Bezier Curves

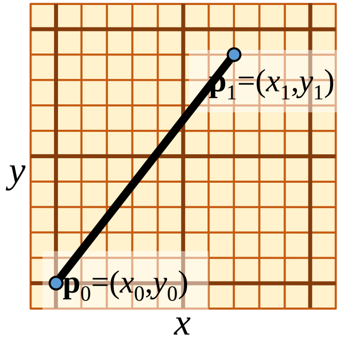
CS 418

Interactive Computer Graphics

John C. Hart

Linear Interpolation

$\mathbf{p}(t) = (1\text{-}t)\ \mathbf{p}_0 + t\ \mathbf{p}_1$

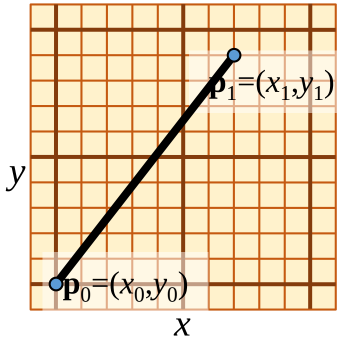


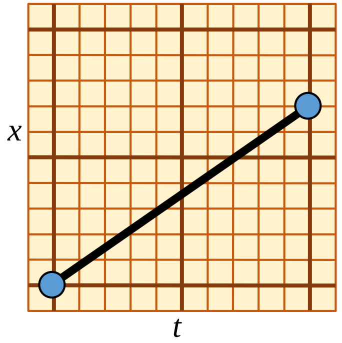
Linear Interpolation

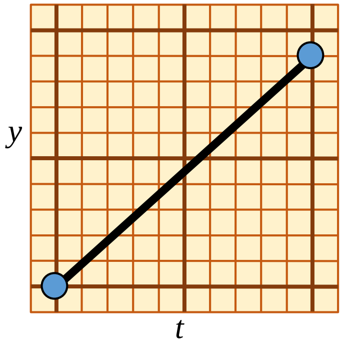
$\mathbf{p}(t) = (1\text{-}t)\ \mathbf{p}_0 + t\ \mathbf{p}_1$

$x(t) = (1\text{-}t)\ x_0 + t\ x_1$

$y(t) = (1\text{-}t)\ y_0 + t\ y_1$





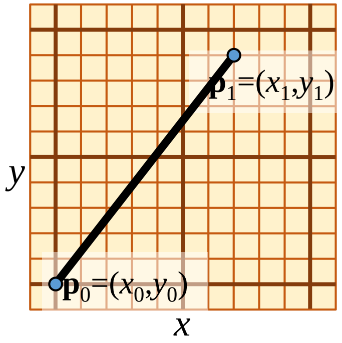


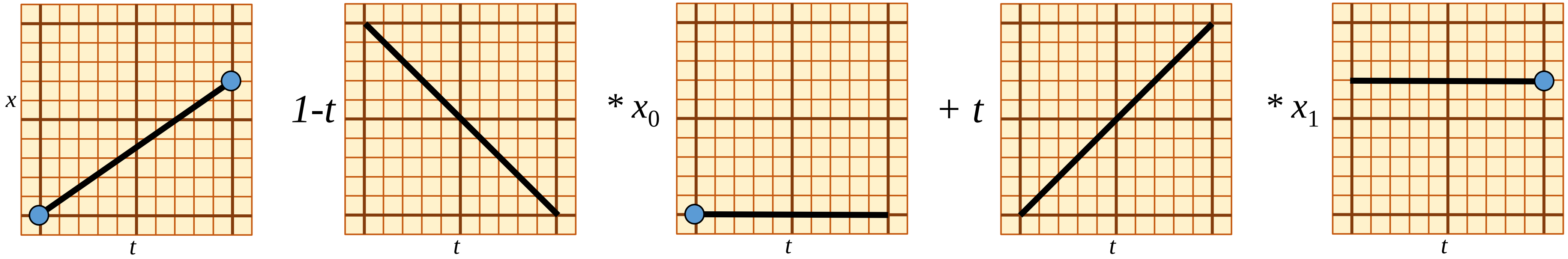
Linear Interpolation

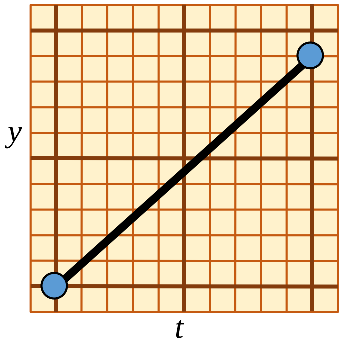
$\mathbf{p}(t) = (1-t)\, \mathbf{p}_0 + t\, \mathbf{p}_1$

