Linear Shift Invariant (LSI) Systems

$$\delta(n_1, n_2) \longrightarrow \boxed{T} \longrightarrow h(n_1, n_2)$$

Shift invariance means

$$\delta(n_1 - k_1, n_2 - k_2) \longrightarrow \boxed{T} \longrightarrow h(n_1 - k_1, n_2 - k_2)$$

Decompose $x(n_1, n_2)$ as,

$$\begin{aligned} x(n_1, n_2) &= \ldots + x(-1,-1)\,\delta(n_1+1, n_2+1) + x(-1,0)\,\delta(n_1+1, n_2) + x(0,0)\,\delta(n_1, n_2) + \ldots \\ &= \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} x(k_1, k_2)\,\delta(n_1 - k_1, n_2 - k_2) \end{aligned}$$

Thus,

$$T\{x\} = y(n_1, n_2) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} x(k_1, k_2)\,h(n_1 - k_1, n_2 - k_2)$$

Zero Phase Filters

A filter $h(n_1, n_2)$ is said to be *zero-phase* when its frequency response $H(\omega_1, \omega_2)$ is a real function, i.e. when

$$H(\omega_1, \omega_2) = H^*(\omega_1, \omega_2). \tag{1}$$

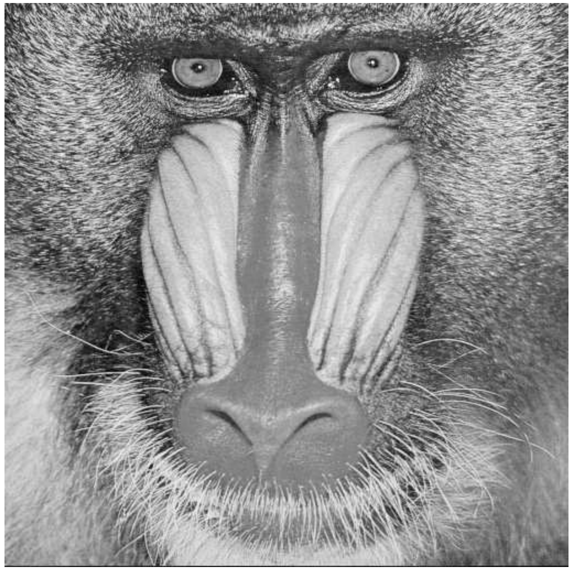
For real $h(n_1, n_2)$, (1) is equivalent to

$$h(n_1, n_2) = h(-n_1, -n_2).$$

In audio applications, the phase characteristics of filters are not very critical.

The situation for images is different. – Recall the 'Magnitude vs. Phase' experiment.

Importance of Phase for Images

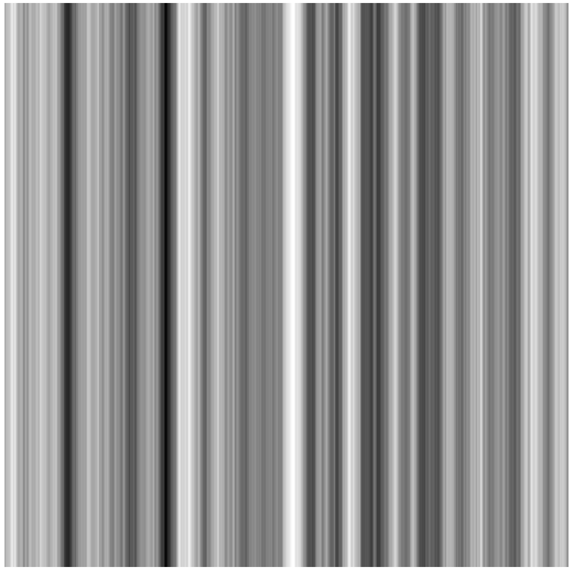


$X(\omega_1, \omega_2)$

$Y(\omega_1, \omega_2)$

$|X| \exp(j \angle Y)$

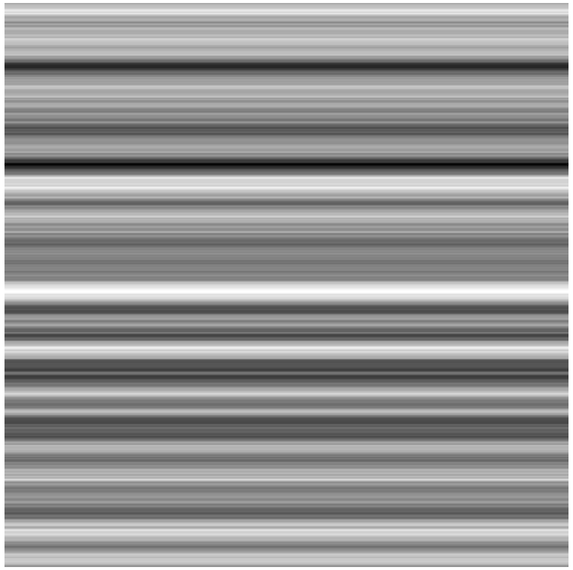
Importance of Phase for Images



$\Theta(\omega_1, \omega_2)$ $\quad$ $Y(\omega_1, \omega_2)$ $\quad$ $Y \; \exp(j\,\Theta)$

Modifying the phase along the horizontal axis creates ghost images along the horizontal axis.

Importance of Phase for Images



$\Theta(\omega_1, \omega_2)$

$Y(\omega_1, \omega_2)$

$Y\ \exp(j\,\Theta)$

Modifying the phase along the vertical axis creates ghost images along the vertical direction.

Importance of Phase for Images

$\Theta(\omega_1, \omega_2)$ $\qquad$ $Y(\omega_1, \omega_2)$ $\qquad$ $Y\ \exp(j\,\Theta)$

For a more general phase modification, we get a combination of ghosts, which is still disturbing.

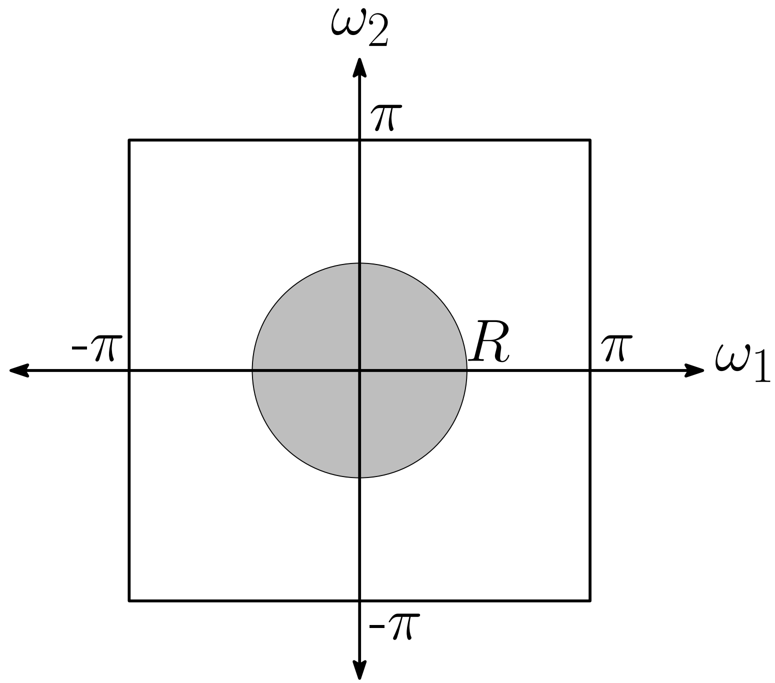
Zero-Phase Filters

Even though these examples are exaggerations, using zero-phase filters completely avoids this problem.

