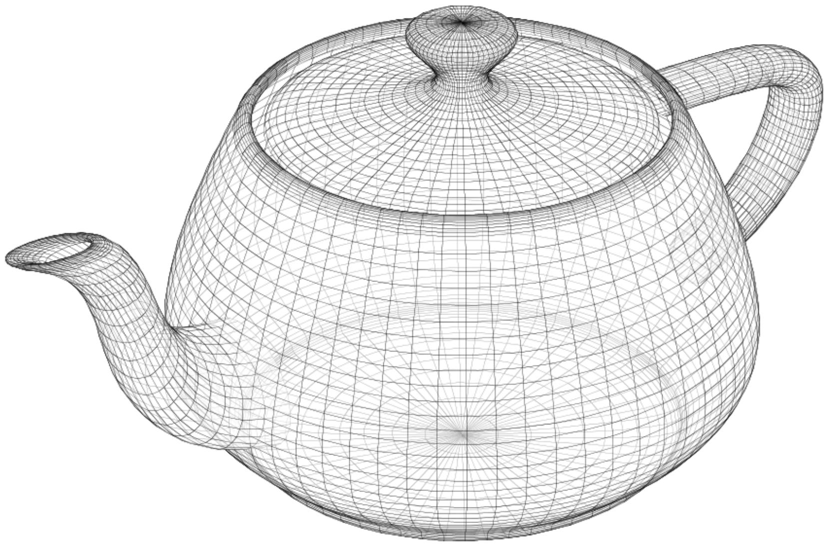

Ray-Sphere Intersection

Production Computer Graphics

Professor Eric Shaffer

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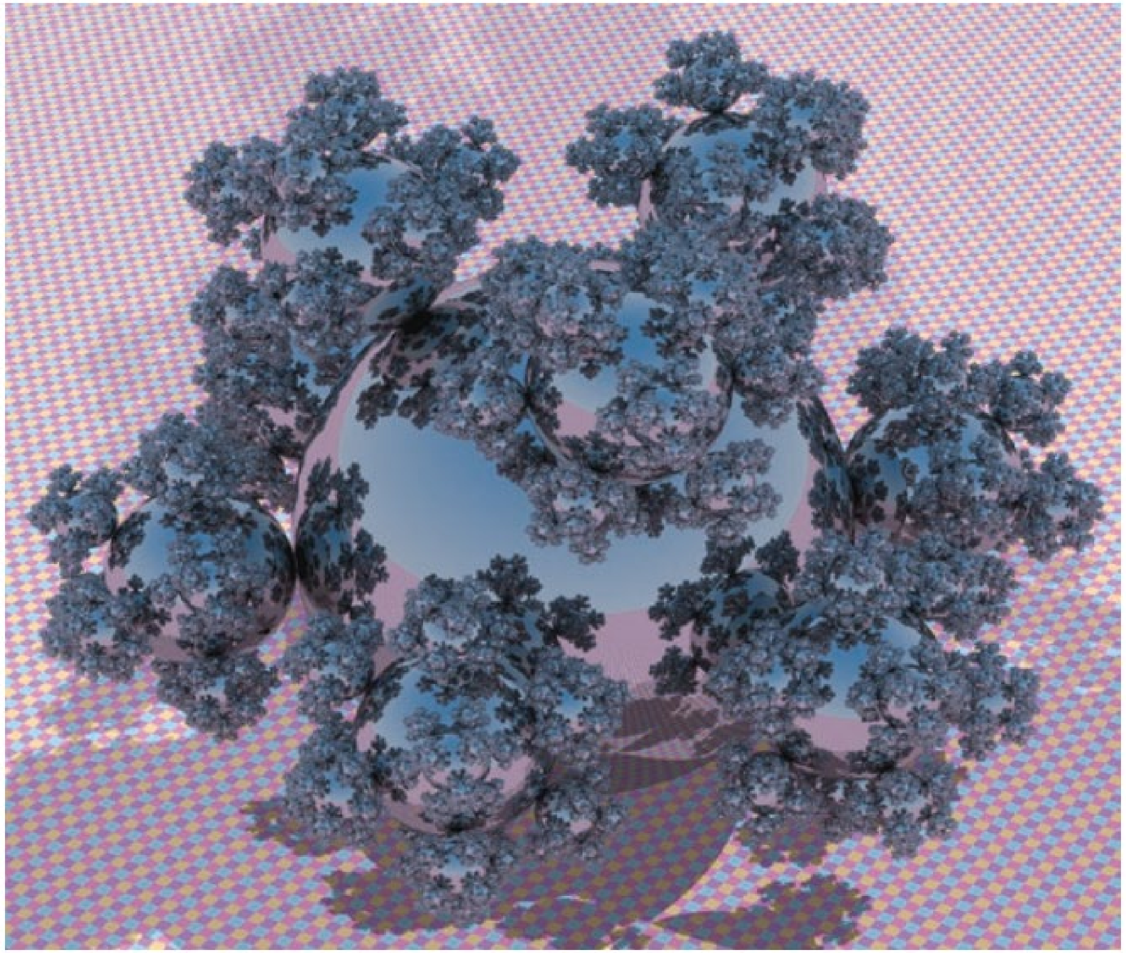
Ray Tracing Spheres



