2D DTFT

For 2D signals, DTFT Analysis and Synthesis Relations are,

$$\text{(S)} \quad x(n_1,n_2) = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\omega_1,\omega_2)\, e^{j\,(\omega_1 n_1+\omega_2 n_2)}\, d\omega_1\, d\omega_2$$

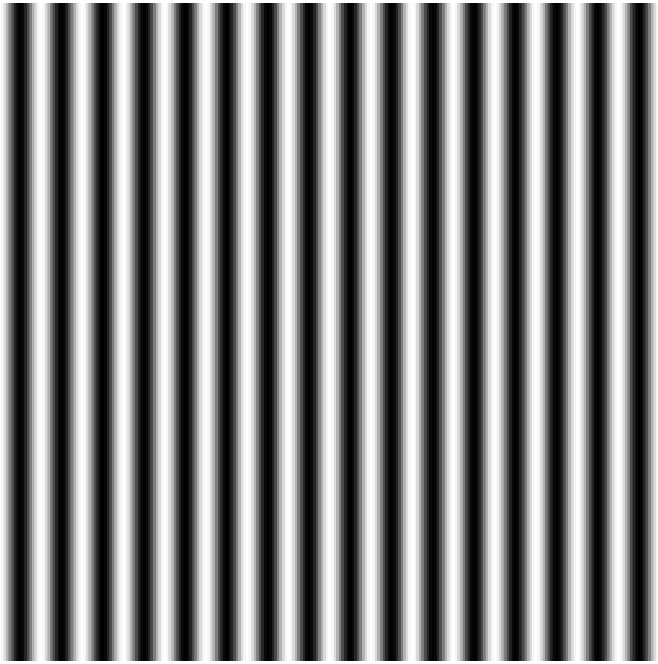
$$\text{(A)} \quad X(\omega_1,\omega_2) = \sum_{n_1}\sum_{n_2} x(n_1,n_2)\, e^{-j\,(\omega_1 n_1+\omega_2 n_2)}$$

(S) represents the function $x(n_1,n_2)$ as a linear combination of complex exponentials $\exp\big(-j\,(\omega_1 n_1 + \omega_2 n_2)\big)$ with weights $(4\pi^2)^{-1} X(\omega_1,\omega_2)\, d\omega_1\, d\omega_2$.

Complex Exponentials

$$\exp\left(j\,\omega\ (\cos\theta\, x + \sin\theta\, y) \right)$$

$$\omega = \pi/8 \quad \theta = 0$$



Complex Exponentials

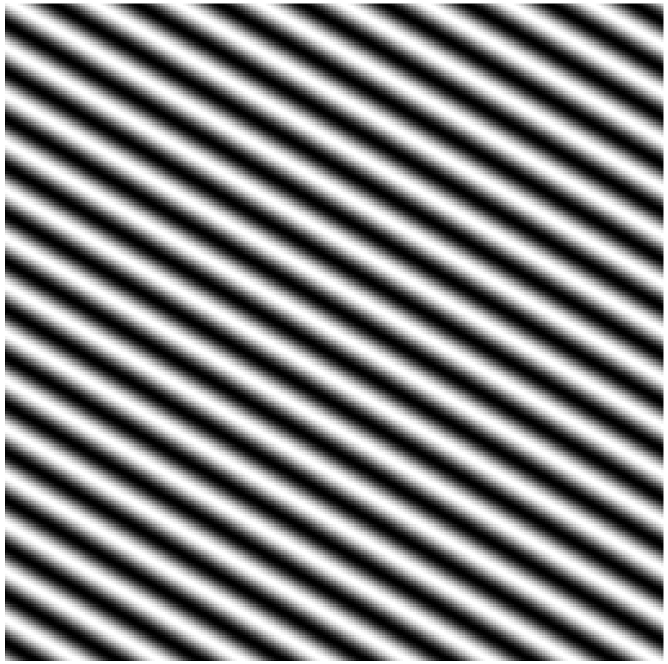
$$\exp\left(j\,\omega\ (\cos\theta\,x + \sin\theta\,y)\right)$$

$$\omega = \pi/8 \quad \theta = \pi/6$$

Complex Exponentials

$$\exp\left(j\,\omega\;(\cos\theta\,x + \sin\theta\,y)\right)$$

$$\omega = \pi/8 \quad \theta = 2\pi/6$$

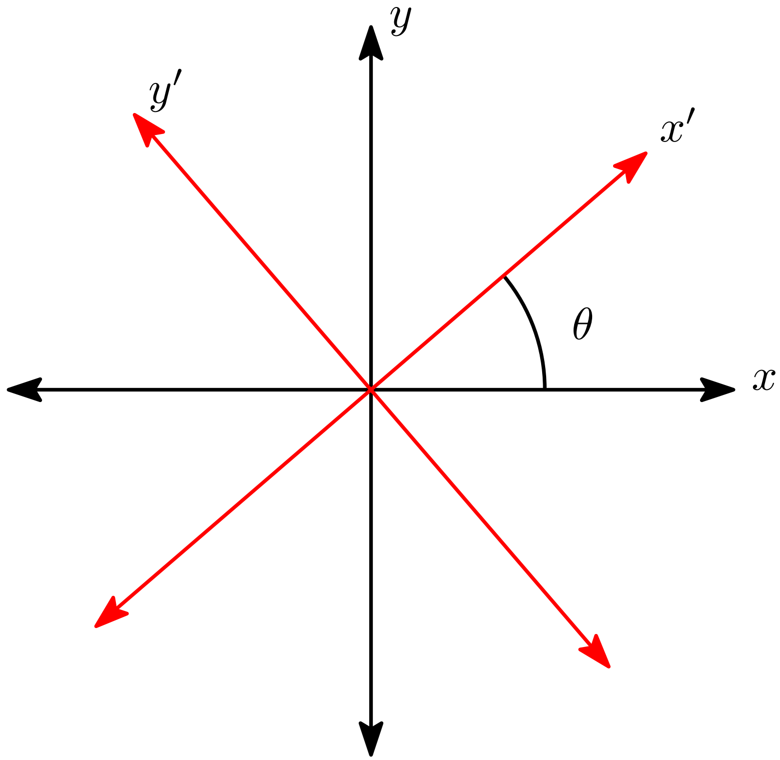


Complex Exponentials

$$\exp\left(j\,\omega\ (\cos\theta\, x + \sin\theta\, y)\right)$$

$$\omega = \pi/8 \quad \theta = 3\pi/6$$

Rotating the Coordinate System

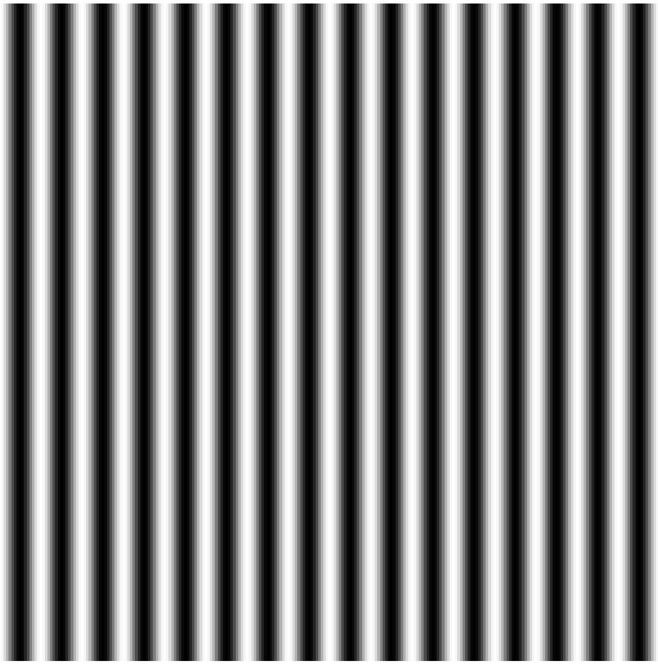


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Complex Exponentials

$$\exp\left(j\,\omega\ (\cos\theta\,x + \sin\theta\,y)\right)$$

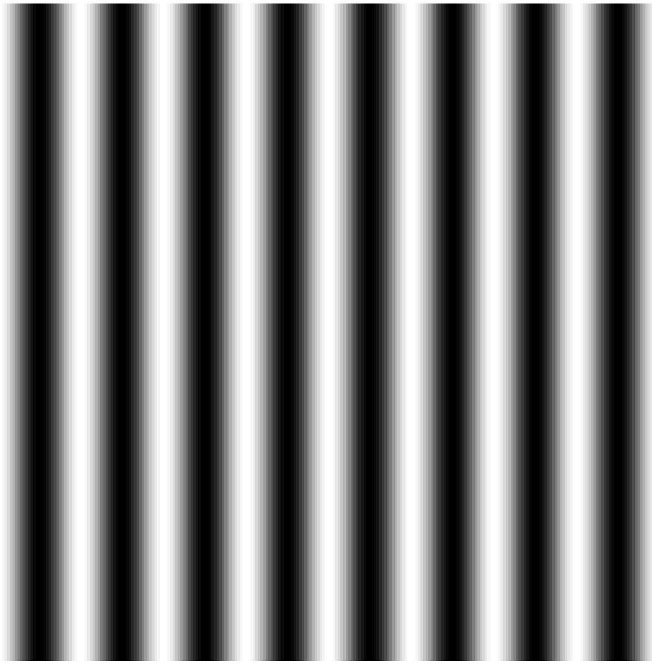
$$\omega = \pi/8 \quad \theta = 0$$



Complex Exponentials

$$\exp\left(j\,\omega\ (\cos\theta\,x + \sin\theta\,y) \right)$$

$$\omega = \pi/16 \quad \theta = 0$$



Complex Exponentials

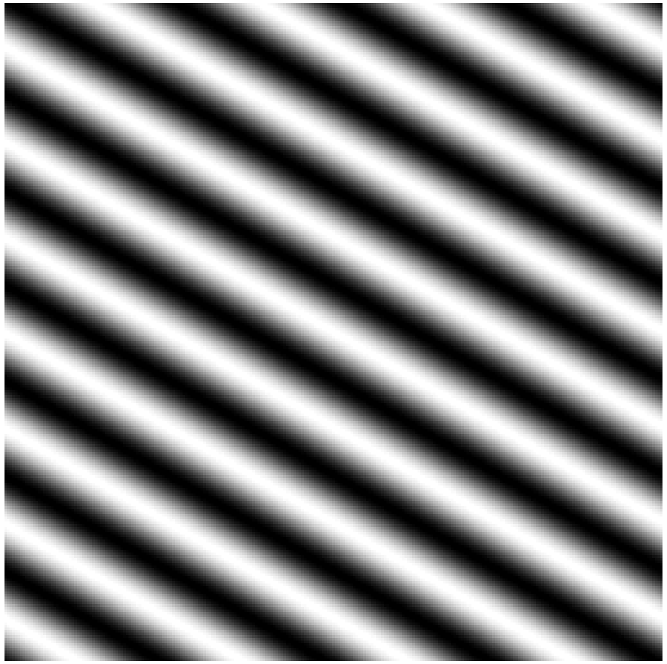
$$\exp\left(j\,\omega\,\left(\cos\theta\,x + \sin\theta\,y\right)\right)$$