$x(t) = (1-t)\, x_0 + t\, x_1$

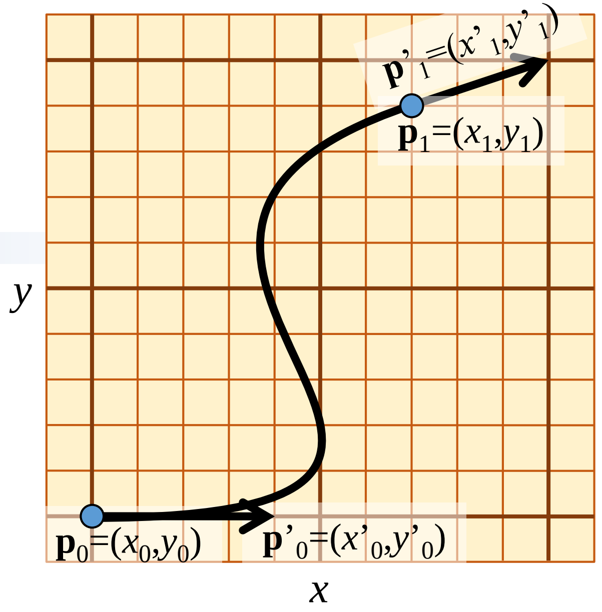
$y(t) = (1-t)\, y_0 + t\, y_1$





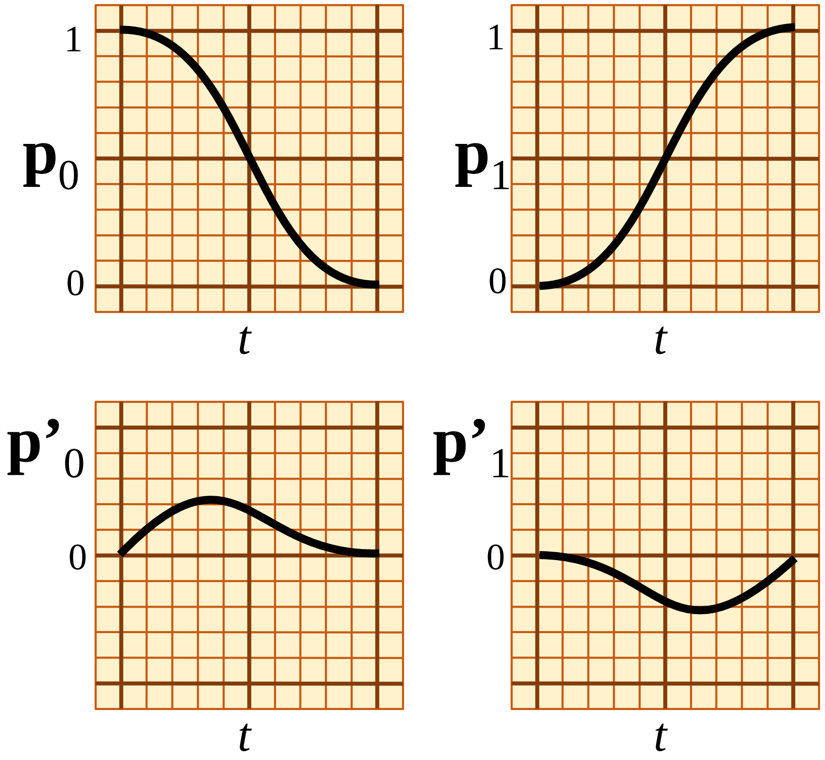


Hermite Interpolation



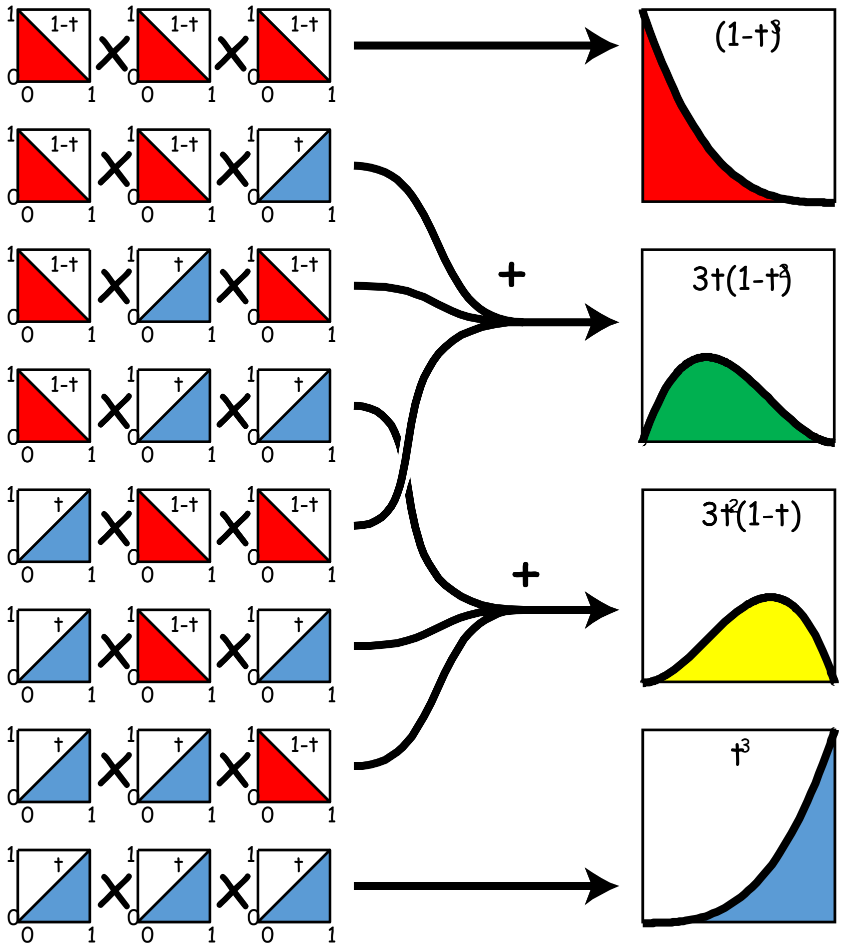
$$
\begin{array}{lll}
\mathbf{p}(t) = & (2\mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}'_0 + \mathbf{p}'_1) & t^3 + \\
 & (-3\mathbf{p}_0 + 3\mathbf{p}_1 - 2\mathbf{p}'_0 - \mathbf{p}'_1) & t^2 + \\
 & \mathbf{p}'_0 & t + \\
 & \mathbf{p}_0 & (1)
\end{array}
$$

$$
\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}'_0 \\ \mathbf{p}'_1 \end{bmatrix}
$$



$$
\begin{array}{lll}
\mathbf{p}(\mathrm{t}) = & (2t^3 - 3t^2 + 1) & \mathbf{p}_0 + \\
 & (-2t^3 + 3t^2) & \mathbf{p}_1 + \\
 & (t^3 - 2t^2 + t) & \mathbf{p}'_0 + \\
 & (t^3 - t^2) & \mathbf{p}'_1
\end{array}
$$

Bernstein Polynomials



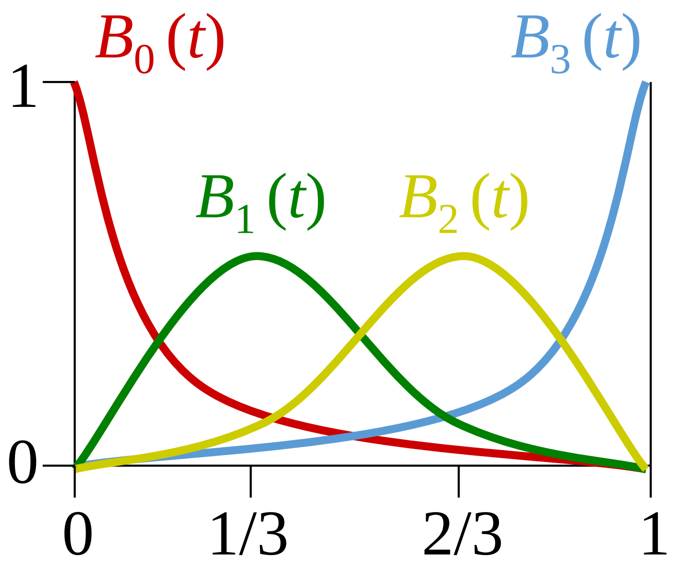
Bernstein Interpolation

$B_0^{3}(t) = (1-t)^3$

$B_1^{3}(t) = 3\,(1-t)^2\,t$

$B_2^{3}(t) = 3\,(1-t)\,t^2$

$B_3^{3}(t) = t^3$



Bernstein Interpolation

$B_0^3(t) = (1-t)^3$

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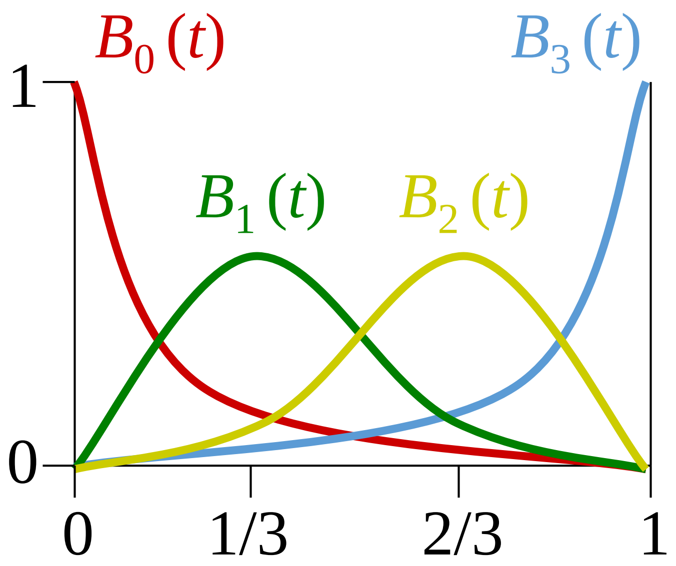
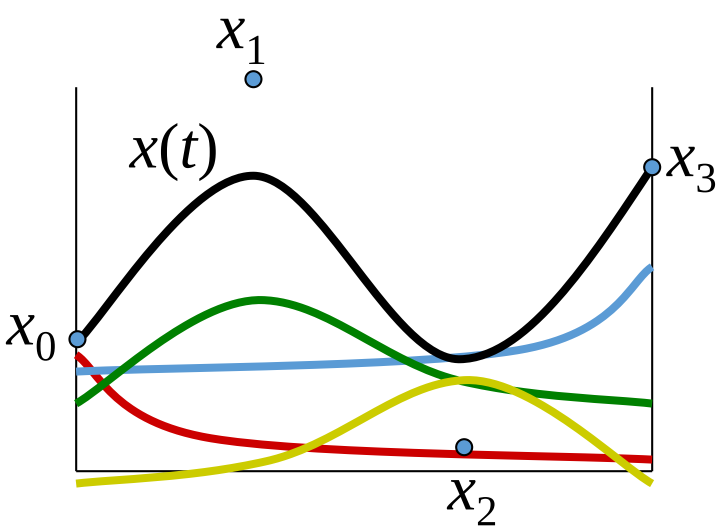


