MAT 205E – Theory of Complex Functions

Fall 2014

Instructor : İlker Bayram
EEB 1103
ibayram@itu.edu.tr

Class Meets : 13.30 – 16.30, Wednesday
EEB 5204

Office Hours : 10.00 – 12.00, Monday

Textbook : E. B. Saff, A. D. Snider, 'Fundamentals of Complex Analysis', 3^{rd} Edition, Pearson.
Supp. Text : J. W. Brown, R. V. Churchhill, 'Complex Variables and Applications'.

Grading : 2 Midterms (30% each), Final (40%).

Webpage : There's a 'Ninova' page. Please log in and check.

Tentative Course Outline

- Complex Numbers
  Basic properties, polar representation, the notion of a domain.
- Analytic Functions
  Limit in the complex plane, continuity of a complex valued function, analyticity, Cauchy-Riemann conditions, harmonic functions.
- Basic Functions
  Rational functions, exponential and trigonometric functions, the logarithm, the notion of branch, the power function.
- Complex Integration
  Contours and integrals on contours, Cauchy-Goursat theorem, Cauchy's Integral Formula, the fundamental theorem of algebra.
- Series Representations
  Taylor series, Laurent series, zeros and singularities.
- Residue Theory
  The residue theorem, integrals involving trigonometric functions, indented paths, integration along a branch cut.
- Miscellaneous
  The complex gradient, functions of a matrix.

MAT 205E – Homework 1

Due 24.09.2014

1. For an integer n, suppose $z^n = 1$ but $z \neq 1$. Show that $z^{n-1} + z^{n-2} + \ldots + z + 1 = 0$.

   Solution. Observe that

   $$\underbrace{(z-1)}_{P(z)}\underbrace{(z^{n-1} + z^{n-2} + \ldots + z + 1)}_{Q(z)} = z^n - 1.$$

   Now if $z^n = 1$ then $P(1)\,Q(1) = 0$, but since $z \neq= 1$, we have $P(1) \neq 0$, therefore $Q(1) = 0$.

2. Suppose z_n and u_n are two convergent complex-valued sequences and their limits are z and u respectively. Using the definition of limit given in class, show that $t_n = z_n\,u_n$ is also convergent and the limit is $t = z\,u$.

   Solution. Suppose $\epsilon > 0$ is given. We need to find N such that if $n \geq N$, then $|t_n - t| \leq \epsilon$.

   Now since z_n is convergent, for any given $\epsilon_1 > 0$, we can find N_1 (possibly depending on ϵ_1) such that if $n \geq N_1$, then $|z_n - z| < \epsilon_1$.

   Also, since u_n is convergent, for any given $\epsilon_2 > 0$, we can find N_2 (possibly depending on ϵ_2) such that if $n \geq N_2$, then $|u_n - u| < \epsilon_2$ (we will specify ϵ_1 and ϵ_2 below).

   Therefore, if $n \geq \max(N_1, N_2)$, we have

   $$\begin{aligned} |z\,u - z_n\,u_n| = |z\,u - z_n\,u + z_n\,u - z_n\,u_n| &\leq |z\,u - z_n\,u| + |z_n\,u - z_n\,u_n| \\ &\leq |u|\epsilon_1 + |z_n|\,\epsilon_2 \\ &\leq |u|\epsilon_1 + |z_n - z + z|\,\epsilon_2 \\ &\leq |u|\epsilon_1 + |z_n - z|\epsilon_2 + |z|\,\epsilon_2 \\ &\leq |u|\epsilon_1 + \epsilon_1\,\epsilon_2 + |z|\,\epsilon_2 \qquad (1) \end{aligned}$$

   We want to make the sum of the three terms less than ϵ by a careful choice of ϵ_1 and ϵ_2, in a way that is independent of the sequences z_n and u_n. For that, if we let $M = \max\{|u|, |z|\}$, and set

   $$\epsilon_1 = \epsilon_2 = \sqrt{\epsilon + M^2} - M, \qquad (2)$$

   then the sum of the terms on the right hand side of (1) is less than ϵ. Now we go back and choose N_1, N_2 accordingly. Then for $N = \max(N_1, N_2)$, if $n \geq N$, then $|t_n - t| < \epsilon$.

3. Using the definition of continuity, show that the following functions are continuous.

   (a) $f(z) = z^2$.

   (b) $g(z) = \bar{z}$.

   (Note that $g(z)$ is continuous but not analytic.)

   Solution. (a) Let $\epsilon > 0$ be given. Let us try to find $\delta > 0$ such that if $|z - z_0| \leq \delta$, then $|f(z) - f(z_0)| \leq \epsilon$. This will imply that f is continuous at z_0.

   Now if $|z - z_0| < \delta$, then $z = z_0 + u$ with $|u| < \delta$ (we will specify δ below). Then,

   $$|f(z) - f(z_0)| = |(z_0 + u)^2 - z_0^2| = |u^2 + 2\,z_0\,u| < |u|^2 + 2\,|z_0|\,|u| < \delta^2 + 2\,|z_0|\,\delta$$

Now if

$$0 < \delta < -|z_0| + \sqrt{|z_0|^2 + \epsilon}, \tag{3}$$

then $\delta^2 + 2\,|z_0|\,\delta < \epsilon$. Thus, for instance if $\delta = \left(\sqrt{|z_0|^2+\epsilon} - |z_0|\right)/2$, we have $|f(z) - f(z_0)| < \epsilon$.

Since z_0 was arbitrary, it follows that $f(z)$ is continuous everywhere.

(b) Let $\epsilon > 0$ be given. Suppose also that $|z - z_0| < \delta$, with $\delta = \epsilon$.

$$|g(z) - g(z_0)| = |\bar{z} - \bar{z}_0| = |z - z_0| < \epsilon.$$

Thus g is continuous at z_0. Since z_0 is arbitrary, it follows that $g(z)$ is continuous everywhere.