48 million spheres

Spheres are immensely popular in ray traced scenes

Determining a ray-sphere intersection is pretty simple…

Involves solving a quadratic equation

We will look at

- A derivation of this solution
- How to find the intersection robustly

Robustly *implies that the solution does not produce wildly different results for a small change in the input and so is resistant to error*

Ray-Sphere Intersection

A sphere can be defined by a center G and radius r.

The surface of the sphere is all points P for which

$$(P-G)\cdot(P-G)=r^2.$$

We can replace P with the ray equation $R(t) = O + t\mathbf{d}$

And let $\mathbf{f} = O - G$

Then, solve for t $\underbrace{(\mathbf{d}\cdot\mathbf{d})}_{a}t^2 + \underbrace{2(\mathbf{f}\cdot\mathbf{d})}_{b}t + \underbrace{\mathbf{f}\cdot\mathbf{f} - r^2}_{c} = at^2 + bt + c = 0$

$$t_{0,1} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

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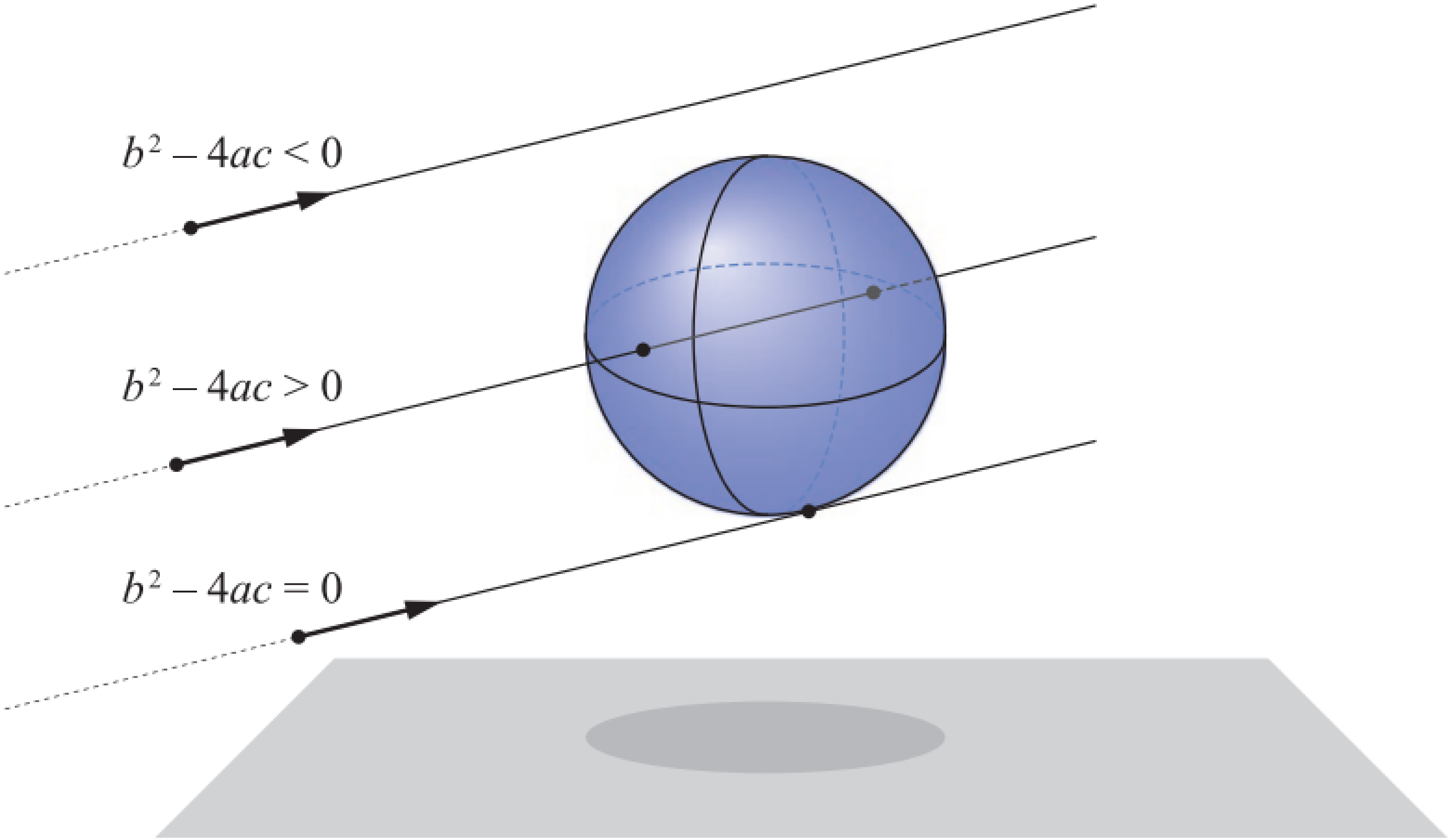
How Many Intersections?



Computing the Intersection Point and Normal

Plug the t value into the ray equation to find the intersection point(s)

$$P_{0,1} = R(t_{0,1}) = O + t_{0,1}\mathbf{d}$$

t_0 generates intersection point P_0
t_1 generates intersection point P_1

We can compute a unit length normal vector at the intersection point

Here, we are using P_0

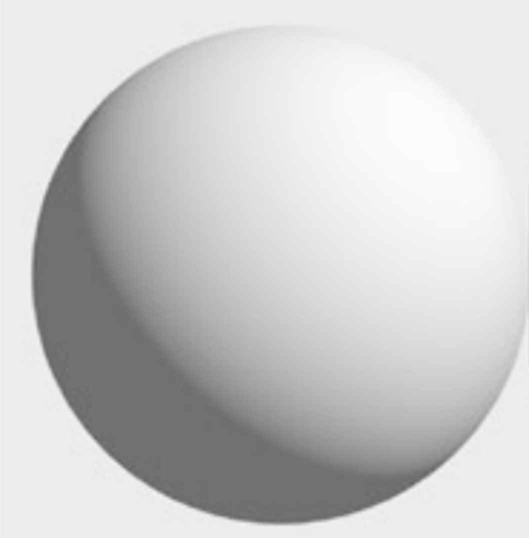
$$\hat{\mathbf{n}} = \frac{P_0 - G}{r}$$

G is the sphere center and the radius is r

Floating Point Precision Considerations

32 bit floating can fail to yield correct answers for ray-sphere intersection

One case is when sphere is small relative to distance to the ray origin



Unit sphere 100 units of distance away from an orthographic camera.

