TEL 311 HW-6 Solutions

1 (a) $X(k) = \sum_{n=0}^{N-1} \delta(n) e^{-j\frac{2\pi}{N}kn} = 1$

(b) $X(k) = \sum_{n=0}^{N-1} \delta(n-m) e^{-j\frac{2\pi}{N}kn} = e^{-j\frac{2\pi}{N}km}$

(c) $X(k) = \sum_{n=0}^{N/2-1} e^{-j\frac{2\pi}{N}k\cdot 2n} = \sum_{n=0}^{N/2-1} e^{-j\frac{2\pi}{N/2}k\cdot n} = \begin{cases} 1 & \text{if } k=0, N/2 \\ 0 & \text{otherwise} \end{cases}$

(d) $X(k) = \sum_{n=0}^{N/2-1} e^{-j\frac{2\pi}{N}k(2n+1)} = e^{-j\frac{2\pi}{N}k} \sum_{n=0}^{N/2-1} e^{-j\frac{2\pi}{N/2}kn} = \begin{cases} 1 & \text{if } k=0 \\ -1 & \text{if } k=N/2 \\ 0 & \text{otherwise} \end{cases}$

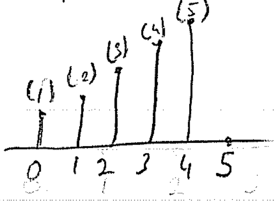
(f) $X(k) = \sum_{n=0}^{N-1} \left(a\cdot e^{-j\frac{2\pi}{N}k}\right)^n = \dfrac{\left(a e^{-j\frac{2\pi}{N}k}\right)^N - 1}{a e^{-j\frac{2\pi}{N}k} - 1} = \dfrac{a^N - 1}{a e^{-j\frac{2\pi}{N}k} - 1}$

(g) $X(k) = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(m-k)n} = \delta(k-m)$

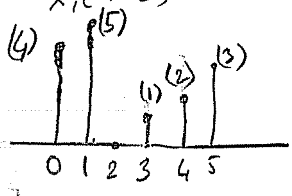
(h) $x(n) = \dfrac{e^{j\frac{2\pi}{N}mn} - e^{j\frac{2\pi}{N}(N-m)n}}{2j} = \dfrac{\delta(k-m) - \delta(k-N+m)}{2j}$

2 (a) The signals are periodic by $N=6$. $x_1(n) \circledast x_2(n) = x_1(n) + 2x(n-3)$

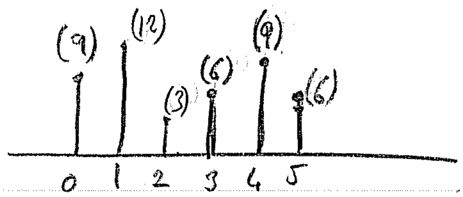
$x_1(n)$



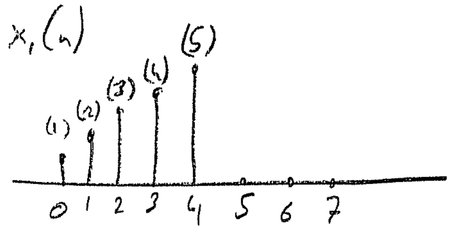
$x_1(n-3)$

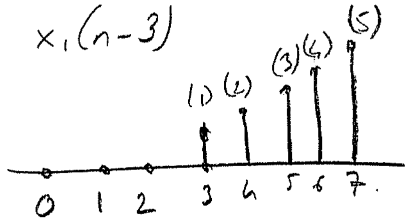


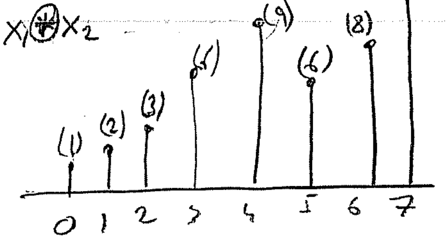
$\Rightarrow x_1 + 2x(n-3) \Rightarrow$



(b) The signals are periodic by $N=8$. $\Rightarrow$ In this case we have







(4.) We will express $Y(k)$ in terms of $x(k)$.

$$Y(k) = \sum_{n=0}^{N-1} y(n)\, e^{-j\frac{2\pi}{N}nk} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk}\left(\sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{N}mn}\right)$$

$$= \sum_{m=0}^{N-1} x(m) \underbrace{\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(k+m)}}_{=N\cdot\delta(k+m) = N\cdot\delta(m-(N-k))}$$

by periodicity.

$$= N\sum_{m=0}^{N-1} x(m)\,\delta(m-(N-k))$$

$$= N\cdot x(N-k).$$

③ First let us define

$$\tilde{y}(n) = x(2n) \quad \text{for } 0 \le n \le N-1.$$ (The signals are periodic by N).

Notice that $y(n) = \tilde{y}(n) \cdot w(n)$ where

$$w(n) = \begin{cases} 1 & \text{if } 0 \le n \le N/2 - 1, \\ 0 & \text{if } N/2 \le n \le N-1. \end{cases}$$

Thus, $Y(k) = \frac{1}{N}\left(\tilde{Y}(k) \circledast W(k)\right)$ (Exercise: Find W).

Let us express $\tilde{Y}$ in terms of X.

$$\tilde{y}(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j \frac{2\pi}{N} 2nk}$$

$$= \frac{1}{N} \sum_{k=0}^{N/2-1} X(k)\, e^{j \frac{2\pi}{N} n\, 2k} + \underbrace{\frac{1}{N} \sum_{k=N/2}^{N-1} X(k)\, e^{j \frac{2\pi}{N} n\, 2k}}_{= \frac{1}{N} \sum_{k=0}^{N/2-1} X(N/2+k)\, e^{j \frac{2\pi}{N} n\, 2(N/2+k)}}$$

$$= e^{j \frac{2\pi}{N} n\, 2k}$$

$$= \frac{1}{N} \sum_{k=0}^{N/2-1} \left[X(k) + X(N/2+k) \right] e^{j \frac{2\pi}{N} n 2k}$$

$$= \frac{1}{N} \sum_{\substack{l=0 \\ l:\text{even}}}^{N-1} \left[X(l/2) + X(N/2 + l/2) \right] e^{j \frac{2\pi}{N} n l}$$

Now define

$$C(k) = \begin{cases} X(k) + X(N/2 + k) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

$$\Rightarrow \tilde{y}(n) = \frac{1}{N} \sum_{k=0}^{N-1} C(k) e^{j \frac{2\pi}{N} nk}$$

$$\Rightarrow C(k) = \tilde{Y}(k).$$