1D Discrete Fourier Transform

Let $x(n)$ be a signal of length N.

$$X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) \, \exp\left(-j \frac{2\pi}{N} k \, n\right)$$

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DFT may be regarded as Sampled DTFT...

$$\mathsf{DTFT}\left\{x\right\} = \sum_{n=0}^{N-1} x(n) \, \exp\left(-j \, \omega \, n\right).$$

Properties of 1D DFT

(1) Linearity

$$a\, x_1(n) + b\, x_2(n) \longleftrightarrow a\, X_1(k) + b\, X_2(k)$$

(2) Periodic Convolution

$$x_1(n) \circledast x_2(n) \longleftrightarrow \sqrt{N}\, X_1(k)\, X_2(k)$$

(3) Multiplication

$$\sqrt{N}\, x_1(n)\, x_2(n) \longleftrightarrow X_1(k) \circledast X_2(k)$$

(4) Shift

$$x(n-m) \longleftrightarrow X(k)\, \exp\left(-j\frac{2\pi}{N}\, k\, m\right)$$

Properties of 1D DFT

(5) Parseval's Theorem

$$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |X(k)|^2$$

(6) Symmetry for real $x(n)$

$$X(k) = X^*(N-k)$$

2D Discrete Fourier Transform

Let $x(n_1, n_2)$ be a signal of length $N_1 \times N_2$.

$$X(k_1, k_2) = \frac{1}{\sqrt{N_1 N_2}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) \exp\left[-j\left(\frac{2\pi}{N_1} k_1 n_1 + \frac{2\pi}{N_2} k_2 n_2\right)\right]$$

$$x(n_1, n_2) = \frac{1}{\sqrt{N_1 N_2}} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X(k_1, k_2) \exp\left[j\left(\frac{2\pi}{N_1} k_1 n_1 + \frac{2\pi}{N_2} k_2 n_2\right)\right]$$

DFT may be regarded as Sampled DTFT...

$$\text{DTFT}\{x\} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) \exp\left[-j\ (\omega_1 n_1 + \omega_2 n_2)\right].$$

2D Discrete Fourier Transform

Notice that the DFT is a separable transform.

To obtain $X(k_1, k_2)$,

(1) Apply (length N_1) 1D DFT on the first (horizontal) variable of $x(n_1, n_2)$ to get $\tilde{X}(k_1, n_2)$.

(2) Apply (length N_2) 1D DFT on the second (vertical) variable of $\tilde{X}(k_1, n_2)$ to get $X(k_1, k_2)$.

Properties of 2D DFT

(1) Linearity

$$a\,x_1(n_1+n_2)+b\,x_2(n_1+n_2)\longleftrightarrow a\,X_1(k_1,k_2)+b\,X_2(k_1,k_2)$$

(2) Periodic Convolution

$$x_1(n_1,n_2)\circledast x_2(n_1,n_2)\longleftrightarrow \sqrt{N_1\,N_2}\quad X_1(k_1,k_2)\,X_2(k_1,k_2)$$

(3) Multiplication

$$\sqrt{N_1\,N_2}\quad x_1(n_1,n_2)\,x_2(n_1,n_2)\longleftrightarrow X_1(k_1,k_2)\circledast X_2(k_1,k_2)$$

(4) Shift

$$x(n_1-m_1,n_2-m_2)\longleftrightarrow X(k_1,k_2)\,\exp\left[-j\left(\frac{2\pi}{N_1}k_1\,m_1+\frac{2\pi}{N_2}k_2\,m_2\right)\right]$$

Properties of 2D DFT

(5) Separability

$$x(n_1, n_2) = x_1(n_1)\, x_2(n_2) \Longleftrightarrow X(k_1, k_2) = X_1(k_1)\, X_2(k_2)$$

(6) Parseval's Theorem

$$\sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} |x(n_1, n_2)|^2 = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} |X(k_1, k_2)|^2$$

(7) Symmetry for real $x(n)$

$$X(k_1, k_2) = X^*(N_1 - k_1, N_2 - k_2)$$

1D Discrete Cosine Transform

Let $x(n)$ be a signal of length N.

$$C(k) = \sum_{n=0}^{N} x(n)\, 2\, \cos\left[\frac{\pi}{2N}\, k\,(2n+1)\right]$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N} C(k)\, w(k)\, \cos\left[\frac{\pi}{2N}\, k\,(2n+1)\right]$$

where

$$w(k) = \begin{cases} 1/2, & \text{if } k = 0, \\ 1, & \text{if } 1 \leq k \leq N. \end{cases}$$

