TEL 311E – Digital Signal Processing

Fall 2010

Instructor :	İlker Bayram EEB 1103 `ibayram@itu.edu.tr`
Class Meets :	13.30 – 16.30, Monday EEB, Devreler ve Sistemler Seminer Odası (next to 2311)
Office Hours :	10.00 – 12.00, Monday
Textbook :	A. V. Oppenheim, R. W. Schafer and J. R. Buck, 'Discrete-Time Signal Processing', 2^{nd} Edition, Prentice Hall.
Grading :	Homeworks (10% total), 2 Midterms (20% each), Final (50%).

Tentative Course Outline

1. Discrete-Time Signals and Systems
    - Basic Sequences (2.1)
    - Properties of Discrete-Time Systems (2.2)
    - Linear Time-Invariant (LTI) Systems (2.3, 2.4)
    - Discrete-Time Fourier Transform (DTFT) (2.6 – 2.9)
2. The z-Transform
    - Definition, Region of Convergence (3.1, 3.2)
    - Inverse z-Transform (3.3)
    - Properties of the z-Transform (3.4)
3. Sampling
    - Sampling of Continuous-Time Signals (4.1, 4.2)
    - Reconstruction from Samples (4.3)
    - Discrete-Time Processing of Continuous-Time Signals (4.4)
    - Changing the Sampling Rate (4.6)
4. Linear Time-Invariant Systems
    - Frequency Response of LTI Systems (5.1)
    - Linear Constant Coefficient Difference Equations (5.2)
    - Frequency Response of Rational Systems (5.3)
    - Magnitude and Phase (5.4)
5. Discrete Fourier Transform

- Discrete Fourier Series (8.1, 8.2)
- Relation with the Fourier Transform (8.3,8.4)
- Discrete Fourier Transform (8.5, 8.6, 8.7)
- Fast Fourier Transform (9.3)

6. Filter Design
   - Impulse Invariance (7.1)
   - Bilinear Transformation (7.1)
   - Windowing (7.2)

TEL 311E – Homework 1

Due 11.10.2010

1. Consider the system given by,

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)\, x^2(k),$$

where $x(n)$ is the input and $y(n)$ is the output. Assume that $h(n) = 0$ for $n < 0$ and $n > 50$. Specify whether the system is
(a) Memoriless, (b) Linear, (c) Time-invariant, (d) Causal, (e) Stable.
Please explain your answers. If information is insufficient, write 'insufficient information' (and explain why you think so).

2. Repeat the first question for the system given by

$$y(n) = \sum_{k=-\infty}^{\infty} h^2(n+k)\, x(k),$$

where $x(n)$ is the input and $y(n)$ is the output (notice the sign change in 'k' in argument of 'h'). Assume now that $h(n) = 0$ for $n > 0$.

3. Consider a cascade of two LTI systems as shown below.

$$x(n) \longrightarrow \boxed{T_1} \xrightarrow{\quad z(n) \quad} \boxed{T_2} \longrightarrow y(n)$$

We mentioned without proof that the overall system is also LTI. Here you will show it in two steps. You can make use of the intermediate signal $z(n)$ if you like.

(a) Assuming T_1 and T_2 are time-invariant, show that the overall system is time-invariant.

(b) Assuming T_1 and T_2 are linear, show that the overall system is linear.

TEL 311E – Homework 2

Due 18.10.2010

1. Let $H(e^{j\omega})$ be the ideal filter with cutoff at $\pi/2$ given by,

$$H(e^{j\omega}) = \begin{cases} 0 & \text{for} \quad -\pi \leq \omega < -\pi/2, \\ 1 & \text{for} \quad -\pi/2 \leq \omega < \pi/2, \\ 0 & \text{for} \quad \pi/2 \leq \omega < \pi. \end{cases}$$

   (a) Compute the convolution of $H(e^{j\omega})$ with itself, i.e.,

$$G(e^{j\omega}) = \int_{-\pi}^{\pi} H(e^{j(\omega-\theta)})\, H(e^{j\theta})\, d\theta.$$

   Sketch $H(e^{j\omega})$ and $G(e^{j\omega})$.

   (b) We derived in class the discrete-time sequence $h(n)$ associated with $H(e^{j\omega})$ through the inverse DTFT relation. Specify the inverse-DTFT of $G(e^{j\omega})$. (Hint: Make use of the DTFT theorems.)

2. In this question, you will derive a more general form of Parseval's relation, that states

$$\sum_{n=-\infty}^{\infty} x(n)\, y^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, Y^*(e^{j\omega})\, d\omega. \tag{1}$$

   You can show this result in two steps.

