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Physics of a Hydrogen Fuel Cell Powered VTOL UAV

Five-Motor Configuration with Tilt-Rotor Propulsion

Energy System Sizing, Aerodynamic Analysis, and Mission Planning

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Abstract

This document presents a complete physics-based model for a vertical takeoff and landing (VTOL) unmanned aerial vehicle (UAV) powered by a hydrogen fuel cell. The aircraft employs five electric motors: three tiltable rotors for thrust vectoring and two fixed vertical rotors mounted in the wings for vertical lift and roll control. The model encompasses kinematics, dynamics, propulsion, aerodynamics, energy management, and control allocation across all flight phases including hover, transition, and cruise.

Contents

1 Introduction

This physics model describes a hybrid VTOL aircraft that combines helicopter-like vertical flight capabilities with efficient fixed-wing cruise performance. The propulsion system consists of:

- **Three tilt rotors** (rotors 1, 2, 3): Capable of rotating from vertical (hover) to horizontal (cruise) orientation
- **Two fixed wing rotors** (rotors 4, 5): Mounted in the wings, oriented vertically for hover thrust and roll control
- **Hydrogen fuel cell**: Primary power source providing electrical energy to all motors

The model is designed to support mission planning, energy budget analysis, and control system design.

1.1 Nomenclature and Constants

1.1.1 Universal Constants

$$\begin{aligned} g &= 9.81\ \text{m/s}^2 && \text{gravitational acceleration} \\ \rho_{\text{air}} &= 1.225\ \text{kg/m}^3 && \text{air density at sea level (ISA)} \\ \text{LHV}_{\text{H}_2} &= 33.33\ \text{kWh/kg} && \text{hydrogen lower heating value} \end{aligned}$$

1.1.2 Reference Frames

- **Body frame**: Origin at center of gravity (CG), $+x$ forward (nose), $+y$ right wing, $+z$ down (standard aerospace convention)
- **Inertial frame**: Earth-fixed, $+z$ down
- **Wind frame**: Aligned with velocity vector

2 Geometry and Mass Properties

2.1 Aircraft Mass

The total aircraft mass is the sum of all components:

$$m_{\text{total}} = m_{\text{airframe}} + m_{\text{FC}} + m_{\text{H}_2} + m_{\text{battery}} + m_{\text{avionics}} + m_{\text{payload}} \tag{1}$$

where:

$$\begin{aligned}
m_{\text{airframe}} &= \text{structural mass (wings, fuselage, motors, ESCs)} \\
m_{\text{FC}} &= \text{fuel cell stack mass} \\
m_{\text{H}_2} &= \text{hydrogen mass (fuel)} \\
m_{\text{battery}} &= \text{backup battery mass} \\
m_{\text{avionics}} &= \text{flight controller, sensors, telemetry} \\
m_{\text{payload}} &= \text{mission payload mass}
\end{aligned}$$

2.2 Inertia Tensor

The moment of inertia tensor about the center of gravity:

$$\mathbf{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \tag{2}$$

For initial analysis, assume principal axes alignment ($I_{xy} = I_{xz} = I_{yz} = 0$).

2.3 Rotor Geometry

Define rotor positions relative to CG:

$$\mathbf{r}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}, \quad i = 1, 2, 3, 4, 5 \tag{3}$$

Typical configuration:

- Rotor 1: Front tilt rotor, $\mathbf{r}_1 = [x_f, 0, 0]^T$
- Rotor 2: Left rear tilt rotor, $\mathbf{r}_2 = [x_r, -y_w, 0]^T$
- Rotor 3: Right rear tilt rotor, $\mathbf{r}_3 = [x_r, y_w, 0]^T$
- Rotor 4: Left wing rotor, $\mathbf{r}_4 = [0, -y_w, 0]^T$
- Rotor 5: Right wing rotor, $\mathbf{r}_5 = [0, y_w, 0]^T$

where $x_f > 0$ (forward), $x_r < 0$ (aft), and $y_w > 0$ (semi-span).

3 Rotor Propulsion Model

3.1 Tilt Angle Convention

For tilt rotors ($i = 1, 2, 3$), define tilt angle ϕ_i:

- $\phi_i = 0$: Vertical orientation (hover mode)
- $\phi_i = 90$: Horizontal orientation (cruise mode)

Fixed wing rotors ($i = 4, 5$) have $\phi_i = 0$ always.

