MAT 271E – Homework 6

Due 06.04.2011

1. Consider an experiment that involves an infinite number of coins being tossed at the same time. Suppose we number the coins and denote the outcome of the n^{th} toss as O_n. We define

$$X_n = \begin{cases} 1, & \text{if } O_n = \text{Head}, \\ 0, & \text{if } O_n = \text{Tail}. \end{cases}$$

   We construct a number (in the base-2 system) as

$$Z = 0.X_1 X_2 X_3 \ldots$$

   (a) Propose a sample space, Ω, for this experiment.
   (b) Consider the events $A = \{Z \geq 1/2\}$, $B = \{Z \geq 1/4\}$. Express these events in terms of the sample space you proposed in (a).
   (c) If the coin tosses are independent of each other and the probability of observing a Head for each coin is given by p, compute the proabablities of A and B.

2. Noise in communication channels at a particular instant is usually modelled as a Gaussian random variable. Let us denote it by X. Suppose for the sake of simplicity that X is a *standard* Gaussian random variable (i.e. it has zero mean and its variance is one). The power of noise is defined as $Y = X^2$. Notice that Y is also a random variable. In this question we will derive the probability distribution function (pdf) of Y. Recall that

$$\Phi(t) = P\big(\{X \leq t\}\big) = \int_{-\infty}^{t} f_X(t)\, dt,$$

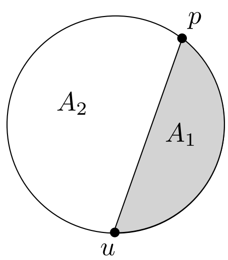
   where

$$f_X(t) = \frac{1}{\sqrt{2\pi}}\, e^{-2t^2}.$$

   (a) Express the cumulative distribution function (cdf) of Y, that is $F_Y(s) = P\big(\{Y \leq s\}\big)$ in terms of the function $\Phi(\cdot)$.
   (b) Obtain the pdf of Y, denoted by $f_Y(s)$, by differentiating $F_Y(s)$ (you can write it explicitly).

3. You choose two points on a line segment independently, according to a uniform distribution and break the segment into three parts. What is the probability that you can form a triangle with the resulting segments?

4. You randomly choose two points p, u, independently, according to a uniform distribution on a circle of radius 1 as shown below.



   The chord between p and u divides the circle into two segments A_1 and A_2, where A_1 is the one with smaller area. Compute the expected areas of A_1 and A_2.

5. Let X_1 be a continuous random variable, uniformly distributed on $[0, 1]$. Let Y be an exponential random variable, i.e.,

$$f_Y(t) = \begin{cases} c\, e^{-ct} & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

   Compute the probability that $X \leq Y$.