   (a) Using the convolution theorem and the symmetry properties of DTFT, determine, in terms of $x(n)$ and $y(n)$, the sequence $z(n)$, whose DTFT is $X(e^{j\omega})\, Y^*(e^{j\omega})$.

   (b) Using the result of part (a) and the inverse DTFT relation, deduce eqn.(1). (Hint : Consider a particular sample of $z(n)$.)

3. Let $x(n) = (3)^n\, u(n+2) - (1/2)^n\, u(-n)$. Find the z-transform of $x(n)$. Sketch the pole-zero diagram and specify the ROC on the diagram.

4. Suppose that $x(n)$ is a causal finite-duration sequence with $x(n) = 0$ for $n > 3$. Suppose we also know that $X(e^{j\pi/4}) = X(e^{j\pi}) = X(e^{-j\pi/4}) = 0$ and $X(e^{j\,0}) = 1$. What is $X(z)$? Sketch the pole-zero plot.
(Hint : How do we express a polynomial in terms of its roots? – See also the 'Fundamental Theorem of Algebra'.)

TEL 311E – Homework 3

Due 25.10.2010

1. Let $x(n) = (3)^n\, u(-n+2) - (1/2)^n\, u(n)$. Find the z-transform of $x(n)$. Sketch the pole-zero diagram and specify the ROC on the diagram.

2. Suppose that the z-transform of the step response (i.e. the response when a unit-step function, $u(n)$, is input to the system) of a stable LTI system is given by

$$X(z) = \frac{1}{1 + \frac{1}{2}z} + \frac{1}{1 - z^{-1}}.$$

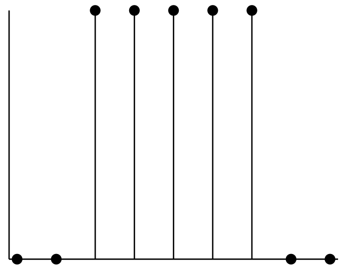
   What should be the ROC? Determine the impulse response, $h(n)$ of this system.

3. Suppose that the z-transform of the impulse response of a stable LTI system is given by

$$H(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})\,(1 - \frac{3}{4}z^{-1})\,(1 - \frac{5}{6}z^{-1})} + \frac{2}{(1 - 2z^{-1})\,(1 - 3z^{-1})\,(1 + 4z^{-1})}.$$

   What is the ROC? Determine $h(1)$.

4. On the second page are shown four signals, their DTFT magnitudes, and the pole-zero diagrams. But they are not in the correct order. Put them in the correct order by matching each signal with its DTFT magnitude and pole-zero diagram.



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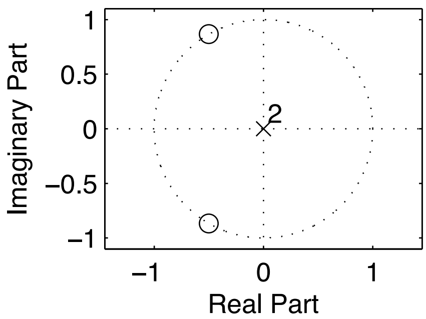

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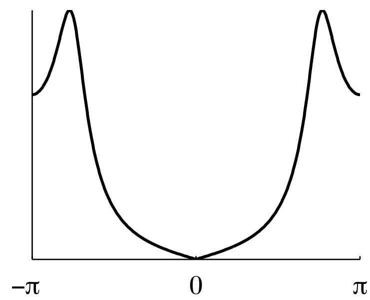

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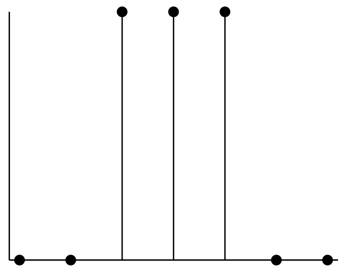

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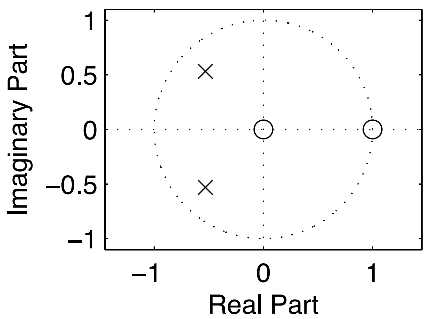

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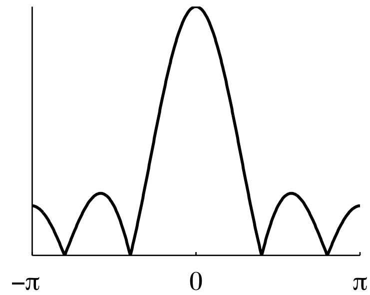