Unit sphere 4100 units of distance away from an orthographic camera…it has totally disappeared

Why?

Floating point number is $2 \times s^e$

s is the significand and e the exponent

- For floating addition and subtraction the exponents must match
- The bits of s for the smaller number are shifted right…so the exponent can be raised.
  - Those rightmost bits are lost, reducing the accuracy of the number
- 32 bit floats have 24 bits in the significand
  - Adding a number $2^{24} \approx 10^7$ smaller to a large number will not change that number

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Diminished Significance in Ray Tracing

$$\underbrace{(\mathbf{d}\cdot\mathbf{d})}_{a}t^2 + \underbrace{2(\mathbf{f}\cdot\mathbf{d})}_{b}t + \underbrace{\mathbf{f}\cdot\mathbf{f}-r^2}_{c} = at^2+bt+c=0$$

The term $c = \mathbf{f}\cdot\mathbf{f} - r^2$ is problematic

Since $\mathbf{f}\cdot\mathbf{f} = \| O - G \|^2$

…both terms are squared which halves the available precision

If a sphere is more than a distance of $2^{12}r = 4096r$ away from the ray origin
…then the radius will have no impact on the intersection solution

Artifacts in the rendered image will appear at shorter distances since so few bits of value remain

A Better Formulation

$$b^2-4ac=4a\left(\frac{b^2}{4a}-c\right)$$

$$=4\mathbf{d}^2\left(\frac{(\mathbf{f}\cdot\mathbf{d})^2}{\|\mathbf{d}\|^2}-\left(\mathbf{f}^2-r^2\right)\right)$$

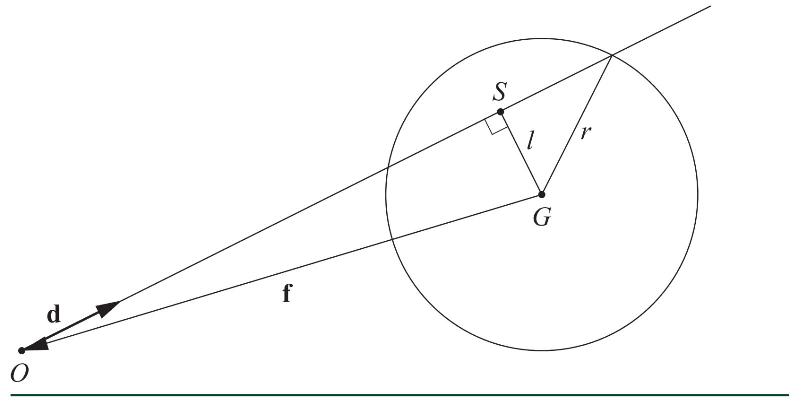
$$=4\mathbf{d}^2(r^2-(\underbrace{\mathbf{f}^2-(\mathbf{f}\cdot\hat{\mathbf{d}})^2}_{l^2}))$$

$$=4\mathbf{d}^2\left(r^2-(\mathbf{f}-(\mathbf{f}\cdot\hat{\mathbf{d}})\hat{\mathbf{d}})^2\right).$$

