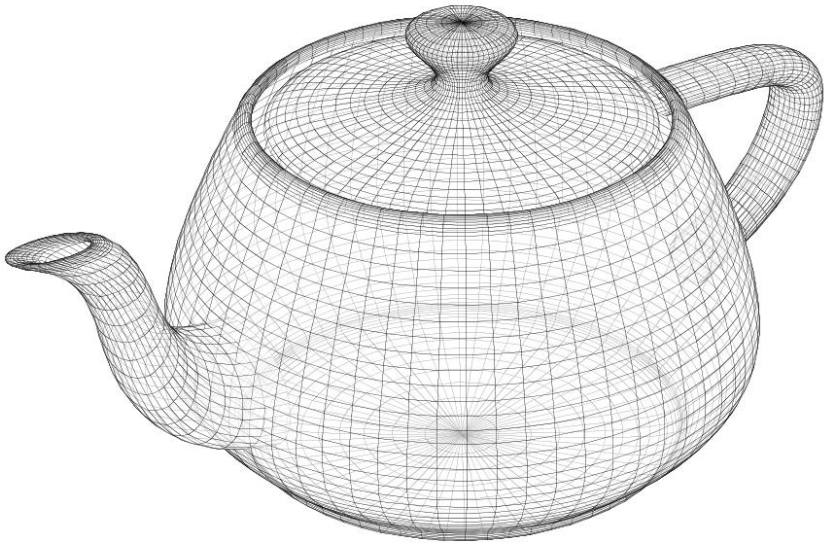

# Animating Fluids: Theory

Production Computer Graphics

Eric Shaffer

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# Fluid Simulation

Imagine the fluid is made of particles

We could animate and render the fluid if we knew some function ***f(t,p)***

The for any time ***t*** could tell us the position of particle ***p***

So…what is that function?

# The Navier-Stokes Equations

Describe the motion of fluid

Same way that $F = ma$ can be said to describe the motion of a particle

The motion of an incompressible fluid can be described by

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla p = \vec{g} + v \nabla \cdot \nabla \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

Can model ocean currents, weather, air flow over a wing…

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# The Navier-Stokes Equations

It is a set of partial differential equation

A solution of the equations would be a function telling us position

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla p = \vec{g} + v \nabla \cdot \nabla \vec{u}$$

$$\nabla \cdot \vec{u} = 0$$

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# Gifted (2017) (arguably the best Navier-Stokes movie ever)

# Very Famous Equations

Despite their wide range of practical uses, it has not yet been proven whether solutions always exist in three dimensions and, if they do exist, whether they are smooth – i.e. they are infinitely differentiable at all points in the domain. These are called the **Navier–Stokes existence and smoothness problems**. The Clay Mathematics Institute has called this one of the seven most important open problems in mathematics and has offered a US$1 million prize for a solution or a counterexample. -- Wikipedia

# Millennium Problems

## Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

## Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

## P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

## Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

## Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

## Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

## Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

The Poincaré Conjecture was solved by Grigori Perelman in 2003…he declined the prize from the Clay Institute. It remains the only solved Millenium Problem.

# Understanding the Equations

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla p = \vec{g} + v \nabla \cdot \nabla \vec{u}$$

$\vec{u}$ is the velocity of the fluid

$\rho$ is the density of the fluid…1000 kg/m$^3$ for water

p is pressure

$\vec{g}$ is body forces…like acceleration due to gravity

$v$ is kinematic viscosity…measures resistance to deformation

# The Momentum Equation

Tells us how the fluid accelerates due to forces acting on it

Let's try to derive it from $F = ma$

We will rewrite this as $m\frac{D\vec{u}}{Dt} = \vec{F}$

The D stands for the ***material derivative***
....which we will totally ignore for right now.