Extension to 2D

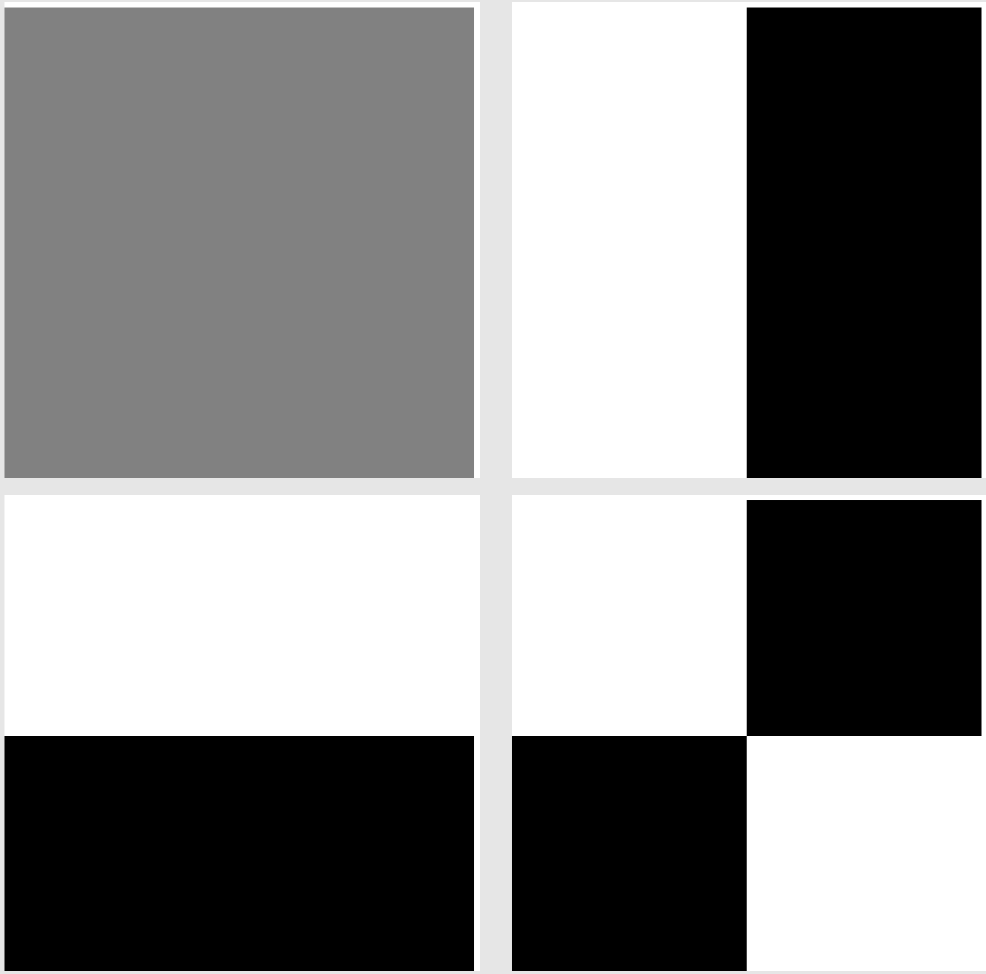
Extends as a separable transform.