The perpendicular distance *l* of center G to the ray can be found

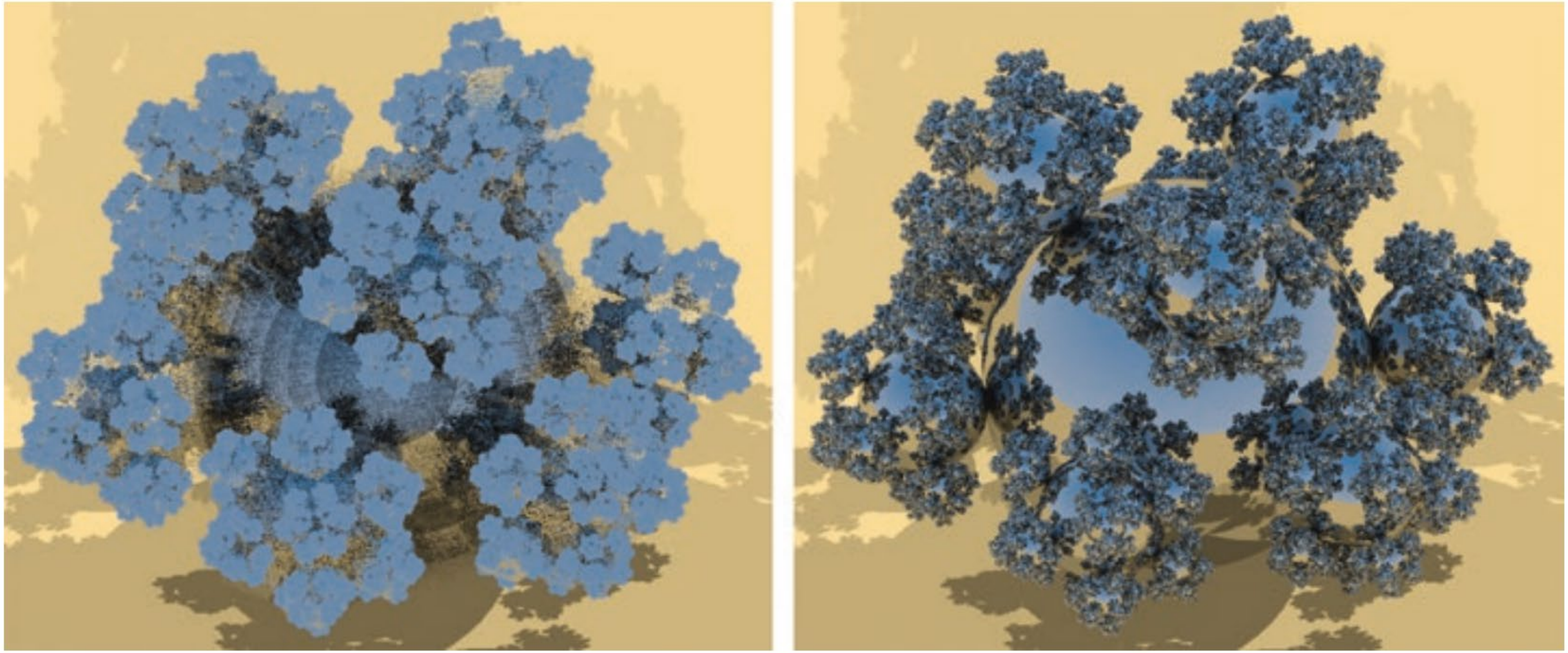
- by the Pythagorean theorem $\mathbf{f}^2=l^2+\left(\mathbf{f}\cdot\hat{\mathbf{d}}\right)^2$
  or
- length of f minus the vector from the ray origin to the foot of the perpendicular

$$S=O+\left(\mathbf{f}\cdot\hat{\mathbf{d}}\right)\hat{\mathbf{d}}$$



This second way is much more precise
The vectors are subtracted before they are squared in the dot product.

Results



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Catastrophic Cancellation

Occurs when subtracting nearly equal numbers

- Many significant bits eliminate each other
- Few meaningful bits remain

We have approximations

$\tilde{x} = x(1 + \delta_x)$ and $\tilde{y} = y(1 + \delta_y)$

Small relative errors

$|\delta_x| = |x - \tilde{x}|/|x|$ and $|\delta_y| = |y - \tilde{y}|/|y|$

Relative error of the difference can be arbitrarily large

$$\left|\frac{x\delta_x - y\delta_y}{x - y}\right|$$

Catastrophic Cancellation Solving Quadratics

Can occur when $b \approx \sqrt{b^2 - 4ac}$ when solving $t_{0,1} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

