



Game Art

Geometric Design: Bezier Curves

CS 415: Game Development

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Images and some text courtesy of
The Essentials of CAGD by Farin and Hansford

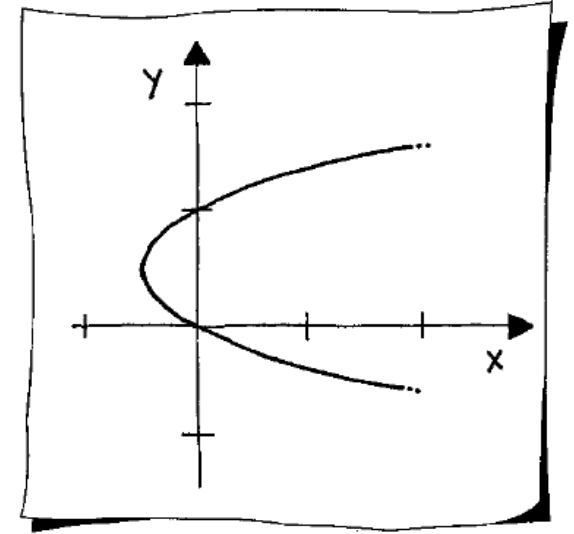
Geometric Modeling

Geometric modeling is typically done by engineers and artists

- Assisted by computational tools (e.g. Maya or Blender or AutoCAD)
- The software provides a mathematical models of curves/surfaces

For rendering, ultimately everything will be turned into triangles.

But modeling triangle-by-triangle would be too tedious



Modeling Curves – Some Questions

Suppose we render curves by approximating them with line segments



How can we generate points on a curve...let's try to do it for a parabola

What would be one possible parametric equation for a simple parabola $y = x^2$?

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$$P(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

We could generate a bunch of line segments using the parametric equation.

What advantages does storing/representing the curve as the equation have over storing the line segments?

Modeling Curves – Some Questions

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We could generate a bunch of line segments using the parametric equation.

What advantages does storing/representing the curve as the equation have over storing the line segments?

- More compact
- Infinite resolution
- Some tasks are easier
e.g. finding derivatives or deforming the geometry

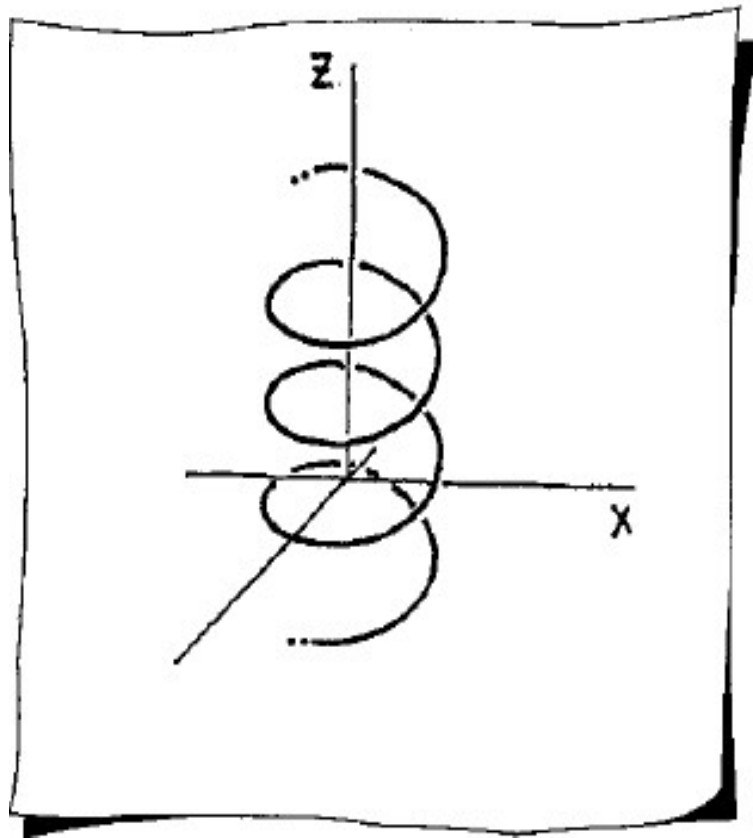
Modeling with Curves

Suppose we render curves by approximating them with line segments

Also, using alternative representations can have other advantages

- More compact
- Infinite resolution
- Some tasks are easier
 - e.g. finding derivatives or deforming the geometry

