CS 418: Interactive Computer Graphics

Bezier Patches

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Some material taken from
The Essentials of CAGD by Gerald Farin and Dianne Hansford

Bezier Patches

The Utah teapot model
Created with Bezier patches by Martin Newell in 1975

Parametric Surfaces

Parametric curve: mapping of the real line into 2- or 3-space

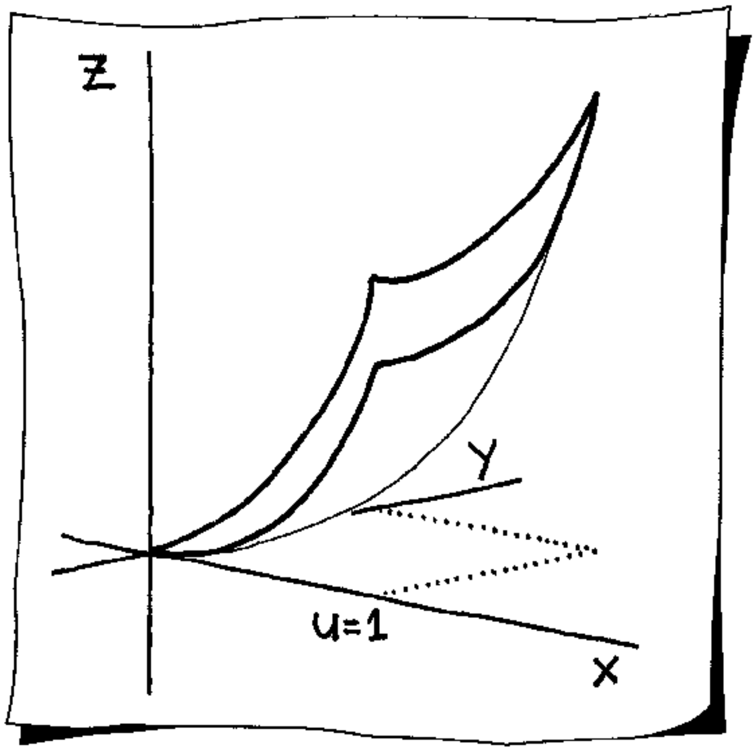
Parametric surface: mapping of the real plane into 3-space

$\mathbb{R}^2$ is the domain of the surface
– A plane with a (u, v) coordinate system

Corresponding 3D surface point:

$$\mathbf{x}(u, v) = \begin{bmatrix} f(u, v) \\ g(u, v) \\ h(u, v) \end{bmatrix}$$

Example: Parametric Surface



$$\mathbf{x}(u,v)=\begin{bmatrix}u\\v\\u^2+v^2\end{bmatrix}$$

Only a portion of surface illustrated

This is also functional surface

- Because two of the coordinate functions are simply u and v

Parametric surfaces may be rotated or moved around

Much more general than bivariate functions $z = f(x, y)$

Why are parametric forms more general? Think about a graph of a function versus a parametric curve..

Bilinear Patches

Typically interested in a finite piece of a parametric surface
– The image of a rectangle in the domain

The finite piece of surface called a patch

Let domain be the *unit square*

$$\{(u, v) : 0 \leq u, v \leq 1\}$$

Map it to a surface patch defined by four points

$$\mathbf{x}(u, v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \mathbf{b}_{0,1} \\ \mathbf{b}_{1,0} & \mathbf{b}_{1,1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}$$

Surface patch is linear in both the u and v parameters
$\Rightarrow$ *bilinear patch*

Bilinear Patches

Bilinear patch:

$$\mathbf{x}(u,v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \mathbf{b}_{0,1} \\ \mathbf{b}_{1,0} & \mathbf{b}_{1,1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}$$

