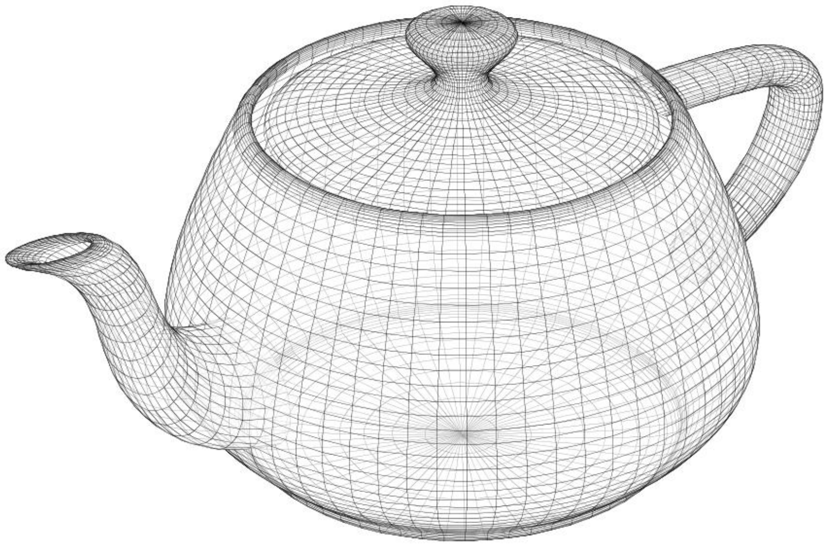
Monte Carlo Methods



Production Computer Graphics

Eric Shaffer

ILLINOIS

directions for making pictures using numbers
(explained using only the ten hundred words people use most often)

position on the stuff

direction towards the eye

light made by the stuff (sometimes because it is very hot)

the answer to how much light from an interesting direction that will keep going in the direction towards the eye, after hitting stuff at the position (this is easy for mirrors, not so easy for everything else)

the light that comes from an interesting direction towards the position on the stuff

how much the light becomes less bright because the stuff leans away from the interesting direction

$$L_o(x, \omega_o) = L_e(x, \omega_o) + \int_\Omega f_r(x, \omega_i \rightarrow \omega_o)\, L_i(x, \omega_i)\, (\omega_i \cdot n)\, d\omega_i$$

light that leaves the position on the stuff and reaches the eye

light can be added said a man who sat under a tree many years ago

for lots of interesting directions inside half a ball facing up from the stuff, add up all the answers in between

this idea came from http://xkcd.com/1133/

@levork

The Power of Randomization

Randomization is an important tool in algorithm design.

Las Vegas algorithm:

- uses randomness but always yields same result for same input

Monte Carlo algorithm:

- give different results depending on “random” inputs used
- …give right answer on average

Expected Value and Variance

- expected value: average value of the variable

$$E[x] = \sum_{j=1}^{n} x_j p_j$$

- variance: variation from the average

$$\sigma^2[x] = E[(x - E[x])^2] = E[x^2] - E[x]^2$$

throwing a die

- expected value: $E[x] = (1 + 2 + \cdots + 6)/6 = 3.5$
- variance: $\sigma^2[x] = 2.916$

ILLINOIS

Estimated E[x]

- to estimate the expected value, choose a set of random values based on the probability and average the results

$$E[x] = \frac{1}{N}\sum_{j=1}^{N} x_i$$

- bigger N gives better estimates

throwing a die

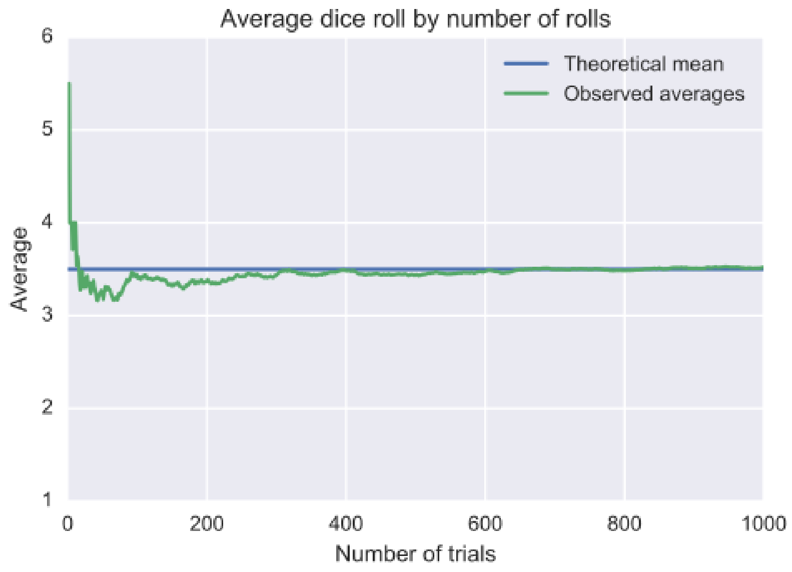
- 3 rolls: $3, 1, 6 \to E[x] \approx (3+1+6)/3 = 3.33$
- 9 rolls:
  $3, 1, 6, 2, 5, 3, 4, 6, 2 \to E[x] \approx (3+1+6+2+5+3+4+6+2)/9 = 3.51$

