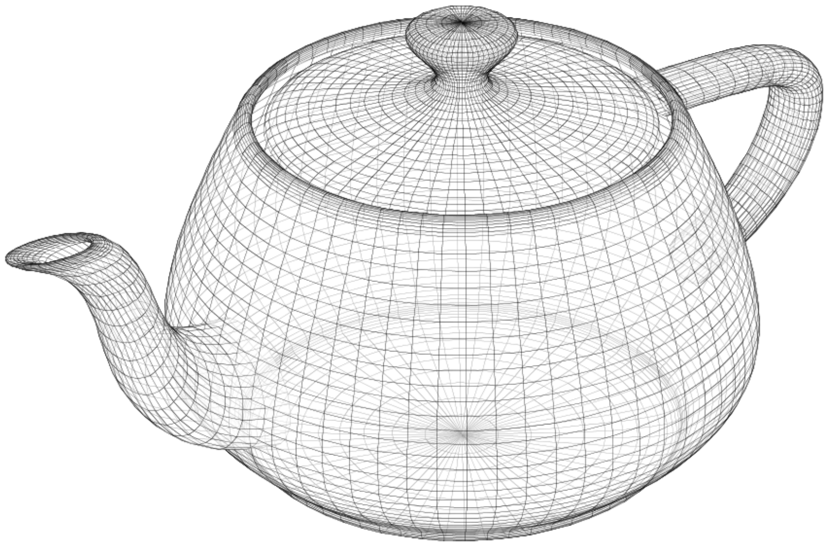
Images and some text courtesy of
The Essentials of CAGD by Farin and Hansford



Geometric Design: Bezier Patches

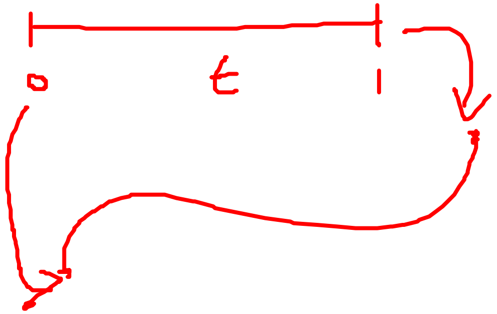
Professor Eric Shaffer

Bezier Patches

The Utah teapot model
Created with Bezier patches by Martin Newell in 1975

Parametric Surfaces

Parametric curve: mapping of the real line into 2- or 3-space

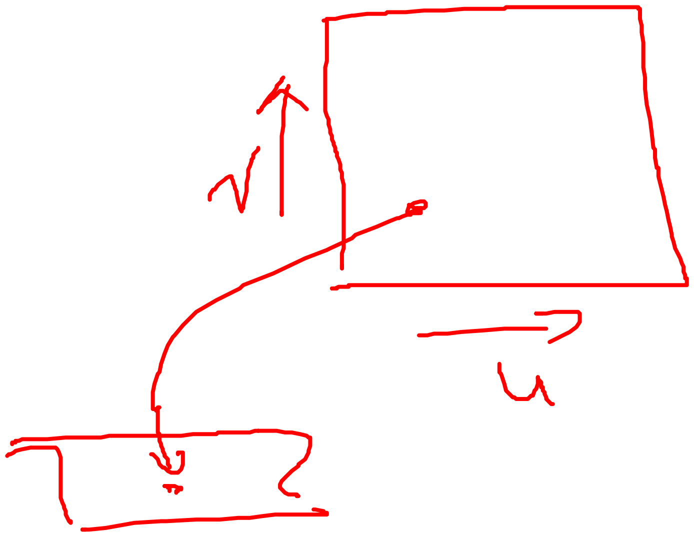


Parametric surface: mapping of the real plane into 3-space

$\mathbb{R}^2$ is the domain of the surface
– A plane with a (u, v) coordinate system

Corresponding 3D surface point:

$$\mathbf{x}(u, v) = \begin{bmatrix} f(u, v) \\ g(u, v) \\ h(u, v) \end{bmatrix}$$