3.2 Thrust Decomposition

Each rotor i produces thrust magnitude T_i (N). The thrust vector in body frame:

$$\mathbf{T}_i = \begin{bmatrix} T_i \sin\phi_i \\ 0 \\ -T_i \cos\phi_i \end{bmatrix} \tag{4}$$

Components:

$$T_i^{(x)} = T_i \sin\phi_i \qquad \text{(longitudinal/forward thrust)} \tag{5}$$

$$T_i^{(z)} = -T_i \cos\phi_i \qquad \text{(vertical thrust, negative is up)} \tag{6}$$

3.3 Rotor Aerodynamics

3.3.1 Momentum Theory for Hover

For rotor i with disk area $A_i = \pi R_i^2$ (where R_i is rotor radius), the induced velocity in hover:

$$v_{i,\mathrm{ind}} = \sqrt{\frac{T_i \cos\phi_i}{2\rho_{\mathrm{air}} A_i}} \tag{7}$$

For hover ($\phi_i = 0$), this simplifies to:

$$v_{i,\mathrm{ind}} = \sqrt{\frac{T_i}{2\rho_{\mathrm{air}} A_i}} \tag{8}$$

3.3.2 Rotor Power

The ideal induced power (momentum theory):

$$P_{i,\mathrm{ind}} = T_i \cos\phi_i \cdot v_{i,\mathrm{ind}} = (T_i \cos\phi_i)^{3/2} \sqrt{\frac{1}{2\rho_{\mathrm{air}} A_i}} \tag{9}$$

Including profile drag and losses with figure of merit $FM \approx 0.75$:

$$P_{i,\mathrm{mech}} = \frac{P_{i,\mathrm{ind}}}{FM} \tag{10}$$

Electrical power with motor/ESC efficiency η_{motor}:

$$P_{i,\mathrm{elec}} = \frac{P_{i,\mathrm{mech}}}{\eta_{\mathrm{motor}}} \tag{11}$$

3.3.3 Forward Flight Corrections

In forward flight with velocity V, the induced velocity is reduced. Using Glauert's formula:

$$v_{i,\text{ind}}(V) = v_{i,\text{hover}} \sqrt{\frac{-V^2 + \sqrt{V^4 + 4v_{i,\text{hover}}^4}}{2v_{i,\text{hover}}^2}} \tag{12}$$

For high-speed cruise where $V >> v_{i,\text{hover}}$:

$$v_{i,\text{ind}} \approx \frac{v_{i,\text{hover}}^2}{V} \tag{13}$$

4 Wing Aerodynamics

4.1 Lift and Drag

The wing provides lift and generates drag during forward flight. Define:

$$\begin{aligned} S &= \text{wing reference area (m}^2\text{)} \\ b &= \text{wingspan (m)} \\ AR &= b^2/S = \text{aspect ratio} \\ V &= \text{airspeed (m/s)} \\ \alpha &= \text{angle of attack (rad)} \end{aligned}$$

4.1.1 Lift

$$L = \frac{1}{2}\rho_{\text{air}} V^2 S C_L(\alpha) \tag{14}$$

The lift coefficient can be modeled as:

$$C_L(\alpha) = C_{L,0} + C_{L,\alpha} \cdot \alpha \tag{15}$$

where $C_{L,0}$ is zero-lift coefficient and $C_{L,\alpha} \approx 2\pi/(1 + 2/AR)$ for finite wings.

4.1.2 Drag

Total drag includes profile drag and induced drag:

$$D = \frac{1}{2}\rho_{\text{air}} V^2 S C_D \tag{16}$$

Drag polar:

$$C_D = C_{D,0} + K C_L^2 \tag{17}$$

where:

$$\begin{aligned} C_{D,0} &= \text{zero-lift drag coefficient (parasite drag)} \\ K &= \frac{1}{\pi e AR} = \text{induced drag factor} \\ e &\approx 0.8\text{–}0.9 = \text{Oswald efficiency factor} \end{aligned}$$

5 Equations of Motion

5.1 Translational Dynamics

In the body frame, Newton's second law:

$$m\dot{\mathbf{V}}_b = \mathbf{F}_{\text{total}} - \boldsymbol{\omega} \times (m\mathbf{V}_b) \tag{18}$$

where $\mathbf{V}_b = [u, v, w]^T$ is velocity in body frame and $\boldsymbol{\omega} = [p, q, r]^T$ is angular velocity.