Parametric Curves



Parametric curves defined in 3D:

$$\mathbf{x}(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$$

Simple example: a *helix*

$$\mathbf{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$$

Bezier Curves

Type of polynomial curve

Curve is defined by a modeler (artist) specifying control points

Can be defined to generate a polynomial of any degree

- Cubics are most common
- Higher degree curve requires more control points

Can be joined together to form piecewise polynomial curves

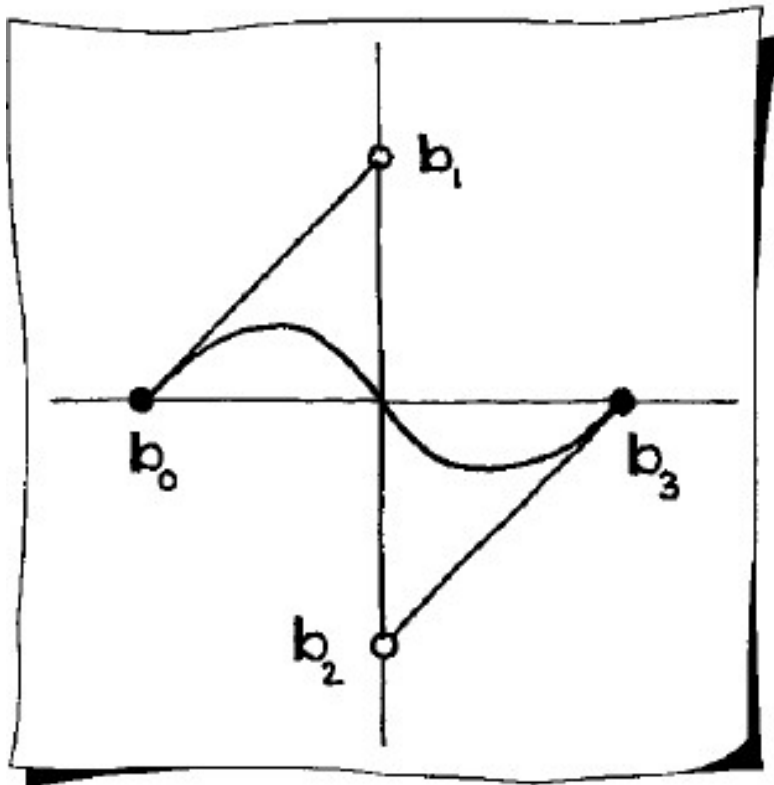
Can form the basis of Bezier patches which define a surface



Named after Pierre Bezier
French Mechanical Engineer
Worked for Renault
Lived 1910-1999

Cubic Bezier Curves

The b_i are control points that an artist picks
In this example they are $(-1,0)$, $(0,1)$, $(0,-1)$ and $(1,0)$



$$\mathbf{x}(t) = \begin{bmatrix} -(1-t)^3 + t^3 \\ 3(1-t)^2t - 3(1-t)t^2 \end{bmatrix}$$

Shape?

Rewrite as a combination of points

$$\begin{aligned} \mathbf{x}(t) = & (1-t)^3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3(1-t)^2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ & + 3(1-t)t^2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Four points form a polygon

– Resembles curve for $t \in [0, 1]$

Cubic Bezier Curves

Define a **cubic Bézier curve** by

$$\mathbf{x}(t) = (1 - t)^3 \mathbf{b}_0 + 3(1 - t)^2 t \mathbf{b}_1 + 3(1 - t) t^2 \mathbf{b}_2 + t^3 \mathbf{b}_3$$

2D or 3D points \mathbf{b}_i are the **Bézier control points**

Control points form the **Bézier polygon** of the curve

Also written as

$$\mathbf{x}(t) = B_0^3(t) \mathbf{b}_0 + B_1^3(t) \mathbf{b}_1 + B_2^3(t) \mathbf{b}_2 + B_3^3(t) \mathbf{b}_3$$