4. We noted in class that if f and g are continuous at z, then $h = f\,g$ is continuous at z. Prove this, using the definition of continuity.

Solution. This is similar to Q2.

Since f is continuous, for a given ϵ_1, we can find δ_1 such that if $|z - z_0| < \delta_1$, then $|f(z) - f(z_0)| < \epsilon_1$.

Similarly, since g is continuous, for a given ϵ_2, we can find δ_2 such that if $|z - z_0| < \delta_2$, then $|g(z) - g(z_0)| < \epsilon_2$.

Now let $\delta = \min(\delta_1, \delta_2)$. If $|z - z_0| < \delta$, we have,

$$\begin{aligned}
|f(z_0)g(z_0) - f(z)\,g(z)| &\le |f(z_0)g(z_0) - f(z_0)\,g(z)| + |f(z_0)g(z) - f(z)\,g(z)| \\
&\le |f(z_0)|\,\epsilon_2 + |g(z)|\,\epsilon_1 \\
&\le |f(z_0)|\,\epsilon_2 + |g(z) - g(z_0) + g(z_0)|\,\epsilon_1 \\
&\le |f(z_0)|\,\epsilon_2 + \epsilon_2\,\epsilon_1 + |g(z_0)|\,\epsilon_1
\end{aligned}$$

Now for $M = \max\{|f(z_0), |g(z_0)||\}$, set

$$\epsilon_1 = \epsilon_2 = \sqrt{\epsilon + M^2} - M.$$

Go back and choose δ_1, δ_2 accordingly and set $\delta = \min(\delta_1, \delta_2)$. Then if $|z - z_0| < \delta$, we have,

$$|f(z_0)g(z_0) - f(z)\,g(z)| < \epsilon.$$

Thus h is continuous at z_0. Since z_0 is an arbitrary point, it follows that h is continuous everywhere.

5. Suppose $f(z) = z^n$, where n is an integer. Show that $f'(z) = n\,z^{n-1}$.

Solution. We will show the claim above by induction. Let $P_n(z) = z^n$. Observe that $P_1'(z) = 1$ so the claim holds for $n = 1$. Suppose that $P_n'(z) = n\,z^{n-1}$. Noting that $P_{n+1}(z) = P_n(z)\,P_1(z)$, we have,

$$\begin{aligned}
P_{n+1}(z) &= P_n'(z)\,P_1(z) + P_n(z)\,P_1'(z) \\
&= n\,z^{n-1}\,z + z^n \\
&= (n+1)\,z^n.
\end{aligned}$$

Thus the claim is valid for $n+1$ too. Since it is true for $n = 1$, by induction, it must therefore be valid for all integers n.

6. Suppose f, g, h are functions which are all differentiable at z_0. Let $d(z) = f(z)\, g(z)\, h(z)$. Show that $d'(z_0) = f'(z_0)\, g(z_0)\, h(z_0) + f(z_0)\, g'(z_0)\, h(z_0) + f(z_0)\, g(z_0)\, h'(z_0)$.

   Solution. Let $u(z) = g(z)\, h(z)$. Note that $u'(z_0) = g'(z_0)\, h(z_0) + g(z_0)\, h'(z_0)$. Thus,

$$\begin{aligned} d'(z_0) &= f'(z_0)\, u(z_0) + f(z_0)\, u'(z_0) \\ &= f'(z_0)\, g(z)\, h(z) + f(z_0)\, g'(z_0)\, h(z_0) + f(z_0)\, g(z_0)\, h'(z_0). \end{aligned}$$

MAT 205E – Homework 2

Due 01.10.2014

1. Consider a second-order polynomial of the form

$$P(x, y) = a_0x^2 + a_1x\,y + a_2y^2,$$

where a_i's are possibly complex valued. Suppose $P(x, y)$ satisfies a Cauchy-Riemann condition of the form $P_y = iP_x$. Show that actually $P(x, y) = a_0(x + iy)^2$.

Solution. Using $P_y = iP_x$, we find

$$P_y = a_1\,x + 2a_2\,y = i\,P_x = i2\,a_0\,x + ia_1\,y.$$

Since this is valid for all values of x and y, we must have,

$$a_1 = 2i\,a_0, \quad 2a_2 = i\,a_1.$$

Combining we have,

$$P(x, y) = a_0x^2 + a_02ixy - a_0y^2 = a_0\,(x + iy)^2.$$

2. Let

$$u(x, y) = x^2 - x - y^2.$$

Find a function $v(x, y)$ such that $f(x, y) = u(x, y) + iv(x, y)$ is an entire function (where $z = x + iy$).

Solution. Note that u is a polynomial so it and its partial derivatives are continous. The function v together with u should satisfy the Cauchy-Riemann conditions :

$$u_x = 2x - 1 = v_y$$
$$u_y = -2y = -v_x.$$

Integratiing these two conditions, we find that

$$v = 2xy - y + h(x) = 2xy + g(y),$$

so that $h(x) - g(y) = y$. This is only possible if $g(y) = -y + c$ and $h(x) = c$ for a constant c. Thus $v(x, y)$ is of the form $v(x, y) = 2xy - y + c$. So we find,

$$\begin{aligned} f(x, y) &= x^2 - x - y^2 + i(2xy - y + c) \\ &= (x + iy)^2 - (x + iy) + ic \\ &= z^2 - z + ic. \end{aligned}$$

3. (From our textbook)

(a) Show that if $f(z)$ is analytic and real-valued in a domain D (recall that domain is an open and connected set), then $f(z)$ is constant throughout D.

(b) Show that if both $f(z)$ and $\overline{f(z)}$ is analytic in a domain D, then $f(z)$ is constant throughout D.

(c) Show that if both $f(z)$ and $|f(z)|$ is analytic in a domain D, then $f(z)$ is constant throughout D.

Solution. (a) Let $f(x,y) = u(x,y) + iv(x,y)$, where u and v are real valued. If f is real valued, $v = 0$. But then by the Cauchy-Riemann conditions, we have, $u_x = v_y = 0$, $u_y = -v_x = 0$. Therefore, u must be a constant. Thus $f(z)$ is a constant.

