Lecture 3

January 20, 2016

1 Review of Linear Algebra

vector in $\mathbb{R}^3$:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Transpose of a vector:

$$\vec{x}^t = (x_1, x_2, \ldots, x_n)$$

Linear combination (span):

$$\vec{x} = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

Linear independence: Vectors $\vec{x_1}, \vec{x_2}, \ldots, \vec{x_k}$ are linearly independent if

$$c_1\vec{x_1} + c_2\vec{x_2} + \cdots + c_k\vec{x_k} = \vec{0}$$

Which implies $c_1 = c_2 = \cdots = c_k = 0$

Dot product as matrices: For a vector $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$ their dot product is:

$$\vec{x} \cdot \vec{y} = \langle x_1, \ldots, x_n \rangle \left\langle \begin{matrix} y_1, \\ \vdots, \\ y_n \end{matrix} \right\rangle = \vec{x}^T \vec{y}$$

Solving a system of equations using a matrix For a given system of equations:

$$\begin{aligned} a - 2b + c &= 0 \\ 2b - 8c &= 8 \\ -4a + 5b + 9c &= -9 \end{aligned}$$

The matrix form of the above system would be:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix}$$

Let A be the 3×3 matrix and $\vec{c}$ be the vector that A is multiplied by.
If $A\vec{c} = \vec{0}$ then this system has a unique solution, which is just $\vec{c} = \vec{0}$

The null space of matrix A

$$N(A) = \{\vec{x} \in \mathbb{R}^k \text{ such that } A\vec{x} = \vec{0}\}$$

The column space of matrix A is

$$R(A) = Col(A) = \{\vec{y} \in \mathbb{R}^n \text{ where } \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^x\}$$

The dimension of $N(A)$ is called the nullity of A.

The dimension of $R(A)$ is called the rank of A.

The rank-nullity theorem

$$\text{Rank}(A) + \text{Nullity}(A) = K = \text{columns in } A$$

The basis of a S where S is a subspace of $\mathbb{R}^n$ is $\vec{v_1}, \ldots, \vec{v_m}$ if S spans $\{\vec{v_1}, \ldots, \vec{v_m}\}$ and $\vec{v_1}, \ldots, \vec{v_m}$ are linearly independent and $m \leq n$. Note that the dimension of S is m.

As an example, $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$ consist of the basis of $\mathbb{R}^2$. Note that the vectors that make up a basis for any space are not unique.

1.1 Example

Find the nullspace, nullity, columnspace, and rank of A.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ -1 & -2 & 2 & -2 & -1 \\ 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 2 & 2 & -1 \end{bmatrix}$$

To the nullspace we need to find $\vec{x}$ so that $A\vec{x} = \vec{0}$. We need to row reduce A augmented with a zero vector (the same as solving $A\vec{x} = \vec{0}$).

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 0 & 0 \\ -1 & -2 & 2 & -2 & -1 & 0 \\ 1 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 2 & -1 & 0 \end{bmatrix}$$