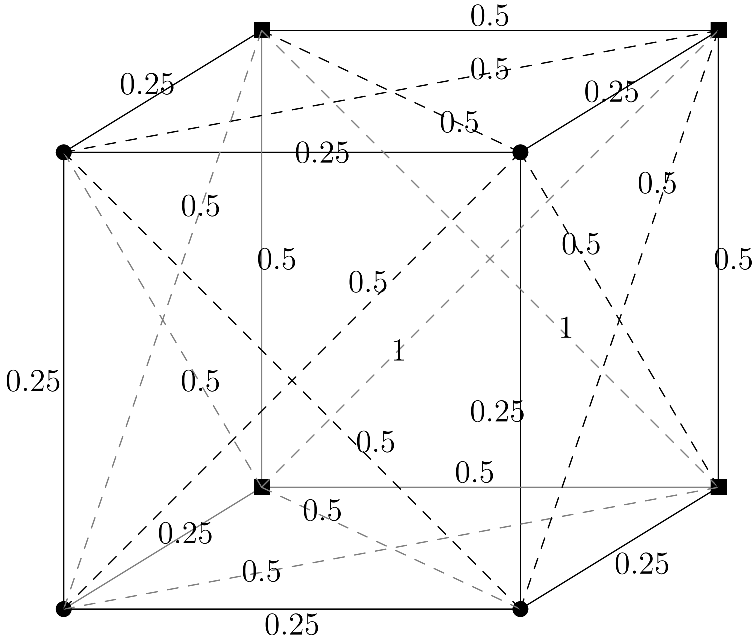

Figure 8: Central interior point.