(b) Let $g(z) = (f(z) - \overline{f(z)})/(2i)$. Then, $g(z)$ is real-valued and since it is a linear combination of analytic functions, it is also analytic. But then by part (a), it follows that g is a constant. But $g(z)$ is equal to the imaginary part of $f(z)$. Let $h(z) = f(z) - i\,g(z)$. Then, h is real-valued and analytic since both f and g are analytic. But again by part (a), it must be constant. Finally we have $f(z) = h(z) + ig(z)$ is a constant function because both h and g are constant functions.

(c) Note that $|f(z)|$ is real valued, so by part (a), it must be a constant.

First note that if $|f(z)| = 0$, we must have $f(z) = 0$ and the claim follows trivially in this case.

Suppose now that $|f(z)| = c \neq 0$, so that $f(z) \neq 0$ on D. Then $|f(z)|^2 = c^2$. But $|f(z)|^2 = f(z)\,\overline{f(z)}$ and since $f(z)$ is analytic and non-zero on D, $\overline{f(z)} = c^2/f(z)$ is also analytic. The claim now follows by part (b).

4. Let

$$
\begin{aligned}
P_1(z) &= (z-2)^2, \\
P_2(z) &= (z-3)\,z.
\end{aligned}
$$

Find two polynomials $Q_1(z)$, $Q_2(z)$ such that

$$P_1(z)\,Q_1(z) + P_2(z)\,Q_2(z) = 1.$$

Solution. Expanding the polynomials, we have,

$$
\begin{aligned}
P_1(z) &= z^2 - 4z + 4 \\
P_2(z) &= z^2 - 3z.
\end{aligned}
$$

By polynomial division, we find

$$P_1(z) = P_2(z) + \underbrace{(-z+4)}_{A_1(z)} \text{ equivalently } A_1 = P_1 - P_2.$$

Then,

$$P_2(z) = A_1(z)\underbrace{(-z-1)}_{A_2(z)} + 4.$$

Thus,

$$
\begin{aligned}
1 &= \frac{1}{4}\Big(P_2 - A_1\,A_2\Big) \\
&= \frac{1}{4}\Big(P_2 - (P_1 - P_2)\,A_2\Big) \\
&= P_1\underbrace{\frac{1}{4}(-A_2)}_{Q_1} + P_2\underbrace{\frac{1}{4}(1 + A_2)}_{Q_2}
\end{aligned}
$$

5. Suppose a degree-k polynomial $P(z)$ satisfies $P(z) \neq 0$ if $|z| \leq r$ for some r. Then it can be shown that we can find positive numbers $0 < c_1 < c_2$ such that $c_1 < |P(z)| < c_2$ if $|z| \leq r$.

Use the fact above to prove the following. Suppose $Q(z)$ is a degree n polynomial and

$$
\begin{aligned}
Q(z_0) &= 0, \\
Q'(z_0) &= 0, \\
Q''(z_0) &\neq 0.
\end{aligned}
$$

Show that we can find $\epsilon > 0$ and two constants $0 < d_1 < d_2$ such that

$$d_1|z - z_0|^2 < |Q(z)| < d_2|z - z_0|^2, \quad \text{if } 0 < |z - z_0| < \epsilon.$$

Solution. Since $Q(z_0) =$, we must have that $Q(z) = (z - z_0)\, Q_1(z)$ where Q_1 is a polynomial of degree $n - 1$. But then,

$$Q'(z) = Q_1(z) + (z - z_0)\, Q_1'(z),$$

so that

$$Q'(z_0) = Q_1(z_0) = 0.$$

Therefore, $Q_1(z) = (z - z_0)\, Q_2(z)$, so that $Q(z) = (z - z_0)^2\, Q_2(z)$. Differentiating twice, we find,

$$Q''(z) = 2Q_2(z) + 2(z - z_0)\, Q_2'(z) + (z - z_0)^2\, Q_2(z),$$

so that

$$Q''(z_0) = 2Q_2(z_0) \neq 0.$$

Thus z_0 is not a zero of $Q_2(z)$. Now let $P(z) = Q(z + z_0)$. P is also a polynomial of degree n and $P(z) = z^2\, P_1(z)$, where $P_1(z) = Q_2(z + z_0)$. Note that $P_1(0) = Q_2(z_0) \neq 0$. Therefore, by the fact stated in the question statement above, we can find r such that if $0 < |z| < r$, then there are constants c_1, c_2 such that $c_1 < |P_1(z)| < c_2$. But this implies that

$$|P(z)| = |z^2|\, |P_1(z)| < c_2|z^2|,$$
$$|P(z)| = |z^2|\, |P_1(z)| > c_1|z^2|,$$

if $0 < |z| < r$. Combining,

$$c_1|z|^2 < |P(z)| < c_2|z|^2 \text{ if } 0 < |z| < r.$$

Let $t = z + z_0$. Then we can write the inequalities above as,

$$c_1|t - z_0|^2 < |\underbrace{P(t - z_0)}_{Q(t)}| < c_2|t - z_0|^2 \text{ if } 0 < |t - z_0| < r.$$

MAT 205E – Homework 3

Due 08.10.2014

1. Suppose $f(z)$ is entire. Show that $g(z) = \overline{f(\bar{z})}$ is also entire. (Notice that $f(z) = \overline{g(\bar{z})}$.)