x $\quad$ $x * (\text{zero-ph.})$ $\quad$ $x * (\text{nonzero-ph.})$

Zero-Phase Filter Design

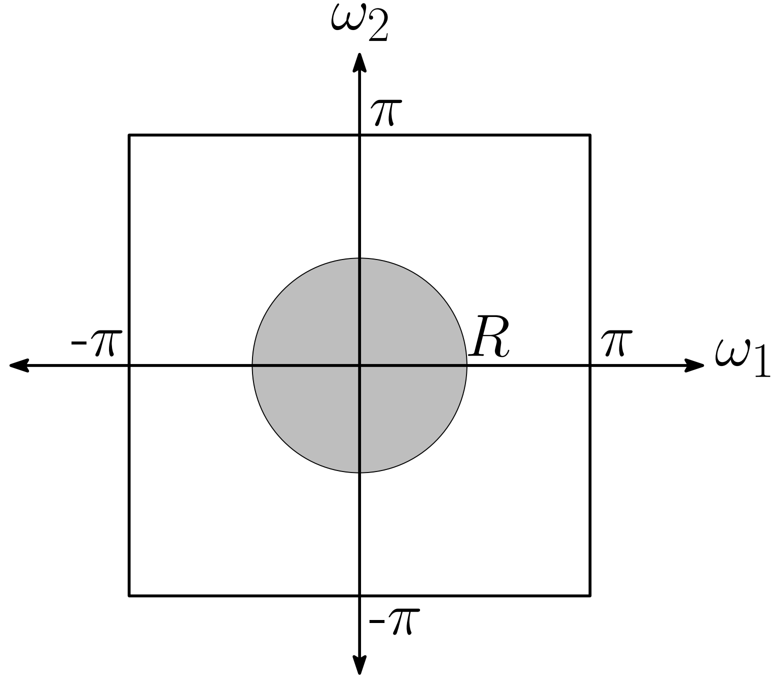
A function $f(u, v)$ is said to have circular symmetry if its value at u, v depends only on $u^2 + v^2$.



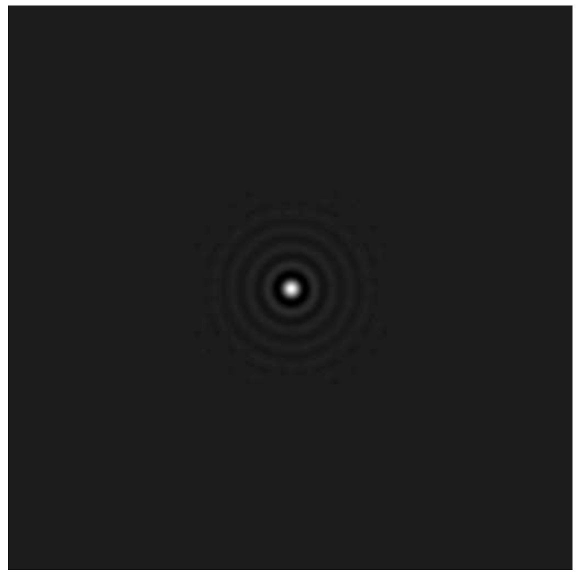
The circularly symmetric frequency response of a lowpass filter.

Zero-Phase Filter Design

Circularly Symmetric $H(\omega_1, \omega_2) \implies$ Circularly Symmetric $h(n_1, n_2)$



