MAT 281E – Homework 6 Solutions

1. True or False? (Notice the correction in (c).)

   (a) An $n \times n$ matrix always has n distinct eigenvalues. (F)

   (b) An $n \times n$ matrix always has n, possibly repeating, eigenvalues. (T)

   (c) An $n \times n$ matrix always has n eigenvectors that span $\mathbf{R}^n$. (F)

   (d) Every matrix has at least 1 eigenvector. (T)

   (e) If A and B have the same eigenvalues, they always have the same eigenvectors. (F)

   (f) If A and B have the same eigenvectors, they always have the same eigenvalues. (F)

   (g) If Q has $1/2$ as an eigenvalue, then it cannot be orthogonal. (T)

   (h) If $A = S\,\Lambda\,S^{-1}$ where Λ is diagonal, then the rows of S have to be the eigenvectors of A. (F)

   (i) If $A = S\,\Lambda\,S^{-1}$ where Λ is diagonal, then the columns of S have to be the eigenvectors of A. (T)

   (j) An arbitrary matrix A can always be diagonalized as $A = S\,\Lambda\,S^{-1}$ where Λ is diagonal. (F)

2. Let A be an $n \times n$ matrix with all entries equal to 1 (i.e. $a_{i,j} = 1$). For $n = 2, 3$, find the eigenvalues and eigenvectors of A.

Here is one way to proceed :

For $n = 2$ we have,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

We recognize the last expression as the scaled version of the projection matrix to the subspace S that contains (α, α). Thus, $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ must be an eigenvector. We note that $A \begin{bmatrix} 1 & 1 \end{bmatrix}^T = 2 \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ so the eigenvalue is 2. The other eigenvector $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$ comes from the orthogonal complement of S, with eigenvalue 0.

For $n = 3$ we have,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

We recognize the last expression as the scaled version of the projection matrix to the subspace S that contains (α, α, α). Thus, $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ must be an eigenvector. We note that $A \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T = 3 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ so the eigenvalue is 3. The other two eigenvectors $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T$, $\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$ come from the orthogonal complement of S, with eigenvalue 0.

Remark : Since A is symmetric, we know without computing anything that we can find n independent (even orthogonal if we like) eigenvectors. Once we do find n such eigenvectors, we can stop, since there can be no more. By the way, for $n = 3$, the eigenvectors (even if we require them to have unit energy) are not unique. Why not?

3. Suppose that A is a 3×3 matrix with eigenvalues λ_1, λ_2, λ_3 where the corresponding eigenvectors are x_1, x_2, x_3. What are the eigenvalues and eigenvectors of $2A - I$?

We have,

$$A\,x_1 = \lambda_1\,x_1, \quad A\,x_2 = \lambda_2\,x_2, \quad A\,x_1 = \lambda_3\,x_3.$$

Multiplying by 2 and subtracting multiples of x_i from both sides of the equations, we have,

$$2\,A\,x_1 - x_1 = (2\lambda_1 - 1)\,x_1, \quad 2\,A\,x_2 - x_2 = (2\lambda_2 - 1)\,x_2, \quad 2\,A\,x_3 - x_3 = (2\lambda_3 - 1)\,x_3.$$

Thus the eigenvalues are $(2\lambda_1 - 1)$, $(2\lambda_2 - 1)$, $(2\lambda_3 - 1)$ with associated eigenvectors x_1, x_2, x_3.

4. Find the eigenvalues of the following matrices.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}.$$

The eigenvalues are given by the diagonal entries. For A, these are 1, 3, 6. Check that $A - \lambda\,I$ is singular if λ is equal to an eigenvalue (these are all the eigenvalues because a 3×3 matrix cannot have more than 3 eigenvalues). Similarly, for B, the eigenvalues are 1, 3, 4, 7, 9.

5. Let $y(n) = 2\,y(n-1) + 3\,y(n-2)$. Suppose that $y(1) = 4$, $y(0) = 0$. Compute $y(101)$.

Define

$$u_n = \begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix}.$$

Then we have,

$$u_n = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}}_{A} u_{n-1}.$$

We can now write

$$u_{101} = \begin{bmatrix} y(101) \\ y(100) \end{bmatrix} = A^{100} \begin{bmatrix} y(1) \\ y(0) \end{bmatrix} = A^{100} \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Let us diagonalize A to compute A^{100}. To find the eigenvalues, we compute the roots of $\det(A - \lambda\,I)$. That is,

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 2\,\lambda - 3 = (\lambda - 3)\,(\lambda + 1).$$

So the eigenvalues are 3 and -1.

To compute the eigenvector for $\lambda = 3$, we look at the nullspace of $A - 3I$:

$$A - 3I = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$$

From this we see that $(3, 1)$ is an eigenvector for $\lambda = 3$.

To compute the eigenvector for $\lambda = -1$, we look at the nullspace of $A + I$:

$$A + I = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$$

From this we see that $(1, -1)$ is an eigenvector for $\lambda = -1$.

Thus, we can write A as,

$$A \underbrace{\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} S.$$

We see that

$$S \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} y(1) \\ y(0) \end{bmatrix}$$

or

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = S^{-1} \begin{bmatrix} y(1) \\ y(0) \end{bmatrix}$$

Since $A^{100} = S\,\Lambda^{100}\,S^{-1}$, we obtain

$$\begin{bmatrix} y(101) \\ y(100) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3^{100} & 0 \\ 0 & (-1)^{100} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3^{101} + 1 \\ 3^{100} - 1 \end{bmatrix}.$$