$$\omega = \pi/16 \quad \theta = \pi/6$$

Complex Exponentials

$$\exp\left(j\,\omega\;(\cos\theta\,x + \sin\theta\,y)\right)$$

$$\omega = \pi/16 \quad \theta = 2\pi/6$$

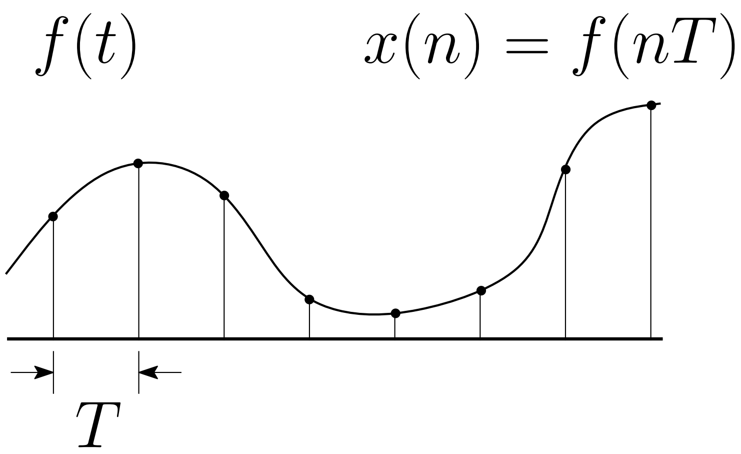


Complex Exponentials

$$\exp\left(j\,\omega\ (\cos\theta\, x + \sin\theta\, y)\right)$$

$$\omega = \pi/16 \quad \theta = 3\pi/6$$

Sampling – 1D



Suppose we know $f(t)$ only for $t \in \{n\,T\}_{n\in\mathbb{Z}}$.

Q_1 : Under what conditions can we reconstruct $f(t)$?

Q_2 : How can we reconstruct $f(t)$?

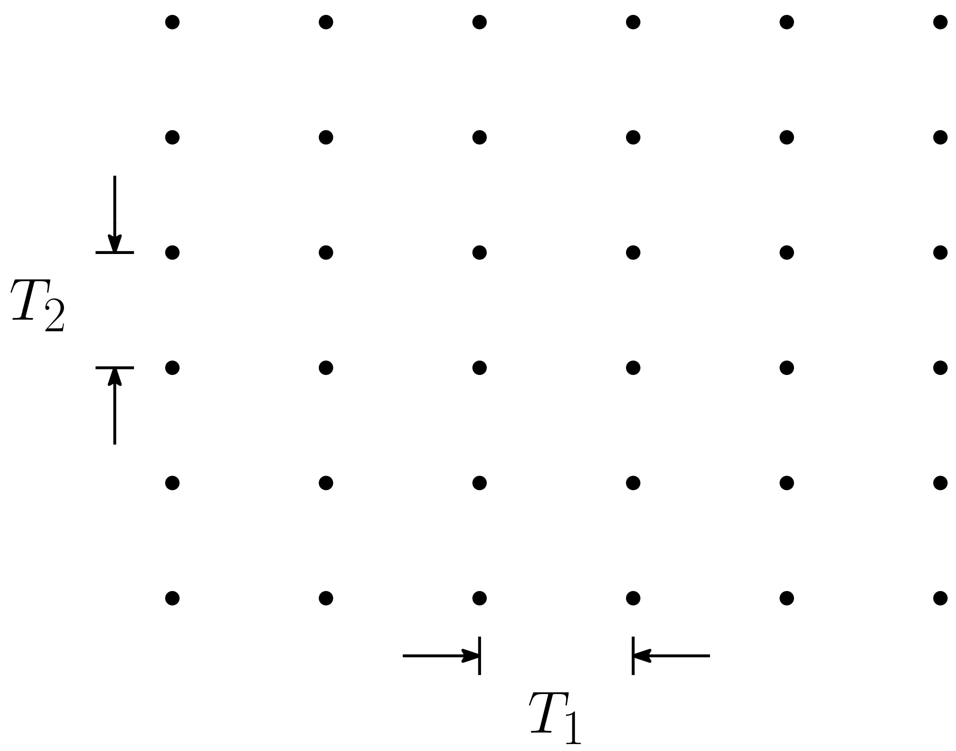
Sampling – 2D

Suppose we know $f(t_1, t_2)$ only the points marked by red dots.

Q_1 : Under what conditions can we reconstruct $f(t_1, t_2)$?

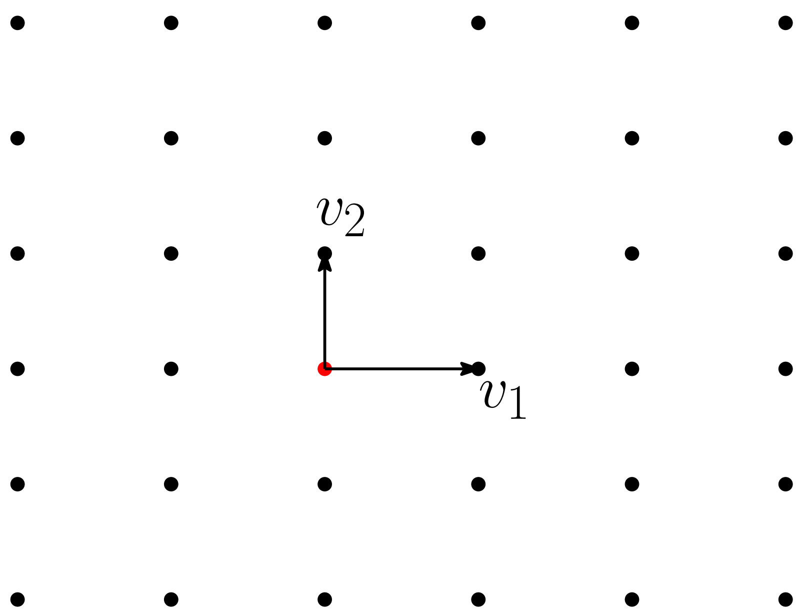
Q_2 : How can we reconstruct $f(t_1, t_2)$?

Rectangular Sampling Lattice



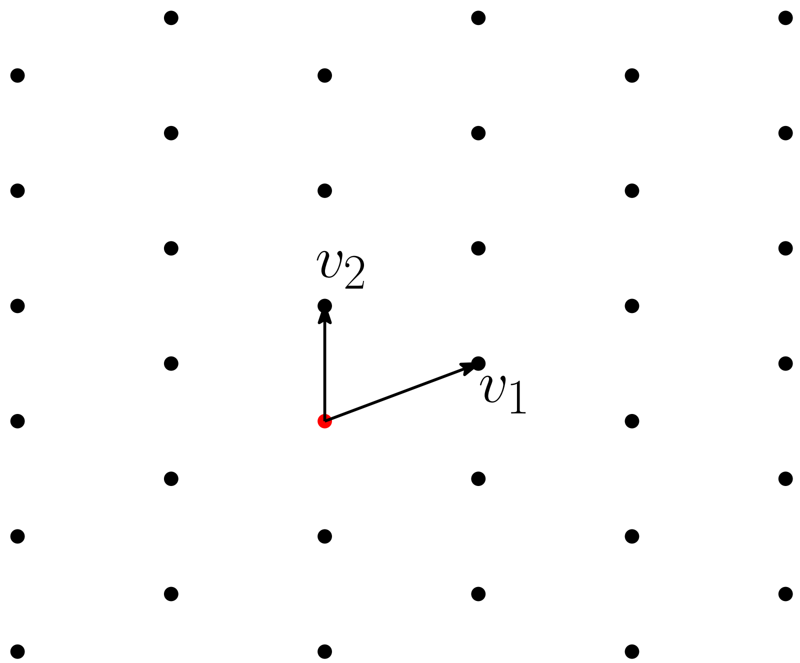
$$\{n_1 T_1, n_2 T_2\}_{n_1, n_2 \in \mathbb{Z}} = \left\{ \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \right\}_{n_1, n_2 \in \mathbb{Z}}$$

Rectangular Sampling Lattice



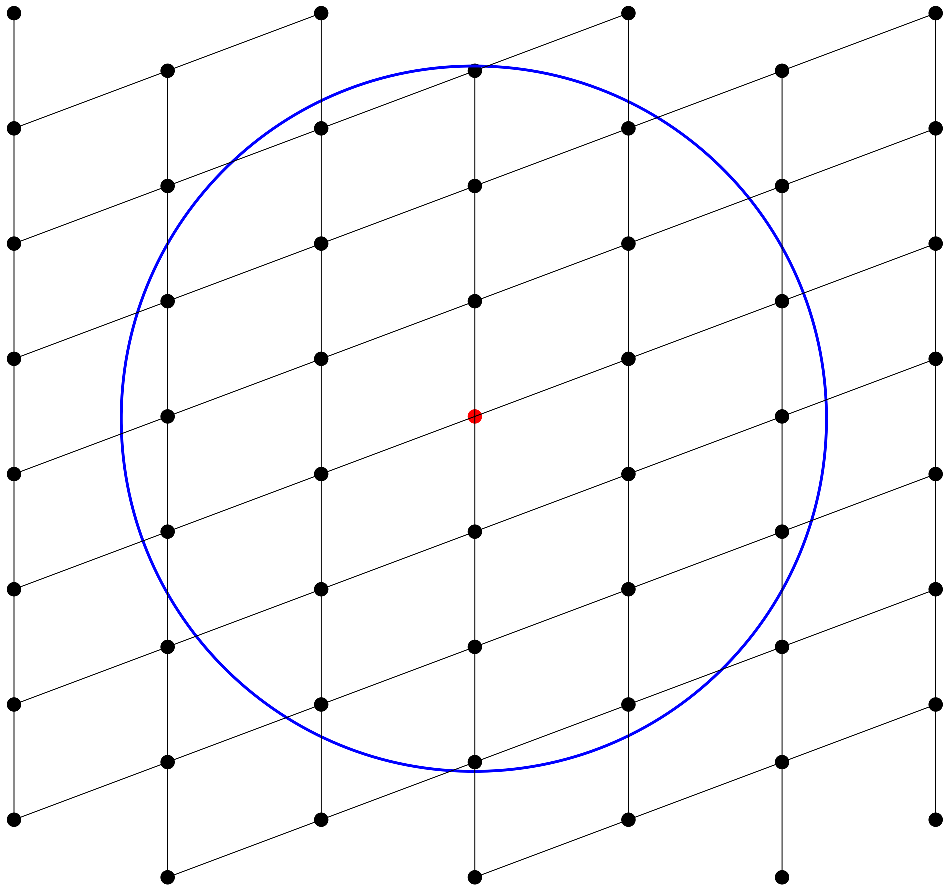
$$\{n_1 T_1, n_2 T_2\}_{n_1,n_2 \in \mathbb{Z}} = \left\{ \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \right\}_{n_1,n_2 \in \mathbb{Z}}$$

