Orthonormal FBs with Rational Sampling Factors and Overcomplete DFT-Modulated FBs: A Connection and Filter Design

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Abstract

Methods widely used to design filters for uniformly sampled filter banks (FB) are not applicable for FBs with rational sampling factors and overcomplete DFT-modulated FBs. In this paper, we show that the filter design problem for these two types of FBs are the same. Following this, we propose two filter design methods for these FBs. The first method considers a constrained case and depends on a parameterization. The second method, which is applicable more generally, uses results from frame theory. Finally, we discuss and provide a motivation for iterated DFT-modulated FBs.

EDICS Categories: DSP-BANK, DSP-MULT, DSP-WAVL

I. Introduction

The spectrum of a signal can be split in various ways using different filter bank (FB) structures. For example, if one is interested in a subband decomposition with uniform subband widths, uniform FBs can be utilized. In particular, a CQF FB splits the signal spectrum in half. If this FB is iterated on its lowpass branch, a decomposition is obtained where the subband widths are halved at each iteration (see Figure 1). In this scheme, the Q-factors of the filters, which are defined for each filter to be the ratio of the center frequency to the bandwidth, stays fixed regardless of how many times the FB is iterated. For this reason, such FBs are called constant-Q FBs. This is a property of the FB structure, i.e. even though using different filters for the base FB changes the particular frequency responses of the subbands, it does not

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substantially alter the Q-factors. Certain applications call for FBs with rational sampling factors (from hereon referred to as rational FBs) (see [3], [5], [12], [16]) which are realized by the structure in Figure 2 (restricting attention to two-band rational FBs). The Q-factor can be controlled by changing p and q in Figure 2. Typical frequency responses for $(p = 2, q = 3)$ and $(p = 5, q = 6)$ are shown in Figure 1.

A different type of FB is the overcomplete DFT-modulated FB, also known as the discrete Gabor or Weyl-Heisenberg frame [9], [14]. These FBs are derived from a single prototype by frequency modulation and provide a uniform subband decomposition. A complete parameterization of perfect reconstruction (PR) FBs of this type was given in [14]. Like rational FBs, the FB structure is determined by two parameters: the number of channels and downsampling factor.

An important point to consider for iterated FBs is the behavior of the lowpass branch as the number of iterations increase. The iterated lowpass branch should analyze lower frequencies as it is iterated. This in turn requires that the lowpass filter possess a number of certain factors, which will be referred to as regularity factors. Unfortunately, designing filters with regularity factors which yield orthonormal rational FBs cannot be carried out as in the M-band uniformly downsampled FBs where one obtains the lowpass filter as the spectral factor of an M^{th}-band filter [28]. The filter design problem for the rational FB is discussed by several authors (see [1], [6], [22]–[25], [31], [32]) but none other than Blu [6] attempts to achieve PR and inclusion of regularity factors using a single filter, and Blu reports that his algorithm diverges when more than one regularity factor is requested. Likewise for DFT-modulated FBs, we are not aware of previous work that attemts to design FBs with corresponding regularity factors (Cvetković and Vetterli are primarily interested in the non-iterated FB in [14]) for rational oversampling factors (for integer oversampling see [8], [9]).

In this paper we show a close connection between orthonormal rational FBs and overcomplete DFT-modulated FBs. In particular we show that under certain additional restrictions on DFT-modulated FBs (which we will argue to be desirable), the filter design problems are the same. More precisely, the lowpass filter of an orthonormal rational FB with K regularity factors can be used to yield a DFT-modulated FB with K regularity factors and vice versa. Following this connection, we provide two methods for filter design. The first method relies on a complete parameterization of orthonormal rational FBs, applicable when only a single regularity factor is required. Using the parameterization, we transform the problem into an unconstrained optimization problem. The second method uses results from the theory of frames. This is an iterative algorithm that preserves the regularity factors at each iteration. This allows us to obtain nearly PR FBs with an arbitrary number of regularity factors.

The outline of the paper is as follows. In Section II, we state the filter design problem for orthonormal

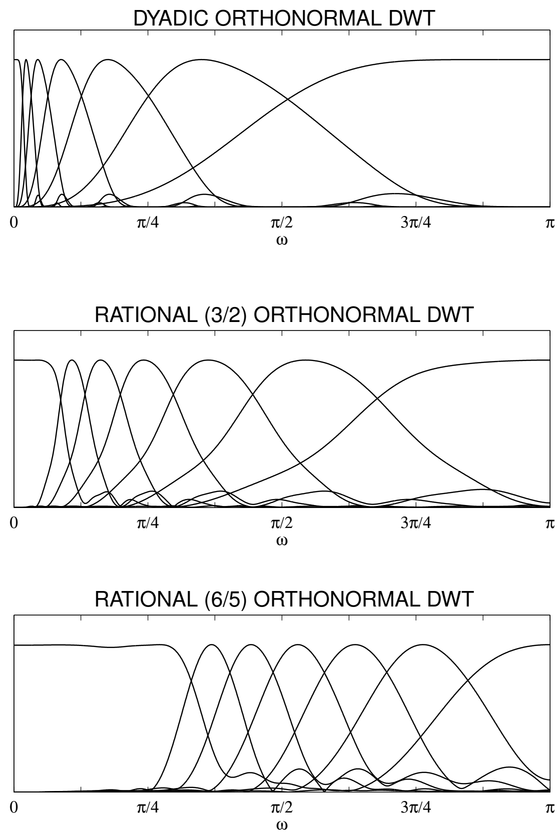


Fig. 1. Typical frequency decompositions performed by a dyadic orthonormal DWT, Orthonormal Rational ($(p = 2, q = 3)$ and $(p = 5, q = 6)$) DWTs (Fig. 2).

rational FBs. We do the same for overcomplete DFT-modulated FBs and show the equivalence of the two problems in Section III. Section IV describes a filter design method when only a single regularity factor is sought. In Section V we present the general method allowing an arbitrary number of regularity factors. A linear phase modification is also provided in this section. We discuss some properties of the iterated overcomplete DFT-Modulated FBs in Section VI. Section VII is the conclusion.

Notation

For a filter transfer function $H(z)$, $\tilde{H}(z)$ denotes

$$\tilde{H}(z) = H^*\left(1/z^*\right). \tag{1}$$

We will refer to the FB in Fig. 2 as a (p, q) rational FB.

Also, an overcomplete DFT-modulated FB with q channels and downsampling factor p will be referred to as a (p, q) DFT-modulated FB (see Fig.4).

II. Paraunitary Filterbanks with rational sampling factors

In this section we will review some facts about orthonormal filter banks with rational sampling factors. We consider two-channel filter banks as shown in Fig. 2, where p and q are coprime. It can be shown that, the structure in Fig. 2 is equivalent to a q-channel filter bank where the polyphase components of $H(z)$ and $G(z)$ constitute the filters (see for example [2], [20]–[22]).

In particular, for $p = 2$, $q = 3$, if we define the polyphase components $H_0(z)$, $H_1(z)$ of $H(z)$ by,

$$H(z) = H_0(z^2) + z^{-3} H_1(z^2), \tag{2}$$

then the two filter banks in Fig. 3 are equivalent. Notice that (2) is different than the usual definiton. This choice simplifies some of the equations in the following. For general (p, q) pairs, we similarly define the polyphase components $H_n(z)$ by

$$H(z) = \sum_{n=0}^{p-1} z^{-qn}\, H_n(z^p). \tag{3}$$

We can always find such a decomposition since p and q are coprime.

The PR conditions in terms of the alias component (AC) matrix imply the following result, which we state as a proposition.

Propositon 1: Suppose the system shown in Fig. 2 has the PR property and that the polyphase components $H_i(z)$ of the filter $H(z)$ are defined by (3). Then,

$$\begin{pmatrix} H_0(z) & H_0(zW) & \dots & H_0(zW^{q-1}) \\ H_1(z) & H_1(zW) & \dots & H_1(zW^{q-1}) \\ & \vdots & & \\ H_{p-1}(z) & H_{p-1}(zW) & \dots & H_{p-1}(zW^{q-1}) \end{pmatrix} \begin{pmatrix} \tilde{H}_0(z) & \tilde{H}_1(z) & \dots & \tilde{H}_{p-1}(z) \\ \tilde{H}_0(zW) & \tilde{H}_1(zW) & \dots & \tilde{H}_{p-1}(zW) \\ & \vdots & & \\ \tilde{H}_0(zW^{q-1}) & \tilde{H}_1(zW^{q-1}) & \dots & \tilde{H}_{p-1}(zW^{q-1}) \end{pmatrix} = pI. \tag{4}$$

where $W = \exp(j2\pi/q)$.

