# MAT281E — HW7 Solutions

(1.) $(1\ 0\ 1)$ cannot be written as a linear combination of $(1\ 1\ 1)$ and $(1\ 1\ 0)$ so, $A=\begin{bmatrix}1&1&0\\1&1&0\\1&0&0\end{bmatrix}$ does it.

(2.) $\ell$ is the nullspace of $B=\begin{bmatrix}1&1&1\\1&2&1\end{bmatrix}$.

To find the nullspace: $\begin{bmatrix}1&1&1\\1&2&1\end{bmatrix}\rightarrow\begin{bmatrix}1&1&1\\0&1&0\end{bmatrix}\rightarrow\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}-1\\0\\1\end{bmatrix}$ is in the null-space of $B$ (it also spans $N(B)$ since $N(B)$ 1-dimensional)

$\Rightarrow A=\begin{bmatrix}-1\\0\\1\end{bmatrix}\begin{bmatrix}1 & a & b\end{bmatrix}$ for arbitrary $a, b$ is a matrix with $C(A)=\ell$.

(3.) The line is not a subspace because it doesn't pass through the origin. We cannot find $A$ with $C(A)=\ell$, since $C(A)$ has to be a subspace.

(4.) (a) We can only say that their dimension will be the same.

(c) If $Ax=0 \Rightarrow Rx=E_1^{-1}Ax=0 \Rightarrow Bx=E_2Rx=0 \Rightarrow N(A)\subset N(B)$

If $Bx=0 \Rightarrow Ax=0$ similarly $\Rightarrow N(B)\subset N(A) \longrightarrow N(A)=N(B)$

(b) $C(A^T) = N(A)^{\perp} = N(B)^{\perp} = C(B^T)$.

(d) $\dim N(A^T) = \dim N(B^T)$. No further conclusion from the information given.

5. $A = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$

$$q_1 = \frac{c_1}{\sqrt{<c_1, c_1>}} = \frac{c_1}{2} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\tilde{q}_2 = c_2 - \underbrace{<c_2, q_1>}_{0} q_1 = c_2 \; ; \; q_2 = \frac{\tilde{q}_2}{\sqrt{<q_2, q_2>}} = \frac{c_2}{2} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\tilde{q}_3 = c_3 - <c_3, q_1> q_1 - <c_3, q_2> q_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix} - 0 \cdot q_1 - 2 \cdot q_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$q_3 = \frac{\tilde{q}_3}{\sqrt{<\tilde{q}_3, \tilde{q}_3>}} = \tilde{q}_3 / 2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$\tilde{q}_4 = c_4 - <c_4, q_1> q_1 - <c_4, q_2> q_2 - <c_4, q_3> q_3$$

$$= \begin{bmatrix} -3 \\ 1 \\ 1 \\ -3 \end{bmatrix} - (-2) \cdot q_1 - 0 \cdot q_2 - 0 \cdot q_3 = \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

$$q_4 = \frac{\tilde{q}_4}{\sqrt{<\tilde{q}_4, \tilde{q}_4>}} = \tilde{q}_4 / 4$$

$$= \begin{bmatrix} -1/2 & 1/2 & 1/2 & -1/2 \end{bmatrix}^T$$

Now $c_1 = 2q_1$ ; $c_2 = 2q_2$ ; $c_3 = 2q_3 + 2q_2$

$c_4 = 4q_4 - 2q_1$

$$\Rightarrow A = \underbrace{\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}}_{Q} \overbrace{\begin{bmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}}^{R}$$

(6.) $Bx_i = \lambda_i x_i$ for $i = 1, 2, 3.$

$\Rightarrow \frac{1}{\lambda_i} x_i = B^{-1} x_i \Rightarrow$ eigvectors: $x_1, x_2, x_3$

eigvalues: $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ (Notice $\lambda_i \neq 0$ since $B$ is invertible)

(7.) $P_1$ is the solution set of $\underbrace{\begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{X} = 2$

The soln. set is described as

$y_p + y_{s_1} \cdot \alpha_1 + y_{s_2} \cdot \alpha_2 + y_{s_3} \cdot \alpha_3$ where $y_{s_i}$'s are special solutions, $y_p$ is a particular soln. and $\alpha_i$'s are scalars.

Free variables: $x_2, x_3, x_4$

Pivot var: $x_1$.

$\Rightarrow y_p = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (Set free var. to zero & solve).

$y_{s_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (Set $x_3 = x_4 = 0$, $x_2 = 1$ and solve $Cx = 0$)

$y_{s_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $y_{s_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ similarly.