General Sampling Lattice



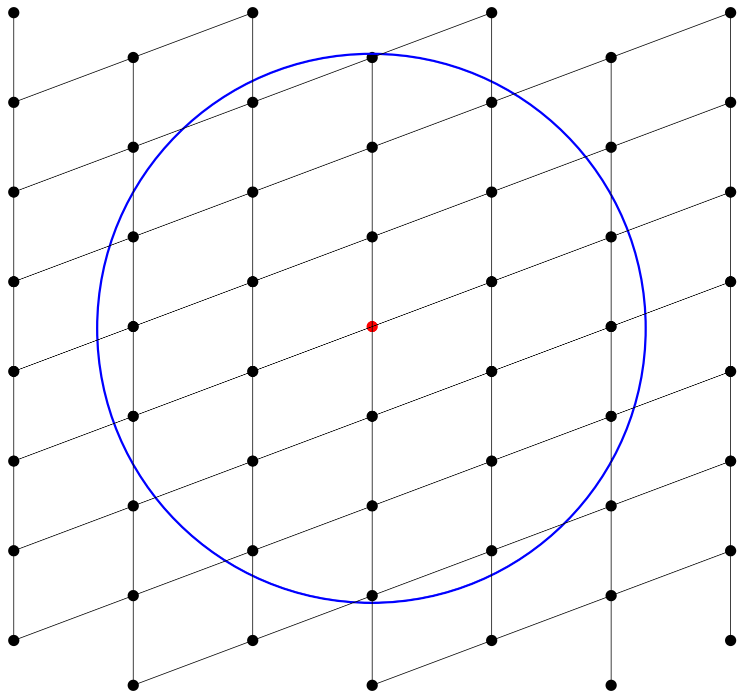
$$\{n_1\,\mathbf{v_1} + n_2\,\mathbf{v_2}\}_{n_1,n_2\in\mathbb{Z}} = \left\{ \underbrace{\begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix}}_{V} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \right\}_{n_1,n_2\in\mathbb{Z}}$$

Sampling Density



Sampling Density = Number of Samples per Unit Area

Sampling Density



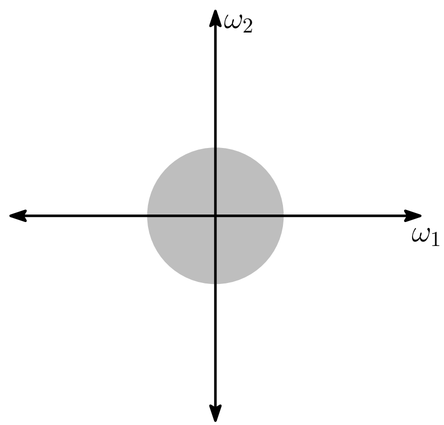
Sampling Density = Number of Samples per Unit Area

$$\rho = \lim_{R \to \infty} \frac{\text{Number of parallelograms}}{\text{Area of Disk}} \propto \frac{1}{|\det(V)|}$$

Rectangular vs. Hexagonal Sampling

Hexagonal Sampling :

$$V = c \begin{bmatrix} 1 & \sqrt{3}/2 \\ 0 & 1/2 \end{bmatrix}$$



For a function with frequency support restricted to a disk,

$$\rho_{\min}(\text{Rectangular Sampling}) \approx 1.15\, \rho_{\min}(\text{Hexagonal Sampling})$$

Irregular Sampling

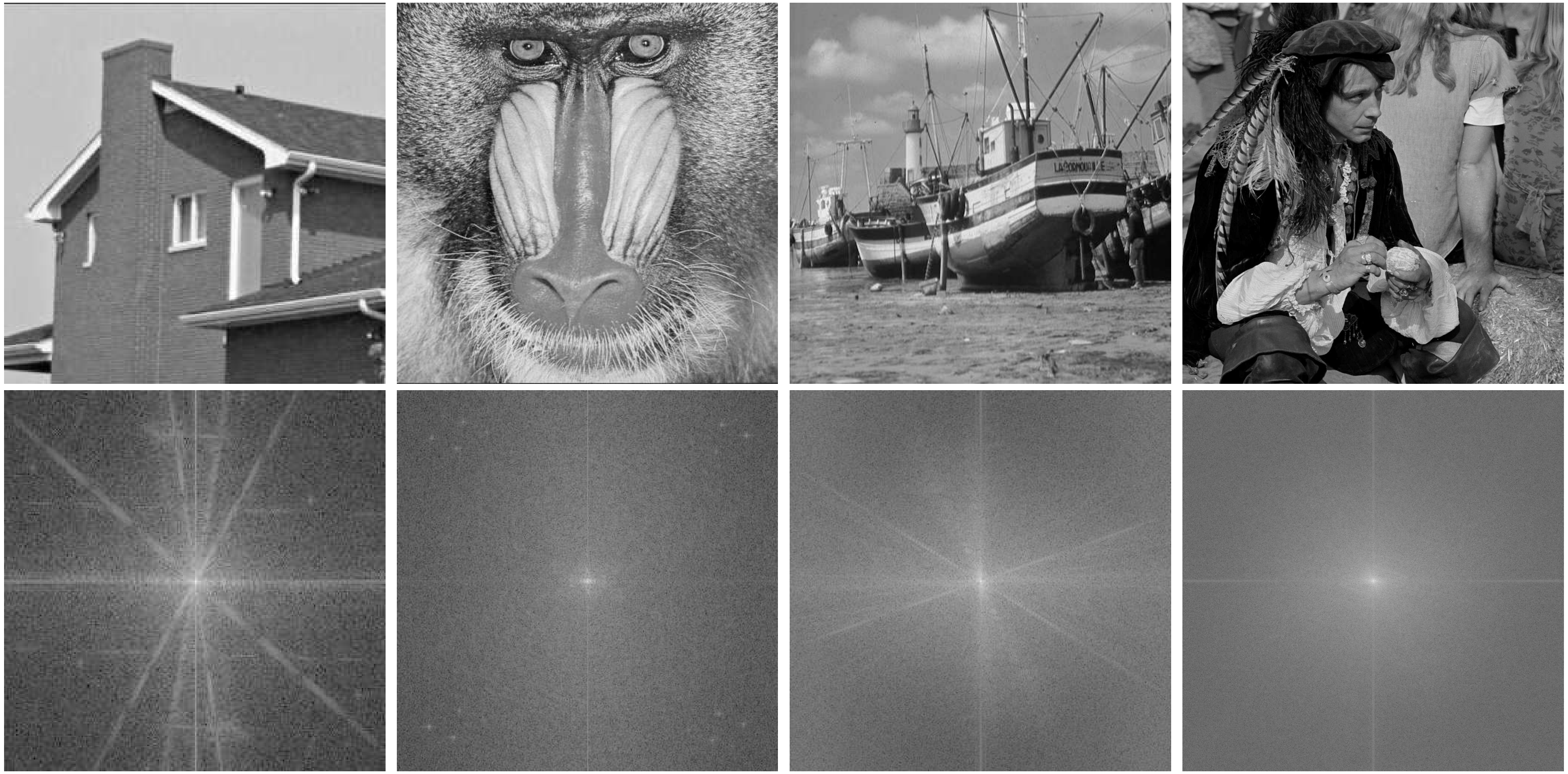
Irregular sampling arises in biomedical imaging problems like MRI, CT, diffraction tomography etc.