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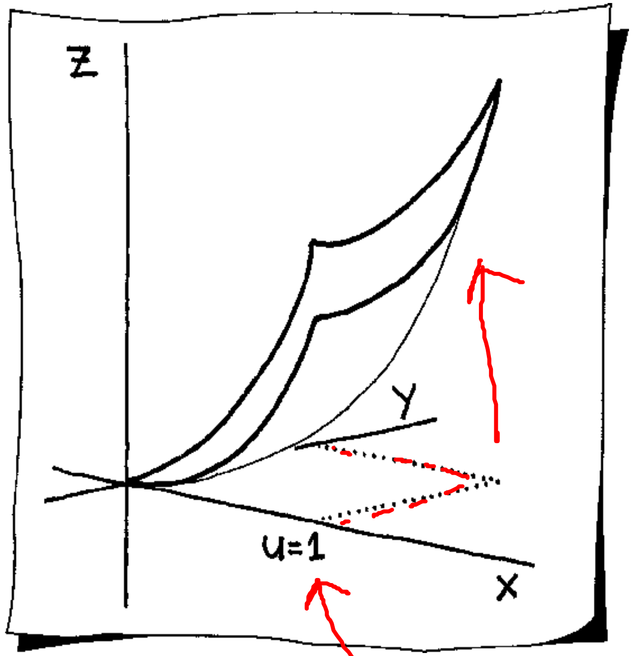
Example: Parametric Surface



$$\mathbf{x}(u,v) = \begin{bmatrix} u \\ v \\ u^2 + v^2 \end{bmatrix}$$

This is also functional surface

- two of the coordinate functions are simply *u* and *v*

Parametric surfaces may be rotated or moved around

Much more general than bivariate functions $z = f(x, y)$

Why are parametric forms more general? Think about a graph of a function versus a parametric curve..

x cos
y sin

Bilinear Patches

Typically interested in a finite piece of a parametric surface
– The image of a rectangle in the domain

The finite piece of surface called a patch

Let domain be the *unit square*

$$\{(u, v) : 0 \leq u, v \leq 1\}$$

Map it to a surface patch defined by four points

$$\mathbf{x}(u, v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \mathbf{b}_{0,1} \\ \mathbf{b}_{1,0} & \mathbf{b}_{1,1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}$$

Surface patch is linear in both the u and v parameters
$\Rightarrow$ *bilinear patch*

Bilinear Patches

Bilinear patch:

$$\mathbf{x}(u,v) = \begin{bmatrix} 1-u & u \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \mathbf{b}_{0,1} \\ \mathbf{b}_{1,0} & \mathbf{b}_{1,1} \end{bmatrix} \begin{bmatrix} 1-v \\ v \end{bmatrix}$$