# The Momentum Equation: Pressure

The first force we will model is pressure

Pressure imbalance results in a force pushing from high to low pressure

We measure this using the negative gradient of the pressure

$$-\nabla p = -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$$

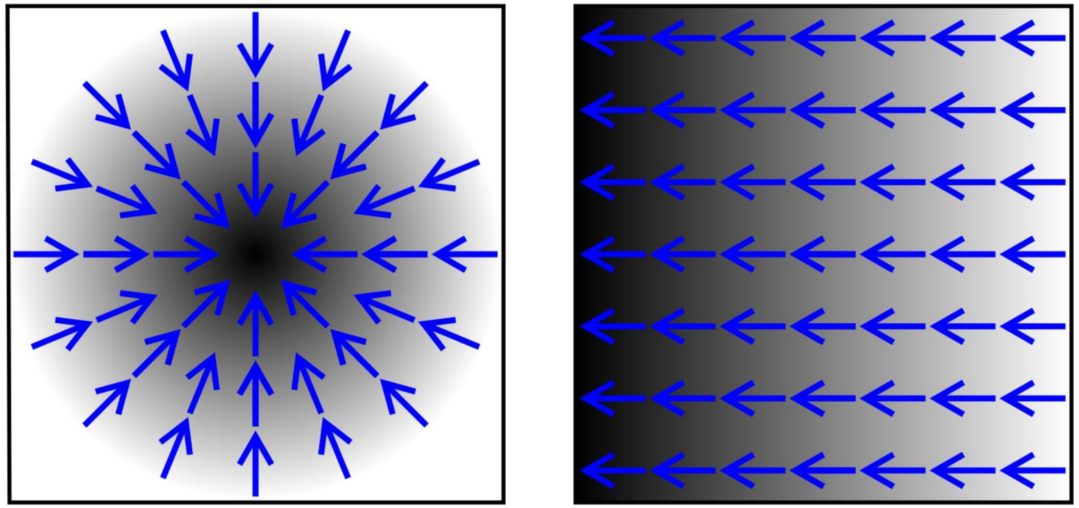
Pressure is a scalar field

# The Momentum Equation: Pressure

Pressure is a scalar field

The negative gradient points in the direction of swiftest descent

$$-\nabla p = -\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$$



# The Momentum Equation: Pressure

To get the total force we need to integrate over volume of the particle

As an approximation, we will multiply by the volume V

$$m\frac{D\vec{u}}{Dt} = -V\nabla p + \vec{F}$$

# The Momentum Equation: Viscosity

Viscosity is another fluid force…it is like friction

A viscous fluid resists deformation

This force tries to make a particle move at same velocity as neighbors

Tries to minimize difference with average velocity of neighbors

# The Laplacian

The Laplacian $\nabla \cdot \nabla$ is a differential operator

Measures how far a quantity is from the average around it

We integrate over the volume of the particle

And multiply by the dynamic viscosity coefficient

So we now have $m\frac{D\vec{u}}{Dt} = -V\nabla p + V\mu\nabla \cdot \nabla\vec{u} + \vec{F}$

***Dynamic*** means we are getting a force out of it, ***kinematic*** means we are getting an acceleration

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# The Momentum Equation: Body Forces

We need to add in the gravitational force…and we're done

$$m\frac{D\vec{u}}{Dt} = m\vec{g} - V\nabla p + V\mu\nabla\cdot\nabla\vec{u}$$

Actually, we'd like to consider a continuum model

Infinite number of particles and particle size approaches 0

Divide the equations by Volume and take the limit

$$\rho\frac{D\vec{u}}{Dt} = \rho\vec{g} - \nabla p + \mu\nabla\cdot\nabla\vec{u}$$

$$\rho = \frac{m}{V}$$

Density is mass divided by volume

# The Momentum Equation

Divide by density

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho}\nabla p = \vec{g} + v\nabla \cdot \nabla\vec{u}$$