Magnitude and Phase

$$\text{Fourier Transform} = \text{Magnitude} \times e^{j\,\text{Phase}}$$

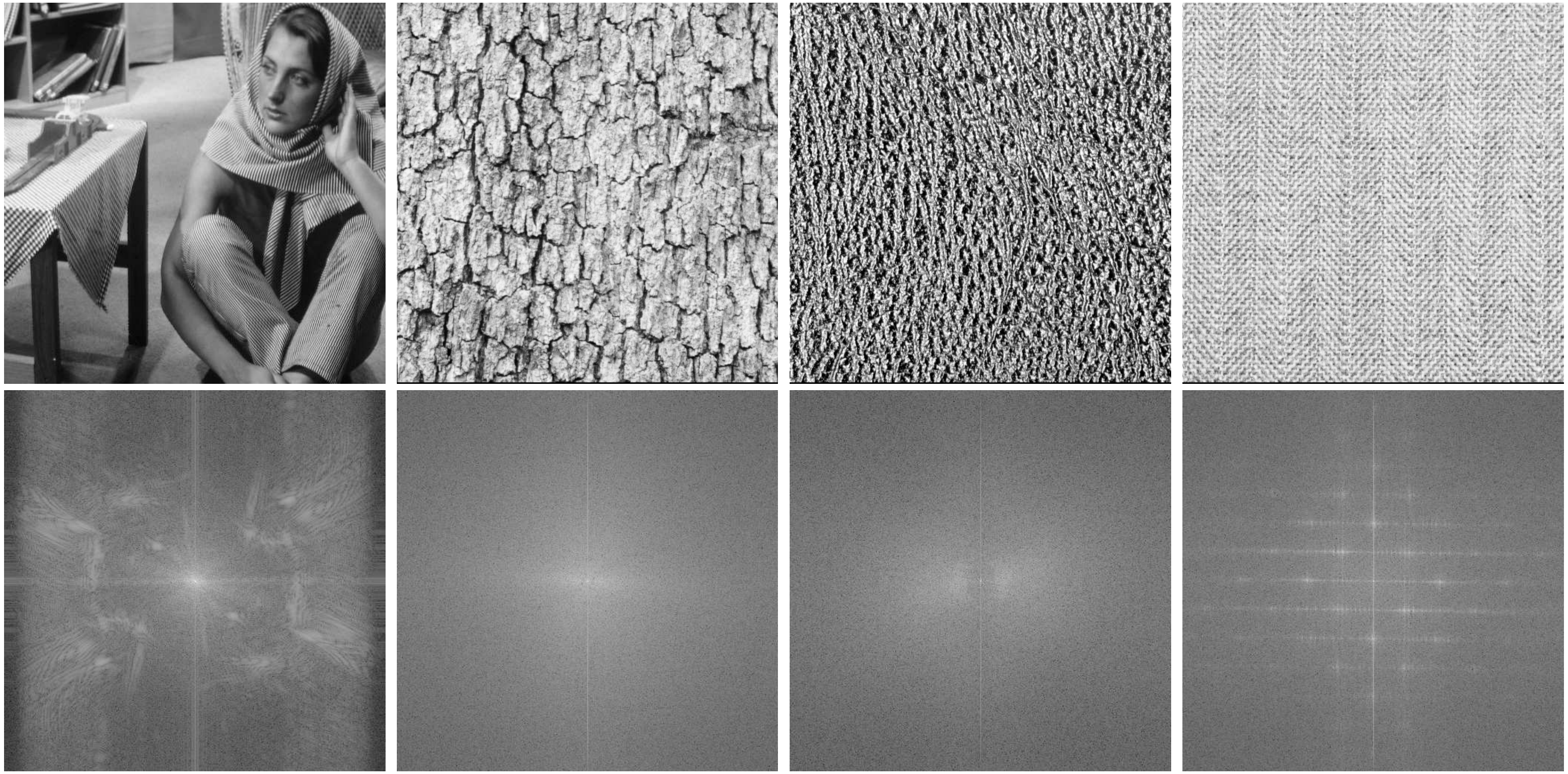
$$X(\omega_1, \omega_2) = |X(\omega_1, \omega_2)|\ \exp(j\,\angle X(\omega_1, \omega_2))$$

Typical Fourier Transform Magnitude of Natural Images

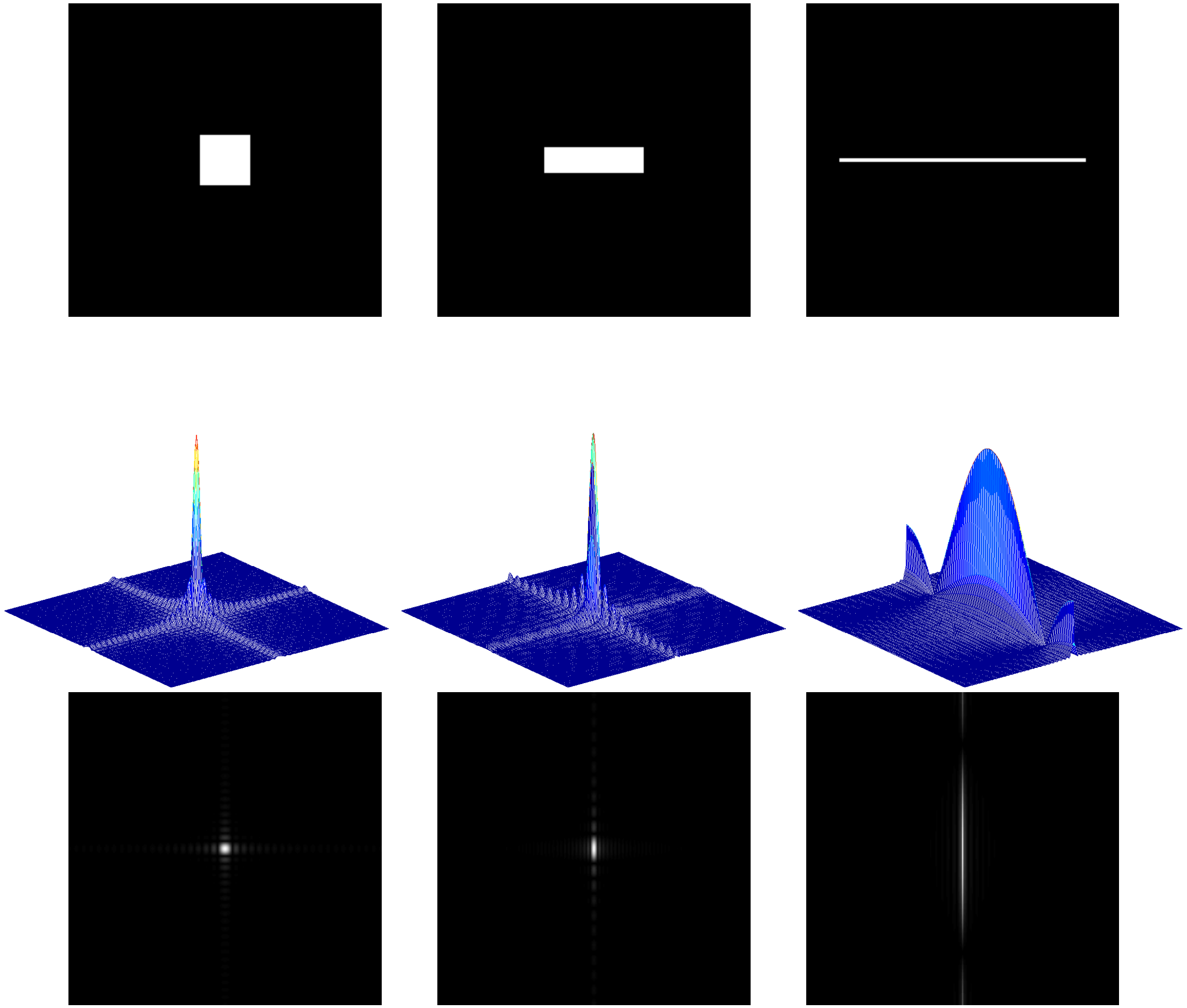


Dominant lowpass behavior...