5.1.1 Force Components

Total force in body axes:

$$\mathbf{F}_{\text{total}} = \mathbf{F}_{\text{thrust}} + \mathbf{F}_{\text{aero}} + \mathbf{F}_{\text{gravity}} \tag{19}$$

Thrust forces:

$$\mathbf{F}_{\text{thrust}} = \sum_{i=1}^{5} \mathbf{T}_i = \begin{bmatrix} \sum_i T_i \sin\phi_i \\ 0 \\ -\sum_i T_i \cos\phi_i \end{bmatrix} \tag{20}$$

Aerodynamic forces (in wind frame, transform to body):

$$\mathbf{F}_{\text{aero}} = \begin{bmatrix} -D\cos\alpha + L\sin\alpha \\ Y \\ -D\sin\alpha - L\cos\alpha \end{bmatrix} \tag{21}$$

Gravity in body frame (with Euler angles ϕ, θ, ψ):

$$\mathbf{F}_{\text{gravity}} = mg \begin{bmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{bmatrix} \tag{22}$$

5.2 Rotational Dynamics

Euler's equation:

$$\mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) = \mathbf{M}_{\text{total}} \tag{23}$$

Expanded form for principal axes:

$$I_{xx}\dot{p} + (I_{zz} - I_{yy})qr = M_x \tag{24}$$
$$I_{yy}\dot{q} + (I_{xx} - I_{zz})pr = M_y \tag{25}$$
$$I_{zz}\dot{r} + (I_{yy} - I_{xx})pq = M_z \tag{26}$$

5.3 Moment Generation

5.3.1 Rotor Moments

Moments from rotor thrust about CG:

$$\mathbf{M}_{\text{thrust}} = \sum_{i=1}^{5} \mathbf{r}_i \times \mathbf{T}_i \tag{27}$$

Expanded:

$$M_x = \sum_{i=1}^{5}(y_i T_i^{(z)} - z_i T_i^{(y)}) = -\sum_{i=1}^{5} y_i T_i \cos\phi_i \tag{28}$$

$$M_y = \sum_{i=1}^{5}(z_i T_i^{(x)} - x_i T_i^{(z)}) = \sum_{i=1}^{5} x_i T_i \cos\phi_i \tag{29}$$

$$M_z = \sum_{i=1}^{5}(x_i T_i^{(y)} - y_i T_i^{(x)}) + Q_i = -\sum_{i=1}^{5} y_i T_i \sin\phi_i + \sum_{i=1}^{5} Q_i \tag{30}$$

where Q_i is the rotor reaction torque:

$$Q_i = C_Q \cdot T_i \cdot R_i \tag{31}$$

with $C_Q \approx 0.01$ (torque coefficient).

5.3.2 Aerodynamic Moments

Include wing and tail contributions (linearized for small angles):

$$M_{x,\text{aero}} = Q_{\text{bar}} S b C_{l,\beta} \beta \tag{32}$$

$$M_{y,\text{aero}} = Q_{\text{bar}} S \bar{c} C_{m,\alpha} \alpha \tag{33}$$

$$M_{z,\text{aero}} = Q_{\text{bar}} S b C_{n,\beta} \beta \tag{34}$$

where $Q_{\text{bar}} = \frac{1}{2}\rho V^2$ is dynamic pressure and $\bar{c}$ is mean aerodynamic chord.

6 Flight Phase Analysis

6.1 Phase 1: Hover

6.1.1 Equilibrium Conditions

For steady hover with zero wind:

$$\sum_{i=1}^{5} T_i = mg \qquad \text{(vertical force balance)} \tag{35}$$

$$\sum_{i=1}^{5} x_i T_i = 0 \qquad \text{(pitch moment balance)} \tag{36}$$

$$\sum_{i=1}^{5} y_i T_i = 0 \qquad \text{(roll moment balance)} \tag{37}$$

All tilt angles: $\phi_i = 0$ for $i = 1, 2, 3$.

6.1.2 Power Requirement

Total hover power:

$$P_{\text{hover}} = \sum_{i=1}^{5} \frac{T_i^{3/2}}{\eta_{\text{motor}} \cdot FM} \sqrt{\frac{1}{2\rho A_i}} + P_{\text{avionics}} \tag{38}$$

6.2 Phase 2: Transition

Transition involves tilting rotors from $\phi = 0$ to $\phi = 90$ while accelerating forward.

6.2.1 Tilt Schedule

A smooth tilt trajectory:

$$\phi(t) = \phi_{\max} \left(\frac{t}{t_{\text{trans}}}\right)^2 \left(3 - 2\frac{t}{t_{\text{trans}}}\right) \tag{39}$$

This provides zero angular velocity at endpoints.