The proposition can be proved by writing the PR conditions in terms of the AC matrices for the analysis and synthesis FBs and switching the matrices.

Now suppose that we iterate the rational FB in Fig. 2 on its lowpass branch. At the n^{th} stage, the lowpass filter may be regarded as a time varying system computing inner products of the input with

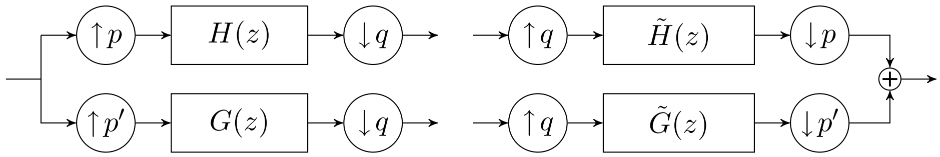


Fig. 2. A filter bank with a rational (q/p) sampling factor where $p' = q - p$.

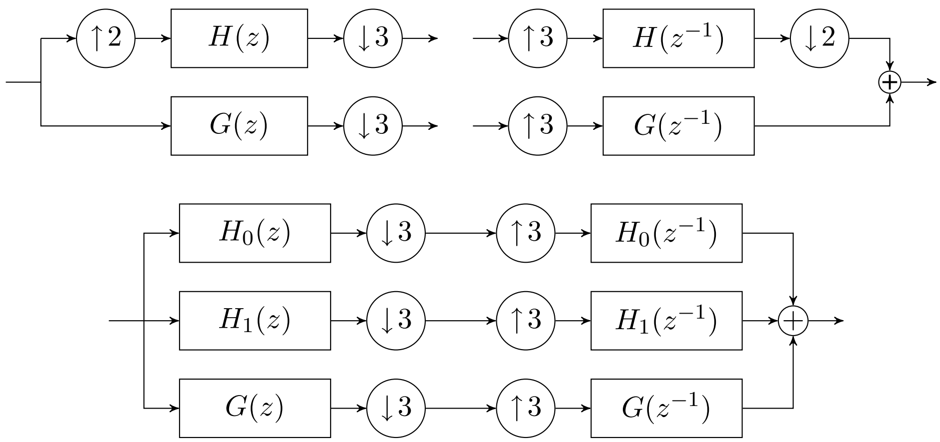


Fig. 3. A filter bank with (2/3) sampling factor and its equivalent. The polyphase components of $H(z)$ is defined via (2).

certain shifts of p^n different discrete-time sequences [26]. As n increases, we would like these discrete-time sequences to be progressively narrower band lowpass sequences. For this, it is necessary that $H(z)$ possess factors of the form $(z^q - 1)/(z - 1)$ [4], [6], [26], which will be referred to as *regularity factors for the rational* (p, q) *rational FB*. For orthonormal systems, this implies that $H(z)$ also has $(z^p - 1)/(z - 1)$ as factors [6].

We remark that given the lowpass filter, the highpass filter can be obtained by using known paraunitary matrix completion procedures [30].

In summary, we would like our lowpass filter to (a) satisfy (4) and (b) have the form

$$H(z) = \left(\frac{z^q - 1}{z - 1} \, \frac{z^p - 1}{z - 1} \right)^K Q(z). \tag{5}$$

III. Overcomplete DFT-Modulated Filter Banks

We now consider overcomplete PR FBs where the filters are obtained by modulating a single prototype. For a given (p, q) pair (once again we assume coprime (p, q)), these are FBs as shown in Figure 4, with

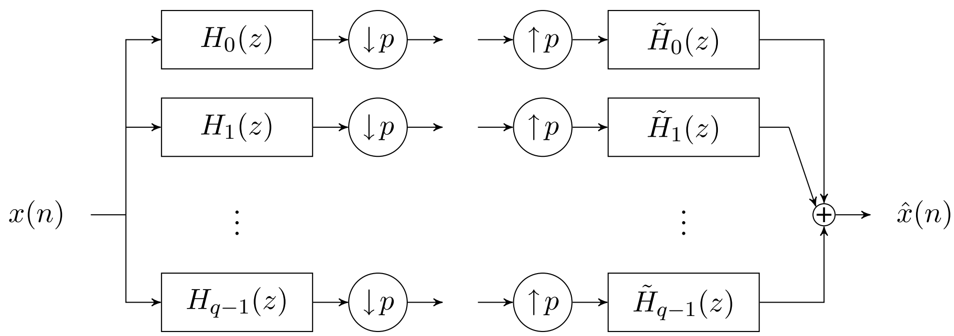


Fig. 4. A uniformly downsampled FB. When $H_i(z)$ is set to be $H_0(zW^i)$ for $W = \exp(j\,2\pi/q)$, this becomes a (p,q) DFT-modulated FB.

filters $H_i(z)$ for $i = 1, 2 \ldots, q-1$ given by,

$$H_i(z) = H_0(zW^i), \tag{6}$$

where $W = \exp(j\,2\pi/q)$.

Let us consider the PR conditions for this FB. In contrast to the FBs in the previous section, we will express the conditions in terms of polyphase matrices. For this, we define the polyphase components as in (3).

$$H_i(z) = \sum_{n=0}^{p-1} z^{-qn}\, H_{i,n}(z^p). \tag{7}$$

This definition yields an interesting relation among the polyphase components. Using the DFT modulation relation (6), and noting that $W^{qn} = 1$, we have

$$\sum_{n=0}^{p-1} z^{-qn}\, H_{i,n}(z^p) = H_i(z) = H_0(zW^i) = \sum_{n=0}^{p-1} z^{-qn}\, H_{0,n}(z^p W^{pi}), \tag{8}$$

and hence

$$H_{i,n}(z) = H_{0,n}(zW^i). \tag{9}$$

Using this relation the PR conditions become (setting $F_n(z) = H_{0,n}(z)$ to avoid cumbersome notation),

$$\begin{pmatrix} F_0(z) & F_0(zW) & \dots & F_0(zW^{q-1}) \\ F_1(z) & F_1(zW) & \dots & F_1(zW^{q-1}) \\ & \vdots & & \\ F_{p-1}(z) & F_{p-1}(zW) & \dots & F_{p-1}(zW^{q-1}) \end{pmatrix} \begin{pmatrix} \tilde{F}_0(z) & \tilde{F}_1(z) & \dots & \tilde{F}_{p-1}(z) \\ \tilde{F}_0(zW) & \tilde{F}_1(zW) & \dots & \tilde{F}_{p-1}(zW) \\ & \vdots & & \\ \tilde{F}_0(zW^{q-1}) & \tilde{F}_1(zW^{q-1}) & \dots & \tilde{F}_{p-1}(zW^{q-1}) \end{pmatrix} = I, \quad (10)$$

This is exactly a scaled version of (4). We state this observation as a proposition.

Propositon 2: Suppose $H(z)$ is the lowpass filter of an orthonormal (p, q) rational FB. Then $H(z)/\sqrt{p}$ generates a PR (p, q) DFT-modulated FB.

Conversely, if $H(z)/\sqrt{p}$ generates a PR (p, q) DFT-modulated FB, then $H(z)$ is the lowpass filter of an orthonormal (p, q) rational FB.

A. Regularity and Vanishing Moments

If we are interested in iterating a (p, q) DFT-modulated FB on its lowpass branch, then it is required that the lowpass filter have factors of the form [28], $(1 - z^p)/(1 - z)$. We will refer to these as *regularity factors for the (p, q) DFT-modulated FB*. Notice that the regularity factors for a (p, q) DFT-modulated FB and a (p, q) rational FB have different forms. However, there exists a unilateral relationship between the two, provided that the rational FB is orthonormal, as explained in the following.

Suppose we have an orthonormal (p, q) rational FB, whose lowpass filter has K regularity factors for this FB structure. We had mentioned in the previous section that if the lowpass filter of an orthonormal (p, q) rational FB K regularity factors, then it also has $[(1 - z^p)/(1 - z)]^K$ as a factor, which is the correct form for the regularity factors of a p, q DFT-modulated FB. Combining with Prop. 2, we see that the lowpass filter (following a normalization) can be used to generate a (p, q) DFT-modulated FB with K regularity factors.