$$v = \mu / V$$

kinematic viscosity is dynamic viscosity divided by volume

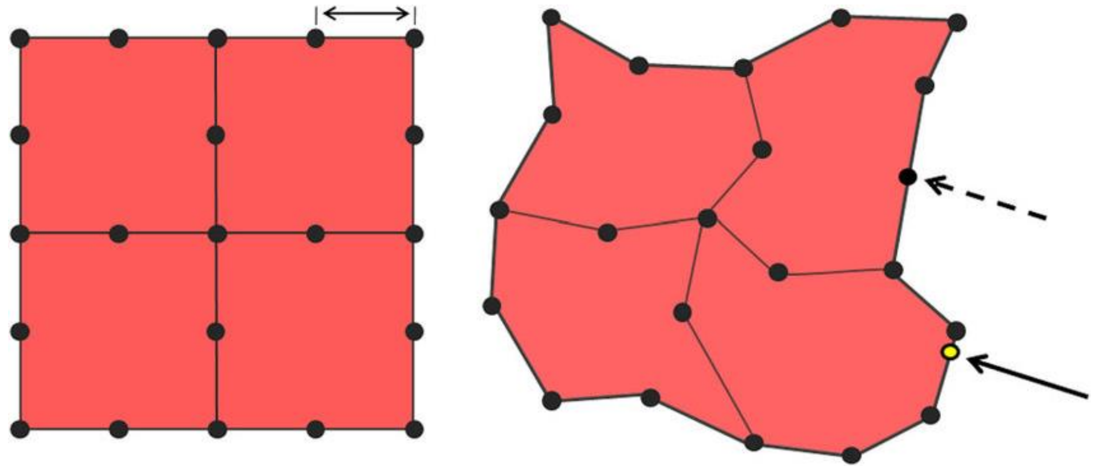
# Lagrangian  Approach

Fluid is made up of particles

Each particle has a position and velocity

Solid are simulated this way…particles connected as a mesh

Fluids can be as well…

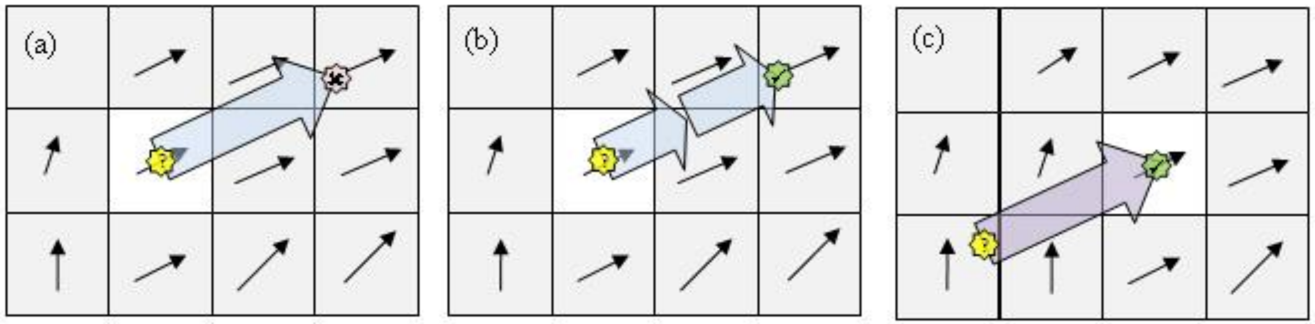


# Eulerian Approach

Used for fluids more often

Space is a grid

Measure fluid quantities in grid cells or at grid vertices

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# The Material Derivative

Suppose each particle has some quantity q

How fast is q changing as a function of time?

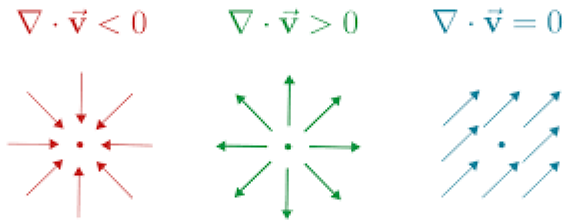
$$\frac{d}{dt}q\big(t,\vec{x}(t)\big) = \frac{\partial q}{\partial t} + \nabla q \cdot \frac{d\vec{x}}{dt} = \frac{\partial q}{\partial t} + \nabla q \cdot \vec{u} = \frac{Dq}{Dt}$$

Can think of the two terms as measuring:

1) How fast q is changing at a fixed point in space
2) A term correcting for change caused by fluid flowing past the point

# Divergence

$$\operatorname{div} \vec{\mathbf{v}} = \nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \cdots$$




In vector calculus, **divergence** is a vector operator that produces a scalar field, representing the volume density of the outward flux of a vector field from an infinitesimal volume around a given point. As an example, consider air as it is heated or cooled. The velocity of the air at each point defines a vector field. While air is heated in a region, it expands in all directions, and thus the velocity field points outward from that region. The divergence of the velocity field in that region would thus have a positive value. While the air is cooled and thus contracting, the divergence of the velocity has a negative value.
--Wikipedia

# Incompressibility

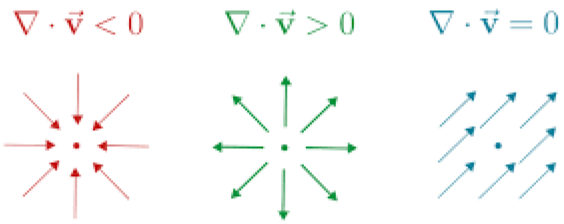
Real fluids change their volume…but usually not by much

Simpler to model incompressible fluids

These are fluids for which the velocity field is divergence-free

$$\nabla \cdot \vec{u} = 0$$

$$\boxed{\operatorname{div} \vec{\mathbf{v}} = \nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \cdots}$$

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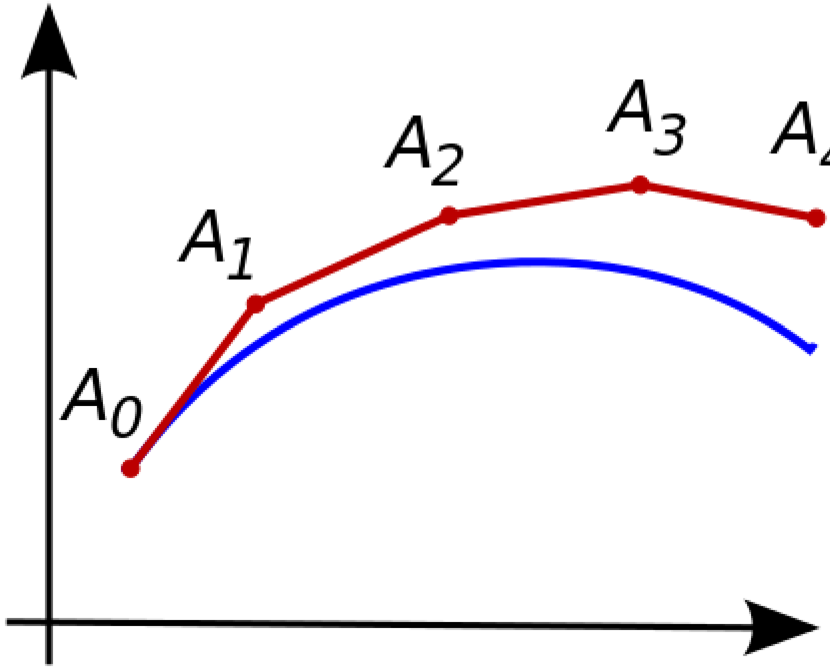
# Solving the Equations

The equations are solved numerically

Like using Euler Integration…but more complicated and better methods




$$y_{n+1} = y_n + hf(t_n, y_n)$$

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