For an $N_1 \times N_2$ signal $x(n_1, n_2)$, to obtain $C(k_1, k_2)$,

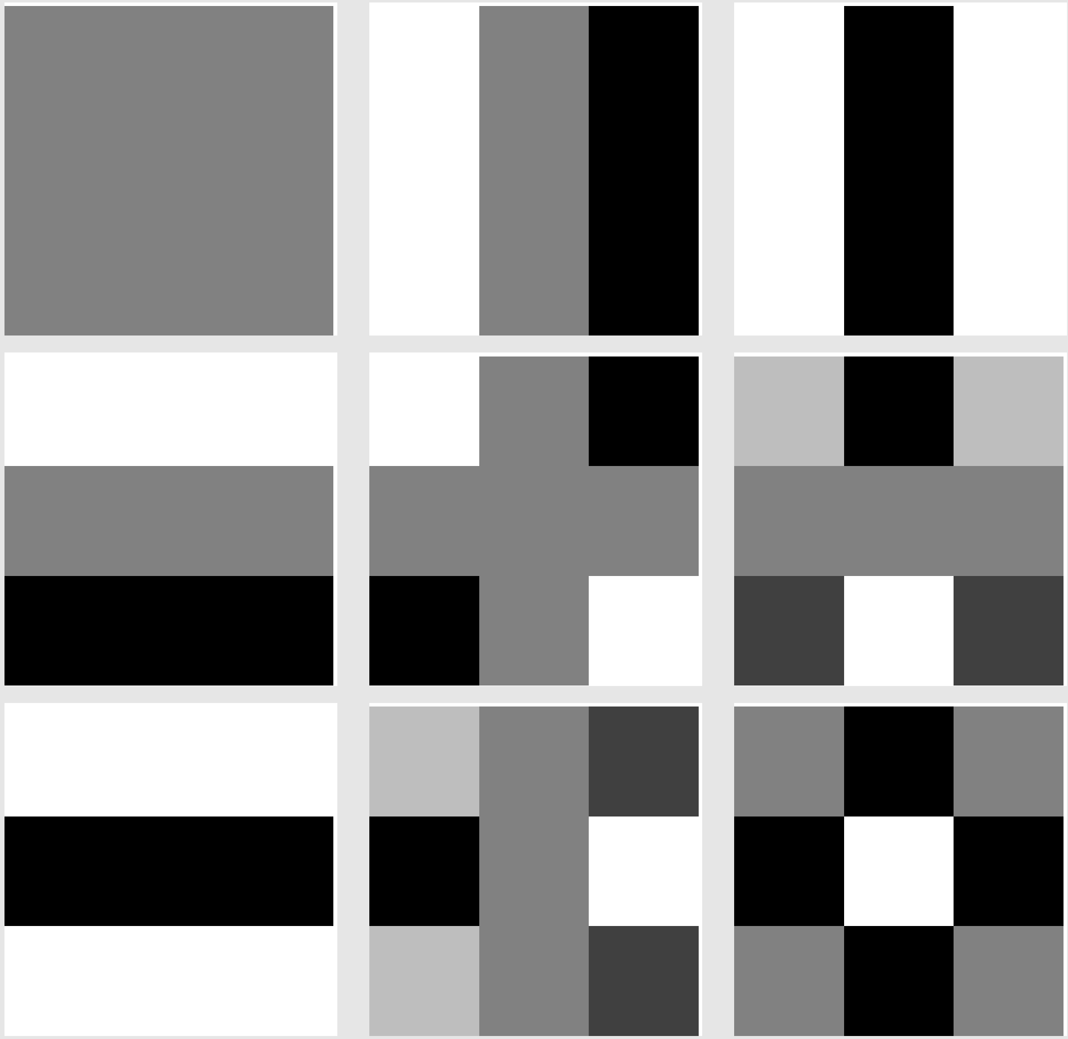
(1) Apply (length N_1) 1D DCT on the first (horizontal) variable of $x(n_1, n_2)$ to get $\tilde{C}(k_1, n_2)$.

(2) Apply (length N_2) 1D DCT on the second (vertical) variable of $\tilde{C}(k_1, n_2)$ to get $C(k_1, k_2)$.

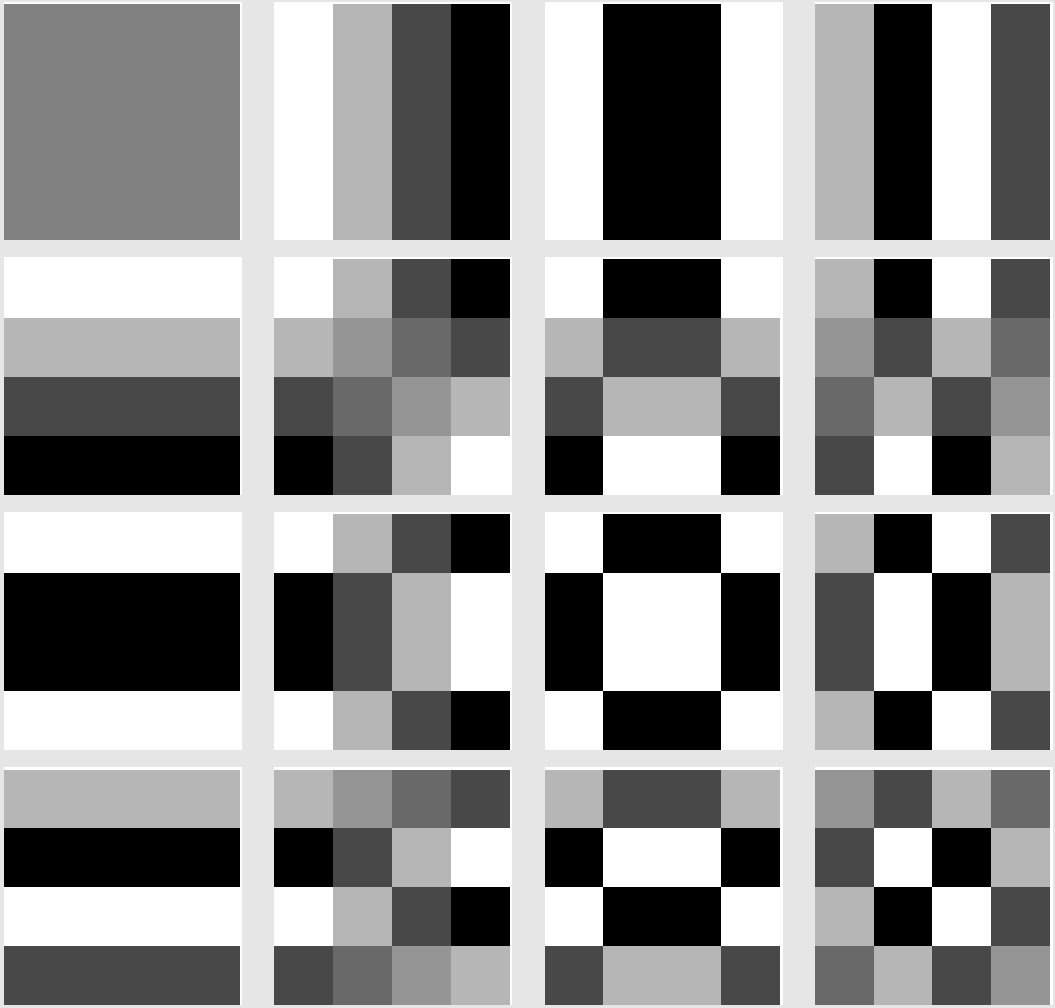
Basis functions for the 2D DCT (2×2)