The converse relation regarding the regularity factors of the two FB structures is not true. That is, suppose we have a (p, q) DFT-modulated FB, where the lowpass filter has K regularity factors. The lowpass filter may be used (following a normalization) as the lowpass filter of an orthonormal (p, q) rational FB. However, in this case, it can be shown that only a single regularity factor (for the rational

FB) is ensured. In other words, an orthonormal (p, q) rational FB might have a lowpass filter with a factor $[(1-z^p)/(1-z)]^K$, but this implies only that the lowpass filter has a factor $(1-z^q)/(1-z)$. In general then, one can scale and use the lowpass filter of a rational FB with K regularity factors to generate an overcomplete DFT-modulated FB with K regularity factors, but not vice versa.

Another property we seek for the overcomplete DFT-modulated FB is that the highpass filters annihilate discrete-time polynomials of degree K. The FB is said to have K vanishing moments in this case (this also implies that the underlying wavelets have K vanishing moments as well). Since a (p, q) DFT-modulated FB is determined by its lowpass filter, this has a direct implication on the lowpass filter. Suppose that the n^{th} highpass filter $H_n(z)$ has K vanishing moments. Then it has $(1-z)^K$ as a factor. Since $H_n(z) = H(zW^{-n})$ (where $W = e^{j2\pi/q}$), this implies that $H(z)$ has $(W^{-n} - z)^K$ as a factor. If all of the highpass filters have K vanishing moments (in which case we say that the FB has K vanishing moments), then the lowpass filter has

$$(W^{-1} - z)^K \, (W^{-2} - z)^K \ldots (W^{-(q-1)} - z)^K = c \left(\frac{1 - z^q}{1 - z} \right)^K \tag{11}$$

as a factor (c is a constant). Notice that this is the correct form for the regularity factor for the (p, q) rational FB. By this discussion, we conclude,

Propositon 3: Suppose $H(z)$ is the lowpass filter of an orthonormal (p, q) rational FB with K regularity factors. Then $H(z)/\sqrt{p}$ generates a PR (p, q) DFT-modulated FB with K regularity factors and K vanishing moments.

Conversely, suppose $H(z)/\sqrt{p}$ generates a PR (p, q) DFT-modulated FB with K regularity factors and K vanishing moments. Then $H(z)$ is the lowpass filter of an orthonormal (p, q) rational FB with K regularity factors.

In other words, the filter design problem for the rational orthonormal FB and the overcomplete DFT-modulated FB is the same. Given some K, we need to find a filter of the form (5) that satisfies (4). Equation (4) calls for the design of p orthonormal lowpass filters which, when combined (as dictated by the polyphase decomposition (3)) possess a specific factor. Unfortunately, this cannot [2] be converted to the problem of designing an autocorrelation sequence with some factor, as is done in the conventional integer sampled FBs [28]. In the following, we will consider two methods for the design of such filters. The first method is based on an observation that the problem can be converted to an unconstrained optimization problem using paraunitary blocks provided only a single regularity factor is required. The second method, applicable for arbitrary number of regularity factors, uses tools from frame theory.

IV. Design of Orthonormal Rational FBs and Overcomplete DFT-Modulated FBs with a Single Regularity Factor

In this section, we restrict our attention to the design of filters which have the form (5) with $K = 1$. Even though this is a relatively modest case, Blu [6] reports that his iterative algorithm (which was also used recently for the design of rational FBs in [12]) converges only for this case. In contrast to the method in [6] which reaches PR only in the limit, our design is basically an unconstrained optimization on paraunitary blocks.

We will begin by looking at the polyphase matrix of the (p, q) rational FB. Let us define the polyphase components $H_n(z)$ of the lowpass filter $H(z)$ as in (3),

$$H(z) = \sum_{n=0}^{p-1} z^{-qn} \, H_n(z^p). \tag{12}$$

The rational FB is equivalent to a uniform integer sampled (by q) FB where the first p filters are given by $H_n(z)$. In order to form the polyphase matrix of this FB, let us denote the polyphase components of $H_n(z)$ as $H_{n,k}$ which are defined through,

$$H_n(z) = \sum_{k=0}^{q-1} z^{-k} \, H_{n,k}(z^q). \tag{13}$$

$H_{n,k}(z)$ give the $(n+1, k+1)$ entry of the polyphase matrix $\mathbf{E}(z)$ for the rational FB. Using this in (12), we get

$$H(z) = \sum_{n=0}^{p-1} z^{-qn} \sum_{k=0}^{q-1} z^{-pk} \, H_{n,k}(z^{pq}). \tag{14}$$

Now if $W_1 = \exp(j2\pi/q)$, since $W_1^q = 1$, we have that

$$H(W_1^m) = \sum_{n=0}^{p-1} \sum_{k=0}^{q-1} W_1^{-pkm} \, H_{n,k}(1). \tag{15}$$

From (15) we conclude that imposing 1 regularity factor constrains only $\mathbf{E}(1)$. Rewriting (15) in matrix form we obtain

$$\underbrace{(\overbrace{1, \ldots, 1}^{p \text{ 1's}}, 0, \ldots, 0)}_{\mathbf{s}} \, \mathbf{E}(1) \, \mathbf{W} = \begin{pmatrix} \sqrt{pq} & 0 & \ldots & 0 \end{pmatrix}. \tag{16}$$

where $\mathbf{s}$ is a $1 \times q$ vector and $\mathbf{W}$ is the unitary $q \times q$ DFT matrix. Multiplying both sides by $\mathbf{W}^H$ from the right, this becomes,

$$\mathbf{s} \, \mathbf{E}(1) = \begin{pmatrix} \sqrt{p} & 0 & \ldots & 0 \end{pmatrix}. \tag{17}$$

Since $\mathbf{E}(1)$ is an orthonormal matrix, this equation states that the first column of $\mathbf{E}(1)$ is equal to $\mathbf{s}^T/\sqrt{p}$. Such orthonormal matrices can be parameterized. In fact, if $\mathbf{C}$ is such a matrix, so is

$$\mathbf{C}\begin{pmatrix} 1 & \mathbf{0}_{1\times q-1} \\ \mathbf{0}_{q-1\times 1} & \mathbf{R} \end{pmatrix} \tag{18}$$

where $\mathbf{R}$ is a $(q-1)\times(q-1)$ orthonormal matrix. Moreover, every $\mathbf{E}(1)$ satisfying (17) can be obtained this way. Next, we will consider factors that do not affect (17) so as to obtain a parameterization of all rational FBs with a single regularity factor.

A. Factorization of Paraunitary Systems Using Householder Type Building Blocks

Consider a system whose $q\times q$ polyphase matrix is given by

$$\mathbf{V}(z) = \left(\mathbf{I} - \mathbf{v}\mathbf{v}^H + z^{-1}\mathbf{v}\mathbf{v}^H\right), \tag{19}$$

where $\mathbf{v}$ is a $q\times 1$ unitary vector. It can be verified that $\mathbf{V}(z)$ is a paraunitary matrix and $\mathbf{V}(1) = \mathbf{I}$. Obviously, we can cascade such systems and obtain new paraunitary systems. We can also control the behavior of the cascaded system at $z = 1$ by using a constant unitary matrix. In fact, any FIR paraunitary system can be factored this way.

Theorem 1: [30] Let $\mathbf{H}(z)$ be a $q\times q$ paraunitary matrix with $\det \mathbf{H}(z) = z^{-N}$. Then, $\mathbf{H}(z)$ can be written as,

$$\mathbf{H}(z) = \mathbf{V}_N(z)\,\mathbf{V}_{N-1}(z)\,\ldots\,\mathbf{V}_1(z)\mathbf{H}(1), \tag{20}$$

where $\mathbf{V}_i(z) = \left(\mathbf{I} - \mathbf{v}_i\mathbf{v}_i^H + z^{-1}\mathbf{v}_i\mathbf{v}_i^H\right)$ with $\mathbf{v}_i^H\,\mathbf{v}_i = 1$. For real coefficient $H(z)$, $\mathbf{v}_i$ can be chosen real.

Parameterizing each $\mathbf{v}_i$ we obtain a parameterization of degree N paraunitary systems. Combining this theorem with the results of the previous subsection, we obtain a parameterization of all degree N systems which give a rational FB with 1 regularity factor. Such a parameterization allows us to do unconstrained optimization on the family of filters, so as to 'optimize' the frequency response of the filter. These are summarized in the algorithm below.