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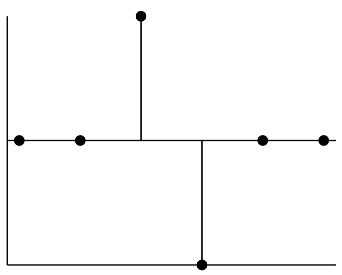

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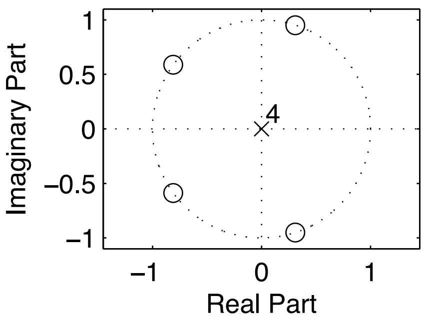

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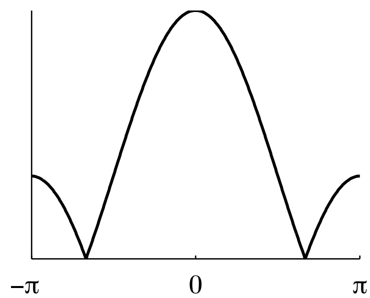

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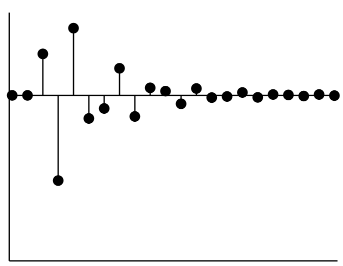

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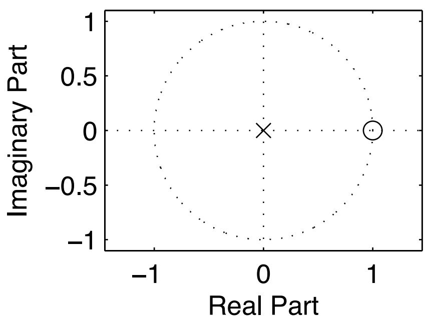

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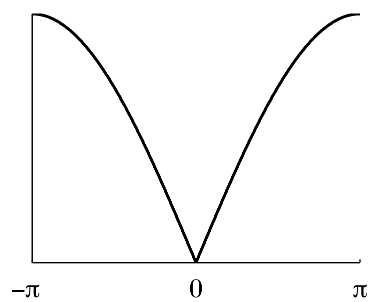

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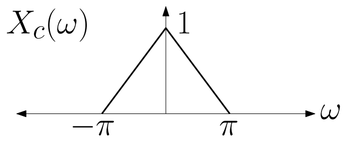
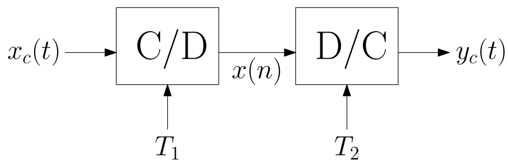
TEL 311E – Homework 4

Due 29.11.2010

1. Let $x_c(t)$ be a continuous-time bandlimited signal whose Fourier Transform is :



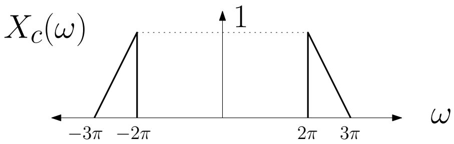
Consider the system composed of a C/D converter followed by a D/C converter.



(a) For a C/D sampling period of 2 sec ($T_1 = 2$), find $X(e^{j\omega})$.

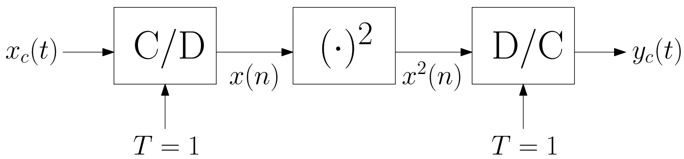
(b) For a D/C sampling period of $T_2 = 1$ sec, determine $Y_c(\omega)$, the Fourier transform of $y_c(t)$.

2. Repeat Question-1 for an input whose Fourier transform is



and $T_1 = 1$, $T_2 = 1$. Determine $X(e^{j\omega})$ and $Y_c(\omega)$.

3. Suppose we apply the following system to $x_c(t)$.



(a) For $x_c(t)$, a bandlimited signal whose Fourier transform is

$$X_c(\omega) = \begin{cases} 1 & \text{if } |\omega| \le \pi/2, \\ 0 & \text{if } |\omega| > \pi/2, \end{cases}$$

determine $Y_c(\omega)$. How is $y_c(t)$ related to $x_c(t)$?