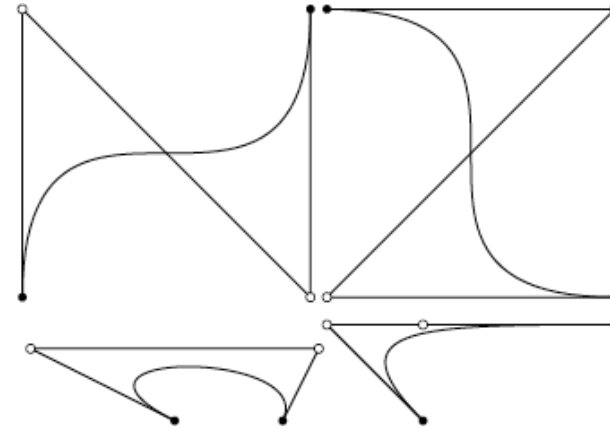
B_i^3 are called the cubic **Bernstein polynomials**

The \mathbf{b}_i are called the **coefficients** of the polynomial $\mathbf{x}(t)$

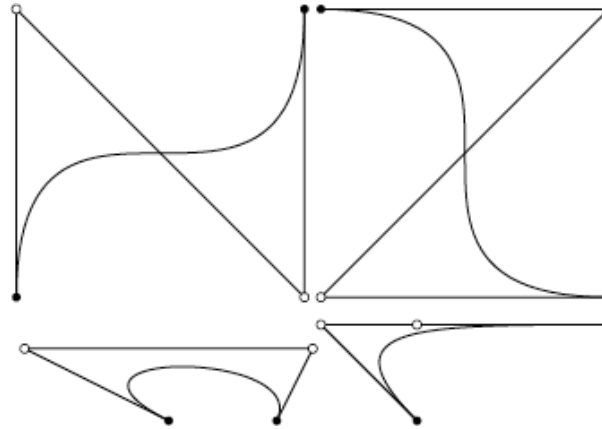
Bezier Curves

Important Properties of Bezier Curves

- Endpoint Interpolation
- Symmetry
- Invariance under affine transformations
- Convex hull property
- Linear precision



Properties of Bezier Curves



Endpoint Interpolation

The curve will pass through the first and last control points:

$$x(0.0) = b_0$$

$$x(1.0) = b_3$$

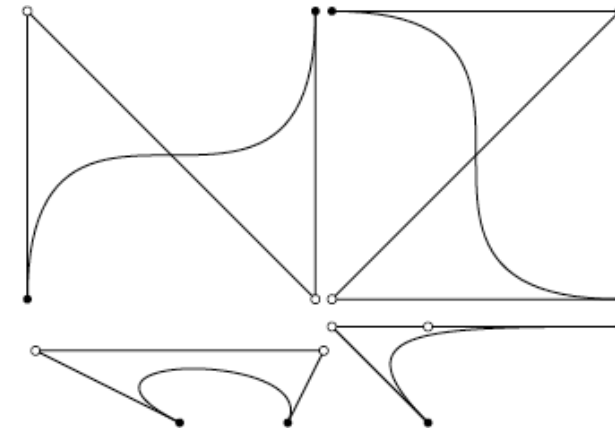
Symmetry

Specifying control points in order b_0, b_1, b_2, b_3 generates same curve as the order b_3, b_2, b_1, b_0

Properties of Bezier Curves

Invariance under affine transformations

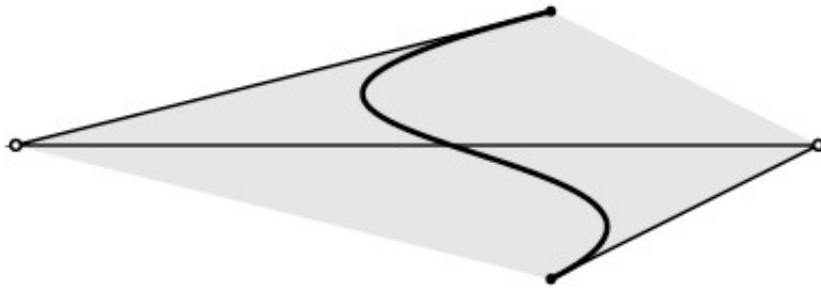
Transforming the control polygon similarly transforms the curve



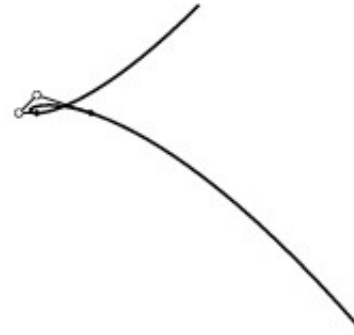
Linear Precision

If b_1 and b_2 are evenly spaced on a straight line, the cubic Bezier curve will be the linear interpolant between b_0 and b_3