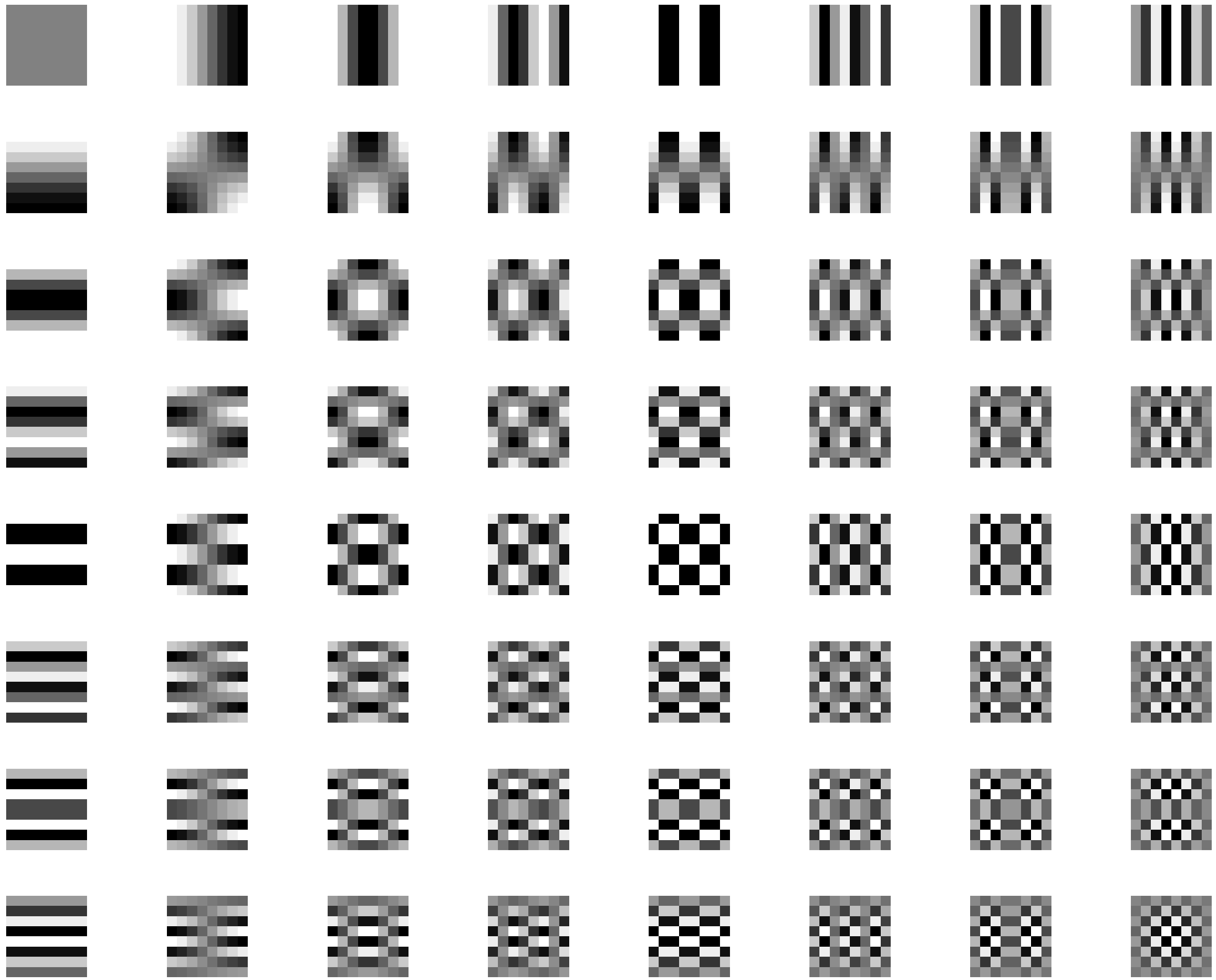
Basis functions for the 2D DCT (3×3)