(b) Repeat part (a) for

$$X_c(\omega) = \begin{cases} 1 & \text{if } |\omega| \le \pi, \\ 0 & \text{if } |\omega| > \pi. \end{cases}$$

TEL 311E – Homework 5

Due 06.12.2010

1. Let a causal, LTI system satisfy the difference equation

$$y(n) - 3\,y(n-1) = x(n) + \left(3 - \frac{2}{3}\right) x(n-1) - 2\,x(n-2).$$

   (a) Find the system function $H(z)$ associated with this function.
   (b) Find the impulse response of the system.
   (c) Is the system stable? Please explain your answer.
   (d) Find the expression for a minimum-phase system $H_1(z)$ and an all-pass system $H_{ap}(z)$ such that

$$H(z) = H_1(z)\,H_{ap}(z).$$

2. Consider an LTI system whose impulse response is

$$\left(\frac{2}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n-1).$$

   Find the difference equation associated with this system.

3. Consider a sequence $x(n)$, whose z-transform is

$$X(z) = \frac{1 - 5\,z^{-1} + 6\,z^{-2}}{1 - \frac{1}{4}\,z}.$$

   For which values of α, is $\alpha^n\,x(n)$ minimum-phase?

TEL 311E – Homework 6

Due 27.12.2010

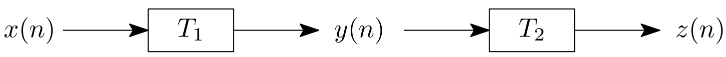
1. Compute the DFTs of the following length-N signals (where N is even).
   (a) $x(n) = \delta(n)$ for $0 \leq n \leq N-1$
   (b) $x(n) = \delta(n-m)$ for $0 \leq n \leq N-1$, where $0 \leq m \leq N-1$
   (c) $x(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$
   (d) $x(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$
   (e) $x(n) = \begin{cases} 0 & \text{if } 0 \leq n \leq N/2-1 \\ 0 & \text{if } N/2 \leq n \leq N-1 \end{cases}$
   (f) $x(n) = a^n$ for $0 \leq n \leq N-1$
   (g) $x(n) = \exp\big(j\frac{2\pi}{N}\,m\,n\big)$ for $0 \leq n \leq N-1$, where $0 \leq m \leq N-1$
   (h) $x(n) = \sin\big(\frac{2\pi}{N}\,m\,n\big)$ for $0 \leq n \leq N-1$, where $0 \leq m \leq N-1$
2. Let $x_1(n)$, $x_2(n)$ be length-N signals, given as,
   $$x_1(n) = \begin{cases} n+1 & \text{for } 0 \leq n \leq 4 \\ 0 & \text{for } 5 \leq n \leq N-1, \end{cases}$$
   $$x_2(n) = \delta(n) + 2\,\delta(n-3).$$
   Let $X_1(k)$, $X_2(k)$ denote their length-N DFTs. Suppose we define $Y(k) = X_1(k)\,X_2(k)$ and let $y(n)$ be the inverse DFT of $Y(k)$. Determine and sketch $y(n)$ for
   (a) $N = 6$,
   (b) $N = 8$.
   What is the minimum value of N such that $y(n)$ is equal to the linear convolution of $x_1(n)$ and $x_2(n)$?
3. Let $x(n)$ be a length-N signal for N even, and let $X(k)$ denote its length-N DFT. Suppose we define
   $$y(n) = \begin{cases} x(2n) & \text{for } 0 \leq n \leq N/2-1 \\ 0 & \text{for } N/2 \leq n \leq N-1, \end{cases}$$
   Let $Y(k)$ denote the length-N DFT of $y(n)$. Express $Y(k)$ in terms of $X(k)$.
4. Let $x(n)$ be a length-N signal for N even, and let $X(k)$ denote its length-N DFT. Suppose we set $y(n) = X(n)$ for $0 \leq n \leq N-1$. Express $y(n)$ in terms of $x(n)$.

TEL 311E – Homework 7

Due 11.01.2011

1. Consider the system given by,
$$y(n) = x(2n) + 1$$
where $x(n)$ is the input and $y(n)$ is the output. Specify whether the system is
(a) Memoriless, (b) Linear, (c) Time-invariant, (d) Causal, (e) Stable in the BIBO sense.
Please explain your answers.