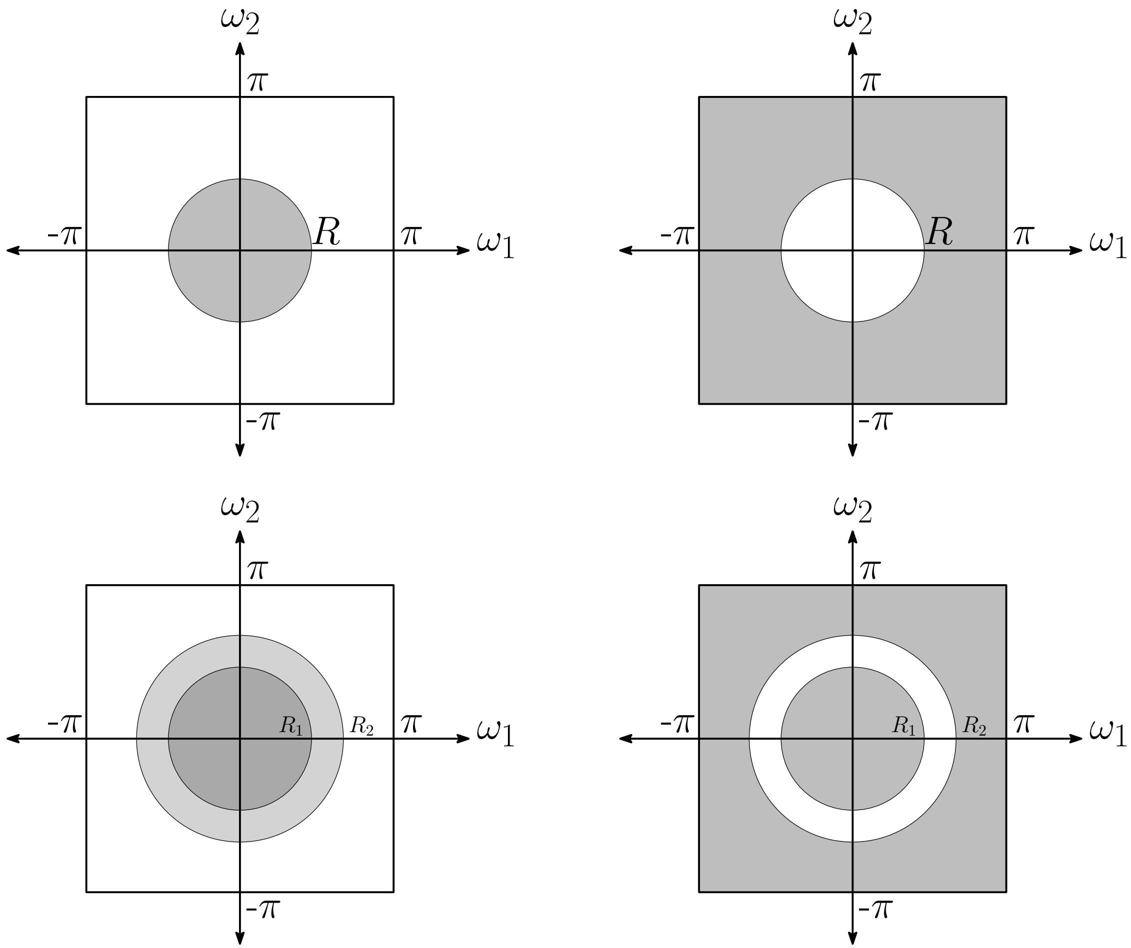
$H_R(\omega_1, \omega_2)$



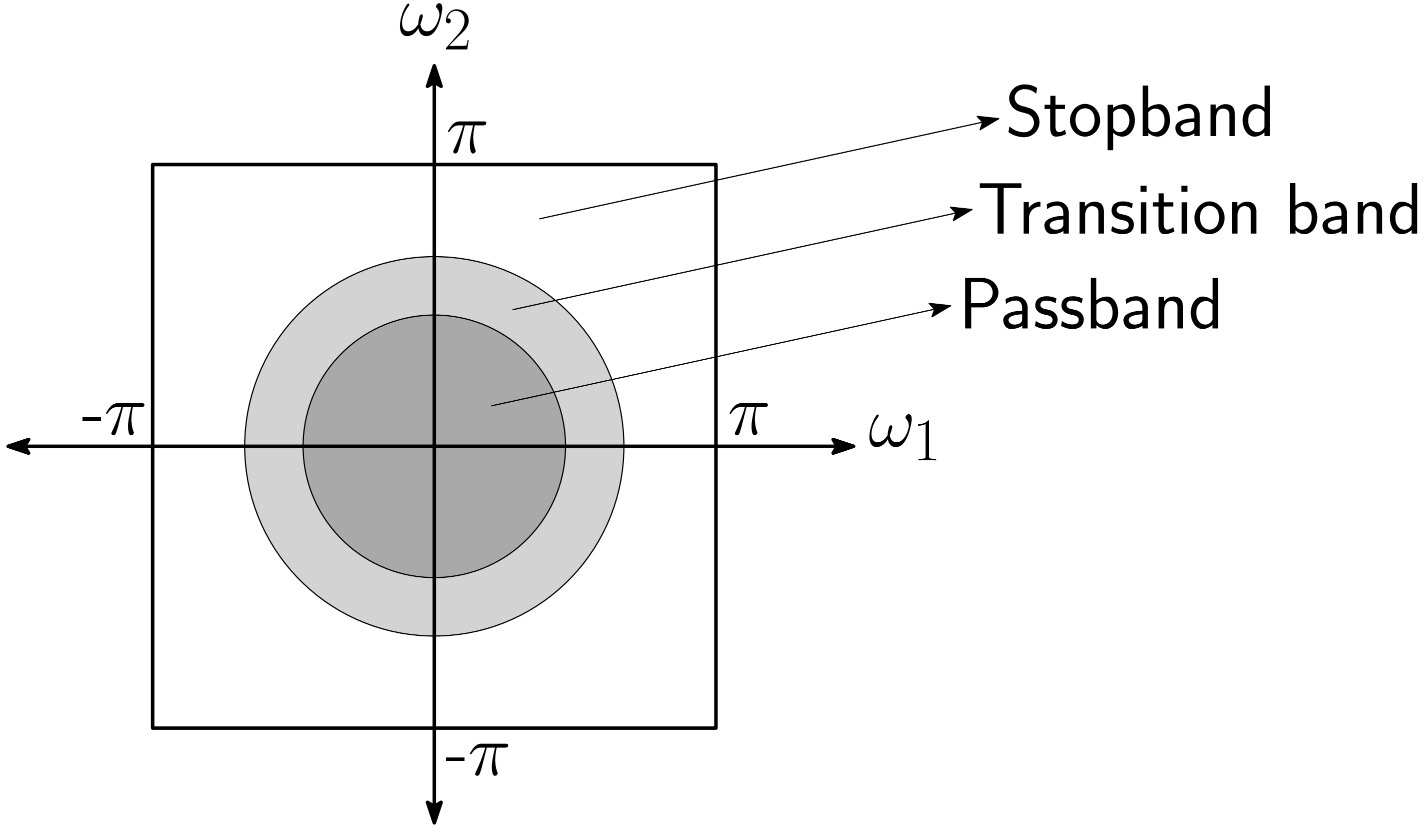
$h_R(n_1, n_2)$

Ideal Circularly Symmetric Filters

What will be the filters in terms of $h_R(n_1, n_2)$?



Design Specifications



The Window Method

- Given : Desired frequency response $H_d(\omega_1, \omega_2)$

The window method consists of two steps :

(1) Obtain $h_d(n_1, n_2)$ by inverse Fourier transforming.

(2) Reduce $h_d(n_1, n_2)$ to an FIR filter by multiplying with a window:

$$h(n_1, n_2) = h_d(n_1, n_2)\, w(n_1, n_2)$$

The Window Method

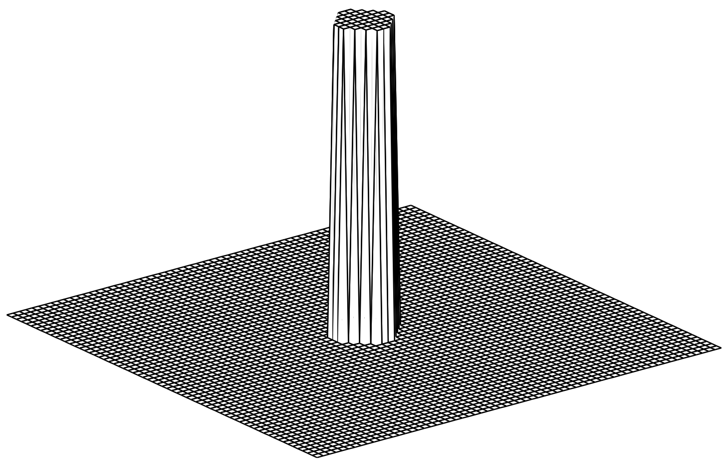
By the convolution theorem,

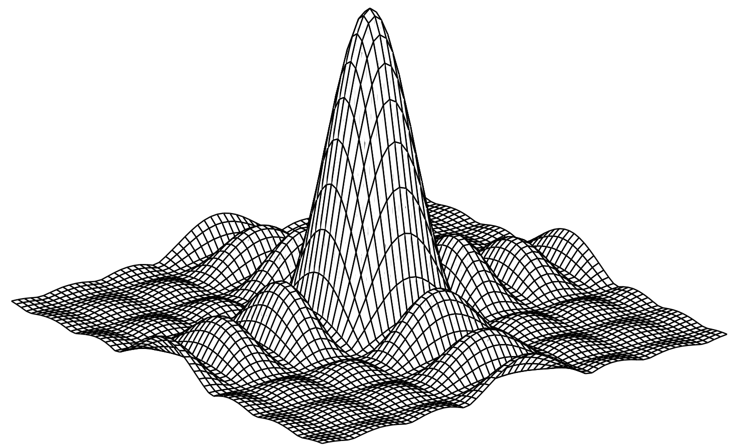
$$H(\omega_1, \omega_2) = H_d(\omega_1, \omega_2) \circledast W(\omega_1, \omega_2).$$

The deviation from $H_d(\omega_1, \omega_2)$ is determined by $W(\omega_1, \omega_2)$.