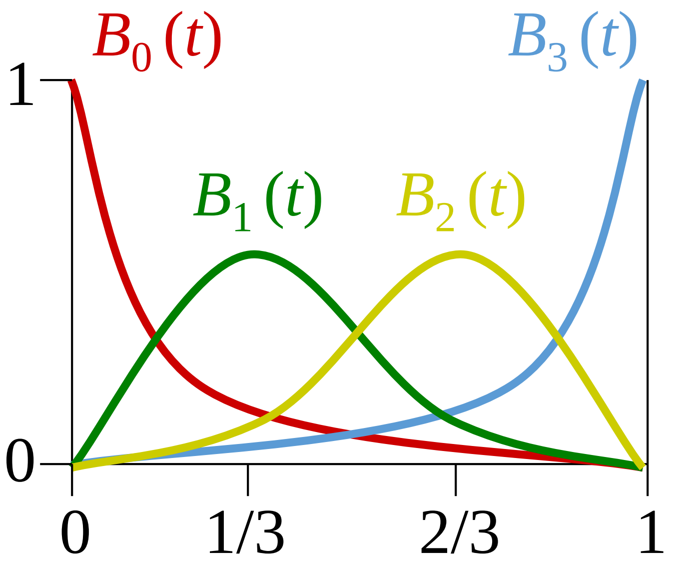
$x(t) = B_0\,(t)\, x_0 + B_1\,(t)\, x_1 + B_2(t)\, x_2 + B_3(t)\, x_3$

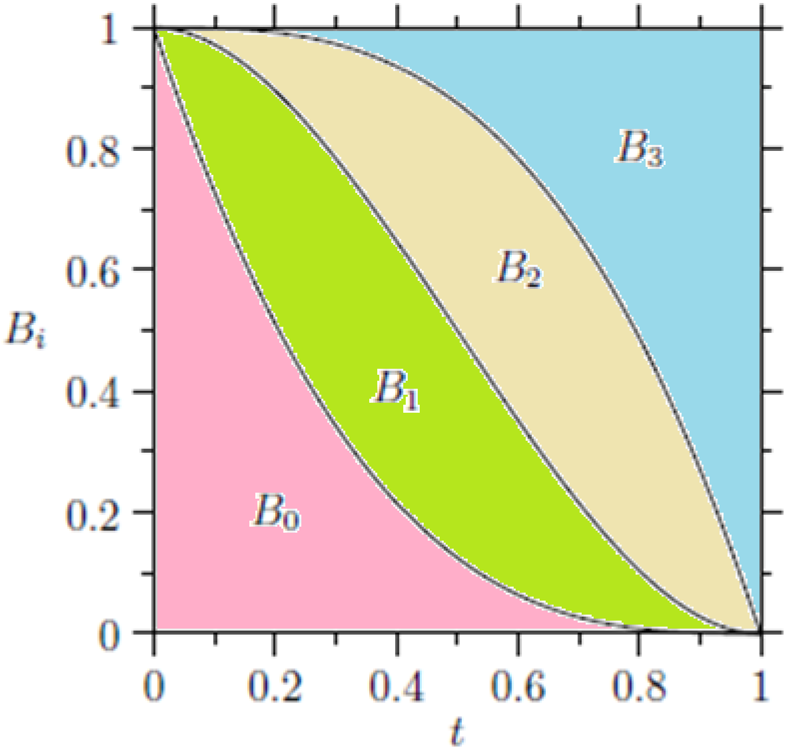


Bernstein Polynomials

- Partition of unity
  - Sum to one for any *t* in [0,1]

$$\Sigma_{i=0..n} B_i^n(t) = 1$$

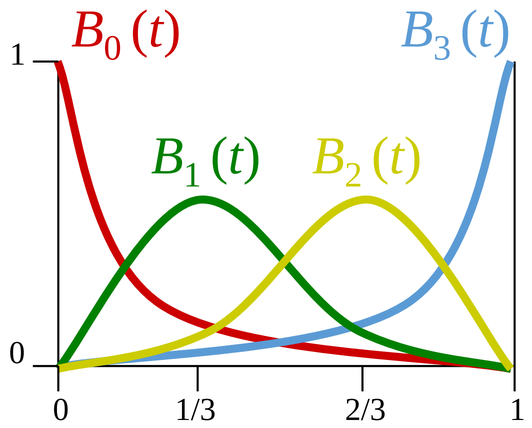




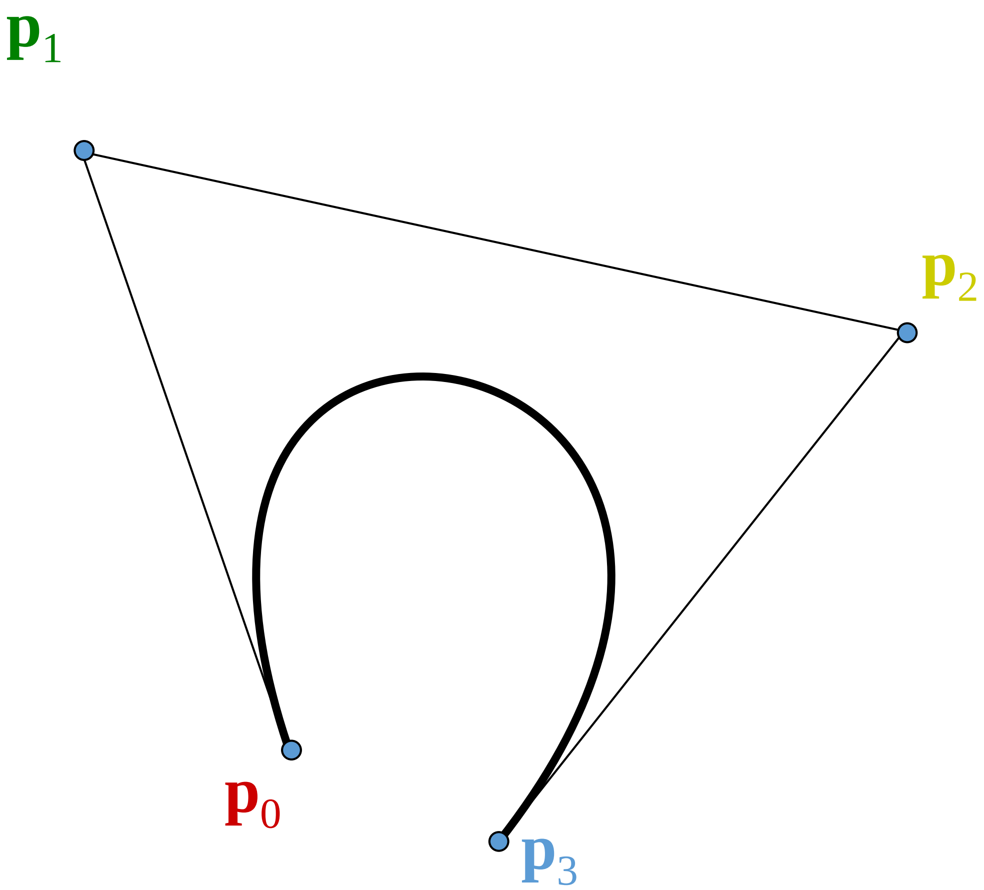
Cubic Bezier Curves

- Bernstein basis applied to points

$$\mathbf{p}(t) = \Sigma_{\mathrm{i}} \, ({}^{3}{}_{\mathrm{i}}) \; t^{i} \, (1\text{-}t)^{3-i} \, \mathbf{p}_{i}$$