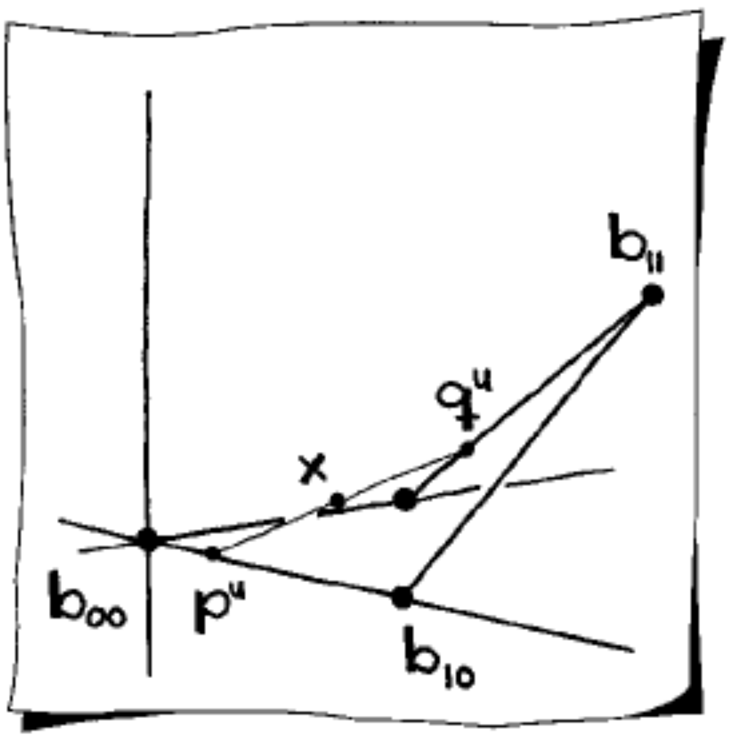
Rewrite as

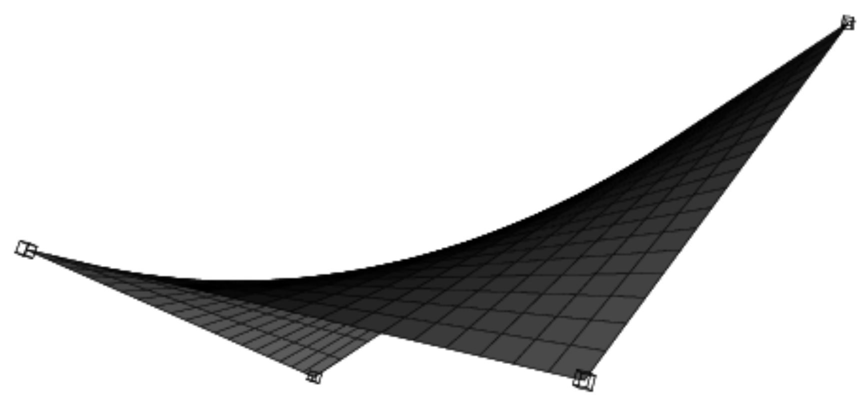
$$\mathbf{x}(u,v) = (1-v)\mathbf{p}^u + v\mathbf{q}^u$$

$$\mathbf{p}^u = (1-u)\mathbf{b}_{0,0} + u\mathbf{b}_{1,0} \quad \text{and} \quad \mathbf{q}^u = (1-u)\mathbf{b}_{0,1} + u\mathbf{b}_{1,1}$$

$\Rightarrow$ Better feeling for the shape of the bilinear patch

Bilinear Patches: Example





Isoparametric Curves

Bilinear patch also called a hyperbolic paraboloid

Isoparametric curve: only one parameter is allowed to vary

Isoparametric curves on a bilinear patch $\Rightarrow$ 2 families of straight lines
$(\bar{u}, v)$: line constant in u but varying in v
$(u, \bar{v})$: line constant in v but varying in u

Four special isoparametric curves (lines):

$$(u, 0) \qquad (u, 1) \qquad (0, v) \qquad (1, v)$$

Curves on Patches

A hyperbolic paraboloid also contains *curves*

Consider the line $u = v$ in the domain – the diagonal
As a parametric line: $u(t) = t$, $v(t) = t$
This domain diagonal is mapped to the 3D curve on the surface

$$\mathbf{d}(t) = \mathbf{x}(t, t)$$

In more detail:

$$\mathbf{d}(t) = \begin{bmatrix} 1-t & t \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \mathbf{b}_{0,1} \\ \mathbf{b}_{1,0} & \mathbf{b}_{1,1} \end{bmatrix} \begin{bmatrix} 1-t \\ t \end{bmatrix}$$