As $w(n_1, n_2) \to 1$, we have $W(\omega_1, \omega_2) \to \delta(\omega_1, \omega_2)$
so that $H(\omega_1, \omega_2) \to H_d(\omega_1, \omega_2)$.

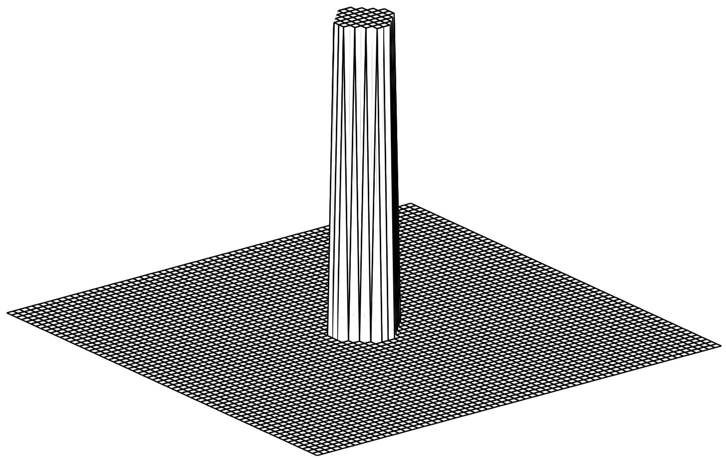
The Window Method – Rectangular Window



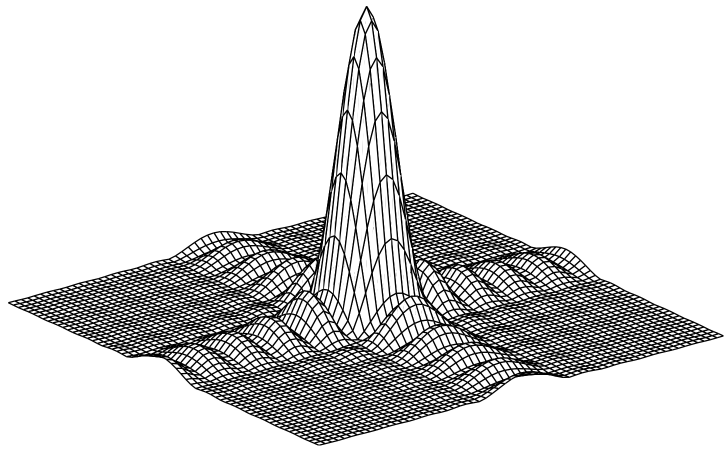


Desired $\quad\quad$ H with 7×7 Rectangular Window

The Window Method – Rectangular Window

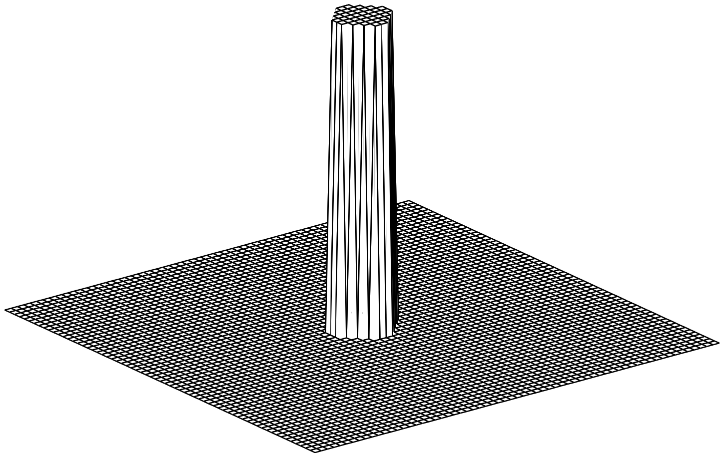


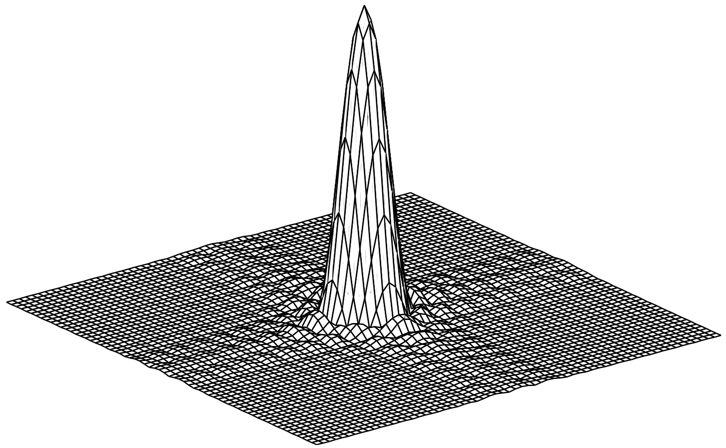
Desired



H with 11×11 Rectangular Window

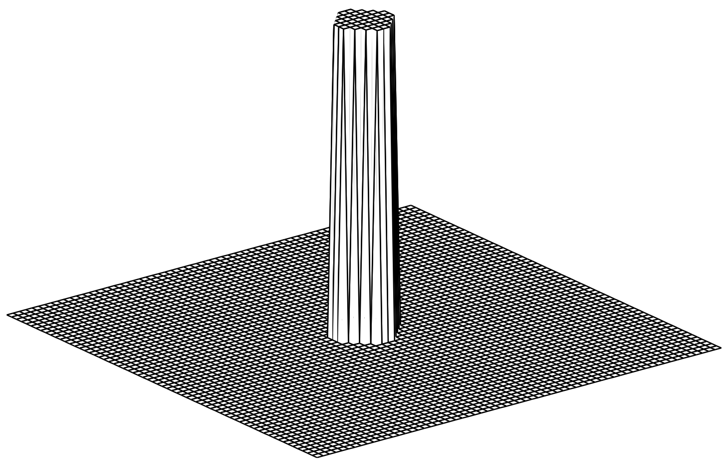
The Window Method – Rectangular Window

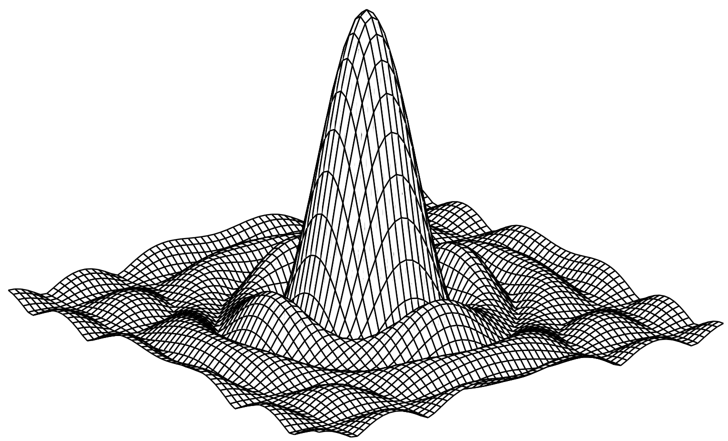