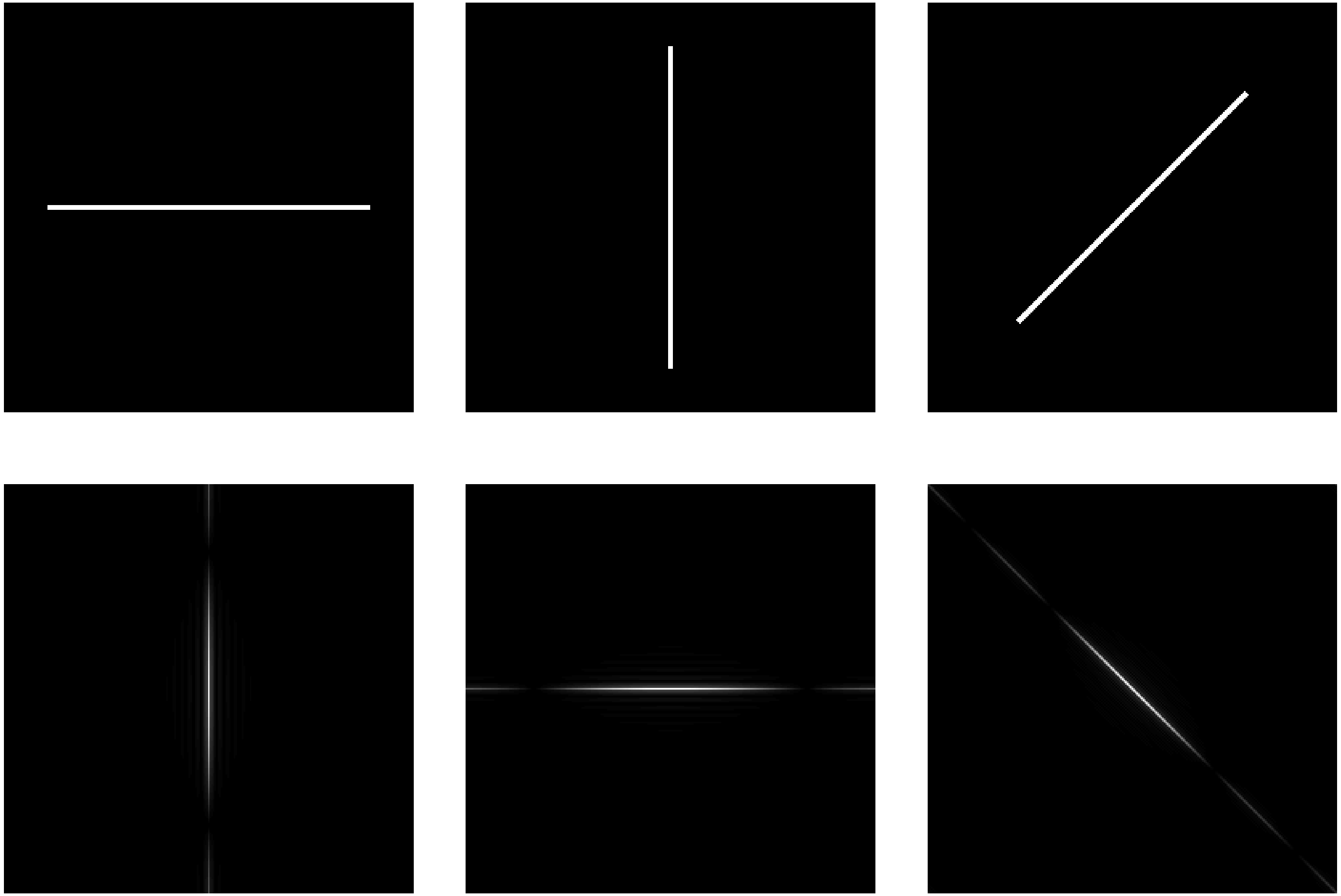
Texture



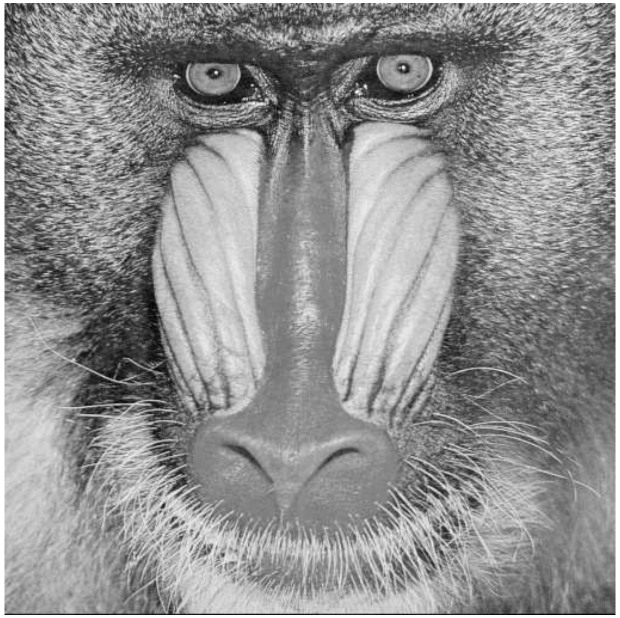
Fourier Transform of a Box / Line



Fourier Transform of a Box / Line



Magnitude vs. Phase

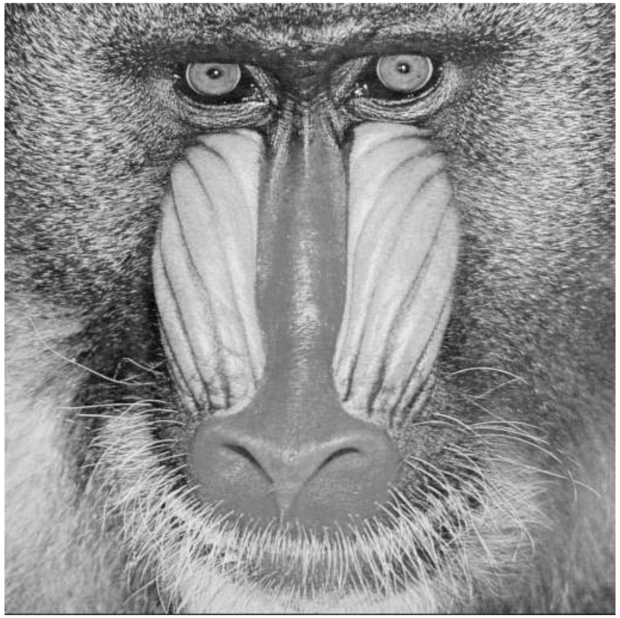


$X(\omega_1, \omega_2)$

$Y(\omega_1, \omega_2)$

$|X| \exp(j \angle Y)$

Magnitude vs. Phase

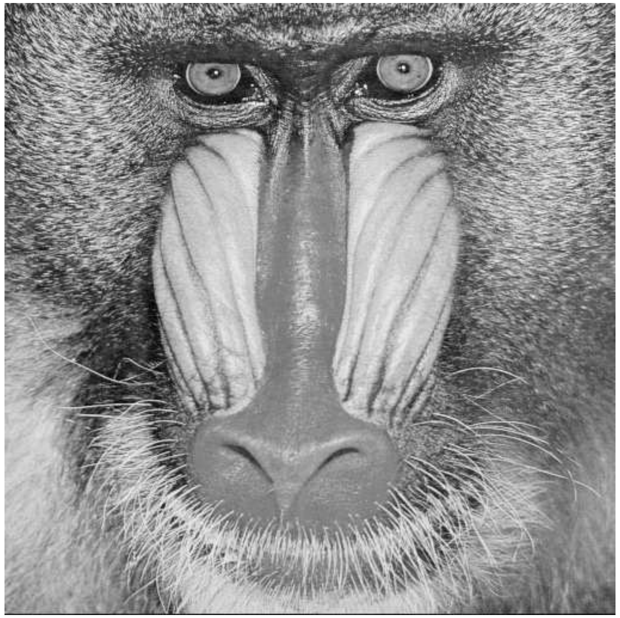


$X(\omega_1, \omega_2)$

$Y(\omega_1, \omega_2)$

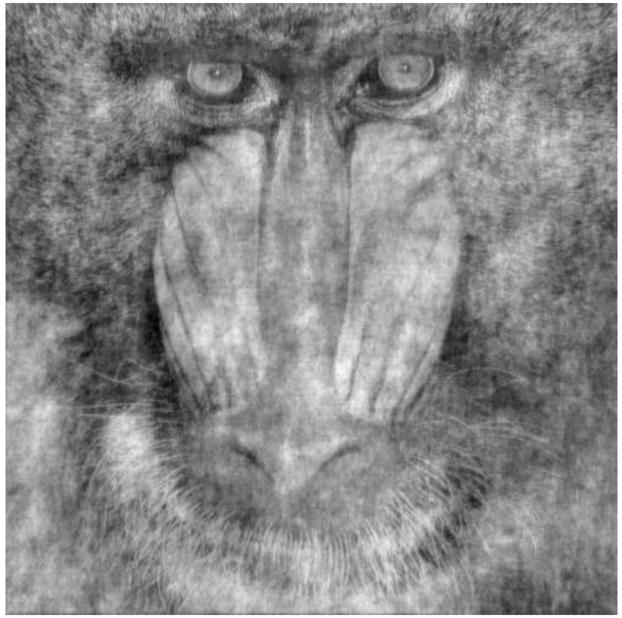
$|X| \exp(j \angle Y)$

Magnitude vs. Phase



$X(\omega_1, \omega_2)$

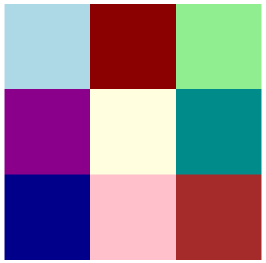
$Y(\omega_1, \omega_2)$



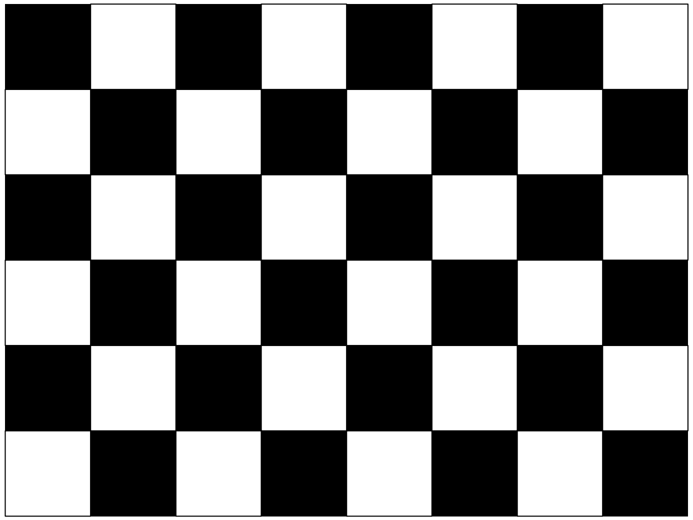
$|Y| \exp(j \angle X)$

Convolution

$$h(n_1, n_2) * *x(n_1, n_2) = \sum_{k_1} \sum_{k_2} x(k_1, k_2)\, h(n_1 - k_1, n_2 - k_2)$$

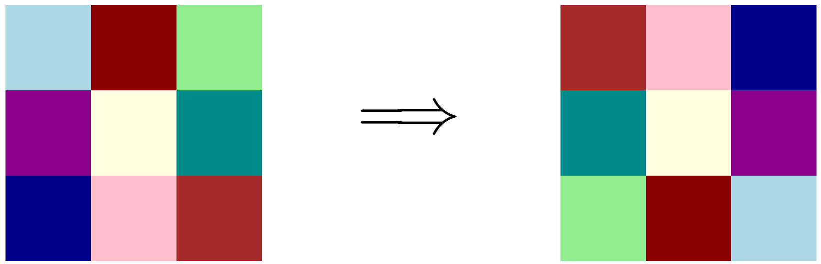


$**$



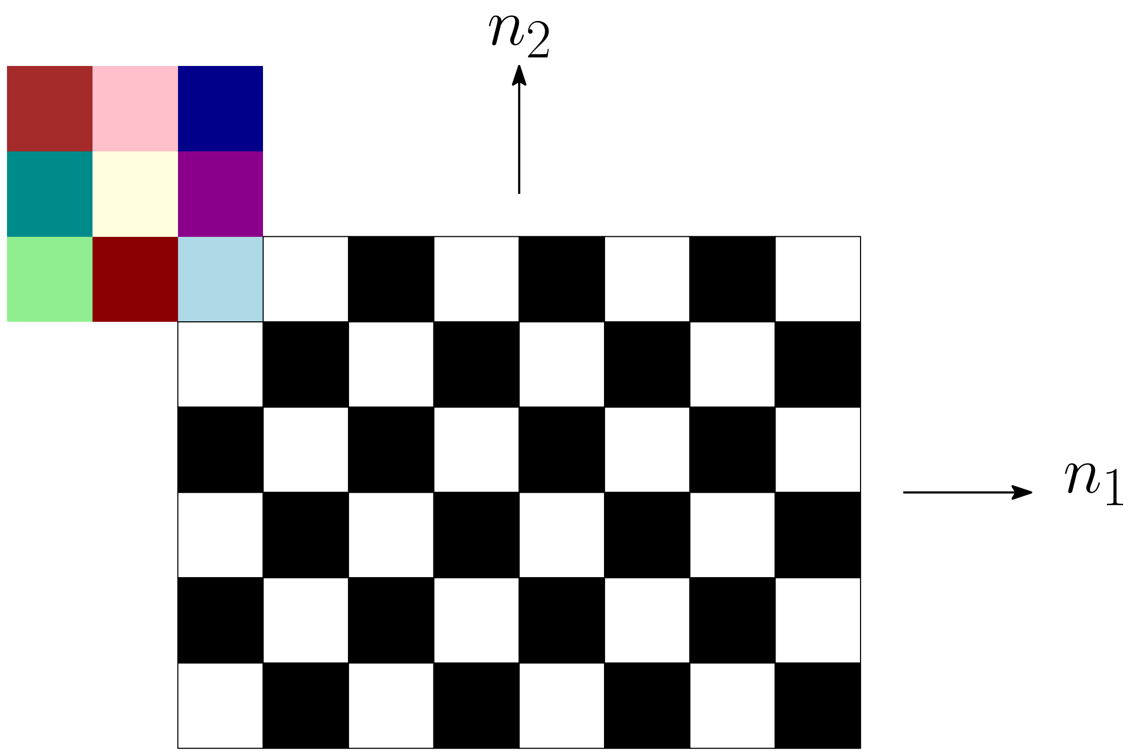
Convolution – Step 1

Flip the kernel in both directions.