6.2.2 Transition Dynamics

The full 6-DOF equations must be integrated numerically during transition. Key constraints:

- Maintain altitude: vertical acceleration $\dot{w} \approx 0$
- Build forward speed: longitudinal acceleration $\dot{u} > 0$
- Attitude stability: limit pitch angle $|\theta| < \theta_{\max}$

6.3 Phase 3: Cruise

6.3.1 Steady Level Flight

Tilt rotors fully horizontal: $\phi_i = 90$ for $i = 1, 2, 3$.

Force equilibrium:

$$\sum_{i=1}^{3} T_i \sin(90) = D \qquad \text{(thrust = drag)} \tag{40}$$

$$L + \sum_{i=4}^{5} T_i = mg \qquad \text{(lift = weight)} \tag{41}$$

Typically in cruise, wing provides most lift:

$$L \approx mg, \quad T_4, T_5 \approx 0 \tag{42}$$

6.3.2 Cruise Power

Power required:

$$P_{\text{cruise}} = \frac{D \cdot V}{\eta_{\text{prop}}} + P_{\text{avionics}} \tag{43}$$

where $\eta_{\text{prop}} \approx 0.8$ is propeller efficiency in forward flight.

Minimum power speed occurs at:

$$V_{P_{\min}} = \sqrt{\frac{2mg}{\rho S}}\sqrt{\frac{K}{C_{D,0}}} \tag{44}$$

7 Control Allocation

7.1 Control Objectives

Four primary control channels:

- **Thrust**: Total vertical/horizontal force
- **Roll**: Moment about x-axis
- **Pitch**: Moment about y-axis
- **Yaw**: Moment about z-axis

7.2 Control Matrix

Define control vector:

$$\mathbf{u}_{\text{cmd}} = [F_z, M_x, M_y, M_z]^T \tag{45}$$

And actuator vector:

$$\mathbf{a} = [T_1, T_2, T_3, T_4, T_5]^T \tag{46}$$

Control effectiveness matrix $\mathbf{B}$ relates actuators to control:

$$\mathbf{u}_{\text{cmd}} = \mathbf{B}\mathbf{a} \tag{47}$$

where for hover ($\phi_i = 0$):

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -y_1 & -y_2 & -y_3 & -y_4 & -y_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ Q_1/T_1 & Q_2/T_2 & Q_3/T_3 & Q_4/T_4 & Q_5/T_5 \end{bmatrix} \tag{48}$$

7.3 Solution Methods

7.3.1 Pseudoinverse

For over-actuated system (5 motors, 4 controls):

$$\mathbf{a} = \mathbf{B}^{+}\mathbf{u}_{\text{cmd}} \tag{49}$$

where $\mathbf{B}^{+} = \mathbf{B}^{T}(\mathbf{B}\mathbf{B}^{T})^{-1}$ is the Moore-Penrose pseudoinverse.

7.3.2 Constrained Optimization

With thrust limits $T_{\min} \leq T_i \leq T_{\max}$:

$$\begin{aligned} \min_{\mathbf{a}} \quad & ||\mathbf{a} - \mathbf{a}_{\text{nom}}||^2 \\ \text{subject to} \quad & \mathbf{B}\mathbf{a} = \mathbf{u}_{\text{cmd}} \\ & T_{\min} \leq T_i \leq T_{\max} \end{aligned} \tag{50}$$

This is a quadratic programming problem solvable in real-time.

8 Hydrogen Fuel Cell Energy System

8.1 Fuel Cell Performance

8.1.1 Electrical Efficiency

Fuel cell efficiency depends on current density:

$$\eta_{\text{FC}}(i) = \frac{V_{\text{cell}}(i)}{V_{\text{ideal}}} \tag{51}$$

where $V_{\text{ideal}} = 1.48$ V (higher heating value basis) or 1.25 V (lower heating value basis).
Typical polarization curve:

$$V_{\text{cell}}(i) = V_{\text{oc}} - b\log(i) - R_{\text{int}}i - m\exp(ni) \tag{52}$$

For system-level analysis, use constant efficiency:

$$\eta_{\text{FC}} \approx 0.5\text{–}0.6 \tag{53}$$

8.1.2 Power Output

Fuel cell electrical power:

$$P_{\text{FC}} = \eta_{\text{FC}} \cdot \dot{m}_{\text{H}_2} \cdot \text{LHV}_{\text{H}_2} \tag{54}$$

8.2 Hydrogen Consumption

8.2.1 Instantaneous Consumption

Hydrogen mass flow rate:

$$\dot{m}_{\text{H}_2}(t) = \frac{P_{\text{elec}}(t)}{\eta_{\text{FC}} \cdot \text{LHV}_{\text{H}_2}} \tag{55}$$