This happens when a huge sphere is intersected close to the ray origin

Catastrophic cancellation happens only for one of the two quadratic solutions
We can compute that solution with higher precision using the identity

$$t_0 t_1 = \frac{c}{a}$$

$$\begin{cases} t_0 = \frac{c}{q}, \\ t_1 = \frac{q}{a}, \end{cases} \quad \text{where} \quad q = -\frac{1}{2}\left(b + \operatorname{sign}(b)\sqrt{b^2 - 4ac}\right)$$

The sign function returns 1 if b>0 and -1 otherwise

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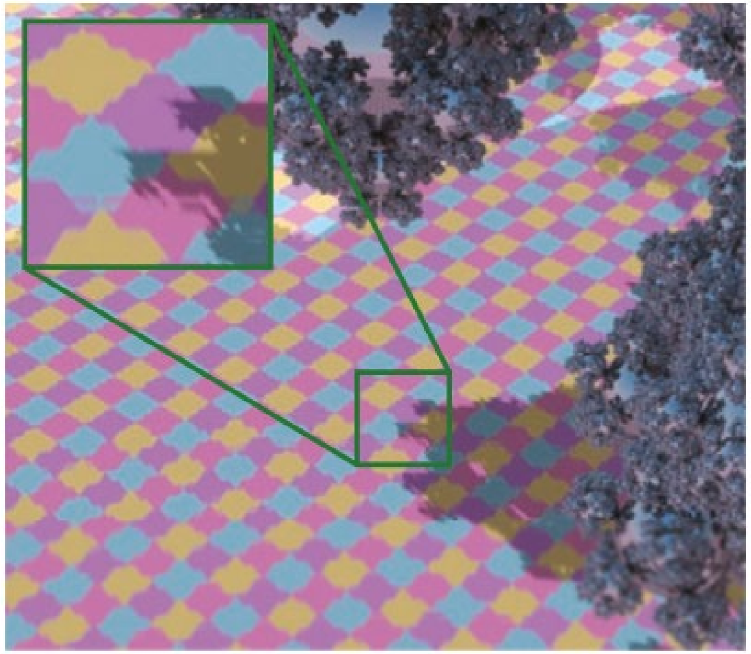
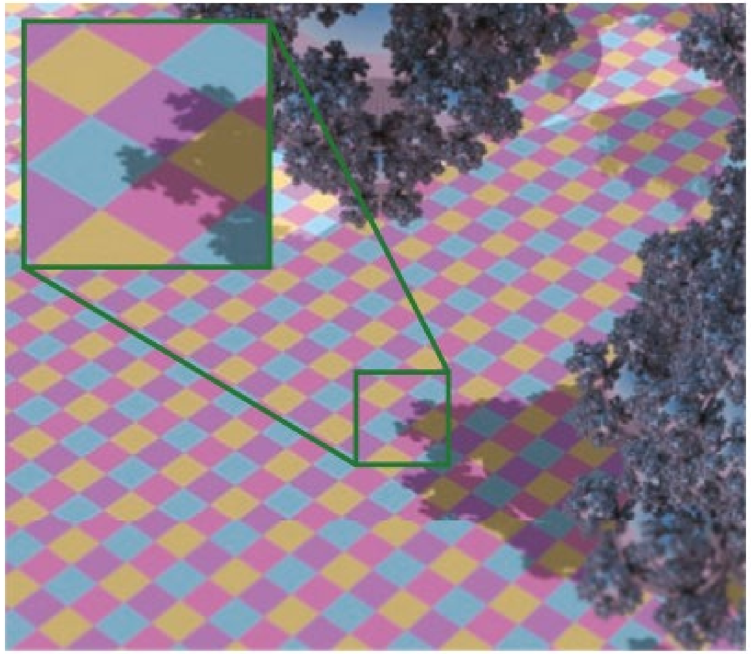
Result

Ground is a huge sphere





Left is traditional solver

Right is the more stable version