CS 418: Interactive Computer Graphics — Spring 2018
In-class Worksheet: Bezier Techniques

Bezier Curves

Cubic Bezier Curves

A Bezier curve is a parametric polynomial curve given by:

$$X(t) = (1-t)^3 b_0 + \overset{t}{3}(1-t)^2 b_1 + 3(1-t)t^2 b_2 + t^3 b_3$$

where b_i are the control points.

The tangent vector of the curve can be found by
$$X(t) = 3(b_1 - b_0)(1-t)^2 + 6(b_2 - b_1)(1-t)t + 3(b_3 - b_2)t^2$$

1. **The de Casteljau Algorithm**
   Suppose our control points are

   $b_0=(-1,0)$ $b_1=(0,1)$ $b_2=(0,-1)$ $b_3=(1,0)$

   Use the de Casteljau algorithm to find the coordinates of X(1/4). Check that you get the same answer from using the parametric expression given above.

See next page

$$b_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \tfrac{3}{4}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \tfrac{1}{4}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3/4 \\ 1/4 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \tfrac{3}{4}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \tfrac{1}{4}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2/4 \end{pmatrix}$$

$$b_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \tfrac{3}{4}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \tfrac{1}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ -3/4 \end{pmatrix}$$

$$b_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\underbrace{\tfrac{3}{4}\begin{pmatrix} -3/4 \\ 1/4 \end{pmatrix} + \tfrac{1}{4}\begin{pmatrix} 0 \\ 2/4 \end{pmatrix}} = \begin{pmatrix} -9/16 \\ 5/16 \end{pmatrix}$$

$$\underbrace{\tfrac{3}{4}\begin{pmatrix} 0 \\ 2/4 \end{pmatrix} + \tfrac{1}{4}\begin{pmatrix} 1/4 \\ -3/4 \end{pmatrix}} = \begin{pmatrix} 1/16 \\ 3/16 \end{pmatrix}$$

$$\boxed{\tfrac{3}{4}\begin{pmatrix} -9/16 \\ 5/16 \end{pmatrix} + \tfrac{1}{4}\begin{pmatrix} 1/16 \\ 3/16 \end{pmatrix} = \begin{pmatrix} -26/64 \\ 18/64 \end{pmatrix}}$$

Check: $\frac{27}{64}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{27}{64}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{9}{64}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \frac{1}{64}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} -26/64 \\ 18/64 \end{pmatrix}$$

Intentionally blank

2. Tangents to a Bezier Curve

a. What are the tangents at the controls b_0 and b_3?
Give the answer as a pair of parameterized functions.

at b_0 $\quad t=0 \quad 3(b_1-b_0)(1-t)^2$

$$= \boxed{3(b_1-b_0)}$$

at b_3 $\quad t=1 \quad ? \quad \Delta \quad 3(b_3-b_2)t^2$

$$= \boxed{3(b_3-b_2)}$$

b. What is the tangent vector at t=0.25 for the curve given in question one?

$b_1-b_0=\begin{pmatrix}1\\1\end{pmatrix} \quad b_2-b_1=\begin{pmatrix}0\\-2\end{pmatrix} \quad b_3-b_2=\begin{pmatrix}1\\1\end{pmatrix}$

$\Delta b_0 \quad \Delta b_1 \quad \Delta b_2$

$$3\left(\Delta b_0\left(9/16\right)+2\Delta b_1\left(\tfrac{3}{16}\right)+\Delta b_2\left(1/16\right)\right)$$

$$=3\left(\begin{pmatrix}9/16\\9/16\end{pmatrix}+\begin{pmatrix}0\\-6/8\end{pmatrix}+\begin{pmatrix}1/16\\1/16\end{pmatrix}\right)=\boxed{\begin{pmatrix}30/16\\-6/16\end{pmatrix}}$$

Simple Newtonian Physics for Graphics
Position update: $p_{new} = p + \dot{p}t$
Velocity update: $\dot{p_{new}} = \dot{p} + \ddot{p}t$
in that order…