Algorithm 1: Given p, q, number of paraunitary blocks K, and maximum number of iterations $M_{\max}$

- Let $\mathbf{c}_1$ be a length-q vector whose first p components are equal to $1/\sqrt{p}$ and the remaining components are 0. Pick a $q\times q$ orthonormal matrix $\mathbf{C}$ whose first column is $\mathbf{c}_1$.
- Set the minimum cost min_c, and the costant $thresh$.
- Repeat the following random restart procedure until the cost min_c is below some preset level.

– Pick randomly, a $q-1 \times q-1$ orthonormal matrix $\tilde{\mathbf{R}}$ and K unitary but otherwise arbitrary vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, K$.

– Set

$$\mathbf{V}_i(z) = \left(\mathbf{I} - \mathbf{v}_i \mathbf{v}_i^H + z^{-1} \mathbf{v}_i \mathbf{v}_i^H\right), \tag{21}$$

$$\mathbf{R} = \begin{pmatrix} 1 & \mathbf{0}_{1 \times q-1} \\ \mathbf{0}_{q-1 \times 1} & \tilde{\mathbf{R}} \end{pmatrix}. \tag{22}$$

Using these obtain the polyphase matrix for the rational FB by,

$$\mathbf{P}(z) = \mathbf{V}_1(z)\, \mathbf{V}_2(z) \ldots \mathbf{V}_K(z)\, \mathbf{C}\, \mathbf{R}. \tag{23}$$

Derive the lowpass filter $H(z)$ from this matrix using the polyphase definitions (3).

– If $\left|H(e^{j\omega})\right| < thresh$ for $\omega \in [2\pi/q, \pi]$,

  * Perform a local optimization on $\tilde{\mathbf{R}}$ (preserving its orthonormality) and $\mathbf{v}_i$ where the cost function is

$$\max_{\omega \in [2\pi/q, \pi]} \left|H(e^{j\omega})\right|. \tag{24}$$

  Let the final cost be c.

  * If $c < min\_c$, update $min\_c = c$, $h_{\text{best}} = h(n)$.

The following examples demonstrate the procedure.

Example 1: For $p = 2, q = 3$, we consider the family of FBs with a single regularity factor for which the polyphase matrix is constant. We set

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{25}$$

Then we consider the family

$$\mathbf{C} \begin{pmatrix} 1 & \mathbf{0}_{1 \times 2} \\ \mathbf{0}_{2 \times 1} & \tilde{\mathbf{R}} \end{pmatrix}, \tag{26}$$

where $\tilde{\mathbf{R}}$ is any 2×2 orthonormal matrix. This family can be parameterized using a single parameter (up to reflections). On this one-dimensional interval, we search for the filter that minimizes the frequency response magnitude on $[2\pi/3, \pi]$. The frequency response magnitude of the (length 8) filter found is shown in Figure 5.

Example 2: Again for $p = 2, q = 3$, we now consider the family of 1 vm FBs where the polyphase matrix has one factor of the type (19). This factor is parameterized by two parameters (used to construct

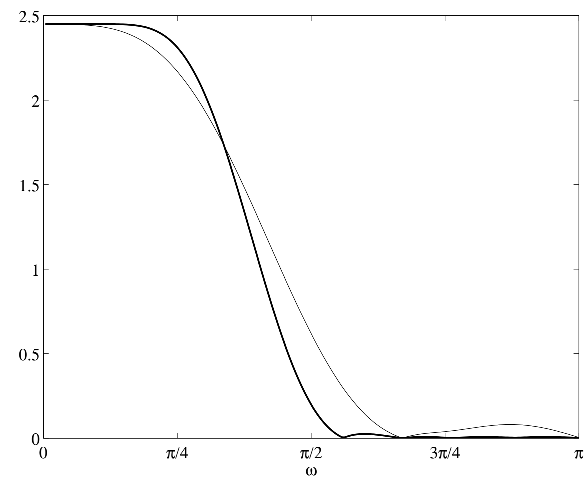


Fig. 5. The frequency response magnitudes of the resulting filters from Example 1 (thin line) and Example 2 (thick line). The lengths of the filters are 8 and 14 respectively.

$\mathbf{v}$). With the single parameter for the constant matrix, we have 3 parameters in total. Using the same cost function as the previous example for optimization, we obtain a longer (length 14) impulse response filter with a more desirable frequency response magnitude, illustrated in Fig. 5.

V. Filter Design for DFT-Modulated FBs and Orthonormal Rational FBs with Arbitrary Number of Regularity Factors

In this section, we will describe an overcomplete iterative method to obtain FIR filters for DFT-modulated FBs (equivalently, lowpass filters for orthonormal rational FBs), with a prescribed number of regularity factors and vanishing moments. For this, we will first review a few results from frame theory.

A. A Brief Review of Frame Theory

The following is a blend of definitions and results from [11], [13].

A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exists constants $A, B > 0$ s.t.

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{27}$$

In this case, A is said to be a *lower frame bound* and B is said to be a *upper frame bound*.
If $A = B$, then the frame is said to be *tight*.
If B is the supremum of the upper frame bounds and A is the infimum of the lower frame bounds, then A, B are called the *optimal* frame bounds.

For a frame $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$, the operator $T : l_2 \rightarrow \mathcal{H}$, defined as

$$T\,\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k\, f_k \tag{28}$$

is called the *synthesis operator*. The adjoint of this operator $T^* : \mathcal{H} \rightarrow l_2$, called the *analysis operator* is given by

$$T^*\, f = \{\langle f, f_k \rangle\}_{k=1}^{\infty}\,. \tag{29}$$

Combining these, the *frame operator* $S : \mathcal{H} \rightarrow \mathcal{H}$ is given by

$$S\, f = T^*\, T\, f = \sum_{k=1}^{\infty} \langle f, f_k \rangle\, f_k. \tag{30}$$

For a tight frame, we have $S = B\, I$,where B is the frame bound and I the identity operator. When a frame $\{f_k\}_{k=1}^{\infty}$ is not tight, it can be made tight by applying the inverse square-root of the frame operator to the sequence:

Theorem 2 ([11]): Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$, and let $S^{-1/2}$ be the operator such that, $S^{-1/2}\, S^{-1/2} = S^{-1}$. Then, $\left\{S^{-1/2}\, f_k\right\}_{k=1}^{\infty}$ is a tight frame with frame bounds equal to 1.

Now let us turn to filter banks. Consider an FB as in Figure 4. The system as a whole acts as the frame operator on l_2. Likewise, the analysis and synthesis parts correspond to the analysis and synthesis operators respectively.

An equivalent representation of the FB is given by the polyphase representation as shown in Figure 6, where $R(z) = E^H(1/z^*)$. Here, the analysis operator is represented by $E(z)$, which maps the polyphase transform of the input $X(z)$ to the z-transforms of the output of each analysis channel. The synthesis operator is represented by $R(z)$, mapping the z-transforms of the channels to the polyphase transform of the output $Y(z)$. Finally, the frame operator is represented by $S(z) = R(z)E(z)$, mapping the polyphase decomposition of the input to that of the output.

When z is constrained to the unit circle, the polyphase decomposition, mapping a filter $F(z)$ to the vector of its polyphase components, is a unitary transform [7]. That a set of filters defines a frame (i.e. the frame operator is invertible) may be investigated by an eigenanalysis of $S(z)$ evaluated on the unit circle. The maximum and minimum of the eigenvalues of the positive semi-definite matrix $S(e^{j\omega})$ give the optimal frame bounds B and A respectively. Notice that if $B = A$, i.e. the frame is tight, the FB has the perfect reconstruction property (upto a multiplicative factor). If the filter is indeed a frame, possibly non-tight, it can be made tight by applying Thm. 2, i.e. calculating $\left(S(e^{j\omega})\right)^{-1/2}\, E(e^{j\omega})$.