8.2.2 Mission Energy Budget

Total hydrogen for mission duration t_{mission}:

$$m_{\mathrm{H}_2} = \int_0^{t_{\text{mission}}} \dot{m}_{\mathrm{H}_2}(t)\, dt = \frac{1}{\eta_{\text{FC}} \cdot \mathrm{LHV}_{\mathrm{H}_2}} \int_0^{t_{\text{mission}}} P_{\text{elec}}(t)\, dt \tag{56}$$

Simplified:

$$m_{\mathrm{H}_2} = \frac{E_{\text{mission}}}{\eta_{\text{FC}} \cdot \mathrm{LHV}_{\mathrm{H}_2}} \tag{57}$$

where E_{mission} is total energy in kWh.

8.2.3 Safety Factor

Include reserves:

$$m_{\mathrm{H}_2,\text{total}} = (1 + f_{\text{reserve}}) \cdot m_{\mathrm{H}_2} \tag{58}$$

Typical: $f_{\text{reserve}} = 0.3$–0.5 (30–50% reserve).

8.3 Hydrogen Storage

8.3.1 Storage Options

- **Compressed gas (350–700 bar)**: Gravimetric density 3–5%
- **Liquid H_2 (20 K)**: Gravimetric density 10–20%, requires cryogenic management
- **Metal hydrides**: Lower density but safer

8.3.2 Tank Mass

Tank mass including pressure vessel:

$$m_{\text{tank}} = \frac{m_{\mathrm{H}_2}}{\mathrm{GD}} \tag{59}$$

where GD is gravimetric density. For 350 bar: GD ≈ 0.04.

9 Mission Analysis

9.1 Mission Profile

Typical mission segments:

1. **Takeoff and hover**: Duration t_1, power P_{hover}
2. **Climb and transition**: Duration t_2, power P_{trans}
3. **Cruise outbound**: Distance d_{out}, speed V_{cruise}

4. **Loiter/payload ops**: Duration t_{loiter}, power P_{loiter}
5. **Cruise return**: Distance d_{ret}
6. **Transition and land**: Duration t_3

9.2 Energy Calculation

Total mission energy:

$$E_{\text{mission}} = P_{\text{hover}} t_1 + P_{\text{trans}} t_2 + P_{\text{cruise}} \left(\frac{d_{\text{out}} + d_{\text{ret}}}{V_{\text{cruise}}} \right) + P_{\text{loiter}} t_{\text{loiter}} + P_{\text{trans}} t_3 \tag{60}$$

9.3 Range and Endurance

9.3.1 Maximum Range

Cruise at maximum lift-to-drag ratio:

$$\left(\frac{L}{D} \right)_{\max} = \frac{1}{2\sqrt{C_{D,0} K}} \tag{61}$$

Range:

$$R_{\max} = \frac{\eta_{\text{prop}} \cdot \eta_{\text{FC}} \cdot \text{LHV}_{\text{H}_2}}{g} \left(\frac{L}{D} \right)_{\max} \ln \left(\frac{m_{\text{initial}}}{m_{\text{final}}} \right) \tag{62}$$

9.3.2 Maximum Endurance

Cruise at minimum power speed $V_{P_{\min}}$:

$$t_{\max} = \frac{\eta_{\text{prop}} \cdot \eta_{\text{FC}} \cdot \text{LHV}_{\text{H}_2} \cdot m_{\text{H}_2}}{P_{\min}} \tag{63}$$

10 Iterative Sizing Procedure

10.1 Coupled Mass-Energy Problem

Aircraft mass affects power required, which affects fuel needed, which affects mass. Requires iteration.

10.2 Algorithm

1. **Initialize**: Guess $m_{\text{H}_2}^{(0)}$, compute $m_{\text{total}}^{(0)}$
2. **Power calculation**:
   - Compute thrust required for each phase: $T_i(m_{\text{total}}^{(k)})$
   - Compute power: $P^{(k)}(t)$ for mission profile

3. **Energy calculation**:

$$E_{\text{mission}}^{(k)} = \int P^{(k)}(t)\,dt \tag{64}$$

4. **Fuel update**:

$$m_{\text{H}_2}^{(k+1)} = \frac{(1 + f_{\text{reserve}}) \cdot E_{\text{mission}}^{(k)}}{\eta_{\text{FC}} \cdot \text{LHV}_{\text{H}_2}} \tag{65}$$

5. **Mass update**:

$$m_{\text{total}}^{(k+1)} = m_{\text{airframe}} + m_{\text{FC}} + \frac{m_{\text{H}_2}^{(k+1)}}{\text{GD}} + m_{\text{battery}} + m_{\text{avionics}} + m_{\text{payload}} \tag{66}$$

6. **Convergence check**:

$$\text{If } |m_{\text{total}}^{(k+1)} - m_{\text{total}}^{(k)}| < \epsilon, \text{ stop; else } k \leftarrow k+1 \tag{67}$$

Typical convergence: 3–5 iterations with $\epsilon = 0.01$ kg.