Properties of Bezier Curves



The convex hull property



A Bézier curve for $t \in [-1, 2]$

- Extrapolation:** t outside $[0, 1]$
- Curve not within convex hull (in general)
 - Unpredictable behavior

Derivatives

Differentiate each component with respect $t \Rightarrow$ the **tangent vector**

$$\frac{d\mathbf{x}(t)}{dt} = -3(1-t)^2\mathbf{b}_0 + [3(1-t)^2 - 6(1-t)t]\mathbf{b}_1 + [6(1-t)t - 3t^2]\mathbf{b}_2 + 3t^2\mathbf{b}_3$$

Group like terms

$$\frac{d\mathbf{x}(t)}{dt} = 3[\mathbf{b}_1 - \mathbf{b}_0](1-t)^2 + 6[\mathbf{b}_2 - \mathbf{b}_1](1-t)t + 3[\mathbf{b}_3 - \mathbf{b}_2]t^2$$

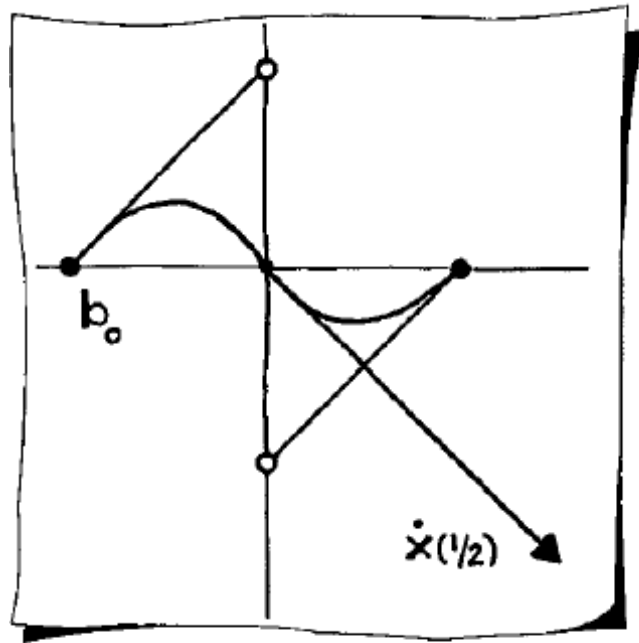
Abbreviated as

$$\frac{d\mathbf{x}(t)}{dt} = 3\Delta\mathbf{b}_0(1-t)^2 + 6\Delta\mathbf{b}_1(1-t)t + 3\Delta\mathbf{b}_2t^2$$

where $\Delta\mathbf{b}_i$ is known as the **forward difference**

Shorten notation: $\dot{\mathbf{x}}(t) \equiv d\mathbf{x}(t)/dt$

Derivatives



Example

$$\mathbf{x}(t) = (1-t)^3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3(1-t)^2 t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ + 3(1-t)t^2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\dot{\mathbf{x}}(t) = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1-t)^2 + 6 \begin{bmatrix} 0 \\ -2 \end{bmatrix} (1-t)t \\ + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^2$$

$$\dot{\mathbf{x}}(0.5) = \begin{bmatrix} 1.5 \\ -1.5 \end{bmatrix}$$

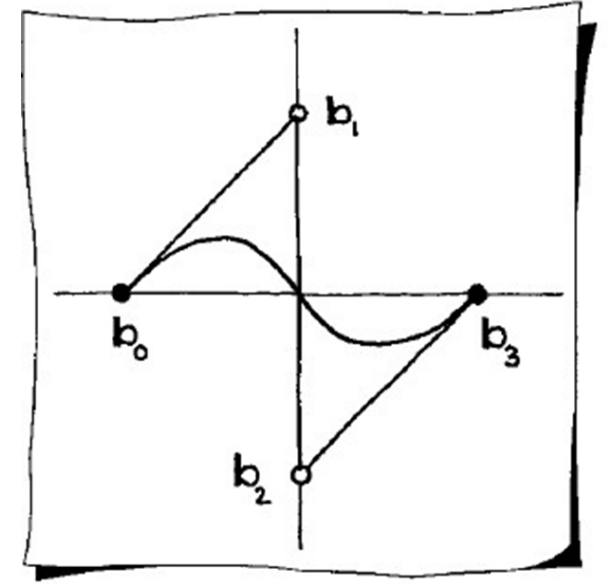
Piecing Curves Together

Tangent vectors at the curve's endpoints:

$$\dot{\mathbf{x}}(0) = 3\Delta\mathbf{b}_0 \quad \dot{\mathbf{x}}(1) = 3\Delta\mathbf{b}_2$$