Basis functions for the 2D DCT (4×4)



Basis functions for the 2D DCT (8×8)



Block DCT

For an $N \times N$ image, one typically does not apply an $N \times N$ DCT.

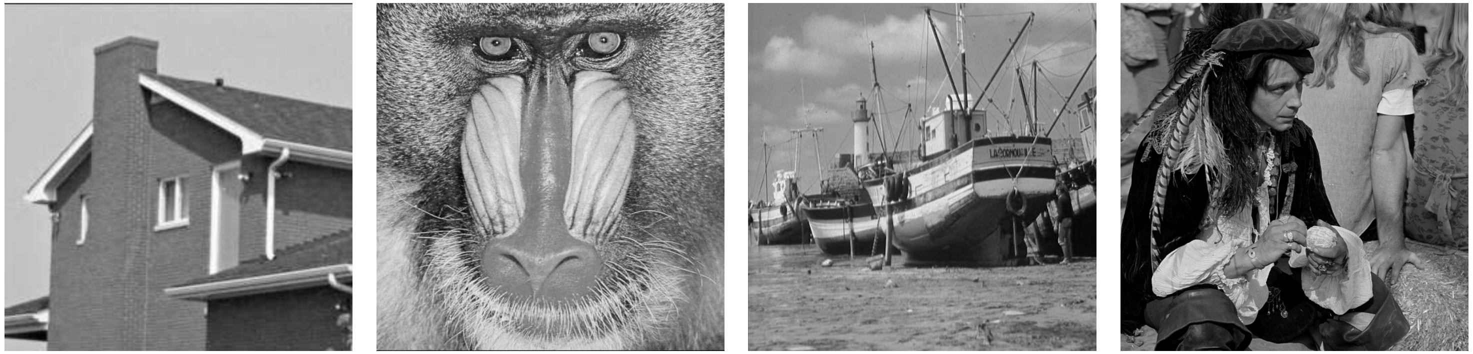
Rather, the image is first cut into smaller blocks (like 8×8) and block-size DCT is applied to each block separately.

Advantage : This yields a more *local* transform.

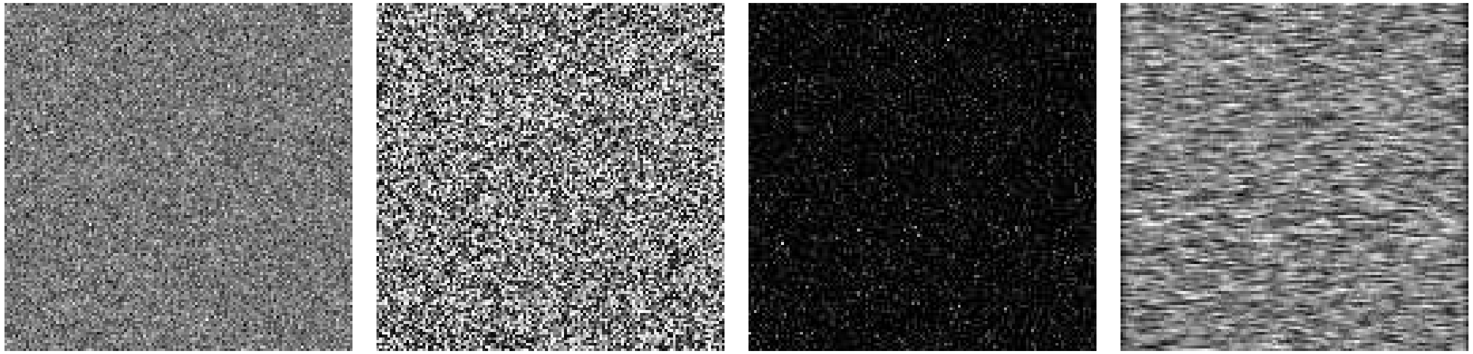
Disadvantage : Blocking artifacts.

Why Use Different Transforms?