- Bezier curve specified by four *control* points including two *endpoints*



Cubic Bezier Curves

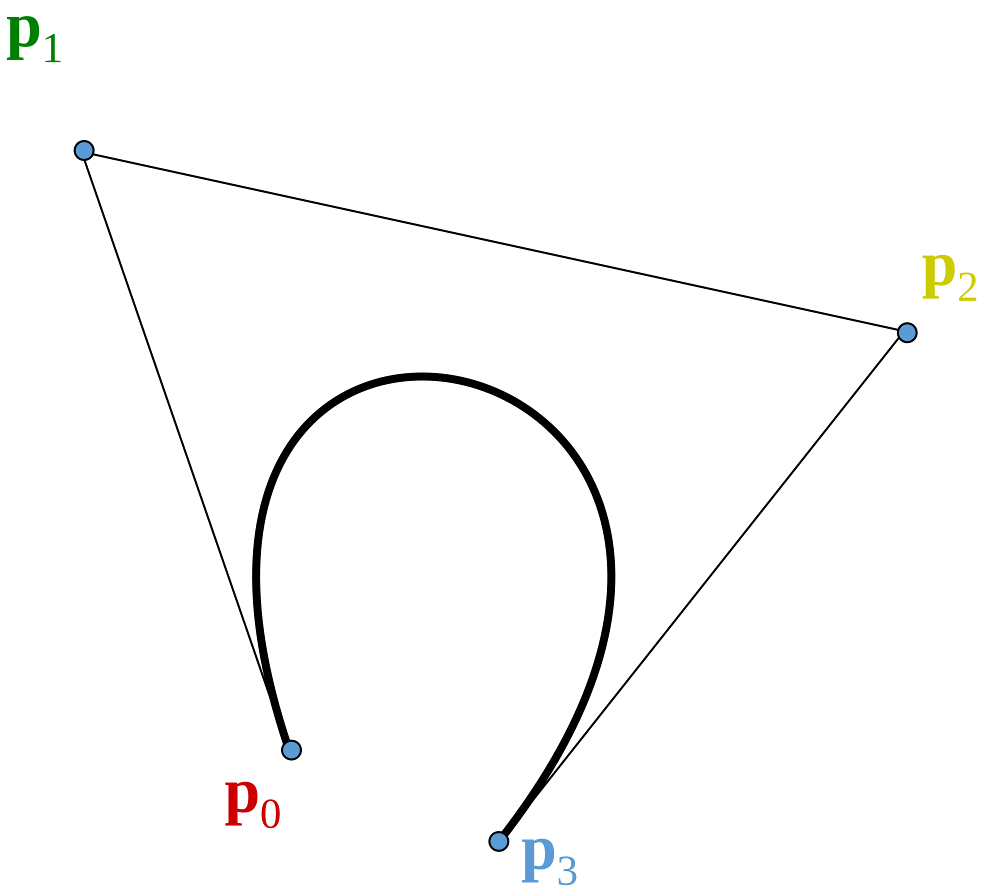
- Bernstein basis applied to points

$$\mathbf{p}(t) = \Sigma_{\mathrm{i}} \binom{3}{\mathrm{i}} t^i (1\text{-}t)^{3\text{-}i}\, \mathbf{p}_i$$

- Affine invariance:

  Let M be a 4x4 transformation

  Then $M\,\mathbf{p}(t) = \Sigma_{\mathrm{i}}\, B_i(t)\, M\,\mathbf{p}_i$

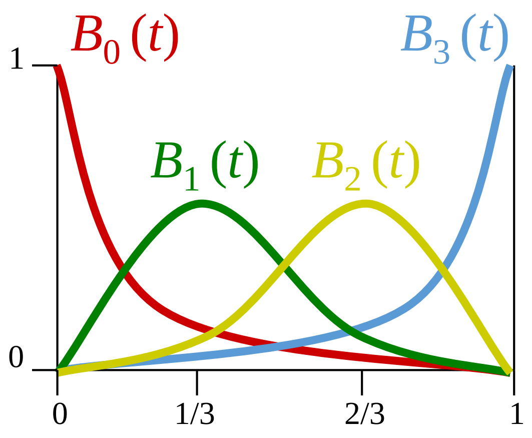


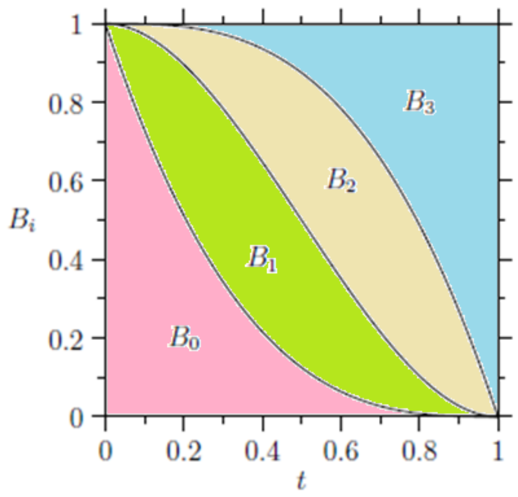
Cubic Bezier Curves

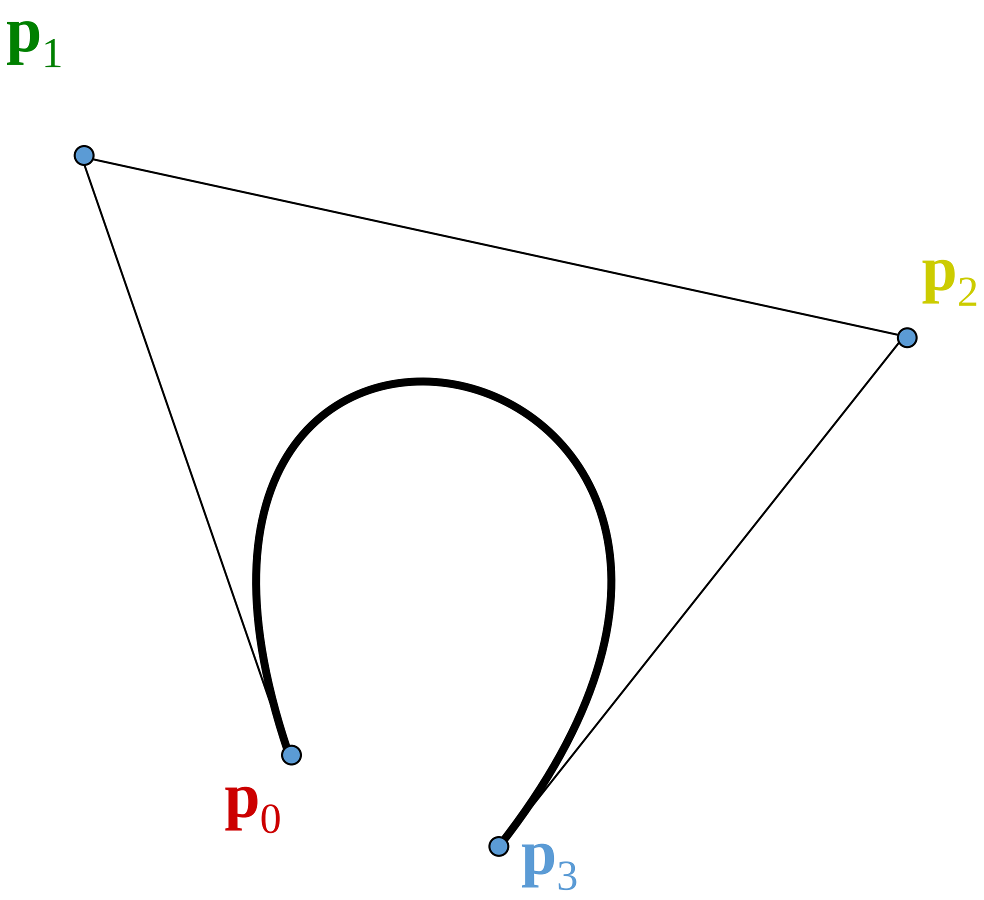
- Bernstein basis applied to points

$$\mathbf{p}(t) = \Sigma_i \binom{3}{i} t^i (1-t)^{3-i} \mathbf{p}_i$$

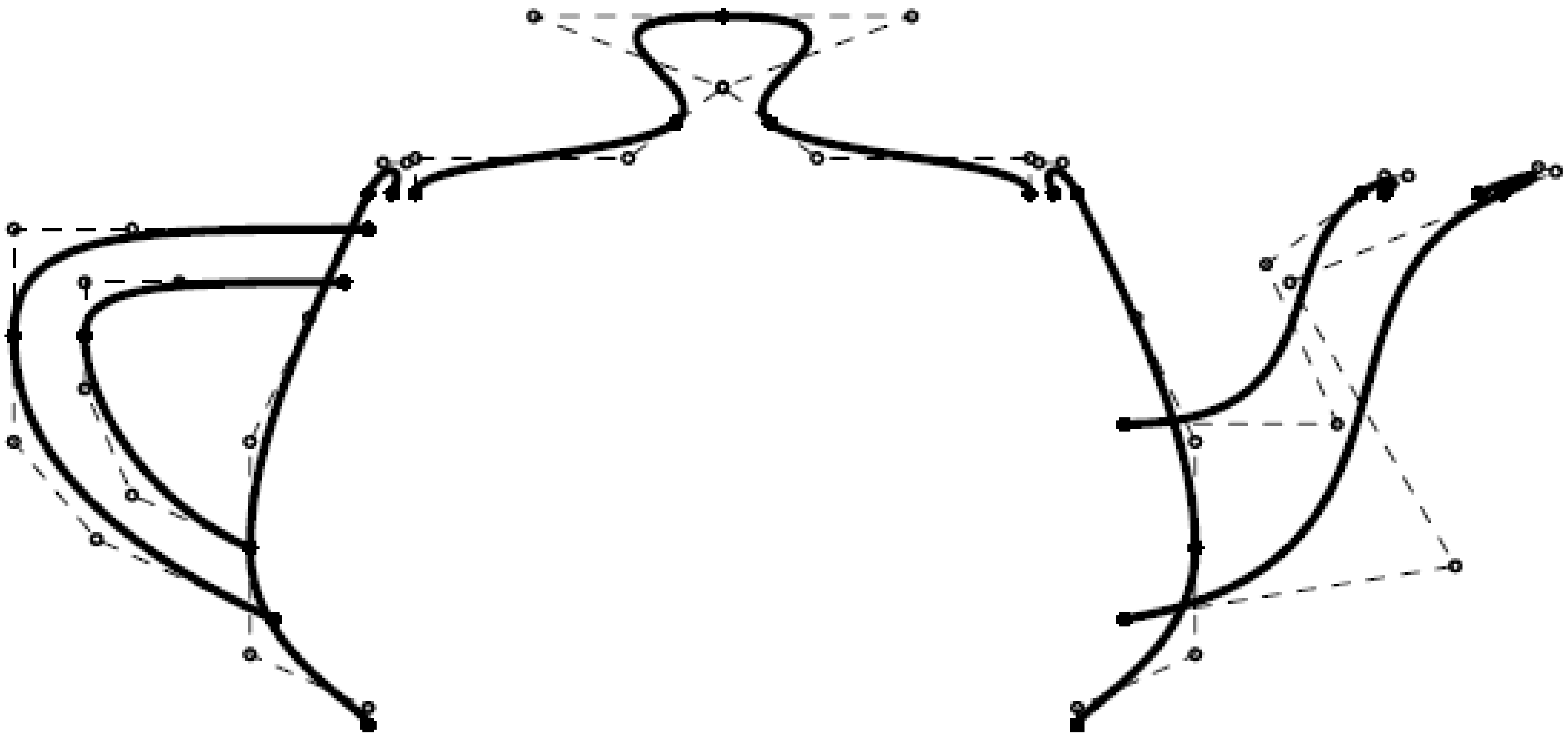
- Curve entirely contained in the convex hull of the control points







Modeling with Bezier Curves



Bezier ≡ Hermite

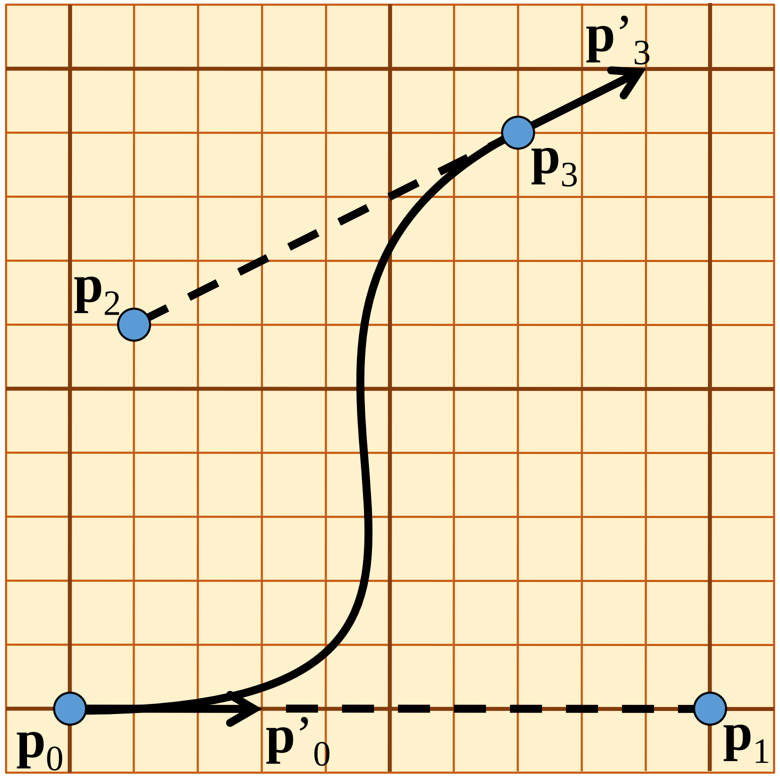
$\mathbf{p}_1 = \mathbf{p}_0 + 3\,\mathbf{p}'_0$, $\mathbf{p}_2 = \mathbf{p}_3 - 3\,\mathbf{p}'_3$

- Bezier

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -3 & 1 \\ 2 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

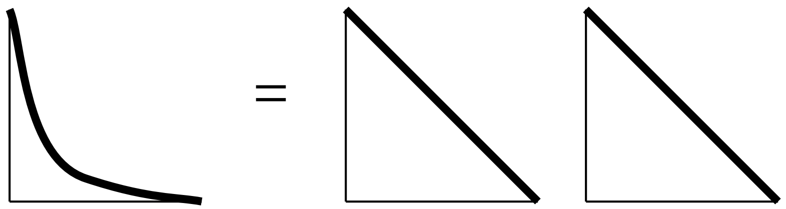
- Hermite

$$\mathbf{p}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_3 \\ \mathbf{p}'_0 \\ \mathbf{p}'_3 \end{bmatrix}$$