⇒ control polygon is tangent to the curve at the endpoints
– property helps with piecing together several Bézier curves

The de Casteljau Algorithm



How do you generate points on a Bezier Curve?

You could just plug values for t into the formula we have already seen and evaluate $\mathbf{x}(t)$

The de Casteljau algorithm is an alternative way to generate points

- More computationally efficient
- Uses repeated linear interpolation
- Can be implemented recursively or iteratively
- Invented by Paul de Faget de Casteljau in 1959

The de Casteljau Algorithm

Given: $\mathbf{b}_0, \dots, \mathbf{b}_3$

and a parameter value t

Find: $\mathbf{x}(t)$

Compute:

$$\mathbf{b}_0^1 = (1 - t)\mathbf{b}_0 + t\mathbf{b}_1$$

$$\mathbf{b}_1^1 = (1 - t)\mathbf{b}_1 + t\mathbf{b}_2$$

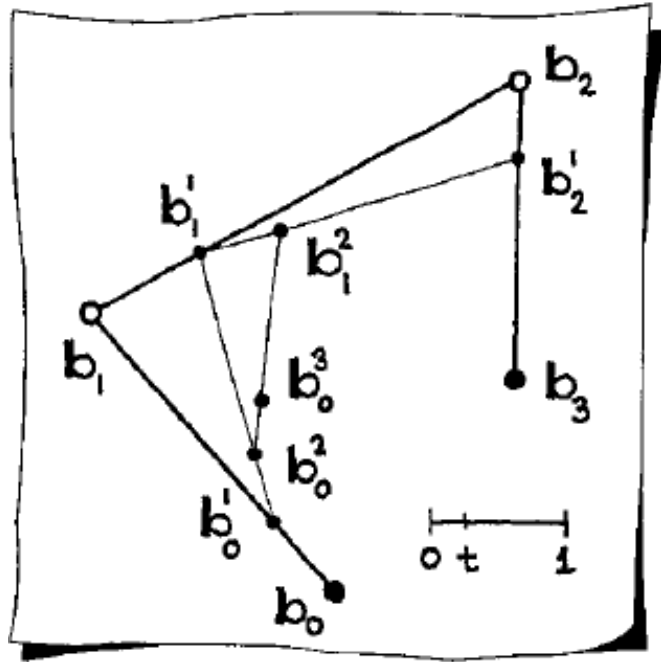
$$\mathbf{b}_2^1 = (1 - t)\mathbf{b}_2 + t\mathbf{b}_3$$

$$\mathbf{b}_0^2 = (1 - t)\mathbf{b}_0^1 + t\mathbf{b}_1^1$$

$$\mathbf{b}_1^2 = (1 - t)\mathbf{b}_1^1 + t\mathbf{b}_2^1$$

$$\mathbf{x}(t) = \mathbf{b}_0^3 = (1 - t)\mathbf{b}_0^2 + t\mathbf{b}_1^2$$

Simply repeated linear interpolation!



The de Casteljau Algorithm

A convenient schematic tool for describing the algorithm

- Arrange the involved points in a triangular diagram

$$\begin{array}{cccc} \mathbf{b}_0 & & & \\ \mathbf{b}_1 & \mathbf{b}_0^1 & & \\ \mathbf{b}_2 & \mathbf{b}_1^1 & \mathbf{b}_0^2 & \\ \mathbf{b}_3 & \mathbf{b}_2^1 & \mathbf{b}_1^2 & \mathbf{b}_0^3 \end{array}$$

In the implementation of the de Casteljau algorithm:

- Not necessary to use a 2D array to simulate the triangular diagram
- A 1D array of control points is sufficient

For example \mathbf{b}_0^1 is calculated and loaded into \mathbf{b}_0

(Must save original control polygon)