Law of Large Numbers

As the number of samples goes to infinity

- the error goes to zero
- the answer converges to the correct number

$$P(E[x] = lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i) = 1$$



Continuous Version

- expected value

$$E[x] = \int_a^b x\rho(x)\,dx$$

$$E[g(x)] = \int_a^b g(x)\rho(x)\,dx$$

Expectation is an integral !

- variance

$$\sigma^2[x] = \int_a^b (x - E[x])^2 \rho(x)\,dx$$

$$\sigma^2[g(x)] = \int_a^b (g(x) - E[g(x)])^2 p(x)\,dx$$

- estimating the expected value

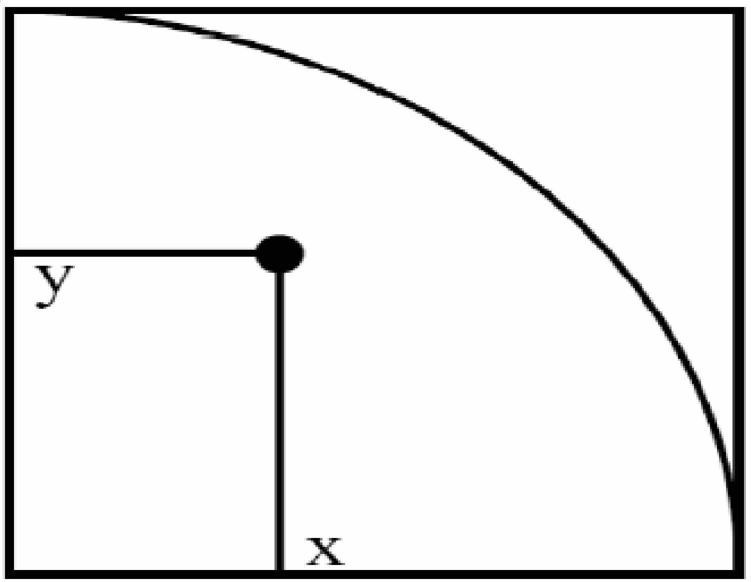
$$E[g(x)] \approx \frac{1}{N}\sum_{i=1}^{N} g(x_i)$$

2D Example…computing π

Use the unit square $[0,1]^2$ with a quarter-circle

$$f(x,y) = \begin{cases} 1 & (x,y) \in \; circle \\ 0 & else \end{cases}$$

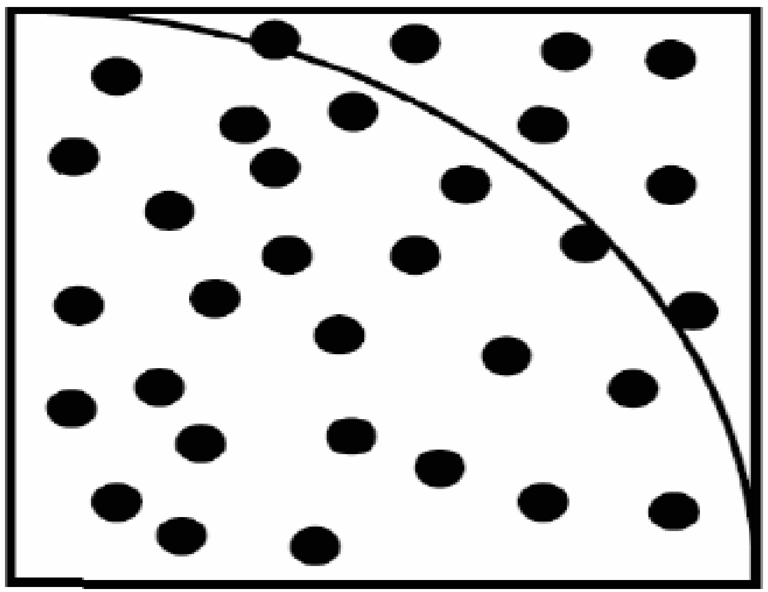
$$A_{quarter-circle} = \int_0^1 \int_0^1 f(x,y)\, dxdy = \frac{pi}{4}$$



2D Example…computing π

Estimate the area of the circle by randomly evaluating $f(x, y)$

$$A_{quarter-circle} \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i, y_i)$$