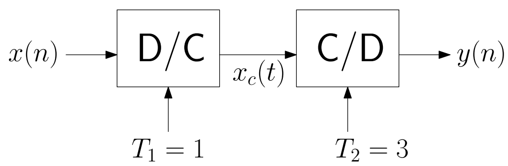
2. Let T_1 be an LTI system with impulse response $h_1(n) = a^n\, u(n)$.
   (a) For which values of 'a' is T_1 stable in the BIBO sense?
   (b) Assume that T_1 is BIBO stable. Suppose we input some $x(n)$ to T_1 and obtain $y(n)$, as shown below. Let T_2 be another LTI system with impulse response $h_2(n) = \delta(n) - a^{-1}\,\delta(n+1)$ and



suppose we input $y(n)$ to this system to obtain $z(n)$. Express $z(n)$ in terms of $x(n)$.

3. Suppose that the z-transform of the step response (i.e. the response when a unit step function $u(n)$, is input to the system) of an LTI system is given by
$$X(z) = \frac{1}{1+\frac{1}{4}\,z^{-1}} + \frac{1}{1-z^{-1}}.$$
Let us denote the z-transform of the impulse response as $H(z)$.
   (a) If we know that the system is stable, what should be the region of convergence for $H(z)$?
   (b) Determine the impulse response $h(n)$ of this stable system.

4. Consider the system below which maps $x(n)$ to $y(n)$.



   (a) Express $y(n)$ in terms of $x(n)$.
   (b) Express $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$.

5. Let $x(n)$ be an N-point signal whose N-point DFT is denoted by $X(k)$. Suppose we circularly shift $X(k)$ by one sample to obtain $\tilde{X}(k)$, i.e.,
$$\tilde{X}(0) = X(N-1),$$
$$\tilde{X}(k) = X(k-1) \quad \text{for } 1 \le k \le N-1.$$
Express $\tilde{x}(n)$, the IDFT of $\tilde{X}(k)$, in terms of $x(n)$.

6. Let $x(n)$ be a 10-point signal with 10-point DFT
$$X(k) = k^2 \quad \text{for} \quad 0 \le k \le 9.$$
Compute
$$s = \sum_{n=0}^{N} x(n)\,\left[\cos\left(\frac{\pi}{N}\,n\right) + 2\sin\left(\frac{6\pi}{N}\,n\right)\right].$$

TEL 311E – Midterm Examination I

01.11.2010

İlker Bayram

5 Questions, 100 Minutes

(30 pts) 1. (a) Consider the system given by,

$$y(n) = x(n) + 1$$

where $x(n)$ is the input and $y(n)$ is the output. Specify whether the system is

(i) Memoriless,
(ii) Linear,
(iii) Time-invariant,
(iv) Causal,
(v) Stable in the BIBO sense.

Please explain your answers. If information is insufficient, write 'insufficient information' (and explain why you think so).

(b) Repeat part (a) for the system given by

$$y(n) = x(n^2 - 3),$$

where $x(n)$ is the input and $y(n)$ is the output.

(20 pts) 2. Let T_1 be an LTI system with impulse response $h_1(n) = a^n\, u(n)$.

(a) For which values of 'a' is T_1 stable in the BIBO sense?

(b) Assume that T_1 is BIBO stable. Suppose we input some $x(n)$ to T_1 and obtain $y(n)$, as shown below.

$$x(n) \longrightarrow \boxed{T_1} \longrightarrow y(n) \longrightarrow \boxed{T_2} \longrightarrow z(n)$$

Let T_2 be another LTI system with impulse response $h_2(n) = \delta(n) - a^{-1}\,\delta(n+1)$ and suppose we input $y(n)$ to this system to obtain $z(n)$. Express $z(n)$ in terms of $x(n)$.

(15 pts) 3. Let $x(n)$ be a causal signal with z-transform given by

$$X(z) = \frac{1}{(1 + 3\,z^{-1})\,(1 - 2\,z^{-1})}.$$

(a) What should be the region of convergence?

(b) Determine $x(n)$.

(20 pts) 4. Let $x(n)$ be a given signal with z-transform $X(z)$ whose ROC is $a < |z| < b$, for some a, b. Consider the sequence $y(n)$ defined as,

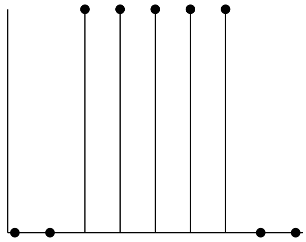
$$y(n) = \begin{cases} x(n) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Notice that $y(n)$ is obtained by setting the odd samples of $x(n)$ to zero. In this question, you will derive the z-transform of $y(n)$ in terms of $X(z)$ in two steps.

(a) Let $x_1(n) = (-1)^n\, x(n)$. Express the z-transform of $x_1(n)$, namely $X_1(z)$, in terms of $X(z)$. What is the ROC for $X_1(z)$?

