MAT 281E – Linear Algebra and Applications

Midterm Examination II

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5 Questions, 120 Minutes

Please Show Your Work!

(10 pts) 1. Consider the space S, spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Construct a matrix A such that $C(A) = S$ (here $C(A)$: the column space of A).

(b) Find a vector from the orthogonal complement of S.

(a) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $x \in S^{\perp} = (C(A))^{\perp} = N(A^T)$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

set $c = 1 \Rightarrow b = -1,\ a = 2$

$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \perp S$

(20 pts) 2. Suppose that A is a 3×3 matrix, whose rank is 2 (i.e. it has 2 independent columns) and

$$\mathbf{v}_1^T A = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix},$$
$$\mathbf{v}_2^T A = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

(a) What is the dimension of $N(A^T)$, the left nullspace of A?

(b) Find a basis for $N(A^T)$.

(c) Find the matrix P that projects any point to $N(A^T)$.

(d) Find the matrix Q that projects any point to $C(A)$, the column space of A.

(a) $\dim N(A^T) = 1$

(b) $A^T(v_1 - 2v_2) = 0$

$N(A^T) = \{\alpha (v_1 - 2v_2)\} = \left\{\alpha \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}\right\} \Rightarrow \underbrace{\left\{\begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}\right\}}_{c}$ is a basis.

(c) $P = \dfrac{cc^T}{c^Tc} = \begin{bmatrix} 9 & 0 & -12 \\ 0 & 0 & 0 \\ -12 & 0 & 16 \end{bmatrix} / 25$

(d) $Q = I - P = \begin{bmatrix} 16/25 & 0 & 12/25 \\ 0 & 1 & 0 \\ 12/25 & 0 & 9/25 \end{bmatrix}$

(15 pts) 3. Consider the lines $l_1 = \left(x,\, 2x,\, x+3,\, -x\right)$, $l_2 = \left(1-y,\, -2y,\, -1-y,\, 2\right)$ in $\mathbb{R}^4$. Find two points $p \in l_1$, $q \in l_2$ that minimize $\|p-q\|$.

$$\underbrace{\begin{bmatrix} x \\ 2x \\ x+3 \\ -x \end{bmatrix}}_{p} - \underbrace{\begin{bmatrix} 1-y \\ -2y \\ -1-y \\ 2 \end{bmatrix}}_{q} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} - \underbrace{\begin{bmatrix} 1 \\ 0 \\ -4 \\ 2 \end{bmatrix}}_{b} = e$$

To minimize $\|e\|$, $\begin{bmatrix} x \\ y \end{bmatrix} = (A^TA)^{-1}A^Tb$

$$A^TA = \begin{bmatrix} 7 & 6 \\ 6 & 6 \end{bmatrix}, \quad A^Tb = \begin{bmatrix} -5 \\ -3 \end{bmatrix} \Rightarrow \begin{matrix} x = -2 \\ y = \frac{9}{6} = \frac{3}{2} \end{matrix}$$

$$\Rightarrow p = \begin{bmatrix} -2 \\ -4 \\ 1 \\ 2 \end{bmatrix}, \quad q = \begin{bmatrix} -1/2 \\ -3 \\ -5/2 \\ 2 \end{bmatrix}$$

$$\left(\text{Notice } p-q = \begin{bmatrix} -3/2 \\ -1 \\ 7/2 \\ 0 \end{bmatrix} \in N(A^T) \right)$$

(25 pts) 4. Let V be a subspace in $\mathbb{R}^3$ spanned by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and l, a line described as $l = \Big(x,\ 1,\ -x\Big)$.

(a) Find two points $p \in V$, $q \in l$ that minimize $\|p - q\|$.

(b) Find two more points $\tilde{p} \in V$, $\tilde{q} \in l$, such that $\tilde{p} \neq p$, $\tilde{q} \neq q$ and $\|p - q\| = \|\tilde{p} - \tilde{q}\|$.

(a)

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}}_{p} - \underbrace{\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)}_{q} = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} - \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{b} = e$$

To minimize $\|e\|$, solve $A^TA\,X = A^Tb$.

$$A^TA = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad A^Tb = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc:c} 2 & -1 & 1 & 0 \\ -1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 2 & -1 & 1 & 0 \\ 0 & 3/2 & 3/2 & 1 \\ 0 & 3/2 & 3/2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 2 & -1 & 1 & 0 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

One solution: $z = 0$, $y = 2/3$, $x = 1/3 \Rightarrow p = \begin{bmatrix} 2/3 \\ 2/3 \\ 0 \end{bmatrix}$, $q = \begin{bmatrix} 1/3 \\ 1 \\ -1/3 \end{bmatrix}$

(Notice $p - q = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}$, $A^T(p-q) = 0$)

(b) For another pair of pts, set $z=1$.

$\Rightarrow y = -1/3, \quad x = -2/3$

$$\Rightarrow p = \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \end{bmatrix}, \quad q = \begin{bmatrix} -2/3 \\ 1 \\ 2/3 \end{bmatrix}$$

$$\left(\text{Notice } p - q = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \end{bmatrix}, \quad A^T(p-q) = 0 \right)$$

(30 pts) 5. (a) Suppose we are given

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

that span $\mathbb{R}^3$.

Let $\mathbf{q}_1 = \alpha\,\mathbf{a}_1$ where α is a scalar. Select α and find two more vectors $\mathbf{q}_2$, $\mathbf{q}_3$, using the Gram-Schmidt procedure, such that $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

(b) Consider the plane P described by the equation $x + y + z = 3$. Find the closest point of P to $(1, 2, 3)$.

(a) $\alpha = \frac{1}{\sqrt{3}}$

$$\tilde{q}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} / \sqrt{6}$$

$$\tilde{q}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} / \sqrt{2}$$

(b) any point $\in \mathbb{R}^3$ is described by

$c = q_1 \alpha_1 + q_2 \alpha_2 + q_3 \alpha_3$ where $\alpha_i = \langle c, q_i \rangle$

If $c \in P \Rightarrow \alpha_1 = \frac{c_1 + c_2 + c_3}{\sqrt{3}} = \sqrt{3}$

$$\Rightarrow \quad c - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} q_2 & q_3 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{= \begin{matrix} 0 \\ -1 \\ -2 \end{matrix}}$$

$\Rightarrow$ we would like to minimize $\left\| \underbrace{\begin{bmatrix} q_2 & q_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix}}_{\alpha} - \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{b} \right\|$

$$A^T A \alpha = A^T b$$

$$A^T A = \begin{bmatrix} q_2^T \\ q_3^T \end{bmatrix} \begin{bmatrix} q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/\sqrt{6} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\Rightarrow p = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3/6 \\ -3/6 \\ +6/6 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\left(\text{Notice} \quad p - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$