2D Example…computing π

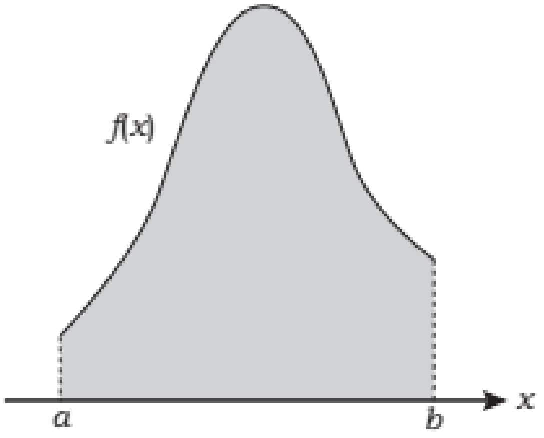
By definition

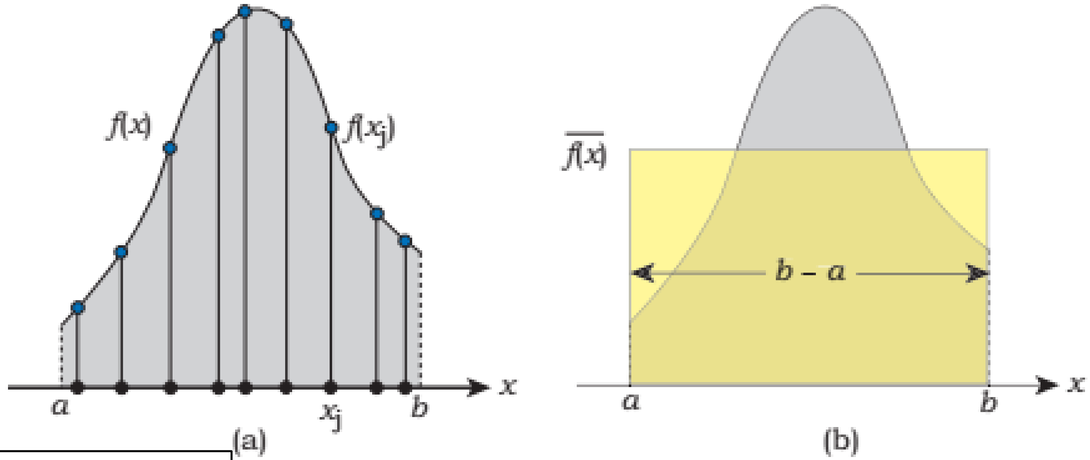
$$A_{quarter-circle} = \pi/4$$

so

$$\pi \approx \frac{4}{N}\sum_{i=1}^{N} f(x_i, y_i)$$

1D Example





$$I = (b - a)\ \overline{f(x)}.$$

$$\overline{f(x)} \approx \frac{1}{n}\sum_{j=1}^{n} f(x_j).$$

$$I = \int_a^b f(x)\,dx$$

$$\langle I \rangle = \frac{b - a}{n} \sum_{j=1}^{n} f(x_j)$$

We can estimate the value of this integral by evaluating f(x) at n uniformly distributed random values of x in the interval [a, b]

The yellow rectangle has the same area as the integral. The height of the rectangle is the average value of f (x) in the interval [a, b].

Monte Carlo estimator for integral I

ILLINOIS

Central Limit Theorem

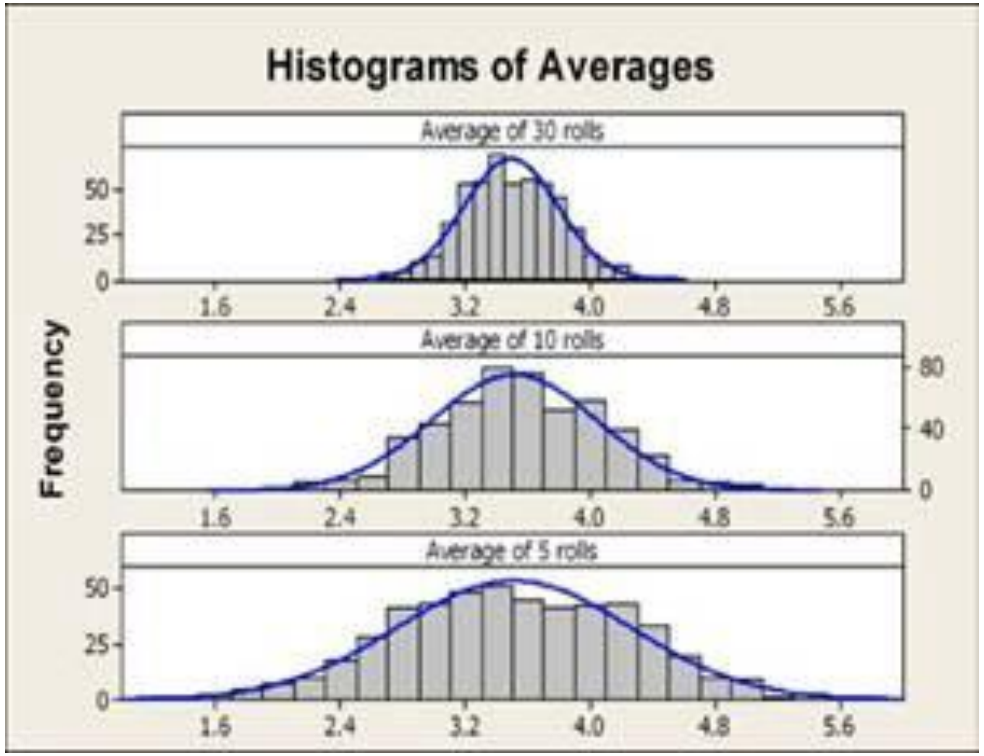
If you are sampling an average, the distribution of the average approaches the normal distribution, even if the the distribution being sampled from is not normal