(b) Write $y(n)$ as a combination of $x(n)$ and $x_1(n)$. Making use of the result of part(a), obtain $Y(z)$ in terms of $X(z)$ alone. What is the ROC for $Y(z)$?

(15 pts) 5. Below are shown four signals, their DTFT magnitudes, and their pole-zero diagrams. But they are not in the correct order. Put them in the correct order by matching each signal with its DTFT magnitude and pole-zero diagram.



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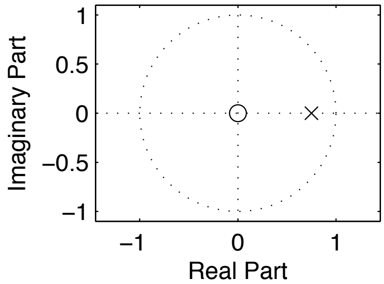

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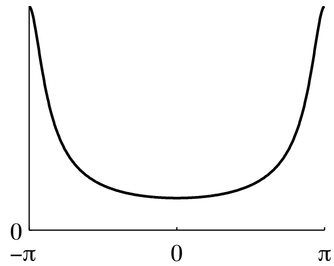

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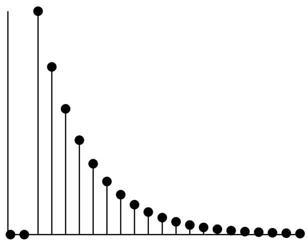

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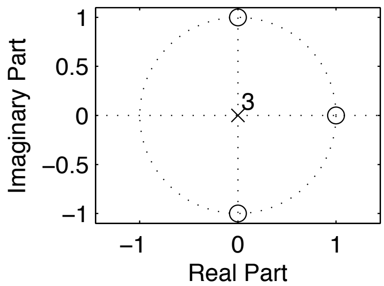

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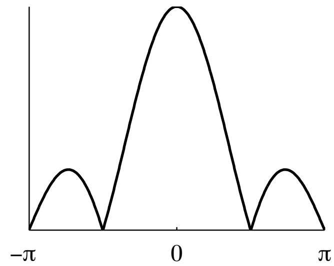

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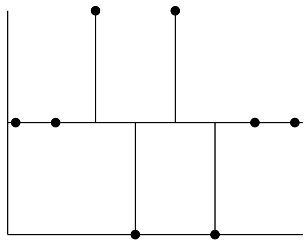

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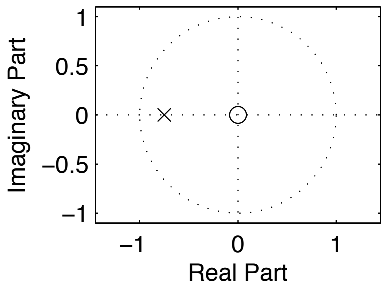

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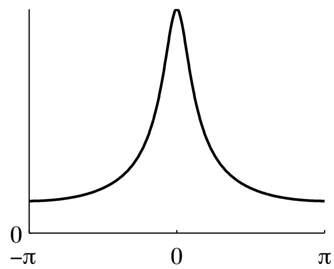

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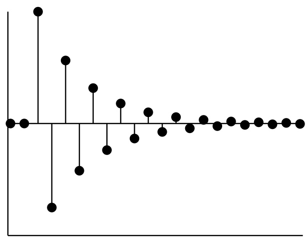
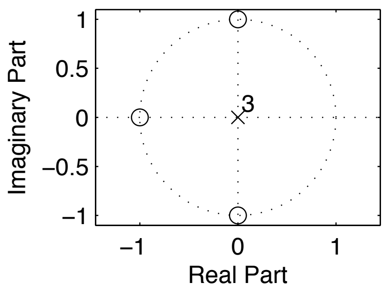
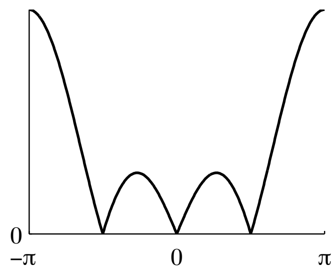


(IV)



(d)



(4)

TEL 311E – Digital Signal Processing

Midterm Examination II

13.11.2010

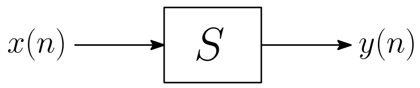
4 Questions, 100 Minutes

(30 pts) 1. Suppose we are given a discrete-time signal $x(n)$ and let $y(n) = x(2n)$.