The de Casteljau Algorithm

Example

$$\mathbf{x}(t) = (1 - t)^3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + 3(1 - t)^2 t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 3(1 - t)t^2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Evaluate at $t = 0.5$

$$\begin{array}{cccc} \begin{bmatrix} -1.0 \\ 0.0 \end{bmatrix} & & & \\ \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} & \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix} & & \\ \begin{bmatrix} 0.0 \\ -1.0 \end{bmatrix} & \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix} & \begin{bmatrix} -0.25 \\ 0.25 \end{bmatrix} & \\ \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} & \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} & \begin{bmatrix} 0.25 \\ -0.25 \end{bmatrix} & \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix} = \mathbf{x}(0.5) \end{array}$$

Modeling with Cubic Bezier Curves

Lots of nice properties...

- Curvy...artistically expressive
- Only 4 control points...control polygon easy for artist to visualize and work with...
- Can be joined piecewise with matching tangents at endpoints

But...can we express any cubic as a Bezier curve? Not immediately obvious....

We can express any cubic as a sum of the monomials t^0, t^1, t^2, t^3

$$P(t) = at^3 + bt^2 + ct^1 + ct^0$$

So...let's see if we can convert between the monomial basis and the Bernstein basis

The Matrix Form and Monomials

A cubic Bézier curve:

$$\mathbf{b}(t) = B_0^3(t)\mathbf{b}_0 + B_1^3(t)\mathbf{b}_1 + B_2^3(t)\mathbf{b}_2 + B_3^3(t)\mathbf{b}_3$$

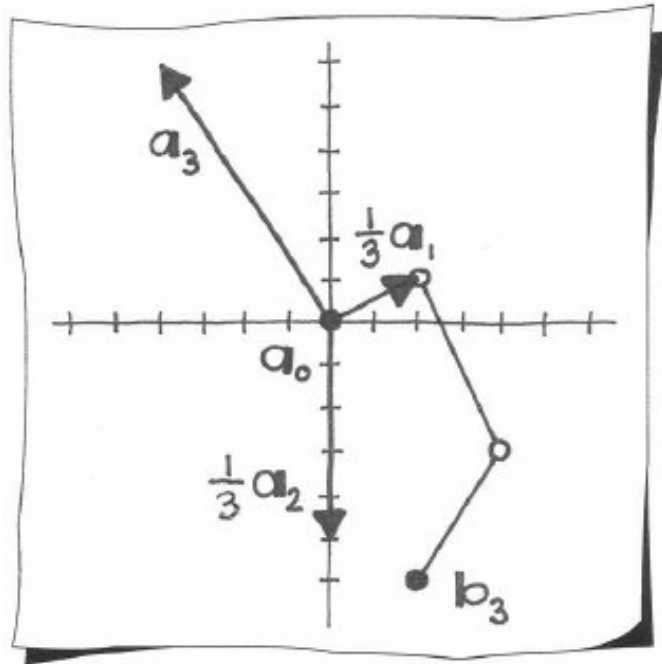
Rewritten in matrix form:

$$\mathbf{b}(t) = [\mathbf{b}_0 \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3] \begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix}$$

A more concise formulation using matrices:

$$\mathbf{b}(t) = [\mathbf{b}_0 \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3] \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

The Matrix Form and Monomials



Monomial polynomials are the most familiar type

– Cubic case: $1, t, t^2, t^3$

Can reformulate a Bézier curve

$$\begin{aligned} \mathbf{b}(t) &= \mathbf{b}_0 + 3t(\mathbf{b}_1 - \mathbf{b}_0) \\ &\quad + 3t^2(\mathbf{b}_2 - 2\mathbf{b}_1 + \mathbf{b}_0) \\ &\quad + t^3(\mathbf{b}_3 - 3\mathbf{b}_2 + 3\mathbf{b}_1 - \mathbf{b}_0) \\ &= \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3 \end{aligned}$$

Geometric interpretation of \mathbf{a}_i and \mathbf{b}_i different

The Matrix Form and Monomials

The monomial coefficients \mathbf{a}_i are defined as

$$[\mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] = [\mathbf{b}_0 \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3] \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse process:

$$[\mathbf{b}_0 \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3] = [\mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3] \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

The square matrix in this equation is nonsingular

\Rightarrow Any cubic curve can be written in Bézier or monomial form