   Solution. Let $f(x,y) = u(x,y) + iv(x,y)$ where $z = x + iy$ and such that x, y, u, v are all real valued. Then, if $h(z) = f(\bar{z})$, we can write $h(x,y) = f(x,-y) = u(x,-y) + iv(x,-y)$. Finally, if $g(z) = g(z) = \overline{f(\bar{z})} = \overline{h(z)}$, then

   $$g(x,y) = \underbrace{u(x,-y)}_{q(x,y)} + i\underbrace{\left(-v(x,-y)\right)}_{w(x,y)}.$$

   Now since f is entire, we have that u and v are differentiable with continuous derivaties and by the Cauchy-Riemann conditions,

   $$u_1(x,y) = v_2(x,y),$$
   $$u_2(x,y) = -v_1(x,y).$$

   Consider now the partial derivatives of w and q. We have,

   $$\begin{aligned} q_1(x,y) &= u_1(x,-y) = v_2(x,-y), \\ q_2(x,y) &= -u_2(x,-y) = v_1(x,-y), \\ w_1(x,y) &= -v_1(x,-y) = u_2(x,-y), \\ w_2(x,y) &= v_2(x,-y) = u_1(x,-y). \end{aligned}$$

   Thus,

   $$q_1(x,y) = w_2(x,y)$$
   $$q_2(x,y) = w_1(x,y).$$

   So, q and w satisfy the Cauchy-Riemann conditions. They also have continuous derivatives, so it follows that g is entire.

2. Compute $\sin(i)$ and $\cos(i)$.

   Solution. Recall

   $$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

   We have,

   $$\sin(i) = \frac{e^{-1} - e}{2i}$$
   $$\cos(i) = \frac{e^{-1} + e}{2}.$$

3. Find all values of $\log(i)$ and $\log(ei)$.

   Solution. Recall,

   $$\log(z) = \ln|z| + i\arg(z).$$

Thus,

$$\begin{aligned}
\log(i) &= \underbrace{\ln|i|}_{=0} + i\arg(i) \\
&= i\left(\frac{\pi}{2} + k\,2\pi\right), \qquad \text{for } k \in \mathbb{Z}, \\
&= \left\{\ldots, -i\frac{7\pi}{2}, -i\frac{3\pi}{2}, i\frac{\pi}{2}, i\frac{5\pi}{2}, i\frac{9\pi}{2} \ldots\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\log(ei) &= \underbrace{\ln|ei|}_{=1} + i\arg(i) \\
&= 1 + i\left(\frac{\pi}{2} + k\,2\pi\right), \qquad \text{for } k \in \mathbb{Z}, \\
&= \left\{\ldots, 1 - i\frac{7\pi}{2}, 1 - i\frac{3\pi}{2}, 1 + i\frac{\pi}{2}, 1 + i\frac{5\pi}{2}, 1 + i\frac{9\pi}{2} \ldots\right\}.
\end{aligned}$$

4. Let $\text{Log}(z)$ denote the principal branch of $\log(z)$ as defined in class (or the textbook). Recall that Log is analytic in the domain $D = \mathbb{C}\backslash I$, where I is the set of non-positive (real) numbers, i.e. $I = \{z \in \mathbb{C} : \text{Re}(z) \leq 0,\ \text{Im}(z) = 0\}$. Also, let f be a function defined as,

$$f(z_1, z_2) = \text{Log}(z_1\, z_2) - \text{Log}(z_1) - \text{Log}(z_2),$$

where $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$.

(a) Compute $f(z_1, z_2)$ when $\text{Re}(z_1) > 0$, $\text{Re}(z_2) > 0$.

(b) Find z_1, z_2 in D such that $f(z_1, z_2) \neq 0$.

Solution. (a) When $\text{Re}(z) > 0$, we have $-\pi/2 < \text{Arg}(z) < \pi/2$. Therefore, if $\text{Re}(z_1) > 0$, $\text{Re}(z_2) > 0$, we have that $-\pi < \text{Arg}(z_1) + \text{Arg}(z_2) < \pi$. It follows that if $\text{Re}(z_1) > 0$, $\text{Re}(z_2) > 0$, then $\text{Arg}(z_1\, z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ (why?). Thus,

$$\begin{aligned}
\text{Log}(z_1\, z_2) &= \ln(|z_1|\,|z_2|) + i\ \text{Arg}(z_1, z_2) \\
&= \Big(\ln|z_1| + i\,\text{Arg}(z_1)\Big) + \Big(\ln|z_2| + i\ \text{Arg}(z_1, z_2)\Big) \\
&= \text{Log}(z_1) + \text{Log}(z_2).
\end{aligned}$$

Therefore, $f = 0$.

(b) Let $z_1 = z_2 = e^{i3\pi/4}$. Then, $\text{Arg}(z_1) = 3\pi/4$. But $\text{Arg}(z_1\, z_2) = -\pi/2$. Thus,

$$\begin{aligned}
\text{Log}(z_1\, z_2) &= -i\pi/2, \\
\text{Log}(z_1) = \text{Log}(z_2) &= i3\pi/4,
\end{aligned}$$

so that $f(z_1, z_2) = -i2\pi$.

5. We know that if x is real valued and positive, then $\log(x^2) = 2\log(x)$. Does the equality $\text{Log}(z^2) = 2\,\text{Log}(z)$ hold in the domain D defined above in Q4?

Solution. No. Take $z = e^{i3\pi/4}$ as above. $\text{Log}(z^2) = -i\pi/2 \neq i3\pi/2 = 2\,\text{Log}(z)$.

6. Determine the domain D' where the function $f(z) = \text{Log}(i - z)$ is analytic.

Solution. Note that $\mathrm{Log}(t)$ is differentiable if $t \notin I$, where I is the branch cut for Log, as described in Q4. Thus, $\mathrm{Log}(i-z)$ is differentiable if $(i-z) \notin I$. Note that $(i-z) \notin I$ is equivalent to requiring $(i-z) \neq r$, where r is any non-positive number. This in turn is equivalent to requiring that $z \neq i - r$, where r is non-positive. If we set F to be the set of complex numbers of the form $(i - r)$ where r is non-positive (sketch the set F), then $f(z)$ is analytic on $\mathbb{C}\backslash F$.

7. (a) Let $\mathrm{Arg}(z)$ denote the principal value of $\arg(z)$ which is analytic in the domain D defined in Q4 above. Note that $\mathrm{Arg}(z)$ is continuous on D but not on all of $\mathbb{C}$. Show however that $|\mathrm{Arg}(z)|$ is continuous on all of $\mathbb{C}$, except the origin, where it is not defined.