Collecting terms now gives

$$\mathbf{d}(t) = (1-t)^2\mathbf{b}_{0,0} + 2(1-t)t[\frac{1}{2}\mathbf{b}_{0,1} + \frac{1}{2}\mathbf{b}_{1,0}] + t^2\mathbf{b}_{1,1}$$

$\Rightarrow$ quadratic Bézier curve

Bezier Patches

Bilinear patch using linear Bernstein polynomials:

$$\mathbf{x}(u,v) = \begin{bmatrix} B_0^1(u) & B_1^1(u) \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \mathbf{b}_{0,1} \\ \mathbf{b}_{1,0} & \mathbf{b}_{1,1} \end{bmatrix} \begin{bmatrix} B_0^1(v) \\ B_1^1(v) \end{bmatrix}$$

Generalization:

$$\mathbf{x}(u,v) = \begin{bmatrix} B_0^m(u) & \dots & B_m^m(u) \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \dots & \mathbf{b}_{0,n} \\ \vdots & & \vdots \\ \mathbf{b}_{m,0} & \dots & \mathbf{b}_{m,n} \end{bmatrix} \begin{bmatrix} B_0^n(v) \\ \vdots \\ B_n^n(v) \end{bmatrix}$$

Examples: $m = n = 1$: bilinear $\quad m = n = 3$: *bicubic*

Expanding:

$$\mathbf{x}(u,v) = \mathbf{b}_{0,0} B_0^m(u) B_0^n(v) + \ldots + \mathbf{b}_{i,j} B_i^m(u) B_j^n(v) + \ldots + \mathbf{b}_{m,n} B_m^m(u) B_n^n(v)$$

Bezier Patches

$$\mathbf{x}(u,v) = \begin{bmatrix} B_0^m(u) & \dots & B_m^m(u) \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \dots & \mathbf{b}_{0,n} \\ \vdots & & \vdots \\ \mathbf{b}_{m,0} & \dots & \mathbf{b}_{m,n} \end{bmatrix} \begin{bmatrix} B_0^n(v) \\ \vdots \\ B_n^n(v) \end{bmatrix}$$

Abbreviated as

$$\mathbf{x}(u,v) = M^{\mathrm{T}}\mathbf{B}N$$

$\Rightarrow$ surface generalization of the curve equation

Evaluate at a parameter pair (u,v):

$$\mathbf{C} = M^{\mathrm{T}}\mathbf{B} = [\mathbf{c}_0, \dots, \mathbf{c}_n]$$

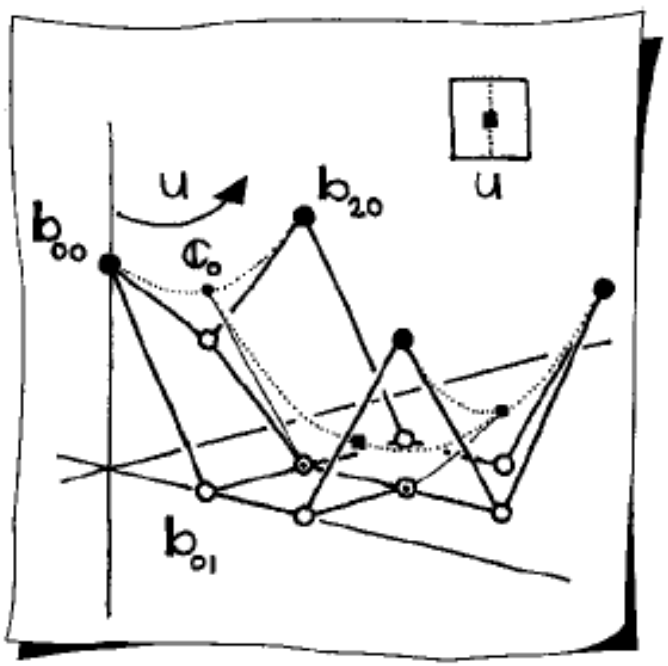
Then final result

$$\mathbf{x}(u,v) = \mathbf{C}N$$

Call this the 2-stage explicit evaluation method
– Bernstein polynomials are explicitly evaluated