(a) Using,

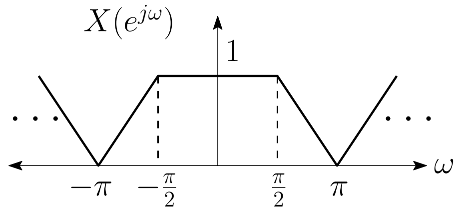
- at least one C/D converter (with sampling period T_1),
- at least one D/C converter (with sampling period T_2),
- as many discrete-time operations as you like,

construct a system S such that,



Specify T_1 and T_2.

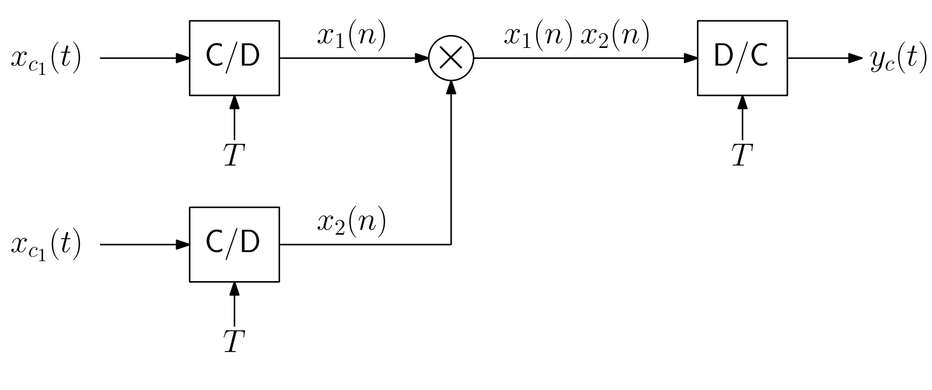
(b) Suppose that $X(e^{j\omega})$, the DTFT of $x(n)$, is given as



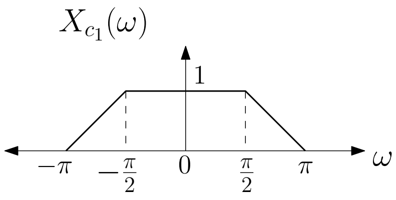
Based on your system in part (a), determine $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$. (Notice that you only need to specify the values for $|\omega| \leq \pi$.)

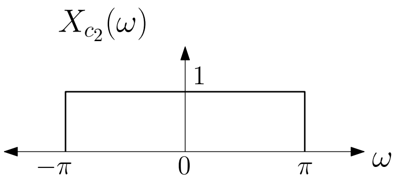
(c) For a general input $x(n)$, express $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$.

(30 pts) 2. Consider the system



(a) Suppose that $X_{c_1}(\omega)$ and $X_{c_2}(\omega)$ are given as





For $T = 1$, determine $Y_c(\omega)$.

(b) Under what conditions on T, $x_{c_1}(t)$, and $x_{c_2}(t)$, is $y_c(t) = x_{c1}(t)\, x_{c2}(t)$?

(20 pts) 3. Consider two causal LTI systems that satisfy the difference equations

$$y_1(n) - \frac{1}{2}\, y_1(n-1) = x(n) + 2\, x(n-1),$$
$$y_2(n) - \frac{1}{6}\, y_2(n-1) - \frac{1}{6}\, y_2(n-2) = x(n),$$

respectively. Let $y(n) = y_1(n) + y_2(n)$

(a) Find the difference equation satisfied $y(n)$ (that should describe $y(n)$ in terms of its past samples and samples of x only – do not use y_1 and y_2).

(b) For $x(n) = \delta(n) - \frac{1}{2}\,\delta(n-1)$, compute $y(100)$.

(20 pts) 4. Consider an LTI system with impulse response

$$h(n) = (2)^n\, u(n) + \delta(n) - 12\,\delta(n-1).$$

(a) Find the impulse response of the minimum phase system $h_{\min}(n)$ that satisfies $|H_{\min}(e^{j\omega})| = |H(e^{j\omega})|$.

(b) Find the impulse response of a causal, stable system, $h_2(n)$, that satisfies $|H_2(e^{j\omega})| = |H(e^{j\omega})|$, but where $h_2(n) \neq h_{\min}(n-k)$ for any value of k – i.e. $h_2(n)$ should not be just a shifted version of $h_{\min}(n)$.

TEL 311E – Digital Signal Processing

Final Examination

19.01.2011

5 Questions, 120 Minutes
Good Luck!

(20 pts) 1. Suppose that a causal LTI system satisfies the difference equation

$$y(n) = 0.5\, y(n-1) + x(n) - 0.1\, x(n-1),$$

where $y(n)$ denotes the output and $x(n)$ denotes the input.

(a) Determine the system function (i.e. the z-transform of the impulse response) of the system.

(b) Determine the impulse response of the system.