Convolution – Step 2

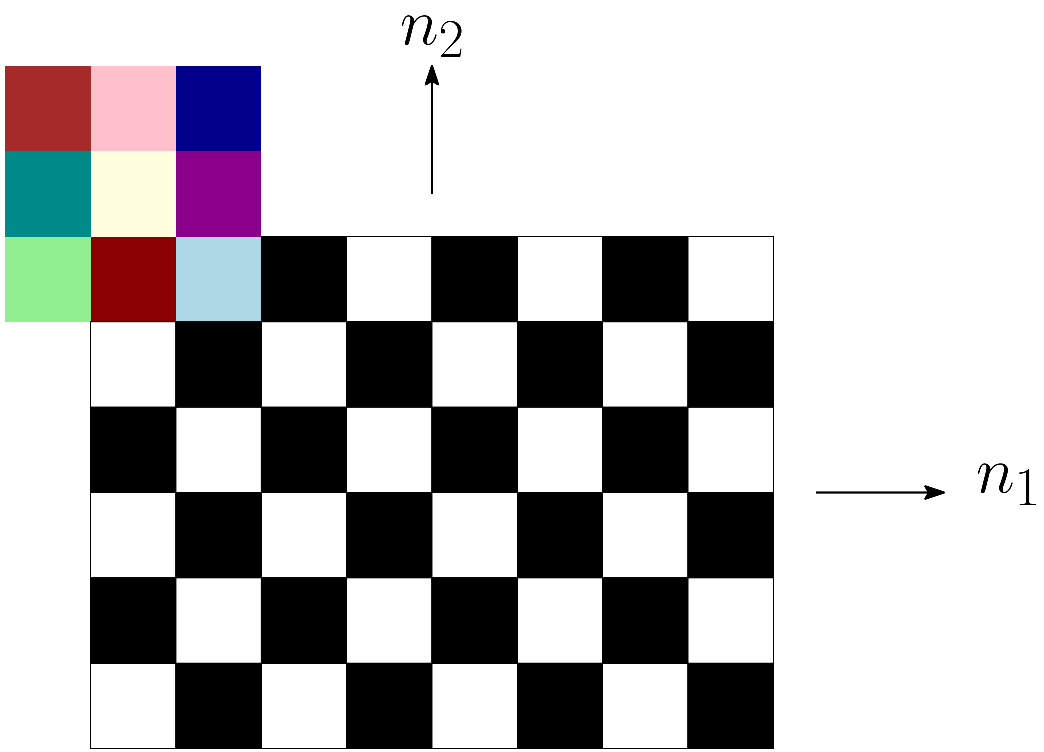
Compute the inner product of the flipped mask with the image.



$$\{h ** x\}(m, l)$$

Convolution – Step 2

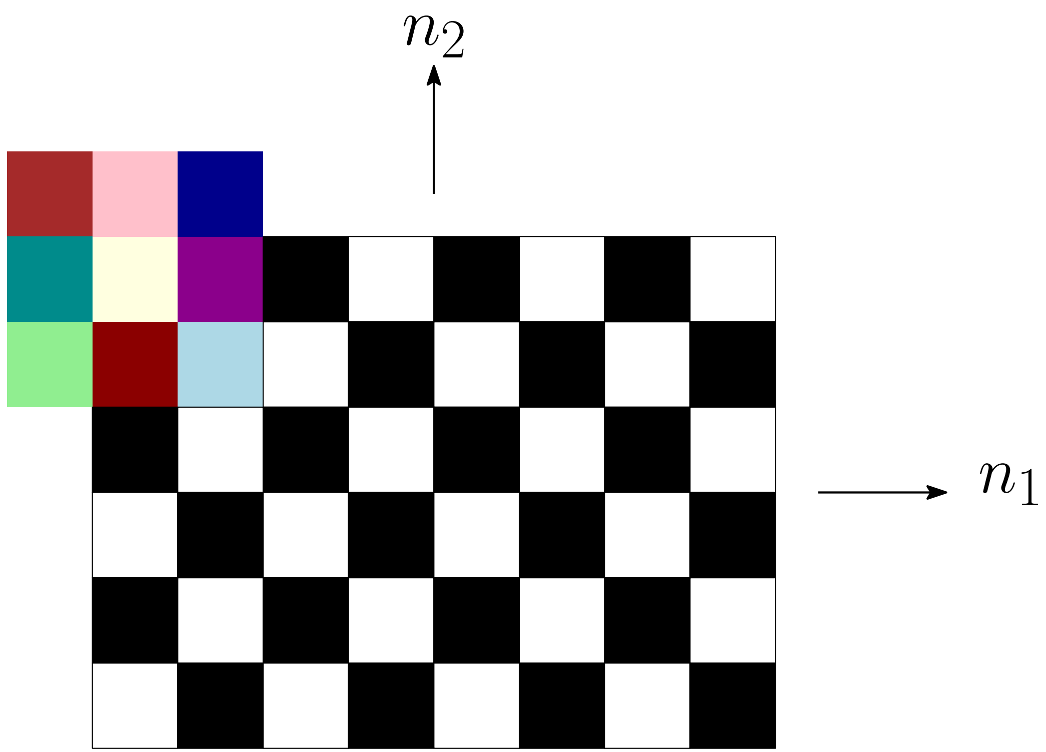
Shift the mask and repeat.



$$\{h * *x\}(m+1, l)$$

Convolution – Step 2

Shift the mask and repeat.



$$\{h**x\}(?,?)$$

Convolution – Some Special Cases

$$\begin{array}{|c|c|c|}\hline 1/9 & 1/9 & 1/9 \\ \hline 1/9 & 1/9 & 1/9 \\ \hline 1/9 & 1/9 & 1/9 \\ \hline \end{array} \quad **$$

$$= ?$$

Convolution – Some Special Cases

Convolution – Some Special Cases

$1/25$	$1/25$	$1/25$	$1/25$	$1/25$
$1/25$	$1/25$	$1/25$	$1/25$	$1/25$
$1/25$	$1/25$	$1/25$	$1/25$	$1/25$
$1/25$	$1/25$	$1/25$	$1/25$	$1/25$
$1/25$	$1/25$	$1/25$	$1/25$	$1/25$

$**$

$= ?$

Convolution – Some Special Cases

Convolution – Some Special Cases

1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49

$**$

$= ?$

Convolution – Some Special Cases

Convolution – Some Special Cases

$$\left(\delta(n_1, n_2) \quad - \quad \begin{array}{|c|c|c|c|c|c|c|} \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline \end{array} \right) \quad ** $$