Desired H with 19×19 Rectangular Window

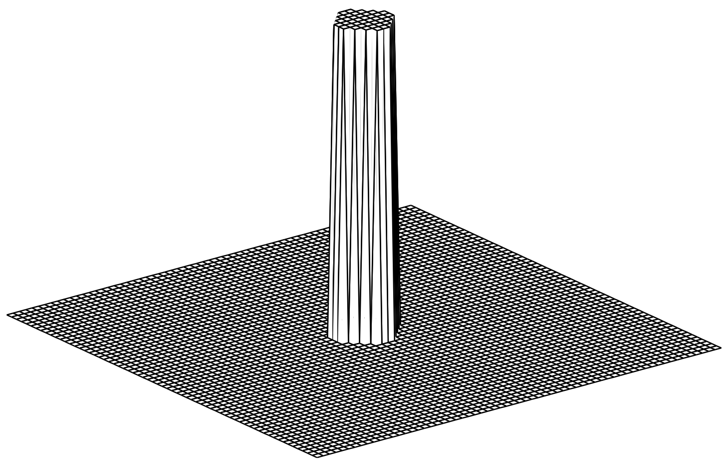
The Window Method – Circular Window

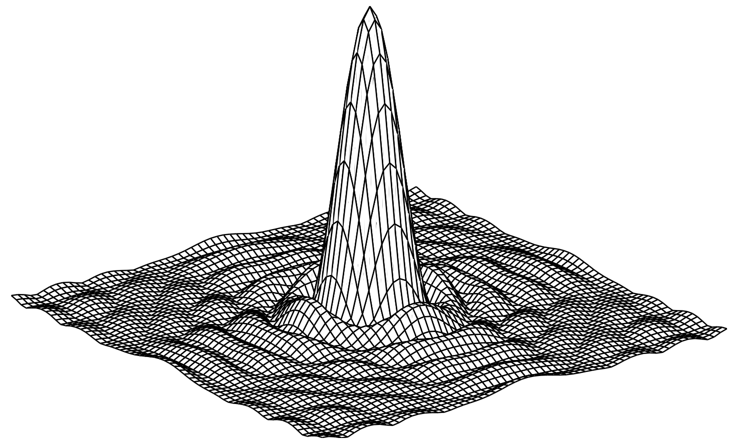




Desired $\qquad$ H with Circular Window (Radius $= 3$)

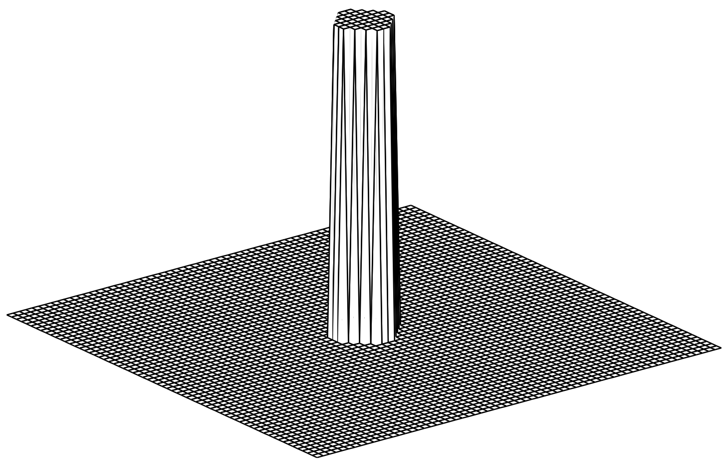
The Window Method – Circular Window

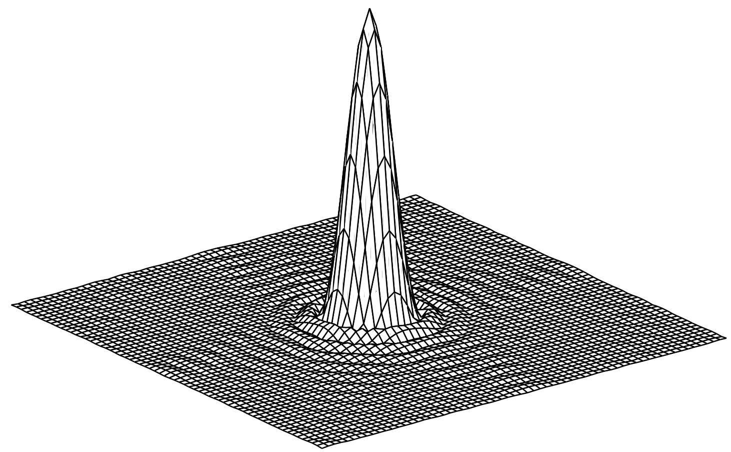




Desired $\qquad H$ with Circular Window (Radius = 5)

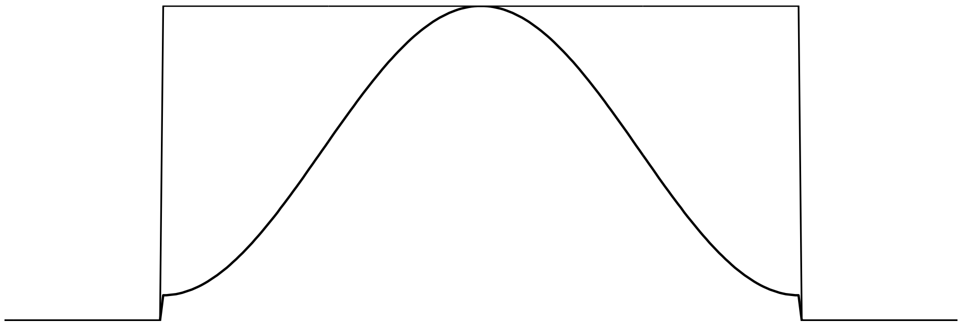
The Window Method – Circular Window

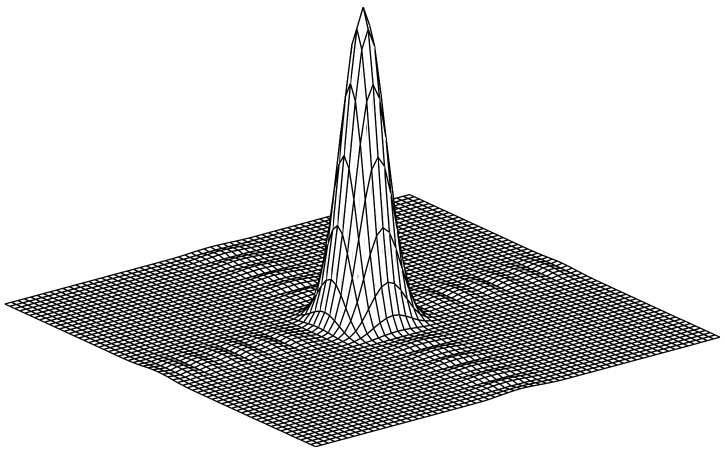




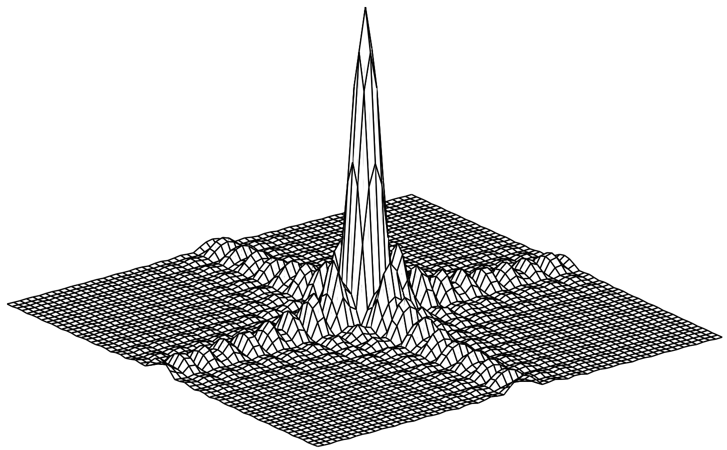
Desired $\qquad$ H with Circular Window (Radius $= 9$)

