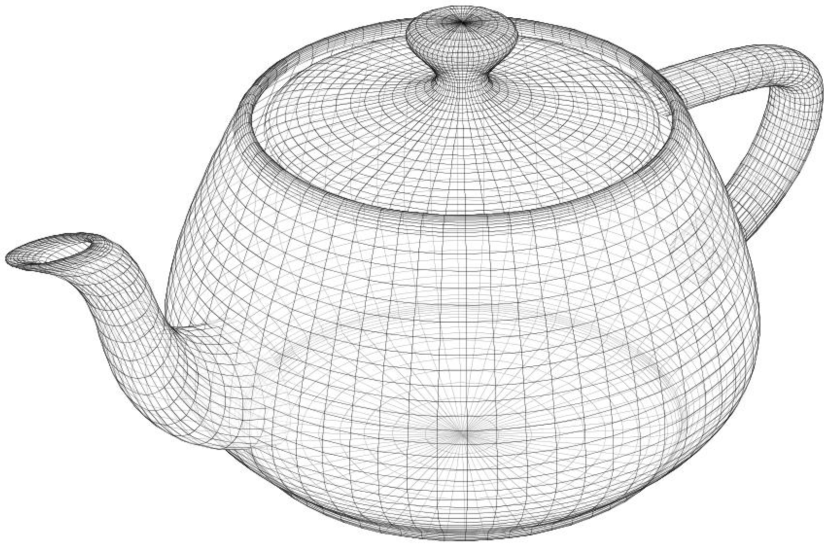


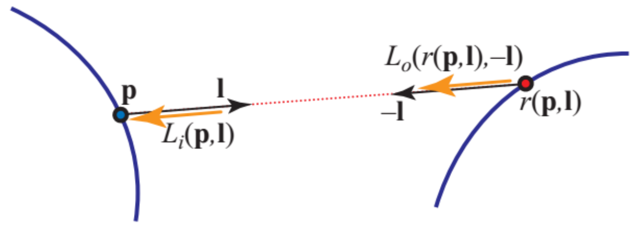
The Rendering Equation

Production Computer Graphics

Eric Shaffer

ILLINOIS

The Rendering Equation



In 1986 James Kajiya developed an integral equation to model rendering

Describes the intensity of light transported between two points

The model is the basis for rendering using Monte Carlo integration

This approach can generate
- Motion blur
- Soft shadows
- Depth of field
- Global illumination
- …and other effects

THE RENDERING EQUATION

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ABSTRACT. We present an integral equation which generalizes a variety of known rendering algorithms. In the course of discussing a monte carlo solution we also present a new form of variance reduction, called Hierarchical sampling and give a number of elaborations shows that it may be an efficient new technique for a wide variety of monte carlo procedures. The resulting rendering algorithm extends the range of optical phenomena which can be effectively simulated.

KEYWORDS: computer graphics, raster graphics, ray tracing, radiosity, monte carlo, distributed ray tracing, variance reduction.

CR CATEGORIES: I.3.3, I.3.5, I.3.7

1. The rendering equation

The technique we present subsumes a wide variety of rendering algorithms and provides a unified context for viewing them as more or less accurate approximations to the solution of a single equation. That this should be so is not surprising once it is realized that all rendering methods attempt to model the same physical phenomenon, that of light scattering off various types of surfaces.

We mention that the idea behind the rendering equation is hardly new. A description of the phenomenon simulated by this equation has been well studied in the radiative heat transfer literature for years [Siegel and Howell 1981]. However, the form in which we present this equation is well suited for computer graphics, and we believe that this form has not appeared before.

The rendering equation is

$$I(x,x') = g(x,x')\left[\epsilon(x,x') + \int_S \rho(x,x',x'')I(x',x'')dx''\right]. \quad (1)$$

where:

$I(x,x')$ is the related to the intensity of light passing from point x' to point x
$g(x,x')$ is a "geometry" term
$\epsilon(x,x')$ is related to the intensity of emitted light from x' to x
$\rho(x,x'x'')$ is related to the intensity of light scattered from x'' to$x$ by a patch of surface at x'

The equation is very much in the spirit of the radiosity equation, simply balancing the energy flows from one point of a surface to another. The equation states that the transport intensity of light from one surface point to another is simply the sum of the emitted light and the total light intensity which is scattered toward x from all other surface points. Equation (1) differs from the radiosity equation of course because, unlike the latter, no assumptions are made about reflectance characteristics of the surfaces involved.

Each of the quantities in the equation are new quantities which we call *unoccluded multipoint transport* quantities. In section 2 we define each of these quantities and relate them to the more conventional quantities encountered in radiometry.

The integral is taken over $S = \bigcup S_i$, the union of all surfaces. Thus the points x, x', and x'' range over all the surfaces of all the objects in the scene. We also include a global background surface S_0, which is a hemisphere large enough to act as an enclosure for the entire scene. Note that the inclusion of a enclosure surface ensures that the total positive hemisphere for reflection and total negative hemisphere for transmission are accounted for.