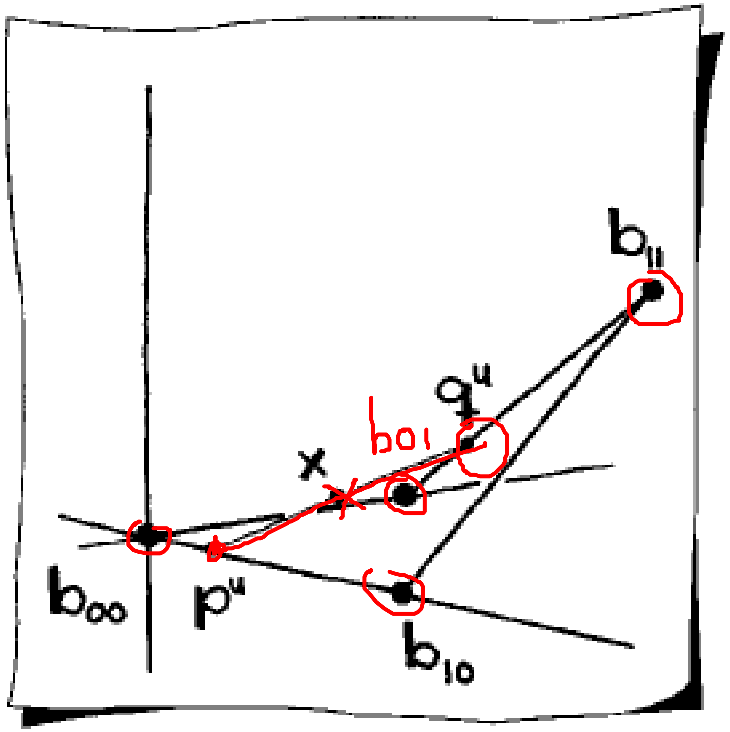
Rewrite as

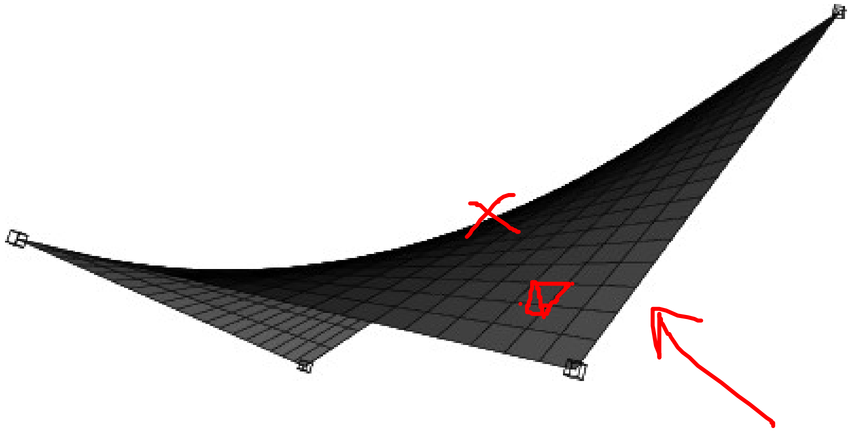
$$\mathbf{x}(u,v) = (1-v)\mathbf{p}^u + v\mathbf{q}^u$$

$$\mathbf{p}^u = (1-u)\mathbf{b}_{0,0} + u\mathbf{b}_{1,0} \quad \text{and} \quad \mathbf{q}^u = (1-u)\mathbf{b}_{0,1} + u\mathbf{b}_{1,1}$$

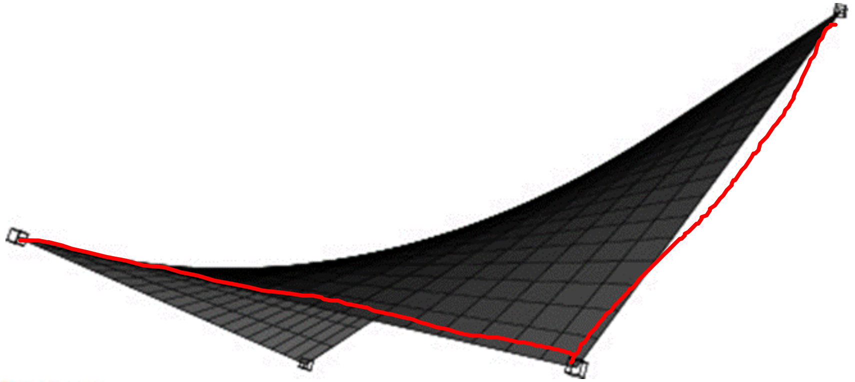
$\Rightarrow$ Better feeling for the shape of the bilinear patch

Bilinear Patch: Example





Isoparametric Curves



Bilinear patch also called a hyperbolic paraboloid

Isoparametric curve: only one parameter is allowed to vary

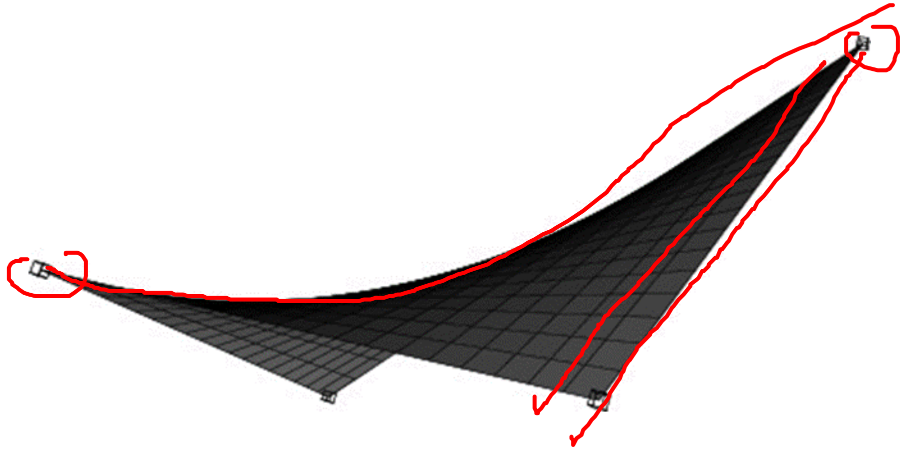
Isoparametric curves on a bilinear patch $\Rightarrow$ 2 families of straight lines
$(\bar{u}, v)$: line constant in u but varying in v
$(u, \bar{v})$: line constant in v but varying in u