$= ?$

Convolution – Some Special Cases

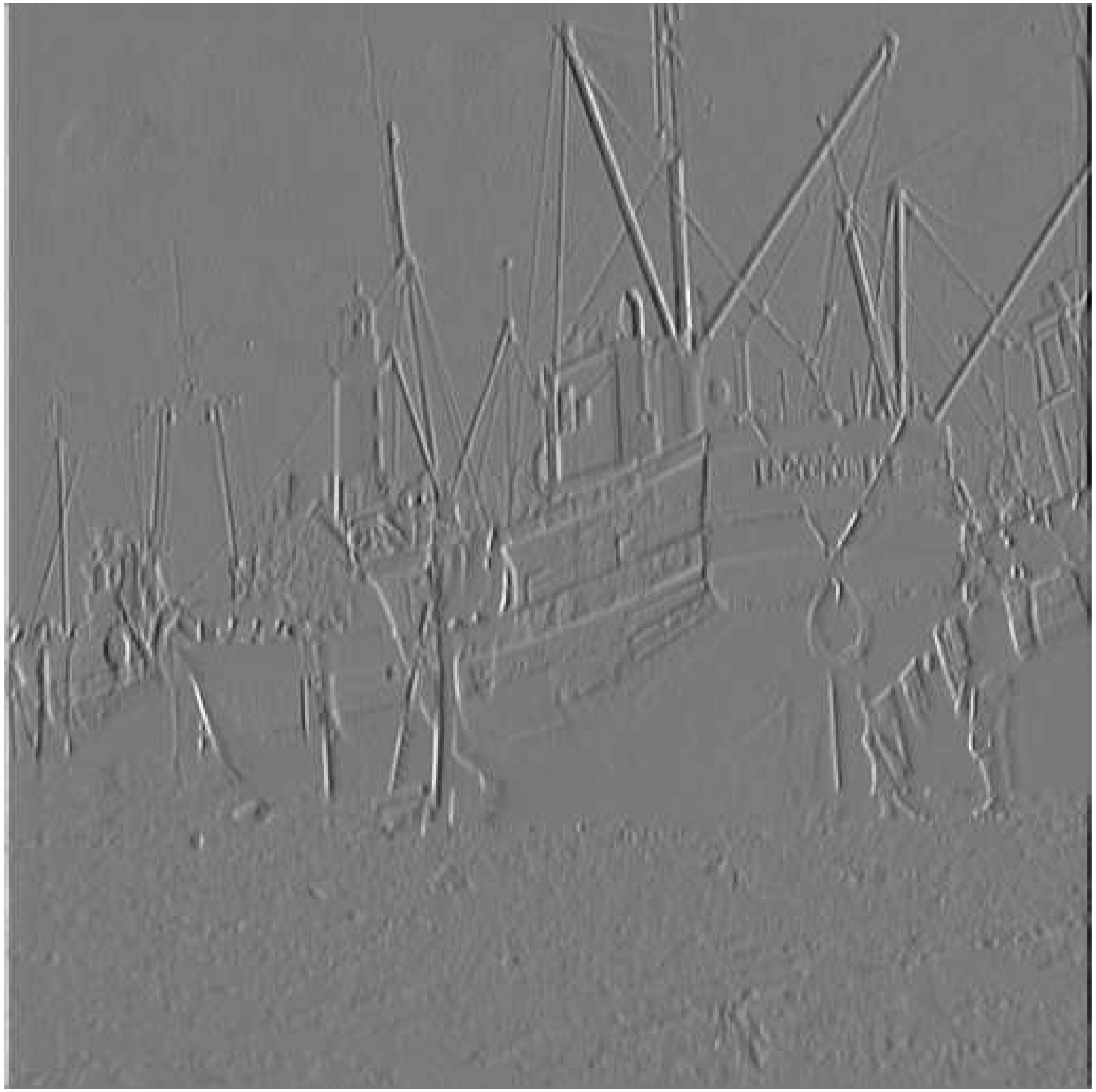
Convolution – Some Special Cases

1	0	−1
1	0	−1
1	0	−1

$**$

$= ?$

Convolution – Some Special Cases



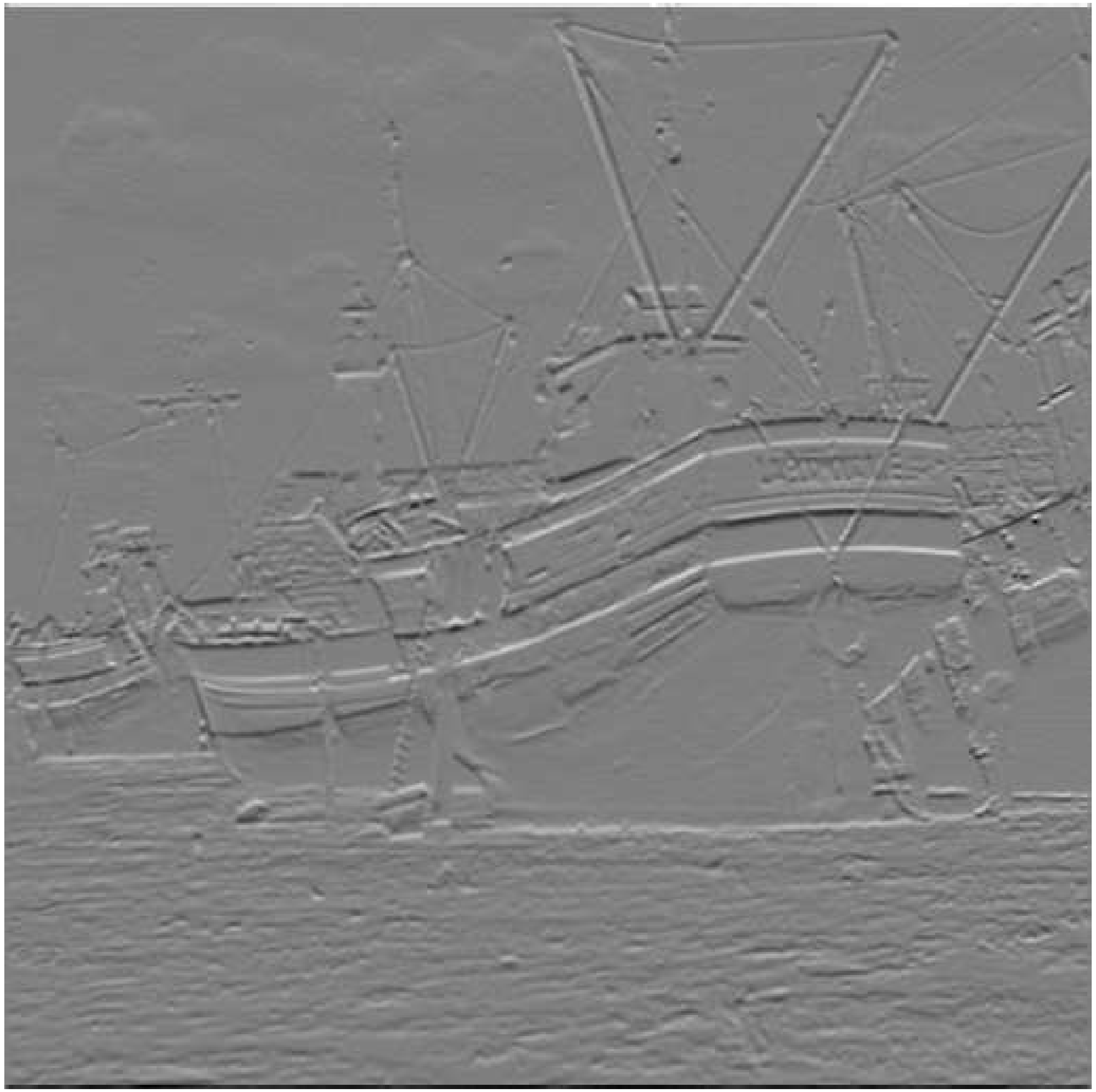
Convolution – Some Special Cases

1	1	1
0	0	0
−1	−1	−1

$**$

$= ?$

Convolution – Some Special Cases



Convolution Theorem

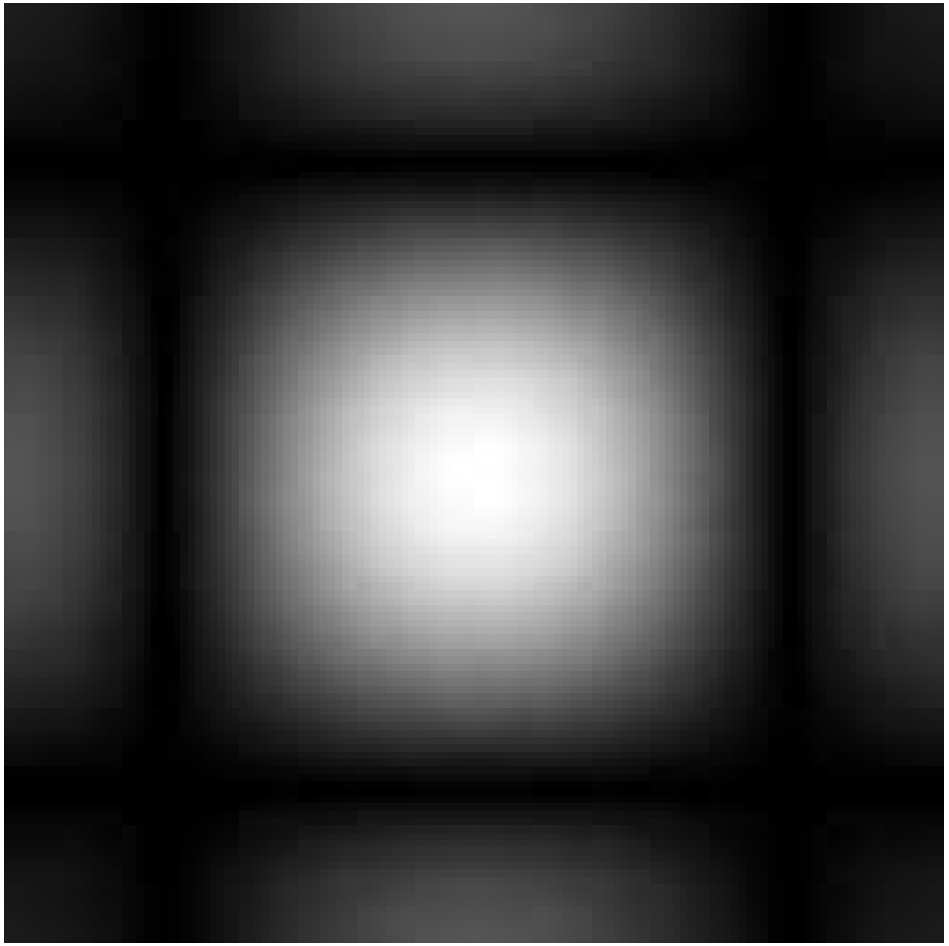
Convolution in the time domain corresponds to multiplication in the frequency domain.

$$y(n_1, n_2) = h(n_1, n_2) \; ** \, x(n_1, n_2)$$
$$\Longleftrightarrow$$
$$Y(\omega_1, \omega_2) = H(\omega_1, \omega_2) \, X(\omega_1, \omega_2)$$

Fourier Transform of Some Masks

$1/9$	$1/9$	$1/9$
$1/9$	$1/9$	$1/9$
$1/9$	$1/9$	$1/9$

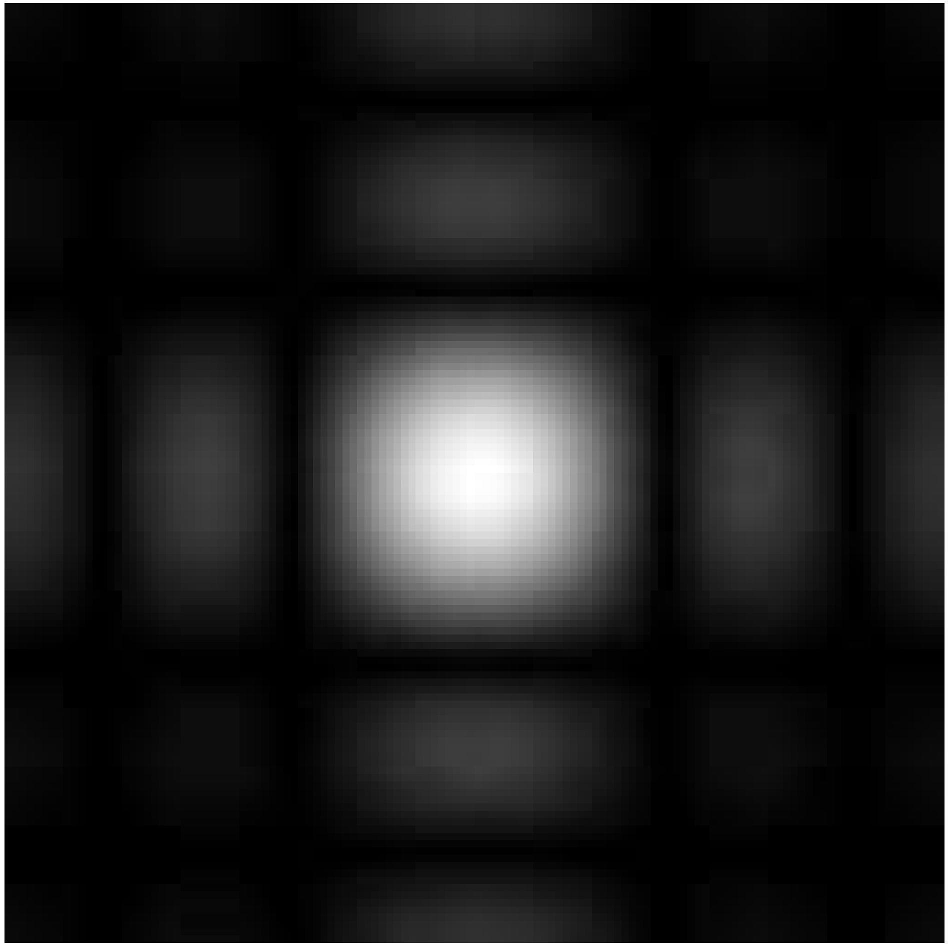
$\Longrightarrow$



Fourier Transform of Some Masks

$1/25$	$1/25$	$1/25$	$1/25$	$1/25$
$1/25$	$1/25$	$1/25$	$1/25$	$1/25$
$1/25$	$1/25$	$1/25$	$1/25$	$1/25$
$1/25$	$1/25$	$1/25$	$1/25$	$1/25$
$1/25$	$1/25$	$1/25$	$1/25$	$1/25$