As an approximation to Maxwell's equation for electromagneticseq. (1) does not attempt to model all interesting optical phenomena. It is essentially a geometrical optics approximation. We only model time averaged transport intensity, thus no account is taken of phase in this equation—ruling out any treatment of diffraction. We have also assumed that the media between surfaces is of homogeneous refractive index and does not itself participate in the scattering light. The latter two cases can be handled by a pair of generalizations of eq. (1). In the first case, simply by letting $g(x,x')$ take into account the eikonal handles media with nonhomogenous refractive index. For participating propagation media, a integro-differential equation is necessary. Extensions are again well known, see [Chandrasekar 1950], and for use in a computer graphics application [Kajiya and von Herzen 1984]. Elegant ways of viewing the eikonal equation have been available for at least a century with Hamilton-Jacobi theory [Goldstein 1950]. Treatments of participatory media and of phase and diffraction can be handled with path integral techniques. For a treatment of such generalizations concerned with various physical phenomena see [Feynman and Hibbs 1965]. Finally, no wavelength or polarization dependence is mentioned in eq. (1). Inclusion of wavelength and polarization is straightforward and to be understood.

2. Discussion of transport quantities

We discuss each of the quantities and terms of equation (1). This equation describes the intensity of photon transport for a simplified model. $I(x,x')$ measures the energy of radiation passing from point x' to point x. We shall name $I(x,x')$ the *unoccluded two point transport intensity* from x' to x, or more compactly the *transport intensity*. The transport intensity $I(x,x')$ is the energy of radiation per unit time per

directions for making pictures using numbers
(explained using only the ten hundred words people use most often)

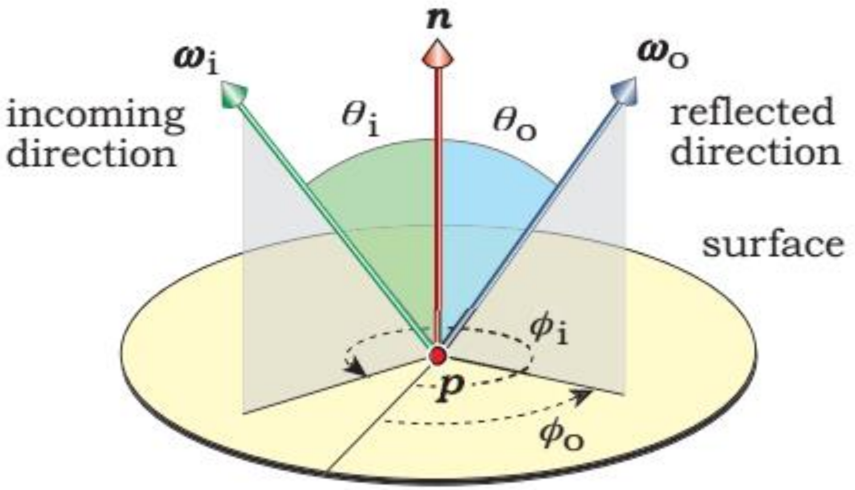
$$\underbrace{L_o(\overset{\substack{\text{position on}\\\text{the stuff}}}{x},\ \overset{\substack{\text{direction towards}\\\text{the eye}}}{\omega_o})}_{\substack{\text{light that leaves the}\\\text{position on the stuff and}\\\text{reaches the eye}}} = \overbrace{L_e(x,\ \omega_o)}^{\substack{\text{light made by the stuff}\\\text{(sometimes because}\\\text{it is very hot)}}} \underset{\substack{\text{light can be added}\\\text{said a man who sat under}\\\text{a tree many years ago}}}{+} \underbrace{\int_\Omega \overbrace{f_r(x,\ \omega_i \rightarrow \omega_o)}^{\substack{\text{the answer to how much light from an}\\\text{interesting direction that will keep going}\\\text{in the direction towards the eye, after}\\\text{hitting stuff at the position (this is easy}\\\text{for mirrors, not so easy for everything else)}}}\ \overbrace{L_i(x,\ \omega_i)}^{\substack{\text{the light that comes from}\\\text{an interesting direction}\\\text{towards the position on the}\\\text{stuff}}}\ \overbrace{(\omega_i \cdot n)}^{\substack{\text{how much the light}\\\text{becomes less bright}\\\text{because the stuff leans}\\\text{away from the}\\\text{interesting direction}}}\ d\omega_i}_{\substack{\text{for lots of interesting directions}\\\text{inside half a ball facing up}\\\text{from the stuff, add up all the}\\\text{answers in between}}}$$

this idea came from http://xkcd.com/1133/

@levork

Bidirectional Reflectance Distribution Functions (BRDFs)