'Natural' Images



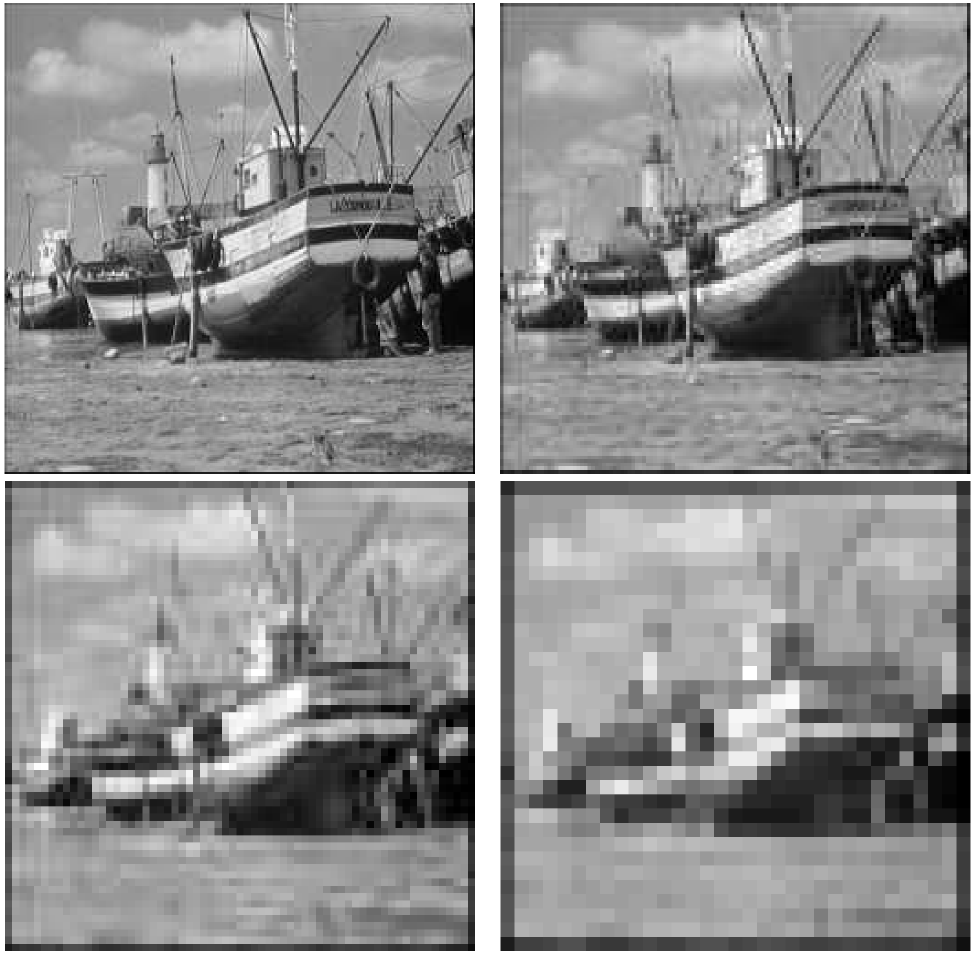
'Unnatural' Images



Laplacian Pyramid

$\Longrightarrow$

g_1 g_2

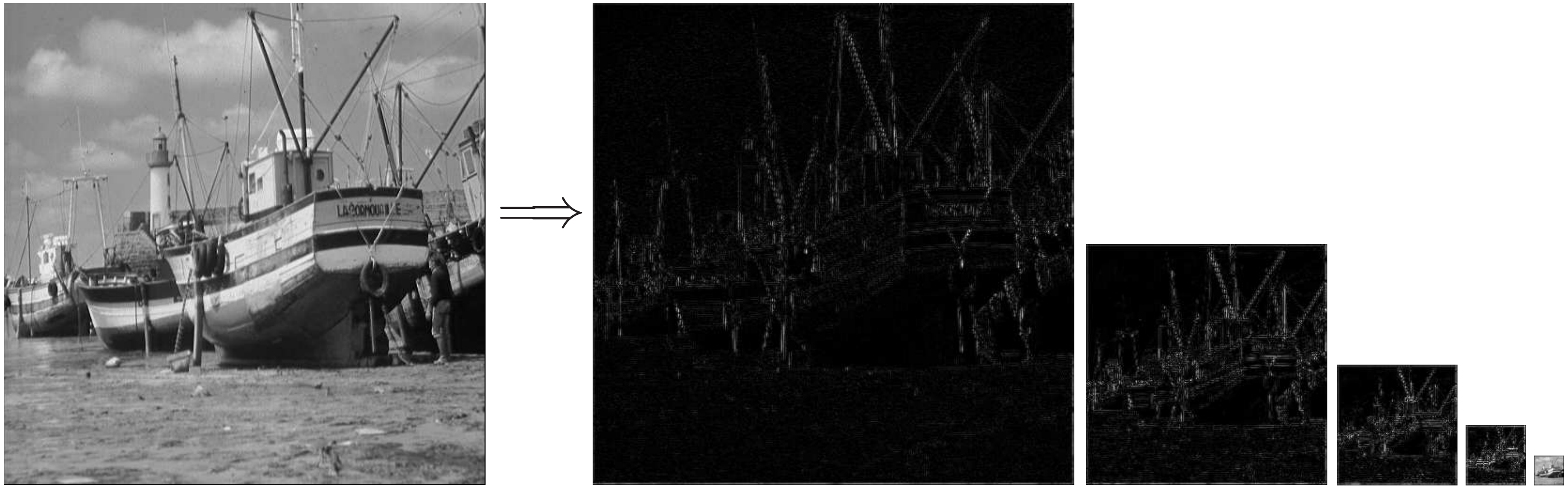


g_3 g_4

Laplacian Pyramid