Bezier Patches

$$\mathbf{x}(u,v) = M^{\mathrm{T}}\mathbf{B}N \qquad \Rightarrow \qquad \mathbf{x}(u,v) = \mathbf{C}N$$



Control points $\mathbf{c}_0, \ldots, \mathbf{c}_n$ of $\mathbf{C}$ do not depend on the parameter value v

Curve $\mathbf{C}N$: curve on surface
– Constant u
– Variable v
$\Rightarrow$ isoparametric curve or isocurve

Bezier Patches: Example

Example: Evaluate the 2×3 control net at $(u, v) = (0.5, 0.5)$

$$\mathbf{B} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} & \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 9 \\ 0 \\ 6 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} & \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} & \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} & \begin{bmatrix} 9 \\ 3 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 6 \\ 6 \end{bmatrix} & \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} & \begin{bmatrix} 6 \\ 6 \\ 0 \end{bmatrix} & \begin{bmatrix} 9 \\ 6 \\ 6 \end{bmatrix} \end{bmatrix}$$

Step 1) Compute quadratic Bernstein polynomials for $u = 0.5$:

$$M^{\mathrm{T}} = \begin{bmatrix} 0.25 & 0.5 & 0.25 \end{bmatrix}$$

$\Rightarrow$ Intermediate control points

$$\mathbf{C} = M^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4.5 \end{bmatrix} & \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} & \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} & \begin{bmatrix} 9 \\ 3 \\ 3 \end{bmatrix} \end{bmatrix}$$

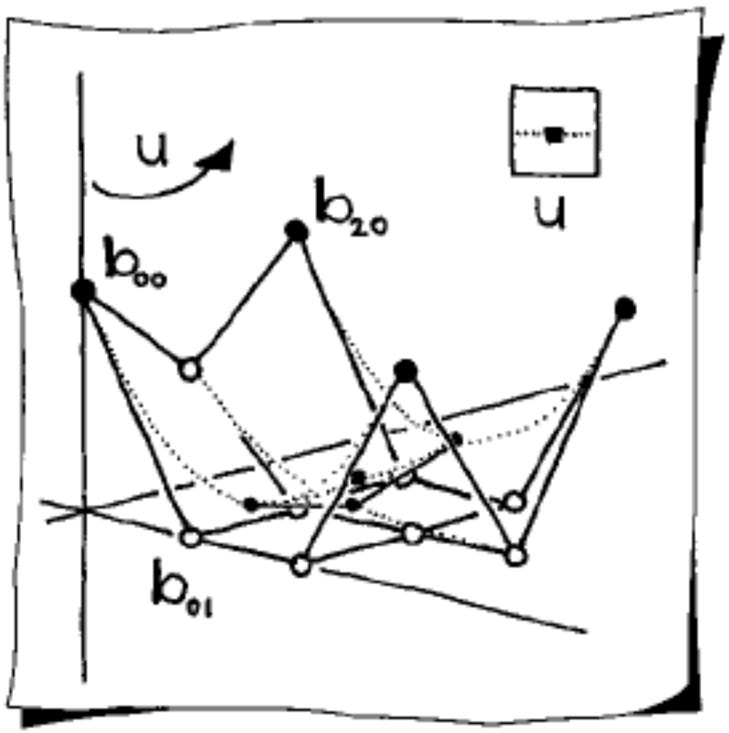
Bezier Patches: Example

Step 2) Compute cubic Bernstein polynomials for $v = 0.5$:

$$N = \begin{bmatrix} 0.125 \\ 0.375 \\ 0.375 \\ 0.125 \end{bmatrix}$$

$$\mathbf{x}(0.5, 0.5) = \mathbf{C}N = \begin{bmatrix} 4.5 \\ 3 \\ 0.9375 \end{bmatrix}$$

