Gatsby Tea: Infinite cardinals

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Question

Q: Consider a mapping S that, to each $x \in [0,1]$, associates a countable subset $S(x) \subset [0,1]$. Can we choose S s.t. for any $(x,y) \in [0,1]^2$, either $x \in S(y)$ or $y \in S(x)$?

Set theory

- foundation of modern mathematics initiated by Georg Cantor (1870s).
- formulated in terms of Zermelo-Fraenkel (ZF) axiom system $+$ axiom of choice (C).
- it is not possible to prove or disprove the axiom of choice from ZF.

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- infitie sets? bijection between sets $\Rightarrow$ same cardinality (Cantor).
- a set A is bigger than a set B ($|B| \leq |A|$) if there is a bijection from a subset of A to B.
- how many different infinite cardinals?

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- we know that $\mathbb{R}$ is bigger than $\mathbb{N}$ (uncountable); let $\mathfrak{c} := |\mathbb{R}|$.
- how about $\mathcal{P}(\mathbb{N})$? Is it bigger than $\mathbb{N}$? how does $2^{\aleph_0} := |\mathcal{P}(\mathbb{N})|$ compare to $\mathfrak{c}$?

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- if $b \notin B = f(b) \Rightarrow b \in B$????
- $\Rightarrow$ such f does not exist $\Rightarrow |\mathcal{P}(S)| > |S|$.
- in particular: $2^{\aleph_0} > \aleph_0$.

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& A \mapsto \sum_{i \in A} 10^{-i} \\
g: \quad &]0,1] \to \mathcal{P}(\mathbb{N}) - \varnothing \\
& 0.d_1 d_2 d_3 \ldots \mapsto \{10^i d_i, i \in \mathbb{N}\}
\end{aligned}
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- f and g are injective $\Rightarrow 2^{\aleph_0} \leq \mathfrak{c}$ and $\mathfrak{c} \leq 2^{\aleph_0}$.

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- $\Rightarrow 2^{\aleph_0} = \mathfrak{c}$ (Cantor-Bernstein Th).

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- f and g are injective $\Rightarrow 2^{\aleph_0} \leq \mathfrak{c}$ and $\mathfrak{c} \leq 2^{\aleph_0}$.
- $\Rightarrow 2^{\aleph_0} = \mathfrak{c}$ (Cantor-Bernstein Th).
- is $2^{\aleph_0}$ the smallest infinity after $\aleph_0$?

Another infinity

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$$0 = \{\}$$
$$n + 1 = n \cup \{n\}$$

we call these sets *ordinal numbers*.

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  $3 = \{0, 1, 2\} = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$, $4 = \{0, 1, 2, 3\}$ ...
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- $n < m \Leftrightarrow n \in m$.
- in reality, those are *successor* ordinals: can be written as $n + 1$ for some n.
- there are other ordinals: *limit* ordinals; for instance supremums of sets of ordinals that have no upper bounds.
- e.g.: $\omega := \sup \mathbb{N}$ (guaranteed to exist by ZF).

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- we can define the next limit ordinal: $2 \cdot \omega := \sup\{\omega + \mathbb{N}\}$, and then the one after that, etc...
- $1, 2, 3, 4, \dots, \omega, \omega + 1, \dots, 2 \cdot \omega, \dots, 3 \cdot \omega, \dots, \omega^2, \dots, \omega^3, \dots,$ $\omega^\omega, \dots, \omega^{\omega^\omega}, \dots$

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- limit ordinals are strictly greater than all the preceding ordinals.
- all these sets are countable.

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- suppose $|\omega_1| = \aleph_0 \Rightarrow \omega_1 \in \omega_1$ a.k.a $\omega_1 < \omega_1$??
- $\Rightarrow |\omega_1| > \aleph_0$; let's call it $\aleph_1$. ω_1 is the smallest uncountable ordinal.

Continuum hypothesis

We've seen two different infinities greater that $\aleph_0$:

- $2^{\aleph_0}$ the cardinal of $\mathbb{R}$ (equal to the cardinal of the power set of $\mathbb{N}$).
- $\aleph_1$ the cardinal of the set of all countable ordinals

Q: Is $2^{\aleph_0} = \aleph_1$? This is the Continuum Hypothesis (CH) (Cantor).

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Q: Is $2^{\aleph_0} = \aleph_1$? This is the Continuum Hypothesis (CH) (Cantor).
A: CH is independent of ZFC: it can be neither proven nor disproven within the context of the ZFC axioms. (Gödel and Cohen).

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A: The answer depends on CH!

- ZFC + CH $\Rightarrow$ yes
- ZFC + not(CH) $\Rightarrow$ no