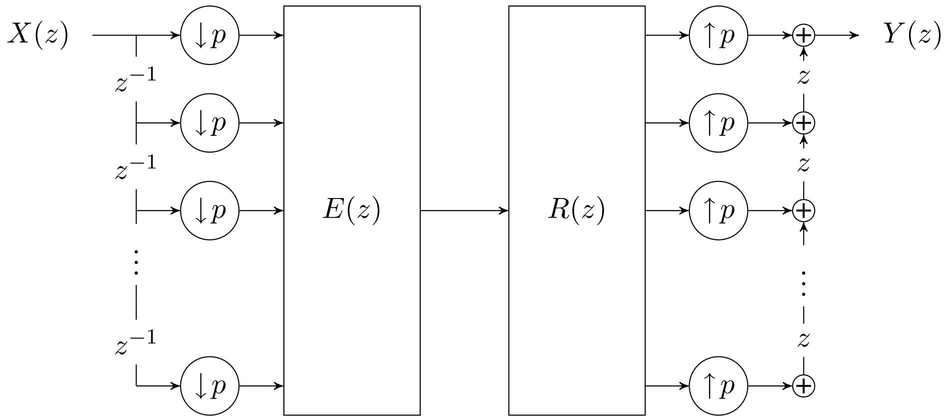


Fig. 6. Polyphase representation of an FB.

B. The approximation of the Inverse Square Root Operator

We would like to obtain an almost tight overcomplete DFT-modulated FB with regularity factors, starting from one with arbitrary frame bounds but with a lowpass filter that has a form as in (5). For this, we will invoke Thm. 2. We need to compute the inverse square root operator.

Given an FB frame, hence some $S(e^{j\omega})$, we have two candidates for approximating the inverse square root operator:

1) The finite section method by Ströhmer [29]
2) Truncating the infinite series representation of $S^{-1/2}$ [15] (also see [10], [11]):

$$S^{-1/2} = \sqrt{\frac{2}{A+B}} \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}\,(k!)^2} \left(I - \frac{2}{A+B}\,S\right)^k \tag{31}$$

where A, B are the frame bounds.

Whichever method we use, we want the new frame to possess the DFT-modulation property (6) and have regularity factors and vanishing moments (i.e. we want the lowpass filter to have the form (5)). In the following we will show that if $H(z)$ is of the form (5), and $E(z)$ represents the FB obtained by modulating $H(z)$, then $S(z)\,E(z)$ represents an overcomplete DFT-modulated FB where the lowpass filter has the form (5) (with a different $Q(z)$). Since the second method above involves a polynomial of S, we see that it satisfies our requirements.

Claim 1: For a given filter $H(z)$, suppose $E(z)$ represents the FB in Figure 4 with $H_i(z) = H(zW^i)$, where $W = e^{j2\pi/q}$. Then,

$$E'(z) = S(z)\,E(z) = \left[E(z)\,E^H(1/z^*)\right] E(z) \tag{32}$$

represents an FB $H_i'(z)$ where $H_i'(z) = H'(zW^i)$ for some $H'(z)$. In words, applying $S(z)$ to a DFT-modulated FB preserves the DFT-modulation property.

Proof: We first show that $S(z) = S'(z^q)$. For this, consider the (j,k) entry of $S(z)$,

$$\sum_{i=0}^{q-1} \tilde{F}_j(zW^i)\, F_k(zW^i). \tag{33}$$

This is a function of z^q. The claim follows since for a q-vector $v(z)$, if

$$c(z) = S(z)\, v(z), \tag{34}$$

then

$$S(z)\, v(zW^k) = S'\left((zW^k)^q\right)\, v(zW^k) = c(zW^k) \tag{35}$$

■

Claim 2: Let

$$H(z) = \left(\frac{z^p-1}{z-1}\,\frac{z^q-1}{z-1}\right)^K Q_1(z). \tag{36}$$

Suppose $E(z)$ represents the DFT-modulated FB. Then, the lowpass filter of $S(z)E(z)$ is of the form

$$H'(z) = \left(\frac{z^p-1}{z-1}\,\frac{z^q-1}{z-1}\right)^K Q_2(z). \tag{37}$$

Proof: Notice that $H'(z)$ is given by

$$H'(z) = \frac{1}{p}\sum_{k=0}^{q-1}\sum_{n=0}^{p-1} \underbrace{H(z\,W_1^k\,W_2^n)\,H(z\,W_2^n)\,\tilde{H}(zW_1^k)}_{G_{n,k}(z)}. \tag{38}$$

where $W_1 = e^{j2\pi/q}$, $W_2 = e^{j2\pi/p}$. By (36), the filter $G_{n,k}(z)$ has K zeros at each of the points:

$$\left\{W_1^{k+r_1}\,W_2^{n+r_2}\right\}_{(r_1,r_2)\in(P\times Q)} \cup \left\{W_1^{r_1}\,W_2^{n+r_2}\right\}_{(r_1,r_2)\in(P\times Q)} \cup \left\{W_1^{k+r_1}\,W_2^{r_2}\right\}_{(r_1,r_2)\in(P\times Q)} \tag{39}$$

where $P = \{1,2,\ldots,p\}$, $Q = \{1,2,\ldots,q\}$. One can collect zeros to verify that

$$G_{n,k}(z) = \left(\frac{z^q-1}{z-1}\right)\left(\frac{z^p-1}{z-1}\right)^K Q_{n,k}(z). \tag{40}$$

Since $H'(z)$ is a linear combination of $G_{n,k}(z)$'s, the claim follows. ■

Truncating the infinite series (31), we obtain quite long filters which generate DFT-modulated FBs with regularity factors and vanishing moments (that are also snug frames). However, the 'practical support' of the filter is indeed much shorter. A desired next step is then to replace the filters with shorter ones. Also, we would like to preserve the number of regularity factors, vanishing moments while loosening the frame bounds as little as possible. The question becomes one of the behavior of the frame bounds when the frame is perturbed, which we consider next.

C. Perturbation of DFT-Modulated FBs

Given an FB, the following theorem provides bounds on the frame bounds when the coefficients of the lowpass filter are altered.

Theorem 3: Let $h(n)$ generate a (p, q) DFT-Modulated FB with frame bounds A, B. Let $g(n) = h(n) + f(n)$. Define

$$R_i = q \sum_{n\in\mathbb{Z}} \left| \sum_{l\in\mathbb{Z}} f\left(pl + i + qn\right)\ f^*(pl + i) \right| \tag{41}$$

for $i = 0, 1, \ldots, p-1$. Set $R = \max\{R_i\}_{i\in\{0,1,\ldots,p-1\}}$. If $R < A$, then $g(n)$ generates a DFT-Modulated FB with bounds

$$A' = A\left(1 - \sqrt{\frac{R}{A}}\right)^2, \quad B' = B\left(1 + \sqrt{\frac{R}{B}}\right)^2. \tag{42}$$

Proof: See the appendix. ■

Roughly, (the proof of) the theorem suggests that each polyphase component of $H(z)$ be simultaneously approximated. This is reasonable since $H(z)$ may be regarded as a p-channel FB. Notice that this is slightly different than approximating $h(n)$ in a least squares sense without paying attention to the errors in each polyphase component. It might turn out that the filter approximating $h(n)$ best in the least squares sense, may be making very small errors in some polyphase components and comparably larger errors in other components. For our purposes an approximation that minimizes the maximum of these errors is more suitable. Such an approximation can be obtained by solving a related problem which is called least squares with a quadratic constraint (LSQI) [17]. Having stated that, we note that in practice, the least squares approximation is adequate for our purposes.

The adequacy of the least squares approximation may be argued as follows. When a long filter is approximated by a shorter one, the approximation is performed in two steps. First, we truncate the filter by selecting a window. This is the first source of error. The truncation (or the selected window) will not have the desired factors (regularity factors and those necessary for vanishing moments). The second source of error is the approximation of this window with a filter that does possess the desired factors. Provided that the same windows are selected for both methods (LSQI approach or the least squares solution), the error from truncation will be the same. If this truncation error is large, it will dominate the total error. Consequently, the choice of the method we use for approximating the truncation will not have much influence on the final frame bounds. Otherwise if the error from truncation is small and if the least squares error is small as well, since the total error (that is, the least squares error) dominates the error of the particular polyphase components' errors, we will end up with a reasonable approximation using the least squares anyway.

In the example that follows, we compare the two approaches and demonstrate that the least squares approach is in fact effective for approximation.

Example 3: In this example we discuss the two methods of approximating a given filter with certain factors. The two methods are namely the LSQI and least-squares methods, to be detailed i the following.