   (b) Show that $|\mathrm{Log}(z)|$ (where $\mathrm{Log}(z)$ is as described in Q4) is continuous on all of $\mathbb{C}$, except the origin.

   (c) Compute $\lim_{z\to 0} |\mathrm{Log}(z)|$.

Solution. (a) Since $\mathrm{Arg}(z)$ is continuous on $z \in D$, $|\mathrm{Arg}(z)|$ is also continuous on $D = \mathbb{C}\backslash I$ (because it is the composition of two continuous functions, namely $\mathrm{Arg}(z)$ and the modulus function $|z|$). All we need to check is the continuity of $|\mathrm{Arg}|$ on I. Suppose z is real valued and negative (so it is non-zero). Also, let $\epsilon > 0$ be given. Now let $\delta > 0$ be such that $\delta < |z|$ and

$$\delta/(|z| - \delta) < \tan(\epsilon).$$

Note that we can find such a δ if we take it small enough because $\tan(\epsilon) > 0$ and $\lim_{\delta\to 0} \delta/(|z| - \delta) = 0$. Now if $|z_0 - z| \leq \delta$, then we have either $\mathrm{Arg}(z_0) \in [\pi, \pi - \epsilon)$ or $\mathrm{Arg}(z_0) \in (-\pi + \epsilon, -\pi]$. In either case,

$$\Big||\mathrm{Arg}(z_0)| - \pi\Big| = \Big||\mathrm{Arg}(z_0)| - |\mathrm{Arg}(z)|\Big| \leq \epsilon.$$

Thus, $|\mathrm{Arg}(z)|$ is continuous on I also.

(b) Observe that

$$|\mathrm{Log}(z)| = \sqrt{(\ln|z|)^2 + |\mathrm{Arg}(z)|^2}.$$

But since $|\mathrm{Arg}(z)|$ is continuous everywhere except the origin, so is $|\mathrm{Arg}(z)|^2$. Finally, since $(\ln|z|)^2 + |\mathrm{Arg}(z)|^2$ is non-negative and the square root function is continuous on the non-negative reals, it follows that $|\mathrm{Log}(z)|$ is continuous everywhere except the origin.

(c) Observe that $|\mathrm{Log}(z)| > \big|\ln|z|\big|$. But given any $M > 0$, if $\delta < e^{-M}$, then for $|z| \leq \delta$, we have $M < \big|\ln|z|\big| < \big|\mathrm{Log}(z)\big|$. Thus, $\lim_{z\to 0}\big|\mathrm{Log}(z)\big| = \infty$.

MAT 205E – Homework 4

Due 12.11.2014

1. Suppose $f(z)$ is continous in a domain that contains z_0. Show that

$$\lim_{\Delta z \to 0} \int_0^1 f(z_0 + t\Delta z)dt = f(z_0).$$

Solution. Suppose $\epsilon > 0$ is given. We want to show that for some $\delta > 0$, if $|\Delta z| < \delta$,

$$\left| \int_0^1 f(z_0 + t\Delta z)dt - f(z_0) \right| = \left| \int_0^1 f(z_0 + t\Delta z) - f(z_0)\, dt \right| \le \epsilon.$$

Now since f is continuous at z_0 we can find a small number $\varepsilon > 0$ such that if $|u| \le \varepsilon$, then

$$|f(z_0 + u) - f(z_0)| \le \epsilon.$$

Set $\delta = \varepsilon$. Note that if $0 \le t \le 1$ and $|\Delta z| \le \delta$, then $|t\Delta z| \le \delta$. Thus,

$$\begin{aligned} \left| \int_0^1 f(z_0 + t\Delta z) - f(z_0)\, dt \right| &\le \int_0^1 |f(z_0 + t\Delta z) - f(z_0)|\, dt \\ &\le \int_0^1 \epsilon\, dt \\ &\le \epsilon. \end{aligned}$$

2. Let Γ be a positively oriented circle of radius r around the point z_0. Compute

$$\int_\Gamma \frac{1}{z - z_0}\, dz.$$

Solution. We can either refer to the Cauchy integral formula or compute the integral explicitly, using a parameterization. For the sake of demonstrating the procedure let us do the latter.

We first need a parameterization for Γ. Note that

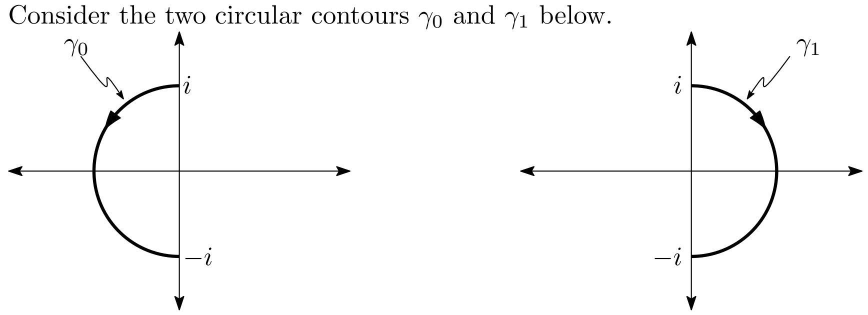
$$z(t) = z_0 + r\, e^{it}, \text{ for } 0 \le t \le 2\pi$$

is an admissable parameterization. Note also that $z'(t) = ir\, e^{it}$. We now compute

$$\begin{aligned} \int_\Gamma \frac{1}{z - z_0}\, dz &= \int_0^{2\pi} \frac{1}{z(t) - z_0}\, z'(t)dt \\ &= \int_0^{2\pi} \frac{1}{r\, e^{it}}\, ire^{it}dt \\ &= 2\pi i. \end{aligned}$$

Observe that the result is independent of r.