Four special isoparametric curves (lines):

$$(u, 0) \qquad (u, 1) \qquad (0, v) \qquad (1, v)$$

Curves on Patches

A hyperbolic paraboloid also contains *curves*



Consider the line $u = v$ in the domain – the diagonal
As a parametric line: $u(t) = t,\ v(t) = t$
This domain diagonal is mapped to the 3D curve on the surface

$$\mathbf{d}(t) = \mathbf{x}(t, t)$$

In more detail:

$$\mathbf{d}(t) = \begin{bmatrix} 1-t & t \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \mathbf{b}_{0,1} \\ \mathbf{b}_{1,0} & \mathbf{b}_{1,1} \end{bmatrix} \begin{bmatrix} 1-t \\ t \end{bmatrix}$$

Collecting terms now gives

$$\mathbf{d}(t) = (1-t)^2\mathbf{b}_{0,0} + 2(1-t)t[\frac{1}{2}\mathbf{b}_{0,1} + \frac{1}{2}\mathbf{b}_{1,0}] + t^2\mathbf{b}_{1,1}$$

$\Rightarrow$ quadratic Bézier curve

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Bezier Patches

Bilinear patch using linear Bernstein polynomials:

$$\mathbf{x}(u,v) = \begin{bmatrix} \overset{1-u}{B_0^1(u)} & \overset{u}{B_1^1(u)} \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \mathbf{b}_{0,1} \\ \mathbf{b}_{1,0} & \mathbf{b}_{1,1} \end{bmatrix} \begin{bmatrix} B_0^1(v) \\ B_1^1(v) \end{bmatrix}$$

Generalization:

$$\mathbf{x}(u,v) = \begin{bmatrix} B_0^m(u) & \dots & B_m^m(u) \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \dots & \mathbf{b}_{0,n} \\ \vdots & & \vdots \\ \mathbf{b}_{m,0} & \dots & \mathbf{b}_{m,n} \end{bmatrix} \begin{bmatrix} B_0^n(v) \\ \vdots \\ B_n^n(v) \end{bmatrix}$$

Examples: $m = n = 1$: bilinear $m = n = 3$: *bicubic*

4x4

Expanding:

$$\mathbf{x}(u,v) = \mathbf{b}_{0,0} B_0^m(u) B_0^n(v) + \ldots + \mathbf{b}_{i,j} B_i^m(u) B_j^n(v) + \ldots + \mathbf{b}_{m,n} B_m^m(u) B_n^n(v)$$

What is the shape of the control point matrix for a bicubic patch (i.e. how many rows and columns?

What kind of data is each entry in the matrix?

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Bezier Patches

$$\mathbf{x}(u,v) = \begin{bmatrix} B_0^m(u) & \dots & B_m^m(u) \end{bmatrix} \begin{bmatrix} \mathbf{b}_{0,0} & \dots & \mathbf{b}_{0,n} \\ \vdots & & \vdots \\ \mathbf{b}_{m,0} & \dots & \mathbf{b}_{m,n} \end{bmatrix} \begin{bmatrix} B_0^n(v) \\ \vdots \\ B_n^n(v) \end{bmatrix}$$

Abbreviated as

$$\mathbf{x}(u,v) = M^{\mathrm{T}}\mathbf{B}N$$

$\Rightarrow$ surface generalization of the curve equation

Evaluate at a parameter pair (u, v):

$$\mathbf{C} = M^{\mathrm{T}}\mathbf{B} = [\mathbf{c}_0, \dots, \mathbf{c}_n]$$

Then final result

$$\mathbf{x}(u,v) = \mathbf{C}N$$

Call this the 2-stage explicit evaluation method
– Bernstein polynomials are explicitly evaluated