For $p = 2, q = 3$, and 4 regularity factors and vanishing moments, we set,

$$H(z) = \underbrace{\left(\frac{1 - z^{-2}}{1 - z^{-1}} \frac{1 - z^{-3}}{1 - z^{-1}}\right)^4}_{V(z)} \underbrace{\left[1 - z^{-1}(0.92\, e^{j\pi 0.05})\right] \left[1 - z^{-1}(0.92\, e^{-j\pi 0.05})\right]}_{F(z)} \tag{43}$$

as our starting filter (see Figure 7(a)). The reason for introducing the factor $F(z)$ is to improve the frame bounds. The length of the filter is 15. The frame bounds are $A = 0.6395$, $B = 32.5969$ and $B/A = 50.9701$. We apply (31), truncating the infinite sum at $k = 15$ and obtain a new filter $h_2(n)$ with frame bounds $A_2 = 8.9491$, $B_2 = 16.6182$, and $B_2/A_2 = 1.8570$. The length of $h_2(n)$ is 269. However, notice in Figure 7(b) that most of the coefficients are practically zero, so we will replace it by a filter of length 50, $\tilde{h}_2(n)$. We want $\tilde{h}_2(n)$ to have 4 regularity factors as well, i.e., it should be of the form,

$$\tilde{H}_2(z) = V(z)\, Q(z). \tag{44}$$

We crop a window of length 50 from $h_2(n)$ and call it [1] $h_c(n)$. This truncation will contribute to the final error. We denote the error energy of the odd and even samples of $h_2(n) - h_c(n)$ by e_1 and e_2. Now we would like $\tilde{H}_2(z) \approx z^K H_c(z)$ (where K is the greatest integer such that $z^K H_c(z)$ is causal – also assume that K is even, otherwise the roles of e_1 and e_2 should be changed in the following). This approximate equality can be written in matrix form as

$$\mathbf{V}\, \mathbf{q} \approx \mathbf{h}_c \tag{45}$$

where $\mathbf{V}$ is the convolution matrix for $V(z)$, $\mathbf{q}$ the coefficient vector of $Q(z)$ and $\mathbf{h}_c$ the coefficient vector of $H_c(z)$. We can express the polyphase components of $\tilde{H}_2(z)$ separately. Let $\mathbf{V}_1$ and $\mathbf{V}_2$ hold the odd and even rows of $\mathbf{V}$ respectively (and define similarly $\mathbf{h}_{c,1}$ and $\mathbf{h}_{c,2}$). Then the final error energy of the polyphase components can be expressed as $\|\mathbf{h}_{c,1} - \mathbf{V}_1\, \mathbf{q}\|_2^2 + e_1$ and $\|\mathbf{h}_{c,2} - \mathbf{V}_2\, \mathbf{q}\|_2^2 + e_2$. We would like to minimize the maximum of these. Now, instead of this, consider the following problem :

$$\min \epsilon_2 = \|\mathbf{h}_{c,2} - \mathbf{V}_2\, \mathbf{q}\|_2^2 \qquad \text{s.t.} \qquad \|\mathbf{h}_{c,1} - \mathbf{V}_1\, \mathbf{q}\|_2^2 \leq \epsilon_1 \tag{46}$$

[1]In fact, this approach is suboptimal but not severely so. One should repeat the approximation procedure for every possible window of the same length to obtain *optimal* behavior.

This problem is called ‘least squares with a quadratic constraint’ (LSQI) and was investigated thoroughly by Gander in [17] (see also [18], [19]). Provided that ϵ_1 is greater than the least squares error ($\min \|\mathbf{h}_{c,1} - \mathbf{V}_1\,\mathbf{q}\|_2^2$), the solution is unique and can be found by solving what is called the secular equation [17]. Our problem can then be solved by searching for the minimum ϵ_1 value s.t. $\epsilon_2 + e_2 - e_1 \leq \epsilon_1$. In fact, one can calculate ϵ_2 without solving (46) [17], but we do not further discuss this.

Turning back to our problem, we find the solution $\mathbf{q}$ that minimizes ϵ_1 with $\epsilon_2 + e_2 - e_1 \leq \epsilon_1$ and form the filter $\tilde{h}_2(n)$. The frame bounds are now $\tilde{A}_2 = 8.9913$, $\tilde{B}_2 = 16.8638$ with $\tilde{B}_2/\tilde{A}_2 = 1.8756$ (see Figure 7(c)).

If instead we calculate the least squares solution of (45), we obtain another filter (denote by $h_{\text{ls}}(n)$). The frame bounds become $A_{\text{ls}} = 8.9818$, $B_{\text{ls}} = 16.8692$, and $B_{\text{ls}}/A_{\text{ls}} = 1.8782$. The bounds are only slightly worse (compared to those of the filter obtained by solving the LSQI problem) in accordance with the previous arguments.

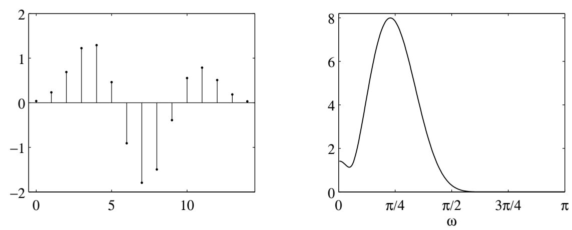
As a final remark, we note that useful characteristics of the LSQI problem like uniqueness and certain monotone behavior in terms of the errors, do not carry over when there is more than one quadratic constraint. This in turn would affect our algorithm for general p, q pairs which could be solved similarly as above by minimizing some least squares error with more than one quadratic constraint. This problem can be attacked by a Lagrange multiplier method but the transformed problem has many local minima. These problems are also sidestepped by using the least square approximation, which can be calculated easily and efficiently.

D. The Algorithm

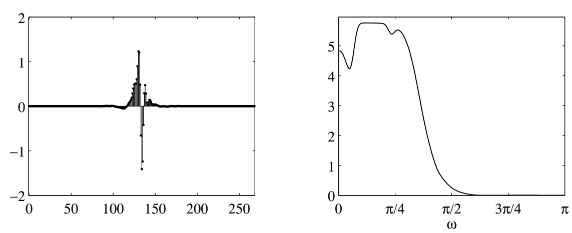
For the final ingredient for a general algorithm let us look at the terms in the series (31). The k^{th} term in this series is

$$\underbrace{\frac{(2k)!}{2^{2k}\,(k!)^2}}_{\leq 1/2^k} \left(I - \frac{2}{A+B}\,S\right)^k . \tag{47}$$

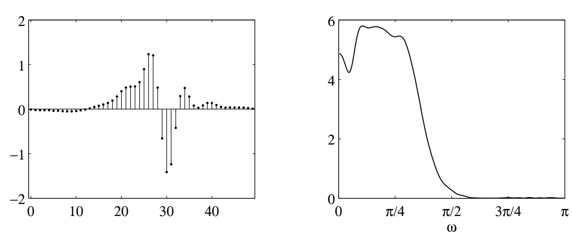
In addition to the rapid decay of the constant term in front, observe that the tighter the frame S, the faster will be the decay of $\left\|I - \frac{2}{A+B}\,S\right\|^k$. That is, the convergence is faster if the starting frame is fairly tight. But how can we find a fairly tight frame? We take an arbitrary frame, apply the series (31) using a *higher than usual* (compared to subsequent stages) number of terms and find the approximating filter that is slightly longer than the filter we started with. This is the first step of our algorithm, which we present below. We do not have a proof of convergence (this is not a hill-climbing algorithm) but we did observe convergence as long as the maximum length allowed for the desired filter is sufficiently high.



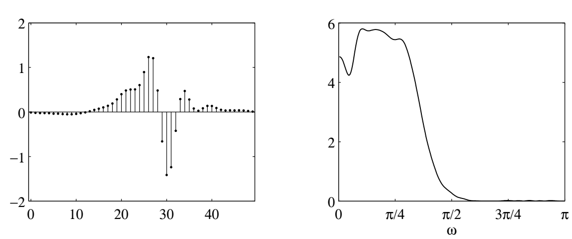
(a)



(b)



(c)



(d)

Fig. 7. The impulse response and the frequency response magnitudes of the filters in Example 3. (a) The starting filter $h(n)$ with ratio of frame bounds $B/A = 50.9701$. (b) The filter $h_2(n)$ after applying (31) to the starting filter; the frame bound ratio is $B_2/A_2 = 1.8570$. (c) 'Least squares with a quadratic inequality' (LSQI) approximation of $h_2(n)$; the frame bound ratio is $\tilde{B}_2/\tilde{A}_2 = 1.8756$. (d) The least squares approximation of $h_2(n)$; the frame bound ratio is $B_{\text{ls}}/A_{\text{ls}} = 1.8782$. The frame bound ratio for the least squares approximation is only slightly worse than that of the LSQI approximation.