3. Consider the two circular contours γ_0 and γ_1 below.



(a) Compute

$$\int_{\gamma_0} \frac{1}{z}\, dz.$$

(b) Make use of the result of part (a) to compute

$$\int_{\gamma_1} \frac{1}{z}\, dz.$$

Note that you don't need to parameterize γ_1 in this case.

Solution. (a) A parameterization for γ_0 is,

$$z(t) = e^{it}, \text{ for } \pi/2 \le t \le 3\pi/2.$$

Note that $z'(t) = ie^{it}$. So,

$$\int_{\gamma_0} \frac{1}{z}\, dz = \int_{\pi/2}^{3\pi/2} \frac{1}{e^{it}}\, ie^{it} dt = \pi i.$$

(b) From Q2, we know that

$$\int_{\gamma_0 - \gamma_1} \frac{1}{z}\, dz = 2\pi i.$$

So,

$$\int_{\gamma_1} \frac{1}{z}\, dz = -\pi i.$$

4. Compute the integral

$$\int_C \frac{z+2}{(z^2-1)}\, dz$$

where C is the circle of radius two around the origin, traversed in the clockwise direction.

Solution. Let us use the Cauchy integral formula for this question. Notice that

$$\frac{z+2}{z^2-1} = \frac{3}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}.$$

Thus,

$$\begin{aligned}\int_C \frac{z+2}{(z^2-1)}\, dz &= \int_C \frac{3}{2}\frac{1}{z-1} dz + \int_C \frac{3}{2}\frac{1}{z-1}\, dz \\ &= 2\pi i \left(\frac{3}{2} - \frac{1}{2}\right).\end{aligned}$$

5. (a) Suppose $g(x, y)$ is a real-valued and continuous function. Show that if

$$h(z) = e^{ig(x,y)},$$

is analytic in a domain D, then the partial derivatives of $g(x, y)$ are zero (so that g is constant).

(b) Suppose $f(z)$ is analytic in a domain D. Show that if $|f(z)|$ is constant, then $f(z)$ is also constant.

Solution. (a) Note that

$$h(x, y) = \underbrace{\cos\big(g(x, y)\big)}_{u(x,y)} + i \underbrace{\sin\big(g(x, y)\big)}_{v(x,y)}.$$

The Cauchy-Riemann conditions for h are,

$$\begin{aligned} u_x &= -\sin(g)\, g_x = \cos(g)\, g_y = v_y \\ u_y &= -\sin(g)\, g_y = -\cos(g)\, g_x = -v_x. \end{aligned}$$

Multiplying these two equations, we obtain,

$$\sin^2\big(g(x, y)\big)\, g_x(x, y)\, g_y(x, y) = \cos^2\big(g(x, y)\big)\, g_x(x, y)\, g_y(x, y).$$

Since g is continuous, these equations can hold in an open set only if $g_x(x, y)\, g_y(x, y) = 0$. Thus at any (x, y) either g_x or g_y must be zero. But plugging this into the Cauchy-Riemann equations, we see that the other partial derivative must also be zero. Thus follows the claim.

(b) If $|f|$ is constant (say M), we can express f as $f(x,y) = M\,e^{ig(x,y)}$ for a continuous and real valued g. Now, by part (a), it follows that $g(x,y)$ is constant, so that f is also constant.

6. (From the textbook) Find all functions f analytic in the unit disk D (i.e. z such that $|z| < 1$) that also satisfy $f(0) = i$ and $|f(z)| \leq 1$ for all $z \in D$.

   Solution. It follows by the maximum modulus principle that $f(z) = i$ is the only analytic function that satisfies the stated conditions.

7. We noted in class that the series

$$\sum_{n=0}^{\infty} c^n$$

   converges if $|c| < 1$. Prove that the series do not converge if $|c| \geq 1$.

   Solution. Let $|c| \geq 1$. Also, let s_n denote the partial sums defined as,

$$s_n = \sum_{k=0}^{n} c^n.$$

   Assume that the series converges and the limit is u. This means that given any $\epsilon > 0$, we can find N such that if $n \geq N$, then $|u - s_n| \leq \epsilon$. Now suppose $\epsilon < 1/4$ and for some n, $|s_n - u| \leq \epsilon$. Then, $|s_{n+1} - u| = |s_n + c^{n+1} - u| \geq |c^{n+1}| - |s_n - u| \geq 3/4 > \epsilon$. Thus we cannot find an integer N that satisfies the stated condition when $\epsilon < 1/4$. So, the series cannot be convergent.

8. We showed in class that $F_n(z) = z^n$ converges uniformly to $F(z) = 0$ on the set of z with $|z| < r$, if $r < 1$. Show that, if $r = 1$, then $F_n(z)$ do not converge to $F(z)$ uniformly.

   Solution. The problem here is that as z approaches 1, convergence slows down. Let us show this rigorously. Suppose convergence is uniform so that given $\epsilon > 0$, we do find N such that if $n \geq N$, then

$$|F_n(z)| \leq \epsilon, \text{ for all } z \text{ with } |z| < 1.$$

   But now consider $u = \exp(\ln(2\epsilon)/N)$. Assuming $\epsilon < 1$, we have that $u < 1$. But $u^{2N} = 2\epsilon$. Thus, we cannot find an integer N that satisfies the condition above. Actually in this case, convergence is not uniform but pointwise (show pointwise convergence as an exercise).

MAT 205E – Homework 5

Due 26.11.2014

1. Remember that the principal branch of the logarithm, namely $\mathrm{Log}(z)$ is defined everywhere except the origin. Can you find a Laurent series expansion of $\mathrm{Log}(z)$ for $\mathbb{C}\backslash\{0\}$ around the origin?

   Solution. No, we cannot find a Laurent series around the origin because we cannot find an annular ring around the origin that does not intersect the branch cut.

2. Note that the Taylor series of e^z is given by

   $$\sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

   Verify that this series is convergent for all $z \in \mathbb{C}$ by showing that

   $$\lim_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0.$$

3. (From the textbook) Does there exist a power series $\sum_{k=0}^{\infty} a_k\, z^k$ which converges at $z_0 = 1 + 3i$ but diverges at $z_1 = 2 + 2i$? If you think so, find one. If not, explain why not.