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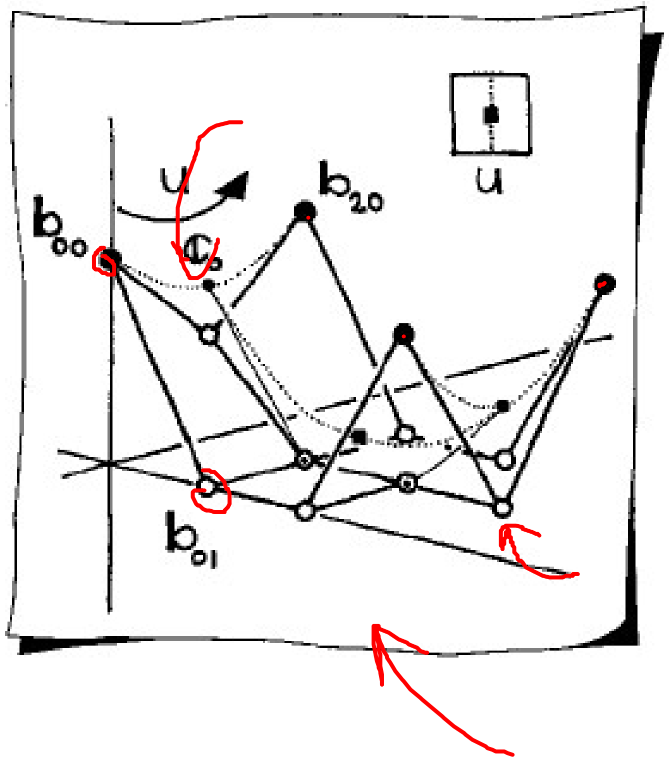
Bezier Patches

$$\mathbf{x}(u, v) = M^{\mathrm{T}}\mathbf{B}N \quad \Rightarrow \quad \mathbf{x}(u, v) = \mathbf{C}N$$



Control points $\mathbf{c}_0, \ldots, \mathbf{c}_n$ of $\mathbf{C}$ do not depend on the parameter value v

Curve $\mathbf{C}N$: curve on surface
– Constant u
– Variable v
$\Rightarrow$ isoparametric curve or isocurve

Meaning that the value of v will determine a point on the curve CN

Bezier Patches: Example

Example: Evaluate the 2×3 control net at $(u, v) = (0.5, 0.5)$

Here, 2x3 refers to the degree of the polynomials…the control points form a matrix with 3 rows and 4 columns

$$\mathbf{B} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \\ 0 \\ 3 \\ 3 \\ 0 \\ 6 \\ 6 \end{bmatrix} & \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \\ 3 \\ 0 \\ 3 \\ 6 \\ 0 \end{bmatrix} & \begin{bmatrix} 6 \\ 0 \\ 0 \\ 6 \\ 3 \\ 0 \\ 6 \\ 6 \\ 0 \end{bmatrix} & \begin{bmatrix} 9 \\ 0 \\ 6 \\ 9 \\ 3 \\ 0 \\ 9 \\ 6 \\ 6 \end{bmatrix} \end{bmatrix}$$

Step 1) Compute quadratic Bernstein polynomials for $u = 0.5$:

$$M^{\mathrm{T}} = \begin{bmatrix} 0.25 & 0.5 & 0.25 \end{bmatrix}$$

$\Rightarrow$ Intermediate control points

$$\mathbf{C} = M^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4.5 \end{bmatrix} & \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} & \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} & \begin{bmatrix} 9 \\ 3 \\ 3 \end{bmatrix} \end{bmatrix}$$

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Bezier Patches: Example

Step 2) Compute cubic Bernstein polynomials for $v = 0.5$:

$$N = \begin{bmatrix} 0.125 \\ 0.375 \\ 0.375 \\ 0.125 \end{bmatrix}$$

$$\mathbf{x}(0.5, 0.5) = \mathbf{C}N = \begin{bmatrix} 4.5 \\ 3 \\ 0.9375 \end{bmatrix}$$

Properties of Bezier Patches

Bézier patches properties essentially the same as the curve ones

1. Endpoint interpolation:
   - Patch passes through the four corner control points
   - Control polygon boundaries define patch boundary curves
2. Symmetry:
   Shape of patch independent of corner selected to be $\mathbf{b}_{0,0}$
3. Affine invariance:
   Apply affine map to control net and then evaluate identical to applying affine map to the original patch
4. Convex hull property:
   $\mathbf{x}(u, v)$ in the convex hull of the control net for $(u, v) \in [0, 1] \times [0, 1]$

One last question:
How would we generate triangles to render from a Bezier patch?