10.3 Relaxation for Stability

If oscillations occur, use under-relaxation:

$$m_{\text{total}}^{(k+1)} = \lambda m_{\text{total}}^{(k+1)} + (1-\lambda) m_{\text{total}}^{(k)} \tag{68}$$

with $\lambda = 0.5$–0.7.

11 Stability and Control Analysis

11.1 Trim Conditions

For steady flight, all accelerations zero:

$$\sum \mathbf{F} = 0 \tag{69}$$

$$\sum \mathbf{M} = 0 \tag{70}$$

Solve for equilibrium thrust values $T_{i,\text{trim}}$ and tilt angles $\phi_{i,\text{trim}}$.

11.2 Linearized Dynamics

Perturb about trim: $\mathbf{x} = \mathbf{x}_{\text{trim}} + \delta\mathbf{x}$

State vector:

$$\mathbf{x} = [u, v, w, p, q, r, \phi, \theta, \psi, x, y, z]^T \tag{71}$$

Linearized system:

$$\delta\dot{\mathbf{x}} = \mathbf{A}\delta\mathbf{x} + \mathbf{B}\delta\mathbf{u} \tag{72}$$

where $\mathbf{A}$ is stability matrix and $\mathbf{B}$ is control matrix.

11.3 Stability Derivatives

Key derivatives for hover:

$$Z_w = \frac{\partial F_z}{\partial w} < 0 \quad \text{(vertical damping)} \tag{73}$$

$$M_q = \frac{\partial M_y}{\partial q} < 0 \quad \text{(pitch damping)} \tag{74}$$

$$L_p = \frac{\partial M_x}{\partial p} < 0 \quad \text{(roll damping)} \tag{75}$$

11.4 Control Authority

11.4.1 Roll Control

Maximum roll moment from wing rotors:

$$M_{x,\max} = 2y_w T_{\max} \tag{76}$$

Roll acceleration:

$$\dot{p}_{\max} = \frac{M_{x,\max}}{I_{xx}} \tag{77}$$

11.4.2 Pitch Control

Pitch moment from differential tilt rotor thrust:

$$M_y = T_1 x_1 - \frac{T_2 + T_3}{2} x_r \tag{78}$$

11.4.3 Yaw Control

Yaw controlled by:

- Differential thrust on rear tilt rotors (hover)
- Rotor reaction torques
- Vertical tail (cruise)

12 Performance Metrics

12.1 Disk Loading

Rotor disk loading (thrust per unit area):

$$DL_i = \frac{T_i}{A_i} = \frac{T_i}{\pi R_i^2} \quad [\text{N/m}^2] \tag{79}$$

Typical values: 100–300 N/m^2 for multirotor, lower is more efficient.

12.2 Power Loading

Power loading (thrust per unit power):

$$PL = \frac{T}{P} = \frac{mg}{P} \quad \text{[N/W or g/W]} \tag{80}$$

Higher is better. Typical: 3–8 g/W.

12.3 Wing Loading

$$WL = \frac{mg}{S} \quad [\text{N/m}^2] \tag{81}$$

Affects stall speed and maneuverability. Typical: 50–150 N/m^2 for small UAV.

12.4 Thrust-to-Weight Ratio

In hover:

$$\frac{T_{\max}}{W} = \frac{\sum_i T_{i,\max}}{mg} \tag{82}$$

Requirement: $T_{\max}/W > 1.5$ for vertical flight, > 2.0 for aerobatics.

12.5 Energy Density

System-level energy density:

$$\text{ED}_{\text{system}} = \frac{\eta_{\text{FC}} \cdot \text{LHV}_{\text{H}_2} \cdot m_{\text{H}_2}}{m_{\text{total}}} \quad \text{[Wh/kg]} \tag{83}$$

Compare to Li-Po battery: ∼150–250 Wh/kg.