The Window Method – Hamming vs. Rectangular



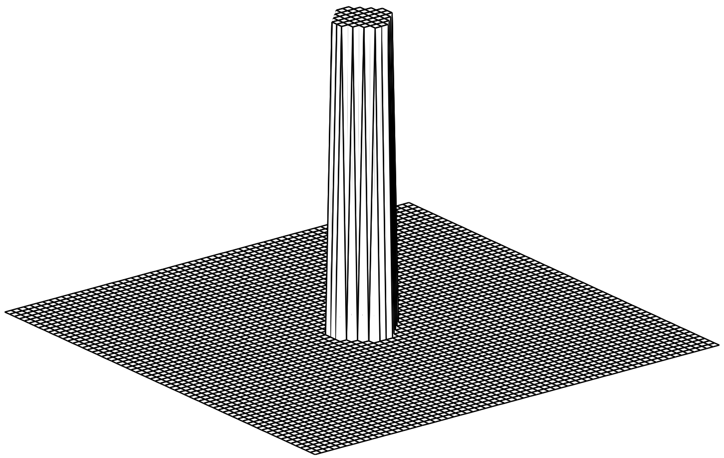


FT of Separable Hamming Window

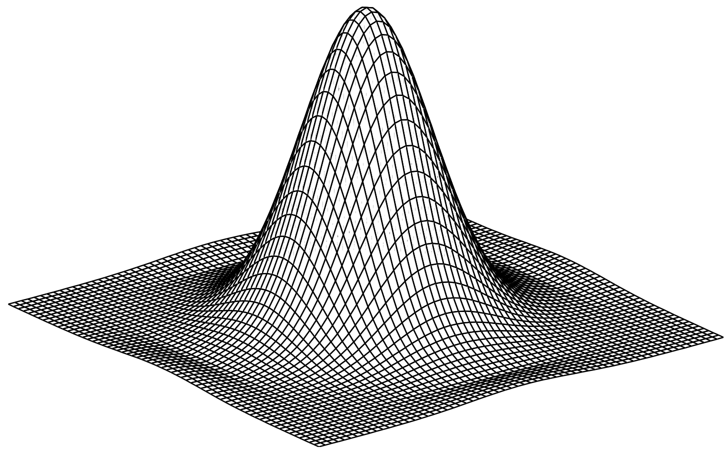


FT of Separable Rectangular Window

The Window Method – Hamming Window

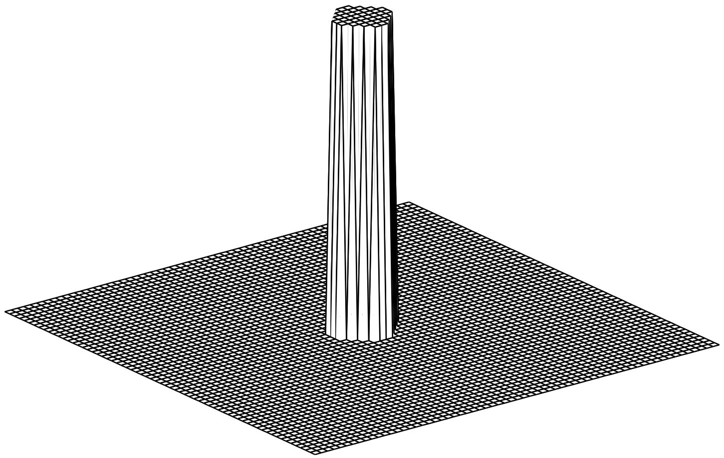


Desired

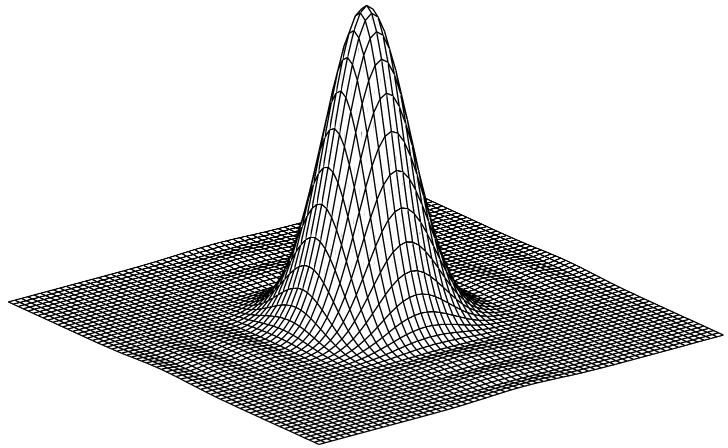


H with 5×5 Hamming Window

The Window Method – Hamming Window

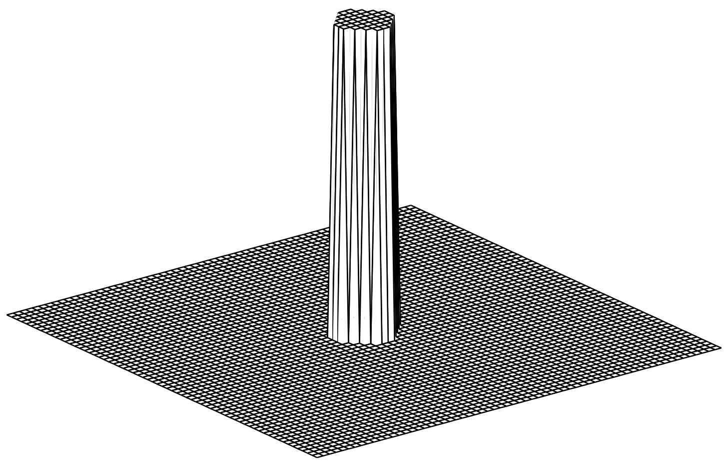


Desired

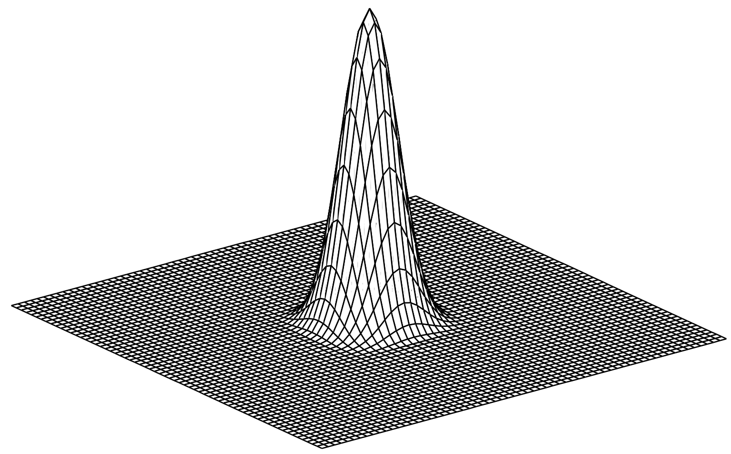


H with 11×11 Hamming Window

The Window Method – Hamming Window



Desired



H with 19×19 Hamming Window

The Frequency Sampling Method

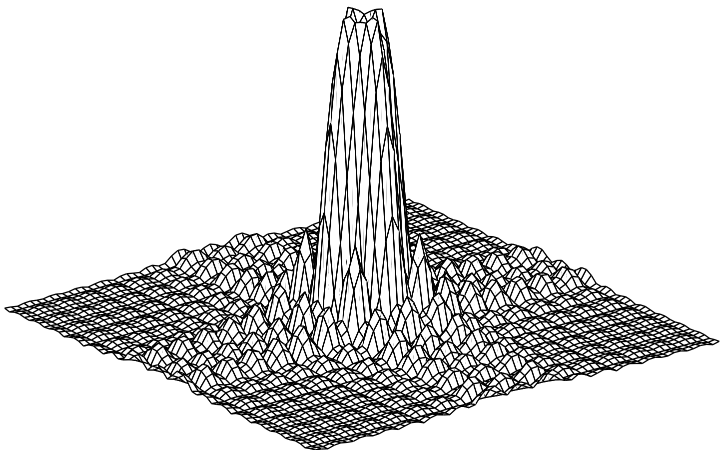
- Given : Desired frequency response $H_d(\omega_1, \omega_2)$

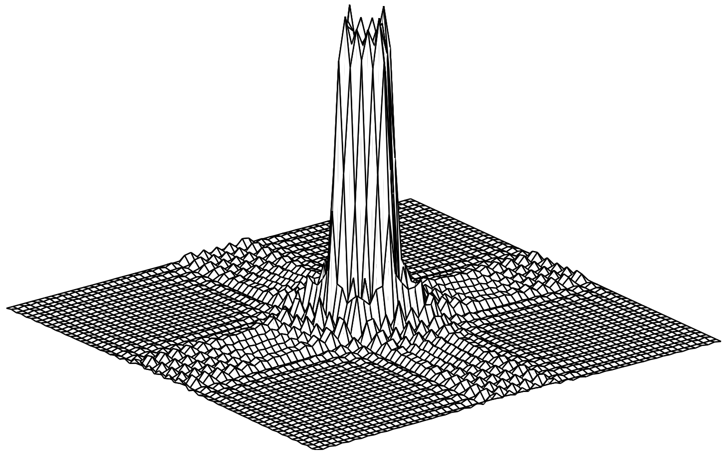
The frequency sampling method consists of two steps :

(1) Sample $H_d(\omega_1, \omega_2)$ to obtain $H(k_1, k_2) = H_d(k_1 \Delta_1, k_2 \Delta_2)$.

(2) Obtain $h(n_1, n_2)$ by applying inverse FFT to $H(k_1, k_2)$.

The Frequency Sampling Method





Frequency response of 17×17 filter 33×33 filter

Frequency Transformation Method