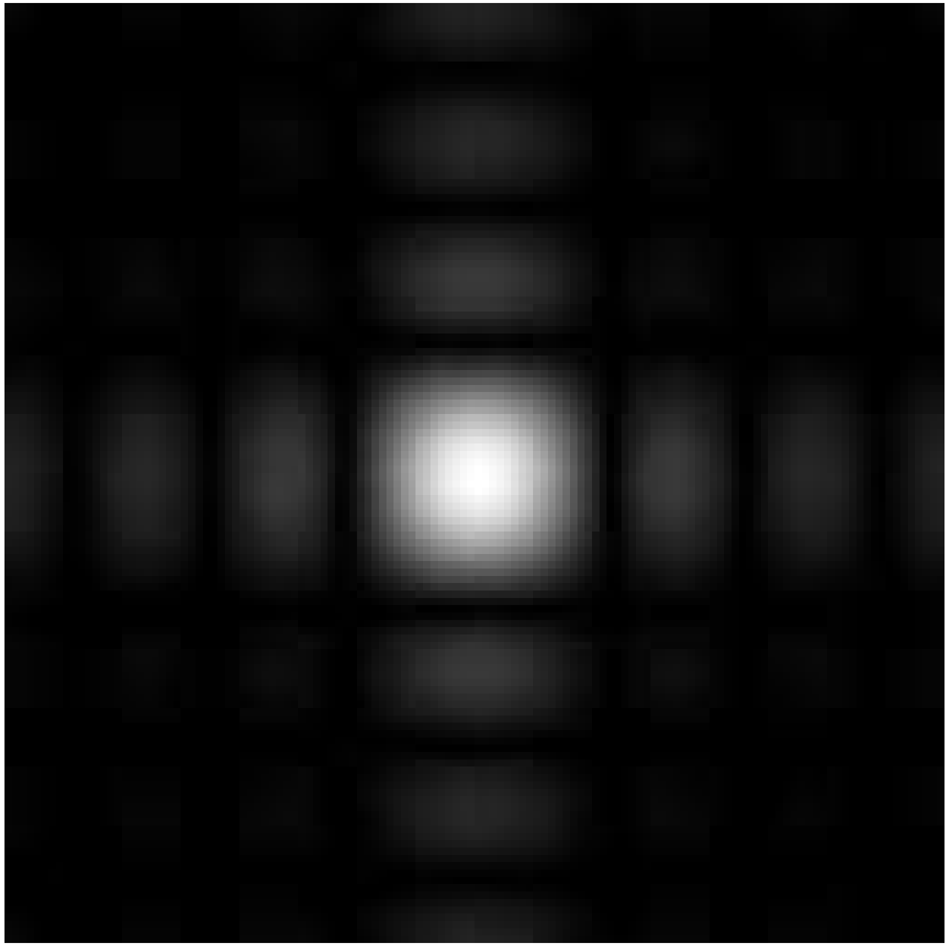
$\Longrightarrow$



Fourier Transform of Some Masks

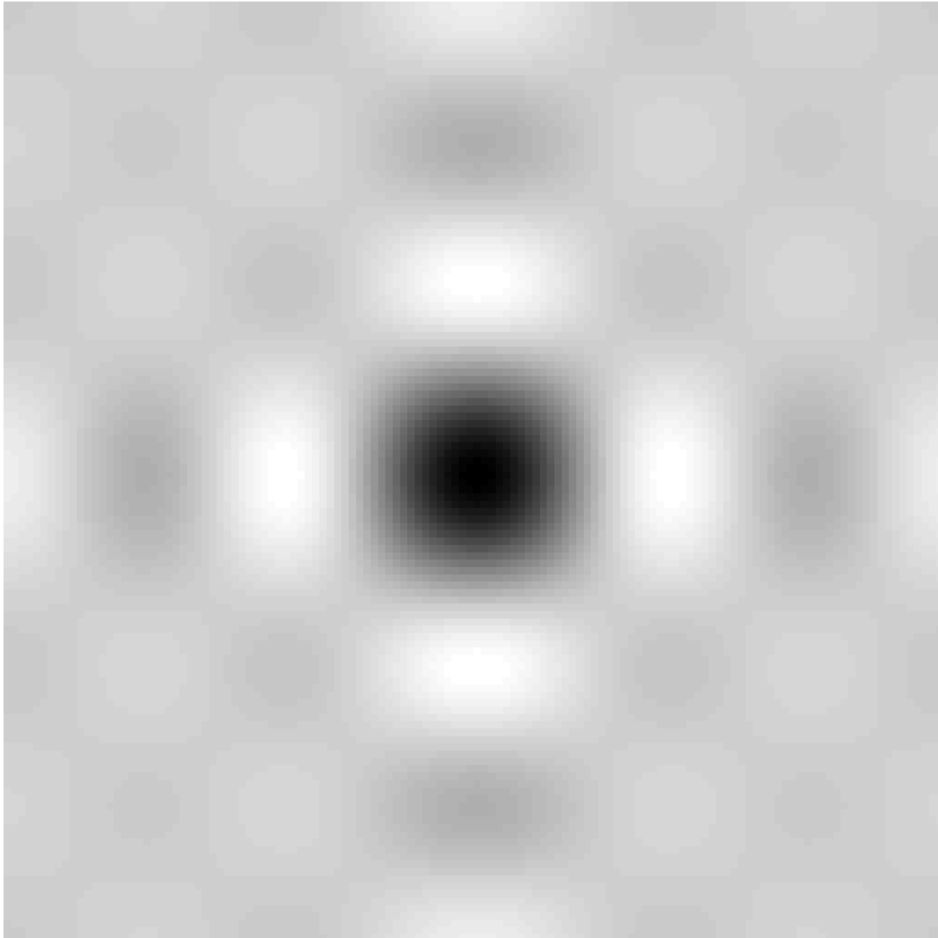
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49
1/49	1/49	1/49	1/49	1/49	1/49	1/49

$\Longrightarrow$



Fourier Transform of Some Masks

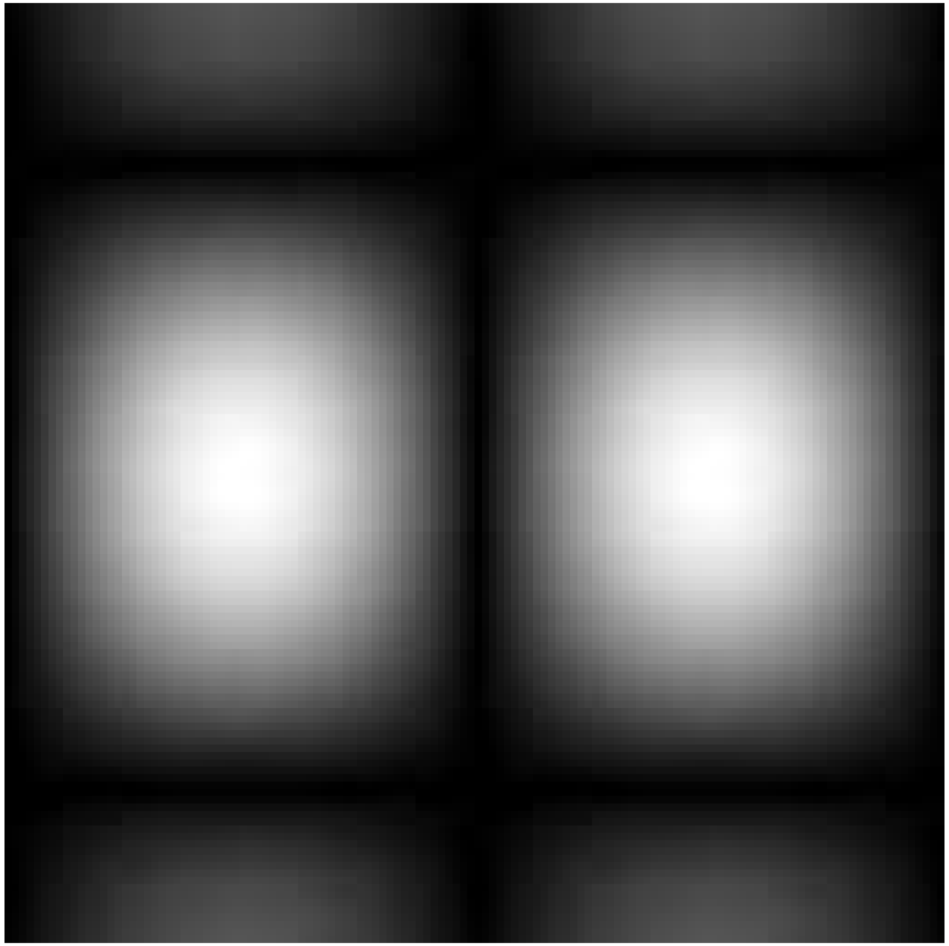
$$\left(\delta(n_1, n_2) \quad - \quad \begin{array}{|c|c|c|c|c|c|c|} \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 & 1/49 \\ \hline \end{array} \right) \Longrightarrow$$



Fourier Transform of Some Masks

1	0	-1
1	0	-1
1	0	-1

$\Longrightarrow$



Fourier Transform of Some Masks

1	1	1
0	0	0
-1	-1	-1

$\Longrightarrow$