# TEL 311E – Homework 7 Solutions

1. Consider the system given by,
$$y(n) = x(2n) + 1$$
where $x(n)$ is the input and $y(n)$ is the output. Specify whether the system is
(a) Memoriless, (b) Linear, (c) Time-invariant, (d) Causal, (e) Stable in the BIBO sense.
Please explain your answers.

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We have $y(1) = x(2) + 1$ so the system is not memoriless and it is not causal.

It is not linear because if we input 0, the output is non-zero.

For $x(n) = \delta(n)$, we get $y(n) = \delta(n) + 1$ but if $x(n) = \delta(n-1)$, we have $y(n) = 1$ and therefore the system is not time-invariant.

If $|x(n)| < M$, then $|y(n)| < M + 1$ so the system is BIBO stable.

2. Let $T_1$ be an LTI system with impulse response $h_1(n) = a^n\, u(n)$.

   (a) For which values of '$a$' is $T_1$ stable in the BIBO sense?

   (b) Assume that $T_1$ is BIBO stable. Suppose we input some $x(n)$ to $T_1$ and obtain $y(n)$, as shown below. Let $T_2$ be another LTI system with impulse response $h_2(n) = \delta(n) - a^{-1}\,\delta(n+1)$ and

$$x(n) \longrightarrow \boxed{T_1} \longrightarrow y(n) \longrightarrow \boxed{T_2} \longrightarrow z(n)$$

   suppose we input $y(n)$ to this system to obtain $z(n)$. Express $z(n)$ in terms of $x(n)$.

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(a) Since the system is LTI, we can check BIBO stability by checking whether the impulse response is absolutely summable or not. In this case,
$$\sum_{n\in\mathbb{Z}} |h_1(n)| = \sum_{n\in\mathbb{Z}} |a^n\, u(n)| = \sum_{n=0}^{\infty} |a|^n$$
converges if $|a| < 1$.

(b) The overall system is LTI with impulse response $h(n) = h_1(n) * h_2(n)$.
$$\begin{aligned}
h(n) &= h_1(n) * h_2(n)\\
&= h_1(n) - a^{-1}\, h_1(n+1)\\
&= a^n\, u(n) - a^{-1}\, a^{n+1}\, u(n+1)\\
&= a^n\, u(n) - a^n\, u(n+1)\\
&= a^n\, u(n) - \big(a^{-1}\,\delta(n+1) + a^n\, u(n)\big)\\
&= -a^{-1}\,\delta(n+1).
\end{aligned}$$
So, $z(n) = -a^{-1}\, x(n+1)$.

3. Suppose that the $z$-transform of the step response (i.e. the response when a unit step function $u(n)$, is input to the system) of an LTI system is given by
$$X(z) = \frac{1}{1 + \frac{1}{4}\, z^{-1}} + \frac{1}{1 - z^{-1}}.$$
Let us denote the $z$-transform of the impulse response as $H(z)$.

(a) If we know that the system is stable, what should be the region of convergence for $H(z)$?

(b) Determine the impulse response $h(n)$ of this stable system.

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Since $X(z)$ is the $z$-transform of the step-response, we have $X(z) = H(z)/(1 - z^{-1})$ where $H(z)$ is the $z$-transform of the impulse response. Therefore, from

$$X(z) = \frac{1}{1 + \frac{1}{4}\,z^{-1}} + \frac{1}{1 - z^{-1}} = \frac{2 - \frac{3}{4}\,z^{-1}}{\left(1 + \frac{1}{4}\,z^{-1}\right)\left(1 - z^{-1}\right)}$$

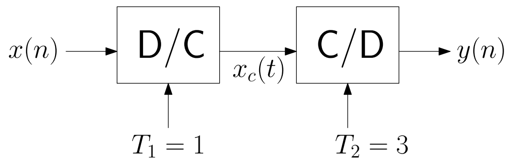
we obtain

$$H(z) = \frac{2 - \frac{3}{4}\,z^{-1}}{1 + \frac{1}{4}\,z^{-1}} = -3 + 5\,\frac{1}{1 + \frac{1}{4}\,z^{-1}}.$$

(a) In order for the system to be stable, ROC must contain the unit circle. $H(z)$ has only a single pole at $z = -1/4$, so the ROC must be $|z| > 1/4$.

(b) From the ROC, we deduce that $h(n)$ is causal. Therefore, $h(n) = (1/4)^n\,u(n)$.

4. Consider the system below which maps $x(n)$ to $y(n)$.




(a) Express $y(n)$ in terms of $x(n)$.

(b) Express $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$.

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(a) Since $x(n) = x_c(n\,T_1) = x_c(n)$, we have $y(n) = x_c(n\,T_2) = x_c(3\,n) = x(3\,n)$.