- Let $F(\omega)$ be the frequency response of a given 1D filter.
- Let $G(\omega_1, \omega_2)$ be a function that maps $[-\pi, \pi] \times [-\pi, \pi]$ to $[-\pi, \pi]$.
- Set

$$H(\omega_1, \omega_2) = F(\omega)|_{\omega = G(\omega_1, \omega_2)}$$

Two issues :

(1) Can we ensure that the resulting filter will be zero-phase?

(2) How do we choose $G(\omega_1, \omega_2)$?

Frequency Transformation Method

Suppose $f(n)$ is a zero-phase filter of length $2N+1$.

$$\begin{aligned} F(\omega) &= \sum_{n=-N}^{N} h(n)\, e^{-j\omega\, n} \\ &= \sum_{n=0}^{N} b(n)\, (\cos\, \omega)^{n} \end{aligned}$$

Now obtain $H(\omega_1, \omega_2)$ through

$$\begin{aligned} H(\omega_1, \omega_2) &= F(\omega)\big|_{\cos\, \omega = T(\omega_1, \omega_2)} \\ &= \sum_{n=0}^{N} b(n)\, \big(T(\omega_1, \omega_2)\big)^{n} \end{aligned}$$

Frequency Transformation Method

If $t(n_1, n_2)$ is a zero-phase FIR sequence, this gives a zero-phase FIR filter.

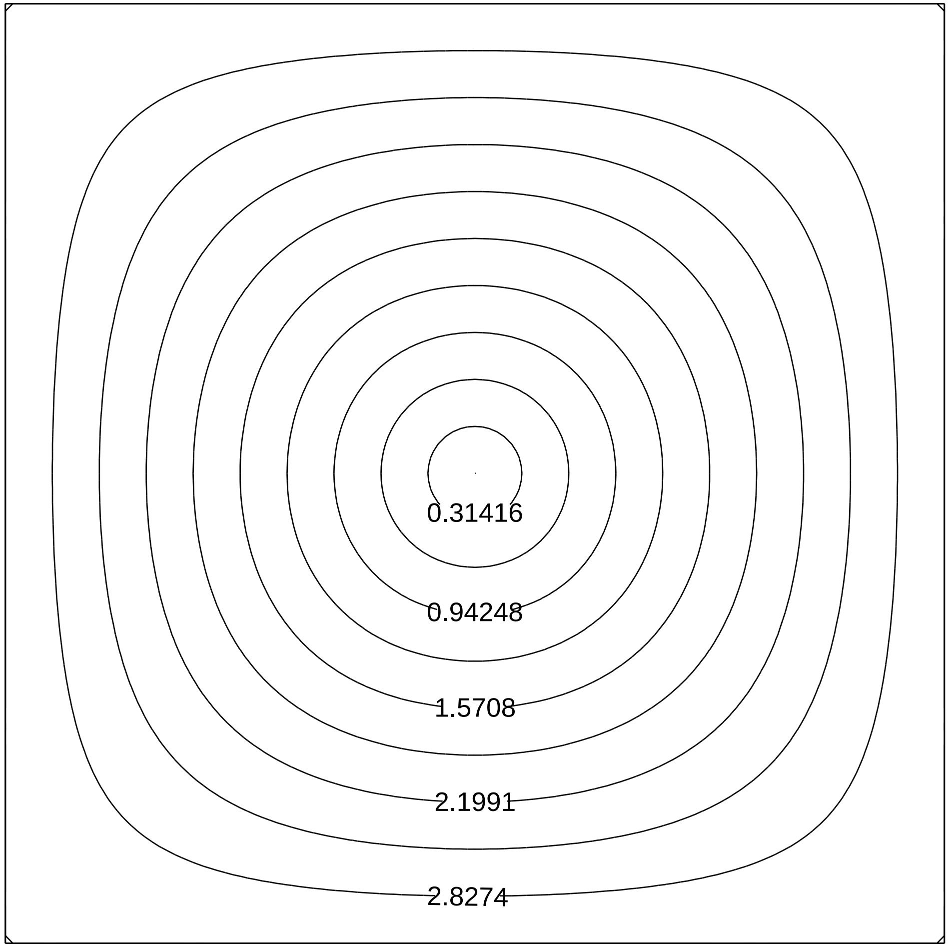
The McClellan transformation,

$$\cos\omega = -\frac{1}{2}+\frac{1}{2}\cos\,\omega_1+\frac{1}{2}\cos\,\omega_2+\frac{1}{4}\cos(\omega_1+\omega_2)+\frac{1}{4}\cos(\omega_1-\omega_2)$$