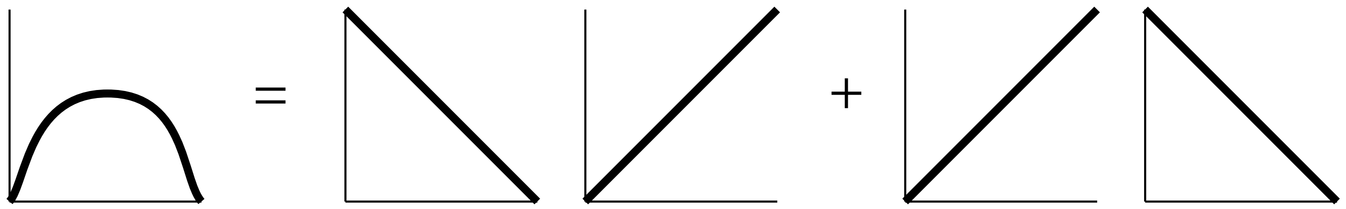


Building Bernsteins

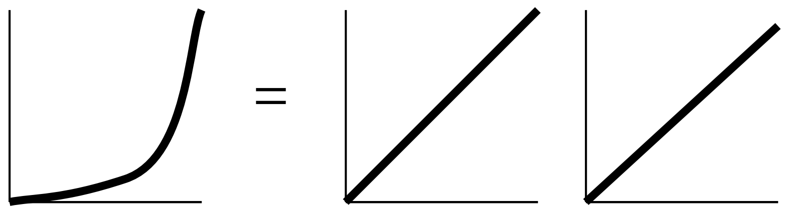
$$\boxed{B_i^n(t) = (1\text{-}t)B_i^{n\text{-}1}(t) + tB_{i-1}^{n-1}(t)}$$



$B_0^2(t) \quad = \quad (1\text{-t}) \quad B_0^1(t)$



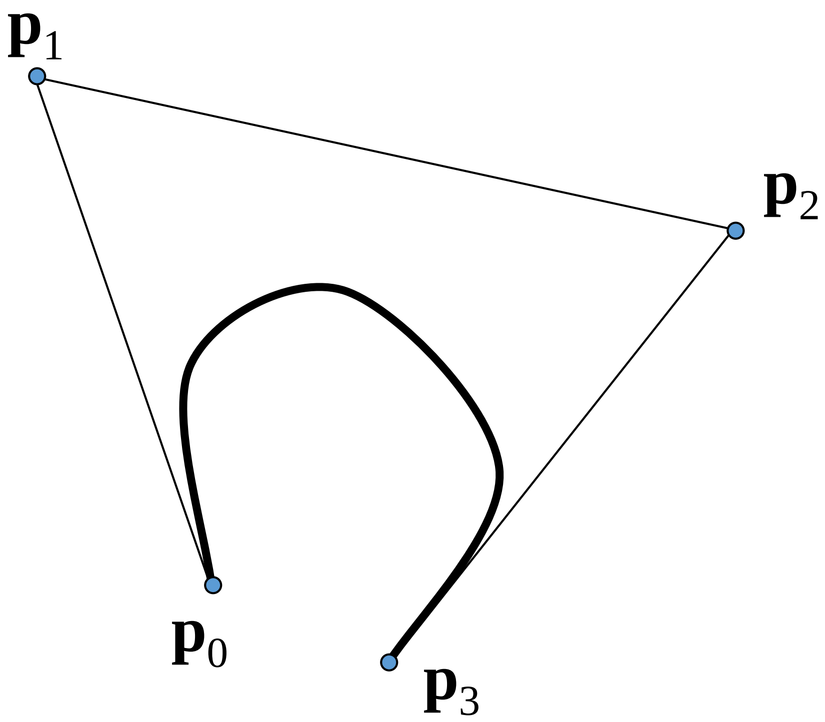
$B_1^2(t) \quad = \quad (1\text{-}t) \quad B_1^1(t) \quad + \quad t \quad B_0^1(t)$