Alternative (not on the exam)
Finding position as the second integral of acceleration:
$$p_{new} = p + \dot{p}t + \ddot{p}\frac{t^2}{2}$$

3. **Newtonian Physics**
Suppose we have an initial particle position of (1,2,3) and velocity of < 1, -1, 2 > per second and constant acceleration < 0, 1, -1 > per second per second.
   a. What is the position after 5 timesteps each with t=1, using the update equations?

p	$(1,2,3)$	$(2,1,5)$	$(3,1,6)$	$(4,2,6)$	$(5,4,5)$
$\dot{p} =$	$\langle 1,-1,2 \rangle$	$\langle 1,0,1 \rangle$	$\langle 1,1,0 \rangle$	$\langle 1,2,-1 \rangle$	$\langle 1,3,-2 \rangle$
$t =$	0	1	2	3	4

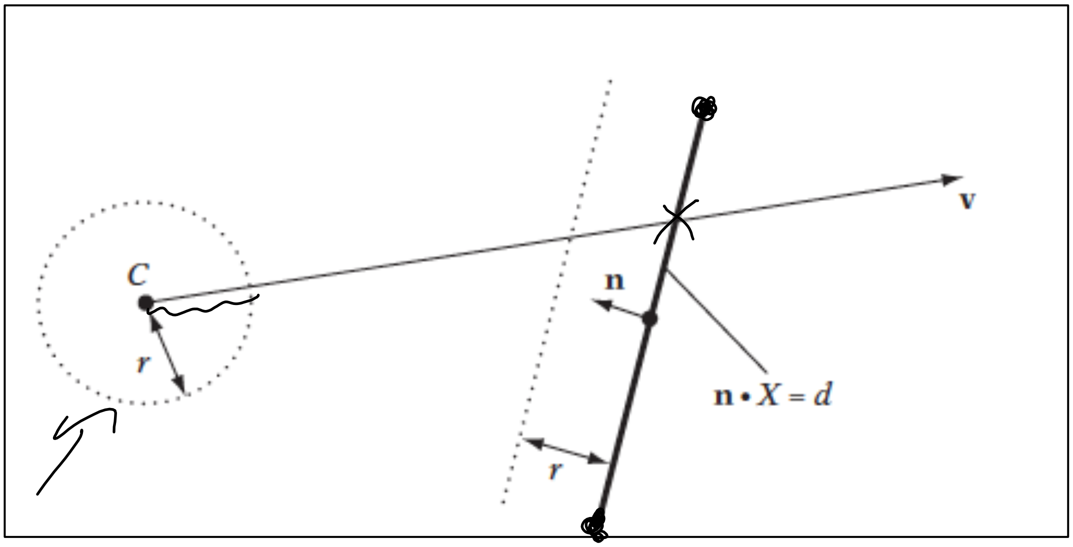
$\boxed{(6,7,3)}$

   b. What is the position given by ~~integrating~~ acceleration twice with t=5?

$$(1,2,3) + \langle 5,-5,10 \rangle + \langle 0, \tfrac{25}{2}, -\tfrac{25}{2} \rangle$$
$$= \left(6, \tfrac{25}{2} - \tfrac{6}{2}, \tfrac{26}{2} - \tfrac{25}{2}\right) = \boxed{\left(6, \tfrac{19}{2}, \tfrac{1}{2}\right)}$$

Different result due to velocity being discrete in a and continuous in b

4. Collision Detection



The position of the sphere center C moving with velocity v is given by $C + tv$ and the equation for a plane can be written as $n \cdot X = d$ where *n* is the normal to the plane and *d* is a constant…meaning that all points *X* satisfying that equation are on the plane.

Without referring to any notes or slides, derive a formula to find the time *t* that the sphere will collide with plane.

$$C + t(v)$$

$$n \cdot (C + tv) = d \pm r$$

$$n \cdot tv = (d \pm r) - (n \cdot C)$$

$$\boxed{t = \frac{(d \pm r) - n \cdot C}{n \cdot v}}$$