(c) Determine the output if the input is $x(n) = u(n)$.

(20 pts) 2. Suppose we input $x(n) = (1/3)^n\, u(n)$ to a <u>stable LTI</u> system and we obtain the output $y(n)$ whose z-transform is,

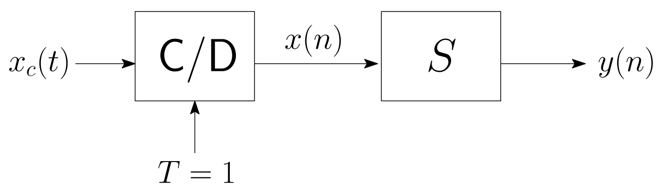
$$Y(z) = \frac{1}{1 - 4\, z^{-1}} + \frac{z^{-1}}{3 - z^{-1}}.$$

(a) Determine the system function (i.e. the z-transform of the impulse response) of the system.

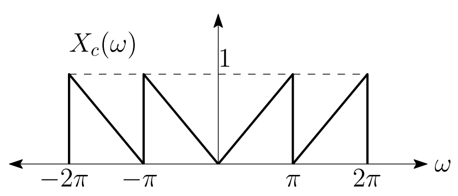
(b) Determine the region of convergence for the system function.

(c) Determine the output $\tilde{y}(n)$, if we input $\tilde{x}(n) = \delta(n) - 4\,\delta(n-1)$ to the same system.

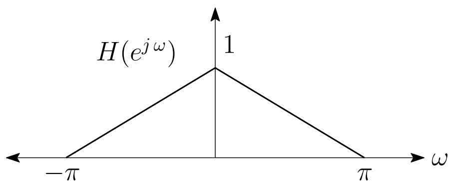
(20 pts) 3. Consider the following system



which samples a continuous-time signal and then applies an LTI system (denoted by S) to the resulting discrete-time signal $x(n)$. Suppose that $X_c(\omega)$, the Fourier transform of $x_c(t)$ is given as



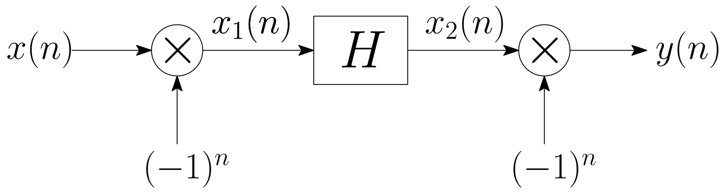
Suppose also that $H(e^{j\omega})$, the frequency response of the system S is given by,



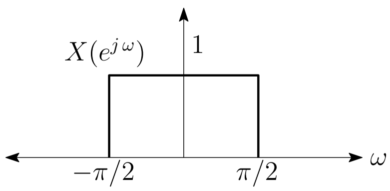
(a) Determine and sketch $X(e^{j\omega})$, the DTFT of $x(n)$.

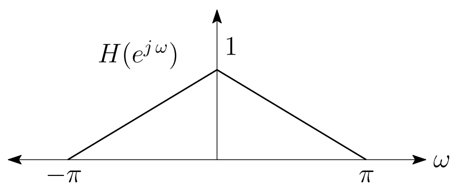
(b) Determine and sketch $Y(e^{j\omega})$, the DTFT of $y(n)$.

(20 pts) 4. Consider the system below, composed of a modulator followed by an LTI filter and another modulator. Note that the modulator outputs are $x_1(n) = x(n)\,(-1)^n$, $y(n) = x_2(n)\,(-1)^n$.



Suppose that the $X(e^{j\omega})$, DTFT of the input and $H(e^{j\omega})$, the frequency response of the LTI system are given by





(a) Determine and sketch $X_1(e^{j\omega})$, the DTFT of $x_1(n)$.
(Hint : Relate $X_1(e^{j\omega})$ to $X(e^{j\omega})$, using the fact that $(-1) = e^{j\pi}$.)

(b) Determine and sketch $X_2(e^{j\omega})$, the DTFT of $x_2(n)$.

(c) Determine and sketch $Y(e^{j\omega})$, the DTFT of $y(n)$.

(20 pts) 5. Let

$$x_1(n) = e^{j\frac{2\pi}{10}3n} + e^{j\frac{2\pi}{10}7n} \quad \text{for } 0 \le n \le 9,$$
$$x_2(n) = 3\,e^{j\frac{2\pi}{10}4n} \quad \text{for } 0 \le n \le 9.$$

(a) Compute the 10-point DFT of $x_1(n)$.

(b) Compute the 10-point DFT of $x_1(n)$.

(c) Compute the 10-point circular convolution of $x_1(n)$ and $x_2(n)$.