$B_2^2(t) \quad = \quad t \quad B_1^1(t)$

de Casteljau Algorithm

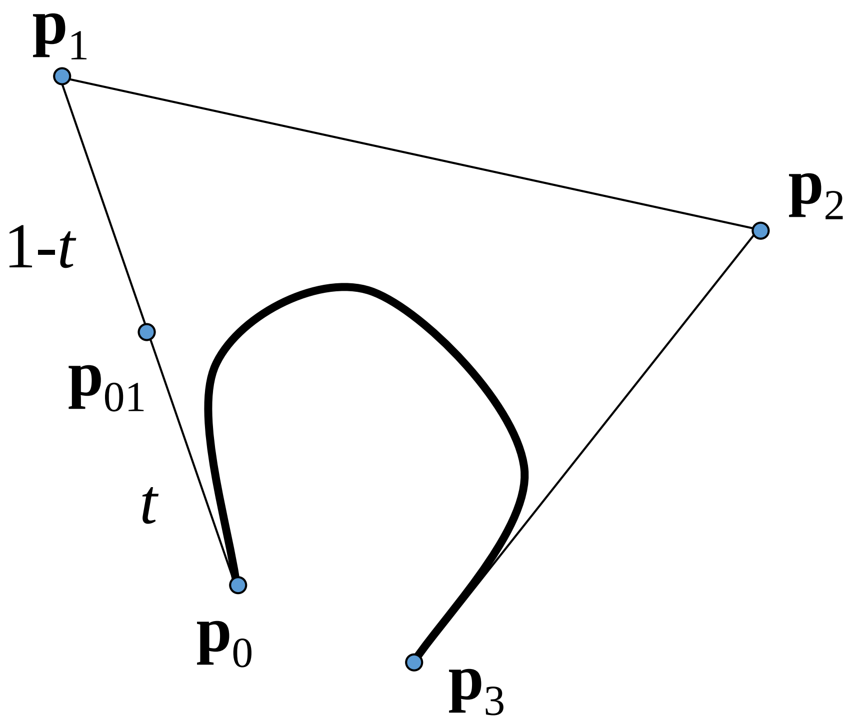
- Cascading lerps



de Casteljau Algorithm

- Cascading lerps

  $\mathbf{p}_{01} = (1\text{-}t)\ \mathbf{p}_0 + t\ \mathbf{p}_1$

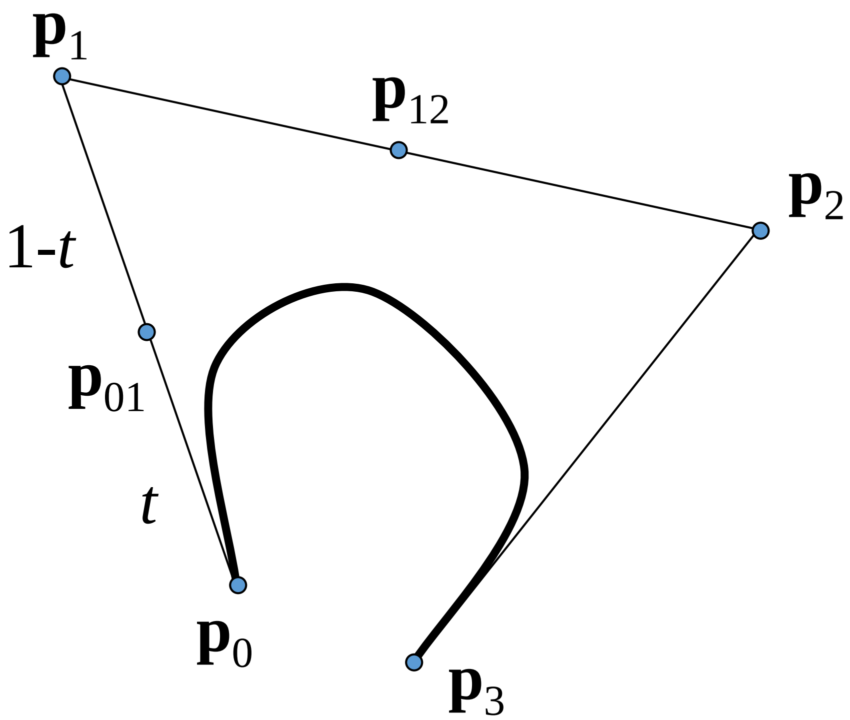


de Casteljau Algorithm

- Cascading lerps

$\mathbf{p}_{01} = (1\text{-}t)\,\mathbf{p}_0 + t\,\mathbf{p}_1$

$\mathbf{p}_{12} = (1\text{-}t)\,\mathbf{p}_1 + t\,\mathbf{p}_2$



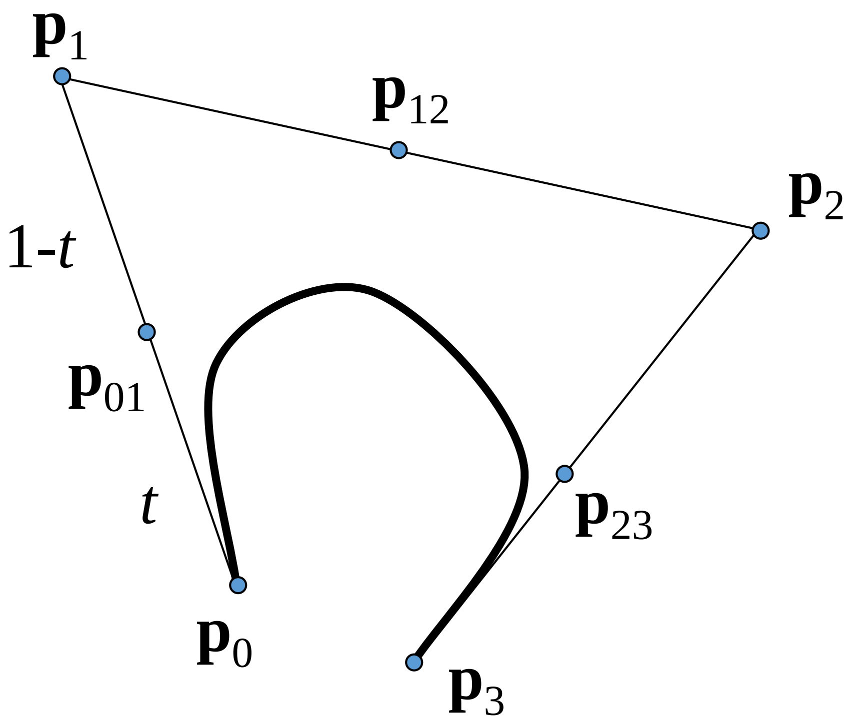
de Casteljau Algorithm

- Cascading lerps

$$\mathbf{p}_{01} = (1\text{-}t)\,\mathbf{p}_0 + t\,\mathbf{p}_1$$

$$\mathbf{p}_{12} = (1\text{-}t)\,\mathbf{p}_1 + t\,\mathbf{p}_2$$

$$\mathbf{p}_{23} = (1\text{-}t)\,\mathbf{p}_2 + t\,\mathbf{p}_3$$



de Casteljau Algorithm

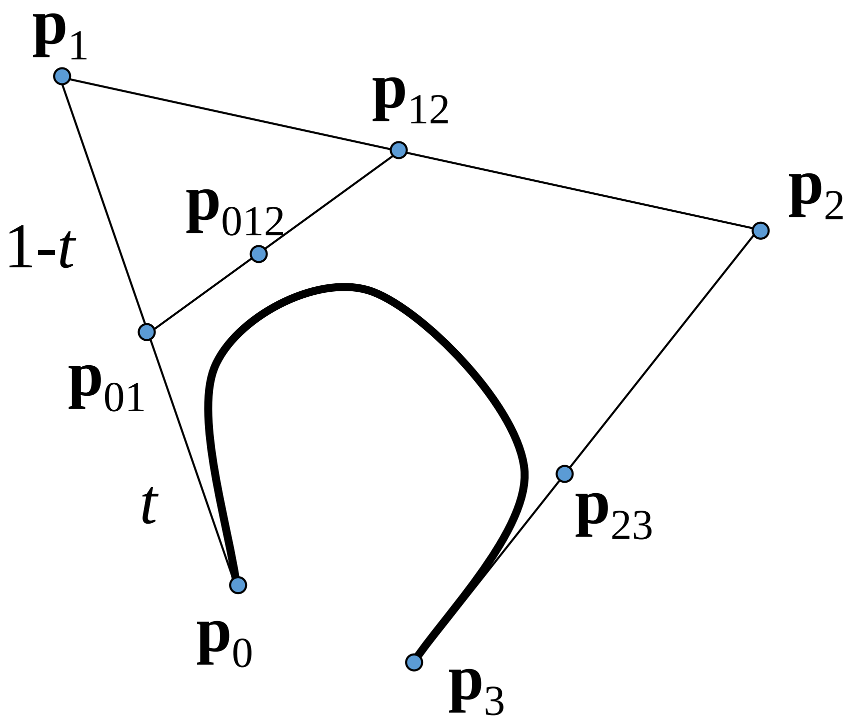
- Cascading lerps

$\mathbf{p}_{01} = (1\text{-}t)\ \mathbf{p}_0 + t\ \mathbf{p}_1$

$\mathbf{p}_{12} = (1\text{-}t)\ \mathbf{p}_1 + t\ \mathbf{p}_2$

$\mathbf{p}_{23} = (1\text{-}t)\ \mathbf{p}_2 + t\ \mathbf{p}_3$

$\mathbf{p}_{012} = (1\text{-}t)\ \mathbf{p}_{01} + t\ \mathbf{p}_{12}$



de Casteljau Algorithm

- Cascading lerps

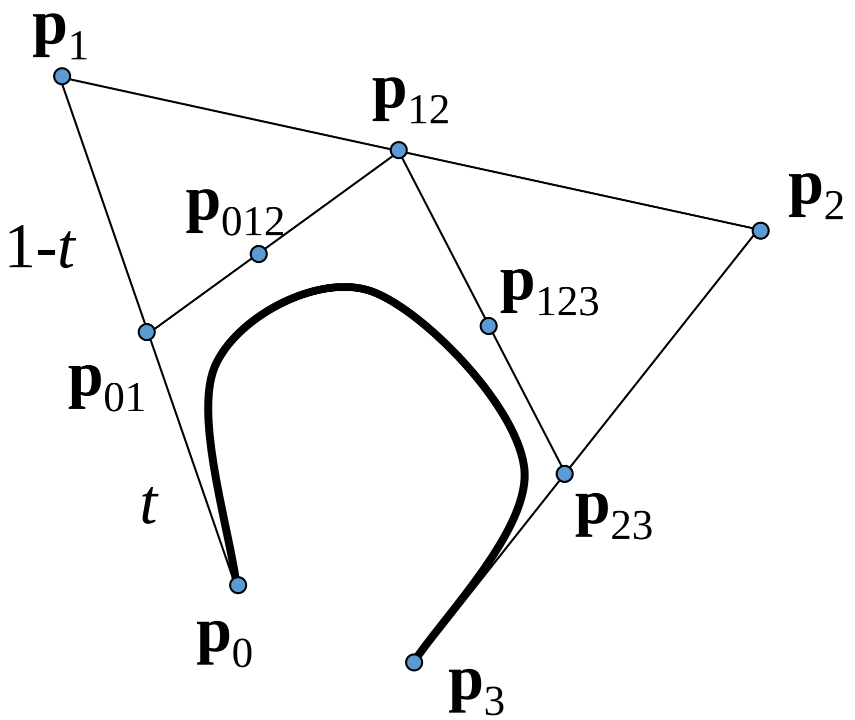
$$\mathbf{p}_{01} = (1-t)\,\mathbf{p}_0 + t\,\mathbf{p}_1$$