(b) First, notice that,

$$X_c(\omega) = \begin{cases} X(e^{j\omega}) & \text{if } |\omega| \le \pi, \\ 0 & \text{if } |\omega| > \pi. \end{cases}$$

and,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} X_c(\omega - n\,2\pi).$$

Now,

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{3} \sum_{n=-\infty}^{\infty} X_c\left(\frac{\omega - n\,2\pi}{3}\right) \\ &= \frac{1}{3} \sum_{n=-\infty}^{\infty} \left\{ X_c\left(\frac{\omega}{3} - n\,2\pi\right) + X_c\left(\frac{\omega}{3} - n\,2\pi - \frac{2\pi}{3}\right) + X_c\left(\frac{\omega}{3} - n\,2\pi - 2\frac{2\pi}{3}\right) \right\} \\ &= \frac{1}{3} \left\{ X\left(e^{j\frac{\omega}{3}}\right) + X\left(e^{j\left(\frac{\omega}{3} - \frac{2\pi}{3}\right)}\right) + X\left(e^{j\left(\frac{\omega}{3} - \frac{4\pi}{3}\right)}\right) \right\} \end{aligned}$$

5. Let $x(n)$ be an $N$-point signal whose $N$-point DFT is denoted by $X(k)$. Suppose we circularly shift $X(k)$ by one sample to obtain $\tilde{X}(k)$, i.e.,

$$\begin{aligned} \tilde{X}(0) &= X(N-1), \\ \tilde{X}(k) &= X(k-1) \quad \text{for } 1 \le k \le N-1. \end{aligned}$$

Express $\tilde{x}(n)$, the IDFT of $\tilde{X}(k)$, in terms of $x(n)$.

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First, we know that,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp\left(j \frac{2\pi}{N} n\, k\right).$$

Now,

$$\begin{aligned}
\tilde{x}(n) &= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) \exp\left(j \frac{2\pi}{N} n\, k\right) \\
&= \frac{1}{N} X(N-1) \exp\left(j \frac{2\pi}{N} n\, 0\right) + \frac{1}{N} \sum_{k=1}^{N-1} \tilde{X}(k-1) \exp\left(j \frac{2\pi}{N} n\, k\right) \\
&= \frac{1}{N} X(N-1) \exp\left(j \frac{2\pi}{N} n\, (N-1)\right) \exp\left(j \frac{2\pi}{N} n\right) + \frac{1}{N} \exp\left(j \frac{2\pi}{N} n\right) \sum_{k=1}^{N-1} \tilde{X}(k-1) \exp\left(j \frac{2\pi}{N} n\, (k-1)\right) \\
&= \frac{1}{N} X(N-1) \exp\left(j \frac{2\pi}{N} n\, (N-1)\right) \exp\left(j \frac{2\pi}{N} n\right) + \frac{1}{N} \exp\left(j \frac{2\pi}{N} n\right) \sum_{k=0}^{N-2} \tilde{X}(k) \exp\left(j \frac{2\pi}{N} n\, k\right) \\
&= \exp\left(j \frac{2\pi}{N} n\right) \quad \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) \exp\left(j \frac{2\pi}{N} n\, k\right)\right] \\
&= \exp\left(j \frac{2\pi}{N} n\right) x(n).
\end{aligned}$$

6. (Notice the correction in the question) Let $x(n)$ be a 10-point signal with 10-point DFT

$$X(k) = k^2 \quad \text{for} \quad 0 \le k \le 9.$$

Compute

$$s = \sum_{n=0}^{N} x(n) \left[\cos\left(\frac{\pi}{5} n\right) + 2 \sin\left(\frac{6\pi}{10} n\right)\right].$$

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Writing

$$\begin{aligned}
\cos\left(\frac{\pi}{5} n\right) &= \frac{1}{2}\left[\exp\left(j \frac{2\pi}{10} n\right) + \exp\left(-j \frac{2\pi}{10} n\right)\right] \\
&= \frac{1}{2}\left[\exp\left(-j \frac{2\pi}{10} n\, 9\right) + \exp\left(-j \frac{2\pi}{10} n\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\sin\left(\frac{6\pi}{10} n\right) &= \frac{1}{2}\left[\exp\left(j \frac{2\pi}{10} n\, 3\right) - j \exp\left(-j \frac{2\pi}{10} n\, 3\right)\right] \\
&= \frac{1}{2}\left[\exp\left(-j \frac{2\pi}{10} n\, 7\right) - j \exp\left(-j \frac{2\pi}{10} n\, 3\right)\right],
\end{aligned}$$

we have,

$$s = \frac{1}{2}\big(X(9) + X(1)\big) + \big(X(7) - jX(3)\big) = (81+1)/2 + (49 - j\, 9) = 90 - 9\, j.$$