BYM-510E - HW3 Solutions

1. $\mathbb{E}\big(y(n)\,y(n+\ell)\big) = \mathbb{E}\big(x(n)\,x(n+\ell)\big) + \frac{1}{2}\mathbb{E}\big(x(n)\,x(n-1+\ell)\big)$

$+\frac{1}{2}\mathbb{E}\big(x(n-1)\,x(n+\ell)\big) + \frac{1}{4}\,\mathbb{E}\big(x(n-1)\,x(n-1+\ell)\big)$

$$= \underbrace{\begin{cases} 2 + \frac{2}{4}, & \text{if } \ell = 0 \\ \frac{2}{2} & \text{if } \ell = 1 \text{ or } -1 \\ 0 & \text{otherwise} \end{cases}}_{R_y(\ell)} \Rightarrow$$

This is a function of 'ℓ' only. Therefore, $y(n)$ is wide-sense stationary.

2. Notice that $y(n) - \frac{1}{2}y(n-1) = x(n)$. If we take the DTFT of both sides, we get,

$$Y(e^{jw}) - \frac{1}{2}\cdot e^{-jw}\cdot Y(e^{jw}) = X(e^{jw}) \quad \text{or}$$

$$Y(e^{jw}) = X(e^{jw})\cdot H(e^{jw}), \quad \text{where } H(e^{jw}) = \frac{1}{1-\frac{1}{2}e^{-jw}}.$$

So, we've

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

In this case: $S_y(e^{jw}) = S_x(e^{jw})\,|H(e^{jw})|^2 = 1 \cdot \dfrac{1}{\frac{3}{2} + \cos w}$

3. (a) $R_y(n_1, n_2) = \mathbb{E}(y(n_1)y(n_2)) = \mathbb{E}(A^2)\cos(n_1)\cos(n_2) + \mathbb{E}(x(n_1)\,x(n_2))$

$$= 1 \cdot \frac{\cos(n_1 - n_2)}{2} + \frac{\cos(n_1 + n_2)}{2} + \delta(n_1 - n_2).$$

(b) $R_y(n_1, n_2)$ is not a function of $n_1 - n_2$ only. $S_y(e^{j\omega})$ is not defined.

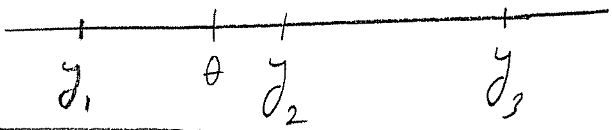
4. z is distributed as $f_z(t;\theta) = \frac{1}{2} e^{-|t-\theta|}$

Since the observations are independent,
the likelihood function becomes: $f(\theta) = \frac{1}{8} \cdot e^{-\sum_{i=1}^{3} |z_i - \theta|}$

The log-likelihood function is:

$$\ln f(\theta) = \ln \frac{1}{8} - (|z_1 - \theta| + |z_2 - \theta| + |z_3 - \theta|)$$

Let
$$\left.\begin{aligned} y_1 &= \min(z_1, z_2, z_3) \\ y_2 &= \max(z_1, z_2, z_3) \\ y_3 &= \text{median}(z_1, z_2, z_3). \end{aligned}\right\} \Rightarrow \begin{aligned} &-\ln(f(\theta)) + c \\ &= \sum_{i=1}^{3} |y_i - \theta| \geq y_3 - y_1 \end{aligned}$$



If we set $\boxed{\theta = \text{median}(z_1, z_2, z_3)}$,
$\sum_{i=1}^{3} |y_i - \theta| = y_3 - y_1$, the minimum value that $\ln f(\theta)$ can take.