Algorithm 2: For given (p,q), number of regularity factors K, and maximum allowed filter length $N_{\max}$;

- Set
$$H(z) = \underbrace{\left(\frac{1-z^{-p}}{1-z^{-1}}\,\frac{1-z^{-q}}{1-z^{-1}}\right)^K}_{V(z)} F(z) \tag{48}$$
as the initial filter.[2]
- Set the maximum length allowed for the end filter N_{max}.
- Set L to be the length of the initial approximation, with $L > \text{length}(H)$.
- Calculate the frame bounds A, B associated with $H(z)$.
- Set $R = B/A$.
- While $R > 1+\epsilon$,
  - Set H_2 as
  $$H_2(z) = \sqrt{\frac{2}{A+B}}\sum_{k=0}^{N}\frac{(2k)!}{2^{2k}\,(k!)^2}\left(I - \frac{2}{A+B}\,S\right)^k H(z). \tag{49}$$
  where N is taken as a 'large' integer (e.g. 60) in the first iteration. (To speed up this step N can be decreased as the frame gets tighter.)
  - Find the segment $w(n)$ of length L in $h_2(n)$ with the greatest energy.
  - Update $H(z)$ to be the filter whose impulse response is given by $\mathbf{V}_L\left(\mathbf{V}_L^T\mathbf{V}_L\right)^{-1}\mathbf{V}_L^T\mathbf{w}$ where $\mathbf{w}$ denotes the samples of $w(n)$.
  - Calculate the new frame bounds A, B for $H(z)$.
  - Update $R = B/A$.
  - If $L < N_{max}$, increment L by one[3].

Example 4: We applied this algorithm for $(p=2,q=3)$, $(p=5,q=6)$, $(p=7,q=8)$ and asked for 4 regularity factors. We set
$$F(z) = \left[1 - z^{-1}(r\,e^{j\theta})\right]\,\left[1 - z^{-1}(r\,e^{-j\theta})\right] \tag{50}$$
with $r = 0.9$ and $\theta = \pi/20$. The results are shown in Figure 8. The frame bound ratios are less than 1.001. In all of the cases, the algorithm took less than 50 iterations. The lengths of the filters are 45, 65 and 100 respectively.

[2]F(z) is introduced to improve the frame bounds of $V(z)$. It can be chosen by performing an optimization on the frame bound ratio, or can be set to 1 otherwise. We remark however that the output of the algorithm does depend on the initialization.

[3]This performs faster than setting $L = N_{max}$ at the beginning of the algorithm.

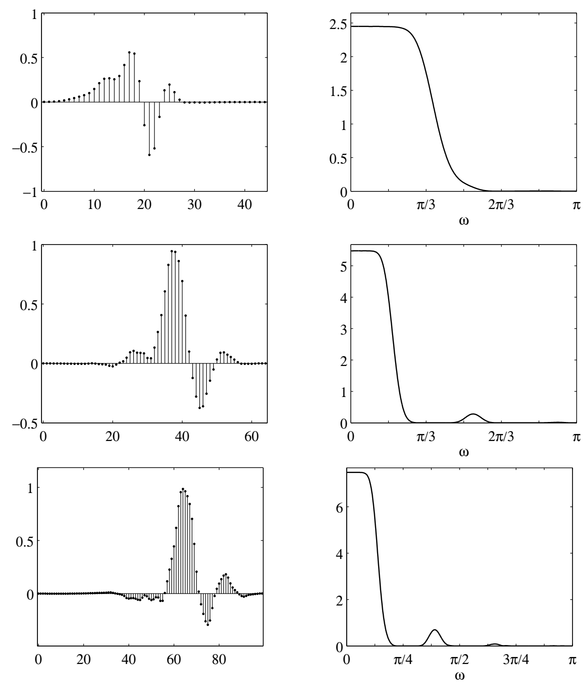


Fig. 8. The result of applying Algorithm 2 with 4 vm. Top panel : $(p = 2, q = 3)$, length = 45; Middle panel: $(p = 5, q = 6)$ length = 65; Bottom panel: $(p = 7, q = 8)$, length = 100.

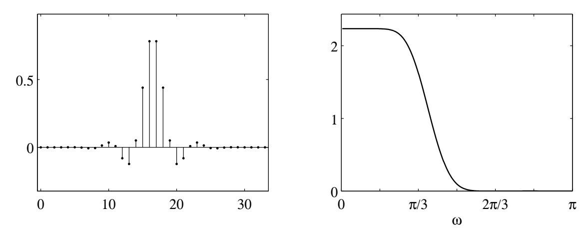


Fig. 9. A symmetric filter for $p = 2, q = 3$ obtained by modifying Algorithm 2. The filter has 4 regularity factors and the length is 34.

E. Linear Phase Filters

One question that might be of importance in certain applications is whether linear phase and DFT-modulation are compatible requests or not. Even though it may not be useful in practice, the filter below suggests that for $p = 2, q = 3$ these requests are not mutually exclusive.

$$\left[\frac{1}{2\sqrt{2}}\ 0\ \frac{1}{2}\ \frac{1}{\sqrt{2}}\ \ -\frac{1}{2\sqrt{2}}\ 0\ \ -\frac{1}{2\sqrt{2}}\ \frac{1}{\sqrt{2}}\ \frac{1}{2}\ 0\ \frac{1}{2\sqrt{2}}\right] \tag{51}$$

In fact, we can slightly modify our algorithm presented in the previous subsection and obtain a

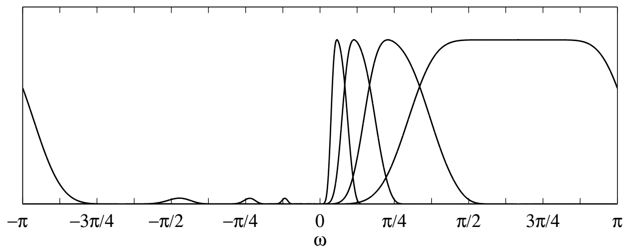


Fig. 10. The frequency responses of $H(z^{2^k}W)\prod_{n=0}^{k-1}H(z^{2^k})$ for $k = 0, 1, 2, 3$ of the iterated DFT-Modulated FB for $p = 2, q = 3$ (and $W = \exp(j2\pi/3)$). The filter with 4 regularity factors from Fig. 9 is used.

symmetric filter for the $p = 2, q = 3$ case with arbitrary number of regularity factors. For this, we set our initialization filter to be symmetric. Then (49) gives a long symmetric filter. Approximating this long filter with a symmetric filter and iterating as in Algorithm 2, we obtain symmetric filters. An example filter is shown in Fig. 9. The number of regularity factors for this filter is 4 and the length is 34.

Less obviously, since the dyadic FB is a special case of the orthonormal rational FB with $p = 1, q = 2$, the algorithm may be used to obtain symmetric filters which are almost PR. However, since this is out of the scope of the paper, we do not further pursue this here.