   Solution. Note that $|z_0| = \sqrt{10}$ and $|z_1| = \sqrt{8}$. But we know that if a Taylor series around the origin converges for $|z| = r$, then it converges for all z with $|z| < r$. Therefore, we cannot find such a series.

4. (a) Suppose $f(z)$ has a Taylor series expansion

   $$f(z) = \sum_{k=0}^{\infty} a_k\, z^k,$$

   uniformly convergent for $|z| < R$. Find the Taylor series expansion of $f'(z)$ in terms of a_k. What is the radius of convergence of the series for f'?

   (b) (From the textbook) Two power series of the form

   $$\sum_{k=0}^{\infty} a_k\, z^k \text{ and } \sum_{k=0}^{\infty} k\, a_k\, z^k$$

   have the same radius of convergence. Explain why.

5. Find the Laurent series of $\cos(1/z)$ around 0 for $|z| > 0$.

   Solution. Note that the Taylor series for $\cos(t)$ is

   $$\cos(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, t^{2k},$$

   valid for all t in $\mathbb{C}$. Replacing t with $1/z$, we obtain the Laurent series around the origin, valid for $z \neq 0$ as,

   $$\cos(1/z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, z^{-2k}.$$

6. Consider the function

   $$f(z) = \frac{1}{z^2 - z}.$$

   (a) Find the Laurent series for $f(z)$ around 0 valid for $0 < |z| < 1$.

   (b) Find the Laurent series for $f(z)$ around 0 valid for $1 < |z|$.

Solution. Notice that

$$\frac{1}{z^2 - z} = \frac{1}{z-1} - \frac{1}{z}.$$

(a) Note that,

$$\frac{1}{z-1} = \sum_{k=0}^{\infty} z^k,$$

for $|z| < 1$. So,

$$\frac{1}{z^2 - z} = -\frac{1}{z} + \sum_{k=0}^{\infty} z^k,$$

for $0 < |z| < 1$.

(b) To obtain a convergent series for $|z| > 1$, note that

$$\frac{1}{z-1} = -\frac{1}{z} \frac{1}{1 - z^{-1}}.$$

Therefore,

$$\frac{1}{z-1} = -\frac{1}{z} \sum_{k=0}^{\infty} z^{-k} = \sum_{k=1}^{\infty} -z^{-k}$$

for $|z^{-1}| < 1$, or $|z| > 1$. So,

$$\frac{1}{z^2 - z} = \sum_{k=1}^{\infty} -z^{-k}$$

7. (From the textbook) Suppose $f(z)$ has an isolated singularity at $z = 0$ but $|f(z)|$ is bounded for $0 < |z| < 1$. Show that the singularity is removable.

   Solution. Let $|f(z)| < M$ for $0 < |z| < 1$ (such an M exists because $|f(z)|$ is given to be bounded in the punctured unit disk). Consider now a Laurent series expansion of f around the origin, valid for $0 < |z| < \epsilon$, where the only singularity of f in this region is the origin.

8. Suppose that $f(z)$ is entire and $f(z) = 0$. Show that $f(z)/z$ is also entire.

9. (From the textbook) Find the smallest r and the greatest R such that the Laurent series

$$\sum_{k=-\infty}^{\infty} \frac{z^k}{2^{|k|}}$$

   is convergent if $r < |z| < R$.

10. What type of a singularity does $f(z) = z\,e^{1/z}$ have at $z = 0$?

MAT 205E – Theory of Complex Functions

Midterm Examination I

15.10.2014

Student Name : ____________________

Student Num. : ____________________

4 Questions, 100 Minutes

Please Show Your Work for Full Credit!

(25 pts) 1. (a) Consider the function

$$f(z) = |z|^2 + 2|z| + 1.$$

Determine whether f is analytic or not. Please briefly explain your answer.

(b) Consider the function (for $z = x + iy$ with x, y real)

$$g(z) = x^3 - 3\,x\,y^2 + y + i\,v(x, y).$$

Find a real-valued function $v(x, y)$ such that $g(z)$ is analytic.

(25 pts) 2. Recall that the principal value of $\arg(z)$ is defined for $z \neq 0$ as,

$$\mathrm{Arg}(z) = \theta \in (-\pi, \pi], \text{ such that } e^{i\theta} = \frac{z}{|z|}.$$

(a) Sketch the largest domain D for which $\mathrm{Arg}(z)$ is continuous.

(b) Find a function $h(z)$ so that

$$f(z) = \mathrm{Arg}(z) + i\,h(z)$$

is analytic in the domain $\tilde{D} = \{z : \mathrm{Re}(z) > 1\}$.

(c) Consider another branch of $\arg(z)$, defined for $z \neq 0$ as

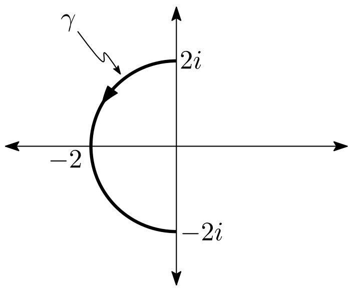
$$A(z) = \theta \in (-\pi/4, 7\pi/4], \text{ such that } e^{i\theta} = \frac{z}{|z|}.$$

Sketch the largest domain $\hat{D}$ for which $A(z)$ is continuous. Is $A(z)$ also analytic on $\hat{D}$? (Please briefly explain your reasoning for credit.)

(25 pts) 3. (a) Write down all values of $f(z) = z^{1/4}$ for $z = 8\sqrt{2}(1 + i)$. How many distinct values can you find?

(b) Write down all values of $g(z) = z^{i/2}$ for $z = (1 + i)/\sqrt{2}$. How many distinct values can you find?

(25 pts) 4. Consider the directed smooth curve γ shown below which is a half circle of radius 2, starting at $\gamma_0 = 2i$ and terminating at $\gamma_1 = -2i$.