$$\mathbf{p}_{12} = (1-t)\,\mathbf{p}_1 + t\,\mathbf{p}_2$$

$$\mathbf{p}_{23} = (1-t)\,\mathbf{p}_2 + t\,\mathbf{p}_3$$

$$\mathbf{p}_{012} = (1-t)\,\mathbf{p}_{01} + t\,\mathbf{p}_{12}$$

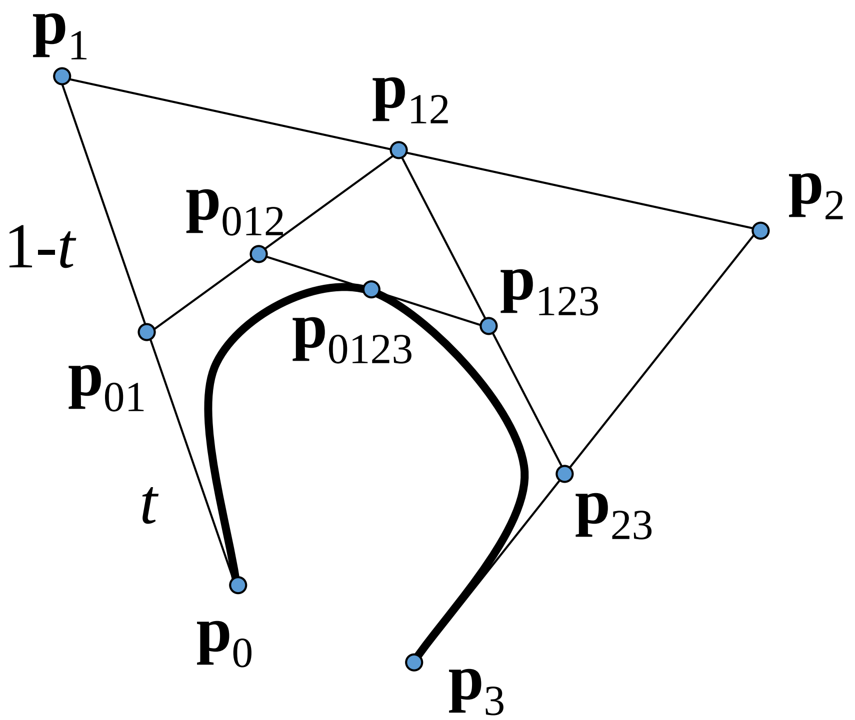
$$\mathbf{p}_{123} = (1-t)\,\mathbf{p}_{12} + t\,\mathbf{p}_{23}$$



de Casteljau Algorithm

- Cascading lerps

$$\mathbf{p}_{01} = (1\text{-}t)\,\mathbf{p}_0 + t\,\mathbf{p}_1$$
$$\mathbf{p}_{12} = (1\text{-}t)\,\mathbf{p}_1 + t\,\mathbf{p}_2$$
$$\mathbf{p}_{23} = (1\text{-}t)\,\mathbf{p}_2 + t\,\mathbf{p}_3$$
$$\mathbf{p}_{012} = (1\text{-}t)\,\mathbf{p}_{01} + t\,\mathbf{p}_{12}$$
$$\mathbf{p}_{123} = (1\text{-}t)\,\mathbf{p}_{12} + t\,\mathbf{p}_{23}$$
$$\mathbf{p}_{0123} = (1\text{-}t)\,\mathbf{p}_{012} + t\,\mathbf{p}_{123}$$



de Casteljau Algorithm

- Cascading lerps

  $\mathbf{p}_{01} = (1\text{-}t)\ \mathbf{p}_0 + t\ \mathbf{p}_1$

  $\mathbf{p}_{12} = (1\text{-}t)\ \mathbf{p}_1 + t\ \mathbf{p}_2$

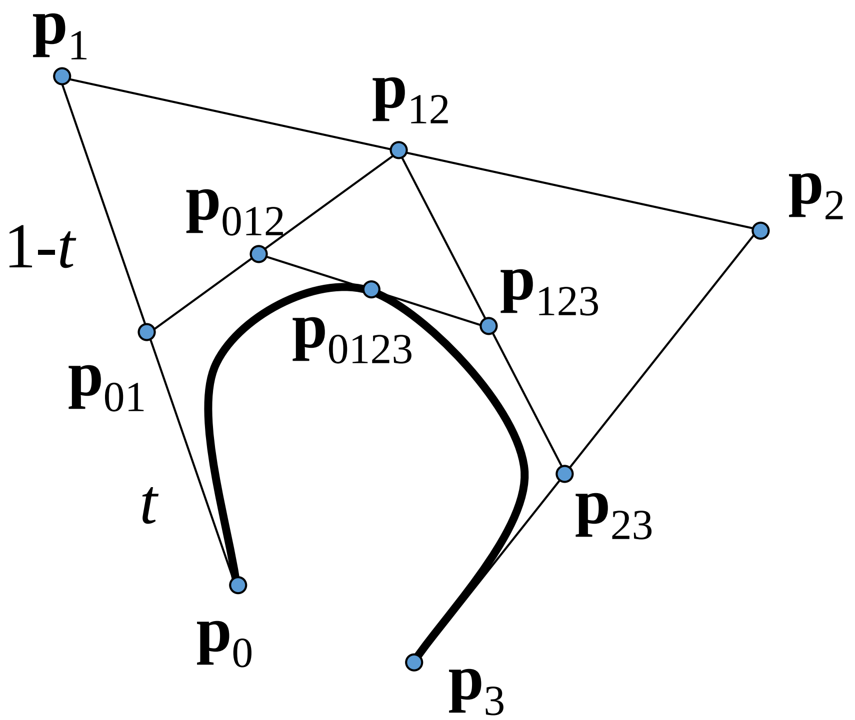
  $\mathbf{p}_{23} = (1\text{-}t)\ \mathbf{p}_2 + t\ \mathbf{p}_3$

  $\mathbf{p}_{012} = (1\text{-}t)\ \mathbf{p}_{01} + t\ \mathbf{p}_{12}$

  $\mathbf{p}_{123} = (1\text{-}t)\ \mathbf{p}_{12} + t\ \mathbf{p}_{23}$

  $\mathbf{p}_{0123} = (1\text{-}t)\ \mathbf{p}_{012} + t\ \mathbf{p}_{123}$

- Subdivides curve at $\mathbf{p}_{0123}$
  - $\mathbf{p}_0\ \mathbf{p}_{01}\ \mathbf{p}_{012}\ \mathbf{p}_{0123}$
  - $\mathbf{p}_{0123}\ \mathbf{p}_{123}\ \mathbf{p}_{23}\ \mathbf{p}_3$
- Repeated subdivision converges to curve



de Casteljau Algorithm

- Cascading lerps

  $\mathbf{p}_{01} = (1\text{-}t)\ \mathbf{p}_0 + t\ \mathbf{p}_1$

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  $\mathbf{p}_{23} = (1\text{-}t)\ \mathbf{p}_2 + t\ \mathbf{p}_3$

  $\mathbf{p}_{012} = (1\text{-}t)\ \mathbf{p}_{01} + t\ \mathbf{p}_{12}$

  $\mathbf{p}_{123} = (1\text{-}t)\ \mathbf{p}_{12} + t\ \mathbf{p}_{23}$

  $\mathbf{p}_{0123} = (1\text{-}t)\ \mathbf{p}_{012} + t\ \mathbf{p}_{123}$

- Subdivides curve at $\mathbf{p}_{0123}$
  - $\mathbf{p}_0\ \mathbf{p}_{01}\ \mathbf{p}_{012}\ \mathbf{p}_{0123}$
  - $\mathbf{p}_{0123}\ \mathbf{p}_{123}\ \mathbf{p}_{23}\ \mathbf{p}_3$
- Repeated subdivision converges to curve