VI. Iterated Overcomplete DFT-Modulated Filter Banks

We finally would like to discuss iterated DFT-Modulated FBs briefly, in an attempt to motivate the use of the filters designed in this paper. Conventionally, modulated FBs are intended to be used in applications which require many channels and the FBs are not iterated. However, iterated overcomplete DFT-modulated FBs with few channels also possess attractive properties. Consider for example the iterated overcomplete DFT-modulated FB with $p = 2, q = 3$. Provided that the frequency response of $H(z)$ is negligible outside $[-\pi/3, \pi/3]$, the highpass filters $H(zW)$, $H(zW^2)$ will have one-sided spectra. As the FB is iterated, these will provide one-sided spectral analyses of the input as shown in Fig. 10. Also, the two wavelets associated with the FB will be approximately analytic (see Fig. 11). This analyticity property can be seen by considering the infinite product formula. We remark that a real transform may be obtained by using $F_0(z)$, $F_1(z)$ instead of $H(zW)$, $H(zW^2)$ where

$$\begin{pmatrix} F_0(z) \\ F_1(z) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -j & j \end{pmatrix} \begin{pmatrix} H(zW) \\ H(zW^2) \end{pmatrix} \Big/ \sqrt{2}\,. \tag{52}$$

Notice that $F_0(z)$ and $F_1(z)$ will be approximately discrete-time Hilbert transform pairs. This discussion links DFT-modulated FBs to the Dual-Tree Complex Wavelet Transform (DT-$\mathbb{C}$WT) [27]. In fact the

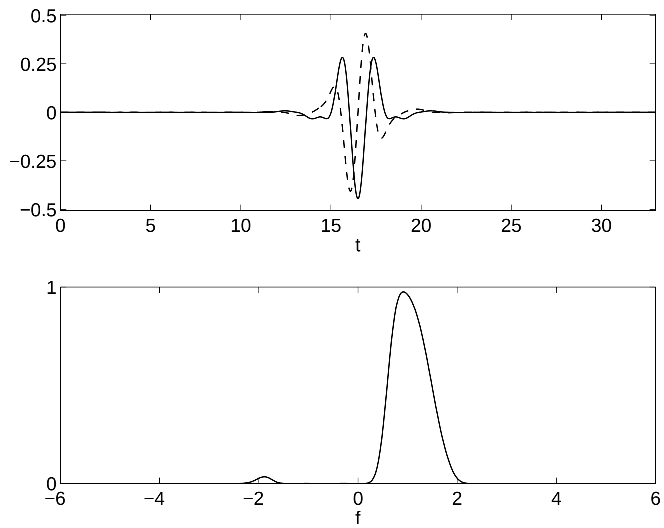


Fig. 11. On the top panel, the real and imaginary parts of one of the wavelets associated with the overcomplete DFT-modulated FB with $p = 2, q = 3$ from Fig. 9 is shown. The real and imaginary parts of the wavelet are symmetric and anti-symmetric respectively, due to the symmetry properties of the underlying filters. On the bottom panel is the Fourier transform of this wavelet. Notice that the wavelet is approximately analytic.

iterated overcomplete DFT-modulated FB with $p = 2, q = 3$ mimicks the DT-$\mathbb{C}$WT, but is less redundant for 1-D signals. Another issue with the DT-$\mathbb{C}$WT is regarding the highpass filters of the first stage. The DT-$\mathbb{C}$WT uses a specific first stage in order to make the frequency responses of the following stages analytic. More precisely, the FBs in the first stage are related by a unit shift. This specific choice makes the frequency response of the first stage of the DT-$\mathbb{C}$WT far from being analytic (see Fig. 12 in [27]) and this undesirable behavior cannot be alleviated through filter design. In contrast, the iterated overcomplete DFT-modulated FB uses the same set of filters in every stage. The first stage's (as well as the other stages') analyticity property can be improved by using a lowpass filter $H(z)$ that is closer to the ideal rectangle filter with frequency support $[-\pi/3, \pi/3]$. The downside of the iterated DFT-Modulated FB is that it will be more redundant in higher dimensions, compared to the dual-tree approach.

VII. Conclusion

In this paper, we showed that the filter design problem posed by the orthonormal rational FBs and overcomplete DFT-modulated FBs are the same. Following this, we provided two methods to obtain filters with regularity factors, which are necessary if the FBs are to be iterated. The first method is based on a parameterization of all such FBs with a single regularity factor. The second method is applicable with an arbitrary number of desired regularity factors. The tradeoff in the second method, which is an iterative algorithm, is between the maximum length allowed and computation time for reaching a snug frame.We also provided a motivation for iterated overcomplete DFT-modulated FBs, namely the implementation of a complex (almost analytic) DWT, that is essentially different from the dual-tree complex wavelet transform but which provides a comparable frequency decomposition.

Appendix

The Proof of Theorem 3

For the proof we will use two lemmas.

Lemma 1: (Corollary 15.1.5 in [13]) Let $\{h_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with frame bounds A, B and let $g_k = h_k + f_k$. If there exists a constant $R < A$ such that,

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq R\|f\|^2, \quad \forall f \in \mathcal{H}, \tag{53}$$

then $\{g_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds

$$A\left(1-\sqrt{\frac{R}{A}}\right)^2, \quad B\left(1+\sqrt{\frac{R}{B}}\right)^2. \tag{54}$$

Proof (Expansion of the sketch in [13]): Since $\{h_k\}_{k=1}^{\infty}$ is a frame with bounds A, B, we have

$$\sqrt{A}\|f\| \leq \sqrt{\sum_{k=1}^{\infty} |\langle f, h_k\rangle|^2} \leq \sqrt{B}\|f\| \qquad \forall f \in \mathcal{H}. \tag{55}$$

By the triangle inequality on l_2, we get

$$\sqrt{\sum_{k=1}^{\infty} |\langle f, h_k\rangle|^2} - \sqrt{\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2} \leq \sqrt{\sum_{k=1}^{\infty} |\langle f, g_k\rangle|^2} \leq \sqrt{\sum_{k=1}^{\infty} |\langle f, h_k\rangle|^2} + \sqrt{\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2} \tag{56}$$

thus,

$$(\sqrt{A}-\sqrt{R})\|f\| \leq \sqrt{\sum_{k=1}^{\infty} |\langle f, g_k\rangle|^2} \leq (\sqrt{B}+\sqrt{R})\|f\|. \tag{57}$$

The result follows by squaring the terms. ■

In the following lemma, $\rho(\mathbf{M})$ denotes the spectral radius of the matrix $\mathbf{M}$. This is a corollary of Gershgorin's Theorem [19].

Lemma 2: Let $\mathbf{M}$ be an $N \times N$ Hermitian matrix with entries denoted by $\mathbf{M}_{j,k}$. If for $B > 0$,

$$\sum_{k=1}^{N} |\mathbf{M}_{j,k}| \leq B, \quad \forall j \in \{1, 2, \ldots, N\} \tag{58}$$

then $\rho(\mathbf{M}) \leq B$.

We can now proceed to the proof of theorem 3.

Proof of Theorem 3: Notice that if $\{h_k(n)\}$ and $\{g_k(n)\}$ are DFT-modulated FBs, then so is $\{f_k(n) = h_k(n) - g_k(n)\}$. We want to bound the frame operator for the DFT-modulated FB generated by $f(n) = h(n) - g(n)$. Let the polyphase components of $F(z)$ be denoted by $F_k(z)$ which are defined through

$$F(z) = \sum_{k=0}^{p-1} z^{-qn}\, F_k(z^p). \tag{59}$$

Then the $(j+1, k+1)^{\text{th}}$ entry of the frame operator for the DFT-modulated FB generated by $F(z)$ is given by

$$\sum_{i=0}^{q-1} \tilde{F}_j(zW^i)\, F_k(zW^i). \tag{60}$$

On the unit circle, this is equal to

$$\mathcal{F}\left\{(\downarrow q)\, \tilde{f}_j(n) * f_k(n)\right\} \tag{61}$$

where $\mathcal{F}$ and $(\downarrow q)$ denote the DTFT and downsampling by q operators respectively. (61) is bounded by,

$$q \sum_{n\in\mathbb{Z}} \left| \left(\tilde{f}_j * f_k\right)(qn) \right| = q \sum_{n\in\mathbb{Z}} \left| \sum_{l\in\mathbb{Z}} f(pqn + pl + qk)\, f^*(pl + qj) \right| \tag{62}$$

$$= q \sum_{n\in\mathbb{Z}} \left| \sum_{l\in\mathbb{Z}} f(pl + i + q(k-j) + pqn)\, f^*(pl + i) \right| \tag{63}$$

where $i = \mod(qj, p)$. Then the $(j+1)^{\text{th}}$ row of $\mathbf{F}(e^{j\omega})$ is bounded by,

$$q \sum_{k=0}^{p-1} \sum_{n\in\mathbb{Z}} \left| \sum_{l\in\mathbb{Z}} f(pl + i + q(k-j) + pqn)\, f^*(pl + i) \right| = q \sum_{n\in\mathbb{Z}} \left| \sum_{l\in\mathbb{Z}} f\left(pl + i + qn\right)\, f^*(pl + i) \right| \tag{64}$$

This is equal to R_i in the statement of the theorem. Since the mapping of j to i via $i = \mod(qj, p)$ is 1-1, by Lemma 2, $\left|\mathbf{F}(e^{j\omega})\right|$ is bounded by $\max R_i$. Applying Lemma 1 finishes the proof. ■

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