$\Longrightarrow$

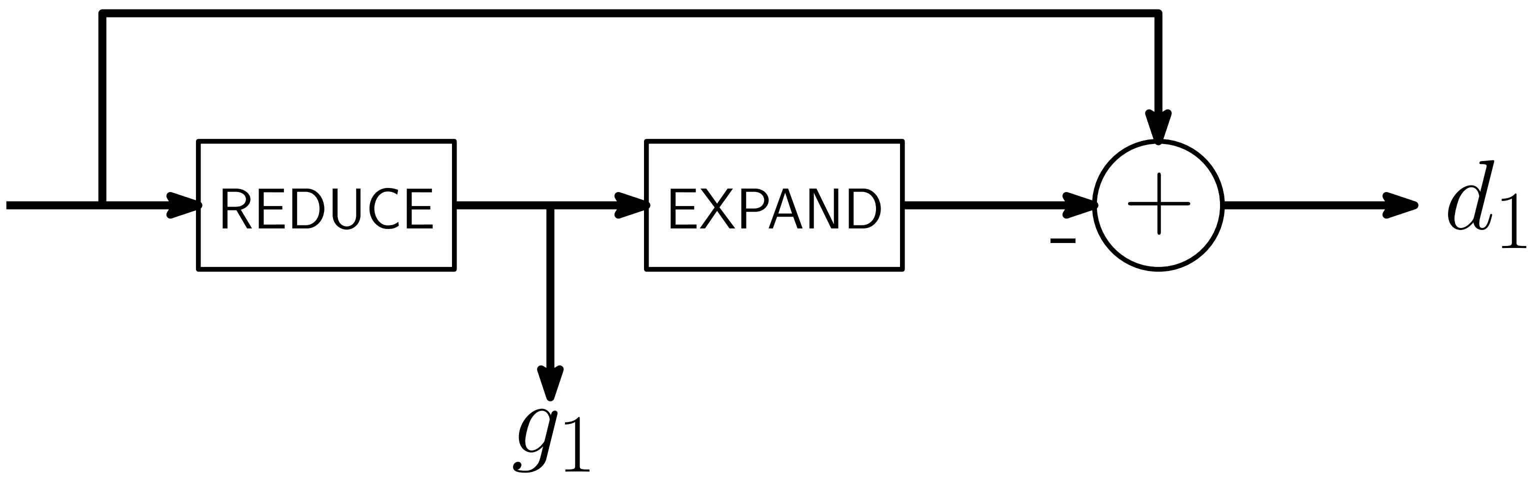
Laplacian Pyramid



Laplacian Pyramid

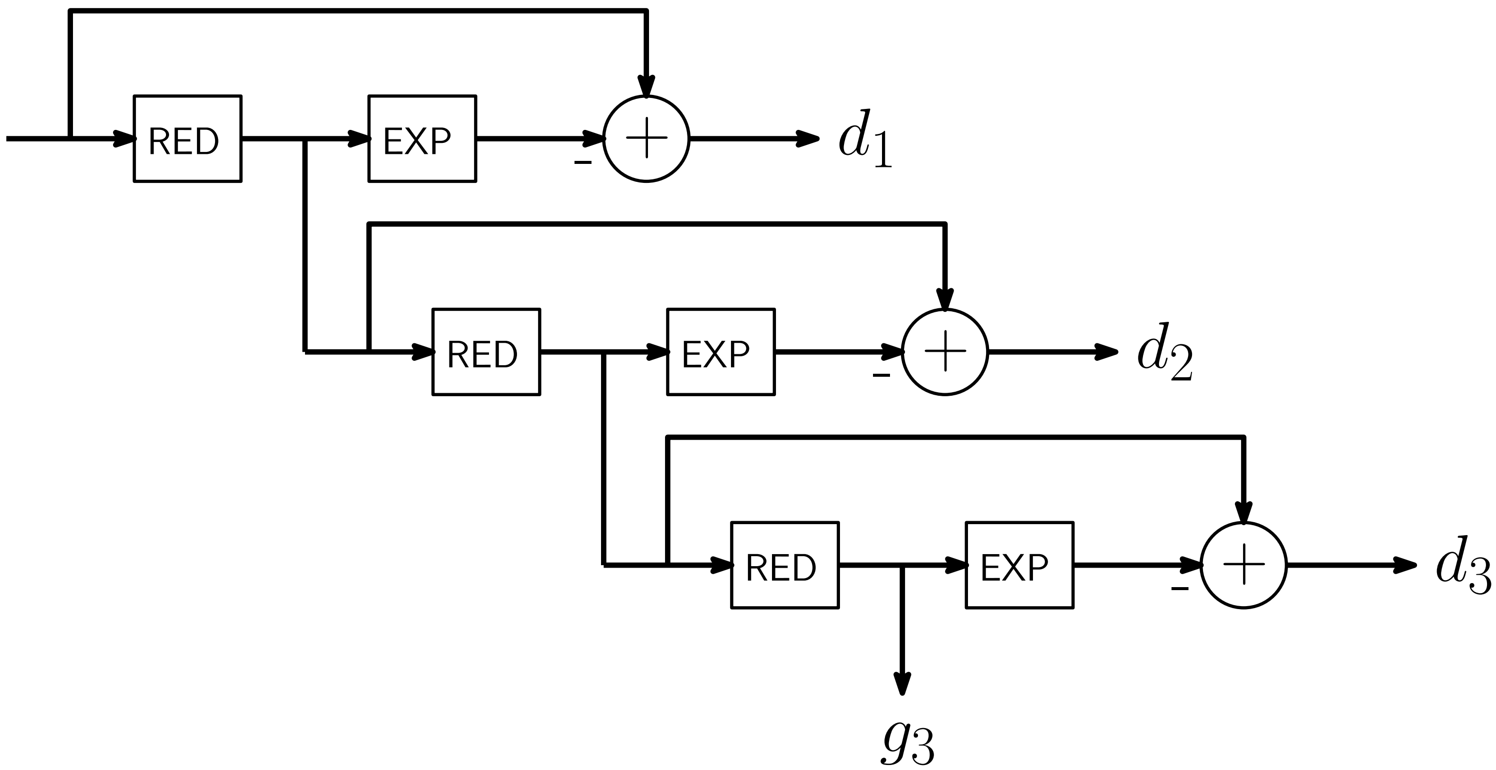
Laplacian Pyramid

Decomposition into a Coarse and Detail Image