13 Example Calculation

13.1 Design Requirements

- Mission range: 50 km
- Cruise speed: 20 m/s
- Hover time: 5 minutes
- Payload: 2 kg

13.2 Initial Sizing

13.2.1 Assumed Parameters

$$
\begin{aligned}
m_{\text{airframe}} &= 5.0 \text{ kg} \\
m_{\text{FC}} &= 1.5 \text{ kg (2 kW stack)} \\
m_{\text{battery}} &= 0.5 \text{ kg} \\
m_{\text{avionics}} &= 0.3 \text{ kg} \\
m_{\text{payload}} &= 2.0 \text{ kg} \\
S &= 0.8 \text{ m}^2 \\
b &= 2.0 \text{ m} \\
C_{L,\text{cruise}} &= 0.6 \\
C_{D,0} &= 0.025 \\
K &= 0.05 \\
\eta_{\text{FC}} &= 0.55 \\
\eta_{\text{prop}} &= 0.75 \\
\text{GD} &= 0.04
\end{aligned}
$$

13.2.2 Iteration 1

Guess: $m_{\text{H}_2} = 0.5$ kg, so $m_{\text{tank}} = 0.5/0.04 = 12.5$ kg

Initial mass:

$$m_{\text{total}}^{(0)} = 5.0 + 1.5 + 0.5 + 0.3 + 2.0 + 12.5 = 21.8 \text{ kg} \tag{84}$$

Hover power:

$$P_{\text{hover}} = \frac{(m_{\text{total}} g)^{3/2}}{\eta_{\text{motor}} \cdot FM} \sqrt{\frac{1}{2\rho A_{\text{total}}}} + P_{\text{av}} \tag{85}$$

Assume 5 rotors, $R = 0.25$ m each, $A_{\text{total}} = 5 \times \pi(0.25)^2 = 0.98$ m^2:

$$P_{\text{hover}} = \frac{(21.8 \times 9.81)^{3/2}}{0.85 \times 0.75} \sqrt{\frac{1}{2 \times 1.225 \times 0.98}} + 20 \approx 1450 \text{ W} \tag{86}$$

Cruise power:

Drag:

$$D = \frac{1}{2}\rho V^2 S(C_{D,0} + KC_L^2) = \frac{1}{2}(1.225)(20)^2(0.8)(0.025 + 0.05 \times 0.6^2) = 11.8 \text{ N} \tag{87}$$

$$P_{\text{cruise}} = \frac{DV}{\eta_{\text{prop}}} + P_{\text{av}} = \frac{11.8 \times 20}{0.75} + 20 = 335 \text{ W} \tag{88}$$

Mission energy:

$$E_{\text{mission}} = P_{\text{hover}} \times \frac{5}{60} + P_{\text{cruise}} \times \frac{50000}{20 \times 3600} \tag{89}$$
$$= 1450 \times 0.0833 + 335 \times 0.694 \tag{90}$$
$$= 121 + 233 = 354 \text{ Wh} = 0.354 \text{ kWh} \tag{91}$$

Hydrogen required:

$$m_{\text{H}_2}^{(1)} = \frac{1.4 \times 0.354}{0.55 \times 33.33} = 0.027 \text{ kg} \tag{92}$$

Tank mass: $m_{\text{tank}}^{(1)} = 0.027/0.04 = 0.675$ kg
New total mass:

$$m_{\text{total}}^{(1)} = 5.0 + 1.5 + 0.5 + 0.3 + 2.0 + 0.675 = 9.98 \text{ kg} \tag{93}$$

Large change! Continue iteration...

13.2.3 Converged Solution

After 4 iterations:

$$\begin{aligned}
m_{\text{total}} &= 10.2 \text{ kg} \\
m_{\text{H}_2} &= 0.029 \text{ kg} \\
m_{\text{tank}} &= 0.73 \text{ kg} \\
P_{\text{hover}} &= 850 \text{ W} \\
P_{\text{cruise}} &= 195 \text{ W} \\
E_{\text{mission}} &= 0.206 \text{ kWh}
\end{aligned}$$

13.3 Performance Summary

$$\begin{aligned}
\text{Hover time} &= \frac{0.55 \times 33.33 \times 0.029}{0.850} = 0.64 \text{ hours} = 38 \text{ minutes} \\
\text{Cruise range} &= \frac{0.55 \times 33.33 \times 0.029 \times 20}{0.195} = 109 \text{ km} \\
\text{Thrust-to-weight} &= 1.8 \\
\text{Wing loading} &= \frac{10.2 \times 9.81}{0.8} = 125 \text{ N/m}^2
\end{aligned}$$