is obtained by using the sequence,

$$t(n_1, n_2) = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}$$

Frequency Transformation Method



The contours obtained by the McClellan Transformation

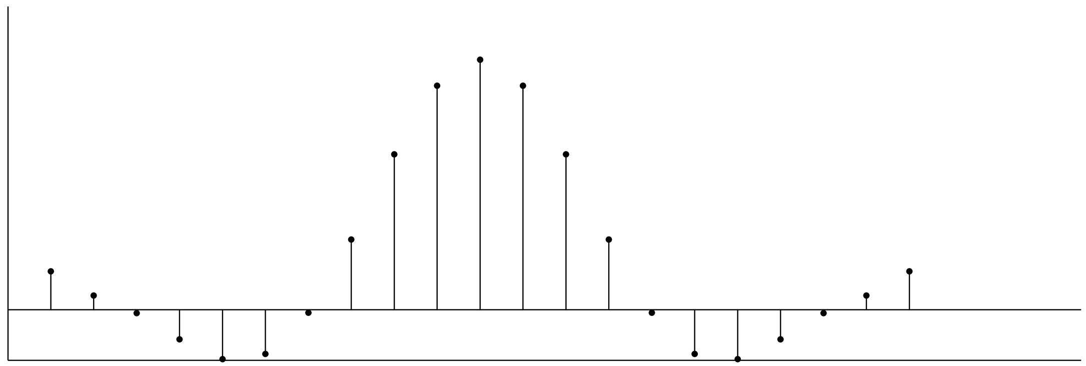
Design Example Using The Frequency Transformation Method

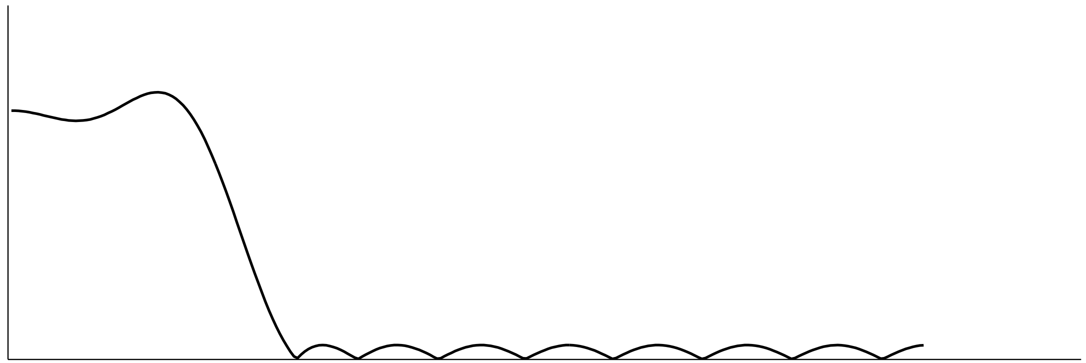
Start from a 1D lowpass zero-phase filter with

- Passband : $\Big[0 \;\; 0.2\pi\Big]$
- Transition Band : $\Big[0.2\pi \;\; 0.3\pi\Big]$
- Stopband : $\Big[0.3\pi \;\; \pi\Big]$

Design Example

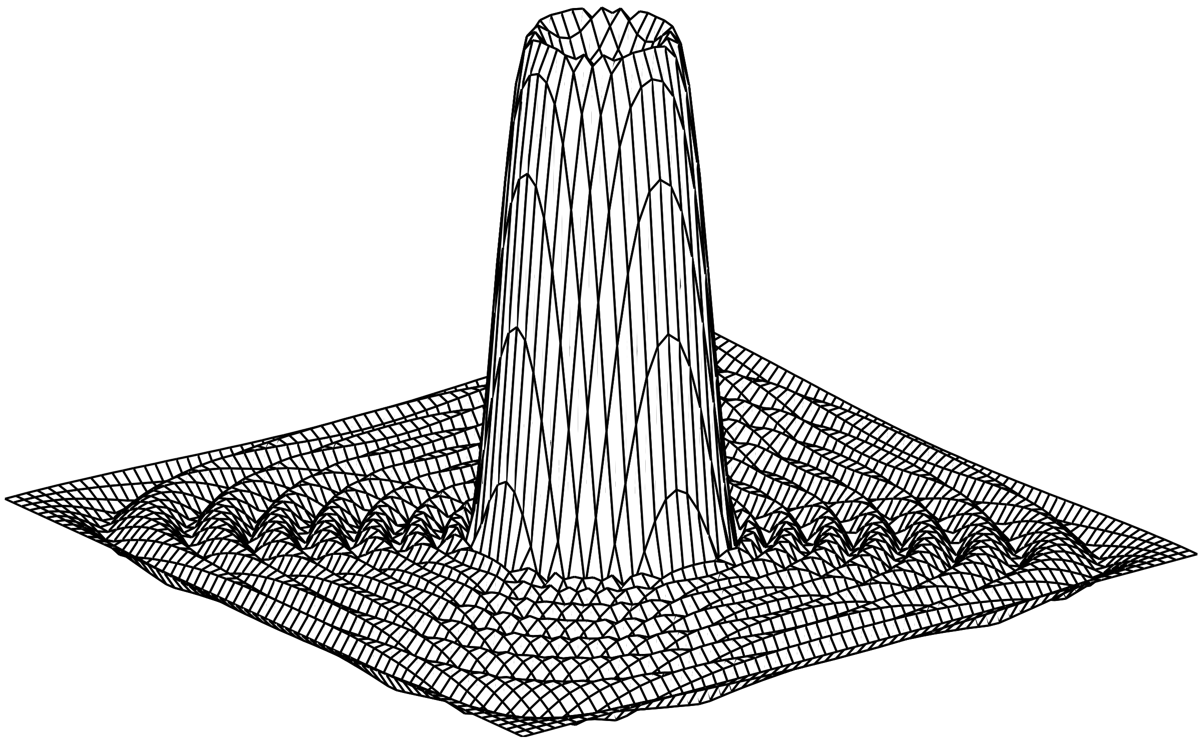
The zero-phase filter





The frequency response of the 1D filter used.

Design Example



Resulting frequency response of the 21$\times$21 filter (after McClellan Transformation)

Lowpass filters and Downsampling

Downsampled by 4

Filtered with H before downsampling