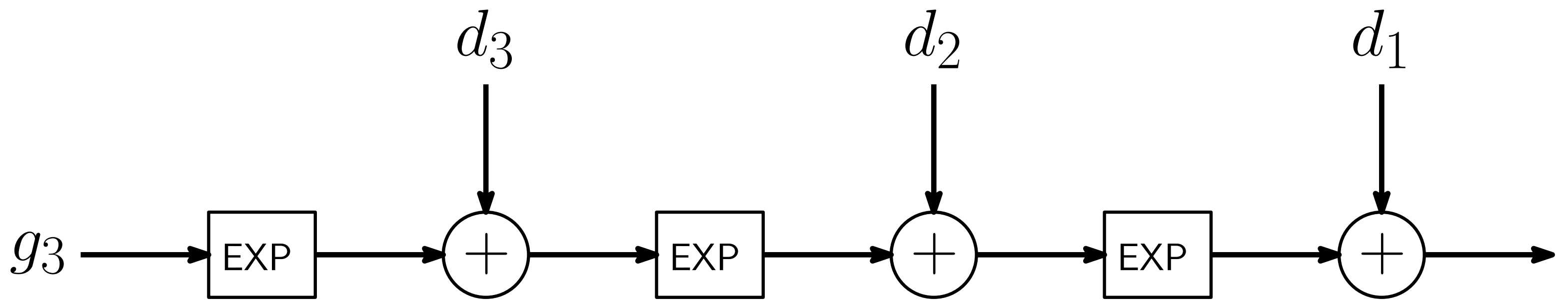
Laplacian Pyramid

Iterate the Basic Block for a Multiresolution Decomposition



Laplacian Pyramid

Reconstruction



Laplacian Pyramid of a Noise Image