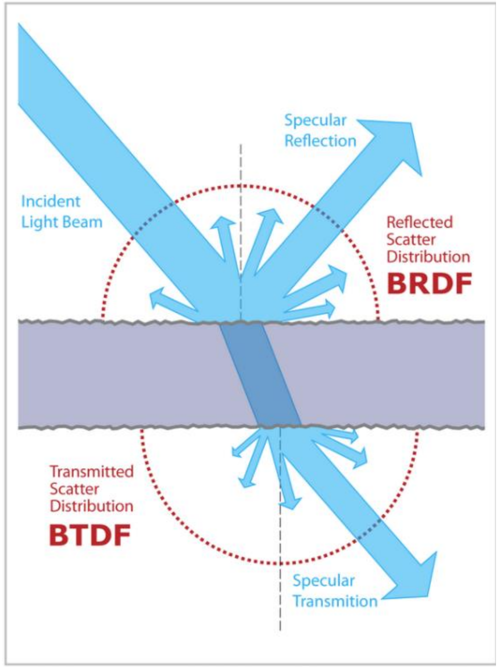
Precise description of surface reflectance at a point



Lots of variations of this term with sometimes slightly different meanings

Spatially Varying Bidirectional Reflectance Distribution Function (SVBRDF) which typically is used interchangeably with BRDF

Or…**BSDF** (**bidirectional scattering distribution function**) which combines transmission and reflection



BRDF is a function that takes:

- incoming light direction ω_i
- outgoing direction ω_o
- a point p on surface

Returns the ratio of reflected radiance exiting along ω_o to the irradiance incident on the surface from direction ω_i

The Rendering Equation

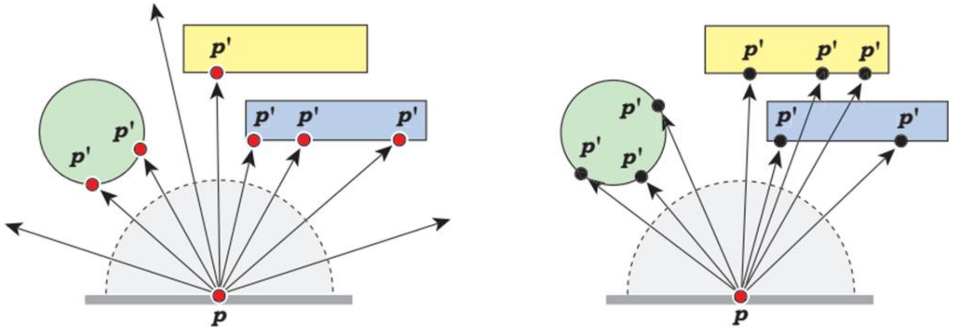
Expresses the steady-state energy-balance in a scene

- We have emissive surfaces
- And reflective surfaces

$$L_o(p,\omega_o) = L_e(p,\omega_o) + \int_{2\pi^+} f_r(p,\omega_i,\omega_o)\, L_i(p,\omega_i) \cos\theta_i \, d\omega_i$$

- Hemisphere form of the rendering equation
- Integration is in terms of solid angle over the hemisphere at point p
- States exitant radiance is the sum of reflected and emitted radiance

The Rendering Equation



$$L_o(\boldsymbol{p},\omega_o) = L_e(\boldsymbol{p},\omega_o) + \int_A f_r(\boldsymbol{p},\omega_i,\omega_o) L_o(\boldsymbol{p}',-\omega_i) V(\boldsymbol{p},\boldsymbol{p}') G(\boldsymbol{p},\boldsymbol{p}')\, dA,$$

We can also express the rendering equation in an *area form*

- Integral over all surfaces in a scene

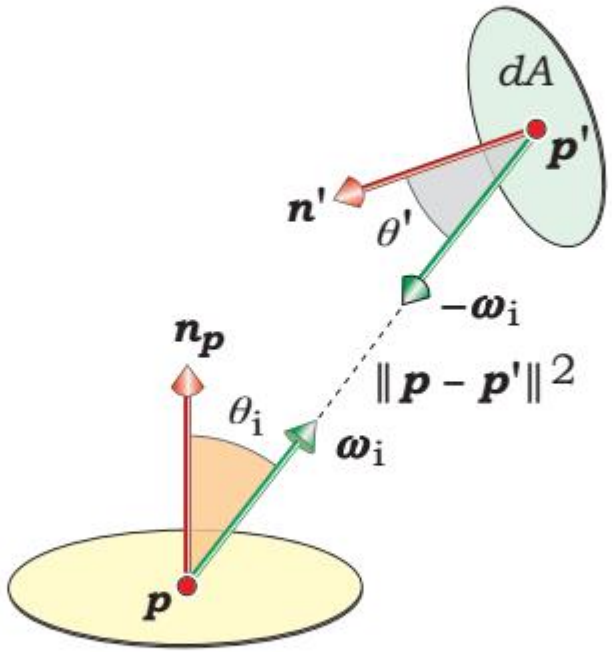
- V is a visibility function…

$$\forall(\boldsymbol{p},\boldsymbol{p}') \in A : V(\boldsymbol{p},\boldsymbol{p}') = \begin{cases} 1 & \text{if } \boldsymbol{p} \text{ and } \boldsymbol{p}' \text{ can see each other,} \\ 0 & \text{if } \boldsymbol{p} \text{ and } \boldsymbol{p}' \text{ cannot see each other.} \end{cases}$$