- N is the number of samples used to compute the average

- As N approaches infinity,
  the estimate lies in a narrower band around the expected value of the integral with higher probability

- Within three standard deviations 99.7% of the time
- Standard deviations vary as $1/\sqrt{N}$

Larger N Reduces Variance



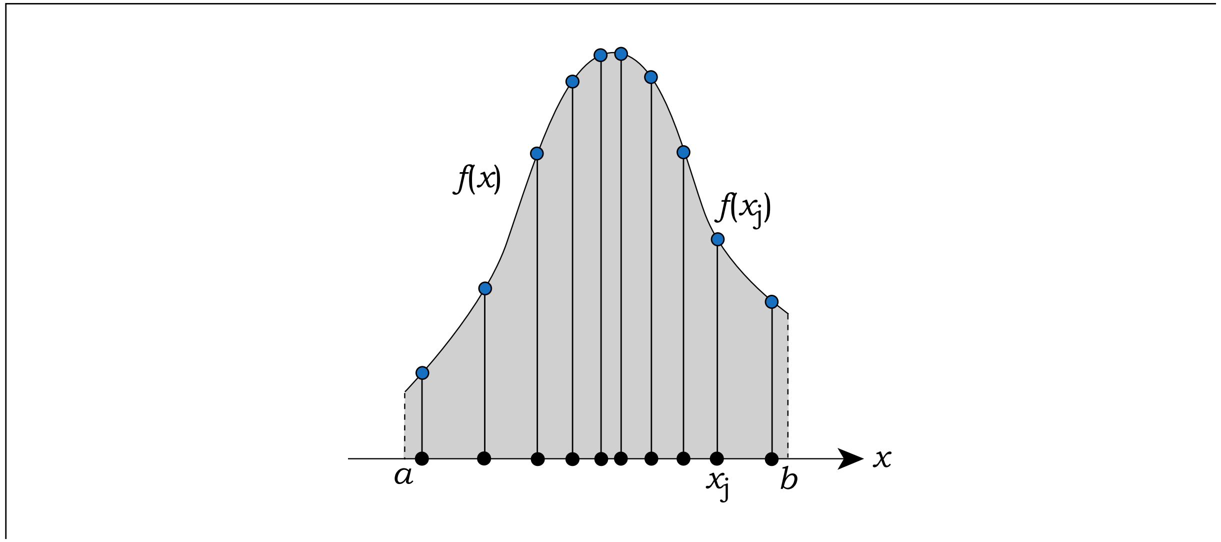
Average value on the N rolls of 1 die

Variance Reduction

- Variance $V=\sigma^2$
- Based on what we've seen, if we wanted to cut error in an N sample estimate in half how many samples would we need to take?
- Monte Carlo methods converge slowly
  - But often are the only realistic option
- Variance reduction techniques can help

Importance Sampling

- Do not sample uniformly
- Sample with a density that has a similar shape to the shape of f(x)



Summary: Monte Carlo Integration

- Computing the expected value of a function is an integral
- We can estimate expected value using random sampling
  - By the Law of Large Numbers
  - So…we can estimate an integral by sampling
- Each integral estimate is an average
- These averages form a normal distribution
  - By the Central Limit Theorem
- The estimate will fall within 1 standard dev. with high probability
  - Can reduce the standard deviation by using more samples in the avg.
  - Error reduction behaves like the function $\frac{1}{\sqrt{N}}$

Stochastic Simulation with MC

Two requirements for Monte Carlo

- Know which probability distribution you need to sample
- Generate sufficiently random numbers

Random Numbers

- Most Random Number Generators (RNGs) are ***pseudo-random***
- You can get faster convergence with a ***quasi-random*** sequence
  - Doesn’t clump, samples more or less uniform across domain
    - Hammersley and Halton are examples