14 Design Considerations

14.1 Advantages of Hydrogen Fuel Cell

- High specific energy: $\sim$1800 Wh/kg (system level with 350 bar storage)

- Long endurance compared to batteries
- Quick refueling (< 5 minutes)
- Zero emissions (water vapor only)
- Constant power output (no voltage sag)

14.2 Challenges

- Storage mass and volume (pressure vessels)
- Cold weather performance
- Humidity management (water production)
- Cost of fuel cell stack
- Hydrogen supply infrastructure
- Safety considerations (hydrogen handling)

14.3 Design Trade-offs

14.3.1 Number of Rotors

More rotors:

+ Better redundancy
+ Lower disk loading, higher efficiency
– More mass (motors, ESCs)
– More complex control

14.3.2 Wing Area

Larger wing:

+ Lower cruise power (more lift)
+ Lower stall speed
– More structural mass
– Higher drag in hover

14.3.3 Cruise Speed

Higher speed:

+ Shorter mission time
+ Better range for given energy
− Higher drag (power $\propto V^3$)
− Requires larger thrust reserve

15 Concluding Remarks

This physics model provides a comprehensive framework for analyzing hydrogen fuel cell powered VTOL UAVs. The model includes:

- Complete 6-DOF rigid body dynamics
- Rotor propulsion with momentum theory
- Wing aerodynamics with lift and drag polars
- Fuel cell energy system modeling
- Control allocation for 5-motor configuration
- Mission analysis and sizing procedures
- Stability and control derivatives

15.1 Implementation Workflow

1. Define mission requirements (range, endurance, payload)
2. Size aircraft components (wing, motors, fuel cell)
3. Perform iterative mass-energy convergence
4. Analyze each flight phase (hover, transition, cruise)
5. Design control system and verify stability
6. Validate with simulation
7. Build and flight test

15.2 Future Extensions

The model can be extended to include:

- Wind disturbances and gusts
- Battery hybrid power management
- Advanced rotor models (blade element theory)
- Structural dynamics and aeroelasticity
- Thermal management of fuel cell
- Trajectory optimization
- Fault detection and redundancy management

16 References

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A Derivation of Rotor Induced Power

From momentum theory, the induced velocity at the rotor disk:

$$v_i = \sqrt{\frac{T}{2\rho A}} \tag{94}$$

Power required to accelerate the air:

$$P = T \cdot v_i = T\sqrt{\frac{T}{2\rho A}} = \frac{T^{3/2}}{\sqrt{2\rho A}} \tag{95}$$

This is the ideal induced power. Real rotors experience profile drag on the blades, so:

$$P_{\text{real}} = \frac{P_{\text{ideal}}}{FM} \tag{96}$$

where figure of merit $FM = 0.6$–0.8 for well-designed rotors.

B Control Matrix Derivation

For the 5-motor configuration in hover with rotor positions:

- Rotor 1: $(x_f, 0, 0)$
- Rotor 2: $(x_r, -y_w, 0)$
- Rotor 3: $(x_r, y_w, 0)$
- Rotor 4: $(0, -y_w, 0)$
- Rotor 5: $(0, y_w, 0)$

The control effectiveness matrix:

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & y_w & -y_w & y_w & -y_w \\ -x_f & -x_r & -x_r & 0 & 0 \\ -C_Q & -C_Q & -C_Q & C_Q & C_Q \end{bmatrix} \tag{97}$$

where signs on yaw depend on rotor rotation direction (clockwise vs counterclockwise).

C Nomenclature Summary

Symbol	Description
m	Total aircraft mass (kg)
g	Gravitational acceleration (m/s^2)
ρ	Air density (kg/m^3)
T_i	Thrust of rotor i (N)
ϕ_i	Tilt angle of rotor i (rad)
S	Wing reference area (m^2)
b	Wingspan (m)
V	Airspeed (m/s)
C_L	Lift coefficient
C_D	Drag coefficient
L	Lift force (N)
D	Drag force (N)
P	Power (W)
η_{FC}	Fuel cell efficiency
η_{prop}	Propeller efficiency
η_{motor}	Motor efficiency
$m_{\mathrm{H_2}}$	Hydrogen mass (kg)
LHV	Lower heating value (kWh/kg)
I_{xx}, I_{yy}, I_{zz}	Moments of inertia (kg·m^2)
p, q, r	Roll, pitch, yaw rates (rad/s)
M_x, M_y, M_z	Roll, pitch, yaw moments (N·m)
A_i	Rotor disk area (m^2)
R_i	Rotor radius (m)
FM	Figure of merit
GD	Gravimetric density of storage