- G is the geometry term…

$$G(p,p') = \frac{\cos q_i \cos q'}{\left\| p' - p \right\|^2}$$

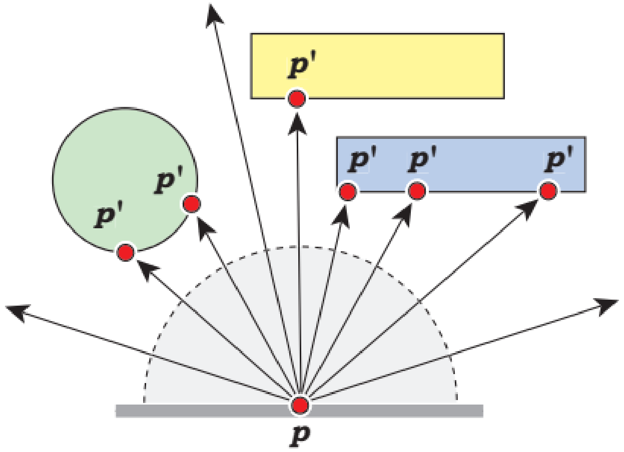
The Geometry Term

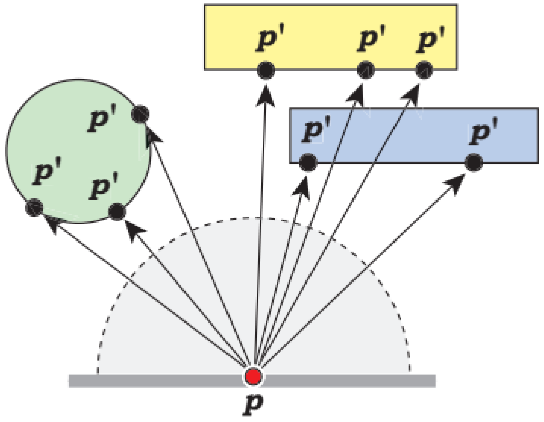


$$G(p,p') = \cos\theta_\mathrm{i} \cos\theta' / \|p' - p\|^2$$

Rendering Equation

Fundamental difference between are and hemisphere formulations





- Hemisphere form samples directions
- Area form has directions determined by sample points on other surfaces

Pragmatically, area form useful for direct illumination from area lights

Solving The Rendering Equation

$$L_o(p,\omega_o) = L_e(p,\omega_o) + \int_{2\pi^+} f_r(p,\omega_i,\omega_o) L_o(r_c(p,\omega_i),-\omega_i)\cos\theta_i \, d\omega_i$$

Seems simple….has L_o as the only unknown

- Fredholm integral equation of the second kind
  - L_o is on both sides and inside the integral
- Equation is recursive
- Not solvable analytically

$$\int_{X_1}^{X_2} a_i x^i = a_i \frac{X_2^{i+1} - X_1^{i+1}}{i+1}.$$

Example of an analytical solution

Approximate Solutions

Define a linear integral operator

$$K \circ L_o = \int_{2\pi^+} f_r(p,\omega_i,\omega_o) L_o(r_c(p,w_i),-w_i)\cos\theta_i \, d\omega_i$$

- L_o is an exitant radiance distribution in solid angle at p
- Operator converts L_o into another radiance distribution
- The rendering equation can be rewritten as

$$L_o = L_e + K \circ L_o$$

Approximate Solutions

The rendering equation can be rewritten as

$$L_o = L_e + K \circ L_o$$

We can use repeated substitution to get

$$L_o = L_e + K \circ (L_e + K \circ L_o)$$

$$L_o = \sum_{j=0}^{\infty} K^j \circ L_e$$

Which can be solved by simulating all light paths in a scene

Approximate Solutions

Ray-casting uses only the first two terms in the series

$$L_o = L_e + K \circ (L_e + K \circ L_o)$$

Approximate solutions can be pretty good

- Surfaces do not reflect all light that hits them
- A finite number of bounces can be a good approximation
- For shading with point lights and perfect specular reflection
    - The integrals educe to simple expressions
    - Whitted-style ray-tracing

Path Tracing is name used to describe better solutions to the series

- We can approximate solutions to the integral using *Monte Carlo integration*