Another Approach to Evaluation



v-isoparametric curve

Another approach to
2-stage explicit evaluation:

$$\mathbf{x}(u, v) = M^{\mathrm{T}}\mathbf{B}N$$

$$\mathbf{D} = \mathbf{B}N$$

$$\mathbf{x} = M^{\mathrm{T}}\mathbf{D}$$

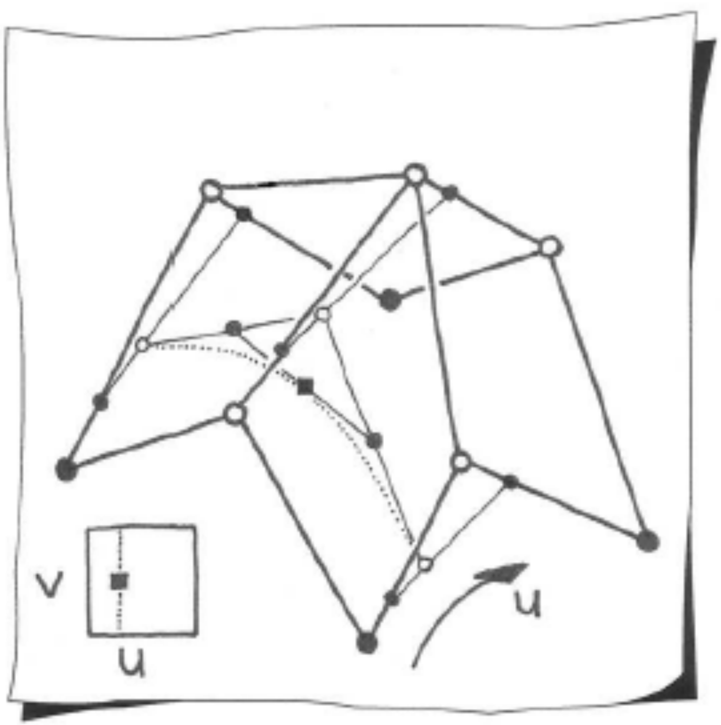
Properties of Bezier Patches

Bézier patches properties essentially the same as the curve ones

1. Endpoint interpolation:
   – Patch passes through the four corner control points
   – Control polygon boundaries define patch boundary curves
2. Symmetry:
   Shape of patch independent of corner selected to be $\mathbf{b}_{0,0}$
3. Affine invariance:
   Apply affine map to control net and then evaluate
   identical to applying affine map to the original patch
4. Convex hull property:
   $\mathbf{x}(u, v)$ in the convex hull of the control net for $(u, v) \in [0, 1] \times [0, 1]$

de Casteljau Algorithm

Evaluation of a Bézier patch: $\mathbf{x}(u,v) = M^T \mathbf{B} N$



Define an intermediate set of points

$$\mathbf{C} = M^{\mathrm{T}}\mathbf{B}$$

$$\mathbf{c}_0 = B_0^m(u)\mathbf{b}_{0,0} + \ldots + B_m^m(u)\mathbf{b}_{m,0}$$
$$\mathbf{c}_1 = B_0^m(u)\mathbf{b}_{0,1} + \ldots + B_m^m(u)\mathbf{b}_{m,1}$$
$$\ldots$$
$$\mathbf{c}_n = B_0^m(u)\mathbf{b}_{0,n} + \ldots + B_m^m(u)\mathbf{b}_{m,n}$$

Evaluate n degree m curves with the de Casteljau algorithm

de Casteljau Algorithm

Final evaluation step: $\mathbf{x}(u, v) = \mathbf{C}N$
$\Rightarrow$ Evaluate this degree n Bézier curve with the de Casteljau algorithm

The 2-stage de Casteljau evaluation method
– Repeated calls to the de Casteljau algorithm for curves

Advantage of this geometric approach:
– Allows computation of a point and derivative

Control polygon $\mathbf{C}$:
– Evaluate point $\mathbf{x}(u, v) = \mathbf{C}N$ and tangent $\mathbf{x}_v$

Control polygon $\mathbf{D} = \mathbf{B}N$:
– Evaluate point $\mathbf{x} = M^{\mathrm{T}}\mathbf{D}$ and tangent $\mathbf{x}_u$