$\Rightarrow$ A pt. on the plane is given by $\underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{e}$.

$$p-q = D\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}+e-\left(\begin{bmatrix}1\\1\\-2\\-1\end{bmatrix}\alpha_4+\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)=\underbrace{\begin{bmatrix}1&1&0&1\\1&0&0&1\\0&1&0&-2\\0&0&1&-1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{bmatrix}}_{x}-\underbrace{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}}_{b}=e$$

Solve $A^TAx = A^Tb$.

$$A^TA = \begin{bmatrix}2&1&0&2\\1&2&0&-1\\0&0&1&-1\\2&-1&-1&7\end{bmatrix}, \quad A^Tb = \begin{bmatrix}-1\\-2\\0\\-1\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}2&1&0&2&-1\\1&2&0&-1&-2\\0&0&1&-1&0\\2&-1&-1&7&-1\end{array}\right]\longrightarrow\begin{bmatrix}2&1&0&2&-1\\0&3/2&0&-2&-3/2\\0&0&1&-1&0\\0&-2&-1&5&0\end{bmatrix}\longrightarrow\begin{bmatrix}2&1&0&2&-1\\0&1&0&-4/3&-1\\0&0&1&-1&0\\0&0&-1&7/3&-2\end{bmatrix}$$

$$\longrightarrow\left[\begin{array}{cccc|c}1&1/2&0&1&-1/2\\0&1&0&-4/3&-1\\0&0&1&-1&0\\0&0&0&4/3&-2\end{array}\right] \Longrightarrow \begin{aligned}\alpha_4 &= -3/2\\ \alpha_3 &= -3/2\\ \alpha_2 &= -1+4/3\,\alpha_4 = -3\\ \alpha_1 &= \frac{-1}{2}-\alpha_4-\frac{\alpha_2}{2} = 5/2\end{aligned}$$

$$\Rightarrow p = D\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}+\begin{bmatrix}2\\0\\0\\0\end{bmatrix}=\begin{bmatrix}3/2\\5/2\\-3\\-3/2\end{bmatrix}, \quad q=\begin{bmatrix}1\\1\\-2\\-1\end{bmatrix}\alpha_4+\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}-3/2\\-1/2\\+3\\3/2\end{bmatrix}$$

$p$ & $q$ are unique because $A^TA$ is invertible (4 pivots).

(Notice: In $\mathbb{R}^3$, if $p$ and $q$ are unique then we have $p=q$. (Why?)
This is not the case in $\mathbb{R}^4$ – Why not? )

(8.) $x^T A x = x^T (Ax) = x^T c = c^T x$

$x^T A x = (x^T A) x = (A^T x)^T x = (-Ax)^T x = -c^T x$

$\Rightarrow x^T A x = -x^T A x \Rightarrow 2(x^T A x) = 0$

(9.) $y_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \quad y_3 = \begin{bmatrix} x_3 \\ 0 \end{bmatrix}, \quad y_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow A y_1 = \begin{bmatrix} Bx_1 \\ 1 \cdot 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \lambda_1 y_1$

Similarly $Ay_2 = \lambda_2 y_2$, $Ay_3 = \lambda_3 y_3$

and $A y_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = y_4$.

$y_1, y_2, y_3, y_4$ span $\mathbb{R}^4$ (Why?)

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(10.) $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ where $A_1 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Eigv of $A_1$ are the solutions of $\lambda^2 - 4 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = -2$

$A_1 - 2I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \Rightarrow$ associated eigvector of $A_1$ $= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1$

$A_1 + 2I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow$ associated eigvector of $A_1$ $= \begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_2$

Notice that

$\underbrace{A\begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}}_{e_1} = \begin{bmatrix} A_1 c_1 \\ 0 \end{bmatrix} = 2\underbrace{\begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}}_{e_1}$

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Similarly $e_2 = \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$ is an eigvector with eigvalue $= -2$.

$\Rightarrow e_1$ is an eigenvector of $A$ with eigenvalue $= 2$.

Eigvectors of $A_2$ are the solutions of $\det(A_2 - \lambda I) = 0$

$$\Rightarrow (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda_3 = 3, \ \lambda_4 = -1$$

$$A_2 - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \Rightarrow \begin{array}{c}\text{associated} \\ \text{eigvector} \\ \text{of } A_2\end{array} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{c_3}$$

$$A_2 + I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \Rightarrow \begin{array}{c}\text{associated} \\ \text{eigvector of } A_2\end{array} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{c_4}$$

$$\Rightarrow e_3 = \begin{bmatrix} 0 \\ 0 \\ c_3 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ c_4 \end{bmatrix}$$ are eigvectors of $A$ with eigenvalue $3$ and $-1$.

$$\Rightarrow A \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{Q} = Q \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\Lambda}$$

$$\Rightarrow A = Q \Lambda Q^T$$

Remark: We can work with submatrices if $A$ is block-diagonal.