(a) Find a parameterization for γ. That is, find a function $z(t)$ such that all of the following hold.

- $z(0) = \gamma_0 = 2i$, and $z(1) = \gamma_1 = -2i$.
- $z(t)$ is a point of γ for $0 \le t \le 1$.
- $z'(t) \neq 0$ for $0 \le t \le 1$.

(b) Compute

$$\int_\gamma \frac{1}{z^2}\, dz.$$

MAT 205E – Theory of Complex Functions

Midterm Examination II

26.11.2014

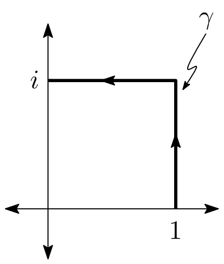
Student Name : ______________________

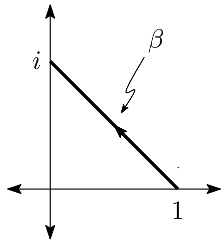
Student Num. : ______________________

5 Questions, 120 Minutes

Please Show Your Work for Full Credit!

(20 pts) 1. Let γ and β be two directed contours (both starting at $z_0 = 1$ and ending at $z_1 = i$) as shown below.





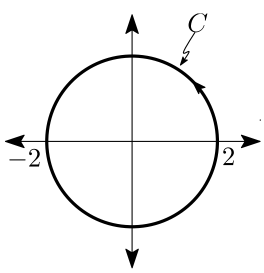
(a) Compute

$$\int_\gamma z^2\, dz.$$

(b) Compute

$$\int_\beta |z|^2\, dz.$$

(25 pts) 2. Let C be the positively oriented circle of radius 2 around the origin as shown below.



(a) Compute

$$\int_C \frac{\cos(z^2)}{z^2 - 3z}\, dz.$$

(b) Suppose $g(z)$ is defined for $|z| < 2$ as,

$$g(z) = \int_C \frac{u^2 - 2\,u - 1}{(u-3)(u-z)}\, du.$$

Find $g(0)$, $g(1)$, $g'(0)$, $g'(1)$.

(20 pts) 3. Consider the function

$$f(z) = \frac{1}{z^2 - z}$$

(a) Find the Laurent series of $f(z)$ around the point $z_0 = 0$, valid for $|z| > 1$.

(b) Find the Laurent series of $f(z)$ around the point $z_1 = 1$, valid for $|z - z_1| < 1$.

(20 pts) 4. Suppose $f(z)$ has a Taylor series expansion of the form

$$f(z) = \sum_{k=0}^{\infty} a_k\, z^k,$$

and the series is known to converge for $|z| < R$, for some $R > 0$. Let $g(z)$ be defined as,

$$g(z) = \left(1 + \frac{1}{z}\right) f'(z).$$

(a) Find a Laurent series for $g(z)$ around the point $z_0 = 0$. Express the Laurent series coefficients in terms of a_k.

(b) What is the region of convergence for the series you found in part (a)? Please briefly explain your answer for full credit.

(15 pts) 5. Let the functions $f(z)$ and $g(z)$ be defined as

$$f(z) = \sin(z) - 1, \text{ and } g(z) = \frac{1}{f(z)}.$$

Notice that $z_0 = \pi/2$ is an isolated singularity of $g(z)$. Please provide your reasoning while answering the following questions.

(a) Is z_0 a removable singularity for $g(z)$?

(b) Is z_0 a pole of order m for $g(z)$? If so, what is m?

(c) Is z_0 an essential singularity for $g(z)$?

MAT 205E – Theory of Complex Functions

Final Examination

25.12.2014

5 Questions, 100 Minutes

Please Show Your Work for Full Credit!

(20 pts) 1. For $z = x + iy$ with x, y real, determine whether the functions below are analytic or not. If they are analytic, compute their derivative at $z_0 = \pi + i\pi$ (that is, evaluate $f'(z_0)$).

(a) $f(x, y) = x^2 + y^2 + i\,2xy$.

(b) $f(x, y) = e^{-y}\left[\cos(x) + i\sin(x)\right]$.

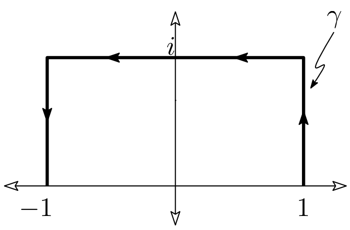
(20 pts) 2. Find all $z \in \mathbb{C}$ that satisfy the following equations.

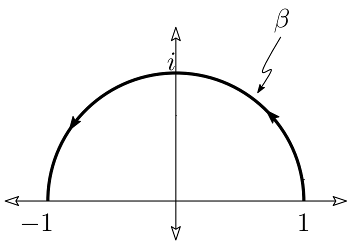
(a) $e^z = 1$.

(b) $z^8 = 1$.

(c) $\mathrm{Log}(z) = i\pi/2$, where $\mathrm{Log}(z)$ denotes the principal value of $\log(z)$.

(20 pts) 3. Let γ and β be two directed contours (both starting at $z_0 = 1$ and terminating at $z_1 = -1$) as shown below.





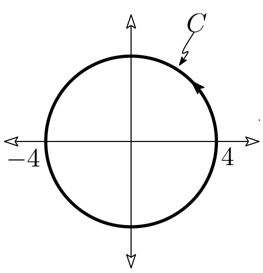
(a) Compute

$$\int_\gamma z^{-2}\,dz.$$

(b) Compute

$$\int_\beta |z|^{-2}\,dz.$$

(20 pts) 4. Let C be the positively oriented circle of radius 4 around the origin as shown below.



Evaluate

$$\int_C \frac{\cos(z^2)}{z^2 - 3z}\,dz.$$

(20 pts) 5. Evaluate the Cauchy principal value of the real valued integral

$$\int_{-\infty}^{\infty} \frac{1}{x^2+2}\, dx.$$